# Hsieh Model - Version 2

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# 1 Introduction

The utility of the individual is given by:

$$U(c,s) = c^{\beta}(1-s) \tag{1}$$

Where:

- $c \rightarrow$ consumption
- $s \rightarrow$ time spent at school
- $\beta \rightarrow$ trade between consumption and accumulation of human capital

People working in a region $r$ and occupation $i$ is paid a net wage of $(1-\tau^w_{ir})w_{ir}$.

- $w_{ir} \rightarrow$ the net wage per efficiency unit of labor
- $\tau^w_{ir} \rightarrow$ is a distortion specific for occupation $i$ and location $r$

Human capital choices are also distorted due to tax on educational goods.

- $\tau^h_{ir} \rightarrow$ tax paid by a person that invest in education

The formation of human capital of a worker in a region $r$ is given by:

$$h_r(e,s) = (p^{\alpha}_{tr} H^{1-\alpha}_{tr})^{\varphi} s^{\phi} e^{\eta} \tag{2}$$

- $e \rightarrow$ consumption of educational goods
- $s \rightarrow$ time spent in school
- $H_{tr} \rightarrow$ aggregate human capital of teachers
- $\phi_i \rightarrow$ elasticity of human capital with respect to time in school
- $\eta \rightarrow$ elasticity of education goods in the human capital function
- $\varphi \rightarrow$ elasticity of teacher human capital in the human capital function
- $\alpha \rightarrow$ parameter between zero and one
- $p_{tr} \rightarrow$ is the fraction of people that work in occupation $t$ in region $r$. This can be view in proposition 1.

Abilities dispersion is modeled as multivariated Fréchet distribution. Let $\epsilon$ the abilities dispersion in modeled as a multivariate Fréchet distribution. So, we have:

$$F(\epsilon_1, ..., \epsilon_N) = exp\left[-\left(\sum_{i=1}^{N} \epsilon_i^{-\frac{\bar{\theta}}{1-\rho}}\right)^{1-\rho}\right] \tag{3}$$

- $\bar{\theta} \rightarrow$ skill dispersion
- $\rho \rightarrow \rho \in [0,1]$ gives the correlation of individual skills

let $\theta = \frac{\bar{\theta}}{1-\rho}$.

The representative firm has the following production function:

$$Y = \sum_{r=1}^{R} \sum_{i=1}^{N} A_r H_{ir} \tag{4}$$

- $Y \rightarrow$ output
- $A_r \rightarrow$ total factor productivity

## 1.1 Firm's problem

The firm's problem can be written as:

$$\underset{H_{ir}}{\text{Max}} \quad \sum_{r=1}^{R} \sum_{i=1}^{N} A_r H_{ir} - \sum_{r=1}^{R} \sum_{i=1}^{N} w_{ir} H_{ir} \tag{5}$$

FOCs:

$$\frac{\partial \Pi}{\partial H_{ir}} = \sum_{r=1}^{R} \sum_{i=1}^{N} A_r - \sum_{r=1}^{R} \sum_{i=1}^{N} w_{ir} = 0$$

$$\sum_{r=1}^{R} \sum_{i=1}^{N} A_r = \sum_{r=1}^{R} \sum_{i=1}^{N} w_{ir} \tag{6}$$

If the condition in equation 6 is satisfied, so $H_{ir}^d = x \in \mathbb{R}_+$. If $A_r < w_{ir}$ the profit function will be negative, so $H_{ir}^d = 0$. If $A_r > w_{ir}$, $H_{ir}^d = \infty$, because the profit function is linear in $H_{ir}$, so, the firms will produce infinitely.

## 1.2 Worker's problem

Given the occupational choice $i$ for which the individual has an idiosyncratic ability $\epsilon$, and taking wage $w_{ir}$ as given, each worker chooses consumption $c$, $e$ and time spent in school $s$ to solve the following problem:

$$\begin{aligned} &\underset{c,s,e}{\text{Max}} \quad c^{\beta}(1-s) \\ st. \quad &c = (1-\tau_{ir}^{w}) h_r(e,s) \epsilon w_{ir} - (1+\tau_{ir}^{h}) e \end{aligned} \tag{7}$$

FOCs:

$$U_s = \beta c^{\beta-1} c_s - \beta c^{\beta-1} c_s - c^{\beta} = 0$$

rearranging the terms we have:

$$\beta c^{\beta-1} c_s (1-s) - c^{\beta} = 0 \tag{8}$$

replacing $h_r(e,s)$ in $c$ we have:

$$c = (1-\tau_{ir}^{w})(p_{tr}^{\alpha} H_{tr}^{1-\alpha})^{\varphi} s^{\phi} e^{\eta} \epsilon w_{ir} - (1+\tau_{ir}^{h}) e$$

Derivating $c$ with respect to $s$ we obtain:

$$\frac{\partial c}{\partial s} = c_s = (1-\tau_{ir}^{w})(p_{tr}^{\alpha} H_{tr}^{1-\alpha})^{\varphi} \phi s^{\phi-1} e^{\eta} \epsilon w_{ir} \tag{9}$$

Plugging equation 9 in 8:

$$\beta c^{\beta-1}[(1-\tau_{ir}^{w})(p_{tr}^{\alpha}H_{tr}^{1-\alpha})^{\varphi}\phi s^{\phi-1}e^{\eta}\epsilon w_{ir}(1-s)]-c^{\beta}=0$$

Rearranging the terms we get $c$:

$$c=\beta[(1-\tau_{ir}^{w})(p_{tr}^{\alpha}H_{tr}^{1-\alpha})^{\varphi}\phi s^{\phi-1}e^{\eta}\epsilon w_{ir}(1-s)] \tag{10}$$

Deriving $U$ with respect $e$ we have:

$$U_e=\beta c^{\beta-1}c_e(1-s)=0 \tag{11}$$

Deriving $c$ with respect $e$ we have:

$$\frac{\partial c}{\partial e}=c_e=(1-\tau_{ir}^{w})(p_{tr}^{\alpha}H_{tr}^{1-\alpha})^{\varphi}s^{\phi}\eta e^{\eta-1}\epsilon w_{ir}-(1+\tau_{ir}^{h}) \tag{12}$$

Plugging equation 12 in 11 we have:

$$\begin{aligned}
\beta c^{\beta-1}[(1-\tau_{ir}^{w})(p_{tr}^{\alpha}H_{tr}^{1-\alpha})^{\varphi}s^{\phi}\eta e^{\eta-1}\epsilon w_{ir}-(1+\tau_{ir}^{h})](1-s)=0\\
(1-\tau_{ir}^{w})(p_{tr}^{\alpha}H_{tr}^{1-\alpha})^{\varphi}s^{\phi}\eta e^{\eta-1}\epsilon w_{ir}=(1+\tau_{ir}^{h})\\
e^{\eta-1}=\frac{(1+\tau_{ir}^{h})}{(1-\tau_{ir}^{w})(p_{tr}^{\alpha}H_{tr}^{1-\alpha})^{\varphi}s^{\phi}\eta\epsilon w_{ir}}\\
e=\left[\eta\left(\frac{1-\tau_{ir}^{w}}{1+\tau_{ir}^{h}}w_{ir}\right)(p_{tr}^{\alpha}H_{tr}^{1-\alpha})^{\varphi}s^{\phi}\epsilon\right]^{\frac{1}{1-\eta}}
\end{aligned} \tag{13}$$

Now, i'll match budget and expression 10.

$$\begin{aligned}
(1-\tau_{ir}^{w})(p_{tr}^{\alpha}H_{tr}^{1-\alpha})^{\varphi}s^{\phi}e^{\eta}\epsilon w_{ir}-(1+\tau_{ir}^{h})e=\beta(1-\tau_{ir}^{w})(p_{tr}^{\alpha}H_{tr}^{1-\alpha})^{\varphi}\phi s^{\phi-1}e^{\eta}\epsilon w_{ir}(1-s)\\
(1-\tau_{ir}^{w})(p_{tr}^{\alpha}H_{tr}^{1-\alpha})^{\varphi}s^{\phi}\epsilon w_{ir}-(1+\tau_{ir}^{h})e^{1-\eta}=\beta(1-\tau_{ir}^{w})(p_{tr}^{\alpha}H_{tr}^{1-\alpha})^{\varphi}\phi s^{\phi-1}\epsilon w_{ir}(1-s)
\end{aligned}$$

Replacing $e^{1-\eta}$ in this last expression we get:

$$\begin{aligned}
(1-\tau_{ir}^{w})(p_{tr}^{\alpha}H_{tr}^{1-\alpha})^{\varphi}s^{\phi}\epsilon w_{ir}-(1-\tau_{ir}^{w})(p_{tr}^{\alpha}H_{tr}^{1-\alpha})^{\varphi}s^{\phi}\eta\epsilon w_{ir}=\beta(1-\tau_{ir}^{w})(p_{tr}^{\alpha}H_{tr}^{1-\alpha})^{\varphi}\phi s^{\phi-1}\epsilon w_{ir}(1-s)\\
(1-\tau_{ir}^{w})(p_{tr}^{\alpha}H_{tr}^{1-\alpha})^{\varphi}s^{\phi}\epsilon w_{ir}(1-\eta)=\beta(1-\tau_{ir}^{w})(p_{tr}^{\alpha}H_{tr}^{1-\alpha})^{\varphi}\phi s^{\phi-1}\epsilon w_{ir}(1-s)\\
s^{\phi}(1-\eta)=\beta s^{\phi-1}\phi(1-s)\\
s^{*}=\frac{\beta\phi}{1-\eta+\beta\phi}
\end{aligned}$$

Finally we obtain $s^*$:

$$s^{*}=\left(1+\frac{1-\eta}{\beta\phi}\right)^{-1} \tag{14}$$

Plugging equation 14 in 13 we get:

$$e=\left[\eta\left(\frac{1-\tau_{ir}^{w}}{1+\tau_{ir}^{h}}w_{ir}\right)(p_{tr}^{\alpha}H_{tr}^{1-\alpha})^{\varphi}\left(1+\frac{1-\eta}{\beta\phi}\right)^{-\phi}\epsilon\right]^{\frac{1}{1-\eta}} \tag{15}$$

## 1.3 Indirect Utility

Now I will get the indirect utility. Recall that utility is given by:

$$U = c^{\beta}(1-s)$$

Replace the budget in this expression we have:

$$\begin{aligned} U &= [(1-\tau_{ir}^{w})(p_{tr}^{\alpha}H_{tr}^{1-\alpha})^{\varphi}s^{\phi}e^{\eta}\epsilon w_{ir} - (1+\tau_{ir}^{h})e]^{\beta}(1-s) \\ &= e^{\beta}[(1-\tau_{ir}^{w})(p_{tr}^{\alpha}H_{tr}^{1-\alpha})^{\varphi}s^{\phi}e^{\eta-1}\epsilon w_{ir} - (1+\tau_{ir}^{h})]^{\beta}(1-s) \end{aligned}$$

Replacing $e^{\eta-1}$:

$$\begin{aligned} &= e^{\beta}\left[\frac{(1-\tau_{ir}^{w})(p_{tr}^{\alpha}H_{tr}^{1-\alpha})^{\varphi}s^{\phi}(1+\tau_{ir}^{h})\epsilon w_{ir}}{(1-\tau_{ir}^{w})(p_{tr}^{\alpha}H_{tr}^{1-\alpha})^{\varphi}s^{\phi}\eta\epsilon w_{ir}} - (1+\tau_{ir}^{h})\right]^{\beta}(1-s) \\ &= e^{\beta}\left[\frac{(1+\tau_{ir}^{h})}{\eta} - (1+\tau_{ir}^{h})\right]^{\beta}(1-s) \\ &= \left[e(1+\tau_{ir}^{h})\left(\frac{1-\eta}{\eta}\right)\right]^{\beta}(1-s) \end{aligned}$$

Replacing $e$ by equation 15:

$$\begin{aligned} &= \left\{\left[\eta\frac{(1-\tau_{ir}^{w})}{(1+\tau_{ir}^{h})}(p_{tr}^{\alpha}H_{tr}^{1-\alpha})^{\varphi}s^{\phi}\epsilon w_{ir}\right]^{\frac{1}{1-\eta}}(1+\tau_{ir}^{h})\left(\frac{1-\eta}{\eta}\right)\right\}^{\beta}(1-s) \\ &= \left[\eta\frac{(1-\tau_{ir}^{w})}{(1+\tau_{ir}^{h})}(p_{tr}^{\alpha}H_{tr}^{1-\alpha})^{\varphi}s^{\phi}w_{ir}(1+\tau_{ir}^{h})^{1-\eta}\left(\frac{1-\eta}{\eta}\right)^{1-\eta}\epsilon\right]^{\frac{\beta}{1-\eta}}(1-s) \\ &= \left[\eta^{\eta}(1-\eta)^{1-\eta}\left(\frac{(1-\tau_{ir}^{w})}{(1+\tau_{ir}^{h})^{\eta}}w_{ir}\right)(p_{tr}^{\alpha}H_{tr}^{1-\alpha})^{\varphi}s^{\phi}\epsilon\right]^{\frac{\beta}{1-\eta}}(1-s) \end{aligned}$$

Finally, we get the indirect utility function:

$$D = \left[\psi\left(\frac{1-\tau_{ir}^{w}}{(1+\tau_{ir}^{h})^{\eta}}w_{ir}\right)(p_{tr}^{\alpha}H_{tr}^{1-\alpha})^{\varphi}s^{\phi}(1-s)^{\frac{1-\eta}{\beta}}\epsilon\right]^{\frac{\beta}{1-\eta}} \tag{16}$$

where $\psi = \eta^{\eta}(1-\eta)^{1-\eta}$.

## 1.4 Propositions

**Proposition 1.** *Aggregating among people, the solution of individual's occupational choice problem allows us to write:*

$$p_{ir} = \frac{\tilde{w}_{ir}^{\theta}}{\sum_{j=1}^{N}\tilde{w}_{jr}^{\theta}} \tag{17}$$

*where $p_{ir}$ is the fraction of people that work in occupation $i$ in region $r$ and:*

$$\tilde{w}_{ir} = \psi\left(\frac{1-\tau_{ir}^{w}}{(1+\tau_{ir}^{h})^{\eta}}w_{ir}\right)(p_{tr}^{\alpha}H_{tr}^{1-\alpha})^{\varphi}s_{i}^{\phi_i}(1-s_i)^{\frac{1-\eta}{\beta}}$$

*We can interpret $\tilde{w}_{ir}$ as a liquid reward for a person with mean ability from region $r$ and occupation $i$. So, $\tilde{w}_{ir}$ is composed by wage per efficiency unit in the occupation $w_{ir}$ schooling, teacher's human capital and frictions.*

*Proof.* Let:

$$\tilde{w}_{ir} = \psi\left(\frac{1-\tau^w_{ir}}{(1+\tau^h_{ir})^\eta} w_{ir}\right)(p^\alpha_{tr} H^{1-\alpha}_{tr})^\varphi s_i^{\phi_i}(1-s_i)^{\frac{1-\eta}{\beta}}$$

We can rewrite equation 16 as:

$$D_{ir} = [\tilde{w}_{ir}\epsilon_i]^{\frac{\beta}{1-\eta}}$$

where $\psi = (\eta^\eta(1-\eta)^{(1-\eta)})$. Therefore, the solution of individual's problem in region r involves picking the occupation with the highest value of $\tilde{w}_{ir}\epsilon_i$.

Without loss of generality, consider the probability of an individual choose occupation 1:

$$\begin{aligned} p_{ir} &= Pr(\tilde{w}_{1r}\epsilon_1 > \tilde{w}_{ir}\epsilon_i) \quad \forall\, i \neq 1 \\ &= Pr\left(\epsilon_i < \frac{\tilde{w}_{1r}}{\tilde{w}_{ir}}\epsilon_1\right) \quad \forall\, i \neq 1 \\ &= \int F_1(\alpha_1\epsilon, \alpha_2\epsilon, ..., \alpha_N\epsilon)d\epsilon \end{aligned} \tag{18}$$

Where $F1$ represents the derivative of equation 3 with respect to its first argument and $\alpha_i = \tilde{w}_{1r}/\tilde{w}_{ir}$ for $i \in \{1, 2, ...N\}$. Taking the derivative of equation 3 with respect to $\epsilon_1$ and evaluating at the appropriate arguments gives:

$$\begin{aligned} \frac{\partial F}{\partial \epsilon_1} = F_1 &= -(1-p)\left(\sum_{i=1}^N \epsilon_i^{-\theta}\right)^{-\rho}\left(\frac{-\theta\epsilon_1^{-\theta-1}}{1-\rho}\right)\exp\left[-\left(\sum_{i=1}^N \epsilon_i^{-\theta}\right)^{1-\rho}\right] \\ &= \theta\epsilon_1^{-(\theta+1)}\left(\sum_{i=1}^N \epsilon_i^{-\theta}\right)^{-\rho}\exp\left[-\left(\sum_{i=1}^N \epsilon_i^{-\theta}\right)^{1-\rho}\right] \\ &= \theta\epsilon_1^{-\theta+1}(\hat{S}\epsilon_1^{-\theta})^{-\rho}\exp\Big[-(\hat{S}\epsilon_1^{-\theta})^{1-\rho}\Big] \\ &= \hat{S}^{-\rho}\theta\epsilon_1^{-\theta(1-\rho)-1}\exp\Big[-(\hat{S}\epsilon_1^{-\theta})^{1-\rho}\Big] \\ F_1(\epsilon) &= \hat{S}^{-\rho}\theta\epsilon^{-\theta(1-\rho)-1}\exp\Big[-(\hat{S}\epsilon^{-\theta})^{1-\rho}\Big] \end{aligned}$$

Where $\hat{S} = \sum_{i=1}^n \alpha_i^{-\theta}$. Then, equation 18 can be written as:

$$\begin{aligned} p_{1r} &= \int \frac{\hat{S}}{\hat{S}}\hat{S}^{-\rho}\theta\epsilon^{-\theta(1-\rho)-1}\exp\Big[-(\hat{S}\epsilon^{-\theta})^{1-\rho}\Big]d\epsilon \\ &= \frac{1}{\hat{S}}\int \hat{S}\hat{S}^{-\rho}\theta\epsilon^{-\theta(1-\rho)-1}\exp\Big[-(\hat{S}\epsilon^{-\theta})^{1-\rho}\Big]d\epsilon \end{aligned}$$

Note that this expression is the derivative of equation 3 with respect to $\epsilon$. So, we have:

$$\begin{aligned} &= \frac{1}{\hat{S}}\int dF(\epsilon) \\ &= \frac{1}{\hat{S}} \\ &= \frac{\tilde{w}_{1r}^\theta}{\sum_{i=1}^N \tilde{w}_{ir}^\theta} \end{aligned} \tag{19}$$

□

**Proposition 2.** *For a given region, the average quality of workers in occupation $i$, including both human capital and idiosyncratic abilities, is:*

$$\mathbb{E}[h(e_{ir}, s_i)\epsilon_i] = \gamma \left[ \left( \frac{1-\tau^w_{ir}}{1+\tau^h_{ir}} w_{ir} \right)^{\eta} \tilde{h}_{ir} p_{ir}^{-\frac{1}{\theta}} \right]^{\frac{1}{1-\eta}} \tag{20}$$

*Where $\gamma = \Gamma(1-(\theta(1-\rho))^{-1}(1-\eta)^{-1})$ is related to the mean of the Fréchet distribution for abilities. And $\tilde{h}_{ir} = [(p^{\alpha}_{tr} H^{1-\alpha}_{tr})^{\varphi} s_i^{\phi_i} \eta^{\eta}]^{\frac{1}{1-\eta}}$.*

*Proof.* Notice that:

$$H_{ir} = p_{ir}\mathbb{E}[h(e_{ir}, s_i)\epsilon_i | \text{person choices } i] \tag{21}$$

and

$$h(e_{ir}, s_i)\epsilon_i = (p^{\alpha}_{tr} H^{1-\alpha}_{tr})^{\varphi} \left[ \eta \left( \frac{1-\tau^w_{ir}}{1+\tau^h_{ir}} w_{ir} \right) (p^{\alpha}_{tr} H^{1-\alpha}_{tr})^{\varphi} \left(1 + \frac{1-\eta}{\beta\phi}\right)^{-\phi} \epsilon_i \right]^{\frac{\eta}{1-\eta}} s_i^{\phi_i} \epsilon_i \tag{22}$$

Where $H_{ir}$ is the total efficiency units of labor supplied to occupation $i$ in region $r$. Then:

$$H_{ir} = p_{ir}\mathbb{E}\left\{ (p^{\alpha}_{tr} H^{1-\alpha}_{tr})^{\varphi} \left[ \eta \left( \frac{1-\tau^w_{ir}}{1+\tau^h_{ir}} w_{ir} \right) (p^{\alpha}_{tr} H^{1-\alpha}_{tr})^{\varphi} \left(1 + \frac{1-\eta}{\beta\phi}\right)^{-\phi} \epsilon_i \right]^{\frac{\eta}{1-\eta}} s_i^{\phi_i} \epsilon_i \middle| \text{person choices } i \right\}$$

$$H_{ir} = p_{ir}\left\{ (p^{\alpha}_{tr} H^{1-\alpha}_{tr})^{\varphi} \left[ \left( \frac{1-\tau^w_{ir}}{1+\tau^h_{ir}} w_{ir} \right) \eta (p^{\alpha}_{tr} H^{1-\alpha}_{tr})^{\varphi} s_i^{\phi_i} \right]^{\frac{\eta}{1-\eta}} s_i^{\phi_i} \mathbb{E}\left[ \epsilon_i^{\frac{1}{1-\eta}} \middle| \text{person choices } i \right] \right\}$$

$$H_{ir} = p_{ir}\tilde{h}_{ir} \left( \frac{1-\tau^w_{ir}}{1+\tau^h_{ir}} w_{ir} \right)^{\frac{\eta}{1-\eta}} \mathbb{E}\left[ \epsilon_i^{\frac{1}{1-\eta}} \middle| \text{person choices } i \right] \tag{23}$$

Where $\tilde{h}_{ir} = [(p^{\alpha}_{tr} H^{1-\alpha}_{tr})^{\varphi} s_i^{\phi_i} \eta^{\eta}]^{\frac{1}{1-\eta}}$.

To calculate this last conditional expectation, we use the Fréchet distribution. For now, we suppress the region index $r$, because this calculation is similar for all regions. Let $yi = \tilde{w}_i \epsilon_i$. Since $y_i$ is the thing we are maximizing, it inherits the extreme value distribution:

$$\begin{aligned} Pr\left(\operatorname*{Max}_i y_i < z\right) &= Pr(\epsilon_i < z/\tilde{w}_i) \quad \forall i \\ &= F(z/\tilde{w}_1, ..., z/\tilde{w}_N) \\ &= \exp\left[ -\left( -\sum_{i=1}^{N} (z/\tilde{w}_i)^{-\theta} \right)^{1-\rho} \right] \\ &= \exp\left[ -\left( \hat{S} z^{-\theta} \right)^{1-\rho} \right] \end{aligned}$$

That is, the extreme value also has a Fréchet distribution. Straightforward algebra then reveals that the distribution of $\epsilon^*$ ,the ability of people in their chosen occupation, is also Fréchet:

$$G(x) = Pr(\epsilon^* < x) = \exp\left[ -\left( \hat{S}^* z^{-\theta} \right)^{1-\rho} \right] \tag{24}$$

Where $\hat{S}^* = \sum_{i=1}^{N} (\tilde{w}_i / \tilde{w}^*)^{\theta}$

Finally, one can then calculate the expectation we needed above, back in equation 23. Let $i$ denote the occupation that the individual chooses, and let $\lambda$ be some positive exponent. So:

$$
\begin{aligned}
E(\epsilon_i^{\lambda}) &= \int_0^{\infty} \epsilon_i^{\lambda} dG(\epsilon) \\
&= \int_0^{\infty} \theta(1-\rho)\hat{S}^{*(1-\rho)}\epsilon^{-\theta(1-\rho)-1+\lambda} \exp\left[-\left(\hat{S}^{*}\epsilon^{-\theta}\right)^{1-\rho}\right] d\epsilon \\
&= \hat{S}^{*\lambda/\theta} \int_0^{\infty} x^{-\frac{\lambda}{\theta(1-\rho)}} \exp(-x) dx
\end{aligned} \tag{25}
$$

Where $x = \left(\hat{S}^{*}\epsilon^{-\theta}\right)^{1-\theta}$. The last part of equation 25 is a gamma function which amounts to $\Gamma(1-\lambda(\theta(1-\rho))^{-1})$. Therefore, we have

$$
\mathbb{E}\left[\epsilon_i^{\frac{1}{1-\eta}} \middle| \text{person choices } i\right] = \left(\frac{1}{p_{ir}}\right)^{\frac{1}{\theta(1-\eta)}} \Gamma\left(1 - \frac{1}{\theta(1-\rho)}\frac{1}{1-\eta}\right) \tag{26}
$$

Using this result in the equation 23 completes the proof.

□

**Proposition 3.** *Let $W_{ir}$ be the gross average earnings in occupation $i$ in region $r$. Then:*

$$
W_{ir} = w_{ir}\mathbb{E}[h(e_{ir}, s_i)\epsilon_i] = \frac{(1-s)^{-1/\beta}}{(1-\tau_{ir}^{w})}\gamma\eta\left(\sum_{s=1}^{N}\tilde{w}_{sr}^{\theta}\right)^{\frac{1}{\theta(1-\eta)}} \tag{27}
$$

## 1.5 Problem

So, the problem consist in minimize equation 28, using Nelder-Mead[1] algorithm.

$$
Dist = \sum_{i=1,r=1}^{N,R}\left(\frac{W_{ir}^{M} - W_{ir}^{T}}{W_{ir}^{T}}\right)^2 + \sum_{i=1,r=1}^{N,R}\left(\frac{p_{ir}^{M} - p_{ir}^{T}}{p_{ir}^{T}}\right)^2 \tag{28}
$$

where $p_{ir}^{M}$ and $W_{ir}^{M}$ are given by equations 17 and 27 respectively. On the other hand, $p_{ir}^{T}$ and $W_{ir}^{T}$ are given by PNAD data. The superscript indicate model and target statistics. We assume that $\tau_{1r}^{h} = 0$, $\tau_{1r}^{w} = \tau_{1}^{w}$, $\forall r$. And $A_R = 1$, i.e, the TPF of the last region is nomalized to 1.

[1]The implementation of Hsieh Model in Python and R language can be view in my GitHub: `https://github.com/mj-ribeiro/College-works/tree/master/Thesis`. Other algorithms can be used, for example genetic algorithm