Let E be a vector space over $\mathbb{C}$.

Definition 0.1 (Hermitian) A map $A \in \operatorname{End} E$ is Hermitian iff A^*, *i.e.*

$$\langle Ax, y \rangle = \langle x, Ay \rangle$$

Theorem 0.2 (finite spectral theorem) Suppose $E \cong \mathbb{C}^n$ is hermitian. Then

- E has eigenvectors that are an orthonormal basis of E.
- All eigenvalues of E are real.

Proof. By the fundamental theorem of algebra, the characterestic polynomial

$$|A - xI|$$

has a root. Hence A has an eigenvalue-eigenvector pair λ, e. Hence

$$\lambda \langle e, e \rangle = \langle e, Ae \rangle = \langle Ae, e \rangle = \overline{\lambda} \langle e, e \rangle$$

thus $\lambda = \overline{\lambda}$. Ergo, $\lambda \in \mathbb{R}$.

Now consider $A|e^{\perp}$. Suppose $\langle x, e \rangle = 0$. Then

$$0 = \lambda \langle x, e \rangle = \langle x, Ae \rangle = \langle Ax, e \rangle$$

Hence $A|e^{\perp} \in \operatorname{End}(e^{\perp})$. Induction on dimension proves the theorem.

Let $\bullet$ denote pointwise multiplication.

Corollary 0.3 (diagonalization) If $A \in \operatorname{End} E$, then

$$\begin{aligned} A &= P^{-1} D P \\ \|A\| &= \left\|P^{-1}\right\| \|D\| \|P\| = \|D\| \end{aligned}$$

where P is orthogonal and $D = \nu \bullet \_$ for some real $\nu \in P(E)$.

Lemma 0.4 (hermitian extension) Suppose E is a subspace of E' with $A \in \operatorname{End} E$ is hermitian. There is some hermitian $A' \in \operatorname{End} E'$ such that $A'|E = A$ and $\|A'\| = \|A\|$.

Proof. Let $\{e_i\}$ be an orthonormal basis of E. Suppose $\operatorname{span} e_j = E^{\perp}$. Then set

$$\begin{aligned} A' e_i &:= A e_i \\ A' e_j &:= \|A\| \, e_j \end{aligned}$$

Definition 0.5 (standard part of operator)

$$\begin{array}{rccc} \mathrm{st}: & \operatorname{End} E & \to & E \\ & (\operatorname{st} T)(x) & := & \mathrm{st}(T({}^*x)) \end{array}$$

Theorem 0.6 (infinite spectral theorem) If $A \in \operatorname{End} E$, then

$$(E, A) \cong (\tilde{E}, \nu \bullet \_)$$

with

$$\|A\| = \|\nu \bullet \_\|$$

where $\nu \in \tilde{E}$ and $\bullet$ is pointwise multiplication.

Proof. Consider the nonstandard enlargement of functional analysis. Consider the hyperfinite-dimensional subspace F such that

$$\operatorname*{span}_{{}^*\mathbb{C}} {}^\sigma E \subseteq F \subseteq {}^*E$$

By hermitian extension, there is some $B \in \operatorname{End} F$ such $B|\,{}^\sigma E = {}^*A|\,{}^\sigma E$. By diagonalization, there is some unitary $P : F \xrightarrow{\sim} \tilde{F}$ and real $\nu \in \tilde{F}$ such that

$$B = P^{-1}(\nu \bullet \_)P \tag{1}$$

By construction, $B({}^\sigma E) \subseteq {}^\sigma E$. Permuting rows of the matrices $\nu \bullet \_$ and P if necessary, assume (without loss of generality) that $P({}^\sigma E) \subseteq {}^\sigma E$.

Then

$$(\operatorname{st} P)^{-1} = \operatorname{st}(P^{-1})$$

similarly,

$$\big(\operatorname{st}(\nu \bullet \_)\big)(x) = \operatorname{st}(\nu \bullet {}^*x) = \operatorname{st} \nu \bullet x = (\operatorname{st} \nu \bullet \_)x$$

By construction, $\operatorname{st} B = A$, hence eq. (1) becomes

$$A = \operatorname{st}(P)^{-1}(\operatorname{st} \nu \bullet \_)\operatorname{st}(P)$$

consequently

$$\operatorname{st} P : (E, A) \xrightarrow{\sim} \Big((\operatorname{st} P)(E), \operatorname{st} \nu \bullet \_\Big)$$

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