Recall, the average velocity of a particle over the time interval $t_1 \le t \le t_2$ is

$$\frac{\Delta x}{\Delta t} = \frac{x(t_2) - x(t_1)}{t_2 - t_1}.$$

Ex: $x(t) = t^3 - 2t$. (in meters) Then the average velocity from $t_1 = 1$ to $t_2 = 2$ seconds is

$$\frac{\Delta x}{\Delta t} = \frac{(8-4) - (1-2)}{2-1}$$

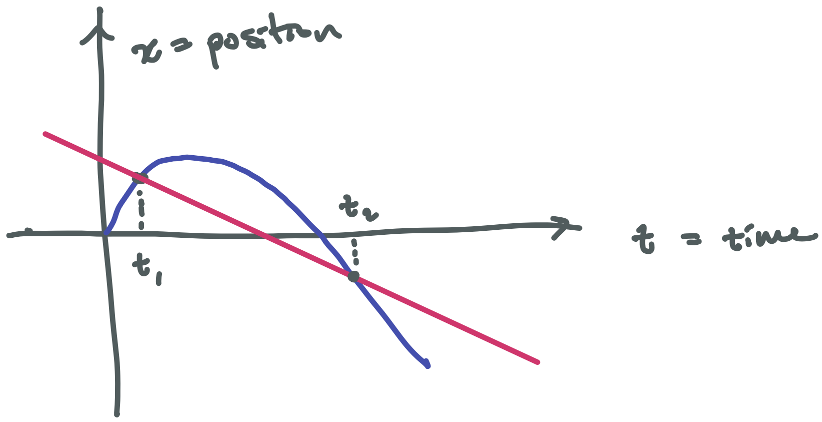
$$= 5 \text{ meters per second}$$

$$\underbrace{\text{meters per second}}_{m/s}.$$

Graphically, average velocity can be understood as the slope of a line. This is the so-called "secant" line through the points

$$(t_1, x(t_1)) \text{ and } (t_2, x(t_2))$$

Slope is $\frac{\text{rise}}{\text{run}} = \frac{x(t_2) - x(t_1)}{t_2 - t_1} =$ avg velocity!



MLM example.

- Instantaneous velocity. So far there is no calculus. This comes up when we go from

$$\text{average} \xrightarrow[\text{limit}]{} \text{instantaneous}$$

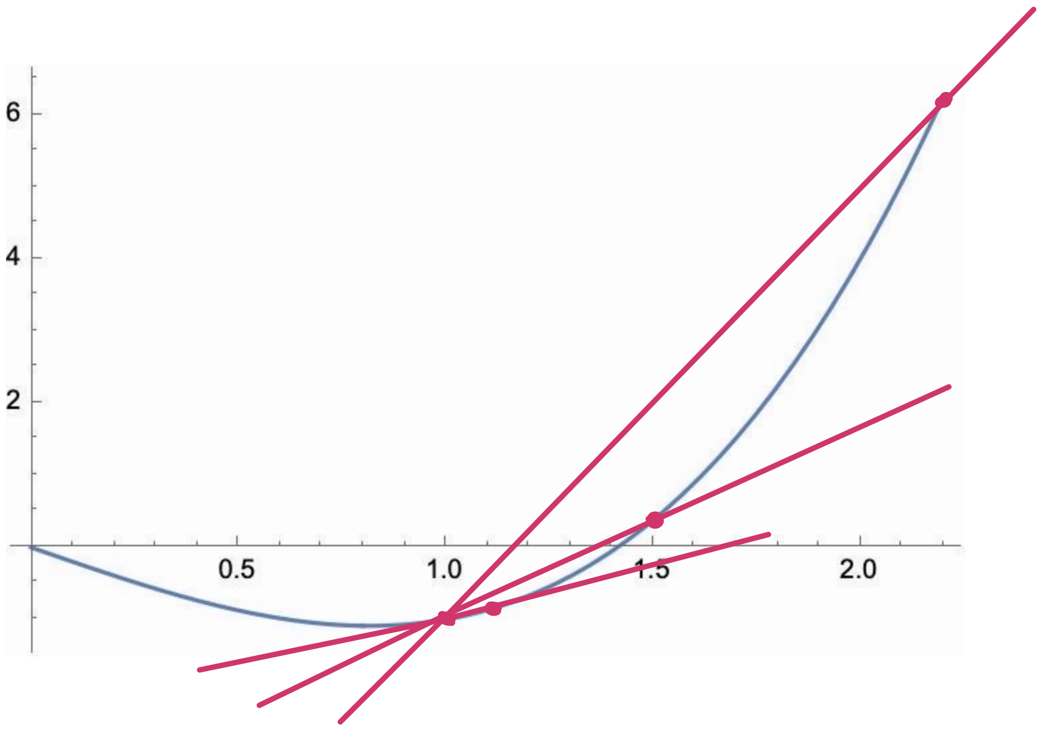
<u>Idea</u>: Instantaneous velocity is average velocity as the time interval gets smaller and smaller....

<u>Ex</u>: $x(t) = t^3 - 2t$. Let's compute the average velocity for the time interval

$$[1, t]$$

as t gets closer and closer to 1.

	Interval	$\frac{\Delta x}{\Delta t}$
$t=2$	$[1, 2]$	5
$t=1.5$	$[1, 1.5]$	2.75
$t=1.1$	$[1, 1.1]$	1.31
$t=1.01$	$[1, 1.01]$	1.0301



Evidence: The instantaneous velocity at $t=1$ is 1 m/s.

MLM exercise

The general formula for the average velocity of $x(t) = t^3 - 2t$ for the time interval

$$[1, t]$$

is
$$\frac{\Delta x}{\Delta t} = \frac{x(t) - x(1)}{t-1}$$
$$= \frac{t^3 - 2t + 1}{t - 1}.$$

So the slope of the secant line is as a function of t.

Notice that if $t \neq 1$ then

$$\frac{t^3 - 2t + 1}{t-1} = \frac{(t-1)(t^2+t-1)}{t-1}$$
$$= t^2 + t - 1.$$

So if we pretend that we can take $t = 1$, then we also find

$$1^2 + 1 - 1 = 1 \ m/s.$$

The notion of a limit justifies this procedure of making sense of expressions like

$$\frac{t^3 - 2t + 1}{t - 1} \quad \text{at} \quad t = 1.$$

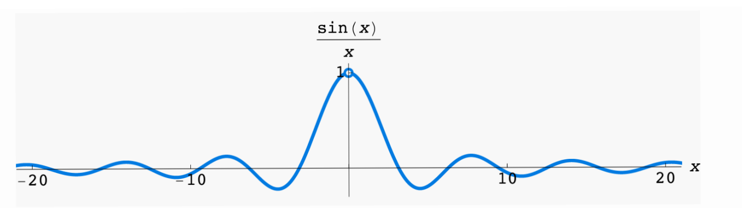
In general we will look at

$$\lim_{x \to a} f(x)$$

$=$ the limit of the function $f(x)$ as x approaches a.

$\underline{Ex}$: $f(x) = \dfrac{\sin x}{x}$. Want to make sense of

$$\lim_{x \to 0} \frac{\sin x}{x}$$



Graphically we see that $\displaystyle\lim_{x \to 0} \frac{\sin x}{x} = 1$.

"Definition" : Sps $f = f(x)$ is defined for all x except possibly $x = a$. If $f(x)$ is arbitrarily close to L for all x sufficiently close to a then

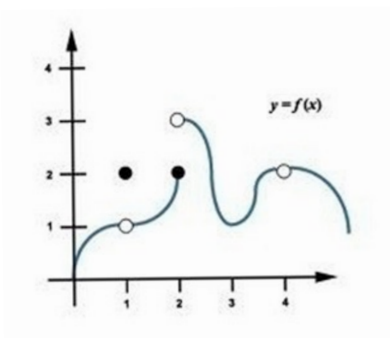
$$\lim_{x \to a} f(x) = L .$$

- One-sided limit : We say

$$\lim_{x \to a^+} f(x) = L \quad \left(\text{resp. } \lim_{x \to a^-} f(x) = L \right)$$

if $f(x)$ is arbitrarily close to L for $x > a$ (resp. $x < a$) sufficiently close to a.

Ex:



- $\lim_{x \to 1} f(x) = 1$
- $\lim_{x \to 2^-} f(x) = 2$
- $\lim_{x \to 2^+} f(x) = 3$.

Theorem : Suppose a function $f(x)$ is defined for all x except possibly $x = a$. Then

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L \Rightarrow \lim_{x \to a} f(x) = L .$$

Ex: $f(x) = \dfrac{x-1}{\sqrt{x}-1}$

What is $\lim_{x \to 1} f(x)$?

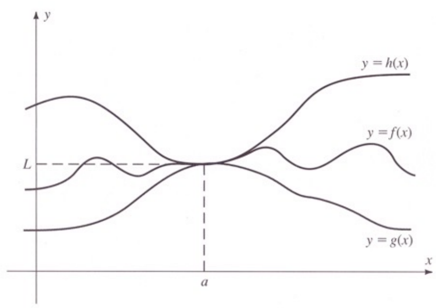
Ex: $\lim_{x \to 0} \dfrac{x+1}{x^2-1} = ?$

Squeeze theorem:

Suppose f, g, h are functions s.t.

$$g(x) \leq f(x) \leq h(x)$$

for all x near a number a.



If $\lim\limits_{x \to a} h(x) = L = \lim\limits_{x \to a} g(x)$

then $\lim\limits_{x \to a} f(x) = L$.

Ex: As an exercise you can show

$$\cos x \leq \frac{\sin x}{x} \leq \frac{1}{\cos x}$$

$$\begin{array}{ccccc} \cos x & \leq & \frac{\sin x}{x} & \leq & \frac{1}{\cos x} \\ \downarrow{\scriptstyle x \to 0} & & \downarrow{\scriptstyle x \to 0} & & \downarrow{\scriptstyle x \to 0} \\ 1 & = & 1 & = & 1 \end{array}$$