- Suppose $T: V \to W$ is a linear transformation. Recall, there is a special vector $\vec{0} = \vec{0}_W \in W$. (we will use $\vec{0}$ for the zero vector in V and W).

Suppose $x, y \in V$ are such that

$$T(x) = \vec{0} \quad \text{and} \quad T(y) = \vec{0}.$$

Then: $T(x+y) = T(x) + T(y) = \vec{0} + \vec{0} = \vec{0}$.

and $T(\lambda x) = \lambda T(x) = \lambda \cdot \vec{0} = \vec{0}$ for all $\lambda \in \mathbb{F}$.

<u>Theorem</u>: The kernel of $T: V \to W$ is the subspace:

$$\ker T = \{ x \in V \mid T(x) = \vec{0} \} \subset V.$$

<u>Ex</u>: Let $T: \mathbb{R}^2 \to \mathbb{R}$ be defined by:

$$T(a_1, a_2) = a_1$$

Then
$$\ker T = \{ (a_1, a_2) \mid T(a_1, a_2) = \vec{0} \}$$
$$= \{ (a_1, a_2) \mid a_1 = 0 \} = \text{"}y\text{-axis"}$$
$$= \operatorname{span}\{ e_2 \}.$$

• Suppose $z, w \in W$ are such that

$$z = T(x) \quad , \quad w = T(y).$$

for some $x, y \in V$. Then

$$z + w = T(x) + T(y) = T(x+y).$$

Similarly: $\lambda z = \lambda T(x) = T(\lambda x)$.

Thus, we see that the set of vectors in the image of T is a subspace.

Theorem: The image of $T : V \to W$ is the subspace

$$\operatorname{Im} T = \left\{ z \in W \;\middle|\; \begin{array}{l} T(x) = z \\ \text{for some } x \in V. \end{array} \right\} \subset W.$$

Ex: Define $T : \mathbb{R} \to \mathbb{R}^3$ by

$$T(a) = (a, a, a)$$

Then

$$\operatorname{Im} T = \operatorname{span}\{(1,1,1)\} = \underbrace{\operatorname{span}}_{\text{A line.}} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Theorem: Let $T : V \longrightarrow W$ be linear and $\{v_i\}_{i=1}^n$ is a basis for V. Then

$$\{T(v_i)\}_{i=1}^n$$

generates $\operatorname{Im} T$.

Pf: Clearly $T(v_i) \in \operatorname{Im} T$ for each i. Sps $w \in \operatorname{Im} T$ is arbitrary vector in image. Then we can find $v \in V$ s.t. $T(v) = w$.

Since $\{v_i\}$ is basis for V we can find $\lambda_1, \dots, \lambda_n$ s.t. $v = \lambda_1 v_1 + \dots + \lambda_n v_n$.

But then

$$\begin{aligned} w = T(v) &= T(\lambda_1 v_1 + \dots + \lambda_n v_n) \\ &= \lambda_1 T(v_1) + \dots + \lambda_n T(v_n). \end{aligned}$$

$\Rightarrow \quad w \in \operatorname{span}\{T(v_1), \dots, T(v_n)\}$. $\blacksquare$

Ex: Let $T : P_2 \longrightarrow \mathbb{R}^3$ be defined as

$$T(f) = \begin{bmatrix} f(1) - f(2) \\ 0 \\ f(0) \end{bmatrix}.$$

Since $\{1, x, x^2\}$ is basis for P_2 we know by the theorem that

$$\operatorname{Im} T = \operatorname{span}\{T(1), T(x), T(x^2)\}$$

$$= \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$= \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} = \{ xz\text{-plane} \}.$$

In particular, $\dim \operatorname{Im} T = 2$.

- Whats $\ker T$? Suppose

$$f = a + bx + cx^2 \in \ker T.$$

Then $f(1) = f(2)$ and $f(0) = 0$. So:

$$a + b + c = a + 2b + 4c \text{ and}$$
$$a = 0$$

$$\Rightarrow \quad b + c = 2b + 4c \quad \Rightarrow \quad \boxed{\begin{aligned} a &= 0 \\ b &= -3c \end{aligned}}$$

$\Rightarrow \quad \ker T = \text{span}\{-3x + x^2\}$.

In particular, $\dim \ker T = 1$.

Notice, in this example $T : P_2 \to \mathbb{R}^3$

$$\dim \ker T + \dim \operatorname{Im} T = 1 + 2 = 3$$

And also $\dim P_2 = 3$. This is no accident.

Theorem: Let $T : V \to W$ be any linear transformation.

Then
$$\boxed{\dim \ker T + \dim \operatorname{Im} T = \dim V.}$$

Pf: Let $n = \dim V$, $k = \dim \ker T$.

Start with a basis $\{u_1, \dots, u_k\}$ for $\ker T$.

Extend it to a basis $\{u_1, \dots, u_k, v_1, \dots, v_{n-k}\}$ for V.

Claim: $\{T(v_1), \dots, T(v_{n-k})\}$ is a basis for $\operatorname{Im} T$.

Proof of claim: Let $w \in \operatorname{Im} T$. So, there is $v \in V$ s.t. $T(v) = w$. Since $\{u_1, \dots, u_k, v_1, \dots, v_{n-k}\}$ is a basis for V we can find $\lambda_1, \dots, \lambda_k, \mu_1 \dots \mu_{n-k}$ s.t.

$$v = \lambda_1 u_1 + \dots + \lambda_k u_k + \mu_1 v_1 + \dots + \mu_{n-k} v_{n-k}.$$

Apply T:

$$w = T(v) = \sum_{i=1}^{k} \lambda_i \cancel{T(u_i)}^{\,0} + \sum_{j=1}^{n-k} \mu_j T(v_j)$$

$$= \vec{0} + \sum_{j=1}^{n-k} \mu_j T(v_j).$$

This shows that $w \in \operatorname{span}\{T(v_j)\}_{j=1}^{n-k}$.

It remains to prove $\{T(v_j)\}_{j=1}^{k}$ is lin. independent.

Suppose $\sum_{j=1}^{n-k} a_j T(v_j) = \vec{0}$. We can write as:

$$\sum_{j=1}^{n-k} a_j T(v_j) = \sum_{j=1}^{n-k} T(a_j v_j) = T\left(\sum_{j=1}^{n-k} a_j v_j\right)$$

$$\Rightarrow \sum_{j=1}^{n-k} a_j v_j \in \ker T.$$

But, since $\{u_1, \dots, u_n\}$ is basis for $\ker T$, we can find $b_1, \dots, b_k$ s.t.:

$$\sum_{j=1}^{n-k} a_j v_j = \sum_{i=1}^{k} b_i u_i .$$

$$\Rightarrow \quad -\sum_{i=1}^{k} b_i u_i + \sum_{j=1}^{n-k} a_j v_j = \vec{0}$$

But, since $\{u_1, \dots, u_k, v_1, \dots, v_{n-k}\}$ is lin. ind.

$$\Rightarrow \quad b_1 = b_2 = \dots = b_k = a_1 = a_2 = \dots = a_{n-k} = 0 .$$

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This completes the proof since

$$\begin{array}{ccccc} \dim V & = & \dim \ker T & + & \dim \operatorname{Im} T \\ \| & & & & \| \\ n & = & k & + & (n-k) . \end{array}$$

$\blacksquare$

• Recall the concept of injective. A function $T: V \to W$ is injective when $T(x) = T(y) \iff x = y$.

Theorem A linear map $T: V \to W$ is injective if and only if $\ker T = \{\vec{0}\}$.

Pf: $(\Rightarrow)$ Suppose T is injective. If $T(x) = \vec{0}$ the observe that $T(x) = T(\vec{0}) = \vec{0}$. Since T is injective we see $x = \vec{0}$. Thus $\ker T = \{\vec{0}\}$.

$(\Leftarrow)$ Sps $\ker T = \{\vec{0}\}$. If $T(x) = T(y)$ we can use linearity to see $T(x-y) = \vec{0}$. Thus $x - y = \vec{0}$ or $x = y$. $\boxtimes$

Theorem: Suppose $T: V \to W$ and $\dim V = \dim W$. The following are equivalent:

1) T is bijective. (injective and surjective).
2) T is injective (equiv. $\ker T = \{\vec{0}\}$).
3) T is surjective. So, $\operatorname{Im} T = W$.

__Pf__: This uses the dimension theorem.

Clearly 1) $\Rightarrow$ 2). We show 2) $\Rightarrow$ 3). So, let T be injective. By the last theorem, we know that $\ker T = \{\vec{0}\}$, or $\dim \ker T = 0$. Thus

$$\overbrace{\dim \ker T}^{0} + \dim \operatorname{Im} T = \dim V \overset{\text{assumption}}{=} \dim W.$$

$$\|$$

$$\dim \operatorname{Im} T$$

Thus $\dim \operatorname{Im} T = \dim W$. We also have a theorem saying if $Z \subset W$ is any subspace with $\dim Z = \dim W$ then $Z = W$. Thus $\operatorname{Im} T = W$ and hence T is surjective. This shows 2) $\Rightarrow$ 3)
(and 2) $\Rightarrow$ 1).)

Finally, we prove 3) $\Rightarrow$ 1).

If T is surjective then reading the dimension formula in reverse we see $\dim \ker T = 0$.

Thus $\ker T = \{\vec{0}\}$. And hence T is injective. $\boxtimes$