

Let E be a Hilbert space over \mathbb{C} .

1 Definition (Hermitian) A map $A \in \text{End } E$ is Hermitian iff

$$\langle Ax, y \rangle = \langle x, Ay \rangle$$

2 Theorem (finite spectral theorem) Suppose $A \in \mathbb{C}^n$ is hermitian. Then

- A has eigenvectors that are an orthonormal basis of A .
- All eigenvalues of A are real.

Proof. By the fundamental theorem of algebra, the characteristic polynomial

$$|A - xI|$$


has a root. Hence A has an eigenvalue-eigenvector pair λ, e . But

$$\lambda \langle e, e \rangle = \langle e, Ae \rangle = \langle Ae, e \rangle = \bar{\lambda} \langle e, e \rangle$$

thus $\lambda = \bar{\lambda}$. Ergo, $\lambda \in \mathbb{R}$.

Now consider $A|e^\perp$. Suppose $\langle x, e \rangle = 0$. Then

$$0 = \lambda \langle x, e \rangle = \langle x, Ae \rangle = \langle Ax, e \rangle$$

Hence $A|e^\perp \in \text{End}(e^\perp)$. Induction on dimension proves the theorem. 

For vectors v, u , let vu denote pointwise multiplication.

3 Corollary (diagonalization) If $A \in \text{End } E$, then

$$A = P^{-1}(v_-)P$$

where P is unitary and $v \in E$ is real.

4 Definition (standard part of map)

$$\begin{aligned} \text{st}_X : (*X \rightarrow *Y) &\rightarrow (X \rightarrow Y) \\ (\text{st } f)(x) &:= \text{st}(f(*x)) \end{aligned}$$

5 Theorem (infinite spectral theorem) If $A \in \text{End } E$ is hermitian, then

$$A = P^{-1}(v_-)P$$

with P unitary and $v \in P(E)$ real.

Proof. This will be proved via hyperfinite approximation. Consider a nonstandard model of functional analysis. Fix a hyperfinite-dimensional subspace F such that

$${}^\sigma E \subseteq F \subseteq {}^*E$$

I claim A extends to a hermitian $B \in \text{End } F$, i.e. $B|_{{}^\sigma E} = {}^*A|_{{}^\sigma E}$. This desired B simultaneously satisfies hermitian-ness and $B({}^*b) = {}^*(Ab)$ for each b in some (standard) basis of E ; such a B exists, internal to a sufficiently saturated model.

By $*$ -transferring [diagonalization](#) (corollary 3), there is some unitary $P \in \text{End } F$ and real $v \in F$ such that

$$B = P^{-1}(v_-)P \tag{1}$$

It suffices to recover the standard diagonalization from the hyperfinite case. Let

$$\tilde{E} = P({}^\sigma E)$$

be a standard Hilbert space. Define $\tilde{\text{st}}(x) = y$ when $y \in \tilde{E}$ and $x \simeq y$. By construction of \tilde{E} ,

$$\begin{aligned} \tilde{\text{st}} P &= P|_E &: E &\rightarrow \tilde{E} \\ \text{st}(P^{-1}) &= P^{-1}|_{\tilde{E}} &: \tilde{E} &\rightarrow E \end{aligned}$$

hence $\text{st}(P^{-1}) = (\tilde{\text{st}} P)^{-1}$ and

$$\tilde{\text{st}}(v_-)x = \tilde{\text{st}}(v^*x) = (\tilde{\text{st}} v)x$$

By construction, $\text{st } B = A$; hence eq. (1) becomes

$$\begin{aligned} A &= \text{st}(P^{-1}(v_-)P) \\ &= \text{st}(P^{-1}v_-) \tilde{\text{st}} P \\ A &= (\tilde{\text{st}} P)^{-1} (\tilde{\text{st}} v_-) \tilde{\text{st}} P \end{aligned}$$

Continuity of P^{-1} and v proves the final equation. 