Part I: Essentials of the fundamentals

Chapter 1

Keeping track of stuff in space

1.1 Units for space, time, and stuff

Before understanding must come labeling and accounting. Mass and momentum and energy ("stuff") are conserved by isolated entities or systems, but we care about different places in the world so we must account for transport. We also care about different categories of stuff (air vs. water mass, heat vs. latent chemical energy, motions of different scales) so we must account for transfers between categories. In our equations, these accounting terms will often outnumber the few key terms expressing fundamental laws of physics. We must understand both.

Space, time, and mass are measured here in *Systeme Internationale* (SI) units based originally on our ten fingers, Earth, and Water.[a] Space is in *meters* (m), devised as the Earth's equator to pole distance divided by 10^7 to be human scale. A *kilogram* is the mass of a cubic meter of water (1 *metric ton*), divided by 10^3 to again fit human bodies and commerce[b]. Time is the contentious domain. The Earth gives us *days*, which could be divided by 10 or 100 for finger-counting convenience. Clocks with decimal face labels were manufactured in 18th century France, but never caught on.

---

[a] Replacing royalty and tradition, in the French Revolution's radical rationality project.

[b] That volume of water is also called the *liter*.

After all, the Six-related numbers of tradition are also rooted in Earthly numerology (almost-12 months and almost-360 days in a year). In the end SI retained the *second*: a day divided by 86400 (60 x 60 x 24). Latitude in degrees (10^7m / 90º = 111.111 km per degree) and subdivisions like nautical miles (1/60 degree) also carry this Six-related history.

Temperature's Celsius scale is also about water and tens: 1 ºC = 1 K = (water's boiling point minus freezing point)/100. This is not quite as fundamental, since the boiling point depends on atmospheric pressure. It is a remarkable coincidence that the weight of a 1 m^2 column Earth's atmosphere (1 Bar of pressure) happens to be so near the weight of a 10 m column of water, making surface pressure nearly 10^5 (the global mean is 101325) in the honorifically named compound SI unit of Pascals (1 Pa = force/area = 1 N m^{-2} = 1 kg m s^{-2} m^{-2} = 1 kg m^{-1} s^{-2}).

Precision science has quietly replaced these original Earth and Water motivations with more fundamental quantum profundities as the root of official SI, without most people even noticing (the kilogram was redefined in 2019). But our 10 fingers still rule numbering in all but the time domain.

1.2 Conservation of the most fundamental “stuff”: mass

In a given volume of space (let’s say 1 m^3 for definiteness), the enclosed mass is customarily labeled ρ, a *mass density* whose inverse is called *specific volume* α=1/ρ. The rate of change of mass in a cubic meter of space is pure spatial accounting: it equals the net inflow of mass into the volume. Physical sources and sinks are zero except for tiny Einsteinian $E=mc^2$ effects. The flow of mass through a two-dimensional (2D) area, like the square face of a 3D cubic volume, is measured by a *flux*. The units of mass flux [kg m^{-2} s^{-1}] embody its meaning better than any further words can elaborate.

Mass flux is $\rho\boldsymbol{V}$ in symbols[c], using bold face for vectors like velocity $\boldsymbol{V} = \boldsymbol{i}u + \boldsymbol{j}v + \boldsymbol{k}w$, with the unit vectors ***i, j, k*** and velocity components *u, v, w* in a Cartesian (*x, y, z*) coordinate system. Readers should verify that the units of $\rho\boldsymbol{V}$ are indeed a flux [(kg m^{-3}) (m s^{-1}) = kg m^{-2} s^{-1}]. Notice with that same units-mind that velocity $\boldsymbol{V}$ is a *volume flux;* and is also a *specific momentum* (where "specific" means *per unit mass*). A feeling for different interpretations of the same quantity is essential to fully appreciating the equations of convection: you need a firm physical grip on the mathematical symbols and their units, but not too tight or exclusive.

A *net inflow* to a volume is called *convergence of flux*, the negative of *divergence*[d]. Translating the sense **change = convergence** into math, with subscripts denoting partial derivatives along the axes in x, y, z, t space,

$$\begin{aligned}\rho_t &= conv(\rho\vec{\mathbf{V}}) = -div(\rho\vec{\mathbf{V}}) = -\vec{\boldsymbol{\nabla}}\cdot(\rho\vec{\mathbf{V}}) \\ &= -(\rho u)_x - (\rho v)_y - (\rho w)_z + mass\ sources \\ &= -(\rho u)_x - (\rho v)_y - (\rho w)_z + 0 \qquad (1.1)\end{aligned}$$

1.2.1 *Aside on mathematical expression culture*

Notice that Equation (1.1) has an ambitious structure:

$$something\ I\ want\ =\ work\ I\ must\ do\ to\ get\ it$$

In the case of a *prognostic equation* like (1.1), what we want is to know the future. Let's call this a *grasping form*, in contrast to a *majestic equality* of the form 0 = (all the terms). Mathematically, these are isomorphic: not different in any meaningful way, just in form. But in seeking this book's top virtue of *appreciation*, special brain hardware can be engaged through the *brainfeel*[e] of equations. To feel a difference, transform (1.1) into the equivalent majestic equality:

---

[c] See Table of Symbols and Notation.

[d] Early meteorology texts simply used the word *vergence*, letting the sign reflect its sense.

[e] This term is motivated by "mouthfeel," a food descriptor distinct from taste or nutrition.

$$0 = (\rho u)_x + (\rho v)_y + (\rho w)_z + (\rho \dot{t})_t \quad (1.2)$$

In the first 3 terms on the RHS, velocity components in space (u = $\dot{x}$, v = $\dot{y}$, w = $\dot{z}$, using Newton's dot notation for univariate time derivatives) measure the futile-looking (zero-equated) spatial journey of moving matter, in units of meters traversed per second of elapsed time. Here an analogous quantity $\dot{t}$ measures the journey of matter through time, in seconds traversed or endured per second of elapsed time. Since both are seconds, $\dot{t}$ is unitless (equal to 1, neglecting relativistic effects). But was no appreciation gained by crystallizing that invisible $\dot{t}$ =1 in (1.2) compared to (1.1)? Your brainfeel may vary, but this book (and many others) may better satisfy readers attuned to appreciate the difference.

1.3 Conservation of specific (per unit mass) other stuff:

1.3.1 *Specific momentum and its physical source terms*

Having established the flux convergence form for mass (1.1), other budgets follow straightforwardly by expressing conserved quantities on a *specific* or *per unit mass* basis. To get the budget equation for momentum, simply multiply the mass flux by specific momentum, which (as noticed above) is velocity. In the **k** direction, along which air's position change $\dot{z} = w$ (vertical velocity),

$$\begin{aligned}(w\rho)_t &= -\vec{\boldsymbol{\nabla}} \cdot \left(w\rho\vec{\mathbf{V}}\right) \\ &= -(w\rho u)_x - (w\rho v)_y - (w\rho w)_z + w\ sources \quad (1.3)\end{aligned}$$

The difference from (1.1) is nonzero *sources* on the right. Newton taught us that momentum sources are called *forces,* in the famous equation **F** = m**a** that earned him the honorific SI unit for force (N = kg m s^{-2}).[f] Acceleration could therefore be called *specific force,* although we don't really think of acceleration as "stuff" like we do for momentum.

---

[f] Is force a flux of some definable "stuff"? Not really: 1 N = 1 (kg m^3 s) m^{-2} s^{-1}, so the unit of "stuff" being transported (kg m^3 s) has no intuitive sense to this author's brainfeel.

Two real forces that act on air are needed to appreciate convection:
(1) *Gravity*[g] $-\boldsymbol{k}\rho g$
(2) The *pressure gradient force* $-\vec{\boldsymbol{\nabla}}p$

What is this "pressure" p? Physics class taught me to reply with *force per unit area,* giving units $N/m^2 = Pa$ (Pascals). However, its units also identify it as a *momentum flux* [with units of (stuff) $m^{-2}s^{-1}$ = (momentum) $m^{-2}s^{-1}$ = ($kg\ m/s$) $m^{-2}s^{-1}$]. But it is a strange flux: directionless, or imparting momentum in all directions at once. Since p is a scalar field, not a vector field, the differential operator measuring *net* flux into a cube of space (the flux difference from one side to the other) is not the convergence of a vector field as above, but rather the *gradient* of the scalar field p. Notice that the calculus concept of a "gradient" is only well defined over a spatial length scale where the continuum approximation is valid (smoothing over the lumpy molecular nature of matter).

In a steadily rotating coordinate system, where “motionless” air is nonetheless accelerating, we must add to (1.3) a third “fictitious" force per unit mass $-2\vec{\boldsymbol{\Omega}} \times \vec{\mathbf{V}}$ where $\vec{\boldsymbol{\Omega}}$ is the coordinate rotation vector, as derived in every dynamical meteorology book (or web-search *Coriolis force Wikipedia)*. For simplicity we will neglect the vertical component of the Coriolis force (which is tiny compared to gravity), and retain only the horizontal Coriolis force based on the *Coriolis parameter* $f = 2|\vec{\boldsymbol{\Omega}}|\ sin(latitude)$ in our Earth-tangent Cartesian (x,y,z,t) coordinates.

Gathering the considerations above for every cubic meter of space, we are up to 4 equations including 5 unknowns (u,v,w,p,ρ):

$$(\ \rho)_t = -(\ \rho u)_x - (\ \rho v)_y - (\ \rho w)_z \quad + 0 \qquad (1.4a)$$
$$(\rho u)_t = -(u\rho u)_x - (u\rho v)_y - (u\rho w)_z \quad -p_x + \rho f v \qquad (1.4b)$$
$$(\rho v)_t = -(v\rho u)_x - (v\rho v)_y - (v\rho w)_z \quad -p_y - \rho f u \qquad (1.4c)$$
$$(\rho w)_t = -(w\rho u)_x - (w\rho v)_y - (w\rho w)_z \quad -p_z - \rho g \qquad (1.4d)$$

[g] Strictly speaking, "gravity" in meteorology is Newton's *gravitational force* plus a small force due to the Earth's rotation which bulges the equator out a little relative to a sphere.

Notice that each p in (1.4 b-d) could be moved inside the parentheses of the flux terms. But pressure's momentum flux is transmitted across the borders of a spatial box by elastic collisions of the molecules outside the border with those inside it. This is new physics, *different in character* from pieces of matter carrying their properties bodily as they cross the spatial border. Border crossing by the smallest pieces of matter (molecules) is part of the flux terms like $u\rho w$ in (1.3). That molecule-borne flux acts as a *diffusion* process: *the convergence of a flux of stuff that is proportional to the gradient of that stuff* (revisited in section 3.1). That process is called the *viscosity force* when it is pulled out of the flux transport terms and moved to the RHS as a "source" term for equations like (1.3)[h].

Without density variations, mighty gravity is indiscriminate in (1.4d), and there can be no *convection* in the sense of our title. We simply must allow ρ to vary a little so that the gravity force $-\boldsymbol{k}\rho g$ can do *work*, which drags us into thermodynamics (Chapter 2). Specifying ρ is called an *equation of state.* Before we tackle that complication, we can already gain a key insight into the nature of pressure from the simplest case, ρ = constant (Problem 1.1). Readers should understand the resulting lesson well: *Continuity is the Law, Pressure is the Enforcer, and **F**=m**a** is its Mechanism*.

1.3.2 *Other specific stuff: humidity and 'heat content'*

To close our four-equation but five-variable set (1.4) we need an equation for ρ. Although recognizing *specific momentum* as another name for velocity made momentum budget construction very direct (1.4 b-d), no similar trick of invoking 'specific density' can help us because density per unit mass is not physically incisive[i]. We need another tactic, another physical law. The ideal gas law comes to hand, but that brings in temperature T. Since T is indicative of warmth or heat, we need a quantity

---

[h] The *Navier-Stokes equations* for fluid dynamics have viscosity pulled out in this way.

[i] That would be the inverse of *specific volume* $\alpha = 1/\rho$, which is just density ρ itself!

that measures the "stuff" that heat is made of: an energy perhaps, although that will lead us on to more abstract stuff called entropy.

For cloudy convection, we also need to keep track of the mass of water, which we can measure as a *specific humidity q*, so you know what to do: replace *w* with *q* in (1.3) and replace the physics of the *w sources* term with the physics of *q sources*. It's a trivial extension to the notation to budgets of categorized water (specific cloud water, specific rain water, specific ice water, etc.). However, one subtle bookkeeping issue in this exercise needs to be noticed.

1.3.3 *Specific X, or mass mixing ratio of X?*

There is a subtle difference between *specific water vapor mass* q_v and *water vapor mass mixing ratio* r_v. Both have water vapor mass in the numerator, but mixing ratio has the mass of *dry air* rather than *total mass* in the denominator[j]. To get more precise, one must decide if ρ in (1.1) stands for dry air density or for total mass density. It seems tempting to use total mass, but then the velocity w is hard to define: in rain, it would have to be a mass-weighted mean of the wind velocity of gases and the fall velocity of particles. The complications explode. Alternately, the third step in (1.1) stating *mass sources = 0*, which is truer for dry air mass, could be revised to account for precipitation falling out. The complications explode differently. If the entire text above were edited to replace "specific" with "dry-air mass mixing ratio of", it would be correct and unambiguous, but tedious to read.

Terms of art and their mathematical cousins (symbols) are often thought of as hyper-specific. But their greatest power can actually lie in a *refusal to draw distinctions* that are inessential to a line of reasoning[k]. Our plain symbol ρ papers over the complications above intentionally, for clarity. The slight distinction between q_v and r_v as measures of water content is downplayed deliberately here. If your goal is to mathematically

[j] We have avoided the term *concentration,* which is less clear than *mixing ratio*.

[k] A favorite example is the term *hydrometeor* for any falling condensed water object.

frame a numerical model that will be integrated over time, requiring your equation set to obey large-scale conservation laws punctiliously even as the various types of “stuff” are shuttled among many small spatial boxes and categories like liquid or ice or different particle shapes and sizes, then such decisions must be strictly defined and adhered to. Symbols, subscripts, labels and small terms will proliferate. Get a longer, fussier book. If such a model must assimilate or quantitatively confront absolutely calibrated observations, it must also use *accurate,* as well as precisely conservative versions of thermodynamics, coordinates, and conservation laws. Since our goal here is to facilitate appreciation of conceptual essentials, such details of rigorous fundamentals will be elided whenever possible in this book.

1.3.4 *Advection and the material derivative*

Notice (Problem 1.1) that you can distribute the derivatives in (1.4b) by the chain rule, divide by ρ, and use (1.4a) to rewrite it without approximation as:

$$u_t = -uu_x - vu_y - wu_z - \alpha p_x + fv \quad (1.5)$$

Defining the special new *total* or *Lagrangian* time derivative $du/dt = u_t + \boldsymbol{V} \cdot \boldsymbol{\nabla} u = u_t + uu_x + vu_y + wu_z$, the set (1.4) becomes:

$$d\rho/dt = -\rho \boldsymbol{\nabla} \cdot \boldsymbol{V} \quad (1.6a)$$
$$du/dt = -\alpha p_x + fv \quad (1.6b)$$
$$dv/dt = -\alpha p_y - fu \quad (1.6c)$$
$$dw/dt = -\alpha p_z - g \quad (1.6d)$$

This *advective* form of transport terms in the total derivative is fully general and tempts us to interpret *advection* as being just as real as flux and its divergence. The sense of advection is to *look upwind: Those conditions are coming toward you. Conditions at your location will soon be like that, unless source terms on the RHS intervene*.

In Chapter 2 we will need thermodynamic laws, learned from interrogating a kilogram of air trapped in a piston in a laboratory. To use these laws in our fluid equations, we need to equate laboratory *time derivatives referring to a unit mass of air* (denoted with Newton's univariate derivative notation $\dot{T}$)[1] to the *total derivative* dT/dt as defined above. One way to see this equivalence is to notice that if we truly had the field *T(x,y,z,t)* – temperature everywhere forever – we could extract the temperature history of a moving parcel of unit mass as $T_p(x_p(t), y_p(t), z_p(t), t)$. We can use the chain rule to extract all the temporal change:

$$dT_p/dt = \partial T/\partial t + \partial T/\partial x \cdot \dot{x_p} + \partial T/\partial y \cdot \dot{y_p} + \partial T/\partial z \cdot \dot{z_p}$$

and equate it to $\dot{T}$, the time-only changes for an air parcel that is trapped in the laboratory chamber.

1.4 Problems

1.4.1 *Show the steps from 1.4b (flux) to 1.6b (advection) forms of the budget equations for the u component.*

1.4.2 *Repeat the problem above for v and w, and gather terms to show that the advection of vector momentum can be written as* $-u\vec{V}_x - v\vec{V}_y - w\vec{V}_z = -(\vec{V}\cdot\vec{\nabla})\vec{V}$

1.4.3 *Set density to a constant ρ_0 and simplify the set (1.4) maximally in that case.*

---

[1] see https://en.wikipedia.org/wiki/Notation_for_differentiation

What phenomena could this *incompressible fluid* equation set describe? In other words, how can an incompressible fluid move, and why would it? What could drive motion, at what scales, and how could that motion decay?

What is the speed of compression (sound) waves in such a fluid? That is, if boundary conditions jiggle one edge of an incompressible body of fluid, how soon is fluid motion transmitted to the other side?

1.4.4 *Using the simplified set from 1.4.2, transform the majestic equality form of the mass continuity equation into a grasping equation of the form*

what I want (pressure) = work I must do to construct it

Hint: differentiate mass conservation in time, momentum conservation equations in space, and substitute the latter into the former. Subscripts for partial differentiation will save many redundant hand motions in this process. You may use the symbolic inverse $\boldsymbol{\nabla}^{-2}$ of the Laplacian operator $\boldsymbol{\nabla}^{2}$ in the final answer, even though solving it is a nontrivial job (Chapter 3).

Interpret the result in your own words, elaborating on this summary: *Continuity is the Law, Pressure is the Enforcer,* ***F*** *=* *m****a*** *is its Mechanism.*

1.5 Solutions

Solution to 1.4.3:

Substituting constant ρ=ρ_0, dividing by ρ_0, and negating the sign in (1.4a),

$$
\begin{aligned}
0 &= u_x + v_y + w_z \\
u_t &= -(uu)_x - (uv)_y - (uw)_z \quad -\pi_x + fv \\
v_t &= -(vu)_x - (vv)_y - (vw)_z \quad -\pi_y - fu \\
w_t &= -(wu)_x - (wv)_y - (ww)_z \ -\pi_z - g
\end{aligned}
$$

where we have introduced the pressure variable $\pi = p/\rho_0$.

What phenomena could this incompressible fluid equation set describe?
If a body of such fluid were initially at rest, the only forces that could drive flow within it are pressure or viscous forces (molecular momentum flux in the parenthetical terms) applied at a boundary. Gravity is powerless without density variations, so nothing worth the name "convection" can occur. Pressure drives divergent flows, like the motions inside a water balloon when massaged. Viscous forces could create shears, which could become shear instabilities, leading to fluid motions that would transport momentum deeper into the fluid. Eventually the fluid could contain all sorts of turbulent motions. The energy of such motions could decay into heat (molecular motions) by internal viscosity (diffusion of momentum).

What is the speed of compression (sound) waves in such a medium?
Infinite, since $\rho_t = 0$ makes the medium infinitely stiff.

<u>Solution to 1.4.4:</u>

Differentiating the u equation in x, the v equation in y, and the w equation in z, and summing them,

$$\begin{aligned}
[u_{tx} &= -(uu)_{xx} - (uv)_{yx} - (uw)_{zx} \quad -\pi_{xx} + fv_x] \\
+[v_{ty} &= -(vu)_{xy} - (vv)_{yy} - (vw)_{zy} \quad -\pi_{yy} - fu_y - uf_y] \\
+[w_{tz} &= -(wu)_{xz} - (wv)_{yz} - (ww)_{zz} \quad -\pi_{zz} \ - g]
\end{aligned}$$

Using mass continuity to see that the left side is zero, and packing up terms into a vector form,

$$0 = -\boldsymbol{\nabla} \cdot [(\boldsymbol{V} \cdot \boldsymbol{\nabla})\boldsymbol{V}] \ -\boldsymbol{\nabla}^2\pi \quad + f\zeta - u\beta$$

where ζ is the vertical component of relative vorticity $\zeta = v_y\text{-}u_x$, $\beta = f_y$ is the latitudinal gradient of the Coriolis parameter, and parentheses are carefully used to make the result depend on no notation beyond the familiar vector dot product and the vector differentiation operator $\boldsymbol{\nabla} = \mathbf{i}\, \partial/\partial x() + \mathbf{j}\, \partial/\partial y() + \mathbf{k}\, \partial/\partial z()$.

Hungrily solving for π, using the symbolic inverse of $\boldsymbol{\nabla}^2$,

$$\pi = \nabla^{-2}[-\nabla \cdot [(\boldsymbol{V} \cdot \nabla)\boldsymbol{V}] + f\zeta - u\beta]$$

Continuity is the Law expressed by the equation for pressure. Since flow divergence is zero, to maintain the (assumed) $\rho = \rho_0$ forever ($\rho_t = 0$), pressure must intimately cancel the divergent component of any other field of force. In this case those divergent forces are (i) the "inertial force" implied by the advection of momentum in the inner square brackets, (ii) the inward- or outward-directed Coriolis force on horizontally swirling flow, and (iii) the divergent Coriolis force on zonal flow u, when the Coriolis parameter varies with latitude.

Pressure is the Enforcer of mass continuity. Pressure does that job[m] in the momentum equations, and *F = ma is the Mechanism* of that enforcement. The common exercise in dynamics courses of calculating flows with the pressure field taken as a given is not very sensible: pressure is a cleanup force, the *last* force *logically*, the one that adjusts to intimately respond to the divergence of all the other forces, almost instantly (strictly, at the speed of sound). Here that sound speed is infinite, which is unrealistic, but even in air it is much faster than all the other information-transmitting waves in meteorological flows. Therefore, the point remains relevant: *Continuity is the Law, Pressure is the Enforcer,* $\boldsymbol{F} = m\boldsymbol{a}$ *is the enforcement mechanism.*

[m] This is *teleology,* the explanation of things in terms of the purpose they fulfil.