The ABS construction

In this note we provide an overview of the construction of Atiyah, Bott, and Shapiro which provides, in part, a relationship between topological K-theory and Clifford modules. After a rapid introduction to K-theory we follow parts of the original reference [ABS64]. For a nice textbook review of K-theory see [Hat03].

1. A rapid introduction to K-theory

In this section the field $\mathbf{k}$ is either $\mathbf{R}$ or $\mathbf{C}$, and we consider $\mathbf{k}$-vector bundles on a space X. Write $\underline{\mathbf{k}}^n$ for the trivial bundle of rank n.

We introduce two equivalence relations on the set of vector bundles $\mathrm{Vect}(X)$.

- Stable isomorphism $\simeq_s$. Two vector bundles E_1, E_2 on X are *stably isomorphic*, we write $E_1 \simeq_s E_2$, if there exists an $N \geq 0$ and a bundle isomorphism
$$e_1 \oplus \underline{\mathbf{k}}^N \simeq E_2 \oplus \underline{\mathbf{k}}^N.$$
- Equivalence relation $\sim$. More generally, we say $E_1 \sim E_2$ if there exists $N, M \geq 0$ such that
$$E_1 \oplus \underline{\mathbf{k}}^N \simeq E_2 \oplus \underline{\mathbf{k}}^M.$$

Proposition 1.1. *If X is compact Hausdorff, then the set of $\sim$-equivalence classes of vector bundles over X forms an abelian group with respect to direct sum $\oplus$. This group is denoted $\widetilde{K}(X)$.*

This group $\widetilde{K}(X)$ is called the *reduced* K-group of X. The unreduced version is defined using the equivalence relation $\simeq$, except it is slightly more complicated. The issue is that only the class of the zero vector bundle is invertible in the set of $\simeq_s$-equivalence classes with respect to direct sum. Nevertheless, we can "cancel" bundles in some sense.

Lemma 1.2. *Suppose X is compact and E is a vector bundle on X. There exists a vector bundle E' on X such that $E \oplus E'$ is trivializable.*

PROOF. Suppose $\{U_i\}$ is a finite trivializing cover for E, so we have trivializations $\phi_i\colon U_i \times \mathbf{k}^n \to E$. There is a partition of unity $\{f_i\colon X \to [0,1]\}$ subordinate to this finite cover. For each i this allows us to define a vector bundle homomorphism

$$f_i \cdot \phi_i^{-1}\colon E \to X \times \mathbf{k}^n \tag{1}$$

Together, thus, we get a vector bundle homomorphism

$$\oplus f_i \cdot \phi_i^{-1}\colon E \to X \times \mathbf{k}^N \tag{2}$$

where N is n times the cardinality of the set parametrizing the cover. This morphism is fiberwise injective since f_i is non-vanishing at at least one point of U_i. So, we have embedded E into a trivial vector bundle. Fix an inner product on the trivial vector bundle (this exists by paracompactness). Then $E \oplus E^{\perp} \simeq \underline{\mathbf{k}}^{\oplus N}$ $\square$

For example, if M has a framing and $S \subset M$ is a submanifold, then the sum of TS with the normal bundle $N_M S$ is trivializable.

From this lemma, we see that if $E_1 \oplus E_2 \simeq_s E_1 \oplus E_3$ then we can add $E_1^{\perp}$ to both sides to see that $E_2 \simeq_s E_3$. It follows that the set of $\simeq_s$-equivalence classes forms a semi-group with respect to $\oplus$. The K-group $K(X)$ is the group completion of this semi-group. (Think of the positive rational numbers $\mathbf{Q}_{>0}$ as the group completion of the semi-group of positive natural numbers $\mathbf{Z}_{>0}$ under multiplication.)

1.1. By definition, one represents elements of $K(X)$ as classes of formal differences

$$[E_1 - E_2]$$

where E_1, E_2 are bundles on X. Then $[E_1 - E_2] = [E_1' - E_2']$ if and only if there is a stable equivalence

$$E_1 \oplus E_2' \simeq_s E_1' \oplus E_2.$$

The group operation is the obvious thing

$$[E_1 - E_2] + [E_2 - E_2'] = [(E_1 \oplus E_2) - (E_1' \oplus E_2')].$$

Note that every element in $K(X)$ can be represented by a formal difference $[E - \underline{\mathbf{k}}^n]$ for some bundle E and some integer n.

There is a natural homomorphism $K(X) \to \widetilde{K}(X)$ defined by $[E - \underline{\mathbf{k}}^n] \to [E]$ whose kernel consists of classes of the form $\underline{\mathbf{k}}^0 - \underline{\mathbf{k}}^n$. Hence, $E \simeq_s \underline{\mathbf{k}}^m$ for some m. Thus, the kernel of this homomorphism is $\mathbf{Z}$ and there is an isomorphism $K(X) \simeq \widetilde{K}(X) \oplus \mathbf{Z}$ coming from a splitting of this homomorphism $K(X) \to K(pt)$ whose kernel is exactly

$\widetilde{K}(X)$. The subgroup $\widetilde{K}(X)$ of $K(X)$ is an ideal and hence a ring in its own right with respect to tensor product.

A map $f\colon X \to Y$ determines a ring map on K-theory $f^*\colon K(Y) \to K(X)$ which sends a vector bundle $[E]$ on Y to the vector bundle $[f^*E]$ on X. Likewise, reduced K-theory is also functorial.

One of the main results about K-theory is Bott periodicity. It is easiest to state for complex K-theory, so for now we work over $\mathbf{C}$.

From now on $K(X)$ and $\widetilde{K}(X)$ will denote complex K-theory and reduced complex K-theory. Real K-theory is denoted $KO(X)$ and its reduced version is $\widetilde{KO}(X)$.

Let $L = \mathcal{O}(-1)$ be the tautological line bundle on $\mathbb{P}^1 = S^2$ (this is a holomorphic line bundle, but K-theory only knows its structure as a complex line bundle).

Lemma 1.3. *There is a bundle isomorphism $L \otimes L \oplus \underline{\mathbf{C}} \simeq L \oplus L$.*

PROOF. On $\mathbb{P}^1$ the data of a vector bundle is specified the homotopy class of a map

$$S^1 \to GL(2, \mathbf{C}) \tag{3}$$

Let E_t be a continuous path in $GL(2, \mathbf{C})$ which satisfies

$$E_0 = \mathbb{1}, \quad E_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4}$$

Such a path exists by connectedness. Consider the path in $Map(S^1, GL(2, \mathbf{C}))$:

$$f_t(z) = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} E_t \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} E_t. \tag{5}$$

This is a path from the clutching function for $L \oplus L$ to the clutching function for $L \otimes L \oplus \underline{\mathbf{C}}$. ☐

It follows that there is a ring homomorphism

$$\mathbf{Z}[L]/(L-1)^2 \to K(S^2). \tag{6}$$

THEOREM 1.4. *This is an isomorphism.*

If $a \in K(X)$ and $b \in K(Y)$ then define $a \star b = p_1^*(a) \otimes p_2^*(b) \in K(X \times Y)$. This is called the external product. Without much more work, one can show that

$$L \star (-)\colon \mathbf{Z}[L]/(L-1)^2 \otimes K(X) \to K(S^2 \times X) \tag{7}$$

is an isomorphism of rings.

The reduced version of this homomorphism is of the form

$$\widetilde{K}(X) \to \widetilde{K}(S^2X) \tag{8}$$

and explicitly sends $x \mapsto (L-1) \star x$.

THEOREM 1.5 (Bott periodicity). *This is an isomorphism of rings.*

1.2. We now construct a graded version of K-theory.

The suspension SX of a space X is defined to be the quotient of the cylinder $X \times [0,1]$ where we identify $X \times \{0\}$ to a single point and $X \times \{1\}$ to a single point. For example S^{n+1} is homeomorphic to $S(S^n)$. Equivalently, S^n can be seen as the nth fold suspension of $S^0 = \{-1,+1\}$.

The graded version of K-theory is defined using the suspension. Define, for reduced K-theory

$$\widetilde{K}^{-n}(X) \stackrel{\text{def}}{=} \widetilde{K}(S^nX). \tag{9}$$

In total, define the graded abelian group

$$\widetilde{K}^{-\bullet}(X) \stackrel{\text{def}}{=} \oplus_{n\geq 0}\widetilde{K}^{-n}(X). \tag{10}$$

The negative grading is chosen to match with with cohomological grading. For non-reduced, one defines $K^{-n}(X) = \widetilde{K}^{-n}(X_+)$.

For $A \subset X$ a closed subspace, there is the following exact sequence in reduced K-groups

$$\cdots \to \widetilde{K}(SX) \to \widetilde{K}(SA) \to \widetilde{K}(X/A) \to \widetilde{K}(X) \to \widetilde{K}(A) \tag{11}$$

The right most map is restriction of vector bundles from X to A, and the second to right most map is pulling back along the quotient map $X \to X/A$.

We use this exact sequence to find a relationship of K-theory with products of spaces. Let $X \wedge Y \stackrel{\text{def}}{=} X \times Y/X \vee Y$, where $X \vee Y$ is defined using specified basepoints of X and Y. Then we can apply the above long exact sequence to the pair $(X \times Y, X \vee Y)$ deduce an isomorphism

$$\widetilde{K}(X \times Y) \simeq K(X \wedge Y) \oplus \widetilde{K}(X) \oplus \widetilde{K}(Y). \tag{12}$$

We thus get a different sort of external product, defined by the composition

$$\widetilde{K}(X) \otimes \widetilde{K}(Y) \xrightarrow{\star} \widetilde{K}(X \times Y) \to \widetilde{K}(X \wedge Y) \tag{13}$$

where the last map is projection. Replacing X, Y by $S^i X, S^j Y$ defines a product

$$\widetilde{K}^i(X) \otimes \widetilde{K}^j(Y) \to \widetilde{K}^{i+j}(X \wedge Y). \tag{14}$$

Finally, in the case that $X = Y$ we can additionally compose with the restriction along the diagonal map $X \to X \wedge X$ to get a product

$$\widetilde{K}^i(X) \otimes \widetilde{K}^j(X) \to K^{i+j}(X). \tag{15}$$

Proposition 1.6. *This endows $\widetilde{K}^\bullet(X)$ with the structure of a commutative graded ring.*

From Bott periodicity, one immediately obtains a graded ring isomorphism

$$K^\bullet(\star) \simeq \mathbf{Z}[L] \tag{16}$$

where $L \in \widetilde{K}^{-2}(\star) = \widetilde{K}(S^2)$ is a degree -2 generator and represents the canonical line bundle L.

1.3. Real K-theory is more complicated. To distinguish it from the notation above, use the notation $KO(X)$ (and all of its relative and reduced variants) to denote the real K-theory of X. The same formal properties hold for real K-theory. There is a graded version $KO^\bullet(X)$ and it is endowed with the structure of a commutative graded ring.

Let's get an idea of how the real K-theory of a space can differ from its complex K-theory. Recall that for any embedded submanifold $S \subset M$ the normal bundle $\mathrm{N}S$ on S is defined by the short exact sequence

$$0 \to \mathrm{T}S \to \mathrm{T}M|_S \to \mathrm{N}S \to 0 \tag{17}$$

In particular, in K-theory one has

$$[\mathrm{T}S] + [\mathrm{N}S] = [\mathrm{T}M|_S] . \tag{18}$$

Now, consider the two-sphere S^2 and its natural embedding in $\mathbf{R}^3$. The normal bundle to this embedding is the trivial (real) line bundle. In particular, this implies that in K-theory

$$[\mathrm{T}S^2] + [\underline{\mathbf{R}}] = [\underline{\mathbf{R}}^3] \iff [\mathrm{T}S^2] = 2. \tag{19}$$

Here is the statement of real Bott periodicity.

THEOREM 1.7 (Real Bott periodicity). *There is a graded ring isomorphism*

$$KO^{-\bullet}(\star) \cong \mathbf{Z}[\eta, y, x] / \langle 2\eta, \eta^3, \eta y, y^2 - 4x \rangle \tag{20}$$

where η, y, x are of degrees $-1, -4, -8$, respectively.

This time, multiplication by the generator $x \in KO^{-8}(\star) = \widetilde{KO}(S^8)$ induces a periodicity.

The generator $\eta \in KO^{-1} = \widetilde{KO}(S^1)$ is represented by the tautological real line bundle $\mathcal{O}_{\mathbf{R}}(-1)$ over $\mathbf{R}\mathbb{P}^1 \simeq S^1$. A real line bundle on any topological space X is classified by the homotopy class of a map $X \to BGL(1, \mathbf{R}) = B(\mathbf{R}^\times)$. On the other hand

$$B(\mathbf{R}^\times) \simeq \mathrm{Gr}(1, \infty) = \mathbf{R}\mathbb{P}^\infty = K(\mathbf{Z}/2\mathbf{Z}, 1). \tag{21}$$

This implies that line bundles on a space X, up to isomorphism, are in one-to-one correspondence with

$$[X, K(\mathbf{Z}/2, 1)] \simeq H^1(X; \mathbf{Z}/2). \tag{22}$$

The generator η of $\widetilde{KO}(S^1)$ corresponds to the generator of $H^1(S^1; \mathbf{Z}/2)$. From this expression, the relation $2\eta = 0$.

Using fancier homotopy theory one can show the relation $\eta^3 = 0$. It is hard to see this directly.

We will use the construction of Atiyah, Bott, and Shapiro [ABS64] to come up with explicit representatives for the generators η, y, x in theorem 1.7.

2. "L"-theory

The construction utilizes an alternative description of K-theory. Suppose Y is a closed subspace of X. Everything in this section holds for real or complex vector bundles, so we will be agnostic about the field. There is a relative version of K-theory $\widetilde{K}(X, Y)$ defined by $\widetilde{K}(X/A)$, and more generally we can define $\widetilde{K}^{-n}(X, A) = \widetilde{K}(S^n(X/A))$.

Let $\mathcal{L}_1(X, Y)$ be the set of tuples

$$(V, W; \sigma) \tag{23}$$

where V, W are vector bundles on X and

$$\sigma \colon V|_Y \xrightarrow{\simeq} W|_Y \tag{24}$$

is a bundle isomorphism between the restriction of the vector bundles over Y. Morphisms $(V, W; \sigma) \to (V', W'; \sigma')$ are defined in the obvious way are bundle morphisms (over X) such that the obvious square diagram commutes (over Y). Isomorphism Let $L_1(X, Y)$ be the set of isomorphisms classes $[V, W; \sigma]$.

We can regard $(X,Y) \mapsto L_1(X,Y)$ as a functor from the category of pairs $(X, Y \subset X)$ to the category of vector spaces.

Proposition 2.1. *There is a natural isomorphism of vector spaces*

$$\chi\colon L_1(X,Y) \to K(X,Y).$$

When $Y = \emptyset$, this isomorphism is

$$\chi([V,W]) = [V - W].$$

PROOF. We sketch the proof. Filling in the gaps is part of the worksheet. The main idea is the so-called difference bundle construction. Define $X_k = X \times \{k\}$ where $k = 0, 1$ and let Z be the quotient of $X_0 \sqcup X_1$ obtained by identifying points $y \times \{0\} \in Y \times \{0\} \subset X_0$ with $y \times \{1\} \subset X_1$ for all $y \in Y$. The relative K-theory $K(Z, X_1)$ is isomorphic to $K(X,Y)$. We use this description of $K(X,Y)$ to make the following construction.

Given $[V_0, V_1; \sigma]$ define the vector bundle W on Z by setting

$$W|_{X_k} = V_k \tag{25}$$

and extending over Y using the isomorphism σ. Let $\pi\colon Z \to X_1$ be the retraction which collapses all points onto X_1. Then, $[W - \pi^* V_1]$ is in the kernel of $i^*\colon K(Z) \to K(X_1)$, the map induced from bundle restriction along $i\colon X_1 \hookrightarrow Z$. Hence, it determines a class in $K(Z, X_1) \simeq K(X,Y)$. The isomorphism in the proposition is

$$[V_0, V_1; \sigma] \mapsto [W - \pi^* V_1]\,. \tag{26}$$

$\square$

From now on, we will use the notation $K(X,Y)$ interchangeably with $L(X,Y)$. In particular, we can represent relative K-theory classes as tuples $[V, W; \sigma]$.

3. Clifford modules

We recall the main algebraic players in this topic. We consider both the real Clifford algebras $C\ell_n = C\ell_{n,0}$ and the complex Clifford algebras $\mathbf{C}\ell_n$. Recall that the Clifford algebra associated to any quadratic vector space is $\mathbf{Z}/2$-graded.

Let $\mathfrak{M}_n$ denote the Grothendieck group of graded $C\ell_n$-modules. We have shown how tensor product of modules yields the structure of a commutative graded ring on $\mathfrak{M}_\bullet = \oplus \mathfrak{M}_n$. Let $\mathfrak{M}_n^{gr,\mathbf{C}}$ denote the Grothendieck ring of graded $\mathbf{C}\ell_n$-modules. We also have the corresponding ring $\mathfrak{M}_\bullet^{\mathbf{C}} = \oplus_n \mathfrak{M}_n^{\mathbf{C}}$.

For each n, there is a natural map $i^*\colon \mathfrak{M}^{gr}_{n+1} \to \mathfrak{M}^{gr}_n$ (and the complex version) induced from the inclusion $\mathbf{R}^n \hookrightarrow \mathbf{R}^{n+1}$ given by the first n coordinates. In other words, we can view $i^*\mathfrak{M}_{n+1}$ as a subgroup of $\mathfrak{M}_n$. In fact, $i^*\mathfrak{M}_{\bullet+1}$ is an ideal of $\mathfrak{M}_\bullet$, so we obtain the quotient ring

$$\mathfrak{A}_\bullet = \oplus_{n\geq 0}\mathfrak{M}_n / i^*\mathfrak{M}_{n+1}. \tag{27}$$

Similarly, the complex version is denoted $\mathfrak{A}^{\mathbf{C}}_\bullet$.

In class, we have seen that as abelian groups $KO^{-n}(\star) \simeq \mathfrak{A}_n$ and $K^{-n}(\star) = \mathfrak{A}^{\mathbf{C}}_n$ for all $n \geq 0$. In fact, the rings $\mathfrak{A}_\bullet$ and $KO^\bullet(\star)$ are isomorphic (as are the complex versions $\mathfrak{A}^{\mathbf{C}}_\bullet$ and $K^\bullet(\star)$). In the next section, we make this isomorphism explicit.

4. The main construction

We first consider complex K-theory. The main construction is a homomorphism

$$\Phi_n\colon \mathfrak{M}^{\mathbf{C}}_n \to K(D^n, S^{n-1}) \simeq \widetilde{K}(D^n/S^{n-1}) = \widetilde{K}(S^n) \tag{28}$$

where $S^{n-1} = \partial D^n$ and D^n is the unit disk in $\mathbf{R}^n$.

Let $M = M^{ev} \oplus M^{odd}$ be a graded $\mathbf{C}\ell_n$-module and let $\underline{M}^{ev,odd}$ denote the trivial bundles over D^n with fiber $M^{ev,odd}$. Define

$$\Phi_n(M) = \left[\underline{M}^{ev}, \underline{M}^{odd}; \sigma\right]$$

where σ is the isomorphism $\underline{M}^{ev}|_{S^{n-1}} \xrightarrow{\simeq} \underline{M}^{odd}|_{S^{n-1}}$ given by Clifford multiplication

$$\sigma(x, m) = (x, x\cdot m).$$

Checking that σ is well-defined (only depends on isomorphism classes of representatives) is an easy exercise.

Proposition 4.1. *The restriction of Φ_n to $i^*(\mathfrak{M}^{\mathbf{C}}_{n+1}) \subset \mathfrak{M}^{\mathbf{C}}_n$ is trivial. Thus, Φ_n defines a homomorphism (with respect to $\oplus$)*

$$\Phi_n\colon \mathfrak{A}^{\mathbf{C}}_n \to K(D^n, S^{n-1}). \tag{29}$$

The same construction holds for real K-theory to define a homomorphism

$$\Phi_n O\colon \mathfrak{A}_n \to KO(D^n, S^{n-1}). \tag{30}$$

Ranging over all n, one obtains ring homomorphisms

$$\Phi_\bullet\colon \mathfrak{A}^{\mathbf{C}}_\bullet \to K^{-\bullet}(\star)$$
$$\Phi_\bullet O\colon \mathfrak{A}_\bullet \to KO^{-\bullet}(\star)$$

THEOREM 4.2 ([**abs**]). *The graded ring homomorphisms $\Phi_\bullet$, $\Phi_\bullet O$ are isomorphisms.*

4.1. We can now see explicit generators in K-theory. Let's first consider the complex version. Recall that $\mathbf{C}\ell_{2n}$ has a fundamental $\mathbf{Z}/2$ graded representation $S = S_+ \oplus S_-$. As an algebra $\mathbf{C}\ell_{2n} = \mathrm{End}\,(S)$. The two generators of $\mathfrak{M}^{\mathbf{C}}_{2n}$ are $[S]$ and its parity flop $[\widetilde{S}]$ (where we exchange the roles of $+$ and $-$). The generator of $i^*\mathfrak{M}_{2n+1}$ is $[S] + [\widetilde{S}]$.

Let $\underline{S}_\pm$ denote the trivial line bundle over D^{2n} with fiber $S_\pm$. Clifford multiplication determines a map of vector bundles

$$\mu\colon \underline{S}_+ \to \underline{S}_- \tag{31}$$

Explicitly, over a point $x \in D^{2n} \subset \mathbf{R}^{2n}$ the map μ_x is multiplication by x. Over S^{n-1}, μ is an isomorphism. Thus

$$\Phi(S) = [\underline{S}_+, \underline{S}_-; \mu|_{S^{n-1}}] \in \widetilde{K}(S^{2n}) = \mathbf{Z}. \tag{32}$$

is a generator.

Bibliography

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