Chapter 5

Kinematically possible convective entities

5.1 Thermals, bubbles, starting plumes, plumes

Coherent structures are common in fluids. Bird watchers and glider pilots know that convective ascent is often concentrated in *thermals*[a] (Ludlam and Scorer 1953), even in clear air, which can be formalized a bit as *bubbles* (Scorer and Ludlam 1953). Long-lived visible *plumes* over fires and smoke stacks are another ancient visual paradigm of turbulent ascent, amenable to controlled study in laboratory tanks[b]. A *starting plume* (Turner 1962) is the combined concept: the response to a localized buoyancy source that is initiated and then maintained. A thorough history and discussion is in chapter 7 of Houze (2014).

Besides observation (indoors and out) and description, thinkers before the computation age had only mathematical study of the dynamical PDEs to fall back on for defining quantitative frameworks. One long-known analytic solution, specific but necessarily idealized, was the vortex ring or spherical vortex (Reynolds 1876, Levine 1959). Such entities are occasionally seen in the sky[c], but they require rare initial and boundary conditions, just as skilled lips are required to blow smoke rings. Still, they can serve as a *paradigm* for more realistic and ragged entities, useful for instance as a framework for statistical composites of data from

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[a] Glossary of Meteorology: "A discrete buoyant element…small-scale rising current…"

[b] along with less-relevant vertical *jets* driven by initial momentum rather than buoyancy

[c] e.g. Fig. 8.28 of Markowski and Richardson (2011); the Web surely has many examples

turbulent simulations (like in the fruitful debate between Sherwood et al. 2013 and Romps and Charn 2015).

A second equation-based approach is very different, based on concepts of *similarity* or *self-similarity* and depending only on the scale-independent properties of the fluid set (see problem 3.9.4). An admirable high culture in fluid dynamics prizes formal agnosticism equal to our ignorance. Often this is expressed in purely units-based reasoning, or the assertion that flow can only depend on fundamental *nondimensional parameters*[d]. This approach leads to deceptively simple final formulas, like the empirical result that the geometric slope of conical-shaped turbulent plumes is about 0.2 on average, as measured for instance on long-exposure photographs (Morton et al. 1956). Since it lacks any physical scale, such a number is expected to generalize directly from 1m deep water tanks to 1km deep air layers.

Although such conical turbulent flows are evidently realizable over their layer of similarity, that is not a complete and closed mass circulation. More elaboration is certainly needed the inevitable terminus at the wide end (like the starting plume concept). Synthesizing such overly-idealized bubble and incomplete plume entities into mathematical models for use at larger scales is a craft as much as a science. Might the time average of a sequence of bubbles act like a plume? An opinionated review of the historical literature and the rather blurry and *ad hoc* synthesis in common use today is in Yano (2014).

Below any given rising bubble, air from the turbulent *wake* is inevitably drawn up into the core, such that a bubble turns itself inside out as it 'rolls' upward a distance about equal to its diameter (e.g. Sànchez et al. 1989). It is better for the entity's success if that wake air is buoyancy-enriched, either by the continuing b source under a starting plume, or by moist or otherwise enriched air mixtures left behind from an earlier process. That "earlier process" can include the *shedding* or

[d] https://en.wikipedia.org/wiki/Buckingham_pi_theorem

erosion process on the sheared and turbulent outer flanks of the bubble itself, which created the material that makes up the turbulent wake.

Another possible "earlier process" is enhancement by the dead bodies of prior bubbles that have stalled and mixed out, enriching the area in ingredients for future bubble buoyancy. Indeed, laminar motions (perhaps driven by a prior bubble) can lift the background moisture gradient, at least in a transient way, as the crest of an internal wave. All of this steers toward the view that convection is an inherently multi-bubble process or "thermal chain" (Morrison et al. 2019). Aircraft-reported humidity enhancements around cumulus clouds (Perry and Hobbs 1996) are thought to indicate cloud-caused moistening, but might natural selection and conditional sampling be also at play? Perhaps successful clouds[e] only "get by with a little help from their friends", including ancestors.

While research on simple entities continues today, and some essential aspects are important to appreciation itself, and to later reasoning, the problem of *entity combinatorics* is so central that it may set a limit on how much concern about precision is warranted in the elemental realm.

One more equation-based approach should be mentioned: the *two-fluid* approach, in which the total flow (satisfying mass continuity) is split into two 'flavors' of air, like convective and environmental (Thuburn et al. 2018, Tan et al. 2018). Bookkeeping equations for the two categories can be written from the pure logic of accounting stuff in space, yielding terms that distinguish the motion *of the boundary* between the labeled flavors of air, distinct from the motion of air across that (moving) boundary. Many complications ensue: a useful definition of boundaries is tricky, but can be devised and calibrated from study of iso-surfaces of some scalar in an uncategorized reference simulation. While fully rigorous, this is at bottom another accounting tool, so its insight and utility depend entirely on how it is deployed.

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[e] Perry and Hobbs selected on criteria of "well isolated," but also "active growth".

5.1.1 Size, geometry, and buoyant updraft acceleration

Understanding the rising of “parcels” in terms of buoyancy b alone (chapter 4) is badly incomplete. Realizable ascent of finite-sized fluid elements importantly involves *pressure drag*, which can be thought of as *pressure acting to enforce of the law of mass continuity* by pushing the air above a buoyant element out of the way and filling the space below as it ascends. The kinetic energy of all this ambient motion comes from a 'tax' on primary energy production by the product bw seen in problem-solution 1.5.1. In other words, the PGF must opposes buoyancy b. This energy required for of the rest of a realizable circulation is sometimes treated as a “virtual mass” (that is, extra inertia of a "parcel") as a factor of about 3, reducing acceleration in the too-simple $dw/dt = b$.

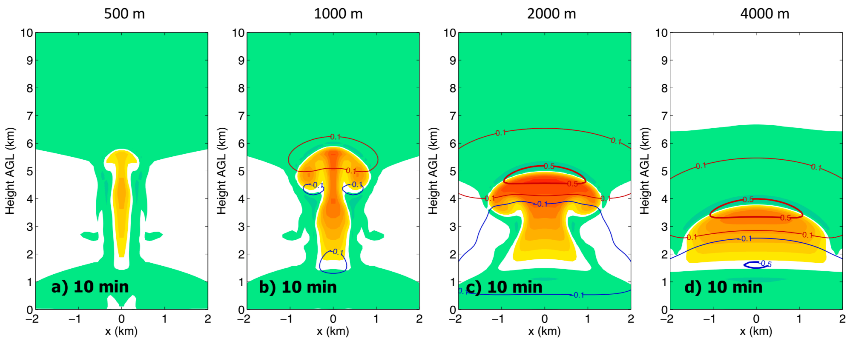

Fig. 5.1. Buoyancy (warm colors for positive values) and π_{buoy} from Eq. (5.1), ten minutes after releasing a bubbles in quiescent air. Narrower drafts feel less opposition from the BPGF, and have ascended further. Fig. 5 of Morrison and Peters (2018).

Pressure’s pushback on convective elements can be quantified with our equation for π (problem 1.4.3), expanding its conceptual "forcing" term $div(\mathbf{F})$ for the complete equation set (2.10) in the Boussinesq case:

$$\begin{aligned}\pi = {} & \boldsymbol{\nabla}^{-2}[b_{\boldsymbol{z}} - \boldsymbol{\nabla}\cdot[(\boldsymbol{V}\cdot\boldsymbol{\nabla})\boldsymbol{V}] + \boldsymbol{\nabla}\cdot\boldsymbol{F}_{Cor}] \\ = {} & \pi_{buoy} \quad + \pi_{dyn} \quad + \pi_{Cor} \end{aligned} \qquad (5.1)$$

The $\boldsymbol{\nabla}^{-2}$ operator is linear (allowing clean additive decomposition). However, it is *nonlocal* and *scale-selective,* bringing in spatial geometry considerations we can no longer ignore. Evaluating $\boldsymbol{\nabla}^{-2}$ involves two anti-derivatives, requiring additional information (constants of integration and/or boundary conditions), but it is useful as a symbolic matter to write it in this way.

It is helpful to write down seudo-code for a simple computational approach, on a periodic domain (codified as computer exercise 5.5.1), using the multi-dimensional Fast Fourier Transform **fft** and its inverse **ifft**, with total wavenumber $k_{tot} = [k^2 + l^2 + m^2]^{1/2}$ for spatial wavenumbers (k,l,m) in the (x,y,z) directions:

$$\pi_{buoy} = \boldsymbol{\nabla}^{-2}(b_{\boldsymbol{z}}) = \mathbf{ifft}[\, {k_{tot}}^{-2}\ \mathbf{fft}(b_{\boldsymbol{z}})] \qquad (5.2)$$

Figure 5.2 shows a toy two-dimensional calculation using (5.2) of the net vertical force b + BPGF, where BPGF is the vertical force per unit mass $-\partial_z(\pi_{buoy})$ for a hypothetical bumpy but assumed uniformly buoyant body (yellow contours over a periodically repeated cloud photo used to define the multiscale bumpy shape).

The feature shape and size dependence of the $\boldsymbol{\nabla}^{-2}$ operation makes the combined vertical force maximize in the growing tips of the tops (brightest reds in lower panel), with the gaps feeling a downward force that acts to enlarge the lobes (pure BPGF, since b=0 is specified there).

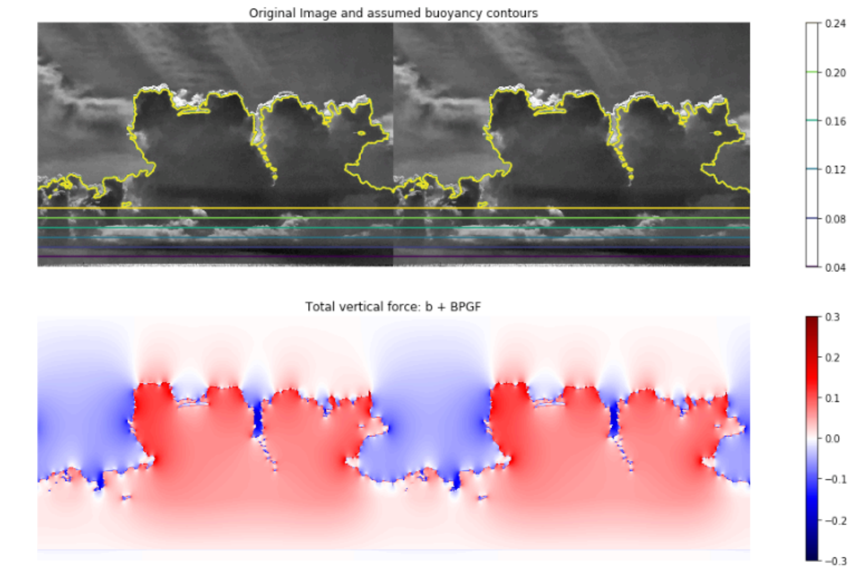

Fig. 5.2. Top: buoyancy (contours), assumed constant and positive in the main cumulus cloud bodies of the underlying photograph). Bottom: Resulting net vertical force *b*+*BPGF* computed from π_{buoy} evaluated by Eq. (5.2) for this two-dimensional periodic scene. Units are $m\ s^{-2}$ but values are arbitrary for this linear problem.

The distinctive texture of growing cumulus tops reflects a vital trade-off between opposition to *b* by the BPGF, which favors narrower entities (Figs. 5.1, 5.2), and vulnerability to mixing which disfavors too-narrow updrafts by thermodynamic destruction of *b* (including *buoyancy reversal*) as well as by viscous 'friction'. Inside even a single cumulus cloud, myriad sub-entities are competing under these influences (along with nonlinear advection and the associated gradients of dynamic pressure π_{dyn}) to create the characteristic cumulus fractal shape, with bumps upon bumps upon bumps[f]. Many an observant outdoors person has made important life and death decisions based on a reading of cumulus top textures. Might this visual texture contain underutilized quantitative information for our science, perhaps extricable by the powerful new tools of machine learning trained on imagery?

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[f] A humorous depiction is at https://xkcd.com/2185/

Dynamic pressure is a formidable puzzle, but some insight can be gained. A decomposition from the Boussinesq equations (problem 5.5.2) has a positive definite squared-deformation "splat" term, plus a negative definite squared-vorticity "spin" term:

$$\pi_{dyn} = -\boldsymbol{\nabla}^{-2}(\boldsymbol{def^2} - \boldsymbol{vor^2}/\boldsymbol{2}) \qquad (5.2)$$

(2.131 of Markowski and Richardson 2011), with vorticity ***vor*** = $|\boldsymbol{\nabla} \times \boldsymbol{V}|$ and ***def***^{2} given in summation notation for Cartesian coordinates by

$$\frac{1}{4}\sum_{i=1}^{3}\sum_{j=1}^{3}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)^2$$

The signs of squared terms in (5.2) make clear that *vortices of any rotation direction have negative-curvature*[g] *features such as pressure minima* in them, to do the job of holding parcels together against the divergent centrifugal force, while *colliding flows in any direction tens to have positive curvature like pressure maxima represent*, pushing back against the convergent "inertial force" of momentum advection.

Unfortunately, the common case of straight shear flow has equal but opposite contributions by ***vor*** and ***def*** as elementary kinematics tells us, even though it has no curvature or collision aspects *per se*. This cancellation makes the decomposition (5.2) unsatisfying in realistic complex flows. Might *curvature vorticity* be isolated usefully with further manipulation, as a way to set aside a residual shear vorticity that is spin-splat balanced, thereby defining some purer 'splat' term for interpretation? The possibility of making more satisfying causality accounts for events within the rigorous but slippery mathematics of Boussinesq or Anelastic fluid equations would be the greatly desirable outcome of such an analytic advance.

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[g] *curvature* is the familiar meaning of a Laplacian, although it is a bit abstract in 3D

Another understandable dynamic pressure contribution is the *linear* interaction of a complex convective entity with a simple mean flow it is superposed upon. Such a component to π_{dyn} can explain how sheared wind impinging on a thermal retards its ascent and creates upshear-downshear asymmetry (Peters et al. 2019), or why right-moving storms are favored by curved wind shear hodographs in splitting supercells (section 8.4 of Markowski and Richardson 2011).

The non-b terms in *dw/dt* invite attempts at *parameterization* for some class of idealized updraft entities such as bubbles, based on prescribed shape or other assumptions. Uncertain coefficients in such parameterization *schemes* can be estimated by controlled numerical simulations (de Roode et al. 2012, Morrison and Peters 2018, Tian et al. 2019). The lengths to which this kind of frontal assault can now be carried are illustrated by Eq. (33) of Morrison (2017), too complex to explain in detail here but illustratively useful:

$$w = \left[\frac{2\mathrm{CAPE}([1/z^2 - 9L_v g k^2 L\Phi/(4c_p P_r R_{\mathrm{HMB}}^2 \mathrm{CAPE})]\{4P_r R_{\mathrm{HMB}}^2 z/(9k^2 L) - 16P_r^2 R_{\mathrm{HMB}}^4/(81k^4L^2)\ln[9k^2 Lz/(4P_r R_{\mathrm{HMB}}^2) + 1]\})}{1 + \alpha^2 R_{\mathrm{HMB}}^2/H^2 + 3k^2 Lz/R_{\mathrm{HMB}}^2} \right]^{1/2}. \tag{33}$$

Ignoring parentheses and the second term in the denominator, we can recognize $w = [2\,CAPE]^{1/2}$ as the conversion of potential energy (CAPE) to vertical kinetic energy ($w^2/2$) for an inviscid undiluted parcel, which certainly needs some correctives for realizability. While somewhat inscrutable, perhaps such a formal dependence on radius R could form the seed of a basis for assessing the competition among bubbles of different sizes, for a simplest ecology (competition among independent entities) in the competitive game considerations of Part III.

Does such mathematical detail become excessive at some stage, given the great idealization of the entity in the first place? Computers may change this doubt's answer. Software for symbolic math manipulation now makes such derivations reliable and trustworthy. While overuse of that convenience may seem to run counter to the goal of reader insight that once spurred such quests, analytic formulae have the great numerical virtues of economy and accuracy. Since formula translation into high-performance code is also becoming easier and more reliable[h], perhaps the old tools of theory could find a renaissance as simulation, testable against traditional brute force numerical grid computations (which can also be criticized as complicated and assumption-laden in their own way), and then implementable in genuinely predictive models.

5.2 Supercell updrafts

Summarizing the above, the vertical velocity equation is at the heart of updraft models, which are in turn at the heart of bubble or thermal theory in cumulus dynamics. The VPGF must be parameterized for such models, usually based primarily on the tractable buoyancy-induced pressure π_{buoy}. However, the dynamical pressure π_{dyn} can become so large that it predominates over the b force in shaping the evolution of a storm. Such is the case in *supercells.* Readers should study an excellent storm-oriented source like Markowski and Richardson (2011) for the loving detail that passionate researchers have assembled on this subject. Science aside, web imagery of supercell convection from storm chasers is abundant and amazing nowadays. Mesmerizing time lapse videos are worth the hours of study they will definitely seduce any reader of this book into staring at.

Ultimately such convection still depends on bw as its energy source, but energy can also be extracted from the strong mean shear flow that makes dynamic pressure competitive with b in such situations. Dynamic pressure arising from shear also creates surprising pathways for long-distance transports of energy across the sky, in the form of terms like $-V \cdot \nabla\pi$ from (2.15)[i] that problem 2.4 conveniently integrated away. The nonlocality of energy, some of it in the form of potential energy as stable air is lifted to saturation, helps create the visually striking quality of laminar cloud structures among convective ones, giving severe and

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[h] in the Julia computer language for instance

[i] this looks like *advection of pressure,* but pressure is not "stuff" that can be advected -- instead this should be pronounced as *work done by flow down a pressure gradient.*

tornadic convective skies their distinctively spooky textures, reminiscent of sculpture galleries strewn with heavy cobwebs.

5.3 Downdrafts: condensation-evaporation asymmetry

Negatively buoyant downdraft entities are often envisioned as analogous to buoyant updrafts (bubbles, plumes, etc.). One distinctly modern phenomenon of downward moist convection is common in jet condensation trails (Fig. 5.3), evocative of a linear instability theory (periodic, smooth) yet with distinctively convective shapes.

Fig. 5.3. Contrail with gentle (periodic, smooth) downward convection of the cloudy air.

Mammatus clouds (darlings of online image and video materials; search also for *asperatus*) showed this processes before the jet age, an upside-down expression of potential instability (section 0.4) on descending anvil bases (Ludlam and Scorer 1953, review in Schultz et al. 2006).

While the above examples are nearly moist adiabatic, a few oddities of evaporation in air + condensate mixtures must be recalled from section 2.3. *Buoyancy reversal* after the mixing of clear and unsaturated air is peculiar to cloudy convection. The fractional mixture that is just saturated (evaporating all the available condensate) has the greatest bulk density of all possible mixtures. However, this density increase occurs only when complete molecular-scale equilibrium is reached by the mixture! Mere turbulent folding (like the *dynamic entrainment* of wake ingestion into a rising bubble discussed above) is not yet mixing, nor is *turbulent entrainment* by shear instabilities on the flanks of narrow updrafts, although that process may lead more quickly to true mixing.

What is the fate of the extra-dense air created by buoyancy reversal? Squires (1958) predicted that it would perforate cumulus clouds with penetrative downdrafts from the mixing at cloud top, while Jonker et al. (2008) emphasize a sheet of descent along cloud edge -- contributing more shear whose instability could drive further mixing. Conceptual models of mixing the subsequent sorting of mixtures are discussed in more detail in section 6.3.

5.4 2D entities: slabs, jumps, squalls

Equation-rooted entity models of 2D convection in steady state have been devised, featuring "jump" updrafts and "overturning" flow branches that satisfy mass continuity and other Lagrangian (d/dt) conservation laws along trajectories, which are the same as streamlines for steady flow (Moncrieff 1978, 2006). These entities have become paradigms or "archetypes" for squall momentum flux. Unfortunately, their virtue of rigor as solutions to the dynamical equations is nontrivial to weave into larger-scale flows. Their far-field asymptote of these 2D flows implies (or requires) a pressure difference that extends out to infinity.

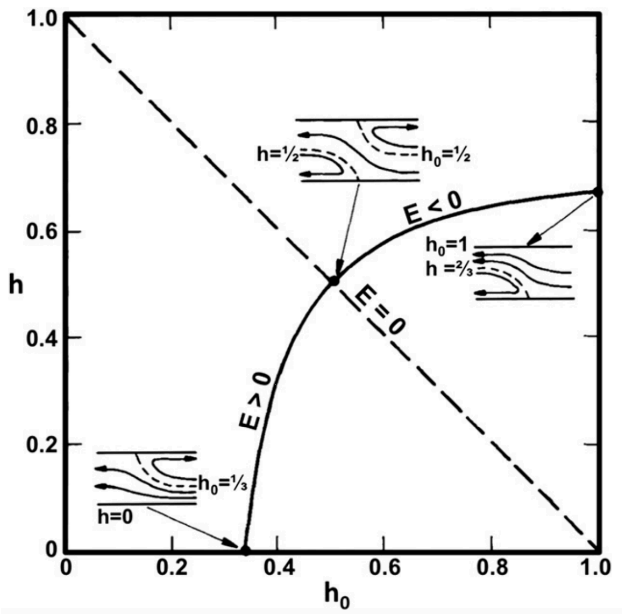

Fig. 5.4. Regime diagram for 2D steady jump and overturning flows bases in rear and front inflow altitudes. Adapted from Fig. 13 of Moncrieff and Lane (2015) where E is defined as a "Bernoulli number" measuring the net pressure drop across the system

The theory of atmospheric convection suffers from a lack of self-consistent, semi-quantitative models for mesoscale convection, in which the ‘dynamic entrainment’ implied by outside-the-updraft motions is reconciled with the main updraft's buoyancy and the $[wb]>0$ requirement for the system as a whole to be energy-generating. Just as the useful plume concept discussed above had to be capped with a bubble as a starting plume, "a craft as much as a science", partial models and partial considerations must be stitched together in *ad hoc* ways at the edges of mesoscale convective entities, whose relevance to realistic reasoning about convection are just as great as plumes and bubbles.

5.5 Problems and exercises

5.5.1 *Dynamic pressure derivation*

Derive Eq. (5.2). Using subscript notation for differentiation will save you many redundant hand motions. Can you spot the terms that go into (5.2)? Do any other decompositions suggest themselves, as discussed in that section?

5.5.2 *Jupyter notebook on inverse Laplacian*

Reproduce Fig. 5.2 from the Jupyter notebook at github.com/brianmapes/ConvectionShortCourse/BPGF.ipynb. Construct other 2D periodic buoyancy patterns, and use the notebook to solve for the implied π_{buoy} and associated BPGF.

5.5.3 *Coalescence and repulsion of buoyant updrafts in 2D*

Explore the effects of interactions by experimenting with the release times of 3 bubbles of thermal buoyancy *b,* in a 2D convective flow model representing an unstratified fluid. The Jupyter notebook at github.com/brianmapes/ConvectionShortCourse/ThreeBubbles.ipynb contains instructions. Optional: relate your work to early model work on multi-bubble clumping and mergers (Wilkins et al. 1976, Orville et al. 1980).

5.5.4 *IDV exploration of Giga-LES data*

Explore the 3D spatial patterns of spatial-eddy temperature (proportional to thermal buoyancy b), perturbation pressure, and other fields in three time steps (5 minutes apart) from a very large and high resolution (giga-LES) doubly-periodic deep convection simulation. *Capture images that illustrate the relationship between perturbation pressure and terms in (5.2), as best you can find them by focusing on regions where a single term likely dominates.*

Model details are as described at Khairoutdinov et al. (2009), but here the model was forced with domain-mean winds derived from the TWP-ICE program on Jan 20, 2006 over northern Australia (Glenn and Krueger 2017), leading to a ragged south-to-north propagating squall comprising several arc-like segments with more 3D structure. Use Unidata's free IDV software, on a computer with at least 8GB of available RAM (Before attempting this, set the IDV's memory allocation in Edit→Preferences menu, System tab, then restart the IDV.) Instructions and orienting images of the basic fields displays are at github.com/brianmapes/ConvectionShortCourse/GigaLES.ipynb.