

# Vectors in $\mathbb{R}^2$ , $\mathbb{R}^3$ (and $\mathbb{R}^n$ )

- Euclidean space  $\mathbb{R}^n$  is the set of  $n$ -tuples of real numbers

$$\{(a_1, \dots, a_n) \mid a_i \in \mathbb{R}\}$$

- On the other hand consider the set  $V$  of all "arrows" which begin at  $\underline{0} = (0, \dots, 0) \in \mathbb{R}^n$  and end at some other point. We call  $V$  the set of vectors.
- There is an bijection \* of sets

$$\mathbb{R}^n \xrightarrow{\cong} V$$

which sends a point  $P = (a_1, \dots, a_n)$  to the vector which ends at  $P$ . In other words

$$P \xrightarrow{\quad} \overrightarrow{OP}$$

[We will review the concept of isomorphism in discussion section. Also, see Appendix A].

Write  $\vec{OP} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in V$  where

$$\vec{P} = (a_1, \dots, a_n) \in \mathbb{R}^n.$$

Recall some familiar operations :

- Vector addition

$$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix}$$

- Scalar Multiplication

If  $\lambda \in \mathbb{R}$  then

$$\lambda \cdot \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \lambda a_1 \\ \vdots \\ \lambda a_n \end{bmatrix}.$$

We will now abstract these operations (and their intrinsic properties) into the notion of a

\* Vector space . \*



• Comment on \*fields\*. A field  $\overset{v}{F}$  is an algebraic object for which we will do linear algebra "over". We will spend some more time w/ fields in this course, but for now we usually take  $F = \mathbb{R}$  the real numbers.

{ See Appendix C for a background on fields }

A field  $F$  forms the "scalars". Other examples of fields include  $F = \mathbb{C}$ , the complex numbers, or  $F = \mathbb{F}_2 = \{0, 1\}$  (the field w/ two elements).

We move on to our main definition.

Dfn: A vector space over a field  $F$  is a set

$$V = \{ \text{the set of "vectors"} \}$$

together with operations

- Addition: if  $v, w \in V$  then  $v + w \in V$ .

- Scalar multiplication: if  $v \in V, \lambda \in F$  then  $\lambda \cdot v \in V$ .

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These operations must satisfy the following V axioms:

VS 1) For all  $v, w \in V$  have  $v + w = w + v$ .

VS 2) For all  $v, w, u \in V$  have

$$(v + w) + u = v + (w + u).$$

VS 3) There is  $0 \in V$  s.t.  $v + 0 = v$   
for all  $v \in V$ . (The zero vector)

VS 4) For all  $v \in V$  there exists  $\tilde{v} \in V$   
s.t.  $v + \tilde{v} = 0$ .

VS 5) For all  $v \in V$ ,  $1 \cdot v = v$ , where  
 $1 \in F$  is the unit.

VS 6) For all  $\lambda, \mu \in F$  and  $v \in V$  have

$$(\lambda\mu) \cdot v = \lambda \cdot (\mu \cdot v).$$

VS 7) For all  $\lambda \in F$ , and  $v, w \in V$  have

$$\lambda \cdot (v + w) = \lambda \cdot v + \lambda \cdot w$$

VS 8) For all  $\lambda, \mu \in F$  and  $v \in V$  have

$$(\lambda + \mu) \cdot v = \lambda \cdot v + \mu \cdot v.$$

Ex: For any integer  $n$  we can look at

$$\mathbb{F}^n = \left\{ (a_1, \dots, a_n) \mid a_i \in \mathbb{F}, i=1, \dots, n \right\}.$$

In the familiar case  $\mathbb{F} = \mathbb{R}$  this is  $\mathbb{R}^n$ . Write elements of  $\mathbb{F}^n$  in "column vector" notation

$$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{F}^n$$

Then the familiar operations

$$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix}$$

$$\lambda \cdot \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \lambda a_1 \\ \vdots \\ \lambda a_n \end{bmatrix}, \quad \lambda \in \mathbb{F}.$$

Endow  $\mathbb{F}^n$  w/ the structure of a vector space over  $\mathbb{F}$ .

Q: What is the zero vector?

Ex: A sequence in  $\bar{F}$  is an ordered, countably infinite collection of elements of  $\bar{F}$ :

$$\underline{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \dots)$$

where each  $\alpha_i \in \bar{F}$ . Denote the set of all such sequences by  $F^\infty$ . Then  $F^\infty$  has the natural structure of a vector space over  $\bar{F}$ .

[Heuristically : " $\lim_{n \rightarrow \infty} F^n = F^\infty$ " ]

Ex: A polynomial in  $\bar{F}$  is an expression of the form

$$\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \dots$$

where  $\alpha_i \in \bar{F}$  and all but finitely many of the  $\alpha_i$ 's are zero. [So only a finite power of  $x$  can appear.]

Then, we can define the "addition"

$$(a_0 + a_1 x + a_2 x^2 + \dots) + (b_0 + b_1 x + b_2 x^2 + \dots)$$

$$= (a_0 + b_0) + (a_1 + b_1) x + (a_2 + b_2) x^2 + \dots$$

And "scalar multiplication"

$$\lambda \cdot (a_0 + a_1 x + a_2 x^2 + \dots)$$

$$= \lambda a_0 + \lambda a_1 x + \lambda a_2 x^2 + \dots$$

These operations satisfy VS1 - VS8. So the set of all polynomials form a vector space.

We denote it

$$\mathbb{F}[x]$$

[Can you extend this to polynomials in more than one variable:

$$\mathbb{F}[x, y, z, \dots] ? ]$$