Let E be a vector space over $\mathbb{C}$.

Definition 1 (Hermitian) A map $A \in \operatorname{End} E$ is Hermitian iff

$$\langle Ax, y \rangle = \langle x, Ay \rangle$$

Theorem 2 (finite spectral theorem) Suppose $E \cong \mathbb{C}^n$ is hermitian. Then

- E has eigenvectors that are an orthonormal basis of E.
- All eigenvalues of E are real.

Proof. By the fundamental theorem of algebra, the characterestic polynomial

$$|A - xI|$$

has a root. Hence A has an eigenvalue-eigenvector pair λ, e. But

$$\lambda \langle e, e \rangle = \langle e, Ae \rangle = \langle Ae, e \rangle = \overline{\lambda} \langle e, e \rangle$$

thus $\lambda = \overline{\lambda}$. Ergo, $\lambda \in \mathbb{R}$.

Now consider $A|e^\perp$. Suppose $\langle x, e \rangle = 0$. Then

$$0 = \lambda \langle x, e \rangle = \langle x, Ae \rangle = \langle Ax, e \rangle$$

Hence $A|e^\perp \in \operatorname{End}(e^\perp)$. Induction on dimension proves the theorem.

Let $\bullet$ denote pointwise multiplication.

Corollary 3 (diagonalization) If $A \in \operatorname{End} E$, then

$$A = P^{-1}(\nu \bullet \_)P$$

where P is unitary and $\nu \in P(E)$ is real.

Definition 4 (standard part of operator)

$$\begin{array}{rccc} \mathrm{st}: & \operatorname{End} E & \to & E \\ & (\mathrm{st}\, T)(x) & := & \mathrm{st}(T(^*x)) \end{array}$$

Theorem 5 (infinite spectral theorem) Suppose E is a Hilbert space. If $A \in \operatorname{End} E$ is hermitian, then

$$A = P^{-1}(\nu \bullet \_)P$$

with P unitary and $\nu \in \tilde{E}$ real.

Proof. Consider a nonstandard model of functional analysis. Fix a hyperfinite-dimensional subspace F such that

$$^{\sigma}E \subseteq F \subseteq {}^{*}E$$

There is some hermitian $B \in \operatorname{End} F$ such that $B|\,^{\sigma}E = {}^{*}A|\,^{\sigma}E$. This B simultaneously satisfies hermitian-ness and $B({}^{*}e) = {}^{*}(Ae)$ for each e in some (standard) basis of E. Such a B exists, internal to a sufficiently saturated model.

By $*$−transferring diagonalization, there is some unitary $P : F \xrightarrow{\sim} \tilde{F}$ and real $v \in \tilde{F}$ such that

$$B = P^{-1}(v \bullet \_)P \tag{1}$$

By construction, $B(^{\sigma}E) \subseteq {}^{\sigma}E$. Permuting rows of the matrices $v \bullet \_$ and P if necessary, assume (without loss of generality) that $P(^{\sigma}E) \subseteq {}^{\sigma}E$.

Then

$$(\operatorname{st} P)^{-1} = \operatorname{st}(P^{-1})$$

similarly,

$$\big(\operatorname{st}(v \bullet \_)\big)(x) = \operatorname{st}(v \bullet {}^{*}x) = \operatorname{st} v \bullet x = (\operatorname{st} v \bullet \_)x$$

By construction, $\operatorname{st} B = A$, hence eq. (1) becomes

$$A = \operatorname{st}(P)^{-1}(\operatorname{st} v \bullet \_)\operatorname{st}(P)$$