Let X be a topological space. For $x \in *X$, define

$$U_x := \{A \subseteq X : x \in *A\}$$

Theorem 1. *U_x is a (proper) filter on X.*

Proof. First, note that $\emptyset \notin U_x$ as $x \notin *\emptyset$. Pick $A, B \in U_x$. Then as $x \in *A \cap *B$, $*A \cap *B$ is nonempty, so by transfer $A \cap B$ is nonempty. Pick $A \in U_x$ and suppose $A \subseteq B$. Then $x \in *A \subseteq *B$, so $x \in *B$, so $B \in U_x$. □

Corollary 1. *U_x is an ultrafilter on X.*

Proof. We have shown that U_x is a filter, so it suffices to show that for all $A \subseteq X$, either $A \in U_x$ or $A^c \in U_x$. Pick an arbitrary $A \subseteq X$. Suppose $x \in *A$. Then $A \in U_x$, and we are done. Otherwise, suppose $x \notin *A$. Therefore $x \in (*A)^c$. But $(*A)^c = *(A^c)$, so $A^c \in U_x$. □

Theorem 2 (Ultrafilter lemma). *Every filter F can be completed to an ultrafilter U_x*

Proof. By definition F has the finite intersection property. Thus for a suitably saturated nonstandard model, there is some $x \in \bigcap_{U \in F} *U$. Then take U_x: by construction, $F \subseteq U_x$. □

This tells us that if we have access to a sufficiently saturated nostandard model of our theory, we automatically have the ultrafilter lemma

Corollary 2. *Any ultrafilter $\mathfrak{U}$ on X is of the form U_x for some $x \in *X$.*

Proof. Applying theorem 2 to $\mathfrak{U}$ guarantees there is some $x \in *X$ such that $\mathfrak{U} \subseteq U_x$. By maximality of ultrafilters, $\mathfrak{U} = U_x$. □

Now we can move on to issues of convergence

Theorem 3. *Let $x \in *X$. The filter U_x converges to the standard point $y \in X$ iff $x \approx y$.*

Proof. Suppose that $x \approx y$. Let U be an arbitrary standard neighborhood of y. Then note that $\mu(y)$ is contained in $*U$, so $U \in U_x$. Thus $U_x \to y$.

Suppose $U_x \to y$. Then for any standard neighborhood U of y, there is some $N \in U_x$ such that $N \subseteq U$. But then $x \in *N \subseteq *U$, so $x \in *U$. Thus $x \in \mu(y)$. □

This tells us that a nonprincipal ultrafilter converging to a point x can be thought of as honing in infinitely close to x, but not quite on x. This gives a way to investigate the structure of ultrafilters converging to the same point:

Lemma 1. *For $x, y \in *X$, $U_x = U_y$ iff there is no standard set $A \subseteq X$ such that $x \in *A$ and $y \notin *A$.*

Proof. Suppose $U_x = U_y$. Then by definition, every standard open set containing x contains y.

Suppose there is a standard set $A \subseteq X$ such that $x \in *A$ and $y \notin *A$. Then $A \in U_x$ but $A \notin U_y$, so $U_x \neq U_y$. □

We say that a standard set A with $x \notin A$ splits the monad $\mu(y)$ iff $\mu(y)$ is not contained in A but A intersects $\mu(y)$. We say that the sets A and B split $\mu(y)$ equivalently iff $A \cap \mu(y) = B \cap \mu(y)$.

Corollary 3. *The set of nonprincipal ultrafilters converging to some $x \in X$ is in bijective correspondence with the different ways of splitting $\mu(x)$ up to equivalence*

Proof. todo □