

Solutions to selected exercises from §1.6

Question 4

No, these vectors do not generate P_3 (the set of polynomials, with real coefficients, which are at least cubic). The reason is that $\dim P_3 = 4$ as we have shown that a basis for this vector space consists of four vectors (for example $\{1, x, x^2, x^3\}$). Since a basis is the minimum sized subset which can generate a vector space, a collection of vectors consisting of less than four vectors can not be generating.

For practice, we will show that these three vectors are linearly independent, however. Suppose $\lambda_1, \lambda_2, \lambda_3$ are scalars such that

$$\mathbf{0} = \lambda_1(x^3 - 2x^2 + 1) + \lambda_2(4x^2 - x + 3) + \lambda_3(3x - 2). \quad (1)$$

This equation can be written as

$$\mathbf{0} = \lambda_1x^3 + (-2\lambda_1 - 4\lambda_2)x^2 + (-\lambda_2 + 3\lambda_3)x + (\lambda_1 + 3\lambda_2 - 2\lambda_3). \quad (2)$$

For this to be zero, each coefficient must be zero. Thus $\lambda_3 = 0$ comes from vanishing of the cubic term. From the vanishing of the linear term we see that $\lambda_2 = 0$. From the vanishing of the quadratic term we see $\lambda_1 = 0$. Thus, the collection is linear independent.

Question 7

The vectors u_1, u_2 are clearly not parallel, hence $\{u_1, u_2\}$ is linearly independent. Next, we check if adjoining u_3 results in a linearly independent set. Suppose $\lambda_1, \lambda_2, \lambda_3$ are scalars such that

$$\lambda_1u_1 + \lambda_2u_2 + \lambda_3u_3 = \mathbf{0}.$$

This is equivalent to the system of three equations

$$2\lambda_1 + \lambda_2 - 8\lambda_3 = 0 \quad (3)$$

$$-3\lambda_1 + 4\lambda_2 + 12\lambda_3 = 0 \quad (4)$$

$$\lambda_1 - 2\lambda_2 - 4\lambda_3 = 0. \quad (5)$$

Let's add two times row one to row three to get

$$5\lambda_1 - 20\lambda_3 = 0 \quad (6)$$

or $\lambda_1 = 4\lambda_3$. Plugging back into the original third equation this means $4\lambda_3 - 2\lambda_2 - 4\lambda_3 = 0$ or $\lambda_2 = 0$. This simplifies the second equation to $-3\lambda_1 + 12\lambda_3 = 0$. In other words, the second equation gives us no additional conditions on λ_1, λ_3 . Thus, as long as $\lambda_2 = 0$ and $\lambda_3 = \lambda$ and $\lambda_1 = 4\lambda$ for some scalar λ then we have solved these equations. In particular, $(\lambda_1, \lambda_2, \lambda_3) = (4, 0, 1)$ is a nonzero solution, therefore $\{u_1, u_2, u_3\}$ is *linearly dependent*. One can similarly check that $\{u_1, u_2, u_4\}$ is also linearly dependent while $\{u_1, u_2, u_5\}$ is linearly independent.

Question 11

First, we will show that $\{u+v, au\}$ is a basis. Notice, since $\dim V = 2$ we need to only show that $u+v, au$ are linearly independent. So, suppose that $\lambda_1(u+v) + \lambda_2(au) = \mathbf{0}$ for λ_1, λ_2 scalars. This simplifies to $(\lambda_1 + a\lambda_2)u + \lambda_1v = \mathbf{0}$. Since $\{u, v\}$ is a basis, hence linearly independent, we see that this implies $\lambda_1 + a\lambda_2 = 0$ and $\lambda_1 = 0$. But, these equations together imply that $\lambda_1 = \lambda_2 = 0$ since $a \neq 0$.

Next, we show that $\{au, bv\}$ is a basis where a, b are nonzero scalars. Again, it suffices to show that this set is linearly independent. Suppose $\lambda_1(au) + \lambda_2(bv) = \mathbf{0}$. This is the same as $(a\lambda_1)u + (b\lambda_2)v = \mathbf{0}$ which implies $a\lambda_1 = b\lambda_2 = 0$. Since a, b are both nonzero this implies $\lambda_1 = \lambda_2 = 0$ hence the set is linearly independent.

Question 20

(a) If $V = \{0\}$ then the statement is tautological. Suppose $V \neq \{0\}$. Then, there exists a nonzero vector u_1 in S . So, $\{u_1\}$ is linearly independent. Now, suppose we continue this process until we build a subset $\{u_1, \dots, u_k\} \subset S$ which is linearly independent and such that adjoining any other vector to $\{u_1, \dots, u_k\}$ results in a linear dependent subset. We know such a process truncates at finite k since V is finite-dimensional.

We claim that $\{u_1, \dots, u_k\} \subset S$ is the desired basis. To see this it suffices to show that it generates V . We will first prove the following lemma.

Lemma 0.1. *Suppose $v \in S$. Then $v \in \text{span}\{u_1, \dots, u_k\}$.*

Proof. If $v \in \{u_1, \dots, u_k\}$ then the statement is tautological. If $v \in S \setminus \{u_1, \dots, u_k\}$ then we know that $\{u_1, \dots, u_k, v\}$ is a linear *dependent* subset, by our construction. Hence, by theorem 1.7 we see that $v \in \text{span}\{u_1, \dots, u_k\}$. \square

We return to part (a). Suppose now that $w \in V$ is an arbitrary vector. Since S generates, we know that there exists scalars $\lambda_1, \dots, \lambda_m$ and vectors $v_1, \dots, v_m \in S$ such that

$$v = \lambda_1 v_1 + \dots + \lambda_m v_m. \quad (7)$$

By the lemma we know that $v_i \in \text{span}\{u_1, \dots, u_k\}$. Hence $v \in \text{span}\{u_1, \dots, u_k\}$ as well. Thus $\{u_1, \dots, u_k\}$ generates and it is a basis.

(b) If S is finite then we can just apply corollary 2(a) from the book. If S is infinite then it certainly contains at least n vectors. In either case, the statement is proven.

Question 22

Observe that $W_1 \cap W_2$ is always a subspace of W_1 . Hence $\dim(W_1 \cap W_2) \leq \dim W_1$ by theorem 1.11 in the book. Furthermore, the other part of theorem 1.11 says that this is an equality if and only if $W_1 \cap W_2 = W_1$. We know this can only be true when $W_1 \subset W_2$. Thus, the necessary and sufficient condition is that W_1 must be a subspace of W_2 .

Question 24

Since f is of degree n it is of the form

$$f = ax^n + h \quad (8)$$

where $a \neq 0$ is a nonzero number and h is a polynomial of degree strictly less than n . We claim that $\{f, f', f'', \dots, f^{(n)}\}$ is a basis for P_n . The solution of the problem follows immediately.

Note that $f' = nax^{n-1} + h'$ where h' is now a polynomial of degree strictly less than $n - 1$. In particular, f' is a polynomial of degree $n - 1$. More generally, for $0 \leq k \leq n$ we see that the k th derivative is

$$f^{(k)} = n(n - 1) \cdots (n - k + 1)x^{n-k} + h^{(k)} \quad (9)$$

where $h^{(k)}$ is of degree strictly less than $n - k$.

Now, suppose that $\lambda_0, \dots, \lambda_n$ are scalars such that

$$\lambda_0 f + \lambda_1 f' + \cdots + \lambda_n f^{(n)} = 0. \quad (10)$$

In order for the polynomial on the left hand side to be zero each coefficient must be zero. The leading coefficient (coefficient of the highest power of x) is the x^n coefficient which is $\lambda_0 a$. For this to be zero we see that $\lambda_0 = 0$ (since $a \neq 0$ by assumption). Thus, the equation simplifies to

$$\lambda_1 f' + \lambda_2 f'' + \cdots + \lambda_n f^{(n)} = 0. \quad (11)$$

This time, after our simplification, the leading coefficient of the polynomial on the left hand side is x^{n-1} and it is read off as $\lambda_1 na$. Again, this is only zero when $\lambda_1 = 0$. Thus, our equation simplifies to

$$\lambda_2 f'' + \lambda_3 f''' + \cdots + \lambda_n f^{(n)} = 0. \quad (12)$$

Continuing we see that each coefficient must be zero. Thus, the set is linearly independent. Since it is of size $n + 1 = \dim P_n$ it must be a basis.

Question 29

I will state part (a) as a theorem.

Theorem 0.2. *Let W_1, W_2 be subspaces of a vector space V . Then*

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2). \quad (13)$$

Proof. Start with a basis $\{u_1, \dots, u_k\}$ for the intersection $W_1 \cap W_2$. By corollary 2 of §1.6, we know that we can extend this to a basis of W_1 call this $\{u_1, \dots, u_k, v_1, \dots, v_r\}$. Similarly, applying the same corollary, extend the original set to a basis for W_2 as well; call this $\{u_1, \dots, u_k, w_1, \dots, w_s\}$. Notice that the combined set

$$\{u_1, \dots, u_k, v_1, \dots, v_r, w_1, \dots, w_s\} \quad (14)$$

is of size $k + r + s$. Also, this combined set certainly generates $W_1 + W_2$. Thus $\dim(W_1 + W_2) \leq k + r + s$. On the other hand, $\dim W_1 = k + r$, $\dim W_2 = k + s$. Thus

$$\dim W_1 + \dim W_2 - \dim(W_1 \cap W_2) = (k + r) + (k + s) - k = k + r + s.$$

Thus, we have shown the \leq part of the theorem.

We now finish by showing \geq and hence the theorem follows. We do this by showing that the combined set in equation (14) is linearly independent (hence a basis for $W_1 + W_2$). Indeed, assume that

$$0 = a_1 u_1 + \cdots + a_k u_k + b_1 v_1 + \cdots + b_r v_r + c_1 w_1 + \cdots + c_s w_s \quad (15)$$

for scalars, a_i, b_j, c_k . Notice that this equation is of the form

$$\mathbf{0} = \mathbf{u} + \mathbf{v} + \mathbf{w} \quad (16)$$

for vectors $\mathbf{u} = a_1 u_1 + \cdots + a_k u_k \in W_1 \cap W_2$, $\mathbf{v} = b_1 v_1 + \cdots + b_r v_r \in W_1$, and $\mathbf{w} = c_1 w_1 + \cdots + c_s w_s \in W_2$. Suppose that $\mathbf{v} \in W_1 \cap W_2$. Then it can be written as a linear combination of the u_i 's. On the other hand, since it is a linear combination of the v_i 's we see that $\mathbf{v} = \mathbf{0}$. Similarly, we see that if $\mathbf{w} \in W_1 \cap W_2$ then it must be zero as well.

We can rearrange (16) in the form

$$\mathbf{w} = -\mathbf{u} - \mathbf{v}. \quad (17)$$

Clearly, $\mathbf{u} \in W_1 \cap W_2 \subset W_1$ and $\mathbf{v} \in W_1$ by definition. By the subspace property, we know that the right hand side is a vector in W_1 . Thus $\mathbf{w} \in W_1 \cap W_2$ and hence it is zero. Similarly, we can rearrange to see that $\mathbf{v} = \mathbf{0}$ as well. By linear independence of all three subsets, we see that this implies all the scalars in (15) must vanish and hence the combined set is a basis. \square