September 6

Welcome to MA 721. The course starts w/ a brief excursion into the world of topological manifolds.

A top^l manifold is, roughly, a top^l space (henceforth just called "space") which looks like $\mathbb{R}^n$, locally.

This is not a place where we can speak of derivatives, like in calculus, but its a step along the way.

Next week, we will introduce the concept of a smooth structure which will allow us to generalize all of those familiar ideas in calculus/analysis.

First, here is a rapid review of topology.

- A top^{l} space is a set X equipped w/ a collection of <u>open</u> subsets s.t.

  1) $\emptyset, X$ are open.

  2) $\bigcup_{\alpha} U_\alpha$ open if $\{U_\alpha\} \subset X$ open.

  3) $\bigcap_{i=1}^{N} U_i$ open if $\{U_i\}_{i=1}^{N} \subset X$ open.

- A function $f: X \to Y$ is <u>continuous</u> if $f^{-1}(U)$ is open for all open $U \subset Y$.

- A <u>homeomorphism</u> is a cts isomorphism.

- The <u>subspace</u> topology on a subset $Y \subset X$ of a space is s.t. $U \subset Y$ open $\Leftrightarrow$ $\exists\, \tilde{U} \subset X$ open s.t. $U = \tilde{U} \cap Y$.

Here are some further ideas from topology we will use this week. (By "space" we will mean a topological space.)

We say a space M is:

1) <u>Hausdorff</u>: if $\forall$ $x \neq y \in M$ there are opens $\mathcal{U}, V \subset M$ s.t.

$$x \in \mathcal{U}, \quad y \in V, \quad \mathcal{U} \cap V = \emptyset .$$

2) <u>Second countable</u>: if $\exists$ a countable topological basis for M.

3) <u>Locally Euclidean</u>: if $\forall$ $x \in M$ $\exists$ a nbd $\mathcal{U}$ of x s.t.

$$\underset{\substack{\uparrow \\ \text{homeomorphic}}}{\mathcal{U} \cong \hat{\mathcal{U}}} .$$

where $\hat{U}$ is an open subset $\hat{U} \subset \mathbb{R}^n$ for same n. (We require that n be the same for all $x \in M$).

Dfn: A *topological manifold* is a space satisfying ① - ③ above.

Perhaps the most important property is ③, let's unpack it. We say a coordinate chart at $x \in M$ is a pair

$$\left(U,\ \varphi : U \longrightarrow \mathbb{R}^n \right)$$

where

- $U \ni x$ is an open subset, containing the point x.
- $\varphi : U \longrightarrow \mathbb{R}^n$ is a cts map s.t. $\varphi : U \xrightarrow{\cong} \varphi(U)$ is a homeomorphism.

Thus (3) is equivalent to the existence of coordinate charts at every $x \in M$.

A coordinate chart gives local coordinates $\{x^i\}_{i=1}^n$ where $x^i : U \to \mathbb{R}$ are :

$$\varphi(p) = \left(x^1(p), \dots, x^n(p) \right) \in \mathbb{R}^n .$$

Ex : $\mathbb{R}^n$, and any open subset of $\mathbb{R}^n$ is a top^l manifold.

Ex : $S^n = \{ x \in \mathbb{R}^{n+1} \mid \|x\| = 1 \}$ is a

top^l manifold. Being a subspace of $\mathbb{R}^{n+1}$ it is automatically Hausdorff and 2^{nd} ctble. We now construct coordinate charts. Let

$$U^{\pm} = \left\{ (x^1 \dots x^{n+1}) \mid x^{n+1} \gtrless 0 \right\} .$$

and define the charts

$$\varphi^{\pm} : U^{\pm} \cap S^n \longrightarrow \mathbb{R}^n$$

$$(x^1, \dots, x^{n+1}) \longmapsto (x^1, \dots, x^n) .$$

* check that this is a homeomorphism onto its image.

More examples.

Prop: If M, N are top'l manifolds, then so is $M \times N$.

Pf: Sps $(p, q) \in M \times N$, and let

$$\begin{aligned} \varphi : \; & U \longrightarrow \mathbb{R}^m \\ & \;\,\cup\!\!\!\shortmid \\ & \;\, p \end{aligned}$$

$$\begin{aligned} \psi : \; & V \longrightarrow \mathbb{R}^n \\ & \;\,\cup\!\!\!\shortmid \\ & \;\, q \end{aligned}$$

be coordinate charts for M, N.

Then

$$\varphi \times \psi : U \times V \longrightarrow \mathbb{R}^{n+m}$$

is a coordinate chart for $(p,q) \in M \times N$. $\boxed{//}$

We won't spend any more time in the wild world of top^{l} manifolds, and we now turn to smooth structures.

In calculus, we typically study functions

$$f : \underset{\mathbb{R}^n}{\overset{\cap}{U}} \longrightarrow \mathbb{R}.$$

We say f is smooth if all partial derivatives

$$\partial^k_{x_i} f : U \longrightarrow \mathbb{R} \quad \forall\, i, k.$$

exist and are continuous.

Given a top^l manifold M what does it mean for a fn

$$f : M \longrightarrow \mathbb{R}$$

to be smooth ? (Think about what sort of properties smoothness should have)

Attempt : We know M admits a cover by coordinate charts. If $p \in M$, let $(U \ni p, \phi)$ be a chart

$$\phi : U \longrightarrow \mathbb{R}^n$$

Then the composition

$$\begin{tikzcd}
\phi(U) \arrow[r, "\phi^{-1}"] \arrow[rr, bend right, "f \circ \phi^{-1}"'] & U \arrow[r, "f"] & \mathbb{R} \\
\mathbb{R}^n \arrow[u, phantom, "\subseteq" rotate=90]
\end{tikzcd}$$

is of the form we studied in calculus. So, one dfn of "smooth" would be to say that $f \circ \phi^{-1}$ is smooth for any coordinate chart (U, ϕ).

Problem: Why is this independent of the chosen chart?

In general it is not! So we need some refinement of a covering by charts ...

Let's give a precise definition. Sps M is a top^{l} manifold and let

$$(U, \phi), (V, \psi) \qquad U \cap V \neq \emptyset$$

be charts. The transition map is the composition:

$$\begin{tikzcd} \phi(U \cap V) \arrow[r, "\phi^{-1}"] \arrow[rr, bend right, "\psi \circ \phi^{-1}"'] & U \cap V \arrow[r, "\psi"] & \psi(U \cap V) \end{tikzcd}$$

$\mathbb{R}^n \supset \phi(U \cap V)$, $\psi(U \cap V) \subset \mathbb{R}^n$.

We say (U, ϕ), (V, ψ) are <u>smoothly compatible</u> if $\psi \circ \phi^{-1}$ is a smooth homeomorphism

$$\text{smooth homeomorphism} = \text{diffeomorphism}.$$

We will continue w/ smoothness next time.