Let E be a vector space over $\mathbb{C}$.

1 Definition (Hermitian) A map $A \in \operatorname{End} E$ is Hermitian iff

$$\langle Ax, y \rangle = \langle x, Ay \rangle$$

2 Theorem (finite spectral theorem) Suppose $A \in \mathbb{C}^n$ is hermitian. Then

- A has eigenvectors that are an orthonormal basis of A.
- All eigenvalues of A are real.

Proof. By the fundamental theorem of algebra, the characterestic polynomial

$$|A - xI|$$

has a root. Hence A has an eigenvalue-eigenvector pair λ, e. But

$$\lambda \langle e, e \rangle = \langle e, Ae \rangle = \langle Ae, e \rangle = \overline{\lambda} \langle e, e \rangle$$

thus $\lambda = \overline{\lambda}$. Ergo, $\lambda \in \mathbb{R}$.

Now consider $A|e^{\perp}$. Suppose $\langle x, e \rangle = 0$. Then

$$0 = \lambda \langle x, e \rangle = \langle x, Ae \rangle = \langle Ax, e \rangle$$

Hence $A|e^{\perp} \in \operatorname{End}(e^{\perp})$. Induction on dimension proves the theorem.

For vectors v, u, let vu denote pointwise multiplication.

3 Corollary (diagonalization) If $A \in \operatorname{End} E$, then

$$A = P^{-1}(v\_)P$$

where P is unitary and $v \in E$ is real.

4 Definition (standard part of map)

$$\begin{aligned} \mathrm{st}_X : \quad (*X \to *Y) &\to (X \to Y) \\ (\mathrm{st}\, f)(x) &:= \mathrm{st}\big(f(^*x)\big) \end{aligned}$$

5 Theorem (infinite spectral theorem) Suppose E is a Hilbert space. If $A \in \operatorname{End} E$ is hermitian, then

$$A = P^{-1}(v\_)P$$

with P unitary and $v \in P(E)$ real.

Proof. This will be proved via hyperfinite approximation. Consider a nonstandard model of functional analysis. Fix a hyperfinite-dimensional subspace F such that

$$^{\sigma}E \subseteq F \subseteq {}^{*}E$$

There is some hermitian $B \in \operatorname{End} F$ such that $B|{}^{\sigma}E = {}^{*}A|{}^{\sigma}E$. This B simultaneously satisfies hermitian-ness and $B({}^{*}e) = {}^{*}(Ae)$ for each e in some (standard) basis of E. Such a B exists, internal to a sufficiently saturated model.

By $*$−transferring diagonalization (corollary 3), there is some unitary $P \in \operatorname{End} F$ and real $\nu \in F$ such that

$$B = P^{-1}(\nu\_)P \tag{1}$$

It suffices to recover the standard diagonalization from the hyperfinite case. Let

$$\widetilde{E} = P({}^{\sigma}E)$$

be a standard Hilbert space. Define $\widetilde{\operatorname{st}}(x) = y$ when $y \in \widetilde{E}$ and $x \simeq y$. By construction of $\widetilde{E}$,

$$\begin{array}{rclcccc} \widetilde{\operatorname{st}}\,P & = & P|_{E} & : & E & \to & \widetilde{E} \\ \operatorname{st}\left(P^{-1}\right) & = & P^{-1}|_{\widetilde{E}} & : & \widetilde{E} & \to & E \end{array}$$

hence $\operatorname{st}\left(P^{-1}\right) = \left(\widetilde{\operatorname{st}}\,P\right)^{-1}$ and

$$\widetilde{\operatorname{st}}(\nu\_)x = \widetilde{\operatorname{st}}(\nu\,{}^{*}x) = (\widetilde{\operatorname{st}}\,\nu)x$$

By construction, $\operatorname{st} B = A$, hence eq. (1) becomes

$$\begin{aligned} A &= \operatorname{st}\left(P^{-1}(\nu\_)P\right) \\ &= \operatorname{st}\left(P^{-1}\nu\_\right)\widetilde{\operatorname{st}}\,P \\ A &= \left(\widetilde{\operatorname{st}}\,P\right)^{-1}\left(\widetilde{\operatorname{st}}\,\nu\_\right)\widetilde{\operatorname{st}}\,P \end{aligned}$$

Continuity of P^{-1} and ν proves the final equation. 👍