Theorem 1 (polynomial remainder theorem). *Suppose* $P(x) : \mathbb{C}[x]$. *Then*

$$P(x) = Q(x)(x-k) + P(k)$$

In particular, $(x-k)|P(x)$ *iff* $P(k) = 0$

Proof. By synthetic division,

$$\begin{array}{lllll} a_n & a_{n-1} & a_{n-2} & \dots & a_0 \\ & ka_n & a_{n-1}k + a_nk^2 & \dots & ka_1 + \dots k^n a_n \\ \hline a_n & a_{n-1} + ka_n & a_{n-2} + a_{n-1}k + a_nk^2 & \dots & a_0 + ka_1 + \dots k^n a_n \end{array}$$

□

Definition 2 (Vandermonde matrix).

$$V :\equiv \begin{bmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 1 & x_1 & x_1^2 & \dots & x_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^n \end{bmatrix}$$

[Wik15]

Lemma 3 (Vandermonde determinant).

$$\det V = \prod_{1 \le i < j \le n} (a_j - a_i)$$

Proof. Consider

$$\begin{aligned} |V_n| &= \begin{vmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 1 & x_1 & x_1^2 & \dots & x_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^n \end{vmatrix} \\ &= \begin{vmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 0 & x_1 - x_0 & x_1^2 - x_0^2 & \dots & x_1^n - x_0^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & x_n - x_0 & x_n^2 - x_0^2 & \dots & x_n^n - x_0^n \end{vmatrix} \\ &= \begin{vmatrix} 1 & x_0 & x_0^2 & \dots & 0 \\ 0 & x_1 - x_0 & x_1^2 - x_0^2 & \dots & x_1^n - x_0 x_1^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & x_n - x_0 & x_n^2 - x_0^2 & \dots & x_n^n - x_0 x_n^{n-1} \end{vmatrix} \\ &= \begin{vmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & x_1 - x_0 & (x_1 - x_0)x_1 & \dots & (x_1 - x_0)x_1^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & x_n - x_0 & (x_n - x_0)x_n & \dots & (x_n - x_0)x_n^{n-1} \end{vmatrix} \end{aligned}$$

$$= (x_n - x_0)\dots(x_1 - x_0)\begin{vmatrix} 1 & x_1 & \dots & x_1^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \dots & x_n^{n-1} \end{vmatrix}$$
$$= (x_n - x_0)\dots(x_1 - x_0)(x_n - x_1)\dots(x_n - x_{n-1})\dots$$

[Pro15] □

Theorem 4 (unisolvence theorem). *The $n+1$ points $(x_0, y_0)\dots(x_n, y_n)$ with distinct x_i determine a unique n-degree polynomial.*

Proof.

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 1 & x_1 & x_1^2 & \dots & x_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix} \tag{1}$$

By Vandermonde determinant, V is isomorphic, as each x_i is distinct. Hence (1) has a unique solution. □

References

[Pro15] ProofWiki. *Vandermonde Determinant*. 2015. URL: https://www.proofwiki.org/wiki/Vandermonde_Determinant.

[Wik15] Wikipedia. *Vandermonde matrix*. 2015. URL: http://en.wikipedia.org/wiki/Vandermonde_matrix.