

THE HOLOMORPHIC BOSONIC STRING

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1. FROM CLASSICAL TO QUANTUM: ANOMALIES IN THE BV FORMALISM

BW: A rapid overview of classical BV and effective quantizations. Stress how obstructions appear, where they live, and how to compute them.

OG: I think we should articulate here the structural features of our BV package that make the arguments below more conceptual. For instance:

- Linear BV quantization is determinantal, which explains why we'll produce determinant line bundles when we do free $\beta\gamma$ system.
- "Gauging" a theory corresponds to a stacky quotient of the original fields. Hence, obstruction to quantizing a gauged theory corresponds to descending the quantization to the quotient.
- If a classical theory makes sense on a class (=site) of manifolds, then to quantize the whole class, it suffices to check on a generating cover (typically given by disks with geometric structure) but compatibly with all automorphisms. This often explains the appearance of characteristic classes as anomalies.
- Every BV theory produces a factorization algebra. The local structure encodes the OPE algebra (and hence recovers a vertex algebra in chiral CFT situation). On compact manifolds, solutions to EoM typically form finite-dimensional space, and the global observables encode a volume form on this space. (An example is conformal blocks for the free $bc\beta\gamma$ system.)

Please add others as you think of them!

OG: We might also add that we view the BV formalism as the analogue in field theory of derived geometry in geometry. That is, in ordinary algebraic geometry, one first builds geometry and then adds (sheaf) cohomology on top: in ordinary physics, one first builds field theories and then adds (BRST) cohomology on top. But derived geometry (respectively, BV formalism) builds the cohomological aspect into the foundations.

2. THE CLASSICAL HOLOMORPHIC BOSONIC STRING

BW: First define the holomorphic theory we will work with. Then show how it's related to more familiar models for the string, eg the Polyakov action. Level of detail depending on the space we have.

There is a basic format for a string theory, at least in the perturbative approach. One starts with a nonlinear σ -model, whose fields are smooth maps from a Riemann surface to a target manifold X ; in this setting we want the theory to make sense for an arbitrary Riemann surface as the source manifold. In the usual bosonic string theory, this nonlinear σ -model picks out the harmonic maps from a Riemannian 2-manifold to a Riemannian manifold. In our holomorphic setting, the nonlinear σ -model picks out holomorphic maps from a Riemann surface to a complex manifold. One then quotients the space of fields (and solutions to the equations of motion) with respect to reparametrization. OG: This description is a bit opaque. We should find a better one. In the usual bosonic string, one quotients by diffeomorphisms, which can thus change the metric on the source. In our setting, we quotient by diffeomorphisms as well, which can thus change the complex structure on the source.

In this section we begin by describing our theory in the BV formalism. We do not expect the reader to find the action functional immediately clear, so we devote some time to analyzing what it means and how it arises from concrete questions. We then turn to interpreting this classical BV theory using dg Lie algebras and derived geometry (i.e., we identify the moduli space it encodes). Finally, we conclude by sketching how our theory appears as the chiral sector of a degeneration of the usual bosonic string when the target is a complex manifold with a Hermitian metric. Our theory thus does provide insights into the usual bosonic string; moreover, it clarifies why so many aspects of the bosonic string, like the anomalies or B -fields, have holomorphic analogues.

2.1. The theory we study. Let V denote a complex vector space (the target), and let $\langle -, - \rangle_V$ denote the evaluation pairing between V and its linear dual V^\vee . Let Σ denote a Riemann surface (the source). Let $T_\Sigma^{1,0}$ denote the holomorphic tangent bundle on Σ , let $\langle -, - \rangle_T$ denote the evaluation pairing between $T_\Sigma^{1,0}$ and its vector bundle dual $T_\Sigma^{1,0*}$. OG: Correct terminology? These are the key geometric inputs.

In a BV theory, the fields are \mathbb{Z} -graded; we call this the *cohomological grading*. We have four kinds of fields:

field	-1	0	1	2
γ		$\Omega^{0,0}(\Sigma) \otimes V$	$\Omega^{0,1}(\Sigma) \otimes V$	
β		$\Omega^{1,0}(\Sigma) \otimes V^\vee$	$\Omega^{1,1}(\Sigma) \otimes V^\vee$	
c	$\Omega^{0,0}(\Sigma, T_\Sigma^{1,0})$	$\Omega^{0,1}(\Sigma, T_\Sigma^{1,0})$		
b			$\Omega^{1,0}(\Sigma, T_\Sigma^{1,0*})$	$\Omega^{1,1}(\Sigma, T_\Sigma^{1,0*})$

More accurately, we have eight different kinds of fields, but we view each row as constituting a single type since each given row consists of the Dolbeault forms of a holomorphic vector bundle. For instance, the field γ is a $(0, *)$ -form with values in the trivial bundle with fiber V , and the field b is a $(0, *)$ -form with values in the bundle $T^{1,0*} \otimes T^{1,0*}$.

To orient oneself it is helpful to start by examining the fields of cohomological degree zero, since these typically have a manifest physical meaning. For instance, the degree zero γ field is a smooth V -valued function and hence the natural field for the nonlinear σ -model into V . The degree zero c field is a smooth $(0, 1)$ -form with values in vector field “in the holomorphic direction,” and hence encodes an infinitesimal change of complex structure of Σ . They thus constitute the obvious fields to introduce for a holomorphic version of the bosonic string. The fields $a\beta$ and b are less obvious but appear as “partners” (or antifields) whose role is clearest once we have the action functional and hence equations of motion.

The action functional is

$$(1) \quad S(\gamma, \beta, c, b) = \int_\Sigma \langle \beta, \bar{\partial}\gamma \rangle_V + \int_\Sigma \langle b, \bar{\partial}c \rangle_T + \int_\Sigma \langle \beta, [c, \gamma] \rangle_V + \int_\Sigma \langle b, [c, c] \rangle_T.$$

(We discuss below how to think about fields with nonzero cohomological degrees as inputs.) The equations of motion are thus

$$\begin{aligned} 0 &= \bar{\partial}\gamma + [c, \gamma] & 0 &= \bar{\partial}\beta + [c, \beta] \\ 0 &= \bar{\partial}c + \frac{1}{2}[c, c] & 0 &= \bar{\partial}b + [c, b] \end{aligned}$$

Note that these equations are familiar in complex geometry. For instance, the equation purely for c encodes a deformation of complex structure on Σ , concretely by modifying the $\bar{\partial}$ operator to $\bar{\partial} + c$. The other equations then amount to solving for holomorphic sections (of the relevant bundle) with respect to this deformed complex structure. For instance, the equation in γ picks out holomorphic maps from Σ , with the c -deformed complex structure, to V .

OG: Add something about the funny degrees. E.g., does b ever appear?

Remark 2.1. Just looking at this action functional, one might notice that if one drops the last two terms, which are cubic in the fields, then one obtains a free theory

$$S_{free}(\gamma, \beta, c, b) = \int_\Sigma \langle \beta, \bar{\partial}\gamma \rangle_V + \int_\Sigma \langle b, \bar{\partial}c \rangle_T,$$

which is known as the *free $bc\beta\gamma$ system*. Thus, one may view the holomorphic bosonic string as a deformation of this free theory by “turning on” those interaction terms. We will repeatedly try a construction first with this free theory before tackling the string itself, as it often captures

important information with minimal work. For instance, we will see that the anomaly can be identified already at the level of the free theory.

2.2. From the perspective of derived geometry. We would like to explain what this theory is about in more conceptual terms, rather than simply by formulas and equations. Thankfully this theory is amenable to such a description. We will be informal in this section and not specify a particular geometric context (e.g., derived analytic stacks), except when we specialize to the deformation-theoretic situation (i.e., perturbative setting) that is our main arena.

Let \mathcal{M} denote the moduli space of Riemann surfaces, so that a surface Σ determines a point in \mathcal{M} . Let $\text{Maps}_{\bar{\partial}}(\Sigma, V)$ denote the space of holomorphic maps from Σ to V , and hence a bundle $\text{Maps}_{\bar{\partial}}(-, V)$ over \mathcal{M} by varying Σ . For our equations of motion, the γ and c fields of a solution determine a point in this bundle $\text{Maps}_{\bar{\partial}}(-, V)$.

It is better to work with the derived version of these spaces. One important feature of derived geometry is that the appropriate version of a tangent space at a point is, in fact, a cochain complex. In our setting, the tangent complex of $\text{Maps}_{\bar{\partial}}(-, V)$ at a point (c, γ) —so that $\bar{\partial} + c$ is a complex structure and γ is a holomorphic map—is precisely

$$\Omega^{0,*}(\Sigma_c, T^{1,0})[1] \oplus \Omega^{0,*}(\Sigma_c, V).$$

The first summand is the usual answer from the theory of the moduli of surfaces (recall, for example, that the ordinary tangent space is the sheaf cohomology $H^1(\Sigma, \mathcal{T}_{\Sigma})$ of the holomorphic tangent sheaf), and the second is usual elliptic complex encoding holomorphic maps.

Remark 2.2. It is useful to bear in mind that the degree zero cohomology of the tangent complex will recover the “naive” tangent space. The negative degree cohomology detects infinitesimal automorphisms (and automorphisms of automorphisms, etc) of the space. For instance, we see $H^0(\Sigma, \mathcal{T}_{\Sigma})$ appear in degree -1, since a holomorphic vector field is an infinitesimal automorphism of a complex curve. The positive degree cohomology detects **OG: finish**

Note that the underlying graded spaces of this tangent complex are the c and γ fields from the BV theory described above. We emphasize that the tangent complex is only specified up to quasi-isomorphism, but it is compelling that a natural representative is the BV theory produced by the usual physical arguments. This behavior, however, is typical of the relationship between derived geometry and BV theories: when physicists write down a classical BV theory, the underlying free theory is essentially always the tangent complex of a nice derived stack.

The reader has probably noticed that, yet again, we have postponed discussing the β and b fields. From a derived perspective, the full BV theory describes the shifted cotangent bundle $\mathbb{T}^*[-1] \text{Maps}_{\bar{\partial}}(-, V)$. At the level of a tangent complex, the shifted cotangent direction contributes

$$\Omega^{1,*}(\Sigma_c, T^{1,0*})[-1] \oplus \Omega^{1,*}(\Sigma_c, V^{\vee}),$$

whose underlying graded spaces are the β and b fields.

2.3. Relationship to the Polyakov action functional. Let us fix a 2-dimensional Riemannian manifold (Σ, g_0) and a Hermitian vector space (V, h) . In this section we start with a description of the classical Polyakov model for the bosonic string as a classical BV theory. This is the ordinary σ -model of maps $\Sigma \rightarrow V$ coupled to a metric on Σ . More precisely, this is a perturbative model for the Polyakov string, since we only look at deformations of the fixed metric g_0 . We will show that after a reparametrization of the space of fields that it makes sense to take a certain “infinite volume limit” as $h \rightarrow \infty$. In this limit we will show that the Polyakov model splits into a certain holomorphic theory plus its complex conjugate. The holomorphic theory is what we call the *holomorphic bosonic string*.

Remark 2.3. A similar analysis has appeared in [?] where one does not consider deformations of the metric: the infinite volume limit of the bare σ -model of maps $\Sigma \rightarrow V$ splits into the free $\beta\gamma$ system plus its complex conjugate. In the case of the string we find an interacting theory that can be thought of as a deformation of a $\beta\gamma$ system.

OG: Give explanation of what this section will be about: writing down a holomorphic theory that appears as the chiral part of a large volume limit of the usual bosonic string. We should advertise that we start with conventional ways of writing a theory and explain the algorithm by which one extracts a BV action.

OG: After looking at Wikipedia, I feel like Polyakov may not be enough, since it seems just like the action for the sigma model. Part of it may be the way that standard expositions go (and that is followed below). Anyway, I'd like to verify this stuff. Related issue: this is for the *critical* string, and Liouville action seems to be for *noncritical* string.

We recall the most familiar form of the classical Polyakov string and show how to write it down in terms of a classical BV theory. The fields of the Polyakov model consist of a C^∞ function $\varphi : \Sigma \rightarrow V$ and a metric g on Σ . Since we are doing perturbation theory, we assume that g is infinitesimally close to the fixed metric g_0 in the space of all metrics on Σ . There is an identification of the tangent space of the space of all metrics $T_{[g_0]} \text{Met}(\Sigma) \cong \text{Sym}^2(T_\Sigma)$. Thus, we can take the metric g to be of the form $g = g_0 + \alpha$ where $\alpha \in \text{Sym}^2(T_\Sigma)$. OG: Should we include comments about “formal (derived) spaces”?

The naive action functional, before accounting for any gauge symmetries, is of the form

$$S^{\text{naive}}(\varphi, \alpha) = \int_{\Sigma} h(\varphi, \Delta_{g_0 + \alpha} \varphi) \, \text{dvol}_h$$

where $\Delta_g : C^\infty(\Sigma; V) \rightarrow \Omega^2(\Sigma; V)$ is the 2-form valued operator equal to the ordinary Laplacian on functions times metric the top form dvol_g . OG: I think this redefinition of the Laplacian might be more confusing than the savings we recover in typing. At the very least, I'd use a different symbol than the usual Laplace symbol.

The functional S^{naive} is invariant under the group of diffeomorphisms $\text{Diff}(\Sigma)$. Infinitesimally, this means that if X is a vector field on Σ the action is left invariant under the transformation $(\varphi, \alpha) \mapsto (\varphi + X \cdot \varphi, \alpha + L_X \alpha)$, where $L_X(-)$ denotes the Lie derivative on tensors.

There is another symmetry, namely Weyl rescalings of the metric, which reflects the fact that the theory is classically conformal. Infinitesimally, this means that for an arbitrary function $f \in C^\infty(\Sigma)$ the action is left invariant under the transformation $\alpha \mapsto \alpha + f\alpha$. In fact this symmetry is compatible with the symmetry by vector fields in an obvious way: if $f \in C^\infty(\Sigma)$ and X is a vector field on Σ then $L_X(f\alpha) = X(f)\alpha + fL_X\alpha$ for any $\alpha \in \text{Sym}^2(T_\Sigma)$.

The BRST operator for the gauge symmetries can be summarized via the following elliptic complex, that we denote $\mathcal{E}^{\text{Polyakov}}$:

$$\begin{array}{ccccccc}
 \underline{-1} & & \underline{0} & & \underline{1} & & \underline{2} \\
 & & & & & & \\
 & & \Omega^0(\Sigma) \otimes V & \xrightarrow{\Delta_{g_0}} & \Omega^2(\Sigma) \otimes V & & \\
 & & & & & & \\
 \text{Vect}(\Sigma) \oplus C^\infty(\Sigma) & \xrightarrow{d_{g_0}} & \text{Sym}^2(T_\Sigma) & & & & \\
 & & & & & & \\
 & & & & \Omega^2(\Sigma; \text{Sym}^2(T_\Sigma^*)) & \xrightarrow{d_{g_0}} & \Omega^2(\Sigma; T_\Sigma^*) \oplus \Omega^2(\Sigma).
 \end{array}$$

In the definition of a classical BV theory we must prescribe the data of a (-1) -shifted symplectic pairing on the BRST complex together with an interaction which is a local functional on the complex. The pairing can be described as follows. If $\varphi \in \Omega^0(\Sigma; V)$ and $\psi \in \Omega^2(\Sigma; V)$ then

$$\langle \varphi, \psi \rangle = \int h(\varphi, \psi).$$

The fields $(X, f) \in W_n(\Sigma) \oplus C^\infty(\Sigma)$ pair with the conjugate fields $(X', f') \in \Omega^2(\Sigma; T_\Sigma^*) \oplus \Omega^2(\Sigma)$ via

$$\langle (X, f), (X', f') \rangle = \int \text{ev}(X, X') + \int f f'$$

where ev denotes the evaluation pairing between the tangent and cotangent bundles.

BW: Start with Polyakov action and explain how the chiral theory emerges in the infinite volume limit. There should also be an explanation for the theory we write down as a twist of 2d supergravity (in the same way that CDO's are a twist of a $(0, 2)$ theory), not sure if you want to get into that. OG: I don't know anything about the supergravity thing you mention. It sounds interesting.

OG: When we write out the whole BV shebang, we should point out how it relates to the usual physics description. Namely, the physicists do the following:

- write a free $bc\beta\gamma$ system as a $\mathbb{Z}/2$ -graded theory
- lift to a \mathbb{Z} -grading such that the bc fields are ghosts/antighosts for holomorphic vector fields (with no action on $\beta\gamma$ fields or themselves)
- deform the action to encode the action of vector fields on functions etc.

We should then see our BV action on the nose. (I think this is correct, but we should double-check, of course.) One nice thing about this observation is that it verifies the identification with semi-infinite homology, which is often explained in these kinds of terms. (See the nice, short, readable Voronov note I've put in our folder.)

3. DEFORMATIONS OF THE THEORY AND STRING BACKGROUNDS

OG: Maybe the Gelfand-Fuk discussion can be anticipated in Section 1? I think versions of it are easy to motivate: "We want to study Lagrangian densities, which are functions on jets of fields. Hence the simplest case is to consider functions on jets at a point, which we recognize as a version of Gelfand-Fuks ..." Then we invoke that discussion to work with formal vector fields and simply quote GF.

OG: We should observe that we see the deformations of the action, such as B -fields and dilatons. Observe we've rediscovered "string backgrounds."

BW: Might be good to hint at the curved sigma model here.

4. QUANTIZING THE HOLOMORPHIC BOSONIC STRING ON A DISK

BW: Gauge fixing condition. The theory is finite, no counterterms. Review Gelfand-Fuksy stuff. Local local deformation complex calculation. Do the anomaly calculation to obtain $\dim_{\mathbb{C}} = 13$. Argue why this produces a quantization on any source Riemann surface.

OG: I'm not sure what is the optimal order: diagrams then cohomology group, or other way? If we do the concrete Feynman diagrams and show that something vanishes exactly when you'd expect from the literature, then people might feel assured. However, it's not clear where this anomaly lives until you do the obstruction-deformation computation. Perhaps we just indicate how the reader can pick her preferred order.

5. OPE AND THE STRING VERTEX ALGEBRA

BW: Write down vertex algebra from quantization above. Possibly state the relationship to semi-infinite cohomology

Proposition 5.1. *Let Obs^q be the factorization algebra on $\Sigma = \mathbb{C}$ of the holomorphic bosonic string. The factorization product of open disks $D \subset \mathbb{C}$ determines the structure of a vertex algebra (see Proposition ?? below) on the cohomology of the factorization algebra on an open disk $H^*(\text{Obs}^q(D))$, that we denote $\text{Vert}(\text{Obs}^q)$. Moreover, there is an isomorphism of vertex algebras*

$$\Phi : V^{\text{string}} \xrightarrow{\cong} \text{Vert}(\text{Obs}^q).$$

OG: I believe these vertex algebras are cohomologically graded, unless we're lucky and the cohomology all sits in degree zero. In which case, we should point out this miracle. Perhaps better would be to extract the dg vertex algebra.

OG: Do you know a citation where the string vertex algebra is already written down? Of course it's almost explicit in any discussion of the "modern"/BRST quantization of the bosonic string, where they write down Q , which ought to be the differential of the dg vertex algebra using our construction.

BW: How deformations discussed in Section 3 gives explicit deformations of the the vertex algebra.

6. QUANTIZATION ON ARBITRARY RIEMANN SURFACES

6.1. Global sections of the factorization algebra. BW: Discuss relationship to conformal blocks

We wish to write the global observables of the holomorphic string in terms of the cohomology of natural holomorphic vector bundles on the Riemann surface.

We introduce a simpler theory defined for any (graded) holomorphic vector bundle \mathcal{V} on Σ . First, consider the elliptic complex

$$\Omega^{0,*}(\Sigma; \mathcal{V})$$

of $(0, *)$ -forms with values in \mathcal{V} . The differential is simply the $\bar{\partial}$ operator. We consider its shift $\Omega^{0,*}(\Sigma; \mathcal{V})[-1]$ as an abelian local dg Lie algebra. In this way, the associated cotangent theory of the elliptic complex is defined as

$$T^*[-1](\Omega^{0,*}(\Sigma; \mathcal{V})) = \Omega^{0,*}(\Sigma; \mathcal{V}) \oplus \Omega^{1,*}(\Sigma; \mathcal{V}^\vee).$$

We write the fields as $\gamma \in \Omega^{0,*}(\Sigma; \mathcal{V})$ for the base direction and $\beta \in \Omega^{1,*}(\Sigma; \mathcal{V}^\vee)$ for the fiber direction. With this notation, the action functional is defined as

$$S(\beta, \gamma) = \int_{\Sigma} \langle \beta, \bar{\partial} \gamma \rangle_{\mathcal{V}}$$

where $\langle -, - \rangle_{\mathcal{V}}$ denotes the evaluation pairing between \mathcal{V} and its dual. This theory is called the $\beta\gamma$ (or sometimes $\beta\gamma - bc$ OG: what are the bc fields?) system with values in \mathcal{V} . This is a free field theory and hence has a natural quantization. We will denote the factorization algebra of quantum observables on Σ by $\text{Obs}_{\mathcal{V}}^q$.

Lemma 6.1. *The cohomology of the global observables of the $\beta\gamma$ system with values in \mathcal{V} is given by*

$$H^* \left(\text{Obs}_{\mathcal{V}}^q(\Sigma) \right) \cong \det \left(H^*(\Sigma; \mathcal{V}) \right) [d(\mathcal{V})]$$

where $d(\mathcal{V}) = \dots$

OG: It would be good here to point out the GRR argument for identifying the first Chern class of this determinant line bundle. We can then point out that it resonates with our computation on the formal disk and give a reminder that this is a manifestation of "Virasoro uniformization." This then leads into the string case: the holomorphic vector fields (rather the free bc system) also contribute a determinant line (we should include that computation) and we can ask when those determinant lines tensor to a trivial line. Note that deforming the action from free $bc\beta\gamma$ system to holomorphic bosonic string doesn't affect the line bundles, since those are continuous parameters and Chern classes are discrete.

Now, consider the global observables of the bosonic string $\text{Obs}^q(\Sigma)$. There is a spectral sequence converging to the cohomology of the global observables $H^*\text{Obs}^q(\Sigma)$ with E_2 page given by the cohomology of the global observables of the $\beta\gamma$ system with values in the holomorphic vector bundle $\mathcal{V} = \mathcal{T}_\Sigma[1] \oplus \mathbb{C}^{13}$. By Lemma 6.1 this E_2 page is concentrated in a single degree and is given by

$$\begin{aligned} E_2 &\cong \det(H^*(\Sigma; \mathcal{T}_\Sigma[1])) \otimes \det(H^*(\Sigma; \mathcal{O}_\Sigma)^{\oplus 13}) \\ &\cong \det(H^1(\Sigma; \mathcal{T}_\Sigma)) \otimes \det(H^0(\Sigma; \mathcal{T}_\Sigma))^{-1} \otimes \det(H^0(\Sigma; K_\Sigma))^{-13} \end{aligned}$$

where we have used the fact that $H^0(\Sigma; \mathcal{O}) \cong \mathbb{C}$ for any Σ .

Let Σ_g be a surface of genus g . Then for $g = 1$ the above simplifies to

$$\det(H^1(\Sigma_1; \mathcal{T}_{\Sigma_1})) \otimes \det(H^0(\Sigma_1; K))^{-14}.$$

If $g \geq 2$ one has

$$\det(H^1(\Sigma_1; \mathcal{T}_{\Sigma_1})) \otimes \det(H^0(\Sigma_1; K))^{-13}.$$

It is clear that the spectral sequence degenerates at this page. Thus the above expressions give the global observables for the holomorphic string for genus $g = 1$ and $g \geq 2$, respectively.

7. THE STRING PARTITION FUNCTION ON AN ELLIPTIC CURVE

BW: Sketch how to see the mumford form

OG: If possible, it would be cool to explain how one can extract the differential equations (=flat connection) governing the partition function from our construction. This might be too hard right now ...

8. LOOKING AHEAD: CURVED TARGETS

OG: I think here we can mention our CDO work and assert that it's compatible with the discussion here. Then we state the corresponding theorems.

BW: State the quantization condition for curved target.