

Solutions to selected exercises from §1.6

Question 4

No, these vectors do not generate P_3 (the set of polynomials, with real coefficients, which are at least cubic). The reason is that $\dim P_3 = 4$ as we have shown that a basis for this vector space consists of four vectors (for example $\{1, x, x^2, x^3\}$). Since a basis is the minimum sized subset which can generate a vector space, a collection of vectors consisting of less than four vectors can not be generating.

For practice, we will show that these three vectors are linearly independent, however. Suppose $\lambda_1, \lambda_2, \lambda_3$ are scalars such that

$$\mathbf{0} = \lambda_1(x^3 - 2x^2 + 1) + \lambda_2(4x^2 - x + 3) + \lambda_3(3x - 2). \quad (1)$$

This equation can be written as

$$\mathbf{0} = \lambda_1x^3 + (-2\lambda_1 - 4\lambda_2)x^2 + (-\lambda_2 + 3\lambda_3)x + (\lambda_1 + 3\lambda_2 - 2\lambda_3). \quad (2)$$

For this to be zero, each coefficient must be zero. Thus $\lambda_3 = 0$ comes from vanishing of the cubic term. From the vanishing of the linear term we see that $\lambda_2 = 0$. From the vanishing of the quadratic term we see $\lambda_1 = 0$. Thus, the collection is linear independent.

Question 11

First, we will show that $\{u + v, au\}$ is a basis. Notice, since $\dim V = 2$ we need to only show that $u + v, au$ are linearly independent. So, suppose that $\lambda_1(u+v) + \lambda_2(au) = \mathbf{0}$ for λ_1, λ_2 scalars. This simplifies to $(\lambda_1 + a\lambda_2)u + \lambda_1v = 0$. Since $\{u, v\}$ is a basis, hence linearly independent, we see that this implies $\lambda_1 + a\lambda_2 = 0$ and $\lambda_1 = 0$. But, these equations together imply that $\lambda_1 = \lambda_2 = 0$ since $a \neq 0$.

Next, we show that $\{au, bv\}$ is a basis where a, b are nonzero scalars. Again, it suffices to show that this set is linearly independent. Suppose $\lambda_1(au) + \lambda_2(bv) = \mathbf{0}$. This is the same as $(a\lambda_1)u + (b\lambda_2)v = 0$ which implies $a\lambda_1 = b\lambda_2 = 0$. Since a, b are both nonzero this implies $\lambda_1 = \lambda_2 = 0$ hence the set is linearly independent.

Question 22

Observe that $W_1 \cap W_2$ is always a subspace of W_1 . Hence $\dim(W_1 \cap W_2) \leq \dim W_1$ by theorem 1.11 in the book. Furthermore, the other part of theorem 1.11 says that this is an equality if and only if $W_1 \cap W_2 = W_1$. We know this can only be true when $W_1 \subset W_2$. Thus, the necessary and sufficient condition is that W_1 must be a subspace of W_2 .

Question 29

I will state part (a) as a theorem.

Theorem 0.1. *Let W_1, W_2 be subspaces of a vector space V . Then*

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2). \quad (3)$$

Proof. Start with a basis $\{u_1, \dots, u_k\}$ for the intersection $W_1 \cap W_2$. By corollary 2 of §1.6, we know that we can extend this to a basis of W_1 call this $\{u_1, \dots, u_k, v_1, \dots, v_r\}$. Similarly, applying the same corollary, extend the original set to a basis for W_2 as well; call this $\{u_1, \dots, u_k, w_1, \dots, w_s\}$. Notice that the combined set

$$\{u_1, \dots, u_k, v_1, \dots, v_r, w_1, \dots, w_s\} \quad (4)$$

is of size $k+r+s$. Also, this combined set certainly generates $W_1 + W_2$. Thus $\dim(W_1 + W_2) \leq k+r+s$. On the other hand, $\dim W_1 = k+r$, $\dim W_2 = k+s$. Thus

$$\dim W_1 + \dim W_2 - \dim(W_1 \cap W_2) = (k+r) + (k+s) - k = k+r+s.$$

Thus, we have shown the \leq part of the theorem.

We now finish by showing \geq and hence the theorem follows. We do this by showing that the combined set in equation (4) is linearly independent (hence a basis for $W_1 + W_2$). Indeed, assume that

$$\mathbf{0} = a_1u_1 + \dots + a_ku_k + b_1v_1 + \dots + b_rv_r + c_1w_1 + \dots + c_sw_s \quad (5)$$

for scalars, a_i, b_j, c_k . Notice that this equation is of the form

$$\mathbf{0} = \mathbf{u} + \mathbf{v} + \mathbf{w} \quad (6)$$

for vectors $\mathbf{u} = a_1u_1 + \dots + a_ku_k \in W_1 \cap W_2$, $\mathbf{v} = b_1v_1 + \dots + b_rv_r \in W_1$, and $\mathbf{w} = c_1w_1 + \dots + c_sw_s \in W_2$. Suppose that $\mathbf{v} \in W_1 \cap W_2$. Then it can be written as a linear combination of the u_i 's. On the other hand, since it is a linear combination of the v_i 's we see that $\mathbf{v} = \mathbf{0}$. Similarly, we see that if $\mathbf{w} \in W_1 \cap W_2$ then it must be zero as well.

We can rearrange (6) in the form

$$\mathbf{w} = -\mathbf{u} - \mathbf{v}. \quad (7)$$

Clearly, $\mathbf{u} \in W_1 \cap W_2 \subset W_1$ and $\mathbf{v} \in W_1$ by definition. By the subspace property, we know that the right hand side is a vector in W_1 . Thus $\mathbf{w} \in W_1 \cap W_2$ and hence it is zero. Similarly, we can rearrange to see that $\mathbf{v} = \mathbf{0}$ as well. By linear independence of all three subsets, we see that this implies all the scalars in (5) must vanish and hence the combined set is a basis. \square