MA 725 - Differential Geometry, I

Final exam

Problem 1. Suppose that $(V, \langle -, - \rangle)$ is a $2n$-dimensional Euclidean vector space equipped with a compatible complex structure J and denote $\omega \in \wedge^2 V^*$ its fundamental (Kähler) form. Let $L\colon \wedge^\bullet V^* \to \wedge^\bullet V^*[2]$ be the Lefshetz operator, $L\alpha = \omega \wedge \alpha$, and let $\Lambda\colon \wedge^\bullet V^* \to \wedge^\bullet V^*[-2]$ be the dual Lefshetz operator defined by the condition $\langle \Lambda\alpha, \beta \rangle = \langle \alpha, L\beta \rangle$.

Show that as degree zero endomorphisms of $\wedge^\bullet V^*$ one has the equality

$$H \stackrel{\text{def}}{=} L \circ \Lambda - \Lambda \circ L = \sum_{k=0}^{2n} (k-n)\pi^k \tag{1}$$

where π^k is projection onto $\wedge^k V^*$. In other words, $[L, \Lambda]$ acts diagonally with eigenvalue $(k-n)$ on $\wedge^k V^*$.

Together, $\{L, \Lambda, H\}$ form a representation of $\mathfrak{sl}(2)$ on $\wedge^\bullet V^*$. Can a compact, simply connected six-manifold with $\chi = 7$ be equipped with a Kähler metric?

Suppose that M is a compact, four-dimensional complex manifold with $\dim H^{1,1} > \dim H^{2,2}$? Can M be equipped with a compatible Kähler metric?

Problem 2. Prove that all spheres are formal.

Show that any Hopf surface $(\mathbf{C}^2 \setminus 0)/(q^{\mathbf{Z}})$ is formal.

Produce a minimal model for $\mathbf{CP}^n$.

Describe the Hopf fibration $S^3 \to \mathbf{CP}^1$ at the level of their minimal models.

Problem 3. Let M be a compact Kähler manifold. Suppose that $\alpha \in \Omega^{1,1}(M)$ is a d-closed $(1,1)$ form which is primitive $\Lambda(\alpha) = 0$. Show that $\Delta\alpha = 0$.

Prove that the Kähler form ω satisfies $\Delta\omega = 0$.

Show that if ω' is another Kähler form on M with $[\omega] = [\omega'] \in H^2(M, \mathbf{R})$ that there exists a real function $f \in C^\infty(M)$ such that $\omega' = \omega + i\partial\bar{\partial} f$.