HOMEWORK 03

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PROBLEM 1

For problem 1, we will apply Hori-Lie-Deprit (HLD) theory to the Hamiltonian expressed in Delaunay variables to obtain the new Kamiltonian and associated equations of motion. First, we will define the Hamiltonian of the system and action-angle variables.

$$\mathcal{H}_d = -\frac{1}{2L^2} + \omega H \tag{1}$$

$$\begin{aligned} L &= na^2 \\ G &= L\sqrt{(1-e^2)} \\ H &= G\cos i \\ \ell &= M \\ g &= \omega_p \\ h &= \omega_p + \Omega \end{aligned} \tag{2}$$

In the expressions in Equation [15], n is the mean motion, a is the semi-major axis, e is the eccentricity, i is the inclination, M is the mean anomaly, ω_p is the argument of perigee, and Ω is the right ascension of the ascending node. The Hamiltonian in Equation [1] is expressed in terms of the Delaunay variables and ω which is the rotation of the frame.

Applying HLD, we have the following,

$$\mathcal{H}_0 = -\frac{1}{2L^2} \tag{3}$$

and

$$\mathcal{H}_1 = H. \tag{4}$$

Next, we have the following expression that must be satisfied,

$$\{\mathcal{H}_0, S\} + \mathcal{H}_1 = 0, \tag{5}$$

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where, $\{.,.\}$ is the Poisson bracket of the internal expressions. Computing the Poisson brackets of the expression gives the following,

$$\left[\frac{\partial \mathcal{H}_0}{\partial \ell}\frac{\partial S}{\partial L} - \frac{\partial \mathcal{H}_0}{\partial L}\frac{\partial S}{\partial \ell}\right] + \left[\frac{\partial \mathcal{H}_0}{\partial g}\frac{\partial S}{\partial G} - \frac{\partial \mathcal{H}_0}{\partial G}\frac{\partial S}{\partial g}\right] + \left[\frac{\partial \mathcal{H}_0}{\partial h}\frac{\partial S}{\partial H} - \frac{\partial \mathcal{H}_0}{\partial H}\frac{\partial S}{\partial h}\right] = -\mathcal{H}_1 = -H. \quad (6)$$

Evaluating the partial derivatives and simplifying gives the remaining expression,

$$-\frac{\partial \mathcal{H}_0}{\partial L}\frac{\partial S}{\partial \ell} = -\left(\frac{1}{L^3}\right)\frac{\partial S}{\partial \ell} = -H. \quad (7)$$

Separating the terms and solving the differential equation we obtain the generating function as follows,

$$\partial S = L^3 H \partial \ell \quad (8)$$

$$S = L^3 H \ell. \quad (9)$$

Now that we have obtained our generating function, S, we can continue to transform the set of action-angles to the new set of variables $[L^{'}, G^{'}, H^{'}, \ell^{'}, g^{'}, h^{'}]$.

$$\begin{aligned}
L^{'} &= L + \omega\{L, S\} = L + \omega\left[\frac{\partial L}{\partial \ell}\frac{\partial S}{\partial L} - \frac{\partial L}{\partial L}\frac{\partial S}{\partial \ell}\right] = L - \omega\frac{\partial S}{\partial \ell} \\
G^{'} &= L + \omega\{G, S\} = G + \omega\left[\frac{\partial G}{\partial g}\frac{\partial S}{\partial G} - \frac{\partial G}{\partial G}\frac{\partial S}{\partial g}\right] = G + \omega(0) \\
L^{'} &= H + \omega\{H, S\} = H + \omega\left[\frac{\partial H}{\partial h}\frac{\partial S}{\partial H} - \frac{\partial H}{\partial H}\frac{\partial S}{\partial h}\right] = H + \omega(0) \\
\ell^{'} &= \ell + \omega\{\ell, S\} = \ell + \omega\left[\frac{\partial \ell}{\partial \ell}\frac{\partial S}{\partial L} - \frac{\partial \ell}{\partial L}\frac{\partial S}{\partial \ell}\right] = \ell + \omega\frac{\partial S}{\partial L} \\
g^{'} &= g + \omega\{g, S\} = g + \omega\left[\frac{\partial g}{\partial g}\frac{\partial S}{\partial G} - \frac{\partial g}{\partial G}\frac{\partial S}{\partial g}\right] = g + \omega(0) \\
h^{'} &= h + \omega\{h, S\} = h + \omega\left[\frac{\partial h}{\partial h}\frac{\partial S}{\partial H} - \frac{\partial h}{\partial H}\frac{\partial S}{\partial h}\right] = h + \omega\frac{\partial S}{\partial H}.
\end{aligned} \quad (10)$$

The new set of variables is then written as the following,

$$\begin{aligned}
L^{'} &= L - \omega L^3 H \\
G^{'} &= G \\
H^{'} &= H \\
\ell^{'} &= \ell + \omega 3L^2 H \ell \\
g^{'} &= g \\
h^{'} &= h + \omega L^3 \ell.
\end{aligned} \quad (11)$$

The new Kamiltonian can then be written,

$$\mathcal{K}_0 = -\frac{1}{2L'^2} + \mathcal{O}(\epsilon^2). \tag{12}$$

Finally, applying Hamilton's equations for the new Kamiltonian, we can obtain the equations of motion,

$$\begin{aligned}
\dot{L}' &= \frac{dL^{'}}{dt} = -\frac{\partial \mathcal{K}_0}{\partial \ell} = 0 \\
\dot{G}' &= \frac{dG^{'}}{dt} = -\frac{\partial \mathcal{K}_0}{\partial g} = 0 \\
\dot{H}' &= \frac{dH^{'}}{dt} = -\frac{\partial \mathcal{K}_0}{\partial h} = 0 \\
\dot{\ell}' &= \frac{d\ell^{'}}{dt} = \frac{\partial \mathcal{K}_0}{\partial L} = \frac{1}{L'^3} \\
\dot{g}' &= \frac{dg^{'}}{dt} = \frac{\partial \mathcal{K}_0}{\partial G} = 0 \\
\dot{h}' &= \frac{dh^{'}}{dt} = \frac{\partial \mathcal{K}_0}{\partial H} = 0.
\end{aligned} \tag{13}$$

PROBLEM 2

Using the Hamiltonian expressed in the Delaunay action-angle variables, we can apply Hamilton's equation to obtain the equations of motion as follows,

$$\mathcal{H}_d = -\frac{1}{2L^2} + \omega H, \tag{14}$$

where,

$$\begin{aligned}
L &= na^2 \\
G &= L\sqrt{(1-e^2)} \\
H &= G\cos i \\
\ell &= M \\
g &= \omega_p \\
h &= \omega_p + \Omega
\end{aligned} \tag{15}$$

$$
\begin{aligned}
\dot{L} &= \frac{dL}{dt} = -\frac{\partial \mathcal{H}_d}{\partial \ell} = 0 \\
\dot{G} &= \frac{dG}{dt} = -\frac{\partial \mathcal{H}_d}{\partial g} = 0 \\
\dot{H} &= \frac{dH}{dt} = -\frac{\partial \mathcal{H}_d}{\partial h} = 0 \\
\dot{\ell} &= \frac{d\ell}{dt} = \frac{\partial \mathcal{H}_d}{\partial L} = \frac{1}{L^3} \\
\dot{g} &= \frac{dg}{dt} = \frac{\partial \mathcal{H}_d}{\partial G} = 0 \\
\dot{h} &= \frac{dh}{dt} = \frac{\partial \mathcal{H}_d}{\partial H} = \omega.
\end{aligned} \tag{16}
$$

Using these given constants, $a = 1$, $e = 0.5$, $i = 45°$, we can then compute the Delaunay variables and propagate the state of the body in our system for 100 time units (TU). Since no initial angles were given, an assumption was made that $\ell = g = h = 0°$ for the initial condition. Additionally, the frame rotation from Problem 1 was chosen as ω = 0.01. MATLAB's *ode89*() was used to perform the numerical integration. After integrating the EOMs given above, the trajectory must be converted back to it's Keplerian orbital elements. The Delaunay variables can be rearranged to solve for the relationships to the Keplerian orbital elements as follows,

$$
\begin{aligned}
&a = \sqrt{L/n} \\
&e = \sqrt{1 - \left(\frac{G}{L}\right)^2} \\
&i = \cos^{-1}\left(\frac{H}{G}\right) \\
&\omega_p = g \\
&\Omega = h - \omega_p \\
&M = \ell.
\end{aligned} \tag{17}
$$

Finally, the Keplerian orbital elements can be converted back to the Cartesian state and plotted to obtain the resulting trajectory seen in Figure [1].

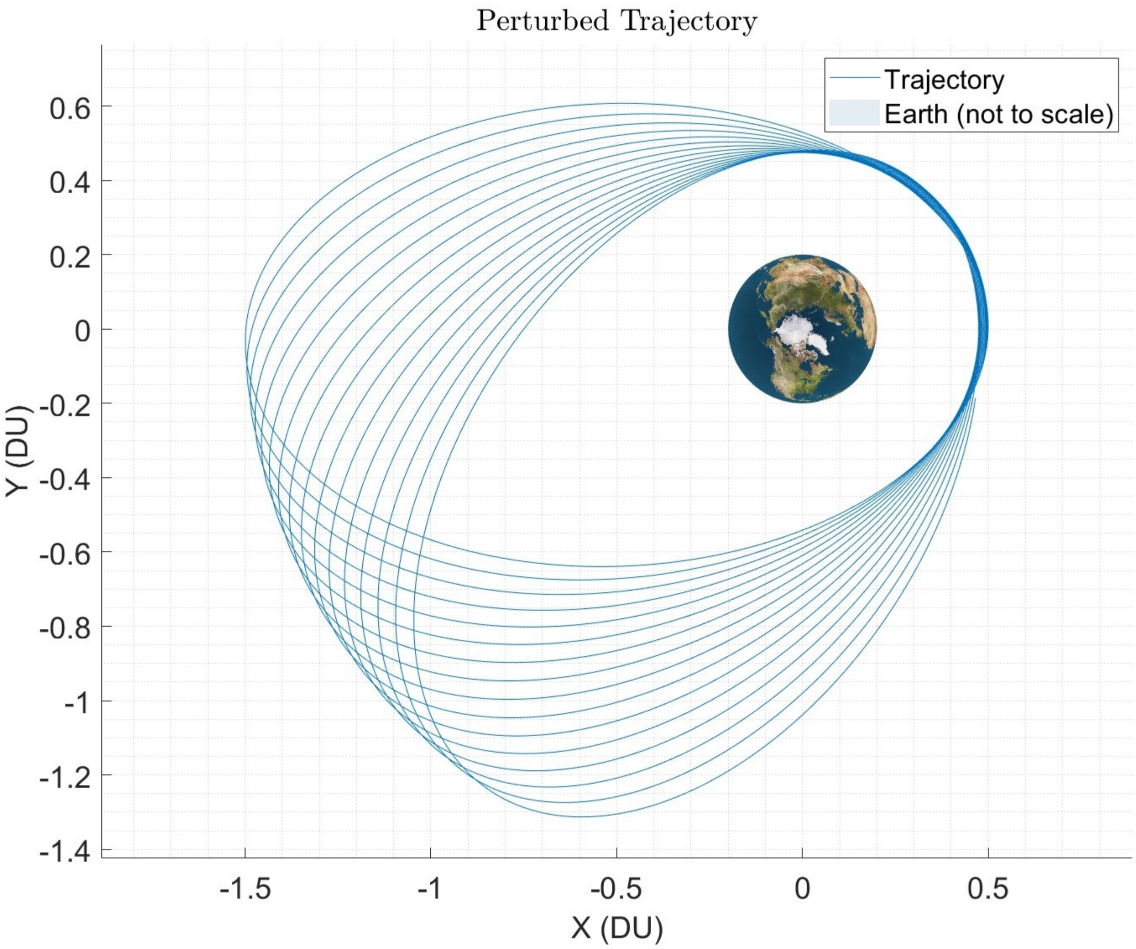

Figure 1: Resulting trajectory, integrated for 100 TU, of the perturbed system using the EOMs derived from the Hamiltonian given in the Delaunay action-angle variables.

GITHUB

Please find the code related to this assignment at the following link to my GitHub repository: https://github.com/brianpatrick3/Advanced-Orbital-Mechanics. The script used to propagate the Delaunay variable EOMs is the *propagateDeluanayVariables.m* script. In order to run the code properly, please make sure you have the CSPICE toolkit for MATLAB (mice) downloaded. If you do not, please go to the Homework1 folder and run the *setup.m* script (this script will automatically download and unzip CSPICE) and add the mice toolkit to the path.