# 1 Equations of motion

## 1.1 Lagrangian

1 Definition (action) Let $L(q, \dot{q}, t)$ be a Lagrangian. Let $q$ : time $\to$ configuration space. Define action by

$$S(q) := \int_a^b L(q, \dot{q}, t)dt$$

Mechanics will minimize action of the appropriate Lagrangian.

2 Theorem (euler-lagrange)

$$\frac{dS}{dq} = \left\langle \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{d\dot{q}}, - \right\rangle$$

Proof. Let $h$ be an infinitesimal perturbation (endpoints are $0$).

$$\begin{aligned}
S(q+h) &= \int L(q+h, \dot{q}+d\dot{q}, t)dt \\
&= \int L(q, \dot{q}, t)dt + \int DL(q, \dot{q}, t)(h, d\dot{q}, 0) + o(h) \\
&= \int \frac{\partial L}{\partial q} h dt + \underbrace{\frac{\partial L}{\partial \dot{q}}}_{u} \underbrace{d\dot{q}dt}_{dv} \\
&= \int \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right) h dt + \underbrace{\left[h\frac{\partial L}{\partial \dot{q}}\right]_a^b}_{0 \text{ as } h(a) = h(b) = 0}
\end{aligned}$$

3 Corollary (stationary action)

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{d\dot{q}} = 0$$

# 2 conservation laws

## 2.1 energy

1 Definition (energy)

$$E :\equiv \left\langle Dq, \frac{\partial L}{\partial \dot{q}} \right\rangle - L$$

2 Theorem When $D(t \mapsto L(\_, \_, t)) = 0$ (*e.g.* closed system or constant field), energy is conserved.

Proof. By hypothesis,

$$\frac{dL}{dt} = \left\langle \frac{\partial L}{\partial q}, \dot{q} \right\rangle + \left\langle \frac{\partial L}{\partial \dot{q}}, \ddot{q} \right\rangle$$

By euler-lagrange,

$$\frac{dL}{dt} = \left\langle \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}, \dot{q} \right\rangle + \left\langle \frac{\partial L}{\partial \dot{q}}, \ddot{q} \right\rangle$$

By product rule,

$$\frac{dL}{dt} = \frac{d}{dt}\left\langle \dot{q}, \frac{\partial L}{\partial \dot{q}} \right\rangle$$

Subtracting $dL/dt$ finishes the proof.

## 2.2 momentum

3 Definition (momentum)

$$P :\equiv \sum_a \frac{\partial L_a}{\partial \boldsymbol{v}_a}$$

where $a$ indexes over particles.

4 Theorem (conservation of momentum) In a closed system, $\dot{P} = 0$.

Proof. This follows from homogeneity of space. Suppose every particle moves by infinitesimal $\varepsilon$ with velocities unchanged. Then

$$dL = \sum_a \left\langle \frac{\partial L}{\partial L} \partial \boldsymbol{r}_a, \varepsilon \right\rangle$$

which must be $0$ by homogeneity. As $\varepsilon$ is arbitrary, we conclude

$$\sum_a \frac{\partial L}{\partial \mathbf{r}_a} = 0$$

By euler-lagrange,

$$\frac{d}{dt} \sum_a \frac{\partial L}{\partial v_a} = 0$$

5 Corollary (Newton's third law)

$$\sum_a \mathbf{F}_a = 0$$