1 Equations of motion

1.1 Lagrangian

1 Definition (action) Let $L(q, \dot{q}, t)$ be a Lagrangian. Let q : time $\to$ configuration space. Define action by

$$S(q) := \int_a^b L(q, \dot{q}, t)dt$$

Mechanics will minimize action of the appropriate Lagrangian.

2 Theorem (euler-lagrange)

$$\frac{dS}{dq} = \left\langle \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{d\dot{q}}, - \right\rangle$$

Proof. Let h be an infinitesimal perturbation (endpoints are 0).

$$\begin{aligned} S(q+h) &= \int L(q+h, \dot{q} + d\dot{q}, t)dt \\ &= \int L(q, \dot{q}, t)dt + \int DL(q, \dot{q}, t)(h, d\dot{q}, 0) + o(h) \\ &= \int \frac{\partial L}{\partial q} h dt + \underbrace{\frac{\partial L}{\partial \dot{q}}}_{u} \underbrace{d\dot{q}dt}_{dv} \\ &= \int \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} \right) h dt + \underbrace{\left[h \frac{\partial L}{\partial \dot{q}} \right]_a^b}_{0 \text{ as } h(a) = h(b) = 0} \end{aligned}$$

3 Corollary (stationary action)

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{d\dot{q}} = 0$$