OBSTRUCTION FOR FLAT STRING

Obstruction calculation for the flat string. Throughout we work on $\Sigma = \mathbb{C}$. Let V denote the target vector space.

1. Deformation complex

The full deformation complex is

$$\mathrm{Def} = \left(\mathscr{O}_{\mathrm{loc}}(\mathscr{E}_{\mathrm{string}}), \overline{\partial} + \{I, -\}\right)$$

There is a subcomplex $\mathrm{Def}' \subset \mathrm{Def}$ consisting of those functionals that only depend on $\mathscr{E}_{\mathrm{grav}}$.

Proposition 1.1. *The obstruction lies in* Def'.

1.1. **Heat Kernels.** this might fit better in an appendix The heat kernel is a degree 1 element $\mathbb{K}_t \in (\mathscr{E}_{\mathrm{string}})^{\otimes 2}$ that depends smoothly on $t \geq 0$. It splits as

$$\mathbb{K}_t = \left(\mathbb{K}_t^{\beta\gamma} \otimes 1\right) \oplus (1 \otimes \mathbb{K}_t^{\mathrm{grav}}) \in \left(\left(\mathscr{E}_{\beta\gamma}^{\oplus 13}\right)^{\otimes 2} \otimes 1\right) \oplus (1 \otimes \mathscr{E}_{\mathrm{grav}})$$

For they single free $\beta\gamma$-theory we have the explicit form

$$\mathbb{K}_t^{\beta\gamma}(z, w) =$$

2. Anomaly cancelation

The machinery of [C1] allows us to define a pre-theory

$$I[L] := \lim_{\epsilon \to 0} W(\mathbb{P}_\epsilon^L, I - I^{\mathrm{CT}}(\epsilon)).$$

Lemma 2.1. *The counterterms vanish. In other words, the limit*

$$\lim_{\epsilon \to 0} W(\mathbb{P}_\epsilon^L, I)$$

exists, and we denote it by $I[L]$.

The obstruction has the form

$$\Theta[L] := \overline{\partial} I[L] + \{I[L], I[L]\}_L + \Delta_L I[L].$$

Lemma 2.2. *The obstruction is computed by*

$$\Theta[L] = \lim_{\epsilon} W(\mathbb{P}_\epsilon^L, \mathbb{K}_\epsilon, I).$$

Lemma 2.3. *Only wheels (of valency 3) contribute to the obstruction. Further, tadpole and wheels with vertices greater than 2 don't contribute. That leaves two wheels of with two vertices each. One of them has inputs $\beta\gamma$ the other has inputs vector fields.*

Thus, the obstruction is

$$\begin{aligned} \Theta[L] &= \lim_{\epsilon \to 0} W_{\gamma_2}(\mathbb{P}_\epsilon^L, \mathbb{K}_\epsilon, I) \\ &= \dim_{\mathbb{C}}(V) \cdot \lim_{\epsilon \to 0} W_{\gamma_2}(\mathbb{P}_\epsilon^{\beta\gamma, L}, \mathbb{K}_\epsilon^{\beta\gamma}, I) + \lim_{\epsilon \to 0} W_{\gamma_2}(\mathbb{P}_\epsilon^{\mathrm{string}, L}, \mathbb{K}_\epsilon^{\mathrm{string}}, I). \end{aligned}$$

We already know from the DEFORMATION section that these functionals are only nonzero on vector fields. Denote by $\Theta^{\beta\gamma}_\epsilon[L](\xi_0,\xi_1)$ the first term, which uses the $\beta\gamma$ edge/propagator on the interior wheel. Similarly, denote the second term by $\Theta^{\mathrm{grav}}_\epsilon[L](\xi_0,\xi_1)$, which uses the vector field edge/propagator on the interior wheel.

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We have

$$\Theta^{\beta\gamma}_\epsilon[L](\xi_0,\xi_1) \quad = \quad \int_{(z_0,z_1)\in\mathbb{C}^2} \left(\xi_0(z_0)\cdot \mathbb{P}^{\beta\gamma,L}_\epsilon(z_0,z_1)\right)\left(\xi_1\cdot \mathbb{K}^{\beta\gamma}_\epsilon(z_0,z_1)\right).$$

We assume that ξ_0,ξ_1 are homogenous. Since the obstruction is a degree one element it is nonzero provided $\deg(\xi_0)+\deg(\xi_1)=1$. For concreteness, suppose

$$\xi_0 = f(z,\overline{z})\partial_z \quad , \quad \xi_1 = g(z,\overline{z})\mathrm{d}\overline{z}\partial_z.$$

Then

$$\Theta^{\beta\gamma}_\epsilon[L](\xi_0,\xi_1) = \int_{(z_0,z_1)\in\mathbb{C}\times\mathbb{C}} \left(f(z_0)\partial_{z_0}P^L_\epsilon(z_0,z_1)\right)\left(g(z_1)\partial_{z_1}K_\epsilon(z_0,z_1)\right).$$

After making the change of coordinates this is

$$\int_{y_0,y_1}\int_\epsilon^L fg\epsilon^{-2}t^{-3}y_0^3\exp\left(-\left(\frac{1}{t}+\frac{1}{\epsilon}\right)y_0\right)\mathrm{d}t\mathrm{dvol}_{\mathbf{y}}.$$

Taylor expanding fg in y_0 and performing Wick in the y_0-variable this reduces to

$$\int_{y_1}\partial_{y_1}^3(fg)\mathrm{d}y_1\mathrm{d}\overline{y}_1\int_\epsilon^L \epsilon^{-2}t^{-3}\frac{\epsilon t}{\epsilon+t}\left[\left(\frac{\epsilon t}{\epsilon+t}\right)^3+\text{higher order terms}\right].$$

Taking the limit as $\epsilon\to 0$ we see that

$$\Theta^{\beta\gamma}[L] = \frac{1}{12}\int_{y_1}\partial^3_{y_1}(fg)\mathrm{d}y_1\mathrm{d}\overline{y}_1.$$

3. Computing Θ^{grav}

Note that $\nu\wedge[\xi,\xi] = \nu L_\xi\xi$ where L_ξ denotes Lie derivative.

$$\Theta^{\mathrm{grav}}_\epsilon[L](\xi_0,\xi_1) = \int_{(z_0,z_1)\in\mathbb{C}^2}\left(L_{\xi_0}\mathbb{P}^{\beta\gamma,L}_\epsilon(z_0,z_1)\right)\left(L_{\xi_1}\cdot\mathbb{K}^{\beta\gamma}_\epsilon(z_0,z_1)\right). \tag{1}$$

Again, we assume that ξ_0,ξ_1 are homogenous and we must have $\deg(\xi_0)+\deg(\xi_1)=1$. For concreteness, suppose

$$\xi_0 = f(z,\overline{z})\partial_z \quad , \quad \xi_1 = g(z,\overline{z})\mathrm{d}\overline{z}\partial_z.$$

Expanding out, the integral (1) splits into four parts

$$\begin{aligned}
\Theta^{\mathrm{grav}}_\epsilon[L](\xi_0,\xi_1) \quad &= \quad \int_{(z_0,z_1)\in\mathbb{C}\times\mathbb{C}} f(z_0)[\partial_{z_0}P^L_\epsilon(z_0,z_1)]g(z_1)[\partial_{z_1}K_\epsilon(z_0,z_1)] && (2)\\
&- \quad \int_{(z_0,z_1)\in\mathbb{C}\times\mathbb{C}}[\partial_{z_0}f(z_0)]P^L_\epsilon(z_0,z_1)g(z_1)[\partial_{z_1}K_\epsilon(z_0,z_1)] && (3)\\
&- \quad \int_{(z_0,z_1)\in\mathbb{C}\times\mathbb{C}} f(z_0)[\partial_{z_0}P^L_\epsilon(z_0,z_1)][\partial_{z_1}g(z_1)]K_\epsilon(z_0,z_1) && (4)\\
&+ \quad \int_{(z_0,z_1)\in\mathbb{C}\times\mathbb{C}}[\partial_{z_0}f(z_0)]P^L_\epsilon(z_0,z_1)[\partial_{z_1}g(z_1)]K_\epsilon(z_0,z_1). && (5)
\end{aligned}$$

We make the change of coordinates $y_0 = z_0 - z_1$, $y_1 = z_1$. Before evaluating the obstruction we compute the three basic integrals.

(I) First consider

$$\int_{(z_0,z_1)\in\mathbb{C}\times\mathbb{C}} \left(f(z_0)\partial_{z_0} P_\epsilon^L(z_0,z_1)\right)\left(g(z_1)\partial_{z_1} K_\epsilon(z_0,z_1)\right).$$

After making the change of coordinates this is

$$\int_{y_0,y_1}\int_\epsilon^L fg\epsilon^{-2}t^{-3}y_0^3 \exp\left(-\left(\frac{1}{t}+\frac{1}{\epsilon}\right)y_0\right) \mathrm{d}t\mathrm{dvol}_{\mathbf{y}}.$$

Taylor expanding fg in y_0 and performing Wick in the y_0-variable this reduces to

$$\int_{y_1} \partial_{y_1}^3(fg)\mathrm{d}y_1\mathrm{d}\overline{y}_1 \int_\epsilon^L \epsilon^{-2}t^{-3}\frac{\epsilon t}{\epsilon+t}\left[\left(\frac{\epsilon t}{\epsilon+t}\right)^3 + \text{higher order terms}\right].$$

(II) Next we evaluate

$$\int_{(z_0,z_1)\in\mathbb{C}\times\mathbb{C}} \left(f(z_0)\partial_{z_0}\partial_{z_1} P_\epsilon^L(z_0,z_1)\right)\left(g(z_1)K_\epsilon(z_0,z_1)\right) = \int_{z_0,z_1} fg(\partial_{z_0}\partial_{z_1}PK).$$

After changing coordinates and Wick expanding as above this has the form

$$\int_{y_1} \partial_{y_1}^3(fg)\mathrm{d}y_1\mathrm{d}\overline{y}_1 \int_\epsilon^L \epsilon^{-1}t^{-4}\frac{\epsilon t}{\epsilon+t}\left[\left(\frac{\epsilon t}{\epsilon+t}\right)^3 + \text{higher order terms}\right].$$

(III) Finally we evaluate

$$\int_{(z_0,z_1)\in\mathbb{C}\times\mathbb{C}} \left(f(z_0)P_\epsilon^L(z_0,z_1)\right)\left(g(z_1)\partial_{z_0}\partial_{z_1} K_\epsilon(z_0,z_1)\right) = \int_{z_0,z_1} fg(P\partial_{z_0}\partial_{z_1}K).$$

Changing coordinates and Wick expanding this reduces to

$$\int_{y_1} \partial_{y_1}^3(fg)\mathrm{d}y_1\mathrm{d}\overline{y}_1 \int_\epsilon^L \epsilon^{-3}t^{-2}\frac{\epsilon t}{\epsilon+t}\left[\left(\frac{\epsilon t}{\epsilon+t}\right)^3 + \text{higher order terms}\right]$$

Since only the lowest order terms in the above integrals contribute in the limit as $\epsilon \to 0$ we will forget about the higher order terms in the following calculation. Consider the first integral in the second line above. We integrate by parts to get

$$\int (\partial_{z_0} fP)(g\partial_{z_1}K) = -\left(\int fg\partial_0 P\partial_{z_1}K + \int fgP\partial_{z_0}\partial_{z_1}K\right).$$

Integration by parts applied to the second line gives

$$\int (f\partial_{z_0}P)(\partial_{z_1}gK) = -\left(\int fg\partial_{z_0}\partial_{z_1}PK + \int fg\partial_{z_0}P\partial_{z_1}K\right).$$

Finally, we can integrate by parts twice for the term in the last line to get

$$\begin{aligned}
\int (\partial_{z_0} fP)(\partial_{z_1}gK) &= -\left(\int f\partial_{z_0}P(\partial_{z_1}gK) + \int (fP)(\partial_{z_1}g\partial_{z_0}K)\right)\\
&= 2\int fg\partial_{z_0}P\partial_{z_1}K + \int fg\partial_{z_0}\partial_{z_1}P + \int fgP\partial_{z_0}\partial_{z_1}K.
\end{aligned}$$

Thus, we see the obstruction has the form

$$\begin{aligned}
\Theta_\epsilon^{\mathrm{grav}}[L](\xi_0,\xi_1) &= 5\mathrm{I} + 2(\mathrm{II}+\mathrm{III}) + [\text{higher } \epsilon \text{ terms}]\\
&= \int_{y_1} \partial_{y_1}^3(fg)\mathrm{d}y_1\mathrm{d}\overline{y}_1 \int_{t=\epsilon}^L \frac{\epsilon}{(\epsilon+t)^4}\left(\epsilon t + 2(2\epsilon t + \epsilon^2 + t^2)\right)\mathrm{d}t + [\text{higher } \epsilon \text{ terms}]\\
&= \int_{y_1} \partial_{y_1}^3(fg)\mathrm{d}y_1\mathrm{d}\overline{y}_1 \int_{t=\epsilon}^L \frac{\epsilon}{(\epsilon+t)^4}\left(\epsilon t + 2(\epsilon+t)^2\right)\mathrm{d}t + [\text{higher } \epsilon \text{ terms}].
\end{aligned}$$

In the limit as $\epsilon \to 0$ we get

$$\Theta^{\mathrm{grav}}[L](\xi_0, \xi_1) = \int_{y_1} \partial_{y_1}^3 (fg) \mathrm{d}y_1 \mathrm{d}\overline{y}_1 \left(\frac{1}{12} + 1 \right) = \frac{13}{12} \int_{y_1} \partial_{y_1}^3 (fg) \mathrm{d}y_1 \mathrm{d}\overline{y}_1 .$$