Yeshiva University- Katz School of Science Health

Mathematics Department

Mathematics of Finance

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Project 1

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Contents

1 Abstract

The use of mathematics as an application to finance has historically been a core part of the economy. Mortgage loans is a great example of this, allowing individuals to borrow money effectively. The ability to purchase homes is a great investment for wealth generation and social mobility of the middle class. Mathematics plays a role in mortgage lending in order to determine interest rates, monthly interest payments and monthly amortization payments. This project aims to derive the formulas for these components of the mortgage loan, and simulate multiple scenarios of mortgage monthly payments. We derive these formulas and implement the simulations using Python.

2 Introduction

Mathematics plays a vital role in the application of finance and the economy as as whole. Mortgage lending is one specific part of finance which mathematics plays a key role in. Mortgage loans involve multiple parameter. Most are fixed, however, the borrower has some control over the terms the deal. The parameters involved which are out of the borrowers control are: the interest rate, r, and the compounding period, m. The parameters which he does have control over are: the amount borrowed, P, the term length, T. Weighing these is vital when considering the mortgage coupon. In this project, we derive formulas for the mortgage coupon, and present examples of simulating mortgage payments. By doing this, we study the relationship between R(n) (interest payments) and A(n) (amortization payments).

3 Project Body

The method of calculating mortgage payments involves compounding interest as opposed to simple interest. Simple interest involves interest which is applied to only the principle investment, which remains unchanged during the investment duration. Noting r as the interest rate, the value of the investment at time t, and time s as the time at which the the principle is invested, we can denote the value of the investment as: $V(t) = (1 + (t - s)r)P$. Where the principle investment is denoted as $V(0) = P$.

As opposed to simple interest, compounding interest includes interest added to the principle periodically, resulting in interest being compounded not only by the principle investment but also by the all the interest earned so far. Noting r as the interest rate, the value of the investment at time t, time s as the time at which the the principle is invested and m as the number of periods within t (usually 1 year) in which interest will be compounded to the loan. We can denote the value of the investment as: $V(t) = P(1+\frac{r}{m})^{tm}$.

Note the initial investment is denoted as: $V(0) = P$.

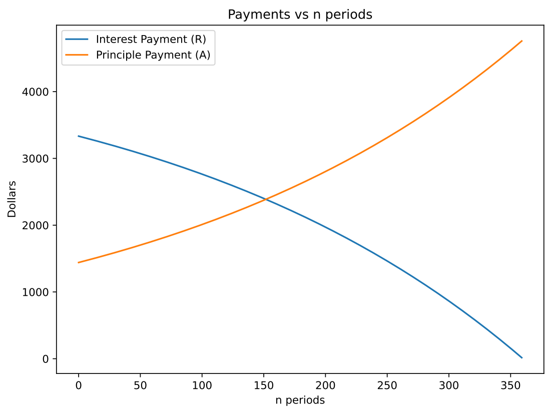
Mortgage loans are a deviation of traditional compounding interest investments called an annuity- a type of loan where the borrower pays a constant coupon each period, C, making sure the interest does not get too high. C is also describes as a "stream of payments", being a fixed amount at equal length time periods. The present value of payments at time N is denoted as: $V(N) = \frac{C}{1+\frac{r}{m}} + \frac{C}{(1+\frac{r}{m})^2} + ... + \frac{C}{(1+\frac{r}{m})^N}$. Using the geometric progression formula, this can be written as: $V(N) = C\frac{1-(1+\frac{r}{m})^{-N}}{\frac{r}{m}}$. Writing this formula for C give: $C = \frac{P\frac{r}{m}}{1-(1+\frac{r}{m})^{-N}}$

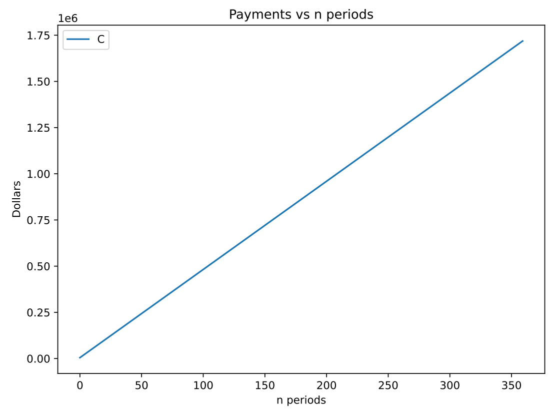
While we now have a formula to find the constant payment for mortgages, we would still like to know how much of the payment is going towards interest vs amortization. First, let's figure out the present value for the first few months. The value of the first month is: $V(1) = V(0) - \frac{C}{1+\frac{r}{m}}$. The value of the second month is: $V(2) = V(0) - \frac{C}{1+\frac{r}{m}} - \frac{C}{(1+\frac{r}{m})^2}$. The value of the nth month gives: $V(n) = V(0) - \frac{C}{1+\frac{r}{m}} - ... - \frac{C}{(1+\frac{r}{m})^n}$ which simplifies to $P\frac{(1+\frac{r}{m})^{(N-n)}-1}{(1+\frac{r}{m})^N-1}$. In order to find the amount owed at that point in time we divide by $(1+\frac{r}{m})^{-n}$ we get the important formula $P\frac{(1+\frac{r}{m})^N-(1+\frac{r}{m})^n}{(1+\frac{r}{m})^N-1}$. The amount of interest paid every period is just last month's actual value times the periodic interest rate: $R(n) = P\frac{(1+\frac{r}{m})^N-(1+\frac{r}{m})^{n-1}}{(1+\frac{r}{m})^N-1}(\frac{r}{m})$. To get the amortization rate take the difference between n value and n-1 value giving us formula $A(n) = P\frac{(1+\frac{r}{m})^n-(1+\frac{r}{m})^{n-1}}{(1+\frac{r}{m})^N-1} = P\frac{\frac{r}{m}(1+\frac{r}{m}^{n-1})}{(1+\frac{r}{m})^N-1}$

4 Computer Simulation & Results

Using Python, we run the simulation of mortgage payments with multiple realistic scenarios. Next, we plot the graphs for interest and amortization payments vs n periods for each scenario.

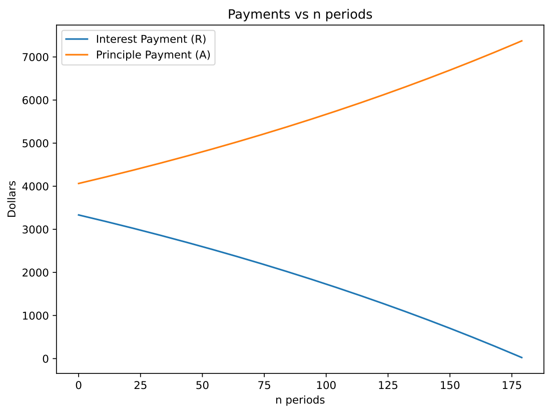
Scenario 1: P = \$1,000,000; r = 4%; T = 30 years; m = 12 months;

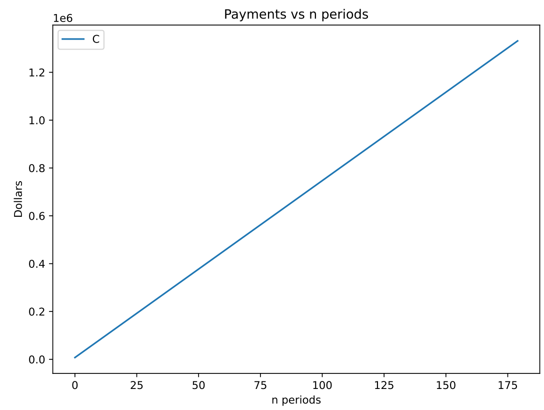




Note that as n periods increase, the interest payments decrease because the total value of money owed is less. Meanwhile, principle payments increase due to the constraint of C being constant (C = \$4,744). In other words, the property of C being constant causes the overall payments to have a linear trend. Additionally, this scenario of a 30 year mortgage leads to a total payment of \$1.72MM, with \$1MM going towards the principle and \$720K going towards interest payments.

Scenario 2: P = \$1,000,000; r = 4%; T = 15 years; m = 12 months;





Note that when T is reduced from 30 to 15 years, the amount of interest payed is significantly reduced. In this case, the a higher percentage of payments goes towards the principle, making 15 year mortgages more efficient that 30 year. As opposed to 30 year mortgage, the 15 year mortgage leads to a total payment of \$1.33MM, with \$1MM going towards the principle and \$331K going towards interest payments. Again, in this case the the overall payments graph is linear due to C being constant (C = \$7,396). Although the total payment in scenario 2 is less than scenario 1, C (the monthly payment) is 55% higher. This trade off is a decision the borrower must make- whether to pay a higher total payment, or a higher monthly payment.

5 Conclusion

The decision in taking a mortgage loan to buy a home is very important. Mathematics provides a valuable tool not only in structuring a mortgage loan, but also in deciding on its parameters. In this project, we derived the formulas in which a mortgage loan is based off of. Additionally, we ran scenarios showing the difference between loans of different lengths. Future simulations can show other scenarios such as changing the interest rates, changing the principle borrowed amount, etc.

6 Literature Cited

[1] M. Capinski, T. Zastawniak, Mathematics for Finance, Springer Undergraduate Mathematics Series, Springer - Verlang Limited 2011

Appendix

```
def c(p, r, T, m):
    N = m*T
    rm = r/m
    return (p*(rm)) / (1-(1+(rm))**-N)
def Rn(p, r, n, T, m):
    N = m*T
    rm = r/m
    return p * (rm) * (((1+ rm)**N) - (1+rm)**(n-1)) / (((1+rm)**N) -1)
def An(p, r, n, T, m):
    N = m*T
    rm = r/m
    return p * (rm*(1 +rm)**(n-1)) / ( ((1+rm)**N) -1 )
def r(r, m, T):
    N = m*T
    return (((1+r)**(1/m)) - 1)*m

p = 1000000
r= .04
T = 30
m = 365
N = T * m
R = []
A = []

C = c(p, r, T, m)

for n in range(1,N +1):
    R.append(Rn(p, r, n, T,m))
    A.append(An(p, r, n, T,m))
df = pd.DataFrame({
    'R' : R,
    'A' : A
})


figure(figsize = (18,6))
plt.subplot(1,2,1)
plt.plot(df['R'], label = 'Interest Payment (R)')
plt.plot(df['A'], label = 'Principle Payment (A)')
plt.ylabel('Dollars')
plt.xlabel('n periods')
plt.title('Payments vs n periods')
plt.legend()
plt.subplot(1,2,2)
plt.plot(np.cumsum(df['R'] + df['A']), label = 'C')
plt.ylabel('Dollars')
plt.xlabel('n periods')
plt.legend()
plt.title('Payments vs n periods')
plt.savefig(r'C:\Users\brian.livian\OneDrive - Interpublic\Desktop\Figure3.pdf')
plt.show()
df
```