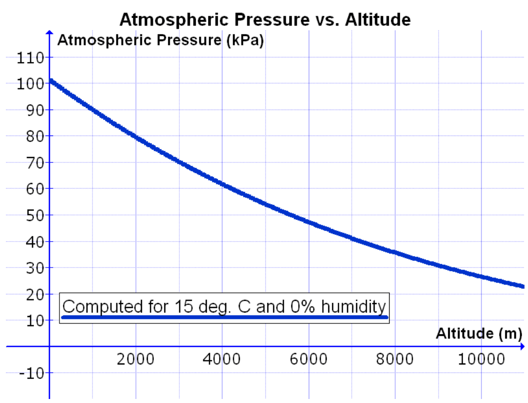
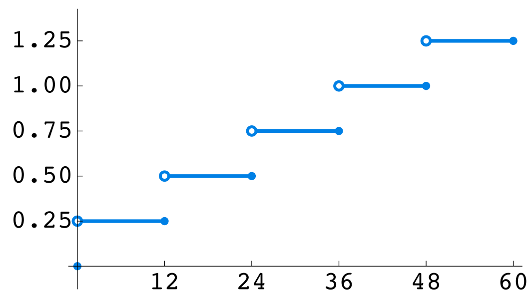
Learning Catalytics exercise #1: Here's some room for a brief calculation.

Continuous functions

Some phenomena are continuous while others are discontinuous. For example, air pressure as a function of altitude is continuous. The amount you pay to park at a parking meter as a function of time is discontinuous.





First we discuss continuity at a number a "inside" the domain of a function. Then we discuss continuity on intervals in the domain.

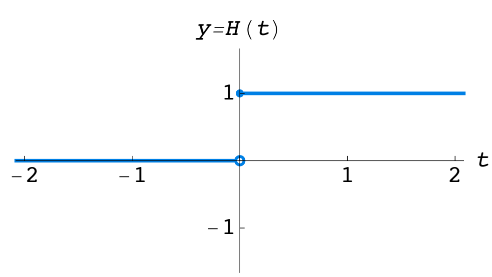
Definition. Suppose that the function f is defined on an open interval containing the number a. Then f is continuous at a if

$$\lim_{x \to a} f(x) = f(a).$$

If f is not continuous at a, then a is a point of discontinuity.

Example. The Heaviside function (the "light switch" function) is

$$H(t) = \begin{cases} 0 & \text{if } t < 0; \\ 1 & \text{if } t \geq 0. \end{cases}$$



The Heaviside function is discontinuous at $a = 0$ because the limit of $H(t)$ as $t \to 0$ does not exist.

Continuity Checklist: In order for f to be continuous at a, the following three conditions must hold:

1. The value $f(a)$ is defined, that is, a is in the domain of f.
2. The limit of $f(x)$ as $x \to a$ exists.
3. The limit must equal $f(a)$.

Examples. The following three examples are discontinuous at $a = 0$.

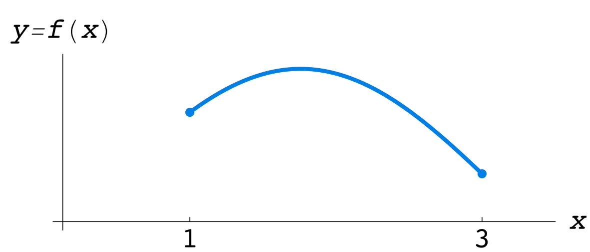
1. The function $f(x) = 1/x$.
2. The function
$$f(x) = \begin{cases} \dfrac{|x|}{x} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$
3. The function $H(-t^2)$.

Theorems 2.9, 2.10, and 2.11 tell us that many functions are continuous for all numbers a in their domains. For example,

1. the sum of two continuous functions is continuous;
2. the ratio of two continuous functions is continuous as long as the value in the denominator is not zero;
3. polynomials are continuous at all numbers a;
4. rational functions are continuous for all numbers a in their domains; and
5. the composition of two continuous functions is continuous.

Continuity on intervals

We also talk about continuity of functions on intervals, and to do so we must think about what to do about intervals with endpoints. For example, let's consider the function f graphed below. Its domain is the interval $[1, 3]$.



We know what it means to be continuous at all numbers in the open interval $(1, 3)$, but we cannot talk about the two-sided limits at 1 or at 3. So we need to make a special convention about continuity at the endpoints of the interval. We say that f is continuous from the right at the endpoint 1 if

$$\lim_{x\to 1^+} f(x) = f(1).$$

Similarly, we say that f is continuous from the left at the endpoint 3 if

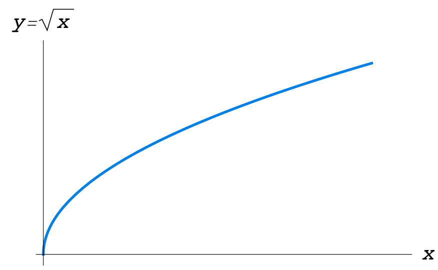
$$\lim_{x\to 3^-} f(x) = f(3).$$

If f is continuous at all numbers in the interval $(1, 3)$ and it is continuous from the right at 1 and from the left at 3, we say that f is continuous on the closed interval $[1, 3]$.

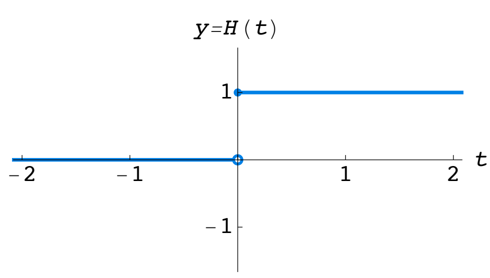
Functions can be continuous on closed intervals, on open intervals, and on half-open intervals. The various possibilities are $[\alpha, \beta]$, $[\alpha, \beta)$, $(\alpha, \beta]$, (α, β), $[\alpha, \infty)$, (α, ∞), $(-\infty, \beta]$, $(-\infty, \beta)$, and $(-\infty, \infty)$, where α and β represent real numbers such that $\alpha < \beta$.

Examples.

1. The function $f(x) = \sqrt{x}$ is continuous on the interval $[0, \infty)$.



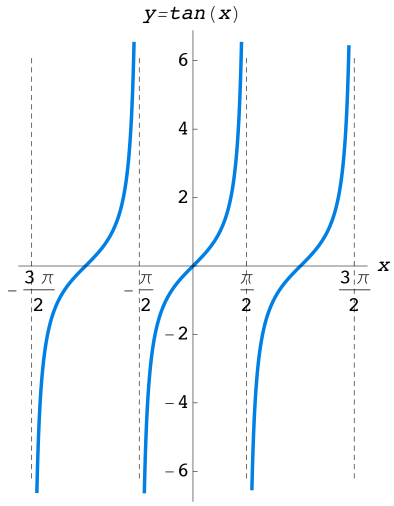
2. The Heaviside function is continuous on the interval $[0, \infty)$ and on the interval $(-\infty, 0)$, but it is not continuous on the interval $(-\infty, \infty)$.



Learning Catalytics exercise #2: Here's some space so that you can organize your thoughts.

The trig functions, exponential functions, and logarithmic functions are continuous at all numbers in their domains, but we must be careful when we talk about continuity on intervals for the trig functions such as $\tan x$ and $\sec x$.

Example.



Intermediate Value Theorem. Suppose the function f is continuous on the closed interval $[a, b]$ and L is a number between $f(a)$ and $f(b)$. Then there is at least one number c between a and b such that $f(c) = L$.

Derivatives

Derivatives are rates of change. We discussed three examples during our first class: velocity, the rate of change of the unemployment rate, and the rate that air pressure changes as a function of altitude.

On September 2 and 7 we discussed average velocity over the time interval $t_0 \leq t \leq t_1$. If $s(t)$ is the position function, then the average velocity is

$$\frac{s(t_1) - s(t_0)}{t_1 - t_0}.$$

The instantaneous velocity at time t_0 is the limit

$$\lim_{t_1 \to t_0} \frac{s(t_1) - s(t_0)}{t_1 - t_0}.$$

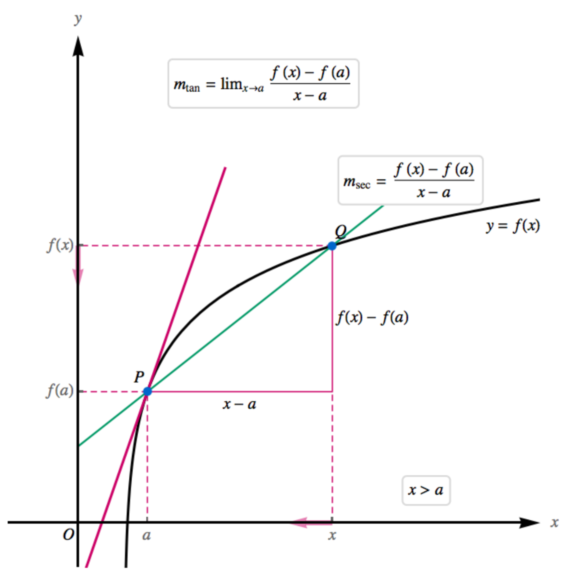
We also discussed why this limit is the same as the slope of the line tangent to the graph of $s(t)$ at $t = t_0$.

We can perform the same limiting process on any function f.

Definition. The instantaneous rate of change in f at $x = a$ is the limit

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

provided this limit exists.



Interactive Figure 3.5 from your textbook

Example. Consider the function $f(x) = 6x - x^2$. What is its instantaneous rate of change at $a = 1$?

Compare this calculation to the informal discussion that we had on September 2 and 7.

Learning Catalytics exercise #3: Use the following space for some algebra: