Part I: Essentials of the fundamentals

Chapter 1

Keeping track of stuff in space

1.1 Units for space, time, and "stuff"

Before understanding must come labeling and accounting. Mass, momentum, and energy are examples of *extensive* quantities (different kinds of "stuff") that are conserved by isolated systems like the whole universe, as well as in interactions between sub-systems. We care about where stuff is, so we must label regions of space and account for transport. We also care about different categories of stuff (such as air vs. water mass, heat vs. latent (chemical) energy, motions of different scales, etc.), so we need accounting notations for transfers between categories. In our equations, these accounting terms will often outnumber and out-complicate the very few key terms expressing the fundamental laws of physics.

Space, time, and mass are measured here in *Système International* (SI) units, based originally on our ten fingers, Earth, and water.[a] Space is in *meters* (m), devised as the Earth's equator to pole distance divided by 10^7 to be human scale. A *kilogram* is the mass of a cubic meter of water (1 *metric ton*), divided by 10^3, again to scale it to human bodies and

[a] Replacing old traditions, in the 18th century French Revolution's radical rationalizing.

commerce[b]. Time is the contentious realm of measurement. The Earth gives us *days*, which could be divided by 10 or 100 for finger-counting convenience. Clocks with decimal face labels were indeed manufactured in 18th century France, but never caught on. After all, the six-related numbers of traditional time and angle measures have their own Earth-based numerology (almost-12 moons and almost-360 days in a year). In the end, SI retained the *second*: a day divided by 86400 (60 x 60 x 24). Latitude in degrees (10^7m / 90° = 111.111 km per degree) and subdivisions like nautical miles (1/60 degree) also carry this six-related history.

Temperature's Celsius scale is also about water and tens: 1 °C = 1 K = (water's boiling point minus freezing point)/100. This is not strictly about water alone, since the boiling point depends on atmospheric pressure. It is a remarkable coincidence that the weight of a 1 m^2 column of Earth's atmosphere (1 bar or atmosphere of pressure) happens to be so near the weight of a 10 m column of water. This coincidental amount of air makes surface pressure fall within 1.5% of an exact power of 10 (the global mean is 101325), in the compound SI unit of Pascals (1 Pa = force/area = 1 N m^{-2} = 1 kg m s^{-2} m^{-2} = 1 kg m^{-1} s^{-2}).

Precision science has quietly replaced these original Earth and Water motivated units with equivalent-sized re-definitions rooted in more fundamental quantum profundities, without most people ever noticing. The kilogram was formally redefined in 2019, for instance. But our 10 fingers still rule numbering in all but the time domain.

1.2 Conservation of the most fundamental "stuff": mass

In a given volume of space (let's say 1 m^3 for definiteness), the enclosed mass is customarily labeled ρ, a *mass density* whose inverse is called *specific volume,* $\alpha=1/\rho$. Because mass is so closely conserved[c], its rate of change in a cubic meter of space ($\partial\rho/\partial t$) is a pure transport

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[b] That volume of water is also called the *liter*.

[c] The only exception is a tiny Einsteinian $E=mc^2$ conversion to energy in radioactive decay.

accounting: it equals the *net inflow of mass into the volume*. When a conservation law is so dominated by its transport terms (with no sources or sinks), it is sometimes called a *continuity* equation. This is *mass continuity*, the heart of fluid dynamics, the quietly powerful equation that shapes all flow through its secret agent, pressure (as elaborated below and in problem 1.4.1).

The flow of mass through a two-dimensional (2D) area, like the square face of a 3D cubic volume, is measured by a *flux*. The units of mass flux [kg m^{-2} s^{-1}] embody its meaning better than any further words can elaborate. As those units say, mass flux is proportional to velocity and density, symbolically expressed as $\rho \boldsymbol{V}$, or sometimes $\rho\vec{\mathbf{V}}$ to extra-emphasize the vector with an arrow[d]. Here velocity $\boldsymbol{V} = \vec{\mathbf{V}} = \boldsymbol{i}u + \boldsymbol{j}v + \boldsymbol{k}w,$ in a Cartesian (x, y, z) coordinate system with its unit vectors $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ and scalar velocity components (u, v, w). Those components could also be written as $(\dot{x}, \dot{y}, \dot{z})$, using Newton's dot notation[e] for univariate time derivatives of the coordinate position variables (x, y, z) of any given air parcel.

Readers should verify that the units of $\rho \boldsymbol{V}$ are indeed a flux [(kg m^{-3}) (m s^{-1}) = kg m^{-2} s^{-1}]. Notice with that same units awareness that velocity $\boldsymbol{V}$ is also a *volume flux;* and it is also *specific momentum* (where "specific" means *per unit mass*). A feeling for different interpretations of the same quantity is essential to fully appreciate the equations of flow and convection: a firm physical grip is needed on the mathematical symbols and their units, but not too tight or exclusive.

As stated above, mass's *rate of change* in a volume of space equals the *net mass inflow*. That *net* is called the *convergence* of mass flux, the negative of *divergence*[f]. Translating words into math, with subscripts denoting partial derivatives along the axes in (x, y, z, t) space,

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[d] See Table of Symbols and Notation.

[e] reviewed in https://en.wikipedia.org/wiki/Notation_for_differentiation

[f] Early meteorology texts simply used the word *vergence*, letting the sign reflect its sense.

$$\begin{aligned}\rho_t &= conv(\rho\vec{\boldsymbol{V}}) + mass\ source\ terms\\ &= conv(\rho\vec{\boldsymbol{V}})\\ &= -div(\rho\vec{\boldsymbol{V}})\\ &= -\boldsymbol{\nabla}\cdot(\rho\vec{\boldsymbol{V}})\\ &= -(\rho u)_x - (\rho v)_y - (\rho w)_z \end{aligned} \quad (1.1)$$

A diagram with a cube (a *control volume*) and flux arrows is sometimes used to illustrate this *vergence* logic, but seems unnecessary to repeat here.

1.2.1 *Aside on mathematical expression culture*

Notice that the structure of (1.1) embodies an ambitious spirit:

$$something\ we\ want\ =\ complications\ we\ must\ face\ to\ get\ it$$

In the case of a *prognostic equation* like (1.1), what we want is to know the future, so we put that on the left-hand side (LHS). Let us call this the *grasping form* of equation writing, in contrast to a *majestic equality* like 0 = (sum of all terms). Mathematically, these are isomorphic: not different in any meaningful way, just in form, by custom. But equations do have a *brainfeel*[g] in addition to their content. To feel a difference, write (1.1) as the equivalent majestic equality:

$$0 = (\rho u)_x + (\rho v)_y + (\rho w)_z + (\rho\dot{t})_t \quad (1.2)$$

In the first 3 terms on the RHS, velocity components in space ($u = \dot{x}$, $v = \dot{y}$, $w = \dot{z}$, using Newton's dot notation for univariate time derivatives of each coordinate's value) measure the spatial journey of moving matter, in units of meters traversed per second of elapsed time. In (1.2), an analogous quantity $\dot{t}$ measures the journey of matter through time in seconds traversed or endured per second of elapsed time. Since both are seconds, $\dot{t}$ is unitless and equal to unity. This form merely re-uses our familiar notation to treat time as just another coordinate in (*x,y,z,t*) space,

[g] This term is motivated by "mouthfeel," a food descriptor distinct from taste or nutrition.

adding no new meaning or conceptual content, just a less familiar symbol $\dot{t}$. But to this author, a distinctly new and equally valid appreciation or brainfeel is gained through the exercise.

1.3 Conservation of specific (per unit mass) other stuff:

1.3.1 *Specific momentum and its physical source terms*

Having established the flux convergence form for mass (1.1), other budgets follow straightforwardly by accounting other intensive properties of air on a *specific* or *per unit mass* basis. For instance, to get a budget equation for momentum, simply multiply the mass flux by specific momentum, which (as noticed above) is velocity, and then reconsider the sources and sinks. In the vertical or **k** direction, along which air's specific momentum is $\dot{z} = w$ (vertical velocity), multiply (1.1) by *w* to get:

$$\begin{aligned}(w\rho)_t &= -\vec{\boldsymbol{\nabla}} \cdot \left(w\rho\vec{\mathbf{V}}\right) + momentum\ source\ terms \\ &= -(w\rho u)_x - (w\rho v)_y - (w\rho w)_z \\ &\quad + momentum\ source\ terms \end{aligned} \tag{1.3}$$

The big novelty is nonzero source terms on the RHS. Newton called such momentum sources *forces,* in the famous equation **F** = m**a** deserving of his honorific SI unit for force (N = kg m s^{-2}).[h] Acceleration could also be called *specific force,* but that doesn't add much useful brainfeel: we don't really think of acceleration as conserved "stuff" like we do of momentum.

Two forces (momentum sources) are needed to appreciate convection:

[h] Is force a flux of some definable "stuff"? Not really: translating into the flux form (stuff) (m^{-2} s^{-1}), we have 1 N = 1 (kg m^3 s^{-1}) (m^{-2} s^{-1}). The unit of implied stuff being transported (kg m^3 s^{-1}) makes no sense to this author's brainfeel.

(1) *Gravity*[i] $-\boldsymbol{k}\rho g$
(2) The *pressure gradient force* $-\vec{\boldsymbol{\nabla}}p$

What is this "pressure" p? Most students learn it as *force per unit area,* implying units $N/m^2 = Pa$ (Pascals). However, those units would also identify it as a *momentum flux,* with flux units of (stuff) $m^{-2}s^{-1}$ = (momentum) $m^{-2}s^{-1}$ = ($kg\ m/s$) $m^{-2}s^{-1}$. But it is a strange flux: directionless, imparting momentum in all directions at once. Since p is a scalar field, not a vector field, the differential operator measuring the net flux into a cube of space (the flux difference from one side to the other) is not the convergence, like of the directional vector mass flux field $\rho\vec{\mathbf{V}}$ above, but rather the *gradient* of the scalar field p.

In a steadily rotating coordinate system, where air we call "motionless" is nonetheless accelerating, we must add to (1.3) a compensating corrective term, the fictitious Coriolis acceleration (force per unit mass) $-2\vec{\boldsymbol{\Omega}} \times \vec{\mathbf{V}}$ where $\vec{\boldsymbol{\Omega}}$ is the coordinate rotation vector, pointing toward the North Star in the case of Earth. For simplicity we will neglect the vertical component of the Coriolis force (because it is tiny compared to gravity), and retain only the horizontal Coriolis force based on the *Coriolis parameter* $f = 2|\vec{\boldsymbol{\Omega}}|$ *sin(latitude)* in our Earth-tangent Cartesian plane with its (x,y,z,t) coordinates.

Gathering the considerations above for every cubic meter of this Cartesian space with strictly-vertical gravity, we are already up to 4 equations for our 5 unknowns (u,v,w,p,ρ)!

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\begin{aligned}
(\ \rho)_t &= -(\ \rho u)_x - (\ \rho v)_y - (\ \rho w)_z \quad + 0 && \text{(1.4a)}\\
(u\rho)_t &= -(u\rho u)_x - (u\rho v)_y - (u\rho w)_z \quad -p_x + \rho f v && \text{(1.4b)}\\
(v\rho)_t &= -(v\rho u)_x - (v\rho v)_y - (v\rho w)_z \quad -p_y - \rho f u && \text{(1.4c)}\\
(w\rho)_t &= -(w\rho u)_x - (w\rho v)_y - (w\rho w)_z \quad -p_z - \rho g && \text{(1.4d)}
\end{aligned}
$$

[i] Strictly speaking, "gravity" in meteorology is the *gravitational force* plus a small centrifugal force from Earth's rotation, slightly bulging the equator relative to a sphere.

Notice that each p in (1.4 b-d) could be moved inside the parentheses of a flux term, emphasizing its interpretation as a momentum flux. But pressure's momentum flux is transmitted across the borders of a spatial box by elastic collisions of the molecules outside the border with those inside it. This is new physics, *different in character* from pieces of matter carrying their properties with them as they cross a border.

Equal and opposite border crossing (swapping) by the smallest pieces of matter (molecules) is part of flux terms like $u\rho v$ in (1.4). If that swap involves *randomly selected* molecules from different sides of the border, then logic dictates that their difference in terms of the stuff they are carrying (momentum, in this example) is proportional to the gradient of that stuff across the border. This random swapping then acts as a *diffusion* process: *the convergence of a flux of stuff that is proportional to the gradient of that stuff*. For the case of momentum, that diffusion process is called the *viscosity force,* when it is pulled out of the flux terms and moved to the RHS as a "source" term for equations like (1.3)[j]. Viscosity is revisited in section 3.1.

Without density variations, the gravity force in (1.4d) is indiscriminate, exerting its mighty pull equally on all air. In order to account for *convection* in the sense of our title, we must allow ρ to vary spatially, at least a little bit, so that the gravity force $-\boldsymbol{k}\rho g$ can do work and drive motion. But letting ρ vary drags us into thermodynamics (Chapter 2). Specifying ρ is called an *equation of state,* of which an example is the *ideal gas law.* Before we tackle those complications, though, we can already gain a key insight into the nature of pressure from the simplest case, ρ = constant (Problem 1.4.1). Readers should understand the resulting lesson well: *Continuity is the Law, pressure is the cop, and* $\boldsymbol{F} = m\boldsymbol{a}$ *is its baton.*

[j] The *Navier-Stokes equations* for fluid dynamics have viscosity pulled out in this way.

1.3.2 *Other specific stuff: humidity and 'heat content'*

To close our four-equation but five-variable set (1.4) we need an equation for ρ. Although recognizing *specific momentum* as another name for velocity made momentum budget construction very direct (1.4 b-d), no analogous trick of invoking a 'specific density' can help us[k]. We need another approach, another physical law. The ideal gas law comes to hand, but that brings in a new variable (temperature T), so the equation set becomes 6 equations in 5 unknowns, still unclosed. Since T is indicative of warmth or heat, we need an equation for the "stuff" that T measures: a heat-energy, or perhaps a more abstract kind of stuff called *entropy*.

For moist convection, we also need to keep track of the mass of water. We can measure that in terms of a *specific humidity* q, so by now you know what to do: multiply (1.l) by q, and express the physics of *q sources*. Further trivial extensions to the notation give us separate budgets of categorized water (specific cloud water, specific rain water, specific ice water, etc.). However, one subtle bookkeeping issue in this exercise needs to be noticed, before we go back to ignoring it for clarity.

1.3.3 *Specific X, or mass mixing ratio of X?*

There is a subtle difference between *specific water vapor mass* q_v and *water vapor mass mixing ratio*[l] r_v. Both have water vapor mass in the numerator, but mixing ratio has the mass of *dry air* rather than *total mass* in the denominator. To get more precise, one must decide whether ρ in (1.1) stands for dry air mass density or for total mass density. It seems tempting to use total mass, because that is the most conserved, but then the velocity w is harder to define: in rain, w would have to be a mass-weighted mean of the wind velocity of gases and the fall velocity of particles. The small complications explode. Alternately, the third step in (1.1) stating *mass sources = 0*, which is truer for dry air mass than for all gases or all mass, could be revised slightly to account for condensation, or for the new

[k] That would be the inverse of *specific volume* $\alpha = 1/\rho$, which is just density ρ itself.

[l] We also avoid the chemistry term *concentration,* which seems ambiguous.

flux divergence that arises due to precipitation falling out. The small complications explode again, but differently, from that choice.

Strictly speaking, this book accepts ρ in (1.1) to be conserved, as if for dry air, yet keeps the word *specific* (instead of *mixing ratio*) for brevity. Terms of art and their mathematical cousins (symbols) are often thought of as being hyper-specific. But sometimes their greatest power can actually lie in being vague, refusing to draw distinctions that are inessential to a line of reasoning[m]. Our plain symbol ρ papers over the complications intentionally, eliding the slight distinction between q_v and r_v as measures of water vapor content.

If your goal were to mathematically frame a numerical model that will be integrated over long times, requiring an equation set that obeys integral conservation laws punctiliously even as the various types of "stuff" are shuttled many times among spatial boxes and categories, then more precise terms must be strictly defined and adhered to. Symbols, subscripts, and small terms will proliferate. You should work from a longer, more detailed book (many of which exist). If your model must also quantitatively assimilate absolutely-calibrated observations, then your equation sets need to be *accurate* as well as precisely conservative. Coordinates must account for Earth's curvature, and the small Coriolis terms should be included. Here, since our goal here is to facilitate appreciation of conceptual essentials, such details of the rigorous fundamentals will be elided freely.

1.3.4 *Advection and the material derivative*

The derivatives of products in (1.4b) can be distributed using the chain rule. One set of the resulting terms vanishes exactly, by (1.4a), and the remaining ones when divided by ρ can be rewritten (Problem 1.4.1) as:

$$u_t = -uu_x - vu_y - wu_z - \alpha p_x + fv \qquad (1.5)$$

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[m] An example is the term *hydrometeor* for any falling condensed water object.

The *advective* form of transport in the first 3 terms on the RHS is fully general, derived *without any approximation* from (1.4). We can therefore interpret transport as *advection*, with validity absolutely equal to the interpretation via flux and its divergence. The conceptual sense of advection is to *look upwind: those air properties are coming toward you, so conditions at your location will soon be like that, unless source terms on the RHS intervene*. For local situations, this is much clearer than looking all around at all the flux vectors, and somehow trying to guess at their often very slight convergence and divergence. We almost always prefer the advective form, except when integrating over volume. The Divergence Theorem (Gauss' Theorem) states that the integral of a divergence over the entire atmosphere is exactly zero, so for integral derivations like that the flux form is preferred.

Defining a special new *total* or *Lagrangian* or *material* time derivative, $du/dt = u_t + \boldsymbol{V} \cdot \boldsymbol{\nabla} u \; = \; u_t + uu_x + vu_y + wu_z$, the set (1.4) attains a wonderful brevity:

$$d\rho/dt \; = -\rho \boldsymbol{\nabla} \cdot \boldsymbol{V} \qquad (1.6a)$$
$$du/dt \; = -\alpha p_x + fv \qquad (1.6b)$$
$$dv/dt \; = -\alpha p_y - fu \qquad (1.6c)$$
$$dw/dt \; = -\alpha p_z - g \qquad (1.6d)$$

In Chapter 2 we will need to invoke thermodynamic laws learned from interrogating a kilogram of air trapped in a piston in a laboratory. To use these laws in our fluid equations, we need to equate laboratory *time derivatives referring to a unit mass of air* (denoted with Newton's univariate derivative notation like $\dot{T}$) to this Lagrangian total derivative *dT/dt* as defined above. One way to see their equivalence is to notice that if we truly had the field or function *T*(*t,x,y,z*) – temperature everywhere forever – we could extract the temperature history of an arbitrary moving parcel of unit mass as $T(t, x_p(t), y_p(t), z_p(t))$. Using the chain rule to extract all the temporal dependence in the function's argument,

$$\begin{aligned} dT_p/dt \; &= \; \partial T/\partial t + \partial T/\partial x \cdot \dot{x}_p + \partial T/\partial y \cdot \dot{y}_p \; + \partial T/\partial z \cdot \dot{z}_p \\ &= \; T_t + uT_x + vT_y + wT_z \end{aligned}$$

For a parcel trapped motionless in a chamber, this dT/dt equates to $\dot{T}$ as measured by the experimenter's thermometer readings.

1.4 Problems:

1.4.1 ***Show the steps from 1.4b (flux) to 1.6b (advection) forms of the budget equations, following the steps in the opening sentences of section 1.3.4.***

1.4.2 ***Repeat the problem above for v and w, and gather terms to show that the advection of vector momentum can be written*** $-\boldsymbol{u}\partial/\partial\boldsymbol{x}(\vec{\boldsymbol{V}}) \; -\boldsymbol{v}\partial/\partial\boldsymbol{y}(\vec{\boldsymbol{V}}) \; -\boldsymbol{w}\partial/\partial\boldsymbol{x}(\vec{\boldsymbol{V}}) = -\left(\vec{\boldsymbol{V}}\cdot\vec{\boldsymbol{\nabla}}\right)\vec{\boldsymbol{V}}$

1.4.3 ***Set density to a constant*** ρ_0, ***and divide it out to simplify the set (1.4) maximally for that kind of fluid.***

What phenomena could this *incompressible fluid* (constant density) equation set describe? In other words, how can an incompressible fluid move, and why would it? What could drive motion, at what scales, and how could that motion decay? Think of the fluid deep inside a frictionless water balloon, for instance.

What is the speed of compression (sound) waves in such a fluid? That is, if boundary conditions jiggle one edge of an incompressible body of fluid, how soon is the motion transmitted to the other side?

1.4.4 *Using the simplified set of component equations with constant density ρ_0 from 1.4.3, transform the majestic equality form of the mass continuity equation (0 = terms) into a grasping equation of the form:*

what I want (pressure) = complicated effort needed to construct it

Hint: differentiate mass conservation in time, momentum conservation equations in space, and substitute the latter into the former. Subscripts for partial differentiation will save many redundant hand motions in this process. You may use the symbolic inverse $\boldsymbol{\nabla}^{-2}$ of the Laplacian operator $\boldsymbol{\nabla}^{2}$ in the final answer, even though solving it is a nontrivial job (Chapter 3).

Interpret the result in your own words, elaborating on this summary: *Continuity is the Law, pressure is the cop, and **F** = m**a** is its baton*.

1.5 Solutions:

Solution to 1.4.3:

Substituting constant $\rho=\rho_0$, dividing by ρ_0, and negating the sign in (1.4a),

$$\begin{aligned}
0 &= u_x + v_y + w_z \\
u_t &= -(uu)_x - (uv)_y - (uw)_z \quad -\pi_x + fv \\
v_t &= -(vu)_x - (vv)_y - (vw)_z \quad -\pi_y - fu \\
w_t &= -(wu)_x - (wv)_y - (ww)_z \; -\pi_z
\end{aligned}$$

where we have introduced the pressure variable $\pi = p/\rho_0$.

What phenomena could this incompressible fluid equation set describe?

If a body of such fluid were initially at rest, the only forces that could drive coherent (larger than molecular) motions within it are coherent momentum sources applied at its boundary. Without density variations, gravity cannot discriminate and nothing worth the name "convection" can occur. Pressure can push divergent (irrotational) internal flows, like the motions inside a water balloon that make some part bulge out when another part is pressed in. If groups of molecules at the fluid's boundary are somehow given a coherent momentum tangential to the boundary, like by a *stress* we could call "friction", they could diffuse that momentum inward (viscosity)[n]. Such viscous forces could create internal shear, which could break down due to shear instabilities into smaller-scale fluid motions that could transport momentum still deeper into the fluid, so that eventually the fluid could contain all sorts of turbulent motions. The energy of such motions would decay into heat (molecular motions) by internal viscous dissipation (diffusion of momentum down its gradient). The speed of sound (compressional waves) in an incompressible fluid is infinite: with constant density $\rho_t = 0$, the medium is infinitely stiff and any motion or vibration is transmitted throughout the body instantly.

Solution to 1.4.4:

Differentiating the u equation in x, the v equation in y, and the w equation in z, and summing them,

$$\begin{aligned} [u_{tx} &= -(uu)_{xx} - (uv)_{yx} - (uw)_{zx} \quad -\pi_{xx} + f v_x] \\ +[v_{ty} &= -(vu)_{xy} - (vv)_{yy} - (vw)_{zy} \quad -\pi_{yy} - f u_y - u f_y] \\ +[w_{tz} &= -(wu)_{xz} - (wv)_{yz} - (ww)_{zz} \quad -\pi_{zz}\,] \end{aligned}$$

Using mass continuity to see that the left side is zero since $\rho_t = 0$, and packing up terms into a vector form,

$$0 = -\boldsymbol{\nabla}\cdot[(\boldsymbol{V}\cdot\boldsymbol{\nabla})\boldsymbol{V}] \; -\boldsymbol{\nabla}^2\pi \quad + f\zeta - u\beta$$

where ζ is the vertical component of relative vorticity $\zeta = v_x - u_y$, $\beta = f_y$ is the latitudinal gradient of the Coriolis parameter, and parentheses are carefully used to make the result not depend on tensor notation that goes beyond the familiar vector dot product, and the vector differentiation operator $\boldsymbol{\nabla} = \mathbf{i}\,\partial/\partial x() + \mathbf{j}\,\partial/\partial y() + \mathbf{k}\,\partial/\partial z()$.

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[n] The vorticity films at http://web.mit.edu/hml/ncfmf.html have excellent illustrations.

Hungrily solving for π, using the symbolic inverse of $\boldsymbol{\nabla}^2$,

$$\pi = \boldsymbol{\nabla}^{-2}[-\boldsymbol{\nabla}\cdot[(\boldsymbol{V}\cdot\boldsymbol{\nabla})\boldsymbol{V}] + f\zeta - u\beta]$$

This inverse-Laplacian pseudo-operator $\boldsymbol{\nabla}^{-2}$ is two anti-derivatives. These two integrals imply two undetermined *constants of integration*. The curvature or second spatial derivative $\boldsymbol{\nabla}^2$ of a function remains the same despite arbitrary offsets in both the first derivative (slope or gradient) and the absolute value of that function. For instance, a rigid bowl of a given curvature can be held up at any height or tilt.

Continuity is the Law expressed by this equation for pressure. To maintain the assumed $\rho=\rho_0$ forever (that is, so that $\rho_t=0$), the divergence of the PGF must intimately cancel the divergence of each and every other field of force (terms in the square brackets). Those divergent forces are (i) divergence of the "inertial force" implied by the advection of momentum in the inner square brackets, (ii) the inward- or outward-directed Coriolis force on cyclonic or anticyclonic vorticity ζ, and (iii) the divergent Coriolis force on zonal flow u when the Coriolis parameter varies with latitude. Other forces like viscosity could be easily added. The surface value of π, and the rate of its linear decrease with height for this constant-ρ fluid, are set by nonlocal facts (the weight of the fluid body). But actually, that *hydrostatic* component of pressure is also doing the same job[o] of maintaining constancy of density: it prevents gravity from collapsing all the mass into an infinitely dense puddle at the surface.

Pressure is the cop enforcing the law of mass continuity. Its baton is the momentum equation, $F = ma$. From this standpoint, the common exercise in dynamics courses of calculating flows with the pressure field taken as a given is not very sensible: pressure is a cleanup force, the *last* force *logically*, one that adjusts to intimately respond to the divergence of all the other forces, almost instantly (strictly, at the speed of sound).

*Continuity is the Law, pressure is the cop, and **F** = m**a** is its baton.*

[o] This is *teleology,* the explanation of things in terms of the purpose they fulfil.