PhD Thesis Defense

Some Hyperbolic $Out(F_N)$-Graphs and Non-Unique Ergodicity of Very Small F_N-trees

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Sections

1. Preliminaries
2. A sketch of the proof of the first main theorem
3. More cyclic splitting complex
4. More results

You can get all these slides on
`http://www.github.com/brianmannmath/thesis_defense`

Goal

Fix $N \geq 3$. Let F_N be the free group of rank N. We are trying to study $Out(F_N) = Aut(F_N)/Inn(F_N)$ by finding well-behaved actions on interesting spaces.

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Motivated by results of Masur and Minksy (the curve complex is hyperbolic [8]), Bestvina and Feighn (the free factor complex is hyperbolic [1]), and Handel and Mosher (the free splitting graph is hyperbolic [4]) as well as many others.

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What's an example of such a space? Consider the simplicial graph FZ_N defined as follows:

- The vertices of FZ_N are one-edge splittings of F_N (up to conjugacy) with $\mathbb{Z}$ or trivial edge group.
- There is an edge between splittings T and S if there exists a two-edge splitting R refining both T and S.

FZ_N is the *cyclic splitting graph*. It has a natural $Out(F_N)$-action.

Example

The splittings of F_5

$$\langle a, b\rangle \xrightarrow[\langle b\rangle]{} \langle b, c, d, e\rangle$$

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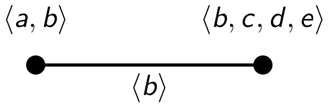

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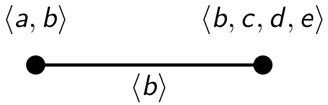
Example

The splittings of F_5



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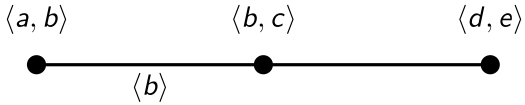
are commonly refined by



First Main Result

Theorem (M [6])

The cyclic splitting graph with the simplicial metric (every edge is length 1) is δ-hyperbolic.

Sections

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The main tool

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Theorem (Kapovich-Rafi [5])

Suppose X and Y are connected graphs, X is hyperbolic, and $f : X \to Y$ is Lipschitz. Suppose there is $S \subseteq V(X)$ such that

1. $f(S) = V(Y)$
2. *S is D-dense in $V(X)$ for some $D \geq 0$.*
3. *There is an $M > 0$ such that if $x, y \in S$ with $d(f(x), f(y)) \leq 1$ then $diam(f[x, y]) \leq M$.*

Then Y is hyperbolic.

How do we use it?

By Handel-Mosher [4] we know that the free splitting graph FS_N is hyperbolic.

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We use a quasi-isometric model of FZ_N and a natural, $Out(F_N)$-equivariant map $FS_N \to FZ_N$ which clearly satisfies the first two conditions of the above theorem.

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Essentially, it boils down to proving the following claim:

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If R and R' are marked roses (free splittings of F_N with 1 vertex and N edges) whose images in FZ_N are close, then the Handel-Mosher folding path from R to R' stays uniformly bounded in FZ_N.

Sections

It really is something new!

Ok, the title is a little bold, but...

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Proposition

FS_N and FZ_N are not $Out(F_N)$-equivariantly quasi-isometric.

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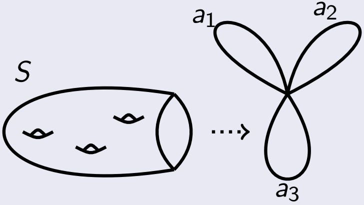
Proof.



(Don't worry, I'll explain on the board.) □

Sections

Define a graph I_N whose vertices are one-edge very small F_N-trees, and where two trees T and T' are connect by and edge if there exists a measured current μ such that $\langle T, \mu \rangle = 0 = \langle T', \mu \rangle$.

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Theorem (M 2014)

I_N is hyperbolic, and furthermore the action of $Out(F_N)$ on I_N satisfies Bestvina and Fujiwara's Weak Proper Discontinuity condition [2].

Nonuniquely ergodic F_N-trees

An arational F_N-tree T in ∂CV_N is *nonuniquely ergodic* if there exist distinct non-homothetic length measures on T.

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Theorem (M-Reynolds [7] 2013)

Given two curves (a certain type of one-edge very small $\mathbb{Z}$-splitting of F_N) with neighborhoods U and U' in ∂CV_N, there is a 1-simplex of nonuniquely ergodic, arational, nongeometric trees with one endpoint in each of U, U'.

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