Let X be a normed space space over $\mathbb{R}$ and suppose the parallelogram law holds, *i.*e.

$$|x+y|^2 + |x-y|^2 = 2|x|^2 + 2|y|^2 \tag{*}$$

for all $x, y \in X$. This is a necessary condition for X to be an inner product space. I will show it is also a sufficient condition

Let $\langle x, y\rangle := |x+y|^2 - |x|^2 - |y|^2$. Note that $\langle x, y\rangle = \langle y, x\rangle$, so this is symmetric. Note also that $\langle x, y\rangle$ is continuous as a map from $X^2 \to \mathbb{R}$.

Lemma 1. $\langle -x, y\rangle = -\langle x, y\rangle$

Proof. Observe that

$$\langle x, y\rangle = |x+y|^2 - |x|^2 - |y|^2$$
$$\langle -x, y\rangle = |-x+y|^2 - |x|^2 - |y|^2$$

So

$$\langle x, y\rangle + \langle -x, y\rangle = |x+y|^2 + |x-y|^2 - 2|x|^2 - 2|y|^2$$

By (*), this is zero. □

Lemma 2. $\langle a+b, c\rangle = \langle a, c\rangle + \langle b, c\rangle$

Proof. Observe that

$$\langle x+y, z\rangle = |x+y+z|^2 - |x+y|^2 - |z|^2$$
$$\langle x-y, z\rangle = |x-y+z|^2 - |x-y|^2 - |z|^2$$

and consider the sum

$$\langle x+y, z\rangle + \langle x-y, z\rangle = |x+z+y|^2 + |x+z-y|^2 - (|x+y|^2 + |x-y|^2) - 2|z|^2$$

By (*), $|x+y|^2 + |x-y|^2 = 2|x|^2 + 2|y|^2$ and $|x+y+z|^2 + |x+y-z|^2 = 2|x+y|^2 + 2|z|^2$, so we get

$$\langle x+y, z\rangle + \langle x-y, z\rangle = 2|x+y|^2 - 2|x|^2 - 2|y|^2$$

which by (*) is

$$|x+y|^2 - |x-y|^2 = |x+y|^2 - 2|x|^2 - 2|y|^2 - (|x-y|^2 - 2|x|^2 - 2|y|^2)$$

By (*) a final time, we get

$$\langle x+y, z\rangle + \langle x-y, z\rangle = \langle x, z\rangle - \langle -x, z\rangle = 2\langle x, z\rangle$$

Now note that $\langle 2x, y\rangle = \langle x+x, y\rangle - \langle x-x, y\rangle = 2\langle x, y\rangle$ by the above. Thus

$$\langle x+y, z\rangle + \langle x-y, z\rangle = \langle 2x, y\rangle$$

Letting $x = \frac{a+b}{2}$, $y = \frac{a-b}{2}$, and $z = c$ proves the lemma. □

Theorem 1 (Linearity). $\langle kx + y, z \rangle = k\langle x, z \rangle + \langle y, z \rangle$

Proof. By lemma 2, it suffices to show that $\langle kx, y \rangle = k\langle x, y \rangle$. To see this, first note it is trivial when $k = 0$. Then consider the case where k is a positive integer. We have

$$\langle kx, y \rangle = \langle x + (k-1)x, y \rangle = \langle x, y \rangle + \langle (k-1)x, y \rangle$$

so it follows by induction that $\langle kx, y \rangle = k\langle x, y \rangle$. Now consider $\langle k^{-1}x, y \rangle$. By the previous result, we know $k\langle k^{-1}x, y \rangle = \langle x, y \rangle$, concluding $\langle k^{-1}x, y \rangle = k^{-1}\langle x, y \rangle$.

As any rational number $q = ab^{-1}$ where $a, b \in \mathbb{Z}$, $b \neq 0$, we see that $\langle qx, y \rangle = \langle ab^{-1}x, y \rangle = ab^{-1}\langle x, y \rangle = q\langle x, y \rangle$. By continuity, we can extend this to the case where $r \in \mathbb{R}$ is arbitrary, giving $\langle rx, y \rangle = r\langle x, y \rangle$. $\square$