Lemma 0.1 (ultrafilter lemma) Let S be a set. Every filter on S is contained in an ultrafilter.

Corollary 0.2 (nonprincipal ultrafilter) For any infinite set S, there is an ultrafilter U on S that is not generated by a singleton. Such a U is nonprincipal.

Proof. Consider the cofinite filter on S:

$$F = \{X \subseteq S : |X| \geqslant \aleph_0\}$$

Then F is contained in a nonprincipal ultrafilter.

Definition 0.3 (ultrapower) Suppose M_i is a model over language $\mathcal{L}$ for each $i \in I$. Let U be a nonprincipal ultrafilter on I. Consider

$$\prod M_i/U$$

where quotienting by U is understood to mean quotienting by the equivalence relation $\sim$ where

$$(x_i)_{i\in I} = (y_i)_{i\in I}$$

iff the set of indices i such that $x_i = y_i$ is in U. (Equality by vote).

We can make $\prod M_i/U$ into a model over $\mathcal{L}$. Interpret the constant symbol c as the equivalence class of

$$(c|_{M_i})_{i\in I}$$

with $c|_{M_i}$ indicating interpretation (of c) in M_i.

Similarly, interpret the function $f(x)$ by

$$f\big((x_i)_{i\in I}\big) = \big[\big(f(x_i)|_{M_i}\big)\big]$$

where $[-]$ denotes 'equivalence class of'.

Finally, interpret the predicate $\phi(x)$ by

$$\phi\big((x_i)_{i\in I}\big) \iff \{i \in I : \phi(x_i)\} \in U$$

which is truth by vote. Because U is an ultrafilter, the law of the excluded middle will in fact hold as either

$$\{i \in I : p|_i \text{ holds}\} \in U$$

or

$$\{i \in I : (\neg p)|_i \text{ holds}\} = U - \{i \in I : p|_i \text{ holds}\} \in U \in U$$

Theorem 0.4 (compactness theorem) If every finite subtheory of Φ has a model, then Φ has a model.

Proof. Let $\mathtt{fin}(\Phi)$ be the set of finite subtheories of Φ. By assumption, for each $F \in \mathtt{fin}(\Phi)$, there is a model M_F of F. For appropriate choice of utlrafilter U on $\mathtt{fin}(\Phi)$, I claim

$$\prod_{F \in \mathtt{fin}(\Phi)} M_F / U$$

models Φ. To see this, pick an arbitrary $\phi \in \Phi$. Define

$$H_\phi = \{f \subseteq \mathtt{fin}(\Phi) : M_f \models \phi\}$$

to be the set of models where ϕ holds. Consider

$$F = \{H_\phi : \phi \in \Phi\}$$

Then F is a filter on $\mathtt{fin}(\Phi)$. By the ultrafilter lemma, F is contained in some ultrafilter U. Hence we have chosen a U such that, for each $\phi \in \Phi$, the set $\{f \in \mathtt{fin}(\Phi) : M_f \models \phi\}$ is in U. Then

$$\prod_{F \in \mathtt{fin}(\Phi)} M_F / U \models \Phi$$

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Corollary 0.5 The ultrafilter lemma implies the compactness theorem.

The converse is also true:

Theorem 0.6 The compactness theorem implies the ultrafilter lemma.

Proof. Let S be a set. Suppose F is a filter on S. Then the theory

$$\begin{aligned}
\Phi = \mathsf{set} \cup \{ & \\
& S \in F \\
& \forall A, B (A \in F \wedge B \in S \implies A \cup B \in F) \\
& \varnothing \notin F \\
& \forall A, B (A \in F \wedge A \subseteq B \implies B \in F) \\
& \} \qquad\qquad\qquad\qquad \cup \{x_f \in F : f \in F\}
\end{aligned}$$

where x_f is a constant symbol naming f for each $f \in F$, has a model M. And $F \subseteq F|_M$.

I claim there is a model of Φ such that $F|_M$ is an ultrafilter. Consider the theory

$$\begin{aligned}
\Phi' = \Phi & \cup \{y_s \subseteq S : s \subseteq S\} \\
& \cup \{y_s \in F \vee S - x_s \in F : s \subseteq S\}
\end{aligned}$$

where y_s is a constant symbol naming s for each $s \subseteq S$.

By the compactness theorem, Φ' has a model. Call it M'. Then

$$\{f \cap S : f \in F|_{M'}\} - \varnothing$$

is an ultrafilter containing F.

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