

January 23

A algebra $\leadsto A^\times \subset A$ group of units.

The adjoint action is

$$\text{Ad} : A^\times \longrightarrow \text{Aut}(A)$$

$$a \longmapsto (x \longmapsto axa^{-1}) \\ \parallel \\ \text{Ad}_a(x).$$

Sps $v \in V \subset \text{Cl}(V, q)$ is s.t. $q(v) \neq 0$.
Then

$$v \cdot \left(-\frac{v}{q(v)} \right) = -\frac{v^2}{q(v)} = 1.$$

So $q(v) \neq 0 \Rightarrow v \in \text{Cl}(V, q)^\times$.

Moreover we have:

Prop: Sp. $v \in V$, $q(v) \neq 0$. Then

$$\text{Ad}_v(w) = -w + 2 \frac{\langle v, w \rangle}{\langle v, v \rangle} v.$$

Pf: $v^{-1} = -\frac{v}{\langle v, v \rangle}$. Thus

$$\text{Ad}_v(w) = v w v^{-1}$$

$$= v w \left(-\frac{v}{\langle v, v \rangle} \right)$$

$$= -\frac{1}{\langle v, v \rangle} v w v$$

$$= -\frac{1}{\langle v, v \rangle} \left(-v^2 w - 2\langle v, w \rangle v \right).$$

$$= -w + 2 \frac{\langle v, w \rangle}{\langle v, v \rangle} v.$$

□

Note:

$\text{Ad}_v(w)$ = reflection of w wrt to the hyperplane v^\perp .

As a corollary, we see that for $q(v) \neq 0$ that the automorphism A_{2v} preserves the subspace

$$V \subset Cl(V, q).$$

$$\text{Let } \mathcal{P}(V, q) \subset Cl(V, q)^{\times}$$

be the subgroup generated by $v \in V$ w/ $q(v) \neq 0$.

$$\mathcal{P}(V, q) = \{v_1 \cdots v_k \mid q(v_i) \neq 0\}.$$

Def: Let $\text{Pin}(V, q) \subset \mathcal{P}(V, q)$

be the subgroup generated by $v \in V$ with $q(v) = \pm 1$.

Let

$$\text{Spin}(V, q) = \text{Pin}(V, q) \cap Cl^+(V, q).$$

Thus

$$\text{Pin}(V, q) = \left\{ v_1 \cdots v_k \mid \begin{array}{l} k \text{ integer} \\ q(v_i) = \pm 1 \end{array} \right\}.$$

$$\text{Spin}(V, q) = \left\{ v_1 \cdots v_k \in \text{Pin}(V, q) \mid k \text{ even} \right\}.$$

• We next define an homomorphism

$$\tilde{\text{Ad}} : \text{Cl}(V, q)^* \longrightarrow \text{GL}(\text{Cl}(V, q))$$

which agrees with Ad_φ for $\varphi \in \text{Cl}^+(V, q)$,
but when $v \in V$,

$$\begin{aligned} \tilde{\text{Ad}}_v &= \text{reflection about } v^\perp. \\ &= -\text{Ad}_v. \end{aligned}$$

Explicitly

$$\tilde{\text{Ad}}_\varphi(x) = \alpha(\varphi) x \varphi^{-1}.$$

$$\text{where } \alpha|_{\text{Cl}^+} = 1, \quad \alpha|_{\text{Cl}^-} = -1.$$

Define

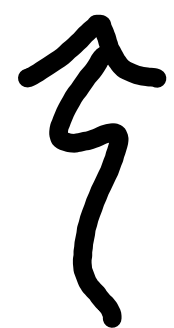
$$\tilde{\mathcal{P}}(V, \mathfrak{g}) = \left\{ \varphi \in \mathcal{C}\ell(V, \mathfrak{g})^+ \mid \text{Im } \tilde{A}\tilde{\downarrow}_{\varphi} = V \right\}.$$

$$\cup \\ \mathcal{P}(V, \mathfrak{g}).$$

Prop: Consider the homomorphism

$$\tilde{A}\tilde{\downarrow} : \tilde{\mathcal{P}}(V, \mathfrak{g}) \longrightarrow GL(V)$$

$$\text{Then } \ker \tilde{A}\tilde{\downarrow} = k^{\times} 1 \cong k^{\times}$$



Nonzero Multiples of unit.

Pf: Write $\varphi = \varphi_+ + \varphi_- \in \ker \tilde{A}\tilde{\downarrow}$.

$$\text{Then } \varphi_+ v = v \varphi_+, \varphi_- v = -v \varphi_-$$

for all $v \in V$.

Let $\{e_i\}$ be a.n.b for V .

Then by using the Clifford alg. relation, we can assume that

$$\varphi_+ = a_0 + a_1 e_1$$

where a_0, a_1 are polynomial expressions in $\{e_2, \dots, e_n\}$. Since φ_+ is even,

a_0 is even and a_1 is odd. So:

$$a_0 e_1 + a_1 e_1^2 = e_1 a_0 + e_1 a_1 e_1$$

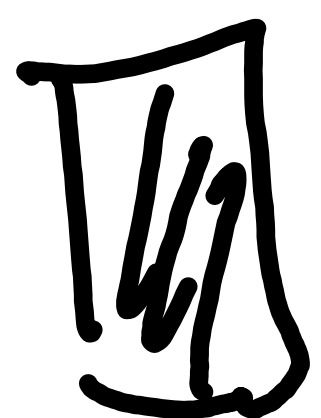
$$= a_0 e_1 - a_1 e_1^2$$

$$= a_0 e_1 - a_1 e_1^2$$

$$\Rightarrow a_1 = 0.$$

Repeating, we see $\varphi_+ = a \cdot 1$ for $a \in k^\times$.

Similarly $\varphi_- = 0$.



Next we will show that $\tilde{\text{Ad}}_\varphi$ is an orthogonal transformation.

For $\varphi \in \text{Cl}(V, q)$ define

$$N(\varphi) = \varphi \cdot \alpha(\varphi^t).$$

Note: for $v \in V$, $N(v) = v(-v) = q(v)$.

Sps $\varphi \in \tilde{\Phi}(V, q)$, then by definition

$$\text{Ad}_\varphi(v) = \alpha(\varphi) v \varphi^{-1} \in V.$$

Apply $(-)^t \rightsquigarrow$

$$\begin{array}{ccc} \text{Ad}_\varphi(v)^t & = & \text{Ad}_\varphi(v) \\ \parallel & & \parallel \end{array}$$

$$(\varphi^t)^{-1} v \alpha(\varphi^t) = \alpha(\varphi) v \varphi^{-1}$$

$$\Rightarrow v = \varphi^t \alpha(\varphi) v \varphi^{-1} \alpha(\varphi^t)^{-1}.$$

$$= \alpha(\alpha(\varphi^t)\varphi) v (\alpha(\varphi^t)\varphi)^{-1}$$

$$= \tilde{A} \downarrow_{N(\varphi)}(\nu).$$

$$\Rightarrow N(\varphi) \in \ker \tilde{A} \downarrow \cong k^x.$$

We conclude that N defines:

$$N: \tilde{P}(\nu, \eta) \longrightarrow k^x.$$

This is, in fact, a homomorphism:

$$\begin{aligned} N(\varphi \eta) &= \varphi \eta (\varphi \eta)^t \\ &= \varphi \eta \eta^t \varphi^t \\ &= \varphi N(\eta) \varphi^t \\ &= \varphi \varphi^t N(\eta) \\ &= N(\varphi) N(\eta). \end{aligned}$$

Also, notice that since $1^t = 1$:

$$\begin{aligned} N(\alpha(\varphi)) &= N(\alpha(\varphi))^t \\ &= (\alpha(\varphi) \varphi^t)^t \\ &= \varphi \alpha(\varphi^t) = N(\varphi). \end{aligned}$$

So: for $v \in V$, $\varphi \in \tilde{\mathcal{P}}(V, \mathbb{F})$:

$$\begin{aligned} \mathbb{F}(\tilde{A}_{\varphi}(v)) &= N(\tilde{A}_{\varphi}(v)) \\ &= N(\alpha(\varphi) v \varphi^{-1}) \\ &= N(\alpha(\varphi)) N(v) N(\varphi^{-1}) \\ &= N(\varphi) N(v) N(\varphi^{-1}) \\ &= N(v). \end{aligned}$$

$$\Rightarrow \tilde{A}_{\varphi} \in \mathcal{O}(V, \mathbb{F}).$$