Chapter 4

TRIGONOMETRIC FUNCTIONS

4.1 The Unit Circle

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- **Definition**:

  The unit circle is a circle of radius 1 centered at the origin.

- **Equation**:
  - Recall that any circle can be written as

$$(x-h)^2+(y-k)^2=r^2$$

  - Since the center is at (0, 0) and the radius is 1

$$x^2+y^2=1$$

4.1 The Unit Circle

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- **Example**:

  Find the point(s) on the unit circle whose y-coordinate is 1/3.

- **Example**:

  Find the point(s) on the unit circle whose two coordinates are equal.

4.1 The Unit Circle

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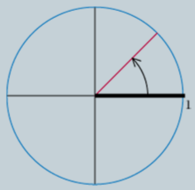
- **Angles in Standard Position**:



  - Start on the positive horizontal axis.
  - Move in a counterclockwise motion for positive angles.
  - Angle represented by the Greek letter "theta" (θ)

4.1 The Unit Circle

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- **Coordinates on the unit circle**:
  - Special Angles

    0° : (1, 0)

    90°: (0, 1)

    180°: (-1, 0)

    270°: (0,-1)

    360°: (1, 0)

4.1 The Unit Circle

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- **Question**: What do you think the coordinates are for the angles, 45°, 135°, 225°, and 315°? Why?

4.1 The Unit Circle

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- **Negative Angles**:
  - Start from standard position and move clockwise.
- **Relationship between negative and positive angles**:
  - Which angles represent the same radius?

4.1 The Unit Circle

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- **Angles greater than 360°**:
  - Represent multiple rotations around the unit circle.
- **Example**:
  - Name two other angle measurements that represent the same radius as 70°.

4.1 The Unit Circle

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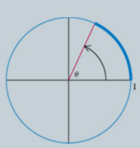
- **Length of a Circular Arc:**
  - Represents the distance along the edge of the circle travelled while moving from standard position to a given degree angle (to a given radius).
  - Recall that the circumference of the unit circle is $2\pi r$ or just 2π.



- **Example**: Find the length corresponding to 65°.

4.1 The Unit Circle

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- **Length of a Circular Arc:**
  - In general, the formula to find the length of a circular arc is

$$\frac{\theta \pi r}{180}$$

  - This is for any circle, but for the unit circle, r would be replaced with 1.

4.1 The Unit Circle

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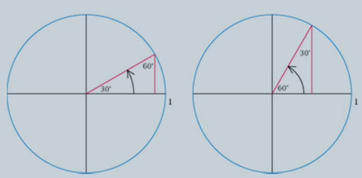
- **Special Points on the Unit Circle:**
  - Consider drawing a right triangle inside the unit circle.



4.1 The Unit Circle

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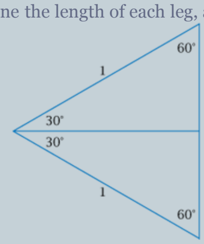
- **Special Points on the Unit Circle:**
  - To determine the length of each leg, and thus the coordinates:



4.1 The Unit Circle

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- **Special Points on the Unit Circle:**

Angle	Endpoint of Radius
0°	(1, 0)
30°	$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
45°	$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
60°	$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
90°	(0, 1)

4.2 Radians

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- **Def:** Radians is a unit of measure used to represent angles.
  - Measures the length of the arc associated with an angle.
  - One-to-one correspondence to degree measures.
- **Example**: If 2π radians corresponds to 360 degrees, how many radians is 180 degrees? 90 degrees? 45 degrees?

4.2 Radians

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- Converting radians to degrees

$$\theta radians = \left(\frac{180\theta}{\pi}\right)^{\circ}$$

- Converting degrees to radians

$$\theta \deg rees = \frac{\theta\pi}{180} radians$$

4.2 Radians

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- Degree/Radian Conversions for Common Angles:

Degrees	Radians
30	π/6
45	π/4
60	π/3
90	π/2
180	π
360	2π

4.2 Radians

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- **Example**: Convert 20 degrees to radians.
- **Example**: Convert –π/12 radians to degrees.
- Positive vs. Negative radians

4.2 Radians

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- Multiple Rotations

$$\theta + 2\pi n$$

- Length of a circular arc
  - If $0 < \theta < 2\pi$, then a circular arc on the unit circle corresponding to θ radians has length θ.
- Area of a slice (for θ in radians)

$$\frac{1}{2}\theta r^2$$

4.2 Radians

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- **Example**: Suppose a slice of a 12 inch pizza has an area of 20 square inches. What is the angle of the slice?

4.2 Radians

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- Special Points on the Unit Circle:

Radians	Points
$\pi/6$	$\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$
$\pi/4$	$\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$
$\pi/3$	$\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$
$\pi/2$	(0 , 1)
π	(-1 , 0)
2π	(1 , 0)

4.3 Cosine and Sine

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- **Definitions:**
  - The **cosine** of any angle is the x-coordinate of where the corresponding radius intersects the unit circle.
  - The **sine** of any angle is the y-coordinate of where the corresponding radius intersects the unit circle.
    - The point where any radius intersects the unit circle has coordinates

$$(\cos\theta, \sin\theta)$$

4.3 Cosine and Sine

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- **Cosine and Sine of Common Angles:**

Θ (radians)	*Θ (degrees)*	cosθ	sinθ
0	0°	1	0
$\frac{\pi}{6}$	30°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{4}$	45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	60°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	90°	0	1
π	180°	-1	0
2π	360°	1	0

4.3 Cosine and Sine

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- **Properties of Cosine and Sine:**
  - These are functions like *f(x)*.
  - Since the equation of the unit circle is $x^2 + y^2 = 1$ then

$$(\cos\theta)^2 + (\sin\theta)^2 = 1$$

  - Notation: $(\cos\theta)^2 = \cos^2\theta$ and $(\sin\theta)^2 = \sin^2\theta$
  - $-1 \le \cos\theta \le 1$ and $-1 \le \sin\theta \le 1$ for all values of θ .

4.3 Cosine and Sine

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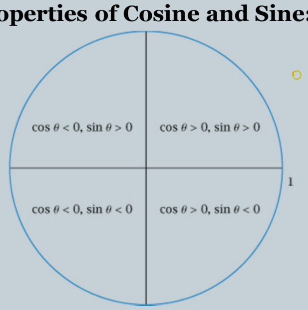
- **Properties of Cosine and Sine:**



  - The sign of the cosine and sine of an angle depends on the quadrant the radius falls in.

4.3 Cosine and Sine

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- **Example:**
  - Suppose that θ is an angle such that sin θ = -0.4. Evaluate cosθ assuming that $\pi \le \theta \le \frac{3\pi}{2}$.

4.3 Cosine and Sine

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- **Domain and Range:**
  - The domain of both cosine and sine is the set of all real numbers.
  - The range of both cosine and sine is the interval [-1, 1].

4.3 Cosine and Sine

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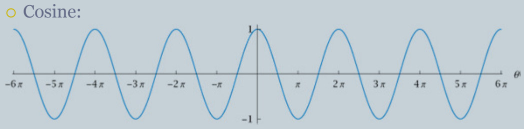
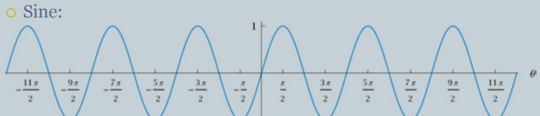
- **Graphs of Cosine and Sine:**
  - Cosine:



  - Sine:



4.4 More Trigonometric Functions

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- **Tangent:**
  - Definition: The tangent of an angle θ, written tanθ, is defined as

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

where $\cos\theta \neq 0$.

  - Represents the slope of the radius.

4.4 More Trigonometric Functions

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- **Example:**
  - Find the equation of a line that goes through the point (2,7) and makes a 55° angle with the positive x-axis.

4.4 More Trigonometric Functions

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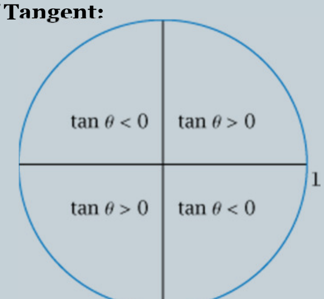
- **Sign of Tangent:**



4.4 More Trigonometric Functions

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- **Relationship between Sine, Cosine, and Tangent:**
  - Recall: $\cos^2\theta + \sin^2\theta = 1$

    and $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

4.4 More Trigonometric Functions

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- **Example:**
  - Find $\cos\theta$ and $\sin\theta$ for θ between π and $3\pi/2$ that has $\tan\theta=4$.

4.4 More Trigonometric Functions

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- **Domain and Range:**
  - Domain: real numbers that are not odd multiples of $\pi/2$.
  - Range: all real numbers

4.4 More Trigonometric Functions

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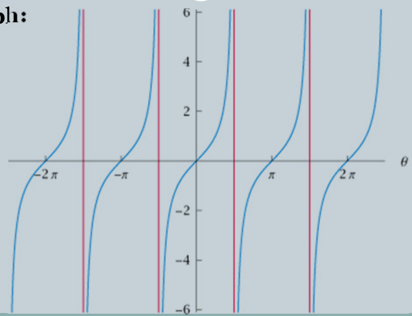
- **Graph:**



4.4 More Trigonometric Functions

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- **Secant:**
  - $\sec\theta = \dfrac{1}{\cos\theta}$
- **Cosecant**
  - $\csc\theta = \dfrac{1}{\sin\theta}$

4.4 More Trigonometric Functions

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- **Cotangent:**
  - $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$
  - Is the multiplicative inverse of $\tan\theta$.
    - $\cot\theta = \dfrac{1}{\tan\theta}$

4.5 Trigonometry in Right Triangles

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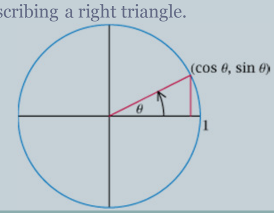
- **Inscribed Right Triangles and the Unit Circle:**
  - We have already showed that the x and y coordinates of points on the unit circle (and thus the cosine and sine of an angle) can be found by inscribing a right triangle.



4.5 Trigonometry in Right Triangles

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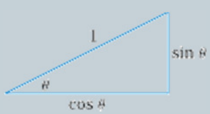
- **Inscribed Right Triangles and the Unit Circle:**
  - It has also been showed that the equation for the unit circle is equivalent to using the Pythagorean theorem where the hypotenuse is 1.



$$x^2 + y^2 = 1$$

$$\cos^2\theta + \sin^2\theta = 1$$

4.5 Trigonometry in Right Triangles

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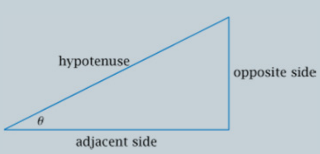
- **Right Triangles and Trig Functions:**
  - Identify the angle of interest.
  - Label the sides of the triangle as being either adjacent or opposite to the angle.



4.5 Trigonometry in Right Triangles

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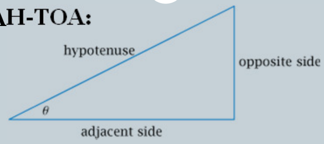
- **SOH-CAH-TOA:**



$$\sin\theta = \frac{opp}{hyp} \qquad \tan\theta = \frac{opp}{adj}$$

$$\cos\theta = \frac{adj}{hyp}$$

4.5 Trigonometry in Right Triangles

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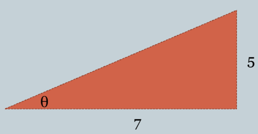
- **Example:**
  - Find the cosine, sine, and tangent of the angle from the triangle below:



4.5 Trigonometry in Right Triangles

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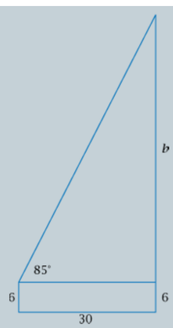
- **Example:**
  - Standing 30 feet from the base of a tall building, you aim a laser pointer at the closest part of the top of the building. You measure that the laser pointer is 5° tilted from pointing straight up. The laser pointer is held 6 feet above the ground. How tall is the building?



4.6 Trigonometric Identities

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- **Relationship Between Cosine and Sine:**
  - Already know $\cos^2\theta + \sin^2\theta = 1$
  - Results in:

$$\cos\theta = \pm\sqrt{1-\sin^2\theta}$$

$$\sin\theta = \pm\sqrt{1-\cos^2\theta}$$

4.6 Trigonometric Identities

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- **Dividing by *cos²θ* and *sin²θ* :**
  - Already know $\cos^2\theta + \sin^2\theta = 1$
  - Results in:

$$1+\tan^2\theta = \sec^2\theta$$

$$\cot^2\theta + 1 = \csc^2\theta$$

4.6 Trigonometric Identities

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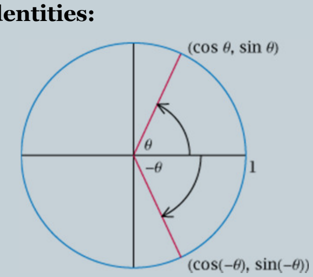
- **Negative Angle Identities:**

$$\cos(-\theta) = \cos\theta$$

$$\sin(-\theta) = -\sin\theta$$

$$\tan(-\theta) = -\tan\theta$$



4.6 Trigonometric Identities

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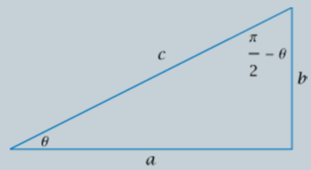
- **Identities with π/2:** $\left(0<\theta<\dfrac{\pi}{2}\right)$

$$\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta$$

$$\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta$$



$$\tan\left(\frac{\pi}{2}-\theta\right)=\frac{1}{\tan\theta}=\cot\theta$$

4.6 Trigonometric Identities

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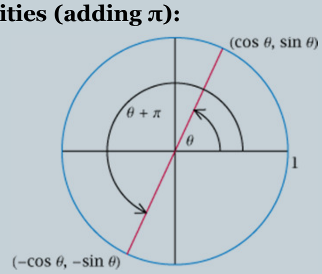
- **Half Circle Identities (adding π):**

$$\cos(\theta+\pi)=-\cos\theta$$

$$\sin(\theta+\pi)=-\sin\theta$$

$$\tan(\theta+\pi)=\tan\theta$$



4.6 Trigonometric Identities

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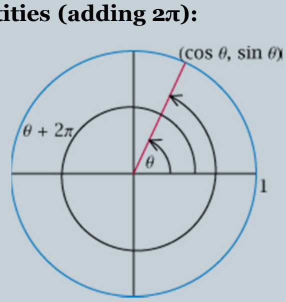
- **Whole Circle Identities (adding 2π):**

$$\cos(\theta+2\pi)=\cos\theta$$

$$\sin(\theta+2\pi)=\sin\theta$$

$$\tan(\theta+2\pi)=\tan\theta$$



4.6 Trigonometric Identities

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- **Example:**
  - Find the smallest positive value of x so that

$$\left(\cos\left(x+\pi\right)\right)\left(\cos x\right)+\frac{1}{2}=0$$

  - If tan$u$ = 2, find cos$u$ assuming that u falls in the first quadrant.