SOLUTIONS TO HOMEWORK 3

Problem 1. It suffices to show that for any pair of coordinate charts (U, ϕ) for M and (V, ψ) for N (with $F(U) \subset V$) that the map $\widehat{F} = \psi \circ F \circ \phi^{-1} \colon \phi(U) \to \psi(V)$ is constant. By chain rule, the differential of this map at $\phi(p) \in \mathbf{R}^m$ is

$$\mathrm{d}\widehat{F}_{\phi(p)} = \mathrm{d}\psi_{F(p)} \circ \mathrm{d}F_p \circ \mathrm{d}\phi^{-1}_{\phi(p)} \tag{1}$$

which is zero by assumption. In other words, with respect to the coordinates determined by ϕ, ψ the matrix of partial derivatives of $\widehat{F}$ at $\phi(p)$ is the zero matrix. For fixed $i = 1, \ldots, n$ consider the component function $\widehat{F}^i \colon \phi(U) \to \mathbf{R}$. Fix an integer $j = 1, \ldots, m$ and real numbers $y_1, \ldots, y_{m-1}$ such that for appropriate $t \in \mathbf{R}$ we have $(y_1, \ldots, y_{j-1}, t, y_{j+1}, \ldots, y_{m-1}) \in U$. Necessarily the set of all such t is an open interval which we call J. For $j = 1, \ldots, m$ let $f^i_j \colon J \to \mathbf{R}$ be the function

$$f^i_j(t) = F^i(y_1, \ldots, y_{j-1}, t, y_j, \ldots, y_{m-1}) \tag{2}$$

By assumption we know that

$$\frac{\mathrm{d}\widehat{f^i_j}}{\mathrm{d}t} = 0 \tag{3}$$

on J. By the (ordinary, single variable) mean value theorem this implies that f^i_j is constant. Doing this for all i, j we see that $\widehat{F}$ is constant.

Problem 2.

(a) Consider the map

$$\mathbf{C}^2 \setminus \{0\} \to \mathbf{CP}^1 \tag{4}$$

which sends a nonzero vector to the line that it spans. This map descends to the set of equivalence classes

$$(\mathbf{C}^2 \setminus \{0\}) / \sim \to \mathbf{CP}^1. \tag{5}$$

The inverse to this map sends a line to equivalence class of any non-zero vector which lies on the line.

(b) By definition $U_z = \pi(V_z)$ is the set of classes $[(z, w)]$ such that $z \neq 0$. Any such class can be written as $[(1, w')]$ for $w' \in \mathbf{C}$. Thus $\pi^{-1}(U_z)$ is the set of all nonzero vectors (z, w) such that $[(z, w)] = [(1, w')]$ for some w'. This means $z = \lambda$, $w = \lambda w'$ for some nonzero complex number λ which shows that $\pi^{-1}(U_z) = V_z$. This set is open by definition. Similarly U_w is open.

(c) We show that ϕ_z is a homeomorphism onto its image. In this case the image is $\mathbf{C}$. The inverse is defined by

$$\phi^{-1}(a) = [1, a]. \tag{6}$$

This is clearly an inverse at the level of sets. It is continuous because both π and $\phi \circ \pi$ are continuous. Similarly ϕ_w is a homeomorphism onto its image.

(d) Note that

$$\phi_z(U_z \cap U_w) = \mathbf{C}^\times = \phi_w(U_z \cap U_w). \tag{7}$$

We need to show that $\phi_z \circ \phi_w^{-1}, \phi_w \circ \phi_z^{-1}$ are smooth as functions $\mathbf{C}^\times \to \mathbf{C}^\times$. Let's consider the first composition. We have

$$\phi_z(\phi_w^{-1}(a)) = \phi_z([a, 1]) = \frac{1}{a}. \tag{8}$$

This map is smooth. To make this completely clear, let us write this out in real coordinates. If $a = x + \mathrm{i}y \neq 0$ then

$$\frac{1}{a} = \frac{1}{x^2 + y^2} - \mathrm{i}\frac{1}{x^2 + y^2}. \tag{9}$$

Thus, as a map $\mathbf{R}^2 \setminus \{0\}$ to itself, this composition is

$$(x, y) \mapsto \left(\frac{1}{x^2 + y^2}, -\frac{1}{x^2 + y^2}\right). \tag{10}$$

This is certainly smooth.

Problem 3.

(a) Consider the coordinate map $\phi_w \colon U_w \to \mathbf{C}$ and the composition $\phi_w \circ \pi$. The differential of this composition at a point $(z, w) \in \mathbf{C}^2 \setminus \{w = 0\}$ is of the form

$$\mathrm{d}(\phi_w \circ \pi)_{(z,w)} \colon \mathbf{C}^2 \to \mathbf{C}. \tag{11}$$

Using the formula $\phi_w([z, w]) = z/w$ we find that with respect to the standard basis the differential is represented by the 1×2 matrix

$$\mathrm{d}(\phi_w \circ \pi)_{(z,w)} = \begin{pmatrix} w^{-1} & -zw^{-1} \end{pmatrix}. \tag{12}$$

This is clearly full rank since we are working in a locus where $w \neq 0$. Similarly one can show that $\mathrm{d}(\phi_z \circ \pi)_{(z,w)}$ is full rank for any $(z, w) \in \mathbf{C}^2 \setminus \{z = 0\}$.

(b) By definition

$$S^3 = \{(x_1, x_2, x_3, x_4) \in \mathbf{R}^4 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\} \subset \mathbf{R}^4. \tag{13}$$

In complex notation this is equivalent to

$$S^3 = \{(z, w) \in \mathbf{C}^2 \mid |z|^2 + |w|^2 = 1\} \subset \mathbf{C}^2. \tag{14}$$

Let $p = \pi|_{S^3}$ be the restriction of π to the subset $S^3 \subset \mathbf{C}^2 \setminus \{0\}$. This is a composition of smooth maps hence smooth.

Consider the point $[1, 0] \in \mathbf{CP}^1$. Then $\pi^{-1}([1, 0])$ is the subspace $\{(z, 0) \mid z \in \mathbf{C}\} \subset \mathbf{C}^2 \setminus \{0\}$. The intersection of this subspace with S^3 is the subspace $\{(z, 0) \mid |z|^2 = 1\} \cong S^1$. A similar argument shows that $p^{-1}([z, w]) \cong S^1$ for any $[z, w]$.

(c) We check that F is well-defined. It is clear that each of the components of F is a real number, so the image of F certainly lies in $\mathbf{R}^3$. From the relation

$$|z/w + \bar{z}/\overline{w}|^2 + |z/w - \bar{z}/\overline{w}|^2 + ||z/w|^2 - 1|^2 = (1 + |z/w|^2)^2 \tag{15}$$

we see that the image of F lands in S^2 as desired.

Now, let z, w be complex numbers with $w \neq 0$. Then

$$\frac{z/w + \bar{z}/\overline{w}}{1 + |z/w|^2} = z\overline{w} + \bar{z}w. \tag{16}$$

This is a simple manipulation:

$$\frac{z/w + \bar{z}/\overline{w}}{1 + z\bar{z}/(w\overline{w})} = \frac{z\overline{w} + \bar{z}w}{w\overline{w} + z\bar{z}}. \tag{17}$$

Using this we see that if $(z, w) \in S^3$ with $w \neq 0$ we see that the first component of $F(z, w)$ can be written as $z\overline{w} + \bar{z}w$. Similarly, the second component can be written as $-\mathrm{i}(z\overline{w} - \bar{z}w)$, and the last component as $z\bar{z} - w\overline{w}$. In total we see that F can be written

$$F(z, w) = (z\overline{w} + \bar{z}w, -\mathrm{i}(z\overline{w} - \bar{z}w), z\bar{z} - w\overline{w}). \tag{18}$$

This expression makes it manifest that F extends to a map $\widetilde{F}$ as in the problem.

(d) (This was a bonus. There are many ways to prove this. The proof here follows part (b)) Consider the map $q \colon \mathbf{C}^2 \setminus \{0\} \to S^3$ which sends a nonzero vector v to the vector pointing in the same direction as v but with unit norm. That is

$$q(z, w) = \frac{(z, w)}{|z|^2 + |w|^2} \tag{19}$$

The composition $\widetilde{F} \circ q$ is a smooth map $\mathbf{C}^2 \setminus \{0\} \to S^2$ with the property that it is constant along the fibers of $\pi \colon \mathbf{C}^2 \setminus \{0\} \to \mathbf{CP}^1$. By theorem 4.30 in the textbook $\widetilde{F} \circ q$ descends to a smooth map $G \colon \mathbf{CP}^1 \to S^2$ with the property that the diagram below commutes:

$$\begin{tikzcd} \mathbf{C}^2 \setminus \{0\} \arrow[r, "q"] \arrow[d, "\pi"] \arrow[dr, "F\circ q"] & S^3 \arrow[d, "F"] \\ \mathbf{CP}^1 \arrow[r, dotted, "G"] & S^2. \end{tikzcd} \tag{20}$$

Since G is the unique smooth map making this diagram commute, we see that it is a diffeomorphism.