OFF-SHELL FORMULATION OF N=2 NON-LINEAR σ -MODELS

ALEXANDER SEVRIN and JAN TROOST¹

Theoretische Natuurkunde, Vrije Universiteit Brussel Pleinlaan 2, B-1050 Brussel, Belgium

ABSTRACT

We study d=2, N=(2,2) non-linear σ -models in (2,2) superspace. By analyzing the most general constraints on a superfield, we show that through an appropriate choice of coordinates, there are no other superfields than chiral, twisted chiral and semi-chiral ones. We study the resulting σ -models and we speculate on the possibility that all~(2,2) non-linear σ -models can be described using these fields. We apply the results to two examples: the $SU(2) \times U(1)$ and the $SU(2) \times SU(2)$ WZW model. Pending upon the choice of complex structures, the former can be described in terms of either one semi-chiral multiplet or a chiral and a twisted chiral multiplet. The latter is formulated in terms of one semi-chiral and one twisted chiral multiplet. For both cases we obtain the potential explicitely.

1. Introduction

Supersymmetric non-linear σ -models, in particular those with two or more supersymmetries, are building blocks for stringtheories. In [1] torsion free supersymmetric non-linear σ -models were studied and this resulted in a full classification of models within this class: N=1 is always possible, N=2 is in one-to-one correspondence with the target manifold being Kähler, N=3 implies N=4, N=4 requires the target manifold to be hyper-Kähler and N>4 is not possible. Later [2], σ -models which do have torsion were studied but, except for the fact that N=1 is always possible, no general classification has been given. In [3] a subclass of torsionful σ -models were studied: the Wess-Zumino-Witten models, where the target manifold is a semi-simple Lie groupmanifold. There the classification of extended supersymmetries has been done: again $N \leq 4$ and N=2 is possible on all even-dimensional groups, $N=3 \Rightarrow N=4$, N=4 is only possible on those groups which could be decomposed as "products" of Wolf spaces.

A second problem arises when one considers supersymmetry transformations of both chiralities, where one finds that the supersymmetry algebra closes only *on-shell*. Finding auxiliary fields such that the supersymmetry algebra closes off-shell, *i.e.* gets realized in a model independent way, is equivalent to finding a superspace formulation for these models. Both the N = (1, 1) and N = (2, 1), [4], cases are easily solved in

¹Aspirant NFWO

their full generality while for torsionful models with $N \geq (2,2)$ this remains largely unsolved. Having an extended superspace description of these models yields various simplifications and applications. The geometry becomes transparant and duality transformations can be performed which keep the extended supersymmetry manifest [5, 6].

In this paper we study the N=(2,2) case. The action in (2,2) superspace is just the integral of a potential, so in this way one might get a handle on the geometry of N=(2,2) supersymmetric non-linear σ -models. As the lagrange density is a potential, the dynamics is now also determined by the choice of superfields. Till now several types of superfields were discovered. The simplest case corresponds to models formulated solely in terms of chiral fields. These σ -models are Kähler. A generalization thereof is obtained by using both chiral and twisted chiral superfields [7]. The resulting model is then a mild generalization of the Kähler case. These models do have torsion. A last type of superfields are the semi-chiral fields [8]. Though discovered several years ago, they have barely been studied. As will become clear in this paper, models which include semi-chiral superfields represent a radical departure of the two previously mentioned types of σ -models.

In this paper we show by analyzing the most general constraints one can impose on superfields, that the previously mentioned three types of fields is all there is. Subsequently we study the most general model and provide several arguments which support the hope that all N = (2, 2) non-linear σ -models can be described using these fields. Finally we give two explicit examples: $SU(2) \times U(1)$ and $SU(2) \times SU(2)$, the former being described either by a single semi-chiral multiplet or by a chiral and a twisted chiral multiplet and the latter by one semi-chiral and one chiral multiplet.

2. Supersymmetric non-linear σ -models

Omitting the dilaton term, a general supersymmetric non-linear σ -model in N = (1,1) superspace is given by²

$$S = \int d^2x d^2\theta \left(g_{ab} + b_{ab}\right) D\phi^a \bar{D}\phi^b, \tag{2.1}$$

The metric on the target manifold is g_{ab} while $b_{ab} = -b_{ba}$ is a potential for the torsion T,

$$T^{a}{}_{bc} \equiv -\frac{3}{2}g^{ad}b_{[bc,d]}.$$
 (2.2)

The equations of motion follow from eq. (2.1):

$$D\bar{D}\phi^a + \Gamma^a_{-cb}D\phi^b\bar{D}\phi^c = 0, (2.3)$$

where we used one of the two natural connections Γ_{\pm} :

$$\Gamma^a_{\pm bc} \equiv \left\{ {}^a_{bc} \right\} \pm T^a_{bc}, \tag{2.4}$$

²We take $D \equiv \frac{\partial}{\partial \theta} + \theta \partial$ and $\bar{D} \equiv \frac{\partial}{\partial \bar{\theta}} + \bar{\theta} \bar{\partial}$, with $\partial \equiv \frac{\partial}{\partial z}$ and $\bar{\partial} \equiv \frac{\partial}{\partial \bar{z}}$.

the first term being the standard Levi-Cevita connection. The action is invariant under the supersymmetry transformations

$$\delta\phi^a = \varepsilon Q\phi^a + \bar{\varepsilon}\bar{Q}\phi^a,\tag{2.5}$$

where

$$Q \equiv \frac{\partial}{\partial \theta} - \theta \partial, \qquad \bar{Q} \equiv \frac{\partial}{\partial \bar{\theta}} - \bar{\theta} \bar{\partial}. \tag{2.6}$$

The commutator of two supersymmetries closes on a translation

$$[\delta_1, \delta_2] \phi^a = 2\varepsilon_2 \varepsilon_1 \partial \phi^a + 2\bar{\varepsilon}_2 \bar{\varepsilon}_1 \bar{\partial} \phi^a. \tag{2.7}$$

A second supersymmetry is necessarily of the form

$$\delta\phi^a = \eta J^a{}_b D\phi^b + \bar{\eta} \bar{J}^a{}_b \bar{D}\phi^b. \tag{2.8}$$

The action, eq. (2.1), is invariant under the transformations eq. (2.8) provided³

$$\nabla_c^+ J^a{}_b = \nabla_c^- \bar{J}^a{}_b = 0, \qquad g_{bc} J^c{}_a = -g_{ac} J^c{}_b, \qquad g_{bc} \bar{J}^c{}_a = -g_{ac} \bar{J}^c{}_b, \tag{2.9}$$

hold. One obtains the standard on-shell supersymmetry algebra, i.e. the first and the second supersymmetry commute, and the second supersymmetry satisfies the same algebra as the first, if J and \bar{J} obey

$$J^2 = \bar{J}^2 = -1, \qquad N^a{}_{bc}[J, J] = N^a{}_{bc}[\bar{J}, \bar{J}] = 0,$$
 (2.10)

with the Nijenhuis tensor, N[A, B], given by

$$N^{a}_{bc}[A, B] \equiv A^{d}_{[b}B^{a}_{c],d} + A^{a}_{d}B^{d}_{[b,c]} + B^{d}_{[b}A^{a}_{c],d} + B^{a}_{d}A^{d}_{[b,c]}.$$
(2.11)

In other words the model is (2,2) supersymmetric iff. the manifold allows for two complex structures for both of which the metric is hermitean. Each of these complex structures is covariantly constant, however w.r.t. different connections. Note that though each of the complex structures is *individually* integrable, they are not necessarily *simultanously* integrable.

A few more interesting formulae can be obtained. Using the constancy of the complex structures, we can rewrite the Nijenhuis condition as

$$3T_{ef[a}J^{e}{}_{b}J^{f}{}_{c]} = T_{abc}, (2.12)$$

and similarly for \bar{J} . Another consequence of the constancy of the complex structures is:

$$J_{[ab,c]} = 2T^d_{[ab}J_{c]d}, \qquad \bar{J}_{[ab,c]} = -2T^d_{[ab}\bar{J}_{c]d},$$
 (2.13)

³By ∇^{\pm} we denote covariant differentiation using the Γ_{\pm} connection.

which gives the exterior derivative of the two fundamental two-forms associated with J and \bar{J} . Combining eqs. (2.13) and (2.12) results in an expression for the torsion in terms of the complex structures:

$$T_{abc} = \frac{3}{2} J^d{}_a J^e{}_b J^f{}_c J_{[de,f]} = -\frac{3}{2} \bar{J}^d{}_a \bar{J}^e{}_b \bar{J}^f{}_c \bar{J}_{[de,f]}. \tag{2.14}$$

The supersymmetry algebra is standard but for the commutator of the left-handed second supersymmetry with the right handed second one:

$$[\delta(\eta), \delta(\bar{\eta})]\phi^a = \eta \bar{\eta} [J, \bar{J}]^a{}_b (D\bar{D}\phi^b + \Gamma^b{}_{-cd} D\phi^d \bar{D}\phi^c). \tag{2.15}$$

So as long as the left and right complex structure commute, the supersymmetry algebra closes off-shell and we expect that we can formulate the model in (2,2) superspace without introducing any new fields. However, if they do not commute, the supersymmetry algebra closes only on-shell, hence the algebra is model dependent and a manifest (2,2) supersymmetric formulation will require the introduction of additional auxiliary fields.

Consider now the case where $[J, \bar{J}] = 0$. In the appendix we show that $N[J, J] = N[\bar{J}, \bar{J}] = 0$ imply that $N[J, \bar{J}] = N[J, \Pi] = N[\bar{J}, \Pi] = N[\Pi, \Pi] = 0$, where $\Pi \equiv J\bar{J}$ is a product structure, [7]: $\Pi^2 = 1$. As was to be expected, the fact that the complex structures commute, guarantees that they are simultaneously integrable. The product structure allows us to introduce projection operators:

$$P_{\pm} \equiv \frac{1}{2} (1 \pm \Pi).$$
 (2.16)

From

$$\ker\left[J,\bar{J}\right] = \ker(J+\bar{J}) \oplus \ker(J-\bar{J}),\tag{2.17}$$

we get that P_+ projects on $\ker(J+\bar{J})$, and P_- on $\ker(J-\bar{J})$. If we have that $J=\pm\bar{J}$, then it follows from eq. (2.13) that the fundamental two form is closed and eq. (2.14) implies that the torsion vanishes: the manifold is Kähler [1]. Having $J\neq\pm\bar{J}$, gives a manifold with a product structure. The subspaces obtained by the projection operators P_\pm are Kähler and the manifold has torsion given by eq. (2.14). We come back to this in the next section. Conversely, one can also show that any model for which $[J,\bar{J}]=0$ can be described by chiral and twisted chiral fields [7]. One example of a manifold which can be described by such a "twisted Kähler" geometry is $SU(2)\times U(1)$ [5].

3. N=(2,2) superfields

3.1. Chiral and twisted chiral superfields

We now turn to a manifest (2,2) supersymmetric formulation of non-linear σ -models. To achieve this we introduce (2,2) superspace which consists of two fermionic directions for each chirality, hence the name. Its left-handed (right-handed) sector is

parametrized by the coordinates (z, θ^+, θ^-) $((\bar{z}, \bar{\theta}^+, \bar{\theta}^-))$. Covariant derivatives are given by

$$D_{\pm} \equiv \frac{\partial}{\partial \theta^{\pm}} + \theta^{\mp} \partial, \qquad \bar{D}_{\pm} \equiv \frac{\partial}{\partial \bar{\theta}^{\pm}} + \bar{\theta}^{\mp} \bar{\partial}. \tag{3.1}$$

Introducing

$$Q_{\pm} = \frac{\partial}{\partial \theta^{\pm}} - \theta^{\mp} \partial; \qquad \bar{Q}_{\pm} = \frac{\partial}{\partial \bar{\theta}^{\pm}} - \bar{\theta}^{\mp} \bar{\partial}, \qquad (3.2)$$

we get the N=(2,2) supersymmetry transformation on a general superfield Φ :

$$\delta\Phi = \varepsilon^{+} Q_{+} \Phi + \varepsilon^{-} Q_{-} \Phi + \bar{\varepsilon}^{+} \bar{Q}_{+} \Phi + \bar{\varepsilon}^{-} \bar{Q}_{-} \Phi. \tag{3.3}$$

Passing from (1,1) to (2,2) superspace is facilitated through the use of the original fermionic coordinates θ and $\bar{\theta}$:

$$\theta = \frac{1}{\sqrt{2}}(\theta^+ + \theta^-), \qquad \bar{\theta} = \frac{1}{\sqrt{2}}(\bar{\theta}^+ + \bar{\theta}^-),$$
 (3.4)

and the extra fermionic coordinates

$$\hat{\theta} = \frac{1}{\sqrt{2}}(\theta^+ - \theta^-), \qquad \hat{\bar{\theta}} = \frac{1}{\sqrt{2}}(\bar{\theta}^+ - \bar{\theta}^-).$$
 (3.5)

In this way we get e.g.

$$D = \frac{1}{\sqrt{2}}(D_{+} + D_{-}) = \frac{\partial}{\partial \theta} + \theta \partial, \qquad \widehat{D} = \frac{1}{\sqrt{2}}(D_{+} - D_{-}) = \frac{\partial}{\partial \widehat{\theta}} - \widehat{\theta} \partial, \tag{3.6}$$

and similarly for \overline{D} and $\widehat{\overline{D}}$. The supersymmetry generators are

$$Q \equiv \frac{1}{\sqrt{2}}(Q_{+} + Q_{-}) = \frac{\partial}{\partial \theta} - \theta \partial \qquad \widehat{Q} = \frac{1}{\sqrt{2}}(Q_{+} - Q_{-}) = \frac{\partial}{\partial \widehat{\theta}} + \widehat{\theta} \partial, \tag{3.7}$$

and similar expressions for \bar{Q} and $\hat{\bar{Q}}$. In particular one gets then that when passing to N=(1,1) superspace $\hat{Q}=\widehat{D}$ and $\hat{\bar{Q}}=\widehat{\bar{D}}$.

On dimensional grounds we know that the Lagrange density has to be a function of scalar fields. The dynamics is then largely determined by the superfields we use or said in a different way, our choice of representations.

Starting from a set of general (2,2) superfields Φ^a , $a \in \{1, \dots, d\}$, we can impose constraints of the form⁴

$$\widehat{D}\Phi^a = iJ^a{}_b D\Phi^b, \tag{3.8}$$

with $J^a{}_b$, some (1,1) tensor. This eliminates half of the degrees of freedom of the superfields. However some integrability conditions have to be met. Computing $\widehat{D}^2\Phi^a$ using the r.h.s. of previous equation, we get:

$$\widehat{D}^{2}\Phi^{a} = (J^{2})^{a}{}_{b}\partial\Phi^{b} + \frac{1}{2}N^{a}{}_{bc}[J,J]D\Phi^{c}D\Phi^{b}, \tag{3.9}$$

⁴For given l.h.s., we get that Lorentz invariance and counting dimensions, does give eq. (3.8) as the most general constraint.

which is consistent with $\widehat{D}^2 = -\partial$ iff. $J^2 = -1$ and the Nijenhuis tensor vanishes, hence J needs to be a complex structure.

Additional constraints of opposite chirality can be imposed and have the form

$$\widehat{\bar{D}}\Phi^a = i\bar{J}(\Phi)^a{}_b\bar{D}\Phi^b. \tag{3.10}$$

Writing Φ^a as an expansion in $\hat{\theta}$ and $\hat{\bar{\theta}}$, one verifies that eqs. (3.8) and (3.10) eliminate three of the four components of Φ^a , thus effectively reducing Φ^a to a (1,1) superfield. Consistency of the constraint (3.10) with $\widehat{\bar{D}}^2 = -\bar{\partial}$ requires \bar{J} to be a complex structure but $\{\widehat{D}, \widehat{\bar{D}}\} = 0$ yields additional conditions:

$$\{\widehat{D}, \widehat{\overline{D}}\}\Phi^a = [\bar{J}, J]^a{}_b \bar{D}D\Phi^a + M^a{}_{bc}[J, \bar{J}]\bar{D}\Phi^c D\Phi^b.$$
(3.11)

The (1,2) tensor $M^a{}_{bc}[J,\bar{J}]$ is discussed in the appendix, where we show that as long as $[J,\bar{J}]=0$, the vanishing of N[J,J] and $N[\bar{J},\bar{J}]$ implies the vanishing of this tensor. So imposing both constraints eqs. (3.8) and (3.10) yields an additional integrability condition: the two complex structures have to commute! In the previous section we showed that a second supersymmetry required the existence of two complex structures, one for each chirality. Furthermore we saw that off-shell closure of the algebra entails the two complex structures to commute, which implies that only in this case no further auxiliary fields are needed. Here we get the same result from a purely kinematic point of view: imposing constraints of both chiralities on a general (2, 2) superfield reduces the degrees of freedom of that field to those of a (1, 1) superfield but integrability of these constraints requires the existence of two, mutually commuting complex structures.

Take now J and \bar{J} two commuting complex structures. This is sufficient to obtain full integrability of both J and \bar{J} simultanously. Then we can always find a coordinate transformation such that both J and \bar{J} are diagonal. As the eigenvalues, $\pm i$, of J and \bar{J} can be combined in four different ways, we get the four basic superfields:

1. chiral superfield:

$$\widehat{D}\Phi = -D\Phi, \qquad \widehat{\overline{D}}\Phi = -\overline{D}\Phi.$$
 (3.12)

2. anti-chiral superfield:

$$\widehat{D}\Phi = +D\Phi, \qquad \widehat{\bar{D}}\Phi = +\bar{D}\Phi.$$
 (3.13)

3. twisted chiral superfield:

$$\widehat{D}\Phi = -D\Phi, \qquad \widehat{\overline{D}}\Phi = +\overline{D}\Phi.$$
 (3.14)

4. twisted anti-chiral superfield:

$$\widehat{D}\Phi = +D\Phi, \qquad \widehat{\bar{D}}\Phi = -\bar{D}\Phi.$$
 (3.15)

From the previous, it follows that $\ker(J-\bar{J})$ corresponds to (anti-)chiral superfields and $\ker(J+\bar{J})$ to twisted (anti-)chiral superfields. So we arrive at the main conclusion of this section: constraining both chiralities of a general (2,2) superfield reduces the degrees of freedom to those of a (1,1) superfield and there is always a coordinate transformation such that these fields reduce to (anti-)chiral and twisted (anti-)chiral fields. In particular, there is no way to mimick the spectral flow at the level of fields, i.e. to continously interpolate between chiral and twisted chiral fields.

Consider now a real potential $K(\phi)$ which is a function of m chiral fields ϕ^{α} , m anti-chiral fields $\phi^{\bar{\alpha}}$, n twisted fields ϕ^{μ} and n twisted anti-chiral fields $\phi^{\bar{\mu}}$, $\alpha, \bar{\alpha} \in \{1, \dots, m\}$, $\mu, \bar{\mu} \in \{1, \dots, n\}$. Using eq. (3.15), we immediately obtain the action in (1, 1) superspace:

$$S = \int d^{2}z d^{2}\theta d^{2}\hat{\theta} K(\phi)$$

$$= -2 \int d^{2}z d^{2}\theta \left(K_{\alpha\bar{\beta}} (D\phi^{\alpha}\bar{D}\phi^{\bar{\beta}} + D\phi^{\bar{\beta}}\bar{D}\phi^{\alpha}) - K_{\mu\bar{\nu}} (D\phi^{\mu}\bar{D}\phi^{\bar{\nu}} + D\phi^{\bar{\nu}}\bar{D}\phi^{\mu}) \right) +$$

$$+2 \int d^{2}z d^{2}\theta \left(K_{\alpha\bar{\nu}} (D\phi^{\alpha}\bar{D}\phi^{\bar{\nu}} - D\phi^{\bar{\nu}}\bar{D}\phi^{\alpha}) - K_{\mu\bar{\beta}} (D\phi^{\mu}\bar{D}\phi^{\bar{\beta}} - D\phi^{\bar{\beta}}\bar{D}\phi^{\mu}) \right),$$
(3.16)

where

$$K_{ab} \equiv \frac{\partial^2 K(\phi)}{\partial \phi^a \partial \phi^b},\tag{3.17}$$

and $\alpha, \bar{\alpha}, \beta, \bar{\beta} \in \{1, \dots, m\}$, $\mu, \bar{\mu}, \nu, \bar{\nu} \in \{1, \dots, n\}$. Comparing this with eq. (2.1), we can read off the metric g_{ab} and the torsion potential b_{ab} . Restricting ourselves to the case where either the chiral or the twisted chiral fields are absent yields the standard Kähler geometry. When both types of fields are present we recognize from the fact that only $g_{\alpha\bar{\beta}}$ and $g_{\mu\bar{\nu}}$ are non-vanishing the product structure⁵. Note that the potential $K(\phi)$ in the (2, 2) action eq. (3.16) is only determined modulo a generalized Kähler transformation:

$$K(\phi) \simeq K(\phi) + f(\phi^{\alpha}, \phi^{\nu}) + g(\phi^{\alpha}, \phi^{\bar{\nu}}) + \bar{f}(\phi^{\bar{\alpha}}, \phi^{\bar{\nu}}) + \bar{g}(\phi^{\bar{\alpha}}, \phi^{\nu}). \tag{3.18}$$

Having exhausted the case where both chiralities of a general (2, 2) superfield were constrained, we turn in the next section to fields where only one of the chiralities gets constrained.

3.2. Semi-chiral superfields

Having a set of general superfields ϕ^a , $a \in \{1, \dots, 2d\}$, we constrain only one chirality as in eq. (3.8). We perform a coordinate transformation such that J becomes diagonal: $J^{\alpha}{}_{\beta} = i\delta^{\alpha}_{\beta}$ and $J^{\bar{\alpha}}{}_{\bar{\beta}} = -i\delta^{\bar{\alpha}}_{\bar{\beta}}$. Thus we get from eq. (3.8):

$$\widehat{D}\phi^{\alpha} = -D\phi^{\alpha}, \qquad \widehat{D}\phi^{\bar{\alpha}} = D\phi^{\bar{\alpha}}.$$
 (3.19)

⁵However, this doesn't imply that the resulting manifold can be written as the product of two manifolds. This would imply that e.g. $g_{\alpha\bar{\beta},\mu}=g_{\alpha\bar{\beta},\bar{\mu}}=g_{\mu\bar{\nu},\alpha}=g_{\mu\bar{\nu},\bar{\alpha}}=0$ which is not necessarily true here.

Introduce the notation

$$\chi^{\alpha} \equiv \widehat{\bar{D}}\phi^{\alpha}, \qquad \chi^{\bar{\alpha}} \equiv \widehat{\bar{D}}\phi^{\bar{\alpha}}, \tag{3.20}$$

with $\alpha, \bar{\alpha} \in \{1, \dots, d\}$. We take a real potential $K(\phi)$ and we pass from (2, 2) to (1, 1) superspace,

$$S = \int d^2z d^2\theta d^2\hat{\theta} K(\phi)$$

$$= -2 \int d^2z d^2\theta K_{\alpha\bar{\beta}} \left(\chi^{\alpha} D \phi^{\bar{\beta}} - \chi^{\bar{\beta}} D \phi^{\alpha} \right). \tag{3.21}$$

We see that the χ fields appear algebraically, but eliminating them through their equations of motion does not lead to a non-linear σ -model. So we take a set of fields ϕ^a , $a \in \{1, \dots d\}$ and another set $\phi^{a'}$, $a' \in \{d+1, \dots d+d'\}$ on which we impose the constraints:

$$\widehat{D}\phi^a = iJ^a{}_B D\phi^B, \qquad \widehat{\bar{D}}\phi^{a'} = iJ^{a'}{}_B \bar{D}\phi^B, \qquad (3.22)$$

where $B \in \{1, \dots d + d'\}$. Integrability of these constraints gives the conditions:

$$J^{a}{}_{b'} = \bar{J}^{a'}{}_{b} = 0, \qquad J^{a}{}_{b,c'} = \bar{J}^{a'}{}_{b',c} = 0,$$
 (3.23)

and $J^a{}_b$ and $\bar{J}^a{}'_{b'}$ are complex structures. Through a coordinate transformation we can diagonalize these structures. A similar reasoning as at the beginning of this section gives that only when d=d' do we get a non-linear σ -model. Furthermore the fact that both J and \bar{J} are complex structures requires d to be even. So one semichiral multiplet corresponds with four real dimensions. Bringing all this together leads to the second main result of this paper: the most general non-linear σ -model in standard N=(2,2) superspace is formulated in terms of chiral, twisted chiral and semi-chiral superfields.

We now study such a model. We take m chiral, ϕ^{μ} and anti-chiral, $\phi^{\bar{\mu}}$, n twisted chiral $\phi^{\bar{\mu}}$ and twisted anti-chiral $\phi^{\bar{\mu}}$ superfields. In addition we take d semi-chiral multiplets $\{\phi^{\alpha}, \phi^{\bar{\alpha}}, \phi^{\bar{\alpha}}, \phi^{\bar{\alpha}}\}$, with $\mu, \bar{\mu} \in \{1, \dots, m\}$, $\tilde{\mu}, \tilde{\bar{\mu}} \in \{1, \dots, n\}$, $\alpha, \bar{\alpha}, \tilde{\alpha}, \tilde{\alpha} \in \{1, \dots, d\}$. The defining relations of the (anti-)chiral and twisted (anti-)chiral fields are given in eq. (3.12-3.15) and the semi-chiral fields are defined by:

$$\widehat{D}\phi^{\alpha} = -D\phi^{\alpha} \qquad \& \qquad \psi^{\alpha} \equiv \widehat{\overline{D}}\phi^{\alpha}
\widehat{D}\phi^{\bar{\alpha}} = D\phi^{\bar{\alpha}} \qquad \& \qquad \psi^{\bar{\alpha}} \equiv \widehat{\overline{D}}\phi^{\bar{\alpha}}
\chi^{\tilde{\alpha}} \equiv \widehat{D}\phi^{\tilde{\alpha}} \qquad \& \qquad \widehat{\overline{D}}\phi^{\tilde{\alpha}} = \bar{D}\phi^{\tilde{\alpha}}
\chi^{\tilde{\alpha}} \equiv \widehat{D}\phi^{\tilde{\alpha}} \qquad \& \qquad \widehat{\overline{D}}\phi^{\tilde{\alpha}} = -\bar{D}\phi^{\tilde{\alpha}}.$$
(3.24)

Taking an arbitrary potential K which is a function of all the previously mentioned fields, we get the σ -model in (1,1) superspace:

$$S = \int d^2z d^2\theta d^2\hat{\theta} K(\phi)$$

$$= -2 \int d^{2}z d^{2}\theta \Big\{ D\Phi_{c} N_{1} \bar{D}\Phi_{c} - D\Phi_{t} N_{2} \bar{D}\Phi_{t} - D\Phi_{c} N_{3} \bar{D}\Phi_{t} + D\Phi_{t} N_{3}^{T} \bar{D}\Phi_{c} \Big\} - \int d^{2}z d^{2}\theta \Big\{ \chi^{T} L\psi - \bar{D}\eta^{T} P L P D\phi - \chi^{T} (L\psi - P L \bar{D}\phi - 2P \widetilde{M} \bar{D}\eta + 2R_{3} \bar{D}\Phi_{c} + 2R_{4} \bar{D}\Phi_{t}) - (\chi^{T} L + D\eta^{T} L P - 2D\phi^{T} P M + 2D\Phi_{c}^{T} R_{1}^{T} P + 2D\Phi_{t}^{T} R_{2}^{T} P)\psi + 2D\phi^{T} R_{1} \bar{D}\Phi_{c} - 2D\phi^{T} R_{2} \bar{D}\Phi_{t} + 2D\Phi_{c}^{T} P S_{1}^{T} P \bar{D}\eta + D\Phi_{t}^{T} P S_{2}^{T} P \bar{D}\eta + D\eta^{T} S_{1} \bar{D}\Phi_{c} - D\eta^{T} S_{2} \bar{D}\Phi_{t} \Big\},$$
(3.25)

where N_1 , N_2 and N_3 are $2m \times 2m$, $2n \times 2n$ and $2m \times 2n$ matrices,

$$N_1 \equiv \begin{pmatrix} 0 & K_{\mu\bar{\nu}} \\ K_{\bar{\mu}\nu} & 0 \end{pmatrix}, \qquad N_2 \equiv \begin{pmatrix} 0 & K_{\tilde{\mu}\tilde{\nu}} \\ K_{\tilde{\tilde{\mu}}\tilde{\nu}} & 0 \end{pmatrix}, \qquad N_3 \equiv \begin{pmatrix} 0 & K_{\mu\tilde{\nu}} \\ K_{\bar{\mu}\tilde{\nu}} & 0 \end{pmatrix}, (3.26)$$

where L, M and \widetilde{M} are $2d \times 2d$ matrices

$$L \equiv \begin{pmatrix} K_{\widetilde{\alpha}\beta} & K_{\widetilde{\alpha}\overline{\beta}} \\ K_{\widetilde{\alpha}\beta} & K_{\widetilde{\alpha}\overline{\beta}} \end{pmatrix}, \qquad \widetilde{M} \equiv \begin{pmatrix} 0 & K_{\widetilde{\alpha}\overline{\beta}} \\ K_{\widetilde{\alpha}\widetilde{\beta}} & 0 \end{pmatrix}, \qquad M \equiv \begin{pmatrix} 0 & K_{\alpha\overline{\beta}} \\ K_{\overline{\alpha}\beta} & 0 \end{pmatrix}, \quad (3.27)$$

 $S_1,$ a $2d\times 2m$ matrix, S_2 a $2d\times 2n$ matrix,

$$S_{1} \equiv \begin{pmatrix} K_{\widetilde{\alpha}\mu} & K_{\widetilde{\alpha}\overline{\mu}} \\ K_{\widetilde{\widetilde{\alpha}}\mu} & K_{\widetilde{\widetilde{\alpha}}\overline{\mu}} \end{pmatrix}, \qquad S_{2} \equiv \begin{pmatrix} K_{\widetilde{\alpha}\mu} & K_{\widetilde{\alpha}\overline{\mu}} \\ K_{\widetilde{\widetilde{\alpha}}\mu} & K_{\widetilde{\widetilde{\alpha}}\overline{\mu}} \end{pmatrix}, \tag{3.28}$$

and the $2d \times 2m$ matrices R_1 and R_3 and the $2d \times 2n$ matrices R_2 and R_4 together with P:

$$R_{1} \equiv \begin{pmatrix} 0 & K_{\alpha\bar{\mu}} \\ K_{\bar{\alpha}\mu} & 0 \end{pmatrix}, \qquad R_{2} \equiv \begin{pmatrix} 0 & K_{\alpha\tilde{\mu}} \\ K_{\bar{\alpha}\tilde{\mu}} & 0 \end{pmatrix}, \qquad R_{3} \equiv \begin{pmatrix} -K_{\tilde{\alpha}\mu} & 0 \\ 0 & K_{\tilde{\alpha}\bar{\mu}} \end{pmatrix},$$

$$R_{4} \equiv \begin{pmatrix} 0 & -K_{\tilde{\alpha}\tilde{\mu}} \\ K_{\tilde{\alpha}\tilde{\mu}} & 0 \end{pmatrix}, \qquad P \equiv \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}.$$
(3.29)

 Φ_c and Φ_t are $2m \times 1$ and $2n \times 1$ matrices resp. and ϕ and η are $2d \times 1$ matrices:

$$\Phi_c \equiv \begin{pmatrix} \phi^{\mu} \\ \phi^{\bar{\mu}} \end{pmatrix}, \qquad \Phi_t \equiv \begin{pmatrix} \phi^{\widetilde{\mu}} \\ \phi^{\widetilde{\mu}} \end{pmatrix}, \qquad \phi \equiv \begin{pmatrix} \phi^{\alpha} \\ \phi^{\bar{\alpha}} \end{pmatrix}, \qquad \eta \equiv \begin{pmatrix} \phi^{\widetilde{\alpha}} \\ \phi^{\widetilde{\alpha}} \end{pmatrix}. \tag{3.30}$$

Also in this case we get that the potential is determined modulo a generalized Kähler transformation:

$$K(\phi) \simeq K(\phi) + f(\phi^{\mu}, \phi^{\widetilde{\mu}}, \phi^{\alpha}) + g(\phi^{\mu}, \phi^{\widetilde{\overline{\mu}}}, \phi^{\widetilde{\alpha}}) + \bar{f}(\phi^{\overline{\mu}}, \phi^{\widetilde{\overline{\mu}}}, \phi^{\widetilde{\alpha}}) + \bar{g}(\phi^{\overline{\mu}}, \phi^{\widetilde{\mu}}, \phi^{\widetilde{\alpha}}). (3.31)$$

One finds that the equations of motion for χ and ψ are

$$L\psi = PL\bar{D}\phi + 2P\tilde{M}\bar{D}\eta - 2R_3\bar{D}\Phi_c - 2R_4\bar{D}\Phi_t,$$

$$\chi^T L = -D\eta^T LP + 2D\phi^T PM - 2D\Phi_c^T R_1^T P - 2D\Phi_t^T R_2^T P.$$
 (3.32)

Assuming that L is invertible, one solves eq. (3.32) for the auxiliary fields ψ and χ . The action in the second order formalism is given by

$$S = -2 \int d^{2}z d^{2}\theta \Big\{ D\Phi_{c}N_{1}\bar{D}\Phi_{c} - D\Phi_{t}N_{2}\bar{D}\Phi_{t} - D\Phi_{c}N_{3}\bar{D}\Phi_{t} + D\Phi_{t}N_{3}^{T}\bar{D}\Phi_{c} \Big\} + \int d^{2}z d^{2}\theta \Big\{ \bar{D}\eta^{T}PLPD\phi - 2D\phi^{T}R_{1}\bar{D}\Phi_{c} + 2D\phi^{T}R_{2}\bar{D}\Phi_{t} - 2D\Phi_{c}^{T}PS_{1}^{T}P\bar{D}\eta - D\Phi_{t}^{T}PS_{2}^{T}P\bar{D}\eta - D\eta^{T}S_{1}\bar{D}\Phi_{c} + D\eta^{T}S_{2}\bar{D}\Phi_{t} + \Big(D\eta^{T}LP - 2D\phi^{T}PM + 2D\Phi_{c}^{T}R_{1}^{T}P + 2D\Phi_{t}^{T}R_{2}^{T}P\Big) L^{-1}\Big(PL\bar{D}\phi + 2P\bar{M}\bar{D}\eta - 2R_{3}\bar{D}\Phi_{c} - 2R_{4}\bar{D}\Phi_{t}\Big) \Big\}.$$
(3.33)

Introducing the notation

$$\Phi \equiv \begin{pmatrix} \phi \\ \eta \\ \Phi_c \\ \Phi_t \end{pmatrix},$$
(3.34)

one reads off the complex structures:

$$J = \begin{pmatrix} iP & 0 & 0 & 0 & 0 \\ -2iL^{-1T}MP & iL^{-1T}PL^{T} & 2iL^{-1T}PR_{1} & 2iL^{-1T}PR_{2} \\ 0 & 0 & iP & 0 \\ 0 & 0 & 0 & iP \end{pmatrix}$$

$$\bar{J} = \begin{pmatrix} -iL^{-1}PL & -2iL^{-1}P\widetilde{M} & 2iL^{-1}R_{3} & 2iL^{-1}R_{4} \\ 0 & -iP & 0 & 0 \\ 0 & 0 & iP & 0 \\ 0 & 0 & 0 & -iP \end{pmatrix}.$$
(3.35)

4. N=(2,2) non-linear σ -models

An obvious question which now arises is whether all N=(2,2) non-linear σ -models can be described using the previously constructed superfields. An important result obtained in [9] states that $\ker(J-\bar{J})$ and $\ker(J+\bar{J})$ are always integrable. This allows us to parametrize these kernels by chiral and twisted-chiral fields resp. If we now choose such preferred coordinates we can restate the result by saying that the levels where these coordinates are constant yield submanifolds where $(J\pm\bar{J})$ are non-degenerate. So whatever is left should then be described by semi-chiral fields.

For simplicity, we focus on manifolds where $\ker[J, \bar{J}] = \emptyset$. One would expect those to be described by semi-chiral superfields. Let us first determine what the implications are of a σ -model solely described by semi-chiral fields and where $\ker[J, \bar{J}] \neq \emptyset$. Given an arbitrary vector ξ , we decompose it as $\xi = \xi_0 + \xi_1$ where $[J, \bar{J}]\xi_0 = 0$ and $[J, \bar{J}]\xi_1 \neq 0$. Writing the ξ_0^a components as ζ and the $\xi_0^{\tilde{a}}$ components as $\tilde{\zeta}$, one derives from eq. (3.35) that ξ_0 satisfies either

$$\{P, L\}\zeta = -2P\widetilde{M}\widetilde{\zeta}, \qquad \{P, L^T\}\widetilde{\zeta} = -2PM\zeta,$$
 (4.1)

or

$$[P, L]\zeta = -2P\widetilde{M}\widetilde{\zeta}, \qquad [P, L^T]\widetilde{\zeta} = -2PM\zeta.$$
 (4.2)

Using this and the explicit form for the action, eq. (3.33), one shows that the metric is degenerate: $g(\xi_0, \xi') = 0$, where ξ' is an arbitrary vector. Presumably, having $\ker[J, \bar{J}] \neq \emptyset$ points towards the existence of gauge invariances such as were studied in [10].

From now on we assume that the metric is non-degenerate, and as such that $\ker[J, \bar{J}] = \emptyset$. A necessary and sufficient condition for the latter is

$$\det \mathcal{N}_1 \neq 0, \qquad \det \mathcal{N}_2 \neq 0, \tag{4.3}$$

with

$$\mathcal{N}_{1} \equiv \begin{pmatrix} K_{\widetilde{\alpha}\beta} & K_{\widetilde{\alpha}\widetilde{\beta}} \\ K_{\bar{\alpha}\beta} & K_{\bar{\alpha}\widetilde{\beta}} \end{pmatrix}, \qquad \mathcal{N}_{2} \equiv \begin{pmatrix} K_{\widetilde{\alpha}\overline{\beta}} & K_{\widetilde{\alpha}\widetilde{\beta}} \\ K_{\alpha\bar{\beta}} & K_{\alpha\tilde{\beta}} \end{pmatrix}. \tag{4.4}$$

We turn back to a complex manifold where $\ker[J, \bar{J}] = \emptyset$. A necessary condition for a description of such a manifold in terms of semi-chiral fields is that its (real) dimension is a multiple of 4. Let us give a local argument why this is true. Consider the following problem: given a $2n \times 2n$ matrix J_+ , such that $J_+^2 = -1$. We choose the canonical form for it:

$$J_{+} = i \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}. \tag{4.5}$$

Consider now a second $2n \times 2n$ matrix J_{-} , such that the standard (flat) metric is hermitean. Its most general form is given by

$$J_{-} = \begin{pmatrix} A & B \\ B^* & A^* \end{pmatrix}, \quad \text{with } A^{\dagger} = -A, \quad B^T = -B.$$
 (4.6)

We define \mathcal{C} as

$$C \equiv [J_+, J_-] = \begin{pmatrix} 0 & 2iB \\ -2iB^* & 0 \end{pmatrix}, \tag{4.7}$$

and assume that $\ker \mathcal{C} = \emptyset$, or in other words $\det \mathcal{C} \neq 0$. One computes,

$$\det \mathcal{C} = (-1)^n 4^n |\det B|^2. \tag{4.8}$$

As C is anti-hermitean, it has imaginary eigenvalues. Furthermore from the form of C, eq. (4.7), one easily gets that if λ is an eigenvalue, then so is $-\lambda$. Combining all of this yields $\det C > 0$. Comparing this to eq. (4.8), requires n to be even.

Having established that the dimension of our manifold is a multiple of 4, we will now put J in its standard form

$$J = i \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}. \tag{4.9}$$

The constancy of J implies

$$\Gamma^{\alpha}_{-c\bar{\beta}} = 0,$$

$$\Gamma_{-\alpha\beta\bar{\gamma}} = -2T_{\alpha\beta\bar{\gamma}},$$

$$\Gamma_{-\bar{\alpha}\beta\gamma} = g_{\bar{\alpha}\beta,\gamma},$$

$$T_{\alpha\beta\gamma} = 0.$$
(4.10)

The vanishing of the Nijenhuis tensor for J imposes no further restrictions. From eq. (4.10), one obtains that locally, the metric and the torsion can be expressed in terms of a vector potential k:

$$g_{\alpha\bar{\beta}} = \frac{1}{2} (k_{\alpha,\bar{\beta}} + k_{\bar{\beta},\alpha}),$$

$$T_{\alpha\beta\bar{\gamma}} = -\frac{1}{4} (k_{\alpha,\beta} - k_{\beta,\alpha})_{,\bar{\gamma}},$$

$$(4.11)$$

and the complex conjugates of these expressions. The vector field k_a is determined modulo

$$k_{\alpha} \simeq k_{\alpha} + f_{\alpha} + ig_{,\alpha},$$
 (4.12)

where $f_{\alpha,\bar{\beta}} = 0$ and g is a real scalar function.

We now analyze the consequences for \bar{J} . As $[J, \bar{J}]$ is non-degenerate, we can parametrize \bar{J} as

$$\bar{J} = \begin{pmatrix} a & b \\ -b^{-1}(1+a^2) & -b^{-1}ab \end{pmatrix}, \tag{4.13}$$

and $a^2 \neq -1$. Hermiticity of the metric is satisfied iff.

$$b_{\alpha\beta} = -b_{\beta\alpha}, \qquad a^{\alpha}{}_{\gamma}b^{\gamma\beta} + a^{\beta}{}_{\gamma}b^{\gamma\alpha} = 0.$$
 (4.14)

The constancy of \bar{J} gets translated into the following conditions:

$$b^{\alpha\beta}_{,\bar{\gamma}} = 0.$$

$$T_{\alpha\beta\bar{\gamma}} = -\frac{1}{2}(b^{-1}a)_{\alpha\beta,\bar{\gamma}},$$
(4.15)

and their complex conjugate. Finally, the vanishing of the Nijenhuis tensor imposes further restrictions. Introducing the notation

$$x_{\alpha\beta} \equiv \left(b^{-1}(a+i)\right)_{\alpha\beta}, \quad x_{\bar{\alpha}\bar{\beta}} \equiv x_{\alpha\beta}^* = \left((a-i)^{-1}b\right)_{\bar{\alpha}\bar{\beta}},$$
 (4.16)

one concisely writes the Nijenhuis conditions as

$$x^{\bar{\delta}}_{[\alpha} x_{\beta\gamma],\bar{\delta}} = x_{\bar{\delta}\bar{\varepsilon},[\alpha} x^{\bar{\delta}}_{\beta} x^{\bar{\varepsilon}}_{\gamma]}. \tag{4.17}$$

We now want to compare this with what we get from a semi-chiral description. The easiest way to obtain the vectors k is to pass to (2,1) superspace. In general, if

one has a N=(2,1) non-linear σ -model, one gets by combining eqs. (2.1) and (4.11) that it is given in N=(2,1) superspace by

$$S = \frac{1}{2} \int d^2z d^2\theta d\hat{\theta} \left(k_{\alpha} \bar{D} \phi^{\alpha} - k_{\bar{\alpha}} \bar{D} \phi^{\bar{\alpha}} \right), \tag{4.18}$$

where ϕ are (2,1) chiral fields:

$$\widehat{D}\phi^{\alpha} = -D\phi^{\alpha}, \qquad \widehat{D}\phi^{\bar{\alpha}} = D\phi^{\bar{\alpha}}.$$
 (4.19)

Starting now from a potential in terms of semi-chiral fields, we perform the integration over $\hat{\bar{\theta}}$ and obtain

$$S = \int d^2z d^2\theta d\hat{\theta} \left(K_a \psi^a + K_{\widetilde{\alpha}} \bar{D} \phi^{\widetilde{\alpha}} - K_{\widetilde{\alpha}} \bar{D} \phi^{\widetilde{\alpha}} \right). \tag{4.20}$$

The equations of motion for ψ imply that $\hat{\phi}_a \equiv K_a$ is a chiral (2,1) field. This is nothing but a reflection of the fact that the coordinate transformation

$$\phi^{\alpha} \to \hat{\phi}^{\alpha} = \phi^{\alpha} \qquad \phi^{\tilde{\alpha}} \to \hat{\phi}_{\alpha} = K_{\alpha}$$

$$\phi^{\bar{\alpha}} \to \hat{\phi}^{\bar{\alpha}} = \phi^{\bar{\alpha}} \qquad \phi^{\tilde{\alpha}} \to \hat{\phi}_{\bar{\alpha}} = K_{\bar{\alpha}}, \tag{4.21}$$

diagonalizes J. Obviously this coordinate transformation is not compatible with the original semi-chiral nature of the fields. Integrating over ψ in eq. (4.20), we get the (2,1) action:

$$S = \int d^2z d^2\theta d\hat{\theta} \left(K_{\tilde{a}} P^{\tilde{a}}_{\tilde{b}} (K^{-1})^{\tilde{b}c} (\bar{D}\hat{\phi}_c - K_{cd}\bar{D}\hat{\phi}^d) \right), \tag{4.22}$$

from which one reads the vectors k. In terms of the *original* semi-chiral coordinates, they become particularly simple

$$\widetilde{k} = LPL^{-1}P\widetilde{K}, \qquad k = 2ML^{-1}\widetilde{k},$$

$$(4.23)$$

where

$$\widetilde{K} \equiv \begin{pmatrix} K_{\widetilde{\alpha}} \\ K_{\widetilde{\widetilde{\alpha}}} \end{pmatrix}, \quad \widetilde{k} \equiv \begin{pmatrix} k_{\widetilde{\alpha}} \\ k_{\widetilde{\widetilde{\alpha}}} \end{pmatrix}, \quad k \equiv \begin{pmatrix} k_{\alpha} \\ k_{\bar{\alpha}} \end{pmatrix}.$$
 (4.24)

So in order to show that a description in terms of semi-chiral fields is possible, one has to show that there exists a coordinate system such that the solution to eqs. (4.14), (4.15) and (4.17) is given by eq. (4.23).

Though we are not able to show this, we focus on the special case where the Nijenhuis conditions are trivially satisfied. This requires that the torsion vanishes, which, as eq. (4.15) shows, is so if a in eq. (4.13) vanishes. Combining this with $\{J, \bar{J}\} = 0$ and eq. (2.13) gives that the manifold is hyper-Kähler. In fact for any hyper-Kähler manifold, one has $\ker[J, \bar{J}] = \emptyset$. Note however that we do know how to put σ -models on hyper-Kähler manifolds in (2, 2) superspace, after all they are

Kähler manifolds and can be described in terms of chiral fields. However this is only possible if we choose the left complex structure to be the same as the right one. It is clear that on a hyper-Kähler manifold, the left and right complex structure can be chosen differently. Precisely this case is being investigated here. Using eq. (3.35), one shows that the necessary and sufficient conditions for J and \bar{J} to anti-commute are given by:

$$L^{-1}PLP + PL^{-1}PL = 4L^{-1}P\widetilde{M}L^{-1T}MP$$

$$\{P, L^{-1T}MPL^{-1}\} = \{P, L^{-1}P\widetilde{M}L^{-1T}\} = 0.$$
 (4.25)

Restricting ourselves to d = 4, we find that the two latter eqs. are trivially satisfied while the former becomes:

$$|K_{\phi\eta}|^2 + |K_{\phi\bar{\eta}}|^2 = 2K_{\eta\bar{\eta}}K_{\phi\bar{\phi}}.$$
(4.26)

It is known that a 4-dimensional Kähler manifold is hyper-Kähler iff. the Kähler potential satisfies the Monge-Ampère equation. So a concrete way to test our hypothesis would be to show that there is a coordinate system where eq. (4.26) is equivalent to the Monge-Ampère equation. In [11], we checked explicitly that a non-trivial class of hyper-Kähler manifolds, the special hyper-Kähler manifolds [12], allows for a semi-chiral parametrization. If it turns out to be true that all hyper-Kähler manifolds can be described with semi-chiral coordinates, then this would provide a non-trivial result: not only the metric can be computed from the semi-chiral potential but all three complex structures as well!

To end this section, we answer the question whether for hyper-Kähler manifolds, the Kähler and the semi-chiral potential are identical. An easy manifold to verify this is flat 4n-dimensional space. The Kähler potential is given by

$$K_{Kahler} = \sum_{i=1}^{n} (x^{i} \bar{x}^{i} + v^{i} \bar{v}^{i}),$$
 (4.27)

and labeling rows and columns as x, \bar{x}, v, \bar{v} we have the complex structures:

$$J = \begin{pmatrix} i\sigma_3 & 0\\ 0 & i\sigma_3 \end{pmatrix}, \quad \bar{J} = \begin{pmatrix} 0 & -\sigma_2\\ \sigma_2 & 0 \end{pmatrix}. \tag{4.28}$$

A coordinate transformation which brings us to the semi-chiral parametrization is

$$x^{i} \to \phi^{i} = x^{i}, \qquad \bar{x}^{i} \to \bar{\phi}^{i} = \bar{x}^{i}$$

$$v^{i} \to \eta^{i} = \bar{x}^{i} + v^{i}, \qquad \bar{v}^{i} \to \bar{\eta}^{i} = x^{i} + \bar{v}^{i}, \tag{4.29}$$

and labeling rows and columns as ϕ , $\bar{\phi}$, η and $\bar{\eta}$ we get the complex structures in semi-chiral coordinates:

$$J = \begin{pmatrix} i\sigma_3 & 0 \\ 2\sigma_2 & i\sigma_3 \end{pmatrix}, \quad \bar{J} = \begin{pmatrix} -i\sigma_3 & -\sigma_2 \\ 0 & -i\sigma_3 \end{pmatrix}. \tag{4.30}$$

Comparing the complex structures and metric to the general expressions yields the potential:

$$K_{semi} \propto \sum_{i=1}^{n} (2\phi^{i}\bar{\phi}^{i} + \eta^{i}\bar{\eta}^{i} - 2\phi^{i}\eta^{i} - 2\bar{\phi}^{i}\bar{\eta}^{i}).$$
 (4.31)

Taking into account that the Kähler potential is only defined modulo a Kähler transformation and the semi-chiral potential has an ambiguity expressed in eq. (3.31), one verifies that both potential are *not* equivalent.

5. Examples

Supersymmetric Wess-Zumino-Witten models on even dimensional group manifolds are interesting examples of (2,2) supersymmetric non-linear σ -models. They are conformally invariant, they have a large number of isometries and they are torsionful. We can write $J=L^{-1}jL$ and $\bar{J}=R^{-1}\bar{j}R$ where L and R are the left, right resp., invariant vielbeins. The (constant) complex structures j and \bar{j} act on the Lie algebra.

In [3], the integrability conditions for a complex structure on a semi-simple Lie algebra were solved. The action of the complex structure is almost completely determined by a Cartan decomposition: the complex structure has eigenvalue +i (-i resp.) on generators corresponding with positive (negative resp.) roots. The only freedom left is the action of the complex structure on the Cartan subalgebra. Except for the requirement that the structure maps the CSA bijectively to itself, no further conditions have to be imposed. As any two Cartan decompositions are related through an inner automorphism, we can state that the complex structure on a Lie algebra is uniquely determined except for its action on the CSA which is left invariant by the complex structure.

However the resulting left and right complex structures on the *group* do not necessarily commute. In fact, in [5] it was shown that only on $SU(2) \times U(1)$ a choice for j and \bar{j} can be made such that $[J, \bar{J}] = 0$. There a description in terms of a chiral and a twisted chiral field is possible.

In the next we give 3 examples of (2,2) WZW models in superspace: the $SU(2) \times U(1)$ model in terms of a chiral and a twisted chiral field, making a different choice for the complex structures we obtain the same model but now in terms of a semi-chiral multiplet and finally $SU(2) \times SU(2)$ as an example of a model having both a chiral and a semi-chiral multiplet.

5.1. The $SU(2) \times U(1)$ WZW model in terms of a chiral and a twisted chiral multiplet We give a brief summary of the results of [5]. There it was shown that for a particular choice of the complex structures one could formulate the model in terms of a chiral ϕ , an anti-chiral $\bar{\phi}$, a twisted chiral χ and a twisted anti-chiral $\bar{\chi}$ superfield. A group

element is written as

$$g = \frac{e^{i\theta}}{\sqrt{|\phi|^2 + |\chi|^2}} \begin{pmatrix} \chi & \bar{\phi} \\ -\phi & \bar{\chi} \end{pmatrix}, \tag{5.1}$$

with

$$\theta = -\frac{1}{2} \ln \left(|\phi|^2 + |\chi|^2 \right). \tag{5.2}$$

The potential is then given by

$$K = -\int_{|\phi|^2} \frac{|\chi|^2}{\zeta} \frac{d\zeta}{\zeta} \ln(1+\zeta) + \frac{1}{2} \left(\ln(\phi\bar{\phi})\right)^2.$$
 (5.3)

In order to achieve this one has to make a different choice for the left and the right complex structures on the *Lie algebra*. The only difference resides in its action on the CSA: there it differs by a sign.

5.2. The $SU(2) \times U(1)$ WZW model in terms of a semi-chiral multiplet

In [9] an implicit description of the $SU(2) \times U(1)$ WZW model was given but now in terms of a semi-chiral multiplet. Here we give the explicit description of the model using one semi-chiral multiplet. For this we now choose the complex structures on the Lie algebra to be equal.

We parametrize $SU(2) \times U(1)$ by:

$$g = e^{\frac{i}{2}\theta_L \sigma_3} e^{\frac{i}{2}\zeta \sigma_2} e^{\frac{i}{2}\theta_R \sigma_3} e^{\frac{i}{2}\chi \sigma_0}, \tag{5.4}$$

with σ_i (i = 1, 2, 3) the Pauli matrices en σ_0 the 2×2 unit matrix. We now introduce a semi-chiral multiplet parametrized by ϕ , $\bar{\phi}$, η and $\bar{\eta}$. These fields are related to the previously introduced coordinates by

$$\phi = \bar{z}_1, \qquad \eta = z_1 - \bar{z}_2, \qquad \bar{\phi} = z_1, \qquad \bar{\eta} = \bar{z}_1 - z_2,$$
 (5.5)

where,

$$z_1 = i\chi - 2i\ln\cos\zeta/2 + \theta_L + \theta_R,$$

$$z_2 = -i\chi + 2i\ln\sin\zeta/2 + \theta_L - \theta_R,$$
(5.6)

and \bar{z}_1 , \bar{z}_2 are given by the complex conjugate of these. The inverse transformations are,

$$\zeta = 2 \arctan(\exp{-\frac{i}{4}(z_1 - \bar{z}_1 + z_2 - \bar{z}_2)}),$$

$$\theta_L = \frac{1}{4}(z_1 + \bar{z}_1 + z_2 + \bar{z}_2),$$

$$\theta_R = \frac{1}{4}(z_1 + \bar{z}_1 - z_2 - \bar{z}_2),$$

$$\chi = -\ln(\exp{\frac{i}{2}(z_1 - \bar{z}_1)} + \exp{\frac{i}{2}(-z_2 + \bar{z}_2)}).$$
(5.7)

Ordening the coordinates as $(\phi, \bar{\phi}, \eta, \bar{\eta})$, one computes the complex structures:

$$J = i \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 2 & 0 & 0 & -1 \end{pmatrix}, \tag{5.8}$$

$$\bar{J} = i \begin{pmatrix} -1 & 0 & 0 & 2\sin^2 \zeta/2 \\ 0 & 1 & -2\sin^2 \zeta/2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$
 (5.9)

The potential can be calculated and this gives the following simple expression involving a dilogarithm,

$$K = -\phi\bar{\phi} + \bar{\phi}\bar{\eta} + \phi\eta - 2i\int^{\bar{\eta}-\eta} dx \ln(1 + \exp\frac{i}{2}x). \tag{5.10}$$

5.3. The $SU(2) \times SU(2)$ WZW model

As a last example we consider the $SU(2) \times SU(2)$ WZW model, which has not been put in superspace yet. Each SU(2) factor gets parametrized using Eulerangles, $g_j = e^{\frac{i}{2}\theta_{Lj}\sigma_3} e^{\frac{i}{2}\phi_j\sigma_2} e^{\frac{i}{2}\theta_{Rj}\sigma_3}$, with $j \in \{1,2\}$. Again we choose the left and right complex structures on the Lie algebra to be equal. The final coordinates are the semi-chiral multiplet $(\phi, \bar{\phi}, \eta, \bar{\eta})$ and the chiral multiplet $(\zeta, \bar{\zeta})$. We introduce auxiliary coordinates $x_i, i \in \{1, \dots 6\}$, related to the original ones by

$$x_{1} = 2 \ln \sin \frac{\phi_{1}}{2} - \theta_{L2} + \theta_{R2},$$

$$x_{2} = \theta_{L1}, \quad x_{3} = \theta_{R1},$$

$$x_{4} = 2 \ln \sin \frac{\phi_{2}}{2} - \theta_{R1} + \theta_{L1},$$

$$x_{5} = \theta_{L2}, \quad x_{6} = \theta_{R2},$$
(5.11)

with inverse given by

$$\phi_1 = 2 \arcsin \exp(\frac{1}{2}(x_1 + x_5 - x_6)),$$

$$\phi_2 = 2 \arcsin \exp(\frac{1}{2}(x_4 + x_3 - x_2)).$$
(5.12)

The final coordinates are related to these by

$$\zeta = -ix_1 + x_4,
\phi = \frac{x_2}{2} + \frac{i}{4}\ln(1 - \exp(-(x_1 + x_5 - x_6)),
\eta = \frac{x_6}{2} + \frac{i}{4}\ln(1 - \exp(-(x_3 + x_4 - x_2)),
(5.13)$$

with $\bar{\zeta}$, $\bar{\phi}$ and $\bar{\eta}$ given by the complex conjugate of these expressions. The inverse transformations are

$$x_{1} = \frac{i}{2}(\zeta - \bar{\zeta}), \qquad x_{2} = \phi + \bar{\phi},$$

$$x_{4} = \frac{1}{2}(\zeta + \bar{\zeta}), \qquad x_{6} = \eta + \bar{\eta},$$

$$x_{5} = -\frac{i}{2}(\zeta - \bar{\zeta}) + \eta + \bar{\eta} - \ln(1 - \exp(2i(\bar{\phi} - \phi))),$$

$$x_{3} = -\frac{1}{2}(\zeta + \bar{\zeta}) + \phi + \bar{\phi} - \ln(1 - \exp(2i(\bar{\eta} - \eta))).$$

Ordering the coordinates as $(\phi, \bar{\phi}, \eta, \bar{\eta}, \zeta, \bar{\zeta})$, we get the complex structures in a recognizable form, where $\alpha_1 = \frac{1}{2} \sec^2 \phi_1/2$ and $\alpha_2 = \frac{1}{2} \sec^2 \phi_2/2$:

$$J_{a}^{b} = \begin{pmatrix} i & 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 1/\alpha_{1} & i & 0 & 0 & -\frac{1}{2} \\ 1/\alpha_{1} & 0 & 0 & -i & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & -i \end{pmatrix},$$
 (5.14)

$$\bar{J}^{b}{}_{a} = \begin{pmatrix} -i & 0 & 0 & 0 & 0 & -i / \\ -i & 0 & 0 & -1/\alpha_{2} & i/2 & 0 \\ 0 & i & -1/\alpha_{2} & 0 & 0 & -i/2 \\ 0 & 0 & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & -i \end{pmatrix}.$$
 (5.15)

Comparing the structures and the action with the general expressions, we get second order differential equations which can be solved for the potential, which again gets expressed in terms of dilogarithmic integrals:

$$K = -\zeta \bar{\zeta} + \zeta \bar{\phi} + \bar{\zeta} \phi + i\eta \zeta - i\bar{\eta}\bar{\zeta} + i\bar{\eta}\bar{\phi} - i\eta \phi$$
$$-i \int^{\bar{\phi} - \phi} dy \ln(1 - \exp iy) - i \int^{\bar{\eta} - \eta} dy \ln(1 - \exp iy), \qquad (5.16)$$

where we performed a trivial rescaling of the fields by a factor 2.

6. Conclusions

We showed that chiral, twisted chiral and semi-chiral superfields provide the exhaustive list of (2,2) superfields. A very interesting question is whether this is sufficient to describe any N=(2,2) non-linear σ -model. We provided several pieces of evidence for an affirmative answer to this question. An important clue is the commutator of the left and right complex structures. One can show that $\ker[J,\bar{J}] = \ker(J-\bar{J}) \oplus \ker(J+\bar{J})$ can always be integrated to chiral and twisted chiral superfields resp.

Remains the subspace where the commutator is non-degenerate. This should then be integrable to semi-chiral fields. The requirement that the metric of the resulting σ -model is non-degenerate imposes that $\ker[J,\bar{J}] = \emptyset$ in the semi-chiral directions. Furthermore one has that one semi-chiral multiplet corresponds to four real dimensions. We showed that the dimension of the subspace of a complex manifold where the commutator is non-degenerate is indeed a multiple of four. We obtained explicitly the conditions on the complex manifold with $\ker[J,\bar{J}] = \emptyset$, under which the σ -model can be described by semi-chiral superfields. It remains to be shown that all such manifolds indeed satisfy these conditions. We pointed out a particularly interesting and simple subcase where this hypothesis can be tested: 4 dimensional hyper-Kähler manifolds, where one makes a different choice for the left and the right complex structures. A necessary and sufficient condition for a 4 dimensional Kähler manifold to be hyper-Kähler, is that the Kähler potential satisfies the Monge-Ampère equation. We derived the condition on the semi-chiral potential which guarantees it to be hyper-Kähler. It turns out to be similar to the Monge-Ampère equation. If hyper-Kähler manifolds can indeed be described by semi-chiral coordinates, then one has a potential, different from the Kähler potential, which does not only allow for the computation of the metric but of the three fundamental two-forms as well.

If all σ -models can indeed be described by (2,2) superfields, then all N=(2,2) manifolds are locally characterized by a scalar potential. Besides the mathematically very intriguing implication that the geometry of a large class of complex manifolds is locally determined by a single potential, this opens interesting physics perspectives as well. In particular, it would allow the systematic study of (2,2), (2,1) and (2,0) strings. Up to now, the only N=2 strings studied are those described solely by chiral fields [14] and those described by chiral and twisted chiral fields [15]. The geometry of N=2 strings with semi-chiral fields is presently being investigated. An interesting question which arises in this context, in particular for those manifolds, e.g. the hyper-Kähler ones, which allow for different choices for the left and right complex structures, is in how far the geometry of an N=2 string depends on the choice of the complex structures. Such a study would be relevant for the recent proposals in [13] relating the D=11 membrane to the type IIB stringtheory.

Another point which certainly deserves further attention is a systematic study of T-duality, such as was done in [6] for chiral and twisted chiral fields, which includes semi-chiral fields.

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A. Appendix

In this appendix we investigate the various integrability conditions which appeared in section 3 of this paper, see also [16]. From two given (1,1) tensors R and S, we can construct a (1,2) tensor N[A,B], the Nijenhuis tensor:

$$N[R, S](U, V) = \frac{1}{2} ([RU, SV] - R[U, SV] - S[RU, V] + RS[U, V] + R \leftrightarrow S).$$
 (A.1)

For 3 (1,1) tensors R, S and T, one can verify by direct computation the following identity discovered by Frochlicher and Nijenhuis:

$$\mathcal{T}[R, S, T](U, V) \equiv N[R, ST](U, V) + N[S, RT](U, V) - RN[S, T](U, V) - SN[R, T](U, V) - N[R, S](TU, V) - N[R, S](U, TV) = 0.$$
(A.2)

If we have two *commuting* (1,1) tensors R and S, we can construct another (1,2) tensor:

$$M[R, S](U, V) = [RU, SV] - R[U, SV] - S[RU, V] + RS[U, V].$$
(A.3)

In particular we get that

$$N[R, S](U, V) = \frac{1}{2} \left(M[R, S](U, V) + M[S, R](U, V) \right). \tag{A.4}$$

Given three mutually commuting (1,1) tensors R, S and T, one obtains through direct computation the identity

$$\mathcal{U}[R, S, T](U, V) \equiv M[R, ST](U, V) - SM[R, T](U, V) - M[R, S](U, TV) = 0.$$
(A.5)

We now prove the following lemma:

<u>Lemma 1</u>: Given J and \bar{J} , two complex structures which commute, $[J, \bar{J}] = 0$, then $N[J, J](U, V) = N[\bar{J}, \bar{J}](U, V) = 0$ imply that $N[\Pi, \Pi](U, V) = N[\Pi, J](U, V) = N[\Pi, \bar{J}](U, V) = N[J, \bar{J}](U, V) = 0$ with $\Pi \equiv J\bar{J}$.

Using the definition of the Nijenhuis tensor, one gets

$$N[J, J](\Pi U, V) + N[J, J](U, \Pi V) + \Pi N[J, J](U, V) - \Pi N[\bar{J}, \bar{J}](\Pi U, \Pi V) = \Pi N[\Pi, \Pi](U, V) - N[J, \bar{J}](U, V).$$
(A.6)

Using $N[J, J](U, V) = N[\bar{J}, \bar{J}](U, V) = 0$, we get from this:

$$\Pi N[\Pi, \Pi](U, V) = N[J, \bar{J}](U, V).$$
 (A.7)

From now on we continuously use $[J, \bar{J}] = 0$ and $N[J, J](U, V) = N[\bar{J}, \bar{J}](U, V) = 0$. From $\mathcal{T}[J, J, \Pi](U, V) = 0$, we get

$$N[J,\Pi](U,V) = J N[J,\bar{J}](U,V),$$
 (A.8)

and from $\mathcal{T}[\bar{J}, \bar{J}, \Pi](U, V) = 0$,

$$N[\bar{J},\Pi](U,V) = \bar{J} N[J,\bar{J}](U,V).$$
 (A.9)

Using this in $\mathcal{T}[J, \bar{J}, \bar{J}](U, V) = 0$ yields,

$$N[J, \bar{J}](\bar{J}U, V) + N[J, \bar{J}](U, \bar{J}V) = 0.$$
(A.10)

Using eqs. (A.7), (A.8) and (A.10) in $\mathcal{T}[J,\Pi,\bar{J}](U,V)=0$, gives

$$N[J, \bar{J}](U, V) = 0.$$
 (A.11)

Combining this with eqs. (A.7-A.9) shows that also all other Nijenhuistensors vanish. Which proves the lemma. As the complex structures commute, this results in additional (1, 2) tensors which can be constructed out of J and \bar{J} : $M[J, \bar{J}](U, V)$, $M[\Pi, J](U, V)$ and $M[\Pi, \bar{J}](U, V)$. We now prove an additional lemma:

<u>Lemma 2</u>: For J and \bar{J} two commuting complex structures, we have that $N[J,J](U,V) = N[\bar{J},\bar{J}](U,V) = 0$ imply the vanishing of the tensors $M[J,\bar{J}](U,V)$, $M[\Pi,J](U,V)$ and $M[\Pi,\bar{J}](U,V)$.

We use throughout the previous lemma. From $\mathcal{U}(\bar{J}, \bar{J}, J) = 0$ we get

$$M[\bar{J}, \Pi](U, V) = \bar{J}M[\bar{J}, J](U, V),$$
 (A.12)

 $\mathcal{U}(\bar{J}, J, \bar{J}) = 0$ implies

$$M[\bar{J}, \Pi](U, V) = M[\bar{J}, J](U, \bar{J}V),$$
 (A.13)

and from $\mathcal{U}(J, \bar{J}, \bar{J}) = 0$ we obtain:

$$M[\bar{J}, J](U, \bar{J}V) = -\bar{J}M[\bar{J}, J](U, V).$$
 (A.14)

Using eq. (A.14) in eq. (A.13) yields

$$M[\bar{J}, \Pi](U, V) = -\bar{J}M[\bar{J}, J](U, V)$$
 (A.15)

Comparing this with eq. (A.12) gives $M[\bar{J},\Pi](U,V)=M[\bar{J},J](U,V)=0$. Finally, using this result in $\mathcal{U}(J,J,\bar{J})=0$ results in the vanishing of $M[J,\Pi](U,V)$, which proves the lemma.

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