PROMYS - Quivers and their representations
Week 1, sheet 1

Problem 1

Two-dimensional vectors. Let $\mathbf{R}^2$ be the set of pairs of real numbers, we will denote such a pair by (a, b) where a, b are real numbers.

(a) Let V be the set of arrows, or *vectors*, which start at $(0, 0)$ and end somewhere in the plane. Write down a bijection between V and $\mathbf{R}^2$.

So, every pair (a, b) uniquely corresponds to an element of the set V (a vector), which we will denote (in this problem) by $\langle a, b\rangle \in V$.

(b) For $\lambda \in \mathbf{R}$ a real number, define the map

$$\lambda\cdot : V \longrightarrow V$$

by $\lambda \cdot \langle a, b\rangle = \langle \lambda a, \lambda b\rangle$. Geometrically, describe the effect of the map $\lambda\cdot$ acting on the vector $\langle a, b\rangle \in V$.

(c) Consider a pair of vectors $\langle a, b\rangle \in V$ and $\langle c, d\rangle \in V$. Define the new element

$$\langle a, b\rangle + \langle c, d\rangle = \langle a + c, b + d\rangle.$$

Describe, geometrically, the relationship between the three vectors:

$$\langle a, b\rangle, \quad \langle c, d\rangle, \quad \langle a + c, b + d\rangle.$$

(d) Simplify the following expression

$$\lambda \cdot (\langle a, b\rangle + \langle c, d\rangle)$$

by writing it as a vector of the form $\langle e, f\rangle$ for some real numbers e, f that can be expressed in terms of λ, a, b, c, d.

(e) Consider the vectors $\langle 1, 2\rangle$ and $\langle 2, 1\rangle$. Find all numbers λ, μ such that:

$$\lambda\langle 1, 2\rangle + \mu\langle 2, 1\rangle = \langle 0, 0\rangle.$$

Remark. Here is a remark on notation. In this problem we wrote an element $v \in V$ as a pair $v = \langle a, b\rangle$. Notationally, this is sometimes written as

$$v = \begin{bmatrix} a \\ b \end{bmatrix}.$$

We refer to this form as a "column vector".

Problem 2

Linear combinations. Let V be as in the previous problem. A *linear combination* of vectors $v_1, v_2, \dots, v_m \in V$ is a vector of the form

$$\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_m v_m \tag{1}$$

where $\lambda_1, \lambda_2, \dots, \lambda_m$ are real numbers (called the *coefficients*).

(a) Solve the following system of equations by any means necessary:

$$\begin{aligned} 3x + 2y &= -2 \\ -x + y &= 4 \end{aligned}$$

(b) Consider the vectors $v_1 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. For which real numbers λ_1, λ_2 (as in equation (1)) is the vector $\begin{bmatrix} -2 \\ 4 \end{bmatrix}$ a linear combination of v_1, v_2?

(c) Let v_1, v_2 be as in part (2). Find coefficients λ_1, λ_2 such that $\lambda_1 v_1 + \lambda_2 v_2 = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$.

Problem 3

Solving systems of equations Consider the $2x2$ system of equations from problem (2.1). The data of this equation can be summarized in the following table:

$$\begin{bmatrix} 3 & 2 & | & -2 \\ -1 & 1 & | & 4 \end{bmatrix} \tag{2}$$

This table is called an *augmented matrix*. Consider the following "row" operations which lead to the solution of this system of equations.

(1) Multiply row two by 3 to obtain the new augmented matrix.

$$\begin{bmatrix} 3 & 2 & | & -2 \\ -3 & 3 & | & 12 \end{bmatrix} \tag{3}$$

(2) Add row one to row two to obtain.

$$\begin{bmatrix} 3 & 2 & | & -2 \\ 0 & 5 & | & 10 \end{bmatrix} \tag{4}$$

(3) Divide row two by 5 to obtain

$$\begin{bmatrix} 3 & 2 & | & -2 \\ 0 & 1 & | & 2 \end{bmatrix} \tag{5}$$

(4) Subtract 2 times row two from row one to obtain

$$\begin{bmatrix} 3 & 0 & | & -6 \\ 0 & 1 & | & 2 \end{bmatrix} \tag{6}$$

(5) Finally, divide row one by 3 to obtain

$$\begin{bmatrix} 1 & 0 & | & -2 \\ 0 & 1 & | & 2 \end{bmatrix} \tag{7}$$

Translating this back to the variables x, y we see that $x = -2, y = 2$—the expected answer.

(a) Perform similar operations to solve the system of equations

$$\begin{aligned} -3x + y &= 4 \\ x + 2y &= 1 \end{aligned}$$

(b) Set up an ”augmented matrix” and perform analogous steps to solve the following 3×3 system of equations (three equations and three unkowns)

$$\begin{aligned} -3x + 2y - 6z &= 6 \\ 5x + 7y - 5z &= 6 \\ x + 4y - 2z &= 8. \end{aligned}$$

(c) Can you perform similar row operations to solve the following system of equations

$$\begin{aligned} x - y &= 0 \\ x + z &= 1 \qquad ? \end{aligned}$$

Can you describe the solution(s) to the above system of equations?

Problem 4

Subspaces. Recall the definition of a subspace of a vector space from lecture.

(a) Let $V = \mathbf{R}[x]$ be the vector space of polynomials in one variable (whose coefficients are real numbers). Consider the set S of polynomials with nonzero constant term. Is S a subspace of $\mathbf{R}[x]$? Explain why or why not. Let S' be the set of all polynomials with vanishing constant term. Is S' a subspace? Explain why or why not.

(b) Let $V = \mathbf{R}^2$ be a two-dimensional real vector space. Find subspaces W, W' of $\mathbf{R}^2$ with the property that $W \cup W'$ is *not* a subspace of $\mathbf{R}^2$.

(c) This problem refers to the system of equations in problem 3(c). Describe the set of *all* solutions to this equation and prove that it defines a subspace of $\mathbf{R}^3$.

(d) Show that a system of linear equations in n-variables $x_1, \ldots, x_n$ defines a subspace of $\mathbf{R}^n$. (This fact is independent of the number of equations in the system.)