THE HOLOMORPHIC BOSONIC STRING

Contents

1. From classical to quantum: anomalies in the BV formalism

BW: A rapid overview of classical BV and effective quantizations. Stress how obstructions appear, where they live, and how to compute them.

OG: I think we should articulate here the structural features of our BV package that make the arguments below more conceptual. For instance:

- Linear BV quantization is determinantal, which explains why we'll produce determinant line bundles when we do free $\beta\gamma$ system.
- "Gauging" a theory corresponds to a stacky quotient of the original fields. Hence, obstruction to quantizing a gauged theory corresponds to descending the quantization to the quotient. BW: Possibly also add the mechanism in which antifields and antighosts are introduced.
- If a classical theory makes sense on a class (=site) of manifolds, then to quantize the whole class, it suffices to check on a generating cover (typically given by disks with geometric structure) but compatibly with all automorphisms. This often explains the appearance of characteristic classes as anomalies.
- Every BV theory produces a factorization algebra. The local structure encodes the OPE algebra (and hence recovers a vertex algebra in chiral CFT situation). On compact manifolds, solutions to EoM typically form finite-dimensional space, and the global observables encode a volume form on this space. (An example is conformal blocks for the free $bc\beta\gamma$ system.)

Please add others as you think of them!

OG: We might also add that we view the BV formalism as the analogue in field theory of derived geometry in geometry. That is, in ordinary algebraic geometry, one first builds geometry and then adds (sheaf) cohomology on top: in ordinary physics, one first builds field theories and then adds (BRST) cohomology on top. But derived geometry (respectively, BV formalism) builds the cohomological aspect into the foundations.

1.1. **OG: description of algorithm.** For us, quantization will mean that we use perturbative constructions in the setting of the BV formalism. Concretely, this means that we enforces the gauge symmetries using the homological algebra of the BV formalism and that we use Feynman diagrams and renormalization to obtain an expression for the desired, putative path integral. OG: Be more careful about saying path integral. It's an approximation. There are toy models for this approach where one can see very clearly how it gives asymptotic expansions for finite-dimensional integrals OG: add references. In particular, these toy models show that this approach need not recover the true integral but does know important information about it; a similar relationship should hold between this quantization method and the putative path integral, but in this case there is no *a priori* definition of the true integral in most cases.

This notion of quantization applies to any field theory arising from an action functional, and the algorithm one applies to obtain a quantization is the following:

(1) Write down the integrals labeled by Feynman diagrams arising from action functional.
(2) Identify the divergences that appear in these integrals and add "counterterms" to the original action that are designed to cancel divergences.
(3) Repeat these steps until no more divergences appear in Feynman diagrams. We call this the "renormalized action."
(4) Check if the renormalized action satisfies the quantum master equation. If it does, you have a well-posed BV quantum theory, and we call the result a *quantized action*. If not, guess a way to adjust the renormalized action and begin the whole process again.

It should be clear that along the way, one makes many choices; hence if a quantization exists, it may not be unique. It is also possible that a BV quantization may not exist.

2. The classical holomorphic bosonic string

BW: First define the holomorphic theory we will work with. Then show how it's related to more familiar models for the string, eg the Polyakov action. Level of detail depending on the space we have.

There is a basic format for a string theory, at least in the perturbative approach. One starts with a nonlinear σ-model, whose fields are smooth maps from a Riemann surface to a target manifold X; in this setting we want the theory to make sense for an arbitrary Riemann surface as the source manifold. In the usual bosonic string theory, this nonlinear σ-model picks out the harmonic maps from a Riemannian 2-manifold to a Riemannian manifold. In our holomorphic setting, the nonlinear σ-model picks out holomorphic maps from a Riemann surface to a complex manifold. One then quotients the space of fields (and solutions to the equations of motion) with respect to reparametrization. OG: This description is a bit opaque. We should find a better one.

In the usual bosonic string, one quotients by diffeomorphisms and Weyl scalings, which can thus change the metric on the source. In our setting, we quotient by biholomorphisms, which act on the complex structure on the source.

In this section we begin by describing our theory in the BV formalism. We do not expect the reader to find the action functional immediately clear, so we devote some time to analyzing what it means and how it arises from concrete questions. We then turn to interpreting this classical BV theory using dg Lie algebras and derived geometry (i.e., we identify the moduli space it encodes). Finally, we conclude by sketching how our theory appears as the chiral sector of a degeneration of the usual bosonic string when the target is a complex manifold with a Hermitian metric. Our theory thus does provide insights into the usual bosonic string; moreover, it clarifies why so many aspects of the bosonic string, like the anomalies or B-fields, have holomorphic analogues.

2.1. **The theory we study.** Let V denote a complex vector space (the target), and let $\langle -,-\rangle_V$ denote the evaluation pairing between V and its linear dual $V^\vee$. Let Σ denote a Riemann surface (the source). Let $T^{1,0}_\Sigma$ denote the holomorphic tangent bundle on Σ, let $\langle -,-\rangle_T$ denote the evaluation pairing between $T^{1,0}_\Sigma$ and its vector bundle dual $T^{1,0*}_\Sigma$. These are the key geometric inputs.

In a BV theory, the fields are $\mathbb{Z}$-graded; we call this the *cohomological grading*. We have four kinds of fields:

field	-1	0	1	2
γ		$\Omega^{0,0}(\Sigma)\otimes V$	$\Omega^{0,1}(\Sigma)\otimes V$	
β		$\Omega^{1,0}(\Sigma)\otimes V^\vee$	$\Omega^{1,1}(\Sigma)\otimes V^\vee$	
c	$\Omega^{0,0}(\Sigma, T^{1,0}_\Sigma)$	$\Omega^{0,1}(\Sigma, T^{1,0}_\Sigma)$		
b			$\Omega^{1,0}(\Sigma, T^{1,0*}_\Sigma)$	$\Omega^{1,1}(\Sigma, T^{1,0*}_\Sigma)$

More accurately, we have eight different kinds of fields, but we view each row as constituting a single type since each given row consists of the Dolbeault forms of a holomorphic vector bundle. For instance, the field γ is a $(0,*)$-form with values in the trivial bundle with fiber V, and the field b is a $(0,*)$-form with values in the bundle $T^{1,0*}\otimes\ T^{1,0*}$.

To orient oneself it is helpful to start by examining the fields of cohomological degree zero, since these typically have a manifest physical meaning. For instance, the degree zero γ field is a smooth V-valued function and hence the natural field for the nonlinear σ-model into V. The degree zero c field is a smooth $(0,1)$-form with values in vector field "in the holomorphic direction," and hence encodes an infinitesimal change of complex structure of Σ. The degree -1 part of c contains the gauge fields of the theories, vector fields. The equations of motion dictate that these vector fields be holomorphic, so we are seeing the infinitesimal version of the symmetry by biholomorphisms we mentioned above. These constitute the obvious fields to introduce for a holomorphic version of the bosonic string. The fields β and b are less obvious but appear as "partners" (or antifields) whose role is clearest once we have the action functional and hence equations of motion.

The action functional is

$$S(\gamma,\beta,c,b) = \int_\Sigma \langle \beta, \bar\partial\gamma\rangle_V + \int_\Sigma \langle b, \bar\partial c\rangle_T + \int_\Sigma \langle \beta, [c,\gamma]\rangle_V + \int_\Sigma \langle b, [c,c]\rangle_T. \tag{1}$$

(We discuss below how to think about fields with nonzero cohomological degrees as inputs.) The equations of motion are thus

$$\begin{aligned} 0 &= \overline{\partial}\gamma + [c,\gamma] \qquad & 0 &= \overline{\partial}\beta + [c,\beta] \\ 0 &= \overline{\partial}c + \tfrac{1}{2}[c,c] \qquad & 0 &= \overline{\partial}b + [c,b] \end{aligned}$$

Note that these equations are familiar in complex geometry. For instance, the equation purely for c encodes a deformation of complex structure on Σ; concretely, it modifies the $\overline{\partial}$ operator to $\overline{\partial} + c$. The other equations then amount to solving for holomorphic sections (of the relevant bundle) withe respect to this deformed complex structure. For instance, the equation in γ picks out holomorphic maps from Σ, with the c-deformed complex structure, to V.

Note that b essentially appears as a Lagrange multiplier, so it doesn't have any intrinsic meaning physical meaning by itself. The field b can be understood as an "antifield" to the ghost field c, in other words an *antighost*.

Remark 2.1. Just looking at this action functional, one might notice that if one drops the last two terms, which are cubic in the fields, then one obtains a free theory

$$S_{free}(\gamma,\beta,c,b) = \int_\Sigma \langle \beta, \overline{\partial}\gamma\rangle_V + \int_\Sigma \langle b, \overline{\partial}c\rangle_T, \tag{2}$$

which is known as the *free $bc\beta\gamma$ system*. Thus, one may view the holomorphic bosonic string as a deformation of this free theory by "turning on" those interaction terms. We will repeatedly try a construction first with this free theory before tackling the string itself, as it often captures important information with minimal work. For instance, we will examine the vertex algebra for the free theory before seeing how the interaction affects the operator products. Similarly, one can identify the anomaly already at the level of the free theory.

Remark 2.2. It is easy to modify this action functional to allow a curved target, i.e., one can replace the complex vector space V with an arbitrary complex manifold X. The fields b, c remain the same. The degree 0 field γ still encodes smooth maps into X, but now the degree 1 field is a section of $\Omega^{0,1}(\Sigma, \gamma^* T_X^{1,0})$. Similarly, β is now a section of $\Omega^{1,*}(\Sigma, \gamma^* T_X^{1,0*})$. The action is then

$$S(\gamma,\beta,c,b) = \int_\Sigma \langle \beta, \overline{\partial}\gamma\rangle_{T_X} + \int_\Sigma \langle b, \overline{\partial}c\rangle_{T_\Sigma} + \int_\Sigma \langle \beta, [c,\gamma]\rangle_{T_X} + \int_\Sigma \langle b, [c,c]\rangle_{T_\Sigma}. \tag{3}$$

In Section 8 we will indicate how the results with linear target generalize to this situation.

2.2. **From the perspective of derived geometry.** We would like to explain what this theory is about in more conceptual terms, rather than simply by formulas and equations. Thankfully this theory is amenable to such a description. We will be informal in this section and not specify a particular geometric context (e.g., derived analytic stacks), except when we specialize to the deformation-theoretic situation (i.e., perturbative setting) that is our main arena.

Let $\mathcal{M}$ denote the moduli space of Riemann surfaces, so that a surface Σ determines a point in $\mathcal{M}$. Let $\mathrm{Maps}_{\overline{\partial}}(\Sigma, V)$ denote the space of holomorphic maps from Σ to V, and hence a bundle $\mathrm{Maps}_{\overline{\partial}}(-, V)$ over $\mathcal{M}$ by varying Σ. For our equations of motion, the γ and c fields of a solution determine a point in this bundle $\mathrm{Maps}_{\overline{\partial}}(-, V)$. The commutative algebra $\mathcal{O}(\mathrm{Maps}_{\overline{\partial}}(\Sigma, V))$ of functions on the space encodes the observables of the classical theory.

This construction makes sense on noncompact Riemann surfaces as well. Let $\mathcal{RS}$ denote the category whose objects are Riemann surfaces and whose morphisms are holomorphic embeddings. There is a natural site structure: a cover is a collection of maps $\{S_i \to \Sigma\}_i$ such that the union of the images is all of Σ. Then $\mathrm{Maps}_{\overline{\partial}}(-, V)$ defines a sheaf of spaces over $\mathcal{RS}$. The observables for the classical theory is, in essence, the *co*sheaf of commutative algebras $\mathcal{O}(\mathrm{Maps}_{\overline{\partial}}(-, V))$, and hence provides a factorization algebra.

In fact, it is better to work with the derived version of these spaces. One important feature of derived geometry is that the appropriate version of a tangent space at a point is, in fact, a cochain complex. In our setting, a point (c, γ) in $\mathrm{Maps}_{\overline{\partial}}(-, V)$ determines a a complex structure $\overline{\partial} + c$ on Σ—we denote this Riemann surface by Σ_c—and γ a V-valued holomorphic function on Σ_c. The tangent complex of $\mathrm{Maps}_{\overline{\partial}}(-, V)$ at (c, γ) is precisely

$$\Omega^{0,*}(\Sigma_c, T^{1,0})[1] \oplus \Omega^{0,*}(\Sigma_c, V).$$

The first summand is the usual answer from the theory of the moduli of surfaces (recall, for example, that the ordinary tangent space is the sheaf cohomology $H^1(\Sigma, T_\Sigma)$ of the holomorphic tangent sheaf), and the second is usual elliptic complex encoding holomorphic maps.

Remark 2.3. It is useful to bear in mind that the degree zero cohomology of the tangent complex will recover the "naive" tangent space. In our case, we have

$$H^1(\Sigma_c, T_{\Sigma_c}) \oplus H^0(\Sigma_c, V),$$

which encodes deformations of complex structure and holomorphic maps. Negative degree cohomology of the tangent complex detects infinitesimal automorphisms (and automorphisms of automorphisms, etc) of the space. For instance, here we see $H^0(\Sigma_c, T_{\Sigma_c})$ appear in degree -1, since a holomorphic vector field is an infinitesimal automorphism of a complex curve. These negative directions are called "ghosts" (or ghosts for ghosts, and so on) in physics. The positive degree cohomology detects infinitesimal relations (and relations of relations, and so on). For instance, here we see $H^1(\Sigma_c, V)$, the cokernel of $\overline{\partial} + c$. OG: ??? We should explain how we see relations of relations in this example

Note that the underlying graded spaces of this tangent complex are the c and γ fields from the BV theory described above. We emphasize that the tangent complex is only specified up to quasi-isomorphism, but it is compelling that a natural representative is the BV theory produced by the usual physical arguments. This behavior, however, is typical of the relationship between derived geometry and BV theories: when physicists write down a classical BV theory, the underlying free theory is essentially always the tangent complex of a nice derived stack.

The reader has probably noticed that, yet again, we have postponed discussing the β and b fields. From a derived perspective, the full BV theory describes the shifted cotangent bundle $\mathbb{T}^*[-1]\,\mathrm{Maps}_{\overline{\partial}}(-, V)$. At the level of a tangent complex, the shifted cotangent direction contributes

$$\Omega^{1,*}(\Sigma_c, T^{1,0*})[-1] \oplus \Omega^{1,*}(\Sigma_c, V^\vee),$$

whose underlying graded spaces are the β and b fields. These "antifields" are added so that the overall space of fields has a 1-shifted symplectic structure when Σ is closed, and a shifted Poisson structure when Σ is open.

2.3. **Relationship to the Polyakov action functional.** This holomorphic bosonic string has a natural relationship with the usual bosonic string. We sketch it briefly, only considering a linear target.

We begin with a bosonic string theory where the source is a 2-dimensional smooth oriented Riemannian manifold Σ and the target is a Hermitian vector space (V, h). The "naive" action functional is

$$S^{\text{naive}}_{Poly}(\varphi, g) = \int_\Sigma h(\varphi, \Delta_g \varphi)\, \mathrm{dvol}_g$$

where the field g is a Riemannian metric on Σ and the field φ is a smooth map from Σ to V. The notation Δ_g denotes the Laplace-Beltrami operator on Σ.

Note that S^{naive}_{Poly} is invariant under the diffeomorphism group $\mathrm{Diff}(\Sigma)$ and under rescalings of the metric (i.e., the theory is classically conformal). Typically we express rescaling as $g \mapsto e^f g$ with $f \in C^\infty(\Sigma)$. As we are interested in a string theory, we want to gauge these symmetries. In geometric language, we want to think about the quotient stack obtained by taking solutions to the equations of motion and quotienting by these symmetry groups.

Our focus is perturbative, so that we want to study the behavior of this action near a fixed solution to the equations of motion. In other words, we want to work with the Taylor expansion of the true action near some solution. Hence, we work around a fixed metric g_0 on Σ, and we substitute for the field g, the term $g_0 + \alpha$ where $\alpha \in \Gamma(\Sigma, \mathrm{Sym}^2(T_\Sigma))$. That is, we will consider deformations of g_0. As φ is linear, we just consider expanding around the zero map. Thus our new fields are $\varphi \in C^\infty(\Sigma, V)$ and $\alpha \in \Gamma(\Sigma, \mathrm{Sym}^2(T_\Sigma))$.

There are also ghost fields associated to the symmetries we gauge. First, there are infinitesimal diffeomorphisms, which are described by vector fields on Σ. We denote this ghost field by $X \in \Gamma(\Sigma, T_\Sigma)$. It acts on the fields by the transformation

$$(\varphi, \alpha) \mapsto (\varphi + X \cdot \varphi, \alpha + L_X \alpha),$$

where L_X denotes the Lie derivative on tensors. Second, there are infinitesimal rescalings of the metric, such as $\alpha \mapsto \alpha + f\alpha$, with ghost field $f \in C^\infty(\Sigma)$. The rescaling does not affect φ. The two symmetries are compatible: given f and X, then $L_X(f\alpha) = X(f)\alpha + f L_X \alpha$ for any $\alpha \in \mathrm{Sym}^2(T_\Sigma)$.

To summarize, we have the following graded vector space of fields:

field/antifield	-1	0	1	2
$\varphi, \varphi^\vee$		$\Omega^0(\Sigma) \otimes V$	$\Omega^2(\Sigma) \otimes V$	
$\alpha, \alpha^\vee$		$\Omega^0(\Sigma, \mathrm{Sym}^2(T_\Sigma))$	$\Omega^2(\Sigma; \mathrm{Sym}^2(T^*_\Sigma))$	
$X, X^\vee$	$\mathrm{Vect}(\Sigma)$			$\Omega^2(\Sigma; T^*_\Sigma)$
$f, f^\vee$	$C^\infty(\Sigma)$			$\Omega^2(\Sigma)$.

The BV action functional is of the form:

$$S^{BV}_{Poly}(\varphi, \alpha, X, f) = \int_\Sigma h(\varphi, \Delta_{g_0}\varphi)\, \mathrm{dvol}_{g_0} + \sum_{n\geq 1} \frac{1}{n!} \int_\Sigma h(\varphi, D_n(\alpha)\varphi) \mathrm{dvol}_{g_0} \tag{4}$$

$$+ \int_\Sigma h(\varphi, X \cdot \varphi) \mathrm{dvol}_{g_0} \tag{5}$$

$$+ S'(X, f, \alpha) \tag{6}$$

The right hand side of the first line amounts to expanding out the Laplace-Beltrami operator $\Delta_{g_0+\alpha}$ as a function of α. Hence, the D_n are differential operators of the form $D_n : \left(\mathrm{Sym}^2(T_\Sigma)\right)^{\otimes n} \to \mathrm{Diff}^{\leq 2}(\Sigma)$ where $\mathrm{Diff}^{\leq 2}(\Sigma)$ are order ≤ 2 differential operators on Σ. Thus, for each section α of $\mathrm{Sym}^2(T_\Sigma)$, we get a second-order differential operator $D_n(\alpha)$ acting on functions on Σ. (This term is the nth term in the Taylor expansion, so its homogeneous of order n: $D_n(t\alpha) = t^n D_n(\alpha)$ for a scalar t.) The second line encodes how vector fields act on the maps of the σ-model. The third line $S'(X, f, \alpha)$ is independent of φ and only depends on the fields f, X, α and their antifields (denoted with checks $(-)^\vee$). It is of the form

$$S'(f, X, \alpha) = \int_\Sigma \left\langle \alpha^\vee, L_X(g_0 + \alpha) + f(g_0 + \alpha) \right\rangle + \int_\Sigma \left\langle X^\vee, [X, X] \right\rangle + \int_\Sigma \left\langle f^\vee, X \cdot f \right\rangle.$$

The first term encodes how vector fields and Weyl transformations act on the perturbed metric $g_0 + \alpha$ and the remaining terms are required to ensure the gauge symmetry is consistent (satisfies the classical master equation).

An explicit formula for $D_n(\alpha, \ldots, \alpha)$ is a rather involved exercise (and not needed here). For instance, if we are working locally on $\Sigma = \mathbb{R}^2$ with the g_0 the flat metric, then the operator $D_1(\alpha)$ is sum of a first-order and a second-order differential operator

$$D_1(\alpha) = \frac{1}{2} \frac{\partial}{\partial x^i}(\mathrm{tr}(\alpha)) \frac{\partial}{\partial x^i} + \frac{1}{2} \mathrm{tr}(\alpha) \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^i},$$

or in a more coordinate-free notation,

$$D_1(\alpha) = \frac{1}{2} \star \mathrm{d}\left(\mathrm{tr}(\alpha) \star \mathrm{d}\right).$$

Here, we use the natural trace map $\mathrm{tr} : \mathrm{Sym}^2 T\Sigma \to C^\infty(\Sigma)$ of symmetric 2×2 matrices.

There is an important parameter in this action functional: the Hermitian inner product h. We can consider scaling it $h \to th$, with $t \in (0, \infty)$. The "infinite volume limit" as $t \to \infty$ admits a nice description, provided one rewrites the action functional in a first-order formalism (i.e., adjoins fields so that only first-order differential operators appear in the action, which is a sort of action functional analogue of working with phase space).

Lemma 2.4. *In this infinite volume limit the bosonic string becomes equivalent to a BV theory whose action functional has the form*

$$S(\beta, \gamma, b, c) + \overline{S}(\overline{\beta}, \overline{\gamma}, \overline{b}, \overline{c}),$$

where $S(\beta, \gamma, b, c)$ is the action functional for the holomorphic bosonic string in Equation (1) and $\overline{S}$ is its anti-holomorphic conjugate.

Remark 2.5. The action functional $\overline{S}$ is similar to S where the fields γ, β, b, c are replaced by sections in the the relevant conjugate bundles. For example, $\beta \in \Omega^{1,*}(\Sigma)$ becomes $\overline{\beta} \in \Omega^{*,1}(\Sigma)$. Moreover, the operator $\overline{\partial}$ is replaced by the holomorphic Dolbeault operator ∂. Another way of saying this is that $\overline{S}$ is the holomorphic string on $\overline{\Sigma}$, which is the conjugate complex structure to Σ.

Outline of proof. There are two things that may cause alarm in the statement of the claim. First, the space of fields of the Polyakov string (in the BV language) and those of the holomorphic bosonic string do not match up. Second, the infinite volume limit $t \to \infty$ is naively ill-defined using the action functional (4). It turns out that these two issues are solved by the same maneuver.

We begin with the first term in the first line of (4). Notice that it is simply the action functional for the σ-model of maps from (Σ, g_0) to (V, h). It is shown in Appendix ?? of [?] how to make sense of the infinite volume limit of this usual σ-model. The idea is to rewrite this theory in the *first order formalism*. This amounts to introducing a new field $B \in \Omega^1(\Sigma) \otimes V^\vee$ and action functional

$$\int_\Sigma \langle B, \mathrm{d}\varphi \rangle_V - \frac{1}{2} \int_\Sigma h^\vee(B, \star B)$$

where $\langle -, - \rangle_V$ represents the evaluation pairing between V and its dual, $\star$ is the Hodge star operator for the metric g_0, and $h^\vee$ denotes the dual metric on V. This action functional is equivalent to the original σ-model; one can compare the equations of motion. Moreover, the limit $(th)^\vee = (1/t)h^\vee$, and so in the infinite volume limit $t \to \infty$, the dual $(th)^\vee$ goes to 0, which kills the second term in the first order action. The remaining theory splits as the direct sum of the free $\beta\gamma$ system with target V and its anti-holomorphic conjugate. At the level of fields, the original field φ corresponds to $\gamma + \overline{\gamma}$ in the first order description, and B corresponds to $\beta + \overline{\beta}$.

Of the remaining terms on the first line of the action, only the D_1 term survives in this infinite volume limit. OG: Is there some way to see this from the concrete computation of D_1 above? Furthermore, the metric g_0 determines an injective map

$$\Omega^{0,1}(\Sigma; T^{1,0}_\Sigma) \oplus \Omega^{1,0}(\Sigma; T^{0,1}_\Sigma) \hookrightarrow \mathrm{Sym}^2(T_\Sigma).$$

Restricting the term $\int_\Sigma h(\varphi, D_1(\alpha)\varphi)$ to the image, we obtain a term whose infinite volume limit is

$$\int_\Sigma \langle \beta, [c, \gamma] \rangle_V + \int_\Sigma \langle \overline{\beta}, [\bar{c}, \bar{\gamma}] \rangle_V .$$

This first term is precisely the third term in the holomorphic string action functional (1), which describes how deformations of complex structure couple to the fields of the σ-model.

In the infinite volume limit, the term $S'(f, X, \alpha)$ recovers the terms

$$\int_\Sigma \langle b, \overline{\partial} c \rangle_T + \int_\Sigma \langle b, [c, c] \rangle_T$$

in the action of the holomorphic string, plus their conjugates. The arguments are similar to those we have just sketched. □

Remark 2.6. Another approach to arrive at the holomorphic theory we consider comes from considering supersymmetry. Without gravity, the pure holomorphic σ-model can be viewed as the *holomorphic twist* of the $N = (2, 0)$ supersymmetric σ model (in this case the target is required to be Kähler). Moreover, the $\beta\gamma bc$ system is the holomorphic twist of the $N = (2,2)$ model. Conjecturally, we expect the holomorphic theory of gravity we consider to be the holomorphic twist of two-dimensional $N = 2$ supergravity.

3. Deformations of the theory and string backgrounds

Whenever one is studying a theory, it is helpful to understand how it can be modified and how features of the theory change as one adjusts natural parameters of the theory, such as coupling constants of the action functional. In other words, one wants to understand the theory in the moduli space of classical theories.

In the BV formalism, because we are working homologically, this moduli space is derived, and there is a tangent complex to our theory in the moduli of classical BV theories. We call it the *deformation complex* of the theory. A systematic discussion can be found in Chapter 5 of [**?**].

As a gloss, the underlying graded vector space of this deformation complex consists of the local functionals on the jets of fields, i.e., Lagrangian densities. (Note that we allow local functionals of arbitrary cohomological degree.) There is also a shifted Lie bracket $\{-,-\}$, which arises from the pairing $\int_\Sigma \langle -,- \rangle$ on the fields. It is, in essence, the shifted Poisson bracket corresponding to that shifted symplectic pairing on the fields. The differential on the local functionals is then $\{S,-\}$, where S is the classical action. All together, the deformation complex forms a shifted dg Lie algebra. Observe that if we find a degree zero element I such that

$$0 = \{S + I, S + I\} = 2\{S, I\} + \{I, I\},$$

then I is a shifted Maurer-Cartan element and hence determines a new classical BV theory whose action functional is $S + I$. In particular, degree 0 cocycles determine first-order deformations of the classical BV theory. Cocycles in degree -1 encode local symmetries of the classical theory; and obstructions to satisfying the quantum master equation end up being degree 1 cocycles.

In this section, we will explain why the deformation complex $\mathrm{Def}_{\mathrm{string}}$ of the holomorphic string can be expressed in terms of Gelfand-Fuks cohomology. Along the way we will see how the usual backgrounds for the bosonic string (a target metric, dilaton term, and so on) appear as elements in this complex of local functionals and hence as deformations of the classical action.

Right now, we will focus on the case $\Sigma = \mathbb{C}$, and in Section 6 we will consider arbitrary Riemann surfaces. We restrict ourselves to examining *translation-invariant* local functionals (which will allow us to descend to a theory defined on an elliptic curve). Unpacking what this means will lead swiftly to Gelfand-Fuks cohomology.

3.1. Deformations for the classical theory.

As a local functional is given by integration of a Lagrangian density, translation invariance requires the density to be the Lebesgue measure d^2z, up to rescaling, and requires the Lagrangian to be specified by its behavior at one point. Hence, a translation-invariant local functional on $\mathbb{C}$ is determined by a function of the jet (i.e., Taylor expansion) of the fields at the origin in $\mathbb{C}$.

It is particularly easy to understand what we mean in the case of the free $bc\beta\gamma$ system. For instance, the γ fields live in the Dolbeault complex $\Omega^{0,*}(\mathbb{C};V)$, and their jets at the origin are $(V[[z,\bar{z}]][\mathrm{d}\bar{z}],\bar{\partial})$, where $\bar{\partial}$ is the formal Dolbeault differential. An example of an element is thus $\widehat{\gamma} = \sum_{m,n} \frac{1}{m!n!} g_{mn} z^m \bar{z}^n$, which is just a formal power series with values in V. An example of a functional is

$$F(\widehat{\gamma}) = g_{10} + g_{21} = (\partial_z \widehat{\gamma})\,|_0 + \left(\partial_z^2 \partial_{\bar{z}} \widehat{\gamma}\right)|_0,$$

which corresponds to the local functional

$$F(\gamma) = \int_{\mathbb{C}} \partial_z \gamma + \partial_z^2 \partial_{\bar{z}} \gamma \, \mathrm{d}^2 z.$$

We call the first kind of term a *chiral* interaction, as it only depends on holomorphic derivatives.

By the $\bar{\partial}$-Poincaré lemma, this complex $(V[[z,\bar{z}]][\mathrm{d}\bar{z}],\bar{\partial})$ is quasi-isomorphic to $V[[z]]$, concentrated in degree zero. This observation is actually quite concrete: it simply says that for a solution γ to the equation of motion $\bar{\partial}\gamma = 0$, its Taylor expansion is just a power series in z and it is independent of $\bar{z}$. In consequence, if we consider translation-invariant Lagrangians depending only on the γ field, then up to quasi-isomorphism these are $\mathrm{Sym}(V^\vee[z^\vee])$. In other words, only chiral interactions yield distinct modifications of the action, when one takes into account the equation of motion.

Note that we have chosen to work with functionals of the fields that are polynomials built out of continuous linear functionals $V^\vee[z^\vee]$ of the jets. This choice is the standard and natural one for variational problems. We note as well that constant functionals are irrelevant, so we want to use $\mathrm{Sym}^{>0}(V^\vee[z^\vee])$ to describe translation-invariant local functionals.

An analogous argument applies to the c field. It shows there is a quasi-isomorphism of dg Lie algebras between the jet at the origin of the Dolbeault complex $\Omega^{0,*}(\mathbb{C};T^{1,0}_{\mathbb{C}})$ of holomorphic vector fields and the Lie algebra of formal vector fields $\mathrm{W}_1 = \mathbb{C}[[z]]\partial_z$. The translation-invariant Lagrangians depending only on the c field are thus quasi-isomorphic to $\mathrm{C}^*_{\mathrm{Lie,red}}(\mathrm{W}_1)$, by which we mean the (reduced) *continuous* Lie algebra cohomology, often known as the Gelfand-Fuks cohomology Similar arguments work for the β and b fields.

If we take all the fields into account together and consider the full equations of motion for the holomorphic string, which couple the c field to the others, then these arguments yield the following.

Lemma 3.1. *There is a quasi-isomorphism*

$$\mathrm{Def}_{\mathrm{string}}(\mathbb{C},V)^{\mathbb{C}} \simeq \mathrm{C}^*_{\mathrm{Lie,red}}(\mathrm{W}_1, \mathrm{Sym}(V^\vee[z^\vee] \oplus V[z^\vee]\mathrm{d}z^\vee \oplus W_1^{\mathrm{ad}}[2]))[2]$$

between the deformation complex of translation-invariant local functionals for the holomorphic string and a certain Gelfand-Fuks cochain complex.

This lemma already substantially simplifies our lives, as one can invoke the literature on Gelfand-Fuks cohomology. But before we do, we will take advantage of another symmetry condition to simplify the situation.

3.2. **Dilating cotangent fibers.** We have already seen how to think of the holomorphic bosonic string theory as corresponding to the shifted cotangent bundle $\mathbb{T}^*[-1]\mathrm{Maps}_{\bar\partial}(-,V)$, as a bundle over the moduli of Riemann surfaces. There is a natural action of the group $\mathbb{C}^\times$ on this space by scaling the shifted cotangent fibers, and we will use the notation $\mathbb{C}^\times_{\mathrm{cot}}$ to indicate this appearance of the multiplicative group.

This group action can be seen on the level of the field theory as follows: we give the γ and c fields—the base of the cotangent bundle—weight 0 and give the β and b fields—the cotangent fiber—weight 1. Note that, in consequence, the pairing $\langle -,-\rangle$ on fields thus has weight -1. In these terms, the classical action functional is weight 1. Thus, we focus on weight 1 deformations of the action for the holomorphic bosonic string, as we are interested in local functionals of the same kind. That means we consider the subcomplex of weight 1 local functionals inside the deformation complex.

Remark 3.2. Although this action S has weight 1, its role in the cochain complex of classical observables is to define the differential $\{S,-\}$. Observe that the shifted Poisson bracket $\{-,-\}$ has weight -1, because it is determined by the pairing, and so the differential has weight 0.

This subcomplex admits a nice description in terms of the geometry of the target.

Lemma 3.3. *There is a* $\mathrm{GL}(V)$*-equivariant quasi-isomorphism*

$$\mathrm{Def}_{\mathrm{string}}(\mathbb{C})^{\mathbb{C},\mathrm{wt}(1)} \simeq \mathrm{Sym}(V^*)\otimes V[1]$$

between the weight 1, translation-invariant deformation complex and the polynomial vector fields on V*, placed in degree -1.*

Concretely, this result says that there are no weight zero interactions that are not not trivialized by an automorphism of the theory. This claim is a consequence of the fact that the zeroth cohomology group vanishes. On the other hand, this lemma says the theory admits a large group of symmetries, namely diffeomorphisms of the target, which appears as the degree -1 cohomology.

The $\mathrm{GL}(V)$ equivariance takes into account the natural symmetries of the target. It also is the first step in the approach to studying the deformation complex with general curved target. We will discuss this further in the section on string backgrounds.

3.3. **Interaction terms that appear at one loop.** As we will see in Section 4, the quantization of the holomorphic string only involves local functional of weight zero for this $\mathbb{C}^\times_{\mathrm{cot}}$-action. (Concretely, this restriction appears because the one-loop Feynman diagrams only have external legs for c and γ fields.) Hence, it behooves us to compute the weight zero subcomplex of the deformation complex as well.

Lemma 3.4. *There is a* $\mathrm{GL}(V)$*-equivariant quasi-isomorphism*

$$\mathrm{Def}_{\mathrm{string}}(\mathbb{C})^{\mathbb{C},\mathrm{wt}(0)} \simeq \mathbb{C}[-1]\oplus\Omega^2_{cl}(V)[1]\oplus\Omega^1(V)\oplus\Omega^1_{cl}(V)[-1]$$

between the weight 0, translation-invariant deformation complex and natural complexes related to the geometry of the target.

Before explaining the key steps of the proof, we remark that there is another, more structural way to see that only weight zero local functionals should be relevant. A quick physical argument would say that we want the path integral measure $\exp(-S/\hbar)$ to be weight zero, which forces $\hbar$ to have weight one to cancel out with the weight of the classical action. But the one-loop term I_1 in the quantized action $S^{\mathrm{q}} = S + \hbar I_1 + \cdots$ must then have weight zero.

There is a BV analogue of this argument. It notes that the differential of the quantum observables has the form $\{S^{\mathrm{q}}, -\} + \hbar\Delta$, where Δ denotes the BV Laplacian. (See Section 4.2 for a discussion of these objects.) As the BV Laplacian has weight -1 because it is determined by the bracket, we must give $\hbar$ weight 1 to ensure the total differential has weight zero. Again the one-loop interaction is forced to have weight zero.

3.3.1. *Sketch of proof.* We have already seen in Lemma 3.5 that we can identify the full translation invariant deformation complex with a certain Gelfand-Fuks cohomology. In terms of this Gelfand-Fuks cohomology we see that the cotangent weight zero piece is identified with the subcomplex

$$\mathrm{Def}_{\mathrm{string}}(\mathbb{C})^{\mathbb{C},\mathrm{wt}(0)} = \mathrm{C}^*_{\mathrm{Lie,red}}\left(\mathrm{W}_1; \mathrm{Sym}(V^\vee[z^\vee])\right)[2].$$

We will drop the overall shift by 2 until the end of the proof.

Any symmetric algebra has a natural maximal ideal: for any vector space W,

$$\mathrm{Sym}(W) = \mathbb{C} \oplus \mathrm{Sym}^{\geq 1}(W).$$

Thus, we can decompose our complexes as

$$\mathrm{C}^*_{\mathrm{Lie,red}}\left(\mathrm{W}_1; \mathrm{Sym}(V^\vee[z^\vee])\right) = \mathrm{C}^*_{\mathrm{Lie,red}}(\mathrm{W}_1) \oplus \mathrm{C}^*_{\mathrm{Lie}}\left(\mathrm{W}_1; \mathrm{Sym}^{\geq 1}(V^\vee[z^\vee])\right).$$

The first summand is the reduced Gelfand-Fuks cohomology of formal vector fields with values in the trivial module. It is well-known that $H^3_{\mathrm{red}}(\mathrm{W}_1) \cong \mathbb{C}[-3]$, i.e., this cohomology is one-dimensional and concentrated in degree 3.

We now proceed to computing the second summand. Denote by $\{L_n = z^{n+1}\partial_z\}$ the standard basis for the Lie algebra of formal vector fields W_1. Notice that the Euler vector field $L_0 = z\partial_z$ induces a grading on W_1, that we will call *conformal dimension*. Note that L_n has conformal dimension n. This grading extends naturally to the Chevalley-Eilenberg complex of W_1 with coefficients in any module.

Let $\lambda_n \in \mathrm{W}_1^\vee$ be the dual vector to L_n. (We work with the continuous dual vector space, as in the setting of Gelfand-Fuks cohomology.) An arbitrary element of $V[[z]]$ is linear combination of vectors of the form $v \otimes z^k$. Write ζ_k for the dual element $(z^k)^\vee$. Thus an element of $(V[[z]])^\vee$ is a linear combination of the vectors of the form $v^\vee \otimes \zeta_k$.

Lemma 3.5. *Let M be any W_1-module. The inclusion of the subcomplex of conformal dimension zero elements*

$$\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_1; M)^{\mathrm{wt}(0)} \xrightarrow{\simeq} \mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_1; M)$$

is a quasi-isomorphism. BW: Is this true?

Proof. For each $p-1 \geq 0$, define the operator $\iota_{L_0} : \mathrm{C}^p_{\mathrm{Lie}}(\mathrm{W}_1; M) \to \mathrm{C}^{p-1}_{\mathrm{Lie}}(\mathrm{W}_1; M)$ by sending a cochain φ to the cochain

$$(\iota_{L_0}\varphi)(X_1, \ldots, X_p) = \varphi(L_0, X_1, \ldots, X_p).$$

Let d be the differential for the complex $\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_1; M)$. It is easy to check that the difference $\mathrm{d}\iota_{L_0} - \iota_{L_0}\mathrm{d}$ is equal to the projection onto the dimension zero subspace. $\square$

The underlying graded vector space of this conformal dimension zero subcomplex splits as follows:

$$\mathrm{C}^{\#}_{\mathrm{Lie}}(\mathrm{W}_1)^{\mathrm{wt}(0)} \otimes \left(\mathrm{Sym}^{\geq 1}\left(V[[z]]\right)^{\vee}\right)^{\mathrm{wt}(0)} \oplus \mathrm{C}^{\#}_{\mathrm{Lie}}(\mathrm{W}_1)^{\mathrm{wt}(1)} \otimes \left(\mathrm{Sym}^{\geq 1}\left(V[[z]]\right)^{\vee}\right)^{\mathrm{wt}(-1)} \tag{7}$$

In the first component, the purely dimension zero part of the reduced symmetric algebra is simply $\mathrm{Sym}^{\geq 1}(V^{\vee})$, i.e., power series OG: Do you want power series? You wrote polynomials earlier on V with no constant term. We denote this algebra concisely as $\mathcal{O}_{red}(V)$, for reduced functions on V. Similarly, in the second component, the dimension one part of $\mathrm{Sym}^{\geq 1}\left(V[[z]]\right)^{\vee}$ is of the form $\mathrm{Sym}(V^{\vee}) \otimes z^{\vee}V^{\vee}$, which is naturally identified with $\Omega^1(V)$.

The differential in this Gelfand-Fuks complex has the form

$$\begin{tikzcd}
\underline{0} & \underline{1} & \underline{2} & \underline{3} \\
1 \otimes \mathcal{O}_{red}(V) \arrow[dr, "\mathrm{d}_{dR}"] & \lambda^0 \otimes \mathcal{O}_{red}(V) \arrow[r] \arrow[dr, "\mathrm{d}_{dR}"] & \lambda^{-1} \wedge \lambda^1 \otimes \mathcal{O}_{red}(V) & \lambda^{-1} \wedge \lambda^1 \wedge \lambda^0 \otimes \mathcal{O}_{red}(V) \\
 & \lambda^{-1} \otimes \Omega^1(V) \arrow[r] & \lambda^{-1} \wedge \lambda^0 \otimes \Omega^1(V) &
\end{tikzcd}$$

The top line comes from the first summand in (7) and the bottom line corresponds to the second summand. The top horizontal map sends λ^0 to $2 \cdot \lambda^{-1} \wedge \lambda^1$, and the bottom horizontal map sends λ^{-1} to $\lambda^{-1} \wedge \lambda^0$ (both are the identity on V). The diagonal maps are given by the de Rham differential $\mathrm{d}_{dR} : \mathcal{O}_{red}(V) \to \Omega^1(V)$. This complex is quasi-isomorphic to

$$\begin{tikzcd}
1 \otimes \mathcal{O}_{red}(V) \arrow[dr, "\mathrm{d}_{dR}"] & & & \lambda^{-1} \wedge \lambda^1 \wedge \lambda^0 \otimes \mathcal{O}_{red}(V) \\
 & \lambda^{-1} \otimes \Omega^1(V) & \lambda^{-1} \wedge \lambda^0 \otimes \Omega^1(V) &
\end{tikzcd}$$

which, in turn, is identified with $\Omega^2_{cl}(V)[-1] \oplus \Omega^1(V)[-2] \oplus \Omega^1_{cl}(V)[-3]$. After accounting for the overall shift by 2, we arrive at the identification of the $\mathbb{C}^{\times}_{\mathrm{cot}}$-weight zero component of the translation-invariant deformation complex.

3.4. **Interpretation as string backgrounds.** We now discuss, in light of the calculations above, how to interpret string backgrounds in our approach. Since V is flat, we will see that the following deformations will be trivializable. Note that this trivializations will *not* be equivariant for the obvious $\mathrm{GL}(V)$ action (or for non-flat targets, general diffeomorphisms of the target). Thus, these deformations are relevant for the case of a curved target, and we can give an interpretation of them in terms of the usual perspective of *string backgrounds*. OG: So now lets add some words providing such an interpretation.

We have already mentioned that we should think of the $\mathbb{C}^\times_{\text{cot}}$ weight 1 local functionals as deformations of the classical theory as a cotangent theory. The cohomological degree zero deformations of the weight one deformations is $H^1(V; T_V)$. Given any such element $\mu \in H^1(V; T_V)$ we can consider the following local functional

$$\int_\Sigma \langle \beta, \mu(\gamma) \rangle_V .$$

The element μ determines a deformation of the complex structure of V, and we have prescribed an action functional encoding this deformation. We propose that this an appearance of the ordinary curved background in bosonic string theory from the perspective of the holomorphic model we work with.

There are interesting deformations that go outside of the world of cotangent theories. Consider the cohomological degree zero part of the weight 0 complex. There is a term of the form $H^1(V; \Omega^2_{cl}(V))$. It is shown in Part 2 Section 8.5 of [**?**] how closed holomorphic two-forms determine local functionals of the $\beta\gamma$ system with curved target. A sketch of this construction goes as follows. Locally we can write a closed holomorphic 2-form as $\mathrm{d}\theta$ for some holomorphic one-form $\theta \in \Omega^1(V)$. If $\gamma : \Sigma \to V$ is a map of the σ-model there is an induced map (when γ satisfies the equations of motion) $\gamma^* : \Omega^1(V) \to \Omega^1(\Sigma)$. We can then integrate $\gamma^*\theta$ along any closed cycle C in Σ and one should think of this as a residue along C. In [**?**] we write down a local functional that realizes this residue, and one can show that it only depends on the corresponding class in $H^1(V; \Omega^2_{cl}(V))$. We posit that this is the appearance of the B-field deformation of the ordinary bosonic string.

4. Quantizing the holomorphic bosonic string on a disk

We will apply the algorithm described in Section 1.1 in the case of $\Sigma = \mathbb{C}$. For this theory we are lucky, however: the integrals that appear from the Feynman diagrams do not have divergences, so that renormalized action is easy to compute. This aspect is the subject of the first part of this section. (In Section 6 we will provide an argument based on deformation theory as to why quantizations exist on arbitrary Riemann surfaces.) Moreover, it is easy to check whether the quantum master equation is satisfied, and the answer is simple. This aspect is the subject of the second part. The results can be summarized as follows.

Proposition 4.1. *The holomorphic bosonic string with source $\mathbb{C}$ and target $\mathbb{C}^d$ admits a BV quantization if $d = 13$. This quantized action only has terms of order $\hbar^0$ and $\hbar$ (i.e., it quantizes at one loop).*

4.1. The Feynman diagrams.

Let us describe the combinatorics of the Feynman diagrams that appear here before we describe the associated integrals.

4.1.1. The procedure constructs graphs out of a prescribed type of vertices and edges; we must consider all graphs with such local structure. The classical action functional determines the allowed kinds of vertices and edges. The quadratic terms of the action tell us the edges; each quadratic term yields an edge whose boundary is labeled by the two fields appearing in the term.

FIGURE 1. The $\beta\gamma$ and bc propagators

FIGURE 2. The trivalent vertices for $\int \langle \beta, [c, \gamma] \rangle$ and $\int \langle b, [c, c] \rangle$

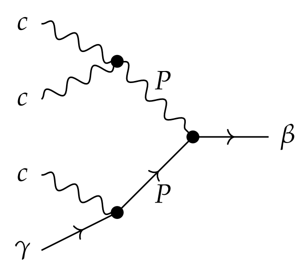

FIGURE 3. An example of a tree with four inputs and one output

For us there are thus two types of edges: an edge that flows from β to γ, and an edge that flows from b to c displayed in Figure 1.

The nonquadratic terms tell us the vertices: each n-ary term yields a vertex with n legs, and the legs are labeled by the n types of fields appearing in the term. For us there are thus two types of trivalent vertices: a vertex with two c legs and a b leg, and a vertex with a c leg, a γ leg, and a β leg. It helpful to picture these legs as directed, so that c and γ legs flow into a vertex and b and β legs flow out. These vertices are displayed in Figure 2.

The kinds of graphs one can build with such vertices and edges are limited. We focus on connected graphs, since an arbitrary graph is just a union of connected components.

A tree (i.e., a connected graph with no loops) must have at most one outgoing leg, which must be either a b or a β; the other legs are incoming, so each must be labeled by a c or a γ. An example of such a tree is given in Figure 3.

Note that there are two types of trees. If there is a γ leg, then there is a β leg, and there is a chain of $\gamma\beta$ edges connecting them; all other external legs are of c type. If there is a b leg, then the only other legs are c type.

A 1-loop graph will consist of a wheel (i.e., a sequence of edges that form an overall loop) with trees attached. The outer legs are all of c type. Every edge along a wheel will have the same type.

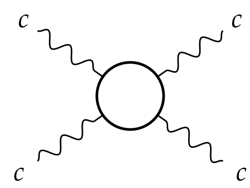

FIGURE 4. An example of a wheel with four inputs

It is not possible to build a connected graph with more than one loop. This combinatorics is the essential reason that we can quantize at one loop. For an example of such a wheel see Figure 4.

We write $\mathbf{Graph}_{\text{string}}$ for the collection of connected graphs just described, namely the directed trees and 1-loop graphs allowed by the string action functional. Let $\mathbf{Graph}^{(0)}_{\text{string}}$ denote the 0-loop graphs (i.e., trees) and let $\mathbf{Graph}^{(1)}_{\text{string}}$ denote the 1-loop graphs (i.e., wheels with trees attached).

4.1.2. These graphs describe linear maps associated to the field. More precisely, a graph with k legs describes a linear functional on the k-fold tensor product of the space of fields. One builds this linear functional out of the data of the action functional.

As an example, a k-valent vertex corresponds to a k-ary term in the action, which manifestly takes in k copies of the fields and outputs a number. Thus, the vertex labels an element of a (continuous) linear dual of the k-fold tensor product of fields. In fact, one restricts to *compactly-supported* fields, since the action functional is rarely well-defined on all fields when the source manifold is non-compact. (Note this domain of compactly-supported fields is all one needs for making variational arguments or for constructing a BV quantization.)

An edge corresponds an element P of the 2-fold tensor product of the space of fields, often called a *propagator*. More precisely, the edge should correspond to the Green's function for the linear differential operator appearing in the associated quadratic term of the action; hence the propagator is an element of the *distributional completion* of the 2-fold tensor product. For us the $\beta\gamma$ leg should be labeled by $\overline{\partial}^{-1} \otimes \mathrm{id}_V$, where $\overline{\partial}^{-1}$ denotes an inverse to the Dolbeault operator on functions. The bc leg should be labeled by $\overline{\partial}_T^{-1}$, the inverse of the Dolbeault operator on the bundle $T^{1,0}$.

Given a graph Γ, one should contract the tensors associated to the vertices and edges. We denote the linear functional for this graph by $w_\Gamma(P, I)$, where w stands for "weight," the term P indicates we label edges by the propagator P, and the term I indicates we label vertices by the "interaction" term of the action S (i.e., the terms that are cubic and higher).

This contraction is not always well-posed, unfortunately. Each vertex labels a distributional section of some vector bundle on Σ, and each edge labels a distributional section of a vector bundle on Σ^2. Thus the desired contraction can be written *formally* as an integral over the product manifold Σ^v, where v denotes the number of vertices. In most situations this contraction is ill-defined, since one cannot (usually) pair distributions. Concretely, one sees that the integral expression is divergent.

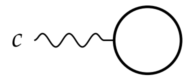

FIGURE 5. The tadpole diagram Γ_{tad}

Thus, to avoid these divergences, one labels the edges by a smooth replacement of the Green's functions. (Imagine replacing a delta function δ_0 by a bump function.) Since one can pair smooth functions and distributions, each graph yields a linear functional on fields using these mollified edges. Thus we have *regularized* the divergent expression.

But now this linear functional depends on the choice of mollifications. Hence the challenge is to show that if one picks a sequence of smooth replacements that approaches the Green's function, there is a well-defined limit of the linear functionals.

4.1.3. We will now sketch one method well-suited to complex geometry that allows us to see that no divergences appear for the holomorphic bosonic string. Our approach is an example of the renormalization method developed by Costello in [**?**], which applies to many more situations.

Our primary setting in this section is $\Sigma = \mathbb{C}$. For this Riemann surface, a standard choice of Green's function for the $\overline{\partial}$ that acts on functions is

$$P(z,w) = \frac{1}{2\pi i}\frac{\mathrm{d}z + \mathrm{d}w}{z-w}.$$

It is a distributional one-form on $\mathbb{C}^2$ that satisfies $\overline{\partial} \otimes 1(P) = \delta_\Delta$, where δ_Δ is the delta-current supported along the diagonal $\Delta : \mathbb{C} \hookrightarrow \mathbb{C}^2$ and providing the integral kernel for the identity. In terms of our discussion above, we view this one-form as a distributional section of the fields γ and β: for example, for fixed w, the one-form $\mathrm{d}z/(z-w)$ is a β field in the z-variable as it is a $(1,0)$-form. (This propagator is for the $\beta\gamma$ fields—and one must tensor with a kernel for the identity on V—but a similar formula provides a propagator for the bc fields.)

4.1.4. We will now describe the integral associated to a simple diagram. For simplicity, we assume $V = \mathbb{C}$ so that the γ and β fields are simply functions and 1-forms on $\mathbb{C}$, respectively. Consider a "tadpole" diagram, Figure 5, Γ_{tad} whose outer legs are c fields (i.e., vector fields on $\mathbb{C}$).

There is only one vertex here, corresponding to the cubic function on fields

$$w_{\Gamma_{\text{tad}}}(P, I_{\text{string}}) = \int_{z\in\mathbb{C}} \beta \wedge c\gamma.$$

If the field c is of the form $f(z)\mathrm{d}\overline{z}\partial_z$, with f compactly supported, then our integral is

$$\int_{z\in\mathbb{C}} \beta \wedge f(z)(\partial_z\gamma)\mathrm{d}\overline{z}.$$

(Note that a general cubic function could be described as an integral over $\mathbb{C}^3$, but our function is supported on the small diagonal $\mathbb{C} \hookrightarrow \mathbb{C}^3$.) The linear functional for this tapole diagram should be given by inserting the propagator P in place of the β and γ fields. Hence it ought to be given by the following integral over $\mathbb{C}$:

$$\int_{z\in\mathbb{C}} c(z)P(z,w)|_{z=w} = \int_{z\in\mathbb{C}} f(z)\partial_z\left(\frac{1}{2\pi i}\frac{\mathrm{d}z+\mathrm{d}w}{z-w}\right)|_{z=w}\,\mathrm{d}\overline{z}.$$

This putative integral is manifestly ill-defined, since the distribution is singular along the diagonal.

4.1.5. We smooth out the propagator P using familiar tools from differential geometry. Fix a Hermitian metric on Σ, which then associates provides an adjoint $\overline{\partial}^*$ to the Dolbeault operator $\overline{\partial}$. For the usual metric on $\mathbb{C}$, we have

$$\overline{\partial}^* = -2\frac{\partial}{\partial(\mathrm{d}\overline{z})}\frac{\partial}{\partial z}.$$

In physics one calls a choice of the operator $\overline{\partial}^*$ a *gauge-fix*. The commutator $[\overline{\partial}, \overline{\partial}^*]$, which we will denote D, is equal to $\frac{1}{2}\Delta$, where Δ is the Laplace-Beltrami operator for this metric.

We introduce a smoothed version of the propagator using the heat kernel e^{-tD}, which is a notation that denotes a solution to the heat equation $\partial_t f(t,z) + Df(t,z) = 0$. For $\mathbb{C}$ with the Euclidean metric, the standard heat kernel is

$$e^{-tD}(z,w) = \frac{1}{4\pi t}e^{-|z-w|^2/4t}(\mathrm{d}z - \mathrm{d}w) \wedge (\mathrm{d}\overline{z} - \mathrm{d}\overline{w}).$$

For $0 < \ell < L < \infty$, we define

$$P_\ell^L = \overline{\partial}^* \int_\ell^L e^{-tD}\mathrm{d}t.$$

We compute

$$\overline{\partial}P_\ell^L = D\int_\ell^L e^{-tD}\mathrm{d}t = \int_\ell^L \frac{d}{dt}e^{-tD}\mathrm{d}t = e^{-LD} - e^{-\ell D}.$$

In the limit as $\ell \to 0$ and $L \to \infty$, the operator P_ℓ^L goes to a propagator (or Green's function) P for $\overline{\partial}$. To see this, consider an eigenfunction f of D where $Df = \lambda f$ where λ is a non-negative real number. Then

$$(\overline{\partial}P_\ell^L)f = (e^{-L\lambda} - e^{-\ell\lambda})f,$$

which goes to f as $L \to \infty$ and $\ell \to 0$. Thus, if one works with the correct space of functions, P_ℓ^L is almost an inverse to $\overline{\partial}$; moreover, it is a smooth function on $\Sigma \times \Sigma$.

BW: I've fixed our convention for eigenvalues and some small typos above. Is there something you want to be careful above?

4.1.6. We now return to the tadpole diagram and put P_ℓ^L on the edge instead of P. (We again assume $V = \mathbb{C}$ for simplicity.) The propagator is

$$P_\ell^L(z,w) = \int_\ell^L \mathrm{d}t\,\frac{\partial}{\partial(\mathrm{d}\overline{z})}\frac{\partial}{\partial z}\left(\frac{1}{4\pi t}e^{-|z-w|^2/4t}(\mathrm{d}z - \mathrm{d}w) \wedge (\mathrm{d}\overline{z} - \mathrm{d}\overline{w})\right) \tag{8}$$

$$= \int_\ell^L \mathrm{d}t\frac{1}{4\pi t}\frac{\overline{z} - \overline{w}}{2t}e^{-|z-w|^2/4t}(\mathrm{d}z - \mathrm{d}w). \tag{9}$$

Note that it is smooth everywhere on $\mathbb{C}^2$. The integral for the tadpole diagram is

$$\begin{aligned} w_{\Gamma_{\text{tad}}}(P_\ell^L, I_{\text{string}}) &= \int_{z\in\mathbb{C}} c(z) P_\ell^L(z,w)|_{z=w} \\ &= \int_{z\in\mathbb{C}} \int_\ell^L \mathrm{d}t f(z)\partial_z \left(\frac{1}{4\pi t} \frac{\overline{z}-\overline{w}}{2t} e^{-|z-w|^2/4t} (\mathrm{d}z - \mathrm{d}w) \right) |_{z=w} \, \mathrm{d}\overline{z} \\ &= \int_{z\in\mathbb{C}} \int_\ell^L \mathrm{d}t f(z) \left(\frac{1}{4\pi t} \left(\frac{\overline{z}-\overline{w}}{2t} \right)^2 e^{-|z-w|^2/4t} (\mathrm{d}z - \mathrm{d}w) \right) |_{z=w} \, \mathrm{d}\overline{z} \\ &= 0, \end{aligned}$$

since the integrand vanishes along the diagonal. Note that this integral is independent of ℓ and L and hence the limit is zero.

4.1.7. By explicitly analyzing the $\ell \to 0$ limit for the integral associated to every Feynman diagram, we find the following result.

Proposition 4.2. *For any graph $\Gamma \in \mathbf{Graph}_{\text{string}}$ allowed by the combinatorics of the string action functional and for any $L > 0$, there is a well-defined limit $\lim_{\ell\to 0} w_\Gamma(P_\ell^L, I_{\text{string}})$.*

We denote this limit by $w_\Gamma(P_0^L, I_{\text{string}})$.

The necessary manipulations and inequalities are very close to those used in []. We recommend looking at OG: exact location for model arguments.

Outline of proof. When Γ is a tree, there is never an issue with divergences; we could even use the Green's function $\overline{\partial}^{-1}$ on each edge. To see this, note that one can view a tree as having a distinguished root, given by the leg that is either of β or b type. One can then see the tree as describing a multilinear map from the leaves (i.e., legs that are not roots) to the root. Indeed, one can view each cubic vertex as such an operator. For instance, $\langle b, [c,c]\rangle$ corresponds to the Lie bracket of vector fields, since we view $\langle b, -\rangle$ as an element of the c fields. For a tree, one can then input arbitrary elements into the leaves, apply the operations labeled by the vertices, apply the operator labeled by the edge, and so on, until one reaches the root. The composite multilinear operator sends smooth sections to smooth sections, even if the edges are labeled by distributional sections, since the associated operator sends smooth sections to smooth sections.

When Γ is a one-loop graph, it consists of a wheel with trees attached to the outer legs. By the preceding argument, we know those trees do not introduce singularities; hence any divergences are due solely to the wheel. It thus suffices to consider pure wheels (i.e., those with no trees attached).

Let the wheel have n vertices. The kth vertex has a coordinate z_k on $\mathbb{C}$; the kth external leg has input $c_k = f_k(z_k, \overline{z}_k)\mathrm{d}\overline{z}_k\,\partial_{z_k}$, where f_k is a compactly-supported smooth function. Then the integral has the form

$$\int_{(z_1,\ldots,z_n)\in\mathbb{C}^n} \mathrm{d}^n\overline{z}\,(f_1\partial_{z_1}P_\ell^L(z_1,z_n))(f_2\partial_{z_2}P_\ell^L(z_2,z_1))\cdots(f_n\partial_{z_n}P_\ell^L(z_n,z_{n-1})),$$

since the kth input will act on one of the propagators entering the kth vertex. One needs to show that this expression has a finite $\ell \to 0$ limit.

Let us prove this limit exists for the case $n = 2$. Then we have

$$\int_{z_1,z_2\in\mathbb{C}} d\bar{z}_1 d\bar{z}_2 \int_\ell^L dt_1 \int_\ell^L dt_2\, f_1(z_1) f_2(z_2) \partial_{z_1}\left(\frac{1}{4\pi t_1}\frac{\bar{z}_1-\bar{z}_2}{2t_1} e^{-|z_1-z_2|^2/4t_1}(dz_1 - dz_2)\right)$$
$$\times\, \partial_{z_2}\left(\frac{1}{4\pi t_2}\frac{\bar{z}_1-\bar{z}_2}{2t_2} e^{-|z_1-z_2|^2/4t_2}(dz_2 - dz_1)\right),$$

which is already a bit lengthy. As our focus is on showing a limit exists, we will throw out unimportant factors and simplify the expression. First, note that taking the partial derivative ∂_{z_i} will simply multiply the integrand by $(\bar{z}_1 - \bar{z}_2)/2t_i$. Moreover, we change coordinates to $u = z_1 - z_2$ and $v = z_2$. Then the integral is proportional to

$$\int_\ell^L dt_1 \int_\ell^L dt_2 \int_{\mathbb{C}^2} d^2u\, d^2v\, f_1 f_2 \frac{\bar{u}^4}{t_1^3 t_2^3} e^{-|u|^2(\frac{1}{t_1}+\frac{1}{t_2})}.$$

We take the integral over v last; it will be manifestly well-behaved after we take the other integrals.

Thus consider the integral just over $u \in \mathbb{C}$, so that we are computing the expected value of $F = f_1 f_2$ against a Gaussian measure whose variance is determined by t_1 and t_2. (Namely, the variance is $t_1 t_2/(t_1 + t_2)$.) We might as well focus on values of t_i that are very small, as those would be the source of divergences when $\ell \to 0$. For small t_i, we only care about the behavior of F near the origin as the measure is concentrated near the origin. Thus, consider a partial Taylor expansion of F. The polynomial part can be computed quickly since the expected values of monomials against a Gaussian measure (i.e., the moments) have a simply expression in terms of the variance. The first nonzero contribution would come from the u^4 term in the Taylor expansion of F, and it contributes a factor of the form $(t_1 t_2/(t_1 + t_2))^5$, up to constant that we ignore. We are left with

$$\int_\ell^L dt_1 \int_\ell^L dt_2 \frac{(t_1 t_2)^3}{(t_1+t_2)^5} \le \int_\ell^L dt_1 \int_\ell^L dt_2\, 2^{-5}\sqrt{t_1 t_2} = 2^{-5}(L^{3/2} - \ell^{3/2})^2,$$

where we use the arithmetic-geometric mean inequality $\sqrt{t_1 t_2}/(t_1 + t_2) \le 1/2$ in the middle. This expression has a finite limit as $\ell \to 0$. The higher terms in the Taylor expansion contribute bigger powers of the variance and hence have $\ell \to 0$ limits. Finally, the expected value of the error term of our partial Taylor expansion, which vanishes to some positive order at the origin, can be bounded in such as way that an $\ell \to 0$ limit exists. □

We can now define the effective theory that we consider for the string.

Definition 4.3. The *renormalized action functional* at scale L for the holomorphic bosonic string is

$$I[L] = \sum_{\Gamma\in\mathbf{Graph}^{(0)}_{\text{string}}} w_\Gamma(P_0^L, I_{\text{string}}) + \hbar \sum_{\Gamma\in\mathbf{Graph}^{(1)}_{\text{string}}} w_\Gamma(P_0^L, I_{\text{string}}).$$

We denote the first summand—the tree-level expansion—by $S_0[L]$ and the second summand—the one-loop expansion—by $S_1[L]$. We use the notation $S[L] = S_{free} + I[L]$ where S_{free} is the classical free part of the action functional.

Remark 4.4. For any functional J, let $w(P_\ell^L, J)$ denote the sum over all graphs as above with the smooth propagator P_ℓ^L placed at the edges and J placed at the vertices. Then, the family $\{I[L]\}$ satisfies the *homotopy RG equation*

$$I[L] = w(P_\ell^L, I[\ell]).$$

The operator $w(P_\ell^L, -)$ defines a homotopy equivalence between the theory at scale ℓ, defined using $S[\ell]$, and the theory at scale S, defined using $S[L]$.

4.2. **The quantum master equation.** In the BV formalism the basic idea is to replace integration against a path integral measure $e^{-S(\phi)/\hbar}\mathcal{D}\phi$ with a cochain complex. In this cochain complex, we view a cocycle as defining an observable of the theory, and its cohomology class is viewed as its expected value against the path integral measure. For toy models of finite-dimensional integration, see []; these examples are always cryptomorphically equivalent to a de Rham complex, which is a familiar homological approach to integration.

Hence the content of the path integral, in this approach, is encoded in the differential. A key idea is that the differential is supposed to behave like a divergence operator for a volume form: recall that given a volume form μ on a manifold, its divergence operator maps vector fields to functions by the relationship

$$\mathrm{div}_\mu(\mathcal{X})\mu = L_{\mathcal{X}}\mu.$$

This relationship, in conjunction with Stokes lemma, implies that if a function f is a divergence $\mathrm{div}_\mu(\mathcal{X})$, then $\int f\mu = 0$, i.e., its expected value against the measure μ is zero. The BV formalism axiomatizes general properties of divergence operators; a putative differential must satisfy these properties to provide a BV quantization.

When following the algorithm of Section 1.1, we want the renormalized action

$$S = S^{\mathrm{cl}} + \hbar S_1 + \hbar^2 S_2 + \cdots$$

to determine a putative differential d_S^q on the graded vector space of observables. To explain this operator, we need to describe further algebraic properties on the observables that the BV formalism uses.

First, in practice, the observables are the symmetric algebra generated by the continuous linear duals to the vector spaces of fields. There is also a pairing on fields that is part of the data of the classical BV theory, between each field and its "anti-field." (This pairing is a version of the action of constant vector fields on functions in the toy models.) In our case, there is the pairing between b and c and between β and γ, respectively. It behaves like a "shifted symplectic" pairing as it has cohomological degree -1, and hence it determines a degree 1 Poisson bracket $\{-,-\}$ on the graded algebra of observables. Finally, the pairing also determines a second-order differential operator Δ_{BV} on the algebra of observables by the condition that

$$\Delta_{BV}(FG) = (\Delta_{BV}F)G + (-1)^F F(\Delta_{BV}G) + \{F, G\}.$$

(This equation is a characteristic feature of divergence operators with respect to the product of polyvector fields.)

With these structures in hand, we can give the formula

$$\mathrm{d}^q_S = \{S, -\} + \hbar\Delta_{BV}$$

for the putative differential. As S has cohomological degree 0, the operator $\{S, -\}$ has degree 1. We remark that modulo $\hbar$, one recovers the differential $\{S^{\mathrm{cl}}, -\}$ on the classical observables; the zeroth cohomology of the classical observables is functions on the critical locus of the classical action S^{cl}.

By construction, this putative differential d^q_S satisfies the conditions of behaving like a divergence operator. The only remaining condition to check is that it is square-zero. This condition ends up being equivalent to S satisfying the *quantum master equation*

$$\hbar\Delta_{BV} S + \frac{1}{2}\{S, S\} = 0. \tag{10}$$

More accurately, d^q_S is a differential if and only if the right hand side is a constant.

4.2.1. We now turn to examining this condition in our setting. It helps to understand it is diagrammatic terms.

As the bracket is determined by a linear pairing, it admits a simple diagrammatic description as an edge. For instance, given an observable F that is a homogeneous polynomial of arity m and an observable G of arity n, then $\{F, G\}$ has arity $m + n - 2$. It can be expressed as a Feynman diagram BW: pic where the edge connecting F and G is labeled by a 2-fold tensor K.

The BV Laplacian acts by attaching an edge labeled by K as a loop in all possible ways. OG: Add picture. This diagrammatic behavior corresponds to the fact that Δ_{BV} is a constant-coefficient second-order differential operator.

The tensor K determined by the pairing on fields is distributional. As one might expect from our discussion of divergences above, these diagrammatic descriptions of the BV bracket and Laplacian are thus typically ill-defined. In other words, the quantum master equation is *a priori* ill-posed for the same reason that the initial Feynman diagrams are ill-defined. We can apply, however, the same cure of mollification.

4.2.2. Costello's framework **[?]** provides an approach to renormalization built to be compatible with the BV formalism. A key feature is that for each "length scale" $L > 0$, there is a BV bracket $\{-, -\}_L$ and BV Laplacian Δ_L. The scale L renormalized action $S[L]$ satisfies the scale L quantum master equation (QME)

$$\hbar\Delta_L S[L] + \frac{1}{2}\{S[L], S[L]\}_L = 0$$

if and only if $S[L']$ satisfies the scale L' quantum master equation for every other scale L', see Lemma 9.2.2 in **[?]**. Hence, we say a renormalized action satisfies the quantum master equation if its solves the scale L equation for some L.

Thus it remains for us to describe the scale L bracket and BV Laplacian in our setting, so that we can examine whether the renormalized action satisfies the quantum master equation.

Definition 4.5. The *scale L bracket* $\{-,-\}_L$ is given by pairing with the scale L heat kernel

$$K_L(z,w) = \frac{1}{4\pi L} e^{-|z-w|/4L} (\mathrm{d}z - \mathrm{d}w) \wedge (\mathrm{d}\overline{z} - \mathrm{d}\overline{w}).$$

The *scale L BV Laplacian* Δ_L is given by the contraction ∂_{K_L}.

These definitions mean that testing the quantum master equation leads to diagrams whose integrals are similar to those we encountered earlier. We explain the diagrammatics and sketch the relevant integrals in the proof of the following result, which characterizes when the string action admits a BV quantization.

We emphasize that up to now, we have not indicated explicitly which vector space V is the target space for our string. But the action functional explicitly depends on this choice, so here we will write S_V for the action with target V.

Proposition 4.6. *The obstruction to satisfying the quantum master equation is the functional*

$$Ob_V[L] = \hbar \Delta_L S_V[L] + \frac{1}{2}\{S_V[L], S_V[L]\}_L.$$

It has the form

$$Ob_V[L] = \hbar(\dim_{\mathbb{C}}(V) - 13)F[L],$$

where $F[L]$ is a functional independent of V.

In short, the failure to satisfy the QME is a linear function of the dimension of the target space V. In particular, when $V \cong \mathbb{C}^{13}$, the obstruction vanishes and the renormalized action *does* satisfy the QME, giving us an immediate corollary. (Note that we do *not* need to know $F[L]$ to recognize that the obstruction vanishes!)

Corollary 4.7. *When the target vector space is 13-dimensional (i.e., has 26 real dimensions), the holomorphic bosonic string admits a BV quantization.*

Proof. It is a general feature of Costello's formalism that the tree-level term $S_0[L]$ of the renormalized action satisfies the scale L equation

$$\{S_0[L], S_0[L]\}_L = 0,$$

known as the classical master equation. Hence the first obstruction to satisfying the QME can only appear with positive powers of $\hbar$. We can also see quickly that no terms of $\hbar^2$ appear: the one-loop term $S_1[L]$ is only a function of the c field, so

$$\{S_1[L], S_1[L]\}_L = 0 \quad \text{and} \quad \Delta_L S_1[L] = 0.$$

Hence the obstruction to satisfying the QME is precisely

$$\hbar \left(\{S_0[L], S_1[L]\} + \Delta_L S_0[L] \right).$$

Thus we see that the obstruction is a multiple of $\hbar$. For simplicity, we will divide out that factor and let Ob_V denote the term inside the parenthesis.

Consider the term $\{S_0[L], S_1[L]\}_L$. Diagrammatically, it corresponds to attaching a tree with a b "root" to a wheel using an edge labeled by K_L. OG: Need to reference/explain the results of Si and KC.

Now consider the term $\Delta_L S_0[L]$. Diagrammatically, it corresponds to turning a tree into a wheel by using an edge—labeled by K_L—to attach the root to an incoming leaf. There are thus two kinds of wheels that appear, since there are two kinds of trees. There are the wheels where the K edge is for bc fields. Note that these wheels are the same for every choice of target V as they only depend on the bc fields, i.e., are independent of the $\beta\gamma$ fields. These will contribute a term $F[L]$ to the obstruction. On the other hand, there are the wheels where the K edge is for $\beta\gamma$ fields. These depend on V but in a very simple way: the distribution K is just the heat kernel tensored with the identity on V, and hence the contraction amounts to taking $\dim_{\mathbb{C}}(V)$ copies of the $V = \mathbb{C}$ value. In other words, the $\beta\gamma$ wheels contribute a term $\dim_{\mathbb{C}}(V)G[L]$ to the obstruction, where $G[L]$ is the value for $V = \mathbb{C}$. The last part of the proof of the theorem is a direct calculation of the functionals $F[L]$ and $G[L]$. So as to not diverge from our track of thought we include this calculation in Section **??** where we show that $F[L], G[L]$ are both independent of L and satisfy $F = -13G$, thus completing the proof. □

Remark 4.8. One can consider coupling the $\beta\gamma$ system to any tensor bundle on the Riemann surface. For instance, suppose γ is a section of $T_\Sigma^{\otimes n}$ and hence β is a section of $T_\Sigma^{*\otimes n+1}$. In this case, one can show that the part of the obstruction with internal edges labeled by the $\beta\gamma$ propagators contributes a factor $(6n^2 + 6n + 1)G$, with G the same functional above. BW: should I say more?

5. OPE AND THE STRING VERTEX ALGEBRA

Vertex algebras are mathematical objects that axiomatize the behavior of local observables (i.e., point-like observables) of a chiral conformal field theory (CFT), such as the $bc\beta\gamma$ system or the holomorphic bosonic string. The vertex operator of a vertex algebras encodes the operator product expansions (OPE) for local observables, which is of central interest in understanding a chiral CFT. (We will not review vertex algebras here as there are many nice expositions OG: cite some.)

In this section we will explain how to extract the vertex algebra of the holomorphic bosonic string, using machinery developed in [**?**, **?**, **?**]. The answer we recover is precisely the chiral sector of the usual bosonic string. OG: Is that true?

5.1. **A reminder on the chiral algebra of the string.**

We provide a brief background on the vertex algebra for the chiral sector of the bosonic string. For a detailed reference we refer the reader to the series of papers [**?**, **?**]. It is easiest to introduce this as a *differential graded vertex algebra*. This is simply a vertex algebra internal to the category of chain complexes. The underlying graded vertex algebra has state space of the form

$$\mathcal{V}_{\beta\gamma}^{\otimes 13} \otimes \mathcal{V}_{bc}$$

where $\mathcal{V}_{\beta\gamma}$ and $\mathcal{V}_{bc}$ are the $\beta\gamma$ and bc vertex algebras, respectively. The β and γ generators are in grading degree zero, the c generator is in grading degree -1, and the b is in grading degree $+1$. In the physics literature this is referred to as the *BRST* grading.

Forgetting the cohomological (or BRST) grading, this vertex algebra is a conformal vertex algebra of central charge zero (by construction). In particular, this means that the vertex algebra has a stress energy tensor. Explicitly, it is of the form

$$T_{\text{string}}(z) = \left(\sum_{i=1}^{13} \beta_i(z)\partial_z\gamma_i(z) + \partial_z\beta_i(z)\gamma_i(z)\right) + \left(b(z)\partial_z c(z) + 2\partial_z b(z)c(z)\right).$$

Note that T_{string} is of cohomological degree zero. The first parenthesis is interpreted as the stress energy tensor of the vertex algebra $\mathcal{V}_{\beta\gamma}^{\otimes 13}$ and the second term is the stress energy tensor of $\mathcal{V}_{bc}$.

We have not yet described the differential on the graded vertex algebra. The BRST differential is defined to be the vertex algebra derivation obtained by taking the following residue

$$Q^{BRST} = \oint c(z)T_{\text{string}}(z). \tag{11}$$

By construction this operator satisfies $(Q^{BRST})^2 = 0$.

Definition 5.1. The *string vertex algebra* is the dg vertex algebra

$$\mathcal{V}_{\text{string}} = \left(\mathcal{V}_{\beta\gamma}^{\otimes 13} \otimes \mathcal{V}_{bc},\ Q^{BRST}\right).$$

There is another grading on $\mathcal{V}_{\text{string}}$ coming from the eigenvalues of the vertex algebra derivation c_0 called the *conformal dimension*. In particular, this determines a filtration and we can consider the associated graded Gr $\mathcal{V}_{\text{string}}$. The conformal weight grading preserves the cohomological grading so that this object still has the structure of a dg vertex algebra.

Note that the cohomology of a dg vertex algebra is an ordinary (graded) vertex algebra. The cohomology of the string vertex algebra is called the *BRST cohomology* of the bosonic string. In the remainder of this section we will show how we recover the string vertex algebra from the quantization of the holomorphic bosonic string.

5.2. **Some context.** In the BV formalism one constructs a cochain complex of observables, for both the classical and the quantized theory, if it exists. The cochain complexes are local on the source manifold of a theory: on each open set U in that manifold Σ, one can pick out the observables with support in U by asking for the observables that vanish on fields with support outside U. It is the central result of [**?**, **?**] that the observables also satisfy a local-to-global property, akin to the sheaf gluing axiom, and hence form a *factorization algebra* on Σ.

We will not need that general notion here. Instead, we will use vertex algebras. Theorem 5.2.3.1 of [**?**] explains how a factorization algebra F on $\Sigma = \mathbb{C}$ yields a vertex algebra $\mathbb{V}\text{ert}(F)$, under natural hypotheses on F. It assures us that the observables of a chiral CFT determine a vertex algebra.

In particular, Section 5.3 of [**?**] examines the free $\beta\gamma$ system in great detail. Its main result is that the well-known $\beta\gamma$ vertex algebra is recovered by the two-step process of BV quantization, which yields a factorization algebra, and then the extraction of a vertex algebra.

The exact same arguments apply to the free $bc\beta\gamma$ system, where the $\beta\gamma$ sector is valued in a vector space V, as we introduced in Section **??** Let $\text{Obs}^{\mathrm{q}}_{free}$ denote the observables of this theory on $\Sigma = \mathbb{C}$. As a quantization of a free field theory, it is a factorization algebra valued in the category of $\mathbb{C}[\hbar]$-modules. In particular, the associated vertex algebra $\mathbb{V}\mathrm{ert}(\mathrm{Obs}^{\mathrm{q}}_{free})$ is also valued in $\mathbb{C}[\hbar]$-modules.

Proposition 5.2. *Let $n = \dim_{\mathbb{C}}(V)$. Then, there is an isomorphism of vertex algebras*

$$\mathbb{V}\mathrm{ert}(\mathrm{Obs}^{\mathrm{q}}_{free})_{\hbar=2\pi i} \cong \mathcal{V}_{bc} \otimes \mathcal{V}_{\beta\gamma}^{\otimes n}$$

where on the left-hand side we have set $\hbar = 2\pi i$.

5.3. **The case of the string.** The holomorphic bosonic string is a chiral CFT and so the machinery of [**?**] applies to it. One can extract a vertex algebra directly by this method.

But there is a slicker approach, using Li's work [**?**], which studies chiral deformations of *free* chiral BV theories such as the $bc\beta\gamma$ system. Recall that a deformation of a classical field theory is given by a local functional. We have seen that this is essentially the data of a Lagrangian density, which is a density valued multilinear functional that depends on (arbitrarily high order) jets of the fields. In other words, for a field φ, a Lagrangian density is of the form

$$\mathcal{L}(\varphi) = \sum (D_{k_1}\varphi) \cdots (D_{k_m}\varphi) \cdot \mathrm{vol}_{\Sigma}$$

for $C^\infty(\Sigma)$-valued differential operators D_{k_i}. By a *chiral* Lagrangian density we mean a Lagrangian for which the differential operators D_{k_i} are all holomorphic. For instance, on $\Sigma = \mathbb{C}$, we require D_{k_i} to be a sum of operators of the form $f(z)\partial_z^n$ where $f(z)$ is a holomorphic function. On $\Sigma = \mathbb{C}$ we will also require the chiral Lagrangian to be translation invariant. This means that all differential operators D_{k_i} are of the form ∂_z^n. Thus, a *translation-invariant chiral deformation* is a local functional of the form

$$I(\varphi) = \sum \int (\partial_z^{k_1}\varphi) \cdots (\partial^{k_m}\varphi) \mathrm{d}^2 z.$$

One of Li's main results is that for a free chiral BV theory with action S_{free} and associated vertex algebra $\mathcal{V}_{\mathrm{free}}$, one has the following:

- For any chiral interaction I, the action $S_{\mathrm{free}} + I$ needs no counterterms, and yields a renormalized interaction $\{S[L]\}$.
- If the renormalized action $\{S[L]\}$ satisfies the quantum master equation, then it determines a vertex algebra derivation D_I of $\mathcal{V}_{\mathrm{free}}$ of the form

  $$D_I = \oint I^q \mathrm{d}z$$

  of cohomological degree one, where $I^q = \lim_{L\to 0} I[L]$.
- The dg vertex algebra $\mathcal{V}_I$ for such an action $\{I[L]\}$ has the same underlying graded vertex algebra $\mathcal{V}_{\mathrm{free}}$ but it is equipped with the differential $\oint I^q \mathrm{d}z$.

Remark 5.3. The fact that I satisfies the quantum master equation implies that one has a map, for each open set $U \subset \mathbb{C}$, from the free factorization algebra evaluated on U to the factorization algebra of the deformed theory evaluated on U:

$$e^{I/\hbar} : \mathrm{Obs}^q_{free}(U) \to \mathrm{Obs}^q_I(U).$$

This map sends an observable $O \in \mathrm{Obs}^q_{free}(U)$ to $O \cdot e^{I/\hbar}$. In fact, this map is an isomorphism with inverse given by $O \mapsto O \cdot e^{-I/\hbar}$. So, open by open, the factorization algebra assigns the same vector space for the deformed theory. This isomorphism is *not* compatible with the factorization product, so we do get a different factorization algebra in the presence of a deformation.

The holomorphic bosonic string with target $V = \mathbb{C}^{13}$ provides a concrete example of this situation. The free theory is the $bc\beta\gamma$ system, and we have seen that the renormalized action satisfies the QME. Hence we obtain the following.

Proposition 5.4. *Let $\mathrm{Obs}^{\mathrm{q}}_{\mathrm{string}}$ be the factorization algebra on $\Sigma = \mathbb{C}$ of the holomorphic bosonic string with target $V = \mathbb{C}^{13}$. And let $\mathbb{V}\mathrm{ert}(\mathrm{Obs}^q_{\mathrm{string}})$ be the dg vertex algebra (defined over $\mathbb{C}[\hbar]$) obtained via Li's construction. There is an isomorphism of vertex algebras $\mathcal{V}_{\mathrm{string}} \cong \mathbb{V}\mathrm{ert}(\mathrm{Obs}^q_{\mathrm{string}})_{\hbar=2\pi i}$. Moreover, this vertex algebra is isomorphic to the chiral sector of the bosonic string as in Section 5.1.*

The factorization algebra $\mathrm{Obs}^{\mathrm{q}}_{\mathrm{string}}$ is a quantization of the factorization algebra $\mathrm{Obs}^{\mathrm{cl}}_{\mathrm{string}}$ of classical observables of the free $bc\beta\gamma$ system. We have noted that the classical observables of any theory has the structure of a P_0 factorization algebra, and the $\hbar \to 0$ limit of $\mathrm{Obs}^{\mathrm{q}}_{\mathrm{string}}$ is isomorphic to $\mathrm{Obs}^{\mathrm{cl}}_{\mathrm{string}}$ as P_0 factorization algebras. By definition, the classical observables are simply functions on the solutions to the classical equations of motion. The P_0 structure is induced from the symplectic pairing of degree (-1) on the fields. The classical factorization algebra still has enough structure to determine a vertex algebra $\mathbb{V}\mathrm{ert}(\mathrm{Obs}^{\mathrm{cl}}_{\mathrm{string}})$. Moreover, the P_0 bracket on the classical observables determines the structure of a *Poisson vertex algebra* on $\mathbb{V}\mathrm{ert}(\mathrm{Obs}^{\mathrm{cl}}_{\mathrm{string}})$.

Corollary 5.5. *In the classical limit, there is an isomorphism of Poisson vertex algebras $\mathbb{V}\mathrm{ert}(\mathrm{Obs}^{\mathrm{cl}}_{\mathrm{string}}) \cong \mathrm{Gr}\, \mathcal{V}_{\mathrm{string}}$.*

Proof of Proposition 5.4. By Proposition 5.2 we know that the vertex algebra of the associated free theory is identified with the $bc\beta\gamma$ vertex algebra. The thing we need to check is that the differential induced from the quantization of the holomorphic string agrees with the differential of the string vertex algebra. In fact, we observe that the induced differential $\oint I dz$ from the classical interaction of the holomorphic bosonic string agrees with the BRST charge in Equation (11). To see that this persists at the quantum level we need to check that there are no quantum corrections. Indeed, this follows from the fact that the quantum master equation holds identically (as opposed to holding up to an exact term in the deformation complex) provided $\dim_{\mathbb{C}} V = 13$. □

5.4. **Relation to semi-infinite cohomology.** BW: O to take a crack at this

5.5. **The E_2 algebra and descent.** We continue to consider the theory on the Riemann surface $\Sigma = \mathbb{C}$. In this section we show how to produce, from the point of view of factorization algebras, the structure of a Gerstenhaber algebra on the BRST cohomology of the bosonic string. A Gerstenhaber algebra is equivalent to an algebra over the homology of the framed little 2-disk operad. It is a well-known result of Lurie [**?**] that a *locally constant* factorization algebra on $\mathbb{R}^n$ is equivalent to an algebra over the little n-disks operad, or an E_n-algebra. We will show that the cohomology of the factorization algebra $\mathrm{Obs}^{\mathrm{q}}_{\mathrm{string}}$ on $\mathbb{C} \cong \mathbb{R}^2$ has the structure of a Gerstenhaber algebra, which implies that $\mathrm{Obs}^{\mathrm{q}}_{\mathrm{string}}$ is equivalent to an E_2-algebra.

Another occurrence of E_n-algebras is as the observables of topological field theories in (real) dimension n. At this level, this implies that the theory of the holomorphic bosonic string is equivalent to a topological field theory. In fact, we can see directly that the factorization algebra outputted by our construction is topological.

Proposition 5.6. *The factorization algebra* $\mathrm{Obs}^{\mathrm{q}}_{\mathrm{string}}$ *is locally constant.*

Proof. We need to show that for any inclusion of open disks $D \hookrightarrow D'$ that natural map

$$\mathrm{Obs}^{\mathrm{q}}_{\mathrm{string}}(U) \to \mathrm{Obs}^{\mathrm{q}}_{\mathrm{string}}(U')$$

is a quasi-isomorphism. We first show that this is true when we replace the quantum observables by the classical observables. We have already mentioned that the classical observables are functions on solutions to the classical equations of motion. It is convenient to phrase this in terms of Lie algebra cohomology. The classical interaction induces the structure of a sheaf of dg Lie algebras on the shift of the space of fields shifted up by one. This sheaf of dg Lie algebras is of the form

$$\Omega^{0,*}(\Sigma; \mathcal{T}_\Sigma) \ltimes \left(\Omega^{0,*}(\Sigma; V)[-1] \oplus \Omega^{1,*}(\Sigma; V^*)[-1] \oplus \Omega^{1,*}(\Sigma; \mathcal{T}^*_\Sigma)[-2]\right).$$

We view this as a square zero extension of the dg Lie algebra $\Omega^{0,*}(\Sigma; \mathcal{T}_\Sigma)$ by the dg module inside the parentheses where the module structure is determined by the Lie derivative. For simplicity we denote $\mathcal{L} = \Omega^{0,*}(\Sigma; \mathcal{T}_\Sigma)$ and $\mathcal{M}$ the module inside the parentheses. In this language, the space of classical observables supported on an open set $U \subset \Sigma$ is the following Chevalley-Eilenberg cochain complex

$$\mathrm{Obs}^{\mathrm{cl}}_{\mathrm{string}}(U) = \mathrm{C}^*_{\mathrm{Lie}}\left(\mathcal{L}(U);\ \mathrm{Sym}(\mathcal{M}(U)^*[-1])\right)$$

where $\mathcal{M}(U)^*$ denotes the continuous dual of $\mathcal{M}(U)$. We consider the case that $U = D$ a disk centered at zero. By the $\bar{\partial}$-Poincaré lemma there is a quasi-isomorphism of dg Lie algebras $\mathcal{T}_{hol}(D) \hookrightarrow \mathcal{L}(D)$ where $\mathcal{T}_{\mathrm{hol}}(D)$ is the vector space of holomorphic vector fields on D. Thus, the classical observables on D are quasi-isomorphic to $\mathrm{C}^*_{\mathrm{Lie}}\left(\mathcal{T}_{hol}(U);\ \mathrm{Sym}(\mathcal{M}(U)^*[-1])\right)$. Now, consider the composition of Lie algebras

$$\mathrm{W}_1^{\mathrm{poly}} \hookrightarrow \mathcal{T}_{hol}(D) \to \mathrm{W}_1$$

where $\mathrm{W}_1^{\mathrm{poly}}$ are the holomorphic vector fields with polynomial coefficients, and W_1 is the Lie algebra of formal vector fields. Moreover, via the inclusion $D \hookrightarrow D'$ we have an induced commutative diagram

$$\begin{tikzcd}
\mathrm{C}^*_{\mathrm{Lie}}\left(\mathrm{W}_1^{\mathrm{poly}}; \mathrm{res}\,\mathrm{Sym}(\mathcal{M}(D)^*[-1])\right) \arrow[d] & \mathrm{C}^*_{\mathrm{Lie}}\left(\mathcal{T}_{\mathrm{hol}}(D); \mathrm{Sym}(\mathcal{M}(D)^*[-1])\right) \arrow[l] \arrow[d] & \mathrm{C}^*_{\mathrm{Lie}}\left(\mathrm{W}_1; \mathrm{Sym}(j_0^\infty \mathcal{M}(D)^*[-1])\right) \arrow[l] \arrow[d] \\
\mathrm{C}^*_{\mathrm{Lie}}\left(\mathrm{W}_1^{\mathrm{poly}}; \mathrm{res}\,\mathrm{Sym}(\mathcal{M}(D')^*[-1])\right) & \mathrm{C}^*_{\mathrm{Lie}}\left(\mathcal{T}_{\mathrm{hol}}(D'); \mathrm{Sym}(\mathcal{M}(D')^*[-1])\right) \arrow[l] & \mathrm{C}^*_{\mathrm{Lie}}\left(\mathrm{W}_1; \mathrm{Sym}(j_0^\infty \mathcal{M}(D')^*[-1])\right) \arrow[l]
\end{tikzcd}$$

Here, res denotes the restriction of the module along the inclusion $\mathrm{W}_1^{\mathrm{poly}} \hookrightarrow \mathcal{T}_{\mathrm{hol}}(D)$ and j_0^∞ denotes the infinite jet of the sheaf $\mathcal{M}$ at zero. For instance, if $\Omega^1_{\mathrm{hol}}(D)$ is the $\mathcal{T}_{\mathrm{hol}}(D)$-module of holomorphic one-forms then $j_0^\infty \Omega^1_{\mathrm{hol}}$ is the W_1-module of formal one-forms (one-forms with coefficients in formal power series).

By Lemma 3.5 (and an analogous result for polynomial vector fields) we know that $\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_1; M)$ and $\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_1^{\mathrm{poly}}; M)$ are quasi-isomorphic to the conformal dimension zero subcomplex, that is, the constants. The conformal zero subcomplex does not depend on the size of the disk, so we conclude that vertical arrows on the outside of the diagram above are quasi-isomorphisms. It follows that the middle vertical arrow is as well, thus showing that $\mathrm{Obs}^{\mathrm{cl}}_{\mathrm{string}}(D) \to \mathrm{Obs}^{\mathrm{cl}}_{\mathrm{string}}(D')$ is a quasi-isomorphism, as desired.

To finish the proof we consider the spectral sequence induced from the filtration of the module $\mathrm{Sym}\ \mathcal{M}(D)$ by symmetric polynomial degree. The E_1 page of this spectral sequence is the classical observables above and it converges to the cohomology of the quantum observables. The map of factorization algebras induced by the inclusion $D \hookrightarrow D'$ preserves this filtration, hence by the convergence of this spectral sequence we conclude that $\mathrm{Obs}^{\mathrm{q}}_{\mathrm{string}}(D) \to \mathrm{Obs}^{\mathrm{q}}_{\mathrm{string}}(D')$ is also a quasi-isomorphism. □

5.5.1. *Gerstenhaber bracket via descent.* There is another link to topological field theories as defined in [?] that we recall here. Consider the action of the differential operators $\frac{\mathrm{d}}{\mathrm{d}z}$ and $\frac{\mathrm{d}}{\mathrm{d}\bar{z}}$ on the Dolbeault complex $\Omega^{0,*}(\mathbb{C})$. This extends to an action of the differential operators to the holomorphic bosonic string, and hence to the observables as well. By Noether's theorem the symmetry of the theory determined by these operators define observables: these are simply the zz and $\overline{zz}$ components of the stress energy tensor $T_{zz}, T_{\overline{zz}}$. We will see that in the case of the string the stress energy tensors are cohomologically trivial. This is the general definition of a topological theory given in [?].

For each open $U \subset \mathbb{C}$ we can have the operators

$$\frac{\mathrm{d}}{\mathrm{d}z}, \frac{\mathrm{d}}{\mathrm{d}\bar{z}} : \mathrm{Obs}^{\mathrm{q}}_{\mathrm{string}}(U) \to \mathrm{Obs}^{\mathrm{q}}_{\mathrm{string}}(U).$$

In fact, these operators define *derivations* of the factorization algebra, in the sense that they are compatible with the factorization product. Note that these operators preserve the cohomological degree.

Consider the operator acting on Dolbeault forms

$$\bar{\eta} = \frac{\partial}{\partial(\mathrm{d}\bar{z})}.$$

This extends to a derivation of degree -1 on the factorization algebra $\mathrm{Obs}^{\mathrm{q}}_{\mathrm{string}}$. Moreover, we have the following relation

$$[\bar{\partial} + \hbar\Delta + \{I^{\mathrm{q}}, -\}, \bar{\eta}] = \frac{\mathrm{d}}{\mathrm{d}\bar{z}} \tag{12}$$

as endomorphisms of the factorization algebra. Indeed, one immediately obvserves $[\bar{\partial}, \bar{\eta}] = \frac{\mathrm{d}}{\mathrm{d}\bar{z}}$. Moreover, since I^{q} is a chiral deformation one has $\bar{\eta} \cdot I^{\mathrm{q}} = 0$. Finally, since the pairing defining the (-1) shifted symplectic structure is holomorphic we see that $\bar{\eta}$ also commutes with the BV Laplacian $[\bar{\eta}, \Delta] = 0$. This means that the operator $\frac{\mathrm{d}}{\mathrm{d}\bar{z}}$ acts cohomologically trivial on $\mathrm{Obs}^{\mathrm{q}}_{\mathrm{string}}$.

Recall that the b-fields of the bosonic string are concentrated in degree $+1$ and $+2$. A field in degree $+1$ is of the form $f(z,\bar{z})\mathrm{d}z^{\otimes 2} \in \Omega^{1,0}(\Sigma, T_\Sigma^{1,0*})$. Define the following linear observable on $\mathbb{C}$,

$$b_{-1} : f(z,\bar{z})\mathrm{d}z^{\otimes 2} \mapsto f(0),$$

which vanishes on all fields besides the degree $+1$ component of the b-fields. This observable is a closed element of degree -1 in $\mathrm{Obs}^{\mathrm{q}}_{\mathrm{string}}$. Given any other observable $O \in \mathrm{Obs}^{\mathrm{q}}_{\mathrm{string}}$ we can define the observable $b_{-1} \cdot O$ using the symmetric product. Denote the derivation $\eta : O \mapsto b_{-1} \cdot O$

Lemma 5.7. *The derivation η is degree -1 and satisfies*

$$[\bar{\partial} + \hbar\Delta + \{I^{\mathrm{q}}, -\}, \eta] = \frac{\mathrm{d}}{\mathrm{d}z} \tag{13}$$

Proof. Immediately we see that both $\bar{\partial}$ and Δ commute with η. Thus, it suffices to show that $[\{I^{\mathrm{q}}, -\}, \eta] = \frac{\mathrm{d}}{\mathrm{d}z}$. Consider, for example the observable $O_{\gamma,0} : \gamma \mapsto \gamma(0)$ which we can write as the following residue $O_{\gamma,0} = \oint \frac{\gamma(z)}{z}$. Then, we see that the only part of the interaction that contributes is $\int \langle \beta, c \cdot \gamma \rangle$ and

$$[\{\int \langle \beta, c \cdot \gamma \rangle, -\}, \eta] O_{\gamma,0} = \{\int \langle \beta, \partial_z \gamma \rangle, O_{\gamma,0}\} = \oint \frac{\partial_z \gamma}{z} \mathrm{d}z = \frac{\mathrm{d}}{\mathrm{d}z} O_{\gamma,0}.$$

The calculation is similar for a general linear observable. To get the result for an arbitrary observable we use the fact that the bracket is a derivation. BW: I tried to come up with a more conceptual proof by interpreting $\{I, -\}$ as the Chevalley-Eilenberg differential for vector fields, but failed. $\square$

This shows that the derivation $\frac{\mathrm{d}}{\mathrm{d}z}$ acts trivial up to homotopy on the factorization algebra. Given any observable O, the relations (12) and (13) allow us to define the following differential form valued observable $\widetilde{O}$ as follows. The zero form part $\widetilde{O}^0$ is just the observable O. The one form part $\widetilde{O}^1$ is equal to the linear combination

$$\mathrm{d}z\ (\eta \cdot O) + \mathrm{d}\bar{z}\ (\bar{\eta} \cdot O).$$

Similarly, the two form part is $\widetilde{O}^2 = \mathrm{d}z\mathrm{d}\bar{z}\eta\bar{\eta} \cdot O$. An easy calculation shows that if O is closed for the quantum differential $\mathrm{d}^{\mathrm{q}} = \bar{\partial} + \hbar\Delta + \{I^{\mathrm{q}}, -\}$ then $\widetilde{O} = \widetilde{O}^0 + \widetilde{O}^1 + \widetilde{O}^2$ satisfies $(\mathrm{d}_{dR} + \mathrm{d}^{\mathrm{q}})\widetilde{O} = 0$. This implies that given any closed observable O and submanifold $C \subset \Sigma$ we can define the closed observable

$$\int_C \widetilde{O} \in \mathrm{Obs}^{\mathrm{q}}(C).$$

If O has cohomological degree k and C is of dimension l then $\int_C \widetilde{O}$ has degree $k - l$.

Remark 5.8. A priori the factorization algebra is only well-defined on open sets $U \subset \Sigma$. One can define the value on a closed submanifold $C \subset \Sigma$ by taking the direct limit of what the factorization algebra assigns to open submanifolds containing C.

We are now able to define the bracket. Suppose O, O' are closed observables on a disk D, D'. For simplicity, we assume that $D \subsetneq D'$. Then, if $C = \partial D'$ we can define $\int_C \widetilde{O}'$ as above. Finally, for D''

another disk containing D and C consider the factorization product $\star : \mathrm{Obs}^{\mathrm{q}}(D) \otimes \mathrm{Obs}^{\mathrm{q}}(C) \to \mathrm{Obs}^{\mathrm{q}}(D'')$ and define

$$\{O, O'\}_{\mathrm{Ger}} := O \star \int_C \widetilde{O}' \in \mathrm{Obs}^{\mathrm{q}}(D'').$$

We note that if $\deg(O) = k$ and $\deg(O') = k'$ then $\deg(\{O, O'\}_{\mathrm{Ger}}) = k + k' - 1$.

The last piece of structure we need to define is that of a commutative product. In this context, this is simply given by the factorization product of two disjoint disks including into a larger disk:

$$\cdot : \mathrm{Obs}^{\mathrm{q}}(D) \otimes \mathrm{Obs}^{\mathrm{q}}(D') \to \mathrm{Obs}^{\mathrm{q}}(D'').$$

Proposition 5.9. *The bracket $\{-,-\}_{\mathrm{Ger}}$ together with the product $\cdot$ determine the structure of a Gerstenaber algebra on the cohomology of the observables on an open disk $H^*\mathrm{Obs}^{\mathrm{q}}_{\mathrm{string}}(D)$.*

BW: Should we prove this or just argue that the bracket and product above agree with what Lian-Zuckerman get.

6. The holomorphic string on closed Riemann surfaces

Thus far we have discussed the local behavior of the holomorphic string, such as its quantization on a disk and the concomitant vertex algebra. Now we turn to its global behavior, particularly the observables on a closed Riemann surface, and the relationship with certain natural holomorphic vector bundles on the moduli space of Riemann surfaces. This local-to-global transition is where the BV/factorization package really shines. On the one hand, the theory of factorization algebras provides a conceptual characterization of the local-to-global relationship, much like the understanding of sheaf cohomology as the derived functor of global sections. On the other hand, the examples from BV quantization provide computable, convenient models for the global sections, much as the de Rham or Dolbeault complexes do for the cohomology of sheaves that arise naturally in differential or complex geometry.

As we will explain, the answers we recover for the holomorphic string can be related quite cleanly to natural determinant lines on the moduli of Riemann surfaces, hence providing a bridge from the Feynman diagrammatic anomaly computations to the index-theoretic computations.

6.1. **The free case.** Before jumping to the holomorphic string, we will work out the global observables in the simpler case of the $bc\beta\gamma$ system, introduced in Remark 2.1. The global *classical* observables on a Riemann surface Σ are given by the symmetric algebra on the continuous linear dual to the fields,

$$\mathrm{Sym}\left(\Omega^{0,*}(\Sigma, V)^\vee \oplus \Omega^{1,*}(\Sigma, V^\vee)^\vee \oplus \Omega^{0,*}(\Sigma, T[1])^\vee \oplus \Omega^{1,*}(\Sigma, T^*_\Sigma[-2])^\vee\right),$$

with the differential $\overline{\partial}$ extended as a derivation. Hence the cohomology is

$$\mathrm{Sym}\left(H^*(\Sigma, V)^\vee \oplus H^*(\Sigma, \omega \otimes V^\vee)^\vee \oplus H^*(\Sigma, T[1])^\vee \oplus H^*(\Sigma, \omega^{\otimes 2}[-2])^\vee\right),$$

where ω denotes the canonical bundle. Although this expression might look complicated, it can be readily unpacked in the setting of algebraic geometry, particularly when Σ is closed. In that case, this graded commutative algebra is a symmetric algebra on a finite-dimensional graded vector space, which encodes the derived tangent space of the moduli of Riemann surfaces at Σ and of holomorphic functions to V.

As this theory is free, it admits a canonical BV quantization. Denote by Obs^{q}_{free} be the corresponding factorization algebra. One can compute its global sections on Σ by using a spectral sequence whose first page is the global classical observables. As mentioned in OG: cite the relevant lesson about determinantsBW: not sure where exactly you wanted to cite, sorry!, when Σ is closed, the cohomology is the determinant of the cohomology of the fields:

$$H^*\left(\mathrm{Obs}^{q}_{free}(\Sigma)\right) \cong \det\left(H^*(\Sigma; \mathcal{O}_\Sigma)\right)^{\otimes \dim(V)} \otimes \det\left(H^*(\Sigma; T^{1,0}_\Sigma)\right)^{-1}[d(\Sigma)]$$

where

$$d(\Sigma) = \dim(V)\left(\dim H^0(\Sigma; \mathcal{O}_\Sigma) + \dim H^1(\Sigma; \mathcal{O}_\Sigma)\right) + \dim(H^0(\Sigma; T^{1,0}_\Sigma)) - \dim(H^1(\Sigma; T^{1,0}_\Sigma)).$$

(See Proposition 8.1.4.1 in **[?]** for the proof of this fact.)

Remark 6.1. The shift $d(\Sigma)$ here likely looks funny. In this case at least, the meaning can be unpacked pretty straightforwardly. The BV complex for an ordinary finite-dimensional vector space is equivalent to the de Rham complex shifted down by the dimension of the vector space, so that the top forms are in degree 0. (Abstracting this situation is one way to "invent" the BV formalism.) For the σ-model, the global solutions to the equations of motion are $H^0(\Sigma, \mathcal{O}) \otimes V$ for the γ fields and $H^0(\Sigma, \omega) \otimes V^\vee$ for the β fields. For Σ closed, these are finite-dimensional, and thus we get the shift

$$\dim(V)\left(\dim H^0(\Sigma; \mathcal{O}_\Sigma) + \dim H^1(\Sigma; \mathcal{O}_\Sigma)\right).$$

For the ghost system (the bc fields), the BV complex recovers the Euler characteristic

$$\dim(H^0(\Sigma; T^{1,0}_\Sigma)) - \dim(H^1(\Sigma; T^{1,0}_\Sigma))$$

as it encodes the de Rham complex on the formal quotient stack $B\mathfrak{g} = */\mathfrak{g}$ for the Lie algebra of symmetries $\mathfrak{g}$.

The computation here works for any Riemann surface Σ and, indeed, for any family of Riemann surfaces. Hence it implies that the global observables of the free $bc\beta\gamma$ system determine a determinant line bundle on the moduli $\mathcal{M}$ of Riemann surfaces.

We can work out the first Chern class of this determinant line bundle using the Grothendieck-Riemann-Roch theorem as follows. Consider the universal Riemann surface $\pi\colon C \to \mathcal{M}$ over the moduli space, and consider the bundles $\mathcal{O}_C \otimes V$ and $\mathcal{T}_\pi = \mathcal{T}_{C/\mathcal{M}}$, which one can view the universal γ fields and c fields, respectively. The first Chern class of the derived pushforward $R\pi_*(\mathcal{O}_C \otimes V)$ is given by the first Chern class of $\det(H^*(\mathcal{O}_C \otimes V)) \cong \det(\mathcal{O}_C)^{\otimes \dim V}$, since the first Chern class of a vector bundle is the first Chern class of its determinant bundle. The Grothendieck-Riemann-Roch theorem states that for a complex of coherent sheafs $\mathcal{F} = \mathcal{F}^*$ on C, the Chern character $\mathrm{ch}(R\pi_*\mathcal{F})$ of its derived pushforward $R\pi_*\mathcal{F}$ is given by

$$\begin{aligned}\pi_*(\mathrm{ch}(\mathcal{F})\mathrm{Td}(T_\pi)) &= \pi_*\left(\left(\sum_i(-1)^i\mathrm{ch}(\mathcal{F}^i)\right)\mathrm{Td}(T_\pi)\right)\\ &= \pi_*\left(\left(\sum_i(-1)^i(\mathrm{rk}(\mathcal{F}^i)+c_1(\mathcal{F}^i)+\frac{1}{2}(c_1(\mathcal{F}^i)^2)+\cdots)\right)(1+\frac{1}{2}c_1(T_\pi)+\frac{1}{12}c_1(T_\pi)^2)\right)\end{aligned}$$

where T_π denotes the relative tangent bundle along π, which is here just the tangent line bundle of a Riemann surface. The first Chern class is the component of cohomological degree 2. For instance, when $\mathcal{F}=\mathcal{F}^0$ is concentrated in degree zero, the above simplifies to:

$$\frac{1}{12}\mathrm{rk}(\mathcal{F})c_1(T_\pi)^2+\frac{1}{2}c_1(\mathcal{F})c_1(T_\pi)+\frac{1}{2}c_1(\mathcal{F})^2.$$

When $\mathcal{F}=T_\pi^{\otimes n}[1]$, the expression for the first Chern class is $-\frac{1+6n+6n^2}{12}c_1(T_\pi)^2$. When $\mathcal{F}=\mathcal{O}\otimes V$ we simply get $\dim(V)$.

Hence, when $\mathcal{F}=T[1]\oplus\mathcal{O}\otimes V$, for the determinant line of global observables $H^*\left(\mathrm{Obs}^{\mathrm{q}}_{free}(C)\right)$ as a bundle over C we obtain

$$c_1\left(H^*\left(\mathrm{Obs}^{\mathrm{q}}_{free}(\Sigma)\right)\right)=\frac{1}{12}(\dim(V)-13)c_1(T_\pi)^2.$$

It is worthwhile to point out that the above argument based on GRR for identifying the first Chern class of this determinant line bundle resonates with our computation of the anomaly of the bosonic string on the disk. Indeed, this is a manifestation of "Virasoro uniformization." Also, notice that the above calculation assumed that there was no deformation, so that we were working with a free theory. However, deforming the action from free $bc\beta\gamma$ system to holomorphic bosonic string doesn't affect the line bundles, since those are continuous parameters and Chern classes are discrete.

6.2. **The anomaly and moduli of quantizations on an arbitrary Riemann surface.** We have already seen that the holomorphic string *on a disk* admits a BV quantization if and only if the target is a complex vector space of dimension 13. Here we will explain why this anomaly calculation is actually enough to show the existence of a quantization on an *arbitrary* Riemann surface. An argument using GRR was given in the above section. In this section we give a proof using only the perspective of BV quantization. One can view this as giving a proof of GRR using Feynman diagrams (and will be the topic of future work). BW: Does that sound contrived?

Our diagrammatic arguments showed that only wheels with c legs appear in the anomaly, and these arguments did not depend on the choice of Σ. Hence the anomaly will be purely a functional on the c fields. So we restrict ourselves to the piece of the deformation complex only involving such fields. By arguments analogous to those in Section 3, when Σ is the disk, this component is quasi-isomorphic to $\mathrm{C}^*_{\mathrm{Lie,red}}(W_1)[2]$, whose cohomology is $\mathbb{C}$ concentrated in degree 1. More generally, the deformation complex is a sheaf of cochain complexes on Σ, and Proposition 5.3 of **??** shows that this sheaf of complexes is quasi-isomorphic to the constant sheaf $\mathbb{C}_\Sigma[-1]$ concentrated in degree $+1$. A main result of BV quantization **[?]** OG: maybe give precise reference is that anomalies are *local*: the anomaly computed on an open set $U\subset\Sigma$ is equal to the anomaly of the theory on Σ restricted to U. In our case, the anomaly on some Riemann surface Σ must match with the anomaly we have already computed if we take U to be a disk in Σ. This global anomaly is a 1-cocycle for the derived global sections of the shifted constant sheaf, and hence, because of the shift, this cocycle is determined by a constant function on Σ. Thus, it suffices to compute the anomaly on an arbitrary open, in particular, it suffices to compute it on a flat disk. But this is precisely the context in which we computed the anomaly in Section **??**, so we know the anomaly is simply the dimension of the target vector space. Thus, a quantization of the holomorphic string exists on any Riemann surface provided $\dim_{\mathbb{C}}(V) = 13$.

Now we ask how many such quantizations are possible, i.e., what is the moduli of theories. By the calculation in Section 3, we know that, up to BV equivalence, the possible one-loop terms in the quantized action functional are parametrized by

$$H^0(\Sigma) \otimes \Omega^1(V) \oplus H^1(\Sigma) \otimes \Omega^2_{cl}(V).$$

(That is, these vector spaces are the first cohomology group of the relevant deformation complex.) This space of deformations corresponds to continuous parameters we can vary in the action functional OG: such as [now we spell out what they are]. As the isomorphism classes of line bundles form a discrete set, varying these continuous parameters will not change the class of the line bundle of global observables. In conclusion, no matter what one-loop quantization we choose, the cohomology of the global observables will be the same. OG: On the other hand, it does affect the flat connection on this line bundle, aka partition function. Should we mention that here?

OG: It might be worth noting that the second direct summand is invisible except in genus > 0, but they are relevant if you're trying to define a theory that makes sense on all Riemann surfaces.

6.3. **Global observables for the holomorphic string.** Now, let us consider the global observables of the holomorphic string $\mathrm{Obs}^{\mathrm{q}}(\Sigma)$. There is a spectral sequence OG: you have to say what it is! converging to the cohomology of the global observables $H^*\mathrm{Obs}^{\mathrm{q}}(\Sigma)$ with E_2 page given by the cohomology of the global observables of the $bc\beta\gamma$ system which we have already computed:

$$\begin{aligned} E_2 &\cong \det\left(H^*(\Sigma; T_\Sigma[1])\right) \otimes \det\left(H^*(\Sigma; \mathcal{O}_\Sigma)^{\oplus 13}\right) \\ &\cong \det\left(H^1(\Sigma; T_\Sigma)\right) \otimes \det\left(H^0(\Sigma; T_\Sigma)\right)^{-1} \otimes \det\left(H^0(\Sigma; K_\Sigma)\right)^{-13} \end{aligned}$$

where we have used the fact that $H^0(\Sigma; \mathcal{O}) \cong \mathbb{C}$ for any Σ. It is clear that the spectral sequence degenerates at this page. OG: Don't say "it is clear": just say why! It's not much more writing.

OG: We should cite Witten, since that's why you're using this notation. We could also unpack what these lines are in terms of the "central charge" (meaning what power we get of the canonical line on the moduli of Riemann surfaces.

Let Σ_g be a surface of genus g. Then for $g = 1$ the above simplifies to

$$\det\left(H^1(\Sigma_1; T_{\Sigma_1})\right) \otimes \det\left(H^0(\Sigma_1; K)\right)^{-14}.$$

OG: You can simplify further: these are trivial lines! If $g \geq 2$ one has

$$\det\left(H^1(\Sigma_1; T_{\Sigma_1})\right) \otimes \det\left(H^0(\Sigma_1; K)\right)^{-13}.$$

OG: Again, I think you can trivialize further and identify these with a line bundle on the moduli space of Riemann surfaces. Thus the above expressions give the global observables for the holomorphic string for genus $g = 1$ and $g \geq 2$, respectively.

7. The string partition function on an elliptic curve

We will evaluate the partition function of the theory on an elliptic curve. Every $\tau \in \mathbb{H}$, the upper half plane, determines an elliptic curve as a quotient $E_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$. Let $\mathrm{d}^2 z = \mathrm{d}z\mathrm{d}\overline{z}$ be the standard volume form on $\mathbb{C}$, which descends to one on E_τ; we denote it by the same name. Note that in terms of τ, the volume of the elliptic curve is

$$\int_{E_\tau} \mathrm{d}^2 z = \mathrm{Im}\ \tau.$$

This presentation of an elliptic curve allows us to take advantage of constructions we've introduced over $\mathbb{C}$. For instance, we have already discussed the gauge fixing operator $\overline{\partial}^*$ and constructed the heat kernel on $\mathbb{C}$. We can make the same choice of gauge fixing operator on E_τ and hence obtain a heat kernel for the operator $e^{-tD^{E_\tau}}$, where D^{E_τ} is the commutator $[\overline{\partial}, \overline{\partial}^*] = \frac{1}{2}\Delta$ on E_τ. This heat kernel admits a nice description when pulled back from the elliptic curve E_τ to its universal cover $\mathbb{C}$, namely it becomes the sum over lattice points

$$K_t^{E_\tau}(z, w) = \sum_{\lambda \in \mathbb{Z} + \tau\mathbb{Z}} \frac{1}{2\pi t} e^{-|z-w-\lambda|^2/4t} (\mathrm{d}z - \mathrm{d}w)(\mathrm{d}\overline{z} - \mathrm{d}\overline{w}).$$

From $K_t^{E_\tau}$ we obtain the propagator $P_{\ell<L}^{E_\tau} = \overline{\partial}^* \int_\ell^L e^{-tD^{E_\tau}} \mathrm{d}t$ as earlier. This propagator regularizes the operator $\overline{\partial}^{-1}$ on E_τ. Explicitly, it is of the form

$$P_{\ell<L}^{E_\tau}(z, w) = \sum_{\lambda \in \mathbb{Z} + \tau\mathbb{Z}} \int_{t=\ell}^{L} \frac{1}{2\pi t} \left(\frac{\overline{z} - \overline{w} + \overline{\lambda}}{4t} \right) e^{-|z-w-\lambda|/4t} \mathrm{d}t (\mathrm{d}z - \mathrm{d}w).$$

The full propagator of the holomorphic string on E_τ has one component for the paired fields β and γ and another component for the paired fields b and c. Since γ is a section of the trivial vector bundle labeled by the vector space V, the first term in the full propagator is of the form $P_{\ell<L}^{E_\tau} \otimes \frac{1}{2}(\mathrm{id}_V + \mathrm{id}_{V^*})$. Now, since the tangent bundle is trivial on an elliptic curve, we can choose a canonical framing and write the second piece of the propagator describing the pairing between b and c as $P_{\ell<L}^{E_\tau} \otimes \frac{1}{2}(\partial_z \otimes \partial_z^\vee + \partial_z^\vee \otimes \partial_z)$, where $\partial_z, \partial_z^\vee$ denotes the canonical framing of $T_{E_\tau}, T_{E_\tau}^*$ respectively.

Just as in the case of the calculation of the anomaly, the propagator is a sum over wheels that are functionals of the c-fields. Unlike the case of the anomaly, the internal edges of the wheels are all labeled by the propagator. The vertices are labeled by the interaction terms which have two types: a $\beta c \gamma$-type interaction and a bcc-type interaction. Thus, the partition function splits up into a sum of two types of diagrams (**??**):

(A) all input legs labeled by the fields c with each internal edge labeled by the $\beta\gamma$ propagator $P_{0<\infty}^{E_\tau} \otimes \frac{1}{2}(\mathrm{id}_V + \mathrm{id}_{V^*})$, and

(B) all input legs labeled by the fields c with each internal edge labeled by the bc propagator $P^{E_\tau}_{0<\infty} \otimes \frac{1}{2}(\partial_z \otimes \partial_z^\vee + \partial_z^\vee \otimes \partial_z)$.

BW: draw diagrams Let $A_j^{E_\tau}$ (respectively $B_j^{E_\tau}$) be the weight of the graph of type A (respectively B) with $j \geq 1$ incoming legs. Since we are working globally on the elliptic curve it suffices to evaluate the diagrams on the generator $\mathrm{d}\bar{z}\partial_z$ of $H^1(E_\tau; \mathcal{T})$. For simplicity, we denote this generator by $\mathfrak{t} = \mathrm{d}\bar{z}\partial_z$ and we will compute the partition function as a power series expansion in $\mathfrak{t}$:

$$W(P^{E_\tau}_{0<\infty}, I)(\mathfrak{t}) = \sum_{j\geq 1} \mathfrak{t}^j (A_j + B_j).$$

We introduce the following modular quantity that will play a role in the below analysis. Write $q = e^{2\pi i \tau}$. The renormalized second Eistenstein series E_2^{ren} is defined by

$$E_2^{ren}(\tau) = E_2(\tau) - \frac{3}{\pi}\frac{1}{\mathrm{Im}\,\tau}.$$

where $E_2(\tau) = 1 - 24\sum_{n=1}^\infty \frac{nq^n}{1-q^n}$. The quantity $E_2(\tau)$ is *not* modular, but it is holomorphic. Conversely, $E_2^{ren}(\tau)$ *is* modular but not holomorphic.

First we consider the tadpole graphs of each type corresponding to $j = 1$.

Lemma 7.1. *The weight of the type A tadpole diagram is*

$$A_1^{E_\tau} = -(2\pi i)\frac{13}{12} \cdot E_2^{ren}(\tau)$$

when evaluated on the generator $c = \mathrm{d}\bar{z}\partial_z$. BW: WORK OUT FACTOR . Similarly, the weight of the type B tadpole diagram evaluated on the generator is

$$B_1^{E_\tau} = (2\pi i)\frac{1}{12}E_2(\tau).$$

Thus, in sum we have $\epsilon A_1^{E_\tau} + \epsilon B_1^{E_\tau} = -(2\pi i) \cdot E_2(\tau)$.

Proof. The vertex of the tadpole of type A is labeled by the interaction $\int \langle \beta, [\mathrm{d}\bar{z}\partial_z, \gamma\rangle$. The z-derivative of $P^{E_\tau}_{\ell<L}(z,w)$ is given by

$$\sum_{\lambda \in \mathbb{Z}+\tau\mathbb{Z}} \int_{t=\ell}^{L} \frac{1}{2\pi t}\left(\frac{\bar{z} - \overline{w} + \overline{\lambda}}{4t}\right)^2 e^{-|z-w+\lambda|/4t} \mathrm{d}t(\mathrm{d}z - \mathrm{d}w).$$

By Lemma 2.2 of [?] one has the following

$$\lim_{\epsilon\to 0}\lim_{L\to\infty}\lim_{z\to w}\frac{\mathrm{d}}{\mathrm{d}z}P^{E_\tau}_{\ell<L}(z,w) = \frac{1}{12\pi}E_2^{ren}(\tau)$$

Thus, the contribution for the tadpole of type A is given by

$$\dim(V)\frac{1}{12\pi}E_2^{ren}(\tau)\int_{E_\tau}\frac{\mathrm{d}^2 z}{\mathrm{Im}\,\tau} =$$

□

There is a simple relationship between the quasi-modular form $E_2(\tau)$ and the discriminant $\Delta(\tau)$.

$$\frac{1}{2\pi i}\frac{\Delta'(\tau)}{\Delta(\tau)} = E_2(\tau)$$

where the prime denotes the derivative with respect to τ. Since Δ'/Δ is equal to the derivative of $\log(\Delta)$ we see that by Lemma 7.1 we can write the contribution from the tadpole diagrams as

$$\epsilon A_1^{E_\tau} + \epsilon B_1^{E_\tau} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\log \Delta(\tau)\right).$$

Lemma 7.2.

There is then a useful inductive formula to determine the weight of diagrams with two or more vertices.

Lemma 7.3. *One has* $\partial_\tau A_j^{E_\tau} = \frac{1}{j+1} A_{j+1}^{E_\tau}$ *and* $\partial_\tau B_j^{E_\tau} = \frac{1}{j+1} B_{j+1}^{E_\tau}$.

Proof. Let's consider diagrams of type A. By Lemma 7.2 weight of the wheel with j inputs all given by $\mathrm{d}\bar{z}\partial_z$ is equal to the trace over $C^\infty(E_\tau)$ of the operator $\frac{1}{\mathrm{Im}\,\tau}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\mathrm{D}^{-1}$.

To compute this trace we proceed in a similar way as in [?]. Introduce the following smooth functions on $\mathbb{C}$

$$F_{m,n}(z,\bar{z}) = ??.$$

The operator $\frac{1}{\mathrm{Im}\,\tau}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\mathrm{D}^{-1}$ applied to $F_{m,n}(z,\bar{z})$ is given by

$$\frac{1}{\mathrm{Im}\,\tau}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\mathrm{D}^{-1}F_{m,n}(z,\bar{z}) = \frac{1}{\mathrm{Im}\,\tau}\frac{m+n\bar{\tau}}{m+n\tau}F_{m,n}(z,\bar{z}).$$

Thus, the weight of the diagram of type A with j inputs is given by

$$\epsilon^j A_j^{E_\tau} = \sum_{(m,n)\in\mathbb{Z}\times\mathbb{Z}}\left(\frac{1}{\mathrm{Im}\,\tau}\frac{m+n\bar{\tau}}{m+n\tau}\right)^{2k}.$$

□

We conclude the following.

Proposition 7.4. *The partition function of the holomorphic bosonic string is the power series expansion of* $-24\log\eta(t)$ *around the point* $\tau \in \mathbb{H}$.

Proof. We have already seen that the first jet expansion of the partition function, the part that is linear in t, is equal to □

OG: If possible, it would be cool to explain how one can extract the differential equations (=flat connection) governing the partition function from our construction. This might be too hard right now . . .

8. Looking ahead: curved targets

OG: I think here we can mention our CDO work and assert that it's compatible with the discussion here. Then we state the corresponding theorems.

BW: State the quantization condition for curved target.

9. APPENDIX: CALCULATION OF ANOMALY

BW: Any direct proof I know for showing that the anomaly reduces to a one-loop calculation reduces to an argument in Si's B-twist paper or Kevin's Witten genus.

Proof. BW: Here I've already assumed we've reduced the calculation to two wheel diagrams: A) with internal edges labeled by the bc heat kernel and propagator, respectively. B) with internal edges labeled by the $\beta\gamma$ heat kernel and propagator, respecitvely.

We start with the weight of diagram A. Use coordinates z, w to denote the coordinates at each of the vertices. Denote the inputs of the weight by the compactly supported vector fields $f(z)\partial_z$ and $g(w)\mathrm{d}\overline{w}\partial_w$. (Note that the diagram is only nonzero if the total degree of the elements is $+1$.) If $c(z)\partial_z$ is another vector field, the action by $f(z)\partial_z$ is given by

$$[f(z)\partial_z, c(z)\partial_z] = f(z)\partial_z c(z)\partial_z - c(z)\partial_z f(z)\partial_z.$$

Thus, the weight of diagram A can be written as the $\ell \to 0$ limit of

$$\begin{array}{rl} & \int_{z,w} f(z)\partial_z P_\ell^L(z,w) g(w)\partial_w K_\ell(z,w) \\ - & \int_{z,w} \partial_z f(z) P_\ell^L(z,w) g(w)\partial_w K_\ell(z,w) \\ - & \int_{z,w} f(z) P_\ell^L(z,w)\partial_w g(w) K_\ell(z,w) \\ + & \int_{z,w} \partial_z f(z) P_\ell^L(z,w)\partial_w g(w) K_\ell(z,w). \end{array} \tag{14}$$

We label each line above as I,II, III, IV.

Using the form of the propagator in (8) we see that line I is given by

$$\mathrm{I} = \frac{1}{(4\pi)^2}\int_{(z,w)\in\mathbb{C}\times\mathbb{C}}\int_{t=\ell}^{L} f(z)g(w)\frac{1}{\epsilon^2}\frac{1}{t^3}\frac{(\overline{z}-\overline{w})^3}{8}\exp\left(-\frac{1}{4}\left(\frac{1}{\ell}+\frac{1}{t}\right)|z-w|^2\right)$$

(we are omitting volume factors for simplicity). To evaluate this integral we change variables and apply the Wick expansion, Lemma **??** to one of the variables of integration. Indeed, introduce $\xi = z - w$, and notice that the integral simplifies to

$$\mathrm{I} = \frac{1}{(4\pi)^2}\int_{w\in\mathbb{C}}\int_{\xi\in\mathbb{C}}\int_{t=\ell}^{L} f(\xi+w)g(w)\frac{1}{\epsilon^2}\frac{1}{t^3}\frac{\overline{\xi}^3}{8}\exp\exp\left(-\frac{1}{4}\left(\frac{1}{\ell}+\frac{1}{t}\right)|\xi|^2\right).$$

Applying Lemma **??** to the ξ-integral we see that this simplifies to

$$\mathrm{I} = \frac{1}{4\pi}\#\int_{w\in\mathbb{C}}\partial_w^3 f(w)g(w)\int_{t=\ell}^{L}\frac{\ell^2 t}{(\ell+t)^4} + O(\ell)$$

where the terms $O(\ell)$ are of order ℓ so are zero in the limit $\ell \to 0$. On the other hand, we can evaluate the remaining t-integral and see that in the limit $\ell \to 0$ Line I becomes

$$\lim_{\ell\to 0}\mathrm{I} = \frac{1}{4\pi}\#\frac{1}{12}\int_{w\in\mathbb{C}}\partial_w^3 f(w)g(w)\mathrm{d}^2 w.$$

Similarly, we can evaluate line II. After changing coordinates as in I and performing the Wick type integral we obtain

$$\mathrm{II} = \frac{1}{4\pi}\#\int_{w\in\mathbb{C}}\partial_w^3 f(w)g(w)\int_{t=\ell}^{L}\frac{\ell t}{(\ell+t)^3} + O(\ell).$$

Evaluating the remaining t integral and taking $\ell \to 0$ this becomes

$$\lim_{\ell \to 0} \mathrm{II} = \frac{1}{4\pi} \# \frac{3}{8} \int_{w \in \mathbb{C}} \partial_w^3 f(w) g(w) \mathrm{d}^2 w.$$

BW: Is this too much detail, or not enough? I'll wait for your feedback before going further, but the last two integrals are very similar.

$\square$

BW: ***BELOW MAY GO IN APPENDIX***

We will utilize the following version of Wick expansion to evaluate the integrals below.

Lemma 9.1. *Let Φ be a smooth compactly supported function on $\mathbb{C}$ and suppose $\tau > 0$. Then*

$$\int_{\xi \in \mathbb{C}} \Phi(\xi) e^{-\tau |\xi|^2/4} \mathrm{d}^2 \xi = 4\pi \cdot \tau^{-1} \left(\exp\left(\tau^{-1} \frac{\partial}{\partial \xi} \frac{\partial}{\partial \bar{\xi}} \Phi \right)_{\xi = 0} \right).$$

BW: Should we prove this?