

• Local coordinates. Recall, in local coords

$$g(x, \gamma) = g_{ij} x^i \gamma^j.$$

Note
$$\begin{aligned} X^\flat &= g(X, \cdot) = g_{ij} dx^i(x) dx^j(\cdot) \\ &= g_{ij} x^i dx^j. \end{aligned}$$

We denote the inverse matrix to $[g_{ij}]$ by $[g^{ij}]$.

$$g^{ik} g_{kj} = \delta^i_j.$$

Sp. $\Theta = \Theta_j dx^j$, is dual to $X = x^i \partial_i$.

That is $X^\flat = \Theta$.

Then:
$$X^k = g^{kj} \Theta_j$$

and

$$\Theta_j = g_{kj} X^k.$$

•
$$\nabla f = g^{ij} \partial_i f \partial_j, \quad \downarrow f = \partial_j f dx^j.$$

• Local formula for L.C. connection.

$$\nabla_Y X = \nabla_{\gamma^i \partial_i} (X^j \partial_j)$$

$$= \gamma^i \nabla_{\partial_i} (X^j \partial_j)$$

$$= \gamma^i \left(\partial_i X^j \partial_j + X^j \nabla_{\partial_i} \partial_j \right).$$

Denote $\nabla_{\partial_i} \partial_j = \Gamma_{ij}^k \partial_k$.

Thus: $g(\nabla_{\partial_i} \partial_j, \partial_\ell) = g(\Gamma_{ij}^k \partial_k, \partial_\ell)$

$= \Gamma_{ij}^k g_{k\ell}$.

$$\frac{1}{2} \left\{ (L_{\partial_j} g)(\partial_i, \partial_\ell) + d(\partial_j^\flat)(\partial_i, \partial_\ell) \right.$$

$$\left. = \frac{1}{2} \left\{ \partial_j g_{i\ell} + \partial_i (\partial_j^\flat(\partial_\ell)) - \partial_\ell (\partial_j^\flat(\partial_i)) \right\} \right.$$

$$= \frac{1}{2} \{ \partial_j g_{i\ell} + \partial_i g_{j\ell} - \partial_\ell g_{ji} \}$$

Thus:

$$\Gamma_{ij}^m = \frac{1}{2} (\partial_j g_{il} + \partial_i g_{jl} - \partial_l g_{ij}) g^{lm}.$$

- Γ_{ij}^k is called the "Christoffel symbol" (of the second kind). Related, to the expression

$$\begin{aligned} \Gamma_{ij,k} &= \frac{1}{2} (\partial_j g_{ik} + \partial_i g_{jk} - \partial_k g_{ij}) \\ &= g(\nabla_{\partial_i} \partial_j, \partial_k). \end{aligned}$$

Note $\Gamma_{ij}^k = g^{kl} \Gamma_{ij,l}$.

- It is always possible to choose coordinates $\{x^i\}$ near $p \in M$ s.t.

$$\begin{cases} g_{ij}|_p = \delta_{ij} \\ \partial_k g_{ij}|_p = 0 \end{cases}.$$

Called "normal coordinates".

Thus, in these coordinates $\Gamma_{ij}^k|_p = 0$ and thus

$$\nabla_Y X = Y^i(p) \partial_i X^j|_p \partial_j.$$

in normal coordinates.

(see HW ...)

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• Torsion-free \Leftrightarrow $\boxed{\Gamma_{ij}^k = \Gamma_{ji}^k}$

$$\Gamma_{ij}^k \partial_k = \nabla_{\partial_i} \partial_j = \nabla_{\partial_j} \partial_i = \Gamma_{ji}^k \partial_k.$$

• Metric compatibility:

$$\partial_k g_{ij} = g(\nabla_{\partial_k} \partial_i, \partial_j) + g(\partial_i, \nabla_{\partial_k} \partial_j).$$

$$= \Gamma_{ki,j} + \Gamma_{kj,i}.$$

\Rightarrow Christoffel symbols determine all derivatives of the metric.

• Can obtain more local formulas.

Pwp:

$$\text{Hess } f(\partial_i, \partial_j)$$

$$= \partial_i \partial_j f - \Gamma_{ij}^k \partial_k f$$

• Curvature: Let

$$R \in \Gamma(T \otimes (T^*)^{\otimes 3})$$

$$\parallel \\ \sim (1, 3)$$

defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z.$$

Why is this tensorial in "Z"? For $f \in C^\infty$:

$$R(X, Y)(fZ) = \nabla_X \nabla_Y (fZ) - \nabla_Y \nabla_X (fZ) - \nabla_{[X, Y]}(fZ)$$

$$= \nabla_X ((Y \cdot f)Z + f \nabla_Y Z)$$

$$- \nabla_Y ((X \cdot f)Z + f \nabla_X Z)$$

$$- ([X, Y] \cdot f + f \nabla_{[X, Y]} Z).$$

$$= f R(x, y) z$$

$$+ x \cdot (y \cdot f) - y \cdot (x \cdot f) - [x, y] \cdot f$$

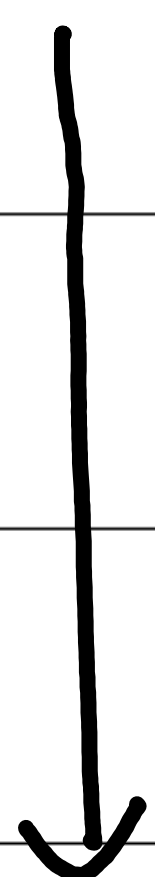
$$= f R(x, y) z \quad . \quad \checkmark$$

Skew
symm.

$$R \in \Gamma(T^* \otimes T^* \otimes T^* \otimes T)$$



$g \cong$



$$\tilde{R} \in \Gamma\left(\underbrace{T^* \otimes T^*}_{\text{Skew symm}} \otimes \underbrace{T^* \otimes T^*}_{\text{Skew symm}}\right).$$

Skew
symm

Skew
symm

$$\tilde{R}(x, y, z, w).$$