1 Equations of motion

1.1 Lagrangian

1 Definition (action) Let $L(q, \dot{q}, t)$ be a Lagrangian. Let q : time $\to$ configuration space. Define action by

$$S(q) := \int_a^b L(q, \dot{q}, t) dt$$

Mechanics will minimize action of the appropriate Lagrangian.

2 Theorem (euler-lagrange)

$$\frac{dS}{dq} = \left\langle \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{d\dot{q}}, - \right\rangle$$

Proof. Let h be an infinitesimal perturbation (endpoints are 0).

$$\begin{aligned} S(q+h) &= \int L(q+h, \dot{q}+d\dot{q}, t) dt \\ &= \int L(q, \dot{q}, t) dt + \int DL(q, \dot{q}, t)(h, d\dot{q}, 0) + o(h) \\ &= \int \frac{\partial L}{\partial q} h dt + \underbrace{\frac{\partial L}{\partial \dot{q}}}_{u} \underbrace{d\dot{q} dt}_{dv} \\ &= \int \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} \right) h dt + \underbrace{\left[h \frac{\partial L}{\partial \dot{q}} \right]_a^b}_{0 \text{ as } h(a) = h(b) = 0} \end{aligned}$$

□

3 Corollary (stationary action)

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{d\dot{q}} = 0$$

2 Conservation laws

2.1 Energy

1 Definition (energy)

$$E \coloneqq \left\langle Dq, \frac{\partial L}{\partial \dot{q}} \right\rangle - L$$

2 Theorem When $D(t \mapsto L(\_, \_, t)) = 0$ (*e.g.* closed system or constant field), energy is conserved.

Proof. By hypothesis,

$$\frac{dL}{dt} = \left\langle \frac{\partial L}{\partial q}, \dot{q} \right\rangle + \left\langle \frac{\partial L}{\partial \dot{q}}, \ddot{q} \right\rangle$$

By euler-lagrange,

$$\frac{dL}{dt} = \left\langle \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}, \dot{q} \right\rangle + \left\langle \frac{\partial L}{\partial \dot{q}}, \ddot{q} \right\rangle$$

By product rule,

$$\frac{dL}{dt} = \frac{d}{dt}\left\langle \dot{q}, \frac{\partial L}{\partial \dot{q}} \right\rangle$$

Subtracting dL/dt finishes the proof. □

2.2 Momentum

3 Definition (momentum)

$$P \coloneqq \sum_a \frac{\partial L_a}{\partial \boldsymbol{v}_a}$$

where a indexes over particles.

4 Theorem (conservation of momentum) In a closed system, $\dot{P} = 0$.

Proof. This follows from homogeneity of space. Suppose every particle moves by infinitesimal ε with velocities unchanged. Then

$$dL = \sum_a \left\langle \frac{\partial L}{\partial L} \partial \boldsymbol{r}_a, \boldsymbol{\varepsilon} \right\rangle$$

which must be 0 by homogeneity. As ε is arbitrary, we conclude

$$\sum_a \frac{\partial L}{\partial \boldsymbol{r}_a} = 0$$

By euler-lagrange,

$$\frac{d}{dt}\sum_a \frac{\partial L}{\partial \boldsymbol{v}_a} = 0$$

□

5 Corollary (Newton's third law)

$$\sum_a \mathbf{F}_a = 0$$

6 Definition (generalized momenta)

$$p_i \coloneqq \frac{\partial L}{\partial \dot{q}_i}$$

7 Definition (generalized force)

$$F_i \coloneqq \dot{p}_i$$

2.3 center of mass

8 Definition (Center of mass)

$$\mathbf{R} \coloneqq \frac{\sum m_a \mathbf{r}_a}{\sum m_a}$$

A *system* is *at rest* ($P = 0$) in a frame of reference where $R = 0$.

9 Definition (internal energy) The energy of a system at rest, denoted E_i.

10 Theorem Energy in different reference frames. Put $\boldsymbol{v}'_a = \boldsymbol{v}_a + \mathbf{V}$. Then

$$E = E' + \left\langle \mathbf{V}, P' \right\rangle + \frac{1}{2}\mu \mathbf{V}^2$$

Proof.

$$\begin{aligned} E &= \frac{1}{2}\sum m_a |\boldsymbol{v}_a|^2 + U \\ &= \frac{1}{2}\sum m_a |\boldsymbol{v}'_a + \mathbf{V}|^2 + U \\ &= \frac{1}{2}\mu \mathbf{V}^2 + \left\langle \mathbf{V}, \sum m_a \boldsymbol{v}'_a \right\rangle + \frac{1}{2}\sum m_a |\boldsymbol{v}'_a|^2 + U \end{aligned}$$

□

11 Corollary

$$E = \frac{1}{2}\mu |\mathbf{V}|^2 + E_i$$

12 Corollary

$$L = L' + \left\langle \boldsymbol{v}, P' \right\rangle + \frac{1}{2}\mu |\mathbf{V}|^2$$

13 Corollary

$$S = S' + \mu \left\langle \boldsymbol{v}, P' \right\rangle + \frac{1}{2}\mu |\mathbf{V}|^2 t$$

2.4 Angular momentum

14 Theorem SO(3) is a Lie group with Lie algebra

$$\mathfrak{so}(3) \cong (\mathbb{R}^3)^* \wedge \mathbb{R}^3$$

Proof. Let

$$f(x) \coloneqq x^T x$$

Then

$$SO(3) \cong f^{-1}(I)$$

But

$$\begin{aligned} f(I + dx) &= (I + dx)^T (I + dx) \\ &= I^T I + dx^T I + I^T dx + dx^T dx \end{aligned}$$

Hence

$$\begin{aligned} Df|_I(h) &= h + h^T \\ Df|_x &= (x\_) Df|_I(x^{-1}\_) \end{aligned}$$

so

$$\mathfrak{so}(3) \cong \ker Df \cong \mathbb{R}^3 \wedge \mathbb{R}^3$$

□

15 Definition (angular momentum)

$$\mathbf{M} \coloneqq \sum \mathbf{r}_a \times \mathbf{p}_a$$

16 Theorem (conservation of angular momentum) In a closed system,

$$\dot{\mathbf{M}} = 0$$

Proof. Let $d\phi \in \mathfrak{so}(3)$ be an infinitesimal rotation.

$$d\mathbf{r} = d\phi \times \mathbf{r}$$

But the Lagrangian is unchanged by $d\phi$, hence

$$\begin{aligned} dL = \sum_a \left(\frac{\partial L}{\partial \mathbf{r}_a} d\mathbf{r}_a + \frac{\partial L}{\partial \mathbf{v}_a} d\mathbf{v}_a \right) &= 0 \\ \sum \langle \dot{\mathbf{p}}_a, d\phi \times \mathbf{r}_a \rangle + \langle \mathbf{p}_a, d\phi \times \mathbf{v}_a \rangle &= 0 \\ \left\langle d\phi, \sum (\mathbf{r}_a \times \dot{\mathbf{p}}_a + \mathbf{v}_a \times \mathbf{p}_a) \right\rangle &= 0 \\ \left\langle d\phi, \frac{d}{dt} \sum \mathbf{r}_a \times \mathbf{p}_a \right\rangle &= 0 \end{aligned}$$

But $d\phi$ is arbitrary, hence

$$\frac{d}{dt} \sum \mathbf{r}_a \times \mathbf{p}_a = 0$$

□

17 Theorem (M in a different frame) If $r_a = r'_a + \mathbf{a}$, then

$$\mathbf{M} = \mathbf{M}' + a \times \mathbf{P}$$

Proof.

$$\begin{aligned}\mathbf{M} &= \sum \mathbf{r}_a \times \mathbf{p}_a \\ &= \sum \mathbf{r}'_a \times \mathbf{p}_a + \mathbf{a} \times \sum \mathbf{p}_a \\ &= \mathbf{M}' + \mathbf{a} \times \mathbf{P}\end{aligned}$$ □

18 Corollary If L (equivalently, U) is symmetric around the axis with unit $\boldsymbol{e}$, then $\langle \mathbf{M}, \boldsymbol{e} \rangle$ is conserved.

19 Corollary If L is symmetric around the origin, then $\mathbf{M}$ is conserved.

3 Integration of the equations of motion

3.1 One dimension

$$L = \frac{1}{2} m\dot{x} - U(x)$$

so

$$t = \int_{x_0}^{x_1} dx \sqrt{\frac{m/2}{E - U(x)}}$$

3.2 Pendulum

Consider the pendulum

$$z = -ir \exp(i\phi)$$

with

$$\begin{aligned}K &= \frac{1}{2} m|\dot{z}|^2 = \frac{1}{2} mr^2|\dot{\phi}|^2 \\ U &= -mgr\cos\phi \\ L &= r|\dot{\phi}|^2 + g\cos\phi\end{aligned}$$

Let $\omega^2 = g/r$. By euler-lagrange

$$\begin{aligned} \ddot{\phi} + \omega^2 \sin\phi &= 0 \\ \dot{\phi}\ddot{\phi} + \dot{\phi}\omega^2 \sin\phi &= 0 \\ \frac{d}{dt}\left(\frac{\dot{\phi}^2}{2} - \omega^2\cos\phi\right) &= 0 \end{aligned}$$

Integrating gives

$$\begin{aligned} 0 &= \left(\frac{\dot{\phi}^2}{2} - \omega^2\cos\phi\right)_0^t \\ \dot{\phi}^2 &= 2\omega^2 \cos\phi\big|_0^t + \dot{\phi}(0)^2 \end{aligned}$$

Hence

$$\pm t = \int_{\phi(0)}^{\phi} \frac{d\phi}{\sqrt{2\omega^2\cos\phi|_0^t + \dot{\phi}(0)^2}}$$

But $\cos\phi|_0^t = 2\sin^2(\phi/2)|_t^0$, so

$$\begin{aligned} \pm t &= \int_{\phi(0)}^{\phi} \frac{d\phi}{\sqrt{4\omega^2\sin^2\frac{\phi}{2}\Big|_t^0 + \dot{\phi}(0)^2}} \\ \pm 2\omega t &= \int_{\phi(0)}^{\phi} \frac{d\phi}{\sqrt{\sin^2\frac{\phi(0)}{2} + \frac{\dot{\phi}(0)^2}{4\omega^2} - \sin^2\frac{\phi}{2}}} \end{aligned}$$

Let

$$k = \left(\sin^2\frac{\phi(0)}{2} + \frac{\dot{\phi}(0)^2}{4\omega^2}\right)^{-1/2}$$

Then

$$\begin{aligned} \pm\frac{2\omega t}{k} &= \int_{\phi(0)}^{\phi} \frac{d\phi}{\sqrt{1 - k^2\sin^2\frac{\phi}{2}}} \\ \pm\frac{\omega t}{k} &= \int_{\phi(0)/2}^{\phi/2} \frac{d\theta}{\sqrt{1 - k^2\sin^2\theta}} \end{aligned}$$

1 Definition (Incomplete elliptic: 1st kind)

$$F(u,k) = \int_0^u \frac{d\theta}{1 - k^2\sin^2\theta}$$

2 Definition (sn)

$$\mathrm{sn}(-, k) = \sin\left(F^{-1}(-, k)\right)$$

So

$$\phi(t) = 2 \arcsin \mathrm{sn}\left(F\left(2^{-1}\phi(0), k\right) \pm \frac{\omega t}{k}, k\right)$$

3.3 Two body problem

As the system is closed, the center of mass is inertial. Set this as the origin.

$$\frac{m_1 z_1 + m_2 z_2}{m_1 + m_2} = 0$$

We would like to switch to the reference frame where the Sun is stationary:

$$z = z_2 - z_1$$

Solving in terms of z gives

$$\begin{aligned} z_1 &= \frac{m_2 z}{m_1 + m_2} \\ z_2 &= \frac{m_1 z}{m_2 + m_1} \end{aligned}$$

To find K,

$$\begin{aligned} K &= \frac{1}{2} m_1 |\dot{z}_1|^2 + \frac{1}{2} m_2 |\dot{z}_2|^2 \\ &= \frac{1}{2} m_1 \left| \frac{m_2 \dot{z}}{m_1 m_2} \right|^2 + \frac{1}{2} m_2 \left| \frac{m_1 \dot{z}}{m_2 + m_1} \right|^2 \\ &= \frac{1}{2} \frac{m_1 m_2 (m_1 + m_2)}{(m_1 + m_2)^2} |\dot{z}|^2 \\ &= \frac{1}{2} \underbrace{\frac{m_1 m_2}{m_1 + m_2}}_{M} |\dot{z}|^2 \\ &= \frac{1}{2} M |\dot{z}|^2 \end{aligned}$$

The Lagrangian is

$$\begin{aligned} LM/2 &= \frac{1}{2} M |\dot{z}|^2 + \frac{G m_1 m_2}{|z|} \\ L &= |\dot{z}|^2 + 2G \frac{m_1 + m_2}{|z|} \\ L &= |\dot{z}|^2 + \frac{2g}{|z|} \end{aligned}$$

where $g = G(m_1 + m_2)$.

By conservation of angular momentum, $z \times \dot{z}$ is constant. Suppose $|z \times \dot{z}| = \omega$. Without loss of generality, $z \in \mathbb{C}$. Furthermore, if $z = re^{i\phi}$, then on time interval dt,

$$\omega dt = z \times dz = r \times ird\phi = r^2 d\phi$$

So

$$d\phi = \frac{\omega}{r^2} dt$$

Hence

$$\begin{aligned} |dz|^2 &= |dr + ird\phi|^2 = dr^2 + \frac{\omega^2}{r^2} dt^2 \\ |\dot{z}|^2 &= \dot{r}^2 + \frac{\omega^2}{r^2} \end{aligned}$$

By conservation of energy,

$$\begin{aligned} E &= \underbrace{\dot{r}^2}_{T} + \underbrace{\frac{\omega^2}{r^2} - \frac{2g}{r}}_{V} \\ \phi &= \int \frac{\omega dr/r^2}{\sqrt{E - V}} \\ &= \int \frac{dr/r^2}{\sqrt{\frac{E}{\omega^2} + \frac{2g}{\omega^2} r^{-1} - r^{-2}}} \\ &= \int \frac{dr/r^2}{\sqrt{\underbrace{\frac{E}{\omega^2} + \frac{g^2}{\omega^4}}_{k^2} - \left(r^{-1} - \frac{g}{\omega^2}\right)^2}} \end{aligned}$$

Substituting

$$k \cos u = r^{-1} - \frac{g}{\omega^2} k \sin u du = dr/r^2$$

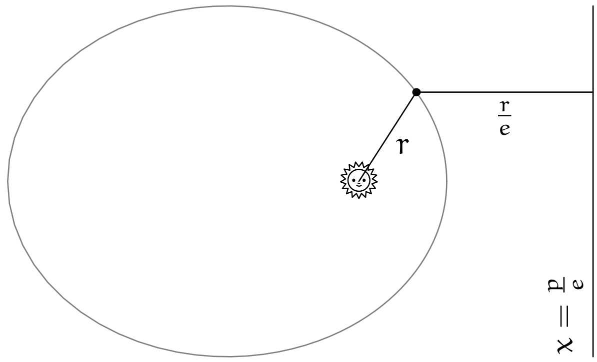
gives

$$\begin{aligned} \phi &= \int \frac{k \sin u}{k \sin u} du = u \\ \sin\phi &= \frac{\omega^2/r - g}{\sqrt{E\omega^2 + g}} + C \\ \sin\phi &= \frac{\omega^2/r}{\sqrt{E\omega^2 + g}} + C \end{aligned}$$

Pick an origin for ϕ such that

$$\frac{\omega^2}{r} = 1 + (\cos\phi)\sqrt{E\omega^2 + g}$$

This is a conic section with focus 0 and eccentricity $e = \sqrt{E\omega^2 + g}$, as the above implies $r/e = p - er\cos\phi$.



4 Hamiltonian

4.1 Legendre Transforms

1 Definition (Legendre transform) Given $f : V \to \mathbb{R}$ define

$$\begin{array}{rccl} \hat{f} & : & V^* & \to & \mathbb{R} \\ & & p & \mapsto & \sup_x(px - f(x)) \end{array}$$

2 Definition Define

$$H(q, p, t) = \sup_{\dot{q}}(p(\dot{q}) - L(q, \dot{q}, t))$$

3 Theorem (conjugate momenta)

$$p = \frac{\partial L}{\partial \dot{q}}$$

Proof. Suppose $\dot{q}$ is maximal, *i.e.*

$$H = p\dot{q} - L(q, \dot{q}, t)$$

Hold p, q, t constant. Take partials in $\dot{q}$. By maximality,

$$0 = p - \frac{\partial L}{\partial \dot{q}}$$

□

4 Note

$$dH = p\,d\dot{q} + dp\,\dot{q} - \frac{\partial L}{\partial q}dq - \frac{\partial L}{\partial \dot{q}}d\dot{q} - \frac{\partial L}{\partial t}dt \tag{4.1}$$

5 Theorem (Hamilton's equations)

$$\begin{aligned}\frac{\partial H}{\partial q} &= -\dot{p}\\ \frac{\partial H}{\partial p} &= \dot{q}\\ \frac{\partial H}{\partial t} &= \frac{-\partial L}{\partial t}\end{aligned}$$

Proof. All but the first follow from directly inspecting eq. (4.1). The first gives

$$-\frac{\partial L}{\partial q} = -\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = -\dot{p}$$

□