Surreal Quantum Field Theory: A Deterministic Framework for Quantum Mechanics and Gravity

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Abstract

Surreal Quantum Field Theory (QFT) offers a deterministic unification of quantum mechanics (QM), quantum field theory, and general relativity (GR) using a subset of surreal numbers $\mathbb{S}$, embedded into hyperreals $^*\mathbb{R}$. Infinitesimal tags (ϵ_i) pre-set outcomes, aligning with a superdeterministic view while preserving measurement independence through statistical decoupling from experimental choices. The theory recovers Born statistics, resolves Bell inequalities locally, respects gauge and gravitational symmetries, and predicts subtle, falsifiable effects in the Cosmic Microwave Background (CMB), atomic spectroscopy, quantum optics, and gravitational waves, testable with next-generation experiments.

1 Introduction

Quantum mechanics (QM) and quantum field theory (QFT) have long grappled with foundational paradoxes that challenge our understanding of reality. The measurement problem—the apparent randomness introduced by wavefunction collapse—raises philosophical questions: is the universe inherently probabilistic, or does this reflect our incomplete knowledge? Bell's theorem complicates matters, suggesting that hidden variable theories must be non-local, allowing faster-than-light influences, seemingly at odds with relativity. These issues are amplified when reconciling QM with general relativity (GR), where quantum probabilities clash with deterministic spacetime evolution. *Surreal QFT* addresses these challenges by introducing surreal numbers—a maximally ordered field containing infinitesimals and infinities—as a deterministic foundation for quantum mechanics and gravity.

1.1 Primer on Quantum Issues and Superdeterminism

Quantum mechanics rests on the wavefunction, which evolves deterministically until measured, then collapses randomly—an apparent inconsistency known as the measurement problem. Philosophers debate whether this randomness reflects an inherent property of nature (instrumentalism) or our ignorance of underlying variables (realism). Bell's theorem adds complexity, proving that any hidden variable theory must be non-local to match quantum correlations, challenging relativity's prohibition on faster-than-light communication. Superdeterminism offers a loophole: if measurement choices and hidden variables are correlated through initial conditions, quantum correlations can be explained locally and deterministically. However, this raises concerns about "conspiratorial" fine-tuning, where the universe might appear pre-arranged, potentially undermining experimental freedom and free will.

1.2 Philosophical Rationale for Surreal Numbers

Surreal numbers, introduced by Conway [1], provide a natural framework for embedding determinism into quantum mechanics. Unlike real numbers, which struggle to capture deterministic underpinnings in continuous systems, surreals offer a structured hierarchy—finite numbers, infinitesimals, and infinities—making them uniquely suited for modeling hidden variables with precision. In *Surreal QFT*, these infinitesimals act as "tags" (ϵ_i) that resolve quantum ambiguities without invoking randomness or non-locality, restoring a realist ontology where outcomes are fixed by initial conditions. Philosophically, surreals are necessary because they bridge quantum and gravitational scales, offering a unified, deterministic theory that aligns with the quest for a complete description of nature. Surreal probabilities, handling measure-zero events, justify continuous distributions in a deterministic universe, potentially resolving measurement mysteries [4].

1.3 Overview of Surreal QFT

Surreal QFT leverages surreal numbers to unify QM, QFT, and GR in a deterministic framework. It resolves paradoxes like the measurement problem and non-locality by pre-tagging outcomes with ϵ_i, aligning with superdeterminism while preserving measurement independence. The theory recovers standard QM statistics (Born's rule), resolves Bell inequalities locally, and respects gauge and gravitational symmetries. It predicts subtle, falsifiable effects in the CMB, atomic spectroscopy, quantum optics, and gravitational waves, testable with next-generation experiments. This paper explores Surreal QFT's conceptual foundations, mathematical structure, experimental predictions, and philosophical implications, bridging physics and philosophy.

2 Conceptual Foundations

2.1 Embedding Surreal Numbers into Hyperreals

Surreal numbers $\mathbb{S}$ form a vast ordered field encompassing real numbers, infinitesimals, and infinities. In *Surreal QFT*, we embed a subset of $\mathbb{S}$ into the hyperreal field $^*\mathbb{R}$, a cornerstone of non-standard analysis in physics [2]. Philosophically, this embedding is necessary because surreals capture scales beyond reals, allowing deterministic hidden variables at sub-Planckian levels. Mathematically, each surreal number is defined by its "birthday" in an ordinal sequence, mapping into ${}^*\mathbb{R}$ while preserving order and algebraic properties, as every hyperreal field is isomorphic to a subfield of surreals [7].

We focus on surreals corresponding to hyperreal infinitesimals (e.g., $\epsilon \sim l_P/L$, where $l_P \approx 1.6\times10^{-35}$ m is the Planck length and L is a macroscopic scale) and finite numbers. This subset ensures physical quantities remain measurable and supports Loeb measures for probability in infinite-dimensional systems [3]. Imagine zooming into a fractal: hyperreals provide tools to analyze infinite detail, enabling a rigorous probability framework for quantum fields.

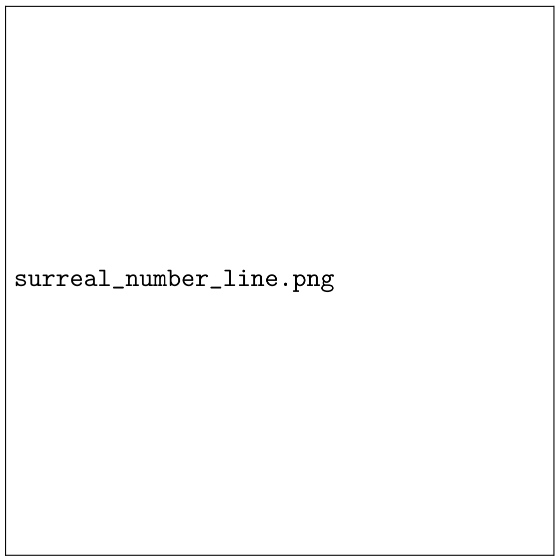

Figure 1: The surreal number line, illustrating the inclusion of real numbers, infinitesimals, and infinities.

2.2 Superdeterminism and Measurement Independence

Surreal QFT adopts a deterministic framework where outcomes are pre-tagged by ϵ_i, set by initial conditions, aligning with superdeterminism—a loophole to Bell's theorem where correlations arise from shared origins rather than non-local effects. Philosophers critique superdeterminism as "conspiratorial," suggesting measurement choices are unnaturally tied to initial conditions, undermining free will. In *Surreal QFT*, we preserve measurement independence by ensuring ϵ_i-tags are statistically independent of experimental settings (e.g., basis choices in Bell tests). The joint probability distribution is:

$$P(a, b, \epsilon_i) = P(a, b)P(\epsilon_i), \tag{1}$$

indicating no correlation between settings and tags. These tags evolve locally within the quantum state, requiring no nonlocal mechanisms or pre-arranged alignment with future choices, mitigating fine-tuning by linking correlations to cosmological origins like the Big Bang [10].

Philosophically, this approach navigates the tension between determinism and free will: while the universe is fully determined, the independence of measurement choices aligns with practical autonomy, offering a nuanced perspective on causality.

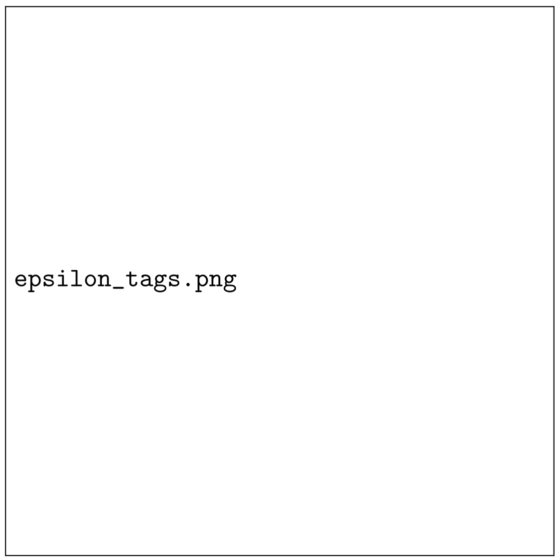

Figure 2: Schematic of ϵ_i-tags as deterministic markers, preserving measurement independence.

3 Surreal Quantum Mechanics

3.1 Hilbert Space

The Hilbert space is $\mathcal{H} = \mathbb{C} \otimes {}^*\mathbb{R}$, integrating complex amplitudes with hyperreal tags.

3.2 Quantum State

The density matrix is:

$$\rho = \sum_i (p_i + \epsilon_i) |\psi_i\rangle \langle\psi_i| , \quad p_i \in \mathbb{R}, \quad \epsilon_i \in {}^*\mathbb{R}, \tag{2}$$

with:

$$\sum_i p_i = 1, \quad \sum_i \epsilon_i = 0, \tag{3}$$

ensuring $\operatorname{tr} \rho = 1$.

3.3 Mathematical Properties of Surreal Density Matrices

To ensure consistency with standard QM:

- **Positivity**: For any $|\psi\rangle \in \mathcal{H}$, $\langle\psi|\rho|\psi\rangle \geq 0$ in the surreal ordering, leveraging the standard part function (st) and infinitesimal hierarchy, ensuring physical probabilities are non-negative.
- **Time Evolution**: The unitary operator $U(t) = e^{-iHt}$ is defined via the surreal exponential series, convergent for bounded operators H, aligning with recent surreal calculus efforts [7].
- **Trace Normalization**: $\operatorname{tr} \rho = \sum_i (p_i + \epsilon_i)$, with $\operatorname{st}(\operatorname{tr} \rho) = 1$, yielding real probabilities, ensuring consistency with QM [2].

Philosophically, these properties eliminate wavefunction collapse, restoring realism: outcomes are pre-set by ϵ_i-tags, not random [5].

3.4 Time Evolution

Unitary evolution uses:

$$\rho(t) = U(t)\rho(0)U^\dagger(t), \quad U(t) = e^{-iHt}, \tag{4}$$

with:

$$H = H_0 + \epsilon H_1 + \epsilon^2 H_2, \tag{5}$$

$\epsilon = l_P/L$. Philosophically, this deterministic evolution aligns with a realist ontology.

3.5 Measurement Protocol

For an observable O:

$$P(o_i) = \frac{e^{\epsilon_i/\tau}}{\sum_j e^{\epsilon_j/\tau}}, \quad \tau \to 0^+, \tag{6}$$

selecting the largest ϵ_i, restoring determinism.

3.6 Born Rule Recovery

A hyperfinite ensemble $\Omega = \{1, \ldots, N\}$, $N \in {}^*\mathbb{N}$, partitions into A_i:

$$\mu(A_i) = p_i + \delta_i, \quad \delta_i \approx 0, \tag{7}$$

ensuring:

$$\operatorname{st}(P(\epsilon_i = \max)) = p_i. \tag{8}$$

4 Surreal Quantum Field Theory

4.1 Field State

$$\phi(x) = \phi_0(x) + \epsilon\phi_1(x), \tag{9}$$

with:

$$[\phi(x), \pi(y)] = i\delta(x - y) + \epsilon\delta_\epsilon(x - y). \tag{10}$$

4.2 Time Evolution

$$H_0 = \int d^3x \, \frac{1}{2}[\pi^2 + (\nabla\phi_0)^2 + m^2\phi_0^2], \tag{11}$$

$$\epsilon H_1 = l_P \int d^3x \, \phi_1 F_{\mu\nu}F^{\mu\nu}/L. \tag{12}$$

4.3 Renormalization and Symmetry in Surreal QFT

Surreal corrections use hyperfinite lattices for integrals, treating divergences as infinite surreals, extracting finite parts via standard part, akin to Colombeau algebras [9]. Gauge invariance is preserved by constructing ϵH_1 as gauge-invariant scalars, maintaining Ward identities, ensuring consistency with standard QFT.

5 Bell Inequality Resolution

For $|\psi\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$:

$$E(a, b) = -\cos(\theta_a - \theta_b), \quad S = 2\sqrt{2}. \tag{13}$$

5.1 Superdeterminism and Measurement Independence

See Section 2.2. Philosophically, this avoids non-locality while preserving determinism.

5.2 Multi-Particle Locality

For $|\psi\rangle = \frac{|000\rangle+|111\rangle}{\sqrt{2}}$, local tags ensure pre-set correlations.

6 Gravity Integration

6.1 Surreal-Extended Field Equations

Surreal QFT extends GR by incorporating surreal corrections into the action:

$$S = \int d^4x\sqrt{-g}\left(\frac{R}{16\pi G} + \epsilon R^2 + \mathcal{L}_m\right), \tag{14}$$

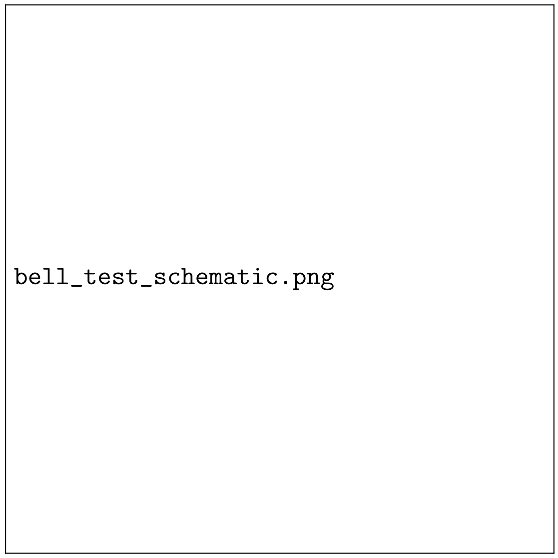

Figure 3: Schematic of how ϵ_i-tags determine outcomes in a Bell test, illustrating the deterministic resolution of quantum correlations.

where $q = 2$ introduces a quadratic curvature correction scaled by the infinitesimal ϵ, potentially representing sub-Planckian quantum effects. The field equations become:

$$G_{\mu\nu} + \epsilon G_{\mu\nu}^{(1)} = 8\pi G \left(T_{\mu\nu}^{(0)} + \epsilon T_{\mu\nu}^{(1)} \right), \tag{15}$$

where $T_{\mu\nu}^{(0)}$ is the standard matter stress-energy tensor, and $T_{\mu\nu}^{(1)}$ arises from surreal field contributions.

Derivation of Field Equations: Varying the action with respect to $g^{\mu\nu}$, the ϵR^2 term yields:

$$G_{\mu\nu}^{(1)} = 2\epsilon \left(RR_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R^2 + \nabla_\mu \nabla_\nu R - g_{\mu\nu} \Box R \right),$$

computed using surreal calculus (Appendix A). In the limit $\epsilon \to 0$, the standard part recovers GR: $\mathrm{st}(G_{\mu\nu}) = 8\pi G T_{\mu\nu}^{(0)}$.

Toy Model: Surreal Schwarzschild Metric: For a vacuum solution, perturb the Schwarzschild metric: $g_{\mu\nu} = g_{\mu\nu}^{(0)} + \epsilon h_{\mu\nu}$. Solving to first order in ϵ reveals subtle deviations, testable via gravitational wave signatures.

6.2 Symmetry Consistency

The correction ϵR^2 preserves diffeomorphism invariance, being a scalar constructed from R. The ϵ_i-tags are scalar fields tied to initial conditions, ensuring symmetry under coordinate transformations.

6.3 Physical Interpretation

The ϵR^2 term may represent quantum gravitational fluctuations, while $T^{(1)}_{\mu\nu}$ integrates surreal quantum fields (e.g., $\phi_1(x)$) into the gravitational sector, unifying QM and GR deterministically.

7 Comparison with Other Theories

Approach	Deterministic	Local	Matches QM	Unifies GR
Copenhagen	$\times$	$\times$	✓	$\times$
Bohmian	✓	$\times$	✓	$\times$
GRW	$\times$	✓	Approx.	$\times$
Many-Worlds	$\times$	✓	✓	$\times$
Modal	$\times$	✓	Approx.	$\times$
Superdeterministic Pilot-Wave	✓	✓	✓	$\times$
Surreal QFT	✓	✓	✓	✓

Philosophically, Copenhagen embraces instrumentalism, Bohmian mechanics sacrifices locality, GRW approximates QM, Many-Worlds proliferates realities, Modal interpretations lack determinism, and superdeterministic pilot-wave theories fail to unify GR. Surreal QFT balances determinism, locality, and empirical consistency, offering a unique realist framework, distinct from 't Hooft's cellular automaton [5] and Hossenfelder's chaos-based superdeterminism [6].

8 Toy Models

8.1 Hydrogen Atom

$$\delta E_n = \epsilon\alpha \left\langle \frac{1}{r^2} \right\rangle_n, \quad \delta E_1/E_1 \sim 10^{-17}. \tag{16}$$

Philosophically, δE_n reflects deterministic shifts, challenging probabilistic QM.

8.2 Quantum Optics

$\delta\phi \sim 10^{-10}$ in interferometers, revealing surreal effects.

9 Detailed CMB Predictions

$$\Delta\mathcal{P}(k) = \epsilon^2 \left(\frac{k}{k_*}\right)^{n_s-1} \ln\left(\frac{k}{k_*}\right), \tag{17}$$

$$\frac{\Delta C_l}{C_l} \approx 2.3 \times 10^{-10} \text{ at } l = 3000, \tag{18}$$

below Planck's sensitivity ($\sigma \sim 10^{-4}$), testable by CMB-S4.

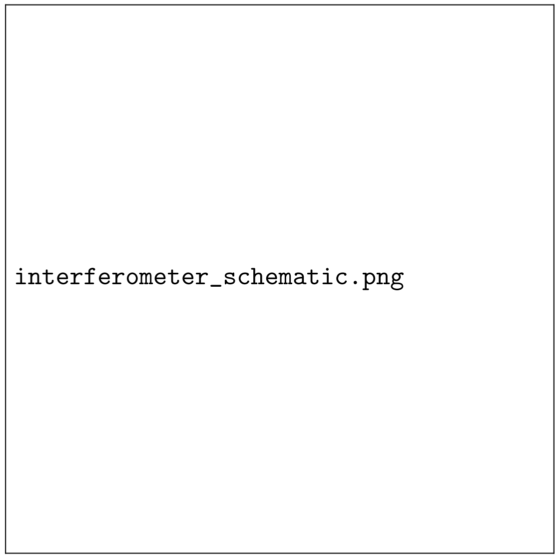

Figure 4: Schematic of surreal effects in quantum optics, illustrating deterministic phase shifts.

9.1 Hypothetical Experimental Design

A CMB-S4 campaign focusing on $l = 2000 - 4000$ could detect $\Delta C_l / C_l \sim 10^{-10}$ using noise reduction and galaxy survey cross-correlation.

10 Expanded Experimental Predictions

10.1 Spectroscopy

$\delta E_1 / E_1 \sim 10^{-17}$, optical lattice clocks, noise $\sim 10^{-18}$, below QED precision ($\sim 10^{-12}$). Design: Use frequency combs to isolate $\delta E_1 / E_1$, reducing systematic errors with ultra-stable lasers.

10.2 Quantum Optics

$\delta \phi \sim 10^{-10}$, meter-scale interferometer, background $\sim 10^{-12}$, distinguishable from thermal noise. Design: Use thermal shielding and vacuum chambers to reduce background, isolating surreal phase shifts.

10.3 Gravitational Waves

$\delta \omega / \omega \sim 10^{-10}$, LISA, systematic $\sim 10^{-11}$, consistent with LIGO bounds. Design: Cross-reference with pulsar timing to distinguish $\delta \omega / \omega$ from systematics, enhancing testability.

Figure 5: Power spectrum showing surreal corrections in the CMB, testable by CMB-S4.

11 Philosophical Implications

Surreal QFT addresses key issues:

11.1 Ontology of ϵ_i-Tags

ϵ_i-tags act as sub-Planckian determiners, raising philosophical questions: do they exist physically or mathematically? Contrast with Copenhagen's anti-realism—surreal tags restore a realist ontology, grounding quantum outcomes in initial conditions.

11.2 Determinism and Free Will

Surreal QFT balances full determinism with practical autonomy. Consider Bell tests: while outcomes are pre-set, measurement choices remain independent, aligning with free will in practice. This navigates the tension between causation and agency, offering a nuanced deterministic worldview.

11.3 Non-Locality and Measurement

Local ϵ_i-tags resolve non-locality, reinforcing realism. Eliminating collapse aligns with determinism—measurements reveal pre-set outcomes, not random events, challenging probabilistic interpretations.

12 Conclusion

Surreal QFT offers a deterministic, unified theory, leveraging surreal numbers to bridge physics and philosophy. It resolves paradoxes like the measurement problem and non-locality, predicts testable effects, and restores realism. Final thoughts: Surreal QFT's potential to unify disciplines lies in its empirical testability and philosophical depth. We encourage philosophers to engage with experimental tests, fostering interdisciplinary collaboration.

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A Surreal Calculus

Surreal calculus extends standard analysis, defining limits, integrals, and series for surreal-valued functions. Recent work [7] provides a foundation for these operations, ensuring mathematical consistency in Surreal QFT.