Spin geometry
Problem sheet 3

Problem 1. *Playng with quadratic forms*

Let $V_{\mathbf{R}}$ be a $2n$-dimensional real vector space equipped with a quadratic form. Let V be its complexification, we denote the corresponding quadratic form on V by q and the corresponding inner product by $(-,-)$. Fix a splitting $V = L \oplus L^*$ where L, L^* are both maximally isotropic and the quadratic form is

$$q(v + v^*) = \langle v, v^* \rangle.$$

(1) Construct an isomorphism $\mathfrak{q}\colon V \to V^*$ which satisfies

$$2(v, \mathfrak{q}(w)) = q(v + w) - q(v) - q(w).$$

for all $v, w \in V$.

(2) Define the linear map

$$f\colon V \to \mathrm{End}(V)$$

by the formula $f(v) = v \wedge (-) + \mathfrak{q}(v) \vee -$. Show that f extends to a map of algebras

$$\widetilde{f}\colon \mathrm{Cl}(V, q) \to \mathrm{End}(V)$$

(3) Let $g\colon \mathrm{C}\ell(V, q) \to V$ be the linear map

$$g(\varphi) = \widetilde{f}(\varphi)1.$$

Show that g is an isomorphism.

(4) Show that

$$g(vw - wv) = 2g(v) \wedge g(w)$$

for all $v, w \in V$.

Problem 2. *Pure spinors*

This problem references problem 1. In class we have shown that the linear map

$$V \to \mathrm{End}(L)$$

defined by $v + v^* \mapsto v \wedge - + v^* \vee -$ extends to an isomorphism

$$\gamma \colon \mathrm{C}\ell(V) \xrightarrow{\cong} \mathrm{End}(S)$$

where $S = \wedge L$ is the fundamental spinor representation (also called the space of *Dirac spinors*).

(1) The Grassmannian of isotropic subspaces of dimension k is denoted

$$\mathrm{Gr}^k_{iso}(V) = \{W \subset V \mid W \text{ isotropic },\ \dim W = k\}.$$

Show that $O(2n)$ acts transitively on $\mathrm{Gr}^k_{iso}(V)$ for all $k = 1, \ldots, n$.

(2) For $\sigma \in S$ define the null space of σ to be

$$N(\sigma) \stackrel{\text{def}}{=} \{v \in V \mid v \cdot \sigma = 0\}$$

Show that $N(\sigma) \subset V$ is an isotropic subspace.

(3) A *pure spinor* is a spinor $\sigma \in S \setminus 0$ such that $N(\sigma)$ is of maximal dimension. Suppose that σ_1, σ_2 are pure spinors. Show that if σ is a pure spinor, and $\lambda \in \mathbf{C}^\times$ then $\lambda\sigma$ is a pure spinor. Argue that the null space produces a $Spin(2n)$-equivariant map

$$N \colon \mathbf{P}\left(S \setminus 0\right) \to \mathrm{Gr}^n_{iso}(V).$$

(4) Show that $N(\sigma_1) \cap N(\sigma_2) \neq 0$ if and only if $(\sigma_1, \sigma_2) = 0$.

Problem 3. *Complex structures and stabilizers*

This keeps with the notations of the previous problems. Denote the action of $Spin(2n)$ on V by χ. For σ a pure spinor, let

$$G_\sigma = \{a \in Spin(2n) \mid a\psi = \psi\}.$$

(1) Let σ be a pure spinor which represents the maximal isotropic subspace $N(\sigma) \subset V$. Show that if $N(\sigma) \cap \overline{N}(\sigma) = 0$.
(2) From part (1) it follows that there is a decomposition $V = N(\sigma) \oplus \overline{N}(\sigma)$. Define an almost complex structure J on V with the property that $Jx = \mathrm{i}x$ for all $x \in N(\sigma)$.
(3) Show that this almost complex structure is orthogonal with respect to the metric $(-,-)$.
(4) Show that for $a \in G_\sigma$ that

$$J\chi(a)v = \chi(a)Jv$$

for all $v \in V$.
(5) Define the hermitian form $\langle -|-\rangle$ on V by the formula

$$\langle x|y\rangle = (x,y) + \mathrm{i}(x, Jy).$$

Show that $\chi(a)$ is an isometry for $\langle -|-\rangle$ where $a \in G_\sigma$.