1 Euler-Lagrange equation

Consider the manifold $C^1[a,b]$ with tangent space $([a,b],a,b) \to_{C^1} (\mathbb{R},0,0)$. We give the tangent space the norm $\|f\| = \max|f|,|f'|$. We will show

`Theorem 1.1 (functional derivative)` The functional

$$S(f) = \int_a^b L(t,f(t),f'(t))dt : C^1[a,b] \to \mathbb{R} \tag{1}$$

has derivative

$$\frac{dS}{df} = \left\langle \frac{\partial L}{\partial f} - \frac{d}{dt}\frac{\partial L}{\partial f'}, - \right\rangle \tag{2}$$

`Proof.` Suppose $h \in T_f C^1[a,b]$. Then

$$S(f+h) = \int_a^b L\left(t, f(t)+h(t), f'(t)+h'(t)\right) dt$$
$$S(f+h) = \int_a^b L\left(t,f,h\right) dt + \int_a^b DL\left(t,f,f'\right)\left(0,h,h'\right) dt + \int_a^b o\left(\|h\|\right) dt$$
$$S(f+h) = S(f) + \int_a^b DL\left(t,f,f'\right)\left(0,h,h'\right) dt + o\left(\|h\|\right)$$

So it suffices to show the middle term is linear in h. By linearity,

$$DL\left(t,f,f'\right)\left(0,h,h'\right) = \frac{\partial L}{\partial f}h(t) + \frac{\partial L}{\partial f'}h'(t)$$

Hence

$$\int_a^b DL\left(t,f,f'\right)\left(0,h,h'\right) dt = \int_a^b \frac{\partial L}{\partial f}h(t) + \underbrace{\frac{\partial L}{\partial f'}}_{u}\underbrace{h'(t)dt}_{dv}$$

Integrating by parts gives

$$\int_a^b \left(\frac{\partial L}{\partial f} - \frac{d}{dt}\frac{\partial L}{\partial f'}\right) h(t) + \left[h(t)\frac{\partial L}{\partial f'}\right]_a^b$$

This is linear in h and by definition of $T_f S$, $h(b) - h(a) = 0$. □

`Corollary 1.2 (euler-lagrange)` The extrema of $S(f)$ as defined in eq. (1) must satisfy

$$\frac{\partial L}{\partial f} - \frac{d}{dt}\frac{\partial L}{\partial f'} = 0$$