Thesis

Zoe Siegelnickel

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Background

Definition 1 (Prefactorization Algebra). *Let M be a topological space. A prefactorization algebra $\mathcal{F}$ on M of vector spaces is a rule that assigns a vector space $\mathcal{F}(U)$ for every open set $U \subseteq M$ such that:*

- *For each inclusion of open sets $U \subseteq V$, there is a linear map $\mathcal{F}(U) \mapsto \mathcal{F}(V)$.*
- *If $\{U_i\}_{i=1}^n$ is a collection of open sets such that $U_i \subseteq V$ and the U_i are pairwise disjoint, then there is a linear map $\mathcal{F}(U_1) \otimes .... \otimes \mathcal{F}(U_n) \mapsto \mathcal{F}(V)$*
- *These maps are compatible: if $U_{i,1},,,,U_{i,n} \subseteq V_i$ with $U_{i,j}$ pairwise disjoint and $V_1, ...V_k \subseteq W$ with V_i pairwise disjoint, then the following diagram commutes:*

$$\begin{tikzcd}
\otimes_{i=1}^k \otimes_{j=1}^n \mathcal{F}(U_j) \arrow[rr] \arrow[dr] & & \otimes_{i=1}^k \mathcal{F}(V_i) \arrow[dl] \\
& \mathcal{F}(W) &
\end{tikzcd}$$

A prefactorization is unital if $\mathcal{F}(\phi)$ is unital.

Definition 2 (Factorization Algebra). *A prefactorization algebra $\mathcal{F}$ on M taking values in a multicategory $\mathcal{C}$ is a factorization algebra if it satisfies the following properties: for every open set $U \subseteq M$ and every Weiss cover $\{U_i\}_{i \in I}$ of U, the following sequence is exact:*

$$\bigoplus_{i,j} F(U_i \cap U_j) \to \bigoplus_k \mathcal{F}(U_k) \to \mathcal{F}(U) \to 0$$

$\mathcal{F}$ is multiplicative if $\mathcal{F}(U) \otimes \mathcal{F}(V) \to \mathcal{F}(U \sqcup V)$ is an isomorphism. A factorization algebra is locally constant if for each inclusion of open sets $U \subseteq U'$, $\mathcal{F}(U) \mapsto \mathcal{F}(U')$ is a quasi-isomoprhism.

The definition of a Weiss cover is not really necessary in this context; refer to [**cg**] for the definition.

Definition 3 (Vertex Algebra, [**bzf**])**.** *A vertex algebra is the following data:*

- *A vector space V*
- *A vector $|0\rangle \in V$ (vacuum) vector*
- *A translation operator $F : V \to V$*
- *A linear operator $Y(\cdot, z) : V \to \operatorname{End} V[[z^{\pm 1}]]$ which takes each $A \in V$ to a field $\sum_{n \in \mathbb{Z}} A_{(n)} z^{-n-1}$ which acts on V*

This data is subject to the following axioms:

- *$Y(|0\rangle, z) = Id_V$ and $Y(A, z)|0\rangle \in V[[z]]$, $Y(A, z)|0\rangle|_{z=0} = A$ (vacuum axiom)*
- *For any $A \in V$, we have that $[T, Y(A, z)] = \partial_z Y(A, z)$ and $T|0\rangle = 0$ (translation)*
- *For all fields $Y(A, z), B(Y, w)$, there exists some $N \in \mathbb{Z}$ such that $(z - w)^N [Y(A, z), Y(B, w)] = 0$ (locality)*

A vertex algebra V is conformal of central charge $c \in \mathbb{C}$ if there exists a non-zero conformal vector $\omega \in V_2$ such that the Fourier coefficients L_n^V of $Y(\omega, z) = L_n^V z^{-n-1}$ satisfy the relations of the Virasoro vertex algebra with central charge c, and $L_{-1}^V = T, L_0^V|_{V_n} = nId$. A vertex alegbra is called quasi-conformal if it carries an action of $Der\ \mathcal{O} = \mathbb{C}[[z]]\partial_z$ such that the relation $[v, Y(A, w)] = -\sum_{n \geq -1} \frac{1}{(m+1)!} (\partial_w^{m+1} v(w)) Y(L_m A, w)$ holds for any $A \in V$, the translation operator is $L_{-1} = -\partial_z$, $L_0 = -z\partial_z$ acts semi-simply, and $z^2\mathbb{C}[[z]]\partial_z = Der_+\mathcal{O}$ acts locally nilpotently.

If V is a vertex algebra, a vector space M is a V-module if it is equipped with an operation $Y_M : V \to$ $\operatorname{End} M[[z^{\pm 1}]]$ *such that $Y_M(A, z) = \sum_{n \in \mathbb{Z}} A_{(n)} z^{-n-1}$ and*

- *$Y_M(|0\rangle, z) = Id_M$*
- *For all $A, B \in V$, $C \in M$, $Y_M(A, z)Y_M(B, w)C \in M((z))M((w))$, $Y_M(B, w)Y_M(A, Z)C \in M((w))((z))$ and $Y_M(Y(A, z - w)B, w)C \in M((w))((z - w))$ are all expansions of the same element of*

$$M[[z, w]][z^{-1}, w^{-1}, (z - w)^{-1}]$$

To define conformal blocks, we must first introduce the notion of vertex algebra bundles

Vertex Algebra Bundles and Conformal Blocks

The vertex operation yields a collection of matrix elements $\langle \phi, Y(A,z)v\rangle \in \mathbb{C}((z))$. For fixed $v \in V$, $\phi \in V^*$, and varying $A \in V$, this can be viewed as a section of a vector bundle over the punctured disk $D^\times = \operatorname{Spec}\mathbb{C}((z))$ with fiber V^*. This section can be defined canonically (see [**bzf**]):

Definition 4 (Vertex Algebra Bundles)**.** *Let $\mathcal{O} = \mathbb{C}[[z]]$, and let V be an $\operatorname{Aut}\mathcal{O}$ vertex algebra module. Then, $\mathcal{V}_X = \operatorname{Aut}_X \times_{\operatorname{Aut}\mathcal{O}} V$ is the vector bundle associated with V, whose fiber at $x \in X$ is $\mathcal{V}_x$. Let z be a coordinate on D_x, and denote by i_z the trivialization $i_z : V[[z]] \cong \Gamma(D_x, \mathcal{V})$. Let y_x be an $\operatorname{End}\mathcal{V}_x$ valued section of V^* on $D_x^\times$ by $\langle (z,\phi), y_x(i_z(A)) \cdot (z,v)\rangle = \langle \phi, Y(A,z)v\rangle$, where $(z,v) \in \mathcal{V}_x$. This section is independent of the choice of coordinate z on D_x.*

For any vector bundle $\mathcal{V}$ attached to a smooth projective curve X, and for any $x \in X$, ϕ is a conformal block if the canonical section $\langle (z,\phi), y_x(i_z(A)) \cdot A\rangle$ can be extended to a regular section of $\mathcal{V}^*$ on X/x for all $A \in \mathcal{V}_x$. The space of conformal blocks forms a vector space;

Definition 5 (Space of Conformal Blocks for a general vertex algebra)**.** *This is equivalent to the following definition: Given a vertex algebra V and an algebraic curve X, we define the space of conformal blocks associated to a point $x \in X$ is $C(X,x,V) = \operatorname{Hom}_{U_{X/x}(\mathcal{V}_x)}(\mathcal{V}_x, \mathbb{C})$, the space of $U_{X/x}(\mathcal{V}_x)$-invariant functionals.*

Passing from Factorization Algebras to vertex algebras

Definition 6 (Holomorphic Translation Invariance, [**cg**])**.** *Let $\mathcal{F}$ be a prefactorization algebra defined over $\mathbb{C}^n$. Then, $\mathcal{F}$ carries an action of the lie algebra $\mathbb{R}^{2n}$ by derivations, which extends to the an action of $\mathbb{R}^{2n} \otimes_{\mathbb{R}} \mathbb{C}$ by translation, which consists of the following two action maps; $\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z_j}} : \mathcal{F}(U) \mapsto \mathcal{F}(U)$. Then, $\mathcal{F}$ is holoimorphically translation invariant if it is equipped with derivations $\eta_i : \mathcal{F} \mapsto \mathcal{F}$ of cohomological degree -1 for $i = 1, ..., n$ such that the following properties hold:*

- $d\eta_i = \frac{\partial}{\partial \overline{z_j}}$
- $[\eta_i, \eta_j] = 0$
- $\left[\eta_i, \frac{\partial}{\partial \overline{z_j}}\right] = 0$

where d is the differential on the dg lie algebra $Der\mathcal{F}$.

Theorem 1. *Let $\mathcal{F}$ be a unital factorization algebra that is S^1-equivariant on $\mathbb{C}$ that is valued in differentiable vector spaces, such that the action of S^1 on $\mathcal{F}(D(0,r))$ is tame for each disk $D(0,r)$ around the origin. Let $\mathcal{F}_k(D(0,r))$ be the weight K eigenspace of the S^1 action on $\mathcal{F}(D(0,r))$. Additionaly, assume the following:*

- *Assume that $r < r', \mathcal{F}_k(D(0,r)) \to \mathcal{F}_k(D(0,r'))$ is a quasi-isomorphism.*
- *For $k >> 0$, the vector space $H^*(\mathcal{F}_k(D(0,r)) = 0$.*
- *For each k, r, $H^*(\mathcal{F}_k(D(0,r))$ is isomorphic as a sheaf to a countable sequential colimit of finite dimensional graded vector spaces.*

Let $V_k = H^*(\mathcal{F}_k(D(0,r))$, and $V = \bigoplus_{k\in\mathbb{Z}} V_k$. Then, V has the structure of a vertex algebra, determined by the structure maps of $\mathcal{F}$.

This construction produces a functor between factorization algebras and vertex algebras, since maps respecting equivaraince conditions produces a map between associated vertex algebras. Then, given any Riemann surface Σ and a factorization algebra $\mathcal{F}$ that satisfies the conditions of the theorem, we can obtain a vertex algebra by restricting Σ to an open disk. In the case that the resulting vertex algebra is quasi-conformal, we can obtain the space of conformal blocks by taking factorization homology. This yields the following diagram;

$$\begin{tikzcd}
\mathrm{Fact}_\Sigma \arrow[rr, leftrightarrow, "\int_\Sigma"'] & & \mathrm{Vect} \\
\mathrm{Fact}_{\mathbb{C}}^{hol} \arrow[u, "inc"] \arrow[rr, "F"] & & \mathrm{Vert} \arrow[u, "Conf"']
\end{tikzcd}$$

Theorem 2. *Factorization homology provides an isomorphism between global sections of $\mathcal{F}$ and the space of conformal blocks of V as vector spaces.*

This is certainly true in the following examples, although the computations are done only by considering the dimensions of the resulting spaces, and therefore does not yield any insight about the functorial nature of this map.

Example 3. *Let V denote that Kac-Moody algebra $\mathrm{Sym}(\Omega_c^{0,*} \otimes \mathfrak{g}[1])$. In genus 0, we have that the space of conformal blocks $C(\mathbb{P}^1, x, V) \cong C(\mathbb{P}^1, 0, V) \cong \mathrm{Span}_{\mathbb{C}}\, \phi_0$, where ϕ_0 is a linear functional that takes V to $\mathbb{C}$ such that $\phi_0(|0\rangle) = 1$ and $\phi_0(A) = 0$ for all $A \in V$. The global sections of the factorization algebra $\mathcal{F}^k$ is computed by $H^0(\mathrm{Sym}(\Omega_c^{0,*}(\mathbb{P}^1) \otimes \mathfrak{g}[1]), \bar{d} + d_{CE})$. Note that the $\bar{d}$ comes from the Dolbeault complex, which* *does not act on* $\mathfrak{g}$*. We compute this using the following spectral sequence;*

$$H^*_{d_{CE}}(H^*_{\bar{d}}(\mathrm{Sym}(\Omega^{0,*}(\mathbb{P}^1)\otimes\mathfrak{g}[1]))) \implies E_2^{0,0}$$

Recall that Sym *commutes with taking cohomology, so we can instead compute*

$$H^*_{d_{CE}}\,\mathrm{Sym}(H^*_{\bar{d}}((\Omega^{0,*}_c(\mathbb{P}^1)\otimes\mathfrak{g})))$$

. Recall the well-known fact that $H_{\bar{d}}(\mathbb{P}^1) = \mathbb{C}$*, since* $\mathbb{P}^1$ *is compact. Then,*

$$H_{\bar{d}}(\Omega^{0,*}_c(\mathbb{P}^1)\otimes\mathfrak{g}[1]) = \mathbb{C}\otimes\mathfrak{g}[1] \cong \mathfrak{g}[1]$$

and

$$H^0_{d_{CE}}(\mathrm{Sym}(\mathfrak{g}[1])) = H^0_{d_{CE}}(\mathrm{Sym}(\mathfrak{g}[1]),\mathbb{C}) \cong \mathbb{C}$$

, since $\mathbb{C}$ *is the trivial module. This is also one dimensional, and since all vector spaces of the same dimension, we have the desired isomorphism.*

This generalizes to genus g:

Example 4. *Let* M *be a compact genus* n *Riemann surface.*

$$H^{\cdot}_{CE}(H^{\cdot}_{\bar{\partial}}(\mathrm{Sym}(\Omega^{0,*}(M)\otimes\mathfrak{g}[1])))$$

$$H^{\cdot}_{CE}(\mathrm{Sym}(H^{\cdot}_{\bar{\partial}}(\Omega^{0,*}(M)\otimes\mathfrak{g}[1])))$$

Compute the inner homology in degree -1*:*

$$H^{0,0}_{\bar{\partial}}(M) \cong \hat{H}^0(M,\Omega^0(M)) \cong \mathbb{C}$$

since $\mathcal{O}_M = \Omega^0$*.*

In degree 0*:*

$$H^{0,1}(M) \cong \hat{H}^1(M,\Omega^0(M)) \cong \mathfrak{C}^n$$

Tensoring with $\mathfrak{g}$ *in degree* -1*:*

$$\mathbb{C}\otimes\mathfrak{g}\cong\mathfrak{g}$$

Tensoring with $\mathfrak{g}$ in degree 0:

$$\mathfrak{C}^n \otimes \mathfrak{g} \cong \mathfrak{g}^{\otimes n}$$

Then

$$H^0_{Lie}(\mathrm{Sym}(\mathfrak{g}^{\otimes n}) \cong (\mathrm{Sym}(\mathfrak{g}^{\otimes n}))^{\mathfrak{g}}$$

Generalizing the factorization homology functor

Factorization Algebra modules

Definition 7 (E_n-Alg, [**lurie**])**.** *Let $\mathcal{C}$ be a symmetric monoidal category. An $\mathbb{E}_n$ object of $\mathcal{C}$ consists of the following:*

- *For every open disk in $U \subseteq \mathbb{R}^n$ which is homeomorphic to a disk, an object $A(U) \in \mathcal{C}$*
- *If $U_1, ..., U_n$ are pairwise disjoint and contained in a larger open set V, then there is a multiplication map $\mu : \otimes_{1 \leq i \leq n} A(U_i) \mapsto A(U)$, all of which are compatible with composition.*

The collection of these objects form a symmetric monoidal ∞-category $\mathbb{E}_n$-Alg($\mathcal{C}$).

Theorem 5 ([**ginot**])**.** *There is an equivalence of categories between the category of locally constant factorization algebras on $\mathbb{R}^n$ and $\mathbb{E}_n$-algebras. The equivalence is given by*

Then, modules over factorization algebras are modeled by modules over $\mathbb{E}_n$-algebras.

Definition 8 (Translation Invariance, [**cg**])**.** *A prefactorization algebra $\mathcal{F}$ on $\mathbb{R}^n$ is translation invariant if for every translation map τ_x, $\mathcal{F}(U) \to \mathcal{F}(\tau_x U)$ is an isomorphism.*

Example 6. *Consider the map $\pi : z \to z^n$ on $\mathbb{C}$. On $\mathbb{C}^*$, this map is a covering map, but has a ramification point at the origin. Let $\mathcal{F} = (\mathrm{Sym}(\Omega^*_c \otimes g), d_{dR})$ be a factorization algebra over $\mathbb{R}^2$, where $\mathfrak{g}$ is lie algebra. Note that $\mathcal{F}$ is locally constant, since by the Poincare lemma with compact support, $\Omega^*_c(U) \mapsto \Omega^*_c(U)$ is a quasi-isomorphism, since $H^k_c(U) = 0 \forall k \geq 1$.*

Then, on $\mathbb{C}^*$, $\pi^{-1}(U) = \coprod_n U$ for an open disk in $\mathbb{C}$ since π is a covering map. Around the origin, we have that $\pi^{-1}(D_r(0)) = D_{n\sqrt{r}}(0)$. Since Ω_c is a sheaf, $\Omega_c(\coprod U) = \bigoplus_n \Omega_c(U)$, and we have that $H^*(\bigoplus_n \Omega_c(U)) = \bigoplus_n H^*(\Omega_c(U))$. Then, there is no isomorphism between $\bigoplus_n H^*(\Omega^*_c(U))$ and $H^*(\Omega^*_c(D_{n\sqrt{r}}(0)$, and so $\pi_*\mathcal{F}$ is not locally constant.

A similar argument demonstrates that $\pi_*\mathcal{F}$ is also not translation invariant; if $U = D_r(a)$, and $T(z) = z - a$, then $TD_r(a)$ is a disk centered at 0. But then, $\pi_*\mathcal{F}(U)$ is of the form $\bigoplus \mathrm{Sym}(\Omega_c(U) \otimes \mathfrak{g})$ and $\pi_*\mathcal{F}(TU)$ is of the form $\mathrm{Sym}(\Omega_c(D_r(0) \otimes \mathfrak{g}))$, which will not be isomorphic.

Theorem 7. *The $\mathbb{Z}/n\mathbb{Z}$ invariance of the origin under π admits an E_2-module structure.*

If we consider only conformal vertex algebras, the space of conformal blocks can easily be generalized to a smooth projective curve with multiple marked points, each with an inserted conformal vertex algebra module. Then, it is a natural assumption that factorization homology of a factorization algebra with marked points will yield this higher dimensional analog of conformal blocks.