Yeshiva University- Katz School of Science  Health

Mathematics Department

Mathematics of Finance

Instructor: Marian Gidea

# Project 2

# Minimum Risk Stock Portfolio

Daniel Cohen

Brian Livian

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# Contents

# 1 Abstract

The stock market is one of most essential parts of any country's economy. This project's goal is to find all the dominant stocks between two stocks. A dominant stock is a stock that has the higher expected returns for its level of risk. It must be stated that some do not mind raising their risk if it comes with some benefit of increased returns. We start with finding formulas for the weights, mean (of the expected value) and standard deviation for the minimum variance portfolio. Next, we increase the portfolio variance and compare the results of new weights and mean with the original minimum risk portfolio's. We find that increasing a portfolios risk increases its expected return.

## 2 Introduction

The stock market is one of the greatest places to invest in. However, not all stocks are created equal. In order to optimize a stock portfolio, one must have an understanding of the quantitative relationship of these two assets. Portfolio management from a quantitative viewpoint can help ensure a minimized risk to one's investment. Minimized risk portfolio management can be applied any arbitrary amount of stocks. In this project we analyze minimum risk portfolio management applied to the case of only 2 stocks, in order to demonstrate the methodology.

## 3 Project Body

Stock portfolio investments involved some degree of speculation, being that nobody truly knows how a stock will behave on a given day. However, analyzing a stocks quantitative history (aka in a time series) can reveal insights on what will likely occur on a given day. In stock investments, the true value of a stock has little relevance. Rather, one must analyzing the returns of the stock because the change of a stocks value is only important relative to its value. There are multiple ways in calculating stock returns, such as the simple percent return and logarithmic return. In this project, we calculate daily simple percent returns of a 2 stock portfolio: $K(t) = \frac{S(t)-S(t-1)}{S(t-1)}$

Analyzing the statistics of returns provides the tools to minimizing a portfolio. The summary statistics computed are means, standard deviations, covariance and correlation of the two stock returns. The portfolio's expected return is defined as: $E(K) = \frac{\Sigma K}{T} = \mu(S)$.

In order to calculate portfolio's estimated return and risk, we first calculate the weights of each stock and multiply them by their means. $\mu_v = w_1\mu_1 + w_2\mu_2$. In order to tell how likely our actual stock portfolio is to converge from our expected mean, we need to get the variance of each stock: $Var_V = E(V - \mu_V)^2 = \sigma_V^2$, and the standard deviation:

$Sd_V = \sqrt{Var_V} = \sigma_V$.

The covariance and correlation of the two stocks say how much the two stocks vary with each other. The formula for covariance is given by: $Cov(X,Y) = ((X - \mu x)(Y - \mu y)) = \Sigma_i (x_i - \mu_x)(y_i - \mu_y)p_i = c_{12}$ The correlation formula: $Cor(X,Y) = \frac{Cov(X,Y)}{\sigma x * \sigma y}$ The variance of portfolio is: $Var(V) = w_1\sigma_1^2 + w_2\sigma_2^2 + 2w_1w_2c_{12}$.

In order to calculate the minimum risk portfolio, we first calculate the weights of each stock, calling the weights $S_0$ and $(1 - S_0)$. The values for these weights are calculated by taking the derivative of $\sigma_v$ with respect to s, and solving for 0 (ie minimizing the function) [1]. This leads to the formula of minimized risk weights to be: $S_0 = \frac{\sigma_2^2 - c_{12}}{\sigma_1^2 + \sigma_2^2 - 2c_{12}}$. Plugging the minimum weights for the expectation and variance yields: $\mu_0 = \frac{\mu_1\sigma_2^2 + \mu_2\sigma_1^2 - (\mu_1 + \mu_2)c_{12}}{\sigma_1^2 + \sigma_2^2 - 2c_{12}}$ and $\sigma_0 = \frac{\sigma_1^2\sigma_2^2 - c_{12}^2}{\sigma_1^2 + \sigma_2^2 - 2c_{12}}$.

Now that we have the values for minimum risk portfolio, we can have better understand of what a good investment looks like. For example, we use these values to ensure not increasing risk without having higher expected returns. The weights of a portfolio, $W_1$ and $W_2$, are given by $W_1 = \frac{\mu_V - \mu_2}{\mu_1 - \mu_2}$ and $W_2 = 1 - W_1$. When raising standard deviation/ risk of a portfolio, we utilize the variance formula to calculate the updated weights. Writing the variance formula as a quadratic yields: $0 = (\sigma_1^2 + \sigma_2^2 - 2c_{12})s^2 + (-\sigma_2^2 + 2c_{12})s + (\sigma_2^2 - \sigma_V^2)$. Then, applying the quadratic formula: $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ yields: $a = \sigma_1^2 + \sigma_2^2 - 2c_{12}$, $b = -\sigma_2^2 + 2c_{12}$ and $c = \sigma_2^2 - \sigma_V^2$. Ultimately, these values lead to the weight value for $W_1$, and thus leads to $\mu_V$.

Alternatively, $\mu_V$ can be calculated using the equation of a hyperbola given in equation (3.13): $x^2 - A^2(y - \mu_0)^2 = \sigma_0^2$, where $-1 < \rho_{12} < 1, \mu_1 \neq \mu_2, x = \sigma_v^2, y = \mu_v, A^2 = \frac{\sigma_1^2 + \sigma_2^2 + 2c_{12}}{(\mu_1 - \mu_2)^2}$ [1]. Solving for y yields: $y = \mu_v = \mu_0 \pm \sqrt{\frac{x^2 - \sigma_0^2}{A^2}}$. Now knowing y, we solve for weights $w_1$ and $w_2 = (1 - w_1)$ using expectation formula $\mu_V = w_1\mu_1 + w_2\mu_2$.

# 4 Computer Simulation & Results

We run two simulations to demonstrate the portfolio management equations explained in the methodology section. The first experiment consists of TSLA and AAPL stocks for a period of 1/1/21 - 10/19/22. The second consists of WMT and AAPL stocks for a period of 1/1/21 - 10/19/22.

## 4.1 Experiment 1: Tesla & Apple

See the image below for the adjusted close and simple returns for these assets:

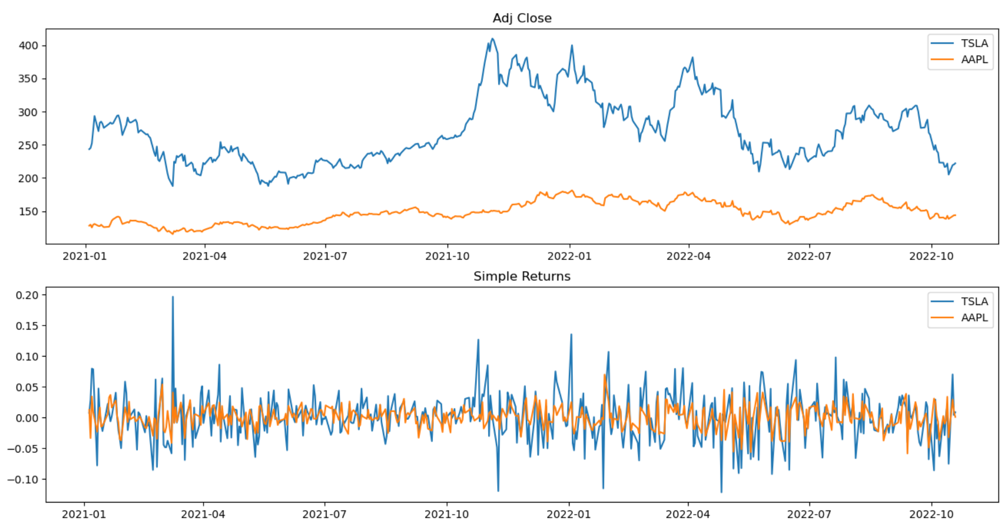


Figure 1: Assets Historical Adjusted Close & Returns

| | AAPL | TSLA |
|---|---|---|
| **count** | 452.000000 | 452.000000 |
| **mean** | 0.000429 | 0.000508 |
| **std** | 0.018549 | 0.037778 |
| **min** | -0.058680 | -0.121841 |
| **25%** | -0.010086 | -0.021575 |
| **50%** | 0.000557 | 0.001637 |
| **75%** | 0.012758 | 0.020518 |
| **max** | 0.069778 | 0.196412 |

Figure 2: Summary Statistics

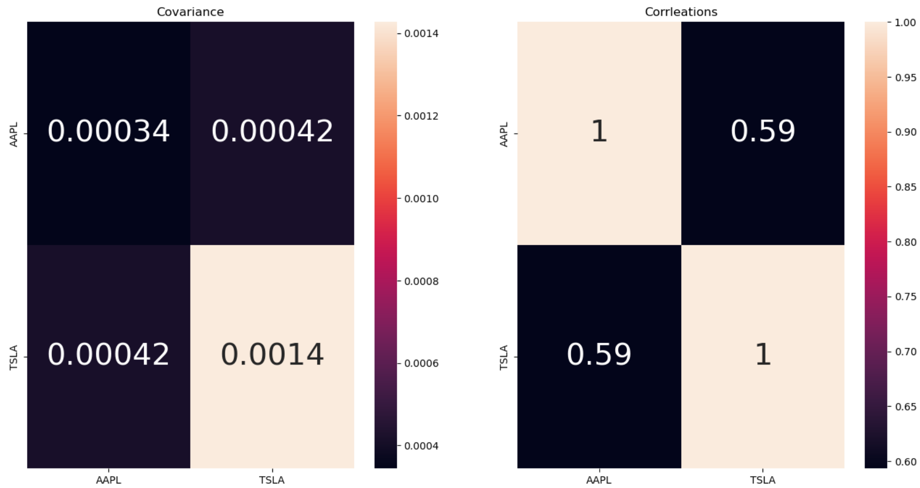


Figure 3: Covariances & Correlations

| | Asset | Minimum Portfolio Risk | Minimum Portfolio Weights | Expectation | 120% Risk Increase (Std) | Weights 2 | Expectation 2 |
|---|---|---|---|---|---|---|---|
| **0** | TSLA | 0.018401 | -0.076287 | 0.000423 | 0.022081 | 0.321868 | 0.000454 |
| **1** | AAPL | 0.018401 | 1.076287 | 0.000423 | 0.022081 | 0.678132 | 0.000454 |

Figure 4: Simulation Results Summary

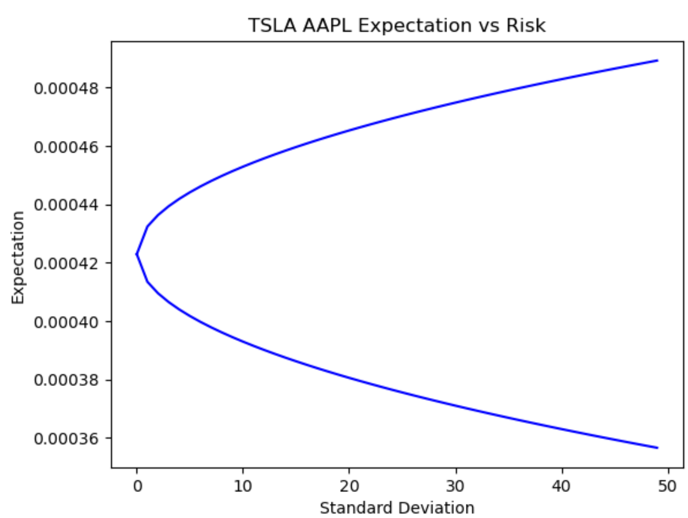


Figure 5: Expectation vs Risk Hyperbola

The minimum portfolio risk weights shown in figure 4 imply the portfolio requires short selling because one of the weights is negative. However, after increasing the risk by 20%, the portfolio no longer requires short selling. Furthermore, the expectation increases after after increasing the risk (of course the increased risk is a decision the investor must make). Figure 5 shows the hyperbola of Expectation vs Risk, with the vertex of the graph being the minimum risk portfolio standard deviation and expectation $(\sigma_0, \mu_0) = (.018401, .000423)$.

## 4.2 Experiment 2: Walmart & Apple

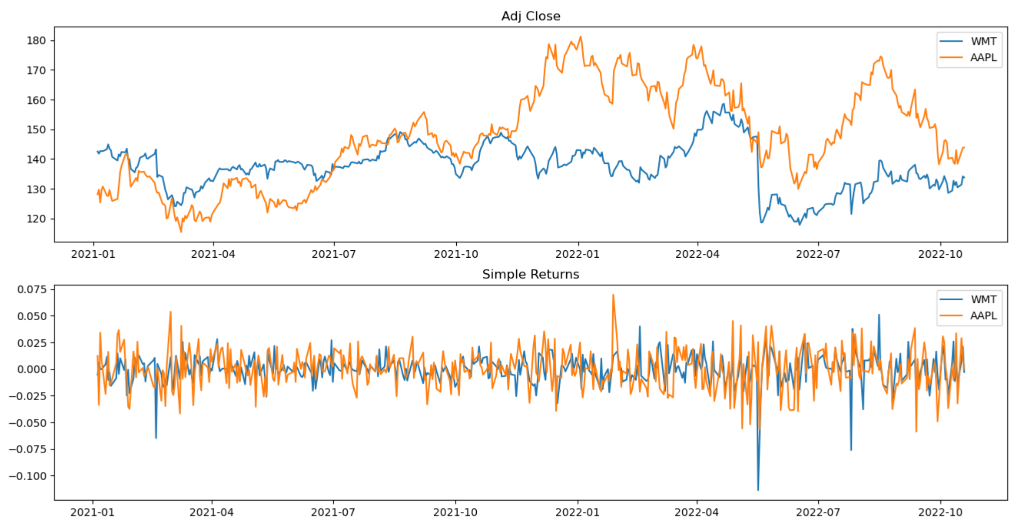


Figure 6: Assets Historical Adjusted Close & Returns

| | AAPL | WMT |
|---|---|---|
| **count** | 452.000000 | 452.000000 |
| **mean** | 0.000429 | -0.000041 |
| **std** | 0.018549 | 0.013997 |
| **min** | -0.058680 | -0.113758 |
| **25%** | -0.010086 | -0.006602 |
| **50%** | 0.000557 | 0.000411 |
| **75%** | 0.012758 | 0.007934 |
| **max** | 0.069778 | 0.051056 |

Figure 7: Summary Statistics

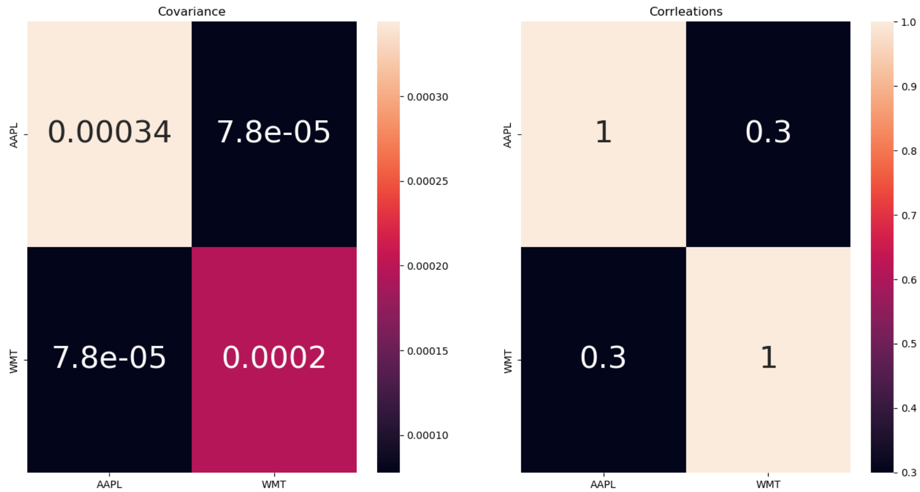


Figure 8: Covariances & Correlations

| | Asset | Minimum Portfolio Risk | Minimum Portfolio Weights | Expectation | 120% Risk Increase (Std) | Weights 2 | Expectation 2 |
|---|---|---|---|---|---|---|---|
| **0** | WMT | 0.012633 | 0.692625 | 0.000103 | 0.01516 | 0.26526 | 0.000304 |
| **1** | AAPL | 0.012633 | 0.307375 | 0.000103 | 0.01516 | 0.73474 | 0.000304 |

Figure 9: Simulation Results Summary

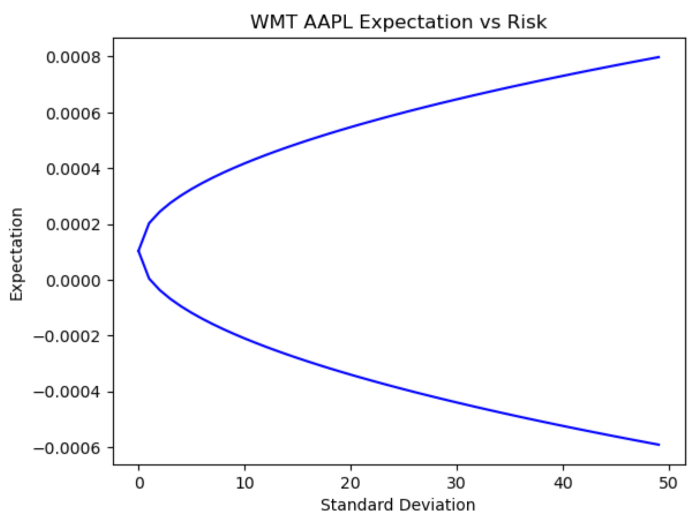


Figure 10: Expectation vs Risk Hyperbola

As opposed to the first experiment, experiment two's minimum risk portfolio does not require short selling (both weights are positive). Walmart stock risk is lower than Tesla's due to it having a smaller standard deviation of returns: .01 compared to .03 (see figures 2 and 7). In addition, the volatility can be seen visually in the return charts of figures 1 and 6, where Walmart's volatility is visually less than Tesla's. After increasing the risk by 20%, the weights shift in Apple's favor being that AAPL is a riskier asset. Furthermore, the expectation increases after the risk increase (again, a trade off the investor must make). Figure 10 shows the hyperbola of Expectation vs Risk, with the vertex of the graph being the minimum risk portfolio standard deviation and expectation $(\sigma_0, \mu_0) = (.012633, .000103)$.

# 5 Conclusion

In summary, we ran two experiments each with two sets of two stocks. The first set being Apple (AAPL) and Tesla (TSLA), and the second set being Apple (AAPL) and Walmart (WMT). Noting tesla as a riskier stock, it is initially given a negative weight, while WMT is given a positive weight. However, after increasing the risk of the portfolio, we see the weights shift in Tesla favor, while it shifts against Walmart. This is due to the increase of portfolio risk favoring riskier yet higher return stocks. Furthermore, although this project only involves two stock portfolios, this methodology can be extended for multiple stock portfolios.

# 6 Literature Cited

[1] M. Capinski, T. Zastawniak, Mathematics for Finance, Springer Undergraduate Mathematics Series, Springer - Verlang Limited 2011

# Appendix

```
import yfinance as yf
import pandas as pd
import matplotlib.pyplot as plt
import seaborn as sns
import numpy as np
```

```
stock1 = 'TSLA'
stock2 = 'AAPL'
data = yf.download(stock1 + ' ' + stock2, start="2021-01-01")
data = data['Adj Close']
```

```
dataPct = data.pct_change().dropna()
```

```
data.head()
```

```
dataPct.head()
```

```
plt.figure(figsize = (16,8), facecolor = 'white')
plt.subplot(2,1,1)
plt.plot(data[stock1], label = stock1)
plt.plot(data[stock2], label = stock2)
plt.title('Adj Close')
plt.legend()

plt.subplot(2,1,2)
plt.plot(dataPct[stock1], label = stock1)
plt.plot(dataPct[stock2], label = stock2)
plt.legend()
plt.title('Simple Returns')
plt.show()
```

```
dataPct.describe()
```

```
plt.figure(figsize = (16,8))
plt.subplot(1,2,1)
sns.heatmap(dataPct.cov(), annot = True, annot_kws={"fontsize":30})
plt.title('Covariance')
plt.subplot(1,2,2)
sns.heatmap(dataPct.corr(), annot = True, annot_kws={"fontsize":30})
plt.title('Corrleations')
plt.show()
```

```
dataPct.describe()
```

```
plt.figure(figsize = (16,8))
plt.subplot(1,2,1)
sns.heatmap(dataPct.cov(), annot = True, annot_kws={"fontsize":30})
plt.title('Covariance')
plt.subplot(1,2,2)
sns.heatmap(dataPct.corr(), annot = True, annot_kws={"fontsize":30})
plt.title('Corrleations')
plt.show()
```

```
var1 = dataPct.var()[stock1]
var2 = dataPct.var()[stock2]
c12 = np.cov(dataPct[stock1], dataPct[stock2])[0][1]

s0 = (var2 - c12) / (var1 + var2 - 2*c12)
s0compliment = 1 - s0

print('Minimum Weight 1 (s0) ', stock1, s0)
print('Minimum Weight 2 ', stock2, s0compliment)

mu1 = dataPct[stock1].mean()
mu2 = dataPct[stock2].mean()

mu0 = (mu1* var2 + mu2*var1 - (mu1 +mu2) * c12) / (var1 + var2 - 2*c12)
print('Expectation at minimum risk = ' + str(mu0))

varmin = (var1*var2 - c12**2) / (var1 + var2 - 2*c12)
sdmin = np.sqrt(varmin)
print('Minimum portfolio standard deviation = ' + str(sdmin))
```

```
sd2 = (1.2)*sdmin
# Calculate A (straight from book)
A = np.sqrt((var1 + var2 -2*c12) / ((mu1 - mu2)**2))
# Calculate expectation (derivation from equation 3.13)
muv = mu0 + np.sqrt((sd2**2 - varmin) / (A**2))
w1_1 = (muv - mu2) / (mu1 - mu2)
w2_1 = 1 - w1_1
```

```python
d = {
    'Asset':[stock1, stock2],
    'Minimum Portfolio Risk' : [sdmin, sdmin],
    'Minimum Portfolio Weights' : [s0, s0compliment],
    'Expectation': [mu0, mu0],
    '120% Risk Increase (Std)' : np.array([sd2, sd2]),
    'Weights 2': [w1_1, w2_1],
    'Expectation 2': [muv, muv]

}
pd.DataFrame(d)
```

```python
plt.figure(facecolor = 'white')
x = np.linspace(varmin, .001)
plt.plot(mu0 + np.sqrt((x - varmin) / (A**2)), color = 'blue')
plt.plot(mu0 - np.sqrt((x - varmin) / (A**2)), color = 'blue')
plt.xlabel('Standard Deviation')
plt.ylabel('Expectation')
plt.title(stock1 + ' ' + stock2 + ' Expectation vs Risk')
```

## Alternative method in calculating expectation using quadratic formula of variance equation:

```python
# sd2 = (1.2)*sdmin

# var2_1 = sd2**2

# a = var1 + var2 - 2*c12
# b = -2*var2 + 2*c12
# c = var2 - var2_1

# x1 = (-b + np.sqrt((b**2) - 4*a*c)) / (2*a)
# x2 = (-b - np.sqrt((b**2) - 4*a*c)) / (2*a)

# mux1 = x1*mu1 + (1-x1)*mu2
# mux2 = x2*mu1 + (1-x2)*mu2

# mu0_1 = max(mux1, mux2)

# w1_1 = (mu0_1 - mu2)/ (mu1- mu2)
# w2_1 = 1 - w1_1
# print('Weights ', w1_1, w2_1)
# print('Expected Return ', mu0_1)
# # Portfolio requires short selling
```