Solutions to selected exercises from §1.5

Question 8

This question uses the concept of a field of characteristic 2 which we have not yet covered. Please skip this for now.

Question 11

This question uses the concept of a field of characteristic 2 which we have not yet covered. Please skip this for now.

Question 14

Suppose that S is linearly dependent and that $S \neq \{\mathbf{0}\}$. Our first case is that $\mathbf{0} \in S$. If this is the case, then we can take $v = \mathbf{0}$ and trivially $v = \mathbf{0}$ is a linear combination of any other vector in S, namely $\mathbf{0} = 0 \cdot u$ for any $u \in S$. The next case is that $\mathbf{0} \notin S$. Since the set S is linearly dependent, there exists vectors $\{u_0, u_1, \dots, u_n\}$ and scalars $\lambda_0, \dots, \lambda_n$, not all zero, such that

$$\lambda_0 u_0 + \lambda_1 u_1 + \cdots + \lambda_n u_n = \mathbf{0} \tag{1}$$

We can assume, without loss of generality, that $u_i \neq u_j$ for all $i, j = 1, \dots, n$. Indeed, suppose for example that $u_{n-1} = u_n$. Then, we can write this expression as

$$\lambda_0 u_0 + \cdots + (\lambda_{n-1} + \lambda_n) u_{n-1} = \mathbf{0} \tag{2}$$

Keep proceeding this way until all vectors u_i are distinct.

Question 20

Suppose that there exists a linear dependence relation. That is, scalars λ, μ such that

$$\lambda e^{rt} + \mu e^{st} = \mathbf{0} \tag{3}$$

where $\mathbf{0}$ is the zero function. In other words, for each $t \in \mathbb{R}$, we have the equation

$$\lambda e^{rt} + \mu e^{rt} = 0. \tag{4}$$

Plugging in $t = 0$ we find that $\lambda + \mu = 0$, or $\lambda = -\mu$. Plugging in $t = 1$ we find that

$$\lambda e^{r} + \mu e^{s} = \lambda(e^{r} - e^{s}) = 0. \tag{5}$$

Since we started with a linear dependence relation, we know that $\lambda \neq 0$. Thus, for this equation to be true at $t = 1$ we see that $e^r - e^s = 0$, or $e^r = e^s$. But, the exponential is an *injective* function, so this would imply that $r = s$; a contradiction! Thus, the functions e^{rt}, e^{st} are linearly independent.