Hyperbolic $Out(F_N)$-graphs

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Sections

Hyperbolic metric spaces

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Study $Out(F_N)$ via its action on hyperbolic graphs.

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Let $\delta > 0$. A geodesic metric space X is δ-*hyperbolic* if for any geodesic triangle with vertices a, b, c, the geodesic segment ab is contained in a δ-neighborhood of the union of bc and ac.
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Some well-known examples are: trees ($\mathbb{R}$-trees, if you know what those are) and hyperbolic space.

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- for each hyperbolic isometry $g \in G$, $x \in X$, and $C > 0$ there exists an $N > 0$ such that

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In English: sufficiently long segments of quasi-axes for hyperbolic $g \in G$ are coarsely stabilized by only finitely many elements of G

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Theorem (Sisto [6])

Hyperbolic isometries are generic in G (in some sense involving a random walk on the Cayley graph).

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The free splitting graph

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The *free splitting graph* FS_N has vertex set = one edge graph of groups decompositions of F_N with trivial edge group (up to some equivalence). Two vertices are connected by an edge if there is a two-edge graph of groups decomposition collapsing to both.

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Theorem (Handel-Mosher [4])

FS_N is hyperbolic.

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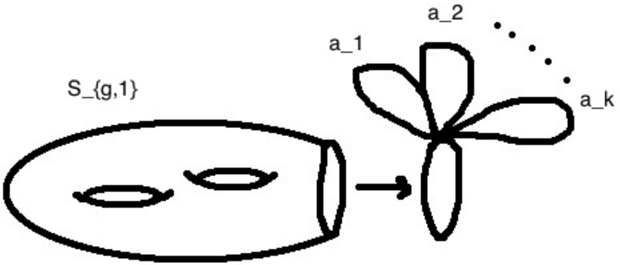

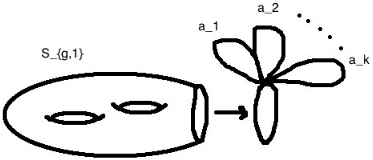

Choose an automorphism ϕ of F_{2g+k} which maps $a_1, \ldots, a_k$ over a_{k+1} in some complicated way and which is the identity on S. If we do this the right way, we ensure that ϕ acts hyperbolically on FS_N. But any surface automorphism of S commutes with ϕ since it is the identity on ∂S, so the centralizer of ϕ is not virtually cyclic!

The Cyclic Splitting Graph

Definition

The *cyclic splitting graph* FZ_N has vertex set $=$ one-edge graph of group decompositions of F_N with cyclic ($\mathbb{Z}$ or trivial) edge groups. Two vertices are connected by an edge if there exists a two-edge decomposition collapsing to both.

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Q: Is it WPD? Note that the automorphisms above where WPD fails for FS_N fix a cyclic splitting.

The free factor graph

Definition

The *free factor graph* FF_N has vertex set = one-edge graph of groups decompositions of F_N with trivial edge group. Two vertices are connected by an edge if there exists a proper free factor which (up to conjugacy) contained in vertex groups of both decompositions.

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Theorem (Bestvina-Feighn [1], [2])

FF_N is hyperbolic and the action of $Out(F_N)$ is WPD.

The Intersection Graph

Definition

The *intersection graph* I_N has vertex set = one-edge graph of groups decompositions of F_N with cyclic edge group. Two vertices are connected by an edge if the corresponding Bass-Serre trees share a common elliptic conjugacy class (which might not be contained in a free factor! $[a, b] \in F_2$).

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Corollary

Fully irreducible automorphisms with no periodic conjugacy classes are generic in $Out(F_N)$.

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