

Linear combinations

Dfn: Let V be a vector space over \mathbb{F} . A vector $v \in V$ is said to be a linear combination of vectors $\{u_1, \dots, u_n\}$ if it can be written

$$v = a_1 u_1 + \dots + a_n u_n.$$

(or
$$v = \sum_{i=1}^n a_i u_i. \quad)$$

n-terms.

where $a_1, \dots, a_n \in \mathbb{F}$.

Ex: The vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$:

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

The problem of solving linear systems of equations is the problem of finding if and how a vector v is a linear combination of other vectors u_1, \dots, u_n .

Ex: Take $V = \mathbb{R}^3$. Is $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \in \mathbb{R}^3$ a linear combination of the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$?

In other words, can we find $a_1, a_2 \in \mathbb{R}$ such that

$$a_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$$\Leftrightarrow \begin{cases} a_1 + a_2 = 2 \\ a_1 + 2a_2 = 1 \\ a_1 - a_2 = 3 \end{cases}$$

\nearrow
This is a system of linear equations

Can we solve this example? To do this, let's add the third equation to the first:

$$2a_1 = 5 \Rightarrow a_1 = 5/2.$$

But then eqn 1 says $5/2 + a_2 = 2$ or

$a_2 = -1/2$. So eqn 1, 2 force $a_1 = 5/2, a_2 = -1/2$.

Does this solve eqn 2?

$$a_1 + 2a_2 = 1$$

$$\parallel$$

$$5/2 - 1$$

$$\parallel$$

$$3/2$$

$$3/2 \neq 1.$$

Thus, $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ is not a linear combination

of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$.

Span : Let V be a vector space. Let

$$S \subset V$$

be a subset of vectors. The span of S is the set

$$\text{Span}(S) = \left\{ \begin{array}{l} \text{all linear combinations} \\ \text{of vectors in } S \end{array} \right\}.$$

Theorem : For any $S \subset V$ the span

$$\text{span}(S) \subset V$$

is a subspace. Moreover, $\text{span}(S)$ is the smallest subspace which contains S .

Pf : If $S = \emptyset$ then

$$\text{span}(\emptyset) = \{0\}, \text{ which is subspace.}$$

If $S \neq \emptyset$ then S contains a vector $z \in S$.

So $0 \cdot z = \vec{0}$ is in $\text{Span } S$. If $x, y \in \text{Span}(S)$ then

$$x = a_1 u_1 + \dots + a_n u_n$$

$$y = b_1 v_1 + \dots + b_n v_n$$

for some $a_i, b_j \in \mathbb{F}$, $u_i, v_j \in S$.

Then

$$x + y = a_1 u_1 + \dots + a_m u_m + b_1 v_1 + \dots + b_n v_n$$

is still a linear combination of vectors in S .

Thus $x + y \in \text{Span}(S)$. Similarly $\lambda \cdot x \in \text{Span}(S)$ for any $\lambda \in \mathbb{F}$. Thus $\text{Span}(S)$ is subspace.

Now, suppose $W \subset V$ is a subspace which contains S .

If $w \in \text{Span}(S)$ then

$$w = a_1 w_1 + \dots + a_k w_k$$

for some $a_1, \dots, a_k \in \mathbb{F}$ and $w_1, \dots, w_k \in S$. Since

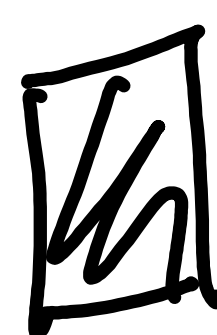
$S \subset W$ we see $w_1, \dots, w_k \in W$ as well. But,

Since W is subspace we see that $w \in W$.

Thus $\text{Span}(S) \subset W$. Since W was

an arbitrary subspace which contains S , we

see $\text{Span}(S)$ is the smallest subspace which contains S .



Dfn: A subset $S \subset V$ spans V if

$$\text{span}(S) = V.$$

Ex: $V = \mathbb{R}^3$. Then, the three element subset:

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

spans \mathbb{R}^3 .

Pf: For any $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{R}^3$ we need to

find a_1, a_2, a_3 s.t.

$$a_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + a_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

You can directly check that

$$\begin{cases} a_1 = \frac{1}{2} (v_1 + v_2 - v_3) \\ a_2 = \frac{1}{2} (v_1 - v_2 + v_3) \\ a_3 = \frac{1}{2} (-v_1 + v_2 + v_3) \end{cases}$$

solves this eqn. So S spans \mathbb{R}^3 . \square

Q: Typically, one can find many subsets which span a given subspace. The goal in the next lecture is deciding how to find such a subset which is as "small as possible".

- Sum of subspaces.

Given two subspaces $W_1, W_2 \subset V$ define

$W_1 + W_2$ to be the smallest subspace which contains W_1 and W_2 .

Prop: Let $S_1, S_2 \subset V$ be arbitrary subsets.

Then

$$\text{Span}(S_1 \cup S_2) = \text{Span}(S_1) + \text{Span}(S_2).$$