April 22

Let M be a Riemannian 4-manifold. Locally, we have the bundle S_- (globally, we would need a spin structure).

Look at $S_-^* \setminus \underline{0}$ = dual bundle minus zero section.

Any section $\varphi \in \Gamma(S_-)$ defines

$$\varphi^\vee \in C^\infty(\mathrm{Tot}(S_-^* - \underline{0})).$$

The key idea of Atiyah-Hitchin-Singer is that:

$\bar{\partial}_1 \varphi = 0 \iff d\varphi^\vee$ lies in a sub-bundle

$$V \subset T^2(\mathrm{Tot}(S^* - \underline{0})) \otimes_{\mathbb{R}} \mathbb{C}$$

Moreover V is involutive and determines a cplx structure on $S_-^* \setminus \underline{0} \iff W_- = 0$.

The point is that $V \cong$ bundle of $(1,0)$ forms on $S_-^* \setminus \underline{0}$.

In other words $\bar{\not{\partial}}_1 \varphi = 0 \iff \bar{\partial} \varphi^\vee = 0$.

Since $\varphi^\vee$ is linear in the fiber of $S_-^* \setminus 0$

$\Rightarrow \varphi^\vee \in H^0(\underbrace{\mathbb{P}(S_-^*)}_{\text{twistor space} := Z(M)}, \mathcal{O}(1))$

Similarly:

$$\left\{ \begin{array}{c} \text{solutions to} \\ \not{\partial}_m \varphi = 0 \\ \text{on } M \end{array} \right\} \cong H^0(Z(M), \mathcal{O}(m)).$$

Since $\varphi \in \Gamma(S^m S_-)$ defines a fn $\varphi^\vee$ of polynomial degree m on fibers of S_-^*.

- Let's unpack this "transformation" between solutions to conformal PDE's on M and sheaf cohomology on $Z(M)$.

- The fibers of $Z(M) \longrightarrow M$ are projective lines in $Z(M)$.

$(S_-^*)_x \subset Z(M)$ has liner coordinates λ_1, λ_2.

By integrability, the co-normal bundle N^* to this fiber is spanned by $\sigma_1^\vee, \sigma_2^\vee$ where

$$\sigma_\alpha^\vee = \sum_i \langle e_i \cdot \psi_\alpha, \varphi \rangle e_i$$

where $\varphi \in (S_-^*)_x$, $\psi_\alpha \in (S_+)_x$.

These sections provide a trivialization of N^* over $(S_-^*)_x$. Since the sections are liner in φ they provide trivialization

$$\mathcal{O}(1) \otimes N^* \simeq \mathcal{O}^{\oplus 2}$$

over $Z(M)$.

$$\Rightarrow \quad N \simeq \mathcal{O}(1) \oplus \mathcal{O}(1).$$

↑ Normal bundle to fiber of $\begin{array}{c} Z(M) \\ \downarrow \\ M \end{array}$.

Here $\mathcal{O}(1)$ is the vector bundle on

$$\mathbb{P}(S_-^*)$$

whose underlying principal $\mathbb{C}^\times$-bundle is $S_-^* - \underline{0}$.

Given $s \in H^0(\mathbb{P}(S_-^*), \mathcal{O}(m))$ we can restrict

$$s\Big|_{\mathbb{P}(S_-^*)_x \cong \mathbb{P}^1} \in H^0(\mathbb{P}(S_-^*)_x, \mathcal{O}(1))$$

$$\updownarrow \cong$$

$$(S_-)_x .$$

Varying $x \in M$ we get

$$Ts \in \Gamma(M, S_-)$$

$$\text{s.t.} \quad \bar{\partial}_m (Ts) = 0 .$$

<u>Q</u>: What about $H^1(Z(M), \mathcal{O}(-m-2))$ for $m \geq 0$?

I.e. what sort of conformal geometry does this higher sheaf cohomology "see"?

By Serre duality

$$H^1(\mathbb{P}^1, \mathcal{O}(-m-2)) \simeq (H^0(\mathbb{P}^1, \mathcal{O}(m)))^*$$

$\Rightarrow$ Restriction to twistor fiber defines

$$T : H^1(Z, \mathcal{O}(-m-2)) \longrightarrow \Gamma(S^m S_-).$$

<u>Thm</u>: This defines isomorphism

$$T : H^1(Z, \mathcal{O}(-m-2)) \xrightarrow{\simeq} \left\{ \begin{array}{l} \psi \in \Gamma(S^m S_-) \\ \not{\partial}_m \psi = 0 \end{array} \right\}.$$

To prove this, we reformulate. Take twistor space

$$Z = Z(M).$$

There is a complex 4-dimensional manifold $M^{\mathbb{C}}$ of projective lines in Z.

We recover M by looking at the real

part of this complex manifold, $M \subset M^{\mathbb{C}}$.

There is a projectivezed spin bundle

$$\underbrace{\mathbb{P}(S_-^*)^{\mathbb{C}}}_{Y =} \text{ over } M^{\mathbb{C}} = \left\{ \begin{matrix} \text{cplx lines} \\ \text{in } Z \end{matrix} \right\}.$$

5-dim^l complex manifold.

We have:

$$\begin{tikzcd} & Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ Z & & M^{\mathbb{C}} \end{tikzcd}.$$

- Here, the p_1 is defined since the fiber over $x \in M^{\mathbb{C}}$ is naturally identified w/ the corresponding line L_x in Z.

Now consider $H = \mathcal{O}(1)$ as a holomorphic vector bundle on Z.

$\rightsquigarrow$ $H^1(Z_x, H^{-m-2})$ is constant rank as x varies in Z.

ns) bundle S_-^m on M. So, if $\alpha \in H^1(Z, H^{-m-2})$ get section $T\alpha$ of S_-^m. by restriction.

But, restriction to arbitrary lines defines a holomorphic vector bundle W_m on $M^{\mathbb{C}}$, and a holomorphic section of W_m. The restriction of this holomorphic section to real pts is exactly $T\alpha$.

$$
\begin{tikzcd}
H^1(Z, H^{-m-2}) \arrow[r, "p_1^*"] \arrow[ddr, dotted, bend right, "T"'] & H^1(Y, p_1^* H) \arrow[d, "\text{restrict to fibors of } p_2"] \\
& H^0(M^{\mathbb{C}}, W_m) \arrow[d, "\text{restrict to real pts}"] \\
& \Gamma(M, S_-^m)
\end{tikzcd}
$$

$x \in S^4 \rightsquigarrow \pi^{-1}(x) \simeq \mathbb{CP}^1 \subset \mathbb{CP}^3$ has normal bundle

$$\begin{array}{c} N \simeq \mathcal{O}(1) \oplus \mathcal{O}(1) \\ \downarrow \\ \mathbb{CP}^1 . \end{array}$$