

January 30:

Prop: Let  $(V, q)$  be such that

$$\left\{ v \in V \mid q(v) = \pm 1 \right\}$$

is path connected. Then  $\text{Spin}(V, q)$  is as well.

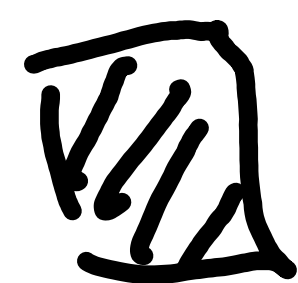
Pf:  $v_1, \dots, v_{2\ell} \in \text{Spin}(V, q)$ ,  $q(v_i) = 1$ .

Let  $\gamma_i$  be a path  $v_{2i-1} \rightsquigarrow -v_{2i}$ .

for  $i = 1, \dots, \ell$ . Then  $\prod_{i=1}^{\ell} \gamma_i$  is a

path from  $v_1, \dots, v_{2\ell}$  to

$$(-1)^{\ell} v_1^2 v_3^2 \dots v_{2\ell-1}^2 = 1.$$



Next, we turn to Clifford modules.

First, some basis observations. Let  $\{e_i\}$  be an orthonormal basis for  $V = \mathbb{R}^n$ , where we use

$$q(x) = \|x\|^2 = x_1^2 + \dots + x_n^2.$$

Then

$$Cl_n = \langle e_i \mid e_i e_j + e_j e_i = -2\delta_{ij} \rangle.$$

Suppose that  $A, B$  are  $\mathbb{Z}/2$  graded algebras. Define the  $\mathbb{Z}/2$  graded alg.

$$A \overset{gr}{\otimes} B$$

as follows:

$$1) \text{ As v.s. } A \otimes^{\text{gr}} B = A \otimes B.$$

2) Product is:

$$(a \otimes b) \cdot (a' \otimes b') = (-1)^{|b||a'|} (aa') \otimes (bb').$$

$$\begin{aligned} \text{Prop: } \text{Cl}(V_1 \oplus V_2, \mathfrak{f}_1 \oplus \mathfrak{f}_2) \\ \cong \text{Cl}(V_1, \mathfrak{f}_1) \hat{\otimes} \text{Cl}(V_2, \mathfrak{f}_2). \end{aligned}$$

Pf: Consider

$$f: V_1 \oplus V_2 \longrightarrow \text{Cl}(V_1) \hat{\otimes} \text{Cl}(V_2)$$

$$v_1 + v_2 \longmapsto v_1 \otimes 1 + 1 \otimes v_2.$$

$$\text{Then } f(v_1 + v_2)^2 = (v_1 \otimes 1 + 1 \otimes v_2)^2$$

$$\begin{aligned} &= v_1^2 \otimes 1 + (v_1 \otimes 1) \cdot (1 \otimes v_2) \\ &\quad + (1 \otimes v_2) \cdot (v_1 \otimes 1) \\ &\quad + 1 \otimes v_2^2 \end{aligned}$$

$$= v_1^2 \otimes 1 + 1 \otimes v_2^2.$$

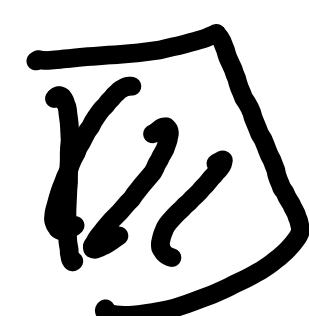
$$= -\left(q_1(v_1) + q_2(v_2)\right) 1 \otimes 1$$

$$= -q(v_1 + v_2) 1 \otimes 1.$$

So  $f$  extends uniquely to homomorphism

$$cl(v_1 \oplus v_2) \xrightarrow{\tilde{f}} cl(v_1) \hat{\otimes} cl(v_2)$$

This is, in fact, an isomorphism.



Cor:  $cl_n \cong cl_1^{\hat{\otimes} n}$

$$\parallel$$

$$cl_1 \hat{\otimes} \dots \hat{\otimes} cl_1.$$

Back to o.n.b  $\{e_i\}$  for  $\mathbb{R}^n$ . Let

$$\stackrel{\text{def}}{\omega} = e_1 \cdots e_n \in \text{Cl}_n.$$

From the relation  $e_i e_j = -e_j e_i$ , we see that  $\omega$  is actually independent of the choice of basis.

lem:  $\omega^2 = (-1)^{n(n+1)/2} \in \mathbb{R}^\times \subset \text{Cl}(V).$

$$v\omega = (-1)^{n-1} \omega v \quad \forall v \in V = \mathbb{R}^n.$$

Prop:  $\text{SpS } \frac{n(n+1)}{2}$  is even ( $n = 4, 8, \dots$ ),

and that  $V$  is a  $\text{Cl}_n$ -module. Then

$$V = V^+ \oplus V^-$$

where  $V^\pm = \left\{ v \in V \mid \omega v = \pm v \right\}.$

Further, if  $\alpha \neq 0$  then it defines

$$\alpha \cdot (-) : V^{\pm} \xrightarrow{\sim} V^{\mp}.$$

Pf : For such  $\alpha$  have  $\alpha^2 = 1$ . Let

$$\pi^{\pm} = \frac{1}{2} (1 \pm \alpha) \in \mathcal{C}l(V).$$

Then let  $V^{\pm} = \pi^{\pm} \cdot V$ .

Note :

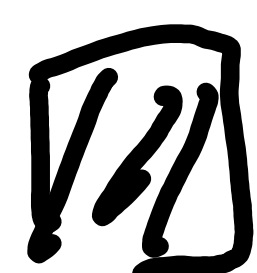
$$\left\{ \begin{array}{ll} \pi^{+} + \pi^{-} = 1 & (1) \\ \pi^{+} \pi^{-} = \pi^{-} \pi^{+} = 0 & (2) \\ (\pi^{\pm})^2 = \pi^{\pm} & (3). \end{array} \right.$$

(1)  $\Rightarrow V = V^{+} + V^{-}$ . Now, suppose :

$$\pi^{+} v = \pi^{-} v$$

$$\Rightarrow \pi^{+} v = \pi^{+} \pi^{-} v = 0. \quad \text{So}$$

$$V = V^{+} \oplus V^{-}.$$



We have mentioned (but not proved) that w/ respect to the natural filtration that

$$\text{gr } \text{Cl}(V, q) \cong \wedge^\bullet V.$$

The resulting canonical isomorphism of vector spaces

$$\wedge^\bullet V \xrightarrow{\cong} \text{Cl}(V, q)$$

can be made explicit:

$$v_1 \wedge \cdots \wedge v_\ell \longmapsto v_1 \cdots v_\ell.$$

Q: What does the Clifford multiplication look like in terms of operation on  $\wedge^\bullet V$ ?

Prop: For  $v \in V$ ,  $\varphi \in \text{Cl}_n \cong \wedge^\bullet \mathbb{R}^n$ .

Then

$$v \cdot \varphi = v \wedge \varphi - v \lrcorner \varphi.$$

$\uparrow$   
Clifford

$\uparrow$   
wedge product/  
exterior mult.

$\uparrow$   
interior product/  
contraction.

Recall:

$$\nu \nu (\nu_1 \wedge \dots \wedge \nu_r) = \sum_{j=1}^r (-1)^{j+1} \langle \nu, \nu_j \rangle \nu_1 \wedge \dots \wedge \hat{\nu}_j \wedge \dots \wedge \nu_r$$

Pf of Prop: Choose o.n.b.  $\{e_i\}$  for  $\mathbb{R}^n$ . We will

prove for  $\nu = e_1$ ,  $\varphi = e_{i_1} \wedge \dots \wedge e_{i_r}$ , where

$$i_1 < \dots < i_r.$$

Case 1:  $e_{i_1} = e_1$ . Then

$$e_1 (e_1 e_{i_2} \dots e_{i_r}) \approx - e_{i_2} \wedge \dots \wedge e_{i_r}$$

$$e_1 \wedge (e_1 \wedge \dots \wedge e_{i_r}) = e_1 \wedge (e_1 \wedge \dots \wedge e_{i_r})$$

Case 2:  $e_{i_1} \neq e_1$ . Then  $e_1 \neq e_{i_j}$  for any  $j=1, \dots, r$

$$\text{So } e_1 (e_{i_1} \wedge \dots \wedge e_{i_r}) \approx e_1 \wedge (e_{i_1} \wedge \dots \wedge e_{i_r}) - e_1 \wedge (e_{i_1} \wedge \dots \wedge e_{i_r})$$

✓

□



We now turn towards classification.

Let:

$$Cl_{r,s} \stackrel{\text{def}}{=} Cl(\mathbb{R}^{r+s}, \sum_{i=1}^r x_i^2 - \sum_{j=1}^s x_{r+j}^2).$$

Have seen  $Cl_1 = Cl_{1,0} \cong \mathbb{C}$ .

$$Cl_2 = Cl_{2,0} \cong \mathbb{H}.$$

Also easy:

- $Cl_{0,1} \cong \mathbb{R}[x]/(x^2 = -1) = \mathbb{R} \oplus \mathbb{R}.$

- $Cl_{0,2}$  is spanned by  $1, x, y, xy$

where  $x^2 = y^2 = 1$ ,  $xy = -yx$ .

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$\begin{matrix} 2 \\ 1 \end{matrix}$

$\begin{matrix} 2 \\ x \end{matrix}$

$\begin{matrix} 2 \\ y \end{matrix}$

$\cdot$

$$\cong \mathbb{R}(2).$$

This defines  $Cl_{0,2} \cong Mat_{2 \times 2}(\mathbb{R}).$

•  $Cl_{1,1}$  is spanned by  $1, x, y, xy$  w/

$$x^2 = -y^2 = 1, \quad xy = -yx.$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$\begin{matrix} 2 \\ 1 \end{matrix}$                        $\begin{matrix} 2 \\ x \end{matrix}$                        $\begin{matrix} 2 \\ y \end{matrix}$  .

So, also  $Cl_{1,1} \cong \mathbb{R}(2)$ .

Theorem:  $Cl_{n,0} \otimes Cl_{0,2} \cong Cl_{0,n+2}$ .

$$Cl_{0,n} \otimes Cl_{2,0} \cong Cl_{n+2,0}$$

$$Cl_{r,s} \otimes Cl_{1,1} \cong Cl_{r+1,s+1}.$$

where  $\otimes$  is ordinary, ungraded  $\otimes$ -product.

Pf :  $\{e_i\}_{i=1}^{n+2}$  o.n.b for  $\mathbb{R}^{n+2}$  w/  $f(x) = -\|x\|^2$ .

Let  $\{e'_i\}_{i=1}^n$  denote generators for  $Cl_{n,0}$ .

$\{e''_1, e''_2\}$  denote generators for  $Cl_{0,2}$ .

Define  $f: \mathbb{R}^{n+2} \rightarrow Cl_{n,0} \otimes Cl_{0,2}$

$$e_i \mapsto \begin{cases} e'_i \otimes e''_1 e''_2, & 1 \leq i \leq n. \\ 1 \otimes e''_{i-n}, & i = n+1, n+2. \end{cases}$$

So if  $1 \leq i, j \leq n$  then

$$f(e_i)^2 = (e'_i e'_i) \otimes (e''_1 e''_2 e''_1 e''_2).$$

$$= (-1) \cdot (-e''_1 e''_1 e''_2 e''_2) = +1.$$

and

$$f(e_{n+1})^2 = 1 \otimes e''_1 e''_1 = 1 = 1 \otimes e''_2 e''_2 = f(e_{n+2})^2.$$

So,  $f(x)^2 = \|x\|^2 \Rightarrow f$  extends to

$$\tilde{f}: Cl_{0,n+2} \rightarrow Cl_{n,0} \otimes Cl_{0,2}.$$

By counting dimensions this is  $\cong$ . Other cases proved similarly.  $\square$

Theorem: [PERIODICITY] There is an iso of  $\mathbb{R}$ -algebras:

$$Cl_{n+4} \cong Cl_n \otimes Cl_4.$$

Pf: From above:

$$Cl_{0,2} \otimes Cl_{2,0} \stackrel{\cong}{=} Cl_{4,0} \stackrel{\cong}{=} Cl_{0,4}.$$

$\cong$

$$\mathbb{R}(2) \otimes \mathbb{H} = \mathbb{H}(2)$$

On the other hand

$$Cl_{n,0} \otimes Cl_{0,2} \otimes Cl_{2,0}$$

$\underbrace{\hspace{10em}}$

$$\cong Cl_{0,n+2} \otimes Cl_{2,0} \cong Cl_{n+4,0}.$$

$\square$

So, we really only need to characterize

$Cl_n$ ,  $n=0,1,2,3,4$ . Note

$$Cl_3 = Cl_{2,0} \otimes Cl_{0,1} \cong H \oplus H..$$

So, we have

<u><math>n</math></u>	<u><math>Cl_n</math></u>	<u><math>Cl_{0,n}</math></u>
1	$\mathbb{C}$	$\mathbb{R} \oplus \mathbb{R}$
2	$H$	$\mathbb{R}(2)$
3	$H \oplus H$	$\mathbb{C}(2)$
4	$H(2)$	$H(2)$

Using the above periodicity we obtain all other Clifford algebras, for example:

$$Cl_5 \cong Cl_{0,4} \otimes Cl_{1,0} \cong H(2) \otimes \mathbb{C}$$

$$= \mathbb{R}(2) \otimes \mathbb{H} \otimes \mathbb{C}$$

$$\cong \mathbb{R}(2) \otimes \mathbb{C}(2)$$

$$\cong \mathbb{R}(2) \otimes \mathbb{R}(2) \otimes \mathbb{C}$$

$$\cong \mathbb{R}(4) \otimes \mathbb{C} \cong \mathbb{C}(4).$$

So:  $\mathbb{C}\ell_5 \cong \mathbb{C}(4).$

On worksheet  
 $\mathbb{H} \otimes \mathbb{C} \cong \mathbb{C}(2).$