Monmouth COLLEGE

- Name: ____________________
- Date: ____________________
- Section: ____________________

ECON 300: Intermediate Price Theory

Problem Set #6

INSTRUCTIONS:

- This problem set is not graded.

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Problem 1. Budget Constraints

Suppose that you as the consumer are participating in a market for good x and good y. Each unit of good x is \$5, and each unit of good y is \$4. Your income to spend on purchasing good x and y is \$200.

1.A Find the mathematical expression for your budget constraint.

Solution:

$$P_x x + P_y y = M \quad \Rightarrow \quad \boxed{5x + 4y = 200}$$

1.B If you spent all of your money on good y, how many units can you purchase?

Solution:

$$\frac{M}{P_y} = \frac{200}{4} = 50$$

1.C Calculate the slope of the budget constraint.

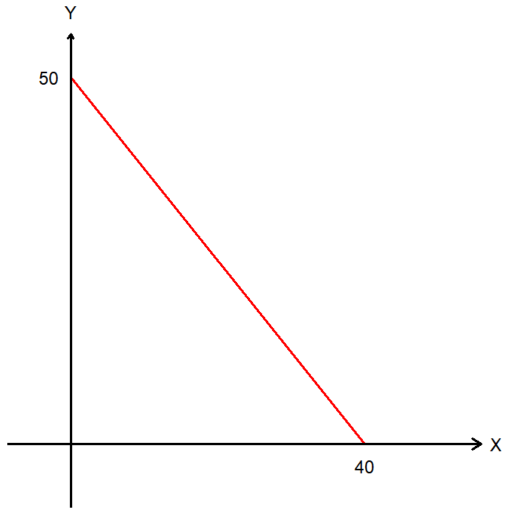
Solution:

$$\frac{P_x}{P_y} = \frac{5}{4}, \quad \text{or} \quad \frac{\text{Rise}}{\text{Run}} = \frac{50}{40} = \frac{5}{4}$$

1.D Interpret the meaning of the slope of the budget constraint by completing the following statement:

`If I am already spending all of my money,`
`should I purchase one extra unit of good` x,
`I must sell off` <u>1.25</u> `units of good` y.

1.E Plot your budget constraint below:



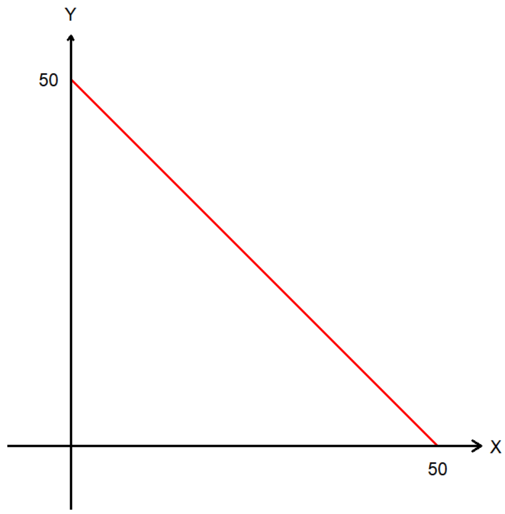
- The budget line.
- The x intercept value.
- The y intercept value.

1.F Find the mathematical expression for your budget constraint when the price of good x falls to 4.

Solution:

$$P_x'x + P_y y = M \quad \Rightarrow \quad \boxed{4x + 4y = 200}$$

1.G Plot the updated budget constraint from 1.F below, and describe how it is different from your answer in 1.E.



The budget set has expanded in the form of the budget line pivoting.

Problem 2. Preferences and Utility

2.A Fill out the following table using the definition of utility and marginal utility.

Quantity	Utility	Marginal Utility
1	100	-
2	190	90
3	270	80
4	340	70
5	350	10

2.B Describe the concept of the Law of Diminishing Marginal Utility.

Solution:

The law of diminishing marginal utility states that the more of a certain good that the consumer is consuming, the marginal utility will be lower. That is, an extra $100 will make anyone happy, but the additional happiness this $100 brings is much greater to a middle-class individual compared to a multi-millionare.

2.C Describe what it means for utility to be an ordinal concept.

Solution:

Ordinal utility means that it is not the magnitude of the utility values, but their relative rankings that matters. If some bundle A gives the consumer 10 utility, and another bundle B gives 100 utility, it does not mean this consumer prefers B 10 times more than A. Simply that the consumer prefers B over A.

2.D Suppose you have two utility functions $u_1(x, y) = 2xy$ and $u_2(x, y) = 4xy$. Can these two utility functions represent the same underlying preference relation?

Solution:

Yes, since utility is ordinal, as long as the relative ordering of the utility values are preserved they will represent the same underlying utility.

2.E Identify and briefly explain the two axioms that are required for a preference relationship to be rational?

Solution:

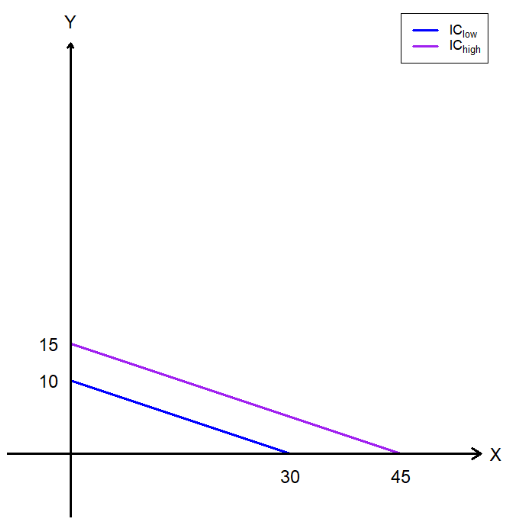
Preference relations are rational if it obeys the axioms of completeness and transitivity. When a preference relationship is complete, the consumer must be able to tell which they prefer out of any two bundles. When a preference relationship obeys transitivity, it means that when $A \succsim B$, and $B \succsim C$, then $A \succsim C$.

2.F Suppose that our preferences follow the rule that "when $A \succsim B$, and $B \succsim C$, then $A \succsim C$." Which axiom does this refer to?

Solution:

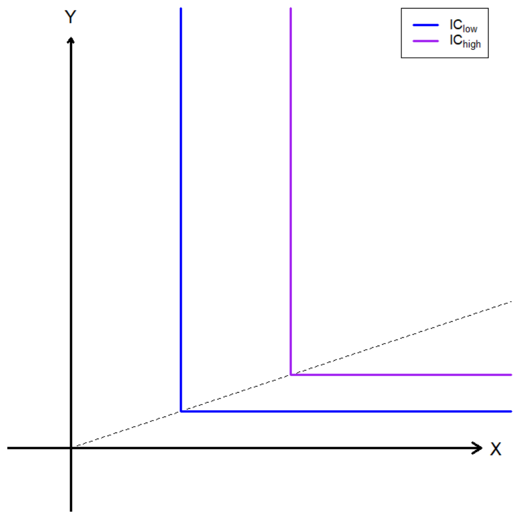
Transitivity.

2.G Suppose that consumers believe that goods x and y are completely interchangeable at a 3:1 ratio. That is, every 3 units of good x is perfectly substituted by 1 unit of good y. Provide a rough plot of the indifference curves.



- Two levels of indifference curves.
- Label the indifference curve representing the lower utility as IC_{low}.
- Label the indifference curve representing the greater utility as IC_{high}.
- Intercept values / ratio indicators.

2.H Suppose that consumers believe that goods x and y must be used strictly following a 3:1 ratio. That is, every 3 units of good x must be accompanied by 1 unit of good y. Provide a rough plot of the indifference curves.



- Two levels of indifference curves.
- Label the indifference curve representing the lower utility as IC_{low}.
- Label the indifference curve representing the greater utility as IC_{high}.
- Intercept values / ratio indicators.

2.I What is the definition of the marginal rate of substitution? How would you find the marginal rate of substitution on a graph?

Solution:

The marginal rate of substitution is defined as $MRS_{xy} = MU_x/MU_y$, and it measures the maximum units of good y that the consumer is willing to give up for one extra units of good x, conditional on utility remaining constant. On a graph, it is the slope of the indifference curve.

2.J Suppose that the MRS_{xy} is measured to be 2. Complete the following statement:

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While keeping the consumer's utility constant,
when the consumer gets one extra unit of good x,
they will be willing to give up _2_ units of good y.
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Problem 3. Utility Maximization: Leontief

Suppose you are participating in a market with goods x and y. The market price of good x and y are P_x and P_y, respectively and your income is M. Assume that your utility function is given as:

$$u(x, y) = min\{2x, 3y\}$$

3.A What happens to utility when $2x > 3y$?

Solution:

When $2x > 3y$, then the consumer's utility is $u(x, y) = 3y$, and some units of good x are being "wasted" in the sense that it is not capable of generating utility for the consumer. Thus, this is not an optimal way to allocate resources.

3.B What happens to utility when $2x < 3y$?

Solution:

When $2x < 3y$, then the consumer's utility is $2x$, and some units of good y are being "wasted".

3.C Is there an optimal ratio of goods x and y?
If so, what is the optimal ratio? If not, what is the rule to follow?

Solution:

From the answers to 3.A and 3.B, we found that $2x = 3y$ is the only way to allocate resources such that there is no "waste" of goods. So, the optimality condition to follow is to set $2x = 3y$.

3.D What is the demand function for good x?

Solution:

We know that the optimal ratio is $2x = 3y$, and that the budget constraint is $P_x x + P_y y = M$. Use these two equations to solve a system of two equations and two unknowns:

$$\begin{aligned}
P_x x + P_y y = M \quad &\Rightarrow \quad P_x x + P_y \left(\frac{2}{3}x\right) = M \qquad \because 2x = 3y \\
&\Rightarrow \quad x \cdot \frac{3P_x + 2P_y}{3} = M \\
&\Rightarrow \quad x^*(P_x, P_y, M) = \frac{3M}{3P_x + 2P_y}
\end{aligned}$$

Problem 4. Utility Maximization: Linear

Suppose you are participating in a market with goods x and y. The market price of good x and y are $P_x = 10$ and $P_y = 5$, respectively and your income is $M = 100$. Assume that your utility function is given as:

$$u(x, y) = 5x + 2y$$

4.A Calculate the marginal utility of x and y.

Solution:

- $MU_x = \frac{\partial}{\partial x}(5x + 2y) = \frac{\partial}{\partial x}5x + \frac{\partial}{\partial x}2y = 5 + 0 = 5$
- $MU_y = \frac{\partial}{\partial y}(5x + 2y) = \frac{\partial}{\partial y}5x + \frac{\partial}{\partial y}2y = 0 + 2 = 2$

4.B Is there an optimal ratio of goods x and y?
If so, what is the optimal ratio? If not, what is the rule to follow?

Solution:

Since goods x and y are perfect substitutes, there is no "optimal ratio" to follow. Instead:

$$\frac{MU_x}{P_x} = \frac{5}{10} > \frac{2}{5} = \frac{MU_y}{P_y}$$

Since good x gives our consumer more utility per dollar, they should exclusively purchase good x.

4.C What is the amount of good x that the consumer should purchase?

Solution:

$$\frac{M}{P_x} = \frac{100}{10} = 10$$

4.D What happens to the optimal amount of good x when P_y falls to 3?

Solution:

$$\frac{MU_x}{P_x} = \frac{5}{10} < \frac{2}{3} = \frac{MU_y}{P'_y}$$

Now good y gives our consumer more utility per dollar, they should shift over to exclusively purchasing good y.

Problem 5. Utility Maximization: Cobb-Douglas

Suppose you are participating in a market with goods x and y. The market price of good x and y are P_x and P_y, respectively and your income is M. Assume that your utility function is given as:

$$u(x, y) = 3xy^2$$

5.A Calculate the marginal utility of x and y.

Solution:

- $MU_x = \frac{\partial}{\partial x} 3xy^2 = 3y \cdot 1 \cdot x^{1-1} = 3y^2$
- $MU_y = \frac{\partial}{\partial y} 3xy^2 = 3x \cdot 2 \cdot y^{2-1} = 6xy$

5.B Assuming that $MU_x = y$ and $MU_y = 2x$, find the marginal rate of substitution.

Solution:

$$MRS_{xy} = \frac{MU_x}{MU_y} = \frac{y}{2x}$$

5.C Is there an optimal ratio of goods x and y?
If so, what is the optimal ratio? If not, what is the rule to follow?

Solution:

When the consumer's utility function is Cobb-Douglas:

$$MRS_{xy} = \frac{P_x}{P_y} \quad \Rightarrow \quad \frac{y}{2x} = \frac{P_x}{P_y} \quad \Rightarrow \quad \boxed{P_y y = 2P_x x}$$

5.D What is the demand function for good x?

Solution:

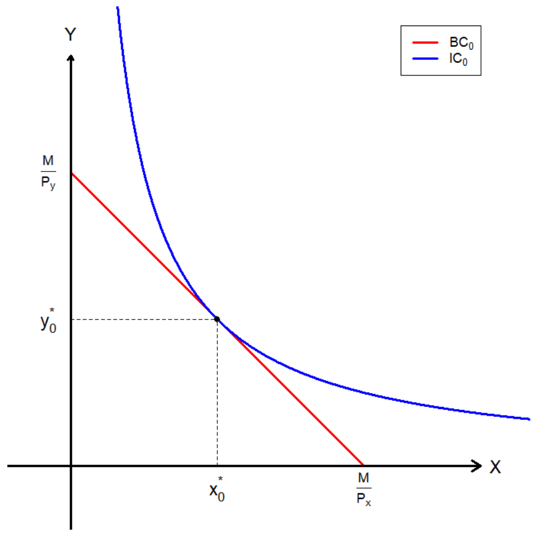
Using the optimal ratio from 5.C, and the budget constraint, we find:

$$\begin{aligned} P_x x + P_y y = M \quad &\Rightarrow \quad P_x x + 2P_x x = M && \because P_y y = 2P_x x \\ &\Rightarrow \quad 3P_x x = M \\ &\Rightarrow \quad x^*(P_x, P_y, M) = \frac{M}{3P_x} \end{aligned}$$

Problem 6. Comparative Statics of the UMP

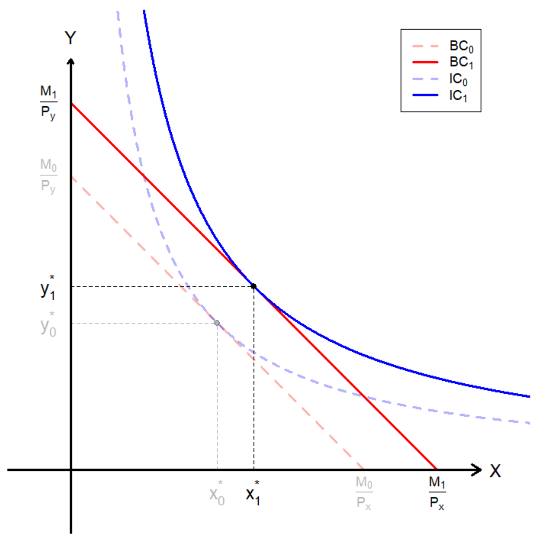
Suppose that the consumer's utility function is Cobb-Douglas, and that the price of good x is P_x, the price of good y is P_y, and income is M.

6.A Depict the consumer's utility maximization problem in the empty chart below. You must plot and label the following elements:



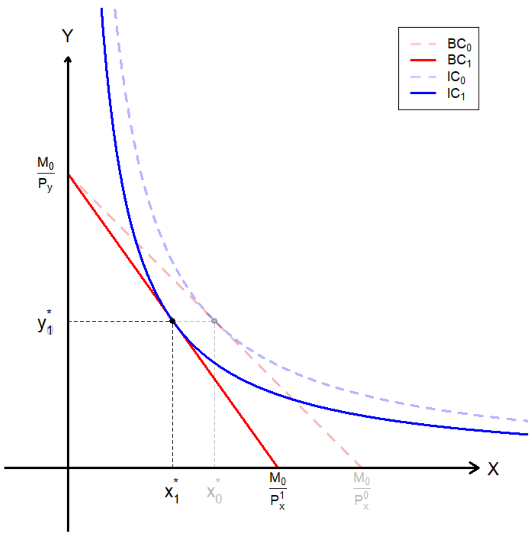
- The budget constraint (approximate).
- The indifference curve (approximate).
- The optimal bundle. The bundle's coordinates need not be exact.
- All intercept values.

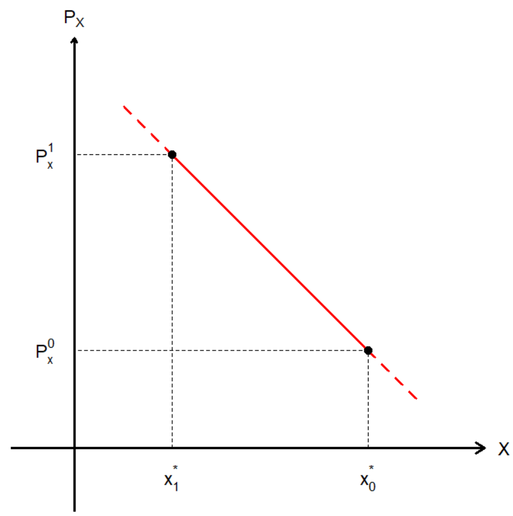
6.B Suppose good x is a normal good, and that that the consumer's income increased to M'. Depict what would happen to the visualization of the consumer's utility maximization problem in the empty chart below:



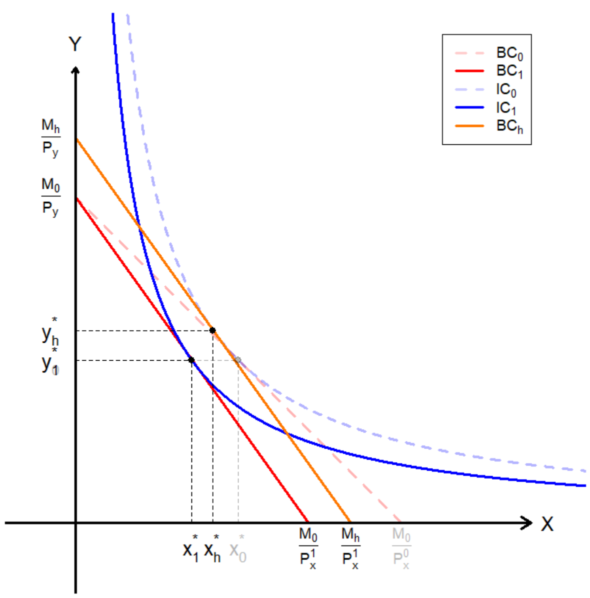
- The original and new budget constraint (approximate).
- The original and new indifference curve (approximate).
- The original and new optimal bundle. The bundle's coordinates need not be exact.
- All intercept values.

6.C Suppose that good x is an ordinary good, and that the price of good x increased. In the diagram to the left, plot the individual consumer's utility maximization problem following the rules of 6.B. To the right, plot the individual consumer's demand curve based on your answers on the diagram to the left. The diagrams need not be to scale.





6.D Suppose good x is a ordinary good, and that that the price of good x increased. Plot the diagram that demonstrates the Hicksian income and substitution effects in the empty chart below:



- The original and new budget constraint.
- The "Hicksian" budget constraint.
- The original and new indifference curve (approximate).
- The original and new optimal bundle. The bundle's coordinates need not be exact.
- The "Hicksian" optimal bundle.
- The substitution and income effects in arrows.

Problem 7. Production Functions: Linear

Suppose you are a producer with production technology that allows you to perfectly substitute between labor and capital at a 1:2 ratio. Specifically, your production function takes the form:

$$F(L, K) = 2L + K$$

7.A How many units of capital is required to replace one unit of labor?

Solution:

The output when using either $(L = 0, K = 2)$ or $(L = 1, K = 0)$ are both 2. Therefore, two units of capital is required to replace one unit of labor.

7.B Calculate the marginal product of labor and capital.

Solution:

- $MP_L = \frac{\partial}{\partial L}(2L + K) = \frac{\partial}{\partial L}2L + \frac{\partial}{\partial L}K = 2 + 0 = 2$
- $MP_K = \frac{\partial}{\partial K}(2L + K) = \frac{\partial}{\partial K}2L + \frac{\partial}{\partial K}K = 0 + 1 = 1$

7.C Calculate the marginal rate of technical substitution. How does this relate to your answer in 7.A?

Solution:

$$MRTS_{LK} = \frac{MP_L}{MP_K} = \frac{2}{1} = 2$$

$MRTS$ measures how many units of capital is required to replace one unit of labor.

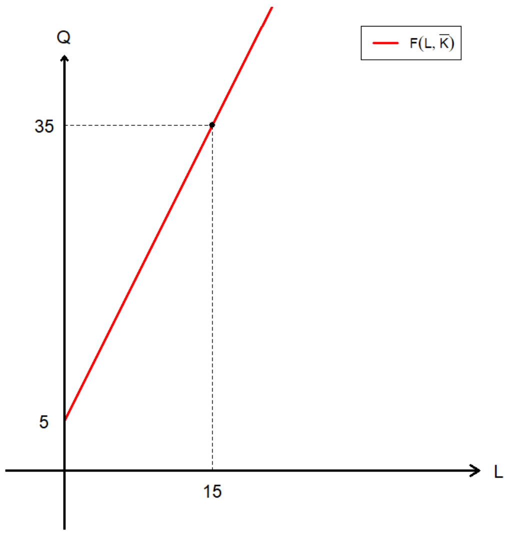
7.D Does this production technology exhibit increasing, decreasing, or constant returns to scale?

Solution:

$$F(\lambda L, \lambda K) = 2(\lambda L) + (\lambda K) = \lambda(2L + K) = \lambda F(L, K)$$

Therefore this production function exhibits constant returns to scale. As an alternative method, consider comparing the output when $L = 1, K = 1$ and when $L = 2, K = 2$.

7.E Suppose that in the short run, capital is fixed at 5. Plot the short run production function in the empty chart below:



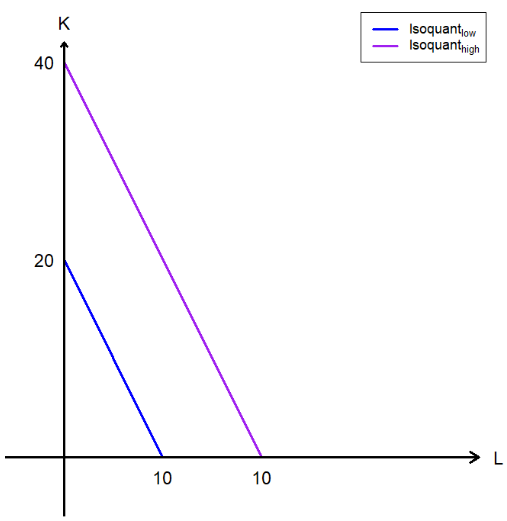
- The short run production function.
- The intercepts.

Solution:

$$F(L, \bar{K}) = 2L + 5 \quad \Rightarrow \quad Q = 2L + 5$$

Therefore the short run production function is a straight line with slope 2, passing through the point $Q = 5, L = 0$.

7.F Suppose now in the long run, capital is flexible. Plot the isoquant of the production technology in the empty chart below:



- Two levels of isoquants.
- Label the isoquant representing the lower level of production as IQ_{low}.
- Label the isoquant representing the greater level of production as IQ_{high}.
- Intercept values / ratio indicators.

Problem 8. Production Functions: Leontief

Suppose you are a producer with production technology that allows requires you to use labor and capital following a strict 3:2 mix. Specifically, for each 3 units of labor and 2 units of capital, you get 6 units of output. That is, your production function takes the form:

$$F(L, K) = min\{2L, 3K\}$$

8.A How many units of capital is required to replace one unit of labor?

Solution:

Given the current production technology, labor cannot replace any capital. Likewise, no amount of capital can replace labor. The inputs labor and capital are perfect complements in production.

8.B Does this production technology exhibit increasing, decreasing, or constant returns to scale?

Solution:

$$F(\lambda L, \lambda K) = min\{2(\lambda L), 3(\lambda K)\} = \lambda \cdot min\{2L, 3K\} = \lambda F(L, K)$$

Therefore this production function exhibits constant returns to scale. As an alternative method, consider comparing the output when $L = 1, K = 1$ and when $L = 2, K = 2$.

8.C Suppose that your production technology shifts to $F_1(L, K) = min\{4L, 6K\}$. Does this change represent technical progress?

Problem 9. Cost Functions

Suppose you are a producer of some good, and the only two inputs you use are labor and capital. Labor is paid wages (w), and capital is paid rent (r).

9.A What is the mathematical expression of your total cost function $TC(Q)$?

Solution:

$$TC(Q) = w \cdot L(w, r, Q) + r \cdot K(w, r, Q)$$

9.B How would you calculate your average cost function $ATC(Q)$?

Solution:

$$ATC(Q) = \frac{TC(Q)}{Q} = \frac{w \cdot L(w, r, Q) + r \cdot K(w, r, Q)}{Q}$$

9.C How would you calculate your marginal cost function $MC(Q)$?

Solution:

$$MC(Q) = \frac{dTC(Q)}{dQ} = \frac{d}{dQ}\{w \cdot L(w, r, Q) + r \cdot K(w, r, Q)\}$$

9.D In your own words, explain why the marginal cost curve will pass through the lowest point of the average cost curve.

Solution:

The marginal cost is the cost of producing one extra unit of output. The average cost is the per unit cost of producing output. When the cost of producing the very last unit of output (MC) is lower than the average cost of producing output, it will "pull down" the average cost. Meanwhile, then the cost of producing the very last unit of output is higher than the current average cost of production, it will increase the average cost.

Problem 10. Cost Minimization

Suppose you are a producer of some good, and the only two inputs you use are labor and capital. Wage is $w = 100$ and rent is $r = 50$. Your production technology is given as:

$$F(L, K) = L^2 K$$

10.A Find the marginal product of labor and capital.

Solution:

- $MP_L = \dfrac{\partial}{\partial L} L^2 K = K \cdot 2 \cdot L^{2-1} = 2LK$
- $MP_K = \dfrac{\partial}{\partial K} L^2 K = L^2 \cdot 1 \cdot K^{1-1} = L^2$

10.B Assuming $MP_L = 2K$ and $MP_K = L$, find the marginal rate of technical substitution.

Solution:

$$MRTS_{LK} = \frac{MP_L}{MP_K} = \frac{2K}{L}$$

10.C Is there an optimal ratio of inputs L and K?
If so, what is the optimal ratio? If not, what is the rule to follow?

Solution:

$$MRTS_{LK} = \frac{w}{r} \quad \Rightarrow \quad \frac{2K}{L} = \frac{100}{50} \quad \Rightarrow \quad 100K = 100L \quad \Rightarrow \quad \boxed{L = K}$$

10.D If your target output is $Q = 1,000$, what is your total cost of production?

Solution:

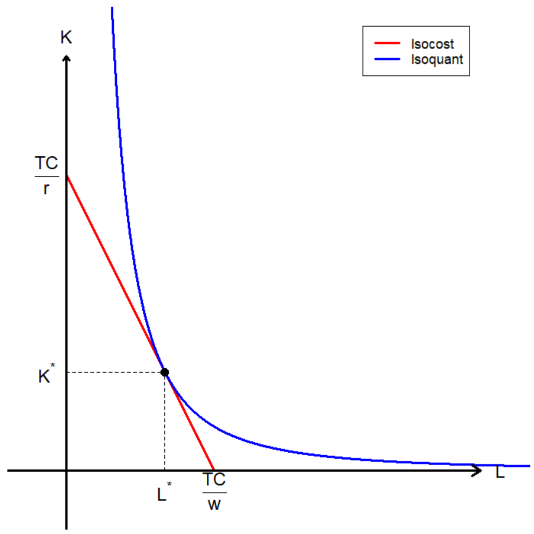
Using the optimal ratio from 10.C, and the production quota, we find:

$$\begin{aligned} L^2 K = 1000 \quad &\Rightarrow \quad L^2 \cdot L = 1000 && \because L = K \\ &\Rightarrow \quad L^3 = 1000 \\ &\Rightarrow \quad L^* = 10 \quad \Rightarrow \quad K^* = 10 && \because L = K \end{aligned}$$

Then, the cost function will be:

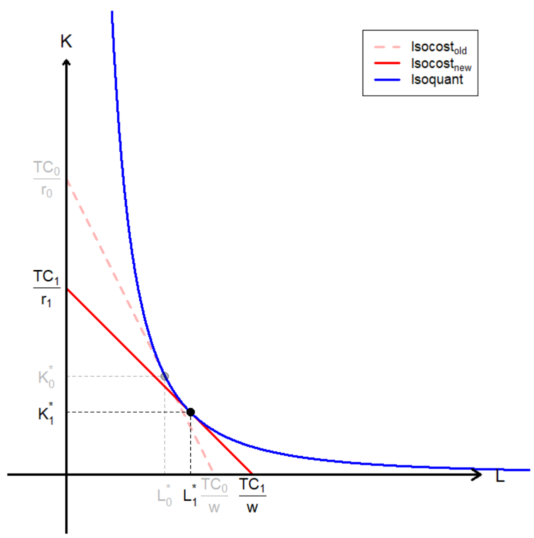
$$TC(Q = 1000) = 100 \cdot 10 + 50 \cdot 10 = 1,500$$

10.E Plot the producer's cost minimization problem below:



- The isocost.
- The isoquant (approximate).
- The optimal input bundle.

10.F For this question only, assume that rent increased. Plot the producer's cost minimization problem below:



- The old and new isocosts.
- The isoquant (approximate).
- The old and new optimal input bundle.

10.G If your target output is some arbitrary Q, what is your total cost of production?

Solution:

Use the optimal ratio found in 10.C, and start off with the production quota:

$$\begin{aligned} Q = L^2K \quad &\Rightarrow \quad Q = L^2 \cdot L && \because L = K \\ &\Rightarrow \quad Q = L^3 \\ &\Rightarrow \quad L^* = Q^{\frac{1}{3}} \quad \Rightarrow \quad K^* = Q^{\frac{1}{3}} && \because L = K \end{aligned}$$

Then, the cost function will be:

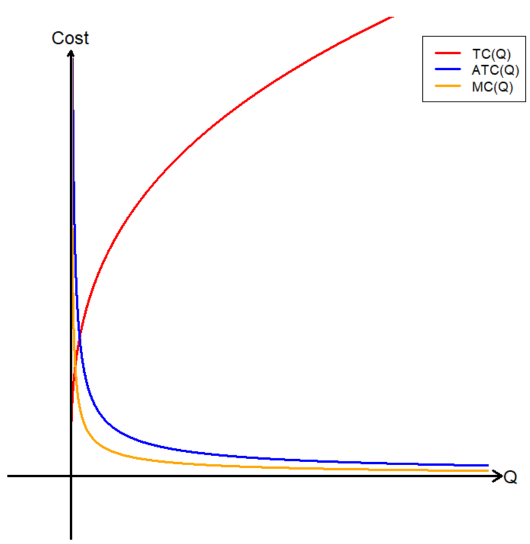
$$TC(Q) = 100 \cdot Q^{\frac{1}{3}} + 50 \cdot Q^{\frac{1}{3}} = 150Q^{\frac{1}{3}}$$

10.H Find the average total cost function and marginal cost function.

Solution:

- $ATC(Q) = \dfrac{TC(Q)}{Q} = \dfrac{150Q^{\frac{1}{3}}}{Q} = 150Q^{-\frac{2}{3}}$
- $MC(Q) = \dfrac{dTC(Q)}{dQ} = \dfrac{d}{dQ}150Q^{\frac{1}{3}} = 150 \cdot \frac{1}{3} \cdot Q^{-\frac{2}{3}} = 50Q^{-\frac{2}{3}}$

10.I (ADVANCED) Plot the total, average, and marginal cost curves below:



- The total cost curve.
- The average total cost curve.
- The marginal cost curve.

Problem 11. Cost Minimization

Suppose you are a producer of some good, and the only two inputs you use are labor and capital. Wage is $w = 50$ and rent is $r = 100$. Your production technology is given as:

$$F(L, K) = min\{2L, K\}$$

11.A Identify the type of production function we are given.

Solution:

The Leontief production function.

11.B Is there an optimal ratio of inputs L and K?
If so, what is the optimal ratio? If not, what is the rule to follow?

Solution:

The optimal ratio is:[1] $2L = K$

11.C If your target output is some arbitrary Q, what is your total cost function?

Solution:

The optimal ratio is $2L = K$, and then we start with the production quota:

$$\begin{aligned} Q = min\{2L, K\} \quad &\Rightarrow \quad Q = min\{2L, 2L\} && \because 2L = K \\ &\Rightarrow \quad Q = 2L \\ &\Rightarrow \quad L^* = \frac{Q}{2} \quad \Rightarrow \quad K^* = Q && \because 2L = K \end{aligned}$$

Then the total cost function is:

$$TC(Q) = wL + rK = 50 \cdot \frac{Q}{2} + 100 \cdot Q = 125Q$$

11.D Find the average total cost function and marginal cost functions.

Solution:

- $ATC(Q) = \dfrac{TC(Q)}{Q} = \dfrac{125Q}{Q} = 125$
- $MC(Q) = \dfrac{dTC(Q)}{dQ} = \dfrac{d}{dQ}125Q = 125 \cdot 1 \cdot Q^{1-1} = 125$

[1] We examine the logic in detail in **Problem 3** of this problem set.

Problem 12. Cost Minimization

Suppose you are a producer of some good, and the only two inputs you use are labor and capital. Wage is $w = 50$ and rent is $r = 100$. Your production technology is given as:

$$F(L, K) = L + 3K$$

12.A Identify the type of production function we are given.

Solution:

The Linear production function.

12.B Find the marginal product of labor and capital.

Solution:

- $MP_L = \frac{\partial}{\partial L}(L + 3K) = \frac{\partial}{\partial L}L + \frac{\partial}{\partial L}3K = 1 \cdot 1 \cdot L^{1-1} + 0 = 1$
- $MP_K = \frac{\partial}{\partial K}(L + 3K) = \frac{\partial}{\partial K}L + \frac{\partial}{\partial K}3K = 0 + 3 \cdot 1 \cdot K^{1-1} = 3$

12.C Is there an optimal ratio of inputs L and K?
If so, what is the optimal ratio? If not, what is the rule to follow?

Solution:

When we have the linear production function, we compare the per dollar marginal product, and exclusively use the input that provides a greater amount of MP per dollar.

$$\frac{MP_L}{w} = \frac{1}{50} < \frac{3}{100} = \frac{MP_K}{r}$$

Therefore we should exclusively use capital as an input.

12.D If your target output is some arbitrary Q, what is your total cost function?

Solution:

Apply the rule "exclusively purchase capital," and consider the production quota:

$$Q = L + 3K \quad \Rightarrow \quad Q = 0 + 3K \qquad \because \text{Rule from 12.C}$$

$$\Rightarrow \quad K^* = \frac{Q}{3} \quad \Rightarrow \quad L^* = 0$$

So our total cost function should be:

$$TC(Q) = wL + rK = 50 \cdot 0 + 100 \cdot \frac{Q}{3} = \frac{100Q}{3}$$

Problem 13. Profit Maximization

Suppose you are a producer of some good, and the only two inputs you use are labor and capital. The output market is perfectly competitive and the market price of your output is $P_x = 100$. Furthermore, your total cost function was calculated as:

$$TC(Q) = 1000 + 30Q + 2Q^2$$

13.A In your own words, explain why $MR(Q) = MC(Q)$ would be the optimization condition for profit maximizing producers. Think about what happens when $MR(Q) > MC(Q)$, and what happens when $MR(Q) < MC(Q)$.

Solution:

When $MR > MC$, we are earning more in revenue than it costs when producing one extra unit. Therefore, the profit maximizer should increase their production. When $MR < MC$, it costs more to produce one extra unit compared to the amount that it brings in when it is sold. Thus, the profit maximizer should scale down production. Therefore, only when $MR = MC$, the producer's profit will be maximized.

13.B What is the expression for total revenue $TR(Q)$?

Solution:

In a perfectly competitive output market where no individual has influence over prices:

$$TR(Q) = P \cdot Q = 100Q$$

13.C What is the expression for marginal revenue $MR(Q)$?

Solution:

Marginal revenue is the first derivative of the total revenue function:

$$MR(Q) = \frac{d}{dQ}TR(Q) = \frac{d}{dQ}100 \cdot Q = 100 \cdot 1 \cdot Q^{1-1} = 100$$

13.D What is the expression for marginal cost $MC(Q)$?

Solution:

Marginal cost is the first derivative of the total cost function:

$$MC(Q) = \frac{d}{dQ}TC(Q) = \frac{d}{dQ}(1000 + 30Q + 2Q^2) = 30 + 4Q$$

13.E What is the optimal amount to produce if you are a profit maximizer?

Solution:

According to our conclusion in 13.A, the profit maximizer will produce up to the point where $MR = MC$:

$$MR(Q) = MC(Q) \quad \Rightarrow \quad 100 = 30 + 4Q \quad \Rightarrow \quad Q^* = \frac{35}{2}$$

13.F Suppose that the market price of your output increased to $P'_x = 150$. What would be your updated optimal amount to produce as a profit maximizer?

Solution:

The rule remains identical at $MR = MC$:

$$MR(Q) = MC(Q) \quad \Rightarrow \quad 150 = 30 + 4Q \quad \Rightarrow \quad Q^* = 30$$