<u>Prop</u>: Let V be a v.s. w/ $\dim V = n$.

a) Any finite generating set for V contains at least n vectors. A generating set which contains exactly n vectors is a basis.

b) Any linearly independent subset of V that contains exactly n vectors is a basis.

c) Any linearly independent subset of V can be extended to a basis by adding vectors to it.

<u>Pf</u>: (a) Sps $G \subset V$ generates V and is finite. By the theorem some subset $H \subset G$ is a basis for V. The result follows.

(b) Sps $L \subset V$ is lin. independent, $\# L = n$. Let $\mathcal{B}$ be any basis for V. By the replacement theorem, there is $H \subset \mathcal{B}$ containing $n - n = 0$ vectors s.t. $L \cup H$ generates V. So $H = \phi$, $L \cup H = L$ generates V.

(c) If $L \subset V$ is lin. ind., $\# L = m \leq n$, the replacement theorem $\Rightarrow$ there is $H \subset B$ containing $n-m$ vectors s.t. $L \cup H$ generates V. Now, $L \cup H$ contains at most $m + n - m = n$ vectors, so (a) $\Rightarrow$ $L \cup H$ contains exactly n vectors and is a basis. ∎

Ex: $V = P_2 \subset \mathbb{R}[x]$ is the vector space of polynomials which are at most quadratic.
Consider the subset

$$G = \{ x^2 + 3x - 2,\ 2x^2 + 5x - 3,\ -x^2 - 4x + 4 \}.$$

For any $a, b, c \in \mathbb{R}$ we have:

$$\begin{aligned} ax^2 + bx + c &= (-8a + 5b + 3c)(x^2 + 3x - 2) \\ &+ (4a - 2b - c)(2x^2 + 5x - 3) \\ &+ (-a + b + c)(-x^2 - 4x + 4) \end{aligned}$$

So, G generates P_2.

On the other hand, by (a) we see that G is a basis for P_2 since $\dim P_2 = 3$.

- Dimension of subspaces

Theorem: Let $W \subset V$ be a subspace of V a finite-dimensional vector space. Then

$$\dim W \leq \dim V.$$

Moreover, if $\dim W = \dim V$ then $V = W$.

Pf: Set $n = \dim V$. If $W = \{0\}$ then $\dim W = 0$, and of course $0 \leq n$. If $W \neq \{0\}$ then W contains a nonzero vector $u_1 \in W$. The set $\{u_1\}$ is linearly independent. Continue choosing vectors $\{u_1, u_2, \ldots, u_m\}$ so that this set is linearly independent and adjoining any other vector results in a linear dependent set. Since no linearly independent set can contain more than n vectors, $m \leq n$. On the other hand, we proved that such a subset also generates W. So, it is a basis. Thus $m = \dim W$.

Finally, sps $\dim W = n$. Then any basis for W automatically is linearly independent and contains n vectors. By Corollary, this is also a basis for V. Thus $W = V$. $\boxtimes$

Ex: $W = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \;\middle|\; \begin{matrix} a_1 + a_3 + a_5 = 0 \\ a_2 = a_4 \end{matrix} \right\}$.

Claim: $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

is a basis for W.

Pf: Since $a_2 = a_4$ we are looking at vectors of the form $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_2 \\ a_5 \end{bmatrix}$ s.t. $a_1 + a_3 + a_5 = 0$.

It is easy to see each of these vectors are in W.

Sps

$$\lambda_1 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = 0$$

$\Leftrightarrow$ $-\lambda_1 - \lambda_2 = 0$

$\lambda_3 = 0$ $\Rightarrow$ set is linearly independent.

$\lambda_1 = 0$, $\lambda_2 = 0$

Finally, we check these vectors generate W.

Since $a_1 + a_3 + a_5 = 0$, $a_2 = a_4$.

$$\Rightarrow a_1 = -a_3 - a_5.$$

Thus, we are free to choose any $a_3 = r$, and $a_5 = s$. Also free to take $a_4 = t = a_2$ where $r, s, t \in \mathbb{R}$. So an arbitrary vector in W is of the form

$$r\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -r-s \\ t \\ r \\ t \\ s \end{pmatrix}.$$

So the set is a basis for W. Thus

$\dim W = 3$.

Cor: If $W \subset V$ is a subspace, then any basis $\{u_1, \ldots, u_m\}$ for W can be extended to a basis for V.

Pf: $\{u_1, \ldots, u_m\}$ is lin. independent. By Corollary 2 of replacement then it can extended to a basis for V. $\boxed{\text{Pf}}$

Ex: Let $c_0, c_1, \ldots, c_n$ be distinct scalars. Define the polynomials:

$$f_i(x) = \prod_{\substack{k=0 \\ k \neq i}}^{n} \frac{x - c_k}{c_i - c_k}$$

do not include

$$= \frac{(x - c_0) \cdots \widehat{(x - c_i)} \cdots (x - c_n)}{(c_i - c_0) \cdots \widehat{(c_i - c_i)} \cdots (c_i - c_n)}.$$

do not include.

Then

- $f_i(x)$ is a polynomial.

- $f_i(c_j) = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j. \end{cases}$

Claim: $\{ f_0, f_1, \dots, f_n \}$ is a basis for

$$P_n = \left\{ \begin{matrix} \text{polynomials} \\ \text{of deg} \le n \end{matrix} \right\}.$$

Pf: