

Hsieh Model

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1 Introduction

The utility of the individual is given by:

$$U(c, s) = c^\beta(1 - s) \quad (1)$$

Where:

- $c \rightarrow$ consumption
- $s \rightarrow$ time spent at school
- $\beta \rightarrow$ trade between consumption and accumulation of human capital

People working in a region r and occupation i is paid a net wage of $(1 - \tau_{ir}^w)w_{ir}$.

- $w_{ir} \rightarrow$ the net wage per efficiency unit of labor
- $\tau_{ir}^w \rightarrow$ is a distortion specific for occupation i and location r

Human capital choices are also distorted due to tax on educational goods.

- $\tau_{ir}^h \rightarrow$ tax paid by a person that invest in education

The formation of human capital of a worker in a region r is given by:

$$h_r(e, s) = H_{tr}^\varphi s^\phi e^\eta \quad (2)$$

- $e \rightarrow$ consumption of educational goods
- $s \rightarrow$ time spent in school
- $H_{tr} \rightarrow$ aggregate human capital of teachers
- $\phi_i \rightarrow$ elasticity of human capital with respect to time in school
- $\eta \rightarrow$ elasticity of education goods in the human capital function
- $\varphi \rightarrow$ elasticity of teacher human capital in the human capital function

Abilities dispersion is modeled as multivariate Fréchet distribution. Let ϵ the abilities dispersion in modeled as a multivariate Fréchet distribution. So, we have:

$$F(\epsilon_1, \dots, \epsilon_N) = \exp \left[- \left(\sum_{i=1}^N \epsilon_i^{-\frac{\bar{\theta}}{1-\rho}} \right)^{1-\rho} \right] \quad (3)$$

- $\bar{\theta} \rightarrow$ skill dispersion
- $\rho \rightarrow \rho \in [0, 1]$ gives the correlation of individual skills

let $\theta = \frac{\bar{\theta}}{1-\rho}$.

The representative firm has the following production function:

$$Y = \sum_{r=1}^R \sum_{i=1}^N A_r H_{ir} \quad (4)$$

- $Y \rightarrow$ output
- $A_r \rightarrow$ total factor productivity

1.1 Firm's problem

The firm's problem can be written as:

$$\text{Max}_{H_{ir}} \sum_{r=1}^R \sum_{i=1}^N A_r H_{ir} - \sum_{r=1}^R \sum_{i=1}^N w_{ir} H_{ir} \quad (5)$$

CPO:

$$\begin{aligned} \frac{\partial \Pi}{\partial H_{ir}} &= \sum_{r=1}^R \sum_{i=1}^N A_r - \sum_{r=1}^R \sum_{i=1}^N w_{ir} = 0 \\ \sum_{r=1}^R \sum_{i=1}^N A_r &= \sum_{r=1}^R \sum_{i=1}^N w_{ir} \end{aligned} \quad (6)$$

If the condition in equation 6 is satisfied, so $H_{ir}^d = x \in \mathbb{R}_+$. If $A_r < w_{ir}$ the profit function will be negative, so $H_{ir}^d = 0$. If $A_r > w_{ir}$, $H_{ir}^d = \infty$, because the profit function is linear in H_{ir} , so, the firms will produce infinitely.

1.2 Worker's problem

Given the occupational choice i for which the individual has an idiosyncratic ability ϵ , and taking wage w_{ir} as given, each worker chooses consumption c , e and time spent in school s to solve the following problem:

$$\begin{aligned} \text{Max}_{c,s,e} \quad & c^\beta (1-s) \\ \text{st.} \quad & c = (1 - \tau_{ir}^w) h_r(e, s) \epsilon w_{ir} - (1 + \tau_{ir}^h) e \end{aligned} \quad (7)$$

CPOs:

$$U_s = \beta c^{\beta-1} c_s - \beta c^{\beta-1} c_s - c^\beta = 0$$

rearranging the terms we have:

$$\beta c^{\beta-1} c_s (1-s) - c^\beta = 0 \quad (8)$$

replacing $h_r(e, s)$ in c we have:

$$c = (1 - \tau_{ir}^w) H_{tr}^\varphi s^\phi e^\eta \epsilon w_{ir} - (1 + \tau_{ir}^h) e$$

Derivating c with respect to s we obtain:

$$\frac{\partial c}{\partial s} = c_s = (1 - \tau_{ir}^w) H_{tr}^\varphi \phi s^{\phi-1} e^\eta \epsilon w_{ir} \quad (9)$$

Plugging equation 9 in 8:

$$\beta c^{\beta-1} [(1 - \tau_{ir}^w) H_{tr}^\varphi \phi s^{\phi-1} e^\eta \epsilon w_{ir} (1-s)] - c^\beta = 0$$

Rearranging the terms we get c :

$$c = \beta [(1 - \tau_{ir}^w) H_{tr}^\varphi \phi s^{\phi-1} e^\eta \epsilon w_{ir} (1-s)] \quad (10)$$

Deriving U with respect e we have:

$$U_e = \beta c^{\beta-1} c_e (1-s) = 0 \quad (11)$$

Deriving c with respect e we have:

$$\frac{\partial c}{\partial e} = c_e = (1 - \tau_{ir}^w)H_{tr}^\varphi s^\phi \eta e^{\eta-1} \epsilon w_{ir} - (1 + \tau_{ir}^h) \quad (12)$$

Plugging equation 12 in 11 we have:

$$\begin{aligned} \beta c^{\beta-1} [(1 - \tau_{ir}^w)H_{tr}^\varphi s^\phi \eta e^{\eta-1} \epsilon w_{ir} - (1 + \tau_{ir}^h)](1 - s) &= 0 \\ (1 - \tau_{ir}^w)H_{tr}^\varphi s^\phi \eta e^{\eta-1} \epsilon w_{ir} &= (1 + \tau_{ir}^h) \\ e^{\eta-1} &= \frac{(1 + \tau_{ir}^h)}{(1 - \tau_{ir}^w)H_{tr}^\varphi s^\phi \eta \epsilon w_{ir}} \\ e &= \left(\frac{(1 - \tau_{ir}^w)H_{tr}^\varphi s^\phi \eta \epsilon w_{ir}}{1 + \tau_{ir}^h} \right)^{\frac{1}{1-\eta}} \end{aligned} \quad (13)$$

Now, i'll match budget and expression 10.

$$\begin{aligned} (1 - \tau_{ir}^w)H_{tr}^\varphi s^\phi e^\eta \epsilon w_{ir} - (1 + \tau_{ir}^h)e &= \beta(1 - \tau_{ir}^w)H_{tr}^\varphi \phi s^{\phi-1} e^\eta \epsilon w_{ir}(1 - s) \\ (1 - \tau_{ir}^w)H_{tr}^\varphi s^\phi \epsilon w_{ir} - (1 + \tau_{ir}^h)e^{1-\eta} &= \beta(1 - \tau_{ir}^w)H_{tr}^\varphi \phi s^{\phi-1} \epsilon w_{ir}(1 - s) \end{aligned}$$

Replacing $e^{1-\eta}$ in this last expression we get:

$$\begin{aligned} (1 - \tau_{ir}^w)H_{tr}^\varphi s^\phi \epsilon w_{ir} - (1 - \tau_{ir}^w)H_{tr}^\varphi s^\phi \eta \epsilon w_{ir} &= \beta(1 - \tau_{ir}^w)H_{tr}^\varphi \phi s^{\phi-1} \epsilon w_{ir}(1 - s) \\ 1 - \tau_{ir}^w)H_{tr}^\varphi s^\phi \epsilon w_{ir}(1 - \eta) &= \beta(1 - \tau_{ir}^w)H_{tr}^\varphi \phi s^{\phi-1} \epsilon w_{ir}(1 - s) \\ s^\phi(1 - \eta) &= \beta s^{\phi-1} \phi(1 - s) \\ s^* &= \frac{\beta \phi}{1 - \eta + \beta \phi} \end{aligned}$$

Finally we obtain s^* :

$$s^* = \left(1 + \frac{1 - \eta}{\beta \phi} \right)^{-1} \quad (14)$$

Plugging equation 14 in 13 we get:

$$e = \left[\eta \frac{(1 - \tau_{ir}^w)}{(1 + \tau_{ir}^h)} H_{tr}^\varphi \left(1 + \frac{1 - \eta}{\beta \phi} \right)^{-\phi} \epsilon w_{ir} \right]^{\frac{1}{1-\eta}} \quad (15)$$

1.3 Indirect Utility

Now I will get the indirect utility. Recall that utility is given by:

$$U = c^\beta (1 - s)$$

Replace the budget in this expression we have:

$$\begin{aligned} U &= [(1 - \tau_{ir}^w)H_{tr}^\varphi s^\phi e^\eta \epsilon w_{ir} - (1 + \tau_{ir}^h)e]^\beta (1 - s) \\ &= e^\beta [(1 - \tau_{ir}^w)H_{tr}^\varphi s^\phi e^{\eta-1} \epsilon w_{ir} - (1 + \tau_{ir}^h)]^\beta (1 - s) \end{aligned}$$

Replacing $e^{\eta-1}$:

$$\begin{aligned}
&= e^\beta \left[\frac{(1 - \tau_{ir}^w) H_{tr}^\varphi s^\phi (1 + \tau_{ir}^h) \epsilon w_{ir}}{(1 - \tau_{ir}^w) H_{tr}^\varphi s^\phi \eta \epsilon w_{ir}} - (1 + \tau_{ir}^h) \right]^\beta (1 - s) \\
&= e^\beta \left[\frac{(1 + \tau_{ir}^h)}{\eta} - (1 + \tau_{ir}^h) \right]^\beta (1 - s) \\
&= \left[e(1 + \tau_{ir}^h) \left(\frac{1 - \eta}{\eta} \right) \right]^\beta (1 - s)
\end{aligned}$$

Replacing e by equation 15:

$$\begin{aligned}
&= \left\{ \left[\eta \frac{(1 - \tau_{ir}^w)}{(1 + \tau_{ir}^h)} H_{tr}^\varphi s^\phi \epsilon w_{ir} \right]^{\frac{1}{1-\eta}} (1 + \tau_{ir}^h) \left(\frac{1 - \eta}{\eta} \right) \right\}^\beta (1 - s) \\
&= \left[\eta \frac{(1 - \tau_{ir}^w)}{(1 + \tau_{ir}^h)} H_{tr}^\varphi s^\phi \epsilon w_{ir} (1 + \tau_{ir}^h)^{1-\eta} \left(\frac{1 - \eta}{\eta} \right)^{1-\eta} \right]^{\frac{\beta}{1-\eta}} (1 - s) \\
&= \left[\eta^\eta (1 - \eta)^{1-\eta} \left(\frac{(1 - \tau_{ir}^w)}{(1 + \tau_{ir}^h) \eta} \right) w_{ir} \epsilon H_{tr}^\varphi s^\phi \right]^{\frac{\beta}{1-\eta}} (1 - s)
\end{aligned}$$

Finally, we get the indirect utility function:

$$D = \left[\eta^\eta (1 - \eta)^{1-\eta} \frac{1 - \tau_{ir}^w}{(1 + \tau_{ir}^h) \eta} w_{ir} \epsilon H_{tr}^\varphi s^\phi (1 - s)^{\frac{1-\eta}{\beta}} \right]^{\frac{\beta}{1-\eta}} \quad (16)$$

1.4 Propositions

Proposition 1. *Aggregating among people, the solution of individual's occupational choice problem allows us to write:*

$$p_{ir} = \frac{\tilde{w}_{ir}^\theta}{\sum_{j=1}^N \tilde{w}_{jr}^\theta} \quad (17)$$

where p_{ir} is the fraction of people that work in occupation i in region r and:

$$\tilde{w}_{ir} = \frac{1 - \tau_{ir}^w}{(1 + \tau_{ir}^h) \eta} w_{ir} H_{tr}^\varphi s_i^{\phi_i} (1 - s_i)^{\frac{1-\eta}{\beta}}$$

We can interpret \tilde{w}_{ir} as a liquid reward for a person with mean ability from region r and occupation i . So, \tilde{w}_{ir} is composed by wage per efficiency unit in the occupation w_{ir} schooling, teacher's human capital and frictions.

Proof. Let:

$$\tilde{w}_{ir} = \frac{1 - \tau_{ir}^w}{(1 + \tau_{ir}^h) \eta} w_{ir} H_{tr}^\varphi s_i^{\phi_i} (1 - s_i)^{\frac{1-\eta}{\beta}}$$

We can rewrite equation 16 as:

$$D_{ir} = [\tilde{w}_{ir} \epsilon_i]^{\frac{\beta}{1-\eta}}$$

Therefore, the solution of individual's problem in region r involves picking the occupation with the highest value of $\tilde{w}_{ir} \epsilon_i$.

Without loss of generality, consider the probability of an individual choose occupation 1:

$$\begin{aligned}
p_{ir} &= Pr(\tilde{w}_{1r}\epsilon_1 > \tilde{w}_{ir}\epsilon_i) \quad \forall i \neq 1 \\
&= Pr\left(\epsilon_i < \frac{\tilde{w}_{1r}}{\tilde{w}_{ir}}\epsilon_1\right) \quad \forall i \neq 1 \\
&= \int F_1(\alpha_1\epsilon, \alpha_2\epsilon, \dots, \alpha_N\epsilon)d\epsilon
\end{aligned} \tag{18}$$

Where F_1 represents the derivative of equation 3 with respect to its first argument and $\alpha_i = \tilde{w}_{1r}/\tilde{w}_{ir}$ for $i \in \{1, 2, \dots, N\}$. Taking the derivative of equation 3 with respect to ϵ_1 and evaluating at the appropriate arguments gives:

$$\begin{aligned}
\frac{\partial F}{\partial \epsilon_1} &= F_1 = -(1-p) \left(\sum_{i=1}^N \epsilon_i^{-\theta}\right)^{-\rho} \left(\frac{-\theta\epsilon_1^{-\theta-1}}{1-\rho}\right) \exp\left[-\left(\sum_{i=1}^N \epsilon_i^{-\theta}\right)^{1-\rho}\right] \\
&= \theta\epsilon_1^{-(\theta+1)} \left(\sum_{i=1}^N \epsilon_i^{-\theta}\right)^{-\rho} \exp\left[-\left(\sum_{i=1}^N \epsilon_i^{-\theta}\right)^{1-\rho}\right] \\
&= \theta\epsilon_1^{-\theta+1} (\hat{S}\epsilon_1^{-\theta})^{-\rho} \exp\left[-(\hat{S}\epsilon_1^{-\theta})^{1-\rho}\right] \\
&= \hat{S}^{-\rho}\theta\epsilon_1^{-\theta(1-\rho)-1} \exp\left[-(\hat{S}\epsilon_1^{-\theta})^{1-\rho}\right] \\
F_1(\epsilon) &= \hat{S}^{-\rho}\theta\epsilon^{-\theta(1-\rho)-1} \exp\left[-(\hat{S}\epsilon^{-\theta})^{1-\rho}\right]
\end{aligned}$$

Where $\hat{S} = \sum_{i=1}^N \alpha_i^{-\theta}$. Then, equation 18 can be written as:

$$\begin{aligned}
p_{1r} &= \int \frac{\hat{S}}{\hat{S}} \hat{S}^{-\rho}\theta\epsilon^{-\theta(1-\rho)-1} \exp\left[-(\hat{S}\epsilon^{-\theta})^{1-\rho}\right] d\epsilon \\
&= \frac{1}{\hat{S}} \int \hat{S} \hat{S}^{-\rho}\theta\epsilon^{-\theta(1-\rho)-1} \exp\left[-(\hat{S}\epsilon^{-\theta})^{1-\rho}\right] d\epsilon
\end{aligned}$$

Note that this expression is the derivative of equation 3 with respect to ϵ . So, we have:

$$\begin{aligned}
&= \frac{1}{\hat{S}} \int dF(\epsilon) \\
&= \frac{1}{\hat{S}} \\
&= \frac{\tilde{w}_{1r}^\theta}{\sum_{i=1}^N \tilde{w}_{ir}^\theta}
\end{aligned} \tag{19}$$

□

Proposition 2. *For a given region, the average quality of workers in occupation i , including both human capital and idiosyncratic abilities, is:*

$$\mathbb{E}[h(e_{ir}, s_i)\epsilon_i] = \gamma \left[H_{tr}^\varphi \left(\frac{1 - \tau_{ir}^w}{1 + \tau_{ir}^h} \right)^\eta \eta^\eta s_i^\phi p_{ir}^{-\frac{1}{\theta}} \right]^{\frac{1}{1-\eta}} \tag{20}$$

Where $\gamma = \Gamma(1 - (\theta(1 - \rho))^{-1}(1 - \eta)^{-1})$ is related to the mean of the Fréchet distribution for abilities.

Proof. Notice that:

$$H_{ir} = p_{ir} \mathbb{E}[h(e_{ir}, s_i) \epsilon_i | \text{person choices } i] \quad \text{and} \quad (21)$$

$$h(e_{ir}, s_i) \epsilon_i = H_{tr}^\varphi \left[\left(\frac{1 - \tau_{ir}^w}{1 + \tau_{ir}^h} \right) w_i \epsilon_i \eta H_{tr}^\varphi s_i^{\phi_i} \right]^{\frac{\eta}{1-\eta}} s_i^{\phi_i} \epsilon_i \quad (22)$$

Where H_{ir} is the total efficiency units of labor supplied to occupation i in region r . Then:

$$\begin{aligned} H_{ir} &= p_{ir} \mathbb{E} \left\{ H_{tr}^\varphi \left[\left(\frac{1 - \tau_{ir}^w}{1 + \tau_{ir}^h} \right) w_i \epsilon_i \eta H_{tr}^\varphi s_i^{\phi_i} \right]^{\frac{\eta}{1-\eta}} s_i^{\phi_i} \epsilon_i \middle| \text{person choices } i \right\} \\ H_{ir} &= p_{ir} \left\{ H_{tr}^\varphi \left[\left(\frac{1 - \tau_{ir}^w}{1 + \tau_{ir}^h} \right) w_i \eta H_{tr}^\varphi s_i^{\phi_i} \right]^{\frac{\eta}{1-\eta}} s_i^{\phi_i} \mathbb{E} \left[\epsilon_i^{\frac{1}{1-\eta}} \middle| \text{person choices } i \right] \right\} \\ H_{ir} &= p_{ir} \tilde{h}_{ir} \left[\frac{(1 - \tau_{ir}^w)}{(1 + \tau_{ir}^h)} w_i \right]^{\frac{\eta}{1-\eta}} \mathbb{E} \left[\epsilon_i^{\frac{1}{1-\eta}} \middle| \text{person choices } i \right] \end{aligned} \quad (23)$$

Where $\tilde{h}_{ir} = (H_{tr}^\varphi s_i^{\phi_i} \eta^\eta)^{\frac{1}{1-\eta}}$.

To calculate this last conditional expectation, we use the Fréchet distribution. For now, we suppress the region index r , because this calculation is similar for all regions. Let $y_i = \tilde{w}_i \epsilon_i$. Since y_i is the thing we are maximizing, it inherits the extreme value distribution:

$$\begin{aligned} Pr \left(\max_i y_i < z \right) &= Pr(\epsilon_i < z/\tilde{w}_i) \quad \forall i \\ &= F(z/\tilde{w}_1, \dots, z/\tilde{w}_N) \\ &= \exp \left[- \left(- \sum_{i=1}^N (z/\tilde{w}_i)^{-\theta} \right)^{1-\rho} \right] \\ &= \exp \left[- \left(\hat{S} z^{-\theta} \right)^{1-\rho} \right] \end{aligned}$$

That is, the extreme value also has a Fréchet distribution. Straightforward algebra then reveals that the distribution of ϵ^* , the ability of people in their chosen occupation, is also Fréchet:

$$G(x) = Pr(\epsilon^* < x) = \exp \left[- \left(\hat{S}^* x^{-\theta} \right)^{1-\rho} \right] \quad (24)$$

Where $\hat{S}^* = \sum_{i=1}^N (\tilde{w}_i/\tilde{w}^*)^\theta$

Finally, one can then calculate the expectation we needed above, back in equation 23. Let i denote the occupation that the individual chooses, and let λ be some positive exponent. So:

$$\begin{aligned} E(\epsilon_i^\lambda) &= \int_0^\infty \epsilon_i^\lambda dG(\epsilon) \\ &= \int_0^\infty \theta(1-\rho) \hat{S}^{*(1-\rho)} \epsilon^{-\theta(1-\rho)-1+\lambda} \exp \left[- \left(\hat{S}^* \epsilon^{-\theta} \right)^{1-\rho} \right] d\epsilon \\ &= \hat{S}^{*\lambda/\theta} \int_0^\infty x^{-\frac{\lambda}{\theta(1-\rho)}} \exp(-x) dx \end{aligned} \quad (25)$$

Where $x = \left(\hat{S}^* \epsilon^{-\theta} \right)^{1-\rho}$. The last part of equation 25 is a gamma function which amounts to $\Gamma(1 - \lambda(\theta(1-\rho))^{-1})$. Therefore, we have

$$\mathbb{E} \left[\epsilon_i^{\frac{1}{1-\eta}} \middle| \text{person choices } i \right] = \left(\frac{1}{p_i} \right)^{\frac{1}{\theta(1-\eta)}} \Gamma \left(1 - \frac{1}{\theta(1-\rho)} \frac{1}{1-\eta} \right) \quad (26)$$

Using this result in the equation 23 completes the proof. \square

Proposition 3. *Let W_{ir} be the gross average earnings in occupation i in region r . Then:*

$$W_{ir} = w_{ir} \mathbb{E}[h(e_{ir}, s_i) \epsilon_i] = \frac{(1-s)^{-1/\beta}}{(1-\tau_{ir}^w)} \gamma \eta \left(\sum_{s=1}^N \tilde{w}_{sr}^\theta \right)^{\frac{1}{\theta(1-\eta)}} \quad (27)$$

1.5 Problem

So, the problem consist in minimize equation 28, using Nelder-Mead¹ algorithm.

$$Dist = \sum_{i=1, r=1}^{N, R} \left(\frac{W_{ir}^M - W_{ir}^T}{W_{ir}^T} \right)^2 + \sum_{i=1, r=1}^{N, R} \left(\frac{p_{ir}^M - p_{ir}^T}{p_{ir}^T} \right)^2 \quad (28)$$

where p_{ir}^M and W_{ir}^M are given by equations 17 and 27 respectively. On the other hand, p_{ir}^T and W_{ir}^T are given by PNAD data. The superscript indicate model and target statistics. We assume that $\tau_{1r}^h = 0$, $\tau_{1r}^w = \tau_1^w$, $\forall r$. And $A_R = 1$, i.e, the TPF of the last region is nomalized to 1.

¹The implementation of model in Python language can be view in my GitHub: https://github.com/mj-ribeiro/Math-Statistics-in-Python/blob/master/Hsieh_model.py. Other algorithms can be used, for example genetic algorithm