- Linear Transformations

Recall, functions or maps go between sets

$$f : S \longrightarrow R \quad , \quad S, R \text{ sets}$$

Linear transformations are functions between vector spaces.

$$T : V \longrightarrow W \quad , \quad V, W \text{ vector spaces.}$$

They are not just any functions, however. They must be compatible with addition and scalar multiplication.

Dfn : Let V, W be vector spaces (over a field $\mathbb{F}$). A linear transformation is a function

$$T : V \longrightarrow W$$

s.t.

1) $T(x+y) = T(x) + T(y)$ for all $x, y \in V$.

2) $T(\lambda x) = \lambda T(x)$ for all $x \in V, \lambda \in \mathbb{F}$.

<u>Facts</u>: If T is linear then the following are true:

- $T(\vec{0}_V) = \vec{0}_W$.

This single condition is equivalent to being linear.

- $\boxed{T(x + \lambda y) = T(x) + \lambda T(y)}$

- $T(x - y) = T(x) - T(y)$.

- $T\left(\sum_i \lambda_i x_i\right) = \sum_i \lambda_i T(x_i)$

for $\lambda_i \in \mathbb{F}$, $x_i \in V$.

<u>Ex</u>: $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by

$$T(a_1, a_2) = (-a_2, a_1)$$

is linear.

<u>Pf</u>: $T\big((a_1, a_2) + \lambda(b_1, b_2)\big)$

$$= T(a_1 + \lambda b_1, a_2 + \lambda b_2)$$

$$= (-a_2 - \lambda b_2, a_1 + \lambda b_1)$$

$$= T(a_1, a_2) + \lambda T(b_1, b_2).$$

✓ ∎

- This transformation is geometrically familar. More generally, let $\alpha \in [0, 2\pi)$ be an angle.

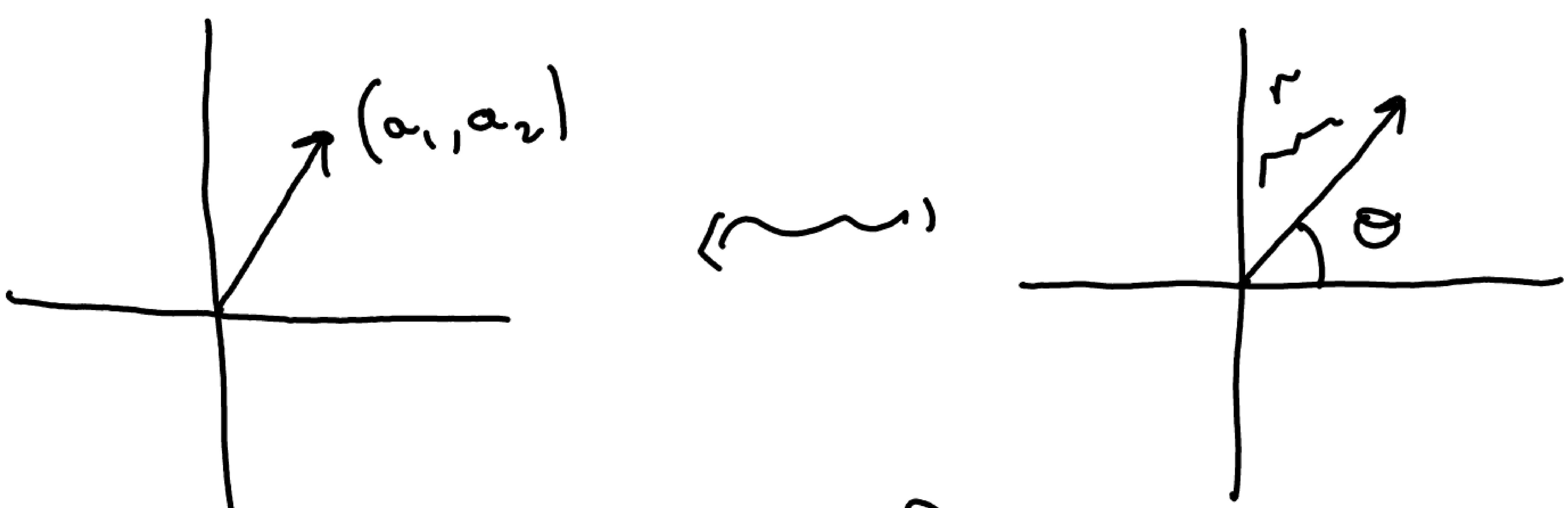
Define $T_\alpha : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$T_\alpha(v) =$ Counterclockwise rotation of v by angle α.

Check this is linear!

We come up w/ an explicit formula for T_α.

To do this, we use polar coordinates.



$$a_1 = r\cos\theta$$
$$a_2 = r\sin\theta.$$

rotation by α counter clockwise.

But then

$$T_\alpha(a_1, a_2) = \big(r\cos(\theta + \alpha),\ r\sin(\theta + \alpha)\big).$$

We need to express this in terms of a_1, a_2.

$$\cos(\theta + \alpha) = \cos\theta \cos\alpha - \sin\theta \sin\alpha$$
$$\sin(\theta + \alpha) = \cos\theta \sin\alpha + \sin\theta \cos\alpha$$

Thus:

$$\big(r\cos(\theta+\alpha),\ r\sin(\theta+\alpha)\big)$$
$$= \boxed{\begin{gathered}\big(a_1\cos\alpha - a_2\sin\alpha,\ a_1\sin\alpha + a_2\cos\alpha\big) \\ \| \\ T_\alpha(a_1, a_2).\end{gathered}}$$

Check when $\alpha = \pi/2$. Then

$$T_{\pi/2}(a_1, a_2) = (-a_2, a_1).$$

This was our first example.

Ex: The reflection function is

$$R : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$R(a_1, a_2) = (a_1, -a_2).$$

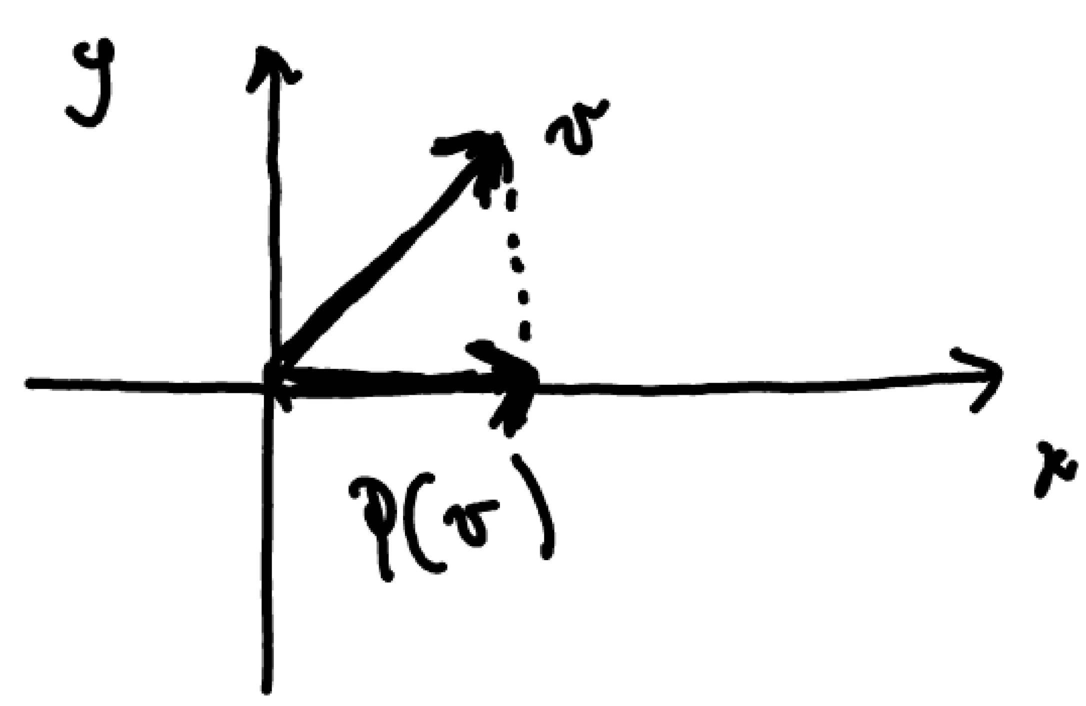
Ex: Dilation is defined for any $c \in \mathbb{R}$.

$$D_c : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$D_c(a_1, a_2) = (ca_1, ca_2).$$

Ex: $P : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $P(a_1, a_2) = (a_1, 0)$.

This is called "the projection onto the x-axis".



Ex: We have discussed $\mathcal{F}(\mathbb{R},\mathbb{R})$ the vector space of all fn's $\mathbb{R} \to \mathbb{R}$.

Let

$$C^1(\mathbb{R},\mathbb{R}) \subset \mathcal{F}(\mathbb{R},\mathbb{R})$$

be the subset of differentiable fns $\mathbb{R} \to \mathbb{R}$.

For example: $x^2, \sin x, e^{3x} \in C^1(\mathbb{R},\mathbb{R})$,

$$\text{but } |x| \notin C^1(\mathbb{R},\mathbb{R})$$

Define

$$T : C^1(\mathbb{R},\mathbb{R}) \longrightarrow \mathcal{F}(\mathbb{R},\mathbb{R}).$$

$$T(f) = f'$$ The derivative of f.

Then $T(f+g) = (f+g)' = f' + g'$

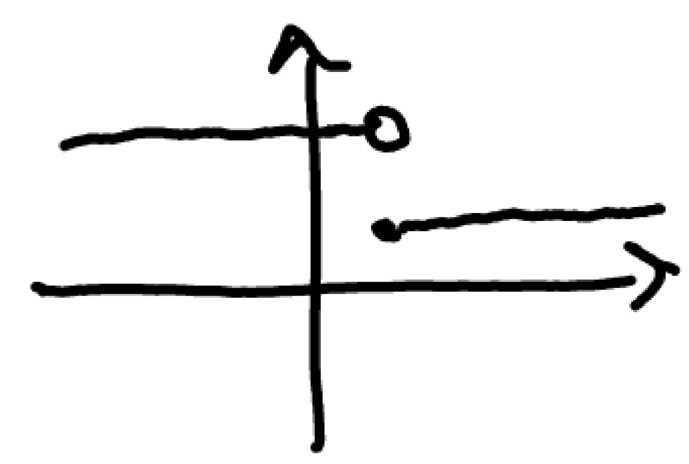
and $T(\lambda f) = (\lambda f)' = \lambda f'$.

So, T is linear.

"The derivative is a linear transformation".

Ex: Let $C^0(\mathbb{R},\mathbb{R}) \subset \mathcal{F}(\mathbb{R},\mathbb{R})$ be the subset of continuous functions

Eg: $|x|,\ x^2,\ \cos x,\ e^{3x} \in C^0(\mathbb{R},\mathbb{R})$

but  $\notin C^0(\mathbb{R},\mathbb{R})$.

$C^0(\mathbb{R},\mathbb{R})$ is a subspace. Define, for $a<b$:

$$T : C^0(\mathbb{R},\mathbb{R}) \longrightarrow \mathbb{R}$$

by the formula $T(f) = \displaystyle\int_a^b f(t)\,dt$.

T is linear. //

Dfn: The identity linear transformation is

$$\mathbb{1} : V \longrightarrow V, \quad \mathbb{1}(x) = x, \text{ for all } x \in V.$$

The zero linear transformation is

$$0 : V \longrightarrow W, \quad 0(x) = 0 \quad \text{for all } x \in V.$$

• Suppose $T: V \to W$ is a linear transformation. Recall, there is a special vector $\vec{0} = \vec{0}_W \in W$. (we will use $\vec{0}$ for the zero vector in V and W).

Suppose $x, y \in V$ are such that

$$T(x) = \vec{0} \quad \text{and} \quad T(y) = \vec{0}.$$

Then: $T(x+y) = T(x) + T(y) = \vec{0} + \vec{0} = \vec{0}$.

and $T(\lambda x) = \lambda T(x) = \lambda \cdot \vec{0} = \vec{0}$ for all $\lambda \in \mathbb{F}$.

<u>Theorem</u>: The kernel of $T: V \to W$ is the subspace:

$$\ker T = \{ x \in V \mid T(x) = \vec{0} \} \subset V.$$

<u>Ex</u>: Let $T: \mathbb{R}^2 \to \mathbb{R}$ be defined by:

$$T(a_1, a_2) = a_1$$

Then

$$\ker T = \{ (a_1, a_2) \mid T(a_1, a_2) = \vec{0} \}$$

$$= \{ (a_1, a_2) \mid a_1 = 0 \} = \text{"} y\text{-axis} \text{"}$$

$$= \text{span}\{e_2\}.$$

- Suppose $z, w \in W$ are such that

$$z = T(x), \quad w = T(y).$$

for some $x, y \in V$. Then

$$z + w = T(x) + T(y) = T(x+y).$$

Similarly: $\lambda z = \lambda T(x) = T(\lambda x)$.

Thus, we see that the set of vectors in the <u>image</u> of T is a subspace.

<u>Theorem</u>: The image of $T : V \to W$ is the subspace

$$\operatorname{Im} T = \left\{ z \in W \;\middle|\; \begin{array}{l} T(x) = z \\ \text{for some } x \in V. \end{array} \right\} \subset W.$$

<u>Ex</u>: Define $T : \mathbb{R} \to \mathbb{R}^3$ by

$$T(a) = (a, a, a)$$

Then

$$\operatorname{Im} T = \operatorname{span}\{(1,1,1)\} = \underbrace{\operatorname{span}}_{\text{A line.}} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$