Part I: Essentials of the fundamentals

Chapter 1

Keeping track of stuff in space

1.1 Units for space, time, and stuff

Before understanding must come labeling and accounting. Mass and momentum and energy (“stuff”) are conserved by closed systems, but we care about different places in the world, so we must account for transport. We also care about different *categories* of stuff (air vs. water mass, heat vs. chemical energy, motions of different scales) so we must account for exchanges between categories. Finally, there are a few physical laws that must be expressed in these "budget" terms.

We measure space, time, and mass in *Systeme Internationale* (SI) units based originally on our Ten fingers, Earth, and Water.[a] Space is in *meters* (m), devised as the Earth’s equator to pole distance divided by 10^7 to be human scale. A *kilogram* is the mass of a cubic meter of water (1 *metric ton*), divided by 10^3 to again match our bodies and commerce (that volume of water is also called the *liter*). The Earth gives us *days* so these could be divided by 10 or 100 in a rational world. Clocks with decimal face labels were manufactured in France, but never caught on. After all, the Sixes of tradition are also rooted in Earthly numerology (almost-12 months and almost-360 days in a year), so SI retained the *second*: a day divided by

[a] Replacing royal “feet” etc. as part of the French Revolution’s anti-aristocracy project.

86400 (60 x 60 x 24). Latitude in degrees (10^7m / 90º = 111.111 km per degree), and subdivisions like nautical miles (1/60 degree), also carry this Six-related history.

Temperature's Celsius scale is also about water and Tens: 1C = 1K = (boiling point minus freezing point)/100. This is not quite as fundamental, since it depends on surface pressure. It is a remarkable coincidence that the weight of a 1 m^2 column Earth's atmosphere (1 Bar of pressure) happens to be so near the weight of a 10 meter column of water, making surface pressure nearly 1000 (really 1013 in a global mean) in the SI unit of hectoPascals (hPa).

Precision science has quietly replaced the original Earth and Water motivations with more fundamental quantum profundities at the root of SI, without most people even noticing (the kilogram was redefined in 2019). But our 10 fingers still rule numbering in all but the messy time domain.

1.2 Conservation of the most fundamental "stuff": mass

In a given volume of space (let's say 1 m^3 for definiteness), the enclosed mass is customarily labeled ρ, a *mass density* with an inverse (*specific volume*) $\alpha=1/\rho$. The rate of change of that mass depends on the net inflow of mass into the volume, plus any physical sources (zero in the case of mass). The flow of mass through a 2D boundary into a 3D volume is measured by a *flux*, whose units [kg m^{-2} s^{-1}] embody its meaning better than any words can elaborate.

Mass flux is $\rho\mathbf{V}$ in symbols[b], using bold face for vector velocity $\mathbf{V} = \mathbf{i}u + \mathbf{j}v + \mathbf{k}w$ with the unit vectors **i, j, k** for a Cartesian (xyz) coordinate system. Readers should verify that the units of $\rho\mathbf{V}$ are indeed a flux [(kg m^{-3}) (m s^{-1}) = kg m^{-2} s^{-1}]. Notice with the same units-mind that velocity **V** is a *volume flux;* and is also a *specific momentum* (where *specific* means *per unit mass*). A feeling for different interpretations of the same quantity

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[b] See Table of symbols and notation at end of book for definitions and equation sets.

is essential to fully appreciating the equations of convection: you need a firm physical grip on the mathematical symbols, but not too tight or exclusive.

A *net inflow* to a volume is called *convergence of flux*, the negative of *divergence*[c]. Translating change = convergence into math, with subscripts denoting partial derivatives along the axes in xyzt space,

$$\begin{aligned} \rho_t &= -div(\rho\vec{\mathbf{V}}) = -\vec{\boldsymbol{\nabla}}\cdot(\rho\vec{\mathbf{V}}) \\ &= -(\rho u)_x - (\rho v)_y - (\rho w)_z + mass\ sources \\ &= -(\rho u)_x - (\rho v)_y - (\rho w)_z + 0 \qquad (1.1) \end{aligned}$$

1.2.1 *Aside on mathematical expression culture*

Notice that Equation (1.1) has an ambitious structure:

$$something\ I\ want\ =\ work\ I\ must\ do\ to\ get\ it$$

In this case, we want to know the future, the point of a *prognostic equation*. We call this a *grasping form*, in contrast to a *majestic equality* of the form 0 = (all the terms). Mathematically, these are isomorphic: not different in any meaningful way except the form. But in seeking our top virtue of *appreciation*, or even science's touchstone of *prediction*[d], special brain hardware can be engaged through the “brainfeel”[e] of equations. To feel a difference, transform (1.1) into the equivalent majestic equality:

$$0 = (\rho u)_x + (\rho v)_y + (\rho w)_z + (\rho\dot{t})_t \qquad (1.2)$$

In the first 3 terms on the RHS, velocities in space (u = $\dot{x}$, v = $\dot{y}$, w = $\dot{z}$ using Newton's dot notation for univariate time derivatives) measure the futile-looking (zero-equated) journey of moving matter, in units of meters traversed per second of elapsed time. Here an analogous quantity $\dot{t}$

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[c] Early meteorology texts simply used the word *vergence*, letting the sign reflect its sense.

[d] *Prediction* perhaps in a logical sense (if X then Y) rather than a temporal forecast sense.

[e] This term is motivated by "mouthfeel" as a food descriptor distinct from taste or nutrition.

measures the journey of matter through time, in seconds traversed or endured per second of elapsed time. Since both are seconds, $\dot{t}$ is unitless, and in fact is unity (neglecting relativistic effects). But was no appreciation gained by crystallizing that invisible $\dot{t}$ =1 in (1.2) compared to (1.1)? Your brainfeel may vary, but this book (and many others) may better satisfy readers attuned to appreciate a difference.

1.3 Conservation of *specific (per unit mass)* other stuff:

1.3.1 *Specific momentum and its physical source terms*

Having established (1.1), other budgets follow straightforwardly by expressing conserved quantities on a *specific* or *per unit mass* basis. To get the budget of momentum, simply multiply the mass flux by specific momentum, which (as noticed above) is velocity. In the **k** direction, along which air's coordinate position change $\dot{z} = w$ (vertical velocity),

$$\begin{aligned}(w\rho)_t &= -\vec{\boldsymbol{\nabla}} \cdot \left(w\rho\vec{\mathbf{V}}\right) \\ &= -(w\rho u)_x - (w\rho v)_y - (w\rho w)_z + w\ sources \quad (1.3)\end{aligned}$$

The difference from (1.1)-(1.2) is nonzero *sources* on the right. Newton taught us that momentum sources are called “forces” **F** = m**a**, earning the honorific SI unit for force (N = kg m s^{-2}).[f] Acceleration could therefore be called *specific force,* although we don't think of acceleration as "stuff" to be transported like momentum is.

Two real forces that act on air are needed to appreciate convection:

(1) Gravity $-\boldsymbol{k}\rho g$

(2) The pressure gradient force $-\boldsymbol{\nabla} p$

What is this *pressure* p? Physics class taught me it is *force per unit area,* with units N/m^2 =Pa (Pascals), but it can be viewed as a *momentum flux*

[f] Is force a flux of some definable “stuff”? Not really: N = (kg m^3 s) m^{-2} s^{-1}, so the unit of "stuff" being transported (kg m^3 s) has no intuitive sense, to this author’s brainfeel.

[with units of (stuff) $m^{-2}s^{-1}$ = (kg m/s) $m^{-2}s^{-1}$]. However it is a strange directionless or all-directional flux, a scalar field not a vector field, so the *net* momentum flux into a cube (the flux difference from one side to the other) cannot be a convergence of a vector field, but rather is a gradient.

In a rotating coordinate system, where "motionless" air is nonetheless accelerating, we must add a fictitious third "fictitious" force per unit mass $2\boldsymbol{\Omega}$ **x V** where $\boldsymbol{\Omega}$ is the coordinate rotation vector on the RHS of (1.3), as derived in every dynamical meteorology book. As is customary, let's retain only the horizontal Coriolis force in our Earth-tangent Cartesian plane xyzt coordinates, burying tiny vertical fictitious forces in the enormous g of the "gravity" (rather than strictly gravitational) force $-\boldsymbol{k}\rho g$. Phenomena that depend on the flow-dependent vertical Coriolis force are not in our scope.

Gathering our considerations for every cubic meter of space, we are up to 4 equations in 5 unknowns (u,v,w,p,ρ):

$$\begin{aligned}
\rho_t &= -(\rho u)_x - (\rho v)_y - (\rho w)_z \quad + 0 && \text{(1.4a)}\\
(\rho u)_t &= -(u\rho u)_x - (v\rho u)_y - (w\rho u)_z \quad -p_x + \rho f v && \text{(1.4b)}\\
(\rho v)_t &= -(u\rho v)_x - (v\rho v)_y - (w\rho v)_z \quad -p_y - \rho f u && \text{(1.4c)}\\
(\rho w)_t &= -(u\rho w)_x - (v\rho w)_y - (w\rho w)_z \quad -p_z - \rho g && \text{(1.4d)}
\end{aligned}$$

The next step, specifying ρ, is called an *equation of state.* One key insight into the nature of pressure can already be learned from the simplest case, ρ = constant (Problem 1.1). Readers should understand the resulting lesson well: *Continuity is the Law, Pressure is the Enforcer, and **F**=m**a** is its enforcement mechanism*.

Without density variations, mighty gravity is indiscriminate in (1.4d), and there can be no *convection* in the sense of our title. We simply must allow ρ to vary a little so that the force $-\boldsymbol{k}\rho g$ can do work, which drags us into thermodynamics (Chapter 2). Notice that each p in (1.4 b-d) could be moved inside the parentheses of the flux terms: *pressure is another kind of momentum flux*, with units (kg m/s) m^{-2} s^{-1}. Pressure's momentum flux is transmitted across the box's border by elastic collisions of the molecules outside the border with those inside it. This is new physics, not just more bookkeeping, *different in character* from the process of pieces of matter carrying their properties bodily as they cross the border.

1.3.1.1 *A glimpse of viscosity*

Border crossing by molecules is merely the smallest-scale category of bodily flux terms like $(u\rho w)$. Molecule-borne flux can be treated as a *diffusion* process: the convergence of a flux of stuff that is proportional to the gradient of that stuff (revisited in section 3.1). Notice that the calculus concept of a gradient is only well defined on a spatial scale long enough for the continuum approximation to be valid (smoothing over the lumpy molecular nature of matter). This diffusive treatment of molecular momentum flux ($\nu\mathbf{\nabla}^2 w$ in a simple case), is called the *viscosity force* if moved to the “sources” end of (1.3). Viscosity is just an interpretation of the smallest-scale part of the fully general stuff-weighted mass flux across borders $(u\rho w)$, as discussed further in chapter 3.

1.3.2 *Other specific stuff: humidity and 'heat content'*

To close our four-equation but five-variable set (1.4) we need an equation of state for ρ. *Specific momentum* as a name for velocity made momentum budget construction (1.4 b-d) very direct, but no budget trick invoking 'specific density' can help us[g]: we truly need another physical law.

The ideal gas law is at hand, but that brings in temperature T. Since T is a measure of "heat", we desire some kind of a specific heat "stuff" with which to construct our thermal energy budget equation. Unfortunately, formal thermodynamics forbids us such a concept, even though everyone thinks that way; we will need a whole half-chapter describing *entropy* and then approximate our way back to a good-enough thermodynamics (section 2.1). Fortunately, we will find that entropy has an information

[g] *Specific volume* $\alpha = 1/\rho$ is sometimes invoked for verbal convenience, but to think of volume as being somehow carried along with mass as it moves in and out of a volume of space is one step too far in brainfeel.

interpretation that is useful (at least conceptually) in Part III, as Chapter 0 has intimated already.

For cloudy convection, we also need to keep track of water mass, which we can measure as a *specific humidity* q, and you know what to do: replace w with q in (1.3) and specify the physics in the *sources* terms. Categorized water budgets are routine extensions (specific cloud water, specific rain water, etc.). Before dismissing that exercise as too trivial to spell out, however, one more bookkeeping issue needs to be confronted.

1.3.3 *Specific X, or mass mixing ratio of X?*

For water vapor, *specific humidity* q_v seems a convenient variable – or should we use *water vapor mixing ratio* r_v? Both have water vapor mass in the numerator, but mass mixing ratio has the mass of *dry air* rather than *total mass* in the denominator[h]. To get more precise, one must decide if ρ in (1.1) stands for dry air density or for total mass. It seems tempting to use ρ for *total mass*, but then the velocity w is hard to define: in rain, it would have to be a mass-weighted mean of the wind velocity of the gases and the fall velocity of particles. The complications explode. Alternately, the third step in (1.1) stating *mass sources = 0*, which is truer for dry air mass, could be revised for raindrops falling out. The complications explode differently. Perhaps the above should be edited to replace "specific" with "dry air mass mixing ratio of". Wouldn't that be fun prose to slog through!

Terms of art and their mathematical cousins (symbols) are often thought of as hyper-specific. But their greatest power can actually lie in a *refusal to draw distinctions* that are inessential to a line of discourse[i]. Our plain symbol ρ papers over these complications *intentionally*, for clarity. The slight distinction between q and r measures of water mass is one of several we shall consciously elide, usually noting the elision for honesty[j].

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[h] We have avoided the trace chemical term *concentration* for its ambiguity.

[i] A favorite example is the term *hydrometeor* for any falling condensed water object.

[j] Sometimes only in terse or cryptic sentence fragment footnotes.

If your goal is to mathematically frame a numerical model that will be integrated over time, requiring your equation set to obey large-scale conservation laws punctiliously even as the various “stuff” of physics is shuttled among many small spatial boxes and categories of form, then all decisions must be strictly defined and adhered to. Symbols, subscripts, labels and small terms will proliferate. Get a longer book: those exist already. If such a model must assimilate or quantitatively confront absolutely calibrated observations, it must also use *accurate as well as precisely conservative* versions of thermodynamics, coordinates, and conservation laws. I hope having this book's essentials in mind may help keep you oriented as your education delves into all that (chapter 10).

1.3.4 *Advection and the material derivative*

Notice (Problem 1.1) that you can distribute the derivatives in (1.4b), divide by ρ, and use (1.4a) to rewrite it without approximation as:

$$u_t = -uu_x - vu_y - wu_z - \alpha p_x + fv \quad (1.5)$$

Defining the *total derivative* $du/dt = u_t + \boldsymbol{V} \cdot \boldsymbol{\nabla} u = u_t + uu_x + vu_y + wu_z$, the set (1.4) becomes:

$$\begin{aligned} d\rho/dt &= -\rho \boldsymbol{\nabla} \cdot \boldsymbol{V} && (1.6a) \\ du/dt &= -\alpha p_x + fv && (1.6b) \\ dv/dt &= -\alpha p_y - fu && (1.6c) \\ dw/dt &= -\alpha p_z - g && (1.6d) \end{aligned}$$

This *advective* form of transport terms in the total derivative is enormously general, and tempts the interpretation of advection as being just as real as flux and its divergence: *Look upwind*[k]*! Those conditions are coming toward you. Conditions at your location will soon be like that, except to the extent that RHS source term effects intervene*. That is advection's logic.

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[k] This asymmetry is false, except in Itô vs. Stratonovich stochastic calculus.

In Chapter 2 we will need thermodynamic laws that were learned from trapping a kilogram of air in a piston in a laboratory. To use these laws in our fluid equations, we need to equate laboratory *time derivatives referring to a unit mass of air* (denoted with the univariate notation $\dot{T}$) to the *total derivative* dT/dt as defined above. One way to see this equivalence is to notice that if we truly had the field T(x,y,z,t) – temperature everywhere forever – we could extract the temperature history of a moving parcel of unit mass as $T_p(x_p(t), y_p(t), z_p(t), t)$. We can use the chain rule to extract all the time change:

$$dT_p/dt = \partial T/\partial t + \partial T/\partial x \cdot \dot{x_p} + \partial T/\partial y \cdot \dot{y_p} + \partial T/\partial z \cdot \dot{z_p}$$

and equate it to $\dot{T}$, the time-only changes in the chamber.

1.4 Now about density… problems

1.4.1 *Show the steps from 1.4 (flux) to 1.5 (advection) forms of the budget equations.*

1.4.2 *Set density to a constant ρ_0 and simplify the set (1.4) maximally in that case.*

What phenomena could this *incompressible fluid* equation set describe? In other words, how can an incompressible fluid move, and why would it? What could drive motion, at what scales, and how could that motion decay?

What is the speed of compression (sound) waves in such a fluid? That is, if boundary conditions jiggle one edge of an incompressible body of fluid, how soon is fluid motion transmitted to the other side?

1.4.3 *Using the simplified set from 1.4.2, transform the majestic equality form of the mass continuity equation into a grasping equation for pressure of the form*

What I want (pressure) = Work I must do to construct it

<u>Hint</u>: differentiate mass conservation in time, momentum conservation equations in space, and substitute the latter into the former. Subscripts for partial differentiation (as in the text and Table of Symbols and Notation) will save many redundant hand motions in this process. You may use the symbolic inverse $\boldsymbol{\nabla}^{-2}$of the Laplacian operator $\boldsymbol{\nabla}^{2}$ in the final answer, even though solving it is a nontrivial job (Chapter 3).

Interpret the result in your own words, elaborating on this summary: *Continuity is the Law, Pressure is the Enforcer,* ***F*** *= m**a** is its Mechanism.*

1.4.4 *Using the simplified set from 1.4.2 with $\rho = \rho_0$ again, show that the flux convergence terms can be wrangled into advection form. In other words, show from Eqs. (1.4) that the flux divergence terms can be rewritten as $-u\vec{V}_x - v\vec{V}_y - w\vec{V}_z = -(\vec{V} \cdot \vec{\nabla})\vec{V}$*

1.5 Solutions

1.5.1 *Show the steps from flux to advective form*

(left as an exercise)

1.5.2 *Set density to constant ρ_0 and simplify the set (1.4).*

Substituting ρ=ρ_0. dividing by ρ_0, and negating the sign in 1.4a yields

$$
\begin{aligned}
0 &= u_x + v_y + w_z \\
u_t &= -(uu)_x - (uv)_y - (uw)_z \quad -\pi_x + fv \\
v_t &= -(vu)_x - (vv)_y - (vw)_z \quad -\pi_y - fu
\end{aligned}
$$

$$w_t = -(wu)_x - (wv)_y - (ww)_z - \pi_z - g$$

where we have introduced the pressure variable $\pi = p/\rho_0$.

What phenomena could this incompressible fluid equation set describe? If a body of such fluid were initially at rest, the only forces that could drive flow within it are pressure or viscous forces (molecular momentum flux in the parenthetical terms) applied at a boundary. Gravity is powerless without density variations, so nothing worth the name "convection" can occur. Pressure would drive divergent flows, like the motions inside a water balloon when massaged. Viscous forces could create shears, which could become shear instabilities, leading to fluid motions that would flux[1] momentum deeper into the fluid. Eventually the fluid could contain all sorts of turbulent motions. The energy of such motions could decay into heat (molecular motions) by diffusion internally.

What is the speed of compression (sound) waves in such a medium? Infinite, since $\rho_t = 0$ makes the medium infinitely stiff.

1.5.3 *2. Using this simplified set, transform the majestic equality form of the mass continuity equation into a grasping equation for pressure*

Differentiating the u equation in x, the v equation in y, and the w equation in z, and summing them,

$$\begin{aligned} [u_{tx} &= -(uu)_{xx} - (uv)_{yx} - (uw)_{zx} \quad -\pi_{xx} + f v_x] \\ +[v_{ty} &= -(vu)_{xy} - (vv)_{yy} - (vw)_{zy} \quad -\pi_{yy} - f u_y - u f_y] \\ +[w_{tz} &= -(wu)_{xz} - (wv)_{yz} - (ww)_{zz} \quad -\pi_{zz} - g] \end{aligned}$$

we obtain (using mass continuity to see that the left side is zero):

$$0 = -\boldsymbol{\nabla} \cdot [(\boldsymbol{V} \cdot \boldsymbol{\nabla})\boldsymbol{V}] \quad -\boldsymbol{\nabla}^2 \pi \quad + f\zeta - u\beta$$

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[1] Here the noun "flux" has been verbified, useful for reasoning sometimes.

where z is the vertical component of relative vorticity $\zeta = v_y - u_x$, β is f_y, and parentheses are carefully used to make the result depend on no notation beyond the familiar vector dot product and the vector differentiation operator $\boldsymbol{\nabla} = \mathbf{i}\,\partial/\partial x + \mathbf{j}\,\partial/\partial y + \mathbf{k}\,\partial/\partial z$.

Hungrily solving for π,

$$\pi = \boldsymbol{\nabla}^{-2}[-\boldsymbol{\nabla}\cdot[(\boldsymbol{V}\cdot\boldsymbol{\nabla})\boldsymbol{V}] + f\zeta - u\beta]$$

Continuity is the Law expressed by the equation for pressure. Since flow divergence is zero to maintain the (assumed) $\rho=\rho_0$ forever ($\rho_t=0$), pressure must intimately cancel the divergent component of any other field of force (in this case only the inward- or outward-directed Coriolis force on horizontally swirling flow, or the divergent Coriolis force on zonal flow on a sphere, or the "inertial force" implied by the transport of momentum in the inner square brackets). *Pressure is the Enforcer* of mass continuity. Pressure does its job[m] in the momentum equations, so *F = ma is the Mechanism* of that enforcement. The common exercise in dynamics courses of calculating flows with the pressure field taken as a given is not very sensible: pressure is a cleanup force, the *last* force logically, the one that intimately responds to the divergence of all the others at the speed of sound. Here that speed is infinite, which is unrealistic, but it is still faster than all the other information-transmitting waves in more realistic fluids, so the point remains relevant: *Continuity is the Law, Pressure is the Enforcer,* ***F*** *= m**a** is the enforcement mechanism.*

1.5.4 *3. Using your simplified set with $\rho = \rho_0$ again, show that the flux convergence terms can be wrangled into advection form. In other words, show from Eqs. (1.4) that the flux divergence terms can be rewritten as $-u\boldsymbol{V}_x - v\boldsymbol{V}_y - w\boldsymbol{V}_z = -(\boldsymbol{V}\cdot\boldsymbol{\nabla})\boldsymbol{V}$.*

Taking just the u component for illustration,

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[m] *Teleology,* the explanation of things by the purpose they fulfil, will come up again.

$$-(uu)_x - (uv)_y - (uw)_z =$$
$$-uu_x - vu_y - wu_z - u(u_x + v_y + w_z)$$

The parenthetical term is zero by mass continuity, so *momentum flux convergence is precisely equal to the advection of momentum in an incompressible fluid.*