

## Linear Combinations

Dfn: Let  $V$  be a vector space over  $\mathbb{F}$ . A vector  $v \in V$  is said to be a linear combination of vectors  $\{u_1, \dots, u_n\}$  if it can be written

$$v = a_1 u_1 + \dots + a_n u_n.$$



( or n-terms.

$$v = \sum_{i=1}^n a_i u_i . )$$

where  $a_1, \dots, a_n \in \mathbb{F}$ .

Ex: The vector  $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$  is a linear combination

of  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ :

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

The problem of solving linear systems of equations is the problem of finding if and how a vector  $v$  is a linear combination of other vectors  $u_1, \dots, u_n$ .

Ex: Take  $V = \mathbb{R}^3$ . Is  $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \in \mathbb{R}^3$  a linear combination of the vectors  $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$  and  $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ ?

In other words, can we find  $a_1, a_2 \in \mathbb{R}$  such that

$$a_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

$$\Leftrightarrow \left\{ \begin{array}{l} a_1 + a_2 = 2 \\ a_1 + 2a_2 = 1 \\ a_1 - a_2 = 3 \end{array} \right.$$

This is a system of linear equations

Can we solve this example? To do this, let's add the third equation to the first:

$$2a_1 = 5 \Rightarrow a_1 = 5/2.$$

But then eqn 1 says  $5/2 + a_2 = 2$  or  $a_2 = -1/2$ . So eqn 1, 2 force  $a_1 = 5/2, a_2 = -1/2$ .

Does this solve eqn 2?

$$a_1 + 2a_2 = 1$$

||

$$\begin{matrix} 5/2 & -1 \\ || & \\ 3/2 & \end{matrix} \quad 3/2 \neq 1.$$

Thus,  $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$  is not a linear combination  
of  $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ .

Span : Let  $V$  be a vector space. Let

$$S \subset V$$

be a subset of vectors. The span of  $S$  is the set

$$\text{Span}(S) = \left\{ \begin{array}{l} \text{all linear combinations} \\ \text{of vectors in } S \end{array} \right\}.$$

Theorem : For any  $S \subset V$  the span

$$\text{Span}(S) \subset V$$

is a subspace. Moreover,  $\text{Span}(S)$  is the smallest subspace which contains  $S$ .

Pf : If  $S = \emptyset$  then

$$\text{Span}(\emptyset) = \{0\}, \text{ which is subspace.}$$

If  $S \neq \emptyset$  then  $S$  contains a vector  $z \in S$ .

So  $0 \cdot z = \vec{0}$  is in  $\text{Span } S$ . If  $x, y \in \text{Span}(S)$  then

$$x = a_1 u_1 + \cdots + a_m u_m$$

$$y = b_1 v_1 + \cdots + b_n v_n$$

for some  $a_i, b_j \in \mathbb{F}$ ,  $u_i, v_j \in S$ .

Then

$$x+y = a_1u_1 + \cdots + a_mu_m + b_1v_1 + \cdots + b_nv_n$$

is still a linear combination of vectors in  $S$ .

Thus  $x+y \in \text{Span}(S)$ . Similarly  $\lambda \cdot x \in \text{Span}(S)$  for any  $\lambda \in F$ . Thus  $\text{Span}(S)$  is subspace.

Now, suppose  $W \subset V$  is a subspace which contains  $S$ .

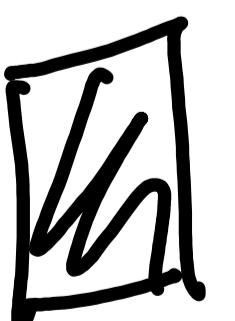
If  $w \in \text{Span}(S)$  then

$$w = a_1w_1 + \cdots + a_kw_k$$

for some  $a_1, \dots, a_k \in F$  and  $w_1, \dots, w_k \in S$ . Since  $S \subset W$  we see  $w_1, \dots, w_k \in W$  as well. But,

Since  $W$  is subspace we see that  $w \in W$ .

Thus  $\text{Span}(S) \subset W$ . Since  $W$  was an arbitrary subspace which contains  $S$ , we see  $\text{Span}(S)$  is the smallest subspace which contains  $S$ .



Dfn: A subset  $S \subset V$  spans  $V$  if

$$\text{Span}(S) = V.$$

Ex:  $V = \mathbb{R}^3$ . Then, the three element subset:

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Spans  $\mathbb{R}^3$ .

Pf: For any  $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{R}^3$  we need to

find  $a_1, a_2, a_3$  s.t.

$$a_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + a_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

You can directly check that

$$\begin{cases} a_1 = \frac{1}{2} (v_1 + v_2 - v_3) \\ a_2 = \frac{1}{2} (v_1 - v_2 + v_3) \\ a_3 = \frac{1}{2} (-v_1 + v_2 + v_3) \end{cases}$$

solves this eqn. So  $S$  spans  $\mathbb{R}^3$ .  $\square$

Q: Typically, one can find many subsets which span a given subspace. The goal in the next lecture is deciding how to find such a subset which is as "small as possible".

• Sum of subspaces.

Given two subspaces  $W_1, W_2 \subset V$  define

$W_1 + W_2$  to be the smallest subspace which contains  $W_1$  and  $W_2$ .

Prop: Let  $S_1, S_2 \subset V$  be arbitrary subsets.

Then

$$\text{Span}(S_1 \cup S_2) = \text{Span}(S_1) + \text{Span}(S_2).$$