1 ultrafilter

`Definition 1.1 (Filter)` F is a filter iff

$$A, B \in F \implies A \cap B \in F \tag{1}$$

$$\emptyset \notin F \tag{2}$$

$$B \supseteq A \in F \implies B \in F \tag{3}$$

`Definition 1.2 (Ultrafilter)`

$$A \subseteq X \implies A \in U \text{ or } X - A \in U \tag{4}$$

`Theorem 1.3` Ultrafilter convergence defines a topology.

`Theorem 1.4` A space is

1. compact iff every ultrafilter converges.
2. Hausdorff iff every ultrafilter converges to at most one point.
3. compact-Hausdorff iff every ultrafilter converges to exactly one point

2 Stone-Čech compactification

Let X be a space.

`Definition 2.1 (ultra)`

`Definition 2.2` $(\mathtt{ultra} : A \to \mathtt{ultra}\, A)$ Suppose $a \in A$. Let $\mathtt{ultra}\, a$ be the ultrafilter generated by a. Suppose $f : A \to B$ is a function. Then $\mathtt{ultra}(f) : \mathtt{ultra}(A) \to \mathtt{ultra}(B)$.

`Definition 2.3` $(\lim : \mathtt{ultra}^2\, X \to \mathtt{ultra}\, X)$ Let $U \in \mathtt{ultra}^2\, X$. Suppose $U = \{U_i\}$ where U_i a subset of $\mathtt{ultra}\, X$. Then define

$$\lim U = \bigcup_{V \in U} \bigcap_{u \in V} u \tag{5}$$

`Theorem 2.4` $\lim U : \mathtt{ultra}^2\, X \to \mathtt{ultra}\, X$

`Proof.` I claim

$$F_i :\equiv \bigcap_{u \in U_i} u \tag{6}$$

is a filter. To see this, note every filter on X contains X. Hence F_i is nonempty. The other filter axioms remain true after taking intersections.

Now pick arbitrary F_α and F_β as defined in eq. (6). I claim there is some F_γ (and corresponding U_γ) such that $F_\alpha \cup F_\beta \subseteq F_\gamma$. It suffices to find the corresponding U_γ. Let $U_\gamma = U_\alpha \cap U_\beta$. Equation (1) guarantees $U_y \in U$. Then

$$\bigcap_{u \in U_\gamma} u \supseteq \left(\bigcap_{u \in U_\alpha} u\right) \cup \left(\bigcap_{u \in U_\beta} u\right)$$
$$F_\gamma \supseteq F_\alpha \cup F_\beta$$

Hence

$$\lim U :\equiv \bigcup_i F_i$$

is a filter.

Finally, I claim $\lim U \in \mathtt{ultra}\, X$. Suppose $A \subseteq X$. Then let $\mathtt{ultra}\, A \subseteq \mathtt{ultra}\, X$ be the set of ultrafilters that contain A. I claim $\mathtt{ultra}(A^C) = (\mathtt{ultra}\, A)^C$; as $(\mathtt{ultra}\, A)^C$, the set of ultrafilters that *do not contain* A, is $\mathtt{ultra}(A^C)$, the set of ultrafilters that *contain* A^C. This is a rewording of eq. (4).

By eq. (4), either $\mathtt{ultra}\, A \in U$ or $\mathtt{ultra}(A^C) \in U$. Suppose, without loss of generality, that $\mathtt{ultra}\, A \in U$. Then

$$F_A := \bigcap_{u \in \mathtt{ultra}\, A} u \subseteq \bigcap_{u \in U} u$$

Note F_A is a filter containing A. Then $\lim U$ contains A. Equation (4) holds. □

`Theorem 2.5` ($\lim \circ \mathtt{ultra} = \mathtt{id}$) Let $u \in \mathtt{ultra}\, X$. Then $\lim \circ \mathtt{ultra}\, u = u$ as every set containing u is in $\mathtt{ultra}\, u$.

3 ultrafilter monad

For this section, we will distinguish between $\mathtt{ultra}$ and η. Here, $\mathtt{ultra} : \mathsf{haus} \to \mathsf{Chaus}$ is the monad's functor and (via abuse of notation) $\mathtt{ultra} : \mathtt{id} \to \mathtt{ultra}$ is the unit and $\lim : \mathtt{ultra}^2 \to \mathtt{ultra}$ is the multiplication.

`Theorem 3.1` The coherence conditions

$$\begin{tikzcd} \mathtt{ultra}^3(X) \arrow[r, "\mathtt{ultra}(\lim_X)"] \arrow[d, "\lim_{\mathtt{ultra}(X)}"] & \mathtt{ultra}^2(X) \arrow[d, "\lim_X"] \\ \mathtt{ultra}^2(X) \arrow[r, "\lim_X"] & \mathtt{ultra}(X) \end{tikzcd} \tag{7}$$

and

$$\begin{tikzcd} \mathtt{ultra}(X) \arrow[r, "\mathtt{ultra}_{\mathtt{ultra}(X)}"] \arrow[d, "\mathtt{ultra}(\mathtt{ultra}_X)"] & \mathtt{ultra}^2(X) \arrow[d, "\lim_X"] \\ \mathtt{ultra}^2(X) \arrow[r, "\lim_X"] & \mathtt{ultra}(X) \end{tikzcd} \tag{8}$$

hold

$\texttt{Proof.}$ Consider eq. (7). Let $\mathcal{U} \in \texttt{ultra}^3\, X$. The goal is to show

$$\lim \circ \texttt{ultra}(\lim)\mathcal{U} = \lim \circ \lim \mathcal{U}$$

First, consider $\lim \circ \texttt{ultra}(\lim)\mathcal{U}$. Then

$$\begin{aligned}
\lim \circ \texttt{ultra}(\lim) &= \bigcup_{U \in \mathcal{U}} \left(\bigcap_{u \in \bigcup_{V \in U} \bigcap_{v \in V} v} u \right) \\
&= \bigcup \bigcap \left(\bigcup_{\bigcap} \right) \\
&= \bigcup_{U \in \mathcal{U}} \bigcap_{V \in U} \bigcup_{u \in V} \bigcap_{v \in u} v \\
\lim \circ \lim &= \bigcup_{U \in \mathcal{U}} \bigcap_{V \in U} \bigcup_{u \in V} \bigcap_{v \in u} v
\end{aligned}$$

Now consider eq. (8). This follows as $\texttt{ultra}$ (unit) $= \texttt{ultra}$ (functor) . ☐

4 Maximality

Consider the space X and its compactification $\bar{X}$. We have the diagram

$$\begin{tikzcd}
X \arrow[r, hook, "i"] \arrow[d, "\texttt{ultra}"] & \bar{X} \arrow[d, equal] \\
\texttt{ultra}\, X \arrow[r, "\texttt{ultra}(i)"] & \texttt{ultra}\, \bar{X}
\end{tikzcd}$$

I claim surjectivity of

$$\lim_{\bar{X}} \circ \texttt{ultra}(i) : \texttt{ultra}\, X \to \bar{X}$$

As $\bar{X}$ is compact, $\lim_{\bar{X}}$ is a homeomorphism. Thus it suffices to show the surjectivity of $\texttt{ultra}(i)$. To see this, consider an ultrafilter $U \in \bar{X}$. As $\bar{X}$ is a compactification, U is uniquely determined by $U|_X$. Thus $\texttt{ultra}(i)\,(U|_x) = U$.