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Up until this point we have mostly been studying functions which are defined explicitly $y = f(x)$. Sometimes, in practical situations, we are know that a variable y depends on a variable x, but the exact dependence is determined *implicitly*. For example, we can study an equation of the form

$$x^2 + y^2 = 1. \tag{61}$$

In this plane, this equation represents the circle. On the other hand, if we think about x as the independent variable, this equation does not determine a function $y = f(x)$ in a unique way. In general, we would need some more information to determine y as a function of x (in this example, assuming that y is non-negative is enough). In this situation, we say that y depends *implicitly* on x.

Even if variables depend implicitly on each other, we can still ask for rates of change of one variable with respect to the other variable. Let's think about the example and ask about the derivative $\mathrm{d}y/\mathrm{d}x$.

The first step is to think about $y = y(x)$ depending on x. Then the equation is

$$x^2 + (y(x))^2 = 1. \tag{62}$$

We then take the derivative of both sides with respect to x:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 + y(x)^2\right) = \frac{\mathrm{d}}{\mathrm{d}x}(1). \tag{63}$$

The right hand side is zero. The left hand side is

$$2x + 2y(x)y'(x) = 2x + 2y(x)\frac{\mathrm{d}y}{\mathrm{d}x} \tag{64}$$

as we can see by applying the chain rule. Thus, the derivative of the equation is

$$x + y(x)y'(x) = 0. \tag{65}$$

Next, we solve for the derivative

$$y'(x) = -\frac{x}{y(x)}. \tag{66}$$

as long as $y(x) \neq 0$. What is happening when $y = 0$?

Example 2.21. What are the slopes of the line tangents to the circle at $x = \frac{1}{2}$.

Example 2.22. Suppose that the variables x, y satisfy

$$x^2 + y^3 = 1. \tag{67}$$

Find $\frac{d^2y}{dx^2}$.

Example 2.23. Find the equations for the vertical and horizontal tangent lines to the graph described by the following equation

$$x^2 + 2y^2 = xy. \tag{68}$$

Example 2.24. Find equations for lines tangent to the graph

$$y^2 - 3xy = 2 \tag{69}$$

when $x = 1/3$.