

Let  $V$  be a vector space.

Then: a)  $0 \cdot v = 0$ . The zero vector.

b)  $(-\lambda) \cdot v = -(\lambda \cdot v) = \lambda(-v)$   
for all  $\lambda \in \mathbb{F}$ ,  $v \in V$ .

c)  $\lambda \cdot 0 = 0$  for all  $\lambda \in \mathbb{F}$   
 $\nearrow$   
zero vector.

Pf: c) By VS8, VS3, VS1:

$$0 \cdot v + 0 \cdot v = (0+0) \cdot v = 0 \cdot v = 0 \cdot v + 0$$

$$\Rightarrow 0 \cdot v = 0 \text{ by the cancellation lemma.}$$

b) From last time, we know  $-(\lambda \cdot v)$  is the unique vector for which

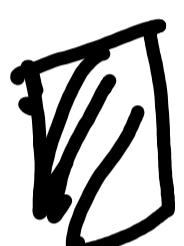
$$\lambda \cdot v + (-(\lambda \cdot v)) = 0.$$

So, if  $\lambda \cdot v + (-\lambda) \cdot v = 0$  we have

that  $(-\lambda) \cdot v = -(\lambda \cdot v)$ . In particular,

$$(-1) \cdot v = -v$$

c) Exercise.



Exercises for § 1.2 :

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1, 4(e)-4(h), 7, 8, 10, 12, 13, 17, 20, 21.

- Subspaces.

A subspace of a vector space is a subset which is, itself, a vector space. Here is the precise definition.

Dfn: A subset  $W \subset V$  is a subspace if  
it is a vector space according to the  
operations of addition and multiplication  
in  $V$ .  
 $\leftarrow V$  is a vector space

Note: if  $W \subset V$  is a subspace then it  
must contain the zero vector!

Ex:  $W = \{0\}$  = the single element set containing  
the zero vector  
 $\cap$   
 $V$   
is a subspace.

Ex:  $W = \bigcap_{V'} V'$  is a subspace.

- To check whether  $W \subset V$  is a subspace:
  - 1)  $v + w \in W$  whenever  $v, w \in W$ .  
(closed under addition)
  - 2)  $\lambda v \in W$  whenever  $v \in W$  and  $\lambda \in F$ .  
(closed under scalar mult.)
  - 3)  $W \ni 0$ .
  - 4) Every vector  $v \in W$  has  $-v \in W$  as well.

In fact, 4) is repetitive.

Theorem:  $W \subset V$  is a subspace if and only if the following three conditions are satisfied:

- a)  $0 \in W$
- b)  $v + w \in W$  whenever  $v, w \in W$
- c)  $\lambda v \in W$  whenever  $v \in W, \lambda \in F$

Pf: ( $\Rightarrow$ ) Assume  $W$  is a subspace. Then  
(b) and (c) automatically hold. Also, we  
know there is  $0' \in W$  s.t.  $v + 0' = v$   
for all  $v \in W$ . But also  $v + 0 = v$   
for all  $v \in V$ . So, if  $v \in W$  then

$$v + 0' = v + 0 \Rightarrow 0 = 0'.$$

Thus  $0 = 0' \in W$ .

( $\Leftarrow$ ) Conversely, suppose (a) - (c) hold. To  
see  $W$  is subspace we need to show that  
the additive inverse of each  $v \in W$  is in  $W$ .  
But by (c) we have  $-v = (-1) \cdot v \in W$ .  
Hence  $W$  is a subspace of  $V$ .  $\square$

Ex:  $V = \mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$ .

We know  $\mathbb{R}^3$  is a vector space using the  
usual rules.

"xy-plane"  
 $\cap$   
 $\mathbb{R}^3$  is a subspace.

Ex:  $\mathbb{R}[x]$  = set of all polynomials with  
real coefficients.

We know this is a vector space.

$$W_d = \left\{ \begin{array}{l} \text{set of all polynomials} \\ \text{of degree } \leq d \end{array} \right\}$$

E.g:  $W_2 = \left\{ a_0 + a_1 x + a_2 x^2 \mid a_0, a_1, a_2 \in \mathbb{R} \right\}$

Then,  $W_d \subset \mathbb{R}[x]$  is a subspace.

• Intersection: If  $\underbrace{A}$  is a set and  
 $B, C \subset A$

are subsets. The intersection is

$$B \cap C = \left\{ \begin{array}{l} \text{elements } a \in A \text{ s.t.} \\ a \in B \text{ and } a \in C \end{array} \right\}.$$

Theorem: If  $V$  is a vector space and

$W, Z \subset V$  are two subspaces. Then

$$W \cap Z \subset V$$

is a subspace.

Pf: Let  $v, w \in W \cap Z$ . Then

$$v + w \in W \text{ and } v + w \in Z$$

$$\Rightarrow v + w \in W \cap Z.$$

Similarly,  $\lambda v \in W \cap Z$  for all  $\lambda \in F$ .

