January 30:

Prop: Let (V, q) be such that

$$\{ v \in V \mid q(v) = \pm 1 \}$$

is path connected. Then $\mathrm{Spin}(V, q)$ is as well.

Pf: $v_1 \cdots v_{2\ell} \in \mathrm{Spin}(V, q)$, $q(v_i) = 1$.

Let γ_i be a path $v_{2i-1} \rightsquigarrow -v_{2i}$.

for $i = 1, \ldots, \ell$. Then $\prod_{i=1}^{\ell} \gamma_i$ is a

path from $v_1 \cdots v_{2\ell}$ to

$$(-1)^{\ell} v_1^2 v_3^2 \cdots v_{2\ell-1}^2 = 1 .$$

$\blacksquare$

Next, we turn to Clifford modules.

First, some basis observations. Let $\{e_i\}$ be an orthonormal basis for $V = \mathbb{R}^n$, where we use

$$q(x) = \|x\|^2 = x_1^2 + \cdots + x_n^2 .$$

Then

$$Cl_n = \langle e_i \mid e_i e_j + e_j e_i = -2\delta_{ij} \rangle .$$

Suppose that A, B are $\mathbb{Z}/2$ graded algebras. Define the $\mathbb{Z}/2$ graded alg.

$$A \overset{gr}{\otimes} B$$

as follows:

1) As v.s. $A \overset{gr}{\otimes} B = A \otimes B$.

2) Product is:

$$(a \otimes b) \cdot (a' \otimes b') = (-1)^{|b||a'|} (aa') \otimes (bb').$$

<u>Prop</u>: $Cl(V_1 \oplus V_2, q_1 \oplus q_2)$

$$\simeq Cl(V_1, q_1) \hat{\otimes} Cl(V_2, q_2).$$

<u>Pf</u>: Consider

$$f : V_1 \oplus V_2 \longrightarrow Cl(V_1) \hat{\otimes} Cl(V_2)$$

$$v_1 + v_2 \longmapsto v_1 \otimes 1 + 1 \otimes v_2 .$$

Then $f(v_1 + v_2)^2 = (v_1 \otimes 1 + 1 \otimes v_2)^2$

$$= v_1^2 \otimes 1 + (v_1 \otimes 1) \cdot (1 \otimes v_2) + (1 \otimes v_2) \cdot (v_1 \otimes 1) + 1 \otimes v_2^2$$

$$= v_1^2 \otimes 1 + 1 \otimes v_2^2 .$$

$$= -\left(q_1(v_1) + q_2(v_2)\right) 1 \otimes 1$$

$$= -q(v_1 + v_2) \, 1 \otimes 1 .$$

So f extends uniquely to homomorphism

$$Cl(V_1 \oplus V_2) \xrightarrow{\tilde{f}} Cl(V_1) \hat{\otimes} Cl(V_2)$$

This is, in fact, an isomorphism. ☒

Cor: $$Cl_n \simeq Cl_1^{\hat{\otimes} n} = Cl_1 \hat{\otimes} \cdots \hat{\otimes} Cl_1 .$$

Back to o.n.b $\{e_i\}$ for $\mathbb{R}^n$. Let

$$\omega \overset{\text{def}}{=} e_1 \cdots e_n \in Cl_n .$$

From the relation $e_i e_j = -e_j e_i$, we see that ω is actually independent of the choice of basis.

<u>**Lem**</u>: $\omega^2 = (-1)^{n(n+1)/2} \in \mathbb{R}^\times \subset Cl(V)$.

$$v\omega = (-1)^{n-1} \omega v \quad \forall\ v \in V = \mathbb{R}^n .$$

<u>**Prop**</u>: Sps $\frac{n(n+1)}{2}$ is even $(n = 4, 8, \dots)$,

and that V is a Cl_n-module. Then

$$V = V^+ \oplus V^-$$

where $V^{\pm} = \{ v \in V \mid \omega v = \pm v \}$.

Further, if $n \neq 0$ then it defines

$$n \cdot (-) : V^{\pm} \xrightarrow{\cong} V^{\mp} .$$

<u>Pf</u>: For such n have $\omega^2 = 1$. Let

$$\pi^{\pm} = \frac{1}{2}(1 \pm \omega) \in Cl(V) .$$

Then let $V^{\pm} = \pi^{\pm} \cdot V$.

<u>Note</u>:
$$\begin{cases} \pi^+ + \pi^- = 1 & (1) \\ \pi^+\pi^- = \pi^-\pi^+ = 0 & (2) \\ (\pi^{\pm})^2 = \pi^{\pm} & (3) \end{cases}.$$

(1) $\Rightarrow$ $V = V^+ + V^-$. Now, suppose:

$$\pi^+ v = \pi^- v$$

$\Rightarrow$ $\pi^+ v = \pi^+\pi^- v = 0$. So

$$V = V^+ \oplus V^- . \qquad \boxed{/\!/}$$

We have mentioned (but not proved) that w/ respect to the natural filtration that

$$\mathrm{gr}\, C\ell(V,q) \cong \Lambda^{\cdot} V.$$

The resulting canonical isomorphism of vector spaces

$$\Lambda^{\cdot} V \xrightarrow{\cong} C\ell(V,q)$$

can be made explicit:

$$v_1 \wedge \cdots \wedge v_\ell \longmapsto v_1 \cdots v_\ell .$$

<u>Q</u>: What does the Clifford multiplication look like in terms of operation on $\Lambda^{\cdot} V$?

<u>Prop</u>: Sps $v \in V$, $\varphi \in C\ell_n \cong \Lambda^{\cdot} \mathbb{R}^n$.
Then

$$v \cdot \varphi = v \wedge \varphi - v \lrcorner \varphi .$$

Clifford

wedge product/
exterior mult.

interior product/
contraction.

Recall:

$$v \vee (v_1 \wedge \dots \wedge v_\ell) = \sum_{j=1}^{\ell} (-1)^{j+1} \langle v, v_j \rangle \, v_1 \wedge \dots \wedge \hat{v}_\ell \wedge \dots \wedge v_\ell$$

Pf of Prop: Choose o.n.b. $\{e_i\}$ for $\mathbb{R}^n$. We will prove for $v = e_1$, $\varphi = e_{i_1} \cdots e_{i_\ell}$, where $i_1 < \dots < i_\ell$.

Case 1: $e_{i_1} = e_1$. Then

$$e_1 (e_1 e_{i_2} \cdots e_{i_\ell}) \overset{\sim}{=} - e_{i_2} \cdots e_{i_\ell} \quad \checkmark$$

$$\|$$

$$e_1 \wedge \overset{0}{\cancel{(e_1 \wedge \dots \wedge e_{i_\ell})}} - e_1 \vee (e_1 \wedge \dots \wedge e_{i_\ell})$$

Case 2: $e_{i_1} \neq e_1$. Then $e_1 \neq e_{i_j}$ for any $j = 1, \dots, \ell$

So $$e_1 (e_{i_1} \cdots e_{i_\ell}) \overset{\sim}{=} e_1 \wedge (e_{i_1} \wedge \dots \wedge e_{i_\ell}) - \overset{=0}{\cancel{e_1 \vee (e_{i_1} \wedge \dots \wedge e_{i_\ell})}}$$

$\checkmark$

$\blacksquare$

We now turn towards classification.

Let:

$$Cl_{r,s} \overset{\text{def}}{=} Cl\left(\mathbb{R}^{r+s}, \sum_{i=1}^{r} x_i^2 - \sum_{j=1}^{s} x_{r+j}^2\right).$$

Have seen $Cl_1 = Cl_{1,0} \cong \mathbb{C}$.

$$Cl_2 = Cl_{2,0} \cong \mathbb{H}.$$

Also easy:

- $Cl_{0,1} \cong \mathbb{R}[x] / (x^2 = 1) = \mathbb{R} \oplus \mathbb{R}$.

- $Cl_{0,2}$ is spanned by $1, x, y, xy$

where $x^2 = y^2 = 1$, $xy = -yx$.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{array}{ccc} 2 & 2 & 2 \\ 1 & x & y. \end{array}$$

$$\equiv \mathbb{R}(2).$$

This defines $Cl_{0,2} \cong Mat_{2\times 2}(\mathbb{R})$.

- $Cl_{1,1}$ is spanned by $1, x, y, xy$ w/

$$x^2 = -y^2 = 1, \quad xy = -yx.$$

$$\begin{array}{ccccc} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & , & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & , & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ 2 & & 2 & & 2 \\ 1 & & x & & y \end{array} .$$

So, also $Cl_{1,1} \cong \mathbb{R}(2)$.

<u>Theorem</u>: $$Cl_{n,0} \otimes Cl_{0,2} \cong Cl_{0,n+2}.$$

$$Cl_{0,n} \otimes Cl_{2,0} \cong Cl_{n+2,0}$$

$$Cl_{r,s} \otimes Cl_{1,1} \cong Cl_{r+1,s+1}.$$

where $\otimes$ is ordinary, ungraded $\otimes$-product.

Pf : $\{e_i\}_{i=1}^{n+2}$ o.n.b for $\mathbb{R}^{n+2}$ w/ $q(x) = -\|x\|^2$.

Let $\{e_i'\}_{i=1}^{n}$ denote generators for $Cl_{n,0}$.

$\{e_1'', e_2''\}$ denote generators for $Cl_{0,2}$.

Define $f : \mathbb{R}^{n+2} \longrightarrow Cl_{n,0} \otimes Cl_{0,2}$

$$e_i \longmapsto \begin{cases} e_i' \otimes e_1'' e_2'' , & 1 \le i \le n. \\ 1 \otimes e_{i-n}'' , & i = n+1, n+2. \end{cases}$$

So if $1 \le i, j \le n$ then

$$f(e_i)^2 = (e_i' e_i') \otimes (e_1'' e_2'' e_1'' e_2'').$$

$$= (-1) \cdot (-e_1'' e_1'' e_2'' e_2'') = +1.$$

and

$$f(e_{n+1})^2 = 1 \otimes e_1'' e_1'' = 1 = 1 \otimes e_2'' e_2'' = f(e_{n+2})^2.$$

So, $f(x)^2 = \|x\|^2 \Rightarrow f$ extends to

$$\tilde{f} : Cl_{0,n+2} \rightarrow Cl_{n,0} \otimes Cl_{0,2}.$$

By counting dimensions this is $\cong$. Other cases proved similarly. $\boxtimes$

<u>Theorem</u>: [PERIODICITY] There is an iso of $\mathbb{R}$-algebras:

$$Cl_{n+4} \cong Cl_n \otimes Cl_4 .$$

<u>Pf</u>: From above:

$$\begin{array}{c} Cl_{0,2} \otimes Cl_{2,0} \cong Cl_{4,0} \cong Cl_{0,4} . \\ \parallel \\ \mathbb{R}(2) \otimes \mathbb{H} = \mathbb{H}(2) \end{array}$$

On the other hand

$$\underbrace{Cl_{n,0} \otimes Cl_{0,2}} \otimes Cl_{2,0}$$

$$\cong Cl_{0,n+2} \otimes Cl_{2,0} \cong Cl_{n+4,0} .$$ $\boxtimes$

So, we really only need to characterize Cl_n, $n = 0,1,2,3,4$. Note

$$Cl_3 = Cl_{2,0} \otimes Cl_{0,1} \cong \mathbb{H} \oplus \mathbb{H}.$$

So, we have

n	Cl_n	$Cl_{0,n}$
1	$\mathbb{C}$	$\mathbb{R} \oplus \mathbb{R}$
2	$\mathbb{H}$	$\mathbb{R}(2)$
3	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{C}(2)$
4	$\mathbb{H}(2)$	$\mathbb{H}(2)$

Using the above periodicity we obtain all other Clifford algebras, for example:

$$Cl_5 \cong Cl_{0,4} \otimes Cl_{1,0} \cong \mathbb{H}(2) \otimes \mathbb{C}$$

$$= \mathbb{R}(2) \otimes \mathbb{H} \otimes \mathbb{C}$$

On worksheet
$\mathbb{H} \otimes \mathbb{C} \cong \mathbb{C}(2)$.

$$\cong \mathbb{R}(2) \otimes \mathbb{C}(2)$$

$$\cong \mathbb{R}(2) \otimes \mathbb{R}(2) \otimes \mathbb{C}$$

$$\cong \mathbb{R}(4) \otimes \mathbb{C} \cong \mathbb{C}(4).$$

<u>So:</u> $Cl_5 \cong \mathbb{C}(4)$.