February 27 :

G = Lie group. M = smooth manifold.

A principal G-bundle on M is

1) P = manifold, $$\begin{tikzcd} P \arrow[d, "\pi"] \\ M \end{tikzcd}$$ smooth surjective map.

2) A smooth G-action on P

$$G \times P \longrightarrow P, \quad (g, p) \longmapsto pg^{-1}.$$

such that $\pi(pg^{-1}) = \pi(p)$.

This data must be s.t. $\exists$ a nbd U about each pt $x \in M$ and a smooth map

$$h : \pi^{-1}(U) = P|_U \longrightarrow G$$

s.t.

1) $h(p \cdot g^{-1}) = h(p)g^{-1}$. ($G$-equivariant)

2) $\varphi = (\pi, h) : \pi^{-1}(U) \longrightarrow U \times G$

is a diffeomorphism.

Equivalent definition : Čech cocycle.

Sps $\mathcal{U} = \{U_\alpha\}$ is an open cover for M. And

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow G \quad , \text{ smooth}$$

are s.t :

1) $g_{\alpha\alpha}(x) = 1 \in G$ for all $x \in U_\alpha$.

2) $g_{\alpha\beta}^{-1} = g_{\beta\alpha}$.

3) $g_{\alpha\beta}\, g_{\beta\gamma}\, g_{\gamma\alpha}(x) = 1$ for all $x \in U_\alpha \cap U_\beta \cap U_\gamma$.

call this "G-cocycle" data. Two such data are equivalent if $\exists$ $h_\alpha : U_\alpha \to G$ such that $h_\alpha|_{U_\beta} = g_{\alpha\beta} \cdot (g'_{\alpha\beta})^{-1}$

Fact: $\{\text{Principal } G\text{-bundle}\}/\simeq \;\cong\; \{G\text{-cocycle data}\}/\simeq$.

Classifying spaces :

Given $f : M \longrightarrow N$, $\begin{array}{c} P \\ \downarrow{\scriptstyle \pi} \\ N \end{array}$ a principal^{G} bundle on N

$\rightsquigarrow$ $f^* P$ is a principal^{G} bundle on M

$$\begin{tikzcd}
f^* P \arrow[d, equal] & \\
\{(x,p) \mid p \in \pi^{-1}(f(x))\} \arrow[r] \arrow[d, "f^*\pi"] & P \arrow[d, "\pi"] \\
M \arrow[r, "N"'] & N
\end{tikzcd}$$

Prop : If $f, g : M \longrightarrow N$ are homotopic,

then $f^* P \cong g^* P$.

Pf (Sketch): Sps $F : [0,1] \times M \longrightarrow N$ is a homotopy $f \underset{F}{\simeq} g$. Then consider $\begin{array}{c} F^* P \\ \downarrow \\ [0,1] \times M \end{array}$.

This bundle has the property that

$$F^* P \big|_{0 \times M} \cong f^* P \quad , \quad F^* P \big|_{1 \times M} \cong g^* P$$

Since $[0,1]$ is contractible it is a consequence that $F^*P|_0 \cong F^*P|_1$. $\square$

As a corollary, if $B \simeq *$, then any principal G-bundle over B is trivial. Can use this to prove:

Thm: Suppose $\begin{tikzcd} EG \arrow[d] \\ BG \end{tikzcd}$ is a principal G-bundle w/ EG (weakly) contractible. Then there is a bijection

$$[X, BG] \xrightarrow[\Phi]{\cong} \left\{ \begin{matrix} \text{Principal G-bundles} \\ \text{on } X \end{matrix} \right\} \Big/ \cong$$

$$\Phi(f) = f^* EG .$$

Theorem [Milnor] $\begin{tikzcd} EG \arrow[d] \\ BG \end{tikzcd}$ exists. (In fact, for any topological group.)

- Principal bundles make sense even when G is discrete. Then, a G-bundle is simply a $\#G$-sheeted covering space w/ G as the group of deck transformations.

In this case we can use the homotopy LES for

$$\begin{tikzcd} EG \simeq * \arrow[d] \\ BG \end{tikzcd}$$

to see that $BG = K(G,1)$. In particular

$$\begin{tikzcd} \left\{ \begin{array}{c} G\text{-covering spaces} \\ \downarrow \\ X \end{array} \right\} \Big/ \cong \arrow[r, leftrightarrow, "\cong"] & {[X, BG]} \arrow[d, equal] & \text{singular cohomology} \arrow[dl, squiggly] \\ & {[X, K(G,1)] = H^1(X;G)} . & \end{tikzcd}$$

Examples: · On $S^2 = \mathbb{P}^1 = N \cup S$ we have the cover

$$\begin{tikzcd} N \cap S \arrow[r, "\sim"] \arrow[rr, bend right, "g_m"'] & S^1 = U(1) \arrow[r, "(-)^m"'] & U(1) \end{tikzcd}$$

$\rightsquigarrow$ Principal $U(1)$ bundle for each $m \in \mathbb{Z}$.

Ex: $SU(2)$ bundles over S^4.

Again look at $S^4 = N \cup S$. Then

$$\begin{tikzcd} N \cap S \arrow[r, phantom, "\simeq"] \arrow[rrr, bend right, "g_m"'] & S^3 \arrow[r, phantom, "\simeq"] & SU(2) \arrow[r, "(-)^m"] & SU(2) \end{tikzcd}$$

For each m g_m defines a principal $SU(2)$ bundle over S^4, call it P_m.

$$\begin{tikzcd} P_1 \arrow[r, phantom, "\cong"] \arrow[d] & S^7 \\ S^4 & \end{tikzcd}$$

diffeomorphic.

[Present $\circlearrowleft SU(2) \subset \mathbb{H}$ acts $g \cdot (p,q) = (pg^{-1}, qg^{-1})$.

$$S^7 = \{ (p,q) \in \mathbb{H} \times \mathbb{H} \mid |p|^2 + |q|^2 = 1 \}$$

Also, recall $S^3 = SU(2) \subset \mathbb{H}$ is the group of unit quaternions.

To see S^4 look at the map:

$$\begin{tikzcd} S^7 \arrow[r] \arrow[dr, dashed, "\pi"'] & \mathbb{R}^5 \\ & S^4 \arrow[u, hook] \end{tikzcd}, \quad (p,q) \overset{\pi}{\longmapsto} \underset{\substack{\uparrow \\ \text{unit length}}}{(2p\bar{q}, |p|^2 - |q|^2)}.$$

Note that π is constant along the $SU(2)$ orbits. This is the bundle projection.

Ex: $H < G$ closed Lie subgroup.

$$\begin{tikzcd} G \arrow[d] \\ G/H \end{tikzcd} \quad \text{is a principal } H\text{-bundle.}$$

Ex: $\mathbb{C}P^n$ has cover $\mathcal{U} = \{U_0, \dots, U_n\}$ s.t.

$U_i \cong \mathbb{C}^n$. $\quad U_i = \{z_i \neq 0\}$.

$$g_{ij} : U_i \cap U_j \longrightarrow U(1)$$

$$[z_0 : \dots : z_n] \longmapsto \frac{|z_j|}{z_j} \cdot \frac{z_i}{|z_i|}.$$

$\rightsquigarrow$ Principal $U(1)$-bundle
$$\begin{tikzcd} P \arrow[d] \\ \mathbb{C}P^n \end{tikzcd} \simeq S^{2n+1}.$$

<u>Back to geometry</u>: Sps that $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$ is a vector bundle of rank r (defined over a field k). There is a bundle Fr_E on M whose fiber over $x \in M$ is $GL(E_x) \overset{\cdot}{\simeq} GL(r)$. There is the natural structure of a principal $GL(r)$-bundle on Fr_E. Called the bundle of frames.

- Let $\begin{matrix} Fr_M \\ \downarrow GL(n) \\ M \end{matrix}$ denote Fr_{T_M} = bundle of linear frames

$n = \dim M$, $k = \mathbb{R}$.

- A Riemannian str on $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$ allows us to

define a $O(r)$-bundle of orthonormal frames of E

$E = T_M$

$\rightsquigarrow$ $\begin{matrix} Fr^O_M \\ \downarrow O(n) \\ M \end{matrix}$ = bundle of orthonormal frames. on Riem. manifold (M, g)

- Consider, on a Riem. vector bundle E:

$$\begin{tikzcd} Fr^O_E / SO(r) \arrow[d] & = & \text{two-sheeted covering of } M \\ w_1(E) & \in & H^1(M; \mathbb{Z}/2) \end{tikzcd}$$

1st Stiefel-Whitney class. $\rightsquigarrow$

Prop: E is orientable iff $w_1(E) = 0$.

Pf: E is orientable $\Leftrightarrow$ $Fr_E^O / SO(r)$

is trivial. ▨

An orientation is a choice of section of $Fr_E^O / SO(r)$

$\rightsquigarrow$ $H^0(X; \mathbb{Z}/2)$.

The class $w_1(E)$ is called a *characteristic class*.

- More generally, if G is any Lie group, a *universal characteristic class* is an element of

$$c \in H^*(BG; \Lambda), \quad \Lambda = \text{any ring}.$$

Given a class c we can pull-back along a classifying map $f_P : X \longrightarrow BG$

$$P \simeq f_P^* EG.$$

$\rightsquigarrow$ $f_P^* c \in H^*(X; \Lambda)$.

The most important feature of char classes is naturality:

$$\begin{tikzcd} & E \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd} \qquad f^* c_E = c_{f^*E} .$$

This is automatic for classes pulled back from universal char classes.

Eg: $\begin{tikzcd} EO(n) \arrow[d] \\ BO(n) \end{tikzcd}$ is the universal $O(n)$-bundle.

$\rightsquigarrow EO(n) / SO(n)$ is the universal orientation bundle

$\rightsquigarrow w_1 = w_1(EO(n)) \in H^1(BO(n); \mathbb{Z}/2)$

is a universal char. class.

In fact

$$H^\bullet(BO(n); \mathbb{Z}/2) \cong \mathbb{Z}/2[w_1, \dots, w_n]$$

$$|w_i| = i .$$