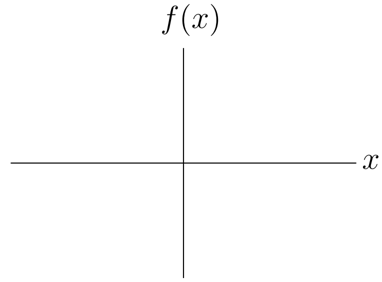
Today we continue our discussion of limits.

Learning Catalytics exercise #1: In our first exercise, you should draw a graph here before you answer the question:



Infinite limits

Textbook Informal Definition. Suppose f is defined for all x near a. If $f(x)$ grows arbitrarily large for all x sufficiently close (but not equal) to a, we write

$$\lim_{x\to a} f(x) = \infty$$

and say the limit of $f(x)$ as x approaches a is infinity.

PB Informal Definition. Suppose that the function f is defined for all numbers x near the number a. We say that

$$\lim_{x\to a} f(x) = \infty$$

if, given any table of numbers x that approaches the number a, the corresponding table of values $f(x)$ grows without bound.

Even though this limit is infinite, we say that *it does not exist.*

There is also an analogous definition for the limit of $f(x)$ being $-\infty$ as $x \to a$. In this case, $f(x) < 0$ and $|f(x)|$ grows without bound as $x \to a$.

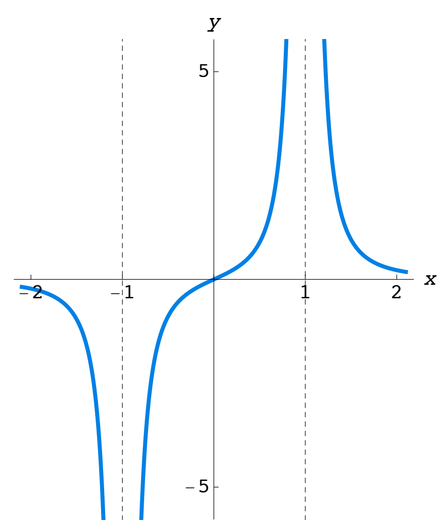
Example. Here's the graph of the function

$$f(x) = \frac{x}{(x^2-1)^2}.$$

This function is a rational function whose domain is the set of all real numbers except -1 and 1. We see that

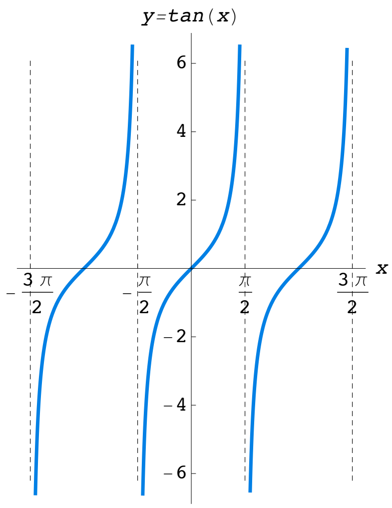
$$\lim_{x\to -1} f(x) = -\infty \text{ and } \lim_{x\to 1} f(x) = \infty.$$

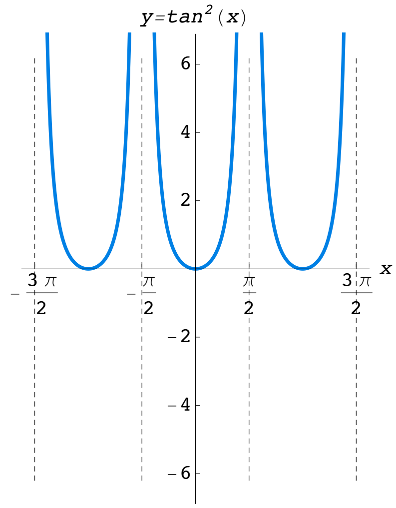
We call the lines $x = -1$ and $x = 1$ *vertical asymptotes.*



Remark. We can also talk about one-sided infinite limits.

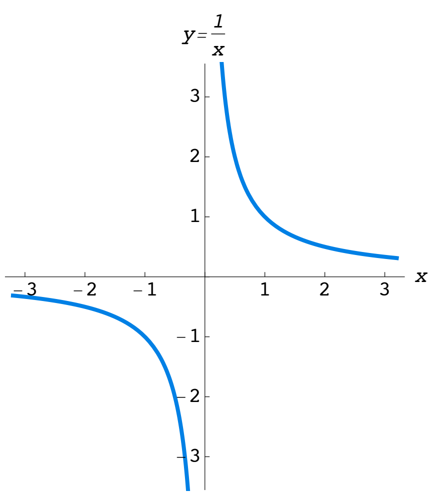
Examples. Consider the graphs of $\tan x$ and $\tan^2 x$.

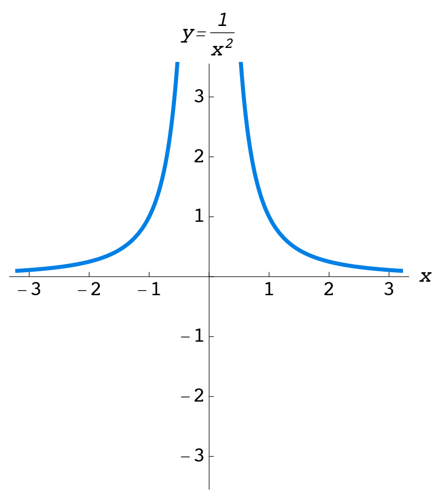




We can calculate many infinite limits if we remember the following two graphs.

Examples. Consider the graphs of $1/x$ and $1/x^2$.





Note that

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty \quad \text{and} \quad \lim_{x\to 0^+} \frac{1}{x} = \infty,$$

but the two-sided limit does not exist and is neither ∞ nor $-\infty$. However,

$$\lim_{x\to 0} \frac{1}{x^2} = \infty.$$

Consequently, both one-sided limits for $1/x^2$ at $a = 0$ are infinite as well.

Using these two graphs as guides, we can remember the (one- or two-sided) limits as $x \to a$ of

$$f(x) = \frac{1}{(x-a)^n}$$

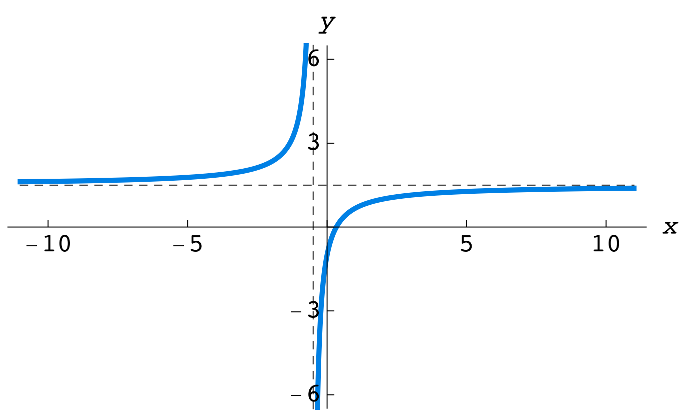
for any positive integer n.

Learning Catalytics exercise #2: Use this space to make a calculation before you answer:

Limits at infinity

Limits at infinity measure the behavior of the dependent variable as the independent variable increases "without bound."

Example. Consider the graph of the function $f(x) = \dfrac{3x-1}{2x+1}$.



We say that $\displaystyle\lim_{x\to\infty} \frac{3x-1}{2x+1} = \frac{3}{2}$.

Informal Definition. Suppose that the function f is defined on the infinite interval (a, ∞). We say that

$$\lim_{x\to\infty} f(x) = L$$

if, given any table of numbers x that go to infinity, the corresponding table of values $f(x)$ approaches the number L.

How do we calculate limits at infinity? Let's go back to the example.

Example. Calculate the limit as $x \to \infty$ of the function $f(x) = \dfrac{3x-1}{2x+1}$.

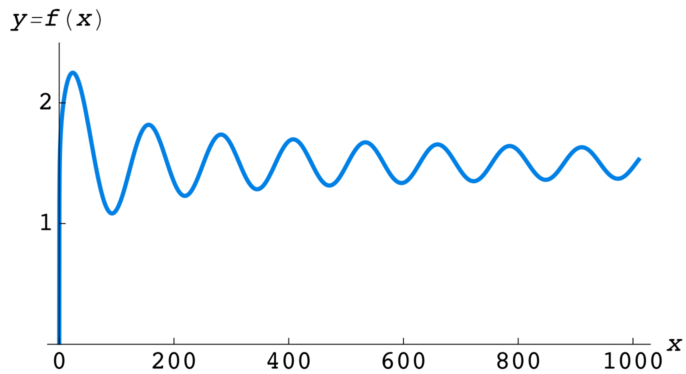
Here is a more complicated example.

Example. Consider $f(x) = x - \dfrac{5x^2}{5x+1}$.

Without using any graphing technology, do you think that $f(x)$ approaches a limit as $x \to \infty$?

Here is a much more complicated example.

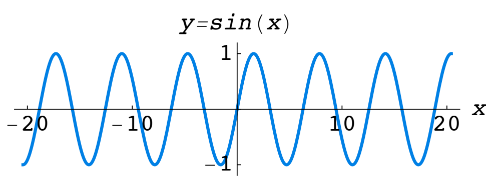
Example. Consider $f(x) = \dfrac{3\sqrt{x} + 8\sin(x/20)}{\sqrt{1+4x}}$.

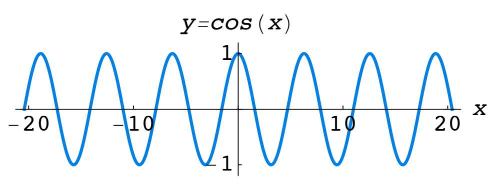


Does $f(x)$ approach some limiting value as $x \to \infty$? If so, what value is it?

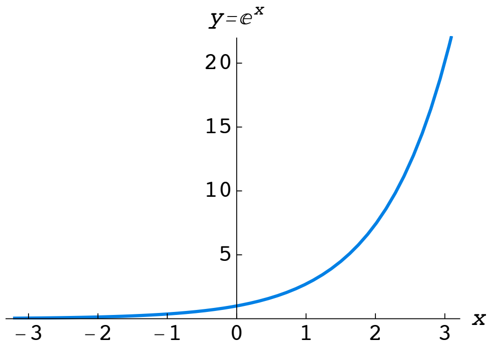
Theorems 2.6 and 2.7 summarize the limit properties for power functions, polynomials, and rational functions. We also need to know these limits for the basic transcendental functions such as $\sin x$, $\cos x$, e^x, and $\ln x$.

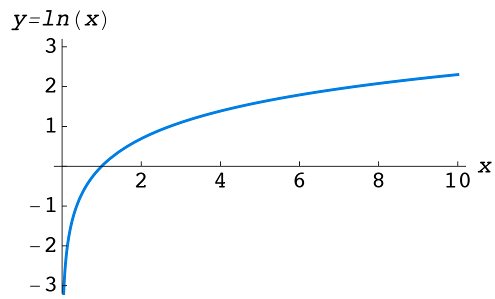
Examples. Here are the graphs of sine and cosine:





Examples. Here are the graphs of $y = e^x$ and $y = \ln x$:





Examples. Here are the graphs of $y = \tan^{-1} x$ and $y = \sec^{-1} x$: