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An accurate, non-iterative approximation for studentized range quantiles

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Abstract

Several statistical procedures require upper quantiles of the studentized range distribution, but few environments for statistical analysis offer on-line calculation of those quantiles. This is because available algorithms require high-precision numerical integration, applied iteratively, to achieve accurate results. Consider instead a simple non-iterative approximate transformation from student t quantiles, which are easily computed and generally available, to studentized range quantiles. Given an accurate quantile from the student t distribution, only a few arithmetic operations yield a studentized range quantile with accuracy sufficient for most data analytic and other practical purposes; in fact, the accuracy is nearly as good as that of the studentized range table that has been in use since 1960. This approach also yields methods for interpolating studentized range quantiles that are more accurate than methods in current use.

Keywords: Computation; Interpolation; Multiple comparisons; Quantile function; Studentized range distribution; Tukey wsd test

1. Introduction and summary

Techniques such as the Tukey wsd and Newman–Keuls multiple comparison procedures require upper quantiles of the studentized range distribution. The source of those quantiles is often still Harter's (1960) table, portions of which have been widely reproduced in textbooks. On-line computation of studentized range quantiles is uncommon because available methods are limited in accuracy or range, or require

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very specialized software. Algorithm AS190 of Lund and Lund (1983) is an example of the former type. One in the latter category is Copenhaver and Holland's (1988) algorithm, which needs high precision routines for the gamma function and the normal distribution function, and which iteratively applies numerical integration based on 16-point Gauss–Legendre quadrature.

According to Harter (1960), usage of the studentized range distribution was first proposed in a letter written by 'Student' to E.S. Pearson, and later published in Pearson (1938). But currently, studentized range quantiles will often be

(a) needed for multiple comparison tests and similiar practical purposes where only a few digits of accuracy are required, and

(b) used in a (possibly interactive) computing environment – say, a statistics package or a spreadsheet – in which accurate quantiles of student's $t(v)$ distribution are already available as a function or command.

This article proposes a non-iterative method of approximating upper quantiles of the studentized range, suitable for this context. Given an accurate quantile from student's $t(v)$ distribution, the method uses only a handful of numeric operations and a table of constants, and is simple enough to implement as a spreadsheet formula or as a user-defined function in a statistics package. The approximate quantiles are nearly as accurate as Harter's (1960) table (they are rarely wrong by more than one unit in the fourth significant digit), over the entire range of that table and beyond – a range much broader than that found in most textbooks.

Section 2 describes the proposed method and an approach to building the required table of constants, and establishes the accuracy of the method. Section 3 examines schemes for accurately interpolating values not directly represented in the table of constants, and Section 4 considers implementation details. Appendix provides details of the computations reported here, and describes available implementations of the proposed method.

2. Approximate studentized range quantiles

The (externally) studentized range is a variable of the form

$$Q = (X_{(r)} - X_{(1)})/S \tag{2.1}$$

where $X_{(1)}$ and $X_{(r)}$ are the smallest and largest order statistics in r independent $N(\mu, \sigma^2)$ random variables, and vS^2/σ^2 is a $\chi^2(v)$ random variable that is independent of $X_{(r)} - X_{(1)}$. The pth quantile $q_p(r, v)$ of the distribution satisfies

$$\Pr(Q \leq q_p(r, v)) = p \quad \text{where} \quad 0 < p < 1, \tag{2.2}$$

but interest here is confined to upper quantiles, where $1/2 \leq p < 1$.

As noted above, computing environments for data analysis seldom offer a function for evaluating $q_p(r, v)$, although Maindonald (1995) provides an exception. But most such environments do include a function for computing the pth quantile of student's $t(v)$ distribution, $t_p(v)$. The present objective is to use Harter's (1960) table as the focal point of a fast, easily implemented, and non-iterative approximate transformation from $t_p(v)$ to $q_p(r, v)$.

Harter's table provides values of $q_p(r,v)$ for p=0.90, 0.95, 0.975, 0.99, 0.995, 0.999, $r = 2(1)20(2)40(10)100$, and $v = 1(1)20$, 24, 30, 40, 60, 120, ∞. Denote by $\mathscr{T}$ the subset of Harter's table where $r = 2(1)20$, 30, 40(20)100; $\mathscr{T}$ contains $6 \times 24 \times 26 = 3744$ values of $q_p(r,v)$ accurate, according to Harter (1960), to the four significant digits tabled. (But see the second paragraph of Section 5.)

The Copenhaver–Holland (1988) algorithm (C-H) in Maindonald (1995) was used to extend the entries in $\mathscr{T}$ to more than four significant digits, and also to obtain values of $q_{0.5}(r,v)$ and $q_{0.75}(r,v)$. Harter's four-digit value of $q_p(r,v)$ was retained at $v = 1$ (where the C-H algorithm is unavailable), and wherever Harter's table and the C-H algorithm differed by more than one unit in the fourth significant digit. (See the Appendix for additional computing details.) Denote by $\mathscr{T}^+$ the table of 4944 quantiles which appends to $\mathscr{T}$ the values of $q_{0.5}(r,v)$ and $q_{0.75}(r,v)$ for $r = 2(1)20$, 30, 40(20)100, and $v = 2(1)20$, 24, 30, 40, 60, 120, ∞. $\mathscr{T}^+$ provides a basis for building a simple approximation to $q_p(r,v)$.

Begin with the obvious relationship that, for any $0 < p < 1$ and all $v \geq 1$,

$$q_p(2,v) = \sqrt{2}t_{p'}(v) \quad \text{where } p' = (1+p)/2. \tag{2.3}$$

The key idea of the present approach is to form the variable

$$y_p(r,v) = q_p(r,v)/\sqrt{2}t_{p'}(v) \tag{2.4}$$

and then model $y_p(r,v)$ as $1 + f(p,r,v)$ where the function f satisfies $f(p,2,v) = 0$ for all p and v. ($y_p(r,v)$ and $q_p(r,v)$ are often written as y, q, y_p, q_p, etc. when there is no risk of confusion.) Transforming to y is useful because both $q = q_p(r,v)$ and $t = t_{p'}(v)$ change rapidly at small v, but their ratio q/t is more stable. Intuitively, this is because a t statistic contains a random variable that is formally the same as S in (2.1), and the quantiles $t_p(v)$ have already captured the effects induced by that variable. This eases the task of approximating q, as can be seen by graphing both q and y as functions of p, r, and v.

Fig. 1 shows such a graph, a plot of values of y versus $\log(r-1)$ for the table $\mathscr{T}^+$ entries at $p = 0.99$. This reveals a series of gently curved and similarly shaped traces radiating from the point $(0,1)$ as v ranges from 1 to ∞; plots at other values of p have a similar appearance. The dashed lines in Fig. 1 show q rather than y as the ordinate, with the traces for $v = 1,2$ suppressed. Even for $v \geq 3$, there is much greater heterogeneity across v in q than in y. In fact, the curves for y can be accurately captured, for a given combination of p and v, by a low-order polynomial in $\log(r-1)$. Specifically, model y as

$$y = 1 + \alpha_1 \log(r-1) + \alpha_2 \log^2(r-1) + \alpha_3 \log^3(r-1) + \alpha_4 \log^4(r-1),$$

at each of the 206 combinations of p and v in the table $\mathscr{T}^+$, using ordinary least-squares estimation. This results in a table $\mathscr{A}$ of 206×4 estimated coefficients $a_j(p,v)$. (Copies of the tables $\mathscr{T}^+$ and $\mathscr{A}$ are available; see the Appendix.)

Then, for p and v represented in table $\mathscr{A}$, approximate $f = f(p,r,v)$ as

$$\hat{f} = \sum_1^4 a_j \log^j(r-1) + b(p) + c(v), \tag{2.5}$$

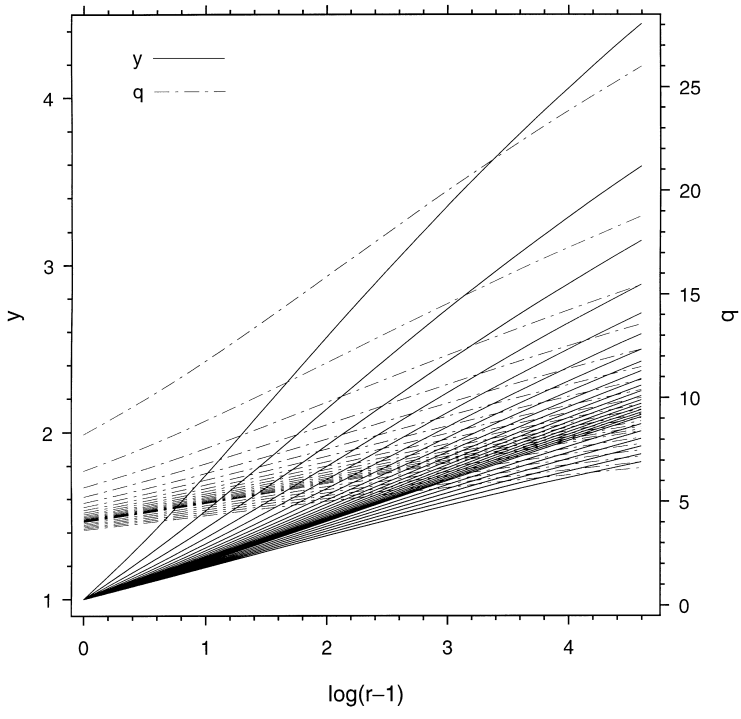

Fig. 1. $y_p(r,v)$ and $q_p(r,v)$ versus $\log(r-1)$ at $p=0.99$. The solid lines trace values of (2.4) for the 624 combinations of r and v in table $\mathscr{T}^+$ at $p=0.99$; the uppermost trace is for $v=1$, the lowermost for $v=\infty$. The dashed lines trace the studentized range quantiles defined by (2.2); the uppermost trace is for $v=3$, the lowermost for $v=\infty$. (The righthand axis would have to extend to $q=400$ to accommodate the trace for $v=1$.)

where $a_j = a_j(p,v)$ is the appropriate entry in $\mathscr{A}$, and approximate $q_p(r,v)$ as

$$\hat{q}_p(r,v) = \sqrt{2}(1+\hat{f})t_{p'}(v). \tag{2.6}$$

(Interpolation for values of p and v not included in $\mathscr{A}$ is considered in Section 3.) The terms $b(p)$ and $c(v)$ are minor corrections applied only at $r=3$:

$$b(p) = \begin{cases} -0.002/(1+12z_p^2) & \text{if } r=3, \\ 0 & \text{otherwise,} \end{cases} \tag{2.7}$$

where $z_p = \Phi^{-1}(p)$ is the pth quantile of the standard normal distribution, and

$$c(v) = \begin{cases} 1/517 - 1/(312v) & \text{if } r=3 \text{ and } v \le 4.364, \\ 1/(191v) & \text{if } r=3 \text{ and } v > 4.364, \\ 0 & \text{otherwise.} \end{cases} \tag{2.8}$$

The corrections $b(p)$ and $c(v)$ can be omitted without great loss, but with those terms in place, approximation (2.5)–(2.6) is rather accurate. (For the remainder of this paper, statements about accuracy refer to the combinations of p, r, and v represented in table $\mathscr{T}^+$, except where otherwise noted – e.g., when interpolating

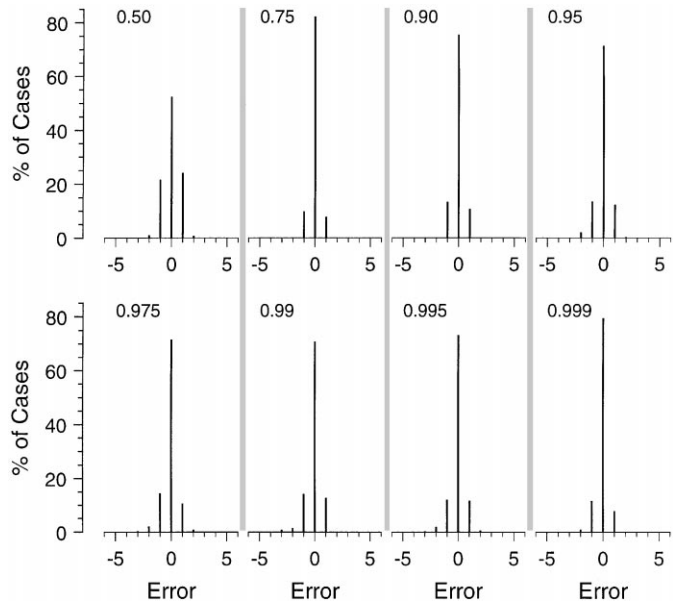

Fig. 2. Distribution of *fourth digit errors* in $\hat{q}$. Each panel plots errors at the value of p shown at the top of the panel. The horizontal axis shows the error in using $\hat{q}_p(r,v)$ (from (2.5)–(2.6)) to approximate $q_p(r,v)$ (from Harter's (1960) table or the C-H algorithm), as units in the fourth significant digit. The vertical axis shows the percentage of cases from the table $\mathscr{T}^+$.

for values not in $\mathscr{T}^+$. Error is expressed ignoring its sign, either in units of the fourth significant digit of $\hat{q}$ or as percent relative error, $100(q-\hat{q})/q$, where q is taken from either Harter's (1960) table or the C-H algorithm. Thus, if $q=29.68$ and $\hat{q}=29.686$, $\hat{q}$ is wrong by one fourth digit unit and by 0.020%.)

Fig. 2 shows that $\hat{q}$ is seldom wrong by more than one fourth digit unit (in fewer than 2% of all cases in $\mathscr{T}^+$); fourth digit error never exceeds four units. Eliminating very small values of v improves the situation slightly: Of 4032 cases where $v>5$, only one error exceeds two units and 99% of the errors are zero or one. Fig. 3 plots the relative error versus v (left panel) and r (right panel), for four choices of p. It can be seen that the largest errors occur at very small v, and that $\hat{q}$ is about equally accurate at small and large values of r. Over the 4738 cases where $r>2$, the relative error never exceeds 0.075%, and is less than 0.02% in 92% of the cases. (At $r=2$, $\hat{q}$ is perfectly accurate if $t_{p'}(v)$ is.)

3. Interpolating studentized range quantiles

Approximation (2.5)–(2.6) adequately reproduces $\mathscr{T}^+$, which covers a large range of possibilities: 8 values of p in $[0.5, 0.999]$, 24 values of r in $[2, 100]$, and 26 values of v in $[1, \infty]$. Consider now interpolation for quantiles $q_p(r,v)$ not included in $\mathscr{T}^+$ (not represented in $\mathscr{A}$), which might be required with respect to r, to v, or to p. Harter (1960, p. 1145) examined each of these types of interpolation, working in

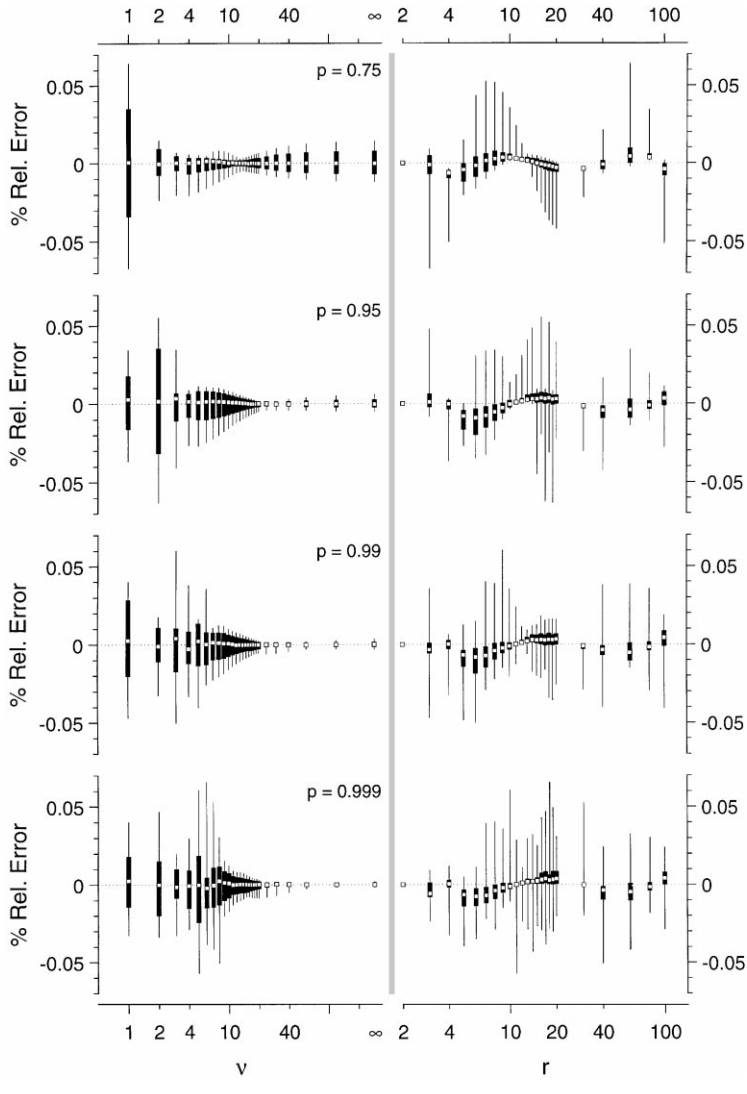

Fig. 3. Distribution of *relative error* in $\hat{q}$. Each layer plots percent relative error $(100(q-\hat{q})/q)$ at the value of p shown at the top of the layer. At each abscissa value, the errors are represented by a box-and-whiskers plot: the whiskers extend from the smallest to the largest error, the dark bar extends from the first to the third quartile, and the open square is at the median of the error distribution. Each plot summarizes 24 errors at a single value of v (left panel), or 26 errors at a single value of r (right panel).

terms of $q_p(r,v)$, but it will be seen that $y_p(r,v)$, defined by (2.4), is better suited to simple interpolation schemes.

3.1. Interpolating in r

Interpolation in r occurs automatically in evaluating (2.5), and the resulting $\hat{q}$ is appreciably more accurate than interpolating q with respect to r. Attempting the latter

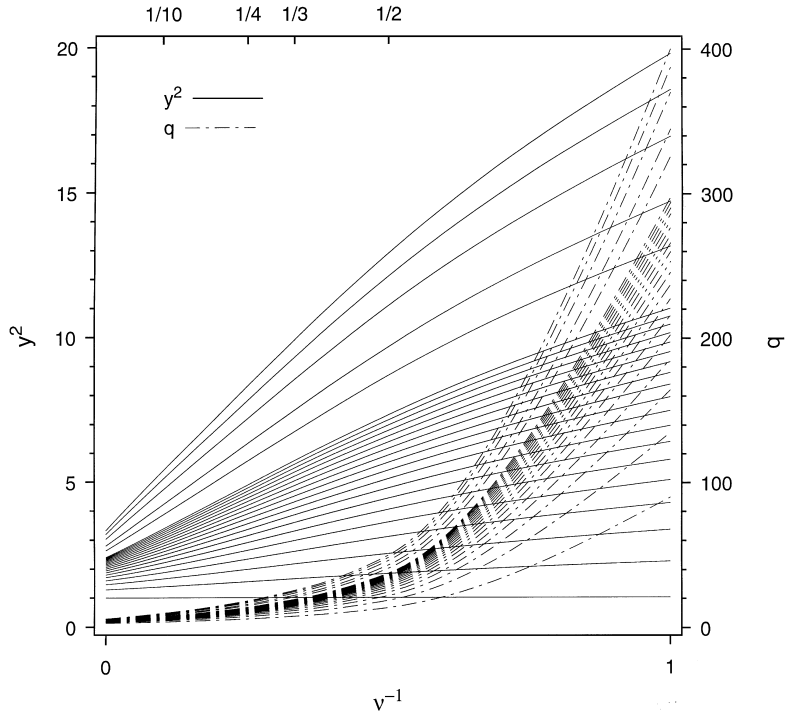

Fig. 4. $y^2(p,r,v)$ and $q_p(r,v)$ versus v^{-1} at $p=0.99$. The solid lines trace values of y^2 where y is given by (2.4), and the dashed lines trace the quantiles defined by (2.2) for the 624 combinations of r and v in table $\mathscr{T}^+$ at $p=0.99$. Within each set of traces, the uppermost trace is for $r=100$, the lowermost for $r=1$.

in Harter's table gives errors as large as three fourth digit units for $20<r<40$, and 11 units for $40<r<100$ (Harter, 1960, p. 1145). By contrast, suppose we interpolate between $r=20$ and $r=30$ for $\hat{q}_p(24,v)$, and between $r=80$ and $r=100$ for $\hat{q}_p(90,v)$. Of the 416 cases in $\mathscr{T}^+$, the error in $\hat{q}$ does not exceed one unit in the fourth digit. In fact, extrapolation beyond $r=100$ works well: At $r=200$, $\hat{q}$ differs from q (obtained from the C-H algorithm) by no more than 11 units in the fourth digit, over all p and v in Harter's table – errors of the same size as interpolating q for $40<r<100$ in that table.

3.2. Interpolating in v

Harter (1960, p. 1145) notes that linear interpolation of $q_p(r,v)$ in v^{-1} gives fourth digit errors of no more than four units at $p=0.999$, two units at $p=0.99, 0.995$, and one unit elsewhere in his table. This statement applies to both integer and non-integer $v>20$, but not to non-integer $v<20$. But, interpolating $y_p^2(r,v)$ linearly in v^{-1} gives much more accurate results, especially for small v. Fig. 4 plots y^2 and q versus v^{-1} for $p=0.99$, and it is clear that linear interpolation in v^{-1} will be more successful with y^2. Increasing p increases the curvature in the traces for q but has little effect on the shape of the traces for y^2, so that the advantage of using y^2 increases with p.

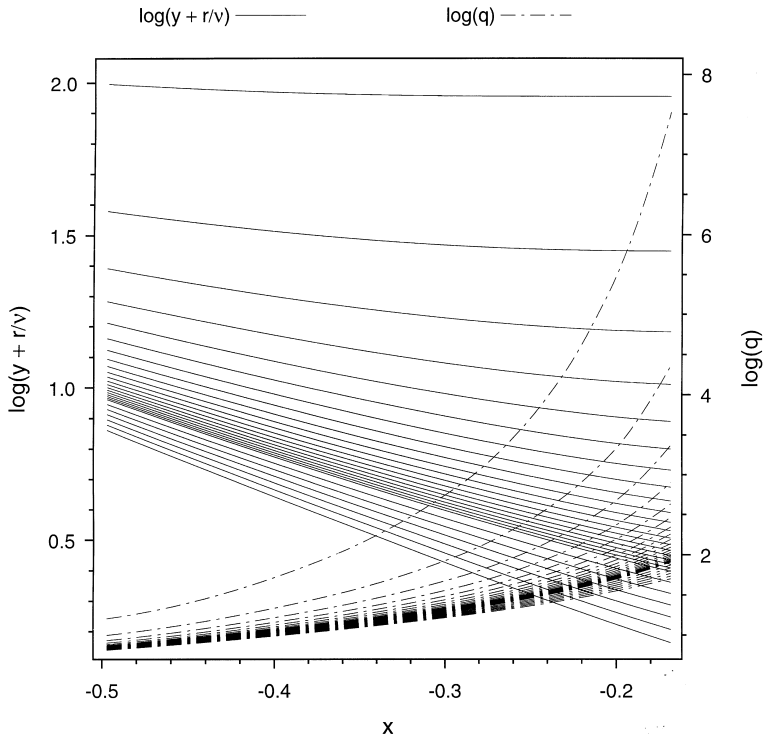

Fig. 5. $\log(y + r/v)$ and $\log(q)$ versus x at $r = 5$. The solid lines trace values of $\log(y + r/v)$ where $y = y_p(r, v)$ is given by (2.4), and the dashed lines trace the quantiles defined by (2.2) for the 208 combinations of p and v in table $\mathscr{T}^+$ at $r = 5$. The abscissa is given by $x = -1/(1 + 1.5\Phi^{-1}(p'))$, $p' = (1 + p)/2$. Within each set of traces, the uppermost trace is for $v = 1$, the lowermost for $v = \infty$.

To illustrate, consider interpolating $\hat{y}^2 = (1 + \hat{f})^2$, with $\hat{f}$ from (2.5), versus v^{-1} between $v = 20$ and $v = 24$ for $\hat{q}_p(r, 22)$, and between $v = 30$ and $v = 40$ for $\hat{q}_p(r, 35)$. This creates just one error as large as two units in the fourth digit over the domain of table $\mathscr{T}^+$ – and the sole error of size two is at $p = 0.50$. The gain from interpolating $\hat{y}^2$ rather than q is even larger at small v. Consider interpolating between $v = 4$ and $v = 5$ for $\hat{q}_p(r, 4.5)$; for all p and r in $\mathscr{T}^+$, this creates no errors larger than three units in the fourth digit.

3.3. Interpolating in p

This case is more challenging than the previous two, and Harter (1960, p. 1145) warned against interpolating $q_p(r, v)$ in p – for good reason. But, interpolation is quite feasible using $\log(y_p(r, v) + r/v)$ as the ordinate, and as the abscissa

$$x = x(p) = -1/(1 + 1.5z_{p'}), \tag{3.1}$$

where $z_{p'} = \Phi^{-1}(p')$ is the $N(0, 1)$ quantile function evaluated at $p' = (1 + p)/2$. Fig. 5 plots $\log(y + r/v)$ versus x at $r = 5$, and reveals a series of nearly linear traces across values of v. By contrast, as Fig. 5 also shows, interpolating $\log q$ versus x

will be far less effective, especially at small v. Increasing r tends to straighten the traces for $\log(y + r/v)$ but not those for $\log q$.

Still, sufficient curvature remains at small r and v to justify using quadratic interpolation, which is nearly as accurate with respect to $x(p)$ as is linear interpolation with respect to v^{-1} or $\log(r-1)$. To demonstrate, consider interpolating $\log(\hat{y}+r/v)$ between $p = 0.75$ and $p = 0.90$ for $\hat{q}_{0.85}(r, v)$, between $p = 0.95$ and $p = 0.975$ for $\hat{q}_{0.965}(r, v)$, and between $p = 0.99$ and $p = 0.995$ for $\hat{q}_{0.9925}(r, v)$. The first of these three is interpolation over a broad interval, and diminished accuracy ought to be expected. Over the domain of table $\mathscr{T}^+$, the largest error in $\hat{q}_{0.85}$ is six units in the fourth digit, and 88% of the errors do not exceed one unit. The largest errors are at small v; for $v \geq 5$, 95% of the errors do not exceed one fourth digit unit, and the largest error is three units.

For $p \geq 0.90$ (i.e., within the range of table $\mathscr{T}$), interpolation in p is still more accurate. Over the domain of table $\mathscr{T}$, the largest error in $\hat{q}_{0.965}$ and $\hat{q}_{0.9925}$ is three fourth digit units, and only 1.6% of the errors exceed one unit – about the same accuracy as for approximation (2.5)–(2.6) without any interpolation. Even extrapolation beyond $p = 0.999$ is tolerable for many purposes. For example, the relative error in $\hat{q}_{0.9999}$ does not exceed 0.16% and the fourth digit errors do not exceed 13 units – about the same as the accuracy given by Harter (1960, p. 1145) for interpolating in his table for $40 < r < 100$.

4. Implementation issues

Given an accurate function for student $t(v)$ quantiles, $\hat{q}$ requires one logarithm and a dozen or so arithmetic operations (ignoring the correction at $r = 3$). The only obstacle is the size of the table $\mathscr{A}$ of constants; $\mathscr{A}$ has 206×4 elements that, in single precision (which is quite adequate), consume 3296 bytes of storage – a small amount of memory or disk space by contemporary standards. Still, it may be possible to reduce the storage demand with little or no effect on accuracy.

First, in the typical application envisioned (say, to perform a Tukey wsd test), one would need only a small part of $\mathscr{A}$ – the four coefficients for a single combination of p and v. In this case, the value of p would likely be one of the cardinal values represented in $\mathscr{A}$, and v would often be large, say $v \geq 20$. This suggests a strategy of subdividing $\mathscr{A}$ and loading only the subset required by the problem at hand; given careful subdivision, no additional portions of $\mathscr{A}$ will be required in a typical data analysis session. For example, one might divide $\mathscr{A}$ into eight submatrices, one for each value of p, and further divide each submatrix at $v = 20$. Then, for a single p, to compute $\hat{q}_p(r, v)$ for any $r \geq 2$ and any $v \geq 20$ requires loading a single submatrix of size 7×4. (Note that in many environments it would be inefficient to store $\mathscr{A}$ as 206 distinct submatrices.)

This approach has the advantage of maintaining the full accuracy of the approximation (2.5)–(2.6), while minimizing the number of constants stored in the most typical applications of the method: 28 constants if p is represented in $\mathscr{A}$, 84 constants if quadratic interpolation in p is required. Gleason (1998) provides a program

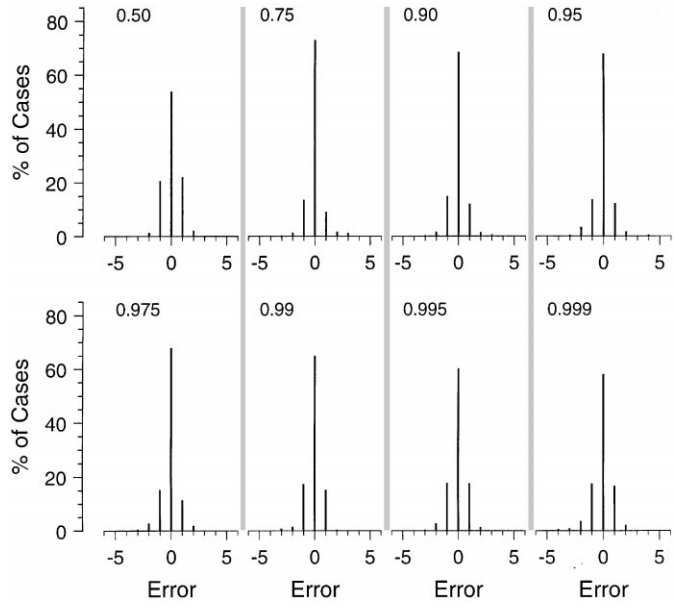

Fig. 6. *Fourth digit errors* in $\hat{q}$, using $\mathscr{A}^*$. The graph is exactly the same as Fig. 2, except that the reduced matrix $\mathscr{A}^*$ has been used in place of $\mathscr{A}$ to calculate $\hat{q}_r(p,v)$, as described at the end of Section 4.

for the statistics package Stata that is designed in this way and uses the interpolation schemes recommended in Section 3; it occupies about 3300 bytes of memory – a small fraction of the amount required for the C-H algorithm in the same situation.

A second approach is to model the elements of $\mathscr{A}$ as functions of p and/or v, and then replace $\mathscr{A}$ with the formulae thus derived. This is possible, though at some loss of accuracy. For example, each of the constants $a_j = a_j(p,v)$ in (2.5) can be approximated as a cubic function in v^{-1},

$$a_j \approx a^*_{j0} + a^*_{j1}v^{-1} + a^*_{j2}v^{-2} + a^*_{j3}v^{-3}, \tag{4.1}$$

where the values of the a^*_{jk}'s are obtained by least-squares estimation. For $v > 5$, this approximation is accurate for all 8 values of p in $\mathscr{A}$; for $v \leq 5$, it is wise to retain the values of the a_j's. So, define the 70×4 matrix $\mathscr{A}^*$ to consist of the elements of $\mathscr{A}$ for $v \leq 5$, combined with the 32×4 matrix of constants a^*_{jk} from (4.1). Then in computing (2.5), use (4.1) and the appropriate entries in $\mathscr{A}^*$.

Matrix $\mathscr{A}^*$ is about one-third the size of $\mathscr{A}$, and it can of course be subdivided along the lines described above for $\mathscr{A}$. The loss of accuracy from using $\mathscr{A}^*$ rather than $\mathscr{A}$ is modest, as Fig. 6 shows. Comparison with Fig. 2 reveals that the main effect is a slight decrease in the frequency of no fourth digit error coupled with more frequent errors of one unit. Most of the degradation occurs at very large values of r; for $r \leq 40$, using $\mathscr{A}^*$ to compute $\hat{q}$ makes no fourth digit errors in about 68% of the cases, and is wrong by more than one unit in about 2.5% of the cases – not much worse than using the larger matrix $\mathscr{A}$.

5. Discussion

Studentized range quantiles can be computed by solving (2.2) numerically for $q_p(r,v)$, and the C-H algorithm does just that. The result is of course 'exact' provided the required numerical integrations are accurate – a source of concern in extreme cases, say when v is small. Somerville (1997) provided a related approach, one that uses a combination of numerical integration and Monte Carlo sampling. The method proposed here sacrifices some exactness for speed and simplicity; it also performs well in cases where numerical integration is difficult. For example, over the 144 entries in table $\mathscr{T}$ at $v=1$, the maximum relative error in $\hat{q}$ is 0.07%, and only one error exceeds two fourth digit units. On the other hand, the C-H code in Maindonald (1995) is restricted to $v>1$, and convergence or precision problems can still arise at other small values of v. Computing (2.5)–(2.6) is also simpler and more accurate than algorithm AS190 of Lund and Lund (1983). AS190 is iterative, valid only for $0.90 \le p \le 0.99$, and accurate to only about two decimal places (Royston, 1987); $\hat{q}$ will typically be accurate to at least one additional decimal place.

In a sense, quantiles from (2.5)–(2.6) are not much less accurate than Harter's (1960) table, which contains some minor errors. Recall that $q_p(2,v) = \sqrt{2}t_{p'}(v)$; see (2.3). $\mathscr{T}$ contains 156 entries at $r=2$, and comparison with accurate values of $t_{p'}(v)$ shows that 30 values of $q_p(2,v)$ are too large by one fourth digit unit. (For example, it can be shown that $q_{.90}(2,2) = 4.129483\ldots$, which Harter tabled as 4.130; the other 29 instances are similar in nature.) Comparison of $\mathscr{T}$ for $r>2$ with $q_p(r,v)$ obtained from the C-H algorithm yields a similar result: About 20% of the entries in $\mathscr{T}$ are too large by one unit in the fourth significant digit. (Apparently, Harter was slightly conservative in rounding to four digits.) By contrast, (2.5)–(2.6) is incorrect in the fourth digit for about 30% of the cases at $r>2$, and only about 2% of those errors exceed one fourth digit unit.

The key to the present approach is the 'absorption' of the effects of the variable S in (2.1) by modeling the ratio $y_p = q_p/t_{p'}$ rather than the quantiles q_p. A similar strategy can be used with a variety of other distributions to approximate quantiles with accuracy similar to or exceeding that of available tabled values.

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Appendix

The table $\mathscr{T}$ is a copy of Harter's (1960) table at $r=2(1)20,30,40(20)100$; its size and precision were extended to obtain table $\mathscr{T}^+$. The C-H algorithm was selected

for this task because of its accuracy in calculating $q_p(r, v)$. (The C-H algorithm was designed to compute quantiles of the maximum of several independent studentized range variates, but it yields values of $q_p(r, v)$ as a special case.) In the following, let $q^{(C)}$ denote the output of the C-H algorithm and $q^{(H)}$ be the corresponding entry in Harter's table. Table $\mathscr{T}^+$ contains $q^{(C)}$, except in the following cases: (a) $v = 1$; (b) $q^{(C)}$ differed from $q^{(H)}$ by more than one unit in the fourth significant digit; (c) $q^{(C)} > 12.5$ and $q^{(H)}$ was unavailable. In these cases, $\mathscr{T}^+$ contains $q^{(H)}$, if available, or else a missing value. Rule (c) was enforced because when $q^{(H)}$ *was* available, large values of $q^{(C)}$ were sometimes slightly inaccurate, due to precision limitations; $q^{(C)} \leq 12.5$ was a conservative criterion for eliminating such cases.

The C-H algorithm was programmed in ANSI C, by adapting the `f2c` conversion of the FORTRAN code in Maindonald (1995). The C program used IEEE double precision arithmetic throughout, and iterated until successive values of $q^{(C)}$ differed by less than 5×10^{-7}. All other computations were done in Stata, Version 5 (Stata, 1997). Gleason (1998) includes a Stata program for approximation (2.5)–(2.6) using table $\mathscr{A}$, as well as a superset of the table $\mathscr{T}$ in PDF (Adobe Acrobat) format. This software is part of the archive `stb46v5.zip` available from Statlib, and by anonymous ftp following the links from `http://www.stata.com/support/`. Plain text copies of tables $\mathscr{T}$, $\mathscr{T}^+$, $\mathscr{A}$, and $\mathscr{A}^*$ will also be available from Statlib.

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