Solutions to selected exercises from §1.6

Question 4

No, these vectors do not generate P_3 (the set of polynomials, with real coefficients, which are at least cubic). The reason is that $\dim P_3 = 4$ as we have shown that a basis for this vector space consists of four vectors (for example $\{1, x, x^2, x^3\}$). Since a basis is the minimum sized subset which can generate a vector space, a collection of vectors consisting of less than four vectors can not be generating.

For practice, we will show that these three vectors are linearly independent, however. Suppose $\lambda_1, \lambda_2, \lambda_3$ are scalars such that

$$\mathbf{0} = \lambda_1(x^3 - 2x^2 + 1) + \lambda_2(4x^2 - x + 3) + \lambda_3(3x - 2). \tag{1}$$

This equation can be written as

$$\mathbf{0} = \lambda_1 x^3 + (-2\lambda_1 - 4\lambda_2)x^2 + (-\lambda_2 + 3\lambda_3)x + (\lambda_1 + 3\lambda_2 - 2\lambda_3). \tag{2}$$

For this to be zero, each coefficient must be zero. Thus $\lambda_3 = 0$ comes from vanishing of the cubic term. From the vanishing of the linear term we see that $\lambda_2 = 0$. From the vanishing of the quadratic term we see $\lambda_1 = 0$. Thus, the collection is linear independent.

Question 7

The vectors u_1, u_2 are clearly not parallel, hence $\{u_1, u_2\}$ is linearly independent. Next, we check if adjoining u_3 results in a linearly independent set. Suppose $\lambda_1, \lambda_2, \lambda_3$ are scalars such that

$$\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 = \mathbf{0}.$$

This is equivalent to the system of three equations

$$2\lambda_1 + \lambda_2 - 8\lambda_3 = 0 \tag{3}$$
$$-3\lambda_1 + 4\lambda_2 - 4\lambda_3 = 0 \tag{4}$$
$$\lambda_1 - 2\lambda_2 - 4\lambda_3 = 0. \tag{5}$$

Let's add two times row one to row three to get

$$5\lambda_1 - 20\lambda_3 = 0 \tag{6}$$

or $\lambda_1 = 4\lambda_3$. Plugging back into the original third equation this means $4\lambda_3 - 2\lambda_2 - 4\lambda_3 = 0$ or $\lambda_2 = 0$. This simplifies the second equation to $-3\lambda_1 - 4\lambda_3 = -16\lambda_3 = 0$. Thus $\lambda_1 = \lambda_3 = 0$ as well. We conclude that $\{u_1, u_2, u_3\}$ form a linearly indepedent set, hence a basis for $\mathbb{R}^3$.

Question 11

First, we will show that $\{u + v, au\}$ is a basis. Notice, since $\dim V = 2$ we need to onlyl show that $u+v, au$ are linearly independent. So, suppose that $\lambda_1(u+v)+\lambda_2(au) = \mathbf{0}$ for λ_1, λ_2 scalars. This simplifies to $(\lambda_1 + a\lambda_2)u + \lambda_1 v = 0$. Since $\{u, v\}$ is a basis, hence linearly independent, we see that this implies $\lambda_1 + a\lambda_2 = 0$ and $\lambda_1 = 0$. But, these equations together imply that $\lambda_1 = \lambda_2 = 0$ since $a \neq 0$.

Next, we show that $\{au, bv\}$ is a basis where a, b are nonzero scalars. Again, it suffices to show that this set is linearly independent. Suppose $\lambda_1(au) + \lambda_2(bv) = \mathbf{0}$. This is the same as $(a\lambda_1)u + (b\lambda_2)v = 0$ which implies $a\lambda_1 = b\lambda_2 = 0$. Since a, b are both nonzero this implies $\lambda_1 = \lambda_2 = 0$ hence the set is linearly independent.

Question 22

Observe that $W_1 \cap W_2$ is always a subspace of W_1. Hence $\dim(W_1 \cap W_2) \leq \dim W_1$ by theorem 1.11 in the book. Furthermore, the other part of theorem 1.11 says that this is an equality if and only if $W_1 \cap W_2 = W_1$. We know this can only be true when $W_2 \subset W_1$. Thus, the necessary and sufficient condition is that W_2 must be a subspace of W_1.

Question 29

I will state part (a) as a theorem.

Theorem 0.1. *Let W_1, W_2 be subspaces of a vector space V. Then*

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2). \tag{7}$$

Proof. Start with a basis $\{u_1, \dots, u_k\}$ for the intersection $W_1 \cap W_2$. By corollary 2 of §1.6, we know that we can extend this to a basis of W_1 call this $\{u_1, \dots, u_k, v_1, \dots, v_r\}$. Similarly, applying the same corollary, extend the original set to a basis for W_2 as well; call this $\{u_1, \dots, u_k, w_1, \dots, w_s\}$. Notice that the combined set

$$\{u_1, \dots, u_k, v_1, \dots, v_r, w_1, \dots, w_s\} \tag{8}$$

is of size $k+r+s$. Also, this combined set certainly generates $W_1 + W_2$. Thus $\dim(W_1 + W_2) \leq k + r + s$. On the other hand, $\dim W_1 = k + r$, $\dim W_2 = k + s$. Thus

$$\dim W_1 + \dim W_2 - \dim(W_1 \cap W_2) = (k + r) + (k + s) - k = k + r + s.$$

Thus, we have shown the $\leq$ part of the theorem.

We now finish by showing $\geq$ and hence the theorem follows. We do this by showing that the combined set in equation (8) is linearly independent (hence a basis for $W_1 + W_2$). Indeed, assume that

$$\mathbf{0} = a_1u_1 + \cdots + a_ku_k + b_1v_1 + \cdots + b_rv_r + c_1w_1 + \cdots + c_sw_s \tag{9}$$

for scalars, a_i, b_j, c_k. Notice that this equation is of the form

$$\mathbf{0} = \mathbf{u} + \mathbf{v} + \mathbf{w} \tag{10}$$

for vectors $\mathbf{u} = a_1u_1 + \cdots + a_ku_k \in W_1 \cap W_2$, $\mathbf{v} = b_1v_1 + \vdots + b_rv_r \in W_1$, and $\mathbf{w} = c_1w_1 + \cdots + c_sw_s \in W_2$. Suppose that $\mathbf{v} \in W_1 \cap W_2$. Then it can be written as a linear combination of the u_i's. On the other hand, since it is a linear combination of the v_i's we see that $\mathbf{v} = 0$. Similarly, we see that if $\mathbf{w} \in W_1 \cap W_2$ then it must be zero as well.

We can rearrange (10) in the form

$$\mathbf{w} = -\mathbf{u} - \mathbf{v}. \tag{11}$$

Clearly, $\mathbf{u} \in W_1 \cap W_2 \subset W_1$ and $\mathbf{v} \in W_1$ by definition. By the subspace property, we know that the right hand side is a vector in W_1. Thus $\mathbf{w} \in W_1 \cap W_2$ and hence it is zero. Similarly, we can rearrange to see that $\mathbf{v} = 0$ as well. By linear independence of all three subsets, we see that this implies all the scalars in (9) must vanish and hence the combined set is a basis. □