

SPIN GEOMETRY
PROBLEM SHEET 1

Problem 1. *Filtered algebras.* Let $0 = F^{-1}A \subset F^0A \subset \cdots \subset F^\infty A = A$ be an increasingly filtered associative algebra.

- (1) Show that the algebra

$$\mathrm{gr} A \stackrel{\mathrm{def}}{=} \bigoplus_{k \geq 0} F^k A / F^{k-1} A$$

is a commutative if and only if $ab - ba \in F^{p+q-1}A$ for all $a \in F^p A, b \in F^q A$.

- (2) Suppose that $\mathrm{gr} A$ is commutative. For $a \in F^p A$, let $\bar{a} = a \bmod F^{p-1}A$ be its image in $\mathrm{gr} A$. Show that the bilinear operation

$$\{-, -\}: \mathrm{gr} A \times \mathrm{gr} A \rightarrow \mathrm{gr} A$$

defined by

$$\{\bar{a}, \bar{b}\} \stackrel{\mathrm{def}}{=} (ab - ba) \bmod F^{p+q-2}A$$

is a Poisson bracket on $\mathrm{gr} A$. (You should first check that $\{-, -\}$ is well-defined.)

- (3) State and prove a “super” version of the above result.
 (4) In class, we defined an increasing filtration on $A = \mathrm{Cl}(V, q)$ —the Clifford algebra associated to the quadratic vector space (V, q) . With respect to this filtration, $\mathrm{gr} \mathrm{Cl}(V, q) = \wedge^\bullet V$. Describe the induced Poisson bracket $\{-, -\}$ in this example.

Problem 2. *Classification of complex Clifford algebras.* Let $\mathrm{Cl}(n)$ be the Clifford algebra, defined over \mathbf{C} , associated to the vector space \mathbf{C}^n equipped with its standard non-degenerate (holomorphic) quadratic form $q(z) = \sum_i z_i^2$.

- (1) Compute $\dim_{\mathbf{C}} \mathrm{Cl}(n)$.
 (2) Show that $\mathrm{Cl}(1) \cong \mathbf{C} \oplus \mathbf{C}$, as algebras.
 (3) Consider the Pauli sigma matrices σ_i which satisfy $\sigma_i \sigma_j = \delta_{ij} \mathbb{1} + i\epsilon_{ijk} \sigma_k$. Show that $i\mathbb{1}, i\sigma_1, i\sigma_2, i\sigma_3$ form a basis for $\mathrm{Cl}(2)$, and hence $\mathrm{Cl}(2) \cong \mathrm{End}_{\mathbf{C}}(\mathbf{C}^2)$.
 (4) Show that

$$\mathrm{Cl}(n+2) \cong \mathrm{Cl}(n) \otimes_{\mathbf{C}} \mathrm{End}_{\mathbf{C}}(\mathbf{C}^2).$$

- (5) Let $\mathrm{Cl}_{r,s}$ be the real Clifford algebra associated to the standard quadratic form on \mathbf{R}^{r+s} of signature (r, s) . Show that there is an isomorphism

$$\mathrm{Cl}_{r,s} \otimes_{\mathbf{R}} \mathbf{C} \cong \mathrm{Cl}(r+s).$$

Problem 3. *Changing base.*

- (1) Consider the algebra $A = \mathbf{C} \otimes_{\mathbf{R}} \mathbf{C}$. Find an element $T \in A$ such that $T^2 = 1 \otimes 1$. By considering $P_{\pm}: A \rightarrow A$, the projections onto the ± 1 eigenspaces for the operator $T \cdot (-): A \rightarrow A$, show that $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C} \cong \mathbf{C} \oplus \mathbf{C}$ as real associative algebras.

- (2) Show that the bilinear map

$$\phi: \mathbf{C} \times \mathbf{H} \rightarrow \text{End}_{\mathbf{C}}(\mathbf{H})$$

defined by $\phi(z, y)x = zx\bar{y}$ determines an isomorphism

$$\mathbf{C} \otimes_{\mathbf{R}} \mathbf{H} \cong \text{End}_{\mathbf{C}}(\mathbf{C}^2)$$

of real associative algebras.

- (3) Show that the bilinear map

$$\phi: \mathbf{H} \times \mathbf{H} \rightarrow \text{End}_{\mathbf{R}}(\mathbf{H})$$

defined by $\phi(y_1, y_2)x = y_1x\bar{y}_2$ determines an isomorphism

$$\mathbf{H} \otimes_{\mathbf{R}} \mathbf{H} \cong \text{End}_{\mathbf{R}}(\mathbf{R}^4)$$

of real associative algebras.

Problem 4. *Pin and spin.*

- (1) A *spinor representation* is the restriction of an irreducible $\text{Cl}(V)_0$ -representation to the group $\text{Spin}(V)$. Show that, as a $\text{Spin}(V)$ -representation, a spinor representation is irreducible.
- (2) Regard $\text{Sp}(1)$ as the group of unit quaternions $\{y \in \mathbf{H} \mid \|y\| = 1\}$, which is diffeomorphic to S^3 . Describe the isomorphism

$$\text{Spin}(3) \cong \text{Sp}(1)$$

by constructing an action of $\text{Sp}(1)$ on $\text{Im } \mathbf{H}$.

- (3) Similarly, describe the isomorphism

$$\text{Spin}(4) \cong \text{Sp}(1) \times \text{Sp}(1)$$

by constructing an action of $\text{Sp}(1) \times \text{Sp}(1)$ on \mathbf{H} .

The next two parts of the problem set up the exceptional isomorphism

$$\text{Spin}(6) \cong \text{SU}(4).$$

- (4) Let V be a four-dimensional complex vector space equipped with an inner product and an element $v \in \wedge^2 V$ such that $\|v\| = 1$. Show that V is equipped with a conjugate-linear isomorphism

$$\star_v: \wedge^2 V \rightarrow \wedge^2 V$$

such that $\star_v^2 = \mathbb{1}$ and that $\dim_{\mathbf{R}} \wedge_{\pm}^2 V = 6$. Here, $\wedge_{\pm}^2 V \subset \wedge^2 V$ are the ± 1 eigenspaces of \star_v .

- (5) Show that the group $\text{SU}(V)$ naturally acts on $\wedge_{\pm}^2 V$. By taking $V = \mathbf{C}^4$, argue that this leads to the isomorphism $\text{Spin}(6) \cong \text{SU}(4)$.

The next two parts of the problem set up the exceptional isomorphism

$$\text{Spin}(5) \cong \text{Sp}(2).$$

- (6) Suppose now that the four-dimensional complex vector space V is equipped with a compatible complex symplectic structure $\omega \in \wedge^2 V$. Argue that one, or both, of the six-dimensional real $\text{Sp}(V)$ -representations $\wedge_{\pm}^2 V$ contains a *five-dimensional* subspace invariant under the action of $\text{Sp}(V)$.
- (7) By considering $V = \mathbf{C}^4$, and using $\text{Spin}(6) \cong \text{SU}(4)$, complete the proof that $\text{Spin}(5) \cong \text{Sp}(2)$.