Today we will review material in sections 2.4–2.5 of the book.

Recall that we say

$$\lim_{x \to a} f(x) = \infty \tag{1}$$

if we can make the value $f(x)$ be infinitely large by choosing x to be sufficiently close to a. Similarly, we can define

$$\lim_{x \to a} f(x) = \pm \infty, \quad \lim_{x \to a^{\pm}} f(x) = \pm \infty. \tag{2}$$

In any of these cases we call the line $x = a$ a vertical asymptote of the function f.

0.0.1. Challenge question. If $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = -\infty$ then what can we say about $\lim_{x \to a}(f(x) + g(x))$?

Example 0.1. Compute the following limit (if it exists)

$$\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{x^2} \right) \tag{3}$$

Example 0.2. Compute the following limit (if it exists)

$$\lim_{x \to 0} \left(\frac{1}{x} - \frac{2x + 1}{x} \right). \tag{4}$$

Example 0.3. Define the function

(5)
$$f(x) = \frac{x^2 - x - 2}{x - a}$$

where a is some number. For which values of a do the following limits exist.

- $\lim_{x \to a^+} f(x)$.
- $\lim_{x \to a^-} f(x)$.
- $\lim_{x \to a} f(x)$.

0.0.2. Recall that we say

$$(6) \qquad \lim_{x \to \infty} f(x) = L$$

if we can make $f(x)$ as close to the value L for x a large enough number. In this case we say that $y = L$ is a horizontal asymptote of the function f.

Example 0.4. Find the limit (if it exists)

$$(7) \qquad \lim_{x \to \infty} \frac{x^2 + 2x + 1}{4x^2 + 5}$$

Example 0.5. Find the limit (if it exists).

$$(8) \qquad \lim_{x \to \infty} \frac{e^x \sin x}{e^{2x} + 1}$$

Definition 0.6. Suppose that a function f is defined on an open subset of $\mathbf{R}$ which contains the point a. We say that f is continuous at a if

$$(9) \qquad\qquad f(a) = \lim_{x \to a} f(x).$$

0.0.3. Examples and non-examples of continuous functions .

0.0.4. Checklist for continuity. For a function f to be continuous at a point a, the following three conditions must hold.

 (1) f must be defined at a.
 (2) the limit $L = \lim_{x \to a} f(x)$ must exist.
 (3) the limit L must equal $f(a)$.

Example 0.7. At which points $a \in \mathbf{R}$ is the function $f(x) = x/|x|$ continuous?

0.0.5. Properties of continuous functions. We say a function is continuous if it is continuous at all points in its domain.

 (1) the sum, product, difference, and ratio (when defined) of two continuous functions is again continuous.
 (2) all polynomials are continuous.
 (3) Rational functions (ratios of polynomials) are continuous at every point in their domain.
 (4) the composition of two continuous functions is again continuous.

0.0.6. Continuity on closed intervals. So far we've covered what it means for a function to be continuous on domains which are open intervals, like $(0, \infty)$. What about the function $f(x) = \sqrt{x}$ which is defined on the closed interval $[0, \infty)$? The problem is at the closed point $0 \in [0, \infty)$. In addition to checking continuity on the open interval $(0, \infty)$ we also need to make sure that the right sided limit at 0 agrees with value of the function at that point

$$(10) \qquad\qquad \lim_{x \to 0^+} f(x) \stackrel{?}{=} f(0).$$

If this is the case then we say that f is continuous on the closed interval $[0, \infty)$.

In the first week we defined the notion of average velocity which is the average rate of change of quantity position. We then motivated the definition of a limit by the idea of instantaneous rate of change. In this lecture we return to the precise definition of the instantaneous rate of change of a function at a point—this is called the *derivative* of the function at a point.

Definition 0.8. The *derivative* of a function f at a point a is the limit

$$f'(a) \stackrel{\text{def}}{=} \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

when it exists.

We say that f is *differentiable* at $x = a$ if the derivative at a exists. Otherwise we say that f is not differentiable at a.

Example 0.9. Using the limit definition above to compute the derivative of the function $f(x) = x^2 - 3x$ at the point $x = 1$.

Graphically, the average rate of change of a function $y = f(x)$ on the interval $(a, a + \Delta x)$ is defined as

(11)
$$\frac{rise}{run} = \frac{\Delta y}{\Delta x} = \frac{f(a + \Delta x) - f(a)}{(a + \Delta x) - \Delta x} = \frac{f(a + \Delta x) - f(a)}{\Delta x}$$

We understood instantaneous rate of change of a function $y = f(x)$ as the limit

(12)
$$\frac{\Delta y}{\Delta x} \xrightarrow{\Delta x \to 0} f'(a)$$

Thus, the derivative at $x = a$ measures the *slope* of the graph at that point $(a, f(a))$.

1. SEPTEMBER 23,2022

Today we'll discuss more examples of derivatives.

Examples.

Example 1.1. Let

$$(13) \qquad\qquad f(x) = \sqrt{x+7}.$$

(a) Is the function $f(x)$ differentiable? If so, on what domain?
(b) Use the limit definition to compute the derivative of $f(x)$ at $x = 2$.
(c) Find the equation of the line tangent to the graph of the function $f(x)$ at $x = 2$.

Here is an example of a function which fails to be differentiable at a point.

Example 1.2. Consider the absolute value function $f(x) = |x|$. At which point is $f(x)$ *not* differentiable. For the values of x that $f(x)$ is differentiable, find $f'(x)$.

Properties of derivatives.

- (Constants) The derivative of a constant function $c(x) = c$ is identically zero, $c'(x) = 0$. If c is any constant and f is a function then

(14)
$$(cf)' = cf'.$$

- (Sum) The derivative of a sum is the sum of derivatives:

(15)
$$(f + g)' = f' + g'.$$

- (Power rule) For any integer n the derivative of $f(x) = x^n$ is $f'(x) = nx^{n-1}$. In fact, if $r = p/q$ is a rational number then we still have

(16)
$$(x^r)' = rx^{r-1}.$$

For example, the derivative of the function $\sqrt{x}$ is $1/\sqrt{x}$.

Warning: The derivative of the *product* $f \cdot g$ is *not* the product of the derivatives:

(17)
$$(f \cdot g)' \neq f' \cdot g'.$$

Soon, we will learn rules for computing the derivative of the product of two functions.

Example 1.3. Where is the function $f(x) = x^{2/3} + 7x^4$ differentiable? For the values of x that f is differentiable, what is $f'(x)$?

Exponentials. Today we continue with rules of derivatives, starting with the exponental function.

Definition 1.4. The exponential function e^x is defined by

(18)
$$e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n$$

This number is defined for all real numbers x. Approximately, one has

(19)
$$e \overset{\text{def}}{=} e^1 = 2.71828\ldots.$$

The exponential function obeys the standard rules of exponentials.[1] For instance, $e^0 = 1$. Another good function to keep in mind is the natural logarithm which is the inverse to the exponential function:

(20)
$$\ln(e^x) = e^{\ln x} = x$$

The natural logarithm $\ln x$ is only defined for $x > 0$. A good limit to keep in mind is:

(21)
$$\lim_{x \to \infty} e^x = \infty.$$

Let's turn to the derivative of the exponential function. First we have the following lemma.

Lemma 1.5. *One has*
$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1.$$

Proof. Let's use equation (**??**) to write

(22)
$$\frac{e^h - 1}{h} = \frac{1}{h} \lim_{n \to \infty} \left(-1 + \left(1 + \frac{h}{n}\right)^n\right).$$

Next, we use the binomial theorem to write
$$-1 + \left(1 + \frac{h}{n}\right)^n = -1 + \left(1 + n\frac{h}{n} + \cdots\right) = h + \cdots$$

where the $\cdots$ stands for terms which are at least quadratic in the parameter h. Plugging this back into (**??**) we obtain

(23)
$$\frac{e^h - 1}{h} = 1 + \cdots$$

where now the $\cdots$ stands for terms that are at least linear in h. Taking the limit $h \to 0$ yields the result. $\qquad\square$

Example 1.6. Show that if $f(x) = e^x$ is the exponential function then

(24)
$$f'(x) = e^x.$$

In other words the exponential is its own derivative $(e^x)' = e^x$. This fact makes it very useful to model population growth, interest, etc. via the exponential function.

---

[1]Actually proving that $e^x = (e^1)^x$ is a good exercise.

If $f = f(x)$ is a function, define the second derivative to be

(25)
$$f''(x) = (f'(x))'.$$

That is, this is the derivative of the derivative. Similarly we can define the third derivative $f'''(x)$, and so on.

Recall that we have introduced the notation

(26)
$$f'(x) = \frac{df}{dx}.$$

We will also use

(27)
$$f''(x) = \frac{d^2f}{dx^2}$$

and so on.

Example 1.7. Let $f(x) = 3x^3 - 12\sqrt{x}$. Find $f''(x)$.

Example 1.8. Find the twenty-first derivative of the function $f(x) = e^x - 3x^{12}$.

Example 1.9. Compute the limit

(28)
$$\lim_{a \to 1} \frac{\sqrt{3+a} - 2}{a - 1}$$

(Hint: Don't compute the limit.)

Today we will cover the product and quotient rules for computing derivatives.

Remember, the derivative $f' = df/dx$ is motived by the concept of a 'rate of change'. For average rate of change we wanted to consider how a function $f(x)$ changed as we change x by some amount Δx:

(29) $$x \rightsquigarrow x + \Delta x.$$

When we change x like this, the function $f(x)$ will then change like

(30) $$f(x) \rightsquigarrow f(x + \Delta x) = f(x) + \Delta f(x)$$

where $\Delta f(x) = f(x + \Delta x) - f(x)$.

We now want to consider what happens when we have a product of two functions $f \cdot g$. Then, we can think of $f \cdot g$ as changing like

(31) $$(f + \Delta f) \cdot (g + \Delta g) = f \cdot g + \Delta f \cdot g + f \cdot \Delta g + (\Delta f) \cdot (\Delta g).$$

In calculus we study the instantaneous rate of change, which is the limit of the rate of change as the interval gets infinitesimally small. So, we can imagine that the quantities Δf and Δg are extremely small. Then, if this is the case, we can think of the quantities $f \cdot \Delta g$ and $\Delta f \cdot g$ as being quite large as compared to $\Delta f \cdot \Delta g$.

In other words, we see that when Δf and Δg are very small, the amount that $f \cdot g$ changes by is approximately

$$\Delta f \cdot g + f \cdot \Delta g.$$

This reasoning can be turned into the following theorem.

Theorem 2.1 (Product rule)**.** *Suppose that f, g are differentiable functions. Then*

(32) $$(f \cdot g)' = f' \cdot g + f \cdot g'.$$

Example 2.2. Using the power rule, "check" the product rule for the product of the two functions $f(x) = x^3$ and $g(x) = 5x^7$.

Example 2.3. Compute the derivative of the function $f(x) = x^2 e^x$.

If f, g are two differentiable functions and g is nonzero (in some domain), then we can consider the derivative of the function $h = f/g$. Notice that we could also write this equation like $f = g \cdot h$. Applying the product rule we see that

$$f' = g' \cdot h + g \cdot h'. \tag{33}$$

But, remember that we really wanted to know what h' is. For this, we can use the above equation to solve for this

$$h' = \frac{f' - g' \cdot h}{g} = \frac{f' - g' \cdot f/h}{g}. \tag{34}$$

Rewriting this equation (in a completely equivalent way) results in the usual formulation of the quotient rule.

Theorem 2.4 (Quotient rule). *If f, g are differentiable functions and f/g is defined, then*

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}. \tag{35}$$

Example 2.5. Where is the following function differentiable?

$$f(x) = \frac{xe^x}{1 + x^2}. \tag{36}$$

Compute the derivative of the function when it is defined.

Example 2.6. Compute the derivative of

$$f(x) = \frac{1 - x^3}{1 - x}. \tag{37}$$

Find the equation of the line tangent to the graph of $f(x)$ at $x = -1$.

Today we will discuss derivatives of trigonometric functions like

$$\text{(38)} \qquad\qquad \sin(x), \quad \cos(x), \quad \tan(x), \quad \text{etc.}$$

Recall the following limit

$$\text{(39)} \qquad\qquad \lim_{x \to 0} \frac{\sin x}{x} = 1.$$

One nice proof of this uses some basic trigonometry for expressions of areas of triangles.

Example 2.7. What is the derivative of the function $\sin x$ at $x = 0$?

Another nice limit is the following

$$\text{(40)} \qquad\qquad \lim_{x \to 0} \frac{\cos x - 1}{x} = 0.$$

To obtain this notice that for $x \neq \pi, 3\pi, \ldots$ one has

$$\text{(41)} \qquad \frac{\cos x - 1}{x} = \frac{\cos x - 1}{x} \cdot \frac{\cos x + 1}{\cos x + 1} = \frac{\cos^2 x - 1}{x(\cos x + 1)}.$$

Now $\sin^2 x + \cos^2 x = 1$, so we can write this expression as

$$\text{(42)} \qquad -\frac{\sin^2 x}{x(\cos x + 1)} = -\frac{\sin x}{x} \cdot \frac{\sin x}{\cos x + 1}.$$

Using the fact that the limit of the product of two functions is the product of the limits, we see that as $x \to 0$ this limit is $1 \cdot 0 = 0$.

Example 2.8. What is the derivative of the function $\cos x$ at $x = 0$?

Let's move on to general formulas for the derivatives of the functions $\sin x$ and $\cos x$.

Proposition 2.9. *One has*

$$\text{(43)} \qquad\qquad (\sin x)' = \cos x, \qquad (\cos x)' = -\sin x.$$

Proof. Let's consider just the derivative of $\sin x$. We appeal to a (hopefully familiar) trigonometric identity

(44) $$\sin(a+b) = \sin a \cos b + \cos a \sin b.$$

Now we compute

$$\begin{aligned}
(\sin x)' &= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\
&= \sin x \cdot \left(\lim_{h \to 0} \frac{\cos h - 1}{h} \right) + \cos x \cdot \left(\lim_{h \to 0} \frac{\sin h}{h} \right) \\
&= \sin x \cdot 0 + \cos x \cdot 1.
\end{aligned}$$

$\square$

Example 2.10. Compute the derivative of the function

(45) $$f(x) = e^x \cos(x) \sin(x).$$

There are other trigonometric functions we can build using sine and cosine. For example, the tangent and cotangent functions are

(46) $$\tan x = \frac{\sin x}{\cos x}, \quad \cot(x) = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$$

And the secant and cosecant functions are

(47) $$\sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}.$$

Example 2.11. When is the function $\tan x$ well-defined? Use the quotient rule to compute the derivative

(48) $$(\tan x)'.$$

Use this to find the equation of the line tangent to the graph $y = \tan x$ at the point $x = \pi/4$.

Let's return to a real-world application of derivatives. We've discussed that if $s(t)$ is the position of a particle as a function of time then the derivative $s'(t)$ represents the velocity as a function of time:

(49) $$s'(t): \quad \text{velocity at time } t.$$

In other words, the rate of change of position is velocity.

As we say last time, we can iterate the process of taking the derivative. In other words, we can consider the second derivative $s''(t)$ This represents the rate of change of velocity as a function of time and is called the *acceleration*.

Example 2.12. Suppose a ball was thrown off of a cliff into the water at time $t = 0$ and its height (in meters) above sea level (before hitting the water) is described by the following function if time (in seconds):

(50) $$s(t) = 10 + 23t - 5t^2.$$

At what time does the ball hit the water? At what time is the ball moving the fastest? Describe the acceleration of the ball. Can you determine the horizontal displacement of the ball?

Chain rule. Now we will move onto the 'chain rule'. First, we need to remind ourselves how to compose functions. Given functions f and g such that the image of g is contained inside the domain of f, the composition $h = f \circ g$ is defined. One also write $h(x) = f(g(x))$.

As an example, consider the functions $f(x) = x + 1$ and $g(x) = \sqrt{x}$. Then

$$f(g(x)) = f(\sqrt{x}) = \sqrt{x} + 1.$$

Notice that the image of g is the non-negative real numbers. Since f is defined everywhere, the function $f \circ g$ is defined wherever g is defined, which is the non-negative real numbers.

In this example we can also consider the composition $g \circ f$ which is

$$g(f(x)) = g(x + 1) = \sqrt{x + 1}.$$

Note here that the image of f is all real numbers, so in order for the composition to be defined we must restrict f to a smaller domain. In this example, the composition $g \circ f$ is defined whenever $x \geq -1$.

Example 2.13. Write the following functions as compositions of two functions

(51) $$e^{x^2}, \quad \sqrt{\sin(x) + 3}, \quad \sin(x^2).$$

The chain rule is a result which expresses the derivative of a composition of two functions $f \circ g$ in terms of the derivatives of the original two functions f and g.

Theorem 2.14. *Suppose that f, g are differentiable functions and that the composition is well-defined. Then $f \circ g$ is differentiable and*

(52) $$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

Sometimes it is useful to introduce more variables. Write $y = g(x)$ so that in this expression x is the independent variable and y is the dependent variable. On the other hand, we write $u = f(y)$ so that in this expression y is the independent variable and u is the dependent variable. Since y depends on x and u depends on y, we see that u depends on x.

Using this notation, we sometimes write derivatives as

$$f' = \frac{du}{dy}, \quad g' = \frac{dy}{dx}.$$

Then, heuristically we can view the chain rule as the following formula

(53) $$\frac{du}{dx} = \frac{du}{dy} \frac{dy}{dx}.$$

Example 2.15. Find the derivative of

(54) $$h(x) = \sqrt{e^x + 1}.$$

Here we write $y = g(x) = e^x + 1$ and $f(y) = \sqrt{y}$. Then $h(x) = f(g(x))$. We compute $g'(x) = e^x$ and $f'(y) = \frac{1}{2\sqrt{y}}$ so that via the chain rule

(55) $$h'(x) = f'(g(x))g'(x) = \frac{e^x}{2\sqrt{e^x + 1}}.$$

Today we will continue with more examples involving the chain rule. Recall that the chain rule says that if $h = f \circ g$ is a composition of two functions then

(56)
$$h'(x) = f'(g(x)) \cdot g'(x).$$

Example 2.16. A function $f(x)$ is *even* if $f(-x) = f(x)$ and is *odd* if $f(-x) = -f(x)$.

- If f is even, is the derivative f' even, odd, or neither?
- If f is odd, is the derivative f' even, odd, or neither?

Example 2.17. Express the function

(57)
$$h(x) = \frac{1}{(2x^2 + 3)^{10}}$$

as a composition of two functions. Compute $h'(x)$ using chain rule.

Let's see an example of chain rule where we do not necessarily know the exact form of one of the functions involved.

Example 2.18. Suppose that f is differentiable and satisfies $f'(0) = 2$, $f'(1/2) = 1/3$. Let

(58)
$$h(x) = f(\sin x).$$

Find $h'(0)$ and $h'(\pi/6)$.

Example 2.19. Let

(59) $$h(x) = x\sqrt{5 - x^2}.$$

(a) Find $h'(x)$.
(b) Determine the equation of the lines tangent to the graph at the values $x = 1$ and $x = -2$.
(c) At which points (if any) do the lines in part (b) intersect? Express your answer as an ordered pair.

Also, chain rule can be used to find limits.

Example 2.20. Find the following limit (if it exists)

$$(60) \qquad \lim_{h \to 0} \frac{\sqrt{4 + \sin(h)} - 2}{h}.$$

Up until this point we have mostly been studying functions which are defined explicitly $y = f(x)$. Sometimes, in practical situations, we are know that a variable y depends on a variable x, but the exact dependence is determined *implicitly*. For example, we can study an equation of the form

(61) $$x^2 + y^2 = 1.$$

In this plane, this equation represents the circle. On the other hand, if we think about x as the independent variable, this equation does not determine a function $y = f(x)$ in a unique way. In general, we would need some more information to determine y as a function of x (in this example, assuming that y is non-negative is enough). In this situation, we say that y depends *implicitly* on x.

Even if variables depend implicitly on each other, we can still ask for rates of change of one variable with respect to the other variable. Let's think about the example and ask about the derivative dy/dx.

The first step is to think about $y = y(x)$ depending on x. Then the equation is

(62) $$x^2 + (y(x))^2 = 1.$$

We then take the derivative of both sides with respect to x:

(63) $$\frac{d}{dx}\left(x^2 + y(x)^2\right) = \frac{d}{dx}(1).$$

The right hand side is zero. The left hand side is

(64) $$2x + 2y(x)y'(x) = 2x + 2y(x)\frac{dy}{dx}$$

as we can see by applying the chain rule. Thus, the derivative of the equation is

(65) $$x + y(x)y'(x) = 0.$$

Next, we solve for the derivative

(66) $$y'(x) = -\frac{x}{y(x)}.$$

as long as $y(x) \neq 0$. What is happening when $y = 0$?

Example 2.21. What are the slopes of the line tangents to the circle at $x = \frac{1}{2}$.

Example 2.22. Suppose that the variables x, y satisfy

(67) $$x^2 + y^3 = 1.$$

Find $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$.

Example 2.23. Find the equations for the vertical and horizontal tangent lines to the graph described by the following equation

(68) $$x^2 + 2y^2 = xy.$$

Let's begin with one more example related to implicit derivatives.

Example 2.24. Find equations for lines tangent to the graph

$$(69) \qquad\qquad y^2 - 3xy = 2$$

when $x = 1/3$.

Next, we will move onto derivatives of logarithms. The natural logarithm $\ln x$ is, by definition, the inverse to the exponential function e^x. Recall that two functions $f(x), g(x)$ are said to be inverses of one another if

$$(70) \qquad\qquad f(g(x)) = x, \quad \text{and} \quad g(f(x)) = x.$$

Thus, the defining properties of the natural logarithm are

$$(71) \qquad\qquad e^{\ln x} = x, \quad \text{and} \quad \ln e^x = x.$$

Some key identities to keep in mind when working with logarithms include $\ln(ab) = \ln a + \ln b$ and $\ln(a^r) = r \ln a$.

Just the knowledge of the properties in (??) will allow us to nail down the derivative of $\ln x$ using the rules that we know. First, introduce a dependent variable $y = \ln x$ into the first expression above:

$$(72) \qquad\qquad e^y = x.$$

Next, we take the implicit derivative with respect to x to get

$$(73) \qquad\qquad e^y y' = 1.$$

In other words, $y' = e^{-y}$. Substituting $y = \ln x$ we then obtain an explicit expression for the derivative.

Proposition 2.25 (Derivative of natural log).

$$(74) \qquad\qquad (\ln x)' = e^{-\ln x} = \frac{1}{x}.$$

Example 2.26. Use the second expression in (??) to 'rederive' the formula $(e^x)' = e^x$.

The expression $\ln x$ is only defined for $x > 0$. On the other hand, the expression $\ln |x|$ is defined for any $x \neq 0$.

Example 2.27. Use the chain rule to show that for all $x \neq 0$ one has

$$(75) \qquad\qquad (\ln |x|)' = \frac{1}{x}.$$

Example 2.28. Compute the derivative of $\ln \left(\sqrt{x+1} \right)$ in two ways.

A simple application of chain rule shows the following.

Proposition 2.29. *Suppose that f is differentiable and $\ln(f(x))$ exists. Then*

$$(76) \qquad\qquad (\ln(f(x)))' = \frac{f'(x)}{f(x)}.$$

Next, we consider functions which are like exponentials but where we use a different base. For example, consider the function $f(x) = 2^x$. To compute the derivative we introduce the dependent variable $y = 2^x$ and write this expression as

$$(77) \qquad\qquad y = 2^x \leftrightarrow \ln y = \ln(2^x) = x \ln 2.$$

Thus, the original equation $y = 2^x$ is equivalent to the implicit representation

$$(78) \qquad\qquad \ln y = x \ln 2.$$

We then apply the derivative to get

$$(79) \qquad\qquad \frac{y'}{y} = \ln 2.$$

In other words $y' = y \ln 2 = 2^x \ln 2$, so that

$$(80) \qquad\qquad (2^x)' = 2^x \ln 2.$$

Without much more difficulty one can show.

Proposition 2.30. *For $0 < b$ and $b \neq 1$ one has*

$$(81) \qquad\qquad (b^x)' = b^x \ln b.$$

Example 2.31. Find the slope of the line tangent to the graph of $f(x) = x^x$ at $x = 1$. Does this graph have any horizontal tangent lines?

PRACTICE FOR MIDTERM 1

The following list is a non-exhaustive of problems on material that we have covered so far. Don't expect to have time to complete all of these problems, this is just meant to be a large source of practice problems in case you run out of other material to prepare with.

- §2.2, 18.
- §2.3, 19-70, 72, 73.
- §2.4, 8, 9, 21-44.
- §2.5, 17-50.
- §2.6, 17-20.
- §3.1, 33-42.
- §3.2, 21-30, 35-40.
- §3.3, 19-40.
- §3.4, 19-60, 61-64, 76-81.
- §3.5, 23-50.
- §3.6, 21-30.
- §3.7, 15-24, 26-76.
- §3.8, 13-40.
- §3.9, 15-48.
- §3.10, 13-22.
- §3.11, 23-30, 42.

Let's begin with an exercise about extrema.

Example 2.32. Find the absolute extrema for the function

(82)
$$f(x) = \frac{2x}{1 + x^2}$$

on the interval $[-2, 2]$.

The **mean value theorem** is one of the most important theorems in calculus. It's the main player behind much of the techniques we will learn in the upcoming weeks about describing the behavior of functions.

An easy case of the mean value theorem is Rolle's theorem. Consider a function f which is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b). Suppose, in addition, the function satisfies $f(a) = f(b)$. In this situation, we can conclude that there is at least one point c with $a < c < b$ such that $f'(c) = 0$.

Example 2.33. Can you give a visual proof of this result? Consider the function $f(x) = 1 - x^2$ defined on the interval $[-1, 1]$.

The mean value theorem is just a skew, or rotated version, of Rolle's theorem. Suppose we take away the condition that $f(a) = f(b)$. We can still consider the average rate of change of the function f on the interval $[a, b]$:

(83)
$$\frac{f(a) - f(b)}{a - b}.$$

Remember, this quantity represents the average slope of tangent lines to the function on the interval.

Theorem 2.34. *Consider a function f which is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b). Then, there exists a number c with $a < c < b$ such that*

(84)
$$f'(c) = \frac{f(a) - f(b)}{a - b}.$$

Example 2.35. Check the mean value theorem for the function $f(x) = x + \frac{1}{x}$ on the interval $[1, 3]$. That is, find the number c which is guaranteed by the mean value theorem.

A nice corollary of the mean value theorem is the following.

Corollary 2.36. *Suppose that $f'(x) = 0$ for all x. Then f is a constant function.*

Proof. Let $a < b$ be two numbers. By the mean value theorem, there exists a c with $a < c < b$ with the property that

$$(85) \qquad f'(c) = \frac{f(a) - f(b)}{a - b}.$$

But $f'(c) = 0$ by assumption, therefore $f(a) = f(b)$. $\qquad\qquad\square$

Example 2.37. This problem involves calculus.

- Show that $\arctan x$ and $\arctan(1/x)$ differ by a constant.
- Show that $\arctan x + \arctan(1/x) = \pi/2$ for $x > 0$.

Let $f(x) = \arctan x + \arctan 1/x$ and consider it on the domain $x > 0$. Then,

$$(86) \qquad f'(x) = \frac{1}{1+x^2} + \frac{-1/x^2}{1+(1/x)^2} = \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0.$$

Thus, by the corollary above we conclude that $f(x) = K$ for some constant K for all $x > 0$. Similarly, we can show that $f(x) = K'$ for some constant K' and all $x < 0$.

Take $x = 1$, then we see

$$(87) \qquad K = f(1) = \pi/4 + \pi/4 = \pi/2.$$

Example 2.38. Suppose that $f(x)$ is defined and differentiable everywhere. If $f'(x) > 0$ for all x is it true that

$$(88) \qquad \lim_{x \to \infty} f(x) = \infty?$$

If not, try to find a counterexample.

Last time we covered the first derivative test. Let's do an example.

Example 2.39. Find the local and absolute extrema of the function

$$(89) \qquad f(x) = \frac{x^2}{1 - x^2}$$

Example 2.40. Find the local and absolute extrema of the function

$$f(x) = x^x \tag{90}$$

defined for $x > 0$.

We've learned what the derivative tells us about the shape of a graph. What about the second derivative? This controls the feature called *concavity.*

- A function is **concave up** on an interval if $f'(x)$ is increasing on the interval.
- A function is **concave down** on an interval if $f'(x)$ is decreasing on the interval.
- A point $x = c$ where f changes concavity is called an **inflection point.**

An easy reformulation of the first two points is the following:

- If $f''(x) > 0$ on some interval, we say that f is concave up on that interval.
- If $f''(x) < 0$ on some interval, we say that f is concave down on that interval.
- If $f''(c) = 0$ and $f''(c)$ changes sign as we go through $x = c$ then $x = c$ is an inflection point of the function.

Example 2.41. Describe the concavity and the inflection points of the following functions.

(a) $f(x) = \arctan(x)$.
(b) $f(x) = \sqrt{x} \ln x$ on the domain $x > 0$.

Next we move on to the second derivative test.

Suppose that $x = c$ is a critical point of a function f which satisfies $f'(c) = 0$.

- If $f''(c) > 0$ then f has a local minimum at $x = c$.
- If $f''(c) < 0$ then f has a local maximum at $x = c$.

If $f''(c) = 0$ then the test is inconclusive—the function may have a local min/max or neither.

Example 2.42. Use the second derivative test to locate and characterize the local extrema of the function

(91) $$f(x) = \frac{e^x}{x+1}$$

defined for $x \neq 1$.

We have many techniques to finding local extrema. Here is one that can be used to locate absolute extrema.

Suppose that $f(x)$ is defined on an interval and has exactly one local extrema at $x = c$.

- if it is a local maximum at $x = c$ then it is an absolute maximum.
- if it is a local minimum at $x = c$ then it is an absolute minimum.

Let's turn to an example that ties in a lot of the concepts we've discussed.

Example 2.43. Consider the polynomial $g(x) = x^2 - x + 1$. Using calculus, show that $g(x)$ has no (real) roots.

Notice that $\lim_{x \to \pm\infty} g(x) = \infty$. Next, we find the local extrema. We have

(92) $$g'(x) = 2x - 1,$$

so there is a critical point at $x = 1/2$. But, $g''(1/2) > 0$, thus $g(x)$ attains a local minimum at $x = 1/2$. Since this is the only extrema, this is in fact an absolute minimum of the function.

Now, notice that

(93) $$g(1/2) = 1/4 - 1/2 + 1 > 0.$$

Thus, $g(x)$ has no roots.

Here is a more involved example that uses similar ideas.

Example 2.44. Consider the function $f(x) = x \ln x - \ln x - x$ defined for $x > 0$. How many roots does the function $f(x)$ have? (Hint: use calculus.)

Notice that

(94)
$$\lim_{x \to 0^+} f(x) = \infty$$

and

(95)
$$\lim_{x \to \infty} f(x) = \infty.$$

Based on this, it is possible that f could have zero, two, four, etc. roots. That is, we know it has an even number of roots. Let's nail this down.

Let's determine if the function has any local extrema. The derivative of $f(x)$ is

(96)
$$f'(x) = \ln x - \frac{1}{x}.$$

Notice that

(97)
$$\lim_{x \to 0^+} f'(x) = -\infty$$

and

(98)
$$\lim_{x \to \infty} f'(x) = \infty.$$

In particular we see that the derivative $f'(x)$ must have at least one zero.

Let's next compute the second derivative

(99)
$$f''(x) = \frac{1}{x} + \frac{1}{x^2}.$$

Notice that this function is always positive. Thus, $f'(x)$ is increasing everywhere. In particular, we see that there is a unique solution to $f'(x) = 0$, let's call it $x = c$. Sine $f''(c) > 0$ we see that f has a local minimum (hence absolute minimum) at $x = c$.

Shuffling terms around, we see that c satisfies

(100)
$$\ln(c) - \frac{1}{c} = 0.$$

Since $c > 0$, this is equivalent to $c \ln c - 1 = 0$. Thus

(101)
$$f(c) = c \ln c - \ln c - c = 1 - \frac{1}{c} - c = \frac{1}{c}\left(-c^2 + c - 1\right).$$

We showed in the last example that $-c^2 + c - 1 < 0$ for all c. Thus $f(c) < 0$.

We conclude that $f(x)$ has two roots.

We move on to optimization problems which generally ask the following:

- *What is the minimum or maximum value of a quantity subject to some class of constraints?*

Here is an example. Suppose we have a rectangle of lengths x and y. Neither x nor y are fixed, but we do impose the constraint that

$$(102) \qquad\qquad x + y = 1.$$

Then, for what values of x, y is the area of the rectangle maximized and minimized?

To solve this, we recall that the area is

$$(103) \qquad\qquad A = xy.$$

Notice that this is a function of two variables x, y. But, the constraint tells us how x, y are related, namely $y = 1 - x$. Plugging this in, we see that the area is a function only of x

$$(104) \qquad\qquad A(x) = x(1 - x) = x - x^2.$$

The critical points of $A(x)$ are ones where $A'(x) = 0$ which is the condition $1 - 2x = 0$. So there is only one critical point at $x = 1/2$. Since $A''(1/2) = -2$ this critical point is an absolute maximum. Notice that for $x = 1/2$ we have $y = 1/2$ by the constraint. This is a square, and the area is $A = 1/4$.

What about the absolute minimum. Naively, we would say that the function $A(x) = x - x^2$ has no absolute minimum. But we need to be careful since we are thinking of x as a length, so we are only considering the function on the domain $x > 0$ and $x < 1$ (the last condition arises from the constraint and the fact that $y > 0$.) Thus, we still see that there is no absolute minimum (we can take x to be closer and closer to zero or one and get a smaller and smaller area).

Example 2.45. Find the point on the curve $y = x^2$ that is closest to the point $(18, 0)$. (Hint: You may use the following fact

$$2 \cdot 2^3 + 2 - 18 = 0 \quad .)$$

(105)

Recall the formula for the distance between $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$

(106) $$d(P_1, P_2)^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2.$$

A point on the curve $y = x^2$ is of the form $P_1 = (x, x^2)$. We want to minimize the distance to the point $P_2 = (18, 0)$. As a function of x, the distance squared is

(107) $$D(x) = d(x)^2 = (x - 18)^2 + (x^2)^2 = (x - 18)^2 + x^4.$$

The derivative of this distance squared is

(108) $$D'(x) = 2(x - 18) + 4x^3 = 2(2x^3 + x - 18).$$

We want to describe solutions to $D'(x) = 0$, so it suffices to look at the equation $2x^3 + x - 18 = 0$. One method is to observe that this can be factored like

(109) $$(x - 2)(2x^2 + 4x + 9) = 0$$

and hence the only solution is $x = 2$.

Thus the point $(2, 4)$ is the closest point on the curve of $y = x^2$ to the point $(18, 0)$.

NOVEMBER 4, 2022

Another optimization problem.

Example 2.46. A fence that is 6 feet high is running parallel to a house and is 3 feet away from the house. What is the minimum length of a ladder which clears the fence and touches the house?

Let's denote by x the distance, in feet, from the base of the ladder to the fence. Then the length between the base of the ladder and the house is $x + 3$. Let y be the height from the top of the ladder to the ground. Then, the length of the ladder ℓ can be computed as

$$(110) \qquad \ell^2 = (x + 3)^2 + y^2.$$

The quantities x, y are related by the following geometric constraint. Using similar triangles we see that we have

$$(111) \qquad \frac{y}{x + 3} = \frac{6}{x}.$$

In other words, $y = \frac{6(x+3)}{x}$.

We can use this to express the length of the ladder just in terms of the quantity x

$$(112) \qquad \ell(x)^2 = (x+3)^2 + \frac{36}{x^2}(x+3)^2 = (x+3)^2 \left(1 + \frac{36}{x^2} \right).$$

We want to minimize the function $L(x) = \ell(x)^2$.

The first derivative is

$$
\begin{aligned}
L'(x) &= 2(x+3) \left(1 + \frac{36}{x^2} \right) + (x+3)^2 \left(-\frac{72}{x^3} \right) \\
&= (x+3) \left(2(1 + 36/x^2) - (x+3)(72/x^3) \right) \\
&= (x+3) \left(2 + 72/x^2 - 72/x^3 - 216/x^3 \right) \\
&= (x+3) \left(2 - 216/x^3 \right).
\end{aligned}
$$

We have potential critical points at $x = -3$ and $x = \sqrt[3]{108} = 3\sqrt[3]{4}$. The point $x = -3$ is not meaningful since $x > 0$ represents a length. Therefore we have only one critical point!

By computing $L''(3\sqrt[3]{4}) > 0$ (or using the first derivative test) we see that this is a local, hence absolute, minimum.

NOVEMBER 7, 2022

If we zoom in, the line tangent to a graph of a smooth function closely approximates the original function. Assume that f is smooth on an interval containing $x = a$. The slope of the line tangent to the curve $y = f(x)$ at the point $(a, f(a))$ is $f'(a)$. The equation of a line tangent to the graph at this point, in slope intercept form, is

$$(113) \qquad y - f(a) = f'(a)(x - a)$$

or

$$(114) \qquad y = f(a) + f'(a)(x - a).$$

Let

$$(115) \qquad L_a(x) = f(a) + f'(a)(x - a)$$

be the function describing this line. Then, we say that that $L_a(x)$ is the linear approximation of $f(x)$ near $x = a$ and will often write

(116)
$$f(x) \approx L(x)$$

for x near a.

Example 2.47. Find the linear approximation for the function $f(x) = \ln x$ and use it to approximate $\ln(1.1)$.

Example 2.48. Find the linear approximation of $f(x) = \tan^3(x)$ and use it to approximate $\tan(11\pi/40)$. (Notice that $11/40 \approx 1/4$.)

We are trying to approximate the value of the function $f(x) = \tan^3(x)$ near $x = \pi/4$. The derivative is

(117)
$$f'(x) = 3\tan^2(x)\sec^2(x).$$

The line $y = L(x)$ tangent to the graph at $x = \pi/4$ is

(118)
$$L(x) - f(\pi/4) = f'(\pi/4)(x - \pi/4).$$

Let's record the relevant values of the function and the derivative:

(119)
$$f(\pi/4) = 1, \quad f'(\pi/4) = 3 \cdot 1 \cdot (\sqrt{2})^2 = 6.$$

So plugging all of this in:

(120)
$$L(x) = 6(x - \pi/4) + 1.$$

Let's move on to L'Hôpital's rule. The most basic application of this rule is to compute limits of the following form

(121)
$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

where $f(a) = g(a) = 0$.

If f, g are continuous and differentiable functions near $x = a$, we can use a linear approximation to compute either of the limits $\lim_{x \to a} f(x)$ or $\lim_{x \to a} g(x)$. Take $f(x)$ for example: near $x = a$ we have seen that $f(x)$ is approximated by the linear function

(122)
$$L_f(x) = f(a) + f'(a)(x - a).$$

Notice that

(123)
$$\lim_{x \to a} f(x) = \lim_{x \to a} L(x) = f(a).$$

Similarly, there is a linear function

(124)
$$L_g(x) = g(a) + g'(a)(x - a)$$

which approximates g near $x = a$.

We can attempt to use these approximations to compute the original limit $\lim_{x \to a} \frac{f(x)}{g(x)}$. Indeed, near $x = a$ we have

(125)
$$\frac{f(x)}{g(x)} \approx \frac{L_f(x)}{L_g(x)} = \frac{f(a) + f'(a)(x - a)}{g(a) + g'(a)(x - a)}.$$

But the assumption was that $g(a) = f(a) = 0$, so

(126)
$$\frac{f(x)}{g(x)} \approx \frac{f'(a)(x - a)}{g'(a)(x - a)}.$$

Notice that for $x \neq a$ we have shown that

(127)
$$\frac{f(x)}{g(x)} \approx \frac{f'(a)}{g'(a)}$$

One can turn this into the following theorem.

Theorem 2.49. *Suppose that f, g are differentiable on an interval containing $x = a$ and assume that $g'(x) \neq 0$ on this interval for $x \neq a$. Also, assume that $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$. Then*

(128)
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$