

Let X be a normed space over \mathbb{R} and suppose the parallelogram law holds, i.e.

$$|x + y|^2 + |x - y|^2 = 2|x|^2 + 2|y|^2 \quad (1)$$

for all $x, y \in X$. This is a necessary condition for X to be an inner product space. I will show it is also a sufficient condition

Let $\langle x, y \rangle := |x + y|^2 - |x|^2 - |y|^2$. Note that $\langle x, y \rangle = \langle y, x \rangle$, so this is symmetric. Note also that $\langle x, y \rangle$ is continuous as a map from $X^2 \rightarrow \mathbb{R}$.

Lemma 1 $\langle -x, y \rangle = -\langle x, y \rangle$

Proof. Observe that

$$\begin{aligned} \langle x, y \rangle &= |x + y|^2 - |x|^2 - |y|^2 \\ \langle -x, y \rangle &= |-x + y|^2 - |-x|^2 - |y|^2 \end{aligned}$$

So

$$\langle x, y \rangle + \langle -x, y \rangle = |x + y|^2 + |-x + y|^2 - 2|x|^2 - 2|y|^2$$

By eq. (1), this is zero. □

Lemma 2 $\langle a + b, c \rangle = \langle a, c \rangle + \langle b, c \rangle$

Proof. Observe that

$$\begin{aligned} \langle x + y, z \rangle &= |x + y + z|^2 - |x + y|^2 - |z|^2 \\ \langle x - y, z \rangle &= |x - y + z|^2 - |x - y|^2 - |z|^2 \end{aligned}$$

and consider the sum

$$\langle x + y, z \rangle + \langle x - y, z \rangle = |x + z + y|^2 + |x + z - y|^2 - (|x + y|^2 + |x - y|^2) - 2|z|^2$$

By eq. (1), $|x + y|^2 + |x - y|^2 = 2|x|^2 + 2|y|^2$ and $|x + y + z|^2 + |x + y - z|^2 = 2|x + y|^2 + 2|z|^2$, so we get

$$\langle x + y, z \rangle + \langle x - y, z \rangle = 2|x + y|^2 - 2|x|^2 - 2|y|^2$$

which by eq. (1) is

$$|x + y|^2 - |x - y|^2 = |x + y|^2 - 2|x|^2 - 2|y|^2 - (|x - y|^2 - 2|x|^2 - 2|y|^2)$$

By (1) a final time, we get

$$\langle x + y, z \rangle + \langle x - y, z \rangle = \langle x, z \rangle - \langle -x, z \rangle = 2\langle x, z \rangle$$

Now note that $\langle 2x, y \rangle = \langle x + x, y \rangle - \langle x - x, y \rangle = 2\langle x, y \rangle$ by the above. Thus

$$\langle x + y, z \rangle + \langle x - y, z \rangle = \langle 2x, y \rangle$$

Letting $x = \frac{a+b}{2}$, $y = \frac{a-b}{2}$, and $z = c$ proves the lemma. □

Theorem 1 (Linearity) $\langle kx + y, z \rangle = k\langle x, z \rangle + \langle y, z \rangle$

Proof. By lemma 2, it suffices to show that $\langle kx, y \rangle = k\langle x, y \rangle$. To see this, first note it is trivial when $k = 0$. Then consider the case where k is a positive integer. We have

$$\langle kx, y \rangle = \langle x + (k-1)x, y \rangle = \langle x, y \rangle + \langle (k-1)x, y \rangle$$

so it follows by induction that $\langle kx, y \rangle = k\langle x, y \rangle$. Now consider $\langle k^{-1}x, y \rangle$. By the previous result, we know $k\langle k^{-1}x, y \rangle = \langle x, y \rangle$, concluding $\langle k^{-1}x, y \rangle = k^{-1}\langle x, y \rangle$.

As any rational number $q = ab^{-1}$ where $a, b \in \mathbb{Z}$, $b \neq 0$, we see that $\langle qx, y \rangle = \langle ab^{-1}x, y \rangle = ab^{-1}\langle x, y \rangle = q\langle x, y \rangle$. By continuity, we can extend this to the case where $r \in \mathbb{R}$ is arbitrary, giving $\langle rx, y \rangle = r\langle x, y \rangle$. \square