MA 725 - DIFFERENTIAL GEOMETRY, I

FINAL EXAM

Problem 1. Let q be a complex number satisfying $0 < q\bar{q} < 1$. Define

$$X_q = \left(\mathbf{C}^2 \setminus 0\right)/\mathbf{Z} \tag{1}$$

where the $\mathbf{Z}$-action is generated by $(z, w) \mapsto (qz, qw)$.

(a) Show that X_q can be endowed with the structure of a complex manifold with the property that the canonical map $\mathbf{C}^2 \setminus 0 \to X_q$ is holomorphic.

(b) Show that $X_q \simeq S^3 \times S^1$ as *smooth* manifolds.

(c) Does there exist a decomposition

$$H^k(X_q; \mathbf{C}) = \oplus_{p+q=k} H^q(X_q, \Omega^{p,hol}) \tag{2}$$

for $k = 0, 1, 2$? Explain why or why not.

(d) Is $\Omega^\bullet(X_{\mathbf{C}})$ formal as a commutative dg algebra?

Problem 2. Let Σ be an oriented smooth manifold of dimension two.

(a) Define a *conformal structure* on Σ

(b) Show that there is a bijective correspondence between conformal structures on Σ and complex structures on Σ.

Problem 3. Show that if G is a complex Lie group (meaning, a Lie group and a complex manifold for which the product and inverse operations are holomorphic) and compact, then G is abelian.

Problem 4. Let X be a compact complex manifold and suppose that ω, ω' are two Kähler forms on X which are cohomologous $[\omega] = [\omega'] \in H^2(X; \mathbf{R})$. Show that there exists a real function $f \in C^\infty(X)$ such that $\omega' = \omega + i\partial\bar{\partial} f$.