1 ultrafilter

Definition 1.1 (Filter). F is a filter iff

$$A, B \in F \implies A \cap B \in F \tag{1}$$

$$\emptyset \notin F \tag{2}$$

$$B \supseteq A \in F \implies B \in F \tag{3}$$

Definition 1.2 (Ultrafilter).

$$A \subseteq X \implies A \in U \text{ or } X - A \in U \tag{4}$$

Theorem 1.3. Ultrafilter convergence defines a topology.

Theorem 1.4. A space is

1. compact iff every ultrafilter converges.
2. Hausdorff iff every ultrafilter converges to at most one point.
3. compact-Hausdorff iff every ultrafilter converges to exactly one point

2 Stone-Čech compactification

Let X be a space.

Definition 2.1 ($\mathtt{ultra}$).

Definition 2.2 ($\mathtt{ultra} : A \to \mathtt{ultra}\, A$). Suppose $a \in A$. Let $\mathtt{ultra}\, a$ be the ultrafilter generated by a.

Definition 2.3 ($\mathtt{lim} : \mathtt{ultra}^2\, X \to \mathtt{ultra}\, X$). Let $U \in \mathtt{ultra}^2\, X$. Suppose $U = \{U_i\}$ where U_i a subset of $\mathtt{ultra}\, X$. Then define

$$\mathtt{lim}\, U = \bigcup_{V \in U} \bigcap_{u \in V} u \tag{5}$$

Theorem 2.4. $\mathtt{lim}\, U : \mathtt{ultra}^2\, X \to \mathtt{ultra}\, X$

Proof. I claim

$$F_i :\equiv \bigcap_{u \in U_i} u \tag{6}$$

is a filter. To see this, note every filter on X contains X. Hence F_i is nonempty. The other filter axioms remain true after taking intersections.

Now pick arbitrary F_α and F_β as defined in eq. (6). I claim there is some F_γ (and corresponding U_γ) such that $F_\alpha \cup F_\beta \subseteq F_\gamma$. It suffices to find the corresponding U_γ. Let $U_\gamma = U_\alpha \cap U_\beta$. Equation (1) guarantees $U_y \in \mathcal{U}$. Then

$$\bigcap_{u \in U_\gamma} u \supseteq \left(\bigcap_{u \in U_\alpha} u\right) \cup \left(\bigcap_{u \in U_\beta} u\right)$$
$$F_\gamma \supseteq F_\alpha \cup F_\beta$$

Hence

$$\mathtt{lim}\,\mathcal{U} :\equiv \bigcup_i F_i$$

is a filter.

Finally, I claim $\mathtt{lim}\,\mathcal{U} \in \mathtt{ultra}\,X$. Suppose $A \subseteq X$. Then let $\mathtt{ultra}\,A \subseteq \mathtt{ultra}\,X$ be the set of ultrafilters that contain A. I claim $\mathtt{ultra}(A^C) = (\mathtt{ultra}\,A)^C$; as $(\mathtt{ultra}\,A)^C$, the set of ultrafilters that *do not contain* A, is $\mathtt{ultra}(A^C)$, the set of ultrafilters that *contain* A^C. This is a rewording of eq. (4).

By eq. (4), either $\mathtt{ultra}\,A \in \mathcal{U}$ or $\mathtt{ultra}(A^C) \in \mathcal{U}$. Suppose, without loss of generality, that $\mathtt{ultra}\,A \in \mathcal{U}$. Then

$$F_A := \bigcap_{u \in \mathtt{ultra}\,A} u \subseteq \bigcap_{u \in \mathcal{U}} u$$

Note F_A is a filter containing A. Then $\mathtt{lim}\,\mathcal{U}$ contains A. Equation (4) holds. □

Theorem 2.5 ($\mathtt{lim} \circ \mathtt{ultra} = \mathrm{id}$). Let $u \in \mathtt{ultra}\,X$. Then $\mathtt{lim} \circ \mathtt{ultra}\,u = u$ as every set containing u is in $\mathtt{ultra}\,u$.

3 ultrafilter monad

For this section, we will distinguish between $\mathtt{ultra}$ and η. Here, $\mathtt{ultra} : \mathsf{haus} \to \mathsf{Chaus}$ is the functor part of the monad, and $\eta : \mathrm{id} \to \mathtt{ultra}$ is the unit, and $\mathtt{lim} : \mathtt{ultra}^2 \to \mathtt{ultra}$ is the multiplication.

Theorem 3.1. The coherence conditions

$$\begin{tikzcd}
\mathtt{ultra}^3(X) \arrow[r, "\mathtt{ultra}(\mathtt{lim}_X)"] \arrow[d, "\mathtt{lim}_{\mathtt{ultra}(X)}"] & \mathtt{ultra}^2(X) \arrow[d, "\mathtt{lim}_X"] \\
\mathtt{ultra}^2(X) \arrow[r, "\mathtt{lim}_X"] & \mathtt{ultra}(X)
\end{tikzcd} \tag{7}$$

and

$$\begin{tikzcd}
\mathtt{ultra}(X) \arrow[r, "\eta_{\mathtt{ultra}(X)}"] \arrow[d, "\mathtt{ultra}(\eta_X)"] & \mathtt{ultra}^2(X) \arrow[d, "\mathtt{lim}_X"] \\
\mathtt{ultra}^2(X) \arrow[r, "\mathtt{lim}_X"] & \mathtt{ultra}(X)
\end{tikzcd} \tag{8}$$

hold

Proof. Consider eq. (7). Let $\mathcal{U} \in \texttt{ultra}^3\, X$. The goal is to show

$$\texttt{lim} \circ \texttt{ultra}(\texttt{lim})\mathcal{U} = \texttt{lim} \circ \texttt{lim}\mathcal{U}$$

First, consider $\texttt{lim} \circ \texttt{ultra}(\texttt{lim})\mathcal{U}$. Then

$$\begin{aligned}
\texttt{lim} \circ \texttt{ultra}(\texttt{lim}) &= \bigcup_{\mathrm{U} \in \mathcal{U}} \left(\bigcap_{u \in \bigcup_{V \in \mathrm{U}} \bigcap_{v \in V} v} u \right) \\
&= \bigcup \bigcap \left(\bigcup_{\bigcap} \right) \\
&= \bigcup_{\mathrm{U} \in \mathcal{U}} \bigcap_{V \in \mathrm{U}} \bigcup_{u \in V} \bigcap_{v \in u} v \\
\texttt{lim} \circ \texttt{lim} &= \bigcup_{\mathrm{U} \in \mathcal{U}} \bigcap_{V \in \mathrm{U}} \bigcup_{u \in V} \bigcap_{v \in u} v
\end{aligned}$$

□