

1 Equations of motion

1.1 Lagrangian

1 Definition (action) Let $L(q, \dot{q}, t)$ be a Lagrangian. Let $q : \text{time} \rightarrow \text{configuration space}$. Define action by

$$S(q) := \int_a^b L(q, \dot{q}, t) dt$$

Mechanics will minimize action of the appropriate Lagrangian.

2 Theorem (euler-lagrange)

$$\frac{dS}{dq} = \left\langle \frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}}, - \right\rangle$$

Proof. Let h be an infinitesimal perturbation (endpoints are 0).

$$\begin{aligned} S(q+h) &= \int L(q+h, \dot{q}+d\dot{q}, t) dt \\ &= \int L(q, \dot{q}, t) dt + \int DL(q, \dot{q}, t)(h, d\dot{q}, 0) + o(h) \\ &= \int \underbrace{\frac{\partial L}{\partial q}}_u h dt + \underbrace{\frac{\partial L}{\partial \dot{q}}}_{dv} d\dot{q} dt \\ &= \int \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) h dt + \underbrace{\left[h \frac{\partial L}{\partial \dot{q}} \right]_a^b}_{0 \text{ as } h(a) = h(b) = 0} \end{aligned}$$

□

3 Corollary (stationary action)

$$\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = 0$$

2 Conservation laws

2.1 Energy

1 Definition (energy)

$$E := \left\langle Dq, \frac{\partial L}{\partial \dot{q}} \right\rangle - L$$

2 Theorem When $D(t \mapsto L(_, _, t)) = 0$ (e.g. closed system or constant field), energy is conserved.

Proof. By hypothesis,

$$\frac{dL}{dt} = \left\langle \frac{\partial L}{\partial q}, \dot{q} \right\rangle + \left\langle \frac{\partial L}{\partial \dot{q}}, \ddot{q} \right\rangle$$

By [euler-lagrange](#),

$$\frac{dL}{dt} = \left\langle \frac{d}{dt} \frac{\partial L}{\partial \dot{q}}, \dot{q} \right\rangle + \left\langle \frac{\partial L}{\partial \dot{q}}, \ddot{q} \right\rangle$$

By product rule,

$$\frac{dL}{dt} = \frac{d}{dt} \left\langle \dot{q}, \frac{\partial L}{\partial \dot{q}} \right\rangle$$

Subtracting dL/dt finishes the proof. □

2.2 Momentum

3 Definition (momentum)

$$P := \sum_a \frac{\partial L_a}{\partial v_a}$$

where a indexes over particles.

4 Theorem (conservation of momentum) In a closed system, $\dot{P} = 0$.

Proof. This follows from homogeneity of space. Suppose every particle moves by infinitesimal ϵ with velocities unchanged. Then

$$dL = \sum_a \left\langle \frac{\partial L}{\partial r_a}, \epsilon \right\rangle$$

which must be 0 by homogeneity. As ϵ is arbitrary, we conclude

$$\sum_a \frac{\partial L}{\partial r_a} = 0$$

By [euler-lagrange](#),

$$\frac{d}{dt} \sum_a \frac{\partial L}{\partial v_a} = 0$$

□

5 Corollary (Newton's third law)

$$\sum_a \mathbf{F}_a = 0$$

6 Definition (generalized momenta)

$$p_i := \frac{\partial L}{\partial \dot{q}_i}$$

7 Definition (generalized force)

$$F_i := \dot{p}_i$$

2.3 center of mass

8 Definition (Center of mass)

$$\mathbf{R} := \frac{\sum m_a \mathbf{r}_a}{\sum m_a}$$

A system is *at rest* ($\mathbf{P} = 0$) in a frame of reference where $\mathbf{R} = 0$.

9 Definition (internal energy) The energy of a system at rest, denoted E_i .

10 Theorem Energy in different reference frames. Put $\mathbf{v}'_a = \mathbf{v}_a + \mathbf{V}$. Then

$$E = E' + \langle \mathbf{V}, \mathbf{P}' \rangle + \frac{1}{2} \mu \mathbf{V}^2$$

Proof.

$$\begin{aligned} E &= \frac{1}{2} \sum m_a |\mathbf{v}_a|^2 + U \\ &= \frac{1}{2} \sum m_a |\mathbf{v}'_a + \mathbf{V}|^2 + U \\ &= \frac{1}{2} \mu \mathbf{V}^2 + \left\langle \mathbf{V}, \sum m_a \mathbf{v}'_a \right\rangle + \frac{1}{2} \sum m_a |\mathbf{v}'_a|^2 + U \end{aligned}$$

□

11 Corollary

$$E = \frac{1}{2} \mu |\mathbf{V}|^2 + E_i$$

12 Corollary

$$L = L' + \langle \mathbf{v}, \mathbf{P}' \rangle + \frac{1}{2} \mu |\mathbf{V}|^2$$

13 Corollary

$$S = S' + \mu \langle \mathbf{v}, \mathbf{P}' \rangle + \frac{1}{2} \mu |\mathbf{V}|^2 t$$

2.4 Angular momentum

14 Theorem $SO(3)$ is a Lie group with Lie algebra

$$\mathfrak{so}(3) \cong (\mathbb{R}^3)^* \wedge \mathbb{R}^3$$

Proof. Let

$$f(x) := x^T x$$

Then

$$SO(3) \cong f^{-1}(I)$$

But

$$\begin{aligned} f(I + dx) &= (I + dx)^T (I + dx) \\ &= I^T I + dx^T I + I^T dx + dx^T dx \end{aligned}$$

Hence

$$\begin{aligned} Df|_I(h) &= h + h^T \\ Df|_x &= (x_-) Df|_I(x^{-1}_-) \end{aligned}$$

so

$$\mathfrak{so}(3) \cong \ker Df \cong \mathbb{R}^3 \wedge \mathbb{R}^3$$

□

15 Definition (angular momentum)

$$\mathbf{M} := \sum \mathbf{r}_a \times \mathbf{p}_a$$

16 Theorem (conservation of angular momentum) In a closed system,

$$\dot{\mathbf{M}} = 0$$

Proof. Let $d\phi \in \mathfrak{so}(3)$ be an infinitesimal rotation.

$$d\mathbf{r} = d\phi \times \mathbf{r}$$

But the Lagrangian is unchanged by $d\phi$, hence

$$\begin{aligned} dL &= \sum_a \left(\frac{\partial L}{\partial \mathbf{r}_a} d\mathbf{r}_a + \frac{\partial L}{\partial \mathbf{v}_a} d\mathbf{v}_a \right) = 0 \\ \sum \langle \dot{\mathbf{p}}_a, d\phi \times \mathbf{r}_a \rangle + \langle \mathbf{p}_a, d\phi \times \mathbf{v}_a \rangle &= 0 \\ \left\langle d\phi, \sum (\mathbf{r}_a \times \dot{\mathbf{p}}_a + \mathbf{v}_a \times \mathbf{p}_a) \right\rangle &= 0 \\ \left\langle d\phi, \frac{d}{dt} \sum \mathbf{r}_a \times \mathbf{p}_a \right\rangle &= 0 \end{aligned}$$

But $d\phi$ is arbitrary, hence

$$\frac{d}{dt} \sum \mathbf{r}_a \times \mathbf{p}_a = 0$$

□

17 Theorem (\mathbf{M} in a different frame) If $\mathbf{r}_a = \mathbf{r}'_a + \mathbf{a}$, then

$$\mathbf{M} = \mathbf{M}' + \mathbf{a} \times \mathbf{P}$$

Proof.

$$\begin{aligned}\mathbf{M} &= \sum \mathbf{r}_a \times \mathbf{p}_a \\ &= \sum \mathbf{r}'_a \times \mathbf{p}_a + \mathbf{a} \times \sum \mathbf{p}_a \\ &= \mathbf{M}' + \mathbf{a} \times \mathbf{P}\end{aligned}$$

□

18 Corollary If L (equivalently, U) is symmetric around the axis with unit \mathbf{e} , then $\langle \mathbf{M}, \mathbf{e} \rangle$ is conserved.

19 Corollary If L is symmetric around the origin, then \mathbf{M} is conserved.

3 Integration of the equations of motion

3.1 One dimension

$$L = \frac{1}{2}m\dot{x} - U(x)$$

so

$$t = \int_{x_0}^{x_1} dx \sqrt{\frac{m/2}{E - U(x)}}$$

3.2 Pendulum

Consider the pendulum

$$z = -ir \exp(i\phi)$$

with

$$K = \frac{1}{2}m|\dot{z}|^2 = \frac{1}{2}mr^2|\dot{\phi}|^2$$

$$U = -mgr \cos \phi$$

$$L = r|\dot{\phi}|^2 + g \cos \phi$$

3 Integration of the equations of motion

Let $\omega^2 = g/r$. By [euler-lagrange](#)

$$\begin{aligned}\ddot{\phi} + \omega^2 \sin \phi &= 0 \\ \dot{\phi} \ddot{\phi} + \dot{\phi} \omega^2 \sin \phi &= 0 \\ \frac{d}{dt} \left(\frac{\dot{\phi}^2}{2} - \omega^2 \cos \phi \right) &= 0\end{aligned}$$

Integrating gives

$$\begin{aligned}0 &= \left(\frac{\dot{\phi}^2}{2} - \omega^2 \cos \phi \right)_0^t \\ \dot{\phi}^2 &= 2\omega^2 \cos \phi \Big|_0^t + \dot{\phi}(0)^2\end{aligned}$$

Hence

$$\pm t = \int_{\phi(0)}^{\phi} \frac{d\phi}{\sqrt{2\omega^2 \cos \phi \Big|_0^t + \dot{\phi}(0)^2}}$$

But $\cos \phi \Big|_0^t = 2 \sin^2(\phi/2) \Big|_t^0$, so

$$\begin{aligned}\pm t &= \int_{\phi(0)}^{\phi} \frac{d\phi}{\sqrt{4\omega^2 \sin^2 \frac{\phi}{2} \Big|_t^0 + \dot{\phi}(0)^2}} \\ \pm 2\omega t &= \int_{\phi(0)}^{\phi} \frac{d\phi}{\sqrt{\sin^2 \frac{\phi(0)}{2} + \frac{\dot{\phi}(0)^2}{4\omega^2} - \sin^2 \frac{\phi}{2}}}\end{aligned}$$

Let

$$k = \left(\sin^2 \frac{\phi(0)}{2} + \frac{\dot{\phi}(0)^2}{4\omega^2} \right)^{-1/2}$$

Then

$$\begin{aligned}\pm \frac{2\omega t}{k} &= \int_{\phi(0)}^{\phi} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \frac{\phi}{2}}} \\ \pm \frac{\omega t}{k} &= \int_{\phi(0)/2}^{\phi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}\end{aligned}$$

1 Definition (Incomplete elliptic: 1st kind)

$$F(u, k) = \int_0^u \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}$$

2 Definition (sn)

$$\text{sn}(-, k) = \sin(F^{-1}(-, k))$$

So

$$\phi(t) = 2 \arcsin \text{sn} \left(F(2^{-1} \phi(0), k) \pm \frac{\omega t}{k}, k \right)$$

3.3 Two body problem

As the system is closed, the center of mass is inertial. Set this as the origin.

$$\frac{m_1 z_1 + m_2 z_2}{m_1 + m_2} = 0$$

We would like to switch to the reference frame where the Sun is stationary:

$$z = z_2 - z_1$$

Solving in terms of z gives

$$\begin{aligned} z_1 &= \frac{m_2 z}{m_1 + m_2} \\ z_2 &= \frac{m_1 z}{m_2 + m_1} \end{aligned}$$

To find K ,

$$\begin{aligned} K &= \frac{1}{2} m_1 |\dot{z}_1|^2 + \frac{1}{2} m_2 |\dot{z}_2|^2 \\ &= \frac{1}{2} m_1 \left| \frac{m_2 \dot{z}}{m_1 + m_2} \right|^2 + \frac{1}{2} m_2 \left| \frac{m_1 \dot{z}}{m_2 + m_1} \right|^2 \\ &= \frac{1}{2} \frac{m_1 m_2 (m_1 + m_2)}{(m_1 + m_2)^2} |\dot{z}|^2 \\ &= \frac{1}{2} \underbrace{\frac{m_1 m_2}{m_1 + m_2}}_M |\dot{z}|^2 \\ &= \frac{1}{2} M |\dot{z}|^2 \end{aligned}$$

The Lagrangian is

$$\begin{aligned} LM/2 &= \frac{1}{2} M |\dot{z}|^2 + \frac{G m_1 m_2}{|z|} \\ L &= |\dot{z}|^2 + 2G \frac{m_1 + m_2}{|z|} \\ L &= |\dot{z}|^2 + \frac{2g}{|z|} \end{aligned}$$

3 Integration of the equations of motion

where $g = G(m_1 + m_2)$.

By [conservation of angular momentum](#), $z \times \dot{z}$ is constant. Suppose $|z \times \dot{z}| = \omega$. Without loss of generality, $z \in \mathbb{C}$. Furthermore, if $z = re^{i\phi}$, then on time interval dt ,

$$\omega dt = z \times dz = r \times ir d\phi = r^2 d\phi$$

So

$$d\phi = \frac{\omega}{r^2} dt$$

Hence

$$|dz|^2 = |dr + ir d\phi|^2 = dr^2 + \frac{\omega^2}{r^2} dt^2$$

$$|\dot{z}|^2 = \dot{r}^2 + \frac{\omega^2}{r^2}$$

By conservation of energy,

$$\begin{aligned} E &= \underbrace{\dot{r}^2}_T + \underbrace{\frac{\omega^2}{r^2} - \frac{2g}{r}}_V \\ \phi &= \int \frac{\omega dr/r^2}{\sqrt{E - V}} \\ &= \int \frac{dr/r^2}{\sqrt{\frac{E}{\omega^2} + \frac{2g}{\omega^2} r^{-1} - r^{-2}}} \\ &= \int \frac{dr/r^2}{\sqrt{\underbrace{\frac{E}{\omega^2} + \frac{g^2}{\omega^4}}_{k^2} - (r^{-1} - \frac{g}{\omega^2})^2}} \end{aligned}$$

Substituting

$$k \cos u = r^{-1} - \frac{g}{\omega^2} k \sin u \quad du = dr/r^2$$

gives

$$\begin{aligned} \phi &= \int \frac{k \sin u}{k \sin u} du = u \\ \sin \phi &= \frac{\omega^2/r - g}{\sqrt{E\omega^2 + g}} + C \\ \sin \phi &= \frac{\omega^2/r}{\sqrt{E\omega^2 + g}} + C \end{aligned}$$

Pick an origin for ϕ such that

$$\frac{\omega^2}{r} = 1 + (\cos \phi) \sqrt{E\omega^2 + g}$$

3.3 Two body problem

This is a conic section with focus 0 and eccentricity $e = \sqrt{E\omega^2 + g}$, as the above implies $r/e = p - er \cos \phi$.

