The ABS construction

In this note we provide an overview of the construction of Atiyah, Bott, and Shapiro which provides, in part, a relationship between topological K-theory and Clifford modules. After a rapid introduction to K-theory we follow parts of the original reference [ABS64].

1. A rapid introduction to K-theory

In this section the field $\mathbf{k}$ is either $\mathbf{R}$ or $\mathbf{C}$, and we consider $\mathbf{k}$-vector bundles on a space X. Write $\underline{\mathbf{k}}^n$ for the trivial bundle of rank n.

We introduce two equivalence relations on the set of vector bundles $\mathrm{Vect}(X)$.

- Stable isomorphism $\simeq_s$. Two vector bundles E_1, E_2 on X are *stably isomorphic*, we write $E_1 \simeq_s E_2$, if there exists an $N \geq 0$ and a bundle isomorphism
$$E_1 \oplus \underline{\mathbf{k}}^N \simeq E_2 \oplus \underline{\mathbf{k}}^N.$$
- Equivalence relation $\sim$. More generally, we say $E_1 \sim E_2$ if there exists $N, M \geq 0$ such that
$$E_1 \oplus \underline{\mathbf{k}}^N \simeq E_2 \oplus \underline{\mathbf{k}}^M.$$

Proposition 1.1. *If X is compact Hausdorff, then the set of $\sim$-equivalence classes of vector bundles over X forms an abelian group with respect to direct sum $\oplus$. This group is denoted $\widetilde{K}(X)$.*

This group $\widetilde{K}(X)$ is called the *reduced* K-group of X. The unreduced version is defined using the equivalence relation $\simeq$, except it is slightly more complicated. The issue is that only the class of the zero vector bundle is invertible in the set of $\simeq_s$-equivalence classes with respect to direct sum. Nevertheless, we can "cancel" bundles in some sense.

Lemma 1.2. *Suppose X is compact and E is a vector bundle on X. There exists a vector bundle E' on X such that $E \oplus E'$ is trivializable.*

PROOF. Suppose $\{U_i\}$ is a finite trivializing cover for E, so we have trivializations $\phi_i \colon U_i \times \mathbf{k}^n \to E$. There is a partition of unity $\{f_i \colon X \to [0,1]\}$ subordinate to this finite cover. For each i this allows us to define a vector bundle homomorphism

$$f_i \cdot \phi_i^{-1} \colon E \to X \times \mathbf{k}^n \tag{1}$$

Together, thus, we get a vector bundle homomorphism

$$\oplus f_i \cdot \phi_i^{-1} \colon E \to X \times \mathbf{k}^N \tag{2}$$

where N is n times the cardinality of the set parametrizing the cover. This morphism is fiberwise injective since f_i is non-vanishing at at least one point of U_i. So, we have embedded E into a trivial vector bundle. Fix an inner product on the trivial vector bundle (this exists by paracompactness). Then $E \oplus E^{\perp} \simeq \underline{\mathbf{k}}^{\oplus N}$ □

From this lemma, we see that if $E_1 \oplus E_2 \simeq_s E_1 \oplus E_3$ then we can add $E_1^{\perp}$ to both sides to see that $E_2 \simeq_s E_3$. It follows that the set of $\simeq_s$-equivalence classes forms a semi-group with respect to $\oplus$. The K-group $K(X)$ is the group completion of this semi-group. (Think of the positive rational numbers $\mathbf{Q}_{>0}$ as the group completion of the semi-group of positive natural numbers $\mathbf{Z}_{>0}$ under multiplication.)

By definition, one represents elements of $K(X)$ as classes of formal differences

$$[E_1 - E_2]$$

where E_1, E_2 are bundles on X. Then $[E_1 - E_2] = [E_1' - E_2']$ if and only if there is a stable equivalence

$$E_1 \oplus E_2' \simeq_s E_1' \oplus E_2.$$

The group operation is the obvious thing

$$[E_1 - E_2] + [E_2 - E_2'] = [(E_1 \oplus E_2) - (E_1' \oplus E_2')].$$

Note that every element in $K(X)$ can be represented by a formal difference $[E - \underline{\mathbf{k}}^n]$ for some bundle E and some integer n.

There is a natural homomorphism $K(X) \to \widetilde{K}(X)$ defined by $[E - \underline{\mathbf{k}}^n] \to [E]$ whose kernel consists of classes of the form $\underline{\mathbf{k}}^0 - \underline{\mathbf{k}}^n$. Hence, $E \simeq_s \underline{\mathbf{k}}^m$ for some m. Thus, the kernel of this homomorphism is $\mathbf{Z}$ and there is an isomorphism $K(X) \simeq \widetilde{K}(X) \oplus \mathbf{Z}$.

Bibliography

[ABS64] M. F. Atiyah, R. Bott, and A. Shapiro. “Clifford modules”. *Topology* 3.suppl (1964), pp. 3–38. URL: https://doi.org/10.1016/0040-9383(64)90003-5.