# Part I: Essentials of the fundamentals

## Chapter 1

# Keeping track of stuff in space

### 1.1 Units for space, time, and stuff

Before understanding must come labeling and accounting. Mass, momentum, and energy ("stuff"[a]) are conserved by isolated entities or systems, but we also care about different places in the world so we must account for transport. We also care about different categories of stuff (air vs. water mass, heat vs. latent (chemical) energy, motions of different scales, etc.) so we must account for transfers between the categories. In our equations, these accounting terms will often outnumber and out-complicate the few key terms expressing the fundamental laws of physics.

Space, time, and mass are measured here in *Systeme Internationale* (SI) units, based originally on our ten fingers, Earth, and water.[b] Space is in *meters* (m), devised as the Earth's equator to pole distance divided by $10^7$ to be human scale. A *kilogram* is the mass of a cubic meter of water (1 *metric ton*), divided by $10^3$ to, again, fit human bodies and commerce[c]. Time is the contentious domain. The Earth gives us *days*, which could be divided by 10 or 100 for finger-counting convenience. Clocks with

---

[a] Quantities which are *extensive* upon aggregation, or *conserved* quantities.

[b] Replacing old traditions, in the 18th century French Revolution's radical rationalizing.

[c] That volume of water is also called the *liter*.

decimal face labels were manufactured in 18th century France, but never caught on. After all, the six-related numbers of traditional time have their own Earth-related numerology (almost-12 months and almost-360 days in a year). In the end SI retained the *second*: a day divided by 86400 (60 x 60 x 24). Latitude in degrees ($10^7$m / 90° = 111.111 km per degree) and subdivisions like nautical miles (1/60 degree) also carry this six-related history.

Temperature's Celsius scale is also about water and tens: 1 °C = 1 K = (water's boiling point minus freezing point)/100. This is not quite as fundamental, since the boiling point depends on atmospheric pressure. It is a remarkable coincidence that the weight of a 1 $m^2$ column of Earth's atmosphere (1 bar or atmosphere of pressure) happens to be so near the weight of a 10 m column of water, making surface pressure nearly $10^5$ (the global mean is 101325) in the compound SI unit of Pascals (1 Pa = force/area = 1 N $m^{-2}$ = 1 kg m $s^{-2}$ $m^{-2}$ = 1 kg $m^{-1}$ $s^{-2}$).

Precision science has quietly replaced these original Earth and Water motivations with more fundamental quantum profundities as the root of official SI, without most people even noticing (the kilogram was redefined in 2019). But our 10 fingers still rule numbering in all but the time domain.

## 1.2 Conservation of the most fundamental "stuff": mass

In a given volume of space (let's say 1 $m^3$ for definiteness), the enclosed mass is customarily labeled $\rho$, a *mass density* whose inverse is called *specific volume,* $\alpha$=1/$\rho$. The rate of change of mass in a cubic meter of space is pure spatial accounting: it equals the net inflow of mass into the volume. Physical sources and sinks are zero except for tiny Einsteinian $E=mc^2$ effects. The flow of mass through a two-dimensional (2D) area, like the square face of a 3D cubic volume, is measured by a *flux*. The units of mass flux [kg $m^{-2}$ $s^{-1}$] embody its meaning better than any further words can elaborate.

Mass flux is $\rho\boldsymbol{V} = \rho\vec{\mathbf{V}}$ in symbols[d], using bold face and optionally arrows for emphasis for vectors like velocity $\boldsymbol{V} = \boldsymbol{i}u + \boldsymbol{j}v + \boldsymbol{k}w$, in a Cartesian (*x, y, z*) coordinate system with its unit vectors (***i, j, k***) and velocity components (*u, v, w*).

Readers should verify that the units of $\rho\boldsymbol{V}$ are indeed a flux [(kg $m^{-3}$) (m $s^{-1}$) = kg $m^{-2}$ $s^{-1}$]. Notice with that same units awareness that velocity $\boldsymbol{V}$ is a *volume flux;* and is also a *specific momentum* (where "specific" means *per unit mass*). A feeling for different interpretations of the same quantity is essential to fully appreciate the equations of convection: a firm physical grip is needed on the mathematical symbols and their units, but not too tight or exclusive.

Logic tells us that for the truly conserved stuff called mass, its *rate of change* in a volume equals the *net mass inflow*. That is called the *convergence of mass flux*, the negative of *divergence*[e]. Translating that into math, with subscripts denoting partial derivatives along the axes in (*x, y, z, t*) space,

$$\begin{aligned}\rho_t &= conv(\rho\vec{\mathbf{V}}) = -div(\rho\vec{\mathbf{V}}) = -\vec{\boldsymbol{\nabla}}\cdot(\rho\vec{\mathbf{V}}) \\ &= -(\rho u)_x - (\rho v)_y - (\rho w)_z + source\ terms \\ &= -(\rho u)_x - (\rho v)_y - (\rho w)_z + 0 \qquad (1.1)\end{aligned}$$

A diagram with a cube (a *control volume*) and flux arrows is sometimes used to illustrate this logic, but seems unnecessary to repeat here. Since mass is truly conserved, *source terms* are zero, except for negligible relativistic conversions to energy via Einstein's famous $E=mc^2$.

### 1.2.1 *Aside on mathematical expression culture*

Notice that the structure of (1.1) embodies an ambitious spirit:

---

[d] See Table of Symbols and Notation.

[e] Early meteorology texts simply used the word *vergence*, letting the sign reflect its sense.

$$something\ we\ want\ =\ complications\ we\ must\ face\ to\ get\ it$$

In the case of a *prognostic equation* like (1.1), what we want is to know the future, so we put that on the left-hand side (LHS). Let us call this the *grasping form* of equation writing, in contrast to a *majestic equality* like 0 = (sum of all terms). Mathematically, these are isomorphic: not different in any meaningful way, just in form, by custom. But equations do have a *brainfeel*[f] in addition to their content. To feel a difference, transform (1.1) into the equivalent majestic equality:

$$0 = (\rho u)_x + (\rho v)_y + (\rho w)_z + (\rho \dot{t})_t \quad (1.2)$$

In the first 3 terms on the RHS, velocity components in space ($u = \dot{x}$, $v = \dot{y}$, $w = \dot{z}$, using Newton's dot notation for univariate time derivatives) measure the spatial journey of moving matter, in units of meters traversed per second of elapsed time. In (1.2) an analogous quantity $\dot{t}$ measures the journey of matter through time, in seconds traversed or endured per second of elapsed time. Since both are seconds, $\dot{t}$ is unitless and equal to unity. But was no appreciation gained?

## 1.3 Conservation of specific (per unit mass) other stuff:

### 1.3.1 *Specific momentum and its physical source terms*

Having established the flux convergence form for mass (1.1), other budgets follow straightforwardly by accounting other intensive properties of air on a *specific* or *per unit mass* basis. For instance, to get the budget equation for momentum, simply multiply the mass flux by specific momentum, which (as noticed above) is velocity, and then reconsider the sources and sinks. In the vertical or **k** direction, along which air's position change $\dot{z} = w$ (vertical velocity) as above,

$$(w\rho)_t = -\vec{\nabla} \cdot (w\rho\vec{\mathbf{V}})$$

---

[f] This term is motivated by "mouthfeel," a food descriptor distinct from taste or nutrition.

$$= -(w\rho u)_x - (w\rho v)_y - (w\rho w)_z \\ + w\ source\ terms \qquad (1.3)$$

The big difference from (1.1) is nonzero source terms on the RHS. Newton calls such momentum sources *forces,* in the famous equation **F** = m**a** that earned him the honorific SI unit for force (N = kg m $s^{-2}$).[g] Acceleration could be called *specific force,* but that doesn't add much sense: we don't really think of acceleration as semi-conserved "stuff" like momentum.

Two forces (momentum sources) are needed to appreciate convection:

(1) *Gravity*[h] $-\boldsymbol{k}\rho g$
(2) The *pressure gradient force* $-\vec{\boldsymbol{\nabla}}p$

What is this "pressure" *p*? Most students learned it as *force per unit area,* implying units $N/m^2$ = *Pa* (Pascals). However, those units would also identify it as a *momentum flux,* with flux units of (stuff) $m^{-2}s^{-1}$ = (momentum) $m^{-2}s^{-1}$ = (*kg m/s*) $m^{-2}s^{-1}$]. But it is a strange flux: directionless, or imparting momentum in all directions at once. Since *p* is a scalar field, not a vector field, the differential operator measuring the net flux into a cube of space (the flux difference from one side to the other) is not the convergence of the directional vector flux field $\rho\vec{\mathbf{V}}$ as above, but rather the *gradient* of the scalar field *p*. Notice also that the calculus concepts of "gradient" and "convergence" are only well defined over a spatial length scale where the continuum approximation is valid (smoothing over the lumpy molecular nature of matter).

In a steadily rotating coordinate system, where air we call "motionless" is actually accelerating, we must add to (1.3) a third corrective term, the fictitious Coriolis force per unit mass $-2\vec{\boldsymbol{\Omega}} \times \vec{\mathbf{V}}$ where $\vec{\boldsymbol{\Omega}}$ is the coordinate

---

[g] Is force a flux of some definable "stuff"? Not really: 1 N = 1 (kg $m^3$ s) $m^{-2}$ $s^{-1}$, so the implied unit of "stuff" being transported (kg $m^3$ s) makes no sense to this author's brainfeel.
[h] Strictly speaking, "gravity" in meteorology is Newton's *gravitational force* plus a small force due to the Earth's rotation which bulges the equator out a little relative to a sphere.

rotation vector. For simplicity we will neglect the vertical component of the Coriolis force (which is tiny compared to gravity), and retain only the horizontal Coriolis force based on the *Coriolis parameter* $f$ = $2|\vec{\boldsymbol{\Omega}}|$ *sin(latitude)* in our Earth-tangent Cartesian (x,y,z,t) coordinates.

Gathering the considerations above for every cubic meter of space, we are up to 4 equations including 5 unknowns (*u,v,w,p,ρ*):

$$
\begin{aligned}
(\ \rho)_t &= -(\ \rho u)_x \ -(\ \rho v)_y - (\ \rho w)_z \quad + 0 && \text{(1.4a)}\\
(\rho u)_t &= -(u\rho u)_x - (u\rho v)_y - (u\rho w)_z \quad -p_x + \rho f v && \text{(1.4b)}\\
(\rho v)_t &= -(v\rho u)_x - (v\rho v)_y - (v\rho w)_z \quad -p_y - \rho f u && \text{(1.4c)}\\
(\rho w)_t &= -(w\rho u)_x - (w\rho v)_y - (w\rho w)_z \ -p_z - \rho g && \text{(1.4d)}
\end{aligned}
$$

Notice that each $p$ in (1.4 b-d) could be moved inside the parentheses of a flux term, emphasizing its role as a flux. But pressure's momentum flux is transmitted across the borders of a spatial box by elastic collisions of the molecules outside the border with those inside it. This is new physics, *different in character* from pieces of matter carrying their properties with them as they cross a border. Border crossing by the smallest pieces of matter (molecules) is part of the flux terms like $u\rho w$ in (1.3). That molecule-borne flux acts as a *diffusion* process: *the convergence of a flux of stuff that is proportional to the gradient of that stuff* (revisited in section 3.1). That process is called the *viscosity force* when it is pulled out of the flux transport terms and moved to the RHS as a "source" term for equations like (1.3)[i].

Without density variations, mighty gravity is indiscriminate in (1.4d), and there can be no *convection* in the sense of our title. We simply must allow $\rho$ to vary a little so that the gravity force -$\boldsymbol{k}\rho g$ can do *work*, which drags us into thermodynamics (Chapter 2). Specifying ρ is called an *equation of state.* Before we tackle that complication, we can already gain a key insight into the nature of pressure from the simplest case, ρ = constant (Problem 1.1). Readers should understand the resulting lesson

---

[i] The *Navier-Stokes equations* for fluid dynamics have viscosity pulled out in this way.

well: *Continuity is the Law, Pressure is the Enforcer, and **F**=m**a** is its Mechanism*.

### 1.3.2 *Other specific stuff: humidity and 'heat content'*

To close our four-equation but five-variable set (1.4) we need an equation for ρ. Although recognizing *specific momentum* as another name for velocity made momentum budget construction very direct (1.4 b-d), no similar trick of invoking 'specific density' can help us because density per unit mass is not physically incisive[j]. We need another tactic, another physical law. The ideal gas law comes to hand, but that brings in temperature T. Since T is indicative of warmth or heat, we need a quantity that measures the "stuff" that heat is made of: an energy perhaps, although that will lead us on to more abstract stuff called *entropy*.

For cloudy convection, we also need to keep track of the mass of water, which we can measure as a *specific humidity q*, so you know what to do: replace *w* with *q* in (1.3) and replace the physics of the *w sources* term with the physics of *q sources*. It's a trivial extension to the notation to budgets of categorized water (specific cloud water, specific rain water, specific ice water, etc.). However, one subtle bookkeeping issue in this exercise needs to be noticed.

### 1.3.3 *Specific X, or mass mixing ratio of X?*

There is a subtle difference between *specific water vapor mass* $q_v$ and *water vapor mass mixing ratio*[k] $r_v$. Both have water vapor mass in the numerator, but mixing ratio has the mass of *dry air* rather than *total mass* in the denominator. To get more precise, one must decide if $\rho$ in (1.1) stands for dry air density or for total mass density. It seems tempting to use total mass, but then the velocity *w* is harder to define: in rain, it would have to be a mass-weighted mean of the wind velocity of gases and the fall

---

[j] That would be the inverse of *specific volume* $\alpha = 1/\rho$, which is just density ρ itself!

[k] We avoid the chemistry term *concentration,* which seems ambiguous.

velocity of particles. The complications explode. Alternately, the third step in (1.1) stating *mass sources = 0*, which is truer for dry air mass, could be revised slightly to account for precipitation falling out. The complications explode differently.

Strictly speaking, this book accepts (1.1) as if for dry air, yet keeps the word *specific* for brevity. Terms of art and their mathematical cousins (symbols) are often thought of as hyper-specific. But their greatest power can actually lie in being vague, refusing to draw distinctions that are inessential to a line of reasoning[1]. Our plain symbol $\rho$ papers over the complications intentionally, eliding the slight distinction between $q_{\mathrm{v}}$ and $r_{\mathrm{v}}$ as measures of water content. If your goal is to mathematically frame a numerical model that will be integrated over long times, requiring an equation set that obeys integral conservation laws punctiliously even as the various types of "stuff" are shuttled among many spatial boxes and categories, then such decisions must be strictly defined and adhered to. Symbols, subscripts, and small terms will proliferate. You should work from a longer, fussier book. If such a model must quantitatively assimilate absolutely calibrated observations, it must also use fully *accurate* as well as precisely conservative thermodynamics, coordinates, and conservation laws. Here, since our goal here is to facilitate appreciation of conceptual essentials, such details of rigorous fundamentals will be elided freely.

### 1.3.4 *Advection and the material derivative*

Notice (Problem 1.4.1) that you can distribute the derivatives in (1.4b) by the chain rule, divide by ρ, and use (1.4a) to rewrite it without approximation as:

$$u_t = -uu_x - vu_y - wu_z - \alpha p_x + fv \quad (1.5)$$

This *advective* form of transport terms on the RHS is fully general, and tempts us to interpret *advection* as being as valid as flux and its divergence. The sense of advection is to *look upwind: those air properties are coming*

[1] An example is the term *hydrometeor* for any falling condensed water object.

*toward you, so conditions at your location will soon be like that unless source terms on the RHS intervene*. Defining a special new *total* or *Lagrangian* time derivative $du/dt = u_t + \boldsymbol{V} \cdot \boldsymbol{\nabla} u \; = \; u_t + uu_x + vu_y + wu_z$, the set (1.4) becomes:

$$d\rho/dt \; = -\rho \boldsymbol{\nabla} \cdot \boldsymbol{V} \qquad (1.6a)$$
$$du/dt \; = -\alpha p_x + fv \qquad (1.6b)$$
$$dv/dt \; = -\alpha p_y - fu \qquad (1.6c)$$
$$dw/dt \; = -\alpha p_z - g \qquad (1.6d)$$

In Chapter 2 we will need to invoke thermodynamic laws learned from interrogating a kilogram of air trapped in a piston in a laboratory. To use these laws in our fluid equations, we need to equate laboratory *time derivatives referring to a unit mass of air* (denoted with Newton's univariate derivative notation like $\dot{T}$)[m] to this Lagrangian total derivative *dT/dt* as defined above. One way to see the equivalence is to notice that if we truly had the field or function $T(t,x,y,z)$ – temperature everywhere forever – we could extract the temperature history of an arbitrary moving parcel of unit mass as $T(t, x_p(t), y_p(t), z_p(t))$. Using the chain rule to extract all the temporal changes in the function's argument,

$$\begin{aligned} dT_p/dt \; &= \; \partial T/\partial t + \partial T/\partial x \cdot \dot{x}_p + \partial T/\partial y \cdot \dot{y}_p \; + \partial T/\partial z \cdot \dot{z}_p \\ &= \; T_t + uT_x + vT_y + wT_z \end{aligned}$$

which can be equated to $\dot{T}$, the time-only changes for an air parcel that is trapped in the laboratory chamber.

---

[m] reviewed in https://en.wikipedia.org/wiki/Notation_for_differentiation

## 1.4 Problems:

**1.4.1** ***Show the steps from 1.4b (flux) to 1.6b (advection) forms of the budget equations for the u component.***

**1.4.2** ***Repeat the problem above for v and w, and gather terms to show that the advection of vector momentum can be written as*** $-u\vec{V}_x - v\vec{V}_y - w\vec{V}_z = -(\vec{V} \cdot \vec{\nabla})\,\vec{V}$

**1.4.3** ***Set density to a constant*** $\rho_0$ ***and simplify the set (1.4) maximally in that case.***

What phenomena could this *incompressible fluid* (constant density) equation set describe? In other words, how can an incompressible fluid move, and why would it? What could drive motion, at what scales, and how could that motion decay?

What would be the speed of compression (sound) waves in such a fluid? That is, if boundary conditions jiggle one edge of an incompressible body of fluid, how soon is the motion transmitted to the other side?

**1.4.4** ***Using the simplified set of component equations with constant density*** $\rho_0$ ***from 1.4.3, transform the majestic equality form of the mass continuity equation (0 = terms) into a grasping equation of the form:***

*what I want (pressure) = complicated effort needed to construct it*

Hint: differentiate mass conservation in time, momentum conservation equations in space, and substitute the latter into the former. Subscripts for partial differentiation will save many redundant hand motions in this process. You may use the symbolic inverse $\boldsymbol{\nabla}^{-2}$of the Laplacian operator $\boldsymbol{\nabla}^{2}$ in the final answer, even though solving it is a nontrivial job (Chapter 3).

Interpret the result in your own words, elaborating on this summary: *Continuity is the Law, Pressure is the Enforcer,* ***F*** *=* *m**a*** *is its Mechanism.*

## 1.5 Solutions:

***<u>Solution to 1.4.3:</u>***

Substituting constant $\rho$=$\rho_0$. dividing by $\rho_0$, and negating the sign in (1.4a),

$$
\begin{aligned}
0 &= u_x + v_y + w_z \\
u_t &= -(uu)_x - (uv)_y - (uw)_z \quad -\pi_x + fv \\
v_t &= -(vu)_x - (vv)_y - (vw)_z \quad -\pi_y - fu \\
w_t &= -(wu)_x - (wv)_y - (ww)_z \ -\pi_z - g
\end{aligned}
$$

where we have introduced the pressure variable $\pi = p/\rho_0$.

***What phenomena could this incompressible fluid equation set describe?***

If a body of such fluid were initially at rest, the only forces that could drive coherent (larger than molecular) motions within it are coherent momentum sources applied at its boundary. Gravity is powerless without density variations, so nothing worth the name "convection" can occur. Pressure can push divergent (irrotational) internal flows, like the motions inside a water balloon that make some part bulge out when another part is pressed in. If groups of molecules at the fluid's boundary are somehow given a coherent momentum tangential to the boundary, like by a *stress* we could call "friction", they could diffuse that momentum inward (viscosity). Such viscous forces could create internal shear, which could break down due to shear instabilities into smaller-scale fluid motions that could transport momentum still deeper into the fluid, so that eventually the fluid could contain all sorts of turbulent motions. The energy of such motions would decay into heat (molecular motions) by internal viscous dissipation (diffusion of momentum down its gradient). The speed of sound (compressional waves) in an incompressible fluid is infinite: with constant density $\rho_t = 0$, the medium is infinitely stiff.

***Solution to 1.4.4:***

Differentiating the $u$ equation in $x$, the $v$ equation in $y$, and the $w$ equation in $z$, and summing them,

$$\begin{aligned} [u_{tx} &= -(uu)_{xx} - (uv)_{yx} - (uw)_{zx} \quad -\pi_{xx} + fv_x] \\ +[v_{ty} &= -(vu)_{xy} - (vv)_{yy} - (vw)_{zy} \quad -\pi_{yy} - fu_y - uf_y] \\ +[w_{tz} &= -(wu)_{xz} - (wv)_{yz} - (ww)_{zz} \; -\pi_{zz} \;\; - g] \end{aligned}$$

Using mass continuity to see that the left side is zero since $\rho_t = 0$, and packing up terms into a vector form,

$$0 \;=\; -\boldsymbol{\nabla}\cdot[(\boldsymbol{V}\cdot\boldsymbol{\nabla})\boldsymbol{V}] \;\; -\boldsymbol{\nabla}^2\pi \quad + f\zeta - u\beta$$

where $\zeta$ is the vertical component of relative vorticity $\zeta = v_y\text{-}u_x$, $\beta = f_y$ is the latitudinal gradient of the Coriolis parameter, and parentheses are carefully used to make the result depend on no notation beyond the familiar vector dot product and the vector differentiation operator $\boldsymbol{\nabla} = \mathbf{i}\,\partial/\partial x() + \mathbf{j}\,\partial/\partial y() + \mathbf{k}\,\partial/\partial z()$.

Hungrily solving for $\pi$, using the symbolic inverse of $\boldsymbol{\nabla}^2$,

$$\pi = \boldsymbol{\nabla}^{-2}[-\boldsymbol{\nabla}\cdot[(\boldsymbol{V}\cdot\boldsymbol{\nabla})\boldsymbol{V}] + f\zeta - u\beta]$$

*Continuity is the Law* expressed by the equation for pressure. Since flow divergence is zero, to maintain the (assumed) $\rho=\rho_0$ forever ($\rho_t$=0), pressure must intimately cancel the divergence of each and every other field of force (terms on the RHS). In this case those divergent forces are (i) the “inertial force” implied by the advection of momentum in the inner square brackets, (ii) the inward- or outward-directed Coriolis force on horizontally swirling flow, and (iii) the divergent Coriolis force on zonal flow $u$, when the Coriolis parameter varies with latitude. Other forces like viscosity could be easily added.

*Pressure is the Enforcer* of mass continuity. Pressure does that job[n] in the momentum equations, and *F = ma is the Mechanism* of that enforcement. From this standpoint, the common exercise in dynamics courses of calculating flows with the pressure field taken as a given is not very sensible: pressure is a cleanup force, the *last* force *logically*, the one that adjusts to intimately respond to the divergence of all the other forces, almost instantly (strictly, at the speed of sound). Here that sound speed is infinite, which is unrealistic, but even in air it is much faster than all the other information-transmitting waves in meteorological flows. Therefore, the point remains relevant: *Continuity is the Law, Pressure is the Enforcer,* ***F*** *=* ***ma*** *is the enforcement mechanism.*

[n] This is *teleology,* the explanation of things in terms of the purpose they fulfil.