THE MAIN THEOREM

Dirac's goal was to find a first-order differential operator D acting whose square is the Laplacian. A ***generalized Laplacian*** H is a second order differential operator acting on sections of a vector bundle E over a Riemannian manifold M with the property that its symbol evaluated at $(x, \xi) \in M \times T_x^*M$ is $|\xi|^2$. In the same spirit as Dirac; Berline, Getzler, and Vergne define a Dirac operator to be any differential operator whose square is a generalized Laplacian.

Definition 0.1. Let $E = E^+ \oplus \Pi E^-$ be a super vector bundle on a Riemannian manifold M. A ***Dirac operator*** on E is an odd first-order differential operator

$$\mathsf{D} \colon \mathcal{E} \to \mathcal{E} \tag{1}$$

such that D^2 is a generalized Laplacian.

A fundamental result is that if M is compact then a Dirac operator D on M has finite dimensional kernel. The Atiyah–Singer index theorem is an expression for the index

$$\operatorname{ind} \mathsf{D} = \dim \ker \mathsf{D}^+ - \dim \ker \mathsf{D}^-. \tag{2}$$

In other words, the index is the super-dimension of $\ker D$. To state the index theorem it is convenient to assume that we have a Dirac operator associated to a so-called Clifford module structure on the bundle E. (We will see that this is at no loss of generality, there is a one-to-one correspondence between Clifford module structures and compatible Dirac operators.)

These notes sketch the proof of the following theorem following the book of Berline, Getzler, and Vergne.

Theorem 0.2. *Let D be the Dirac operator associated to a Clifford module $\mathcal{E}$ over a compact oriented manifold M of even dimension. Then*

$$\operatorname{ind}(\mathsf{D}) = \frac{1}{(2\pi \mathrm{i})^{n/2}} \int\limits_M \widehat{A}(M) \operatorname{ch}(\mathcal{E}/S). \tag{3}$$

1. Heat kernels of generalized Laplacians and their trace

Let E be a vector bundle on a Riemannian manifold M. Let Dens^s be the bundle of s-densities on M; this is the line bundle associated to the one-dimensional representation $|\det|^{-s}$. A ***kernel*** is a section

$$k(x, y) \in \Gamma\big(M \times M, (E^* \otimes \mathrm{Dens}^{1/2}) \boxtimes (F \otimes \mathrm{Dens}^{1/2})\big). \tag{4}$$

A kernel determines an operator

$$K\colon \overline{\Gamma}_c(M, E \otimes \mathrm{Dens}^{1/2}) \to \Gamma(M, F \otimes \mathrm{Dens}^{1/2}) \tag{5}$$

defined by the formula $(Ks)(x) = \int_{y \in M} k(x,y)s(y)$. Here $\overline{\Gamma}(M, -)$ denotes distributional (or generalized) sections. The Schwarz kernel theorem asserts an equivalence between bounded linear operators of the above type and kernels. If K is an operator of this type, we will often write the associated kernel as $\langle x|K|y\rangle$.

We are most interested in making sense of the $\mathbf{R}_+$-family of operators e^{-tH} where H is a generalized Laplacian. A ***heat kernel*** $p_t(x,y)$ axiomatizes the properties that the kernel of such a family of operators must possess. A heat kernel $p_t(x,y)$ for H is of class C^1 in t, of class C^2 in x, y. Importantly, a heat kernel satisfies the heat equation

$$(\partial_t + H_x)p_t(x,y) = 0 \tag{6}$$

together with the initial condition $\lim_{t\to 0} p_t(x,y) = \delta(x-y)$.

On Euclidean space $\mathbf{R}^n$, there is the following explicit expression for the heat

$$q_t(x,y) = \frac{1}{(4\pi t)^{n/2}} e^{-\|x-y\|^2/4t}. \tag{7}$$

To produce the heat kernel associated to an arbitrary generalized Laplacian H one proceeds by the following steps.

(1) First, one constructs a *formal* heat kernel of the form

$$k_t(x,y) = q_t(x,y) \sum_{i=0}^{\infty} t^i \Phi_i(x,y,H)|\mathrm{d}y|^{1/2} \tag{8}$$

By formal one means a few things. The sections Φ_i are defined only in a neighborhood of the diagonal in $M \times M$, and the resulting local section $x \mapsto \Phi_t(x,y)$ satisfies the modified heat equation

$$(\partial_t + t^{-1}\nabla_{Eu} + j^{1/2} \circ H \circ j^{-1/2})\Phi_t(\cdot, y) = 0, \tag{9}$$

where Eu is the Euler vector field defined using normal coordinates in a neighborhood of y, and j is the determinant of the Jacobian matrix in normal coordinates.

(2) From a formal solution $k_t(x,y)$ one uses a cut-off function $\psi\colon \mathbf{R}_+ \to [0,1]$ to define an *approximate* solution of the form

$$k_t^N(x,y) = \psi(d(x,y)^2) q_t(x,y) \sum_{i=0}^{N} t^i \Phi_i(x,y,H)|\mathrm{d}y|^{1/2} \tag{10}$$

which is defined everywhere on $M \times M$ and for each $N \geq 0$. The key property of the approximate heat kernel is that its failure to satisfy the heat equation

$$r_t^N(x,y) \stackrel{\text{def}}{=} (\partial_t + H_x)k_t^N(x,y) \tag{11}$$

satisfies an estimate of the form

$$\|r_t^N(x,y)\|_\ell \leq C(\ell)t^{N-n/2-\ell/2} \tag{12}$$

for each $\ell > 0$.

(3) From the approximate solution one defines a family of kernels

$$q_t^{N,k}(x,y) \stackrel{\text{def}}{=} \int\limits_{t\Delta^k}\int\limits_{M^k} k_{t-t_k}^N(x,z_k)r_{t_k-t_{k-1}}(z_k,z_{k-1})\cdots r_{t_1}(z_1,y) \tag{13}$$

for $k \geq 0$. For N large enough, we can use the above estimate to argue that this integral is well-defined, the sum

$$p_t(x,y) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty}(-1)^k q_t^{N,k}(x,y) \tag{14}$$

converges, and is a heat kernel for H.

The ***Hilbert–Schmidt norm*** of an operator A acting on a Hilbert space with orthonormal basis $\{e_i\}$ is defined as

$$\|A\|_{HS}^2 = \sum_{i,j}(Ae_i, e_j). \tag{15}$$

An operator A is called Hilbert–Schmidt if its Hilbert–Schmidt norm is finite. An operator is ***trace-class*** if it has the form AB where A, B are Hilbert–Schmidt. For such an operator the sum

$$\mathrm{Tr}(AB) \stackrel{\text{def}}{=} \sum_i (ABe_i, e_i), \tag{16}$$

is finite.

Let M be a compact manifold and E a Hermitian vector bundle on M. Given two sections s, s' of $E \otimes \mathrm{Dens}^{1/2})$ then $(s,s')_E = \mathrm{Tr}(s^*s')$ is a section of Dens. Denote

$$\Gamma_{L^2}(M, E \otimes \mathrm{Dens}^{1/2}) \tag{17}$$

the Hilbert space of space of square-integrable sections of $E \otimes \mathrm{Dens}^{1/2}$. If A is an operator acting on sections of $E \otimes \mathrm{Dens}^{1/2}$ with square-integrable kernel

$$\langle x|A|y\rangle \in \Gamma_{L^2}(M \times M, E \otimes \mathrm{Dens}^{1/2} \boxtimes E \otimes \mathrm{Dens}^{1/2}), \tag{18}$$

then A is trace class with

$$\mathrm{Tr}(A) = \int\limits_{x \in M} \mathrm{Tr}(\langle x|A|x\rangle). \tag{19}$$

Here, $\mathrm{Tr}(\langle x|A|x\rangle)$ is the density obtained by restricting $\langle x|A|y\rangle$ to the diagonal and applying the inner product.

If H is a generalized Laplacian acting sections of $E \otimes \mathrm{Dens}^{1/2}$, then the operator P_t associated to the heat kernel $p_t(x,y)$ of H is trace class for any $t > 0$ with trace

$$\mathrm{Tr}(P_t) = \int\limits_{x \in M} \mathrm{Tr}(p_t(x,x)). \tag{20}$$

If E is a Hermitian vector bundle, then a generalized Laplacian H acting on sections of $E \otimes \mathrm{Dens}^{1/2}$ is symmetric if $H = H^*$, the formal adjoint of H. In this case, the operator P_t associated to the heat kernel $p_t(x,y)$ of H is equal to e^{-tH}.[1] Let $P_{(0,\infty)}$ be the projection onto the space of eigensections of H with positive eigenvalue. Then, the kernel $\langle x|P_{(0,\infty)}e^{-tH}P_{(0,\infty)}|y\rangle$ satisfies the following important bound: for t sufficiently large one has

$$\| \langle x|P_{(0,\infty)}e^{-tH}P_{(0,\infty)}|y\rangle \|_\ell \leq C(\ell)e^{-t\lambda_1} \tag{21}$$

where λ_1 is the smallest non-zero positive eigenvalue of H.

[1]More precisely, it is the closure $\overline{H}$ of H acting on $\Gamma(M, E \otimes \mathrm{Dens}^{1/2})$ that should appear here, but we will not distinguish these two operators in what follows.