

ORDERS ON FREE METABELIAN GROUPS

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ABSTRACT. A bi-order on a group G is a total, bi-multiplication invariant order. Such an order is regular if the positive cone associated to the order can be recognised by a regular language. A subset S in an orderable group (G, \leq) is convex if for all $f \leq g$ in S , every element $h \in G$ satisfying $f \leq h \leq g$ belongs to S . In this paper, we study the convex hull of the derived subgroup of a free metabelian group with respect to a bi-order. As an application, we prove that non-abelian free metabelian groups of finite rank do not admit a regular bi-order while they are computably bi-orderable.

1. INTRODUCTION

A group G is *bi-orderable* if there exists a total order \leq which is invariant under multiplication from both sides, i.e., if $g \leq h$ then $f_1 g f_2 \leq f_1 h f_2$ for all $f_1, f_2 \in G$. Such a total order is called a bi-invariant order or bi-order for short on the group G . Similarly, a group G is *left-orderable* (*right-orderable*) if there exists a left-invariant (right-invariant) order on G , in which the order is invariant under left-multiplication (right-multiplication). It is not hard to see right-orders and left-orders have a one-to-one correspondence. Thus in this paper we will only discuss left-orders and bi-orders on a group. For every left-order \leq on G , the positive cone P_{\leq} consists of all positive elements in G under \leq . It is a semigroup and $G = P \sqcup P^{-1} \sqcup \{1\}$. If \leq is bi-invariant, then P is invariant under conjugation. A positive cone completely determines the corresponding order and vice versa. Hence in this paper we will identify positive cones and their associated orders when it is convenient.

It is well-known that left-orderability is preserved under group extensions. Let G be an extension of A by Q , where $\pi : G \rightarrow Q$ is the quotient map, and suppose A, Q are left-orderable. In addition if we assume P_A and P_Q are positive cones of A and Q respectively, then $P := P_A \cup \pi^{-1}(P_Q)$ is a positive cone of a left-order on G , and thus G is also left-orderable. While in general, bi-orderability is not preserved under group extension, unless P_A is invariant under the action of Q , where in this case $P_A \cup \pi^{-1}(P_Q)$ defines a bi-order on G . An order given by such construction is called a *lexicographical order leading by the quotient*. A subgroup H of an orderable group G is *convex* with respect to an order \leq if for any pair of elements $f_1 \leq f_2$ in H , the condition $f_1 \leq g \leq f_2$ implies $g \in H$. Note that under a lexicographical order leading by the quotient, the normal subgroup A is always convex with respect to \leq . Conversely, if A is convex, then \leq is a lexicographical order leading by the quotient (See Proposition 3.2).

In this paper, we study the convex hull of the derived subgroup in a free metabelian group with respect to a bi-order. Recall that the convex hull \overline{H} of a subgroup H is the smallest convex subgroup containing H . Let M_n be the free metabelian group of rank n . We show that the derived subgroup is always convex when $n = 2$.

Theorem A (Theorem 3.1). *M_2' is convex with respect to any bi-invariant order on M_2 .*

When $n \geq 3$, we construct a bi-order such that $\overline{M}'_n \neq M'_n$. But we can still obtain some information about the order from the restriction of the order on the derived subgroup.

Theorem B (Theorem 4.7). *Let \leq be a bi-invariant order on M_n then M_n/\overline{M}'_n is not trivial, where \overline{M}'_n is the convex hull of the derived subgroup with respect to \leq .*

Let $\mathcal{LO}(G)$ be the set of all left-orders on G . It carries a natural topology whose sub-basis is the family of sets of the form $V_g = \{P_\leq \mid 1 \leq g\}$ for $g \in G$. The space $\mathcal{LO}(G)$ is a closed subset of the Cantor set and is metrizable (See, for example, [DNR14], [CR16]). And the space of all bi-orders $\mathcal{O}(G)$ is a closed subspace of $\mathcal{LO}(G)$. A lot have been known for the structure of those spaces. The space of bi-orders of a non-cyclic free abelian group is homeomorphic to the Cantor set [Sik04] and the space of left-orders on a non-abelian free group is also a Cantor set [McC89] while the structure of spaces of bi-orders of non-abelian free groups remains unknown. For the Braid group $B_n, n \geq 2$, the space $\mathcal{LO}(B_n)$ is infinite and has isolated points [DD01]. Tararin gave a complete classification of groups which have finite space of left-orders [KM96, Proposition 5.2.1]. As a consequence of Theorem A we have

Corollary 1.1. *The space $\mathcal{O}(M_2)$ is homeomorphic to the Cantor set.*

Note that the space of left order of a free metabelian group of rank 2 or higher is a Cantor set [RT16].

Let X be a generating set of G . A language \mathcal{L} over X is a subset of X^* , the free monoid (including the empty word) generated by X and X^{-1} . A language is *regular* if it is accepted by a finite state automaton and is *context-free* if it is accepted by a pushdown machine. We refer [HU79] for the definitions of finite state automata and pushdown machines.

An order is computable if there exists an algorithm deciding if $u \leq v$ in G for any pair of words $u, v \in X^*$. An order is regular (context-free) if the positive cone can be recognized by a regular (context-free) language. Computability of left-orders and bi-orders has gained a lot interest in recent years. Harrison-Trainor [HT18] have shown that there exists a left-orderable group with solvable word problem but no computable left-orders, while Darbinyan [Dar20] has constructed an example for the case of bi-orders. Šunić [Š13a, Š13b] showed that there exists a one-counter left-orders on the free groups and later with Hermiller [Hv17], proved that left-orders on free products are never regular which implies that such left-orders constructed by Šunić are the computationally simplest orders in the sense of Chomsky hierarchy. Antolín, Rivas and Su [ARS21] have studied regular and context-free left-orders on groups and have shown that the metabelian Baumslag-Solitar group $BS(1, q), |q| > 1$ does not admit a regular bi-invariant order.

Recall that a group is *computably bi-orderable* if the group admits a computable bi-order. As a consequent of Theorem B, we have that

Theorem C (Theorem 5.1, Theorem 5.4). *Let M_n be the free metabelian group of rank n . Then every M_n is computably bi-orderable. Moreover, M_n admits a regular bi-order if and only if $n = 1$.*

When $n = 2$, it can be shown that M_2 admits a context-free bi-order. It remains unknown if the same holds true for $n \geq 3$. It also worth to mention that if an order is not regular then it is not an isolated point in the space of left-orders [Lin06].

The paper is organized in the following way: in Section 2 we analyse orders on the ring $\mathbb{Z}(x)$ and $\mathbb{Z}(x, y)$ and in Section 3 we prove the main theorem for the case of the free metabelian

group of rank 2 as in Section 4 we will show the remaining cases. In Section 5 we will show that a free metabelian group of finite rank is computably bi-orderable but the order is never regular unless the group is the infinite cyclic group.

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2. Q -INVARIANT ORDERS ON $\mathbb{Z}(x)$ AND $\mathbb{Z}(x, y)$

We begin with a study on orders on the group ring of free abelian group of finite rank, since the derived subgroup M'_2 is isomorphic to $\mathbb{Z}(x, y)$ (See [Bac65], [GM86]). Note that since the order on M'_2 given by the restriction of an order on M_2 is compatible with not only the group operation but also the action of Q , we consider the following ordering structure on modules over the group ring of free abelian group of finite rank.

Definition 2.1. Let Q be a free abelian group of finite rank and M a finitely generated $\mathbb{Z}Q$ -module. By a Q -invariant order \leq on M we mean an order satisfying:

- (i) if $\mu_1 \leq \mu_2$ then $\mu_1 + \mu_3 \leq \mu_2 + \mu_3$ for any $\mu_3 \in M$;
- (ii) if $\mu_1 \leq \mu_2$ then $q \cdot \mu_1 \leq q \cdot \mu_2$ for any $q \in Q$.

In this section, all orders considered on $\mathbb{Z}Q$ -modules will be Q -invariant.

The following is a classification of all Q -invariant orders on $\mathbb{Z}Q$ when $Q \cong \mathbb{Z}$.

Proposition 2.2. Every $\langle x \rangle$ -invariant order on $\mathbb{Z}(x)$ is uniquely determined by two sequences of real numbers (r_1, r_2, \dots, r_k) and $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k)$, where $r_i \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ and $\varepsilon_i \in \{-1, 1\}$, such that r_1, \dots, r_{k-1} are positive algebraic numbers and r_k is positive transcendental number, 0 or ∞ , or if $k = \infty$ then every r_i is positive and algebraic.

Proof. We embed the abelian subgroup $\langle 1, x \rangle$ into \mathbb{R}^2 such that $1, x$ send to $(1, 0), (0, 1)$ respectively. Then every order on the abelian subgroup $\langle 1, x \rangle$ corresponds to a line passing through the origin in the plane, where the line separates positive lattice points and negative lattice points (See [CR16, Section 10.2] or [Teh61]). Equivalently, every order depends on a normal vector which is orthogonal to the line and points to the positive half plane.

We set $\varepsilon_1 = 1$ if and only if $1 > 0$. For the case when $\varepsilon_1 = 1$, since $1 > 0$, we have that x^n for all $n \in \mathbb{Z}$ is positive. Therefore the normal vector must be in the first quadrant (including the positive half of 1 and x). Say the normal vector is $(1, r)$ (if the normal vector is $(0, 1)$ then we will denote r by ∞), which r is uniquely determined by the order. Note that when $1 < 0$, the normal vector will be of the form $(-1, -r)$. One remark is that if the normal vector is $(1, 0)$ or $(1, \infty)$, the order corresponds to one of the lexicographic order. There are in total 4 different lexicographical orders on $\mathbb{Z}(x)$, where each of them corresponds to the choices of $r = 0, \infty$ and $\varepsilon_1 = \pm 1$.

Say the normal vector is $(1, r)$ where $r > 0$ is transcendental. Then $f \in \langle 1, x \rangle$ is positive if and only if $f \cdot (1, r) > 0$. The order extends naturally to $\langle 1, x, x^2 \rangle$ since it is invariant by the action of x (also every plane is uniquely determined by two crossing lines). It is not hard to compute that the normal vector determining the order on $\langle 1, x, x^2 \rangle$ is $(1, r, r^2)$ if we embeds $\langle 1, x, x^2 \rangle$ to \mathbb{R}^3 . Inductively, we may extend the order to any degree even negative degrees. The normal vector for space expanded by $\langle x^{-t}, \dots, x^{-1}, 1, x \dots x^s \rangle$ is $(r^{-t}, \dots, r^{-1}, 1, r, \dots, r^s)$ as we embeds the group to \mathbb{R}^{s+t+1} . Therefore, for any $f(x) \in$

$\mathbb{Z}(x)$ with $\text{supp } f(x) \subset \{x^{-k}, \dots, x^{-1}, 1, x \dots x^k\}$, $f(x)$ is positive if and only if $f(x) \cdot (r^{-k}, \dots, r^{-1}, 1, r, \dots, r^k) > 0$, which is equivalent to $f(r) > 0$.

Now consider the general case. If r is algebraic, we assume the primitive irreducible polynomial in $\mathbb{Z}[x]$ is $p(x)$. Let $\varphi : \mathbb{Z}(x) \rightarrow \mathbb{R}$ be a homomorphism that sends x to r . We need to give an order on the ideal $\ker \varphi$. Note that we have

$$\ker \varphi = \{p(x)f \mid f \in \mathbb{Z}(x)\}.$$

Thus $\ker \varphi \cong \mathbb{Z}(x)$. Hence we need to order another copy of $\mathbb{Z}(x)$. Repeating the process, the statement is proved. \square

An order is *Archimedean* if for every pair of positive elements f, g there exists a natural number n such that $g < f^n$. Hölder's Theorem states that an order on a group is Archimedean if and only if the group embeds in the additive group \mathbb{R} , where the order on \mathbb{R} is given by the usual ordering and the embedding is order preserving [Höl01]. One immediate observation is that

Corollary 2.3. *The $\langle x \rangle$ -invariant order on $\mathbb{Z}(x)$ is Archimedean if and only if r_1 is transcendental.*

Let x, y be elements in a torsion-free abelian group and P_{\leq} an order on the subgroup generated by x, y . Then if $x, y \in P_{\leq}$ we define

$$\mu(x, y; P_{\leq}) = \sup_n \inf_m \left\{ -\frac{m}{n} \mid mx + ny \in P_{\leq} \cup \{0\}, n > 0 \right\}.$$

If the supreme does not exist, we will denote $\mu(x, y; P_{\leq}) = \infty$. Since $y \in P_{\leq}$, the value of μ is always a non-negative real number or ∞ .

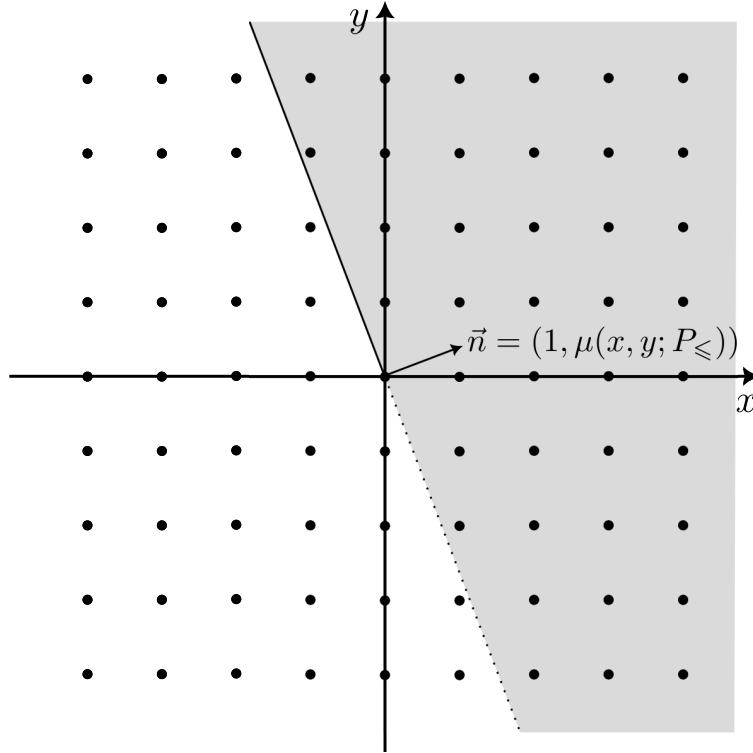


FIGURE 1. A geometric interpretation of $\mu(x, y; P_{\leq})$

We give a geometric interpretation of $\mu(x, y; P_{\leq})$. If x, y generate a free abelian group of rank 2, identify $\mathbb{Z}^2 = \langle x, y \rangle$ with lattice points in the plane \mathbb{R}^2 and x, y with $(1, 0), (0, 1)$ respectively. Every order P_{\leq} corresponds to a line at the origin such that the line separates positive lattice points to negative ones. Then the vector $(1, \mu(x, y; P_{\leq}))$ (or $(0, 1)$ if $\mu(x, y; P_{\leq}) = \infty$) is a normal vector orthogonal to the line such that the vector points to the positive half of the plane. Hence if $(m, n) \cdot (1, \mu(x, y; P_{\leq})) > 0$, then (m, n) is positive in \mathbb{Z}^2 .

It is not hard to check the map μ has following properties:

Proposition 2.4. Let A be a torsion-free abelian group (not necessarily finitely generated) and x, y, z elements in A . Then we have

- (1) If $mx = ny, mn \neq 0$ then $\mu(x, y; P_{\leq}) = \frac{m}{n}$ for any order \leq on $\langle x, y \rangle$ such that $x, y \in P_{\leq}$.
- (2) Let \leq be an order on $\langle x, y \rangle$ such that $x, y \in P_{\leq}$. Then $\mu(x, y; P_{\leq}) = 1/\mu(y, x; P_{\leq})$. When $\mu(x, y; P_{\leq}) = \infty$ then $\mu(y, x; P_{\leq}) = 0$ and vice versa.
- (3) Let \leq be an order on $\langle x, y, z \rangle$ such that $x, y, z \in P_{\leq}$. Then $\mu(x, y; P_{\leq})\mu(y, z; P_{\leq}) = \mu(x, z; P_{\leq})$ where $\mu(x, z; P_{\leq}) = \infty$ if one of $\mu(x, y; P_{\leq}), \mu(y, z; P_{\leq})$ is ∞ .

Proof. (1) Note that $-mx + ny = 0$, then for fixed $k > 0$ we have

$$\inf_j \left\{ -\frac{j}{k} \mid jx + ky \in P_{\leq} \cup \{0\} \right\} = -\frac{1}{k} \left\lceil \frac{-mk}{n} \right\rceil.$$

Hence

$$-\frac{1}{k} \left\lceil \frac{-mk}{n} \right\rceil \leq -\frac{1}{k} \left(\frac{-mk}{n} + 1 \right) = \frac{m}{n} - \frac{1}{k}.$$

Therefore $\mu(x, y; P_{\leq}) = \frac{m}{n}$.

- (2) It is quite obvious from the definition.
- (3) If two or more of x, y, z generate a cyclic group, the statement is easy to check. Now we assume that x, y, z generates \mathbb{Z}^3 . We regard \mathbb{Z}^3 as lattice points in the space \mathbb{R}^3 and map x, y, z to $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ respectively. Then the vectors $\vec{n}_1 = (1, \mu(x, y; P_{\leq}), 0)$ and $\vec{n}_2 = (0, 1, \mu(y, z; P_{\leq}))$ are normal vectors associate with $(\langle x, y \rangle, \leq)$ and $(\langle y, z \rangle, \leq)$. Thus the order \leq on \mathbb{R}^3 is associated with a normal vector

$$\vec{n} = (1, \mu(x, y; P_{\leq}), \mu(x, y; P_{\leq}) \cdot \mu(y, z; P_{\leq})).$$

Restricted to the plane $\langle x, z \rangle$ embeds in, the normal vector is $\vec{n}_3 = (1, 0, \mu(x, y; P_{\leq}) \cdot \mu(y, z; P_{\leq}))$. We are done. □

Definition 2.5. Let x, y be elements in a torsion-free abelian group and \leq an order on the subgroup generated by x, y . If there exists $\varepsilon, \eta \in \{-1, 1\}$ such that $\mu(\varepsilon x, \eta y; P_{\leq})$ is a positive real number, we say that x is *comparable* with y with respect to P_{\leq} , denoted by $x \sim y$; if there exists $\varepsilon, \eta \in \{-1, 1\}$ such that $\mu(\varepsilon x, \eta y; P_{\leq}) = 0$, we say that y is *lexicographically smaller* than x with respect to \leq , denoted by $y \ll x$.

Note that if $\mu(x, y; P_{\leq}) = \infty$ then $x \ll y$.

In the case of Q -invariant orders on the group ring $\mathbb{Z}Q$, every such order induces a relation \sim and preorder \ll on multiplicative group of Q , which is the set of all monomial with coefficient 1 in $\mathbb{Z}Q$. By the property of μ , it is not hard to check that \sim is an equivalent relation and is invariant under multiplication of Q . Thus every order \leq on $\mathbb{Z}Q$ induces a

bi-order \ll on Q/\sim , which is either trivial or torsion-free. If $\mathbb{Z}Q \cong \mathbb{Z}(x, y)$, then Q/\sim is trivial if and only if $1 \sim x \sim y$.

Depending on whether elements are all comparable, there are two types of Q -invariant orders on $\mathbb{Z}Q \cong \mathbb{Z}(x, y)$: Archimedean-type and lexicographical-type.

Definition 2.6. A Q -invariant order on $\mathbb{Z}Q$ is of *Archimedean-type* if $1 \sim q \cdot 1$ for all $q \in Q$. Otherwise the order is of *lexicographical-type*.

Note that under any Q -invariant order on $\mathbb{Z}Q$, 1 and all $q \in Q$ will have the same sign, i.e., they are either all positive or all negative. WLOG in following lemmas we always assume $1 > 0$. And consequently every monomial with coefficient 1 is positive. The following lemma gives a description of Archimedean-type orders on $\mathbb{Z}(x, y)$.

Lemma 2.7. *Let \leq be an Archimedean-type order on $\mathbb{Z}(x, y)$ such that $1 \in P_{\leq}$. Suppose $\mu(1, x; P_{\leq}) = r$, $\mu(1, y; P_{\leq}) = s$. Then r, s are positive real numbers and $\varphi^{-1}((0, \infty)) \subset P_{\leq}$, where φ is the homomorphism to \mathbb{R} that sends $1, x, y$ to $1, r, s$ respectively.*

Proof. The proof goes the same as the proof of Proposition 2.2. For any subspace spanned by $\{1, x^i y^j\}$ the normal vector associated with the order P_{\leq} is $(1, r^i s^j)$. The result can be extended to any space expanded by finite monomials in Q . Then $f(x, y)$ is positive if $\varphi(f(x, y)) = f(r, s) > 0$. Thus $\varphi^{-1}((0, \infty))$ belongs to the positive cone. \square

The lexicographical order in the usual sense on $\mathbb{Z}(x, y)$ with respect to some order on $\mathbb{Z}^2 = \langle x, y \rangle$ is the order under which an element is positive if and only if its leading coefficient is positive, where the leading term is given by the order on \mathbb{Z}^2 . Those orders can be found by the following lemma.

Lemma 2.8. *An order \leq on $\mathbb{Z}(x, y)$ is lexicographical with respect to some order \prec on the free abelian group Q (written in multiplication) if and only if 1 is not comparable with any $x^i y^j$ except $i = j = 0$.*

Proof. WLOG we assume $1 \in P_{\leq}$. If the order is lexicographical with respect to some order \prec on the free abelian group Q then it is not hard to check that 1 is not comparable with any $x^i y^j$ except $i = j = 0$.

Now we prove the other direction. If 1 is comparable with $x^i y^j$ where $(i, j) \neq (0, 0)$, then there exists $N_1, N_2 \in \mathbb{Z}$ such that

$$-N_1 \leq x^i y^j \leq N_1, -N_2 x^i y^j \leq 1 \leq N_2 x^i y^j.$$

Thus \leq is not lexicographical with respect to any order on Q .

If 1 is not comparable with any $x^i y^j$ except $i = j = 0$, then x^i is not comparable with y^{-j} since \leq on $\mathbb{Z}(x, y)$ is invariant under the action of Q . Let $f = \alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_k f_k$ be an arbitrary element in $\mathbb{Z}(x, y)$ such that f_i are monomials with coefficient 1, $\alpha_i \neq 0$, and $f_i \gg f_{i+1}$ for $i = 1, 2, \dots, k-1$. It suffices us to show that f is positive if and only if α_1 is positive. We embeds the abelian subgroup generated by all f_i 's to \mathbb{R}^k where f_i maps to the i -th basis element of \mathbb{R}^k . Then since $f_1 \gg f_i$ for all $i = 2, \dots, k$, we have $\mu(f_1, f_i; P_{\leq}) = 0$. Hence the normal vector associated with \leq in \mathbb{R}^k is $\vec{n} = (1, 0, 0, \dots, 0)$. Thus f is positive if and only if $f \cdot \vec{n} > 0$, equivalently, $\alpha_1 > 0$.

Thus \leq is lexicographical with respect to (Q, \ll) . \square

In general, the order on $\mathbb{Z}(x, y)$ might not be a lexicographical order in the usual sense even when $1 \not\sim x, 1 \not\sim y$, since x^i might be comparable with y^j for some $i, j \in \mathbb{Z}$. But

some of those orders still share a lot of similarities to a usual lexicographical order. Thus we define the concept of a partially lexicographical order, which is a generalization of the usual lexicographical order on $\mathbb{Z}Q$.

Definition 2.9. Let Q be a free abelian group of finite rank. Then a Q -invariant order \leq on the group ring $\mathbb{Z}Q$ is a *partially lexicographical order* if Q/\sim is infinite and \leq is determined by $(\mathbb{Z}Q_0, \leq)$, where Q_0 consist of all elements that are \sim -equivalent to 1 in Q , in the following way

$$f \in P_{\leq} \text{ if and only if } f_l \in P_{\leq}$$

where if we regard elements in $\mathbb{Z}Q$ as finite supported functions from Q to \mathbb{Z} then

$$f_l(q) = \begin{cases} f(q), & \text{when } q \succcurlyeq q' \text{ for all } q' \in \text{supp } f. \\ 0, & \text{otherwise,} \end{cases}$$

and $f_l = q \cdot f_0$ where $q \in Q, f_0 \in \mathbb{Z}Q_0$. The element f_l is called the leading term of f with respect to (Q, \ll) .

Essentially, by a partially lexicographical order on $\mathbb{Z}(x, y)$ with respect to $(Q/\sim, \ll)$ we mean an order under which an element is positive if and only if its leading term is positive.

Note that elements equivalent to 1 form a subgroup in Q . Since Q/\sim is infinite and torsion-free, then the subgroup is generated by a single element $x^i y^j$ where (i, j) are coprime. Thus in the case of $\mathbb{Z}(x, y)$, a partially lexicographical order is determined by an order on $\mathbb{Z}(x^i y^j)$ for i, j coprime, where such an order can be described by Proposition 2.2. Then we have a complete description of all partially lexicographical orders on $\mathbb{Z}(x, y)$.

Proposition 2.10. A partially lexicographical order on $\mathbb{Z}(x, y)$ is determined by an order on the abelian group $Q/\langle x^i y^j \rangle$ and a $\langle x^i y^j \rangle$ -invariant order on $\mathbb{Z}(x^i y^j)$ where i, j are coprime.

Therefore partially lexicographical orders on $\mathbb{Z}(x, y)$ are relatively easy to understand. However there exists another class of Q -invariant lexicographical-type orders. Note that if \leq is partially lexicographical, the abelian subgroup $R_f = \langle g \mid g \ll f \rangle$ is convex for any monomial f . But it is not hard to show that there exists an order such that $y \ll 1 \sim x$ but $y > 1 - x > 0$ and such order cannot be partially lexicographical nor of Archimedean-type. We then define

Definition 2.11. An order is *pseudo-lexicographical* if there exists some f such that R_f is not convex.

The main difference between a partially lexicographical order and a pseudo-lexicographical one is that under a partially lexicographical order the leading term dominates other terms while under a pseudo-lexicographical order it may not be the case although the leading term can still be defined. But pseudo-lexicographical orders still inherit properties of a lexicographical order to some extend.

Lemma 2.12. Let \leq be an order of lexicographical-type on $\mathbb{Z}(x, y)$ and f a monomial with coefficient 1, then $R_f < \sum_{i=1}^k a_i f_i$ where $f_i \sim f$ and $a_i > 0$.

Proof. It is enough to show $f > R_f$. Let $g \in R_f$ and $\text{supp } g = \{g_1, g_2, \dots, g_k\}$. Consider the abelian group A generated by f and $\text{supp } g$. We embed A into Euclidean space \mathbb{R}^{k+1} such that we send f to $(1, 0, \dots, 0)$ and g_i to $(0, \dots, 1, \dots, 0)$ where the $(i+1)$ -th coordinate is 1. The restriction of \leq determines a k -dimensional hyperplane that separates positive and

negative elements. Since $f \gg g_i$ for every i , we have $\mu(f, g_i; P_{\leq}) = 0$. The intersection of the hyperplane and the plane expanded by y and g_i is the line $(0, \dots, t, \dots, 0)$, where the $(i+1)$ -th coordinate is $t \in \mathbb{R}$. Thus the normal vector of this hyperplane is $\vec{n} = (1, 0, \dots, 0)$. It follows that $f > g$ since $f \cdot \vec{n} > g \cdot \vec{n}$. \square

Note that the set of lexicographical-type orders consists of partially lexicographical and pseudo-lexicographical orders.

In summary,

Proposition 2.13. Every order on $\mathbb{Z}(x, y)$ induces an equivalent relation \sim and a preorder \ll on the set of monomials with coefficient 1 of $\mathbb{Z}(x, y)$. An order on $\mathbb{Z}(x, y)$ is

- (1) of Archimedean-type if and only if $1 \sim x \sim y$;
- (2) partially lexicographical if and only if R_f is convex for every monomial f with coefficient 1;
- (3) pseudo-lexicographical otherwise.

Note that the definition of Archimedean-type and lexicographical-type can be extended to group rings of free abelian group of any rank as well as the modules over those group rings. However, pseudo-lexicographical orders dominates other types (Archimedean-type and partially lexicographical) of orderings in those cases. Thus this classification is not as useful as in the case of $\mathbb{Z}(x)$ and $\mathbb{Z}(x, y)$.

3. BI-ORDERS ON THE FREE METABELIAN GROUP OF RANK 2

As we analyse Q -invariant orders on $\mathbb{Z}(x, y)$, which is isomorphic to the derived subgroup M'_2 , we are ready to prove our main theorem for the free metabelian group M_2 of rank 2.

Theorem 3.1. M'_2 is convex with respect to any bi-invariant order on M_2 .

Proof. Let a, b be generators of M_2 , and \bar{a}, \bar{b} be the image of a, b in the quotient $Q \cong \mathbb{Z}^2$.

Let \leq be a bi-order on M_2 such that $[a, b] \in P_{\leq}$. The restriction of \leq on M'_2 gives a Q -invariant order on the free $\mathbb{Z}Q$ -module of rank 1. By replacing a, b by a^{-1} and b^{-1} if necessary, we can always assume that $[a, b] > 1$ and $a > 1$.

To prove the theorem, it suffices us to show that $a^i b^j > M'_2$ whenever $a^i b^j$ is positive for any bi-order \leq . We first claim that it is enough to prove the theorem for the case $a > M'_2$. For $a^i b^j$ where i, j are coprime, there exists an automorphism of M_2 such that it sends $a^i b^j$ to ga where $g \in M'_2$. Note that the image of a positive cone is again a positive cone (possible for a different order). And we observe that $ga > M'_2$ if and only if $a > M'_2$. Moreover if $a^i b^j > M'_2$ for coprime i, j then for every positive integer k we have $a^{ki} b^{kj} > M'_2$. Thus if $ga > M'_2$ under any bi-order, then $a^i b^j$ for every i, j such that $a^i b^j$ is positive. Hence the claim is proved.

We consider two cases: the restriction of \leq on M'_2 is of Archimedean-type and lexicographical-type.

First we suppose the restriction of \leq on M'_2 is of Archimedean-type. Let $\mu([a, b], [a, b]^a; P_{\leq}) = r$, $\mu([a, b], [a, b]^b; P_{\leq}) = s$. Then $\{r, s\} \cap \{0, \infty\} = \emptyset$, and hence, by Lemma 2.7, the homomorphism $\varphi : M'_2 \rightarrow \mathbb{R}$ that sends $[a, b], [a, b]^a, [a, b]^b$ to $1, r, s$ respectively satisfies the condition $\varphi^{-1}((0, \infty)) \subset P_{\leq}$. If there exists $g \in M'_2$ such that $g > a$, then by the bi-invariant of the order we have

$$(g^{-1}a)^{b^n} < 1, n > 0.$$

Simplifying the left hand side we have

$$g^{-b^n}[a, b]^{a^{-1}(1+b)^{n-1}}a < 1.$$

Then

$$\varphi(g^{-b^n}[a, b]^{a^{-1}(1+b)^{n-1}}) = -\varphi(g)s^n + \frac{1}{r}(1+s)^n.$$

Because s is a positive real number, $\varphi(g^{-b^n}[a, b]^{a^{-1}(1+b)^{n-1}})$ will eventually be positive as n goes to ∞ . Thus for n large enough, $(g^{-1}a)^{b^n}$ becomes positive, contradicting the fact that the order is bi-invariant.

Therefore $a > M'_2$ in the case when \leq on M'_2 is of Archimedean-type.

Next we assume that the restriction of \leq on M'_2 is of lexicographical-type. First we consider the case when $[a, b] \not\sim [a, b]^a$.

Again we assume that there exists $g \in M'_2$ such that $1 < a < g$. For any $h \in M'_2$ we have

$$(g^{-1}a)^h = g^{-1}[a^{-1}, h^{-1}]a < 1.$$

Now we regard g^{-1} and $[a^{-1}, h^{-1}]$ as an element in the $\mathbb{Z}(\bar{a}, \bar{b})$ -module M'_2 . Let $g = \alpha_1 g_1 + \alpha_2 g_2 + \dots + \alpha_k g_k + r$ where $g_i \sim g_k$ are the leading monomials and every nomomial in $\text{supp } r$ is strictly lexicographically less than g_i . If $[a, b] \gg [a, b]^a$, we let

$$h = \sum_{i=1}^k (|\alpha_i| + 1) \bar{a} g_i.$$

Then

$$-(1 - \bar{a}^{-1})h - g = \sum_{i=1}^k (|\alpha_i| - a_i + 1)g_i - \sum_{i=1}^k (|\alpha_i| + 1)\bar{a}g_i - r.$$

Note that in M'_2 since $1 \gg \bar{a}$, $g_i \gg \bar{a}g_i$. Then by Lemma 2.12 we have

$$\sum_{i=1}^k (|\alpha_i| - a_i + 1)g_i > \sum_{i=1}^k (|\alpha_i| + 1)\bar{a}g_i + r.$$

Hence $-(1 - \bar{a}^{-1})h - g > 0$.

If $[a, b] \ll [a, b]^a$, we then let

$$h = \sum_{i=1}^k (|\alpha_i| + 1)g_i.$$

It is not hard to check that $-(1 - \bar{a}^{-1})h - g$ is also positive.

Therefore, there exists h such that $g^{-1}[a^{-1}, h^{-1}] > 1$ in M'_2 . On the contrary, we have $g^{-1}[a^{-1}, h^{-1}]a < 1$ in the assumption. Thus $a > M'_2$ when $[a, b] \not\sim [a, b]^a$.

The last case is when $[a, b] \sim [a, b]^a$ and $[a, b] \not\sim [a, b]^b$. If $g < a$ for $g \in M'_2$, then by conjugating ga by b^n and b^{-n} we then have the following (elements in M'_2 are written as elements in $\mathbb{Z}(\bar{a}, \bar{b})$): for any $f(\bar{a}, \bar{b}) < a$ and for $n > 0$

$$(1) \quad \bar{b}^n f(\bar{a}, \bar{b}) - \bar{a}^{-1}(1 + \bar{b})^{n-1} < a,$$

$$(2) \quad \bar{b}^{-n} f(\bar{a}, \bar{b}) + \bar{a}^{-1}\bar{b}^{-1}(1 + \bar{b}^{-1})^{n-1} < a.$$

Since $0 < a$ (where 0 in M'_2 represents trivial element in M_2), then letting $f(\bar{a}, \bar{b}) = 0$ we have that $a > \bar{a}^{-1}\bar{b}^{-1}(1 + \bar{b}^{-1})^{n-1}$ for any n by applying (2). By (1) for $n = 1$ and $f(\bar{a}, \bar{b}) =$

$\bar{a}^{-1}\bar{b}^{-1}(1+\bar{b}^{-1})^{n-1}$ we have $\bar{a}^{-1}((1+\bar{b}^{-1})^{n-1}-1) < a$. Thus let $f(\bar{a}, \bar{b}) = \bar{a}^{-1}((1+\bar{b}^{-1})^{n-1}-1)$ and apply (1) for $k > n$ we have

$$a > \bar{b}^k \bar{a}^{-1}((1+\bar{b}^{-1})^{n-1}-1) - \bar{a}^{-1}(1+\bar{b})^{k-1}.$$

The leading term (with respect to the degree of \bar{b}) of the right hand side is $((\binom{n-1}{2}-1)\bar{a}^{-1}\bar{b}^{k-1}$ for any $k > n$. Since $1 \sim \bar{a} \not\sim \bar{b}$, we have that $a > \bar{b}^n$ and $a > \bar{b}^{-n}$ for all $n > 0$. Thus we conclude $a > M'_2$.

Therefore $a > M'_2$ when \leq on M'_2 is of lexicographical type. The theorem is proved. \square

Note that if M'_2 is convex with respect to an order \leq , then \leq induces an order on the quotient $\mathbb{Z}^2 \cong M_2/M'_2$. In general,

Proposition 3.2. Let G be a finitely generated orderable group that is an extension of A by Q . If A is convex with respect to order \leq , then \leq is a lexicographical order leading by the quotient there the order on Q is induced by \leq .

Proof. Let $\pi : G \rightarrow Q$ be the canonical quotient map. Then we define an order $\tilde{\leq}$ on Q in the following way: $q_1 \tilde{\leq} q_2$ in Q if $\pi^{-1}(q_1) \leq \pi^{-1}(q_2)$ in G . It is well-defined since A is convex. Let P_Q be the positive cone in Q associated with $\tilde{\leq}$ and P_A the positive cone in A associated with the restriction of \leq on A . Then it is not hard to check that $P_A \cup \pi^{-1}(P_Q)$ is the positive cone associated with \leq in G . Hence \leq is a lexicographical order leading by the quotient. \square

Thus we have

Corollary 3.3. Any bi-invariant order \leq on M_2 is a lexicographical order leading by the quotient with respect to the extension of M'_2 by Q .

Let $\mathcal{QO}(M'_2)$ be the space of Q -invariant orders on M'_2 . Note that since $\mathcal{O}(M'_2)$ is a compact Hausdorff space and Q acts on $\mathcal{O}(M'_2)$ by homeomorphism, then $\mathcal{QO}(M'_2)$ is a closed subspace of $\mathcal{O}(M'_2)$. By Theorem 3.1 we have that $\mathcal{O}(M_2)$ is homeomorphic to $\mathcal{QO}(M'_2) \times \mathcal{O}(\mathbb{Z}^2)$. The space of $\mathcal{O}(\mathbb{Z}^2)$ does not have any isolated point, so does $\mathcal{O}(M_2)$. Thus we immediately have

Corollary 3.4. The space $\mathcal{O}(M_2)$ is homeomorphic to the Cantor set.

4. BI-ORDERS ON THE FREE METABELIAN GROUPS OF HIGHER RANK

Let $Q = \mathbb{Z}^n = \langle x_1, x_2, \dots, x_n \rangle$ be the free abelian group of rank n . Now we first consider Q -invariant orders on free $\mathbb{Z}Q$ -modules as a continuation of the discussion in Section 2.

Let F_k be a free $\mathbb{Z}Q$ -module of rank k and $\{e_1, e_2, \dots, e_k\}$ a basis of it.

Lemma 4.1. If under a Q -invariant order \leq we have $e_1 \sim e_2$ and $e_1, e_2 > 0$, then

$$\mu(e_1, qe_1; P_{\leq}) = \mu(e_2, qe_2; P_{\leq})$$

for all $q \in Q$.

Proof. Let $\mu(e_1, e_2; P_{\leq}) = r \neq 0$. Since \leq is Q -invariant, by the definition of μ , it is easy to check that $\mu(qe_1, qe_2; P_{\leq}) = r$ for all q .

By Proposition 2.4, we then have

$$\begin{aligned}
 \mu(e_1, qe_1; P_{\leq}) &= \mu(e_1, e_2; P_{\leq}) \times \mu(e_2, qe_2; P_{\leq}) \times \mu(qe_2, qe_1; P_{\leq}) \\
 &= (\mu(e_1, e_2; P_{\leq}) \times \mu(qe_2, qe_1; P_{\leq})) \times \mu(e_2, qe_2; P_{\leq}) \\
 &= (r \times \frac{1}{r}) \times \mu(e_2, qe_2; P_{\leq}) \\
 &= \mu(e_2, qe_2; P_{\leq}).
 \end{aligned}$$

The lemma is proved. \square

Similar to the case of $\mathbb{Z}(x, y)$, a Q -invariant order on F_k is of *Archimedean-type* if $e_i \sim e_j$ and $e_1 \sim qe_1$ for all $1 \leq i, j \leq k, q \in Q$. Lemma 4.1 allows us to associated an Archimedean-type order to a homomorphism to \mathbb{R} .

Definition 4.2. Let \leq be a Q -invariant Archimedean-type order on the free $\mathbb{Z}Q$ -module F_k . Then its associated homomorphism $\varphi_{\leq} : F_k \rightarrow \mathbb{R}$ is a homomorphism defined by

$$\varphi_{\leq}(e_1) = 1, \varphi_{\leq}(e_i) = \mu(e_1, e_i; \leq), \text{ for } 2 \leq i \leq k, \varphi_{\leq}(x_i) = \mu(e_1, x_i e_1; \leq), \text{ for } 1 \leq i \leq n.$$

Similar to the proof of Proposition 2.2, it is not hard to show that $\varphi_{\leq}^{-1}((0, \infty))$ lies in the positive cone of the order \leq .

In general, when basis elements are not comparable, we have a partition of basis elements as following:

$$\{e_1, e_2, \dots, e_k\} = B_1 \sqcup B_2 \sqcup \dots \sqcup B_s,$$

such that

- (1) $e_i \sim e_j$ for $e_i, e_j \in B_t$ for all i, j and $1 \leq t \leq s$,
- (2) $e_i \gg e_j$ for all $e_i \in B_t, e_j \in B_{t+1}, t = 1, 2, \dots, s-1$.

WLOG we may assume $e_1 \in B_1$. If $e_1 \sim x_i e_1$, then the restriction of \leq on the free submodule $\langle B_1 \rangle$ is of Archimedean-type and hence is associated with a homomorphism $\varphi : \langle B_1 \rangle \rightarrow \mathbb{R}$. We can extend φ to F_k by letting $\varphi(e_i) = 0$ for all $e_i \notin B_1$. We denoted the extended map by φ_{\leq} .

We have

Lemma 4.3. *The extended map $\varphi_{\leq} : F_k \rightarrow \mathbb{R}$ is a homomorphism and $\varphi_{\leq}^{-1}((0, \infty))$ lies in the positive cone of \leq .*

Proof. The map is a homomorphism by definition. Observe that

$$\varphi_{\leq}^{-1}((0, \infty)) = \varphi^{-1}((0, \infty)) + \langle B_2, B_3, \dots, B_s \rangle.$$

Thus to show that $\varphi_{\leq}^{-1}((0, \infty))$ lies in the positive cone of \leq , it is enough to show that for $f \in \langle B_1 \rangle$ if $\varphi(f) > 0$ then $f + g > 0$ for all $g \in \langle B_2, B_3, \dots, B_s \rangle$.

We fix an arbitrary $g \in \langle B_2, B_3, \dots, B_s \rangle$. For an element $h \in F_k$, the support $\text{supp } h$ of h is the set of all terms qe_i with non-trivial coefficients. We enumerate $\text{supp } f$ and $\text{supp } g$ as $\{f_1, f_2, \dots, f_{t_1}\}$ and $\{g_1, g_2, \dots, g_{t_2}\}$ respectively. Then embed the abelian subgroup generated by $\langle \text{supp } f \cup \text{supp } g \rangle$ to \mathbb{R}^d where d is the size of $\text{supp } f \cup \text{supp } g$ such that f_i is sent to the i -th basis element of \mathbb{R}^d and g_i is sent to the $(i + t_1)$ -th basis element of \mathbb{R}^d . Note that $f_i \gg g_j$ and $\mu(f_i, f_j; P_{\leq}) = \varphi(f_j)/\varphi(f_i)$. Following the same procedure as in Lemma 2.12, the normal vector of the hyperplane that divides positive and negative elements is $\vec{n} = (\varphi(f_1), \varphi(f_2), \dots, \varphi(f_{t_1}), 0, \dots, 0)$. Then

$$(f + g) \cdot \vec{n} = f \cdot \vec{n} = \varphi(f) > 0.$$

Therefore $f + g$ is positive. We are done. \square

We then consider the case when the restriction of \leq on the free submodule $\langle B_1 \rangle$ is not of Archimedean-type. Since \leq induces a total order \ll on Q/\sim , it also induces a total order on the set $(Q \cdot B_1)/\sim$. Let f be a term in the submodule $\langle B_1 \rangle$. By R_f we denote the abelian subgroup generated by all terms that are \ll -less than f in submodule $\langle B_1 \rangle$. We have an analog of Lemma 2.12.

Lemma 4.4. *In F_k we have $R_f + \langle B_2, \dots, B_s \rangle < \sum_{i=1}^t a_i f_i$, for all $f_i \sim f, a_i > 0$.*

Let M_n be the free metabelian group of rank $n \geq 3$. Suppose the generating set is $\{a_1, a_2, \dots, a_n\}$. It is known that M'_n is a $\mathbb{Z}Q$ -module (See [Bac65], [GM86]), where $Q \cong \mathbb{Z}^n = \langle \bar{a}_1, \bar{a}_2, \dots, \bar{a}_n \rangle$, with the following module presentation:

$$M'_n = \langle e_{ij}, i < j \mid (1 - \bar{a}_i)e_{jk} - (1 - \bar{a}_j)e_{ik} + (1 - \bar{a}_k)e_{ij} = 0, i < j < k \rangle,$$

where $e_{ij} = [a_i, a_j]$. Equations like $(1 - \bar{a}_i)e_{jk} - (1 - \bar{a}_j)e_{ik} + (1 - \bar{a}_k)e_{ij} = 0$ are called Jacobi identities.

Although M'_n is no longer free when $n \geq 3$, we can still understand orders on M'_n by lift them to a free $\mathbb{Z}Q$ -module.

Lemma 4.5. *Let Q be a free abelian group of finite rank. Then every Q -invariant order on a $\mathbb{Z}Q$ -module M can be lifted to a Q -invariant order on F_k , where k is the size of the generating set of M .*

Proof. Let $S \leq F_n$ be the submodule of F_n such that $M \cong F_n/S$. Denoted by $\pi : F_n \rightarrow S$ the canonical quotient map. Let P_{\leq} be a Q -invariant order on M .

Since S as a set is Q -invariant, then the restriction of any Q -invariant order of F_n gives a Q -invariant order on S . Let \leq' be one of them and $P_{\leq'}$ the positive cone. We then define

$$P = P_{\leq'} \cup \pi^{-1}(P_{\leq}).$$

It is the positive cone of a lexicographical order on F_n . We denoted this order by $\tilde{\leq}$. To show that P_{\leq} is Q -invariant it is enough to show that $\pi^{-1}(P_{\leq})$ is Q -invariant.

Let $f \in \pi^{-1}(P_{\leq})$, then we have

$$\pi(q \cdot f) = q \cdot \pi(f) \in P_{\leq}, \forall q \in Q.$$

The lemma is proved. \square

One remark is that $\tilde{\leq}$ is not unique. Every Q -invariant order on S gives a lift.

Now let us focus on the $\mathbb{Z}Q$ -module M'_n . Let F be the free $\mathbb{Z}Q$ -module with basis $\{e_{ij}, i < j\}$. An order \leq on M'_n can be lifted to an order $\tilde{\leq}$ on F . If the restriction of $\tilde{\leq}$ on $\langle B_1 \rangle$ is of Archimedean-type, then there exists an associated homomorphism $\varphi_{\tilde{\leq}} : F \rightarrow \mathbb{R}$. We then have

Lemma 4.6. *The homomorphism $\varphi_{\tilde{\leq}}$ sends all Jacobi identities to 0. In particular, it induces a homomorphism $\varphi_{\leq} : M'_n \rightarrow \mathbb{R}$ such that $\varphi_{\leq}^{-1}((0, \infty))$ lies in the positive cone of \leq on M'_n .*

Proof. Let S be the submodule generated by all Jacobi identities in F . Suppose there exists $r \in S$ such that $\varphi_{\tilde{\leq}}(r) > 0$. WLOG we assume that $\varphi_{\tilde{\leq}}(\bar{a}_1) \in (0, 1)$. Then there exists $N \in \mathbb{N}$ such that

$$\varphi(\bar{a}_1)^N \varphi(f) < \varphi(r),$$

where f is an element in $F \setminus S$ such that $\varphi_{\leq}(f) > 0$. Hence $0 < \bar{a}_1^N f < r$. It contradicts to the fact that S is convex with respect to \leq .

The case when $\varphi_{\leq}(r) < 0$ is similar. Thus $\varphi_{\leq}(r) = 0$ for all $r \in S$.

Since $M'_n \cong F/S$ and $\varphi_{\leq}(S) = 0$, the induced homomorphism $\varphi_{\leq}(g) := \varphi_{\leq}(\pi^{-1}(g))$ is well-defined. Note that the set $\varphi_{\leq}^{-1}((0, \infty))$ lies in the positive cone of \leq . Thus $\varphi_{\leq}^{-1}((0, \infty))$ lies in the positive cone of \leq on M'_n . \square

Let \overline{M}'_n be the convex hull of M'_n with respect to some order \leq , i.e., \overline{M}'_n is the smallest convex subgroup containing M'_n . Since the order is bi-invariant and M'_n is normal, the subgroup \overline{M}'_n is normal. Then we have

Theorem 4.7. *Let \leq be a bi-invariant order on M_n for $n \geq 3$, then M_n/\overline{M}'_n is not trivial. Equivalently, the quotient is a free abelian group of rank at least 1.*

Proof. Let \leq be an order on M_n . Since \overline{M}'_n is convex, then M_n/\overline{M}'_n is a free abelian group if it is not trivial. To prove the theorem, it is enough to show that some a_i survives under the quotient map.

By Lemma 4.6, the restriction of \leq on M'_n can be lifted an order $\tilde{\leq}$ to F , the free $\mathbb{Z}Q$ -module generated by $\{e_{ij}\}$. We consider two cases depending on the restriction of $\tilde{\leq}$ on $\langle B_1 \rangle$, where B_1 consists of all leading terms in the basis.

The first case is when the restriction of $\tilde{\leq}$ on $\langle B_1 \rangle$ is of Archimedean-type. Then by Lemma 4.6, the order \leq is associated with a homomorphism $\varphi_{\leq} : M'_n \rightarrow \mathbb{R}$. WLOG we may assume $[a_1, a_2] = e_{12} > 1 \in B_1$ and $a_1 > 1$. If there exists $g \in M'_n$ such that $1 < a_1 < g$, then $(g^{-1}a_1)^{a_2^N}$ is negative for any integer N . Note that

$$(g^{-1}a_1)^{a_2^N} = g^{-a_2^N} [a_1^{-1}, a_2^{-N}] a_1 = g^{-a_2^N} [a_1, a_2]^{a_1^{-1}(1+a_2)^{N-1}} a_1.$$

Then we have

$$\varphi_{\leq}(g^{-a_2^N} [a_1, a_2]^{a_1^{-1}(1+a_2)^{N-1}}) = -\varphi_{\leq}(\bar{a}_2)^N \varphi_{\leq}(g) + \varphi_{\leq}(\bar{a}_1^{-1})(1 + \varphi_{\leq}(\bar{a}_2))^{N-1} \varphi_{\leq}([a_1, a_2])$$

Since $\varphi_{\leq}(\bar{a}_i) > 0$ and $\varphi_{\leq}([a_1, a_2]) > 0$ for all i by the definition of φ_{\leq} , then as N goes to ∞ ,

$$\varphi_{\leq}(g^{-a_2^N} [a_1, a_2]^{a_1^{-1}(1+a_2)^{N-1}})$$

will eventually be positive. Thus $g^{-a_2^N} [a_1, a_2]^{a_1^{-1}(1+a_2)^{N-1}} > 1$ and hence $(g^{-1}a_1)^{a_2^N}$ is positive, that leads to a contradiction.

Thus in this case $a_1 > \overline{M}'_n$. Therefore when the restriction of $\tilde{\leq}$ on $\langle B_1 \rangle$ is of Archimedean-type, the quotient M_n/\overline{M}'_n is not trivial.

The second case is when the restriction of $\tilde{\leq}$ on $\langle B_1 \rangle$ is of lexicographical-type. WLOG, we still assume $[a_1, a_2] = e_{12} \in B_1$ and $[a_1, a_2] > 1$. Then there exists a_i such that $e_{12} \not\prec \bar{a}_i e_{12}$ in the free module F with respect to the lifted order $\tilde{\leq}$. In addition, we assume $a_i > 1$. Suppose there exists $g \in M'_n$ such that $1 < a_i < g$, then $(g^{-1}a_i)^h$ is negative for any $h \in M'_n$. Note that $(g^{-1}a_i)^h = g^{-1}[a_i^{-1}, h^{-1}]a_i$. Now we regard g as an element in the free module F , it can be decomposed as

$$g = \alpha_1 g_1 + \alpha_2 g_2 + \cdots + \alpha_k g_k + r$$

where g_i are all the leading terms in $\langle B_1 \rangle$ with respect to \ll and r are the rest of g . Note that it is possible for $k = 0$, i.e., g does not consist any term in the submodule $\langle B_1 \rangle$.

Similar to the proof of Theorem 3.1, if $e_{12} \gg \bar{a}_i e_{12}$, we let

$$h = e_{12} + \sum_{j=1}^k (|\alpha_j| + 1) \bar{a}_j g_j.$$

Then

$$-(1 - \bar{a}^{-1})h - g = (\bar{a}_i^{-1} - 1)e_{12} + \sum_{i=1}^k (|a_i| - a_i + 1)g_i - \sum_{i=1}^k (|\alpha_i| + 1)\bar{a}_i g_i - r.$$

Note that since $e_{12} \gg \bar{a}_i e_{12}$, $g_i \gg \bar{a}_i g_i$ and $(\bar{a}_i^{-1} - 1)e_{12}$ is positive. Then by Lemma 4.4 we have

$$(\bar{a}_i^{-1} - 1)e_{12} + \sum_{i=1}^k (|a_i| - a_i + 1)g_i > \sum_{i=1}^k (|\alpha_i| + 1)\bar{a}_i g_i + r.$$

Hence $-(1 - \bar{a}^{-1})h - g > 0$.

If $e_{12} \ll \bar{a}_i e_{12}$, we then let

$$h = -e_{12} + \sum_{i=1}^k (|\alpha_i| + 1)g_i.$$

It is not hard to check that $-(1 - \bar{a}^{-1})h - g$ is also positive.

The remaining problem is that $-(1 - \bar{a}^{-1})h - g$ may lie in S , the submodule generated by all Jacobi identities, in which case $-(1 - \bar{a}^{-1})h - g$ is the trivial element under the quotient map $\pi : F \rightarrow M'_n$. We can resolve this problem by adding another e_{12} or $-e_{12}$, depending on the case, to h such that the image of $-(1 - \bar{a}^{-1})h - g$ is non-trivial in M'_n .

Thus we show that there exists $h \in M'_n$ such that $g^{-1}[a_i^{-1}, h^{-1}]$ is positive. We have a contradiction. Hence $a_i > \bar{M}'_n$. Therefore when the restriction of \leq on $\langle B_1 \rangle$ is of lexicographical-type, the quotient M_n/\bar{M}'_n is also a free abelian group of rank at least 1.

The theorem is proved. \square

When $n \geq 3$, the derived subgroup may not be convex even under Archimedean-type orders. The next lemma gives a construction of all Archimedean-type orders on the a $\mathbb{Z}Q$ -module M .

Lemma 4.8. *Let F be the free $\mathbb{Z}Q$ -module with basis $\{e_{ij} \mid 1 \leq i < j \leq n\}$ and S a submodule contains all Jacobi identities. Suppose $M \cong F/S$ admits a Q -invariant order. For non-zero real numbers $\lambda_{ij}, 1 \leq i < j \leq n$ and positive real numbers $\gamma_i, 1 \leq i \leq n$, if the homomorphism $\varphi : F \rightarrow \mathbb{R}$ induced by $\varphi(e_{ij}) = \lambda_{ij}, 1 \leq i < j \leq n, \varphi(\bar{a}_i) = \gamma_i, 1 \leq i \leq n$ satisfies $\varphi(s) = 0$ for all $s \in S$, then there exists a Archimedean-type Q -invariant order \leq on M such that its associated homomorphism φ_{\leq} is induced by φ .*

Proof. Since all γ_i are positive real numbers, the map $\varphi : F \rightarrow \mathbb{R}$ is a homomorphism.

Note that S is Q -orderable. Since F/S is Q -orderable then so is $\ker \varphi/S$. Let $P_{\ker \varphi}$ be the lexicographical order leading by the quotient on $\ker \varphi$. S is convex under $P_{\ker \varphi}$. Thus we construct $P := P_{\ker \varphi} \cup \varphi^{-1}((0, \infty))$ defines a Q -invariant order \leq on F and its associated homomorphism is φ . Since S is convex under \leq , the order induces a Q -invariant order \leq on M . Finally since $\varphi(S) = 0$, then φ induces a homomorphism $\varphi_{\leq} : M \rightarrow \mathbb{R}$. \square

With above lemma, we now can construct an order such that the derived subgroup is not convex for $n \geq 3$.

Proposition 4.9. Let $a_1^{t_1} a_2^{t_2} \dots a_n^{t_n}$ be a non-trivial element in M_n for $n \geq 3$. Then there exists a bi-invariant order such that $a_1^{t_1} a_2^{t_2} \dots a_n^{t_n} \in \overline{M}'_n$.

Proof. Fix $a_1^{t_1} a_2^{t_2} \dots a_n^{t_n}$ such that the greatest common divisor of t_i 's is 1. Then we claim that there exists an Archimedean-type order \leq on M'_n such that $\varphi([a_1^{t_1} a_2^{t_2} \dots a_n^{t_n}, a_i]) = 0$ for all i where φ is the associated homomorphism with respect to \leq . Pick $\gamma_i = 1, 1 \leq i \leq n$. Then consider the system of equations for $j = 1, 2, \dots, n$

$$\begin{aligned} \varphi([a_1^{t_1} a_2^{t_2} \dots a_n^{t_n}, a_j]) &= \sum_{i=1, t_i \neq 0}^n \varphi(a_{i+1}^{t_{i+1}} \dots a_n^{t_n} (1 + a_i^{\text{sign}(t_i)})^{|t_i|-1}) \cdot \varphi([a_i^{\text{sign}(t_i)}, a_j]) \\ &= \sum_{i=1, t_i \neq 0}^n \gamma_{i+1}^{t_{i+1}} \dots \gamma_n^{t_n} (1 + \gamma_i^{\text{sign}(t_i)})^{|t_i|-1} \varphi([a_i^{\text{sign}(t_i)}, a_j]) \\ &= \sum_{i=1, t_i \neq 0}^n 2^{|t_i|-1} \varphi([a_i^{\text{sign}(t_i)}, a_j]) \\ &= 0. \end{aligned}$$

With some linear algebra, it is not hard to show that above system of equations has a non-zero solution $\{\varphi([a_i, a_j]) = \lambda_{ij} \neq 0 \mid 1 \leq i < j \leq n\}$. Then by Lemma 4.8 there exists an order \leq on M'_n such that its associated homomorphism is φ .

We denote $a_1^{t_1} a_2^{t_2} \dots a_n^{t_n}$ by q . Now we have $\varphi([q^i, g]) = 0$ for all $i \in \mathbb{Z}, g \in M_n$. Moreover $Q/\langle \bar{q} \rangle$ is torsion-free finitely generated abelian group and hence bi-orderable. We denote $\pi : M_n \rightarrow Q/\langle \bar{q} \rangle$ be the canonical quotient map, the composition of the quotient maps from $M_n \rightarrow Q$ and $Q \rightarrow \langle \bar{q} \rangle$.

Let P_d be the positive cone of \leq and $P_{Q/\langle \bar{q} \rangle}$ a positive cone on $Q/\langle \bar{q} \rangle$. We then let

$$P' := P_d \sqcup \left(\bigsqcup_{i \neq 0} \varphi^{-1}((0, \infty)) q^i \right) \sqcup \left(\bigsqcup_{i > 0} \ker \varphi q^i \right) \sqcup M'_n \pi^{-1}(P_{Q/\langle \bar{q} \rangle}).$$

It is not hard to check that P' is a semigroup, $M_n = P' \sqcup P'^{-1} \sqcup \{1\}$, and is invariant under conjugation with elements in M_n . Therefore P' defines a bi-order $<'$ on the free metabelian group. And by its definition, we immediately have

$$\ker \varphi <' a_1^{t_1} a_2^{t_2} \dots a_n^{t_n} <' \varphi^{-1}((0, \infty)).$$

Therefore M'_n is not convex under $<'$ and any power of $a_1^{t_1} a_2^{t_2} \dots a_n^{t_n} \in \overline{M}'_n$. \square

Remark. When $n = 2$, there is no non-zero solution to the system of equations in the proof.

5. ORDERS ON FREE METABELIAN GROUPS ARE NEVER REGULAR

Let G be a finitely generated group and X a finite generating set of G . An order \leq on G is said to be regular (context-free) if there exists a regular (context-free) language $\mathcal{L} \subset X^*$ such that $\pi(\mathcal{L}) = P_{\leq}$. An order \leq is *computable* if there exists an algorithm to decide if $g \leq h$ for any pair of $g, h \in G$. All those properties are independent of the choice of the finite generating set [ARS21, Lemma 2.11].

By Theorem 3.1 and Lemma 4.8, there are uncountably many bi-orders on M_n for $n \geq 2$. Hence there always exist uncountably many orders that are not computable on M_n .

In this section, we will show that there exist computable orders on M_n but none of them is regular when $n \geq 2$.

Recall that by Magnus embedding (See [Mag39], [Bau73]), a free metabelian group of rank n embeds into the wreath product of two free abelian groups of rank n . It naturally inherits a computable left-order from the wreath product [ARS21]. However, the regular lexicographical left-order on the wreath product where the base group leads is not bi-invariant. One workaround is to replace the lexicographical order by one which leads by the quotient. The order will become computably bi-invariant but no longer regular.

Let M_n be the free metabelian group of rank n and A_n, T_n free abelian groups of rank n . The generating sets of M_n, A_n, T_n are respectively $X = \{x_1, x_2, \dots, x_n\}$, $A = \{a_1, a_2, \dots, a_n\}$ and $T = \{t_1, t_2, \dots, t_n\}$. The Magnus embedding is given by the homomorphism $\varphi(x_i) = a_i t_i$.

Let P_A and P_T be regular positive cones of A_n and T_n respectively. We define a bi-order on the base group $B = \oplus_{t \in T_n} T_n$ as follows: note that an element f in B can be uniquely written as a product of conjugates of elements in A_n in the following fashion:

$$f = g_1^{h_1} g_2^{h_2} \dots g_s^{h_s}, g_i \in A_n, h_i \in T_n,$$

such that $h_1 > h_2 > \dots > h_s$ with respect to the order on T_n , then $f > 1$ if and only if the leading term $g_1 > 1$. It is not hard to check that this order on B is invariant under the action of T_n . The lexicographical order on $A_n \wr T_n$ is given by the positive cone $P = P_B \cup \pi^{-1}(P_T)$ where $\pi : A_n \wr T_n \rightarrow T_n$ is the canonical quotient map.

One remark is that while the lexicographical order where the base group leads is regular, the new order we define is not regular (it can be shown using the same idea as [ARS21, Lemma 3.11] and Theorem 5.4). Next we will show that P can be recognised by a context-free language (r -counter to be precise).

Let $\mathcal{L}_A, \mathcal{L}_T$ be the regular languages evaluate onto P_A and P_T respectively. We then define

$$\begin{aligned} \mathcal{L} = & \{vu_1w_1u_2w_2 \dots u_nw_nz \mid u_1 \in \mathcal{L}_A, w_i \in \mathcal{L}_T, u_2, \dots, u_n \in (A \cup A^{-1})^*, \\ & v, z \in (T \cup T^{-1})^*, z =_{F(T)} (vw_1w_2 \dots w_n)^{-1}, n \neq 0\} \\ & \sqcup \{vu_1w_1u_2w_2 \dots u_nw_nzz' \mid w_i, z' \in \mathcal{L}_T, u_i \in (A \cup A^{-1})^*, z =_{F(T)} (vw_1w_2 \dots w_n)^{-1}\}. \end{aligned}$$

The first part of \mathcal{L} covers the positive elements in the base group and the second part covers the rest. The actual pushdown machine is not hard to construct as it needs a stack to store the information of $vw_1w_2 \dots w_n$. Thus \mathcal{L} is context-free.

The membership problem of M_r in $A_r \wr T_r$ is not hard to solve. Therefore $P \cap M_r$ is recursive at least. Whether the set $P \cap M_r$ can be recognised as a context-free language remains unknown.

But for M_2 , we can indeed construct a context-free bi-order on it. Let $X = \{a, b, c\}$, where a, b generate M_2 and $c = [a, b]$. The quotient is generated by the set $T := \{\bar{a}, \bar{b}\}$, where \bar{a}, \bar{b} are the images of a, b respectively. Let P_Q be a regular positive cone on Q and \mathcal{L}_Q be the corresponding regular language. We then define

$$\begin{aligned} \mathcal{L} = & \{vc^{t_1}w_1c^{t_2}w_2 \dots c^{t_n}w_nz \mid t_1 \in \mathbb{N}, w_i \in \mathcal{L}_Q, t_2, \dots, t_n \in \mathbb{Z} \setminus \{0\}, \\ & v, z \in (T \cup T^{-1})^*, z =_{F(T)} (vw_1w_2 \dots w_n)^{-1}, n \neq 0\} \\ & \sqcup \{vc^{t_1}w_1c^{t_2}w_2 \dots c^{t_n}w_nzz' \mid w_i, z' \in \mathcal{L}_Q, t_i \in \mathbb{Z} \setminus \{0\}, z =_{F(T)} (vw_1w_2 \dots w_n)^{-1}\}. \end{aligned}$$

It is not hard to check \mathcal{L} is context-free and recognises a bi-invariant positive cone of M_2 .

In summary,

Theorem 5.1. *Free metabelian groups of finite rank are computably bi-orderable. Moreover, $M_1 \cong \mathbb{Z}$ admits a regular bi-order and M_2 admits a context-free bi-order.*

Recall that a subset S of a metric space (X, d) is coarsely connected if there is $R > 0$ such that the R -neighbourhood of S is connected. The following lemma gives a description of a regular positive cone from a geometric point of view.

Lemma 5.2 ([AABR22, Proposition 7.2]). *Let G be a finitely generated group. If \leq is a regular order on G , then P_{\leq} and P_{\leq}^{-1} are coarsely connected subsets of the Cayley graph of G .*

Next recall that for a finitely generated group G , a non-trivial homomorphism $\varphi : G \rightarrow \mathbb{R}$ belongs to $\Sigma^1(G)$, the Bieri-Neumann-Strebel invariant (BNS invariant for short), if and only if $\varphi^{-1}((0, \infty))$ is coarsely connected. And $\ker \varphi$ is finitely generated if both φ and $-\varphi$ belong to $\Sigma^1(G)$. For details of BNS invariant, we refer [BS80].

As a direct consequence of [Bau67, Theorem 1], we have the following:

Lemma 5.3. *Let M_n be the free metabelian group of rank n for $n \geq 2$ and $\pi : M_n \rightarrow Q \cong \mathbb{Z}^n$ be the quotient map such that $\ker \pi = M'_n$. Suppose Q_0 is a subgroup of Q such that Q/Q_0 is infinite. Then $\pi^{-1}(Q_0)$ is not finitely generated.*

Following the idea of [ARS21, Lemma 3.11], we have

Theorem 5.4. *Let M_n be the free metabelian group of rank n for $n \geq 2$. Then no bi-invariant order on M_n is regular.*

Proof. Let $\pi : M_n \rightarrow M_n/\overline{M}'_n$ be the canonical quotient map and P_{\leq} be a bi-invariant order on M_n . Since \overline{M}'_n is convex, P_{\leq} induces an order $\tilde{\leq}$ on M_n/\overline{M}'_n . By Theorem 3.1 and Theorem 4.7, M_n/\overline{M}'_n is a free abelian group of rank at least 1.

We claim that there exists a homomorphism $\varphi : M_n/\overline{M}'_n \rightarrow \mathbb{R}$ such that $\varphi^{-1}((0, \infty))$ consists of positive elements with respect to $\tilde{\leq}$. There are two cases. First if all of basis elements of M_n/\overline{M}'_n are comparable, then there exists a homomorphism sends every basis element to a non-zero real number such that any element in $\varphi^{-1}((0, \infty))$ is positive. Otherwise, there exists a basis element b such that it is amongst the maximal equivalent class under the induced \ll . Hence there exists a homomorphism, that kills all elements which are strictly lexicographically less than b under \ll , onto $\langle [b] \rangle$ where $[b]$ consists of all elements \sim -equivalent to b . The abelian rank of $\langle [b] \rangle$ is strictly less than n and all basis element in $\langle [b] \rangle$ of it is comparable. By composing the map from the first case we are done.

Now let $f := \varphi \circ \pi : M_n \rightarrow \mathbb{R}$. Let P_1 be the positive cone of M'_n and g_1, g_2, \dots, g_{t-1} be the generating set of the subgroup $\ker \varphi$. We identify g_i and one of their preimage in M_n . Then we have

$$P_{\leq} = f^{-1}((0, \infty)) \cup P_1 \cup \left(\bigcup_{g_1^{s_1} g_2^{s_2} \dots g_{t-1}^{s_{t-1}} \lesssim 1} g_1^{s_1} g_2^{s_2} \dots g_{t-1}^{s_{t-1}} \overline{M}'_n \right).$$

Since there exists a generator that is sent to a positive number, thus in the Cayley graph of M_n the distance between $f^{-1}((0, \infty))$ and P_1 or any coset $g_1^{s_1} g_2^{s_2} \dots g_{t-1}^{s_{t-1}} \overline{M}'_n$ is 1. Thus $f^{-1}((0, \infty))$ is coarsely connected if and only if P_{\leq} is coarsely connected. By BNS theory,

$f \in \Sigma^1(M_n)$ if and only P_{\leq} is coarsely connected. Since \overline{M}'_n contains the derived subgroup, we have

$$(g_1^{s_1} g_2^{s_2} \cdots g_{n-1}^{s_{n-1}} \overline{M}'_n)^{-1} = g_1^{-s_1} g_2^{-s_2} \cdots g_{n-1}^{-s_{n-1}} \overline{M}'_n.$$

Thus $-f \in \Sigma^1(M_n)$ if and only P_{\leq}^{-1} is coarsely connected.

If P_{\leq} is a regular positive cone, both P_{\leq} and P_{\leq}^{-1} are coarsely connected by Lemma 5.2. From BNS theory, $\ker f$ must be finitely generated, which contradicts Lemma 5.3. \square

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