CRITICAL CASE

We compute the critical wheel. In dimension d this is the wheel with $d+1$ vertices. On $\mathbb{C}^d$ we have seen the obstruction lives in

$$\mathrm{H}^1_{\mathrm{loc}}(\mathscr{L}^{\mathbb{C}^d})$$

which we can identify with $\mathrm{H}^{2d+2}(\mathrm{BU}(d))$. Write the generators of this space as c_σ where σ labels degree $2d+2$ characteristic classes in d-(complex) dimensions. For instance when $d=2$ we have two admissible indices and they correspond to

$$c_{\sigma_1} = c_1^3 \ , \ c_{\sigma_2} = c_1 c_2.$$

For coefficients in the tensor bundle $E = T_X^{\otimes n} \otimes (T_X^\vee)^{\otimes m}$ we will find polynomials $p_\sigma(n,m) \in \mathbb{Q}[n,m]$ such that under the above isomorphism the obstruction has the form

$$\Theta(d) = \sum_\sigma p_\sigma(n,m) c_\sigma.$$

1. $\dim_{\mathbb{C}} = 1$

We take coefficients in the tensor bundle $T_{\mathbb{C}}^{\otimes n}$, i.e. $m=0$ in the above notation.

Recall that

$$f(z)\partial_z \cdot (\phi(z)\partial_z^{\otimes n}) := f(z)[\partial_z \phi(z)]\partial_z^{\otimes n} - n\phi(z)[\partial_z f(z)]\partial_z^{\otimes n},$$

extending the adjoint action to the n-fold power.

The obstruction has the form $\Theta(1) = \lim_{\epsilon\to 0} \Theta_\epsilon(1)$ where $\Theta_\epsilon(1)$ is the functional that sends $(f\partial_z, g\mathrm{d}\overline{z}\partial_z)$ to

$$\begin{aligned} & \int_{(z_0,z_1)\in\mathbb{C}\times\mathbb{C}} f(z_0)[\partial_{z_0} P_\epsilon^L(z_0,z_1)] g(z_1)[\partial_{z_1} K_\epsilon(z_0,z_1)] \\ - \ & n\left(\int_{(z_0,z_1)\in\mathbb{C}\times\mathbb{C}} [\partial_{z_0} f(z_0)] P_\epsilon^L(z_0,z_1) g(z_1)[\partial_{z_1} K_\epsilon(z_0,z_1)] - \int_{(z_0,z_1)\in\mathbb{C}\times\mathbb{C}} f(z_0)[\partial_{z_0} P_\epsilon^L(z_0,z_1)][\partial_{z_1} g(z_1)] K_\epsilon(z_0,z_1)\right) \\ + \ & n^2\left(\int_{(z_0,z_1)\in\mathbb{C}\times\mathbb{C}} [\partial_{z_0} f(z_0)] P_\epsilon^L(z_0,z_1)[\partial_{z_1} g(z_1)] K_\epsilon(z_0,z_1)\right). \end{aligned}$$

Recall that

$$K_t(z_0,z_1) = \frac{1}{t} e^{-|z_0-z_1|/t} \ , \ P_\epsilon^L(z_0,z_1) = \int_\epsilon^L (\overline{\partial}^* \otimes 1) K_t(z_0,z_1).$$

As in the previous sections we will utilize the change of coordinates $y_0 = z_0 - z_1$, $y_1 = z_1$. We compute the three basic integrals.

(I) First consider

$$\int_{(z_0,z_1)\in\mathbb{C}\times\mathbb{C}} \left(f(z_0)\partial_{z_0} P_\epsilon^L(z_0,z_1)\right)\left(g(z_1)\partial_{z_1} K_\epsilon(z_0,z_1)\right).$$

After making the change of coordinates this is

$$\int_{y_0,y_1}\int_\epsilon^L fg\epsilon^{-2}t^{-3}y_0^3 \exp\left(-\left(\frac{1}{t}+\frac{1}{\epsilon}\right)y_0\right) \mathrm{d}t\mathrm{dvol}_{\mathbf{y}}.$$

Taylor expanding fg in y_0 and performing Wick in the y_0-variable this reduces to

$$\int_{y_1} \partial_{y_1}^3 (fg) \mathrm{d}y_1 \mathrm{d}\overline{y}_1 \int_\epsilon^L \epsilon^{-2} t^{-3} \frac{\epsilon t}{\epsilon + t} \left[\left(\frac{\epsilon t}{\epsilon + t} \right)^3 + \text{higher order terms} \right].$$

(II) Next we evaluate

$$\int_{(z_0,z_1)\in\mathbb{C}\times\mathbb{C}} \left(f(z_0)\partial_{z_0}\partial_{z_1}P_\epsilon^L(z_0,z_1)\right)\left(g(z_1)K_\epsilon(z_0,z_1)\right) = \int_{z_0,z_1} fg(\partial_{z_0}\partial_{z_1}PK).$$

After changing coordinates and Wick expanding as above this has the form

$$\int_{y_1} \partial_{y_1}^3 (fg) \mathrm{d}y_1 \mathrm{d}\overline{y}_1 \int_\epsilon^L \epsilon^{-1} t^{-4} \frac{\epsilon t}{\epsilon + t} \left[\left(\frac{\epsilon t}{\epsilon + t} \right)^3 + \text{higher order terms} \right].$$

(III) Finally we evaluate

$$\int_{(z_0,z_1)\in\mathbb{C}\times\mathbb{C}} \left(f(z_0)P_\epsilon^L(z_0,z_1)\right)\left(g(z_1)\partial_{z_0}\partial_{z_1}K_\epsilon(z_0,z_1)\right) = \int_{z_0,z_1} fg(P\partial_{z_0}\partial_{z_1}K).$$

Changing coordinates and Wick expanding this reduces to

$$\int_{y_1} \partial_{y_1}^3 (fg) \mathrm{d}y_1 \mathrm{d}\overline{y}_1 \int_\epsilon^L \epsilon^{-3} t^{-2} \frac{\epsilon t}{\epsilon + t} \left[\left(\frac{\epsilon t}{\epsilon + t} \right)^3 + \text{higher order terms} \right]$$

Since only the lowest order terms in the above integrals contribute in the limit as $\epsilon \to 0$ we will forget about the higher order terms in the following calculation. Consider the first integral in the second line above. We integrate by parts to get

$$\int (\partial_{z_0} fP)(g\partial_{z_1}K) = -\left(\int fg\partial_0 P\partial_{z_1}K + \int fgP\partial_{z_0}\partial_{z_1}K\right).$$

Integration by parts applied to the second line gives

$$\int (f\partial_{z_0}P)(\partial_{z_1}gK) = -\left(\int fg\partial_{z_0}\partial_{z_1}PK + \int fg\partial_{z_0}P\partial_{z_1}K\right).$$

Finally, we can integrate by parts twice for the term in the last line to get

$$\begin{aligned}\int (\partial_{z_0} fP)(\partial_{z_1}gK) &= -\left(\int f\partial_{z_0}P(\partial_{z_1}gK) + \int (fP)(\partial_{z_1}g\partial_{z_0}K)\right)\\ &= 2\int fg\partial_{z_0}P\partial_{z_1}K + \int fg\partial_{z_0}\partial_{z_1}P + \int fgP\partial_{z_0}\partial_{z_1}K.\end{aligned}$$

Thus we see the obstruction has the form

$$\begin{aligned}\Theta_\epsilon(1)(f\partial_z, g\mathrm{d}\overline{z}\partial_z) &= (1+2n+2n^2)\mathrm{I} + (n^2+n)(\mathrm{II}+\mathrm{III})\\ &= \int_{y_1} \partial_{y_1}^3 (fg) \mathrm{d}y_1 \mathrm{d}\overline{y}_1 \int_{t=\epsilon}^L \frac{\epsilon}{(\epsilon+t)^4}\left(\epsilon t + (n+n^2)(2\epsilon t + \epsilon^2 + t^2)\right)\mathrm{d}t\\ &= \int_{y_1} \partial_{y_1}^3 (fg) \mathrm{d}y_1 \mathrm{d}\overline{y}_1 \int_{t=\epsilon}^L \frac{\epsilon}{(\epsilon+t)^4}\left(\epsilon t + (n+n^2)(\epsilon+t)^2\right)\mathrm{d}t.\end{aligned}$$

In the limit as $\epsilon \to 0$ we get

$$\Theta(1) = \int_{y_1} \partial_{y_1}^3 (fg) \mathrm{d}y_1 \mathrm{d}\overline{y}_1 \left(\frac{1}{12} + \frac{1}{2}(n^2+n)\right) = \frac{1}{12}\int_{y_1} \partial_{y_1}^3 (fg) \mathrm{d}y_1 \mathrm{d}\overline{y}_1 (1+6n+6n^2).$$

Identifying the cocycle

$$\frac{1}{12}\int_{y_1} \partial_{y_1}^3 (fg) \mathrm{d}y_1 \mathrm{d}\overline{y}_1$$

with the generator for the obstruction group we conclude that the polynomial for $d=1$ and $m=0$ is

$$p(n,0) = 1 + 6n + 6n^2.$$

I know this isn't exactly the argument you had in mind but this is a slightly different way of using integration by parts and shows that we really only need to know how to evaluate TWO rational integrals. I think something similar should happen in higher dimensions.

In another direction, I think you are right in that we could try to just look at polynomial inputs and utilize a "recursive" Wick's lemma. I'm working on writing that up now for dimension 1.

2. General dimension

Now, we work on $\mathbb{C}^d$. We modify a calculation of Si to calculate the obstruction. As we have already seen, only the wheel with $d+1$ vertices contributes nontrivially. Thus, the obstruction is described by a functional

$$\Theta : \Omega^{0,*}(\mathbb{C}, T\mathbb{C})^{\otimes(d+1)} \to \mathbb{C}$$

which is explicitly described by the weight $W_{\gamma_d}(P_\epsilon^L, K_\epsilon)$. On a $d+1$-tuple of vector fields $(\vec{\xi}_0, \ldots, \vec{\xi}_d)$ its value is

$$\Theta(\vec{\xi}_0, \ldots, \vec{\xi}_d) = \int_{(z_\alpha)\in(\mathbb{C}^d)^{d+1}} \left(\vec{\xi}_d \cdot K_\epsilon\right) \prod_{\alpha=0}^{d-1} \left(\vec{\xi}_\alpha \cdot P_{\alpha,\alpha+1}\right)$$

where $P_{\alpha,\alpha+1} = P(z_\alpha, z_{\alpha+1})$ and $K_\epsilon = K_\epsilon(z_0, z_d)$.

We have a canonical framing

$$\Omega^{0,*}(\mathbb{C}, T\mathbb{C}) = \Omega^{0,*}(\mathbb{C}) \otimes \mathbb{C}\left\langle \partial_{z^1}, \ldots, \partial_{z^d}\right\rangle.$$

Fix a collection of integers $i_0, \ldots, i_d \in \{1, \ldots, d\}$. We will compute

$$\Theta(A_0 \partial_{z_0^{i_0}}, \ldots, A_d \partial_{z_d^{i_d}})$$

where $A_\alpha \in \Omega^{0,*}(\mathbb{C})$ for $0 \le \alpha \le d$.

First, we compute $\partial_{z_\alpha^{i_\alpha}} P_{\alpha,\alpha+1} = (\overline{z}_\alpha^{i_\alpha} - \overline{z}_{\alpha+1}^{i_\alpha})\tilde{P}_{\alpha,\alpha+1}$ where

$$\tilde{P}_{\alpha,\alpha+1} = \int_{t_\alpha\in[\epsilon,L]} t_\alpha^{-1} \overline{\partial}^*_{z_\alpha} K_{t_\alpha}(z_\alpha, z_{\alpha+1})\, \mathrm{d}t_\alpha.$$

Similarly, $\partial_{z_d^{i_d}} K_\epsilon = (\overline{z}_d^{i_d} - \overline{z}_0^{i_d})\tilde{K}_\epsilon$ where $\tilde{K}_\epsilon = \epsilon^{-1} K_\epsilon$.

We also employ the following change of coordinates

$$w_\alpha = w_\alpha - w_{\alpha+1} \quad , \quad 0 \le \alpha < d$$

and $w_d = z_d$.

Up to a sign, this weight is

$$\int_{(w_\alpha)\in(\mathbb{C}^d)^{d+1}} \left(\prod_{\alpha=0}^{d-1} \overline{w}_\alpha^{i_\alpha} A_\alpha \wedge \mathrm{dvol}_\alpha^{\mathrm{hol}}\right) (\overline{w}_0^{i_d} + \cdots + \overline{w}_{d-1}^{i_d})\, \tilde{K}_\epsilon \int_{\mathbf{t}\in[\epsilon,L]^d} \prod_{\alpha=0}^{d} \tilde{P}_{\alpha,\alpha+1}\, \mathrm{dvol}_{\mathbf{t}}.$$

We compute the term

$$\begin{aligned} \tilde{K}_\epsilon \int_{\mathbf{t}\in[\epsilon,L]^d} \prod_{\alpha=0}^{d} \tilde{P}_{\alpha,\alpha+1}\, \mathrm{dvol}_{\mathbf{t}} \quad &= \quad \pm\epsilon^{-d-1} \int_{\mathbf{t}\in[\epsilon,L]^d} \left(\prod_{\alpha=0}^{d-1} t_\alpha^{-d}\right) \\ &\sum_{\mathbf{k}=(k_0,\ldots,k_{d-1})} \epsilon_{\mathbf{k}} \left(\prod_{\alpha=0}^{d-1} t_\alpha^{-1} \overline{w}_\alpha^{k_\alpha}\right) \exp(-\overline{w}^T M w) \left(\prod_{\alpha=0}^{d-1} \mathrm{d}^d \overline{w}_\alpha\right) \mathrm{dvol}_{\mathbf{t}}. \end{aligned}$$

Here, $\overline{w}^T M w$ denotes the quadratic form

$$\sum_{\alpha=0}^{d-1} |w_\alpha|^2/t_\alpha - |w_0 + \cdots + w_{d-1}|/\epsilon.$$

Thus, when we write A_α in the new w-coordinates only terms with $\mathrm{d}\overline{w}_d^j$ factors will contribute.