THE HOLOMORPHIC BOSONIC STRING

Contents

## 1. From classical to quantum: anomalies in the BV formalism

BW: A rapid overview of classical BV and effective quantizations. Stress how obstructions appear, where they live, and how to compute them.

OG: I think we should articulate here the structural features of our BV package that make the arguments below more conceptual. For instance:

- Linear BV quantization is determinantal, which explains why we'll produce determinant line bundles when we do free $\beta\gamma$ system.
- "Gauging" a theory corresponds to a stacky quotient of the original fields. Hence, obstruction to quantizing a gauged theory corresponds to descending the quantization to the quotient.
- If a classical theory makes sense on a class (=site) of manifolds, then to quantize the whole class, it suffices to check on a generating cover (typically given by disks with geometric structure) but compatibly with all automorphisms. This often explains the appearance of characteristic classes as anomalies.
- Every BV theory produces a factorization algebra. The local structure encodes the OPE algebra (and hence recovers a vertex algebra in chiral CFT situation). On compact manifolds, solutions to EoM typically form finite-dimensional space, and the global observables encode a volume form on this space. (An example is conformal blocks for the free $bc\beta\gamma$ system.)

Please add others as you think of them!

OG: We might also add that we view the BV formalism as the analogue in field theory of derived geometry in geometry. That is, in ordinary algebraic geometry, one first builds geometry and then adds (sheaf) cohomology on top: in ordinary physics, one first builds field theories and then adds (BRST) cohomology on top. But derived geometry (respectively, BV formalism) builds the cohomological aspect into the foundations.

1.1. **OG: description of algorithm.** For us, quantization will mean that we use perturbative constructions in the setting of the BV formalism. Concretely, this means that we enforces the gauge symmetries using the homological algebra of the BV formalism and that we use Feynman diagrams and renormalization to obtain an expression for the desired, putative path integral. OG: Be more careful about saying path integral. It's an approximation. There are toy models for this approach where one can see very clearly how it gives asymptotic expansions for finite-dimensional integrals OG: add references. In particular, these toy models show that this approach need not recover the true integral but does know important information about it; a similar relationship should hold between this quantization method and the putative path integral, but in this case there is no *a priori* definition of the true integral in most cases.

This notion of quantization applies to any field theory arising from an action functional, and the algorithm one applies to obtain a quantization is the following:

(1) Write down the integrals labeled by Feynman diagrams arising from action functional.

(2) Identify the divergences that appear in these integrals and add "counterterms" to the original action that are designed to cancel divergences.

(3) Repeat these steps until no more divergences appear in Feynman diagrams. We call this the "renormalized action."

(4) Check if the renormalized action satisfies the quantum master equation. If it does, you have a well-posed BV quantum theory, and we call the result a *quantized action*. If not, guess a way to adjust the renormalized action and begin the whole process again.

It should be clear that along the way, one makes many choices; hence if a quantization exists, it may not be unique. It is also possible that a BV quantization may not exist.

## 2. THE CLASSICAL HOLOMORPHIC BOSONIC STRING

BW: First define the holomorphic theory we will work with. Then show how it's related to more familiar models for the string, eg the Polyakov action. Level of detail depending on the space we have.

There is a basic format for a string theory, at least in the perturbative approach. One starts with a nonlinear $\sigma$-model, whose fields are smooth maps from a Riemann surface to a target manifold $X$; in this setting we want the theory to make sense for an arbitrary Riemann surface as the source manifold. In the usual bosonic string theory, this nonlinear $\sigma$-model picks out the harmonic maps from a Riemannian 2-manifold to a Riemannian manifold. In our holomorphic setting, the nonlinear $\sigma$-model picks out holomorphic maps from a Riemann surface to a complex manifold. One then quotients the space of fields (and solutions to the equations of motion) with respect to reparametrization. OG: This description is a bit opaque. We should find a better one. In the usual bosonic string, one quotients by diffeomorphisms, which can thus change the metric on the source. In our setting, we quotient by diffeomorphisms as well, which can thus change the complex structure on the source.

In this section we begin by describing our theory in the BV formalism. We do not expect the reader to find the action functional immediately clear, so we devote some time to analyzing what it means and how it arises from concrete questions. We then turn to interpreting this classical BV theory using dg Lie algebras and derived geometry (i.e., we identify the moduli space it encodes). Finally, we conclude by sketching how our theory appears as the chiral sector of a degeneration of the usual bosonic string when the target is a complex manifold with a Hermitian metric. Our theory thus does provide insights into the usual bosonic string; moreover, it clarifies why so many aspects of the bosonic string, like the anomalies or $B$-fields, have holomorphic analogues.

2.1. **The theory we study.** Let $V$ denote a complex vector space (the target), and let $\langle -,-\rangle_V$ denote the evaluation pairing between $V$ and its linear dual $V^\vee$. Let $\Sigma$ denote a Riemann surface (the source). Let $T^{1,0}_\Sigma$ denote the holomorphic tangent bundle on $\Sigma$, let $\langle -,-\rangle_T$ denote the evaluation pairing between $T^{1,0}_\Sigma$ and its vector bundle dual $T^{1,0*}_\Sigma$. OG: Correct terminology? These are the key geometric inputs.

In a BV theory, the fields are $\mathbb{Z}$-graded; we call this the *cohomological grading*. We have four kinds of fields:

| field | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|
| $\gamma$ | | $\Omega^{0,0}(\Sigma)\otimes V$ | $\Omega^{0,1}(\Sigma)\otimes V$ | |
| $\beta$ | | $\Omega^{1,0}(\Sigma)\otimes V^\vee$ | $\Omega^{1,1}(\Sigma)\otimes V^\vee$ | |
| $c$ | $\Omega^{0,0}(\Sigma, T^{1,0}_\Sigma)$ | $\Omega^{0,1}(\Sigma, T^{1,0}_\Sigma)$ | | |
| $b$ | | | $\Omega^{1,0}(\Sigma, T^{1,0*}_\Sigma)$ | $\Omega^{1,1}(\Sigma, T^{1,0*}_\Sigma)$ |

More accurately, we have eight different kinds of fields, but we view each row as constituting a single type since each given row consists of the Dolbeault forms of a holomorphic vector bundle. For instance, the field $\gamma$ is a $(0,*)$-form with values in the trivial bundle with fiber $V$, and the field $b$ is a $(0,*)$-form with values in the bundle $T^{1,0*}\otimes\ T^{1,0*}$.

To orient oneself it is helpful to start by examining the fields of cohomological degree zero, since these typically have a manifest physical meaning. For instance, the degree zero $\gamma$ field is a smooth $V$-valued function and hence the natural field for the nonlinear $\sigma$-model into $V$. The degree zero $c$ field is a smooth $(0,1)$-form with values in vector field "in the holomorphic direction," and hence encodes an infinitesimal change of complex structure of $\Sigma$. They thus constitute the obvious fields to introduce for a holomorphic version of the bosonic string. The fields a$\beta$ and $b$ are less obvious but appear as "partners" (or antifields) whose role is clearest once we have the action functional and hence equations of motion.

The action functional is

$$S(\gamma,\beta,c,b) = \int_\Sigma \langle \beta, \bar\partial\gamma\rangle_V + \int_\Sigma \langle b, \bar\partial c\rangle_T + \int_\Sigma \langle \beta, [c,\gamma]\rangle_V + \int_\Sigma \langle b, [c,c]\rangle_T. \tag{1}$$

(We discuss below how to think about fields with nonzero cohomological degrees as inputs.) The equations of motion are thus

$$\begin{aligned} 0 &= \bar\partial\gamma + [c,\gamma] & \qquad 0 &= \bar\partial\beta + [c,\beta] \\ 0 &= \bar\partial c + \tfrac{1}{2}[c,c] & \qquad 0 &= \bar\partial b + [c,b] \end{aligned}$$

Note that these equations are familiar in complex geometry. For instance, the equation purely for $c$ encodes a deformation of complex structure on $\Sigma$; concretely, it modifies the $\bar\partial$ operator to $\bar\partial + c$. The other equations then amount to solving for holomorphic sections (of the relevant bundle) withe respect to this deformed complex structure. For instance, the equation in $\gamma$ picks out holomorphic maps from $\Sigma$, with the $c$-deformed complex structure, to $V$.

OG: Add something about how to understand the degrees. E.g., does $b$ ever appear?

OG: Add explanation of writing BV theory from ordinary action.

*Remark* 2.1. Just looking at this action functional, one might notice that if one drops the last two terms, which are cubic in the fields, then one obtains a free theory

$$S_{free}(\gamma, \beta, c, b) = \int_\Sigma \langle \beta, \overline{\partial}\gamma \rangle_V + \int_\Sigma \langle b, \overline{\partial} c \rangle_T, \tag{2}$$

which is known as the *free bc$\beta\gamma$ system*. Thus, one may view the holomorphic bosonic string as a deformation of this free theory by "turning on" those interaction terms. We will repeatedly try a construction first with this free theory before tackling the string itself, as it often captures important information with minimal work. For instance, we will examine the vertex algebra for the free theory before seeing how the interaction affects the operator products. Similarly, one can identify the anomaly already at the level of the free theory.

*Remark* 2.2. It is easy to modify this action functional to allow a curved target, i.e., one can replace the complex vector space $V$ with an arbitrary complex manifold $X$. The fields $b, c$ remain the same. The degree 0 field $\gamma$ still encodes smooth maps into $X$, but now the degree 1 field is a section of $\Omega^{0,1}(\Sigma, \gamma^* T^{1,0}_X)$. Similarly, $\beta$ is now a section of $\Omega^{1,*}(\Sigma, \gamma^* T^{1,0*}_X)$. The action is then

$$S(\gamma, \beta, c, b) = \int_\Sigma \langle \beta, \overline{\partial}\gamma \rangle_{T_X} + \int_\Sigma \langle b, \overline{\partial} c \rangle_{T_\Sigma} + \int_\Sigma \langle \beta, [c, \gamma] \rangle_{T_X} + \int_\Sigma \langle b, [c, c] \rangle_{T_\Sigma}. \tag{3}$$

In Section **??** we will indicate how the results with linear target generalize to this situation.

2.2. **From the perspective of derived geometry.** We would like to explain what this theory is about in more conceptual terms, rather than simply by formulas and equations. Thankfully this theory is amenable to such a description. We will be informal in this section and not specify a particular geometric context (e.g., derived analytic stacks), except when we specialize to the deformation-theoretic situation (i.e., perturbative setting) that is our main arena.

Let $\mathcal{M}$ denote the moduli space of Riemann surfaces, so that a surface $\Sigma$ determines a point in $\mathcal{M}$. Let $\mathrm{Maps}_{\overline{\partial}}(\Sigma, V)$ denote the space of holomorphic maps from $\Sigma$ to $V$, and hence a bundle $\mathrm{Maps}_{\overline{\partial}}(-, V)$ over $\mathcal{M}$ by varying $\Sigma$. For our equations of motion, the $\gamma$ and $c$ fields of a solution determine a point in this bundle $\mathrm{Maps}_{\overline{\partial}}(-, V)$.

This construction makes sense on noncompact Riemann surfaces as well. Let $\mathcal{RS}$ denote the category whose objects are Riemann surfaces and whose morphisms are holomorphic embeddings. There is a natural site structure: a cover is a collection of maps $\{S_i \to \Sigma\}_i$ such that the union of the images is all of $\Sigma$. Then $\mathrm{Maps}_{\overline{\partial}}(-, V)$ defines a sheaf of spaces over $\mathcal{RS}$. The observables for the classical theory is, in essence, the *co*sheaf of commutative algebras $\mathcal{O}(\mathrm{Maps}_{\overline{\partial}}(-, V))$, and hence provides a factorization algebra.

In fact, it is better to work with the derived version of these spaces. One important feature of derived geometry is that the appropriate version of a tangent space at a point is, in fact, a cochain complex. In our setting, a point $(c, \gamma)$ in $\mathrm{Maps}_{\overline{\partial}}(-, V)$ determines a a complex structure $\overline{\partial} + c$ on $\Sigma$—we denote this Riemann surface by $\Sigma_c$—and $\gamma$ a $V$-valued holomorphic function on $\Sigma_c$. The tangent complex of $\mathrm{Maps}_{\overline{\partial}}(-, V)$ at $(c, \gamma)$ is precisely

$$\Omega^{0,*}(\Sigma_c, T^{1,0})[1] \oplus \Omega^{0,*}(\Sigma_c, V).$$

The first summand is the usual answer from the theory of the moduli of surfaces (recall, for example, that the ordinary tangent space is the sheaf cohomology $H^1(\Sigma, \mathcal{T}_\Sigma)$ of the holomorphic tangent sheaf), and the second is usual elliptic complex encoding holomorphic maps.

*Remark* 2.3. It is useful to bear in mind that the degree zero cohomology of the tangent complex will recover the "naive" tangent space. In our case, we have

$$H^1(\Sigma_c, \mathcal{T}_{\Sigma_c}) \oplus H^0(\Sigma_c, V),$$

which encodes deformations of complex structure and holomorphic maps. Negative degree cohomology of the tangent complex detects infinitesimal automorphisms (and automorphisms of automorphisms, etc) of the space. For instance, here we see $H^0(\Sigma_c, \mathcal{T}_{\Sigma_c})$ appear in degree -1, since a holomorphic vector field is an infinitesimal automorphism of a complex curve. These negative directions are called "ghosts" (or ghosts for ghosts, etc) in physics. The positive degree cohomology detects infinitesimal relations (and relations of relations, etc). For instance, here we see $H^1(\Sigma_c, V)$, the cokernel of $\overline{\partial} + c$. OG: ???

Note that the underlying graded spaces of this tangent complex are the $c$ and $\gamma$ fields from the BV theory described above. We emphasize that the tangent complex is only specified up to quasi-isomorphism, but it is compelling that a natural representative is the BV theory produced by the usual physical arguments. This behavior, however, is typical of the relationship between derived geometry and BV theories: when physicists write down a classical BV theory, the underlying free theory is essentially always the tangent complex of a nice derived stack.

The reader has probably noticed that, yet again, we have postponed discussing the $\beta$ and $b$ fields. From a derived perspective, the full BV theory describes the shifted cotangent bundle $\mathbb{T}^*[-1]\,\mathrm{Maps}_{\overline{\partial}}(-, V)$. At the level of a tangent complex, the shifted cotangent direction contributes

$$\Omega^{1,*}(\Sigma_c, T^{1,0*})[-1] \oplus \Omega^{1,*}(\Sigma_c, V^\vee),$$

whose underlying graded spaces are the $\beta$ and $b$ fields. These "antifields" are added so that the overall space of fields has a 1-shifted symplectic structure when $\Sigma$ is closed, and a shifted Poisson structure when $\Sigma$ is open.

2.3. **Relationship to the Polyakov action functional.** This holomorphic bosonic string has a natural relationship with the usual bosonic string. We sketch it briefly, only considering a linear target.

We begin with a bosonic string theory where the source is a 2-dimensional smooth oriented manifold $\Sigma$ and the target is a Hermitian vector space $(V, h)$. The "naive" action functional is

$$S^{naive}_{Poly}(\varphi, g) = \int_\Sigma h(\varphi, \Delta_g \varphi)\, \mathrm{dvol}_g$$

where the field $g$ is a Riemannian metric on $\Sigma$ and the field $\varphi$ is a smooth map from $\Sigma$ to $V$. The notation $\Delta_g$ denotes the Laplace-Beltrami operator on $\Sigma$.

Note that $S^{naive}_{Poly}$ is invariant under the diffeomorphism group $\mathrm{Diff}(\Sigma)$ and under rescalings of the metric (i.e., the theory is classically conformal). Typically we express rescaling as $g \mapsto e^f g$ with

$f \in C^\infty(\Sigma)$. As we are interested in a string theory, we want to gauge these symmetries. In geometric language, we want to think about the quotient stack obtained by taking solutions to the equations of motion and quotienting by these symmetry groups.

OG: It might be better to explain the first-order description of sigma model before entering into the perturbative & BV discussion.

Our focus is perturbative, so that we want to study the behavior of this action near a fixed solution to the equations of motion (e.g., the Taylor expansion of the true action near some solution). OG: Might be good here to leverage the derived discussion earlier: it's easy to see what the tangent complex looks like ... which leads to fields we work with. Hence, we fix a metric $g_0$ on $\Sigma$ and substitute for the field $g$, the term $g_0 + \alpha$ where $\alpha \in \Gamma(\Sigma, \mathrm{Sym}^2(T_\Sigma))$. That is, we simply consider deformations of $g_0$. As $\varphi$ is linear, we just consider expanding around the zero map. Thus our initial fields are $\varphi \in C^\infty(\Sigma, V)$ and $\alpha \in \Gamma(\Sigma, \mathrm{Sym}^2(T_\Sigma))$.

There are also ghost fields associated to the symmetries we gauge. First, there are infinitesimal diffeomorphisms, which are described by vector fields on $\Sigma$. We denote this ghost field by $X \in \Gamma(\Sigma, T_\Sigma))$. It acts on the initial fields by the transformation

$$(\varphi, \alpha) \mapsto (\varphi + X \cdot \varphi, \alpha + L_X \alpha),$$

where $L_X$ denotes the Lie derivative on tensors. Second, there are infinitesimal rescalings such as $\alpha \mapsto \alpha + f\alpha$, with ghost field $f \in C^\infty(\Sigma)$. The rescaling does not affect $\varphi$. The two symmetries are compatible: given $f$ and $X$, then $L_X(f\alpha) = X(f)\alpha + f L_X \alpha$ for any $\alpha \in \mathrm{Sym}^2(T_\Sigma)$.

To summarize, we have the following fields:

| $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|
| | $\Omega^0(\Sigma) \otimes V$ | $\Omega^2(\Sigma) \otimes V$ | |
| $\mathrm{Vect}(\Sigma) \oplus C^\infty(\Sigma)$ | $\mathrm{Sym}^2(T_\Sigma)$ | | |
| | | $\Omega^2(\Sigma; \mathrm{Sym}^2(T^*_\Sigma))$ | $\Omega^2(\Sigma; T^*_\Sigma) \oplus \Omega^2(\Sigma)$. |

The BV action functional is then

$$S_{Poly}(\phi, \alpha, X, f) = \int_\Sigma h(\varphi, \Delta_{g_0+\alpha}\varphi)\, \mathrm{dvol}_{g_0+\alpha} + \text{MORE STUFF}$$

OG: Finish writing action This situation is quite a bit more complicated than our holomorphic bosonic string, but it admits a succinct description in geometric terms OG: add statement about quotient of mapping stack

***********

In this section we start with a description of the classical Polyakov model for the bosonic string as a classical BV theory. This is the ordinary $\sigma$-model of maps $\Sigma \to V$ coupled to a metric on $\Sigma$. More precisely, this is a perturbative model for the Polyakov string, since we only look at deformations of the fixed metric $g_0$. We will show that after a reparametrization of the space of fields that it makes sense to take a certain "infinite volume limit" as $h \to \infty$. In this limit we will show that the Polyakov model splits into a certain holomorphic theory plus its complex conjugate. The holomorphic theory is what we call the *holomorphic bosonic string*.

*Remark* 2.4. A similar analysis has appeared in **[?]** where one does not consider deformations of the metric: the infinite volume limit of the bare $\sigma$-model of maps $\Sigma \to V$ splits into the free $\beta\gamma$ system plus its complex conjugate. In the case of the string we find an interacting theory that can be thought of as a deformation of a $\beta\gamma$ system.

OG: Give explanation of what this section will be about: writing down a holomorphic theory that appears as the chiral part of a large volume limit of the usual bosonic string. We should advertise that we start with conventional ways of writing a theory and explain the algorithm by which one extracts a BV action.

We recall the most familiar form of the classical Polyakov string and show how to write it down in terms of a classical BV theory. The fields of the Polyakov model consist of a $C^\infty$ function $\varphi : \Sigma \to V$ and a metric $g$ on $\Sigma$. Since we are doing perturbation theory, we assume that $g$ is infinitesimally close to the fixed metric $g_0$ in the space of all metrics on $\Sigma$. There is an identification of the tangent space of the space of all metrics $T_{[g_0]}\mathrm{Met}(\Sigma) \cong \mathrm{Sym}^2(T_\Sigma)$. Thus, we can take the metric $g$ to be of the form $g = g_0 + \alpha$ where $\alpha \in \mathrm{Sym}^2(T_\Sigma)$. OG: Should we include comments about "formal (derived) spaces"? In the definition of a classical BV theory we must prescribe the data of a $(-1)$-shifted symplectic pairing on the BRST complex together with an interaction which is a local functional on the complex. The pairing can be described as follows. If $\varphi \in \Omega^0(\Sigma; V)$ and $\psi \in \Omega^2(\Sigma; V)$ then

$$\langle \varphi, \psi \rangle = \int h(\varphi, \psi).$$

The fields $(X, f) \in \mathrm{W}_n(\Sigma) \oplus C^\infty(\Sigma)$ pair with the conjugate fields $(X', f') \in \Omega^2(\Sigma; T^*_\Sigma) \oplus \Omega^2(\Sigma)$ via

$$\langle (X, f), (X', f') \rangle = \int \mathrm{ev}(X, X') + \int f f'$$

where ev denotes the evaluation pairing between the tangent and cotangent bundles.

BW: Start with Polyakov action and explain how the chiral theory emerges in the infinite volume limit. There should also be an explanation for the theory we write down as a twist of 2d supergravity (in the same way that CDO's are a twist of a $(0, 2)$ theory), not sure if you want to get into that. OG: I don't know anything about the supergravity thing you mention. It sounds interesting.

## 3. Deformations of the theory and string backgrounds

OG: Maybe the Gelfand-Fuk discussion can be anticipated in Section 1? I think versions of it are easy to motivate: "We want to study Lagrangian densities, which are functions on jets of fields. Hence the simplest case is to consider functions on jets at a point, which we recognize as a version of Gelfand-Fuks . . . " Then we invoke that discussion to work with formal vector fields and simply quote GF.

OG: We should observe that we see the deformations of the action, such as $B$-fields and dilatons. Observe we've rediscovered "string backgrounds."

BW: Might be good to hint at the curved sigma model here.

The local deformation complex is the complex of local functionals on the full BV complex describing the classical field theory. In this section, we will provide an interpretation of this complex of local functionals in terms of Gelfand-Fuks cohomology. Along the way we will see how the usual backgrounds for the bosonic string (a target metric, dilaton term, etc.) appear as elements in this complex of local functionals and hence as deformations of the classical action.

We have already seen that the holomorphic bosonic string is the shifted cotangent bundle on the tangent complex of $\mathrm{Maps}_{\bar\partial}(--, V)$ as a bundle over the moduli of Riemann surfaces. Consider the following action by the group $\mathbb{C}^\times_{\mathrm{cot}} = \mathbb{C}^\times$ on the theory. The base of the cotangent bundle, the tangent complex of $\mathrm{Maps}_{\bar\partial}(--, V)$ has weight 0, and the cotangent fiber has weight $+1$. That is, this action simply comes from rescaling the cotangent fiber. Note that the classical action functional is weight one for this $\mathbb{C}^\times_{\mathrm{cot}}$-action. Thus classical deformations of the holomorphic bosonic string will consist of weight one local functionals in the deformation complex.

The weight of the parameter $\hbar$ is also one with respect to the scaling by $\mathbb{C}^\times_{\mathrm{cot}}$. Thus, for quantum corrections at one loop we consider local functionals that are of weight zero for this $\mathbb{C}^\times_{\mathrm{cot}}$-action. Put simply, these are local functionals that only depend on the base of the shifted cotangent bundle. On a Riemann surface $\Sigma$ the $\mathbb{C}^\times$ invariant local functionals are of the form

$$\left(\mathcal{O}_{\mathrm{loc}}\left(\Omega^{0,*}(\Sigma, V) \oplus \Omega^{0,*}(\Sigma, T^{1,0})[1]\Omega^{1,*}(\Sigma; V) \oplus \Omega^{1,*}(\Sigma, T^{1,0*})[-1]\right), \{S, -\}\right).$$

**Proposition 3.1.** *There are* $\mathrm{GL}(V)$*-equivariant quasi–isomorphisms*

(0)
$$\left(\mathcal{O}_{\mathrm{loc}}(\mathcal{E})^{(0)}, \{S, -\}\right) \simeq \Omega^2_{cl}(V)[1] \oplus \Omega^1(V) \oplus \Omega^1_{cl}(V)[-1]$$

(1)
$$\left(\mathcal{O}_{\mathrm{loc}}(\mathcal{E})^{(1)}, \{S, -\}\right) \simeq ??.$$

3.1. OG: Every subsection should have a label or we should change how the table of contents works to only show section headings.

The jets at $0 \in \mathbb{C}$ of local Lie algebra $\Omega^{0,*}(\mathbb{C}; T^{1,0} \ltimes (V[-1] \oplus \mathrm{d}zV^*[-1] \oplus \mathrm{d}zT^{1,0*})$ is quasi-isomorphic to the Lie algebra

$$\mathrm{W}_1 \ltimes (V[[z]][-1] \oplus \mathrm{d}zV^*[[z]][-1] \oplus \widehat{\Omega}^1_1[-2]) \tag{4}$$

where $\mathrm{W}_1 = \mathbb{C}[[z]]\partial_z$ is the Lie algebra of formal vector fields in one variable and $\widehat{\Omega}^1_1 = \mathbb{C}[[z]]\mathrm{d}z$ is the space of formal one forms. The Lie bracket comes from bracket on $\mathrm{W}_1$ and the natural action of $\mathrm{W}_1$ on $\mathbb{C}[[z]]$ and $\widehat{\Omega}^1_1$.

Denote by $\{L_n = z^{n+1}\partial_z\}$ the standard basis for the Lie algebra of formal vector fields $\mathrm{W}_1$. Let $\lambda_n \in \mathrm{W}_1^\vee$ be the dual vector to $L_n$ (we are using the continuous dual, as in the setting of Gelfand-Fuks cohomology). An arbitrary element of $V[[z]]$ is linear combination of vectors of the form $v \otimes z^k$. Write $\zeta_k$ for the dual element $(z^k)^\vee$. Thus an element of $(V[[z]])^\vee$ is a linear combination of the vectors of the form $v^\vee \otimes \zeta_k$.

3.1.1. *The $\mathbb{C}^\times_{\mathrm{cot}}$-weight zero piece.* The weight $\mathbb{C}^\times_{\mathrm{cot}}$-weight zero sub Lie algebra of the Lie algebra (**??**) is simply $W_1 \ltimes V[[z]][-1]$, where the semi-direct product comes from the natural action of formal vector fields on formal power series. Thus, we have reduced the calculation of the $\mathbb{C}^\times_{\mathrm{cot}}$-weight zero piece of the local deformation complex to calculating the Chevalley-Eilenberg complex of this Lie algebra:

$$\mathrm{C}^*_{\mathrm{Lie,red}}\left(\mathrm{W}_1 \ltimes V[[z]][-1]\right).$$

This splits into two terms $\mathrm{C}^*_{\mathrm{Lie,red}}(\mathrm{W}_1) \oplus \mathrm{C}^*_{\mathrm{Lie}}(W_1; \mathrm{Sym}^{\geq 1}(V[[z]])^\vee)$.

The first term in this summand is the reduced Gelfand-Fuks cohomology of formal vector fields with values in the trivial module. It is well-known that the cohomology is one-dimensional and concentrated in degree 3, $H^3_{\mathrm{red}}(\mathrm{W}_1) \cong \mathbb{C}[-3]$. We will identify the anomaly for the holomorphic string with flat target as multiple of the generator of this space. The remaining piece of the weight zero deformation complex is the home of the anomalies for the holomorphic string placed in a non trivial background: for instance, when the target of the $\sigma$ model is curved. We will not see this in our theory, of course, but the following will hopefully be...

The first step in computing this is to notice that there is a quasi-isomorphic subcomplex. The vector field $L_0 = z^{n+1}\partial_z$ induces a grading on $\mathrm{W}_1$ and hence on the Chevalley-Eilenberg complex of $\mathrm{W}_1$ with coefficients in any module. We will call this grading the *conformal dimension*.

**Lemma 3.2.** *Let $M$ be any $\mathrm{W}_1$-module. Then, the inclusion of the conformal dimension zero subcomplex*

$$\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_1; M)^{(0)} \xrightarrow{\simeq} \mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_1; M)$$

*is a quasi-isomorphism.* BW: Is this true?

*Proof.* For $p-1$ define the operator $\iota_{L_0} : \mathrm{C}^p_{\mathrm{Lie}}(\mathrm{W}_1; M) \to \mathrm{C}^{p-1}_{\mathrm{Lie}}(\mathrm{W}_1; M)$ defined by sending a cochain $\varphi$ to the cochain

$$(\iota_{L_0}\varphi)(X_1, \ldots, X_p) = \varphi(L_0, X_1, \ldots, X_p).$$

Let d be the differential for the complex $\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_1; M)$. It is easy to check that the difference $\mathrm{d}\iota_{L_0} - \iota_{L_0}\mathrm{d}$ is equal to the projection onto the dimension zero subspace. □

The underlying graded vector space of this conformal dimension zero subcomplex splits as follows:

$$\mathrm{C}^\#_{\mathrm{Lie}}(\mathrm{W}_1)^{(0)} \otimes \left(\mathrm{Sym}^{\geq 1}\left(V[[z]]\right)^\vee\right)^{(0)} \oplus \mathrm{C}^\#_{\mathrm{Lie}}(\mathrm{W}_1)^{(1)} \otimes \left(\mathrm{Sym}^{\geq 1}\left(V[[z]]\right)^\vee\right)^{(-1)}$$

Observe that the dimension zero part of the reduced symmetric algebra is simply $\mathrm{Sym}^{\geq 1}(V^\vee)$ which is identified $\mathrm{GL}(V)$-equivariantly with $\mathcal{O}_{red}(V)$. That is, power series on $V$ with no constant term. Similarly, the dimension one part of $\mathrm{Sym}^{\geq 1}\left(V[[z]]\right)^\vee$ is of the form $\mathrm{Sym}(V^\vee) \otimes z^\vee V^\vee$, which is identified $\mathrm{GL}(V)$-equivariantly with $\Omega^1(V)$.

The full dimension zero complex, including the differential is

$$\begin{tikzcd}
1\otimes \mathcal{O}_{red}(V) \arrow[dr, "\mathrm{d}_{dR}"] & \lambda^0\otimes\mathcal{O}_{red}(V) \arrow[r] \arrow[dr, "\mathrm{d}_{dR}"] & \lambda^{-1}\wedge\lambda^1\otimes\mathcal{O}_{red}(V) & \lambda^{-1}\wedge\lambda^1\wedge\lambda^0\otimes\mathcal{O}_{red}(V) \\
& \lambda^{-1}\otimes\Omega^1(V) \arrow[r] & \lambda^{-1}\wedge\lambda^0\otimes\Omega^1(V) &
\end{tikzcd}$$

The top horizontal map sends $\lambda^0 \mapsto 2\cdot\lambda^{-1}\wedge\lambda^1$ and the bottom horizontal map sends $\lambda^{-1}$ to $\lambda^{-1}\wedge\lambda^0$ (both are the identity on $V$). The diagonal maps are given by the de Rham differential $\mathrm{d}_{dR} : \mathcal{O}_{red}(V) \to \Omega^1(V)$. This complex is quasi-isomorphic to

$$\begin{tikzcd}
1\otimes \mathcal{O}_{red}(V) \arrow[dr, "\mathrm{d}_{dR}"] & & & \lambda^{-1}\wedge\lambda^1\wedge\lambda^0\otimes\mathcal{O}_{red}(V) \\
& \lambda^{-1}\otimes\Omega^1(V) & \lambda^{-1}\wedge\lambda^0\otimes\Omega^1(V) &
\end{tikzcd}$$

which, in turn, is identified with $\Omega^2_{cl}(V)[1]\oplus\Omega^1(V)\oplus\Omega^1_{cl}(V)$. This completes the calculation of the $\mathbb{C}^\times_{\rm cot}$-weight zero component.

3.1.2. *The $\mathbb{C}^\times_{\rm cot}$-weight one piece.* The $\mathbb{C}^\times_{\rm cot}$-weight one part of the Lie algebra (**??**) is $\mathrm{d}zV^\vee[[z]][-1]\oplus \mathrm{d}z\widehat{\Omega}^1_1[-2]$.

A totally analogous calculation as in the weight zero case yields the following.

**Proposition 3.3.** *There is a* $\mathrm{GL}(V)$*-equivariant quasi-isomorphism*

$$C^*_{\rm Lie}\left(W_1;\mathrm{Sym}\,(V[[z]])^\vee\otimes(\mathrm{d}zV^*[[z]]\oplus\widehat{\Omega}^{1\otimes 2}_1[-1])^\vee\right)\simeq T_V[-1]\oplus T_V[-2]$$

*where $T_V$ denotes the adjoint representation.*

## 4. Quantizing the holomorphic bosonic string on a disk

BW: Gauge fixing condition. The theory is finite, no counterterms. Review Gelfand-Fuksy stuff. Local local deformation complex calculation. Do the anomaly calculation to obtain $\dim_{\mathbb{C}} = 13$. Argue why this produces a quantization on any source Riemann surface.

We will apply the algorithm described in Section **??** in the case of $\Sigma = \mathbb{C}$. For this theory we are lucky, however: the integrals that appear from the Feynman diagrams do not have divergences, so that renormalized action is easy to compute. This aspect is the subject of the first part of this section. (Later we will explain why these divergences do not appear on an arbitrary Riemann surface. OG: add cross ref) Moreover, it is easy to check whether the quantum master equation is satisfied, and the answer is simple. This aspect is the subject of the second part. The results can be summarized as follows.

**Proposition 4.1.** *The holomorphic bosonic string with source $\mathbb{C}$ and target $\mathbb{C}^d$ admits a BV quantization if $d = 13$. This quantized action only has terms of order $\hbar^0$ and $\hbar$ (i.e., it quantizes at one loop).*

4.1. **The Feynman diagrams.** Let us describe the combinatorics of the Feynman diagrams that appear here before we describe the associated integrals.

4.1.1. The procedure constructs graphs out of a prescribed type of vertices and edges; we must consider all graphs with such local structure. The classical action functional determines the allowed kinds of vertices and edges. The quadratic terms of the action tell us the edges; each quadratic term yields an edge whose boundary is labeled by the two fields appearing in the term. For us there are thus two types of edges: an edge that flows from $c$ to $b$, and an edge that flows from $\gamma$ to $\beta$. OG: Add picture. The nonquadratic terms tell us the vertices: each $n$-ary term yields a vertex with $n$ legs, and the legs are labeled by the $n$ types of fields appearing in the term. For us there are thus two types of trivalent vertices: a vertex with two $c$ legs and a $b$ leg, and a vertex with a $c$ leg, a $\gamma$ leg, and a $\beta$ leg. It helpful to picture these legs as directed, so that $c$ and $\gamma$ legs flow into a vertex and $b$ and $\beta$ legs flow out. OG: Add picture.

The kinds of graphs one can build with such vertices and edges are limited. We focus on connected graphs, since an arbitrary graph is just a union of connected components.

A tree (i.e., a connected graph with no loops) must have at most one outgoing leg, which must be either a $b$ or a $\beta$; the other legs are incoming, so each must be labeled by a $c$ or a $\gamma$. OG: Add picture. Note that there are two types of trees. If there is a $\gamma$ leg, then there is a $\beta$ leg, and there is a chain of $\gamma\beta$ edges connecting them; all other external legs are of $c$ type. If there is a $b$ leg, then the only other legs are $c$ type.

A 1-loop graph will consist of a wheel (i.e., a sequence of edges that form an overall loop) with trees attached. The outer legs are all of $c$ type. OG: Add pictures. Every edge along a wheel will have the same type. It is not possible to build a connected graph with more than one loop. This combinatorics is the essential reason that we can quantize at one loop.

4.1.2. These graphs describe linear maps associated to the field. More precisely, a graph with $k$ legs describes a linear functional on the $k$-fold tensor product of the space of fields. One builds this linear functional out of the data of the action functional.

*Remark* 4.2. Like the action functional, it is rarely well-defined on all fields if the source manifold is non-compact. Instead, like the action functional, it defines a function on compactly-supported fields. This domain is all one needs for making variational arguments or for constructing a BV quantization.

For instance, a $k$-valent vertex corresponds to a $k$-ary term in the action, which manifestly takes in $k$ copies of the fields and outputs a number. Thus, the vertex labels an element of a (continuous) linear dual of the $k$-fold tensor product of fields. An edge corresponds an element of the 2-fold tensor product of the space of fields, often called a *propagator*. More precisely, the edge should correspond to the Green's function for the linear differential operator appearing in the associated quadratic term of the action; hence the propagator is an element of the *distributional completion* of the 2-fold tensor product. For us the $\beta\gamma$ leg should be labeled by $\overline{\partial}^{-1} \otimes \mathrm{id}_V$, where $\overline{\partial}^{-1}$ denotes an inverse to the Dolbeault operator on functions. The $bc$ leg should be labeled by $\overline{\partial}_T^{-1}$, the inverse of the Dolbeault operator on the bundle $T^{1,0}$.

Given a graph, one should contract the tensors associated to the vertices and edges. Each vertex labels a distributional section of some vector bundle on $\Sigma$, and each edge labels a distributional section of a vector bundle on $\Sigma^2$. Thus the desired contraction can be written *formally* as an integral over the product manifold $\Sigma^v$, where $v$ denotes the number of vertices. In most situations this contraction is ill-defined, since one cannot (usually) pair distributions. Concretely, one sees that the integral expression is divergent.

Thus, to avoid these divergences, one labels the edges by a smooth replacement of the Green's functions. (Imagine replacing a delta function $\delta_0$ by a bump function.) Since one can pair smooth functions and distributions, each graph yields a linear functional on fields using these mollified edges. But now this linear functional depends on the choice of mollifications. Hence the challenge is to show that if one picks a sequence of smooth replacements that approaches the Green's function, there is a well-defined limit of the linear functionals.

4.1.3. We will now sketch one method well-suited to complex geometry that allows us to see that no divergences appear for the holomorphic bosonic string. Our approach is an example of the renormalization method developed by Costello in [**?**], which applies to many more situations.

Our primary setting in this section is $\Sigma = \mathbb{C}$. For this Riemann surface, a standard choice of Green's function for $\overline{\partial}$ is

$$P(z, w) = \frac{1}{2\pi i} \frac{\mathrm{d}z + \mathrm{d}w}{z - w}.$$

It is a distributional one-form on $\mathbb{C}^2$ that satisfies $\overline{\partial} \otimes 1(P) = \delta_\Delta$, where $\delta_\Delta$ is the delta-current supported along the diagonal $\Delta : \mathbb{C} \hookrightarrow \mathbb{C}^2$ and providing the integral kernel for the identity. In terms of our discussion above, we view this one-form as a distributional section of the fields $\gamma$ and $\beta$: for example, for fixed $w$, the one-form $\mathrm{d}z/(z - w)$ is a $\beta$ field in the $z$-variable as it is a $(1, 0)$-form.

We will now describe the integral associated to a simple diagram. For simplicity, we assume $V = \mathbb{C}$ so that the $\gamma$ and $\beta$ fields are simply functions and 1-forms on $\mathbb{C}$, respectively. Consider a "tadpole" diagram whose outer legs are $c$ fields (i.e., vector fields on $\mathbb{C}$). OG: Add picture. There is only one vertex here, corresponding to the cubic function on fields

$$F(c, \gamma, \beta) = \int_{z \in \mathbb{C}} \beta \wedge c\gamma.$$

If the field $c$ is of the form $f(z)\mathrm{d}\overline{z}\partial_z$, with $f$ compactly supported, then our integral is

$$\int_{z \in \mathbb{C}} \beta \wedge f(z)(\partial_z \gamma)\mathrm{d}\overline{z}.$$

(Note that a general cubic function could be described as an integral over $\mathbb{C}^3$, but our function is supported on the small diagonal $\mathbb{C} \hookrightarrow \mathbb{C}^3$.) The linear functional for this tapole diagram should be given by inserting the propagator $P$ in place of the $\beta$ and $\gamma$ fields. Hence it ought to be given by the following integral over $\mathbb{C}$:

$$\int_{z \in \mathbb{C}} c(z) P(z, w)|_{z=w} = \int_{z \in \mathbb{C}} f(z) \partial_z \left( \frac{1}{2\pi i} \frac{\mathrm{d}z + \mathrm{d}w}{z - w} \right) |_{z=w} \mathrm{d}\overline{z}.$$

This putative integral is manifestly ill-defined, since the distribution is singular along the diagonal.

4.1.4. We smooth out the propagator $P$ using familiar tools from differential geometry. Fix a Hermitian metric on $\Sigma$, which then associates provides an adjoint $\overline{\partial}^*$ to the Dolbeault operator $\overline{\partial}$. For the usual metric on $\mathbb{C}$, we have

$$\overline{\partial}^* = -2\frac{\partial}{\partial(\mathrm{d}\overline{z})}\frac{\partial}{\partial z}.$$

In physics one calls a choice of $\overline{\partial}^*$ a *gauge-fix* as it OG: not sure how to end this sentence for noncompact manifolds ... maybe this comment should go elsewhere. The commutator $[\overline{\partial}, \overline{\partial}^*]$, which we will denote $D$, is equal to $\frac{1}{2}\Delta$, where $\Delta$ is the Laplace-Beltrami operator for this metric OG: correct?. We can thus call upon Hodge theory and many nice results about finding partial inverses to the Laplacian.

OG: I'm not sure how much to say here.

We introduce a smoothed version of the propagator using the heat kernel $e^{-tD}$, which is a notation that denotes a solution to the heat equation $\partial_t^2 f(t,z) = Df(t,z)$. For $\mathbb{C}$ with the Euclidean metric, the standard heat kernel is

$$e^{-tD}(z,w) = \frac{1}{4\pi t}e^{-|z-w|^2/4t}(\mathrm{d}z - \mathrm{d}w) \wedge (\mathrm{d}\overline{z} - \mathrm{d}\overline{w}).$$

For $0 < \ell < L < \infty$, we define

$$P_\ell^L = \overline{\partial}^* \int_\ell^L e^{-tD}\mathrm{d}t.$$

We compute

$$\overline{\partial}P_\ell^L = D\int_\ell^L e^{-tD}\mathrm{d}t = \int_\ell^L \frac{d}{dt}e^{-tD}\mathrm{d}t = e^{-LD} - e^{-\ell D}.$$

In the limit as $\ell \to 0$ and $L \to \infty$, the operator $P_\ell^L$ goes to a propagator (or Green's function) $P$ for $\overline{\partial}$. To see this, consider an eigenfunction $f$ of $D$ where $Df = \lambda f$. OG: with our conventions, is $\lambda$ positive or negative? Then

$$(\overline{\partial}P_\ell^L)f = (e^{-L\lambda} - e^{-\ell\lambda})f,$$

which goes to $f$ as $L \to \infty$ and $\ell \to 0$. OG: I want to be careful about this since eigenfunction decomposition is subtle on noncompact manifolds ... Thus, if one works with the correct space of functions, $P_\ell^L$ is almost an inverse to $\overline{\partial}$; moreover, it is a smooth function on $\Sigma \times \Sigma$. OG: Should I say why?

4.1.5. We now return to the tadpole diagram and put $P_\ell^L$ on the edge instead of $P$. The propagator is

$$\begin{aligned} P_\ell^L(z,w) &= \int_\ell^L \mathrm{d}t\, \frac{\partial}{\partial(\mathrm{d}\overline{z})}\frac{\partial}{\partial z}\left(\frac{1}{4\pi t}e^{-|z-w|^2/4t}(\mathrm{d}z - \mathrm{d}w) \wedge (\mathrm{d}\overline{z} - \mathrm{d}\overline{w})\right) \\ &= \int_\ell^L \mathrm{d}t \frac{1}{4\pi t}\frac{\overline{z}-\overline{w}}{2t}e^{-|z-w|^2/4t}(\mathrm{d}z - \mathrm{d}w). \end{aligned}$$

Note that it is smooth everywhere on $\mathbb{C}^2$. The integral for the tadpole diagram is

$$\begin{aligned}\int_{z\in\mathbb{C}} c(z)P_\ell^L(z,w)|_{z=w} &= \int_{z\in\mathbb{C}}\int_\ell^L \mathrm{d}t f(z)\partial_z\left(\frac{1}{4\pi t}\frac{\overline{z}-\overline{w}}{2t}e^{-|z-w|^2/4t}(\mathrm{d}z-\mathrm{d}w)\right)|_{z=w}\,\mathrm{d}\overline{z}\\ &= \int_{z\in\mathbb{C}}\int_\ell^L \mathrm{d}t f(z)\left(\frac{1}{4\pi t}\left(\frac{\overline{z}-\overline{w}}{2t}\right)^2 e^{-|z-w|^2/4t}(\mathrm{d}z-\mathrm{d}w)\right)|_{z=w}\,\mathrm{d}\overline{z}\\ &= 0,\end{aligned}$$

since the integrand vanishes along the diagonal. Note that this integral is independent of $\ell$ and $L$ and hence the limit is zero.

4.1.6. By explicitly analyzing the $\ell\to 0$ limit for the integral associated to every Feynman diagram, we find the following result.

**Proposition 4.3.** *OG: Insert statement*

The necessary manipulations and inequalities are very close to those used in []. We recommend looking at OG: exact location for model arguments.

OG: Define the renormalized action at scale $L$ as the sum of the Feynman diagrams labeled by $P_0^L$.

OG: Will also remark that this flow satisfies an exact RG equation, and Kevin calls it "RG flow" (but not to be confused with Wilsonian RG flow)

4.2. **The quantum master equation.** In the BV formalism the basic idea is to replace integration against a path integral measure $e^{-S(\phi)/\hbar}\mathcal{D}\phi$ with a cochain complex. In this cochain complex, we view a cocycle as defining an observable of the theory, and its cohomology class is viewed as its expected value against the path integral measure. For toy models of finite-dimensional integration, see []; these examples are always cryptomorphically equivalent to a de Rham complex, which is a familiar homological approach to integration.

Hence the content of the path integral, in this approach, is encoded in the differential. A key idea is that the differential is supposed to behave like a divergence operator for a volume form: recall that given a volume form $\mu$ on a manifold, its divergence operator maps vector fields to functions by the relationship

$$\mathrm{div}_\mu(\mathcal{X})\mu = L_{\mathcal{X}}\mu.$$

This relationship, in conjunction with Stokes lemma, implies that if a function $f$ is a divergence $\mathrm{div}_\mu(\mathcal{X})$, then $\int f\mu = 0$, i.e., its expected value against the measure $\mu$ is zero. The BV formalism axiomatizes general properties of divergence operators; a putative differential must satisfy these properties to provide a BV quantization.

When following the algorithm of Section **??**, we want the renormalized action

$$S = S^{\mathrm{cl}} + \hbar S_1 + \hbar^2 S_2 + \cdots$$

to determine a putative differential $\mathrm{d}_S^q$ on the graded vector space of observables. To explain this operator, we need to describe further algebraic properties on the observables that the BV formalism uses.

First, in practice, the observables are the symmetric algebra generated by the continuous linear duals to the vector spaces of fields. There is also a pairing on fields that is part of the data of the classical BV theory, between each field and its "anti-field." (This pairing is a version of the action of constant vector fields on functions in the toy models.) In our case, there is the pairing between $b$ and $c$ and between $\beta$ and $\gamma$, respectively. It behaves like a "shifted symplectic" pairing as it has cohomological degree $-1$, and hence it determines a degree 1 Poisson bracket $\{-,-\}$ on the graded algebra of observables. Finally, the pairing also determines a second-order differential operator $\Delta_{BV}$ on the algebra of observables by the condition that

$$\Delta_{BV}(FG) = (\Delta_{BV}F)G + (-1)^F F(\Delta_{BV}G) + \{F,G\}.$$

(This equation is a characteristic feature of divergence operators with respect to the product of polyvector fields.)

With these structures in hand, we can give the formula

$$\mathrm{d}_S^q = \{S,-\} + \hbar\Delta_{BV}$$

for the putative differential. As $S$ has cohomological degree 0, the operator $\{S,-\}$ has degree 1. We remark that modulo $\hbar$, one recovers the differential $\{S^{\mathrm{cl}},-\}$ on the classical observables; the zeroth cohomology of the classical observables is functions on the critical locus of the classical action $S^{\mathrm{cl}}$.

By construction, this putative differential $\mathrm{d}_S^q$ satisfies the conditions of behaving like a divergence operator. The only remaining condition to check is that it is square-zero. This condition ends up being equivalent to $S$ satisfying the *quantum master equation*

$$\hbar\Delta_{BV}S + \frac{1}{2}\{S,S\} = 0. \tag{5}$$

More accurately, $\mathrm{d}_S^q$ is a differential if and only if the right hand side is a constant.

4.2.1. We now turn to examining this condition in our setting. It helps to understand it is diagrammatic terms.

As the bracket is determined by a linear pairing, it admits a simple diagrammatic description as an edge. For instance, given an observable $F$ that is a homogeneous polynomial of arity $m$ and an observable $G$ of arity $n$, then $\{F,G\}$ has arity $m+n-2$. It can be expressed as a Feynman diagram OG: add picture where the edge connecting $F$ and $G$ is labeled by a 2-fold tensor $K$.

The BV Laplacian acts by attaching an edge labeled by $K$ as a loop in all possible ways. OG: Add picture. This diagrammatic behavior corresponds to the fact that $\Delta_{BV}$ is a constant-coefficient second-order differential operator.

The tensor $K$ determined by the pairing on fields is distributional. As one might expect from our discussion of divergences above, these diagrammatic descriptions of the BV bracket and Laplacian are thus typically ill-defined. In other words, the quantum master equation is *a priori* ill-posed for the same reason that the initial Feynman diagrams are ill-defined. We can apply, however, the same cure of mollification.

4.2.2. Costello's framework **[?]** provides an approach to renormalization built to be compatible with the BV formalism. A key feature is that for each "length scale" $L > 0$, there is a BV bracket $\{-,-\}_L$ and BV Laplacian $\Delta_L$. The scale $L$ renormalized action $S[L]$ satisfies the scale $L$ quantum master equation (QME)

$$\hbar \Delta_L S[L] + \frac{1}{2}\{S[L], S[L]\}_L = 0$$

if and only if $S[L']$ satisfies the scale $L'$ quantum master equation for every other scale $L'$. OG: Cite Kevin's lemma in book. Hence, we say a renormalized action satisfies the quantum master equation if its solves the scale $L$ equation for some $L$.

Thus it remains for us to describe the scale $L$ bracket and BV Laplacian in our setting, so that we can examine whether the renormalized action satisfies the quantum master equation.

**Definition 4.4.** The *scale L bracket* $\{-,-\}_L$ is given by pairing with the scale $L$ heat kernel

$$K_L(z,w) = \frac{1}{4\pi t} e^{-|z-w|/4t} (\mathrm{d}z - \mathrm{d}w) \wedge (\mathrm{d}\bar{z} - \mathrm{d}\bar{w}).$$

The *scale L BV Laplacian* $\Delta_L$ is given by the contraction $\partial_{K_L}$.

These definitions mean that testing the quantum master equation leads to diagrams whose integrals are similar to those we encountered earlier. We explain the diagrammatics and sketch the relevant integrals in the proof of the following result.

**Proposition 4.5.** *OG: Describe the obstruction of the bosonic string to solve the QME as a function of the dimension of the target.*

*Sketch of proof.* It is a general feature of Costello's formalism that the tree-level term $S_0[L]$ of the renormalized action satisfies the scale $L$ equation

$$\{S_0[L], S_0[L]\}_L = 0,$$

known as the classical master equation. Hence the first obstruction to satisfying the QME can only appear with positive powers of $\hbar$. We can also see quickly that no terms of $\hbar^2$ appear: the one-loop term $S_1[L]$ is only a function of the $c$ field, so

$$\{S_1[L], S_1[L]\}_L = 0 \quad \text{and} \quad \Delta_L S_1[L] = 0.$$

Hence the obstruction to satisfying the QME is precisely

$$\hbar \left( \{S_0[L], S_1[L]\} + \Delta_L S_0[L] \right).$$

Let $Ob$ denote the term inside the parenthesis (i.e., we drop the $\hbar$ term going forward). OG: Add the rest ... $\square$

## 5. OPE AND THE STRING VERTEX ALGEBRA

BW: Write down vertex algebra from quantization above. Possibly state the relationship to semi-infinite cohomology

**Proposition 5.1.** *Let* $\mathrm{Obs}^q$ *be the factorization algebra on* $\Sigma = \mathbb{C}$ *of the holomorphic bosonic string. The factorization product of open disks* $D \subset \mathbb{C}$ *determines the structure of a vertex algebra (see Proposition* **??** *below) on the cohomology of the factorization algebra on an open disk* $H^*(\mathrm{Obs}^q(D))$*, that we denote* $\mathbb{V}\mathrm{ert}(\mathrm{Obs}^q)$*. Moreover, there is an isomorphism of vertex algebras*

$$\Phi : V^{\text{string}} \xrightarrow{\cong} \mathbb{V}\mathrm{ert}(\mathrm{Obs}^q).$$

OG: I believe these vertex algebras are cohomologically graded, unless we're lucky and the cohomology all sits in degree zero. In which case, we should point out this miracle. Perhaps better would be to extract the dg vertex algebra.

OG: Do you know a citation where the string vertex algebra is already written down? Of course it's almost explicit in any discussion of the "modern"/BRST quantization of the bosonic string, where they write down $Q$, which ought to be the differential of the dg vertex algebra using our construction.

BW: How deformations discussed in Section 3 gives explicit deformations of the the vertex algebra.

## 6. Global sections of the factorization algebra

BW: Discuss relationship to conformal blocks??

We wish to write the global observables of the holomorphic string in terms of the cohomology of natural holomorphic vector bundles on the Riemann surface.

In Section **??** we have introduced the $bc\beta\gamma$ system defined on an Riemann surface $\Sigma$. This theory is free, and hence admits a canonical BV quantization. Denote by $\mathrm{Obs}^{\mathrm{q}}_{free}$ be the corresponding factorization algebra. In fact, Proposition 8.1.4.1 in **[?]**, there is an isomorphism of the cohomology of the global observables of this factorization algebra:

$$H^*\left(\mathrm{Obs}^{\mathrm{q}}_{free}(\Sigma)\right) \cong \det\left(H^*(\Sigma; \mathcal{O}_\Sigma)\right)^{\otimes \dim(V)} \otimes \det\left(H^*(\Sigma; T^{1,0}_\Sigma)\right)^{-1} [d(\Sigma)]$$

where

$$d(\Sigma) = \dim(V) \cdot \left(\dim H^0(\Sigma; \mathcal{O}_\Sigma) + \dim H^1(\Sigma; \mathcal{O}_\Sigma)\right) + \dim(H^0(\Sigma; T^{1,0}_\Sigma)) - \dim(H^1(\Sigma; T^{1,0}_\Sigma))$$

OG: It would be good here to point out the GRR argument for identifying the first Chern class of this determinant line bundle. We can then point out that it resonates with our computation on the formal disk and give a reminder that this is a manifestation of "Virasoro uniformization." This then leads into the string case: the holomorphic vector fields (rather the free $bc$ system) also contribute a determinant line (we should include that computation) and we can ask when those determinant lines tensor to a trivial line. Note that deforming the action from free $bc\beta\gamma$ system to holomorphic bosonic string doesn't affect the line bundles, since those are continuous parameters and Chern classes are discrete.

6.1. **The holomorphic string on a Riemann surface.** We have constructed the quantization of the holomorphic string on $\mathbb{C}$ and derived from its factorization algebra a vertex algebra. The only anomaly cancellation condition to have such a quantization was the restriction that the target be 13 complex dimensional. Here, we see why our anomaly calculation above is actually enough to show the existence of a quantization on an arbitrary Riemann surface. It is clear that in the case of a flat target that the only anomaly is purely a functional of the $c$-fields (holomorphic vector fields on $\Sigma$). Hence, the part of the deformation complex that contains the anomaly is:

$$\Omega^*_\Sigma \otimes_{D_\Sigma} C^*_{\mathrm{Lie,red}}(J\Omega^{0,*}(\Sigma; T_\Sigma))[2].$$

Now Proposition 5.3 of **??** shows that this sheaf of complexes is quasi-isomorphic to the constant sheaf $\mathbb{C}[-1]$ concentrated in degree $+1$. Now, a main result of the Costello-Gwilliam formulation of BV quantization is that anomalies are *local*: the anomaly computed on an open set $U \subset \Sigma$ is equal to the anomaly of the theory restricted to $U \subset \Sigma$. In particular, since the anomaly is an element of the constant sheaf, it suffices to compute the anomaly on an arbitrary open in the Riemann surface, which we may take to be $U \cong \mathbb{C}$ with its flat metric. This is precisely the context in which we computed the anomaly in Section **??**.

Thus, just as in the case of a flat source, a quantization of the holomorphic string exists on any Riemann surface provided $\dim_{\mathbb{C}}(V) = 13$. Now, there is a moduli of quantizations. Indeed, by the calculation of the $\mathbb{C}^\times_{\mathrm{cot}}$-weight zero component of the deformation complex in Section **??** we see that the space of deformations for quantizations at one loop are a torsor for the vector space

$$H^0(\Sigma) \otimes \Omega^1(V) \oplus H^1(\Sigma) \otimes \Omega^2_{cl}(V).$$

Thus, the space of deformations is given by a space of continuous parameters and hence will not change the class of the line bundle of global observables. In conclusion, we are free to choose any one-loop quantization; this will not affect the cohomology of the global observables.

6.2. Now, let us consider the global observables of the bosonic string $\mathrm{Obs}^{\mathrm{q}}(\Sigma)$. There is a spectral sequence converging to the cohomology of the global observables $H^*\mathrm{Obs}^{\mathrm{q}}(\Sigma)$ with $E_2$ page given by the cohomology of the global observables of the $bc\beta\gamma$ system which we have already computed:

$$\begin{aligned} E_2 &\cong \det\left(H^*(\Sigma; \mathcal{T}_\Sigma[1])\right) \otimes \det\left(H^*(\Sigma; \mathcal{O}_\Sigma)^{\oplus 13}\right) \\ &\cong \det\left(H^1(\Sigma; \mathcal{T}_\Sigma)\right) \otimes \det\left(H^0(\Sigma; \mathcal{T}_\Sigma)\right)^{-1} \otimes \det\left(H^0(\Sigma; K_\Sigma)\right)^{-13} \end{aligned}$$

where we have used the fact that $H^0(\Sigma; \mathcal{O}) \cong \mathbb{C}$ for any $\Sigma$.

Let $\Sigma_g$ be a surface of genus $g$. Then for $g = 1$ the above simplifies to

$$\det\left(H^1(\Sigma_1; \mathcal{T}_{\Sigma_1})\right) \otimes \det\left(H^0(\Sigma_1; K)\right)^{-14}.$$

If $g \geq 2$ one has

$$\det\left(H^1(\Sigma_1; \mathcal{T}_{\Sigma_1})\right) \otimes \det\left(H^0(\Sigma_1; K)\right)^{-13}.$$

It is clear that the spectral sequence degenerates at this page. Thus the above expressions give the global observables for the holomorphic string for genus $g = 1$ and $g \geq 2$, respectively.

## 7. The string partition function on an elliptic curve

We will evaluate the partition function of the theory on an elliptic curve. For $\tau \in \mathbb{H}$, the upper half plane, we can write an elliptic curve as a quotient of $\mathbb{C}$ by the lattice $\mathbb{Z} + \tau\mathbb{Z}$

$$E_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}).$$

Let $\mathrm{d}^2 z = \mathrm{d}z\mathrm{d}\bar{z}$ be the volume form on $\mathbb{C}$ which descends to one on $E_\tau$ that we denote by the same name.

We have already discussed the gauge fixing operator and constructed the heat kernel that regularizes the operator $\bar{\partial}^{-1}$ on $\mathbb{C}$. We can make the same choice of gauge fixing operator, namely $\bar{\partial}^*$, on the elliptic curve $E_\tau$ and hence we obtain a heat kernel for the operator $e^{-tD^{E_\tau}}$ where $D^{E_\tau}$ is the commutator $[\bar{\partial}, \bar{\partial}^*] = \frac{1}{2}\Delta$ on $E_\tau$. Explicitly, the heat kernel of the operator $e^{-tD^{E_\tau}}$ on the elliptic curve $E_\tau$ pulled back to the universal cover $\mathbb{C}$ along the $\mathbb{C} \to E_\tau$ is of the form

$$K_t^{E_\tau}(z, w) = \sum_{\lambda \in \mathbb{Z}+\tau\mathbb{Z}} \frac{1}{2\pi t} e^{-|z-w+\lambda|^2/2t} (\mathrm{d}z - \mathrm{d}w)(\mathrm{d}\bar{z} - \mathrm{d}\bar{w}).$$

From $K_t^{E_\tau}$ we obtain the propagator $P_{\ell<L}^{E_\tau} = \bar{\partial}^* \int_\ell^L e^{-tD^{E_\tau}} \mathrm{d}t$ just as above.

BW: Sketch how to see the mumford form

OG: If possible, it would be cool to explain how one can extract the differential equations (=flat connection) governing the partition function from our construction. This might be too hard right now …

## 8. Looking ahead: curved targets

OG: I think here we can mention our CDO work and assert that it's compatible with the discussion here. Then we state the corresponding theorems.

BW: State the quantization condition for curved target.