

Efficient and Scalable Bipartite Matching with Fast Beta Linkage (fabl)

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Abstract

In this paper, we propose the *fast beta linkage* (**fabl**), which extends a recent Bayesian Fellegi Sunter model to allow for parallel computing. In addition, we propose using hashing techniques to hasten calculations and to reduce the overall computational complexity. To achieve a scalable model, we propose independent priors in order to create computational speeds and gains, where we relax the one-to-one matching assumption. Second, we resolve the one-to-one matching issue via a post processing step. Third, we derive conditional distributions, a Gibbs sampler, and corresponding Bayes estimates for our proposed model. Fourth, we explore other computational speeds up based upon storage efficient indexing. Finally, we explore a sensitivity analysis and revisit the case study of the El Salvadoran conflict to understand the practical ramifications regarding our proposed modeling choices, where we provide comparisons to the BRL package in CRAN.

Keywords: record linkage, bipartite record linkage, Bayesian methods, Gibbs sampling, hashing techniques, Markov chain Monte carlo

1 INTRODUCTION

Record linkage is the task of identifying duplicate records across multiple data sources, often in the absence of a unique identifier (Christen, 2012). Record linkage is an increasingly important task in “data cleaning,” that is used for inferential and predictive analyses in fields such as statistics, computer science, machine learning, political science, economics, precision medicine, official statistics, and others. Many statistical record linkage methods and case studies are extensions of seminal work of Fellegi and Sunter (1969) and Newcombe et al. (1959). Specifically, Fellegi and Sunter (1969) created comparison vectors for each pair of records in the data and independently classified those pairs as a match or a non-match using a likelihood ratio test (or hypothesis test). Recent work in the statistical literature has extended the aforementioned work (Winkler and Thibaudeau, 1991; Fair, 2004; Wagner et al., 2014; Gill and Goldacre, 2003).

In this paper, we consider bipartite record linkage, which is commonly known as merging two databases that contain duplication across the two databases, but do not contain any duplications within each database (Sadinle, 2017). Much of the statistical literature focuses on bipartite record linkage (Fellegi and Sunter, 1969; Jaro, 1989; Winkler, 1988; Belin and Rubin, 1995; Larsen and Rubin, 2001; Tancredi and Liseo, 2011; Herzog et al., 2007b; Gutman et al., 2013; Sadinle, 2017).

The Fellegi-Sunter method and its extensions are widely popular mainly due to its simplicity and computational scalability. Despite this, like any method, it has limitations, that are well known in the literature. Specifically, the assumption of no duplications within a database implies a maximum one-to-one constraint. That is, a record from one database can be linked with only one record in the other database. Modern methods of Fellegi-Sunter either ignore this restriction (Winkler, 1988; Belin and Rubin, 1995; Larsen and Rubin, 2001), include this as a post-processing step (Jaro, 1989), or include it directly in the modeling framework (Sadinle, 2017). It is important to note that Fellegi and Sunter (1969) ignored this restriction as well. In practice, it seems that one should either directly include it in the modeling framework or correct for this in a post-processing step.

To our knowledge, Sadinle (2017) was the first to propose correcting the Fellegi and

Sunter model for the one-to-one constraint. Specifically, the authors propose a Bayesian variant of the original model that satisfies two goals. First, it allows one to quantify uncertainty regarding match or non-match status. In addition, posterior distributions are readily available to help estimate point estimates of interest (and quantify such estimates). Second, under the Bayesian paradigm, the one-to-one matching constraint is easily imbedded. The downside to this constraint is an added computational cost. The goal of our paper is proposing an extension of Sadinle (2017) that scales to large databases, but does not compromise much in terms of accuracy on the original case study considered by the author.

Our Contribution In this paper, we propose a Bayesian extension of Sadinle (2017) (BRL) through a modification to the model specification that allows for parallel computing of the linkage parameter, using hashing techniques to hasten calculations, and to reduce computational complexity. Specifically, Sadinle (2017) proposes a prior distribution that strictly enforces one-to-one matching. Instead, we propose independent priors (breaking the one-to-one matching) in order to create computational speeds and gains. Second, we resolve the one-to-one matching issue via a post processing step. Third, we derive conditional distributions, a Gibbs sampler, and corresponding Bayes estimates for our proposed model. Fourth, we explore other computational speeds up based upon storage efficient indexing. Finally, we explore a sensitivity analysis and revisit the case study of the El Salvadoran conflict to understand the practical ramifications regarding our proposed modeling choices. To recognize the lineage from the original BRL method, we name our method *fast beta linkage* (**fabl**, pronounced “fable”)

The remainder of the paper is as follows. Section ?? reviews bipartite record linkage; section ?? reviews the traditional approach of the Fellegi and Sunter framework before introducing our proposed work.

2 BIPARTITE RECORD LINKAGE AND RELATED WORK

As already mentioned, bipartite record linkage is the traditional set up studied in record linkage, where we assume there are two databases \mathbf{X}_1 and \mathbf{X}_2 such that there are duplications across the databases but not within each databases. We assume that each database has n_1

and n_2 records, and that $n_1 \leq n_2$. Denote the number of entities simultaneously recorded by *both databases* by n_{12} such that $0 \leq n_{12} \leq n_2$. Given this set up, as explained by Steorts et al. (2016) and Sadinle (2017), this can be viewed as bipartite record linkage. This is important as it allows one to view the matching matrix or linkage structure in the following way:

$$\Delta_{ij} = \begin{cases} 1 & \text{if records } i \in \mathbf{X}_1 \text{ and } j \in \mathbf{X}_2 \text{ refer to the same entity;} \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

Bipartite matching can also be viewed as a matching labeling $\mathbf{Z} = (Z_1, \dots, Z_{n_2})$ for the records in database \mathbf{X}_2 such that

$$Z_j = \begin{cases} i & \text{if records } i \in \mathbf{X}_1 \text{ and } j \in \mathbf{X}_2 \text{ refer to the same entity;} \\ n_1 + j & \text{if records } j \in \mathbf{X}_2 \text{ does not have a match in database } \mathbf{X}_1. \end{cases} \quad (2)$$

Depending on which representation is the most convenient, we can go back and forth between the two using $\Delta_{ij} = I(Z_j = i)$, where $I(\cdot)$ is the indicator function.

There are many approaches to bipartite record linkage, and we review some of the most common approaches below.

Fellegi and Sunter As already mentioned, the most common approach is the original approach of Fellegi and Sunter (1969). Enamorado et al. (2019) have extended this approach using computational speeds ups to allow for a much more scalable and practical approach, providing open source code for the community.

Directly Modeling the Data Instead of relying on comparison vectors, some authors will incorporate the data directly into their bipartite record linkage model (Fortini et al., 2001; Matsakis, 2010; Liseo and Tancredi, 2011; Tancredi and Liseo, 2011; Gutman et al., 2013; Shan et al., 2020). These approaches have led to extensions regarding downstream tasks in the literature or joint models, such as regression approaches (Dalzell and Reiter, 2018; Steorts et al., 2018; Tancredi et al., 2020).

There have been extensions to directly model the data in this literature at combining many databases, however, we do not review this as it's not directly comparable with the

situation of bipartite record linkage (Steorts et al., 2016; Steorts, 2015; Zanella et al., 2016; Marchant et al., 2019; Tancredi and Liseo, 2015; ?).

Classification Approaches Now state the related work here in this literature and common approaches.

ATTN: These are just notes and need to be edited

1. The most common approach is FS.
2. “A number of approaches directly model the information contained in the datafiles (Fortini et al., 2002; Matsakis, 2010; Liseo and Tancredi, 2011; Tancredi and Liseo, 2011; Gutman et al., 2013; Steorts et al., 2013), which requires crafting specific models for each type of field in the datafile, and are therefore currently limited to handle nominal categorical fields, or continuous variables modeled under normality.” [Update this literature]
3. Classification. “A more common way of tackling this problem is to see it as a traditional classification problem: we need to classify record pairs into matches and non-matches. If we have access to a sample of record pairs for which the true matching statuses are known, we can train a classifier on this sample using comparisons between the pairs of records as our predictors, and then predict the matching statuses of the remaining record pairs (e.g., Cochinwala et al., 2001; Bilenko et al., 2003; Christen, 2008; Ventura et al., 2013).” [Update this literature]

3 THE FELLEGI SUNTER APPROACH TO RECORD LINKAGE

I think this should be reviewed so that the paper is more clear.

4 RELATED WORK

In this section, we review work that is most related to our proposed work. For surveys of the record linkage literature, see (Christen, 2012; Herzog et al., 2007a; Liseo and Tancredi, 2013; Winkler, 2014; Christophides et al., 2021; Jurek-Loughrey and P, 2019).

Fellegi and Sunter Method Most record linkage techniques are derived from the seminal (?) paper "A Theory for Record Linkage". The defining characteristic of their approach was to transform the sets of records, which often contain text data that is difficult to model, into sets of comparison vectors governed by parameters that can be more easily estimated. Concretely, if files A and B have n_A and n_B records respectively, and if the files share F fields in common upon which to base the linkage, the Fellegi and Sunter approach generates an $n_A n_B \times F$ matrix Γ , which contains similarity scores between each pair of records across datasets. We say γ_{ij} is the comparison vector for record $i \in A$ and record $j \in B$, with $\gamma_{ij}^f \in \{1, \dots, L_f\}$ providing their similarity score on the f^{th} field. For ease of modeling and computation, we restrict these similarity scores to be discrete, ordinal variables, and the construction of these is left to the modeler. We adopt the convention that $\gamma_{ij}^f = 1$ corresponds to the highest level of agreement and $\gamma_{ij}^f = L_f$ corresponds to the lowest. It is common to use binary 1-2 variables to indicate exact matching, and 1-2-3 variables to provide an option for partial matching. For text data, we calculate similarity based on Levenstein distance or some other text similarity score, and bin these scores to integers for use in the model. (?) provided guidelines for appropriate binning for the comparison vectors, but details of implementation are generally context specific.

The likelihood in the original Fellegi Sunter model is a mixture model through which each record pair is independently classified as a match or nonmatch. This independent classification often leads to sets of matches that break expectations of transitivity, which is undesirable. Although (Jaro, 1995) devised an procedure to reduce a set of conflicting matches to the mostly likely set of one-to-one matchings, the abundance of false matches can bias the estimation of other parameters of the model, and ultimately leads to poor linkage performance.

(Sadinle, 2017) uses the same comparison vector approach as Fellegi and Sunter, but proposed a prior distribution for the set of matches that strictly enforced one-to-one matching without the need for any post-hoc processing procedures. Specifically, in each iteration of his Gibbs sampler, he considers each record $j \in B$, removes from consideration the records $i \in A$ that have already been matched, and then samples a potential link. This leads to a sampler that is significantly more accurate than the standard Fellegi-Sunter approach. However, accounting for these dependencies throughout the linkage process is computationally burdensome, leaving BRL only suitable for small to moderate linkage problems.

5 A SCALABLE MODEL FOR BAYESIAN RECORD LINKAGE

In this section, we propose a statistical model suitable for efficient computation in large linkage tasks. Denote two files as A and B , with n_A and n_B records respectively, and with records indexed as $i \in \{1, \dots, n_A\}$ in A and $j \in \{1, \dots, n_B\}$ in B . Without loss of generality, label the files such that $n_A \geq n_B$. We assume there are no duplicates within files, only across. For each record pair under consideration, we generate a comparison vector $\gamma_{ij} = \{\gamma_{ij}^1, \dots, \gamma_{ij}^F\}$, where F is the number of fields used in the linkage and each γ_{ij}^f takes on a value $l \in \{1, \dots, L_f\}$ indicating the level agreement between the two records on a specified field.

To indicate matching status, we adopt the *linkage structure parameter* $\mathbf{Z} = (Z_1, \dots, Z_{n_B})$ from Sadinle 2017, defined as

$$Z_j = \begin{cases} i, & \text{if records } i \in A \text{ and } j \in B \text{ refer to the same entity;} \\ n_A + 1, & \text{if record } j \in B \text{ does not have a match in file } A; \end{cases}$$

This parameter is vector of length n_B , providing more memory efficient storage for the linkage information than a $n_A \times n_B$ sparse matrix of indicators.

Following the Fellegi Sunter framework, we define $m^{fl} := P(\gamma_{ij}^f = l | Z_j = i)$ to be the probability of observing agreement level l in field f for records i and j given that the records

are a match, and similarly define $u^{fl} := P(\gamma_{ij}^f = l | Z_j \neq i)$, for non-matches. We also adopt Fellegi and Sunter’s conditionally independent fields assumption that the level of agreement on one field is independent of the level of agreement on another. Though this assumption is often not reasonable (for example, two individuals of a different genders are more likely to have different names than two individuals of the same gender), but it is common within the record linkage literature and generally leads to models that perform well in practice; see discussion for further remarks. Lastly, we define λ to be the (marginal) probability that some record $j \in B$ has a match in A .

5.1 Model Specification

For fields $f \in \{1, \dots, F\}$ and levels $l \in \{1, \dots, L_f\}$ we adopt the following likelihood and prior distributions.

$$P(\Gamma | \mathbf{Z}, \mathbf{m}, \mathbf{u}) = \prod_{j=1}^{n_B} \prod_{i=1}^{n_A} \left[\prod_{f=1}^F \prod_{l=1}^{L_f} m_{fl}^{I(Z_j=i)} u_{fl}^{I(Z_j \neq i)} \right]^{I(\gamma_{ij}^f=l)}$$

$$\mathbf{m}^f \sim \text{Dirichlet}(\alpha^{f1}, \dots, \alpha^{fL_f})$$

$$\mathbf{u}^f \sim \text{Dirichlet}(\beta^{f1}, \dots, \beta^{fL_f})$$

$$Z_j | \lambda = \begin{cases} \frac{1}{n_A} \lambda & z_j \leq n_A; \\ 1 - \lambda & z_j = n_A + 1 \end{cases}$$

$$\lambda \sim \text{Beta}(\alpha_\lambda, \beta_\lambda)$$

The prior for Z_j has equal probability of matching to all records $i \in A$, and non-matching probability governed by λ . Therefore a $\lambda \sim \text{Beta}(1, 1)$ corresponds to a prior belief that nonmatches and matches are equally likely, and a $\lambda \sim \text{Beta}\left(1, \frac{1}{n_A}\right)$ prior corresponds to a uniform prior on the labeling of \mathbf{Z} .

Here the reader should the relationship between our proposed model and that of Sadinle 2017. In his model, Sadinle constructs a prior distribution on the entire \mathbf{Z} vector, which induces a Gibbs sampler that strictly enforces one-to-one matching. In particular,

this sampler removes previously matches records from the set of candidate records when sampling Z_j , creating a dependency that makes the sampler *inherently serial*. We however use independent priors for each Z_j , creating a sampler that is *perfectly parallelizable*, allowing for significant computational gains. More importantly, since only the agreement pattern of Z_j is used for calculations within the Gibbs sampler, and not the particular record label, we can conduct this sampling only at the level of the unique agreement patterns, offering even more computational savings. In doing so however, we thereby weaken the one-to-one requirement from BRL; our sampler does ensure that each record in B can be matched to at most one record in A , but allows for the possibility that multiple records in B match to the same record in A . We then resolve these contradictions through a straightforward post-processing step. We explore the ramifications of this distinction with the El Salvador case study.

5.2 Gibbs Sampler

We work with the following factorization of the joint posterior distribution:

$$\begin{aligned}
p(\mathbf{Z}, \mathbf{m}, \mathbf{u}, \lambda | \Gamma) &\propto p(\Gamma | \mathbf{Z}, \mathbf{m}, \mathbf{u}) p(\mathbf{Z} | \lambda) p(\mathbf{m}, \mathbf{u}) p(\lambda) \\
&\propto \prod_{j=1}^{n_B} \prod_{i=1}^{n_A} \left[\prod_{f=1}^F \prod_{l=1}^{L_f} m_{fl}^{I(Z_j=i)} u_{fl}^{I(Z_j \neq i)} \right] \\
&\times \prod_{f=1}^F \prod_{l=1}^{L_f} m_{fl}^{\alpha_{fl}-1} \times \prod_{f=1}^F \prod_{l=1}^{L_f} u_{fl}^{\beta_{fl}-1} \\
&\times \prod_{j=1}^{n_B} \left[I(Z_j \leq n_A) \frac{1}{n_A} \lambda + I(Z_j = n_A + 1) (1 - \lambda) \right]
\end{aligned}$$

This factorization leads to following Gibbs Sampler:

Sample $\mathbf{m}^{(s+1)}$ $\mathbf{u}^{(s+1)} | \Gamma, \mathbf{Z}^{(s)}$: The \mathbf{m} and \mathbf{u} parameters are updated through standard multinomial-dirichlet mechanics. Thus we have

$$\mathbf{m}_f | \mathbf{Z}, \Gamma \sim \text{Dirichlet}(\alpha_{f1}(\mathbf{Z}), \dots, \alpha_{fL_f}(\mathbf{Z}))$$

$$\mathbf{u}_f | \mathbf{Z}, \Gamma \sim \text{Dirichlet}(\beta_{f1}(\mathbf{Z}), \dots, \beta_{fL_f}(\mathbf{Z}))$$

where $\alpha_{fl}(\mathbf{Z}) = \alpha_{fl} + \sum_{i,j} I(\gamma_{ij}^f = l)I(z_j = i)$ and $\beta_{fl}(\mathbf{Z}) = \beta_{fl} + I(\gamma_{ij}^f = l)I(z_j \neq i)$.

Sample $\lambda^{(s+1)}|\mathbf{Z}^{(s)}$: As a function of λ , the linkage structure parameter \mathbf{Z} is sequence of successes (when $z_j < n_A + 1$) and failures (when $z_j = n_A + 1$), and therefore $p(\mathbf{Z}|\lambda) = \mathcal{L}(\lambda|\mathbf{Z})$ is determined only by the number of duplicates $D = \sum_{i=1}^{n_B} \mathbf{1}_{z_j < n_A + 1}$ encoded by \mathbf{Z} . Thus we have

$$\begin{aligned} p(\lambda|\mathbf{Z}) &\propto p(\mathbf{Z}|\lambda)p(\lambda) \\ &\propto \lambda^D (1 - \lambda)^{n_B - D} \lambda^{\alpha_\lambda - 1} (1 - \lambda)^{\beta_\lambda - 1} \\ &\propto \lambda^{D + \alpha_\lambda - 1} (1 - \lambda)^{n_B - D + \beta_\lambda - 1} \\ \implies \lambda^{(s+1)}|\mathbf{Z}^{(s+1)} &\sim \text{Beta}(D + \alpha_\lambda, n_B - D + \beta_\lambda) \end{aligned}$$

Sample $\mathbf{Z}^{(s+1)}|\Gamma, \mathbf{m}^{(s+1)}, \mathbf{u}^{(s+1)}, \lambda^{(s+1)}$: Because we sample Z_j independently of all other $Z_{j'}$, we use only the full conditional for an individual Z_j . Let $\Gamma_{\cdot j}$ denote the set of n_A comparison vectors with $j \in B$, and note that as a function of Z_j , the likelihood $p(\Gamma_{\cdot j}|Z_j, \mathbf{m}, \mathbf{u}) = \mathcal{L}(Z_j|\Gamma_{\cdot j}, \mathbf{m}, \mathbf{u})$ is a discrete distribution with probabilities proportional to

$$\begin{aligned} p(\Gamma_{\cdot j}|Z_j = z_j, \mathbf{m}, \mathbf{u}) &\propto \prod_{i=1}^{n_A} \left[\prod_{f=1}^F \prod_{l=1}^{L_f} m_{fl}^{I(Z_j=i)} u_{fl}^{I(Z_j \neq i)} \right]^{I(\gamma_{ij}^f=l)} \\ &\propto \prod_{i=1}^{n_A} \left(\prod_{f=1}^F \prod_{l=1}^{L_f} \frac{m_{fl}}{u_{fl}} \right)^{I(z_j=i, \gamma_{ij}^f=l)} \\ &= \begin{cases} w_{ij} & z_j \leq n_A; \\ 1 & z_j = n_A + 1 \end{cases} \end{aligned}$$

where $w_{ij} = \left(\frac{\prod_{f=1}^F \prod_{l=1}^{L_f} m_{fl}}{\prod_{f=1}^F \prod_{l=1}^{L_f} u_{fl}} \right)^{I(\gamma_{ij}^f=l)} = \frac{P(\gamma_{ij}|Z_j=i)}{P(\gamma_{ij}|Z_j \neq i)}$. The interested reader should note that these are precisely the likelihood ratios used in the Fellegi-Sunter model to classify matches and non-matches, and we therefore refer to w_{ij} as the *Fellegi Sunter weights*.

With the likelihood in this form, we can derive the full conditional

$$\begin{aligned}
p(Z_j | \Gamma_{.j}, \mathbf{m}, \mathbf{u}, \lambda) &\propto p(\Gamma_{.j} | Z_j, \mathbf{m}, \mathbf{u}) P(Z_j | \lambda) \\
&\propto \left(\sum_{i=1}^{n_A} w_{ij} \mathbf{1}_{z_j=i} + \mathbf{1}_{z_j=n_A+1} \right) \left(\lambda \sum_{i=1}^{n_A} \frac{1}{n_A} \mathbf{1}_{z_j=i} + (1-\lambda) \mathbf{1}_{z_j=n_A+1} \right) \\
&= \frac{\lambda}{n_A} \sum_{i=1}^{n_A} w_{ij} \mathbf{1}_{z_j=i} + (1-\lambda) \mathbf{1}_{z_j=n_A+1} \\
&\implies Z_j^{(s+1)} | \mathbf{m}, \mathbf{u}, \Gamma, \lambda \propto \begin{cases} \frac{\lambda}{n_A} w_{ij} & z_j \leq n_A; \\ 1-\lambda & z_j = n_A + 1 \end{cases}
\end{aligned}$$

In order to make fair comparisons against the (Sadinle, 2017) model, we integrate over the posterior of λ and rearrange terms to produce the final full conditional:

$$p\left(Z_j^{(s+1)} = i | \mathbf{m}, \mathbf{u}, \mathbf{Z}^{(s)}\right) \propto \begin{cases} w_{ij} & i \leq n_A; \\ n_A \frac{n_B - D + \beta_\lambda}{D + \alpha_\lambda} & i = n_A + 1 \end{cases}$$

5.3 Bayes Estimate

We calculate a Bayes estimate $\hat{\mathbf{Z}}$ for the linkage parameter \mathbf{Z} by assigning different positive losses to different types of errors, and minimizing posterior expected loss. We adopt the loss function proposed in Sadinle 2017, in which $\hat{Z}_j \in \{1, \dots, n_A + 1, R\}$, with R representing the option to leave the matching undetermined by the model. Specifically, we minimize the quantity $L(\hat{\mathbf{Z}}, \mathbf{Z}) = \sum_{j=1}^{n_B} L(\hat{Z}_j, Z_j)$ where

$$L(\hat{Z}_j, Z_j) = \begin{cases} 0 & \text{if } Z_j = \hat{Z}_j; \\ \theta_R, & \text{if } \hat{Z}_j = R; \\ \theta_{10}, & \text{if } Z_j \leq n_A, \hat{Z}_j = n_A + 1; \\ \theta_{01}, & \text{if } Z_j = n_A + 1, \hat{Z}_j \leq n_A; \\ \theta_{11}, & \text{if } Z_j, \hat{Z}_j \leq n_A, Z_j \neq \hat{Z}_j; \end{cases}$$

Here, θ_R is the loss from not making a decision on the linkage status, θ_{10} is the loss from a false non-match, θ_{01} is the loss from a false match, and θ_{11} is the loss from the special case of a false match in which the record has a true match other than the one estimated by the model. This loss function leads to closed form decision rules for minimizing posterior expected loss. In this paper, we adopt losses $\theta_R = \infty, \theta_{10} = 1, \theta_{01} = 1, \theta_{11} = 2$, inducing the intuitive decision rule

$$\hat{Z}_j = \begin{cases} i, & \text{if } P(Z_j = i|\Gamma) > \frac{1}{2}; \\ 0, & \text{otherwise;} \end{cases}$$

For a more in-depth explanation of this function and the induced Bayes estimate, see (Sadinle, 2017).

Since our Gibbs procedure does not strictly enforce one-to-one matching, it is possible for the final Bayes estimate to link multiple records in B to one record in A . The modeler can either report both such matches (with their respective posterior match probabilities), or resolve these conflicts by accepting only the match with highest posterior probability. A similar approach can be seen in the most probable maximal matching sets used by (Steorts, 2013) to match records to latent entities. Such a resolution procedure indeed is equivalent to minimizing posterior risk under the restriction of one-to-one matching, and as thus theoretically justified within the Bayesian framework.

6 EFFICIENT AND SCALABLE IMPLEMENTATION

Broadly speaking, we increase our computational efficiency by recognizing that record pairs contribute to posterior calculations only through the agreement pattern of the γ_{ij} vector. Let \mathcal{H} be the set of unique agreement patterns in the data, let $P = |\mathcal{H}|$ denote the total number of unique agreement patterns. Note that P is bounded above by $\prod_{f=1}^F L_f$, and that this bound does not scale with n_A or n_B . Prior to processing the data, we identify all P patterns in \mathcal{H} and enumerate them h_1, \dots, h_P , and map record pairs to this hashed value. When the (i, j) pair exhibits the p^{th} agreement pattern, we say $(i, j) \in h_p$. Wherever possible, we conduct calculations over these P agreement patterns rather than the $n_A \times n_B$

record pairs. Some of these techniques are similar to those used by (?) to produce **fastlink**, a fast a scalable implementation of the Fellegi-Sunter model, and others have been created here for our particular context.

6.1 Data Representation, Hashing, and Storage

We first hash record pairs of the same agreement pattern to unique integer values. (?) accomplished this efficiently through the hashing function

$$\tilde{\gamma}_{ij} = \sum_{f=1}^F I(\gamma_{ij}^f > 0) 2^{\gamma_{ij}^f + I(k>1) \sum_{e=1}^{k-1} (L_e - 1)}$$

This function maps each agreement pattern to a unique integer, allowing us to store a scalar quantity instead of an entire vector for each record pair. For computational ease, we then map these integers to sequential integers from 1 to P.

The classic Fellegi Sunter method represents the γ_{ij} comparison as a vector of length F , with each component γ_{ij}^f taking on values in $\{1, \dots, L_f\}$. To ease computations, we instead use a one hot encoding of the comparison vector. For example, if $L_1 = L_2 = 2$ and $L_3 = 3$, then $\gamma_{ij} = (2, 1, 3)$ under the classical framework becomes $\gamma_{ij} = (0, 1, 1, 0, 0, 0, 1)$ under our framework. This is a bijective transformation that does not change the meaning of the data, but this representation eases calculations and posterior updates. This is also the form the data takes in the BRL package in R.

In the classic Fellegi Sunter framework, Γ is a $n_A n_B \times F$ matrix, with each row providing the comparison vector for a different (i, j) pair. In contrast, we do not store these comparison vectors themselves, but instead only the hashed value h_p corresponding to the agreement pattern of the (i, j) pair. We store this information in a nested list $\tilde{\Gamma}$ where the p^{th} component of the j^{th} list contains a vector of records in A that share agreement pattern p with record $j \in B$. For each p , we also calculate $H_p = \sum_{i=1}^{n_A} \sum_{j=1}^{n_B} \mathbf{1}_{(i,j) \in h_p}$, the total instances of agreement pattern p throughout the data, and also for each j , we calculate $H_{p_j} = \sum_{i=1}^{n_A} \mathbf{1}_{(i,j) \in h_p}$ the instances of agreement pattern p among the comparison vectors between record $j \in B$ and each of the n_A records in A .

The hashing procedure described above considerably reduces the memory needed to

store the comparison information: instead of storing $n_A \times n_B$ comparison vectors, which are relatively long under either the Fellegi Sunter or our modified framework, we only store the P unique vectors, and then $n_A \times n_B$ scalar quantities relating record pairs to those vectors. However, even storing these $n_A \times n_B$ scalar labels can become burdensome with large data. Worse, the overwhelming majority of these labels relate to record pairs that are clear non-matches.

To address this, we propose a new strategy called *storage efficient indexing* (SEI). In standard indexing, the modeler decides a certain criteria that they expect all true matching pairs to satisfy, and a priori label any record pairs that do not meet that criteria as non-matching. For example, one might only consider pairs with a certain similarity score on a field deemed to be important (like first name), or only pairs with exact matching on a specified number of fields. While generally chosen to be quite loose, establishing these criteria requires knowledge of the problem and invites room for human error. We propose a method of reducing the comparison space (and reducing storage requirements) without these drawbacks. Note that all records of the same agreement pattern have the same probability when sampling Z_j . Therefore we know that records belonging to an h_p such that H_{p_j} is large are very unlikely to be sampled consistently enough to be deemed a match through the Bayes estimate, even without considering the form of the agreement pattern itself.

In SEI, rather than store all of these unlikely record labels, we choose to store only a small number R of them. Posterior calculations still attribute the appropriate weight to all records through the summary statistics H_p , and H_{p_j} . Rather than storing $n_A \times n_B$ record labels, SEI allows us to store at most $n_B \times P \times R$ labels, regardless of how large n_A is.

Lastly, for large data, we can partition the two datasets A and B into smaller blocks $\{A_m\}$ and $\{B_m\}$ for more manageable computations. On a single machine, we can read-in data sequentially, conduct hashing, compress information through SEI, and delete the original data from memory before continuing with the next chunk of data. With multiple cores or multiple machines, this can be done in parallel. Thus the combination of hashing, SEI, and partitioning allows us to conduct linkage tasks over much larger datasets.

7 EFFICIENT POSTERIOR INFERENCE

Updating \mathbf{m} and \mathbf{u} : After receiving matching statuses from \mathbf{Z} , the Sadinle method calculates $\alpha_{fl}(\mathbf{Z})$ and $\beta_{fl}(\mathbf{Z})$ for each field and level. This constitutes $2 \times \sum L_f$ many summations over $n_A \times n_B$ quantities, and becomes computationally burdensome with large data. In contrast, we recognize that each unique agreement pattern contributes to the posterior $\alpha(\mathbf{Z})$ and $\beta(\mathbf{Z})$ vectors in the same way. In fact, if we denote $H_p^m = \sum_{j=1}^{n_B} \mathbf{1}_{(Z_j, j) \in h_p}$ to be the number of matching record pairs with agreement pattern p , then the contribution of pairs of pattern p to the $\alpha(\mathbf{Z})$ vector is simply $H_p^m \times h_p$. Thus our posterior update for the α vector is simply $\alpha(\mathbf{Z}) = \alpha_0 + \sum_{p=1}^P H_p^m \times h_p$. Then, we can easily calculate H_p^u , the number of nonmatching record pairs of agreement pattern p , by subtracting the number of matching pairs from the total present in the data; that is $H_p^u = H_p - H_p^m$. From this, we can update our β parameter through $\beta(\mathbf{Z}) = \beta_0 + \sum_{p=1}^P H_p^u \times h_p$. Note that these constitute P many summations over n_B quantities, and thus avoid the $n_A \times n_B$ summation from the original method.

Updating \mathbf{Z} : Although sampling Z_j from a the full conditional provided earlier is conceptually straightforward, it becomes computational burdensome when n_A is larger. The reader can confirm that sampling a value from a large set of unequal probabilities becomes difficult in most programming languages. To speed up computation, we break this sampling step into two simpler steps. First, we calculate the Fellegi Sunter weight w_p associated with each unique pattern and sample the agreement pattern between j and its potential match. Second, we sample the record label uniformly among records associated with that agreement pattern for that particular $j \in B$. More concretely, define $h(Z_j)$ to be the agreement pattern between j and its potential match, and say $h(Z_j) = h_{P+1}$ when $Z_j = n_A + 1$. Then,

$$P\left(h\left(Z_j^{(s+1)}\right)=p \mid \mathbf{m}, \mathbf{u}, \mathbf{Z}^{(s)}\right) \propto \begin{cases} w_p \times H_{p_j} & p \leq P; \\ n_A \frac{n_B - D + \beta_\lambda}{D + \alpha_\lambda} & p = P + 1 \end{cases}$$

$$P\left(Z_j^{(s+1)}=i \mid h\left(Z_j^{(s+1)}\right)=p\right)=\begin{cases} \frac{1}{H_{p_j}} & (i, j) \in h_p \\ 0 & \text{otherwise} \end{cases}$$

Lastly, we recognize that all posterior updates are governed by the agreement patterns of the record pairs rather than the record labels themselves. Thus we complete the entire Gibbs procedure first at the level of the P agreement patterns with the first equation above. After, we can back-fill the records corresponding to the agreement patterns through the second equation. Sampling uniformly is computationally simple even for large sets of candidate records, but this step can also be parallelized when working with large data.

To aid the reader, we provide summary of the **fabl** method through pseudocode:

Algorithm 1 Summary of fabl algorithm

```

1: procedure HASHING AND PREPROCESSING
2:   Partition files  $A$  and  $B$  into chunks  $\{A_I\}, \{B_J\}$ 
3:   for each  $I, J$  do
4:     Create comparison vectors between  $A_I$  and  $B_J$ 
5:     Hash results and calculate summary statistics
6:     Use SEI to reduce memory usage
7:   end for
8:   Synthesize results across pairings
9: end procedure
10: procedure GIBBS SAMPLING
11:   Initialize  $m$ ,  $u$ , and  $Z$  parameters
12:   for  $t \in \{1, \dots, T\}$  do
13:     Sample  $m^{t+1} | Z^t, \Gamma$  and  $u^{t+1} | Z^t, \Gamma$ 
14:     Sample  $H(Z^{t+1}) | m^{t+1}, u^{t+1}, \Gamma$  ▷ Sample agreement pattern, not record
15:   end for
16:   Sample  $Z | H(Z), \Gamma$  ▷ Fills in record label based on agreement pattern
17: end procedure

```

8 SIMULATION STUDIES

8.1 Accuracy

We first compare `fabl` against `BRL` on several simulated datasets with varying amounts of error and overlap. We use first name, last name, age, and occupation for this linkage, and create comparison vectors according to the default settings of the `compareRecords` function from the `BRL` package. Each simulation identifies duplicated individuals between two datasets, each with 500 records. We conduct linkage when matching records exhibit 1, 2, and 3 errors across the four fields, and when there are 50, 250, and 450 individuals in common across datasets. We use flat priors for all m and u parameters, run the Gibbs Sampler for 1000 iterations, and discard the first 100 as burn-in. This is a near exact replication of the simulation study provided by (?).

In cases when there are only one or two errors in matching records, and in cases with low to moderate duplication across records, we see that `fabl` provides approximately equivalent accuracy as `BRL`. We find that our method only has weakened performance in the most extreme scenario of very high errors and very high overlap across files. In these situations, `BRL` is removing large numbers of records from consideration throughout the Gibbs Sampler, making its implementation most different from `fabl`. We note however that such extreme linkage tasks, which such high amounts of errors and overlap, are extremely rare in practice.

8.2 Speed

To demonstrate speed, we generate comparison vectors from pre-specified distributions so that we can easily increase the size of the linkage problem. Distributions are meant to emulate the behavior of similarity scores across first name, last name, and day, month. For example, $u^{\text{month}, 1} = P(\text{Records have same birth-month} \mid \text{Nonmatch}) = \frac{1}{12}$. For simplicity, we consider only exact matching, so a vector $(1, 0)$ corresponds to agreement and $(0, 1)$ to disagreement. We simulate these data for different values of n_A and n_B , and compare the run-time of `fabl` against `BRL`. Note that the number of unique patterns P is bounded above by $2^5 = 32$, a bound which is consistently attained in the larger simulations.

We see that at low data size, `BRL` outperforms, but that `fabl` is significantly faster at handling larger data. In particular, run-time for `BRL` seems to grow quadratically (or

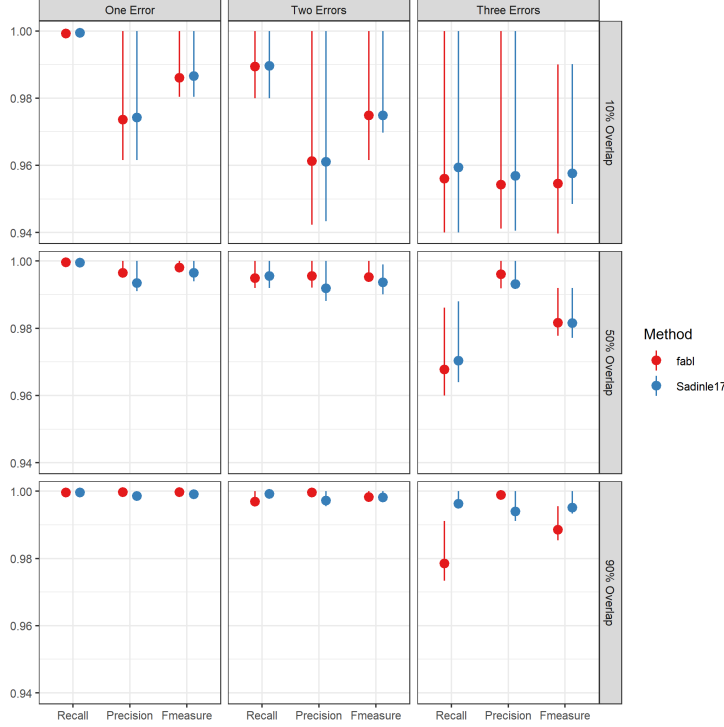


Figure 1: Posterior means and credible intervals for accuracy metrics under the replication of simulation study from Sadinle 2017. For each level of overlap and each level of error, we have 100 paired sets of 500 records.

linearly with the size of both A and B) while run-time for **fabl** seems to grow linearly (in the size of B).

The above discussion suggests that for fixed n_B , computation time should remain mostly constant with growing n_A . Simulation study suggests that this is true. In the plot below, fixing $n_B = 500$, we see linear growth for the run-time under BRL as n_A increases, with much more static run-time under **fabl**. The slight increases in run-time that we do see are due primarily to the hashing step, which again can be run in parallel for large data.

We note here that BRL is coded in C, which makes for unfair comparison against **fabl**, currently only built in R. Additionally, although **fabl** is amenable to parallelization, this simulation was run on a single core. Running **fabl** in C++ with parallelization for the hashing step and sampling the matching status of the record pairs should lead to even more computational gains.

	m	u
fname	$(\frac{19}{20}, \frac{1}{20})$	$(\frac{1}{100}, \frac{99}{100})$
lname	$(\frac{19}{20}, \frac{1}{20})$	$(\frac{1}{100}, \frac{99}{100})$
day	$(\frac{19}{20}, \frac{1}{20})$	$(\frac{1}{30}, \frac{29}{30})$
month	$(\frac{19}{20}, \frac{1}{20})$	$(\frac{1}{12}, \frac{11}{12})$
year	$(\frac{19}{20}, \frac{1}{20})$	$(\frac{1}{12}, \frac{11}{12})$

Table 1: Distributions used for m and u probabilities in simulation studies

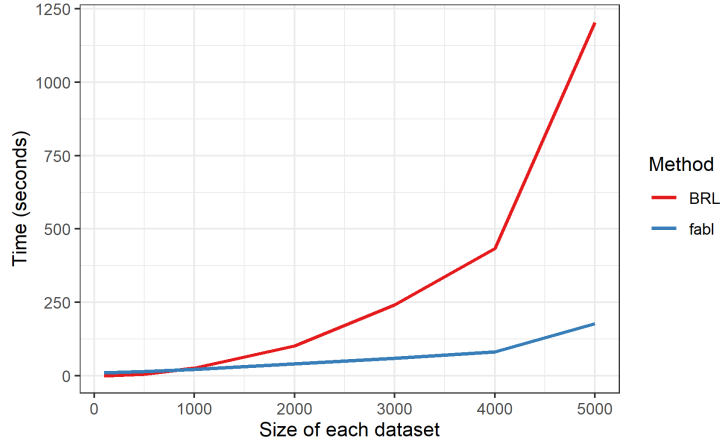


Figure 2: Run-time for BRL and fabl to run 1000 Gibbs iterations, including hashing step for fabl, for increasing values of both n_A and n_B . We see near quadratic growth in runtime for BRL, and near linear growth for fabl.

8.3 Scale

In one final simulation, we demonstrate how partitioning and hashing the data through **fabl** method allows us undertake significantly larger linkage tasks. We concatenate two sets of 40 simulated datasets from the Sardinle simulation to create two larger files, each of 20,000 records. Working on one machine, we sequentially compare records across chunks, hash results, and then synthesize summary statistics for the entire simulation. Under standard Fellegi Sunter procedures, this would require 400,000,000 comparison vectors, each consisting for four integers, resulting a final comparison matrix about 6.4 GB in size.

For simplicity, we partition one dataset into 20 smaller chunks, and leave the second

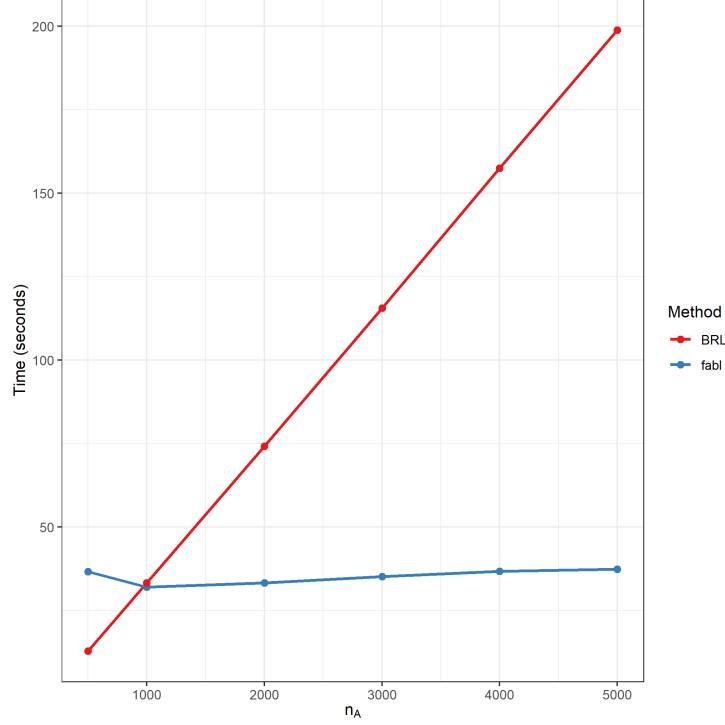


Figure 3: Run-time for BRL and fabl to run 1000 Gibbs iterations, including hashing step for fabl, with increasing n_A , and n_B fixed at 500. We see linear growth in runtime for BRL, and near constant runtime for fabl.

dataset fully intact. We then compare records, hash results, and then synthesize summary statistics for all 20 chunk comparisons. The resulting data object is only 90 MB, about 1% the size of the object required under the standard method. Executed sequentially, these comparisons and the Gibbs sampler take about one hour to run; using distributed computing, it could be completed much faster. This simulation achieved 96.5% recall and 97.7% precision, with an overall F-measure of 97.1% F-measure. The reader will note that this slightly worse performance than witnessed in the smaller simulation studies; this is expected because it is naturally more difficult to link more files with the same amount of information. With more linkage fields **fabl** maintains the high accuracy seen above.

9 CASE STUDIES

We now demonstrate the power of our method through a case study of documented identifiable deaths (DID) from the El Salvadoran Civil War. Though the data files used here are small, this study shows how the computational complexity of `fabl` depends on the number of unique agreement patterns, and how significant computational gains can be achieved by simplifying the construction of the comparison vectors. Secondly, the case study reveals the impact of independently sampling Z_j rather than strictly enforcing one-to-one matching as done by `BRL`.

9.1 El Salvadoran Civil War

The country of El Salvador was immersed in civil war from 1980 to 1991, and throughout the time, several organizations attempted to document casualties of the conflict. When estimating the total number of casualties, one cannot simply sum the numbers recorded by each organization, as it is likely that the same individuals are recorded in multiple casualty lists. To obtain a more accurate estimate of the casualties then, we follow Sadinle’s 2017 paper and link records of El Salvadoran civilian casualties from two sources: El Rescate - Tutela Regal (ERTL) and the Salvadoran Human Rights Commission (CDHES, by its acronym in Spanish). The ERTL dataset consists of digitized reports that had been published throughout the conflict. The CDHES dataset consists of casualties that had been reported directly to the organization, and later digitized.

There are several challenges with working with such data. Firstly, both datasets have been automatically digitized, which inherently leads to some degree of typographical error. Secondly, the CDHES records are all second hand accounts reported by individuals, which can result in additional errors. Lastly, the only fields recorded are given name, last name, date of death, and place of death; it is relatively common for a parent and child to share the same given name, resulting in indistinguishable records for two different individuals. This last point nearly breaks the earlier mentioned assumption that there are no duplicates within files, and reveals a key difference between `BRL` and our proposed method.

We only utilized records with nonmissing entries for given and last name, results in

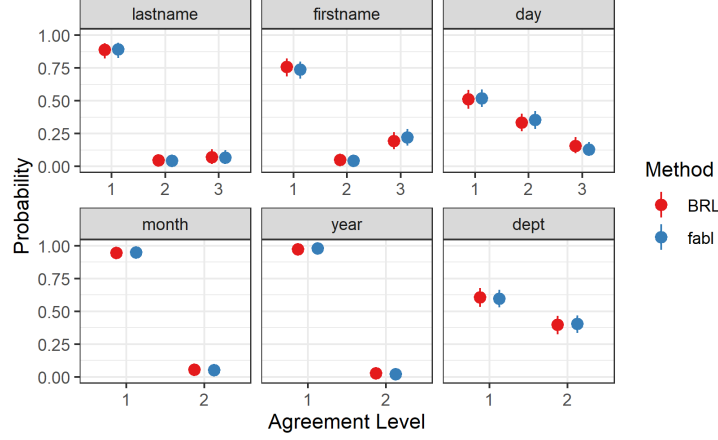


Figure 4: Posterior estimates of \mathbf{m} parameters with 95% credible intervals

$n_A = 4420$ files in CHDES and $n_B = 1323$ files in ERTL. The names were standardized to account for common misspellings in the Spanish language, and then compared using a modified Levenstein distance to account for the fact that second names are often omitted. Place of birth is recorded by municipality and department within that municipality; however, since department was missing in 95% of records in CHDES and 80% of records in ERTL, we excluded it from our linkage process. Thus we conduct linkage using given name, last name, municipality, and day, month, and year of death. We again use flat priors for the \mathbf{m} and \mathbf{u} parameters.

To mirror the original implementation, we constructed the comparison vectors using 4 levels of agreement for each field, according to the thresholds provided in Figure XX. This took 422 seconds for our proposed method and 240 for BRL. However, we observed that posterior distributions of several levels of the \mathbf{m} and \mathbf{u} parameters were nearly identical, and that of the $4^5 \times 2 = 2048$ possible agreement patterns, only 1173 are realized in the data. This leads us to believe that such high number of agreement levels creates unnecessary distinctions in the data and makes the comparison vectors less interpretable. Therefore we re-ran our analysis with fewer agreement levels for each field (see Figure XX), and obtained analogous results. With 216 possible agreement patterns, 159 were realized in the data, and our proposed method became much faster, finishing in 124 seconds. Meanwhile BRL took 239 seconds, relatively unchanged from the first implementation. This demonstrates the

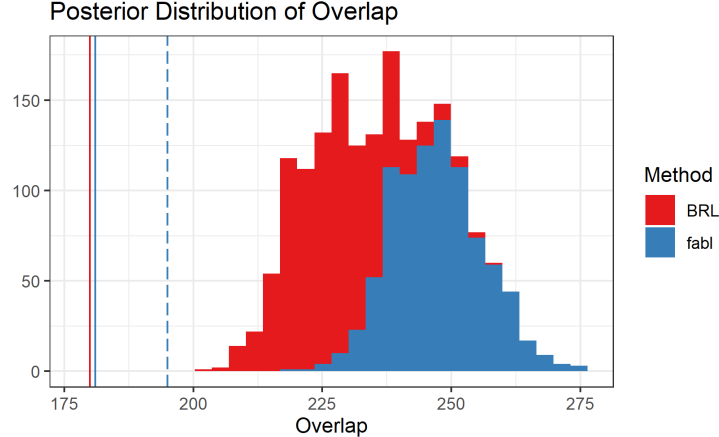


Figure 5: Posterior distribution of overlap across the two files. The solid lines show the Bayes estimate for the amount of overlap, and the dashed line is the Bayes estimate under `fabl` before resolving violations of one-to-one matching

way that the computational complexity of our method depends on the number of unique agreement patterns, and how significant computational gains can be made by simplifying the construction of the comparison vectors. We also note that estimates of the \mathbf{m} and \mathbf{u} parameters under each method are very similar, as shown in Figure 4.

Violations of One-to-One Matching Figure 5 shows the posterior distribution for D , the number of duplicates found across file for both `fabl` and `BRL`. We see that `fabl` consistently overmatches within each Gibbs iteration when compared to `BRL`, which is to be expected because `BRL` explicitly prevents matches that violate one-to-one matching throughout the entire sampler. Most of such matchings are due to the randomness in the sampling procedure, and they occur sporadically throughout the sampler in such a way that does not measurably influence the eventual Bayes estimate.

Overall, both methods presented similar results. `fabl` yielded an initial Bayes estimate of 195 matches found across files, and after resolving matches that violated one-to-one requirements, yielded 180 matches across files. This is acceptably close to the 181 matches found by `BRL`. The main reason for this discrepancy is the difference in how each model handles situations in which one file in B has multiple plausible matches in A .

	lastname	firstname	dataset	day	month	year	dept	muni
825	PINEDA	ROSA	CDHES	6	4	1984	NA	NA
826	PINEDA	ROSA MARIA	CDHES	6	4	1984	NA	NA
2776	PINEDA	ROSA MARIA	ER-TL	4	4	1984	CUSCATLAN	NA

Figure 6: Example of linkage situation with multiple plausible matches

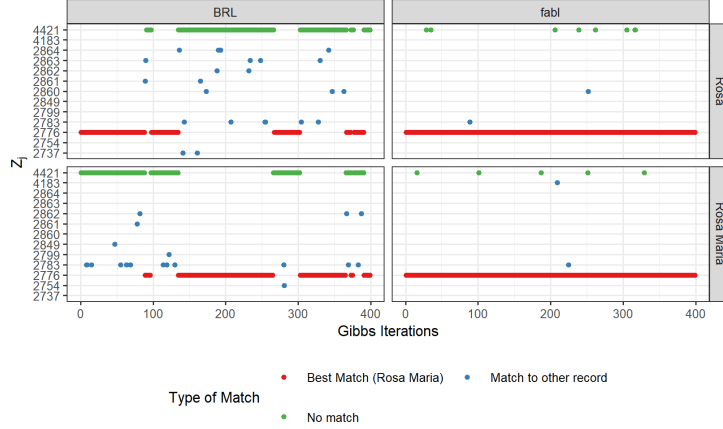


Figure 7: Gibbs sampling in situation with multiple plausible matches.

The records shown in Figure 6 provide one such instance. Note these records present a near violation of our assumption that there are no duplications within files; we continue to assume that records 825 and 826 in ERTL correspond to different individuals, but their records are nearly identical. Additionally, using the modified Levenstein distance from Sadinle 2017, the comparison vectors $\gamma_{2776,825}$ and $\gamma_{2776,826}$ are exactly identical.

Figure 7 shows the values that Z_{825} and Z_{826} take on throughout the Gibbs sampler, and demonstrates how each method handles this situation. Under **fabl**, both records in B match to the same record in A throughout the Gibbs process, creating consistent violations of one-to-one matching. Under **BRL**, the Gibbs process creates one matching configuration stays there for a while. However, if one pair “unmatches,” then the other record has a chance to latch on. Then, the Gibbs process is stuck with that matching status for a while, resulting in a Gibbs process with poor mixing. Additionally, **fabl** allows the modeler to inspect records with multiple plausible matches, and if they desire, to then choose the

record pairing with the highest posterior probability. BRL in contrast, in strictly enforcing one-to-one matching throughout the sampler, can lead to situations where none of the plausible matches reach the threshold to be identified through the Bayes estimate.

10 DISCUSSION

We have presented a method for Bayesian record linkage that is feasible for large datasets. In particular, our hashing procedure and model assumptions allow for a linkage procedure whose computational complexity does not scale with the size of the larger dataset, making this method particularly powerful in linking records when one datafile is substantially smaller than the other.

In our case study, we included an exploration of how to conduct record linkage when modeling assumptions are not met in practice. We explored “one-to-many” scenarios in which one record in A has multiple plausible matches in B , and showed how both **fabl** and BRL demonstrated undesirable qualities. Other issues arise under “many-to-one” scenarios, where one record in B has multiple plausible matches in A , and “many-to-many” scenarios in which there is duplication both across and within datasets. Tuning **fabl** for use in these scenarios is one potential avenue for future work.

11 APPENDIX

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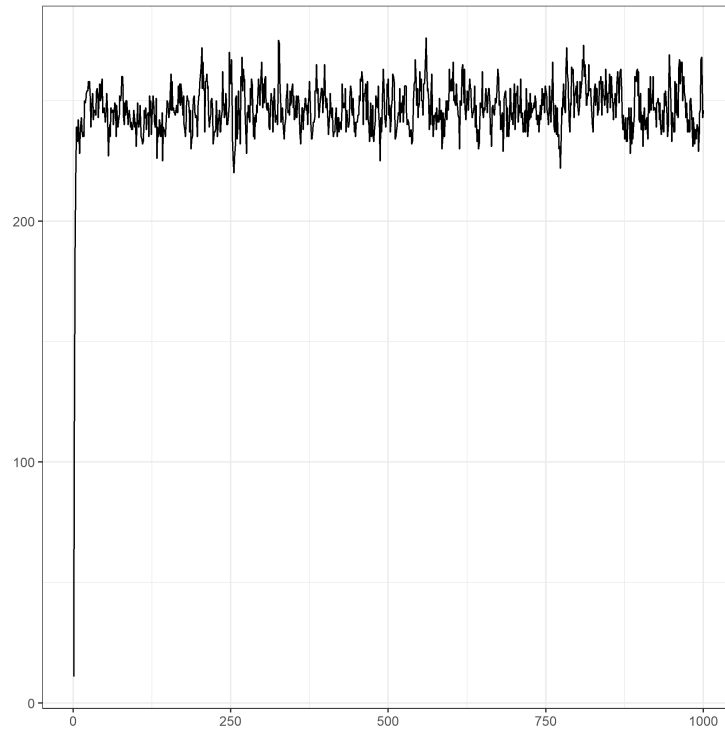


Figure 8: Traceplot for number of matches found across datasets in El Salvador case study

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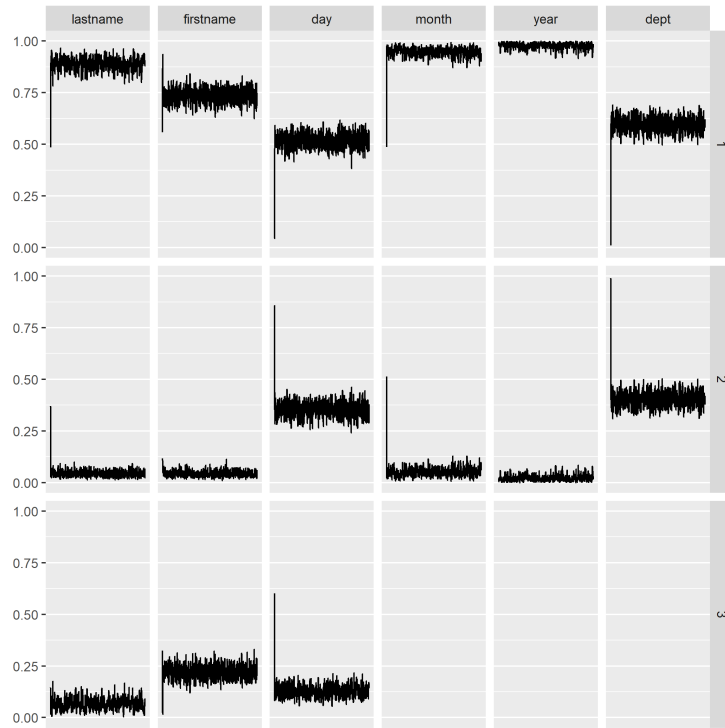


Figure 9: Traceplot for m parameter in El Salvador case study

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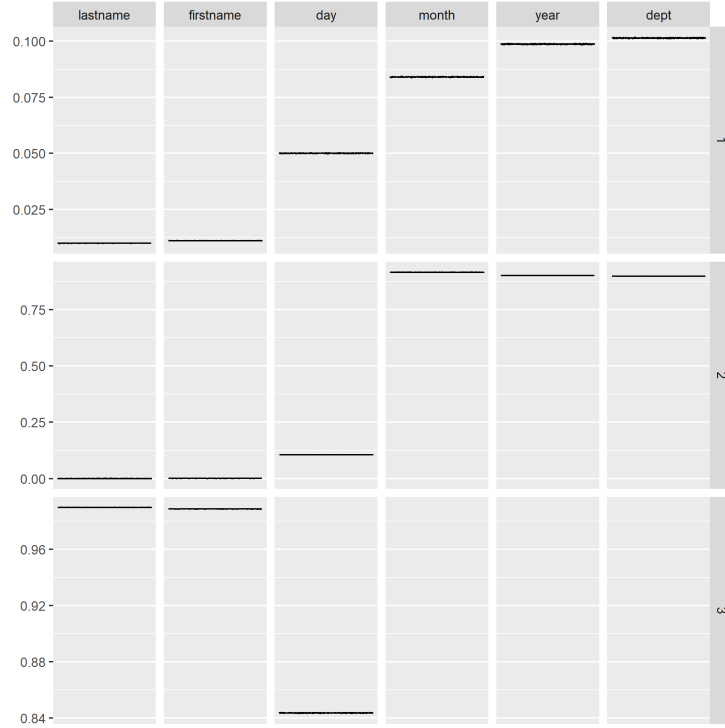


Figure 10: Traceplot for u parameter in El Salvador case study

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