

Let E be a vector space over \mathbb{C} .

Definition 1 (Hermitian) A map $A \in \text{End } E$ is Hermitian iff

$$\langle Ax, y \rangle = \langle x, Ay \rangle$$

Theorem 2 (finite spectral theorem) Suppose $E \cong \mathbb{C}^n$ is hermitian. Then

- E has eigenvectors that are an orthonormal basis of E .
- All eigenvalues of E are real.

Proof. By the fundamental theorem of algebra, the characteristic polynomial

$$|A - xI|$$


has a root. Hence A has an eigenvalue-eigenvector pair λ, e . But

$$\lambda \langle e, e \rangle = \langle e, Ae \rangle = \langle Ae, e \rangle = \bar{\lambda} \langle e, e \rangle$$

thus $\lambda = \bar{\lambda}$. Ergo, $\lambda \in \mathbb{R}$.

Now consider $A|e^\perp$. Suppose $\langle x, e \rangle = 0$. Then

$$0 = \lambda \langle x, e \rangle = \langle x, Ae \rangle = \langle Ax, e \rangle$$

Hence $A|e^\perp \in \text{End}(e^\perp)$. Induction on dimension proves the theorem. 

Let \bullet denote pointwise multiplication.

Corollary 3 (diagonalization) If $A \in \text{End } E$, then

$$A = P^{-1}(v \bullet _)P$$

where P is unitary and $v \in E$ is real.

Definition 4 (standard part of operator)

$$\begin{aligned} \text{st} : \quad \text{End } E &\rightarrow E \\ (\text{st } T)(x) &:= \text{st}(T(*x)) \end{aligned}$$

Theorem 5 (infinite spectral theorem) Suppose E is a Hilbert space. If $A \in \text{End } E$ is hermitian, then

$$A = P^{-1}(v \bullet _)P$$

with P unitary and $v \in P(E)$ real.

Proof. Consider a nonstandard model of functional analysis. Fix a hyperfinite-dimensional subspace F such that

$${}^\sigma E \subseteq F \subseteq {}^*E$$

There is some hermitian $B \in \text{End } F$ such that $B|_{{}^\sigma E} = {}^*A|_{{}^\sigma E}$. This B simultaneously satisfies hermitian-ness and $B({}^*e) = {}^*(Ae)$ for each e in some (standard) basis of E . Such a B exists, internal to a sufficiently saturated model.

By $*$ -transferring [diagonalization](#), there is some unitary $P \in \text{End } F$ and real $v \in F$ such that

$$B = P^{-1}(v \bullet _)P \tag{1}$$

By construction, $B({}^\sigma E) \subseteq {}^\sigma E$.

Let $\tilde{E} = P({}^\sigma E)$ be a standard Hilbert space. With slight abuse of notation, say $\text{st } P : E \rightarrow \tilde{E}$ and $\text{st } v \in \tilde{E}$. Now

$$(\text{st } P)^{-1} = \text{st}(P^{-1})$$

similarly,

$$(\text{st}(v \bullet _))(x) = \text{st}(v \bullet {}^*x) = \text{st } v \bullet x = (\text{st } v \bullet _)x$$

By construction, $\text{st } B = A$, hence eq. (1) becomes

$$A = \text{st}(P)^{-1}(\text{st } v \bullet _) \text{st}(P) \quad \text{👉}$$