1 measureable space

`Definition 1.1 (algebra of sets)` The set $\mathcal{A}$ is an algebra on X iff

1. $\emptyset \in \Sigma$
2. A is closed under finite unions, intersections, and relative complements

`Definition 1.2 (σ-algebra of sets)` The set Σ is a σ-algebra on X iff

1. $\emptyset \in \Sigma$
2. Σ is closed under complements
3. Σ is closed under countable (σ) unions

`Note 1.3` Given a sequence of sets $A_0, A_1, \ldots$ in X, we may define

$$B_0 = A_0$$
$$B_{n+1} = A_{n+1} - \bigcup_{i=0}^{n} B_i$$

Hence, we can assume wologen that $\{A_n\}$ is pairwise-disjoint.

`Category definition 1.4 (measureable spaces)` Spaces are sets with a σ-algebra denoting measurable sets: (X, Σ)
Morphisms are measurable maps: $f : (X, \Sigma) \to (X', \Sigma')$ iff $f^{-1} : \Sigma' \to \Sigma$.

`Theorem 1.5` The pointwise limit f of a sequence $f_0, f_1, \cdots : (X, \Sigma) \to (Y, \mathcal{B})$ into measureable space Y is measurable if each f_i is.

`Proof.` Pick an arbitrary open $U \subseteq Y$. Then

$$f^{-1}(U) \subseteq \bigcup_{k=m}^{\infty} f_k^{-1}(U)$$

hence

$$f^{-1}(U) \subseteq \bigcap_{m=0}^{\infty} \bigcup_{k=m}^{\infty} f_k^{-1}(U) \tag{1}$$

Pick an arbitrary closed set $K \subseteq Y$. We can get the reverse inclusion. Note

$$\bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} f_k^{-1}(K) \subseteq f(K) \tag{2}$$

as a point x in the lhs must be a limit point of $f^{-1}(K)$; hence closure gives eq. (2).

Let V be some fixed open set. Let K_n be the closed set of y s.t. $d(y, V^C) \geq 1/n$ and V_n be the open set of y s.t. $d(y, V^C) > 1/n$. Then $V_n \subseteq K_n$ and

$$V = \bigcup_n K_n = \bigcup_n V_n$$

By eq. (2)

$$f^{-1}(V) = \bigcup_n f^{-1}(K_n) \supseteq \bigcup_n \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} f_k^{-1}(K_n) = \bigcup_n \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} f_k^{-1}(V_n)$$

By eq. (1)

$$f^{-1}(V) = \bigcup_n f^{-1}(V_n) \subseteq \bigcup_n \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} f_k^{-1}(V_n)$$

□

2 positive measure

`Definition 2.1 (positive measure)` A positive measure on (X, Σ) is a map $\mu : \Sigma \to [0, \infty]$ that is countably (σ) additive:

$$\mu\left(\bigcup A_n\right) = \sum_{n=0}^{\infty} \mu(A_n)$$

with A_i disjoint.

3 step maps

Let E be some Banach space.

`Definition 3.1 (Step maps)` Define the space of step maps

$$\mathtt{step}(\mu : (X, \Sigma) \to [0, \infty], E) :\equiv E \otimes \{\chi_S : S \in \Sigma, \mu(S) < \infty\}$$

`Definition 3.2 (integral of step maps)` Define the linear map $\int_X$ on $\mathtt{step}(\mu, E)$ by

$$\int_X \chi_S d\mu :\equiv \mu(S)$$

`Definition 3.3 (L¹(μ))` Let $L^1(\mu)$ be the closure of $\mathtt{step}(\mu, E)$ with norm

$$\|f\|_1 :\equiv \int_X |f| d\mu$$

`Definition 3.4 (almost uniformly convergent)` Let $\{f_n\}$ be a sequence of maps. They are almost uniformly convergent iff they converge almost everywhere and for any $\varepsilon > 0$, they converge absolutely and uniformly outside of a set of measure ε.

`Theorem 3.5 (fundamental theorem of integration)` Every convergent sequence $\{f_n\}$ in L^1 has a subsequence that almost uniformly converges to some map f.

`Proof.` By passing to a subsequence, assume without loss of generality that

$$\|f_n - f_m\|_1 < \frac{1}{2^{2n}}$$

when $m > n$.

Consider the series

$$f_0 + \sum_{n=0}^{\infty} (f_{n+1} - f_n) \tag{3}$$

Let $Y_n = \{x : |f_{n+1}(x) - f_n(x)| \geq 2^{-n}\}$. But

$$\frac{1}{2^n}\mu(Y_n) = \int_{Y_n} \frac{1}{2^n} \leq \int_X |f_{n+1} - f_n| \leq \frac{1}{2^{2n}}$$

Hence

$$\mu(Y_n) \leq \frac{1}{2^n}$$

Let $Z_n = Y_n \cup Y_{n+1} \ldots$. Then $\mu(Z_n) = 2^{-n+1}$ and outside of Z_n,

$$|f_{n+1} - f_n| < \frac{1}{2^n}$$

hence eq. (3) is uniformly and absolutely convergent. It is pointwise convergent outside of $\bigcap Z_n$, which has measure zero. □

`Theorem 3.6` If $\{f_n\}$ is an L^1-cauchy sequence that converges to 0 pointwise a.e., then $\lim \|f_n\|_1 = 0$.

`Proof.` Fix N. Pick an $n > N$ such that $\|f_n - f_N\| < \varepsilon$. As f_N is in $\mathtt{step}(\mu, E)$, there is some set A of finite measure s.t. $f_N|_{A^C} = 0$. By fundamental theorem of integration, pick some Z such that

$$\mu(Z) < \frac{\varepsilon}{1 + \|f_N\|_1} \tag{4}$$

and $f|_{Z^C}$ converges to 0 uniformly, passing to a subsequence if necessary.

Then

$$\|f_N\|_1 = \int_{A^C} |f_n| + \int_{A-Z} |f_n| + \int_Z |f_n| \tag{5}$$

On A^C, $f_N = 0$, hence

$$\int_{A^C} |f_n| = \int_{A^C} |f_n - F_N| < \varepsilon$$

On $A - Z$, for n sufficiently large, we have

$$|f_n(x)| < \frac{\varepsilon}{1 + \mu(A)}$$

so $\|f_n(x)\|_1 \leq \varepsilon$.

On Z, we have, by eq. (4)

$$\int_Z |f_n| \leq \|f_n - f_N\| + \mu(Z)\,\|f_N\|_1 < 2\varepsilon$$

Thus eq. (5) becomes

$$\|f_n\| \leq 4\varepsilon \to 0 \qquad \square$$