MA 442 - Quiz

February 4

Name: ______________________ **BUID:** ______________

There are two graded questions and one, optional, BONUS question.

Question 1. Prove that the set of vectors $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ generates $\mathbb{R}^3$.

Solution. Solution 1. The easier and direct method is presented in the solutions to problem 1.4.6 found on the homepage.

Solution 2. This solution uses the concept of a basis. Recall that the dimension of $\mathbb{R}^3$ is 3. By corollary 2 of §1.6 of the book we know that any linear independent subset of $\mathbb{R}^3$ containing three vectors is automatically a basis, hence generating. Thus, *it suffices to show that the vectors are linearly independent.* To this end, suppose $\lambda_1, \lambda_2, \lambda_3$ satisfy

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}. \tag{1}$$

This gives three equations $\lambda_1 + \lambda_2 = 0, \lambda_1 + \lambda_3 = 0, \lambda_2 + \lambda_3 = 0$. The first equation says that $\lambda_1 = -\lambda_2$. So, the second equation becomes $-\lambda_2 + \lambda_3 = 0$ or $\lambda_2 = \lambda_3$. Plugging into the third and final equation we see that $\lambda_3 = 0$ hence $\lambda_1 = \lambda_2 = 0$ as well. We have proved linear independence.

Solution 3. There is a third and final solution that some students used and I will present here. Denote the vectors in the problem by $\{u_1, u_2, u_3\}$. We refer to the standard basis $\{e_1, e_2, e_3\}$ for $\mathbb{R}^3$. Note that

$$e_1 = \frac{1}{2}(u_1 + u_2 - u_3), \quad e_2 = \frac{1}{2}(u_1 - u_2 + u_3), \quad e_3 = \frac{1}{2}(-u_1 + u_2 + u_3).$$

Thus $\{e_1, e_2, e_3\} \subset \operatorname{span}\{u_1, u_2, u_3\}$. Applying span again we see that

$$\mathbb{R}^3 = \operatorname{span}\{e_1, e_2, e_3\} \subset \operatorname{span}\{u_1, u_2, u_3\},$$

which finishes the proof that $\{u_1, u_2, u_3\}$ generates.

Question 2. Consider the real vector space $V = \mathcal{F}(\mathbb{R}, \mathbb{R})$ of functions $\mathbb{R}$ to $\mathbb{R}$. Show that the subset

$$\{\sin x, \cos x\} \subset \mathcal{F}(\mathbb{R}, \mathbb{R}) \tag{2}$$

is linearly independent.

Solution. Let λ, μ be scalars such that

$$\lambda \sin x + \mu \cos x = \mathbf{0}. \tag{3}$$

This is an equality of functions. When $x = 0$ this equation becomes $\mu = 0$. When $x = \frac{\pi}{2}$ this equation becomes $\lambda = 0$. Thus, $\lambda = \mu = 0$ and hence the vectors are linearly independent.

BONUS: One day, at the end of class, your professor was running out the room and spuriously writes on the board:

$$\text{“} \quad \operatorname{span}(S_1 \cap S_2) = \operatorname{span}(S_1) \cap \operatorname{span}(S_2) \quad \text{”} \tag{4}$$

But, in his impetuousness, he made an **error**! Can you find a counterexample to this assertion?

Solution. Here is a counterexample for the vector space $\mathbb{R}^2$. Let

$$S_1 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

so that $\operatorname{span}(S_1) = \mathbb{R}^2$ (so, S_1 is generating). Let

$$S_2 = \left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\}$$

Note also $\operatorname{span}(S_2) = \mathbb{R}^2$ (so, S_2 is also generating). Then $S_1 \cap S_2 = \emptyset$, so $\operatorname{span}(S_1 \cap S_2) = \{0\}$ (the span of the empty set is just the zero vector). On the other hand, $\operatorname{span}(S_1) \cap \operatorname{span}(S_2) = \mathbb{R}^2 \cap \mathbb{R}^2 = \mathbb{R}^2$. Since $\{0\} \neq \mathbb{R}^2$ this is a counterexample to the false claim.[1]

The statement that **IS ALWAYS** true is

$$\operatorname{span}(S_1 \cap S_2) \subset \operatorname{span}(S_1) \cap \operatorname{span}(S_2). \tag{5}$$

Try to prove this!

[1] For an even easier example consider just $V = \mathbb{R}$, the one-dimensional vector space with $S_1 = \{a\}$ and $S_2 = \{b\}$ with $a \neq b$ and both a, b nonzero.