LECTURE 2

The Hilbert scheme of points on a surface

In this lecture we introduce symmetric products of algebraic varieties, give a scheme-theoretic definition of the Hilbert scheme of points, and introduce an explicit description using geometric invariant theory. The justification of this description is where we will go in the next few lectures. Much of what we do in this course works over an arbitrary algebraically closed field. For the most part we will restrict ourselves to working over $\mathbf{C}$.

2.1. Symmetric products and Hilbert schemes

Let's begin with a simple example. Given any topological space X we can consider the n-fold symmetric product

$$S^n X = X^{\times n} / S_n \tag{1}$$

where the symmetric group S_n acts on the cartesian product $X^{\times n}$ in the natural way. Notice that if X is a smooth manifold it is no longer the case that $S^n X$ is a smooth manifold. The problem is that there are singular points (so-called 'orbifold' points.) In a sense, the Hilbert scheme of points on X is a 'resolution' of these singularities.

There is the following algebraic interpretation of the symmetric product. For example, suppose that X is just (complex) algebraic affine space $\mathbf{A}^1 = \mathrm{Spec}(\mathbf{C}[x])$. Then, we have the following presentation

$$S^n \mathbf{A}^1 = \mathrm{Spec}\left(\mathbf{C}[x_1, x_2, \ldots, x_n]^{S_n}\right) \tag{2}$$

where S_n permutes the variables x_i in the defining way. By classical invariant theory one knows that

$$\mathbf{C}[x_1, \ldots, x_n]^{S_n} \simeq \mathbf{C}[s_1, \ldots, s_n] \tag{3}$$

where s_n are the elementary symmetric polynomials in n-variables. Thus $S^n \mathbf{A}^1 \simeq \mathbf{A}^n$ as algebraic varieties.

More generally, we have the following definition of the symmetric power of an arbitrary affine algebraic variety X as

$$S^n X \stackrel{\mathrm{def}}{=} \mathrm{Spec}\left(\left(\mathbf{C}[X]^{\otimes n}\right)^{S_n}\right). \tag{4}$$

That is, the spectrum of the S_n invariants of the ring $\mathbf{C}[X]^{\otimes n}$, where $\mathbf{C}[X]$ is the ring of regular functions on X.

In higher dimensions, the symmetric powers of a smooth variety can be singular. Take for example the affine algebraic variety $X = \mathbf{A}^2 = \mathrm{Spec}\,\mathbf{C}[z_1, z_2]$. By definition, the symmetric square $S^2\mathbf{A}^2$ is the spectrum of the following ring

$$A = \mathbf{C}[z_1, z_2, w_1, w_2]^{\mathbf{Z}/2} \tag{5}$$

where $\mathbf{Z}/2$ acts by $z_i \leftrightarrow w_i$ for $i = 1, 2$.

Proposition 2.1.1. *There is an isomorphism of rings*

$$A \simeq \mathbf{C}[x, y, u, v, w]/(uv - w^2). \tag{6}$$

In particular

$$S^2\mathbf{A}^2 \simeq \mathbf{A}^2 \times Q \tag{7}$$

where Q is the (singular) quadric in $\mathbf{C}^3$ defined by $uv = w^2$.

PROOF. Make the following change of variables $x = z_1 + w_1, y = z_2 + w_2, s = z_1 - w_1, t = z_2 - w_2$. Then, the $\mathbf{Z}/2$ action on these new variables leaves x, y invariant so that

$$A \simeq \mathbf{C}[x, y] \otimes B \tag{8}$$

where $B = \mathbf{C}[s, t]^{\mathbf{Z}/2}$ and the new $\mathbf{Z}/2$ action is $s \to -s$, $t \to -t$. If we further reparameterize $u = s^2$, $v = t^2$, $w = st$ we see that

$$B \simeq \mathbf{C}[u, v, w]/(uv - w^2). \tag{9}$$

□

We want to do better than the symmetric product. Let $X = \mathrm{Spec}(A)$ be an affine algebraic variety. The ***Hilbert scheme*** of n-points in X has underlying set defined by

$$\mathrm{Hilb}_n(X) \overset{\mathrm{def}}{=} \{J \subset A \mid J \,\text{ideal}, \dim(A/J) = n\}. \tag{10}$$

When $\dim X = 1$ it is easy to see that $\mathrm{Hilb}_n(X) = S^n X$. But more generally, the Hilbert schemes differ from the symmetric powers. There is, however, a natural map

$$\pi_{HC} \colon \mathrm{Hilb}_n(X) \to S^n X \tag{11}$$

called the ***Hilbert–Chow*** morphism. On sets, it sends an ideal $J \subset A$ to the support $\mathrm{supp}(A/J)$.

Remark 2.1.2. Here, if M is an A-module then its support $\mathrm{supp}(M) \subset X = \mathrm{Spec}(A)$ can be thought of as an unordered set of points in X.

Remark 2.1.3. If $\dim X = 2$, and X is nonsingular, then the Hilbert–Chow morphism is a resolution of singularities.

This is the definition of the Hilbert scheme as a set. Below we will see how one endows it with the structure of a scheme.

Here is another useful presentation of the Hilbert scheme as a set.

Lemma 2.1.4. *Let $X = \mathrm{Spec}(A)$ be an affine variety. There is a bijection of $\mathrm{Hilb}_n(X)$ with the set of pairs*

$$(M, v) \tag{12}$$

*where M is an A-module of dimension n and $v \in M$ is a vector which satisfies $A \cdot v = M$ (such a vector is called a **cyclic** vector).*

PROOF. In one direction the correspondence takes an ideal J and sends it to $M = \mathbf{C}[X]/J$ with $v = 1$. $\square$

2.2. REPRESENTABILITY

In the next section we will define the Hilbert scheme using the functor of points perspective. The main objects are functors of the form

$$F\colon (\mathrm{Sch}_{/S})^{op} \to \mathrm{Sets} \tag{13}$$

where $\mathrm{Sch}_{/S}$ is the category of schemes over a fixed scheme S and the op denotes the opposite category.

An important example of a contravariant functor is the following. Suppose that X is any scheme. Then consider the "Yoneda" functor

$$h_{X/S}\colon (\mathrm{Sch}_{/S})^{op} \to \mathrm{Sets} \tag{14}$$

defined by

$$h_{X/S}(U) = \mathrm{Hom}_{\mathrm{Sch}_{/S}}(U, X). \tag{15}$$

When we write U on the left hand side we implicitly remember it is a scheme over S, that is, it comes with a morphism $U \to S$.

Given a functor $F\colon (\mathrm{Sch}_{/S})^{op} \to \mathrm{Sets}$ we want to know whether it is representable. This means that there is an equivalence of functors $h_X \simeq F$ for some scheme X. In this case, we then say that X represents F. Let's unpack what such an equivalence would mean.

First off, an equivalence of functors means we have a natural transformation $\eta\colon h_X \to F$. There is a canonical element in $h_X(X)$ given by the identity $\mathbb{1}_Y$. Via the transformation η we obtain an element

$$\xi \overset{\mathrm{def}}{=} \eta(\mathbb{1}_X) \in F(X). \tag{16}$$

Conversely, given an element $\xi \in F(X)$ we can construct a natural transformation $\eta_\xi\colon h_X \to F$ as follows. For any $f\colon Y \to X$ in $h_X(Y)$ let $\eta_\xi(f) = f^*\xi$. (Here $f^*\xi$ stands for the image of ξ under the map $F(f)\colon F(Y) \to F(X)$.) One can see that these two operations are inverses to one another which gives the "Yoneda lemma"

$$\mathrm{Fun}(h_X, F) \simeq F(X). \tag{17}$$

Thus we can rephrase representability as follows.

Definition 2.2.1. Let F be a functor as above. A pair (X, ξ) where X is a scheme over S and $\xi \in F(X)$ ***represents*** F if the induced natural transformation $\eta_\xi\colon h_{X/S} \to F$ is an equivalence. Equivalently, for any $T \to S$ there is a natural one-to-one correspondence between lifts

$$\begin{tikzcd} & X \arrow[d] \\ T \arrow[ur, dotted, "\phi"] \arrow[r] & S. \end{tikzcd} \tag{18}$$

and elements $\phi^*\xi \in F(T)$.

The element ξ is usually called the *universal family* corresponding to F.

Example 2.2.2 (Point functor for projective space)**.** Let $S = \operatorname{Spec} \mathbf{C}$ for concreteness. Which functor represents projective space $\mathbb{P}^n$? Recall that $\mathbb{P}^n$ is the space of lines in $\mathbf{C}^{n+1}$. Thus, for a scheme X we can think about a map

$$\phi\colon X \to \mathbb{P}^n \tag{19}$$

as giving a family of lines in $\mathbf{C}^{n+1}$ parametrized by X. Conversely, given a family of lines in $\mathbf{C}^{n+1}$ parametrized by X we should be able to construct such a map.

The important question is what does 'family' of lines mean in this context? A first attempt would be to define a family of lines in $\mathbf{C}^{n+1}$ parametrized by X as a sub vector bundle of the trivial rank $n+1$ bundle $\mathcal{O}_X^{\oplus n+1}$. The problem with this is that sub bundles are not so well-behaved sheaf-theoretically. Indeed, if $\mathcal{F}$ is a locally free sheaf with corresponding bundle F and $\mathcal{E} \subset \mathcal{F}$ is a locally free subsheaf with corresponding bundle E, then the map on stalks $E_x \to F_x$ may not be injective. Better, then, is to look at locally free subsheaves.

For a fixed scheme X let $F(X)$ be the set of exact sequences

$$0 \to \mathcal{K} \to \mathcal{O}_X^{\oplus n+1} \to \mathcal{L} \to 0 \tag{20}$$

up to equivalence where $\mathcal{L}$ (or $\mathcal{K}$) is rank one. We can upgrade $X \mapsto F(X)$ to a functor as above; indeed, pulling back sheaves along $X \to Y$ results in a map $F(Y) \to F(X)$.

The functor F is represented by the projective space in the sense that there is a natural bijective correspodence between maps $\phi\colon X \to \mathbb{P}^n$ and elements of $F(X)$.

2.3. Hilbert schemes: formal definition

So far we have only provided the careful definition of the Hilbert scheme for affine varieties. Even in this case we didn't give an argument as to why it has the structure of a scheme. The goal of this section is to remedy these two shortcomings using the functor of points perspective. We will state, but not prove, a very important result that the Hilbert scheme functor is representatable in an extremely general situation. Later on, for Hilbert schemes on $\mathbf{A}^2$ we will come up with an explicit presentation.

First a definition.

Definition 2.3.1. Let X be a scheme over S. An ***algebraic family of closed subschemes*** of X/S parameterized by a scheme T is a closed subscheme

$$Z \subset X_T \overset{\text{def}}{=} X \times_S T. \tag{21}$$

The family is ***flat*** if the induced morphism $Z \to X_T \to T$ is flat.

For the most part we will take $S = \operatorname{Spec} \mathbf{C}$. Fix a projective scheme X over $\operatorname{Spec} \mathbf{C}$ and let $\mathrm{Sch}_{/\mathbf{C}}$ be the category of all schemes over $\operatorname{Spec} \mathbf{C}$. Define the functor

$$\mathcal{H}\mathrm{ilb}_X\colon (\mathrm{Sch}_{/\mathbf{C}})^{op} \to \mathrm{Sets} \tag{22}$$

by sending a scheme T to the set of flat algebraic families of closed subschemes parametrized by T. If $f\colon T \to T'$ is a map of schemes then any T'-family $Z \subset X \times T'$

restricts along f to a T-family $f^*Z \subset X \times T'$. Thus, $\mathcal{H}\mathrm{ilb}_X$ is a contraviant functor from the category of schemes over $\mathbf{C}$ to sets.

Now, we will define a subfunctor of $\mathcal{H}\mathrm{ilb}_X$ with some nice properties. Let $\mathcal{H}\mathrm{ilb}_{X,n}$ be the subfunctor which assigns to a scheme T the set of families with Hilbert polynomial P.

Aside 2.3.2 (Hilbert polynomial)**.** The Euler characteristic of a sheaf $\mathcal{F}$ on X is

$$\chi(X, \mathcal{F}) \stackrel{\text{def}}{=} \sum_i (-1)^i \dim H^i(X, \mathcal{F}). \tag{23}$$

Let $j \colon X \hookrightarrow \mathbb{P}^N$ be a projective scheme. For $m \geq 0$ let $\mathcal{O}_X(m) = j^{-1}\mathcal{O}_{\mathbb{P}^N}(m)$ and $\mathcal{F}(m) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(m)$. The Hilbert polynomial of $\mathcal{F}$ is defined by

$$P_{\mathcal{F}}(m) \stackrel{\text{def}}{=} \chi(X, \mathcal{F}(m)). \tag{24}$$

The fact that this is actually a polynomial requires a bit of work.

If $Z \subset X \times T$ is a closed family of subschemes parametrized by T then we let

$$P_t(m) \stackrel{\text{def}}{=} P_{\mathcal{O}_{Z_t}}(m). \tag{25}$$

By flatness, when T is connected this polynomial is independent of $t \in T$. In this case we simply denoted by P.

THEOREM 2.3.3 ([Gro95]). *Let X be a projective scheme. Then, the functor $\mathcal{H}\mathrm{ilb}_{X,P}$ is representable by a projective scheme that we denote by $\mathrm{Hilb}_P(X)$. In particular, this means that there is a universal family $\mathcal{Z}_{X,P} \to \mathrm{Hilb}_P(X)$ such that every family on a scheme U is determined by restricting this family via a unique morphism $U \to \mathrm{Hilb}_P(X)$.*

Remark 2.3.4. This theorem implies that the full Hilbert scheme $\mathcal{H}\mathrm{ilb}_X$ (with no condition on the Hilbert polynomial) is represented by the scheme

$$\bigsqcup_P \mathrm{Hilb}_P(X). \tag{26}$$

This result allows us to define the Hilbert scheme for any quasi-projective scheme. Indeed, if $Y \subset X$ is an open subscheme of a projective scheme then we have the corresponding open subscheme $\mathrm{Hilb}_P(Y) \subset \mathrm{Hilb}_P(X)$.

Definition 2.3.5. Let P be the constant polynomial $P = n$. Then, we denote $\mathrm{Hilb}_P(X) = \mathrm{Hilb}_n(X)$ and call it the Hilbert scheme of n points on X.

It is worthwhile to see that in the case that X is an affine algebraic variety that this definition agrees with (10). For the most part, we will restrict ourselves to Hilbert schemes of points on schemes of dimension two. We will give an explicit descriptions of the Hilbert scheme $\mathrm{Hilb}_n(\mathbf{A}^2)$. Via a gluing argument, one can define the Hilbert scheme associated to any nonsingular complex surface in the complex analytic category [Dou66]. In particular, for X a complex analytic surface the space $\mathrm{Hilb}_n(X)$ is defined and has the structure of a complex manifold.

2.4. An explicit description

Suppose that $X = \mathrm{Spec}(A)$ is an affine algebraic variety and that G is a linear algebraic group acting algebraically on X (all defined over $\mathbf{C}$). We will also assume that G is *reductive* meaning that its radical is a torus. It is in this case that the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ is a direct sum of semisimple and commutative Lie algebras.

Definition 2.4.1. The ***geometric invariant theory (GIT) quotient*** of $X = \mathrm{Spec}\, A$ by an algebraic G-action is the affine algebraic variety

$$X /\!/ G \stackrel{\mathrm{def}}{=} \mathrm{Spec}\left(A^G\right). \tag{27}$$

Remark 2.4.2. It is a theorem of Hilbert that the algebra $\mathbf{C}[X]^G$ is finitely generated. Therefore the set of maximal ideals indeed defines an affine algebraic variety.

We will study GIT quotients in more detail during the next lecture. For the time being, we will introduce an explicit GIT description of the Hilbert scheme.

Fix an integer n and consider the following (non-linear) subspace

$$H_n \subset \mathrm{Hom}(\mathbf{C}^n, \mathbf{C}^n)^{\otimes 2} \oplus \mathbf{C}^n \oplus (\mathbf{C}^n)^* \tag{28}$$

as the set of tuples (X, Y, i, j) which satisfy

$$[X, Y] - ij = 0. \tag{29}$$

Also let

$$H_n^s \subset H_n \tag{30}$$

be the subspace where i generates $\mathbf{C}^n$ under the action by X, Y. There is a natural action of $GL(n, \mathbf{C})$ on H_n and H_n^s.

The proof of the following result will occupy the next few lectures.

THEOREM 2.4.3. *There are isomorphisms of algebraic varieties*

$$S^n\mathbf{A}^2 \simeq H_n /\!/ GL(n, \mathbf{C})$$
$$\mathrm{Hilb}_n(\mathbf{A}^2) \simeq H_n^s /\!/ GL(n, \mathbf{C}).$$

Moreover, the natural map $H_n^s \hookrightarrow H_n$ induces the Hilbert–Chow morphism

$$\pi_{HC} \colon \mathrm{Hilb}_n(\mathbf{A}^2) \to S^n\mathbf{A}^2. \tag{31}$$

Remark 2.4.4. Actually, one can obtain a slightly more minimal description of $\mathrm{Hilb}_n(\mathbf{A}^2)$. The condition that i generates $\mathbf{C}^n$ under the action of X, Y together with (29) can be shown to imply that $j = 0$. Thus $\mathrm{Hilb}_n(X)$ can be realized as the $GL(n; \mathbf{C})$ quotient of the set of triples (X, Y, i) such that $[X, Y] = 0$ and that i generates $\mathbf{C}^n$ under the action by X, Y.

Bibliography

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