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Let's begin with an exercise about extrema.

Example 2.32. Find the absolute extrema for the function

$$f(x) = \frac{2x}{1+x^2} \tag{82}$$

on the interval $[-2, 2]$.

The **mean value theorem** is one of the most important theorems in calculus. It's the main player behind much of the techniques we will learn in the upcoming weeks about describing the behavior of functions.

An easy case of the mean value theorem is Rolle's theorem. Consider a function f which is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b). Suppose, in addition, the function satisfies $f(a) = f(b)$. In this situation, we can conclude that there is at least one point c with $a < c < b$ such that $f'(c) = 0$.

Example 2.33. Can you give a visual proof of this result? Consider the function $f(x) = 1 - x^2$ defined on the interval $[-1, 1]$.

The mean value theorem is just a skew, or rotated version, of Rolle's theorem. Suppose we take away the condition that $f(a) = f(b)$. We can still consider the average rate of change of the function f on the interval $[a, b]$:

$$\frac{f(a) - f(b)}{a - b}. \tag{83}$$

Remember, this quantity represents the average slope of tangent lines to the function on the interval.

Theorem 2.34. *Consider a function f which is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b). Then, there exists a number c with $a < c < b$ such that*

$$f'(c) = \frac{f(a) - f(b)}{a - b}. \tag{84}$$

Example 2.35. Check the mean value theorem for the function $f(x) = x + \frac{1}{x}$ on the interval $[1, 3]$. That is, find the number c which is guaranteed by the mean value theorem.

A nice corollary of the mean value theorem is the following.

Corollary 2.36. *Suppose that $f'(x) = 0$ for all x. Then f is a constant function.*

Proof. Let $a < b$ be two numbers. By the mean value theorem, there exists a c with $a < c < b$ with the property that

$$f'(c) = \frac{f(a) - f(b)}{a - b}. \tag{85}$$

But $f'(c) = 0$ by assumption, therefore $f(a) = f(b)$. □

Example 2.37. This problem involves calculus.

- Show that $\arctan x$ and $\arctan(1/x)$ differ by a constant.
- Show that $\arctan x + \arctan(1/x) = \pi/2$ for $x > 0$.

Let $f(x) = \arctan x + \arctan 1/x$ and consider it on the domain $x > 0$. Then,

$$f'(x) = \frac{1}{1+x^2} + \frac{-1/x^2}{1+(1/x)^2} = \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0. \tag{86}$$

Thus, by the corollary above we conclude that $f(x) = K$ for some constant K for all $x > 0$. Similarly, we can show that $f(x) = K'$ for some constant K' and all $x < 0$.

Take $x = 1$, then we see

$$K = f(1) = \pi/4 + \pi/4 = \pi/2. \tag{87}$$