

HOMEWORK 2

DUE SEPTEMBER 22

There are three problems to turn in.

- (1) Let $M_n(\mathbf{R})$ denote the vector space of real $n \times n$ matrices.
 - (a) Fix a matrix $C \in M_n(\mathbf{R})$ and define the map $l_C: M_n(\mathbf{R}) \rightarrow M_n(\mathbf{R})$ by the rule $l_C(A) = CA$. Show that l_C is differentiable and find its derivative.
 - (b) Let $\tau: M_n(\mathbf{R}) \rightarrow M_n(\mathbf{R})$ be the transpose map $\tau(A) = A^t$. Show that τ is differentiable and find its derivative.
 - (c) Let $f, g: M_n(\mathbf{R}) \rightarrow M_n(\mathbf{R})$ be differentiable maps. Show that the map $h: M_n(\mathbf{R}) \rightarrow M_n(\mathbf{R})$ defined by $h(A) = f(A)g(A)$ is differentiable and express its derivative in terms of the derivatives of f, g .
 - (d) Let $f: M_n(\mathbf{R}) \rightarrow M_n(\mathbf{R})$ be the map $f(A) = A^t A$. Show that f is differentiable and find its derivative.

- (2) The determinant function

$$\det: M_n(\mathbf{R}) \rightarrow \mathbf{R}$$

is a polynomial in n variables; as such it is a smooth function. Complete the steps below to express the derivative of \det in terms of familiar objects.

- (a) Let $\mathbb{1} \in M_n(\mathbf{R})$ be the identity matrix. For a matrix B compute $D_{\mathbb{1}}(B)$ in terms of a simple invariant of $n \times n$ matrices.
- (b) Using (a), find an expression for $D_A(\det)(B)$ where $A \in GL_n(\mathbf{R}) \subset M_n(\mathbf{R})$ is an invertible $n \times n$ matrix.
- (c) Let $\text{cof}(A)$ denote the cofactor of a square matrix. Using that $GL_n(\mathbf{R}) \subset M_n(\mathbf{R})$ is an open dense subset (you may use this without proof) find a formula for $D_A(\det)(B)$ for arbitrary $A, B \in M_n(\mathbf{R})$ in terms of $\text{cof}(A)$. (Hint: when $A \in GL_n(\mathbf{R})$ one has $\text{cof}(A) = (\det A)A^{-1}$).

- (3) Let M be any topological space and let $C^0(M)$ denote the algebra of continuous functions on M . Given a continuous map between spaces $F: M \rightarrow N$ define $F^*: C^0(N) \rightarrow C^0(M)$ by $F^*(f) = f \circ F$. We say that $F^*(f)$ is the *pullback* (or restriction) of f along F .
 - (a) Show that F^* is an algebra homomorphism.

- (b) Suppose now that M, N are smooth manifolds. Show that $F: M \rightarrow N$ is smooth if and only if

$$F^*(C^\infty(N)) \subset C^\infty(M).$$

- (c) Show that $F^*: C^\infty(N) \rightarrow C^\infty(M)$ is an isomorphism if and only if F is a diffeomorphism.