January 23

A algebra $\rightsquigarrow A^\times \subset A$ group of units.

The adjoint action is

$$\begin{aligned} \mathrm{Ad} : A^\times &\longrightarrow \mathrm{Aut}(A) \\ a &\longmapsto (x \longmapsto \underbrace{axa^{-1}}_{= \mathrm{Ad}_a(x)}). \end{aligned}$$

Sps $v \in V \subset Cl(V, q)$ is s.t. $q(v) \neq 0$. Then

$$v \cdot \left(- \frac{v}{q(v)} \right) = - \frac{v^2}{q(v)} = 1.$$

So $q(v) \neq 0 \Rightarrow v \in Cl(V, q)^\times$.

Moreover we have:

Prop: Sps $v \in V$, $q(v) \neq 0$. Then
$$Ad_v(w) = -w + 2\frac{\langle v, w\rangle}{\langle v, v\rangle} v.$$

Pf: $v^{-1} = -\dfrac{v}{\langle v, v\rangle}$. Thus

$$\begin{aligned}
Ad_v(w) &= v w v^{-1} \\
&= v w \left(-\frac{v}{\langle v, v\rangle}\right) \\
&= -\frac{1}{\langle v, v\rangle} v w v \\
&= -\frac{1}{\langle v, v\rangle}\left(-v^2 w - 2\langle v, w\rangle v\right). \\
&= -w + 2\frac{\langle v, w\rangle}{\langle v, v\rangle} v. \qquad \blacksquare
\end{aligned}$$

Note:

$Ad_v(w)$ = reflection of w wrt to the hyperplane $v^{\perp}$.

As a corollary, we see that for $q(v) \neq 0$ that the automorphism Ad_v preserves the subspace

$$V \subset C\ell(V, q).$$

Let $P(V, q) \subset C\ell(V, q)^\times$

be the subgroup generated by $v \in V$ w/ $q(v) \neq 0$.

$$P(V, q) = \left\{ v_1 \cdots v_k \;\middle|\; q(v_i) \neq 0 \right\}.$$

<u>Def</u>: Let $\mathrm{Pin}(V, q) \subset P(V, q)$

be the subgroup generated by $v \in V$ with $q(v) = \pm 1$.

Let

$$\mathrm{Spin}(V, q) = \mathrm{Pin}(V, q) \cap C\ell^+(V, q).$$

Thus

$$\mathrm{Pin}(V,q) = \left\{ v_1 \cdots v_k \;\middle|\; \begin{array}{l} k \text{ integer} \\ q(v_i) = \pm 1 \end{array} \right\}.$$

$$\mathrm{Spin}(V,q) = \left\{ v_1 \cdots v_k \in \mathrm{Pin}(V,q) \;\middle|\; k \text{ even} \right\}.$$

- We next define an homomorphism

$$\widetilde{Ad} : Cl(V,q)^\times \longrightarrow GL(Cl(V,q))$$

which agrees with Ad_φ for $\varphi \in Cl^+(V,q)$, but when $v \in V$,

$$\widetilde{Ad}_v = \text{reflection about } v^\perp.$$
$$= -Ad_v.$$

Explicitly

$$\widetilde{Ad}_\varphi(x) = \alpha(\varphi)\, x\, \varphi^{-1}.$$

where $\alpha\big|_{Cl^+} = 1$, $\alpha\big|_{Cl^-} = -1$.

Define

$$\tilde{P}(V,q) = \{\varphi \in Cl(V,q)^+ \mid \operatorname{Im} \widetilde{Ad}_\varphi = V\}.$$

$$\cup$$

$$P(V,q).$$

<u>Prop</u>: Consider the homomorphism

$$\widetilde{Ad} : \tilde{P}(V,q) \longrightarrow GL(V)$$

Then $\ker \widetilde{Ad} = \underbrace{k^\times 1}_{\text{Nonzero Multiples of unit.}} \cong k^\times$

<u>Pf</u>: Write $\varphi = \varphi_+ + \varphi_- \in \ker \widetilde{Ad}$.

Then $\varphi_+ v = v\varphi_+$, $\varphi_- v = -v\varphi_-$

for all $v \in V$.

Let $\{e_i\}$ be o.n.b for V.

Then by using the Clifford alg. relation, we can assume that

$$\varphi_+ = a_0 + a_1 e_1$$

where a_0, a_1 are polynomial expressions in $\{e_2, \dots, e_n\}$. Since φ_+ is even, a_0 is even and a_1 is odd. So:

$$a_0 e_1 + a_1 e_1^2 = e_1 a_0 + e_1 a_1 e_1$$

$$= a_0 e_1 - a_1 e_1^2$$

$$= a_0 e_1 - a_1 e_1^2$$

$\Rightarrow$ $a_1 = 0$.

Repeating, we see $\varphi_+ = a 1$ for $a \in k^\times$.

Similarly $\varphi_- = 0$. $\blacksquare$

Next we will show that $\widetilde{Ad}_\varphi$ is an orthogonal transformation.

For $\varphi \in Cl(V, q)$ define

$$N(\varphi) = \varphi \cdot \alpha(\varphi^t).$$

Note: for $v \in V$, $N(v) = v(-v) = q(v)$.

Sps $\varphi \in \tilde{P}(V, q)$, then by definition

$$Ad_\varphi(v) = \alpha(\varphi) v \varphi^{-1} \in V.$$

Apply $(-)^t$ ⇝

$$\begin{array}{ccc} Ad_\varphi(v)^t & = & Ad_\varphi(v) \\ \| & & \| \\ (\varphi^t)^{-1} v \, \alpha(\varphi^t) & = & \alpha(\varphi) v \varphi^{-1} \end{array}$$

$$\Rightarrow v = \varphi^t \alpha(\varphi) v \varphi^{-1} \alpha(\varphi^t)^{-1}.$$

$$= \alpha(\alpha(\varphi^t)\varphi) \, v \, (\alpha(\varphi^t)\varphi)^{-1}$$

$$= \widetilde{Ad}_{N(\varphi)}(v).$$

$$\Rightarrow N(\varphi) \in \ker \widetilde{Ad} \cong k^{\times}.$$

We conclude that N defines:

$$N : \tilde{P}(V, q) \longrightarrow k^{\times}.$$

This is, in fact, a homomorphism:

$$\begin{aligned}
N(\varphi\psi) &= \varphi\psi(\varphi\psi)^t \\
&= \varphi\psi\psi^t\varphi^t \\
&= \varphi N(\psi)\varphi^t \\
&= \varphi\varphi^t N(\psi) \\
&= N(\varphi)N(\psi).
\end{aligned}$$

Also, notice that since $1^t = 1$:

$$N(\alpha(\varphi)) = N(\alpha(\varphi))^t$$

$$= (\alpha(\varphi)\varphi^t)^t$$

$$= \varphi\,\alpha(\varphi^t) = N(\varphi).$$

So: for $v \in V$, $\varphi \in \tilde{P}(V, q)$:

$$q(\widetilde{Ad}_\varphi(v)) = N(\widetilde{Ad}_\varphi(v))$$

$$= N(\alpha(\varphi)\, v\, \varphi^{-1})$$

$$= N(\alpha(\varphi))\, N(v)\, N(\varphi^{-1})$$

$$= N(\varphi)\, N(v)\, N(\varphi^{-1})$$

$$= N(v).$$

$\Rightarrow \widetilde{Ad}_\varphi \in O(V, q).$