Chapter 2
Preliminaries of Sliding Mode Control

Abstract In this chapter, a brief introduction of the concept of sliding mode control and the evolution of discrete-time sliding mode from the continuous-time sliding mode is discussed. The concept of the multirate output feedback-based sliding mode control technique is also discussed.

Keywords Discrete-time sliding mode control · Output feedback · Reaching law · Multirate output feedback

2.1 Variable Structure Control

Variable Structure Control (VSC) with Sliding Mode Control (SMC) was first presented and elaborated in the 1960s in the then Soviet Union by Emelyanov [9, 13] and other researchers [20, 34]. Since then, VSC has been developed into a general design method being examined for a wide spectrum of system types including nonlinear systems, multi-input–multi-output systems, large scale systems, infinite dimensional systems, and stochastic systems. Also, the objectives of VSC have been extended from stabilization to other control functions. The main feature of the VSC is the invariance to a class of bounded disturbance and parameter variations [10, 34]. In Variable Structure Systems, the system is assumed to consist of continuous subsystems known as structures. These structures are changed or switched depending on the state of the system. The gain of a system may be changed or the transfer function of the system may be completely changed in these types of systems. The times (states) at which the structures change contribute to discontinuity surfaces in the phase planes. These surfaces are called as switching surfaces. If the switching surface satisfies the condition of having positive attraction, then such a surface would become a sliding surface [35].

A simple example of such a variable structure system would be a second-order system having system equations

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= ax_1 + bx_2 + u\end{aligned}\tag{2.1}$$

A. Mehta and B. Bandyopadhyay, *Frequency-Shaped and Observer-Based Discrete-time Sliding Mode Control,* SpringerBriefs in Applied Sciences and Technology, DOI 10.1007/978-81-322-2238-5_2

where x_1, x_2 are system states and a, b are system parameters. The system has feedback input given by

$$u = -\Psi x_1$$

The parameter Ψ is a variable parameter that takes values α and β as the structure changes. Suppose the system with input as α has complex eigenvalues with positive real part and the system with input as β has eigenvalues real but one positive and one negative, then the system trajectories in the two structures are both unstable as shown in Fig. 2.1. The complex eigenvalues give an unstable focus, whereas the one positive and one negative real eigenvalue gives a saddle point.

If we observe the phase portrait carefully, we can notice that the two unstable structures have certain regions of stability, like the describing point moves toward the saddle point along the eigenvector corresponding to the negative eigenvalue. To have the desired regions of the two structures in the resultant system, two switching surfaces are selected.

$$x_1 = 0 \tag{2.2}$$

$$s = cx_1 + x_2 = 0 \tag{2.3}$$

Selecting the switching law from these two equations, we get

$$\Psi = \begin{cases} \alpha, & \text{when } \; x_1 s > 0 \\ \beta, & \text{when } \; x_1 s < 0 \end{cases}$$

The phase portrait of the resultant system is as shown in Fig. 2.2. As we can see the switching surface $x_1 = 0$ has attraction properties only on one side of the surface, so no sliding occurs. But the switching surface s has attraction property on both sides of the surface, as a result this surface becomes the sliding surface of the system. If we look at the resultant motion on the sliding surface, the describing point slides toward the equilibrium point and hence the closed-loop system is stable. A variable

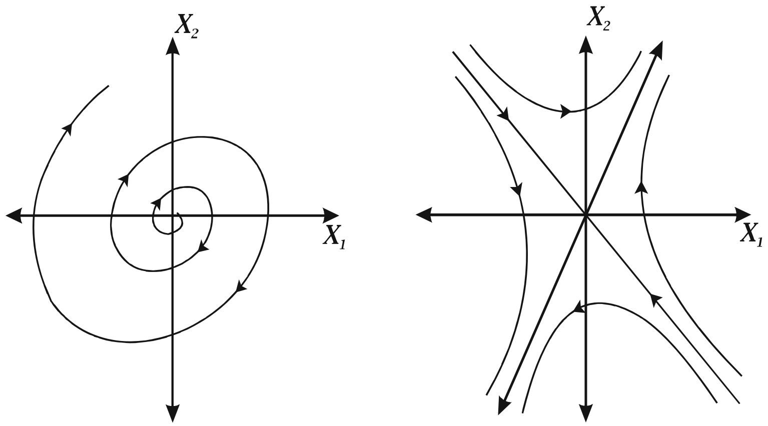

Fig. 2.1 Phase portrait of two unstable structure

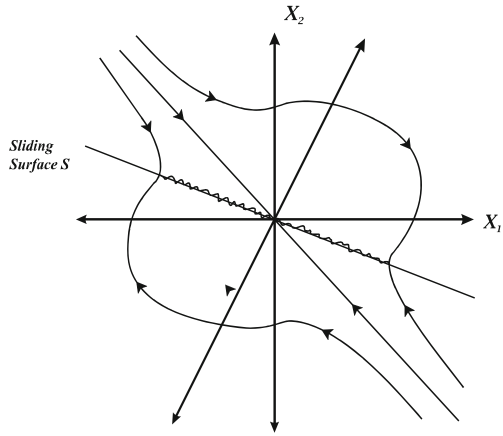

Fig. 2.2 Phase portrait of switched structure

structure system consists of a set of continuous subsystems with a proper switching logic and, as a result, control actions are discontinuous function of system state, disturbance, and reference input. The sliding mode is the principal mode in variable structure systems.

Definition 2.1 Sliding Mode: It is the motion of the system trajectory along a chosen line/plane/surface of the state space.

2.2 Continuous-Time Sliding Mode Control

The sliding mode control can be viewed as a control process consisting of two important phases:

- The reaching phase: The reaching phase is the part where the describing point starts from its initial condition and moves toward the sliding surface. During this period, however, the tracking error cannot be controlled directly and the system response is sensitive to parameter variations and noise. Thus, one would ideally like to shorten the duration or even eliminate the reaching phase. One easy way to minimize the reaching phase and hence the reaching time is to employ a larger control input. This, however, may cause extreme system sensitivity to unmodelled dynamics, actuator saturation, and undesirably higher chattering as well. The robustness of the VSC can be improved by shortening the reaching phase or may be guaranteed during the whole intervals of control action by eliminating the reaching phase. Several methods have been reported in the literature to eliminate the reaching phase completely [7, 8, 24, 30, 38]. The algorithms proposed in these papers employed piecewise constant sliding lines, i.e., the lines which move step by step and not continuously. This causes existence of many short reaching phases after each instant when the line is moved and does not allow to ensure system

insensitivity. This issue was resolved by Bartoszevicz [1] where the piecewise constant lines were replaced with continuously time-varying lines which indeed eliminate the reaching phase. The concept is applied further to several systems [3].
- The sliding phase: This is the phase in which the describing point moves only on the desired sliding surface. In this phase, the describing point does not necessarily follow any system trajectory that was present in the original fixed input system. This is because at the sliding surface the input continuously switches, and the system description is essentially discontinuous.

To find the equation of the system along the sliding surface many methods have been proposed. This is due to the fact that the differential equation has a nonanalytic right-hand side, which is the relay-type discontinuity. Consider a nth order system represented in the phase variable form

$$\dot{x}_i = x_{i+1}, \quad i = 1, 2, \ldots, n-1 \tag{2.4}$$
$$\dot{x}_n = -a_n x_n + \cdots + a_1 x_1 + Bu \tag{2.5}$$

The sliding surface is defined as

$$s(t) = C_s x(t). \tag{2.6}$$

The vector C_s consist of coefficients that describe the sliding surface in terms of the state vector $x(t)$. The sliding surface defined in such a way is called a hyperplane. The surface need not be a plane (or line in case of second-order system) always, the surface can be of any shape. In that case, the vector C_s is the gradient of the sliding surface, let us say G. If the sliding surface is a plane, then the gradient of the matrix is the matrix itself. The value of s specifies the distance of the point from the sliding surface, hence $s = 0$ implies the point that is on the sliding surface.

Defining the sliding surface as

$$s = c_{s1}x_1 + c_{s2}x_2 + \cdots + c_{sn-1}x_{n-1} + x_n = 0 \tag{2.7}$$
$$x_n = -c_{s1}x_{s1} - c_{s2}x_2 - \cdots - c_{sn-1}x_{n-1} \tag{2.8}$$
$$\dot{x}_n = -c_{s1}x_2 - c_{s2}x_3 - \cdots - c_{sn-2}x_{n-1} + \sum_{i=1}^{n-1} c_{sn-1}c_{si}x_i. \tag{2.9}$$

Thus, the entire dynamics of the system is governed by the sliding line/surface parameters only.

$$\dot{x}_i = x_{i+1}, \quad i = 1, 2, \ldots, n-1 \tag{2.10}$$
$$\dot{x}_n = -c_{s1}x_2 - c_{s2}x_3 - \cdots - c_{sn-2}x_{n-1} + \sum_{i=1}^{n-1} c_{sn-1}c_{si}x_i. \tag{2.11}$$

The system dynamics are independent of system parameters and determined by the surface parameters C_s only.

At the outset, two important properties are achieved during the sliding motion, that is, robustness and order reduction. However, to induce the sliding mode the following properties should exist:

- The system stability confined to sliding surface;
- Sliding mode should start in finite time.

The condition for first requirement is obtained as given below. Consider the system in regular form as

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 \tag{2.12}$$

$$\dot{x}_2 = A_{21}x_1 + A_{22}x_2 + Bu \tag{2.13}$$

If the sliding surface is designed as

$$s = \begin{bmatrix} k & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{2.14}$$

then the system dynamics confined to the sliding surface $s = kx_1 + x_2 = 0$ is given by

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 = (A_{11} - A_{12}k)x_1. \tag{2.15}$$

If k is so designed that $A_{11} - A_{12}k$ has eigenvalues on LHP only, then the dynamics of x_1 is stable. Since $kx_1 + x_2 = 0$, the dynamics of x_2 is also stable. Hence, if the sliding surface is designed as $s = C_s x = kx_1 + x_2$, then the system dynamics confined to $s = 0$ is stable.

The second requirement is that the sliding mode should start in finite time. In the sliding phase, the describing point is supposed to move along the chosen surface. This in turn dictates that the sliding surface should be such that it has on both sides state trajectories corresponding to the two structures coming into it. If s is the distance of the describing point from the surface, then positive value of s implies that the point is above the sliding surface, whereas a negative value of s implies the point is below the sliding surface. $\dot{s}$ is the rate of change of distance from the sliding surface. Hence for the sliding motion to exist on the surface, the condition that needs to be satisfied is

$$s\dot{s} < 0. \tag{2.16}$$

This is called the '*reachability condition*'. Next, it is shown here that the reachability condition is also not sufficient for the sliding. To show that, consider the example

$$\dot{s} = -s, \tag{2.17}$$

$$s\dot{s} = -s^2, \forall s \neq 0. \tag{2.18}$$

For which the solution for $s(t)$ is given by

$$s(t) = e^{-t}s(0), \tag{2.19}$$

that gives $s(t) = 0$ as $t \to \infty$. So, it takes infinite time to reach on the surface as it approaches the surface. To overcome the situation another condition is defined as

$$s\dot{s} < -\eta|s|, \quad \eta > 0 \tag{2.20}$$

This condition is known as '*η-reachability condition*' that defines the minimum rate of convergence.

2.2.1 Reaching Law Approach

In the reaching law approach, the dynamics of the sliding function is directly expressed. Let the dynamics of the switching function be specified by the differential equation

$$\dot{s} = -qf(s) - k\mathrm{sgn}(s) \tag{2.21}$$

$$q, k > 0 \tag{2.22}$$

$$sf(s) > 0, \quad \forall s \neq 0. \tag{2.23}$$

The control law may be obtained directly by the condition $\dot{s} = 0$ for the system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{2.24}$$

$$y(t) = Cx(t), \tag{2.25}$$

as

$$u(t) = -(CB)^{-1}(CAx(t) + qf(s) + k\mathrm{sgn}(Cx(t))) \tag{2.26}$$

Similarly, the other reaching laws proposed in the literature are

- Constant rate reaching law

$$\dot{s} = -k\mathrm{sgn}(s)$$

- Constant—proportional rate

$$\dot{s} = -qs - k\mathrm{sgn}(s)$$

- Power-rate reaching law

$$\dot{s} = -k|s|^{\alpha}\mathrm{sgn}(s), \quad 0 < \alpha < 1$$

2.2.2 Fillipov's Condition

The VSS dynamics is characterized by differential equation with discontinuous right-hand side. Filippov [15] first gave the solution for this type of system. The resultant

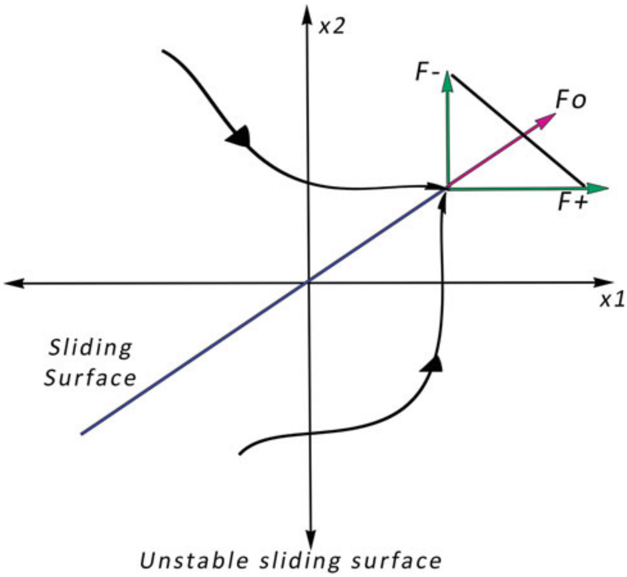

Fig. 2.3 Fillipov condition

direction of motion of the describing point along the sliding surface is specified by the Fillipov's Vector (F_0). According to Fillipov if the state motion vector on one side of the sliding surface is F^+ and on the other side of the surface is F^-, then the resultant vector is given by the convex sum of the two vectors

$$F_0 = \mu F^+ + (1 - \mu)F^-,$$

where $0 < \mu < 1$.

The parameter μ depends on the magnitudes and directions of the vectors F^+, F^- and the gradient of the sliding surface s. In the case of n vectors, this condition is generalized to linear combination of all the vectors where the sum of coefficients of the combination is unity (Fig. 2.3).

2.2.3 Limitations of Continuous-Time Sliding Mode Control

Variable structure control systems are high-speed switching feedback control systems, which are known to be insensitive to matched uncertainties [9]. However, unmatched uncertainties in physical systems may be present and may destroy the stability of the sliding mode. In other words, if the invariance condition (matching condition) is not satisfied, unmatched uncertainties will enter into the dynamics of the system in the sliding mode. Thus, the system behavior in the sliding mode is *not invariant to unmatched uncertainties*. Another obstacle for sliding mode to become useful in practical systems is the high-frequency switching which results in '*chattering phenomenon*'. One of the causes for the chattering phenomenon is the presence of finite time delays for control computation and finite delay in switching. In the absence of switching delays, the switching device switches ideally at an infinite frequency. The second cause is the limitations of physical actuators and sensors, whose dynamics are often neglected. These parasitic dynamics in series with the plant cause small amplitude high-frequency nondecaying oscillations to appear in the neighborhood of the sliding manifold [19]. These oscillations are also referred as chattering. They excite the unmodeled high-frequency dynamics of the system. The chattering is not preferable from a practical point of view because it results in low control accuracy, high heat losses in electrical power circuits, and high wear of moving mechanical parts. Thus, the controller with a high switching frequency will cause fatigue of plant and reduce the service life of a machine. Several solutions are proposed for the reduction of chattering. One of the approaches to reduce chattering is to replace the relay control by saturating continuous approximation [19, 39].

2.3 Discrete-Time Sliding Mode Control

In case of continuous-time sliding mode, once the closed-loop system is driven into the sliding mode, a discontinuous control term switches with infinite frequency and that makes the main difference between a CSMC and a DSMC. The DSMC is automatically constrained to the sampling frequency due to limited sampling frequency. It means the control signal inevitably changes at the sample instances only. Moreover, in DSMC the control input remains constant for the entire sampling period. So, the states can never be on the sliding surface and move in zigzag form called quasi-sliding mode motion [16]. Thus, DSMC does not possess the invariance property which is found in CSMC as the invariance property is achieved only when the system states are exactly on the sliding surface. The robustness issues in DSMC are also still under investigation.

The concept of discrete-time sliding mode was first introduced by Milosavljevic [28] and further extended by Utkin and Drakunow [36]. Since then much work has been done in the field and many new algorithms are proposed.

Similar to CSMC, the design procedure for DSMC includes two steps:

- Computation of sliding surface

$$s(k) = C_s x(k) \tag{2.27}$$

  which has stable internal dynamics and
- Establishing a control law which steers the closed-loop system toward the sliding surface and ensures the system trajectories to stay as close as possible to the surface.

The first step of the design procedure is exactly the same as the design procedure presented for the CSMC. It is assumed that the closed-loop system is kept close enough to the sliding surface to approximate the switching function $s(k)$ by zero. The second step of the design procedure is different for DSMC as compared to CSMC in case of reaching law approach and is presented in the following section. The main difference is in the definition of a reaching law that is not as straightforward as for the continuous-time case.

2.3.1 State-Based Discrete-Time Sliding Mode Control

Consider the continuous-time linear time-invariant system

$$\dot{x}(t) = Ax(t) + Bu(t) + Ef(t), \tag{2.28}$$
$$y(t) = Cx(t), \tag{2.29}$$

where $x \in \Re^n$ is the state variable, $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$ is full rank, $u \in \Re^m$ is the control input, $C \in \Re^{p\times n}$ such that CB is nonsingular and $y \in \Re^p$ is the output. We assume that (A, B) is completely controllable and $m < n$. Let, the system in Eqs. (2.28) and (2.29) be discretized at τ sampling instant, then the discrete-time system is given by

$$x(k+1) = \Phi_\tau x(k) + \Gamma_\tau u(k) + E_\tau f(k), \tag{2.30}$$
$$y(k) = C_\tau x(k). \tag{2.31}$$

Representing the system in regular form as

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ \Gamma_2 \end{bmatrix} u(k) + \begin{bmatrix} 0 \\ E_{\tau 2} \end{bmatrix} f(k) \tag{2.32}$$

with the assumption that the uncertainty changes at the sampling instant only. Various sliding mode control laws have been proposed for the discrete-time system using different reaching laws as given below.

- Sarpturk's Reaching Law: Sarpturk [31] presented a reaching law which is direct discretized version of continuous-time sliding mode is given by

  $$|s(k+1)| < |s(k)|.$$

  Here, the sliding function is always directed toward the surface and also the norm of $s(k)$ monotonically decreases. The reaching law may be written in another way as

  $$(s(k+1) - s(k))\mathrm{sgn}(s(k)) < 0 \tag{2.33}$$
  $$(s(k+1) + s(k))\mathrm{sgn}(s(k)) > 0 \tag{2.34}$$

  The first condition implies that the closed-loop system should be moving in the direction of the sliding surface, whereas the second condition implies that the closed-loop system is not allowed to go too far in that direction. In other words, the condition in Eq. (2.33) results in a lower bound for the control action and the same in Eq. (2.34) results in an upper bound. In [31], following control law is proposed

$$u(k) = -k(x, s)x(k), \tag{2.35}$$

where the gain k is given by

$$k(x, s) = \begin{cases} k^+, & \text{when} \quad x(k)s(k) > 0 \\ k^-, & \text{when} \quad x(k)s(k) < 0 \end{cases}$$

The computation of the coefficients k^+ and k^- is not an easy task. They can be determined by evaluating the conditions (2.33) and (2.34) resulting in an upper and a lower bound for each k^+ and k^-. Indeed, there are circumstances where they do not exist at all.

- Gao's Reaching Law: In order to design a DSMC, the Gao's reaching law [16] is adopted as

$$s(k+1) = (1 - q\tau)s(k) - \rho\tau \operatorname{sgn}(s(k)) \tag{2.36}$$

where, $\tau > 0$ is the sampling time, $q > 0$, $\rho > 0$, and $1 - q\tau > 0$.
The DSMC is required to achieve the following performances [16].

1. Starting from any initial state, the trajectory will move monotonically toward the switching plane and cross it in finite time.
2. Once the trajectory has crossed the switching plane, it will cross the plane again in every successive sampling period, resulting in a zigzag motion about the switching plane.
3. The size of each successive zigzag step is nonincreasing and the trajectory stays within a specified band.

The control law with above reaching law (2.36) is derived for the system (2.30) as

$$u(k) = -(C_s\Gamma_\tau)^{-1}[C_s\Phi_\tau x(k) - (1 - q\tau)s(k) + \rho\tau \operatorname{sgn}(s(k))] \tag{2.37}$$

The magnitude δ_s of quasi-sliding mode band (QSMB) for $s(k)$ that achieves the DSMC performance (2) can be computed by solving Eq. (2.36) for $s(k) = \delta_s$ and $s(k+1) = -s(k)$. So

$$-2\delta_s = -q\tau\delta - \rho\tau$$

That gives,

$$\delta_s = \frac{\rho\tau}{2 - q\tau} \tag{2.38}$$

The control law (2.37) has two parameters ρ and q for tuning the response. From (2.38), ρ is directly proportional to the QSMB, and the system will overshoot when ρ is too large. On the other hand, large ρ could speed up transient response. From

(2.37), $q\tau$ is required to be smaller than one, so q has to be smaller than $1/\tau$, but large q could speed up transient response.

- Bartoszewicz's Reaching Law: Bartoszewicz [2] proposed a reaching law as

$$s(k+1) = d(k) - d_0 + s_d(k+1), \tag{2.39}$$

where the unknown $d(k)$ is defined as $d_1 \leq d(k) = C_s^T \Delta\Phi_\tau x(k) + C_s^T E_\tau f(k) \leq d_u$ with d_l as lower bound and d_u as upper bound. Also d_0 and δ_d is given by

$$d_0 = \frac{d_l + d_u}{2} \text{ and } \delta_d = \frac{d_u - d_l}{2}.$$

$s_d(k)$ is an a priori known function such that the following applies:

– If $s(0) > 2\delta_d$ then

$$s_d(0) = s(0) \tag{2.40}$$
$$s_d(k) \cdot s_d(0) \geq 0 \quad \text{for any} \quad k \geq 0 \tag{2.41}$$
$$s_d(k) \geq 0 \quad \text{for any} \quad k \geq k^* \tag{2.42}$$
$$|s_d(k+1)| < |s_d(k)| - 2\delta_d \quad \text{for any} \quad k \leq k^* \tag{2.43}$$

The above relations state that the time-dependent hyperplane monotonically, and in a finite time, converges from its initial position to the origin of the state space. Furthermore, in each control step, the hyperplane moves by the distance greater than $2\delta_d$. This, together with (2.39), implies that the reaching condition is satisfied, even in the case of the worst combination of disturbance in any two consecutive time steps.

– Otherwise, i.e., if $s(0) < 2\delta_d$ then $s_d(k) = 0$ for any $k \geq 0$.

The constant k^* in the above relations, is a positive integer chosen by the designer in order to achieve good trade off between the fast convergence rate of the system and the magnitude of the control required to achieve this convergence rate. The control law that satisfies the reaching law in Eq. (2.39) can be computed for system in Eq. (2.30) as

$$u(k) = -(C_s^T \Gamma_\tau)^{-1}(C_s^T \Phi_\tau x(k) + d_0 - s_d(k+1)) \tag{2.44}$$

The control law so designed guarantee that for any $k \geq k^*$, the system states satisfy the inequality

$$|s(k)| = |d(k-1) - d_0| \leq \delta_d. \tag{2.45}$$

Hence, the states of the system settle within a quasi-sliding mode band whose width is less than half the width of the band achieved by the control law proposed in [16]

- Linear Reaching Law: Edwards [10] and also Hui and Zak [18] have given reaching law in another way as

$$s(k+1) = \Phi s(k). \tag{2.46}$$

This reaching law is similar to the Gao's reaching law as well to Sarpturk's reaching law. However, the above reaching law gives an exact description of the desired trajectory toward the sliding surface. Despite the fact that this trajectory cannot be achieved due to the unknown disturbance, the design of the controller is fairly straightforward. Using the system Eq. (2.32) and neglecting the unknown disturbance term, the control law is obtained as

$$u(k) = (\Phi - \Phi_{22})s(k) - \Phi_{21}x_1(k) \tag{2.47}$$

The quasi-sliding mode band is given by

$$\frac{1}{1-\Phi}(f_{\max}(k)), \tag{2.48}$$

$f_{\max}(k)$ is the maximum value of the disturbance and Φ has all the eigenvalues inside the unit circle.

- Linear Reaching Law with Disturbance Estimation: The smallest quasi-sliding mode band is obtained with the linear reaching law where $\Phi = 0$, but still the quasi-sliding mode band has the same norm as the upper bound for the disturbance. For the Gao's reaching law, the minimum quasi-sliding mode band is even twice the maximum norm of the disturbance. To overcome this problem, a disturbance estimator is introduced. Define disturbance estimator $\tilde{d}(k)$ by:

$$\tilde{d}(k) = \tilde{d}(k-1) + s(k) - \Phi s(k-1) \tag{2.49}$$

where $\tilde{d}(k)$ is the estimation of disturbance vector projected on $s(k)$. The above control law (2.47) is changed as

$$u(k) = (\Phi - \Phi_{22})s(k) - \Phi_{21}x_1(k) - \tilde{d}(k) \tag{2.50}$$

In this case, the sliding mode band is given as

$$\frac{1}{1-\Phi}\delta f(k), \tag{2.51}$$

where $\delta f(k)$ is the maximum rate of change of the disturbance vector.

2.3.2 Output Feedback-Based Discrete-Time Sliding Mode Control

The VSS approach is quite successful in the design of state feedback controller for robust control. But if only the output is accessible, then one needs to utilize output feedback or state estimator (observer). The continuous-time output feedback VSC systems consists of nonlinear and linear parts for the systems with disturbances and/or uncertainties. There have been fundamentally two approaches to design the linear part under the output feedback scheme. The first one is to use state observers [13, 18] and the second one, direct output-based controllers such as static gains types

[6, 17, 18, 26, 27] and dynamic compensators types [11, 12]. Emelyanov et al. [14] proposed an observer to use the very same method as the state feedback VSC. Hui and Zak [18] also constructed an observer-based output feedback controller and even a simpler controller with a static output feedback structure. Kwan [26, 27], Hsu and Lizarralde [17] maintained the linear part as simple as possible, and instead introduced dynamics into the nonlinear part, which allowed them to handle a larger class of matched uncertainties. Edwards and Spurgeon [11], and Edwards et al. [12] considered dynamic variable structure compensators. Especially, they systematically developed a switching surface design method using a dynamic compensator.

The Output Feedback Discrete-time Sliding Mode control (ODSMC) obtained attention recently [5, 32, 33]. Misawa [29] proposed the Observer-Based Sliding Mode Control (OBDSMC) and applied to the position control of single-stage hard disk drive actuators. The algorithm facilitates assignment of eigenvalues for the system matrix which defines the tracking error dynamics inside the boundary layer. Recently, there have been efforts to design the multirate or fast sampled Output Feedback-Based Sliding Mode Control (MROFSMC) where the available output is measured at a faster rate than the input actuation rate and by means of that the states are obtained implicitly [21, 22]. The MROFSMC is applied to various applications like Nuclear Reactor Control, Power Systems, Stepper Motor, Smart Structure, etc.

2.3.3 Multirate Output Feedback Based Sliding Mode Control

In the design of sliding mode controller based on the state feedabck methods, observers are often used to estimate the state vector. The advantage of using an observer is that the observer design can be separated from the controller design and therefore the complete design is simplified. Nonetheless, the introduction of the observer increases the additional complexity. Recently, much work is done on multirate output feedback-based control which guarantee the closed loop stability, while retaining the structural simplicity of the static output feedback [4, 23, 25]. The term multirate includes the situation wherein the system output is sampled at a faster rate compared to the control input.

Consider the continuous-time linear time-invariant system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{2.52}$$

$$y(t) = Cx(t), \tag{2.53}$$

where $x \in \Re^n$ is the state variable, $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$ is full rank, $u \in \Re^m$ is the control input, $C \in \Re^{p\times n}$ and $y \in \Re^p$ is the output. We assume that (A, B) is completely controllable and $m < n$. Let, the system in Eqs. (2.52) and (2.53) be discretized at τ sampling instant, then the discrete-time system is given by

$$x(k+1) = \Phi_\tau x(k) + \Gamma_\tau u(k), \tag{2.54}$$

$$y(k) = C_\tau x(k). \tag{2.55}$$

It has been shown by Werner [37] and Janardhanan et al. [21, 22] that the fast sampled output data can be used for state estimation instead of using state observer. In this process, the output measurement is done at N-times faster rate than the input updates and they are related as $\Delta = \tau/N$ where $N \geq \nu$, the observability index of (Φ_τ, C). To realize the fast sampled output system, a fictitious lifted system is constructed for which Δ is considered to be the sampling time at which the output is measured.

Let the system in Eqs. (2.54) and (2.55) be sampled at Δ s is given as

$$x(k+1)\Delta = \Phi_\Delta x(k) + \Gamma_\Delta u(k), \tag{2.56}$$

$$y(k) = Cx(k). \tag{2.57}$$

Definition 2.2 Observability Index: The observability index of a system (Φ, Γ, C) is the smallest positive integer ν such that

$$\text{Rank}\left(\begin{bmatrix} C \\ C\Phi \\ \vdots \\ C\Phi^{\nu-1} \end{bmatrix}\right) = \text{Rank}\left(\begin{bmatrix} C \\ C\Phi \\ \vdots \\ C\Phi^{\nu} \end{bmatrix}\right) \tag{2.58}$$

The relationship between the system parameters of the so-called 'τ' system and the 'Δ' system is given as

$$\Phi_\tau = \Phi_\Delta^N; \qquad \Gamma_\tau = \sum_{i=1}^{N-1} \Phi_\Delta^i \Gamma_\Delta. \tag{2.59}$$

Then, the lifted system with the output sampled at an interval Δ s and the control input update interval τ s would be

$$x(k+1) = \Phi_\tau x(k) + \Gamma_\tau u(k), \tag{2.60}$$

$$y_{k+1} = C_0 x(k) + D_0 u(k), \tag{2.61}$$

where y_k, C_0 and D_0 are defined in [37] as

$$y_k = \begin{bmatrix} y((k-1)\tau) \\ y((k-1)\tau + \Delta) \\ y((k-1)\tau + 2\Delta) \\ \vdots \\ y(k\tau - \Delta) \end{bmatrix}; C_0 = \begin{bmatrix} C \\ C\Phi_\Delta \\ C\Phi_\Delta^2 \\ \vdots \\ C\Phi_\Delta^{N-1} \end{bmatrix}; D_0 = \begin{bmatrix} 0 \\ C\Gamma_\Delta \\ C(\Phi_\Delta\Gamma_\Delta + \Gamma_\Delta) \\ \vdots \\ C\sum_{i=1}^{N-2} \Phi_\Delta^i \Gamma_\Delta \end{bmatrix} \tag{2.62}$$

From Eq. (2.61), we may write

$$x(k) = (C_0^T C_0)^{-1} C_0^T (y_{k+1} - D_0 u(k)).$$

Further,

$$\begin{aligned} x(k+1) &= \Phi_\tau[(C_0^T C_0)^{-1} C_0^T (y_{k+1} - D_0 u(k))] + \Gamma_\tau u(k), \\ &= \Phi_\tau (C_0^T C_0)^{-1} C_0^T y_{k+1} + \left(\Gamma_\tau - \Phi_\tau (C_0^T C_0)^{-1} C_0^T D_0\right) u(k) \\ x(k+1) &= L_y y_{k+1} + L_u u(k), \end{aligned} \tag{2.63}$$

where

$$\begin{aligned} L_y &= \Phi_\tau (C_0^T C_0)^{-1} C_0^T, \\ L_u &= \Gamma_\tau - \Phi_\tau (C_0^T C_0)^{-1} C_0^T D_0. \end{aligned}$$

Further from Eq. (2.63),

$$x(k) = L_y y_k + L_u u(k-1). \tag{2.64}$$

Thus, the state $x(k)$ can be expressed using fast sampled output stack and past input. The state computation by fast sampled output measurement is better than the conventional discrete-time state observer as it computes the states just in one sampling instant [21] compared to discrete-time observer that takes at least ν instants (ν is the observability index of the system). Moreover, it does not increase the order of the overall system dimension and so reduces the complexity.

Using the Gao's reaching law (2.36), the state feedback control law for the discrete-time LTI system of form (2.54) can be derived as

$$u(k) = -(C_s \Gamma_\tau)^{-1}((C_s \Phi_\tau - C_s + q\tau C_s)x(k) + \rho\tau \text{sgn}(C_s x(k))) \tag{2.65}$$

The above state feedback control algorithm (2.65) can be converted into an output feedback algorithm by the multirate output feedback. Substituting $x(k)$ from (2.64) into (2.65), the multirate output feedback-based sliding mode control law is derived as

$$u(k) = F_y y_k + F_u u(k-1) - (C_s \Gamma_\tau)^{-1} \rho\tau \text{sgn}(C_s L_y y_k + C_s L_u u(k-1)), \tag{2.66}$$

where

$$\begin{aligned} F_y &= -(C_s \Gamma_\tau)^{-1}(C_s \Phi_\tau - C_s + q\tau C_s) L_y, \\ F_u &= -(C_s \Gamma_\tau)^{-1}(C_s \Phi_\tau - C_s + q\tau C_s) L_u, \\ L_y &= \Gamma_\tau - L_y D_0, \\ L_u &= \Phi_\tau (C_0^T C_0)^{-1} C_0^T \end{aligned}$$

2.4 Conclusion

In this chapter, first we presented the basic concept of the continuous-time sliding mode and variable structure control. It is evident that a discontinuous control in a continuous time induces a sliding motion in some manifold of the state space. The existence condition of the sliding mode and the stability of the system during sliding mode is also discussed. Moreover, a design method for sliding mode control law based on reaching law is presented. Due to the wide use of digital controllers, it is in demand today to develop discrete-time sliding mode control. In this chapter, we presented the evolution of discrete-time sliding mode along with various reaching laws for the state-based discrete-time sliding mode. The output feedback-based discrete-time sliding mode control strategy is also discussed. Lastly, the multirate output feedback technique for state estimation and the multirate output feedback-based sliding mode control design method are discussed.

References

1. Bartoszewicz A (1995) A comment on A time-varying sliding surface for fast and robust tracking control of second-order uncertain systems. Automatica 31(12):1893–1895
2. Bartoszewicz A (1998) Discrete-time quasi sliding mode control strategies. IEEE Trans Ind Electron 45(04):633–637
3. Bartoszewicz A, Nowacka-Leverton A (2009) Time-varying sliding modes for second and third order systems. Springer, Berlin
4. Chammas AB, Leondes CT (1979) Pole assignment by piecewise constant output feedback. Int J Control 29(01):31–38
5. Chan CY (1999) Discrete adaptive sliding mode control of a class of stochastic systems. Automatica 35:1491–1498
6. Choi HH (2002) Variable structure output feedback control design for a class of uncertain dynamic systems. Automatica 38:335–341
7. Choi SB, Cheong CC, Park DW (1993) Moving switching surfaces for robust control of a second order variable structure systems. Int J Control 58(01):229–245
8. Choi S, Park D, Jayasuriya S (1994) A time-varying sliding surface for fast and robust tracking control of second order uncertain systems. Automatica 30:899–904
9. Drazenovic B (1969) The invariance conditions in variable structure systems. Automatica 05:287–295
10. Edwards C (1998) Sliding mode control: theory and applications. Taylor and Francis, London
11. Edwards C, Spurgeon SK (1998) Compensator based output feedback sliding mode control design. Int J Control 71(04):601–614
12. Edwards C, Spurgeon SK, Hebden RG (2003) On the design of sliding mode output feedback controllers. Int J Control 76(09/10):893–905
13. Emelyanov SV (1967) Variable structure control system (In Russian). Nauka, Moscow
14. Emelyanov SV, Korovin SK, Nersisian AL, Nisenzon YE (1992) Output feedback stabilization of uncertain plants: a variable structure approach. Int J Control 55:61–81
15. Filippov AF, FM A (1988) Differential equations with discontinuous right-hand sides: control systems. Kluwer Academic Publishers, Soviet Union
16. Gao W, Wang Y, Homaifa A (1995) Discrete-time variable structure control system. IEEE Trans Ind Electron 42(02):117–122

17. Hsu L, Lizarralde F (1998) Comments and further results on variable structure output feedback controllers. IEEE Trans Autom Control 43(09):1338–1340
18. Hui S, Zak SH (1999) On discrete-time variable structure sliding mode control. Syst Control Lett 38(4–5):283–288
19. Hung JY, Gao W, Hung JC (1993) Variable structure control: a survey. IEEE Trans Autom Control 40(01):2–22
20. Itkis U (1976) Control systems of variable structure. Wiley, New York
21. Janardhanan S, Bandyopadhyay B (2006) Output feedback sliding mode control for uncertain systems using fast output sampling technique. IEEE Trans Ind Electron 53(05):1677–1682
22. Janardhanan S, Bandyopadhyay B (2007) Multirate output feedback based robust quasi-sliding mode control of discrete-time systems. IEEE Trans Autom Control 52(03):499–503
23. Jury E (1967) A note on multirate sampled-data systems. IRE Trans Autom Control 12(03):319–320
24. Kaynak O, Denker A (1993) Discrete-time sliding mode control in the presence of system uncertainty. Int J Control 57(05):1177–1189
25. Kranc GM (1957) InputOutput analysis of multi-rate feedback system. IRE Trans Autom Control 03:21–28
26. Kwan CM (1996) On variable structure output feedback controllers. IEEE Trans Autom Control 41(11):1691–1693
27. Kwan CM (2001) Further results on variable output feedback controllers. IEEE Trans Autom Control 46(09):1505–1508
28. Milosavljevic C (1985) General conditions for the existence of quasi sliding mode on the switching hyperplane in discrete variable structure systems. Autom Remote Control 46:307–314
29. Richter H, Misawa EA (2002) Boundary layer eigenvalues in observer based discrete-time sliding mode control. In: Proceedings of the American control conference, Anchorage, AK, pp 2985–2936
30. Roy RG, Olgac N (1997) Robust nonlinear control via moving sliding surfaces—n-th order case. In: Proceedings of 36th conference on decision and control, San Diego, pp 943–948
31. Sarpturk SZ, Istefanopulos Y, Kaynak O (1987) On the stability of discrete-time sliding mode control systems. IEEE Trans Control Syst 32(10):930–932
32. Sharav-Schapiro N, Palmor J, Steinberg A (1996) Robust output feedback stabilizing control for discrete uncertain SISO systems. lEEE Trans Autom Control 41(09):1377–1381
33. Sharav-Schapiro N, Palmor J, Steinberg A (1998) Output stabilizing robust control for discrete uncertain systems. Automatica 34(06):731–739
34. Utkin VI (1977) Variable structure systems with sliding mode. IEEE Trans Autom Control 22(02):212–222
35. Utkin VI (1992) Sliding modes in control optimization. Springer, New York
36. Utkin VI, Drakunov SV (1989) On discrete-time sliding modes control. In: Proceedings of IFAC conference on nonlinear control, Capri, Italy, pp 484–489
37. Werner H (1998) Multimodel robust control by fast output sampling—an LMI approach. Automatica 34(12):1625–1630
38. Yilmaz C, Hurmuzlu Y (2000) Eliminating the reaching phase from variable structure control. ASME J Dyn Syst, Meas, Control 122(04):753–757
39. Young KD, Utkin VI, Ozguner U (1999) A control engineers guide to sliding mode control. IEEE Trans Autom Control 07(03):328–342

http://www.springer.com/978-81-322-2237-8

Frequency-Shaped and Observer-Based Discrete-time Sliding Mode Control
Mehta, A.; Bandyopadhyay, B.
2015, XX, 95 p. 35 illus., Softcover
ISBN: 978-81-322-2237-8