

Vectors in \mathbb{R}^2 , \mathbb{R}^3 (and \mathbb{R}^n)

- Euclidean space \mathbb{R}^n is the set of n -tuples of real numbers

$$\{(a_1, \dots, a_n) \mid a_i \in \mathbb{R}\}$$

- On the other hand consider the set V of all "arrows" which begin at $\underline{0} = (0, \dots, 0) \in \mathbb{R}^n$ and end at some other point. We call V the set of vectors.

- There is an \approx bijection $*$ of sets

$$\mathbb{R}^n \xrightarrow{\approx} V$$

which sends a point $P = (a_1, \dots, a_n)$ to the vector which ends at P . In other words

$$P \longmapsto \overrightarrow{OP}$$

[We will review the concept of isomorphism in discussion section. Also, see Appendix A].

Write $\vec{OP} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in V$ where

$$\mathcal{P} = (a_1, \dots, a_n) \in \mathbb{R}^n.$$

Recall some familiar operations:

- Vector addition

$$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix}$$

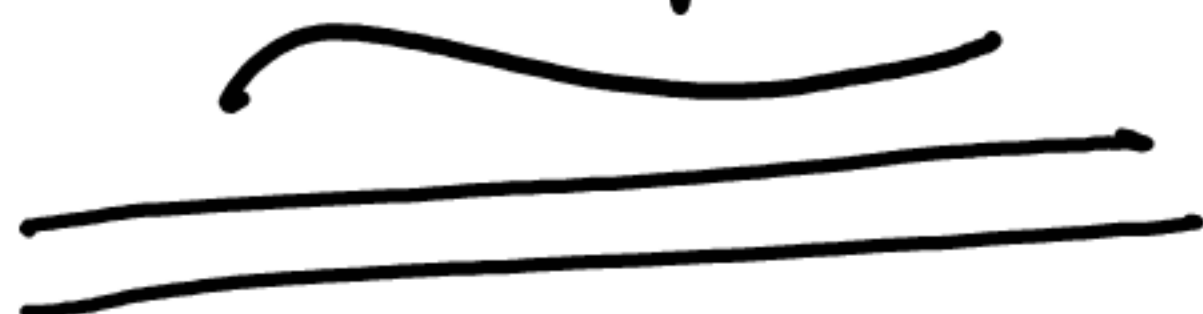
- Scalar Multiplication

If $\lambda \in \mathbb{R}$ then

$$\lambda \cdot \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \lambda a_1 \\ \vdots \\ \lambda a_n \end{bmatrix}.$$

We will now abstract these operations (and their intrinsic properties) into the notion of a

* Vector space . *



- Comment on *fields*. A field ^{\mathcal{F}} is an algebraic object for which we will do linear algebra "over". We will spend some more time w/ fields in this course, but for now we usually take $F = \mathbb{R}$ the real numbers.

[See Appendix C for a background on fields]

A field F forms the "scalars". Other examples of fields include $F = \mathbb{C}$, the complex numbers, or $F = \mathbb{F}_2 = \{0, 1\}$ (the field w/ two elements).

We move on to our main definition.

Dfn: A vector space over a field F is a set

$$V = \{ \text{the set of "vectors"} \}$$

together with operations

- Addition: if $v, w \in V$ then $v + w \in V$.

- Scalar multiplication: if $v \in V, \lambda \in F$
then $\lambda \cdot v \in V$.

These operations must satisfy the following ^{EIGHT} axioms:

VS 1) For all $v, w \in V$ have $v + w = w + v$.

VS 2) For all $v, w, u \in V$ have

$$(v + w) + u = v + (w + u).$$

VS 3) There is $0 \in V$ s.t. $v + 0 = v$
for all $v \in V$. (The zero vector)

VS 4) For all $v \in V$ there exists $\tilde{v} \in V$
s.t. $v + \tilde{v} = 0$.

VS 5) For all $v \in V$, $1 \cdot v = v$, where
 $1 \in F$ is the unit.

VS 6) For all $\lambda, \mu \in F$ and $v \in V$ have

$$(\lambda \mu) \cdot v = \lambda \cdot (\mu \cdot v).$$

VS 7) For all $\lambda \in F$, and $v, w \in V$ have

$$\lambda \cdot (v + w) = \lambda \cdot v + \lambda \cdot w$$

VS 8) For all $\lambda, \mu \in F$ and $v \in V$ have

$$(\lambda + \mu) \cdot v = \lambda \cdot v + \mu \cdot v.$$

Ex: For any integer n we can look at

$$\mathbb{F}^n = \left\{ (a_1, \dots, a_n) \mid a_i \in \mathbb{F}, i=1, \dots, n \right\}.$$

In the familiar case $\mathbb{F} = \mathbb{R}$ this is \mathbb{R}^n . Write elements of \mathbb{F}^n in "column vector" notation

$$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{F}^n$$

Then the familiar operations

$$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix}$$

$$\lambda \cdot \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \lambda a_1 \\ \vdots \\ \lambda a_n \end{bmatrix}, \quad \lambda \in \mathbb{F}.$$

Endow \mathbb{F}^n w/ the structure of a vector space over \mathbb{F} .

Q: What is the zero vector?

Ex: A sequence in \mathbb{F} is an ordered, countably infinite collection of elements of \mathbb{F} :

$$\underline{a} = (a_1, a_2, a_3, \dots)$$

where each $a_i \in \mathbb{F}$. Denote the set of all such sequences by \mathbb{F}^∞ . Then \mathbb{F}^∞ has the natural structure of a vector space over \mathbb{F} .

[Heuristically: " $\lim_{n \rightarrow \infty} \mathbb{F}^n = \mathbb{F}^\infty$ "]

Ex: A polynomial in \mathbb{F} is an expression of the form

$$a_0 + a_1 x + a_2 x^2 + \dots$$

where $a_i \in \mathbb{F}$ and all but finitely many of the a_i 's are zero. [So only a finite power of x can appear.]

Then, we can define the "addition"

$$\begin{aligned} & (a_0 + a_1 x + a_2 x^2 + \dots) + (b_0 + b_1 x + b_2 x^2 + \dots) \\ &= (a_0 + b_0) + (a_1 + b_1) x + (a_2 + b_2) x^2 + \dots \end{aligned}$$

And "scalar multiplication"

$$\begin{aligned} & \lambda \cdot (a_0 + a_1 x + a_2 x^2 + \dots) \\ &= \lambda a_0 + \lambda a_1 x + \lambda a_2 x^2 + \dots \end{aligned}$$

These operations satisfy VS1 - VS8. So the set of all polynomials form a vector space. We denote it

$$\mathbb{F}[x]$$

[Can you extend this to polynomials in more than one variable :

$$\mathbb{F}[x, y, z, \dots] \quad ? \quad]$$