## 1 taylor series

1 **Lemma** (mean value theorem: integration) If $f : [a,b] \to_{\texttt{top}} \mathbb{R}$ and $\phi : [a,b] \to_{\texttt{top}} [0,\infty)$ then there is some $x : [a,b]$ s.t.

$$\int_a^b f(t)\phi(t) = f(x)\int_a^b \phi(t)\,dt$$

[Wik15b]

*Proof.* Consider the case where $\int_a^b \phi = 1$. Then it suffices to show for some $x : [a,b]$, we have $f(x) = \mathbb{E}(f; d\phi)$. By compactness, we can find some $x_{\texttt{max}} : [a,b]$ maximizing $f$ and $x_{\texttt{min}} : [a,b]$ minimizing $f$. As $\mathbb{E}(f; d\phi) : [f(x_{\texttt{min}}), f(x_{\texttt{max}})]$, connectedness gives the desired $x$.

2 **Definition** (Taylor polynomial) Given $f : C^N(\mathbb{R};\mathbb{R})$, define its $N$-degree Taylor polynomial centered at $x_0$ as

$$\texttt{taylor}^N_{x_0} f(x) :\equiv \sum_{n=0}^{N} D^n f(x_0)\frac{(x-x_0)^n}{n!}$$

3 **Theorem** (Taylor's theorem: Lagrange remainder) For $f : C^{N+1}(\mathbb{R};\mathbb{R})$,

$$f(x) = \texttt{taylor}^N f(x) + \frac{1}{N!}\int_{x_0}^{x} D^{N+1} f(t)(x-t)^N\,dt$$

and for some $x^* : (x_0, x)$,

$$f(x) = \texttt{taylor}^N f(x) + D^{N+1} f(x^*)\frac{(x-x^*)^{N+1}}{(N+1)!}$$

*Proof.* Suppose $f : C^{N+1}(\mathbb{R};\mathbb{R})$. By the fundamental theorem of calculus,

$$f(x) = f(x_0) + \int_{x_0}^{x} Df(t)\,dt = f(x_0) + \int_{x_0}^{x} \underbrace{D^1 f(t)}_{u}\underbrace{(x-t)^0\,dt}_{dv}$$

Then $du = D^2 f(t)$ and $v = -(x-t)$. Integrating by parts gives

$$\begin{aligned}
f(x) &= f(x_0) - Df(t)(x-t)\Big|_{t=x_0}^{x} + \int_{x_0}^{x} D^2 f(t)(x-t)\,dt \\
&= f(x_0) + Df(x_0)(x-x_0) + \int_{x_0}^{x} \underbrace{D^2 f(t)}_{u}\underbrace{(x-t)\,dt}_{dv} \\
&= f(x_0) + \cdots + Df(x_0)\frac{(x-x_0)^{n-1}}{(n-1)!} + \frac{1}{(n-1)!}\int_{x_0}^{x} \underbrace{D^n f(t)}_{u}\underbrace{(x-t)^{n-1}\,dt}_{dv}
\end{aligned}$$

$$f(x) = \sum_{n=0}^{N} D^n f(x_0)\frac{(x-x_0)^n}{n!} + \frac{1}{N!}\int_{x_0}^{x} D^{N+1} f(t)(x-t)^N\,dt$$

The $x^*$ is given by mean value theorem: integration.

4 Corollary (Cauchy's formula for repeated integration)

$$D^{-n}f(x) = \frac{1}{(n-1)!}\int_{x_0}^{x} f(t)(x-t)^{n-1}\,dt$$

[Wik15a]

5 Corollary (Taylor's theorem: no remainder) For $f : C^N(\mathbb{R};\mathbb{R})$,

$$f(x_0+h) = \mathtt{taylor}^N f(x_0+h) + o(h^N)$$

Proof. By Taylor's theorem: Lagrange remainder,

$$f(x_0+h) = \mathtt{taylor}^{N-1} f(x_0+h) + D^N f(x^*)\frac{(x^*-x_0)^N}{N!}$$

Suppose $h$ is infinitesimal. Then $x^* = x_0 + h^*$, where $h^* : [0, h]$. By continuity,

$$D^N f(x_0+h^*)\frac{(h^*)^N}{N!} = D^N f(x_0)\frac{(h^*)^N}{N!} + O(h^*)\frac{(h^*)^N}{N!}$$

So

$$D^N f(x^*) = D^N f(x_0) + O(h^{N+1})$$

## 2 analytic functions

1 Definition A function $f : C^\infty(\mathbb{R};\mathbb{R})$ is analytic in an open interval $(a, b)$ iff $f|(a, b) = \mathtt{taylor}^\infty f|(a, b)$. The function is analytic on $\mathbb{R}$ iff for each $x : \mathbb{R}$, there is some open interval containing $x$ on which $f$ is analytic. The space of analytic functions is denoted $C^\omega(\mathbb{R};\mathbb{R})$.

2 Definition A function in $C^\omega(\mathbb{C};\mathbb{C})$ is called entire.

3 Lemma (ratio test) Let $a_n : \mathbb{C}$ be a sequence. If

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1 \tag{1}$$

then $\sum_{n=0}^{\infty} a_n$ converges absolutely. [Wik15d]

Proof. (Adapted from [Wik15d]) For absolute convergence, it suffices to consider the case where $a_n = |a_n| : [0, \infty)$. Suppose eq. (1) holds. Let

$$r :\equiv \lim_{n\to\infty}\frac{\frac{a_{n+1}}{a_n}+1}{2} < 1 \tag{2}$$

For some $N$, for all $n \geqslant N$, eq. (2) gives $a_{n+1} < ra_n$. But then $a_n < r^{n-N}a_N$, so

$$\sum_{n=N}^{\infty} a_n \leqslant \sum_{n=N}^{\infty} r^{n-N}a_N \to 0$$

as $r < 1$.

4 Definition (exp) Let

$$\exp(x) :\equiv \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

By ratio test

$$\left| \frac{x^{n+1}}{(n+1)!} \frac{n!}{x^n} \right| = \left| \frac{x}{n+1} \right| \to 0$$

Hence $\exp : \mathbb{C} \to \mathbb{C}$ is well-defined and entire.

5 Lemma (exp homeomorphism)

$$\exp(x+y) = \exp(x)\exp(y)$$

Proof.

$$\begin{aligned} \exp(x+y) &= \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=1}^{n} \binom{n}{k} \frac{x^{n-k} y^k}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{k=1}^{n} \frac{x^{n-k} y^k}{(n-k)!k!} \\ \exp(x)\exp(y) &= \left( \sum_{\alpha=0}^{\infty} \frac{x^\alpha}{\alpha!} \right) \left( \sum_{\beta=0}^{\infty} \frac{y^\beta}{\beta!} \right) = \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} \frac{x^\alpha}{\alpha!} \frac{y^\beta}{\beta!} \end{aligned}$$

with equivalence given by setting $\alpha = n - k$ and $\beta = k$.

6 Corollary $\exp(xn) = [\exp(x)]^n$

7 Lemma $(\exp(it))^* = \exp(-it)$

Proof.

$$\begin{aligned} \exp(it) &= \sum_{n=0}^{\infty} \frac{(it)^{2n}}{(2n)!} + \frac{(it)^{2n+1}}{(2n+1)!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n (t)^{2n}}{(2n)!} + i \frac{(-1)^n (t)^{2n+1}}{(2n+1)!} \\ \exp(-it) &= \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} - i \frac{(-1)^n t^{2n+1}}{(2n+1)!} \end{aligned}$$

8 Theorem (Euler's theorem) The map $\exp(i-) : \mathbb{R} \to \mathbb{C}$ is a universal covering map of the unit circle. In particular, $\exp(it)$ is the point on the unit circle $t$ radians counterclockwise from 1.

Proof. As $|\exp(it)| = \exp(it)\exp(-it) = 1$, its image is contained in $S = \{z : |z| = 1\}$. Note $D\exp(it) = i\exp(it)$, so $\exp(it)$ moves clockwise. Finally, consider arc length.

$$s(t) = \int_0^t |D\exp(it)| dt = \int_0^t |i\exp(it)| dt = \int_0^t dt = t$$

9 Example (hyperbolic trig)

$$\begin{aligned}
\sinh(z) &= \frac{e^z - e^{-z}}{2} \\
\cosh(z) &= \frac{e^z + e^{-z}}{2} \\
\tanh(z) &= \frac{\sinh(z)}{\cosh(z)} = \frac{e^z - e^{-z}}{e^z + e^{-z}}
\end{aligned}$$

They are entire.

10 Example (trig)

$$\begin{aligned}
\sin(z) &= -i\sinh(iz) \\
\cos(z) &= \cosh(iz) \\
\tan(z) &= \frac{\sin z}{\cos z}
\end{aligned}$$

They are entire. Note that for $x : \mathbb{R}$,

$$\begin{aligned}
\sin(x) &= \Re\exp(ix) \\
\cos(x) &= \Im\exp(ix)
\end{aligned}$$

11 Definition (ln) Let $\ln : [0, \infty) \to_{\texttt{top}} \mathbb{R}$ be the inverse of exp. The inverse function theorem guarantees its existence.

12 Theorem The map $\ln(x)$ is analytic on $(0, 2)$.

Proof. By the inverse function theorem,

$$\begin{aligned}
D\ln(x) &= x^{-1} \\
D^2\ln(x) &= -x^{-2} \\
D^n\ln(x) &= (-1)^{n+1}(n-1)!x^{-n}
\end{aligned}$$

So

$$\texttt{taylor}_1^\infty \ln(x) = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{(x-1)^n}{n}$$

But ratio test gives

$$\left|\frac{(x-1)^{n+1}}{n+1}\frac{n}{(x-1)^n}\right| = |x-1|\frac{n}{n+1} \to |x-1|$$

so $\mathtt{taylor}_1^\infty \ln(x)$ converges for $x : (0,2)$.

By Taylor's theorem: Lagrange remainder, for some $t$ between $1$ and $x$,

$$\begin{aligned}
\ln(x) - \mathtt{taylor}_1^n \ln(x) &= \frac{(-1)^{n+1}t^{-n-1}(x-t)^{n+1}}{n+1} \\
|\ln(x) - \mathtt{taylor}_1^n \ln(x)| &= \left|\frac{t^{-n-1}(x-t)^{n+1}}{n+1}\right| \\
&\leqslant \left|\frac{\max(1, x^{-n-1})(x-1)^{n+1}}{n+1}\right| \to 0
\end{aligned}$$

when $x : (0,2)$.

13 Corollary The map $\sqrt{x}$ is analytic on $(0,2)$.

Proof.

$$\sqrt{x} = \sqrt{\exp \ln x} = \exp\left(\frac{1}{2}\ln x\right)$$

14 Example ($\mathtt{taylor}\sqrt{x}$)

$$\begin{aligned}
D(x)^{1/2} &= \frac{1}{2}x^{-1/2} \\
D^2(x)^{1/2} &= -\frac{1}{4}x^{-3/2} \\
D^n(x)^{1/2} &= (-1)^n\frac{(2n-1)!!x^{1/2-n}}{(1-2n)2^n}
\end{aligned}$$

Thus

$$\mathtt{taylor}_1^\infty \sqrt{x} = \sum_{n=0}^{\infty} \frac{(-1)^n(2n-1)!!(x-1)^n}{(1-2n)2^n n!}$$

15 Lemma (double factorial)

$$(2n-1)!! = \frac{(2n)!}{2^n n!} \tag{3}$$

Proof. Base case: $n = 1$. Note $1!! = 1$. Likewise, note $2!/2 = 1$. Hence eq. (3) holds for $n = 1$.

Suppose for the sake of induction that eq. (3) holds. Note $(2n+1)!! = (2n+1)(2n-1)!!$. Consider

$$\frac{(2n+2)!}{2^{n+1}(n+1)!} = \frac{(2n+2)(2n+1)(2n)!}{2(2^n)(n+1)n!} = (2n+1)n!!$$

16 Corollary

$$\texttt{taylor}_1^\infty \sqrt{x} = \sum_{n=1}^{\infty} \frac{(-1)^n (2n)!(x-1)^n}{(1-2n)(n!)^2(4^n)}$$

as in [Wik15e].

## 3 power series

1 Theorem (radius of convergence) Let $P$ be a power series, $a : \mathbb{C}$. If $P(a)$ converges, then for all $z : \mathbb{C}$ such that $|z| < |a|$, we have $P(z)$ converges absolutely [Nee12].

Proof. Let $P(z) = c_0 + c_1 z^1 + c_2 z^2 \dots$. By hypothesis, $P(a) = c_0 + c_1 a + c_2 a^2 + \dots$ converges. Then $c_n a^n \to 0$, so we can find some $M$ such that $|c_n a^n| \leqslant M$ for all $n$. For $|z| < |a|$, thus $|z|/|a| < 1$,

$$\sum_{n=N}^{\infty} |c_n z^n| = \sum_{n=N}^{\infty} |c_n||a|^n \frac{|z|^n}{|a|^n} \leqslant M \sum_{n=N}^{\infty} \frac{|z|^n}{|a|^n} = \frac{M \frac{|z|^N}{|a|^N}}{1 - \frac{|z|}{|a|}} \to 0$$

Hence $P(z)$ converges absolutely. [Nee12]. 👍

2 Corollary If $P(a)$ diverges, then $P(z)$ diverges for all $z : \mathbb{C}$ such that $|z| > |d|$ [Nee12].

3 Theorem (identity theorem) Let $z_n : \mathbb{C}$ be a sequence such that $z_n \to 0$. Let $P$ and $Q$ be power series. If $P(z_n) = Q(z_n)$, then $P = Q$. [Nee12].

Proof. Let $P(z) = a_0 + a_1 z + a_2 z^2 \dots$ and $Q(z) = b_0 + b_1 z + b_2 z^n \dots$.

$$P(z_n) = a_0 + a_1 z_n + a_2 z_n^2 = b_0 + b_1 z_n + b_2 z_n^2 \cdots = Q(z_n) \tag{4}$$

Then as $\lim P(z_n) = \lim Q(z_n)$, we have $a_0 = b_0$. Hence by eq. (4), we get

$$\begin{aligned} a_1 z_n + a_2 z_n^2 \dots &= b_1 z + b_2 z_n^2 \dots \\ a_1 + a_2 z_n^1 + a_3 z^2 \dots &= b_1 + b_2 z_n^1 + b_3 z_n^2 \dots \end{aligned}$$

Taking a limit gives $a_1 = b_1$. Repeating this inductively gives $P = Q$. 👍

## 4 non-analytic smooth functions

1 Counter example

$$f(x) :\equiv \begin{cases} \exp(-x^{-1}), & x > 0 \\ 0, & x \leqslant 0 \end{cases}$$

Smoothness follows from noting

$$D^n f(0) = \lim_{x\to 0} \frac{f(x)}{x^n} = \frac{\exp(-x^{-1})}{x^n} = \lim_{x\to 0} \frac{x^{-n}}{\exp(x^{-1})}$$
$$= \lim_{x\to 0} \frac{x^{-n}}{1 + x^{-1} + x^{-2}/2 + \dots x^{-n}/n! \dots} = 0$$

with,

$$\mathtt{taylor}_0^\infty f(x) = 0$$

hence $f$ is not analytic at $0$. [Wik15c]

## References


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