Let E be a vector space over $\mathbb{C}$.

Definition 1 (Hermitian) A map $A \in \operatorname{End} E$ is Hermitian iff

$$\langle Ax, y \rangle = \langle x, Ay \rangle$$

Theorem 2 (finite spectral theorem) Suppose $E \cong \mathbb{C}^n$ is hermitian. Then

- E has eigenvectors that are an orthonormal basis of E.
- All eigenvalues of E are real.

Proof. By the fundamental theorem of algebra, the characterestic polynomial

$$|A - xI|$$

has a root. Hence A has an eigenvalue-eigenvector pair λ, e. But

$$\lambda \langle e, e \rangle = \langle e, Ae \rangle = \langle Ae, e \rangle = \overline{\lambda} \langle e, e \rangle$$

thus $\lambda = \overline{\lambda}$. Ergo, $\lambda \in \mathbb{R}$.

Now consider $A|e^{\perp}$. Suppose $\langle x, e \rangle = 0$. Then

$$0 = \lambda \langle x, e \rangle = \langle x, Ae \rangle = \langle Ax, e \rangle$$

Hence $A|e^{\perp} \in \operatorname{End}(e^{\perp})$. Induction on dimension proves the theorem.

For vectors ν, u, let νu denote pointwise multiplication.

Corollary 3 (diagonalization) If $A \in \operatorname{End} E$, then

$$A = P^{-1}(\nu\_)P$$

where P is unitary and $\nu \in E$ is real.

Definition 4 (standard part of map)

$$\begin{array}{rccc} \mathrm{st}_X : & (*X \to *Y) & \to & (X \to Y) \\ & (\mathrm{st}\, f)(x) & \coloneqq & \mathrm{st}\big(f(^*x)\big) \end{array}$$

Theorem 5 (infinite spectral theorem) Suppose E is a Hilbert space. If $A \in \operatorname{End} E$ is hermitian, then

$$A = P^{-1}(\nu\_)P$$

with P unitary and $\nu \in P(E)$ real.

Proof. Consider a nonstandard model of functional analysis. Fix a hyperfinite-dimensional subspace F such that

$$^{\sigma}E \subseteq F \subseteq {}^{*}E$$

There is some hermitian $B \in \operatorname{End} F$ such that $B|\,{}^{\sigma}E = {}^{*}A|\,{}^{\sigma}E$. This B simultaneously satisfies hermitian-ness and $B({}^{*}e) = {}^{*}(Ae)$ for each e in some (standard) basis of E. Such a B exists, internal to a sufficiently saturated model.

By $*$−transferring diagonalization, there is some unitary $P \in \operatorname{End} F$ and real $v \in F$ such that

$$B = P^{-1}(v\_)P \tag{1}$$

It suffices to recover the standard diagonalization from the hyperfinite case. Let

$$\tilde{E} = P({}^{\sigma}E)$$

be a standard Hilbert space. Define $\widetilde{\mathrm{st}}(x) = y$ when $y \in \tilde{E}$ and $x \simeq y$. Then

$$\begin{array}{rclcccc} \widetilde{\mathrm{st}}\,P & = & P|_{E} & : & E & \to & \tilde{E} \\ \widetilde{\mathrm{st}}\left(P^{-1}\right) & = & P^{-1}|_{\tilde{E}} & : & \tilde{E} & \to & E \end{array}$$

hence $\widetilde{\mathrm{st}}\left(P^{-1}\right) = \left(\widetilde{\mathrm{st}}\,P\right)^{-1}$ and

$$\widetilde{\mathrm{st}}(v\_)x = \widetilde{\mathrm{st}}(v\,{}^{*}x) = (\widetilde{\mathrm{st}}\,v)x = (\widetilde{\mathrm{st}}\,v)x$$

By construction, $\mathrm{st}\,B = A$, hence eq. (1) becomes

$$A = \left(\widetilde{\mathrm{st}}\,P\right)^{-1}\left(\widetilde{\mathrm{st}}\,v\_\right)\widetilde{\mathrm{st}}(P)$$