1 AKSZ stuff

1.1 Two-dimensional TFT from GC structures

In this section, we summarize the work of Kapustin-Li in the AKSZ formalism with an eye towards modification in the para geometric setting.

1.1.1 A recollection of generalized complex geometry

Recall, an exact Courant algebroid on a smooth manifold X is determined, up to equivalence, by a class $H \in H^1(X, \Omega^2_{cl})$, called its Severa class, or H-flux. The underlying vector bundle of an exact Courant algebroid is $TX \oplus T^*X$, and the H-flux deforms the standard Dorfman bracket.

An almost generalized complex structure on a manifold X is the data of a smooth bundle map

$$\mathcal{J} : TX \oplus T^*X \to TX \oplus T^*X$$

satisfying $\mathcal{J}^2 = -\mathbb{1}$ and $\langle \mathcal{J}u, \mathcal{J}v\rangle = \langle u, v\rangle$, where $\langle -, -\rangle$ denotes the obvious pairing between the tangent and cotangent bundle. Given an almost generalized complex structure, we denote by $L \subset TX^{\mathbf{C}} \oplus T^*X^{\mathbf{C}}$ the complex $+i$ eigenbundle of $\mathcal{J}$.

Fix an exact Courant algebroid on X with class H. An almost generalized complex structure on X is an *H-twisted generalized complex structure* if the subbundle L is preserved under the H-twisted Dorfman bracket. An H-twisted generalized complex structure is equivalent to an H-twisted complex Dirac structure L [1] satisfying $L \cap \bar{L} = \{0\}$. When $H = 0$, one simply calls this a generalized complex structure.

Example 1.1. Every almost complex structure on X determines an almost generalized complex structure on X. It is a generalized complex structure if the almost complex structure is integrable.

In fact, consider the almost generalized complex structure

$$\mathcal{J} = \begin{pmatrix} I & 0 \\ 0 & -I^* \end{pmatrix},$$

where $I : TX \to TX$ is an almost complex structure, and I^* is the dual bundle map. The $+i$ eigenbundle is $L = T^{(1,0)} \oplus T^{*(0,1)}$ and the $-i$ eigenbundle is $\bar{L} = T^{(0,1)} \oplus T^{*(1,0)}$. Involutivity of L under $[\ ,\ ]_H$ is equivalent to the condition

$$[X + \bar{\alpha}, Y + \bar{\beta}] = [X, Y] + \mathcal{L}_X \bar{\beta} - \imath_Y \mathrm{d}\bar{\alpha} + H(X, Y),$$

where $X, Y \in \mathfrak{X}^{(1,0)}$ and $\bar{\alpha}, \bar{\beta} \in \Omega^{(0,1)}$. This implies that one must have have $[X, Y] \subset T^{(1,0)}$, meaning I must be a complex structure. Moreover, inspecting

[1] A complex Dirac structure is a maximally isotropic involutive subbundle of the exact complex Courant algebroid $TX^{\mathbf{C}} \oplus T^*X^{\mathbf{C}}$

the cotangent component of the above and splitting $\mathrm{d} = \partial + \bar{\partial}$, the non-zero components are

$$\imath_X(\partial\bar{\beta}) - \imath_Y(\partial\bar{\alpha}) + H(X,Y),$$

where $\imath_X\partial\bar{\beta} - \imath_Y\partial\bar{\alpha}$ is always in $T^{*(0,1)}$ and so $H(X,Y)$ must be in $T^{*(0,1)}$ as well, i.e. the $(3,0)$ component of H must vanish. The same argument for $\bar{L}$ gives $H^{(0,3)} = 0$.

Example 1.2. Suppose $\omega \in \Omega^2(X)$ is a nondegenerate 2-form. Then, ω determines an almost generalized complex structure on X. It is a generalized complex structure for $H = \mathrm{d}\omega$. In particular, every symplectic structure determines a generalized complex structure for $H = 0$.

Indeed, define the almost generalized complex structure

$$\mathcal{J} = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix},$$

With eigenbundles given by $X \pm i\omega(X)$ for any $X \in \mathfrak{X}$. Checking the involutivity of these under $[\ ,\ ]_H$:

$$[X \pm i\omega(X), Y \pm i\omega(Y)] = [X,Y] \pm i\omega([X,Y]) \pm i\mathrm{d}\omega(X,Y) + H(X,Y),$$

where some identities for Lie derivatives etc. have been used. Clearly $[X,Y] \pm i\omega([X,Y])$ is of the desired form and the remaining terms are in Ω^1 so they have to vanish (cannot be of the form $Z \pm i\omega(Z)$). This gives

$$i\mathrm{d}\omega(X,Y) + H(X,Y) = 0$$

for all X, Y.

More generally, one can speak of (almost) generalized complex structures in *any* (possibly non exact) Courant algebroid. If E is the underlying vector bundle of the Courant algebroid, then an almost generalized complex structure is a bundle map $\mathcal{J} : E \to E$ satisfying the same conditions as above, namely $\mathcal{J}^2 = -\mathbb{1}$ and $\langle \mathcal{J}u, \mathcal{J}v\rangle = \langle u, v\rangle$. It is a generalized complex structure if it is integrable for the bracket defining the Courant algebroid.

Analogous to the exact case, we have the following equivalent characterization of generalized complex structures in E.

Proposition 1.3. *A generalized complex structure in a Courant algebroid E is equivalent to a Dirac structure L in the complex Courant algebroid $E^{\mathbf{C}}$ satisfying $L \cap \bar{L} = 0$.*

BW: Should I define what a Dirac structure in a general CA is?

1.1.2 The associated Lie algebroid and cohomology

To any (twisted) generalized complex structure there exists a naturally associated (complex) Lie algebroid defined as follows.

Let $\mathcal{J}$ be an H-twisted generalized complex structure and $\bar{L} \subset TX^{\mathbf{C}} \oplus T^*X^{\mathbf{C}}$ the complex $-i$ eigenbundle. BW: I think we want to use the $-i$ eigenbundle. While the Dorfman bracket does *not* define a Lie bracket on $TX^{\mathbf{C}} \oplus T^*X^{\mathbf{C}}$ it does define one on the subbundle L. Indeed, by definition $\bar{L}$ is isotropic and integrable, so the Jacobi identity is satisfied. The anchor map $a : \bar{L} \to TX^{\mathbf{C}}$ is given by the restriction of the natural projection $TX^{\mathbf{C}} \oplus T^*X^{\mathbf{C}} \to TX^{\mathbf{C}}$ to L. We denote this Lie algebroid by $\bar{L}_{\mathcal{J},H}$.

We define *cohomology* of an H-twisted generalized complex structure $\mathcal{J}$ to be the Lie algebroid cohomology of $L_{\mathcal{J},H}$.

Example 1.4. When the generalized complex structure $\mathcal{J}$ is defined using an ordinary complex structure, as in Example 1.1 the resulting Lie algebroid is given by

$$\bar{L}_{\mathcal{J},H} = T^{0,1}X \to TX^{\mathbf{C}}.$$

The cohomology of the twisted generalized complex structure is equal to the Dolbeault cohomology $H^{0,*}(X)$.

More generally, there is a (complex) Lie algebroid associated to a generalized complex structure $\mathcal{J}$ in any (possibly non exact) Courant algebroid E. Again, we define $\bar{L} \subset E^{\mathbf{C}}$ to be the $-i$ eigenspace of $\mathcal{J}$. The same argument as in the exact case shows that $\bar{L}$ has a natural Lie algebroid structure, where bracket is given by restricting the bracket of E defining the Courant algebroid structure, and the anchor map is given by the composition

$$a : \bar{L} \hookrightarrow E^{\mathbf{C}} \to TX^{\mathbf{C}}.$$

We denote the resulting Lie algebroid by $\bar{L}_{\mathcal{J},E}$, and define the cohomology of $\mathcal{J}$ to be the Lie algebroid cohomology of $\bar{L}_{\mathcal{J},E}$.

1.1.3 The AKSZ theory

From the point of view of topological field theory, Courant algebroids are important because they provide geometric examples of 2-shifted symplectic spaces. Via the AKSZ construction, 2-shifted symplectic spaces are the natural home for 3-dimensional topological field theories in the BV formalism.

Example 1.5. Any Lie algebra together with a non-degenerate invariant pairing defines a 2-shifted symplectic structure on the classifying stack $B\mathfrak{g}$. The resulting AKSZ theory is Chern-Simons theory.

To any Courant algebroid E, we can associate a derived stack $\mathfrak{X}_E$ which carries a 2-shifted symplectic structure ω_E. In the case that the Courant algebroid is exact, we denote the 2-shifted symplectic space as $\mathfrak{X}_H$ where H is the Severa class.

To a compact, oriented 3-manifold M^3 the AKSZ construction endows the derived mapping space

$$\mathrm{Map}(M^3, \mathfrak{X}_E)$$

with a BV structure.

BW: Recall Lagrangian of a stack.

An H-twisted generalized complex structure $\mathcal{J}$ defines a complex Dirac structure on $TX^{\mathbf{C}} \oplus T^*X^{\mathbf{C}}$, and hence a 2-shifted Lagrangian inside of the 2-shifted symplectic space $\mathfrak{X}_H$. More generally, if $\mathcal{J}$ is a generalized complex structure in an arbitrary Courant algebroid E, then we obtain a 2-shifted Lagrangian inside of the 2-shifted symplectic space $\mathfrak{X}_E$.

The Lagrangian in $\mathfrak{X}_E$ associated to a generalized complex structure $\mathcal{J}$ defines a boundary condition for the three-dimensional AKSZ theory. In the case the the Courant algebroid is exact, the resulting boundary theory is the generalized A/B-model of [**?**].

1.2 Two dimensional TFT from GpC structures

The goal in this section is to describe a construction of a two-dimensional topological field theory from the data of a generalized para complex structure. This construction is analogous to the generalized A/B-model of [**?**] that we have just recollected.