

Chapter 2

Good enough equations

Budgets in boxes (Chapter 1) got us the mass continuity and momentum equations (1.4 and 1.6), with pressure and gravity as the sources (forces). The molecular part of flux convergence is soon to be treated as a diffusion and moved to the right-hand side as another force. Coriolis forces, although sometimes called "fictitious", are as real as the ground is stationary on our rotating planet. The important distinction among forces is not real vs. fictitious, but primary vs. *adaptive*.

Problem 1.4.2 showed that pressure is the adaptive one, adjusting itself instantly (in the incompressible fluid), or really at the speed of sound, to negate the divergence of all of the other forces. Negating the divergence of gravity gives p a *hydrostatic* component that keeps air from collapsing into a puddle. Negating divergent Coriolis forces gives p a *geostrophic* component. Negating *buoyancy* (defined below) gives a component important for cumulus dynamics, while *dynamic pressure* arises to negate the divergence of the *inertia (or inertial forces)* of moving air. All of that left us with 4 equations for 5 unknowns (u, v, w, p, ρ), and a teleological sense of what pressure is. To close the set, we need equations that define density, allowing it to vary so that gravity can drive convection.

2.1 Good-enough thermodynamics of moist air

2.1.1 Density and the ideal gas law

To close the set, we need an equation predicting the density of air. The ideal gas law springs to mind: $\rho = I/\alpha = p/(RT)$, but that doesn't solve the problem. For one thing, it introduces a new unknown (RT), so now we will need additional equations. Also it is nonlinear (a quotient) so taking a time derivative to get the needed prognostic equation yields 2 terms. Using Newton's dot notation again, for 1 kg of ideal gas trapped in a laboratory chamber, our grasping form is:

$$\dot{\rho} = (\dot{p} - \dot{T}/T)(RT)^{-1} \quad (2.1a)$$

or perhaps more usefully we can grasp at T :

$$R\dot{T} = p\dot{\alpha} + \alpha\dot{p} \quad (2.1b)$$

Here some complications must begin, but let's minimize them.

2.1.2 Virtual temperature, density temperature

The *gas constant* R , which quantifies the partition of energy between the kinetic energy of molecules (proportional to T) and the *elastic energy* $p\alpha$, is defined on a *per molecule or mole* basis, not a *per unit mass* basis. Since H_2O (molecular weight 18 kg/mole) is lighter than Earth's mixture of N_2 and O_2 ("dry air"; average molecular weight 29), R for moist air depends on humidity. It's a slight effect for Earthly humidity values, so meteorology's custom to hide the complication is to use R to denote the truly constant dry-air gas constant R_d , and then elaborate temperature with the subscript v for *virtual temperature* $T_v = T(1 + 0.602 q)^a$, the

^a Derivation is in Wikipedia or many textbooks.

temperature at which dry air would have the same bulk density as the actual *moist air*^b we are talking about.

2.1.3 First Law: internal energy and the quest for warmth

We know that density depends on *warmth*, a kind of energy, so an energy budget or conservation law (the *First Law of Thermodynamics*) is the route to a T equation for our trapped kilogram of air in an insulated piston. But it can still export energy, by doing one kind of *work*: expanding its volume V against the pressure force, as expressed by the term $pdV/dt = p\dot{a}$. Since internal plus external is complete, conservation of added energy for a trapped gas with a controllable (perhaps electric) heater must therefore take the conceptual form:

$$\text{heating rate } Q = d/dt(\text{internal energy}) + \text{work air does}$$

Internal energy is defined by the above conservation statement, and expressed using the *heat capacity at constant volume* C_v :

$$Q = C_v \dot{T} + p\dot{a} \quad (2.2)$$

Hooray for laboratory^c science, but our brainfeel wants a grasping equation for \dot{T} with a single-term RHS, so we desire (2.2) to become:

$$\text{heating} = d/dt(\text{"some stuff called warmth involving T"})$$

Substituting (2.1) into (2.2) doesn't help -- still there are 2 RHS terms:

$$Q = (R + C_v)\dot{T} - \alpha\dot{p} = C_p\dot{T} - \alpha\dot{p} \quad (2.3)$$

^b Inclusion of the mass of suspended particles is sometimes telegraphed by using *density temperature* T_ρ , but anyone who bothers to worry about T_v obviously wants total density.

^c Knowing the structure of molecules (diatomic, triatomic) allows us to estimate from first principles how many internal motions or "degrees of freedom" they have to partition internal energy among, and thus C_v , but (2.2) is really its definition.

We need a trick. Two approaches suggest themselves.

Trick 1: Divide by temperature. Why? Because it works:

$$Q/T = C_p \dot{T}/T - \alpha \dot{p}/T = C_p \dot{T}/T - R \dot{p}/p$$

Since the coefficients are constant (C_p and R), they commute through the time derivative operator. From pure calculus, $T^l dT/dt = d/dt(\ln T)$, so we may write a conservation law in our desired form involving $d/dt(\text{stuff})$, but only for confusing stuff called *specific entropy* S :

$$\begin{aligned} Q/T = \dot{S} &= d/dt(C_p \ln T - R \ln p) \\ &= d/dt(C_p \ln \theta) \end{aligned} \quad (2.4)$$

What is specific entropy? Do the units ($\text{J kg}^{-1} \text{K}^{-1}$) satisfy any intuition? No, because "degrees" of temperature are not fundamental (Ch. 1). What does it mean to divide a heating rate (energy input rate) by T in degrees?

Deeper understanding is available in statistical mechanics (Ben-Naim 2013): entropy is the *missing information* in our bulk thermodynamic description $\{p, T, \rho\}$ of the situation: specifically, it is the log of the *number of redundant microscopic states* that would yield the same bulk thermodynamic state, measured in bits if the log is base 2 (see chapter 0 and study its Problem at the end). The free manifesto chapter 1 of Ben-Naim (2013) shows that if the vigor of molecular motions were expressed in energy units, instead of arbitrary equal ticks along a tube of mercury (degrees), entropy would be a dimensionless quantity, as a count of possible situations (and its logarithm) should be. But culture is hard to change, history is impossible to rewrite: thermal expansion was noticed and measured long before the secret molecular nature of matter and energy could be inferred.

Meteorology has invested heavily in *potential temperatures* like $\theta = T(p_0/p)^{R/C_p}$ in (2.4), expressing *the temperature a parcel of air would have after it undergoes various hypothetical processes* (in this case, adiabatic compression to a reference pressure p_0). Generations of

students from the pre-computer age learned to navigate the mild nonlinearities of Earthly moist thermodynamics by following gently curved lines (such as *adiabats* with constant S) on large paper *aerological diagrams* labeled with the axes of thermo's laboratory observables: pressure and temperature. An entropy version (the *Tephigram*, shorthand for T - ϕ since ϕ was used for entropy) has its pressure axes woozily not-quite parallel to the paper edges, and lost the textbook-mediated multidecadal popularity contest to pseudoadiabatic or skewed- T^d by log- p diagrams (MetEd^e). Despite the end of oversized-paper based science, the easy availability of computers for evaluating mildly nonlinear functions, and the rise of tropical meteorology with its emphasis on humidity for moist convection (section 4.1 below), these anachronistic treatments are in the literature, so students must know them (including gaining a feel for their values), so they are still taught.

Trick 2: Define a new labeling function for pressure, the *hypsometric height* $Z_h(p)$ whose increments satisfy the hydrostatic force balance relationship $\alpha dp = g dZ_h$. Why? Because it works: unlike α , g is time independent so it commutes through the time derivative in (2.3) like in Trick 1, giving our desired form, d/dt ("some warmth stuff") = heating:

$$Q = \dot{s} = d/dt(C_p T + g Z_h) \quad (2.5a)$$

This s is called *dry static energy*. It has the same useful property of potential temperature θ for reasoning: it is conserved during vertical motions of an unsaturated parcel, which makes it uniform in any well-mixed layer. Converting that logical statement into the *dry adiabatic lapse rate of temperature* $\Gamma_d = -g/C_p = -10$ K/km is far easier (taking $dZ_h/dz = 1$, an excellent approximation) than using the constancy of θ . Dividing \dot{s} by C_p to get a warming rate at constant pressure is straightforward, as compared to $\dot{\theta}$ which despite units of K/s must be

^d Not statistical skew (chapter 0.2), merely a non-orthogonal axis that reduces *one* of the diagram's design failures to map importance of distinctions onto feature size on paper.

^e Free course materials at https://www.meted.ucar.edu/training_module.php?id=225

converted using the nonconstant profile factor (T/θ) to get a \dot{T} for density-change purposes, our real goal.

Back to that larger quest. We can easily recast (2.5) as a grasping equation for the future of T , ready to append to the set (1.6):

$$\begin{aligned} dT/dt &= Q/C_p - g/C_p(Z_h) \\ &= Q/C_p - g/C_p(Z_{ht} + \mathbf{V} \cdot \nabla Z_h) \end{aligned} \quad (2.5b)$$

But wait! We indulged in some sleight of hand in Trick 2. For Z_h to depend only on p , so that its increment has only one term $gdZ_h = \alpha dp$, α must also depend only on p . In other words, we tacitly assumed that ρ is approximated by a reference density profile that is a function only of altitude.

This approximation is sufficient for all this book's further purposes, and will be kept, but we backed into it. It eliminates all but vertical advection from the advection term in (2.5b). But before we say goodbye to that horizontal "advection" term, notice that $-\mathbf{V} \cdot \nabla(gZ_h)$ can be interpreted as *kinetic energy gained by flow down the geopotential gradient* (problem 2.1). Later we will also see that the vertical advection term $gw(Z_h)_z$ has a similar term due to vertical flow down the gradient of nonhydrostatic pressure. *Neglecting kinetic energy is the key simplification of a static energy treatment (Betts 1979)^f*. Fortunately, the kinetic energy of wind is far less than that of molecular pandemonium (thermal energy), so the simplification gained by Trick 2 is well worth its cost in accuracy. And turning good-enough (2.5) into a prognostic equation for the *thermal buoyancy force* b will soon require a reference density profile anyway (section 2.2), so it is just a head start.

In the shorthand for convective dynamics implied by one-dimensional soundings analysis (section 4.1), a balloon-measured ρ profile is used as the reference to sensibly define $Z_h=Z=z$. The assumptions in that case are numerous and conceptual: an oversimplified momentum equation

^f Relatedly, MSE minus CAPE is the true lifted-parcel conserved variable (Romps 2015).

(buoyancy only) for hypothetical pure lifted "parcels" within an infinitely broad "environmental" airmass whose conditions the balloon's tiny barometer, thermometer, and hygrometer are supposed to have measured. Neglect of kinetic energy is far from the most dubious of these.

2.1.4 Latent vs. 'diabatic' heating and moist adiabaticity

Equation (2.5) says that s is *conserved for adiabatic flow* ($Q=0$) even when air changes pressure by moving vertically. If we take the *latent heat of vaporization* L as constant^g, the part of Q due to condensation minus evaporation $L(c-e) = L dq_v/dt = -L dq_v/dt$ can be moved to the right-hand side, which barely counts as a third Trick:

$$Q_d = \dot{h} = d/dt(C_p T + gZ + Lq_v) \quad (2.6)$$

This h is called *moist static energy*, interpreted very easily as the sum of a warmth measure called *enthalpy* plus geopotential energy (equating Z to Z_h , the *hydrostatic* approximation) plus latent energy. Equation (2.6) says that h is *conserved for* $Q_d=0$, even when air changes pressure by moving vertically *with or without phase change*. Such flow is sometimes called *moist adiabatic*, leaving Q_d to represent the genuinely 'diabatic' heating rates like radiation, heat conduction from the surface, and viscous dissipation of wind kinetic energy^h.

The $T(p)$ or $T(Z_h(p))$ profile that results from such flow is called a *moist adiabat*, with the fine distinction of a *reversible adiabat* vs. *pseudoadiabat* to express whether the condensed water is carried along as thermal inertia in C_p , keeping the cooling parcel slightly less cold, or is instantly precipitated out. Obviously natural cloud updrafts lie somewhere in that range, depending on highly situational microphysics. We will later meet a more teleological or functional definition: the

^g It varies with T , compensated by the heat capacity of condensate in the proof that a perpetual-motion heat engine is impossible. Neglecting both is a good approximation.

^h Also "apparent" heating or moistening by small scale eddy flux convergence, if " d/dt " in (2.6) has been redefined to include apparent advection by large scales only (chapter 3).

neutral stability profile toward which real moist convection adjusts the atmosphere.

2.1.5 *Static energy vs. entropy vs. potential temperatures*

Entropy (Trick 1 above) was derived without approximation. It connects to the rest of thermophysics and can be extended to very different gases on strange planets, but makes some useful reasoning tactics hard to see. For one thing, it undermines our units sense, although that can be seen past (Ben-Naim 2013). To avoid truly meaningless units like $\ln(\text{K})$ one divides by a reference T, p , so only increments from that arbitrary state have meaning. More costly is the nonlinearity of the $\ln()$ function and the $\exp()$ function that moist potential temperature formulas bristle with. The small range of Earthly states allow a linearization, as the adequacy of $h = C_p T + gZ + Lq$ (warmth-energy plus height-energy plus latent heat) makes obvious on its face, a lucidity that Taylor expansions and discarding higher-order terms can't begin to replicate. While entropy budgets and the Second Law are real, and have their profundities (e.g. Pauluis and Held 2002, Romps 2008) and some surprisesⁱ, in practice, most scientists really think with the brainfeel of energy conservation.

For soundings analysis (lifted parcel games, section 4.1), a side by side graphical comparison can be easily made between static energy, entropy, and potential temperature diagrams (chapter 4 computer exercises). The resulting almost-identical graphic aside from axis units indicates that *static energy is clearly good enough*. Even for numerical model design, static energy can be used as part of a well-considered *suite* of approximations. At that equation-set level of consideration, having a conserved quantity is the point, not the precise form of that quantity. The popular System for Atmospheric Modeling (SAM, Khairoutdinov and Randall 2003) simulates impressively Earth-like convection and even planetary-scale circulations (Bretherton and Khairoutdinov 2015), using static-energy thermodynamics with one more term for the latent heat of

ⁱ For instance, turbulent dissipation in the wakes of raindrops falling at terminal velocity is the main dissipative source of entropy in the atmosphere (Pauluis et al. 2000).

freezing ("frozen" MSE). Framing inaccuracies are rarely the biggest problem^j, except perhaps for quantitatively assimilating absolutely calibrated ("anchor") observations across very different regimes (polar to tropical).

With good-enough thermodynamics, let's return to our main quest: closing a set with a good-enough density equation for moist convection.

2.2 Good-enough fluid dynamics

Recall that we needed a better density equation to complete our mass and momentum budgets (1.4) and (1.6). Let's return to that project, with the First Law and the assumption of a reference density profile.

2.2.1 Gravity becomes buoyancy, PGF is univariate

Besides lucid static energy thermodynamics, linearization for Earth's modest and gentle convection can also open up powerful intuitive access to fluid dynamics. Begin by expressing small density and pressure fluctuations as differences from a reference profile, $\rho \equiv \bar{\rho}(z) + \rho'$ and $p \equiv \bar{p}(z) + p'$. We already accepted the need for a reference density profile, but here we must choose it. The word *buoyancy*^k and its sign will have the most meaning if we choose chapter 1's $\rho_o = \bar{\rho}$, the horizontally averaged density across the whole domain over which we are building an accounting system for the flow: the whole Earth, or some smaller "airmass" within which we are considering convection.

Recalling pressure's job to enforce mass continuity, \bar{p} is the hydrostatic reference profile that holds up the mean $\bar{\rho}$ profile in the face of the mean gravity force $\bar{\rho}g$ that would otherwise converge air into a puddle. Level by level, $\bar{p}_z = -\bar{\rho}g$, making $\bar{p}(z)$ nearly linear for incompressible liquid

^j A fascinating case study is Williamson et al. (2015, doi:[10.1002/2015MS000448](https://doi.org/10.1002/2015MS000448)), discovering a longstanding error in the energetics of a popular atmosphere model.

^k Ancient: see https://en.wikipedia.org/wiki/Archimedes_principle

like in the *Boussinesq*¹ system, and roughly exponentially decreasing for air. Using hydrostatic pressure $\bar{p}(z)$ as our vertical plotting axis is a way to relate the wonderfully simple Boussinesq equations in xyz coordinates to real atmospheric convection. A necessary constant of integration in defining $\bar{p}(z)$ from $\bar{\rho}(z)$ flows from the *non-exploding planet approximation*: the entire fluid is not vertically accelerating.

Adding this hydrostatic relation $0 = \bar{p}_z/\bar{\rho} + g$ to (1.6d) yields:

$$dw/dt = -p'_z/\bar{\rho} - g\rho'/\bar{\rho} = -\pi_z + \hat{b} \quad (2.7)$$

where we have again used the symbols $\pi = p'/\bar{\rho}$ for a spatially varying *dynamic pressure* variable (related to the elastic energy encountered earlier), and $\hat{b} = g(\rho'/\bar{\rho})$ for the *buoyancy force per unit mass*, an acceleration like g as the parentheses emphasize. Expressing the pressure gradient force (PGF) as the gradient of a single scalar π like this makes it vanish from vorticity equations (Problem 2.4.2). Likewise, horizontal derivatives of single scalar \hat{b} retain the meteorologically important effect of the untidy and distracting *baroclinic*^m term $\nabla\rho \times \nabla p$ in unapproximated fluid dynamics.

Buoyancy \hat{b} can further be approximated as *thermal buoyancy* $b = gT'_v/\bar{T}_v \cong gT'/\bar{T}$. The use of virtual temperature here helps emphasize that *density* is the true reference state and essential perturbation, but for notational simplicity the v subscript is often left off and for many purposes virtual effects can be neglected. Thermal buoyancy neglects some untidy product-of-primes terms, and also the slight effect of p' on ρ' , as revealed by simple ideal gas law substitution into $b = g\rho'/\bar{\rho}$. There are no meteorological phenomena in our scope (Earth's very-subsonic convection) for which thermal buoyancy isn't good enough. With (2.7) thus simplified by replacing \hat{b} with b , we can finally bring a dT/dt equation extracted from the First Law (2.5b) or (2.6) into the b

¹ Named for a man: https://en.wikipedia.org/wiki/Joseph_Valentin_Boussinesq

^m The spinup of synoptic cyclones is often called "baroclinic", even though it is mostly vortex stretching: circulation increases even on p surfaces along which ∇p vanishes.

equation, to close a 5-equation set capable of expressing thermal convection.

Applying the *total derivative* d/dt to the definition of b , with virtual effects hidden for clarity with the z subscript for differentiation, gives

$$\begin{aligned} db/dt &= (g/\bar{T}) d/dt[T - \bar{T}(z)] \\ &= (g/\bar{T}) [dT/dt - w\bar{T}_z] \end{aligned} \quad (2.8)$$

Substituting the First Law (2.5b) into the first term, and equating $Z_h(p)=Z=z$ (a hydrostatic assumption used above; and a sensible choice of zero reference for both geopotential height and the z coordinate origin)ⁿ, gives our desired grasping equation:

$$\begin{aligned} db/dt &= (g/\bar{T}) [Q/C_p + w(\Gamma_d - \bar{T}_z)] \\ &= g/(C_p\bar{T}) [Q - w\bar{s}_z] \\ &= Q_b - wN^2 \end{aligned} \quad (2.9)$$

The first term reflects the *production or source of buoyancy by thermal energy addition rate* Q , while the second term is the vertical advection of T plus adiabatic warming/cooling by compression when air changes pressure. The second term can also be expressed in the second line as the effect of *stratification or static stability*, owing to the area-averaged vertical static energy gradient \bar{s}_z . Since that is positive definite for a stably stratified fluid, it is convenient to package as a squared frequency of free vertical parcel oscillations, $N^2 = g\bar{s}_z C_p^{-1} \bar{T}^{-1} = g\bar{\theta}_z (T/\theta) \bar{\theta}^{-1}$ (units s^{-2} ; the second equality is Problem 2.4.4). Virtual effects can be re-inserted for precision by simply inserting T_v and s_v for T and s and θ in all the forms above, so we are glad not to carry them in the algebra.

Gathering the above, we have achieved equation set closure! After substitution of $\rho = \bar{\rho}(z \text{ only}) + \rho'$ and linearization to settle for merely *thermal* buoyancy b , followed by use of the First Law for \dot{T} in good-

ⁿ Deriving (2.8) and (2.9) is a useful student exercise, with a few illuminating steps.

enough static energy form, our equation set (1.6) has grown to five equations in 5 unknowns (u, v, w, π, b):

$$0 = -\nabla \cdot (\bar{\rho} \mathbf{V}) \quad (2.10a)$$

$$du/dt = -\pi_x + fv \quad (2.10b)$$

$$dv/dt = -\pi_y - fu \quad (2.10c)$$

$$dw/dt = -\pi_z + b \quad (2.10d)$$

$$db/dt = -wN^2 + Q_b \quad (2.10e)$$

This set is called *anelastic* when $\bar{\rho}$ is a function of z only (gas-like), and *Boussinesq* when $\bar{\rho}$ is a constant and can be erased from (2.10a) entirely (liquid-like). The power of a closed set of PDEs has to be elicited to be fully appreciated: These equations for local relationships "govern" (or can describe) all possible flows or "solutions", so they are not very discriminating. Let's march through just one simplest solution (for internal or *buoyancy waves*, sometimes called *gravity waves*).

2.2.2 Ubiquitous simplest motions: buoyancy waves

How many complications can we remove from the set (2.10), making it tractable to "solve", short of trivial cases like timeless single-equation balance statements? This was the art of fluid dynamics before computers. Waves are our cleanest time-dependent flow paradigm, with well-developed applied math strategies like complex number methods. The stripped-down simplicity of such problems doesn't make the results irrelevant: quite the contrary! It means that such waves are *ubiquitous*, *almost no matter what happens with all the various complications we ignore and neglect and approximate away*.

Let's (1) neglect advection by replacing d/dt with $\partial/\partial t$. Physically, this assumes the fluid is at rest other than the wave motions, which remain small enough not to advect their own scalars (a *linearization*). For simplicity, (2) neglect north-south flow and gradients, working only in the x - z plane. And (3) make the Boussinesq approximation ($\bar{\rho}$ constant). That is more valid for ocean than the atmosphere, although Boussinesq results can be relevant to the atmosphere on graphs with hydrostatic

pressure as the vertical coordinate^o. For clarity (4) neglect the Coriolis force, which can be added in later fairly easily. Finally (5) take N^2 to be a constant; it is about $2\pi/(10 \text{ minutes})$ in the tropical troposphere. With these assumptions, we have the simplest possible set:

$$0 = u_x + w_z \quad (2.11a)$$

$$u_t = -\pi_x \quad (2.11b)$$

$$w_t = -\pi_z + b \quad (2.11c)$$

$$b_t = -N^2 w \quad (2.11d)$$

One approach is to combine these into a single higher-order equation for w . Pressure can be eliminated with a vorticity approach (see problem 2.4.2) by differentiating and subtracting u and w as $(2.11b)_z - (2.11c)_x$, which leaves b_x as the key right hand side term: *buoyancy gradients are a torque in the vertical plane*. This same equation shows that b and w are 90° out of phase in these waves, so that the *vertical flux of b is zero in internal buoyancy waves*. They are specifically NOT a form of convection, as on a b - w scatterplot waves trace a circle, contributing nothing to the buoyancy flux (covariance $[b'w']$) that Problem 2.4.2 shows to be the source of kinetic energy in fluid motion.

Differentiating in time to eliminate b with (2.11d) leaves only u and w in the equation, then more differentiating allows substitution of (2.11a) to eliminate u , giving a single equation for w :

$$w_{xxtt} + w_{zztt} = -N^2 w_{xx} \quad (2.12)$$

This is clearly a wave equation, because its equality requires that double derivatives act like a negative sign (N^2 being positive), characteristic of sine and cosine functions or their elegant combination in the complex exponential wave form $w = We^{i(kx+mz+\omega t)}$. Here k is horizontal wavenumber ($2\pi/\text{wavelength}$), m is vertical wavenumber, and

^o One phenomenon the Boussinesq equations cannot represent is the breaking of upward-propagating buoyancy waves as density lapses, important in the middle atmosphere.

ω is frequency ($2\pi/\text{period}$). The wave form satisfies (2.12) *if and only if* $\{k, m, \omega\}$ are related in a special way called the *dispersion relation*:

$$\omega = N [k^2/(k^2 + m^2)]^{1/2} \quad (2.13)$$

How can this wave theory result be reasoned with?

Waves are everywhere in the atmosphere and ocean, often satisfying the dispersion relation as *free solutions* far from their forcing and damping processes. The character of observed wave fields must be explained by their drivers and decay mechanisms neglected in (2.11) -- thermal forcing Q_b , or boundary conditions like flow over mountains, or frictional forces \mathbf{F}) -- and by reflection and filtering and screening effects of wind shear and N^2 profile features also neglected in (2.11) between wave-source and observation locations. Expressing these phenomena tractably within equation sets like (2.10) is a mathematical-physics field unto itself (e.g. Sutherland 2010), but a few words here can help launch your efforts in Problem 2.4.5.

Buoyancy waves exist only between the high frequency limit of N (vertical parcel oscillations, problem 2.4.4) and the low frequency limit of purely horizontal motions, whether that is the in-up-out flow of outward-propagating "buoyancy bores" (Mapes 1993) that transport heat laterally away from irreversible latent heat release events, or (adding the the Coriolis force neglected in 2.11) the pendulum-day period of horizontal *inertial oscillations*. Within this range, anything goes -- so what do we expect to observe? That depends not on what is *possible* in the majesty of the governing equation set, but on *what situations have drivers in our world*^p.

The irreversible latent heat bombs of deep convection create motions near the low-frequency limit. In that case, vertical wavenumbers are set by the depth of the heating, perhaps interactively (chapter 6), leaving horizontal wavenumbers to be shaped by the dispersion relation (see

^p And what we are *able* to observe (section 0.6).

Mapes 1993, 1997 for examples and implications). Temporal fluctuations of that heating also drive oscillatory waves that propagate up out of the troposphere (Fritts and Alexander 2003). Beyond these convection-related examples are the wind-over-mountain forcings at the heart of classic gravity wave work (Web-search imagery on *lenticularis* or *lee waves*). Mixed cases include convective clouds acting like moving mountains with possibly sheared wind blowing across the tops (sometimes adding an asymmetric aspect to cap or *pileus* clouds), and gorgeous waves with stable (non-turbulent or non-convective) condensation or evaporation processes (search *asperatus* videos).

Buoyancy waves are everywhere, even where there aren't (yet) cloud droplets to mark them visibly, as revealed by stratigraphy in LIDAR observations (e.g. Parsons et al. 2019). The simplicity of the derivation above means that they are specifically NOT an extraordinary or exotic phenomenological claim requiring extraordinary evidence, as tropospheric meteorology has sometimes treated this poor cousin of visible weather. This cognitive bias (section 0.6) has left the topic mainly to middle-atmosphere science, with its interests skewed toward the importance metric of vertical flux of zonal momentum.

2.3 Good-enough moisture and microphysics (barely)

The First Law was the core of the prognostic b equation (2.10e). To include latent heating and virtual effects in Q_b , we need to carry a sixth equation for q_v or q_c -- not total water q_t which doesn't change upon condensation.

Water phase change can be folded into a *moist adiabatic* view, as in section 2.1.4, by specifying the saturation condition for water when *relative humidity* (RH) is 100%. That is clearer than leaving it in with truly 'diabatic' parts of Q_b , because phase change largely cancels $-wN^2$ in (2.10e) in airmasses (like much of the tropics) whose lapse rate $T(z)$ and thus N^2 is maintained near a state of neutrality by cloudy convection (its main teleological role).

The *saturation vapor pressure* $e_s(T)$ for water is called the *Clausius-Clapeyron relation*, a name easily searched. Its theoretical form involving an exponentiated negated squared inverse of T offers little insight, and real applications use calibrated formulas fit to laboratory steam tables of not-quite-ideal H_2O gas. The important concept for reasoning is merely that $e_s(T)$ is *monotonically increasing*, and *concave upward* (increasing at an increasing rate)^q. This smooth curve, and the $\ln()$ in entropy, are the gentle nonlinearities that forced practical meteorological thermodynamics to be taught graphically in the pre-computer era^r.

Thermal buoyancy b in convective clouds hinges on $\Delta T = T_{parcel} - T_{environment}$. Moist adiabatic ascent conserves h , so (2.6) = 0. For saturated air, $q_v = q_s(T, p)$, so (2.6) becomes a transitive equation determining $T(p)$, although it cannot be solved analytically. Still, the *sign* of ΔT and thus b can be assessed by comparing h of a saturated updraft to h_s of the environment. Specifically, if $h = C_p T_{parcel} + gZ_h(p) + Lq_s(T_{parcel}, p)$ exceeds $h_s = C_p T_{env} + gZ_h(p) + Lq_s(T_{env}, p)$, at a given pressure p , it follows immediately that $T_{parcel} > T_{env}$, because $e_s(T)$ is an increasing function. A clear lifted-parcel buoyancy assessment diagram based on h and h_s is covered in Chapter 4.

Beyond its sign, the actual *value* of ΔT and thus b is not *linearly proportional* to $h(T_{parcel}, p, RH = 100\%) - h_s(T_{env}, p)$, because the upward-curved shape of $e_s(T)$ makes the proportionality depend on T and $q_s(T, p)$ depends a little bit on p too. If your analysis hinges on that precise value of b , under strict assumptions of h conservation (an updraft with no mixing or accounting for condensed water), you are probably being too fussy.

^q Anyone pondering climate should also know its slope at Earthly T : about 7% per K.

^r On oversized diagrams, to compensate for the graphic-design weakness of a poor correspondence between distances on the paper and the importance of differences.

The time has come for a terribly brief mention of the microphysics of condensed water particles (hydrometeors) and the nuclei (*aerosol* particles) they form on. Whole textbooks and careers address the subject, which is very important for electromagnetic observations (section 0.6) including visual beauty, and has some important impacts on how cloudy convective processes play out. The albedo of the Earth (discussed around Figure 0.2) depends crucially on microphysics. Since humans control significant sources of aerosol, that area has drawn an almost unreviewably large and detailed amount of sometimes aggressive study, motivated by worldly interests. But for present purposes, a short list of principles gets us surprisingly far in terms of reasoning power without delving into all that. Chapter 10 will gesture toward more.

- Condensation of liquid droplets when air is brought to saturation occurs very efficiently, far below $RH = 101\%$ and within fractions of a second. Nuclei determine only how many, and thus how large.
- Evaporation of "cloud" droplets (droplets small enough to have negligible fall velocity) upon descent of cloudy air is comparably efficient, again keeping RH very close to 100% , with time delays far shorter than the life cycles of convection's entities (subject of Part II).
- A bulk process called *autoconversion* transforms cloud droplets to precipitation drops (radii $>100\ \mu\text{m}$), whose fall velocity through air is important for subsequent evolution. The time scale is minutes, significantly long compared to meteorological convective air motion processes, and is longer in cloud condensation nuclei (CCN) rich (polluted) airmasses, but it is difficult to be more precise because...
- Falling drops *collect* cloud droplets in a rapid runaway positive feedback that transforms the condensed water field dramatically, leading to precipitation.
- Evaporation of precipitation-sized drops is slow enough that RH is often far below 100% in descending rain shafts, whose air descent rate is itself driven by the finite evaporative cooling of those drops. Such *unsaturated downdrafts* are important to boundary-layer air properties and convection-convection interactions (chapter 8), but it is difficult to be more precise because the details depend on rain shaft size, the distribution of drop sizes, and ambient T and RH .

- Freezing of liquid water releases another kind of *latent heating*, about 1/6 the strength of condensation of the same mass of water.
- Latent heat of freezing is significant to thermal buoyancy b , and can be delayed for a meteorologically significant time (minutes or even hours) by lack of ice nuclei. Only at -40C does water reliably freeze without nuclei. But it is difficult to be more precise, because the availability of ice-nucleating particles is very situational, and ...
- Ice particles nucleate additional ice formation very efficiently, in a runaway feedback (like precipitation development above) called *ice multiplication*, by various mechanisms. It is difficult to be more precise because these are very situational.
- Air that is saturated with respect to liquid (RH = 100%) is supersaturated with respect to ice.
 - For this reason, a rare ice particle can rob mass from more numerous neighboring supercooled cloud droplets by vapor diffusion. Initial ice formation is thus a shortcut to having fewer and larger particles, initiating runaway precipitation processes (collection, riming) and changing cloud optics
- Ice particles grow to precipitation size by vapor deposition and ice collisions (to become snow and crystal aggregates), riming (becoming pellets or graupel or hail). It is difficult to be more precise because particle shape (habit) is involved along with other factors.
- Precipitation-sized ice particles, like liquid ones, evaporate slowly enough that unsaturated downdraft dynamics apply again.
- Melting of precipitating occurs at 0C without nucleation, but can take a meteorologically significant time or fall distance (as surface hail in summer makes obvious).

This concludes the chapter. We walk away an adequate equation set:

$$0 = -\nabla \cdot (\bar{\rho} \mathbf{V}) \quad (2.14a)$$

$$du/dt = -\pi_x + fv \quad (2.14b)$$

$$dv/dt = -\pi_y - fu \quad (2.14c)$$

$$dw/dt = -\pi_z + b \quad (2.14d)$$

$$db/dt = -wN^2 + Q_{b_latent}(q_v) + Q_{b_diabatic} \quad (2.14e)$$

$$dq_v/dt = e - c \quad (2.14f)$$

The representation of water phase changes as Q_{b_latent} and condensation minus evaporation (c-e) is merely symbolic, but a sufficient basis was presented in moist static energy reasoning to understand rising-parcel b illustrations in chapter 4.1. Microphysics principles are too situational to bother crystallizing into equations here, but are also hopefully "good enough" in your mind to proceed.

2.4 Properties of an equation set: problems and solutions

Resource: The Boussinesq set is for a stratified rotating fluid (2.10) may be written, with a horizontal force $\mathbf{F} = iF_x + jF_y$ for completeness:

$$\begin{aligned} 0 &= u_x + v_y + w_z \\ u_t &= -uu_x - vu_y - wu_z - \pi_x + fv + F_x \\ v_t &= -uv_x - vv_y - wv_z - \pi_y - fu + F_y \\ w_t &= -uw_x - vw_y - ww_z - \pi_z + b \\ b_t &= -ub_x - vb_y - wb_z - N^2w + Q_b \end{aligned}$$

As seen in Problem-Solution 1.4.3-1.5.3, continuity (the Law) can be expressed as giving us the pressure (the Enforcer):

$$\pi = \nabla^{-2}[-\nabla \cdot [(\mathbf{V} \cdot \nabla)\mathbf{V}] + (f\zeta - u\beta) + \text{div}(\mathbf{F}) + b_z]$$

2.4.1 Energetics of a whole horizontally unbounded fluid

a. Multiply the 4 prognostic equations by u, v, w, b respectively to generate prognostic equations for $KE = (uu+vv+ww)/2$ and $PE = bb/(2N^2)$. Interpret these: where and how is KE generated? How is it transported away from there to other regions of the fluid?

b. Global energetics: Integrate the energy equations over a whole atmosphere, after transforming from advective to flux so you can express the fact that there are no lateral boundary fluxes (it is unbounded like a

planetary atmosphere although we use Cartesian coordinates for convenience), and that $w=0$ at the top and bottom. Show your work. With brackets $[\]$ denoting the integral, show that

$$[KE]_t = [wb] - \mathbf{F} \cdot \mathbf{V}$$

$$[PE]_t = [Jb] - [wb]$$

$$[PE + KE]_t = [Jb] - \mathbf{F} \cdot \mathbf{V}$$

where $J = Q_b/N^2$

Interpret the terms: what drives fluid motion, what damps it? You may need to postulate a simple assumption about \mathbf{F} (whose symbol should evoke *friction* as well as *force*. What kind of a steady state achieved in a long-lasting, moving (convecting) atmosphere?

How do internal shear instabilities fit into this framework?

2.4.2 Vorticity equation for the nondivergent motions

Since the curl of a gradient vanishes in 3D vector calculus, the PGF when expressed as a gradient of a single scalar like here can be eliminated cleanly to form a vorticity equation. Derive at whatever mathematical detail you are assigned, but with the line of reasoning made clear, one of the horizontal components of vorticity. You may like to consult book derivations. Again the subscript notation for derivatives is a real hand saver.

Emphasize especially: What is the role of the *horizontal gradient of buoyancy*?

2.4.3 Stokes' theorem and vorticity patch reasoning

In the xy plane, write Stokes' theorem, also known as the Circulation Theorem (pure math; look it up), for *every* circle of radius R centered on an isolated small circular patch of vertical vorticity $\zeta = \zeta_0$ of radius r , with $\zeta=0$ elsewhere. In this way, show that the tangential component of flow "induced by" any isolated element of vorticity decays with distance as

1/R. With this rule, the flow from arbitrarily complicated arrangements of vorticity can be constructed by superposition. Try a web search on "contour dynamics" to see the power of this result, and share what you find.

Interpret the result: Does it violate causality somehow that changing vorticity at one point or r -sized local patch instantly "induces" changes in the flow out to arbitrary radius? Think carefully about what was specified in this problem.

2.4.4 Stratification

- a. Confirm the derivations of equations (2.8) and (2.9).
- b. Show that $N^2 = g \bar{s}_z C_p^{-1} \bar{T}^{-1} = g \bar{\theta}_z (T/\theta) \bar{\theta}^{-1}$ for the potential temperature θ .
- c. Show using the w_t equation with b as the only force that N is the frequency of sinusoidal *buoyancy oscillation* of a parcel displaced vertically in a stratified fluid.

2.4.5 Buoyancy waves

- a. Derive equations (2.11) and (2.12), following the text's meaning. Show that the simplified longwave limit ($k \ll m$) is the same answer you get from making the hydrostatic assumption at the outset, by setting the left hand side of (2.11c) to zero.
- b. Discuss applications (situations) in which the forcing mechanism of waves sets every combination of TWO of the set $\{k, m, \omega\}$, leaving the third to be determined by the dispersion relation as a *condition for the existence* of buoyancy-wave motions. Find and share online videos that illustrate these principles.