

# Supplementary Material for “Efficient and Scalable Bipartite Matching with Fast Beta Linkage (fabl)”

Brian Kundinger<sup>\*</sup>, Jerome P. Reiter<sup>\*</sup> and Rebecca C. Steorts<sup>\*</sup>

## Supplement A: Derivations of Full Conditionals

We provide detailed derivations of the full-conditionals provided in Section 3. The  $\mathbf{m}$  and  $\mathbf{u}$  parameters are updated through standard multinomial-Dirichlet distributions. For a particular  $m_{fl}$ , we have

$$\mathcal{L}(m_{fl}|\gamma, \mathbf{u}, \mathbf{Z}, \pi) \propto \prod_{i=1}^{n_A} \prod_{j=1}^{n_B} m_{fl}^{I(Z_j=i)I(\gamma_{ij}^f=l)I_{obs}(\gamma_{ij}^f)} m_{fl}^{\alpha_{fl}-1} = m_{fl}^{\alpha_{fl}(\mathbf{Z})-1}, \quad (1)$$

where  $\alpha_{fl}(\mathbf{Z}) = \alpha_{fl} + \sum_{i=1}^{n_A} \sum_{j=1}^{n_B} I_{obs}(\gamma_{ij}^f)I(\gamma_{ij}^f = l)I(Z_j = i)$ . Analogous procedures lead to  $\mathcal{L}(u_{fl}|\gamma, \mathbf{m}, \mathbf{Z}, \pi) \propto u_{fl}^{\beta_{fl}(\mathbf{Z})-1}$ , where  $\beta_{fl}(\mathbf{Z}) = \beta_{fl} + \sum_{i=1}^{n_A} \sum_{j=1}^{n_B} I_{obs}(\gamma_{ij}^f)I(\gamma_{ij}^f = l)I(Z_j \neq i)$ . Thus, for the vectors of parameters  $\mathbf{m}_f$  and  $\mathbf{u}_f$ , we have

$$\mathbf{m}_f^{(s+1)}|\gamma, \mathbf{Z}^{(s)}, \mathbf{u}^{(s)}, \pi^{(s)} \sim \text{Dirichlet}(\alpha_{f1}(\mathbf{Z}^{(s)}), \dots, \alpha_{fL_f}(\mathbf{Z}^{(s)})), \quad (2)$$

$$\mathbf{u}_f^{(s+1)}|\gamma, \mathbf{Z}^{(s)}, \mathbf{m}^{(s)}, \pi^{(s)} \sim \text{Dirichlet}(\beta_{f1}(\mathbf{Z}^{(s)}), \dots, \beta_{fL_f}(\mathbf{Z}^{(s)})). \quad (3)$$

Since  $\pi$  encodes the rate of matching across the two data files, the full conditional  $p(\pi|\gamma, \mathbf{Z}, \mathbf{m}, \mathbf{u}, \alpha_\pi, \beta_\pi)$  depends only on the number of links  $n_{AB}(\mathbf{Z}) = \sum_{i=1}^{n_B} I(Z_i \leq n_A)$  encoded by  $\mathbf{Z}$  and hyperparameters. We have the full conditional

$$p(\pi|\gamma, \mathbf{Z}, \mathbf{m}, \mathbf{u}) \propto p(\mathbf{Z}|\pi)p(\pi) \quad (4)$$

$$\propto \pi^{n_{AB}(\mathbf{Z})} (1 - \pi)^{n_B - n_{AB}(\mathbf{Z})} \pi^{\alpha_\pi - 1} (1 - \pi)^{\beta_\pi - 1} \quad (5)$$

$$\propto \pi^{n_{AB}(\mathbf{Z}) + \alpha_\pi - 1} (1 - \pi)^{n_A - n_{AB}(\mathbf{Z}) + \beta_\pi - 1}. \quad (6)$$

Thus,  $\pi^{(s+1)}|\gamma, \mathbf{Z}^{(s)}, \mathbf{m}^{(s+1)}, \mathbf{u}^{(s+1)}$  has a Beta( $n_{AB}(\mathbf{Z}^{(s)}) + \alpha_\pi, n_B - n_{AB}(\mathbf{Z}^{(s)}) + \beta_\pi$ ) distribution.

Due to the independence in the fast beta prior in (5), we can obtain the full conditional for  $\mathbf{Z}$  through the full conditionals for each individual  $Z_j$ . Let  $\Gamma_{\cdot j}$  denote the random matrix of  $n_A$  comparison vectors relating to an arbitrary record  $B_j$ , and let  $\gamma_{\cdot j}$  be a realization of  $\Gamma_{\cdot j}$ . We have

$$p(\mathbf{Z}|\gamma, \mathbf{m}, \mathbf{u}, \pi) = \prod_{j=1}^{n_B} p(Z_j|\gamma_{\cdot j}, \mathbf{m}, \mathbf{u}, \pi). \quad (7)$$

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<sup>\*</sup>Department of Statistical Science, Duke University, P.O. Box 90251, Durham, NC 27708, USA  
[brian.kundinger@duke.edu](mailto:brian.kundinger@duke.edu), [jreiter@duke.edu](mailto:jreiter@duke.edu), [beka@stat.duke.edu](mailto:beka@stat.duke.edu)

## 2 Efficient and Scalable Bipartite Matching with Fast Beta Linkage (fabl)

Following the observation of Wortman (2019), when  $B_j$  does not link to any record in  $A$ , the contribution to the likelihood is simply a product of  $u$  parameters, which we will call  $c_j$ :

$$p(\Gamma_{.j} | \mathbf{m}, \mathbf{u}, \pi, Z_j = n_A + j) = \prod_{i=1}^{n_A} \prod_{f=1}^F \prod_{l=1}^{L_f} u_{fl}^{I(\gamma_{ij}^f=l)I_{obs}(\gamma_{ij}^f)} = c_j. \quad (8)$$

When  $Z_j = q$  for some  $q \leq n_A$ , we have

$$p(\Gamma_{.j} | \mathbf{m}, \mathbf{u}, \pi, Z_j = q) = \prod_{f=1}^F \prod_{l=1}^{L_f} m_{fl}^{I(\gamma_{qj}^f=l)I_{obs}(\gamma_{qj}^f)} \prod_{i \neq q} \prod_{f=1}^F \prod_{l=1}^{L_f} u_{fl}^{I(\gamma_{ij}^f=l)I_{obs}(\gamma_{ij}^f)}. \quad (9)$$

We multiply and divide by the  $u$  parameters for the matching record pair to obtain

$$p(\Gamma_{.j} | \mathbf{m}, \mathbf{u}, \pi, Z_j = q) = \prod_{f=1}^F \prod_{l=1}^{L_f} \left( \frac{m_{fl}}{u_{fl}} \right)^{I(\gamma_{qj}^f=l)I_{obs}(\gamma_{qj}^f)} \prod_{i=1}^{n_A} \prod_{f=1}^F \prod_{l=1}^{L_f} u_{fl}^{I(\gamma_{ij}^f=l)I_{obs}(\gamma_{ij}^f)} \quad (10)$$

$$= w_{qj} c_j. \quad (11)$$

We can divide the result of each case by  $c_j$  to get

$$p(\Gamma_{.j} | \mathbf{m}, \mathbf{u}, \pi, Z_j) \propto \begin{cases} w_{qj}, & q \leq n_A; \\ 1, & q = n_A + j. \end{cases} \quad (12)$$

Lastly, we multiply the likelihood by the fast beta prior in (5) to obtain the full conditional

$$p\left(Z_j^{(s+1)} = q | \gamma, \mathbf{m}^{(s+1)}, \mathbf{u}^{(s+1)}, \pi^{(s+1)}\right) \propto \begin{cases} \frac{\pi^{(s+1)}}{n_A} w_{qj}^{(s+1)}, & q \leq n_A; \\ 1 - \pi^{(s+1)}, & q = n_A + j. \end{cases} \quad (13)$$

## 8 Supplement B: Bayes Estimate

We calculate a Bayes estimate  $\hat{Z}$  for the linkage parameter  $Z$  by assigning different positive losses to different types of errors, and minimizing posterior expected loss. We adopt the loss function proposed in Sadinle (2017) in which  $\hat{Z}_j \in \{1, \dots, n_A, n_A + j, R\}$ , with  $R$  representing the option to leave the matching undetermined by the model. Specifically, we have

$$L(\hat{Z}_j, Z_j) = \begin{cases} 0, & \text{if } Z_j = \hat{Z}_j; \\ \theta_R, & \text{if } \hat{Z}_j = R; \\ \theta_{10}, & \text{if } Z_j \leq 1, \hat{Z}_j = n_A + j; \\ \theta_{01}, & \text{if } Z_j = n_A + j, \hat{Z}_j \leq n_A; \\ \theta_{11}, & \text{if } Z_j \leq n_A, \hat{Z}_j \leq n_A, Z_j \neq \hat{Z}_j. \end{cases} \quad (14)$$

<sup>9</sup> Here,  $\theta_R$  is the loss from not making a decision on the linkage status,  $\theta_{10}$  is the loss  
<sup>10</sup> from a false nonmatch,  $\theta_{01}$  is the loss from a false match, and  $\theta_{11}$  is the loss from the

11 special case of a false match in which the record has a true match other than the one  
 12 estimated by the model.

In general, we follow [Sadinle \(2017\)](#) and set  $(\theta_{10}, \theta_{01}, \theta_{11}, \theta_R) = (1, 1, 2, \infty)$  inducing the decision rule

$$\hat{Z}_j = \begin{cases} i, & \text{if } p(Z_j = i|\gamma) > \frac{1}{2}; \\ 0, & \text{otherwise.} \end{cases} \quad (15)$$

13 Since `fabl` does not strictly enforce one-to-one matching, it is possible for this Bayes  
 14 estimate to link multiple records in  $B$  to one record in  $A$ . In the event that we have two  
 15 records  $B_j$  and  $B_{j'}$  such that both  $p(\hat{Z}_j = i|\gamma) > \frac{1}{2}$  and  $p(\hat{Z}_{j'} = i|\gamma) > \frac{1}{2}$ , we accept the  
 16 match with the higher posterior probability, and declare the other to have no match.  
 17 Since each  $Z_j$  is independent, this is equivalent to minimizing the expected loss subject  
 18 to the constraint that  $\hat{Z}_j \neq \hat{Z}_{j'}$  for all  $j \neq j'$ . A similar approach appears in the most  
 19 probable maximal matching sets used by [Steorts et al. \(2016\)](#) to match records to latent  
 20 entities.

21 When we seek a partial estimate of the linkage structure, leaving a portion of record  
 22 pairs to be classified manually in clerical review, we adopt losses  $(\theta_{10}, \theta_{01}, \theta_{11}, \theta_R) =$   
 23  $(1, 1, 2, .1)$ . For a more in-depth explanation of this function and the induced Bayes  
 24 estimate, see [Sadinle \(2017\)](#).

## 25 **Supplement C: Traceplots for Simulation Study**

26 Figures 2, 3, and 4 are traceplots for one of the 900 linkage tasks that comprise the  
 27 simulation in Section 5.2. It is set up with one error across the linkage fields and 50  
 28 duplicates across files. Traceplots across other settings exhibit similar behavior. Note  
 29 that traceplots for  $\mathbf{u}$  parameters show very little variation because the overwhelming  
 30 majority of record pairs are nonmatching.

#### 4 *Efficient and Scalable Bipartite Matching with Fast Beta Linkage (fabl)*

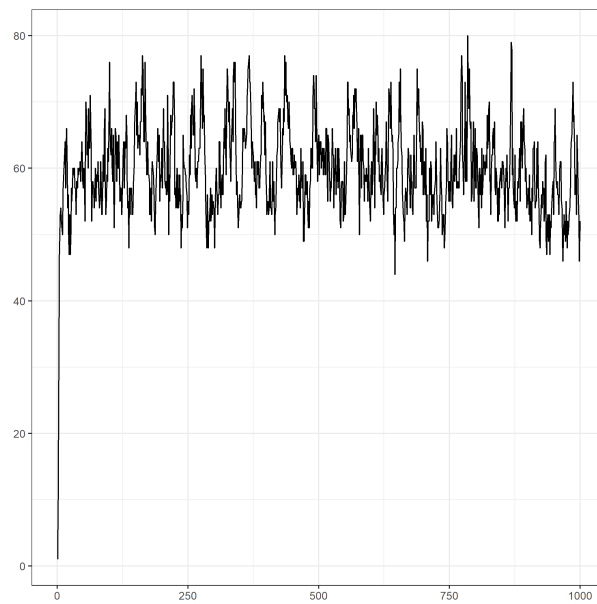


Figure 1: Representative traceplot of overlap between files from simulation study in Section 5.2.

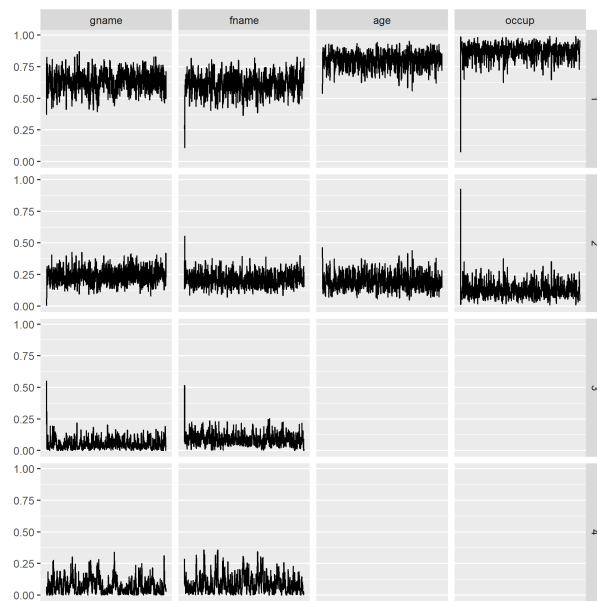


Figure 2: Representative traceplot of  $m$  parameter from simulation study in Section 5.2.

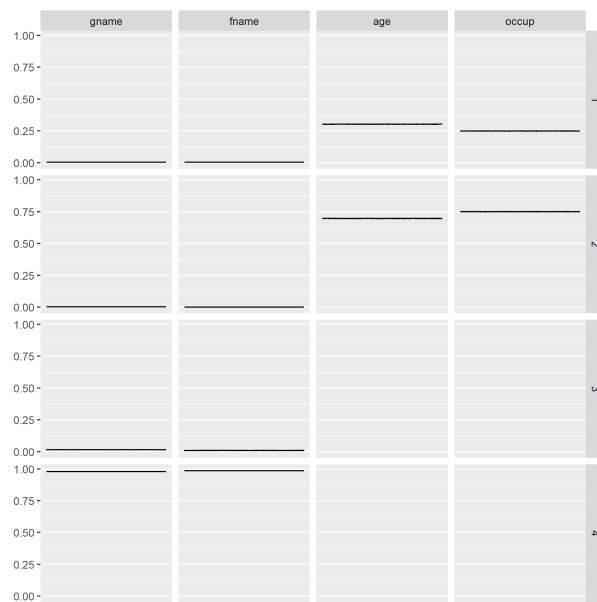


Figure 3: Representative traceplot of  $u$  parameters from simulation study in Section 5.2.

## 31 Supplement D: Accuracy under Partial Estimates

32 In this section, we repeat the simulation study in Section 5.2, allowing for clerical  
 33 review rather than forcing all records to have or not have links. Specifically, by leaving  
 34  $\theta_{10} = \theta_{01} = 1$  and  $\theta_{11} = 2$ , but setting  $\theta_R = 0.1$ , we allow the model to decline  
 35 to decide a match for certain records, with nonassignment being 10% as costly as a  
 36 false match. In this context, we are no longer focused on finding all true matches, but  
 37 rather protecting against false matches. Thus, instead of recall, we use the negative  
 38 predictive value (NPV), defined as the proportion of non-links that are actual nonmatches.  
 39 Mathematically,  $\text{NPV} = \sum_{j=1}^{n_B} I(\hat{Z}_j = Z_j = n_A + j) / \sum_{j=1}^{n_B} I(\hat{Z}_j = n_A + j)$ . We continue  
 40 to use the precision, which is renamed the positive predictive value (PPV) in this context.  
 41 Lastly, we also examine the rejection rate (RR), or how often the model declines to make  
 42 a linkage decision, defined as  $\text{RR} = \sum_{j=1}^{n_B} I(\hat{Z}_j = R) / n_B$ . To convey this information  
 43 alongside NPV and PPV, for which values close to 1 indicate strong performance, we  
 44 report the decision rate (DR), defined as  $\text{DR} = 1 - \text{RR}$ .

45 In Figure 1, we see that **fabl** maintains equivalently strong PPV as **BRL** across all  
 46 linkage settings. However, with high amounts of error, and thus fewer accurate and  
 47 discerning fields of information, the rejection rate under **fabl** rises, leading to a decrease  
 48 in NPV. Since **fabl** does not remove previously matched records from consideration  
 49 for a new record, posterior probabilities of matches at times can be split across more  
 50 records; in contrast, **BRL** is able to maintain higher confidence in matches in this setting.  
 51 If one wishes to use partial estimates, **fabl** will possibly leave more linkages for the  
 52 modeler to match by hand than would be left under **BRL**, but the decisions made by  
 53 each method should have nearly equal accuracy.

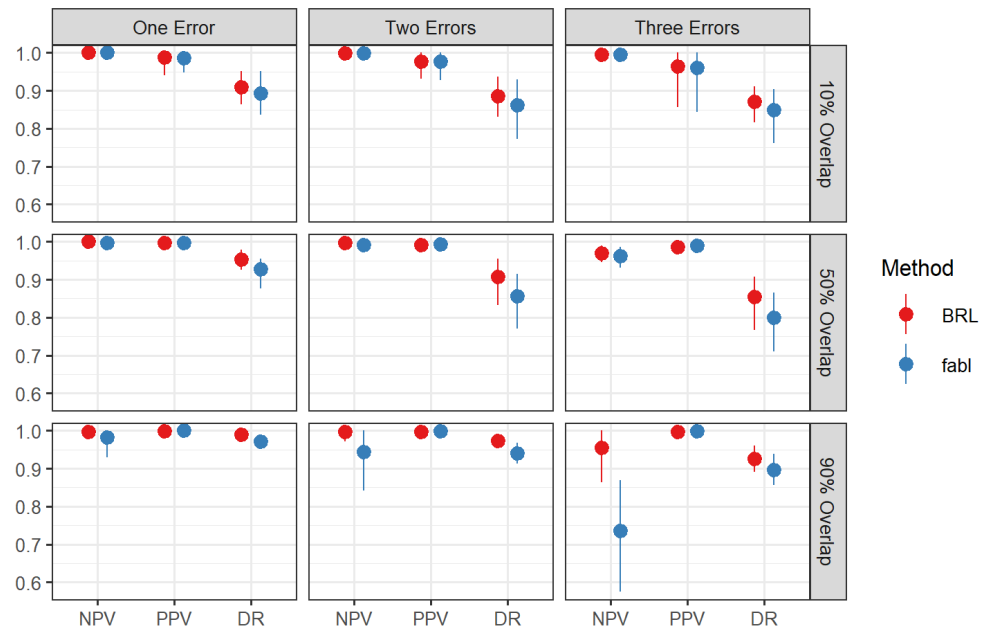


Figure 4: Negative predictive value (NPV), positive predictive value (PPV), and decision rate (DR) on data files in the simulation in Supplement C. We see poorer performance for `fabl` only in situations with high overlap.

## Supplement E: Additional Speed Simulation Study

Figures 5 and 6 illustrate that different constructions of the comparison vectors lead to similar speed gains. We replicate the speed study of Section 5.1 under different settings. Here, we use four fields of comparison, each with three possible levels of agreement, resulting in  $3^4 = 81$  possible patterns. The  $\mathbf{m}$  and  $\mathbf{u}$  parameters for this simulation are shown in Table 1.

	$\mathbf{m}$			$\mathbf{u}$		
	Agree	Partial	Disagree	Agree	Partial	Disagree
Feature 1	$\frac{9}{10}$	$\frac{9}{100}$	$\frac{1}{100}$	$\frac{1}{100}$	$\frac{3}{100}$	$\frac{96}{100}$
Feature 2	$\frac{9}{10}$	$\frac{9}{100}$	$\frac{1}{100}$	$\frac{1}{100}$	$\frac{3}{100}$	$\frac{96}{100}$
Feature 3	$\frac{9}{10}$	$\frac{9}{100}$	$\frac{1}{100}$	$\frac{1}{100}$	$\frac{3}{100}$	$\frac{96}{100}$
Feature 4	$\frac{9}{10}$	$\frac{9}{100}$	$\frac{1}{100}$	$\frac{1}{100}$	$\frac{3}{100}$	$\frac{96}{100}$

Table 1: Probabilities used for  $\mathbf{m}$  and  $\mathbf{u}$  distributions in simulation study in Supplement E.

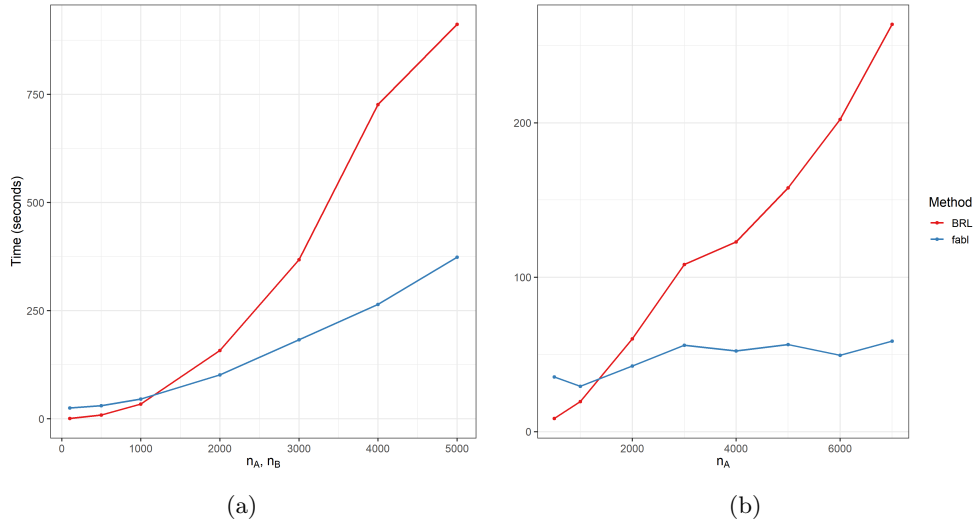
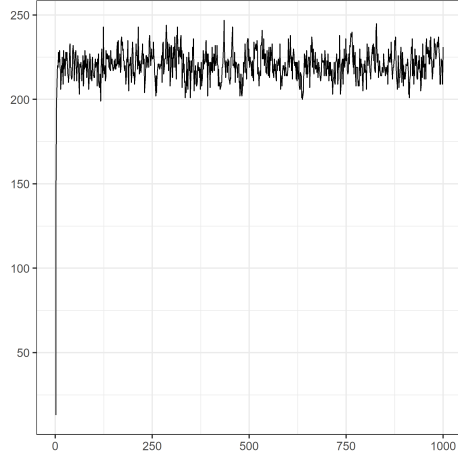


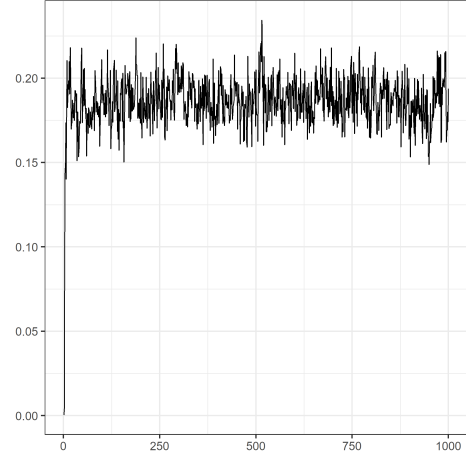
Figure 5: Run-time for BRL and **fabl** to run 1000 Gibbs iterations in simulations described in Supplement E. In (5), both  $n_A$  and  $n_B$  are increasing. We see quadratic growth in BRL and linear growth in **fabl**. In (6), only  $n_A$  only is increasing. We see linear growth in BRL and approximately constant run-time in **fabl**.



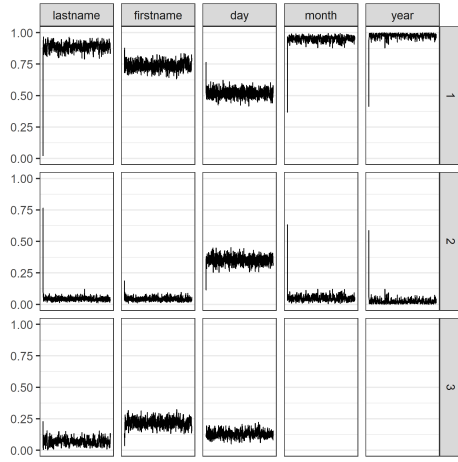
60 **Supplement F: Traceplots for El Salvador Case Study**



(a) Traceplot for overlap between files.



(b) Traceplot for  $\pi$ .



(c) Traceplot for  $m$  parameters.



(d) Traceplot for  $u$  parameters.

Figure 6: Traceplots for parameters of interest in El Salvador case study in Section 6.1.