LECTURE 4

Moduli spaces of sheaves, I

Last time we showed that the Hilbert scheme of n points in $\mathbf{A}^2$ is non-singular and equivalent to the quotient of

$$\widetilde{H}_n^s = \{(X, Y, i) \mid [X, Y] = 0, \quad \text{and stability}\} \subset \mathrm{End}(\mathbf{C}^n)^{\oplus 2} \oplus \mathbf{C}^n. \tag{1}$$

by the natural $GL(n, \mathbf{C})$ action. Today we will wrap up this discussion with a computation of the dimension of $\mathrm{Hilb}_n(\mathbf{A}^2)$ and some examples of Hilbert schemes for small values of n. Then, we turn to a sheaf-theoretic description of the Hilbert scheme.

4.1. Dimension of the Hilbert scheme

For $(X, Y, i) \in \widetilde{H}_n^s$ let $(C^\bullet, \mathrm{d})$ be the following complex

$$\mathrm{End}(\mathbf{C}^n) \xrightarrow{\mathrm{d}_1} \mathrm{End}(\mathbf{C}^n)^{\oplus 2} \oplus \mathbf{C}^n \xrightarrow{\mathrm{d}_2} \mathrm{End}(\mathbf{C}^n) \tag{2}$$

where the first arrow is the derivative of the $GL(n, \mathbf{C})$ action

$$\mathrm{d}_1(A) = ([A, X], [A, Y], Ai) \tag{3}$$

and the second arrow is

$$\mathrm{d}_2(A, B, v) = [X, A] + [Y, B]. \tag{4}$$

Then

$$\mathrm{T}_{(X,Y,i)} \mathrm{Hilb}_n(\mathbf{A}^2) \simeq H^1(C, \mathrm{d}). \tag{5}$$

We have already shown that the dimension of the cokernel of d_2 is n. By the stability condition we have $\ker \mathrm{d}_1 = 0$. This shows that $\dim H^1(C) = 2n$.

4.2. Examples

Let's consider some examples of $\mathrm{Hilb}_n(\mathbf{A}^2)$ for small n. For $n = 1$ we have $X = x, Y = y$ for some numbers $x, y \in \mathbf{C}$. Furthermore, the stability condition implies that $i \neq 0$. Using the $\mathbf{C}^\times$-action we can assume that $i = 1$. The idea corresponding to the pair x, y is

$$I = \{f(z_1, z_2) \in \mathbf{C}[z_1, z_2] \mid f(x, y) = 0\}. \tag{6}$$

This is simply the maximal idea corresponding to $(x, y) \in \mathbf{A}^2$. Thus $\mathrm{Hilb}_1(\mathbf{A}^2) = \mathbf{A}^2$.

Next we look at $n = 2$. Then X, Y are 2×2 matrices. Suppose that at least X, Y have distinct eigenvalues. Since $[X, Y] = 0$ we can assume that

$$X = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}, \quad Y = \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix} \tag{7}$$

with $(x_1, y_1) \neq (x_2, y_2)$. By the stability condition we can take

$$i(1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{8}$$

The corresponding ideal is

$$I = \{f(z_1, z_2) \in \mathbf{C}[z_1, z_2] \mid f(x_i, y_i) = 0\}, \tag{9}$$

which corresponds to two distinct points in $\mathbf{A}^2$. Thus away from the diagonal in $\mathbf{A}^2 \times \mathbf{A}^2$ the Hilbert scheme agrees with $S^2\mathbf{A}^2$.

The interesting stuff happens when we assume that X, Y each have one eigenvalue. We cannot assume that X, Y are both diagonalizable as this violates the stability condition. Thus, we have

$$X = \begin{pmatrix} x & a \\ 0 & x \end{pmatrix}, \quad Y = \begin{pmatrix} y & b \\ 0 & y \end{pmatrix} \tag{10}$$

for some $(a, b) \in \mathbf{A}^2 - 0$. In this basis we can assume that

$$i(1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{11}$$

The corresponding ideal is

$$I = \left\{ f(z_1, z_2) \in \mathbf{C}[z_1, z_2] \mid f(x, y) = \left(a\frac{\partial f}{\partial z_1} + b\frac{\partial f}{\partial z_2} \right)(x, y) = 0 \right\}. \tag{12}$$

This corresponds to two infinitesimally close points in $\mathbf{A}^2$ at (x, y) which point to each other in the direction of the vector field $a\frac{\partial}{\partial z_1} + b\frac{\partial}{\partial z_2}$.

4.3. Torsion-free sheaves

A quasi-coherent sheaf $\mathcal{F}$ on an algebraic variety X is ***torsion-free*** if for every affine open subset $U \subset X$ the space of local sections $\mathcal{F}(U)$ is torsion-free as a module over the ring of functions $\mathcal{O}(U)$ on U. That is, for ever nonzero section $s \in \mathcal{F}(U)$ and nonzero function $f \colon U \to \mathbf{C}$ one has $f \cdot s \neq 0$. A typical example of a torsion-free sheaf is the sheaf of sections of a vector bundle; the condition of being a locally free implies torsion-free. We will mostly be concerned with coherent torsion-free sheaves.

For any quasi-coherent sheaf $\mathcal{F}$ there is a canonical morphism

$$\mathcal{F} \to (\mathcal{F}^\vee)^\vee = \mathcal{F}^{\vee\vee} \tag{13}$$

where $\mathcal{F}^\vee = \mathrm{Hom}_{\mathrm{Mod}_{\mathcal{O}}}(\mathcal{F}, \mathcal{O})$ is the dual sheaf.[1] The main technical result about torsion-free sheaves that we will use is the following.

THEOREM 4.3.1 (**[??]**). *Let X be a non-singular algebraic variety and suppose $\mathcal{F}$ is a coherent torsion-free sheaf on X. Then:*

[1] A quasi-coherent sheaf which is isomorphic to its double dual is called reflexive.

- *there exists a Zariski open set $U \subset X$ of codimension ≥ 2 such that $\mathcal{F}|_U$ is locally free.*
- *If $\dim X = 2$ then the sheaf $\mathcal{F}^{\vee\vee}$ is locally free of finite rank and the morphism $\mathcal{F} \to \mathcal{F}^{\vee\vee}$ is injective. Restriction of this morphism to U results in an isomorphism $\mathcal{F}|_U \xrightarrow{\sim} \mathcal{F}^{\vee\vee}|_U$.*

4.4. Chern classes

Let X be a smooth algebraic variety over $\mathbf{C}$, which you are free to think of as just complex manifold. The jth Chern class of a complex vector bundle E over X is an element

$$c_j(E) \in H^{2j}(X; \mathbf{R}). \tag{14}$$

The total Chern class is usually denoted

$$c(E) = \sum_{j \geq 0} c_j(E) \in H^{2\bullet}(X; \mathbf{R}). \tag{15}$$

The Chern classes are determined by the following axioms.

- *The zeroeth Chern class*. For any bundle $E \to X$ one has $c_0(E) = 1$.
- *Naturality*. For any bundle $E \to X$ and smooth map $f \colon Y \to X$ one has

$$c(f^*E) = f^*c(E) \in H^{2\bullet}(X; \mathbf{R}). \tag{16}$$

- *Whitney sum.* For a finite collection of bundles E_i one has

$$c(\oplus_i E_i) = \sum_i c(E_i). \tag{17}$$

- *Normalization.* Let $\mathcal{O}(1)$ be the dual of the tautological line bundle over $\mathbf{CP}^1$. Then

$$\int_{\mathbf{CP}^1} c_1(\mathcal{O}(1)) = 1. \tag{18}$$

We will need to extend the definition of Chern classes to coherent sheaves. Let $\mathrm{Coh}(X)$ be the category of coherent sheaves on X and let $\mathrm{Vect}(X) \subset \mathrm{Coh}(X)$ be the subcategory of locally free coherent sheaves. This subcategory is equivalent to the category of holomorphic vector bundles on X; the equivalence is obtained by taking the sheaf of holomorphic sections of a given holomorphic vector bundle. Both $\mathrm{Coh}(X)$ and $\mathrm{Vect}(X)$ are abelian categories.

Construction 4.4.1. Given any abelian category $\mathcal{A}$ we can look at the free abelian group $\mathbf{Z}[\mathcal{A}]$ which is generated by the isomorphism classes of objects of $\mathcal{A}$. Given a short exact sequence

$$0 \to A \to B \to C \to 0 \tag{19}$$

in $\mathcal{A}$ we can form the element

$$-[A] + [B] - [C] \in \mathbf{Z}[\mathcal{A}]. \tag{20}$$

Let $E(\mathcal{A})$ be the subgroup of $\mathbf{Z}[\mathcal{A}]$ generated by elements of this form. The ***Grothendieck group*** of the abelian category $\mathcal{A}$ is defined as the quotient group

$$K_0(\mathcal{A}) \stackrel{\text{def}}{=} \mathbf{Z}[\mathcal{A}]/E(\mathcal{A}). \tag{21}$$

By definition, if (19) is a short exact sequence then we have the relation

$$[B] = [A] + [C] \tag{22}$$

in $K_0(\mathcal{A})$.

Consider the free abelian group generated by isomorphism classes of vector bundles on X

$$\mathbf{Z}[\mathrm{Vect}(X)]. \tag{23}$$

This has the structure of a commutative ring where multiplication is the tensor product (over $\mathcal{O}_X$) of coherent sheaves. In the sequence we denote the Grothendieck group of $\mathrm{Vect}(X)$ by $K_0(X) \stackrel{\text{def}}{=} K_0(\mathrm{Vect(X)})$.

Lemma 4.4.2. *Let X be a smooth complex variety or a complex manifold.*

(1) The subring $E(\mathrm{Vect}(X)) \subset \mathbf{Z}[\mathrm{Vect}(X)]$ is an ideal, and therefore $K_0(X)$ has the structure of a commutative ring with unit given by the trivial rank one vector bundle.
(2) The group $K_0(\mathrm{Coh}(X))$ is naturally a module for $K_0(X)$.
(3) The embedding $\mathbf{Z}[\mathrm{Vect}(X)] \hookrightarrow \mathbf{Z}[\mathrm{Coh}(X)]$ determines a group homomorphism

$$i\colon K_0(X) \to K_0(\mathrm{Coh}(X)). \tag{24}$$

Notice that by construction the total Chern class defines a group homomorphism

$$c\colon K_0(X) \to H^\bullet(X). \tag{25}$$

Remark 4.4.3. In fact, there is a more refined relationship between $K_0(X)$ and the cohomology of X.

The Chern character of a complex vector bundle $E \to X$ is an element

$$\mathrm{ch}(E) \in H^{2\bullet}(X; \mathbf{R}) \tag{26}$$

defined formally as follows. Suppose that ξ_i are constants and x is a formal variable such that

$$\sum_i c_i(E) x^i = \prod_i (1 + \xi_i x). \tag{27}$$

Then the Chern character is defined by

$$\mathrm{ch}(E) = \sum_i e^{\xi_i}. \tag{28}$$

The Chern character enjoys a similar sum rule $\mathrm{ch}(\oplus_i E_i) = \sum_i \mathrm{ch}(E_i)$ and also a product identity

$$\mathrm{ch}(\otimes_i E_i) = \prod_i \mathrm{ch}(E_i). \tag{29}$$

Immediately, then, we see that the Chern character defines a ring homomorphism

$$\mathrm{ch}\colon K_0(X) \to H^\bullet(X). \tag{30}$$

Now, we can state how to extend Chern classes to coherent sheaves. Given a coherent sheaf $\mathcal{F}$ on a smooth projective algebraic variety over $\mathbf{C}$ there exists a locally free resolution of $\mathcal{F}$ (that is, a resolution by vector bundles) of the form

$$0 \to \mathcal{E}_{-n} \to \mathcal{E}_{-n+1} \cdots \to \mathcal{E}_{-1} \to \mathcal{E}_0 \to \mathcal{F} \to 0. \tag{31}$$

In the case of a general complex manifold such a resolution is only guaranteed to exist locally. Using such a resolution we define

$$c(\mathcal{F}) \stackrel{\text{def}}{=} \sum_i (-1)^i c(\mathcal{E}_i) \in H^\bullet(X). \tag{32}$$

One can show that this definition does not depend on the resolution.

This construction can be refined to providing an inverse j to the ring homomorphism $i\colon K_0(X) \to K_0(\mathrm{Coh}(X))$ by the formula

$$j([\mathcal{F}]) = \sum_i (-1)^i [\mathcal{E}_i]. \tag{33}$$

The proof of the fact that these homomorphisms are inverses to each other is outside of the scope of these notes.