ON THE EQUINOCTIAL ORBIT ELEMENTS*

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Abstract. This paper investigates the equinoctial orbit elements for the two-body problem, showing that the associated matrices are free from singularities for zero eccentricities and zero and ninety degree inclinations. The matrix of the partial derivatives of the position and velocity vectors with respect to the orbit elements is given explicitly, together with the matrix of inverse partial derivatives, in order to facilitate construction of the matrizant (state transition matrix) corresponding to these elements. The Lagrange and Poisson bracket matrices are also given. The application of the equinoctial orbit elements to general and special perturbations is discussed.

1. Introduction

This paper describes some results related to a remarkable set of orbit elements in the two-body problem. The set of elements contains h and k, which are functions of the classical elements, e, ω, and Ω together with p and q, which are functions of the classical elements i and Ω (see Arsenault *et al.*, 1970). Note that the elements h and k have been discussed previously (see Brouwer and Clemence, 1961, pp. 287–288; Chebotarev, 1967, p. 115; or Moulton, 1970, p. 421) but that elements p and q differ slightly from those used by Brouwer and Clemence, Chebotarev, and Moulton. The name 'equinoctial elements' introduced by Arsenault *et al.* (1970) has been adopted by the present authors.

These orbit elements were used more than a century ago (by Lagrange) in the study of secular effects due to mutual planetary perturbations. This set of elements was chosen because it is especially well adapted to orbits with small eccentricity and small inclination. Recent literature gives the 'variation of parameters' perturbation equations for these elements, generally stating that the equations are valid when eccentricity and inclination are zero. In this paper, these facts are proved in a systematic way and a large number of additional properties of the equinoctial elements are derived. In particular, all of the partial derivatives and the Poisson and Lagrange brackets related to these elements are given. For the particular form of the equinoctial elements studied, all the partial derivatives and the Poisson and Lagrange brackets exist for orbits with an inclination of 90° as well as for orbits with zero eccentricity

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and/or zero inclination. With these results, the equinoctial elements can be used for differential correction in orbit determination, as well as for the integration of orbits with special and general perturbations. The corresponding explicit form of the state transition matrix (also called the matrizant or Green's function) can also be written. The matrizant is useful in many applications, particularly in guidance theory and in general-perturbations planetary theory with rectangular coordinates (Broucke, 1969).

The present set of orbit elements is similar to the so-called set-III elements (see Brouwer and Clemence, 1961, p. 241). However, the present elements have the advantage of being defined by finite explicit relations, while the set-III elements are defined by nonintegrable differential relations. In the terminology of classical mechanics, these nonintegrable differential expressions are nonholonomic (see Goldstein, 1950) and the resulting set-III elements are called quasicoordinates (see Corben and Stehle, 1966). The classical orbit elements, or the equinoctial orbit elements, are ordinary coordinates. In addition, the Poisson brackets and the inverse partials for the set-III elements do not exist for zero eccentricity. Therefore, we recommend using the present set of elements rather than the set-III elements.

For the applications in general perturbations, the equinoctial elements have a marked advantage over the universal variables (see Goodyear, 1965, and Escobal, 1965). In the universal variables, the (6×6) matrices $R(t)$ and $R^{-1}(t)$ contain secular terms in all 72 elements. In the classical elements, the set-III elements, or the equinoctial elements, however, only 12 out of the 72 elements contain secular terms. This fact alone results in a saving of computer time by a factor of three in first-order planetary theory applications. For a small eccentricity application such as the planet Venus, the equinoctial elements give another improvement of a factor of two in running time, compared with the classical orbit elements, thanks to the removal of the singularity in the classical partial derivatives at $e=0$. We found that for these planetary theory applications the characteristics of universal variables are not applicable. For this reason we use a less general but still very satisfactory set of variables: the equinoctial orbit elements. Extensive results to be released in the future substantiate these claims.

The following sections give four important (6×6)-matrices related to the equinoctial elements: the fundamental matrix $R(t)$ with partial derivatives of position and velocity, the two matrices with the Poisson and Lagrange brackets and finally the inverse matrix $R^{-1}(t)$. The matrizant $M(t, \tau)$ is then obtained by multiplying the two matrices $R(t)$ and $R^{-1}(\tau)$. Surprisingly these results, although useful, are not found in this definitive form in the literature for the problem of two bodies.

2. Partial Derivatives of Position and Velocity with Respect to Equinoctial Orbit Elements

Several closely related sets of equinoctial orbit elements have been studied in the literature. We will only describe the results relating to the six orbit elements which are defined as follows:

$$\begin{aligned} a &= a \\ h &= e \sin(\omega + \Omega) \\ k &= e \cos(\omega + \Omega) \\ \lambda_0 &= M_0 + \omega + \Omega \\ p &= \tan(i/2) \sin \Omega \\ q &= \tan(i/2) \cos \Omega \end{aligned} \tag{1}$$

where a, e, i, M_0, ω, Ω are the classical elements for which the matrizant is known in detail (Broucke, 1970). It can be seen in the literature that several variations exist in the definition of p and q. Sometimes the factor tan $(i/2)$ is replaced by $\tan i$ or $\sin i$; however, we have not used these two definitions because a close inspection shows that they lead to singularities for polar orbits (Broucke and Cefola, 1971). In the present discussion we will consider only the elements given in (1); they are the same as those used by Arsenault *et al.* (1970).

The partial derivatives of position and velocity vectors with respect to the equinoctial elements $p_\alpha = (a, \lambda_0, h, k, p, q)$ can be derived from the partials with respect to the classical elements $a_\alpha = (a, e, i, M_0, \omega, \Omega)$ as given by Broucke (1970) by using the well-known chain rules:

$$[\partial \mathbf{x}/\partial p_\beta] = [\partial \mathbf{x}/\partial a_\alpha]\,[\partial a_\alpha/\partial p_\beta]\,. \tag{2}$$

To simplify the resulting expressions we will define the following auxiliary vectors

$$\begin{aligned} \boldsymbol{\beta}_1 &= L\mathbf{P} + M\mathbf{Q} \\ \boldsymbol{\beta}_2 &= (1/e)\left[\mathbf{Q}X - \mathbf{P}Y - (\dot{\mathbf{x}}/n)\right] \\ \boldsymbol{\beta}_3 &= \mathbf{Q}X - \mathbf{P}Y \\ \boldsymbol{\beta}_4 &= \left[X \sin(\omega + \Omega) + Y \cos(\omega + \Omega)\right] \mathbf{R} \\ \boldsymbol{\beta}_5 &= \left[X \cos(\omega + \Omega) - Y \sin(\omega + \Omega)\, \mathbf{R}\right. \\ \boldsymbol{\beta}_6 &= \left[X \sin \omega + Y \cos \omega\right] \mathbf{R} \end{aligned} \tag{3}$$

and

$$\begin{aligned} \dot{\boldsymbol{\beta}}_1 &= \dot{L}\mathbf{P} + \dot{M}\mathbf{Q} \\ \dot{\boldsymbol{\beta}}_2 &= (1/e)\left[\mathbf{Q}\dot{X} - \mathbf{P}\dot{Y} + n(a/r)^3\, \mathbf{x}\right] \\ \dot{\boldsymbol{\beta}}_3 &= \mathbf{Q}\dot{X} - \mathbf{P}\dot{Y} \\ \dot{\boldsymbol{\beta}}_4 &= \left[\dot{X} \sin(\omega + \Omega) + \dot{Y} \cos(\omega + \Omega)\right] \mathbf{R} \\ \dot{\boldsymbol{\beta}}_5 &= \left[\dot{X} \cos(\omega + \Omega) - \dot{Y} \sin(\omega + \Omega)\right] \mathbf{R} \\ \dot{\boldsymbol{\beta}}_6 &= \left[\dot{X} \sin \omega + \dot{Y} \cos \omega\right] \mathbf{R}\,. \end{aligned} \tag{4}$$

The matrix R of the partial derivatives of $\mathbf{x}$ and $\dot{\mathbf{x}}$ is given in Table I. Inspection of these results shows that $\partial \mathbf{x}/\partial h$ and $\partial \mathbf{x}/\partial k$ might be undefined for $e = 0$. However the

TABLE I

Matrix R of the partial derivatives of position and velocity with respect to equinoctial orbit elements

$\partial\mathbf{x}/\partial a = 1/a\,(\mathbf{x} - \frac{3}{2}\dot{\mathbf{x}}t)$	$\partial\dot{\mathbf{x}}/\partial a = -(1/2a)\,(\dot{\mathbf{x}} - 3\mu(\mathbf{x}t/r^3))$
$\partial\mathbf{x}/\partial\lambda_0 = \dot{\mathbf{x}}/n$	$\partial\dot{\mathbf{x}}/\partial\lambda_0 = -n(a/r)^3\mathbf{x}$
$\partial\mathbf{x}/\partial h = \boldsymbol{\beta}_1 \sin(\omega+\Omega) + \boldsymbol{\beta}_2 \cos(\omega+\Omega)$	$\partial\dot{\mathbf{x}}/\partial h = \dot{\boldsymbol{\beta}}_1 \sin(\omega+\Omega) + \dot{\boldsymbol{\beta}}_2 \cos(\omega+\Omega)$
$\partial\mathbf{x}/\partial k = \boldsymbol{\beta}_1 \cos(\omega+\Omega) - \boldsymbol{\beta}_2 \sin(\omega+\Omega)$	$\partial\dot{\mathbf{x}}/\partial k = \dot{\boldsymbol{\beta}}_1 \cos(\omega+\Omega) - \dot{\boldsymbol{\beta}}_2 \sin(\omega+\Omega)$
$\partial\mathbf{x}/\partial p = -\cos\Omega\,\sin i\,\boldsymbol{\beta}_3 - (1+\cos i)\boldsymbol{\beta}_5$	$\partial\dot{\mathbf{x}}/\partial p = -\cos\Omega\,\sin i\,\dot{\boldsymbol{\beta}}_3 - (1+\cos i)\dot{\boldsymbol{\beta}}_5$
$\partial\mathbf{x}/\partial q = \sin\Omega\,\sin i\,\boldsymbol{\beta}_3 + (1+\cos i)\boldsymbol{\beta}_4$	$\partial\dot{\mathbf{x}}/\partial q = \sin\Omega\,\sin i\,\dot{\boldsymbol{\beta}}_3 + (1+\cos i)\dot{\boldsymbol{\beta}}_4$

quantity $\boldsymbol{\beta}_2$ can be expanded as:

$$\begin{aligned}\boldsymbol{\beta}_2 &= \mathbf{Q}\{-a + (a^2/r)\cos E[-\cos E + e/(1+(1-e^2)^{1/2})]\} + \\ &+ \mathbf{P}(a^2/r)\sin E[\cos E(1-e^2)^{1/2} + e/(1+(1-e^2)^{1/2})].\end{aligned} \tag{5}$$

We see thus that the derivatives $\partial\mathbf{x}/\partial h$ and $\partial\mathbf{x}/\partial k$ do exist for $e=0$. Note that $\dot{\boldsymbol{\beta}}_2$ can be treated in a similar manner:

$$\dot{\boldsymbol{\beta}}_2 = (1/e)\{\mathbf{Q}[\dot{X} + n(a/r)^3 Y] + \mathbf{P}[-\dot{Y} + n(a/r)^3 X]\} \tag{6}$$

where:

$$\begin{aligned}\dot{X} + n(a/r)^3 Y = &- nae(a/r)^3 \sin E \times \\ &\times [-4\cos E + (1 + 6\cos^2 E)e - 4e^2\cos^3 E + e^3\cos^4 E]/ \\ &[(r/a)^2 + (1-e^2)^{1/2}]\end{aligned} \tag{7}$$

$$\begin{aligned}&-\dot{Y} + n(a/r)^3 X = -nae(a/r)^3 - nae(a/r)^3\cos E \times \\ &\times\{-e + (1-e^2)\cos E[-4 + 6e\cos E - 4e^2\cos^2 E + e^3\cos^3 E]\}/ \\ &[(r/a)^2(1-e^2)^{1/2} + 1].\end{aligned} \tag{8}$$

These expressions clearly demonstrate the existence of $\dot{\boldsymbol{\beta}}_2$ for $e=0$.

3. Lagrange and Poisson Brackets

The Lagrange and Poisson brackets can be derived from the Lagrange and Poisson brackets for the classical elements (Broucke, 1970) by the formulas

$$[(p_\alpha, p_\beta)] = \left[\frac{\partial a_\lambda}{\partial p_\alpha}\right]^T [(a_\lambda, a_\mu)] \left[\frac{\partial a_\mu}{\partial p_\beta}\right] \tag{9}$$

$$[(p_\alpha, p_\beta)] = \left[\frac{\partial p_\alpha}{\partial a_\lambda}\right] [(a_\lambda, a_\mu)] \left[\frac{\partial p_\mu}{\partial a_\mu}\right]^T. \tag{10}$$

We could have used the direct definitions of Lagrange brackets and thus derive the Lagrange brackets from the partial derivatives; we avoid doing that because the partial derivatives involved are too complex. However, these definitions could be used as a verification of our results. The Poisson bracket matrix could also be obtained by inverting the Lagrange bracket matrix, and this would be another possible verification. To simplify the expressions for the Lagrange and Poisson brackets, we introduce the auxiliary variables:

$$\begin{aligned} \alpha_1 &= e/(1+\alpha_2), \\ a_2 &= (1-e^2)^{1/2}, \\ \alpha_3 &= na^2/\alpha_2, \\ \alpha_4 &= na^2\alpha_2, \\ \alpha_5 &= \sin i/(1+\cos i), \\ \alpha_6 &= \sin i/[\alpha_4(1+\cos i)^2]. \end{aligned} \tag{11}$$

We found that there are 11 basic non-zero Poisson brackets and also 11 non-zero Lagrange brackets. They are given in Table II.

TABLE II

Brackets of equinoctial elements[a]

$[a, \lambda_0] = -na/2$	$(a, \lambda_0) = -(2/na)$
$[a, h] = (na\alpha_1/2)\cos(\omega+\Omega)$	$(\lambda_0, h) = -(\alpha_1\alpha_2/na^2)\sin(\omega+\Omega)$
$[a, k] = -(na\alpha_1/2)\sin(\omega+\Omega)$	$(\lambda_0, k) = -(\alpha_1\alpha_2/na^2)\cos(\omega+\Omega)$
$[a, p] = (na\alpha_2/2)\sin i\cos\Omega$	$(\lambda_0, p) = -\alpha_6\sin\Omega$
$[a, q] = -(na\alpha_2/2)\sin i\sin\Omega$	$(\lambda_0, q) = -\alpha_6\cos\Omega$
$[h, k] = -\alpha_3$	$(h, k) = -(1/\alpha_3)$
$[h, p] = -\alpha_3 e\sin(\omega+\Omega)\sin i\cos\Omega$	$(h, p) = -\alpha_6 e\cos(\omega+\Omega)\sin\Omega$
$[h, q] = \alpha_3 e\sin(\omega+\Omega)\sin i\sin\Omega$	$(h, q) = -\alpha_6 e\cos(\omega+\Omega)\cos\Omega$
$[k, p] = -\alpha_3 e\cos(\omega+\Omega)\sin i\cos\Omega$	$(k, p) = \alpha_6 e\sin(\omega+\Omega)\sin\Omega$
$[k, q] = \alpha_3 e\cos(\omega+\Omega)\sin i\sin\Omega$	$(k, q) = \alpha_6 e\sin(\omega+\Omega)\cos\Omega$
$[p, q] = -\alpha_4(1+\cos i)^2$	$(p, q) = [\alpha_4(1+\cos i)^2]^{-1}$

[a] Left column: Lagrange brackets; right column: Poisson brackets.

4. Partial Derivatives of the Equinoctial Orbit Elements with Respect to Position and Velocity

The partial derivatives of the equinoctial orbit elements with respect to position and velocity (the inverse partials) are computed using the Poisson matrix and the R matrix by the following general formulas (see Broucke, 1970):

$$\frac{\partial p_\alpha}{\partial \mathbf{x}} = \sum_{\beta=1}^{6} (p_\alpha, p_\beta)\frac{\partial \dot{\mathbf{x}}}{\partial p_\beta} \tag{12}$$

$$\frac{\partial p_\alpha}{\partial \dot{\mathbf{x}}} = -\sum_{\beta=1}^{6} (p_\alpha, p_\beta)\frac{\partial \mathbf{x}}{\partial p_\beta}. \tag{13}$$

The results for the equinoctial elements are given in Table III. Note that there are no singularities in Table III for $i=0$ or $\pi/2$ and/or $e=0$.

TABLE III

Matrix R^{-1} of the partial derivatives with respect to position and velocity vectors

$\partial a/\partial \mathbf{x} = 2(1/a)(a/r)^3 \mathbf{x}$	$\partial a/\partial \dot{\mathbf{x}} = (2/n^2 a)\dot{\mathbf{x}}$
$\partial \lambda_0/\partial \mathbf{x} = -(1/na^2)[\dot{\mathbf{x}} - (3\mu \mathbf{x} t/r^3) + \alpha_1\alpha_2 \dot{\boldsymbol{\beta}}_1] - (\alpha_5/\alpha_6)\dot{\boldsymbol{\beta}}_6$	$\partial \lambda_0/\partial \dot{\mathbf{x}} = (1/na^2)[-2\mathbf{x} + 3\dot{\mathbf{x}} t + \alpha_1\alpha_2 \boldsymbol{\beta}_1] + (\alpha_5/\alpha_4)\boldsymbol{\beta}_6$
$\partial h/\partial \mathbf{x} = \sin(\omega+\Omega)[-\alpha_1 n(a/r)^3 \mathbf{x} + \dot{\boldsymbol{\beta}}_2]/\alpha_3 - \cos(\omega+\Omega)[\dot{\boldsymbol{\beta}}_1 + e\alpha_5 \dot{\boldsymbol{\beta}}_6/(1-e^2)]/\alpha_3$	$\partial h/\partial \dot{\mathbf{x}} = -\sin(\omega+\Omega)[\alpha_1(\dot{\mathbf{x}}/n) + \boldsymbol{\beta}_2]/\alpha_3 + \cos(\omega+\Omega)[\boldsymbol{\beta}_1 + e\alpha_5 \boldsymbol{\beta}_6/(1-e^2)]/\alpha_3$
$\partial k/\partial \mathbf{x} = \cos(\omega+\Omega)[-\alpha_1 n(a/r)^3 \mathbf{x} + \dot{\boldsymbol{\beta}}_2]/\alpha_3 + \sin(\omega+\Omega)[\dot{\boldsymbol{\beta}}_1 + e\alpha_5 \dot{\boldsymbol{\beta}}_6/(1-e^2)]/\alpha_3$	$\partial k/\partial \dot{\mathbf{x}} = -\cos(\omega+\Omega)[\alpha_1(\dot{\mathbf{x}}/n) + \boldsymbol{\beta}_2]/\alpha_3 - \sin(\omega+\Omega)[\boldsymbol{\beta}_1 + e\alpha_5 \boldsymbol{\beta}_6/(1-e^2)]/\alpha_3$
$\partial p/\partial \mathbf{x} = -[\alpha_4(1+\cos i)]^{-1}\dot{\boldsymbol{\beta}}_4$	$\partial p/\partial \dot{\mathbf{x}} = [\alpha_4(1+\cos i)]^{-1}\boldsymbol{\beta}_4$
$\partial q/\partial \mathbf{x} = -[\alpha_4(1+\cos i)]^{-1}\dot{\boldsymbol{\beta}}_5$	$\partial q/\partial \dot{\mathbf{x}} = [\alpha_4(1+\cos i)]^{-1}\boldsymbol{\beta}_5$

5. Conversion from Equinoctial Elements to Position-Velocity

When integrating numerically perturbed orbits, it is necessary to convert, at each integration step, the equinoctial variables into position and velocity vectors, because in general the perturbing acceleration $\mathbf{F}$ is a function of $\mathbf{x}$ and $\dot{\mathbf{x}}$. In order to have a variation of parameters program which is valid for all eccentricities and inclinations, it is thus necessary to use for the above-mentioned conversion a formulation which is free of singularities. The key to this formulation is in the use of longitudes, λ, F, L, rather than the more classical anomalies M, E, v:

$$\lambda = M + \omega + \Omega, \tag{14.a}$$

$$F = E + \omega + \Omega, \tag{14.b}$$

$$L = v + \omega + \Omega. \tag{14.c}$$

In particular when λ_0 and the time $(t - t_0)$ are known, the mean longitude λ can be computed without difficulty:

$$\lambda = n(t - t_0) + \lambda_0 = M + \omega + \Omega. \tag{15}$$

However, in order to compute the position vector, it is necessary to solve the Kepler equation. As was said before, it is advantageous to write Kepler's equation in terms of the eccentric longitude F rather than the eccentric anomaly E. Some elementary manipulations show that Kepler's equation and the expression for the radius vector r can be written in terms of the eccentric longitude F

$$\lambda = F + h\cos F - k\sin F, \tag{16.a}$$

$$r = a[1 - h\sin F - k\cos F]. \tag{16.b}$$

Note that (16.a) is similar to the Kepler equation given by Deprit and Rom (1970) except that a different definition of F is required. This difference is connected with the definitions of the equinoctial orbit elements used in this paper. Kepler's Equation (16.a) can be solved for F with the standard Newton-Raphson procedure (or any other interation method) once the value of λ has been determined from (15). Once Kepler's equation has been solved, the three coordinates (x, y, z) are then obtained with the following matrix equation:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{1 + p^2 + q^2} \begin{bmatrix} 1 - p^2 + q^2 & 2pq & 2p \\ 2pq & 1 + p^2 - q^2 & -2q \\ -2p & 2q & 1 - p^2 - q^2 \end{bmatrix} \begin{bmatrix} X_1 \\ Y_1 \\ 0 \end{bmatrix}. \tag{17}$$

The quantities $(X_1, Y_1, 0)$ are the coordinates relative to the equinoctial frame. They can be expressed either in terms of L or in terms of F by:

$$\begin{aligned} X_1 &= r\cos L = a[\cos F - k - h(\lambda - F)/(1 + \sqrt{1 - e^2})], \\ Y_1 &= r\sin L = a[\sin F - h + k(\lambda - F)/(1 + \sqrt{1 - e^2})]. \end{aligned} \tag{18}$$

In practice the expression in terms of F will be most useful. It is seen that the above expressions are valid for all eccentricities and inclinations. It is also seen that all the expression are functions of the equinoctial rather than the classical elements. Similarly, all the expressions in Tables I-III of this article could also be expressed in terms of the equinoctial elements. The basic variation of parameter equations can then be written in the equinoctial form:

$$\dot{p}_\alpha = (\partial p_\alpha / \partial \dot{\mathbf{x}}) \cdot \mathbf{F}, \tag{19}$$

where the right hand side of (19) is the dot product of the partials with respect to velocity together with the rectangular components of the perturbing acceleration $\mathbf{F}$.

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