MAS374 Optimization theory
Homework #4

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Problem 1 (*Exercise 6.1* in [1]: Least-squares and total least-squares)**.**

Let

$$\mathbf{X} := \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ 1 & x_4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \in \mathbb{R}^{4 \times 2} \quad \text{and} \quad \mathbf{y} := \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \in \mathbb{R}^4.$$

We first find the least-squares (LS) line for the given data-set $\left\{(x_i, y_i) \in \mathbb{R}^2 : i \in [4]\right\}$. The LS problem seeks for the line $y = \theta_0^{\mathsf{LS}} + \theta_1^{\mathsf{LS}} x$ that matches with the given data-set most well by minimizing the *sum of squared residuals*:

$$\min_{(\theta_0, \theta_1) \in \mathbb{R}^2} \sum_{i=1}^{4} \{y_i - (\theta_0 + \theta_1 x)\}^2 = \|\mathbf{y} - \mathbf{X}\boldsymbol{\theta}\|_2^2, \tag{1.1}$$

where $\boldsymbol{\theta} = \begin{bmatrix} \theta_0 \\ \theta_1 \end{bmatrix} \in \mathbb{R}^2$. So it suffices to find the optimal solution $\boldsymbol{\theta}^{\mathsf{LS}} = \begin{bmatrix} \theta_0^{\mathsf{LS}} \\ \theta_1^{\mathsf{LS}} \end{bmatrix} \in \mathbb{R}^2$ to the LS problem (1.1). It's clear that $\mathbf{X} \in \mathbb{R}^{4 \times 2}$ has full column-rank, thereby $\mathbf{X}^\top \mathbf{X} \in \mathbb{R}^{2 \times 2}$ is invertible. From the normal equation $\mathbf{X}^\top \left(\mathbf{y} - \mathbf{X}\boldsymbol{\theta}^{\mathsf{LS}}\right) = \mathbf{0}$, one has

$$\boldsymbol{\theta}^{\mathsf{LS}} = \left(\mathbf{X}^\top \mathbf{X}\right)^{-1} \mathbf{X}^\top \mathbf{y} = \begin{bmatrix} \frac{3}{10} \\ \frac{2}{5} \end{bmatrix} \in \mathbb{R}^2. \tag{1.2}$$

Thus, the least-squares (LS) line for the given data-set is given by

$$y = \theta_0^{\mathsf{LS}} + \theta_1^{\mathsf{LS}} x = \frac{3}{10} + \frac{2}{5} x = 0.3 + 0.4x. \tag{1.3}$$

We next look for the total least-squares (TLS) line for the given data-set $\left\{(x_i, y_i) \in \mathbb{R}^2 : i \in [4]\right\}$. To this end, we first consider the linear system $\mathbf{y} = \mathbf{X}\boldsymbol{\theta}$. We note that the LS problem can be interpreted as follows: find a *minimal perturbation* $\delta\mathbf{y} \in \mathbb{R}^4$ in the $\mathbf{y}$ term so that the linear system $\mathbf{y} + \delta\mathbf{y} = \mathbf{X}\boldsymbol{\theta}$ becomes feasible, *i.e.*,

$$\begin{aligned} &\min_{\delta\mathbf{y} \in \mathbb{R}^4} \ \|\delta\mathbf{y}\|_2^2 \\ &\text{subject to} \ \ \mathbf{y} + \delta\mathbf{y} \in \mathcal{R}(\mathbf{X}). \end{aligned} \tag{1.4}$$

This interpretation of the LS problem was covered in our lectures.

On the other hand, the total least-squares (TLS) approach extends this interpretation of the LS problem by allowing the perturbation to act both on the $\mathbf{y}$ term and on the $\mathbf{X}$ matrix. The TLS problem searches for the pair of *minimal perturbations* $\delta\mathbf{X} \in \mathbb{R}^{4\times 2}$ and $\delta\mathbf{y} \in \mathbb{R}^4$ in the $\mathbf{X}$ matrix and the $\mathbf{y}$ term, respectively, so that the linear system $\mathbf{y} + \delta\mathbf{y} = (\mathbf{X} + \delta\mathbf{X})\,\boldsymbol{\theta}$ is feasible, *i.e.*,

$$\begin{aligned} \min_{(\delta\mathbf{X},\delta\mathbf{y})\in\mathbb{R}^{4\times 2}\times\mathbb{R}^4} \ & \left\| \begin{bmatrix} \delta\mathbf{X} & \delta\mathbf{y} \end{bmatrix} \right\|_{\mathsf{F}}^2 \\ \text{subject to} \ & \mathbf{y} + \delta\mathbf{y} \in \mathcal{R}\left(\mathbf{X} + \delta\mathbf{X}\right). \end{aligned} \tag{1.5}$$

Let $\mathbf{A} := \begin{bmatrix} \mathbf{X} & \mathbf{y} \end{bmatrix} \in \mathbb{R}^{4\times 3}$ and $\delta\mathbf{A} := \begin{bmatrix} \delta\mathbf{X} & \delta\mathbf{y} \end{bmatrix}$. For the given data-set, we have

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}.$$

Then, $\mathbf{A}$ satisfies the technical assumptions required for the validity of *Theorem 6.2* in [1]:

(i) $\mathsf{rank}(\mathbf{A}) = 3$;

(ii) $\sigma_3(\mathbf{A}) = \sigma_{\min}(\mathbf{A}) < \sigma_{\min}(\mathbf{X})$.

It is straightforward to see that the assumption (i) holds. So it remains to verify that the assumption (ii) holds. From

$$\mathbf{A}^\top\mathbf{A} = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 6 & 3 \\ 2 & 3 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{X}^\top\mathbf{X} = \begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix}$$

together with the observations $\sigma_i(\mathbf{A}) = \sqrt{\lambda_i\left(\mathbf{A}^\top\mathbf{A}\right)}$ for $i \in [3]$, and $\sigma_j(\mathbf{X}) = \sqrt{\lambda_i\left(\mathbf{X}^\top\mathbf{X}\right)}$ for $j \in [2]$, one can compute both $\sigma_{\min}(\mathbf{A})$ and $\sigma_{\min}(\mathbf{X})$ numerically:

$$\begin{aligned} \sigma_{\min}(\mathbf{A})^2 &= \sigma_3(\mathbf{A})^2 = \lambda_3\left(\mathbf{A}^\top\mathbf{A}\right) \approx 0.15927314; \\ \sigma_{\min}(\mathbf{X})^2 &= \sigma_2(\mathbf{X})^2 = \lambda_2\left(\mathbf{X}^\top\mathbf{X}\right) = 4 - \sqrt{13} \approx 0.39444872, \end{aligned}$$

thereby the assumption (ii) holds. At this moment, we note that the eigenvalues of the 3×3 positive definite matrix $\mathbf{A}^\top\mathbf{A}$ (as well as the singular values of $\mathbf{A}$) should be computed numerically, since the characteristic polynomial $\mathsf{ch}_{\mathbf{A}^\top\mathbf{A}}(x) := \mathsf{det}\left(x\mathbf{I}_3 - \mathbf{A}^\top\mathbf{A}\right)$ of $\mathbf{A}^\top\mathbf{A}$ is irreducible over the rational number field $\mathbb{Q}$:

$$\mathsf{ch}_{\mathbf{A}^\top\mathbf{A}}(x) = x^3 - 12x^2 + 27x - 4.$$

To this end, for instance, I made use of the function `np.linalg.svd` in Python 3.

From *Theorem 6.2* in [1], we find that the TLS problem (1.5) has the unique optimal solution $\left((\delta\mathbf{X})^*, (\delta\mathbf{y})^*\right) \in \mathbb{R}^{4\times 2}\times\mathbb{R}^4$ and it satisfies

$$(\delta\mathbf{A})^* = \begin{bmatrix} (\delta\mathbf{X})^* & (\delta\mathbf{y})^* \end{bmatrix} = -\sigma_3(\mathbf{A})\cdot\mathbf{u}_3\mathbf{v}_3^\top,$$

where $\mathbf{A} = \sum_{i=1}^{3} \sigma_i(\mathbf{A}) \cdot \mathbf{u}_i \mathbf{v}_i^\top$ is the compact-form SVD of $\mathbf{A} \in \mathbb{R}^{4\times 3}$. Moreover, the solution $\boldsymbol{\theta}^{\mathsf{TLS}} \in \mathbb{R}^2$ of the feasible linear system $\mathbf{y} + (\delta\mathbf{y})^* = \{\mathbf{X} + (\delta\mathbf{X})^*\}\,\boldsymbol{\theta}$ uniquely exists, and it is given by

$$\begin{aligned}
\boldsymbol{\theta}^{\mathsf{TLS}} &= \left\{\mathbf{X}^\top\mathbf{X} - \sigma_{\min}(\mathbf{A})^2\mathbf{I}_2\right\}^{-1}\mathbf{X}^\top\mathbf{y} \\
&= \frac{1}{20 - 10\sigma_{\min}(\mathbf{A})^2 + \sigma_{\min}(\mathbf{A})^4}\begin{bmatrix} 6 - 2\sigma_{\min}(\mathbf{A})^2 \\ 8 - 3\sigma_{\min}(\mathbf{A})^2 \end{bmatrix} \\
&\approx \begin{bmatrix} 0.30822795 \\ 0.40809032 \end{bmatrix},
\end{aligned} \tag{1.6}$$

where the approximated value in (1.6) is computed numerically via the functions `np.linalg.inv` and `np.linalg.svd` in Python 3. Therefore, the total-least squares (TLS) line for the given data-set is

$$y = \theta_0^{\mathsf{TLS}} + \theta_1^{\mathsf{TLS}} x \approx 0.30822795 + 0.40809032x. \tag{1.7}$$

Finally, we plot both the LS line (1.3) and the TLS line (1.7) on the same set of axes. It can be done by using the following code in Python 3:

```python
import numpy as np
import matplotlib as mpl
import matplotlib.pyplot as plt

def axes():
    plt.axhline(0, alpha=.1)
    plt.axvline(0, alpha=.1)

dataset_x = np.array([-1, 0, 1, 2])
dataset_y = np.array([0, 0, 1, 1])
X = np.transpose(np.stack([np.ones(len(dataset_x)), dataset_x]))
theta_LS = np.linalg.lstsq(X, dataset_y, rcond=None)[0]
print(theta_LS)

M = np.transpose(np.stack([np.ones(len(dataset_x)), dataset_x, dataset_y]))
u, s, vh = np.linalg.svd(M, full_matrices=False)
min_singular_M = s[2]
theta_TLS = np.dot(np.linalg.inv(np.dot(np.transpose(X), X) - (min_singular_M**2)*np.identity(2)), np.dot(np.transpose(X), dataset_y))
print(theta_TLS)

axes()
_ = plt.plot(dataset_x[0], dataset_y[0], 'o', label='Data point 1', markersize=4)
_ = plt.plot(dataset_x[1], dataset_y[1], 'o', label='Data point 2', markersize=4)
_ = plt.plot(dataset_x[2], dataset_y[2], 'o', label='Data point 3', markersize=4)
_ = plt.plot(dataset_x[3], dataset_y[3], 'o', label='Data point 4', markersize=4)
_ = plt.plot(dataset_x, theta_LS[0] + theta_LS[1]*dataset_x, label='Least-squares line')
_ = plt.plot(dataset_x, theta_TLS[0] + theta_TLS[1]*dataset_x, label='Total least-squares line')
_ = plt.legend()
plt.show()
```

This code results in the following visualization:

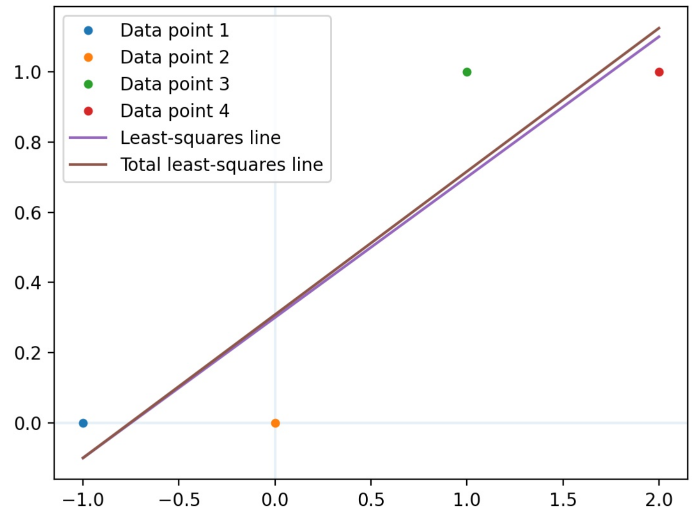

Figure 1: Visualization of both the least-squares line and the total least-squares line in Problem 1

Problem 2 (*Exercise 6.2* in [1]: Geometry of least-squares problems)**.**

To begin with, let $\mathcal{X}_{\mathsf{opt}} := \operatorname{argmin}\{\|\mathbf{y}-\mathbf{A}\mathbf{x}\|_2 : \mathbf{x}\in\mathbb{R}^n\}$. Due to the projection theorem (*Theorem 2.2* in [1]), we have $\mathcal{X}_{\mathsf{opt}} \neq \varnothing$ and if $\mathbf{x}^* \in \mathcal{X}_{\mathsf{opt}}$, then

$$\mathbf{y}-\mathbf{A}\mathbf{x}^* \perp \mathcal{R}(\mathbf{A}). \tag{2.1}$$

In order to discuss the properties of the residual vector $\mathbf{r} := \mathbf{y}-\mathbf{A}\mathbf{x}^* \in \mathbb{R}^m$ at an optimal solution, we first establish the well-definedness of the residual vector. In other words, we claim that

$$\mathbf{y}-\mathbf{A}\mathbf{x}_1^* = \mathbf{y}-\mathbf{A}\mathbf{x}_2^* \quad \text{for any } \mathbf{x}_1^*, \mathbf{x}_2^* \in \mathcal{X}_{\mathsf{opt}}. \tag{2.2}$$

This claim immediately follows from the following useful lemma:

Lemma 1.

$$\mathcal{X}_{\mathrm{opt}} = \left\{\mathbf{x}\in\mathbb{R}^n : \mathbf{A}^\top(\mathbf{y}-\mathbf{A}\mathbf{x}) = \mathbf{0}\right\} = \left\{\mathbf{A}^\dagger\mathbf{y}+\mathbf{z} : \mathbf{z}\in\mathcal{N}(\mathbf{A})\right\}, \tag{2.3}$$

where $\mathbf{A}^\dagger \in \mathbb{R}^{n\times m}$ denotes the Moore-Penrose pseudo-inverse of $\mathbf{A}\in\mathbb{R}^{m\times n}$.

Proof of Lemma 1. It's clear from (2.1) that if $\mathbf{x}^* \in \mathcal{X}_{\mathsf{opt}}$, then $\mathbf{y}-\mathbf{A}\mathbf{x}^* \in (\mathcal{R}(\mathbf{A}))^\perp = \mathcal{N}\left(\mathbf{A}^\top\right)$, which implies

$$\mathcal{X}_{\mathrm{opt}} \subseteq \left\{\mathbf{x}\in\mathbb{R}^n : \mathbf{A}^\top(\mathbf{y}-\mathbf{A}\mathbf{x}) = \mathbf{0}\right\}.$$

On the other hand, let $\mathbf{x}^*\in\mathbb{R}^n$ be a vector satisfying the normal equation, *i.e.*, $\mathbf{A}^\top(\mathbf{y}-\mathbf{A}\mathbf{x}^*) = \mathbf{0}$. Then for any $\mathbf{x}\in\mathbb{R}^n$, one has

$$\begin{aligned}
\|\mathbf{y}-\mathbf{A}\mathbf{x}\|_2^2 &= \|(\mathbf{y}-\mathbf{A}\mathbf{x}^*) + \mathbf{A}(\mathbf{x}^*-\mathbf{x})\|_2^2 \\
&= \|\mathbf{y}-\mathbf{A}\mathbf{x}^*\|_2^2 + \|\mathbf{A}(\mathbf{x}^*-\mathbf{x})\|_2^2 + 2(\mathbf{x}^*-\mathbf{x})^\top\mathbf{A}^\top(\mathbf{y}-\mathbf{A}\mathbf{x}^*) \\
&\overset{\text{(a)}}{=} \|\mathbf{y}-\mathbf{A}\mathbf{x}^*\|_2^2 + \|\mathbf{A}(\mathbf{x}^*-\mathbf{x})\|_2^2 \\
&\geq \|\mathbf{y}-\mathbf{A}\mathbf{x}^*\|_2^2,
\end{aligned}$$

where the step (a) follows from the normal equation $\mathbf{A}^\top(\mathbf{y}-\mathbf{A}\mathbf{x}^*)=\mathbf{0}$. This implies that $\mathbf{x}^*\in\mathcal{X}_{\mathsf{opt}}$, which establishes

$$\mathcal{X}_{\mathrm{opt}}=\left\{\mathbf{x}\in\mathbb{R}^n:\mathbf{A}^\top(\mathbf{y}-\mathbf{A}\mathbf{x})=\mathbf{0}\right\}. \tag{2.4}$$

Now, we are going to prove

$$\left\{\mathbf{x}\in\mathbb{R}^n:\mathbf{A}^\top(\mathbf{y}-\mathbf{A}\mathbf{x})=\mathbf{0}\right\}=\left\{\mathbf{A}^\dagger\mathbf{y}+\mathbf{z}:\mathbf{z}\in\mathcal{N}(\mathbf{A})\right\}. \tag{2.5}$$

Let $\mathbf{A}=\mathbf{U}_r\mathbf{\Sigma}\mathbf{V}_r^\top$ denote the compact-form SVD of $\mathbf{A}$, where $\mathbf{U}_r\in\mathbb{R}^{m\times r}$ and $\mathbf{V}_r\in\mathbb{R}^{n\times r}$ satisfy $\mathbf{U}_r^\top\mathbf{U}_r=\mathbf{V}_r^\top\mathbf{V}_r=\mathbf{I}_r$, and $\mathbf{\Sigma}:=\mathsf{diag}\left(\sigma_1(\mathbf{A}),\sigma_2(\mathbf{A}),\cdots,\sigma_r(\mathbf{A})\right)\in\mathbb{R}^{r\times r}$. Note that $r:=\mathsf{rank}(\mathbf{A})\le\min\{m,n\}$ and $\sigma_1(\mathbf{A})\ge\sigma_2(\mathbf{A})\ge\cdots\ge\sigma_r(\mathbf{A})>0$ are the singular values of $\mathbf{A}$. Then, the Moore-Penrose pseudo-inverse of $\mathbf{A}$ is given by

$$\mathbf{A}^\dagger=\mathbf{V}_r\mathbf{\Sigma}^{-1}\mathbf{U}_r^\top.$$

Also we have

$$\left(\mathbf{A}^\top\mathbf{A}\right)\left(\mathbf{A}^\dagger\mathbf{y}\right)=\left(\mathbf{V}_r\mathbf{\Sigma}\mathbf{U}_r^\top\cdot\mathbf{U}_r\mathbf{\Sigma}\mathbf{V}_r^\top\right)\left(\mathbf{V}_r\mathbf{\Sigma}^{-1}\mathbf{U}_r^\top\mathbf{y}\right)\overset{\text{(b)}}{=}\left(\mathbf{V}_r\mathbf{\Sigma}\mathbf{U}_r^\top\right)\mathbf{y}=\mathbf{A}^\top\mathbf{y}, \tag{2.6}$$

where the step (b) holds by the fact $\mathbf{U}_r^\top\mathbf{U}_r=\mathbf{V}_r^\top\mathbf{V}_r=\mathbf{I}_r$. Therefore, for any $\mathbf{z}\in\mathcal{N}(\mathbf{A})$, we obtain

$$\mathbf{A}^\top\left\{\mathbf{y}-\mathbf{A}\left(\mathbf{A}^\dagger\mathbf{y}+\mathbf{z}\right)\right\}=\mathbf{A}^\top\left(\mathbf{y}-\mathbf{A}\mathbf{A}^\dagger\mathbf{y}\right)\overset{\text{(c)}}{=}\mathbf{0},$$

where the step (c) follows from (2.6). This yields the relation

$$\left\{\mathbf{x}\in\mathbb{R}^n:\mathbf{A}^\top(\mathbf{y}-\mathbf{A}\mathbf{x})=\mathbf{0}\right\}\supseteq\left\{\mathbf{A}^\dagger\mathbf{y}+\mathbf{z}:\mathbf{z}\in\mathcal{N}(\mathbf{A})\right\}.$$

On the other hand, suppose $\mathbf{x}\in\mathbb{R}^n$ obeys the normal equation $\mathbf{A}^\top(\mathbf{y}-\mathbf{A}\mathbf{x})=\mathbf{0}$. Since $\mathbf{A}^\dagger\mathbf{y}\in\mathbb{R}^n$ also satisfies the normal equation, we have

$$\left(\mathbf{A}^\top\mathbf{A}\right)\left(\mathbf{x}-\mathbf{A}^\dagger\mathbf{y}\right)=\mathbf{A}^\top\mathbf{y}-\mathbf{A}^\top\mathbf{y}=\mathbf{0}, \tag{2.7}$$

thereby $\mathbf{x}-\mathbf{A}^\dagger\mathbf{y}\in\mathcal{N}\left(\mathbf{A}^\top\mathbf{A}\right)=\mathcal{N}(\mathbf{A})$. Note that the property $\mathcal{N}\left(\mathbf{A}^\top\mathbf{A}\right)=\mathcal{N}(\mathbf{A})$ was a homework problem in Homework #1 (*Exercise 3.7* in [1]). This proves the relation

$$\left\{\mathbf{x}\in\mathbb{R}^n:\mathbf{A}^\top(\mathbf{y}-\mathbf{A}\mathbf{x})=\mathbf{0}\right\}\subseteq\left\{\mathbf{A}^\dagger\mathbf{y}+\mathbf{z}:\mathbf{z}\in\mathcal{N}(\mathbf{A})\right\},$$

and this completes the proof of the claim (2.5). By combining (2.4) and (2.5) together, we arrive at the desired result.

□

In order to prove (2.2), it suffice to show that $\mathbf{A}(\mathbf{x}_2^*-\mathbf{x}_1^*)=\mathbf{0}$ for all $\mathbf{x}_1^*,\mathbf{x}_2^*\in\mathcal{X}_{\mathsf{opt}}$. By Lemma 1, we have $\mathbf{x}_1^*=\mathbf{A}^\dagger\mathbf{y}+\mathbf{z}_1$ and $\mathbf{x}_2^*=\mathbf{A}^\dagger\mathbf{y}+\mathbf{z}_2$ for some $\mathbf{z}_1,\mathbf{z}_2\in\mathcal{N}(\mathbf{A})$. Thus we have $\mathbf{z}_2-\mathbf{z}_1\in\mathcal{N}(\mathbf{A})$, thereby

$$\mathbf{A}(\mathbf{x}_2^*-\mathbf{x}_1^*)=\mathbf{A}(\mathbf{z}_2-\mathbf{z}_1)=\mathbf{0},$$

as desired. Hence, the residual vector $\mathbf{r}:=\mathbf{y}-\mathbf{A}\mathbf{x}^*\in\mathbb{R}^m$ at an optimal solution $\mathbf{x}^*\in\mathcal{X}_{\mathsf{opt}}$ to the LS problem is well-defined.

Lastly, it remains to prove the properties (i) $\mathbf{r}^\top \mathbf{y} > 0$; (ii) $\mathbf{A}^\top \mathbf{r} = \mathbf{0}$. For (i), it suffices to observe that $\mathbf{y} - \mathbf{r} \in \mathcal{R}(\mathbf{A})$. Since $\mathbf{r} \in (\mathcal{R}(\mathbf{A}))^\perp$, we obtain

$$0 = \mathbf{r}^\top (\mathbf{y} - \mathbf{r}) = \mathbf{r}^\top \mathbf{y} - \|\mathbf{r}\|_2^2,$$

thereby $\mathbf{r}^\top \mathbf{y} = \|\mathbf{r}\|_2^2 > 0$. This is because we have $\mathbf{r} \neq \mathbf{0}$ from the assumption $\mathbf{y} \notin \mathcal{R}(\mathbf{A})$. Also, the property (ii) immediately follows from the fact $\mathbf{r} \in (\mathcal{R}(\mathbf{A}))^\perp = \mathcal{N}\left(\mathbf{A}^\top\right)$. This completes the proof of all the desired results. Now, it's time to provide geometric interpretations of these results. We first present the visualization of the LS problem:

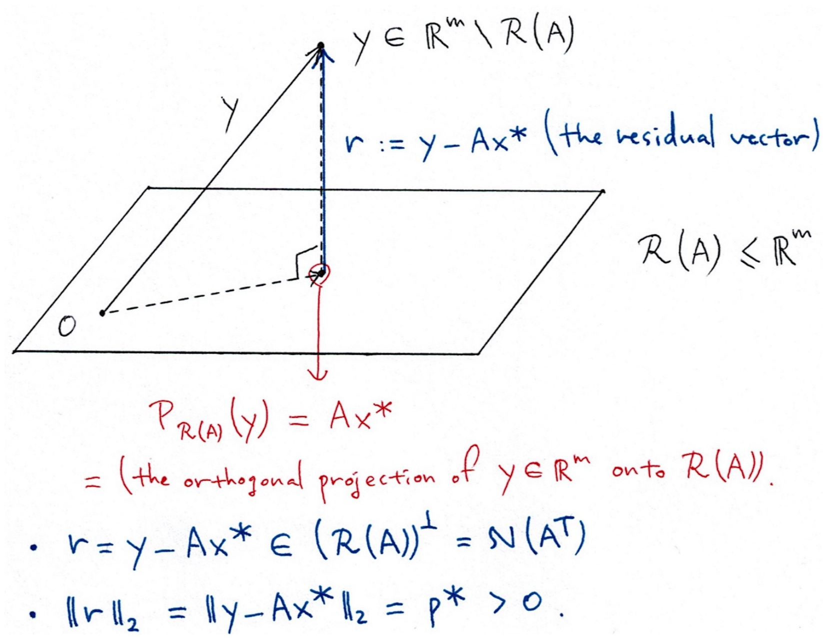

Figure 2: Geometric interpretation of the least-squares problem

As Figure 2 shows, the property (ii) $\mathbf{A}^\top \mathbf{r} = \mathbf{0}$ asserts that the residual vector $\mathbf{r} = \mathbf{y} - \mathbf{A}\mathbf{x}^*$ is orthogonal to the range $\mathcal{R}(\mathbf{A})$ of $\mathbf{A} \in \mathbb{R}^{m \times n}$ since $(\mathcal{R}(\mathbf{A}))^\perp = \mathcal{N}\left(\mathbf{A}^\top\right)$. Also, the property (i) $\mathbf{r}^\top \mathbf{y} > 0$ can be interpreted as follows: let $\theta \in [0, \pi]$ denote the angle between two vectors $\mathbf{y} \in \mathbb{R}^m \setminus \mathcal{R}(\mathbf{A})$ and $\mathbf{r} \in \mathbb{R}^m \setminus \{\mathbf{0}\}$, *i.e.*,

$$\cos\theta = \frac{\mathbf{r}^\top \mathbf{y}}{\|\mathbf{r}\|_2 \cdot \|\mathbf{y}\|_2}.$$

The property (i) $\mathbf{r}^\top \mathbf{y} > 0$ implies $\cos\theta > 0$, which is equivalent to $\theta \in \left(0, \frac{\pi}{2}\right)$. In other words, the property (i) asserts that for two vectors $\mathbf{y}$ and $\mathbf{r}$ in $\mathbb{R}^m$, each vector has a component in the direction of the other.

Problem 3 (*Exercise 6.4* in [1]: Regularization for noisy data)**.**

Let $\hat{\mathbf{a}}_1, \hat{\mathbf{a}}_2, \cdots, \hat{\mathbf{a}}_m \in \mathbb{R}^n$ and $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_m$ be n-dimensional random vectors such that

$$\mathbb{E}[\mathbf{u}_i] = \mathbf{0} \in \mathbb{R}^n \quad \text{and} \quad \mathsf{Cov}[\mathbf{u}_i] = \mathbb{E}\left[\mathbf{u}_i \mathbf{u}_i^\top\right] = \sigma^2 \mathbf{I}_n \tag{3.1}$$

for every $i \in [m]$. Let $\mathbf{a}_i = \hat{\mathbf{a}}_i + \mathbf{u}_i$ for $i \in [m]$, $\mathbf{u} := (\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_m)$, and

$$\hat{\mathbf{A}} := \begin{bmatrix} \hat{\mathbf{a}}_1^\top \\ \hat{\mathbf{a}}_2^\top \\ \vdots \\ \hat{\mathbf{a}}_m^\top \end{bmatrix} \in \mathbb{R}^{m \times n}; \quad \mathbf{U} := \begin{bmatrix} \mathbf{u}_1^\top \\ \mathbf{u}_2^\top \\ \vdots \\ \mathbf{u}_m^\top \end{bmatrix}; \quad \mathbf{A} = \mathbf{A}(\mathbf{u}) := \begin{bmatrix} \mathbf{a}_1^\top \\ \mathbf{a}_2^\top \\ \vdots \\ \mathbf{a}_m^\top \end{bmatrix} = \hat{\mathbf{A}} + \mathbf{U}.$$

Then both $\mathbf{U}$ and $\mathbf{A}$ are $m \times n$ real random matrices. One can observe that the objective function of the given original optimization problem satisfies

$$
\begin{aligned}
\mathbb{E}_{\mathbf{u}}\left[\|\mathbf{A}(\mathbf{u})\mathbf{x}-\mathbf{y}\|_2^2\right] &= \mathbb{E}_{\mathbf{u}}\left[\left\{\left(\hat{\mathbf{A}}+\mathbf{U}\right)\mathbf{x}-\mathbf{y}\right\}^{\top}\left\{\left(\hat{\mathbf{A}}+\mathbf{U}\right)\mathbf{x}-\mathbf{y}\right\}\right] \\
&= \mathbb{E}_{\mathbf{u}}\left[\mathbf{x}^{\top}\left(\hat{\mathbf{A}}+\mathbf{U}\right)^{\top}\left(\hat{\mathbf{A}}+\mathbf{U}\right)\mathbf{x}-2\mathbf{y}^{\top}\left(\hat{\mathbf{A}}+\mathbf{U}\right)\mathbf{x}+\mathbf{y}^{\top}\mathbf{y}\right] \\
&= \mathbf{x}^{\top}\mathbb{E}_{\mathbf{u}}\left[\left(\hat{\mathbf{A}}+\mathbf{U}\right)^{\top}\left(\hat{\mathbf{A}}+\mathbf{U}\right)\right]\mathbf{x}-2\mathbf{y}^{\top}\mathbb{E}_{\mathbf{u}}\left[\hat{\mathbf{A}}+\mathbf{U}\right]\mathbf{x}+\mathbf{y}^{\top}\mathbf{y} \\
&= \mathbf{x}^{\top}\left(\hat{\mathbf{A}}^{\top}\hat{\mathbf{A}}+\mathbb{E}_{\mathbf{u}}[\mathbf{U}]^{\top}\hat{\mathbf{A}}+\hat{\mathbf{A}}^{\top}\mathbb{E}_{\mathbf{u}}[\mathbf{U}]+\mathbb{E}_{\mathbf{u}}\left[\mathbf{U}^{\top}\mathbf{U}\right]\right)\mathbf{x}-2\mathbf{y}^{\top}\left(\hat{\mathbf{A}}+\mathbb{E}_{\mathbf{u}}[\mathbf{U}]\right)\mathbf{x}+\mathbf{y}^{\top}\mathbf{y} \\
&\overset{\text{(a)}}{=} \mathbf{x}^{\top}\left(\hat{\mathbf{A}}^{\top}\hat{\mathbf{A}}+\mathbb{E}_{\mathbf{u}}\left[\mathbf{U}^{\top}\mathbf{U}\right]\right)\mathbf{x}-2\mathbf{y}^{\top}\hat{\mathbf{A}}\mathbf{x}+\mathbf{y}^{\top}\mathbf{y} \\
&\overset{\text{(b)}}{=} \left(\mathbf{x}^{\top}\hat{\mathbf{A}}^{\top}\hat{\mathbf{A}}\mathbf{x}-2\mathbf{y}^{\top}\hat{\mathbf{A}}\mathbf{x}+\mathbf{y}^{\top}\mathbf{y}\right)+m\sigma^2\cdot\mathbf{x}^{\top}\mathbf{I}_n\mathbf{x} \\
&= \left\|\hat{\mathbf{A}}\mathbf{x}-\mathbf{y}\right\|_2^2+m\sigma^2\|\mathbf{x}\|_2^2,
\end{aligned}
$$

where the step (a) holds due to the fact $\mathbb{E}_{\mathbf{u}}[\mathbf{U}] = \mathbf{O}_{m\times n}$, where $\mathbf{O}_{m\times n}$ denotes the $m \times n$ zero matrix, and the step (b) follows from the following observation:

$$
\mathbb{E}_{\mathbf{u}}\left[\mathbf{U}^{\top}\mathbf{U}\right] = \mathbb{E}_{\mathbf{u}}\left[\sum_{i=1}^{m}\mathbf{u}_i\mathbf{u}_i^{\top}\right] = \sum_{i=1}^{m}\mathbb{E}_{\mathbf{u}}\left[\mathbf{u}_i\mathbf{u}_i^{\top}\right] \overset{\text{(c)}}{=} \sum_{i=1}^{m}\sigma^2\mathbf{I}_n = m\sigma^2\mathbf{I}_n,
$$

where the step (c) comes from the assumption (3.1). Hence, the original optimization problem

$$
\min_{\mathbf{x}\in\mathbb{R}^n}\ \mathbb{E}_{\mathbf{u}}\left[\|\mathbf{A}(\mathbf{u})\mathbf{x}-\mathbf{y}\|_2^2\right]
$$

can be written as the following regularized least-squares problem with the regularization parameter $\lambda = m\sigma^2$:

$$
\min_{\mathbf{x}\in\mathbb{R}^n}\ \left(\left\|\hat{\mathbf{A}}\mathbf{x}-\mathbf{y}\right\|_2^2+m\sigma^2\|\mathbf{x}\|_2^2\right).
$$

Problem 4 (*Exercise 8.1* in [1]: Quadratic inequalities)**.**

To begin with, let $\Omega \subseteq \mathbb{R}^2$ be a subset of $\mathbb{R}^2$ defined by

$$
\begin{aligned}
\Omega &:= \left\{(x_1,x_2)\in\mathbb{R}^2 : x_2(x_1-x_2+1)\geq 0\right\} \\
&= \left\{(x_1,x_2)\in\mathbb{R}^2 : x_1\geq x_2-1 \text{ and } x_2\geq 0\right\}\cup\left\{(x_1,x_2)\in\mathbb{R}^2 : x_1\leq x_2-1 \text{ and } x_2\leq 0\right\}.
\end{aligned}
\tag{4.1}
$$

(1) We first plot the region Ω explicitly:

In the above figure, the region (4.1) is filled with grey color. Now, we claim that Ω is not convex in $\mathbb{R}^2$. To this end, assume on the contrary that Ω is a convex subset of $\mathbb{R}^2$. Since $(0,0)\in\Omega$ and $(-2,-1)\in\Omega$, the closed line segment $\{(1-t)(0,0)+t(-2,-1) : t\in[0,1]\}$ connecting these two points should be contained in Ω. However, the midpoint of those two points $\frac{1}{2}\{(0,0)+(-2,-1)\} = \left(-1,-\frac{1}{2}\right)$ does not belong to Ω, *i.e.*,

$$
\left(-1,-\frac{1}{2}\right)\in\mathbb{R}^2\setminus\Omega,
$$

and this yields a contradiction. Hence, the subset Ω of $\mathbb{R}^2$ is not convex!

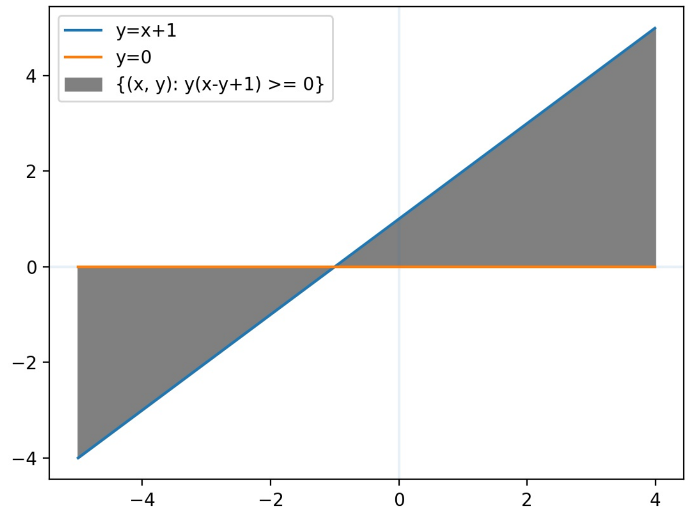

Figure 3: Visualization of the region $\Omega \subseteq \mathbb{R}^2$

(2) We begin with the following expression of the region $\Omega \subseteq \mathbb{R}^2$:

$$\Omega = \left\{ (x_1, x_2) \in \mathbb{R}^2 : x_2 \left(x_2 - x_1 - 1 \right) \leq 0 \right\}.$$

One can observe that

$$\begin{aligned} q(\mathbf{x}) &:= x_2 \left(x_2 - x_1 - 1 \right) \\ &= -x_1 x_2 + x_2^2 - x_2 \\ &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 2 \begin{bmatrix} 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \mathbf{x}^\top \mathbf{A} \mathbf{x} + 2\mathbf{b}^\top \mathbf{x} + c, \end{aligned} \tag{4.2}$$

where $\mathbf{x} := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$, $\mathbf{A} := \begin{bmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix} \in \mathcal{S}^2$, $\mathbf{b} := \begin{bmatrix} 0 \\ -\frac{1}{2} \end{bmatrix} \in \mathbb{R}^2$, and $c := 0 \in \mathbb{R}$. Here, $\mathcal{S}^n$ denotes the set of all $n \times n$ real symmetric matrices. Hence, we have from (4.2) that

$$\Omega = \left\{ \mathbf{x} \in \mathbb{R}^2 : q(\mathbf{x}) = \mathbf{x}^\top \mathbf{A} \mathbf{x} + 2\mathbf{b}^\top \mathbf{x} + c \leq 0 \right\},$$

with the above choice of $(\mathbf{A}, \mathbf{b}, c) \in \mathcal{S}^2 \times \mathbb{R}^2 \times \mathbb{R}$, as desired.

(3) Given any subset S of $\mathbb{R}^2$, let $\mathsf{conv}(S)$ denote the convex hull of S in $\mathbb{R}^2$:

$$\mathsf{conv}(S) := \left\{ \sum_{k=1}^{n} \alpha_k \mathbf{x}_k : n \in \mathbb{N}, \mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n \in S, \text{ and } \alpha_k \geq 0, \ \forall k \in [n] \text{ such that } \sum_{k=1}^{n} \alpha_k = 1 \right\}.$$

We claim that $\mathsf{conv}(\Omega) = \mathbb{R}^2$. To this end, we will show that every point in $\mathbb{R}^2$ can be written as a convex combination of two points in Ω. From the definition of convex hull, it's evident that this assertion establishes our desired conclusion $\mathsf{conv}(\Omega) = \mathbb{R}^2$.

Choose and fix any point $(u_0, v_0) \in \mathbb{R}^2$. If $(u_0, v_0) \in \Omega$, then (u_0, v_0) is clearly a convex combination of two points $(u_0, v_0) \in \Omega$ and $(u_0, v_0) \in \Omega$. So we may assume that $(u_0, v_0) \in \mathbb{R}^2 \setminus \Omega$, *i.e.*,

$$v_0 (u_0 - v_0 + 1) < 0. \tag{4.3}$$

Now, let us consider the affine line $\left\{(x_1, x_2) \in \mathbb{R}^2 : x_2 - v_0 = \frac{1}{2}(x_1 - u_0)\right\}$ in $\mathbb{R}^2$ which passes through the point $(u_0, v_0) \in \mathbb{R}^2 \setminus \Omega$ and has slope $\frac{1}{2}$. Intuitively, this affine line should have intersections with affine lines $\left\{(x_1, x_2) \in \mathbb{R}^2 : x_2 = x_1 + 1\right\}$ as well as $\left\{(x_1, x_2) \in \mathbb{R}^2 : x_2 = 0\right\}$, which form the boundary of the region Ω. To be precise, some straightforward calculations give

$$\left\{(x_1, x_2) \in \mathbb{R}^2 : x_2 - v_0 = \frac{1}{2}(x_1 - u_0)\right\} \cap \left\{(x_1, x_2) \in \mathbb{R}^2 : x_2 = x_1 + 1\right\} = \{(2v_0 - u_0 - 2, 2v_0 - u_0 - 1)\};$$
$$\left\{(x_1, x_2) \in \mathbb{R}^2 : x_2 - v_0 = \frac{1}{2}(x_1 - u_0)\right\} \cap \left\{(x_1, x_2) \in \mathbb{R}^2 : x_2 = 0\right\} = \{(u_0 - 2v_0, 0)\}.$$

It's clear that $(2v_0 - u_0 - 2, 2v_0 - u_0 - 1) \in \Omega$ and $(u_0 - 2v_0, 0) \in \Omega$. Intuitively, one can anticipate that (u_0, v_0) would lie in the closed line segment connecting these two points. In order to make this intuition rigorous, we observe that

$$(u_0, v_0) = \{1 - \theta^* (u_0, v_0)\} (2v_0 - u_0 - 2, 2v_0 - u_0 - 1) + \theta^* (u_0, v_0) (u_0 - 2v_0, 0), \tag{4.4}$$

where $\theta^* (u_0, v_0) := \frac{u_0 - v_0 + 1}{u_0 - 2v_0 + 1}$.

Claim 1. *If $(u_0, v_0) \in \mathbb{R}^2 \setminus \Omega$, then $\theta^* (u_0, v_0) \in (0, 1)$.*

Proof of Claim 1. The inequality (4.3) will play a crucial role in the proof of Claim 1.

- The case $v_0 > 0$: automatically, we have $u_0 - v_0 + 1 < 0$. This leads to $0 < v_0 - u_0 - 1 < 2v_0 - u_0 - 1$, thereby
$$0 < \theta^* (u_0, v_0) = \frac{v_0 - u_0 - 1}{2v_0 - u_0 - 1} < 1.$$
- The case $v_0 < 0$: automatically, we have $u_0 - v_0 + 1 > 0$. This yields $0 < u_0 - v_0 + 1 < u_0 - 2v_0 + 1$, so
$$0 < \theta^* (u_0, v_0) = \frac{u_0 - v_0 + 1}{u_0 - 2v_0 + 1} < 1.$$

This completes the proof of Claim 1.

□

By combining the equation (4.4) together with Claim 1, it is possible to conclude that $(u_0, v_0) \in \mathbb{R}^2 \setminus \Omega$ can be written as a convex combination of two points $(2v_0 - u_0 - 2, 2v_0 - u_0 - 1) \in \Omega$ and $(u_0 - 2v_0, 0) \in \Omega$. Therefore, we have $\mathbb{R}^2 \subseteq \mathsf{conv}(\Omega) \subseteq \mathbb{R}^2$, and this establishes

$$\mathbb{R}^2 = \mathsf{conv}(\Omega),$$

as desired.

References

[1] Giuseppe C Calafiore and Laurent El Ghaoui. *Optimization models*. Cambridge university press, 2014.