Mathematics deals with absolute truth, right? Errr, well, absolute truth is an incredibly loaded term. Properly unpacking it requires a healthy amount of philosophy and mathematics. Wait. Don't go away. It's fun. You'll be grappling with paradoxical statements like 'this sentence is false' in a productive and well-defined way. You'll find statements that are neither true nor false, yet are critical to our modern understanding of mathematics.

1 Philosophical Underpinnings

1.1 won't get fooled again

I'll tip my hat to the new constitution
Take a bow for the new revolution
Smile and grin at the change all around
Pick up my guitar and play
Just like yesterday
Then I'll get on my knees and pray
We don't get fooled again
Don't get fooled again
No, no!

The Who

Rhetoric is an ancient tradition of disguising opinions as sound logic. Through millions of years of evolution, we carbon sack computers have been tuned to take shortcuts. Taking shortcuts in thought is necessary: no one would teach an infant to count by formally proving $2 + 2 = 4$. But with these shortcuts come logical fallacies, passing off emotions as fact, perception as reality.

We could start with assertions and then work toward a conclusion by requiring each new claim we make must logically follow from what we have already established, but English is too ambiguous and irregular. Consider the sentence

> Every mouse fears some cat.

Does this mean there is one terribly frightening cat that every single mouse fears? Or does it mean for each mouse there is a corresponding scary cat? [Wik15c]

We need to move toward logical deductions governed by rules expressed in an appropriate notation (not English). These rules should be straight-forward enough that they can be checked by a computer, but expressive enough that we can work out all mathematics in terms of these logical deductions.

We look to answer a nebulous philosophical question, 'is mathematics absolutely true?' by building a mathematical model that can handle mathematics, and then asking our model what it can tell us about truth in mathematics. Instead of saying, yes it is absolutely true, or no it is not, we will look for algorithms that will check our proofs when possible. Such an approach discards much nuance of the philosophical question; we're punting. This punting is necessary, because the question 'is mathematics absolutely true' is rife with uncertainty. What is mathematics? What is absolute truth? Does absolute truth exist? How can a human judge truth if 'to err is human'? One can object *ad nauseum*.
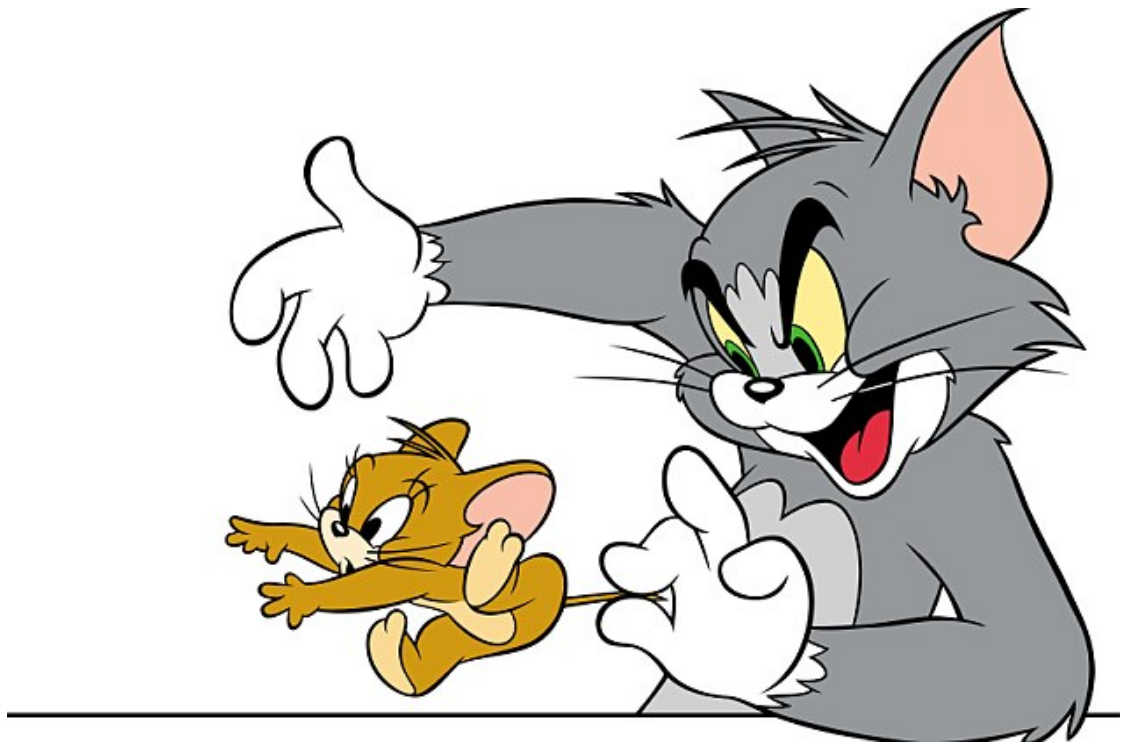
Figure 1.1: A scary cat

A similar problem: we cannot get something from nothing. We'll need to think logically in order to set up our mathematical model of logic. This circular reasoning can't be helped. It does pose a philosophical problem—we may very well be in the matrix, in which case everything we know is a lie, so our mathematical model could be dead wrong. I will assume without proof that we are not in the matrix. I will assume without proof that logic-checking computers do not lie. I will assume without proof that this bit of circular reasoning we're stuck with is innocent. The proof, of course, is left as an exercise.

2 boolean logic

The simplest interesting model of logic we will discuss. There are two possible values, true and false. When it is not ambiguous, we will abbreviate these as 0 and 1. Letters are variables. There are three operators we start with: and ($\wedge$), or ($\vee$), and not ($\neg$).

Definition 2.0.1 (and $\wedge$) $x \wedge y$ is true exactly when x and y are both true. This is also called conjunction.

$\wedge$	0	1
0	0	0
1	0	1

Definition 2.0.2 (or $\vee$) $x \vee y$ is true exactly when either x or y or both are true. This is also called disjunction.

$\vee$	0	1
0	0	1
1	1	1

And

Definition 2.0.3 (not $\neg$) Negation flips 0 and 1:

x	$\neg x$
0	1
1	0

Theorem 2.0.4 (De Morgan's law)

$$\neg(x \wedge y) = (\neg x) \vee (\neg y) \tag{2.1}$$
$$\neg(x \vee y) = (\neg x) \wedge (\neg y) \tag{2.2}$$

(see also [Wik15a])

Proof. First show eq. (2.1) by checking truth tables:

x	y	$x \wedge y$	$\neg(x \wedge y)$	$\neg x$	$\neg y$	$(\neg x) \vee (\neg y)$
0	0	0	1	1	1	1
0	1	0	1	1	0	1
1	0	0	1	0	1	1
1	1	1	0	0	0	0

Then negate both sides of *eq.* (2.1), giving

$$x \wedge y = \neg(\neg x \vee \neg y)$$

Make the substitution $\bar{x} = \neg x$ and $\bar{y} = \neg y$:

$$(\neg\bar{x}) \wedge (\neg\bar{y}) = \neg(\bar{x} \vee \neg\bar{y})$$

or, equivalently, eq. (2.2). $\qquad\square$

2.1 the conditional

We would like to model causality. Causality is vital in logical reasoning. But defining causality is a black hole ready to swallow up even the cleverest of thinkers. So hark the olde adage 'the best mathematician is a lazy one', and punt. We will define the operator implies, a.k.a. the conditional, denoted $\rightarrow$ to act as our stripped-down notion of causality.

Definition 2.1.1 (implies $\rightarrow$) $x \rightarrow y$ is true for all inputs except $1 \rightarrow 0$.

$\rightarrow$	0	1
0	1	1
1	0	1

We like this definition because $x \rightarrow y$ captures the idea y must follow from x being true, while allowing for the possibility that y may occur even when x is false. For example, if x means 'that's a cat', and y means 'that's a mammal', $x \rightarrow y$ means 'that's a cat, hence it's a mammal', with the unspoken caveat that dogs are not cats ($x = 0$), but they are mammals ($y = 1$).

That example is awkward because we have no good way of saying 'all cats are mammals, but not all mammals are cats' in boolean logic. We can say this, but not in a particularly useful way. In the next chapter, we will construct a more sophisticated logic that allows such statements.

We defined the conditional through a truth table; however, our definition of boolean logic does not mention truth tables. We ought to give a definition of $\rightarrow$ using $\wedge$, $\vee$, and $\neg$:

Theorem 2.1.2 $x \rightarrow y$ is equivalent to $y \vee (\neg x)$

Proof. Both expressions are false only when $x = 1$ and $y = 0$. $\qquad\square$

Definition 2.1.3 (logical equivalence $\leftrightarrow$) $x \leftrightarrow y$ exactly when $x = y$.

Likewise, we can define this in terms of $\wedge$, $\vee$, and $\neg$:

Theorem 2.1.4 $x \leftrightarrow y$ is equivalent to $(x \wedge y) \vee (\neg x \wedge \neg y)$.

Table 2.1: some truthtable

x_1	x_2	x_3	f
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	0

2.2 tables to expressions

We have just translated two truth-tables into expressions consisting only of $\wedge$, $\vee$, and $\neg$. You might wonder if we can always do this. We can.

Theorem 2.2.1 (completeness) Every truth table has a corresponding boolean expression.

Proof. It's best to start with an example. Consider the truth table: We want an expression that is true exactly when (x_1, x_2, x_3) is $(0, 0, 1)$ or $(1, 0, 0)$ or $(1, 1, 0)$. A straightforward way to do this is to or together a statement that is true only on the input $(0, 0, 1)$, a statement true only on $(1, 0, 0)$, and a statement that is true only on $(1, 1, 0)$.

Consider $(x_1, x_2, x_3) = (0, 0, 1)$. To find an expression that is true for this input and false for all others, we can simply assert $x_1 = 0$ and $x_2 = 0$ and $x_3 = 1$, *i.e.*

$$\neg x_1 \wedge \neg x_2 \wedge x_3$$

By the same argument, we can find the remaining expressions:

$$\begin{array}{ll} (x_1, x_2, x_3) = (0, 0, 1) & \neg x_1 \wedge \neg x_2 \wedge x_3 \\ (x_1, x_2, x_3) = (1, 0, 0) & x_1 \wedge \neg x_2 \wedge \neg x_3 \\ (x_1, x_2, x_3) = (1, 1, 0) & x_1 \wedge x_2 \wedge \neg x_3 \end{array}$$

Hence table 2.1 can be written as

$$(\neg x_1 \wedge \neg x_2 \wedge x_3) \vee (\wedge \neg x_2 \wedge \neg x_3) \vee (x_1 \wedge x_2 \wedge \neg x_3)$$

To prove the theorem, we just need to generalize the work we've done. Suppose we are given a truth table for the function $f(x_1, \ldots, x_n)$. Mark off the rows in the truth table where $f(x_1, \ldots, x_n) = 1$. For each such row, we construct the expression that asserts the input matches. If the current row is for the input

$$(x_1, \ldots, x_n) = (\alpha_1, \ldots, \alpha_n)$$

then the expression is

$$\begin{pmatrix} x_1 & \text{if } \alpha_1 = 1 \\ \neg x_1 & \text{if } \alpha_1 = 0 \end{pmatrix} \wedge \cdots \wedge \begin{pmatrix} x_n & \text{if } \alpha_n = 1 \\ \neg x_n & \text{if } \alpha_n = 0 \end{pmatrix}$$

Abbreviate this as $(x_1 = \alpha_1) \wedge \cdots \wedge (x_n = \alpha_n)$. This is just a notational convenience; the final answer will not actually contain $=$ or refer to the α_i.

Then we or over each of the above expressions:

$$f(x_1, \ldots, x_n) = \bigvee_{\alpha_1, \ldots, \alpha_n : f(\alpha_1, \ldots, \alpha_n) = 1} (x_1 = \alpha_1) \wedge \cdots \wedge (x_n = \alpha_n)$$

giving a boolean expression with the desired truth table. $\square$

This tells us that boolean logic is big enough to express any truth table. An interesting implication is that general purpose computers can be constructed!

2.3 absolute truth?

We are supposed to shy away from notions of absolute truth, but boolean logic is simple enough that we can talk about it without making a mess. A boolean expression's truth value can be computed directly from the inputs. Since each input can be either true or false, there are only finitely many possible inputs. Hence, everything you need to know is contained in the expression's truth table. These computations can be carried out unambiguously by a machine in finite time, so things are looking pretty good.

We can also show there are no contradictions. In boolean logic, a contradiction would be of the form

$$p \wedge \neg p$$

for some expression p. No matter what the expression p is, it must either evaluate to 1 or 0. Suppose $p = 1$. Then $\neg p = 0$. So $p \wedge \neg p = 1 \wedge 0 = 0$. Likewise, if $p = 0$, we get $0 \wedge 1 = 0$. Thus all contradictions are false.

3 predicate logic

Boolean logic is beautiful, but it is not expressive enough to make deductions like 'Socrates is a man. All men are mortal. Therefore, Socrates is mortal.' Enter Gottlob Frege.

Nineteenth century mathematics was in crisis. For thousands of years, mathematics had remained essentially unchanged: the gold standard was proof by Euclidean geometry. Algebra, arabic numerals, zero were tools that expedite geometric reasoning. Even new results in number theory were assumed to be, at heart geometric truths. But by the 1800s, it was clear this worldview was no longer sustainable. Mathematics had grown too abstract.

Calculus, the most revolutionary discovery this side of the dark ages, stood with embarrassingly poor theoretical footing. Newton and Leibniz through Euler and Cauchy relied on infinitesimals, but could not justify their use *a priori*, certainly not by Euclidean geometry. To make matters worse, if you were not careful, an argument that seemed kosher could result in nonsense. Fourier's work was downright scandalous. By decomposing functions as an infinite series of sin and cos waves, he was able to solve the heat equation, a problem unsolved for all but the most trivial cases. Experiments verified his results, but conservative mathematicians could not come to terms with it.

Abel, and later Galois, proved that degree 5 equations have no general solution that can be written in terms of the basic arithmetic operations and radicals. In particular, the crux Galois's proof were abstract algebraic systems that were not clearly based in Euclidean geometry.

Even Euclidean geometry was proving to be less than the paragon of flawless logic it once seemed. Problems in Euclidean geometry, such as is there a method to trisect an angle using a compass and straight edge, went unsolved for a few millennia. But these were quickly solved using Galois's tools. Euclid's parallel axiom was thought to be provable from his other axioms for thousands of years, but it was shown in the mid 19th century, that it was in fact independent, as Euclid's remaining axioms are satisfied by spherical and hyperbolic geometry, in both of which Euclid's parallel postulate does not hold.

In the nineteenth century, mathematics outgrew its ontology. The situation was so disorienting, Dodgson, an Oxford mathematician better known as Lewis Carol, wrote *Alice in Wonderland*, a veiled satire ridiculing the absurdity of this new abstract mathematics. [Ang15]

Gottlob Frege laid the foundations for mathematics' new ontology. Whereas George Boole's logic was viewed more as a useful trick that cleaned up some ambiguities for the pedants, Frege developed a logic capable of housing all of the new mathematical objects. His work went relatively unnoticed by his contemporaries, but it revolutionized mathematics and philosophy.

3.1 what is predicate logic?

First order predicate logic is the modern descendant of Frege's logic. It is more expressive than boolean logic, making it both critical to our modern understanding of mathematics, and

complicated enough that mathematicians have given up deciding if it is absolutely true.

Statements in predicate logic are made up of the logical symbols:

quantifiers: for every ($\forall$) and there exists ($\exists$)

logical connectives: $\land, \lor, \neg, \rightarrow, \leftrightarrow$

parenthesis

variables: lowercase letters

equality: $=$

and additionally

predicate symbols: capital letters $P(x, y, z)$ or typewriter text `even?(n)`.

function symbols

constant symbols

The new entities are the quantifiers, the predicate symbols, the function symbols, and the constants. Function symbols and constant symbols probably do what you expect: function symbols denote functions, and constant symbols denote constants.

Predicates denote true-or-false propositions of their arguments. For example $\geq$ is a predicate symbol that is understood to mean 'greater than or equal to'.

Finally, quantifiers allow us to form statements like 'every man is mortal'. If `man?` and `mortal?` are the predicates for, well, 'is the argument a man?' and 'is the argument mortal?', this sentence translates to

$$\forall x(\texttt{man?}(x) \rightarrow \texttt{mortal?}(x))$$

which says roughly, for all x in the universe, if x is a man, then x is mortal.

If we refer to Socrates by the constant symbol s, we can state the syllogism 'Socrates is a man. Men are mortal. Hence, Socrates is mortal' as follows:

$$\texttt{man?}(s) \land \forall x\big(\texttt{man?}(x) \rightarrow \texttt{mortal?}(x)\big) \rightarrow \texttt{mortal?}(s)$$

Recall the statement

Every mouse fears some cat

We can now state this unambiguously as

$$\forall m\big(\texttt{mouse?}(m) \rightarrow \exists c(\texttt{cat?}(c) \land \texttt{fears?}(m, c))\big)$$

i.e. 'for every mouse there is a corresponding scary cat', or

$$\exists c\big(\texttt{cat?}(c)\forall m(\texttt{mouse?}(m)\texttt{fears?}(m, c)\big)$$

i.e. 'there is one particularly frightening cat all mice fear'.

3.2 peano arithmetic

I promised this would be sufficient to house modern mathematics. I will show you how the natural numbers can be defined using Peano's axioms [Wik15b].

First, stipulate there is a constant called 0 and a function of one argument called succ. The Peano axioms are as follows:

zero is the smallest

$$\forall x.0 \neq \mathsf{succ}(x)$$

two naturals are equal if their successors are equal

$$\forall x, y.\mathsf{succ}(x) = \mathsf{succ}(y) \rightarrow x = y$$

induction: if Φ is a predicate

$$\Phi(0) \wedge \forall x\big(\Phi(x) \rightarrow \Phi(\mathsf{succ}(x))\big) \rightarrow \forall y.\Phi(y)$$

in English, if $\Phi(0)$ and $\Phi(s)$ implies $\Phi(s+1)$, then Φ must be true for all natural numbers.

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