

Chapter 2

Good enough equations

Budgets in boxes (Chapter 1) got us the mass continuity and momentum equations (1.4 and 1.6), with pressure and gravity as the sources (forces). The molecular part of flux convergence will soon (chapter 3) be treated as a diffusion and moved to the right-hand side as another force. Coriolis forces, although sometimes called "fictitious", are as real as the ground is stationary on our rotating planet. The important distinction among forces is not real vs. fictitious, but non-negotiable vs. *adaptive*.

Pressure is the adaptive force, as Problem 1.4.2 (solution 1.5.2) showed, adjusting itself instantly (in the incompressible fluid), or really at the speed of sound, to negate the divergence of all of the other forces in a fluid whose basic density structure varies little with time. In this chapter we will see that negating the divergence of gravity gives p a *hydrostatic* component that keeps air from collapsing into a puddle, negating divergent Coriolis forces gives p a *geostrophic* component, negating *buoyancy* (defined below) gives a component important for cumulus dynamics, and *dynamic pressure* arises to negate the divergence of the advection of momentum (sometimes called *inertial force*) in moving air. That problem (solution 1.5.2) left us with 4 equations for 5 unknowns (u, v, w, p, ρ), and a teleological sense of what pressure is.

To close our equation set more satisfyingly, we need equations to define density more interestingly than a constant, allowing it to vary horizontally so that gravity can drive convection.

2.1 Good-enough thermodynamics of moist air

2.1.1 Density and the ideal gas law

To close the set, we need an equation predicting the density of air. The ideal gas law springs to mind: $\rho = 1/\alpha = p/(RT)$, but that doesn't solve the problem. For one thing, it introduces a new unknown (RT), so now we will need additional equations. Also it is nonlinear (a quotient), so taking a time derivative to get the needed prognostic equation yields 2 terms. Using Newton's dot notation for the time rate of change in 1 kg of ideal gas trapped in a laboratory chamber, our grasping form predicting ρ is:

$$\dot{\rho} = (\dot{p} - \dot{T}/T)(RT)^{-1} \quad (2.1a)$$

or perhaps more usefully we can grasp at the rate of change of T :

$$R\dot{T} = p\dot{\alpha} + \alpha\dot{p} \quad (2.1b)$$

Here some complications must begin, which we will seek to minimize.

2.1.2 Virtual temperature, density temperature

The *gas constant* R quantifies the partition of energy between the kinetic energy of molecules (proportional to T) and an *elastic energy* implicit within a body of compressed gas $p\alpha$. R is defined on a *per mole* basis, not a *per unit mass* basis. Since H_2O (molecular weight 18 kg/mole) is lighter than Earth's mixture of N_2 and O_2 ("dry air"; average molecular weight 29), with a ratio between these of 0.622 we will see again, R for moist air depends on humidity. That dependence is fairly slight for Earthly humidity values, so meteorology's custom to hide the complication is to use R to denote the truly constant dry-air gas constant R_d , and then elaborate T with the subscript v for *virtual temperature* $T_v =$

$T(1 + 0.602 q_v)^a$, the temperature at which dry air would have the same bulk density as the actual *moist air*^b we are talking about.

2.1.3 First Law: internal energy and the quest for warmth

Air's density depends on *warmth*, a kind of energy measured by T , so an energy budget or conservation law (the *First Law of Thermodynamics*) is surely our route to a T equation for a trapped kilogram of air in an insulated laboratory piston. But that air can still export energy, by doing one kind of *work*: expanding its volume V against the pressure force, as expressed by the term $pdV/dt = p\dot{a}$. The conservation of energy added to a trapped gas with a controllable (perhaps electric) heater must therefore take the conceptual form:

$$\text{heating rate } Q = d/dt(\text{internal energy}) + \text{work air does}$$

Internal energy is defined by the above conservation statement, and is T times the *heat capacity at constant volume* C_v :

$$Q = C_v \dot{T} + p\dot{a} \quad (2.2)$$

Three term equations are uncomfortable: our brainfeel wants an equation with a single-term RHS, something more like:

$$\text{heating} = d/dt(\text{"some stuff called warmth involving } T\text{"})$$

Substituting (2.1) into (2.2) doesn't help -- still there are 2 RHS terms:

$$Q = (R + C_v)\dot{T} - \alpha\dot{p} = C_p\dot{T} - \alpha\dot{p} \quad (2.3)$$

We need a trick. Two approaches suggest themselves.

^a Derivation is in Wikipedia or many textbooks.

^b Inclusion of the mass of suspended particles is sometimes telegraphed by using *density temperature* T_p , but anyone who bothers to worry about T_v obviously wants total density.

Trick 1: Divide by temperature. Why? Because it works:

$$Q/T = C_p \dot{T}/T - \alpha \dot{p}/T = C_p \dot{T}/T - R \dot{p}/p$$

Since the coefficients are constant (C_p and R), they commute through the time derivative operator. From pure calculus, $T^{-1} dT/dt = d/dt(\ln T)$, so we may write a conservation law in our desired form involving $d/dt(\text{stuff})$, but only for confusing stuff called *specific entropy* S :

$$\begin{aligned} Q/T = \dot{S} &= d/dt(C_p \ln T - R \ln p) \\ &= d/dt(C_p \ln \theta) \end{aligned} \quad (2.4)$$

What is this specific entropy S ? Do the units ($\text{J kg}^{-1} \text{K}^{-1}$) satisfy any intuition? No, because "degrees" of temperature are not fundamental (Ch. 1). What does it mean to divide a heating rate (energy input rate) by T in degrees? Deeper understanding is available in statistical mechanics (e.g. Ben-Naim 2013). If the vigor of molecular motions had been expressed in energy units instead of temperature's arbitrary equal ticks along a tube of mercury (degrees), then entropy would be a dimensionless quantity. This dimensionless entropy measures the *missing information* in our bulk thermodynamic description $\{p, T, \rho\}$ of the zillions of molecules in that kg of air. Specifically, it is the *log of the number of redundant microscopic states* that would yield the same bulk thermodynamic state $\{p, T, \rho\}$. That information is measured in bits if the log is base 2 (see chapter 0 and its Problem at the end).

Meteorology has invested heavily in *potential temperatures* like $\theta = T(p_0/p)^{R/C_p}$ in (2.4), expressing *the temperature a parcel of air would have after it undergoes various hypothetical processes* (in this case, adiabatic compression to a reference pressure p_0). Generations of students from the pre-computer age learned to navigate the mild nonlinearities of Earthly moist thermodynamics by following gently curved lines (such as *adiabats* or *isentropic lines* with constant S) on large paper *aerological diagrams*, labeled with the axes of thermodynamics' laboratory observables: pressure and temperature. An entropy version of the diagram (the *Tephigram*, shorthand for T - ϕ since ϕ

was used for entropy) has its pressure axes woozily not-quite parallel to the paper edges. In the USA this lost the textbook-mediated generational popularity contest to pseudoadiabatic or skewed- T^c by log- p diagrams. Despite the end of oversized-paper based science, the ease of evaluating mildly nonlinear functions in the computer age, and with the rise of tropical meteorology with its emphasis on quantitative humidity for moist convection (section 4.1 below), these anachronistic diagrammatic treatments still pervade the literature. Students should learn them (MetEd^d), but this book would gain nothing from covering them.

Trick 2: As an alternative to the entropy approach of Trick 1, define a new labeling function for pressure, the *hypsometric height* $Z_h(p)$ whose increments satisfy the hydrostatic force balance relationship $\alpha dp = g dZ_h$. Why? Because it works. Unlike α in (2.3), g is time independent, so it commutes through the time derivative in (2.3) like in Trick 1, giving our desired form, d/dt ("some kind of warmth stuff") = heating:

$$Q = \dot{s} = d/dt(C_p T + g Z_h) \quad (2.5a)$$

This s is called *dry static energy* (per unit mass). It has the same useful property of potential temperature θ for reasoning: it is conserved during vertical motions of an unsaturated parcel. That makes it uniform with height in any well-mixed layer, which immediately gives the *dry adiabatic lapse rate of temperature* $\Gamma_d = -g/C_p = -10$ K/km (taking $dZ_h/dz = 1$, an excellent approximation). Dividing \dot{s} by C_p to get a warming rate at constant pressure is straightforward, unlike the use of $\dot{\theta}$ which -- despite units of K/s -- must be converted using the nonconstant profile factor (T/θ) to get a true warming rate \dot{T} for density-change purposes, our larger goal.

^c Not statistical skew (chapter 0.2), but merely a graphical skew, a non-orthogonal axis that minimizes *one* of the diagram's failures to match the importance of distinctions to the size of the corresponding feature on the paper.

^d Free course materials at https://www.meted.ucar.edu/training_module.php?id=225

Turning back to that larger quest, we can easily recast (2.5) as a grasping equation for the rate of change of T , ready to append to the set (1.6):

$$\begin{aligned} dT/dt &= Q/C_p - g/C_p(Z_h) \\ &= Q/C_p - g/C_p(Z_{h_t} + \mathbf{V} \cdot \nabla Z_h) \quad (2.5b) \end{aligned}$$

But wait a minute! We indulged in some sleight of hand in Trick 2. For Z_h to depend only on p , so that its increment has only one term $gdZ_h = \alpha dp$, α must also depend only on p . In other words, we tacitly assumed that ρ is approximated by a reference density profile that is a function only of altitude. This approximation is sufficient for all this book's further purposes, and in fact is needed later for other reasons, but we backed into it.

This approximation eliminates all but vertical advection from the advection term in (2.5b). Before we say goodbye to the horizontal component, notice that its form $-\mathbf{V} \cdot \nabla(gZ_h)$ can be reinterpreted in terms of meteorology's usual pressure-coordinate equations as *kinetic energy gained by flow down the PGF* (problem 2.4.1). This interpretation is one glimpse of the fact that *the key simplification of a static energy treatment is neglect of kinetic energy* (Betts 1979)^e. Fortunately, the kinetic energy of winds on Earth is far less than that of molecular pandemonium (thermal energy), so the simplification gained by Trick 2 is well worth its small cost in accuracy. Turning good-enough (2.5) into a prognostic equation for the *thermal buoyancy force* b will soon require a reference density profile anyway (section 2.2), so this is just a head start.

In the shorthand for convective dynamics implied by one-dimensional soundings analysis (section 4.1), a balloon-measured ρ profile is used as the reference to sensibly define $Z_h=Z=z$. The assumptions in that case are numerous and conceptual: they imply an oversimplified vertical momentum equation (buoyancy only) for the ascent of hypothetical pure lifted "parcels" within an infinitely broad "environmental" airmass whose conditions the balloon's tiny barometer, thermometer, and hygrometer are

^e Relatedly, MSE minus CAPE is the true lifted-parcel conserved variable (Romps 2015).

supposed to have measured. Neglect of kinetic energy is far from the most dubious of these idealizations, so (2.5) is more than good enough.

2.1.4 Latent vs. 'diabatic' heating and moist adiabaticity

Equation (2.5a) says that s is *conserved for adiabatic flow* (for which $Q=0$) even when air changes pressure by moving vertically. If we take the *latent heat of vaporization* L as constant^f, the part of Q due to condensation minus evaporation $L(c-e) = L dq_c/dt = -L dq_v/dt$ can be moved to the right-hand side. This almost counts as a third Trick:

$$Q_d = \dot{h} = d/dt(C_p T + gZ + Lq_v) \quad (2.6)$$

This h is called *moist static energy* (per unit mass), interpreted very readily as the sum of a warmth measure called *enthalpy* $C_p T$ plus geopotential energy (equating Z to Z_h , the *hydrostatic* approximation) plus latent heat energy. Equation (2.6) says that h is *conserved for* $Q_d=0$, even when air changes pressure by moving vertically *with or without condensation and evaporation*. Such flow is sometimes called *moist adiabatic*, leaving Q_d to represent the genuinely 'diabatic' heating rates like radiation, heat conduction from the surface, and the small viscous dissipation of wind kinetic energy^g.

The $T(p)$ or $T(Z_h(p))$ profile that results from such saturated but otherwise adiabatic vertical flow is called a *moist adiabat*. One fine distinction is a *reversible adiabat* in which the condensed water is carried along, adding its thermal inertia in C_p and keeping an ascending, cooling parcel slightly less cold; vs. a *pseudoadiabat* in which condensate is instantly precipitated out. Obviously natural convective cloud updrafts must lie somewhere in that range, depending on highly situational microphysics.

^f Its slight increase with T is compensated by the heat capacity of condensate, in the relevant no-free-lunch proof that perpetual-motion heat engines are impossible. Neglecting both (taking L constant and C_p for air only) is a good approximation pair.

^g Also "apparent" heating or moistening by small scale eddy flux convergence, if " d/dt " in (2.6) is defined to include apparent advection by large scales only (chapter 3).

We will later meet a more teleological or functional definition of a moist adiabat: the neutral stability profile toward which real moist convection adjusts the atmosphere. That functional definition is useful for understanding how thunderstorms will respond to climate change for instance (Singh and O'Gorman 2015, Romps et al. 2015).

2.1.5 *Static energy vs. entropy vs. potential temperatures*

Entropy (Trick 1 above) was derived without approximation. It connects to the rest of thermophysics, and can be extended to very different gases on strange planets. Entropy budgets and the Second Law are real, and have their profundities (e.g. Pauluis and Held 2002, Romps 2008) and some surprises^h. But in practice, most scientists really think with the brainfeel of conservation of some form of warmth-energy. Most costly to intuition is the formal nonlinearity of entropy's $\ln()$ function, or the $\exp()$ function that moist or "equivalent" potential temperature formulas bristle with. The clarity of $h = C_p T + gZ + Lq$, with its lucid self-evident meaning, could never be regained by Taylor expansions and discarding higher-order terms to formally linearize the $\ln()$ and $\exp()$ functions.

For soundings analysis (lifted parcel games, section 4.1), a side by side graphical comparison can be easily made between static energy, entropy, and potential temperature diagrams (chapter 4 computer exercises). The resulting almost-identical graphic (aside from axis units) indicates that static energy is clearly good enough, given the other uncertainties around mixing and microphysics and other situational effects in convection. Even for numerical model design, static energy can be used as part of a well-considered *suite* of approximations. At that equation-set level of consideration, having a conserved quantity is the point, not the precise form of that quantity. The popular System for Atmospheric Modeling (SAM, Khairoutdinov and Randall 2003) simulates impressively Earth-like convection and even planetary-scale circulations (Bretherton and Khairoutdinov 2015), using static-energy thermodynamics with one

^h For instance, turbulent dissipation in the wakes of raindrops falling at terminal velocity is the main dissipative source of entropy in the atmosphere (Pauluis et al. 2000).

additional term for the latent heat of freezing ("frozen" MSE). Framing inaccuracies are rarely the biggest problem in modelingⁱ, except perhaps for precision models that must quantitatively assimilate absolutely calibrated observations across very different regimes (polar to tropical).

2.2 Good-enough fluid dynamics

We need a better density equation to close our mass and momentum budgets (1.4) and (1.6). Let's return to that project.

2.2.1 Gravity becomes buoyancy, PGF is univariate

Just as the linear sums of static energy make moist thermodynamics clear, linearized approximations for Earth's gentle convection can also open up powerful intuitive access to fluid dynamics. Begin by expressing density and pressure fluctuations as differences from a *reference* profile, $\rho \equiv \rho_0(z) + \rho'$ and $p \equiv p_0(z) + p'$. We already accepted the need for a reference density profile below (2.5b), but here we must choose it. The word *buoyancy*^j and its sign will have the most meaning if we choose $\rho_0 = \bar{\bar{\rho}}$, with double overbars *the horizontally averaged density across the whole domain* over which we are building an accounting system for the flow: the whole Earth, or a cloud model with cyclic domain, or some conceptual "airmass" (defined by a balloon sounding) within which we are considering convection.

Since pressure's job is to enforce mass continuity in a fluid (like an atmosphere or ocean) where the basic density structure is not very time-dependent, a corresponding $\bar{\bar{p}}$ is the *hydrostatic* reference p_0 profile needed to hold up the mean density profile in the face of the mean gravity force $\bar{\bar{\rho}}g$. Level by level, $\bar{\bar{p}}_z = -\bar{\bar{\rho}}g$, making $\bar{\bar{p}}(z)$ nearly linear

ⁱ A fascinating case is Williamson et al. (2015, doi:[10.1002/2015MS000448](https://doi.org/10.1002/2015MS000448)) exposing a longstanding hidden error in the energy formulation of a popular atmosphere model. The fix incurred a sizable T bias, soon compensated away in tunings and adjustments.

^j An ancient concept: see https://en.wikipedia.org/wiki/Archimedes_principle

for nearly-incompressible liquids in the *Boussinesq^k system*, and roughly exponentially decreasing for air. The necessary constant of integration for defining p springs from the *non-exploding planet approximation*: the entire fluid is not vertically accelerating. Using this hydrostatic pressure $\bar{p}(z)$ as a vertical axis in graphs is a way to relate atmospheric (gaseous, exponential in z) data to the wonderfully clear Boussinesq equations in xyz coordinates, used preferentially in this book.

Adding the hydrostatic relation $0 = \bar{p}_z / \bar{\rho} + g$ to (1.6d) yields:

$$dw/dt = -p'_z / \bar{\rho} - g\rho' / \bar{\rho} = -\pi_z + \hat{b} \quad (2.7)$$

where $\hat{b} = g(\rho' / \bar{\rho})$ is the *buoyancy force per unit mass*, and we have simply redefined the symbol $\pi = p' / \bar{\rho}$, retaining the important spatially varying part of p but setting aside the mean hydrostatic $\bar{p}(z)$.

Expressing the pressure gradient force (PGF) as the gradient of a single scalar π makes it vanish from vorticity equations (Problem 2.4.2). Likewise, horizontal derivatives of single scalar \hat{b} retain the meteorologically important part of the untidy, distracting *baroclinic*^l term $\bar{\nabla}\rho \times \bar{\nabla}p$ in unapproximated fluid dynamics.

Buoyancy \hat{b} can further be approximated as *thermal buoyancy* $b = gT'_v / \bar{T}_v \cong gT' / \bar{T}$. The use of virtual temperature T_v here helps emphasize that density is the true reference state and the essential variable whose perturbation matters, but for notational simplicity the v subscript is often left off, and for many purposes the virtual effect can be neglected. Thermal buoyancy neglects some untidy product-of-primes terms, and also the slight effect of p' on ρ' , as revealed by a simple ideal gas law substitution into $b = g\rho' / \bar{\rho}$. Fortunately, there are no meteorological phenomena in our scope (Earth's very-subsonic convection) for which thermal buoyancy isn't good enough.

^k Named for https://en.wikipedia.org/wiki/Joseph_Valentin_Boussinesq

^l The spinup of synoptic cyclones in meteorology is unhelpfully called "baroclinic", even though it is mostly vortex stretching: circulation increases even on isobaric surfaces.

With (2.7) thus simplified by replacing \hat{b} with b , we can finally bring a dT/dt equation extracted from the First Law (2.5b) or (2.6) into the b equation, to close a 5-equation set capable of expressing thermal convection. Applying the *total derivative* d/dt to the definition of b , with virtual effects hidden to clarity the z subscript for differentiation, gives

$$\begin{aligned} db/dt &= (g/\bar{T}) d/dt[T - \bar{T}(z)] \\ &= (g/\bar{T}) [dT/dt - w\bar{T}_z] \end{aligned} \quad (2.8)$$

Substituting the First Law (2.5b) into the first term, and again equating $Z_h(p)=Z=z$ (the hydrostatic approximation, plus a sensible choice of a zero origin for both geopotential height Z and the z coordinate)^m, gives our desired prognostic (grasping) equation for b :

$$\begin{aligned} db/dt &= (g/\bar{T}) [Q/C_p + w(\Gamma_d - \bar{T}_z)] \\ &= g/(C_p\bar{T}) [Q - w\bar{s}_z] \\ &= Q_b - wN^2 \end{aligned} \quad (2.9)$$

The first term reflects the *production or source of buoyancy by thermal energy addition rate* Q . The second term combines the vertical advection of T plus adiabatic warming/cooling by compression when air changes pressure. In the second line, that combination is expressed as the effect of the area-averaged *stratification* or *static stability* \bar{s}_z . Since that quantity is positive definite for a stably stratified fluid, it is expressive to package it as a squared frequency of free vertical parcel oscillations, $N^2 = g\bar{s}_z C_p^{-1} \bar{T}^{-1} = g\bar{\theta}_z (T/\theta) \bar{\theta}^{-1}$ (units s^{-2} ; the second equality is Problem 2.4.4). Virtual effects can be re-inserted for precision by simply inserting T_v and s_v for T and s or θ in all the forms above, so we are glad not to have carried them in the algebra.

Gathering the above, we have achieved equation set closure, except for the heating rate Q . By substituting $\rho = \bar{\rho}(z \text{ only}) + \rho'$ and linearizing to retain only the *thermal* buoyancy b and neglect ρ' otherwise, and using

^m Deriving (2.8) and (2.9) is a useful student exercise, with a few illuminating steps.

good-enough static energy (Trick 2, Eq. 2.5) for the First Law, our equation set (1.6) has become five equations in 5 unknowns (u, v, w, π, b):

$$0 = -\nabla \cdot (\bar{\rho} \mathbf{V}) \quad (2.10a)$$

$$du/dt = -\pi_x + fv \quad (2.10b)$$

$$dv/dt = -\pi_y - fu \quad (2.10c)$$

$$dw/dt = -\pi_z + b \quad (2.10d)$$

$$db/dt = -wN^2 + Q_b \quad (2.10e)$$

This set is called *anelastic* when $\bar{\rho}$ is a function of z only, and *Boussinesq* when $\bar{\rho}$ is a constant and can be erased from (2.10a) entirely.

The power of a closed set of PDEs has to be elicited to be fully appreciated: These equations for local relationships "*govern*" (or can describe) all possible flows or "*solutions*", so they are not very discriminating. Let's march through just one simplest solution (for *internal or buoyancy waves*, sometimes called *gravity waves*). Other derived properties of this equation set are in Problems below.

2.2.2 Ubiquitous simplest motions: buoyancy waves

How many complications can we remove from the set (2.10), making it tractable to "solve" for non-trivial flows that vary in time? This question defined the art of fluid dynamics before computers. Waves are our cleanest time-dependent flow paradigm, with well-developed applied math strategies like complex number methods. The stripped-down simplicity of such a problem doesn't make the results irrelevant: quite the contrary! It means that such waves are *ubiquitous, almost no matter what happens with all the various complications we ignore and neglect and approximate away*.

(1) Let us neglect advection by replacing d/dt with $\partial/\partial t$. Physically, this assumes the fluid is at rest other than the wave motions, which remain small enough not to advect their own scalars importantly (another *linearization*). For simplicity, (2) neglect north-south flow and gradients, working only in the x - z plane. (3) make the Boussinesq approximation

($\bar{\rho}$ constant). That is more valid for liquids, but Boussinesq results can be made relevant to the atmosphere by simply plotting atmospheric data on graphs with hydrostatic pressure as the vertical coordinateⁿ. For clarity (4) neglect the Coriolis force, which can be added in later fairly easily. Finally (5) take N^2 to be a constant; it is about $2\pi/(10 \text{ minutes})$ in the tropical troposphere. With these assumptions, we have the simplest possible set:

$$0 = u_x + w_z \quad (2.11a)$$

$$u_t = -\pi_x \quad (2.11b)$$

$$w_t = -\pi_z + b \quad (2.11c)$$

$$b_t = -N^2 w \quad (2.11d)$$

To analyze these equations, we combine them into a single higher-order equation for w . Pressure π can be eliminated with a vorticity approach (see problem 2.4.2) by differentiating and subtracting u and w as $(2.11b)_z - (2.11c)_x$ to prognose $\eta = (u_z - w_x)$:

$$\eta_t = -b_x \quad (2.12)$$

That leaves b_x as the key right hand side term: *buoyancy gradients are a torque in the vertical plane*, valid even if the reference state for b is offset from local conditions^o. Equation (2.12) also shows that b and w are 90° out of phase in these propagating waves, so that the *vertical flux of b is zero in internal buoyancy waves*. In other words, they are specifically *not* a form of convection: on a b - w scatterplot, waves trace a circle, contributing nothing to the 'buoyancy flux'^p (covariance $[b'w']$) which Problem 2.4.2 shows to be the source of all kinetic energy.

ⁿ One phenomenon the Boussinesq equations cannot represent is the breaking of upward-propagating buoyancy waves as density lapses, important in the middle atmosphere.

^o *Effective buoyancy* (Jevanjee and Romps 2015) brings this virtue to the w equation.

^p a misleading use of the word "flux": b is not a conserved quantity being transported.

Differentiating (2.12) again in time to eliminate b with (2.11d) leaves only u and w in the equation. More differentiating allows substitution of (2.11a) to eliminate u , giving a single equation for w :

$$w_{xxtt} + w_{zztt} = -N^2 w_{xx} \quad (2.13)$$

This is clearly a wave equation, because its equality requires that double derivatives act like a negative sign (N^2 being positive), characteristic of sine and cosine functions or their elegant combination in the complex exponential wave form $w = We^{i(kx+mz+\omega t)}$. Here k is horizontal *wavenumber* ($2 \times 3.14159/\text{wavelength}$), m is vertical wavenumber, and ω is the wave's frequency ($2 \times 3.14159/\text{period}$). The wave form satisfies (2.13) *if and only if* $\{k, m, \omega\}$ are related in a special way called the *dispersion relation*:

$$\omega = N [k^2/(k^2 + m^2)]^{1/2} \quad (2.14)$$

How can this wave theory result be reasoned with?

Waves are everywhere in the atmosphere and ocean, often satisfying the dispersion relation as *free solutions*, far from their forcing and damping processes. The true character of observed wave fields must then be explained by the driving and decay mechanisms neglected in (2.11) -- thermal forcing Q_b , or boundary conditions like flow over mountains, or mechanical forces) -- as well as by reflection and filtering and screening effects, such as non-constant N^2 profile features and wind shear also neglected in (2.11). Expressing these phenomena tractably with equation sets like (2.10) is a mathematical-physics field unto itself (e.g. Sutherland 2010), but a few more words here can help launch your efforts in Problem 2.4.5 and discussions about convection-environment interactions in Parts II and III.

Buoyancy waves exist only between the high frequency limit N (vertical parcel oscillations, problem 2.4.4) and the low frequency limit of purely horizontal motions. The latter limit is set by the pendulum-day period of horizontal *inertial oscillations* if the Coriolis force is

reintroduced to (2.11). Within this frequency range, anything goes -- so what do we expect to observe? That depends not on what is possible in the majesty of the governing equations, but on *what waves have drivers in our world*⁹.

The irreversible net buoyancy source implied by deep convection's latent heat bombs create motions near the low-frequency limit. In that case, vertical lengths are set by the depth of the heating, interactively as wave dynamics shape environmental b and thereby the local or effective buoyancy force felt by convective updrafts (chapter 6). That leaves horizontal wavenumbers to be determined by the dispersion relation (see Mapes 1993, 1997 for examples and implications). Temporal fluctuations of convective heating also drive oscillatory waves that propagate up out of the troposphere (e.g. Fritts and Alexander 2003). Beyond these convection-related examples are the wind-over-mountain oscillatory forcings at the heart of classic gravity wave work (Web-search imagery on *lenticularis* or *lee waves*). Mixed cases include convective clouds acting like moving mountains with possibly sheared wind blowing across the tops that can sometimes add an asymmetric aspect to cobweb-like cap or *pileus* clouds sometimes seen around cumulonimbus clouds, or the gorgeous newly named wavy yet somewhat turbulent cloud type *asperatus* (web-search for time lapse videos).

Buoyancy waves are everywhere, even where there aren't cloud droplets to mark them visibly, as revealed by stratigraphy in LIDAR observations (chapter 8, Parsons et al. 2019). Again, the simplicity of the derivation above means that internal buoyancy are specifically NOT an extraordinary or exotic phenomenological claim requiring extraordinary evidence, a historical blind spot (literally) in meteorology, where such clear-air (invisible) and transient motions have often been underappreciated, ignored, or even consciously disbelieved.

⁹ filtered by what we are *able* to observe (section 0.6, Fritts and Alexander 2003).

2.3 Good-enough moisture and microphysics (barely)

The First Law was the core of the prognostic b equation (2.10e) that closed the equation set, this chapter's main goal. To treat latent heating effects or virtual effects in Q_b , we need to carry a sixth equation for q_v . Condensation of that vapor can be folded into a *moist adiabatic* view of thermodynamics, as in section 2.1.4, by specifying the saturation condition for water (when *relative humidity RH* exceeds 100%). This "moist-adiabatic" approach is clearer than leaving condensation in with truly 'diabatic' parts of Q_b , because phase change largely cancels $-wN^2$ in (2.10e) in airmasses (like much of the tropics) whose lapse rate $T(z)$ and thus N^2 is maintained near a state of neutrality^r by cloudy convection.

The *saturation vapor pressure* $e_s(T)$ for water is called the *Clausius-Clapeyron relation*, a name easily searched. Its theoretical form, involving an exponentiated negated squared inverse of T , offers little intuitive insight, while real applications use calibrated formulas fit to laboratory steam tables of not-quite-ideal H_2O gas. The important concept for reasoning is merely that $e_s(T)$ is *monotonically increasing, and concave upward* (increasing at an increasing rate)^s. This smooth curve, and the $\ln()$ in entropy that makes tephigrams appear a bit woozy, are the gentle nonlinearities that forced practical meteorological thermodynamics to be taught graphically in the pre-computer era^t.

Thermal buoyancy in convective clouds hinges on $\Delta T = T_{parcel} - T_{environment}$. Moist adiabatic ascent conserves h , so (2.6) = 0. For saturated air, $q_v = q_s(T, p) = 0.622 e_s(T)/p$. Then (2.6) becomes a transitive equation determining $T(p)$, but not as an analytic formula with any helpful clarity. Nevertheless, the *sign* of ΔT and thus of b can be assessed by comparing h of a saturated updraft to h_s of the environment.

^r This is a *teleological* understanding of convection at the largest scales.

^s Anyone pondering climate should also know its slope at Earthly T : about 7% per K.

^t On oversized diagrams, to compensate for the graphic-design weakness of a poor correspondence between distances on the paper and the importance of differences.

Specifically, if at a given pressure $h_{parcel} = C_p T_{parcel} + gZ_h(p) + Lq_s(T_{parcel}, p)$ exceeds $h_s = C_p T_{env} + gZ_h(p) + Lq_s(T_{env}, p)$, it follows immediately that $T_{parcel} > T_{env}$, because $e_s(T)$ is an increasing function. A lifted-parcel buoyancy assessment diagram based on h and h_s is covered in Chapter 4.

Beyond its sign, the actual *value of ΔT* and thus of thermal buoyancy is complicated, not *linearly proportional* to $h(T_{parcel}, p, RH = 100\%) - h_s(T_{env}, p)$ because the upward-curved shape of $e_s(T)$ makes the proportionality depend on T , and $q_s(T, p)$ depends a little bit on p too. The Clausius-Clapeyron equation for $e_s(T)$, easily studied in Wikipedia, allows a mathematically closed approximate equation set for moist convection to be formulated (Vallis et al. 2019), but reproducing it here would not especially illuminate our way forward. Since computing a quantitative value of b for a convective cloud updraft involves understanding mixing and accounting for condensed water, writing down such equations here adds little to graphical methods (chapter 4.1).

The time has come for a terribly brief mention of the microphysics of condensed water particles, and even less about the nuclei (*aerosol* particles) they form on. Whole textbooks and careers address the subject, which is very important for electromagnetic observations at both optical and radar wavelengths (section 0.6), including aspects of visual beauty, and has some impacts on the details of how cloudy convective processes play out. The albedo of the Earth (discussed around Figure 0.2) depends sufficiently much on microphysics that our delicate planetary heat imbalance hinges on the details (as well as on myriad other processes)^u. Because humans control significant sources of aerosol, that area has drawn an almost unreviewably large amount of minutely detailed study.

For present purposes (an appreciation of convection), a short list of principles may get us perhaps surprisingly far in terms of reasoning power, without delving into all that.

^u a condition called overdetermination, <https://en.wikipedia.org/wiki/Overdetermination>

- Condensation of liquid droplets when humid air is lifted past its saturation level occurs very efficiently, far below $RH = 101\%$ and within fractions of a second. Nucleation determines only how many, and thus how large, are the resulting droplets.
- Evaporation of liquid cloud droplets (droplets small enough to have negligible fall velocity) is comparably efficient, again keeping RH very close to 100% upon air descent, with time delays very short compared to the life cycles of convection's entities (subject of Part II).
- A bulk process called *autoconversion* expresses the conversion of small cloud droplets to larger precipitation drops, defined as drops with a fall velocity through air that is important for subsequent evolution ($>100\ \mu\text{m}$, roughly). This time scale is minutes, significantly long compared to meteorological convective air motion processes. It takes longer in cloud condensation nuclei (CCN) rich (polluted) airmasses, but it is difficult to be more specific because...
- Falling drops *collect* cloud droplets in a rapid positive feedback that transforms the condensed water field dramatically, leading to runaway precipitation development.
- Evaporation of precipitation-sized drops is slow enough that RH is often far below 100% in descending rain shafts, whose air descent rate is itself driven by the evaporative cooling of those drops, which requires RH substantially below 100% . Such *unsaturated downdrafts* are important to boundary-layer air properties under convection, and to convection-convection interactions (Part III), but it is difficult to be more specific because the details depend on rain shaft geometry, the distribution of drop sizes, and ambient T and RH .
- Freezing of liquid water releases another kind of *latent heating*, about $1/6$ the strength of condensation of the same mass of water.
- Latent heat of freezing is significant to thermal buoyancy b , and can be delayed for a meteorologically significant time (minutes or even hours) by lack of ice nuclei. Only at -40°C does water reliably freeze without nuclei. But it is difficult to be more specific, because the availability of ice-nucleating particles is very situational, and ...
- Ice particles nucleate additional ice formation very efficiently, in a runaway feedback (like precipitation development above) called *ice*

multiplication, by various mechanisms. It is difficult to be more specific because these are very situational.

- Air that is saturated with respect to liquid (RH = 100%) is supersaturated with respect to ice.
 - For this reason, a rare ice particle can rob mass from more numerous neighboring supercooled cloud droplets by vapor diffusion. Initial formation of a few ice particles is thus a shortcut to having fewer and larger condensate particles, initiating runaway precipitation processes (collection, *riming*) and changing cloud optics importantly.
- Ice particles grow to precipitation size by vapor deposition (becoming snowflakes) and by particle collisions (becoming aggregate snowflakes or liquid-rimed particles like graupel or hail). It is difficult to be more specific because ice particle shape (*habit*) is involved, along with many other situational factors.
- Precipitation-sized particles of ice, like liquid, evaporate so slowly that unsaturated downdraft dynamics are complex and situational.
- Melting of precipitating begins just above 0 C, without the nucleation complications of freezing, but for large particles it can take a meteorologically significant time or fall distance (as surface hail in summer makes obvious, for instance).

This concludes the chapter. We walk away an adequate equation set:

$$0 = -\nabla \cdot (\bar{\rho} \mathbf{V}) \quad (2.15a)$$

$$du/dt = -\pi_x + fv \quad (2.15b)$$

$$dv/dt = -\pi_y - fu \quad (2.15c)$$

$$dw/dt = -\pi_z + b \quad (2.15d)$$

$$db/dt = -wN^2 + Q_{b_latent}(q_v) + Q_{b_diabatic} \quad (2.15e)$$

$$dq_v/dt = e - c \quad (2.15f)$$

The representation of water phase changes as $Q_{b_latent}(q_v)$ and condensation minus evaporation (c-e) is still merely symbolic, but a sufficient basis was presented in moist static energy reasoning to understand the key graphical rising-parcel b illustrations in chapter 4.1. Microphysics principles are too situational to bother crystallizing into

equations here, but are also hopefully "good enough" in your mind to proceed with sensible reasoning.

2.4 Properties of an equation set: problems and solutions

For this problem, the Boussinesq set for a stratified rotating fluid (2.10) may be written with a horizontal "frictional" mechanical force $\mathbf{F} = iF_x + jF_y$ for completeness, as:

$$\begin{aligned} 0 &= u_x + v_y + w_z \\ u_t &= -uu_x - vu_y - wu_z - \pi_x + fv + F_x \\ v_t &= -uv_x - vv_y - wv_z - \pi_y - fu + F_y \\ w_t &= -uw_x - vw_y - ww_z - \pi_z + b \\ b_t &= -ub_x - vb_y - wb_z - N^2w + Q_b \end{aligned} \quad (2.15)$$

As seen in Problem-Solution 1.4.3-1.5.3, continuity (the Law) can be expressed as giving us the pressure (the Enforcer), with $\beta = f_v$

$$\pi = \nabla^{-2}[-\nabla \cdot [(\mathbf{V} \cdot \nabla)\mathbf{V}] + (f\zeta - u\beta) + \text{div}(\mathbf{F}) + b_z]$$

2.4.1 Energetics of a horizontally unbounded atmosphere

a. Multiply the 4 prognostic equations above by u, v, w, b respectively to generate prognostic equations for $KE = (uu + vv + ww)/2$ and $PE = bb/(2N^2)$.

Interpret these: where and how is KE generated? How is it transported away from there to other regions of the fluid?

b. Integrate the energy equations over a whole atmosphere, after transforming from advective to flux so you can express the fact that there are no lateral boundary fluxes (valid for a *cyclic* or *unbounded* air body like a spherical planetary atmosphere, even though we are using Cartesian coordinates for convenience). Also use the fact that $w=0$ at the

top and bottom. Show your work and explain the logic. With brackets $[]$ denoting the integral, show that

$$\begin{aligned}[KE]_t &= [wb] - \mathbf{F} \cdot \mathbf{V} \\ [PE]_t &= [Jb] - [wb] \\ [PE + KE]_t &= [Jb] - \mathbf{F} \cdot \mathbf{V}\end{aligned}$$

where $J = Q_b/N^2$

Interpret the terms: what drives fluid motion, what damps it? You may need to postulate a simple assumption about \mathbf{F} (whose symbol should evoke *friction*). What kind of a steady state achieved in the motions of this atmosphere, deep into its history and over long time scales when the time rate of change of global averages is small?

How do fluid shear instabilities within the fluid fit into this framework?

2.4.2 Vorticity equation for nondivergent motions

Since the curl of a gradient vanishes in 3D vector calculus, expressing the PGF as the gradient of a single scalar like above allows it to be eliminated cleanly to form a vorticity equation. Derive at whatever mathematical detail you are assigned, but with the line of reasoning made clear, one of the *horizontal* components of vorticity like in (2.12). The subscript notation for derivatives is a real hand saver for the proliferating advection terms (since the derivative of three products creates 6 terms).

Emphasize especially: What is the role of the *horizontal gradient of buoyancy*?

2.4.3 Stokes' theorem and vorticity patch reasoning

In the xy plane, write Stokes' theorem, also known as the Circulation Theorem (pure math; look it up), for *every* circle of radius R centered on an isolated small circular patch of vertical vorticity $\zeta = \zeta_0$ of radius r ,

with $\zeta=0$ elsewhere. In this way, show that the tangential component of flow associated with (“induced by”) every isolated element of vorticity decays with distance proportional to R^{-1} . With this rule, the flow from arbitrarily complicated arrangements of vorticity can be constructed by superposition of many localized vortices. (Try a web search on "contour dynamics" to see the power of this result, and share what you find.)

Interpret the result: Does it violate causality somehow that changing vorticity at one point or r -sized local patch instantly “induces” changes in the flow out to any arbitrary larger distance R ? Think carefully about what was specified in this problem.

2.4.4 Stratification

a. Confirm the derivations of equations (2.8) and (2.9).

b. Show that $N^2 = g \bar{s}_z C_p^{-1} \bar{T}^{-1} = g \bar{\theta}_z (T/\theta) \bar{\theta}^{-1}$ as stated below (2.9).

c. Show using the w_t equation with b as the only force that N is the frequency of sinusoidal *buoyancy oscillation* of a parcel displaced vertically in a stratified fluid. You may be helped by examining Wikipedia or another source on the [Brunt–Väisälä frequency](#).

2.4.5 Buoyancy waves

a. Derive equations (2.11) - (2.13), following the text's meaning. Show that the simplified longwave limit ($k \ll m$) is the same answer you get from making the hydrostatic assumption at the outset by setting the left hand side of (2.11c) to zero.

b. Discuss applications (situations) in which the forcing mechanism of waves determines every combination of TWO of the set $\{k, m, \omega\}$, leaving the third to be determined by the dispersion relation as a condition for the existence of buoyancy-wave motions. Find and share online videos that illustrate the forcing and dispersion of these types of internal waves.