- In discussion, we proved that a linear map

$$T: V \longrightarrow W$$

is uniquely determined by its behavior on a basis for V. We use this idea to introduce the concept of a matrix.

<u>Dfn</u>: An <u>ordered basis</u> is a basis where the elements/vectors have a definitive ordering.

Eg: $\{e_1, e_2, e_3\}$ can be considered as an ordered basis.

- Let $\beta = \{u_i\}_{i=1}^{n}$ be any ordered basis for V. Then any vector $v \in V$ has the form

$$v = \sum_{i=1}^{n} a_i u_i$$

for some scalars $a_1, \ldots, a_n$. Define

$$[v]_B \overset{\text{def}}{=} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n.$$

This is the "column vector representation of v with respect to the ordered basis β".

Ex: Consider $V = P_2$. A nice ordered basis for P_2 $\beta = \{1, x, x^2\}$. Then:

$$[1 + 2x - x^2]_\beta = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \in \mathbb{R}^3$$

$$[3 + x]_\beta = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \in \mathbb{R}^3.$$

- Now suppose $\beta = \{v_i\}_{i=1}^n$ is ordered basis for V and $\gamma = \{w_j\}_{j=1}^m$ is ordered basis for W.

Then:

$$T(v_i) = \sum_{j=1}^m a_i^j w_j$$

for some scalars a_i^j, $j = 1, \dots, m$. But, we do this for each i so really we get $m \times n$ scalars

$$\{a_i^j\}_{\substack{i=1\dots n \\ j=1,\dots m}}.$$

We arrange these into a matrix. Define the $m \times n$ matrix:

$$[T]_\beta^\gamma = \begin{bmatrix} a_1^1 & a_2^1 & \cdots & a_n^1 \\ a_1^2 & a_2^2 & & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ a_1^m & a_2^m & \cdots & a_n^m \end{bmatrix}.$$

Matrix representation of T with respect to β, γ.

__Ex__: $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, $T\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a+b \\ 3b \\ 2a-b \end{bmatrix}$.

We will use the standard bases

$\beta = \{e_1, e_2\}$ for $\mathbb{R}^2$

$\gamma = \{e_1, e_2, e_3\}$ for $\mathbb{R}^3$.

__Then__: $T(e_1) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $T(e_2) = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$.

So:

$$[T]_\beta^\gamma = \begin{bmatrix} 1 & 1 \\ 0 & 3 \\ 2 & -1 \end{bmatrix}.$$

This is the 3×2 matrix which represents T using the standard basis vectors of $\mathbb{R}^n$.

Ex: Suppose V is a vector space w/ an ordered

$\beta = \{u_1, u_2\}$. So $\dim V = 2$.

Also, assume that $T: V \to W$ is a linear map s.t. $\{T(u_1), T(u_2)\}$ is a basis for W.

Let $\gamma = \{T(u_1), T(u_2)\}$. Then:

$$[T]_\beta^\gamma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- When we look at maps of the form

$$T : V \longrightarrow V$$

we can consider the same ordered basis β on "both sides".

We let $$[T]_\beta \overset{\text{def}}{=} [T]_\beta^\beta .$$

Ex: If $T = \mathbb{1} : V \longrightarrow V$ is the identity

$$\mathbb{1}(v) = v \quad \text{for all } v.$$

Then

$$[T]_\beta = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}$$

"Identity matrix".

for any basis β.

Ex: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be rotation by $\theta = \pi/3$ counter clockwise.

Let $\beta = \{e_1, e_2\}$ be standard basis.

Then $T(e_1) = \begin{bmatrix} \cos \pi/3 \\ \sin \pi/3 \end{bmatrix}$

$= \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix}$

- $T(e_2) = \begin{bmatrix} \cos(\pi/2 + \pi/3) \\ \sin(\pi/2 + \pi/3) \end{bmatrix}$

$= \begin{bmatrix} -\sqrt{3}/2 \\ 1/2 \end{bmatrix}$

So:

$$[T]_\beta = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$$

More generally, if $T = T_\theta$ rotation by θ then

$$[T]_\beta = \begin{bmatrix} \cos\theta & \cos(\pi/2 + \theta) \\ \sin\theta & \sin(\pi/2 + \theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Ex: Let $\int : P_2 \to P_3$ be the linear map

$$\int f = \int_0^x f(t)\,dt .$$

Let $\beta = \{1, x, x^2\}$ and $\gamma = \{1, x, x^2, x^3\}$.

We compute $[\int]_\beta^\gamma$.

- $\int 1 = \int_0^x 1\,dt = x$.
- $\int x = \int_0^x t\,dt = \frac{1}{2}x^2$.
- $\int x^2 = \int_0^x t^2\,dt = \frac{1}{3}x^3$.

Thus:

$$[\int]_\beta^\gamma = \begin{array}{cc} \begin{array}{ccc} 1 & x & x^2 \end{array} & \\ \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} & \begin{array}{c} 1 \\ x \\ x^2 \\ x^3 \end{array} \end{array} .$$

Prop: Sps $T : V \longrightarrow \mathbb{R}$ is not the zero map. Then, there exists a basis β of V s.t.

$$[T]_{\beta}^{\beta_{std}} = [1 \ \underbrace{0 \ \ 0 \ \cdots \ \ 0}_{}]$$

$n = \dim V$ entries.

Pf: Sps $v \in V$ is such that $T(v) \neq 0$.

Consider the nonzero vector $\dfrac{v}{T(v)} = \dfrac{1}{T(v)} v \in V$.

The set $\left\{ \dfrac{v}{T(v)} \right\}$ is linearly independent, and can hence be extended to a basis:

$$\gamma = \left\{ \overset{u_1}{\overset{\shortparallel}{\frac{v}{T(v)}}}, u_2, u_3, \dots, u_n \right\}.$$

Claim 1: $T(u_1) = 1$. Easy.

The problem is we don't know $T(u_2) = T(u_3) = \dots = 0$.

In general this may not be true. Sps $T(u_2) \neq 0$.

Then, define:

$$\tilde{u}_2 = u_2 - T(u_2) u_1.$$

Notice, $T(\tilde{u}_2) = T(u_2) - T(T(u_2)u_1)$

$$= T(u_2) - T(u_2)T(u_1)$$

$$= T(u_2) - T(u_2) = 0.$$

We can do this for each $u_k \rightsquigarrow \tilde{u}_k = u_k - T(u_k)u_1$.

<u>Claim</u>: $\beta = \{u_1, \tilde{u}_2, \tilde{u}_3, \dots, \tilde{u}_n\}$ is a basis.

[Sps $\lambda_1 u_1 + \lambda_2 \tilde{u}_2 + \dots + \lambda_n \tilde{u}_n = 0$.

$$\| $$

$$\lambda_1 u_1 + \lambda_2 u_2 + \dots + \lambda_n u_n - T(u_2)u_1 - T(u_3)u_1 \dots$$

$$\|$$

$$\left(\lambda_1 - \sum_{i=2}^{n} \lambda_i T(u_i)\right) u_1 + \sum_{j=2}^{n} \lambda_j u_j$$

Therefore, since $\{u_1, u_2, \dots, u_n\}$ is basis we have:

$$\lambda_1 = \sum_{i=2}^{n} \lambda_i T(u_i) \quad \text{and} \quad \lambda_2 = \lambda_3 = \dots = \lambda_n = 0.$$

but then $\lambda_1 = 0$ as well.].

This basis β has the property:

$$[T]_\beta = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$ as desired. ☒