Generalized para-Kähler Geometry, Generalized Structures of Non-Isotropic Type and Born Geometry

May 30, 2019

Abstract

In this paper we study generalized structures, i.e. endomorphisms of the generalized tangent bundle $\mathcal{J} \in \mathrm{End}(T \oplus T^*)$, which square to the identity, $\mathcal{J}^2 = \mathbb{1}$ as opposed to negative identity, which is the case for generalized complex geometry. Such structures are either generalized para-complex or of non-isotropic type (NIT), the latter having eigenbundles which are not isotropic with respect to the natural pairing on $T \oplus T^*$. The NIT structures include the well-known generalized metrics, but also generalized pseudo-metrics and generalized product structures. Commuting pairs of generalized para-complex structures are shown to give rise to generalized para-Kähler geometry (GpK), which is a para-complex version of generalized Kähler geometry. GpK geometry is shown to be equivalent to a bi-para-Hermitian geometry and integrability criteria are derived. On the other hand, pairs of commuting NIT structures give rise to generalized Chiral geometry, which corresponds to bi-pseudo-Riemannian product structures. We further introduce generalized integrability conditions in terms of a generalized Bismut connection, which is suitable for the NIT structures and generalizes the usual Courant integrability in the case of structures with isotropic eigenbundles. We then show that combining the GpK and NIT structures yields generalized Born geometry. We conclude with examples.

1 Product, Para-Complex and Born Geometry

1.1 Product and Para-Complex Geometry

In this section we briefly review important notions in paracomplex geometry. For more details see for example [?] or a review of paracomplex geometry with further references [?].

Definition 1.1. An (almost) **product structure** on a smooth manifold is an endomorphism $K \in \mathrm{End}(T\mathcal{P})$ which squares to identity, $K^2 = \mathbb{1}_{T\mathcal{P}}$, $K \neq \mathbb{1}$.

An (almost) **para-complex structure** is a product structure such that the $+1$ and -1 eigenbundles of K have the same rank.

A direct consequence of above definition is that any para-complex manifold is of even dimension. From now on, the $+1$ and -1 eigenbundles of an almost product/para-complex structure will be denoted L and $\tilde{L}$, respectively.

The use of the word *almost* as usual refers to integrability of the endomorphism, i.e. whether its eigenbundles are involutive with respect to the Lie bracket and therefore define a foliation of the underlying manifold. Similarly to the complex case, the integrability is governed by the **Nijenhuis tensor**

$$\begin{aligned} N_K(X,Y) &= [X,Y] + [KX,KY] - K([KX,Y] + [X,KY]) \\ &= (\nabla_{KX}K)Y + (\nabla_X K)KY - (\nabla_{KY}K)X - (\nabla_Y K)KX \\ &= 4(\mathrm{P}[\tilde{\mathrm{P}}X, \tilde{\mathrm{P}}Y] + \tilde{\mathrm{P}}[\mathrm{P}X, \mathrm{P}Y]), \end{aligned} \tag{1}$$

where ∇ is any torsionless connection and $\mathrm{P} := \frac{1}{2}(\mathbb{1} + K)$ and $\tilde{\mathrm{P}} := \frac{1}{2}(\mathbb{1} - K)$ are the projections onto L and $\tilde{L}$, respectively. We say K is integrable and call K a product/para-complex manifold if $N_K = 0$. From (1) it is apparent that K is integrable if and only if *both* eigenbundles are simultaneously Frobenius integrable, i.e. involutive distributions in $T\mathcal{P}$. This is one of the main differences between complex and para-complex geometry; one of the eigenbundles can be integrable while the other is not. For this reason, it is useful to introduce the notion of **half-integrability**:

Definition 1.2. Let K be an almost para-complex/product structure. If the $+1$ (-1) eigenbundle is Frobenius integrable, we call K a **p-para-complex/product** (**n-para-complex/product**) structure. A K that is both p- and n- para-complex/ is simply a para-complex/product structure.

Let now $(\mathcal{P}, K)$ be an almost para-complex manifold. If K is integrable, we get a set of $2n$ coordinates $(x^i, \tilde{x}_i)$ called **adapted coordinates**, $\mathcal{P}$ locally splits as $M \times \tilde{M}$, and K acts as identity on $TM = L$ and negative identity on $T\tilde{M} = \tilde{L}$. The splitting of the tangent bundle gives rise to a decomposition of tensors analogous to the (p,q)-decomposition in complex geometry. Denote $\Lambda^{(+k,-0)}(T^*\mathcal{P}) := \Lambda^k(L^*)$ and $\Lambda^{(+0,-k)}(T^*\mathcal{P}) := \Lambda^k(\tilde{L}^*)$. The splitting is then

$$\Lambda^k(T^*\mathcal{P}) = \bigoplus_{k=m+n} \Lambda^{(+m,-n)}(T^*\mathcal{P}), \tag{2}$$

with corresponding sections denoted as $\Omega^{(+m,-n)}(\mathcal{P})$. The bigrading (2) yields the natural projections

$$\Pi^{(+p,-q)} : \Lambda^k(T^*\mathcal{P}) \to \Lambda^{(+p,-q)}(T^*\mathcal{P}),$$

so that the de-Rham differential splits as $\mathrm{d} = \partial_+ + \partial_-$, where

$$\begin{aligned} \partial_+ &:= \Pi^{(+p+1,-q)} \circ \mathrm{d} \\ \partial_- &:= \Pi^{(+p,-q-1)} \circ \mathrm{d}, \end{aligned}$$

are the **para-complex Dolbeault operators**, acting on forms as

$$\begin{aligned}\partial_+ &: \Omega^{(+p,-q)}(\mathcal{P}) \to \Omega^{(+p+1,-q)}(\mathcal{P})\\ \partial_- &: \Omega^{(+p,-q)}(\mathcal{P}) \to \Omega^{(+p,-q-1)}(\mathcal{P}),\end{aligned} \tag{3}$$

such that when K is integrable, we have

$$\partial_+^2 = \partial_-^2 = \partial_+\partial_- + \partial_-\partial_+ = 0.$$

One can also introduce the *twisted differential* $\mathrm{d}^p := (\Lambda^{k+1}K) \circ \mathrm{d} \circ (\Lambda^k K)$:

Lemma 1.3. *Let* $(\mathcal{P}, K)$ *be a paracomplex manifold. Then* $\mathrm{d}^p := (\Lambda^{k+1}K) \circ \mathrm{d} \circ (\Lambda^k K)$ *can be expressed as*

$$\mathrm{d}^p = \partial_+ - \partial_-. \tag{4}$$

Proof. Let $\alpha \in \Omega^{+m,-n}(\mathcal{P})$. Then we have

$$\mathrm{d}^p\alpha = (-1)^n(\Lambda^k K)\mathrm{d}\alpha = (-1)^{2n}\partial_+\alpha + (-1)^{2n+1}\partial_-\alpha = (\partial_+ - \partial_-)\alpha,$$

□

Para-Holomorphic structure

We will now explore the para-holomorphic structure of para-complex manifolds, and give important examples of para-holomorphic vector bundles.

We start with the natural definition of a para-Holomorphic map between para-complex vector spaces.

Definition 1.4. Let (M, K_M) and (N, K_N) be para-complex manifolds. A map $f : V \to W$ is called para-holomorphic if

$$K_W \circ f_* = f_* \circ K_V$$

Locally, the definition means the following. Let V and W be $2n$- and $2m$-dimensional vector spaces, respectively. Choose the respective adapted bases for V and W as $\{v_i, \tilde{v}^j\}_{i,j=1\cdots n}$, $\{v_k, \tilde{v}^l\}_{k,l=1\cdots m}$, so that K_V and K_W take the diagonal forms $\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$. It is easy to check that a para-holomorphic map $f : V \to W$ then takes the form

$$f = (w_1(v_i), \cdots, w_m(v_i), \tilde{w}^1(\tilde{v}^i), \cdots, \tilde{w}^m(\tilde{v}^i)),$$

i.e. the first m components of f are independent of the $\tilde{v}$ variables, while the remaining last m components are independent of the v variables, meaning $(w_i, \tilde{w}^j)$ satisfy the para-complex Cauchy-Riemann equations:

$$\frac{\partial}{\partial v_i}\tilde{w}^j = \frac{\partial}{\partial \tilde{v}^i} w_j = 0, \quad \forall i = 1, \cdots, n,\ j = 1, \cdots, m. \tag{5}$$

Therefore, a para-holomorphic map f of para-complex manifolds (M, K_M) and (N, K_N) takes on each pair of patches $U \subset M$ and $V \subset N$ the local form

$$f : \mathbb{R}^{2n} \to \mathbb{R}^{2m} : \; f = (y(x), \tilde{y}(\tilde{x})),$$

where $(x, \tilde{x}) : U \to \mathbb{R}^{2n}$ and $(y, \tilde{y}) : V \to \mathbb{R}^{2m}$ are the local adapted coordinates on U and V.

Now, because on a $2n$-dimensional para-complex manifold, the adapted coordinates $(x, \tilde{x})$ patch into two separate foliations which can be seen as two n-dimensional manifolds, the coordinates along these manifolds transform among themselves, which means the transition functions $\phi_{UV} : U \,|_{U\cap V} \to V \,|_{U\cap V}$ between two patches U and V have to have the form $(x'(x), \tilde{x}'(\tilde{x}))$, $(x, \tilde{x})$ and $(x', \tilde{x}')$ being the adapted coordinates on U and V, respectively. The transition functions on a para-complex manifold are therefore easily seen to be para-holomorphic functions, and we call this the para-holomorphic structure of the manifold.

Let us now explore para-holomorphic vector bundles, starting from the following definition

Definition 1.5. A para-holomorphic vector bundle $E \xrightarrow{\pi} M$ over a para-complex manifold M with is a para-complex vector bundle (i.e. the fibers are para-complex vector spaces V), such that its transition functions $g_{UV} : U \cap V \to Gl(V)$ are para-holomorphic maps. A para-holomorphic section of E is a section of the projection π that is a para-holomorphic map.

Example 1.6 (Tangent bundle of a para-complex manifold). Let (M, K) be a $2n$-dimensional para-complex manifold. Its tangent bundle TM is a para-holomorphic vector bundle. We noted above that due to the para-holomorphic structure of M, the gluing functions between two patches $\phi_{UV} : U \,|_{U\cap V} \to V \,|_{U\cap V}$ take the form $(y(x), \tilde{y}(x))$. The transition function for TM is then given by the push-forward of ϕ_{UV}:

$$(\phi_{UV})_* : U \,|_{U\cap V} \times \mathbb{R}^{2n} \to V \,|_{U\cap V} \times \mathbb{R}^{2n}$$
$$\left(x^i, \tilde{x}_i, \frac{\partial}{\partial x^i} = \partial_i, \frac{\partial}{\partial \tilde{x}_i} = \tilde{\partial}^i \right) \mapsto \left(y^j(x^i), \tilde{y}^j(\tilde{x}_i), \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial y^j}, \frac{\partial \tilde{y}_j}{\partial \tilde{x}_i} \frac{\partial}{\partial \tilde{y}_j} \right),$$

which can also be seen as a map $g_{UV} : U \cap V \to Gl_n^+ \oplus Gl_n^- \subset Gl(V) \simeq Gl^{2n}$

$$g_{UV} : (x, \tilde{x}) \mapsto \begin{pmatrix} \left(\frac{\partial y(x)}{\partial x} \right) & 0 \\ 0 & \left(\frac{\partial \tilde{y}(\tilde{x})}{\partial \tilde{x}} \right) \end{pmatrix},$$

which is holomorphic with respect to the para-complex structure of $Gl_n^+ \oplus Gl_n^-$ that acts diagonally by ± 1 on the copies $Gl_n^{\pm}$.

The reason $TM = L \oplus \tilde{L}$ is para-holomorphic is because we can see the eigenbundles L and $\tilde{L}$ as tangent bundles of two foliations $\mathcal{F}_{\pm}$: $L = T\mathcal{F}_+$, $\tilde{L} = T\mathcal{F}_-$, which can be understood as individual n-dimensional manifolds. It is therefore clear that each factor in the sum $L \oplus \tilde{L}$ transforms with transition functions only depending on the coordinates of the corresponding foliation manifold.

Example 1.7 (Wedge powers of L and $\tilde{L}$). Consider now the vector bundle $E = \Lambda^k(L) \oplus \Lambda^k(\tilde{L})$ with sections the poly-vector fields $\mathfrak{X}^{(k,0)+(0,k)}$ for some $1 < k < n$ over a $2n$-dimensional para-complex manifold (M, K). From the discussion in Example 1.6 we can see that the transition functions of such bundle is going to be given by $g_{UV} = \Lambda^k\left(\frac{\partial y(x)}{\partial x}\right) \oplus \Lambda^k\left(\frac{\partial \tilde{y}(\tilde{x})}{\partial \tilde{x}}\right)$, acting on the basis vectors (e_i, e^j),

$$e_I = \partial_{i_1} \wedge \cdots \wedge \partial_{i_k},\ \tilde{e}^I = \tilde{\partial}^{i_1} \wedge \cdots \wedge \tilde{\partial}^{i_k}, \quad 0 < i_1 < \cdots < i_k < n,\ I = 1, \cdots, r,$$

, where the fibre para-complex structure K_E is given by

$$K_E(e_I) = e_I, \quad K_E(\tilde{e}^I) = -\tilde{e}^I.$$

Again, such vector bundle is para-holomorphic for the same reasons $TM + L \oplus \tilde{L}$ itself is holomorphic. Let $\sigma : M \to E$ be a section of the bundle E. Expanding σ in a local coordinates, we get

$$\sigma = a(x, \tilde{x})^{i_1 \cdots i_r} \partial_{i_1} \wedge \cdots \wedge \partial_{i_r} + \tilde{a}(x, \tilde{x})_{j_1 \cdots j_r} \tilde{\partial}^{j_1} \wedge \cdots \wedge \tilde{\partial}^{j_r}.$$

It is easy to see that in order for σ to be a holomorphic section, the coefficient functions $(a, \tilde{a})$ have to satisfy

$$\tilde{\partial}^i a(x, \tilde{x}) = \partial \tilde{a}(x, \tilde{x}) = 0, \tag{6}$$

meaning the coefficient functions satisfy the para-complex Cauchy-Riemann equations (5).

1.2 Para-Hermitian Geometry

Para-Hermitian geometry should be again thought of as the para-complex version of Hermitian geometry, i.e. an additional metric structure compatible with the para-complex structure is introduced. This metric then induces a non-degenerate two-form, which can be closed, giving rise to a para-Kähler geometry.

Definition 1.8. Let $(\mathcal{P}, K)$ be a para-complex manifold and let η be a pseudo-Riemannian metric that satisfies $\eta(K\cdot, K\cdot) = -\eta$. Then we call $(\mathcal{P}, K, \eta)$ a **para-Hermitian manifold**[1].

The above definition implies that the tensor $\omega \coloneqq \eta K$ is skew

$$\omega(X, Y) = \eta(KX, Y) = -\eta(X, KY) = -\omega(Y, X),$$

and nondegenrate (because η is nondegenerate), therefore ω is an almost symplectic form, sometimes called the **fundamental form**. From $K^2 = \mathbb{1}$ we also have $K = \eta^{-1}\omega = \omega^{-1}\eta$. Another observation is that since the eigenbundles of K have the same rank, η has split signature (n, n). Furthermore, the eigenbundles of K are isotropic with respect to both η and ω. This means that the almost symplectic form ω is of the type $+1, -1$, $\omega \in \Omega^{+1,-1}$.

[1] If K is not integrable, i.e. $(\mathcal{P}, K)$ is almost para-complex, we would call $(\mathcal{P}, K, \eta)$ an almost para-Hermitian manifold.

Remark. As shown above, the data $(\mathcal{P}, K, \eta)$, $(\mathcal{P}, \eta, \omega)$ and $(\mathcal{P}, K, \omega)$ are on a para-Hermitian manifold equivalent and so we may use the different triples interchangeably to refer to a para-Hermitian manifold.

Definition 1.9. Let $(\mathcal{P}, \eta, \omega)$ be a para-Hermitian manifold with $\mathrm{d}\omega = 0$. We call $(\mathcal{P}, \eta, \omega)$ a **para-Kähler manifold**.

1.3 Chiral Geometry

Chiral geometry is, similarly to para-Hermitian geometry, given by a pair of a real endomorphism $J^2 = \mathbb{1}$ and a compatible pseudo-Riemannian metric η:

Definition 1.10. An (almost) **chiral structure**[2] is given by the pair (J, η), where J is an (almost) product structure, i.e. $J \in \mathrm{End}(TM)$, $J^2 = \mathbb{1}_{TM}$ and η is a pseudo-Riemannian metric, such that $\eta(J\cdot, J\cdot) = \eta(\cdot, \cdot)$.

Contrary to para-Hermitian geometry, the structure J need not be para-complex (though in this paper we will mostly treat chiral structures that are para-complex) and J is an *isometry* of η as opposed to anti-isometry. A consequence of this is that the tensor $\mathcal{H} := \eta J$ is now again a (pseudo-) Riemannian structure as opposed to a two-form. The ± 1 eigenbundles $C_\pm$ of J are not isotropic with respect to η or $\mathcal{H}$, but rather orthogonal, $\eta(C_\pm, C_\mp) = \mathcal{H}(C_\pm, C_\mp) = 0$. A full classification of such structures is known in terms of 36 classes characterized by the **fundamental tensor** $F(X, Y, Z) := \mathring{\nabla}_X \mathcal{H}(Y, Z)$, where $\mathring{\nabla}$ denotes the Levi-Civita connection of η. One of the 36 classes is the class $\mathcal{W}_3$ defined as follows:

Definition 1.11. A Chiral structure is of class $\mathcal{W}_3$ if

$$\sum_{Cycl.X,Y,Z} F(X, Y, Z) = 0, \quad \Phi(X, Y) = -\Phi(JX, JY), \tag{7}$$

where $F(X, Y, Z) = \eta((\mathring{\nabla}_X J)Y, Z) = \mathring{\nabla}_X \mathcal{H}(Y, Z)$ is the fundamental tensor of the Chiral structure and $\Phi(X, Y)$ is defined as $\Phi(X, Y) = (\mathring{\nabla}_X - \mathring{\nabla}'_X)Y$, $\mathring{\nabla}$ and $\mathring{\nabla}'$ being the Levi-Civita connections of η and $\mathcal{H}$, respectively.

We can see that the characteristic property (7) of the fundamental tensor F is directly analogous to requiring that $\mathrm{d}\omega = 0$ on the side of para-Hermitian geometry.

1.4 Born Geometry

Born geometry has been introduced in physics [] and its properties later discussed in []. The (overdefined) data of Born geometry is given by an almost para-hypercomplex (or paraquaternionic) triple $\{I, J, K\}$, i.e.

$$-I^2 = J^2 = K^2 = -IJK = \mathbb{1}, \tag{8}$$

[2]The name chiral comes from physics, in mathematical literature such structures are called (almost) product pseudo-Riemannian

along with a compatible, split signature metric η:

$$\eta(I\cdot, I\cdot) = -\eta(J\cdot, J\cdot) = \eta(K\cdot, K\cdot) = -\eta(\cdot, \cdot).$$

This means that (η, K) is (almost) para-Hermitian (i.e. $\eta K = \omega$ is almost symplectic) and (η, J) is (almost) chiral ($\eta J = \mathcal{H}$ is Riemannian). It then follows that $(\mathcal{H}, I)$ is (almost) Hermitian.

Born geometry can therefore be understood as either a chiral structure with an additional compatible two-form ω, as a Hermitian structure with a compatible split metric η, or as a para-Hermitian structure with a compatible Riemannian metric $\mathcal{H}$, which is the point of view we mostly adopt here. We also note that (8) implies $IJ = -K$ and $\{I, J\} = \{J, K\} = \{K, I\} = 0$. To summarize, we cite Table 1 from [1, Sec. 3.1].

$I = \mathcal{H}^{-1}\omega = -\omega^{-1}\mathcal{H}$	$J = \eta^{-1}\mathcal{H} = \mathcal{H}^{-1}\eta$	$K = \eta^{-1}\omega = \omega^{-1}\eta$
$-I^2 = J^2 = K^2 = \mathbb{1}$	$\{I, J\} = \{J, K\} = \{K, I\} = 0$	$IJK = -\mathbb{1}$
$\mathcal{H}(IX, IY) = \mathcal{H}(X, Y)$	$\eta(IX, IY) = -\eta(X, Y)$	$\omega(IX, IY) = \omega(X, Y)$
$\mathcal{H}(JX, JY) = \mathcal{H}(X, Y)$	$\eta(JX, JY) = \eta(X, Y)$	$\omega(JX, JY) = -\omega(X, Y)$
$\mathcal{H}(KX, KY) = \mathcal{H}(X, Y)$	$\eta(KX, KY) = -\eta(X, Y)$	$\omega(KX, KY) = -\omega(X, Y)$

Table 1: Summary of structures in Born geometry.

The following statement shows that if we understand Born geometry as a para-Hermitian geometry along with a compatible metric $\mathcal{H}$, the additional data encoded in $\mathcal{H}$ is in fact given only by a choice of a Riemannian metric g on one of the para-Hermitian eigenbundles:

Corollary 1.12 ([1]). *Let $(\mathcal{P}, \eta, K)$ be an (almost) para-Hermitian structure. The data of a Born geometry on $\mathcal{P}$ compatible with (η, K) is given by a Riemannian metric on L (or $\tilde{L}$), the $+1$ (-1) eigenbundle of K, such that $\mathcal{H}$ takes the diagonal form*

$$\mathcal{H} = \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix},$$

where the identification of $\tilde{L} \simeq L^*$ via contraction with η is implicitly understood.

This fact reflects that from physics point of view, the para-Hermitian structure should be understood as an *extended background geometry* accommodating T-duality, where the $+1$ eigenbundle represents the physical directions of the underlying theory; therefore it should be equipped with a Riemannian structure.

Note that Born geometry therefore exists on any (almost) para-Hermitian manifold and corresponds to a choice of a reduction of the structure group from the para-unitary group

$$PU(n) = \left\{ \begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix} \mid A \in Gl(n) \right\},$$

which is the structure group of an almost para-Hermitian manifold, to $O(n)$. This is in detail discussed in [1] and in [2], where what we call Born geometry is incorrectly called almost hyper-para-Kähler geometry, which is a name already used for different type of geometry, see [3] or Example 3.8 further in Section 3.1.4.

2 Generalized Structures

In this section we introduce different notions of *Generalized structures*, by which we mean endomorphisms of the generalized tangent bundle $\mathcal{J} \in \mathrm{End}(T \oplus T^*)$, which square to ± 1 and are (anti-)orthogonal with respect to the natural pairing $\langle \ , \ \rangle$ on $T \oplus T^*$. This involves four different choices:

- $\mathcal{J}^2 = -\mathbb{1}$, $\langle \mathcal{J}\cdot, \mathcal{J}\cdot \rangle = \langle \ , \ \rangle \longrightarrow$ generalized complex (GC) structures
- $\mathcal{K}^2 = \mathbb{1}$, $\langle \mathcal{K}\cdot, \mathcal{K}\cdot \rangle = -\langle \ , \ \rangle \longrightarrow$ generalized para-complex (GpC) structures
- $\mathcal{A}^2 = \mathbb{1}$, $\langle \mathcal{A}\cdot, \mathcal{A}\cdot \rangle = \langle \ , \ \rangle \longrightarrow$, generalized product (GP) structures
- $\mathcal{A}^2 = -\mathbb{1}$, $\langle \mathcal{A}\cdot, \mathcal{A}\cdot \rangle = -\langle \ , \ \rangle \longrightarrow$, generalized complex product (GcP) structures

While GC and special non-degenerate type of GP structures called generalized metrics have been extensively studied, the remaining structures on the list have been given considerably less attention in the literature. GpC geometry was discussed in [**?**, **?**] and various generalized structure were the subject of [**?**, **?**].

2.1 Review of Dirac Geometry

By *Dirac geometry* we mean a general term for studying the geometry of the bundle $T \oplus T^*$ itself, which involves in particular the study of Dirac structures, the Courant algebroid structure and the induced Lie (bi-)algebroid structures on $T \oplus T^*$. We recall some of these notions relevant for the current discussion below.

The bundle $T \oplus T^*$ has a natural Courant algebroid structure given by the symmetric pairing

$$\langle X + \alpha, Y + \beta \rangle = \alpha(Y) + \beta(Y),$$

the Dorfman bracket[3],

$$[X + \alpha, Y + \beta] = [X, Y] + \mathcal{L}_X \beta - \imath_Y \mathrm{d}\alpha, \tag{9}$$

[3] In the following text we will omit the superscript whenever no confusion with the Lie bracket is possible.

and the anchor $\pi : X+\alpha \mapsto X$. In the above, $X+\alpha$ denotes a section of $T\oplus T^*$ with the splitting to tangent and cotangent parts given explicitly. The Dorfman bracket can be thought of as an extension of the Lie bracket from T to $T\oplus T^*$ and therefore we opt to use the same notation for both brackets; the expression $[X,Y]$ is always the Lie bracket of vector fields whether we think of $[\ ,\ ]$ as the Lie bracket or the Dorfman bracket and no confusion is therefore possible.

The Courant algebroid on $T\oplus T^*$ is exact, meaning that the associated sequence

$$0 \longrightarrow T^* \xrightarrow{\pi^T} T\oplus T^* \xrightarrow{\pi} T \longrightarrow 0, \tag{10}$$

is exact. Here, π^T is the transpose of π with respect to the pairing $\langle\ ,\ \rangle$,

$$\langle \pi^T(\alpha), Y+\beta\rangle = \langle \alpha, \pi(Y+\beta)\rangle = \langle \alpha, Y\rangle$$

i.e. $\pi^T : \alpha \mapsto \alpha + 0$. In fact, all possible Courant algebroid structures on $T\oplus T^*$ are parametrized by a closed three-form $H\in\Omega^3_{cl}$, which enters the definition of the bracket (9), changing it to a *twisted* Dorfman bracket

$$[X+\alpha, Y+\beta]_H = [X,Y] + \mathcal{L}_X\beta - \imath_Y \mathrm{d}\alpha + \imath_Y\imath_X H.$$

Moreover, any isotropic splitting of (10) $s : T \to T\oplus T^*$ is given by a two-form b, such that $X \overset{s}{\mapsto} X + b(X)$. This is equivalent to an action of a b-field transformation on $T\oplus T^*$[4]

$$e^b = \begin{pmatrix} \mathbb{1} & 0 \\ b & \mathbb{1} \end{pmatrix} \in \mathrm{End}(T\oplus T^*),$$

which consequently changes the bracket as

$$[e^b(X+\alpha), e^b(Y+\beta)]_H = e^b([X+\alpha, Y+\beta]_{H+\mathrm{d}b}),$$

which implies that when H is trivial in cohomology, then a choice of a b-field transformation such that $\mathrm{d}b = -H$ brings the twisted bracket $[\ ,\ ]_H$ into the standard form (9). When H is cohomologically non-trivial this can be done at least locally. This also means that any choice of splitting with a non-trivial b-field can be absorbed in the Courant algebroid bracket in terms of the *flux*[5] $\mathrm{d}b$.

We remark here that all the results in this paper remain valid for any exact courant algebroid E (i.e. E fits in the sequence (10)), which can be always identified with $T\oplus T^*$ by the choice of splitting equivalent to a choice of a representative $H\in\Omega^3_{cl}$. This also amounts to setting $b=0$ in all formulas since the b-field appears as a difference of two splittings.

[4]Here we are using the term b-field transformation more liberally as it is customary to use the term only in the cases when $\mathrm{d}b = 0$ so that e^b is a symmetry of $[\ ,\ ]$.

[5]Flux is a term used mainly in physics, in this context simply meaning the "tensorial contribution to the bracket".

A **Dirac structure** L is a maximally isotropic subbundle of $T \oplus T^*$ which is involutive under the Dorfman bracket, i.e.

$$L \subset T \oplus T^*, \quad \langle L, L \rangle = 0, \quad [L, L] \subset L.$$

Any Dirac structure inherits a natural Lie algebroid structure by restricting the Courant algebroid data $([\ ,\ ], \langle\ ,\ \rangle, \pi)$ to L.

We conclude with a useful formula for $[\ ,\ ]$ [4, Prop. 2.7]

$$\begin{aligned}\langle [X+\alpha, Y+\beta], Z+\gamma \rangle = &\ \langle \nabla_X (Y+\beta) - \nabla_Y (X+\alpha), Z+\gamma \rangle \\ &+ \langle \nabla_Z (X+\alpha), Y+\beta \rangle,\end{aligned} \tag{11}$$

where ∇ is any torsionless connection.

2.2 Generalized Para-Complex Geometry

In [5, **?**], the notion of generalized para-complex geometry (GpCG) along with basic integrability conditions and examples were given. All basic facts, which are directly analogous to generalized complex geometry (GCG), will be reviewed here.

Definition 2.1. A **generalized para-complex** (GpC) structure $\mathcal{K}$ is an endomorphism of $T \oplus T^*$, such that $\mathcal{K}^2 = \mathbb{1}$ and $\langle \mathcal{K}, \mathcal{K} \rangle = -\langle\ ,\ \rangle$, whose generalized Nijenhuis tensor,

$$\mathcal{N}_{\mathcal{K}}(u,v) = [\mathcal{K}u, \mathcal{K}v] + \mathcal{K}^2[u,v] - \mathcal{K}([\mathcal{K}u, v] + [u, \mathcal{K}v]), \tag{12}$$

vanishes.

If the integrability condition given by vanishing of (12) is omitted, we talk about **almost** generalized para-complex structures. If the integrability is given by a twisted Dorfman bracket, we call $\mathcal{K}$ a **twisted** GpC structure. Let us note here that the tensoriality of the expression (12) is ensured by the property that $\mathcal{K}$ is $\langle\ ,\ \rangle$-skew.

The most general form of an almost GpC structure is given by

$$\mathcal{K} = \begin{pmatrix} A & \Pi \\ \Omega & -A^* \end{pmatrix}, \qquad \begin{aligned} A^2 + \Pi\Omega &= \mathbb{1} \\ A\Pi - \Pi A^* &= 0 \\ \Omega A + A^* \Omega &= 0\ , \end{aligned} \tag{13}$$

where $A \in \mathrm{End}(T)$ and $\Omega \in \Omega^2(M)$, $\Pi \in \Gamma(\Lambda^2 T)$ are skew tensors.

Similarly to the ordinary tangent bundle geometry, we can use the projections onto the $+1$ and -1 eigenbundles of $\mathcal{K}$, $\mathrm{P} = \frac{1}{2}(\mathbb{1} + \mathcal{K})$ and $\tilde{\mathrm{P}} = \frac{1}{2}(\mathbb{1} - \mathcal{K})$ to decompose $\mathcal{N}_{\mathcal{K}}$ as

$$\mathcal{N}_{\mathcal{K}}(u,v) = 4(\mathrm{P}[\tilde{\mathrm{P}}u, \tilde{\mathrm{P}}v] + \tilde{\mathrm{P}}[\mathrm{P}u, \mathrm{P}v]). \tag{14}$$

Denote L and $\tilde{L}$ the $+1$ and -1 eigenbundles of $\mathcal{K}$, respectively. It is clear that both L and $\tilde{L}$ are (almost) Dirac structures. A direct consequence of the formula (14) is

Proposition 2.2. *A generalized almost para-complex structure is integrable iff its eigenbundles are involutive with respect to* $[\ ,\ ]$.

Similarly to the complex case, we also have a correspondence between Dirac structures and generalized para-complex structures:

Theorem ([5])**.** *There is a one-to-one correspondence between generalized para-complex structures on* M *and pairs of transversal Dirac subbundles of* $T \oplus T^*$.

Combining this result with the well-known result of [6] which states that any pair of transversal Dirac structures $(L, \tilde{L})$ forms a Lie bialgebroid $(L, L^* \simeq \tilde{L})$, one can immediately infer the following

Lemma. *Generalized para-complex structures on* $T \oplus T^*$ *are in one-to-one correspondence with Lie bialgebroid pairs* (L, L^*) *such that* $L \oplus L^* = T \oplus T^*$.

Analogously to GCG we define the notion of a type for GpCG

Definition 2.3. Let $\mathcal{K}$ be a GpC structure over a $2n$-domensional manifold M. A **type** $(k, l) \in \mathbb{Z}^2$, $0 \leq k, l \leq 2n$ of $\mathcal{K}$ is given by the types of the corresponding Dirac structures L, $\tilde{L}$, i.e. the rank of their respective projections onto T.

add the generalized darboux theorem for GP structures We now present main examples. More can be found in [5].

Example 2.4. **Product structures**, i.e. tensor fields $P \in \mathrm{End}(T)$, such that $P^2 = \mathbb{1}$, give the diagonal generalized para-complex structures:

$$\mathcal{K}_P = \begin{pmatrix} P & 0 \\ 0 & -P^* \end{pmatrix}.$$

The corresponding Dirac structures are given by $L = p_+ \oplus p_-^*$ and $\tilde{L} = p_- \oplus p_+^*$, where $p_\pm$ are the ± 1 eigenbundles of P. The integrability of $\mathcal{K}_P$ is equivalent to Frobenius integrability of P , i.e. vanishing of the Nijenhuis tensor of P. The type (k, l) of $\mathcal{K}_P$ is always such that $k + l = 2n$; if $\mathcal{K}_P$ is of type (n, n) then P is a para-complex structure.

Example 2.5. **Symplectic structures** give the off-diagonal generalized para-complex structures:

$$\mathcal{K}_\omega = \begin{pmatrix} 0 & \omega^{-1} \\ \omega & 0 \end{pmatrix}.$$

The ± 1 eigenbundles are given by $\mathrm{graph}(\pm\omega) = \{X \pm \omega(X) \mid X \in \mathfrak{X}\}$, and the integrability of $\mathcal{K}_\omega$ is equivalent to $\mathrm{d}\omega = 0$. Its type is $(2n, 2n)$. This example can also be seen as the *nondegenerate* case of a GpC structure, since both its eigenbundles are isomorphic to T (as well as T^*).

We will close this section with a discussion about the relationship between para-Hermitian and GpC geometry that will be developed in more detail in forthcoming sections. As we have seen, a para-Hermitian structure can be seen as a signature (n,n) metric on a $2n$-dimensional manifold $\mathcal{P}$ along with a choice of splitting $T\mathcal{P} = L \oplus \tilde{L}$ to a pair of maximally isotropic distributions. We can therefore observe that if $(T\mathcal{P}, \eta)$ is equipped with a structure of an exact Courant algebroid (i.e. a Courant algebroid bracket compatible with η), then the para-Hermitian K can be seen as a generalized (almost) para-complex structure in this Courant algebroid. Courant algebroid structures on para-Hermitian manifolds were studied in [4].

2.3 Generalized Product and Metric Type Structures

Definition 2.6. A **generalized product** structure (GP) is an endomorphism $\mathcal{A} \in \mathrm{End}(T \oplus T^*)$, such that $\mathcal{A}^2 = \mathbb{1}$ and $\langle \mathcal{A}, \mathcal{A} \rangle = \langle \cdot, \cdot \rangle$.

From the definition is obvious that even though the GP structures have, similarly to para-complex structures, eigenbundles $A_\pm$ corresponding to the eigenvalues ± 1, the bundles $A_\pm$ are not isotropic with respect to the pairing $\langle\ ,\ \rangle$. The consequence of this is, for example, that the generalized Nijenhuis tensor, defined in (12), is not a tensor if defined for a NIT structure. There is therefore no straightforward notion of integrability as for endomorphisms of $T \oplus T^*$ whose eigenbundles are isotropic.

The main examples are the following:

Example 2.7 (Product structures). Any (almost) product structure $J \in \mathrm{End}(T)$ defines a GP structure $\mathcal{A}_J \in \mathrm{End}(T \oplus T^*)$ in the following way

$$\mathring{A}_J = \begin{pmatrix} J & 0 \\ 0 & J^* \end{pmatrix},$$

and the ± 1 eigenbundles are $A_\pm = C_\pm \oplus C_\pm^* \subset T \oplus T^*$, where $C_\pm \subset T$ are the ± 1 eigenbundles of J.

Example 2.8. (Pseudo-Riemannian structures) Any (pseudo-)Riemannian structure η defines a GP structure $\mathcal{A}_\eta$ in the following way

$$\mathring{A}_\eta = \begin{pmatrix} 0 & \eta^{-1} \\ \eta & 0 \end{pmatrix},$$

and the ± 1 eigenbundles are $\mathrm{graph}(\pm\eta) \subset T \oplus T^*$. When η is a Riemannian structure, $\mathcal{A}$ defines a generalized metric.

Lemma 2.9. *A b-field transformation of a GP structure is again a GP structure.*

A general form of a GP structure is following

$$\mathcal{K} = \begin{pmatrix} A & g \\ \sigma & A^* \end{pmatrix}, \qquad \begin{aligned} A^2 + g\sigma &= \mathbb{1}, \\ Ag + gA^* &= 0, \\ \sigma A + A^*\sigma &= 0 \ , \end{aligned} \tag{15}$$

where $A \in \mathrm{End}(T)$ and $g \in \Gamma(T \otimes T)$, $\sigma \in \Gamma(T^* \otimes T^*)$ are symmetric tensors.

Whenever g is invertible, this system can be solved explicitly in terms of a pseudo-Riemannian metric $\eta \coloneqq g^{-1}$ and a two-form $b \coloneqq -\eta A$. The structure $\mathcal{A}$ is in this case simply a b-transform of $\mathcal{A}_\eta$ from Example 2.8:

$$\mathcal{A} \coloneqq \mathcal{A}(\eta, b) = e^b(A_\eta) = \begin{pmatrix} \mathbb{1} & 0 \\ b & \mathbb{1} \end{pmatrix} \begin{pmatrix} 0 & \eta^{-1} \\ \eta & 0 \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ -b & \mathbb{1} \end{pmatrix}. \tag{16}$$

The eigenbundles of $\mathcal{A}$ are $A_\pm = \mathrm{graph}(g \pm \eta)$ and therefore isomorphic to T. We shall denote the isomorphisms $\pi_\pm$:

$$\begin{aligned} \pi_\pm : C_\pm &\overset{\simeq}{\longleftrightarrow} T \\ X &\overset{\pi_\pm^{-1}}{\longmapsto} X + (b \pm \eta)X, \quad X \in \Gamma(T) \\ X + \alpha &\overset{\pi_\pm}{\longmapsto} X, \quad x + \alpha \in \Gamma(C_\pm). \end{aligned} \tag{17}$$

This is a well-known property first observed in [] for generalized metrics, here simply presented for the non-degenerate GP structures, which are just generalized metrics with the requirement of positive-definiteness relaxed. We will refer to these structures as **indefinite generalized metrics**.

We also recall the following relationship recovering the metric η from $\mathcal{A} = \mathcal{A}(\eta, b)$:

$$\eta(X, Y) = \frac{1}{2}\langle \mathcal{A}\pi_\pm^{-1}X, \pi_\pm^{-1}Y\rangle = \pm\frac{1}{2}\langle \pi_\pm^{-1}X, \pi_\pm^{-1}Y\rangle. \tag{18}$$

2.3.1 Complex Product and Metric Type Structures

For completeness, in order to exhaust all possibilities of sign choices in definitions of endomorphisms of $T \oplus T^*$ that square to ± 1 and are (anti-)orthogonal with respect to $\langle\ ,\ \rangle$, we define

Definition 2.10. A **complex generalized product** structure (CGP) is an endomorphism $\mathcal{A} \in \mathrm{End}(T \oplus T^*)$, such that $\mathcal{A}^2 = -\mathbb{1}$ and $\langle \mathcal{A}, \mathcal{A}\rangle = -\langle \cdot, \cdot \rangle$.

The general form of the CGP structures is the same as (15), except for a sign change in the equations the blocks satisfy

$$\begin{aligned} A^2 + g\sigma &= -\mathbb{1}, \\ Ag + gA^* &= 0, \\ \sigma A + A^*\sigma &= 0. \end{aligned}$$

The examples are similar as for the GP structures, given by complex and pseudo-Riemannian structures:

Example 2.11. Any complex structure I defines a CGP structure by

$$\mathring{A}_I = \begin{pmatrix} I & 0 \\ 0 & I^* \end{pmatrix},$$

and any (pseudo-)Riemannian metric defines a CGP structure by

$$\mathring{A}_\eta = \begin{pmatrix} 0 & -\eta^{-1} \\ \eta & 0 \end{pmatrix},$$

which is the non-degenerate type of CGP structures.

The nondegenerate CGP structures have been previously studied by Vaisman in [7].

2.4 Metric-induced splitting of the Courant bracket

Let $\mathcal{G}(g,b)$ be a (indefinite) generalized metric. As we have seen, this gives a splitting $T \oplus T^* = C_+ \oplus C_- \overset{\pi_+ \oplus \pi_-}{\simeq} T \oplus T$. The (twisted) Dorfman bracket $[\ ,\ ]_H$ on $T \oplus T^*$ can then be decomposed accordingly into a set of 8 in general different maps. We will denote the maps by

$$[\ ,\ ]_H^{ijk} : C_i \times C_j \to C_k,$$

where the symbols i, j, k take value either $+$ or $-$. These can then be identified as maps on T using the isomorphisms $\pi_\pm$ and formula (18):

$$g([X,Y]_H^{ijk}, Z) = \langle [\pi_i^{-1} X, \pi_j^{-1} Y]_H, \pi_k^{-1} Z \rangle.$$

There is a well-know statement about the 6 *mixed* components $[\ ,\ ]_H^{\pm\mp\mp}$, $[\ ,\ ]_H^{\pm\mp\pm}$ and $[\ ,\ ]_H^{\pm\pm\mp}$:

Proposition 2.12. *Let $\mathcal{G}(g,b)$ be an (indefinite) generalized metric and $[\ ,\ ]_H$ the H-twisted Dorfman bracket. Then*

$$\begin{aligned}
\frac{1}{2} g([X,Y]_H^{\pm\mp\mp}, Z) &= \mp g(\nabla_X^{\mp} Y, Z), \\
\frac{1}{2} g([X,Y]_H^{\mp\pm\mp}, Z) &= \pm g(\nabla_Y^{\mp} X, Z), \\
\frac{1}{2} g([X,Y]_H^{\pm\mp\mp}, Z) &= \mp g(\nabla_Z^{\mp} X, Y),
\end{aligned}$$

where $\nabla^\pm = \mathring{\nabla} \pm \frac{1}{2} g^{-1}(\mathrm{d}b + H)$, $\mathring{\nabla}$ being the Levi-Civita connection of g, are g-metric connections with a fully skew torsion given by $T^{\nabla^\pm} = \pm(\mathrm{d}b + H)$.

Proof. We show a short proof using the formula (11). First, since ∇ can be any torsionless connection, we take $\nabla = \mathring{\nabla}$, the Levi-Civita of g, and calculate

$$\begin{aligned}\langle \mathring{\nabla}_X(\pi_\pm^{-1}Y), \pi_\pm^{-1}Z\rangle &= \langle \mathring{\nabla}_X(Y + (b\pm g)Y), Z + (b\pm g)Z\rangle \\ &= \langle \mathring{\nabla}_X Y, (b\pm g)Z\rangle + \langle (b\pm g)\mathring{\nabla}_X Y, Z\rangle + \langle (\mathring{\nabla}_X b)Y, Z\rangle \\ &= \pm 2g(\mathring{\nabla}_X Y, Z) + (\mathring{\nabla}_X b)(Y,Z)\end{aligned} \tag{19}$$

and

$$\begin{aligned}\langle \mathring{\nabla}_X(\pi_\pm^{-1}Y), \pi_\mp^{-1}Z\rangle &= \langle \mathring{\nabla}_X(Y + (b\pm g)Y), Z + (b\mp g)Z\rangle \\ &= \langle \mathring{\nabla}_X Y, (b\mp g)Z\rangle + \langle (b\pm g)\mathring{\nabla}_X Y, Z\rangle + \langle (\mathring{\nabla}_X b)Y, Z\rangle \\ &= (\mathring{\nabla}_X b)(Y,Z).\end{aligned}$$

Using this, we get

$$\begin{aligned}&g([X,Y]_H^{\pm\mp\mp}, Z) = \langle [\pi_\pm^{-1}X, \pi_\mp^{-1}Y]_H, \pi_\mp^{-1}Z\rangle \\ &= \langle \nabla_X(\pi_\mp^{-1}Y), \pi_\mp^{-1}Z\rangle - \langle \nabla_Y(\pi_\pm^{-1}X), \pi_\mp^{-1}Z\rangle + \langle \nabla_Z(\pi_\pm^{-1}X), \pi_\mp^{-1}Y\rangle + H(X,Y,Z) \\ &= \mp 2g(\mathring{\nabla}_X Y, Z) + (\mathring{\nabla}_X b)(Y,Z) - (\mathring{\nabla}_Y b)(X,Z) + (\mathring{\nabla}_Z b)(X,Y) + H(X,Y,Z) \\ &= \mp 2g(\mathring{\nabla}_X Y, Z) + \mathrm{d}b(X,Y,Z) + H(X,Y,Z),\end{aligned}$$

proving the first equality. For the second, we use the following general properties of the (twisted) Dorfman bracket

$$\begin{aligned}\langle [u,v], w\rangle &= -\langle [v,u], w\rangle + \pi(w)\langle u,v\rangle \\ \langle [u,v], w\rangle &= \pi(u)\langle v,w\rangle - \langle [u,w], v\rangle,\end{aligned}$$

so that

$$\begin{aligned}&g([X,Y]_H^{\mp\pm\mp}, Z) = \langle [\pi_\mp^{-1}X, \pi_\pm^{-1}Y]_H, \pi_\mp^{-1}Z\rangle = -\langle [\pi_\pm^{-1}Y, \pi_\mp^{-1}X]_H, \pi_\mp^{-1}Z\rangle \\ &\quad = \pm 2g(\mathring{\nabla}_Y X, Z) - (\mathrm{d}b(Y,X,Z) + H(Y,X,Z))\end{aligned}$$

and

$$\begin{aligned}&g([X,Y]_H^{\mp\mp\pm}, Z) = \langle [\pi_\mp^{-1}X, \pi_\mp^{-1}Y]_H, \pi_\pm^{-1}Z\rangle = -\langle [\pi_\mp^{-1}X, \pi_\pm^{-1}Z]_H, \pi_\mp^{-1}Y\rangle \\ &\quad = \langle [\pi_\pm^{-1}Z, \pi_\mp^{-1}X], \pi_\mp^{-1}Y\rangle = \mp 2g(\mathring{\nabla}_Z X, Y) + \mathrm{d}b(Z,X,Y) + H(Z,X,Y).\end{aligned}$$

□

The pure components of the Dorfman bracket, $[\ ,\ ]_H^{\pm\pm\pm}$ give a certain bracket operation that is closely related to the D-bracket used in the physics literature [**?**, **?**, **?**], which has been recently related to para-Hermitian geometry in [4, 8].

Proposition 2.13. *Let $\mathcal{G}(g,b)$ be an (indefinite) generalized metric and $[\ ,\ ]_H$ the H-twisted Dorfman bracket. Then*

$$\frac{1}{2}g([X,Y]_H^{\pm\pm\pm},Z) = \pm g([\![X,Y]\!]^{\pm}_{H+\mathrm{d}b},Z)$$

where

$$g([\![X,Y]\!]^{\pm}_{H+\mathrm{d}b},Z) = g(\mathring{\nabla}_X Y - \mathring{\nabla}_Y X, Z) + g(\mathring{\nabla}_Z X, Y) \\ \pm \frac{1}{2}(H(X,Y,Z) + \mathrm{d}b(X,Y,Z)),$$

is the **almost D-bracket***.*

Proof. This is again a simple application of the formula (11) and (19):

$$\begin{aligned} g([X,Y]_H^{\pm\pm\pm},Z) =& \langle [\pi_\pm^{-1}X, \pi_\pm^{-1}Y]_H, \pi_\pm^{-1}Z\rangle \\ =& \pm 2(g(\mathring{\nabla}_X Y,Z) - g(\mathring{\nabla}_Y X,Z) + g(\mathring{\nabla}_Z X,Y)) \\ &+ (\mathring{\nabla}_X b)(Y,Z) - (\mathring{\nabla}_Y b)(X,Z) + (\mathring{\nabla}_Z b)(X,Y) + H(X,Y,Z) \\ =& \pm 2g([\![X,Y]\!]^{\pm}_{H+\mathrm{d}b},Z) \end{aligned}$$

□

We will see later that when $\mathcal{G} = \mathcal{G}(\eta,b)$ is a indefinite generalized metric of split signature which commutes with a generalized para-complex structure $\mathcal{K}$ (i.e. $(\mathcal{G},\mathcal{K})$ is a GpK structure, see Section 3), there is a pair of (almost) para-Hermitian structures $(\eta,K_\pm)$ on the tangent bundle and the brackets $[\![\ ,\ ]\!]^{\pm}_{H+\mathrm{d}b}$ become the D-brackets for each of them.

This way of expressing the D-bracket has been implicitly (without the relationship to generalized metrics and GpC structures) used in [**?**].I'd like to describe this in more detail

3 Commuting Pairs of Generalized Structures

In this section we generalize following construction well known from Generalized Kähler geometry. Let $\mathcal{J}_+$ be an almost GC structure and $\mathcal{G}$ a commuting generalized metric. Then $\mathcal{J}_- = \mathcal{G}\mathcal{J}_+$ is another almost *GC* structure and any two of the triple $(\mathcal{J}_+,\mathcal{J}_-,\mathcal{G})$ commute. This then implies that both GC structures preserve the eigenbundles $C_\pm$ of $\mathcal{G}$ and therefore yield endomorphisms of $C_\pm$ that square to $-\mathbb{1}$. It is then easy to see that $\mathcal{J}_+ |_{C_\pm} = \pm\mathcal{J}_- |_{C_\pm}$ and the following

$$J_+ = \pi_+\mathcal{J}_\pm\pi_+^{-1} \quad J_- = \pm\pi_-\mathcal{J}_\pm\pi_-^{-1} \tag{20}$$

where $\pi_\pm$ are the isomorphisms associated to $\mathcal{G}$ by (17), defines a pair of almost complex structures and the metric g defining $\mathcal{G} = \mathcal{G}(g,b)$ is Hermitian with respect to both of them:

$$g(I_\pm X, I_\pm Y) = \frac{1}{2}\langle \pi_+^{-1} I_\pm X, \pi_+^{-1} I_\pm Y\rangle = \frac{1}{2}\langle \mathcal{J}_\pm \pi_+^{-1} X, \mathcal{J}_\pm \pi_+^{-1} Y\rangle = \frac{1}{2}\langle \pi_+^{-1} X, \pi_+^{-1} Y\rangle$$
$$= g(X, Y),$$

where we used (18) and (20). The data $(\mathcal{J}_+, \mathcal{J}_-, \mathcal{G})$ therefore defines an almost **bi-Hermitian structure** with a B-field b [6] (g, b, I_+, I_-) on the tangent bundle. We can observe that the correspondence

$$(\mathcal{J}_+, \mathcal{J}_-, \mathcal{G}) \longleftrightarrow (g, b, I_+, I_-)$$

preserves the *type* of the data; the bi-Hermitian data on T corresponds to bi-Hermitian data on $T \oplus T^*$ once we realize that $\mathcal{G}$ defines a genuine Riemannian structure on the bundle $T \oplus T^*$ by

$$\hat{\mathcal{G}}(u, v) \coloneqq \langle \mathcal{G}u, v\rangle,$$

which is Hermitian with respect to $\mathcal{J}_\pm$ due to $\mathcal{G}$ and $\mathcal{J}_\pm$ commuting. The signature of g then also corresponds to the signature of $\hat{\mathcal{G}}$. A similar construction can be carried out for any triple of generalized structures, under the condition that one of them is of *non-degenerate type*, i.e. plays the role of the metric $\mathcal{G}$ above.

We will label any generalized structure $\mathcal{J}$ by a pair of signs (α, β), according to:

$$\mathcal{J} = \alpha \mathbb{1}, \quad \langle \mathcal{J}\cdot, \mathcal{J}\cdot\rangle = \beta\langle\ ,\ \rangle,$$

i.e. $(-,+)$ represents a GC structure, $(+,-)$ a GpC structure, $(+,+)$ GP or metric type structures and $(-,-)$ their complex counterpart, CGP structures. We summarize all possible combinations of commuting pairs in the following table

$\mathcal{J}_+$ Type	$\mathcal{J}_-$ Type	$\mathcal{G} = \mathcal{J}_+\mathcal{J}_-$ Type	
$(-,+)$	$(-,+)$	$(+,+)$	Generalized Kähler
$(+,-)$	$(+,-)$	$(+,+)$	Generalized para-Kähler
$(+,+)$	$(+,+)$	$(+,+)$	Generalized Chiral
$(-,-)$	$(-,-)$	$(+,+)$	Generalized Complex Chiral
$(-,+)$	$(+,-)$	$(-,-)$	Generalized para-Kähler (Vaisman)

Table 2: Commuting pairs of generalized structures.

[6] The B-field becomes relevant when one formulates the integrability conditions on $\mathcal{J}_\pm$ in terms of the bi-Hermitian data

The first line is the example we discussed above, describing GK structures. To recover the bi-hermitian data, $\mathcal{G}$ has to be a generalized metric. If $\mathcal{G}$ is not positive-definite, we get a generalization to generalized pseudo Kähler structures, where the corresponding tangent bundle data give what we might call a pseudo-bi-hermitian geometry.

The second line describes the direct analogue of a (pseudo) GK structure but for GpC structures, which we call Generalized para-Kähler geometry[7]. The corresponding tangent bundle data is given, as one might expect, by a pair of para-Hermitian structures. We discuss this geometry in Section 3.1.

The third line describes what we call a Generalized Chiral structures, i.e. a commuting pair of GP structures that recovers a pair of Chiral structures on the tangent bundle. We discuss this geometry in Section 4. The complex case in the fourth line is completely analogous but will not be discussed in this paper.

The last line describes a *mixed* geometry that Vaisman in [7] calls a Generalized para-Kähler geometry. Because this type of geometry via the above construction gives a complex and a product structure on the tangent bundle, we will in the present paper coin the name **generalized semi-Kähler** for this geometry and reserve the term generalized para-Kähler for the direct para-complex analogue of GK geometry in the second line.

3.1 Generalized Para-Kähler Structures

We now discuss the Generalized para-Kähler structures in more detail. Because a lot of constructions are entirely analogous to their complex counterpart, we will frequently not give excessive detail. To consult classical literature on generalized Kähler geometry, see [10, 11].

Definition 3.1. An (almost) **Generalized para-Kähler structure** (GpK) is a commuting pair $(\mathcal{G}, \mathcal{K}_+)$ of a split signature generalized metric $\mathcal{G} = \mathcal{G}(\eta, b)$ and a GpC structure $\mathcal{K}_+$. If additionally both $\mathcal{K}_+$ and $\mathcal{K}_- := \mathcal{G}\mathcal{K}_+$ are integrable w.r.t. the (twisted) Dorfman bracket, we call $(\mathcal{G}, \mathcal{K}_+)$ a (twisted) GpK structure.

Since any two structures in the triple $(\mathcal{G}, \mathcal{K}_1, \mathcal{K}_2)$ determine the third, we may refer to the GpK structure $(\mathcal{G}, \mathcal{K}_1)$ by the pair $(\mathcal{K}_1, \mathcal{K}_2)$, in particular when integrability – which is tied with $\mathcal{K}_{1/2}$ – is discussed.

Example 3.2. Let (η, K) be an almost para-Hermitian structure, with $\omega = \eta K$ the fundamental form. Then

$$\mathcal{K}_+ = \begin{pmatrix} K & 0 \\ 0 & -K^* \end{pmatrix}, \ \mathcal{K}_- = \begin{pmatrix} 0 & \omega^{-1} \\ \omega & 0 \end{pmatrix}, \ \mathcal{G} = \begin{pmatrix} 0 & \eta^{-1} \\ \eta & 0 \end{pmatrix},$$

gives an almost generalized para-Kähler structure which is GpK iff (η, K) is Kähler.

[7] Generalized para-Kähler geometry corresponds to a split signature generalized metric, and again generalizations with arbitrary signature can be discussed

Let $C_\pm$ be the eigenbundles of $\mathcal{G}$. As discussed in Section 3 above, $\mathcal{K}_+ |_{C_\pm} = \pm\mathcal{K}_- |_{C_\pm}$ and we can therefore construct two para-complex structures $K_\pm$ as follows:

$$K_+ = \pi_+ \mathcal{K}_\pm \pi_+^{-1} \quad K_- = \pm\pi_- \mathcal{K}_\pm \pi_-^{-1} \tag{21}$$

Using (18), it can be easily checked that $\eta(K_\pm X, K_\pm Y) = -\eta(X,Y)$ and $\eta K_\pm := \omega_\pm$ defines two almost symplectic forms, therefore $(\eta, K_\pm)$ are two almost para-Hermitian structures. We therefore see that any (almost) generalized para-Kähler structure defines an (almost) bi-para-Hermitian structure $(\eta, K_\pm)$ with an extra data given by the two-form b. The converse is also true; given (η, b) we reconstruct the isomorphisms $\pi_\pm$ and use them to define a pair of commuting $\mathcal{K}_\pm$ using $K_\pm$:

$$\mathcal{K}_\pm = \pi_+^{-1} K_+ \pi_+ P_+^C \pm \pi_-^{-1} K_- \pi_- P_-^C, \tag{22}$$

where $P_\pm^C$ are the projections onto $C_\pm$ given by $P_\pm^C = \frac{1}{2}(\mathbb{1} \pm \mathcal{G})$. In matrix form, this yields an expression similar to GK geometry

$$\mathcal{K}_\pm = \frac{1}{2}\begin{pmatrix} \mathbb{1} & 0 \\ b & \mathbb{1} \end{pmatrix} \begin{pmatrix} K_+ \pm K_- & \omega_+^{-1} \mp \omega_-^{-1} \\ \omega_+ \mp \omega_- & -(K_+^* \pm K_-^*) \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ -b & \mathbb{1} \end{pmatrix}, \tag{23}$$

Because the existence of an generalized almost para-Kähler structure implies an existence of two almost para-Hermitian structures, we have the following observation.

Corollary 3.3. *Any generalized almost para-Kähler manifold is of even dimension.*

We remark here that the GpK structures (23) can be also written in a different parametrization. Let now wlog $b = 0$, i.e. $\mathcal{G} = \mathcal{G}(\eta, 0)$. Ten $\mathcal{K}_+$ takes the form (13) and the requirement that $\mathcal{K}_+$ and $\mathcal{G}$ commute imply

$$\Pi = \eta^{-1}\Omega\eta^{-1}, \quad -A^* = \eta A \eta^{-1},$$

so that $\mathcal{K}_+$ takes the form

$$\mathcal{K}_+ = \begin{pmatrix} A & \eta^{-1}\Omega\eta^{-1} \\ \Omega & \eta A \eta^{-1} \end{pmatrix}, \tag{24}$$

and the equation $A^2 + \Pi\Omega = \mathbb{1}$ in (13) then implies that $(A \pm \eta^{-1}\Omega, \eta)$ is a pair of almost para-Hermitian structures. Of course, comparing (24) with (23), we see that $A = \frac{1}{2}(K_+ + K_-)$ and $\eta^{-1}\Omega = \frac{1}{2}(K_+ - K_-)$ so that indeed $K_\pm = A \pm \eta^{-1}\Omega$. When $b \neq 0$, the argument can be generalized so that

$$\mathcal{K}_+ = \begin{pmatrix} \mathbb{1} & 0 \\ b & \mathbb{1} \end{pmatrix} \begin{pmatrix} A & \eta^{-1}\Omega\eta^{-1} \\ \Omega & \eta A \eta^{-1} \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ -b & \mathbb{1} \end{pmatrix}.$$

To summarize, we have

Corollary 3.4. *An almost GpK structure on M is equivalent to the data (A, Ω, η, b), where $A \in \mathrm{End}(TM)$, $\Omega, b \in \Omega^2(M)$ and η is a split signature metric, such that*

$$\eta A = -A^* \eta, \quad (A \pm \eta^{-1}\Omega)^2 = \mathbb{1}.$$

We see that even though A is skew-orthogonal w.r.t. η, it is not in general an (almost) para-Hermitian structure, since $A^2 \neq \mathbb{1}$. However, we have the following

$$2A^2 = \frac{1}{2}(K_+^2 + K_-^2 + K_+K_- + K_+K_-) = \mathbb{1} + \{K_+, K_-\},$$

Therefore, the obstruction to $\sqrt{2}A$ being itself an almost para-Hermitian structure is the anti-commutator $\{K_+, K_-\}$. When this vanishes, $I = K_+K_-$ is an almost pseudo-Hermitian structure and the triple (I, K_+, K_-) defines an almost para-Hyper-Hermitian structure [**?**, **?**]. This case is discussed in Example 3.8.

We now turn to an eigenbundle decomposition of $T \oplus T^*$ induced by the GpK structure $(\mathcal{K}_1, \mathcal{K}_2)$. By the virtue of $\mathcal{K}_1$ and $\mathcal{K}_2$ commuting, their respective eigenbundles will further split into $\pm$ eigenbundles of the other GpC structure, i.e.

$$T \oplus T^* = \ell_+ \oplus \ell_- \oplus \tilde{\ell}_+ \oplus \tilde{\ell}_-, \tag{25}$$

where $L_1 = \ell_+ \oplus \ell_-$ and $\tilde{L}_1 = \tilde{\ell}_+ \oplus \tilde{\ell}_-$ are the ± 1 eigenbundles of $\mathcal{K}_1$, while $L_2 = \ell_+ \oplus \tilde{\ell}_+$ and $\tilde{L}_2 = \ell_- \oplus \tilde{\ell}_-$ are the ± 1 eigenbundles of $\mathcal{K}_2$. Furthermore, because $\mathcal{G} = \mathcal{K}_1\mathcal{K}_2$ has eigenbundles $C_\pm$, we see that

$$C_\pm = \ell_\pm \oplus \tilde{\ell}_\mp.$$

3.1.1 Bi-para-Hermitian Geometry and Integrability

We will now discuss the relationship between the Courant integrability of $\mathcal{K}_{/2}$ and properties of the induced tangent bundle data. We immediately see that the integrability of the GpK structure implies the involutivity of each of the four eigenbundles of (25), since they are intersections of involutive subbundles $L_{1/2}$ and $\tilde{L}_{1/2}$. In fact, this is also a sufficient condition.

Proposition 3.5. *The generalized almost para-Kähler structure $(\mathcal{K}_1, \mathcal{K}_2)$ is integrable iff all eigenbundles in the decomposition (25) are Courant involutive.*

Proof. We will show that the integrability of both $\ell_\pm$ implies integrability of $L_1 = \ell_+ \oplus \ell_-$ and the involutivity of L_2 and $\tilde{L}_{1/2}$ needed for integrability of $\mathcal{K}_{1/2}$ follows analogously. Let $x_+, z_+ \in \Gamma(\ell_+)$ and $y_-, z_- \in \Gamma(\ell_-)$. Because we assume $\ell_\pm$ are involutive, we only need to show that $[\ell_+, \ell_-], [\ell_-, \ell_+] \subset L$. Using the compatibility of the Courant bracket with the pairing $\langle , \rangle_+$ through the anchor π, we have

$$\begin{aligned}
\langle [x_+, y_-], z_+ \rangle &= \pi(x_+)\langle y_-, z_+ \rangle - \langle y_-, [x_+, z_+] \rangle = 0 \\
\langle [x_+, y_-], z_- \rangle &= -\pi(y_-)\langle x_+, z_- \rangle + \langle x_+, [y_-, z_-] \rangle = 0,
\end{aligned}$$

because $\ell_\pm$ are mutually orthogonal and $\ell_\pm$ are involutive. This shows that $[\ell_+, \ell_-] \perp L_1$, therefore $[\ell_+, \ell_-] \subset L_1$ because L_1 is maximally isotropic, proving that L_1 is involutive. □

We will now aim to express the involutivity of $\ell_\pm$ and $\tilde{\ell}_\pm$ in terms of the induced biparahermitian data $(\omega_{1/2}, \eta, b)$. For this, we first notice that the isomorphisms $\pi_\pm$ map the four bundles $\ell_\pm$ and $\tilde{\ell}_\pm$ exactly to the four eigenbundles of $K_{1/2}$ in T. Explicitly, we have

$$\pi_+\ell_+ = T^{(1,0)_+},\ \pi_+\tilde{\ell}_- = T^{(0,1)_+}$$
$$\pi_-\ell_- = T^{(1,0)_-},\ \pi_-\tilde{\ell}_+ = T^{(0,1)_-},$$

where $(p,q)_\pm$ denotes the (p,q) decomposition (2) induced by $K_\pm$. The above can be checked, for example by using (21):

$$K_+(\pi_+\ell_+) = \pi_+\mathcal{K}_\pm\ell_+ = \pi_+\ell,$$

and so $\pi_+\ell$ must be the $+1$ eigenbundle of K_+, $T^{(1,0)_+}$. Next, because of this, the subbundles in the decomposition (25) can be expressed as graphs of $b \pm \eta$, e.g. for $x_+ \in \Gamma(\ell_+)$ (recalling $\eta = \omega_\pm K_\pm$)

$$x_+ = \pi_+^{-1}X = X + (b+\eta)X = X + (b + \omega_+K_+)X = X + (b+\omega_+)X, \quad (26)$$

for some $X \in \Gamma(T^{(1,0)_+})$. All the bundles in (25) can therefore be all expressed as graphs of two-forms $b \pm \omega_\pm$ mapping from the eigenbundles of $K_\pm$. We can now formulate the integrability of $\mathcal{K}_{1/2}$ using the data $(\omega_{1/2}, \eta, b)$.

Theorem 3.6. *A generalized almost para-Kähler structure $(\mathcal{K}_+, \mathcal{K}_-)$, given alternatively by the induced biparahermitian data (K_+, K_-, η, b), is integrable if and only if the following conditions are simultaneously satisfied*

1. *$K_\pm$ are integrable para-Hermitian structures, i.e. their Nijenhuis tensors vanish*
2. *$\mathrm{d}^p_+\omega_+ = -\mathrm{d}^p_-\omega_- = -(H + \mathrm{d}b)$,*

where $\mathrm{d}^p_\pm = (\partial^\pm_{(+)} - \partial^\pm_{(-)})$ are the d^p operators (4).

Proof. We have seen previously that the integrability of $(\mathcal{K}_+, \mathcal{K}_-)$ is equivalent to the bundles $\ell_\pm$ and $\tilde{\ell}_\pm$ being Courant involutive. We have further found that all $\ell_\pm, \tilde{\ell}_\pm$ can be written as $X + (b \pm \omega_\pm)$ for X a vector in eigenbundles of $K_\pm$. We will now use to find the conditions on the involutivity of these bundles.

We start with ℓ_+ which is given by $X + (b+\omega_+)X$, $X \in \Gamma(T^{(1,0)_+})$. The Dorfman bracket of two such sections is

$$\begin{aligned}[X + (b+\omega_+)X, Y + (b+\omega_+)Y]_H = {}& [X,Y] + (b+\omega_+)([X,Y]) \\ &+ \imath_Y\imath_Y(\mathrm{d}(b+\omega_+) + H).\end{aligned}$$

The only tangent component is $[X, Y]$ and so it has to belong to $T^{(1,0)_+}$, meaning the bundle $\Gamma(T^{(1,0)_+})$ is Frobenius integrable. When this is satisfies, we see that $[X, Y] + (b + \omega_+)([X, Y])$ in turn belongs to ℓ_+, which also implies that

$$\imath_Y \imath_Y (\mathrm{d}(b + \omega_+) + H) = 0, \quad X, Y \in \Gamma(T^{(1,0)_+}) \tag{27}$$

must be satisfied. The $(1, 0)_+$ component of this equation yields

$$(\mathrm{d}b + H)^{(3,0)_+} = 0,$$

since $(\mathrm{d}\omega_+)^{(3,0)} = 0$ whenever $T^{(1,0)_+}$ is integrable. The $(0, 1)_+$ component of (27) then translates to

$$(\mathrm{d}b + H)^{(2,1)_+} = -\mathrm{d}\omega_+^{(2,1)_+}.$$

Carrying out the same argument for the bundle $\tilde{\ell}_-$ then tells us $T^{(0,1)_+}$ is integrable and

$$(\mathrm{d}b + H)^{(0,3)_+} = 0,$$
$$(\mathrm{d}b + H)^{(1,2)_+} = +\mathrm{d}\omega_+^{(1,2)_+}.$$

Summing up, the involutivity of ℓ_+ and $\tilde{\ell}_-$ is equivalent to K_+ being integrable and further, since ω_+ is of type $(1, 1)_+$,

$$(\mathrm{d}b + H) = -\partial^+_{(+)}\omega_+ + \partial^+_{(-)}\omega_+ = -\mathrm{d}^p_+\omega_+.$$

Analogous calculation for ℓ_- and $\tilde{\ell}_+$ then completes the proof. □

3.1.2 Para-Holomorphic Poisson structures

In this section we show that every GpK manifold has a Poisson bivector, which in addition is para-holomorphic with respect to both para-Hermitian structures (K_+, K_-). This then gives another pair of GpC structures – which are related by a certain B-field transformation – constructed purely from the GpK data.

Theorem 3.7. *Let $(\mathcal{K}_+, \mathcal{G})$ be a GpK structure and (η, K_+, K_-, b) the corresponding bi-para-hermitian data. Then*

$$Q = \frac{1}{2}[K_+, K_-]\eta^{-1} \tag{28}$$

is a Poisson bivector of type $(2, 0)_\pm + (0, 2)_\pm$, which is para-holomorphic with respect to both $K_\pm$.

Proof. Q being of type $(2, 0) + (0, 2)$ with respect to both $K_\pm$ is equivalent to $Q(K_\pm^*, K_\pm^*) = Q(\cdot, \cdot)$, or $K_\pm Q K_\pm^* = Q$. This can be simply checked by using $[K_+, K_-] = (K_+ + K_-)(K_+ - K_-)$ and para-Hermitian compatibility conditions, $K_\pm \eta^{-1} = -\eta^{-1} K_\pm^*$.

For Q to be para-Holomorphic, the local coefficient functions Q^{ij} of the $(2,0)$ components have to be locally independent of the $\tilde{x}$ coordinates, i.e. $\tilde{\partial}^i Q^{jk} = 0$ and similarly the $(0,2)$ components have to satisfy $\partial_i Q_{jk} = 0$, see Example 1.7 and Equation (6). We now check the condition on the $(2,0)$ component for K_+, i.e. $\tilde{\partial}^{+i} Q_+^{jk} = 0$. The remainder of the calculations are completely analogous. To simplify the notation, we will write $K_+ = K$ and use no $+$ subscript for quantities assigned to $K_+ = K$.

Let $(x^i, \tilde{x}_i)$ be the local adapted coordinates of K. The coordinate functions of the $(2,0)$ component of Q are given by

$$\begin{aligned} Q^{jk} = Q(dx^j, dx^k) &= \frac{1}{2}\langle (KK_- - K_-K)\eta^{-1}(dx^j), dx^k\rangle \\ &= \frac{1}{2}(\langle K_-\eta^{-1}(dx^j), K^* dx^k\rangle + \langle K_-\eta^{-1}(dx^j), dx^k\rangle) \\ &= \omega_-^{-1}(dx^j, dx^k), \end{aligned}$$

because $\eta^{-1}(dx^j)$ is in L.

We now use the following formula for the exterior derivative of a one-form:

$$\mathrm{d}\alpha(X,Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X,Y]). \tag{29}$$

Choosing in (29) $\alpha = dx^j$, $X = \tilde{\partial}^i$ and $Y = \omega_-^{-1}(dx^k)$, we get (because $\mathrm{d}dx^j = 0$ and $dx^j(\tilde{\partial}^i) = 0$)

$$\begin{aligned} \tilde{\partial}^i \omega_-^{-1}(dx^j, dx^k) &= \langle [\tilde{\partial}^i, \omega_-^{-1}(dx^k)], dx^j\rangle = \langle \mathring{\nabla}_{\tilde{\partial}^i}\omega_-^{-1}(dx^k) - \mathring{\nabla}_{\omega_-^{-1}(dx^k)}\tilde{\partial}^i, dx^j\rangle \\ &= \langle (\mathring{\nabla}_{\tilde{\partial}^i}\omega_-^{-1})dx^k + \omega_-^{-1}(\mathring{\nabla}_{\tilde{\partial}^i}dx^k) - \mathring{\nabla}_{\omega_-^{-1}(dx^k)}\tilde{\partial}^i, dx^j\rangle. \end{aligned}$$

The formula (29) in the following form

$$\mathrm{d}\alpha(X,Y) = \mathring{\nabla}_X\alpha(Y) - \mathring{\nabla}_Y\alpha(X),$$

with $\alpha = dx^k$, $X = \tilde{\partial}^j$ and $Y = \omega_-^{-1}(dx^j)$, can be used to derive

$$\langle \mathring{\nabla}_{\tilde{\partial}^i}dx^k, \omega_-^{-1}(dx^j)\rangle = \langle \mathring{\nabla}_{\omega_-^{-1}(dx^j)}dx^k, \tilde{\partial}^i\rangle = -\langle \mathring{\nabla}_{\omega_-^{-1}(dx^j)}\tilde{\partial}^i, dx^k\rangle,$$

so that we have

$$\begin{aligned} \tilde{\partial}^i \omega_-^{-1}(dx^j, dx^k) =& \langle (\mathring{\nabla}_{\tilde{\partial}^i}\omega_-^{-1})dx^k, dx^j\rangle \\ &+ \langle \mathring{\nabla}_{\omega_-^{-1}(dx^j)}\tilde{\partial}^i, dx^k\rangle - \langle \mathring{\nabla}_{\omega_-^{-1}(dx^k)}\tilde{\partial}^i, dx^j\rangle. \end{aligned} \tag{30}$$

We further use $\nabla^\pm K_\pm = 0$, where $\nabla^\pm = \mathring{\nabla} \pm \frac{1}{2}\eta^{-1}H$, $H = h + \mathrm{d}b$, which implies

$$\mathring{\nabla}_X\omega_\pm(Y,Z) = \mp\frac{1}{2}(H(X, K_\pm Y, Z) + H(X, Y, K_\pm Z)).$$

Consequently, this yields

$$\langle (\mathring{\nabla}_{\tilde{\partial}^i}\omega_-^{-1})dx^k, dx^j\rangle = \frac{1}{2}(H(\tilde{\partial}^i, \omega_-^{-1}dx^k, \eta^{-1}dx^j) + H(\tilde{\partial}^i, \eta^{-1}dx^k, \omega_-^{-1}dx^j)),$$

as well as

$$\begin{aligned}\langle\mathring{\nabla}_{\omega_-^{-1}(dx^j)}\tilde{\partial}^i, dx^k\rangle = \langle\mathrm{P}\mathring{\nabla}_{\omega_-^{-1}(dx^j)}\tilde{\partial}^i, dx^k\rangle &= -\frac{1}{2}\mathring{\nabla}_{\omega_-^{-1}(dx^j)}\omega(\tilde{\partial}^i, \eta^{-1}dx^k)\\ &= -\frac{1}{2}H(\omega_-^{-1}dx^j, \tilde{\partial}^i, \eta^{-1}dx^k),\end{aligned}$$

and

$$-\langle\mathring{\nabla}_{\omega_-^{-1}(dx^k)}\tilde{\partial}^i, dx^j\rangle = \frac{1}{2}H(\omega_-^{-1}dx^k, \tilde{\partial}^i, \eta^{-1}dx^j).$$

Summing these terms as in (30), we conclude that $\tilde{\partial}^i\omega_-^{-1}(dx^j, dx^k) = 0$. $\square$

still need to prove Q is Poisson, show the gauge equivalence of two hol. Poisson structures

3.1.3 Bi-para-Hermitian point of view

In [4, 8] it has been shown that on a para-Hermitian manifold $(\mathcal{P}, K, \eta)$, there are two natural Courant algebroid structures on $\mathcal{P}$ given purely by the para-Hermitian data. Here we briefly recall this point of view and explain how it relates to the present discussion.

Recall that the tangent bundle of a para-Hermitian manifold splits to isotropic eigenbundles of K as $T\mathcal{P} = L \oplus \tilde{L}$. Because L and $\tilde{L}$ are isotropic, whenever $\tilde{x} \in \Gamma(\tilde{L})$, $\eta(\tilde{x}, \cdot)$ is an element of $\Gamma(L^*)$ because it contracts only with vectors in L. In fact, any section of L^* is of this form by non-degeneracy of η. We therefore have the following vector bundle isomorphism:

$$\begin{aligned}\rho:\ T\mathcal{P} = L \oplus \tilde{L} &\to L \oplus L^*\\ X = x + \tilde{x} &\mapsto x + \eta(\tilde{x}).\end{aligned}$$

and similarly we can define $\tilde{\rho} : L \oplus \tilde{L} \to \tilde{L} \oplus \tilde{L}^*$. When L is integrable, we have $L \oplus L^* \simeq (T \oplus T^*)\mathcal{F}$, where $\mathcal{F}$ is an n-dimensional integral foliation of L. The tangent bundle then inherits a structure of the exact Courant algebroid of $(T \oplus T^*)\mathcal{F}$ via the map ρ.

From the point of view of the Courant algebroid $(T \oplus T^*)\mathcal{F}$, the para-Hermitian structure K defines the simple GpC structure

$$K = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix},$$

and it is easy to see that any other almost para-Hermitian structure sharing the same metric η, will define another almost GpC structure. I still want to extend this discussion

3.1.4 Examples of GpK structures

We have already seen that the simplest GpK structure is given by a para-Kähler structure. Here we present few more examples.

Example 3.8 (Para-Hyperkähler geometry). This is the para-complex version of the correspondence between hyperkähler Generalized Kähler geometries [10, Example 6.3]. Let $(K_i, \omega_i, \eta)_{i=1,\cdots,3}$ be a para-hyper-Kähler (PhK) structure [3, 12]. This means that K_i are a para-quaternionic triple, $-K_1^2 = K_2^2 = K_3^2 = \mathbb{1}$ and $(K_i, \eta)_{i=2,3}$ are para-Kähler, while (K_1, η) is pseudo-Kähler, i.e. the associated fundamental forms $\omega_i = \eta K_i$ are symplectic. In particular, $(K_i, \eta)_{i=2,3}$ is a bi-Hermitian structure and therefore this defines a generalized para-Kähler structure $(\mathcal{K}_1, \mathcal{K}_2)$ by

$$\mathcal{K}_{1/2} = \frac{1}{2}\begin{pmatrix} K_2 \pm K_3 & \omega_2^{-1} \mp \omega_3^{-1} \\ \omega_2 \mp \omega_3 & -(K_2^* \pm K_3^*) \end{pmatrix},$$

which, just like in the GK case, can be rewritten as B-field transformations by $\pm\omega_1$ of two GpC structures of a symplectic type:

$$\mathcal{K}_{1/2} = \begin{pmatrix} \mathbb{1} & 0 \\ \pm\omega_1 & \mathbb{1} \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2}(\omega_2^{-1} \mp \omega_3^{-1}) \\ \omega_2 \mp \omega_3 & 0 \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ \mp\omega_1 & \mathbb{1} \end{pmatrix}.$$

Example 3.9 (B-transformation of a para-Kähler structure). In [4] a B-transformation of a para-Hermitian structure was introduced. This is an operation on an almost para-Hermitian structure (η, K), which shears the eigenbundle L of K into the direction of $\tilde{L}$ and amounts to adding a $(2,0)$ form to $\omega = \eta K$,

$$\omega \mapsto \omega + 2b,$$

producing a new almost para-Hermitian structure $(\eta, K_B = K + 2B)$, where we denote $B = \eta^{-1}b$. K_B can then be thought of as a finite deformation of K.

From the point of view of the bundle $L \oplus L^*$, the B-transformation of K is the usual b-field transformation, changing the splitting $L \oplus L^* \mapsto e^b(L) \oplus L^*$. Similarly, we can see this operation as a β-field transformation of the bundle $\tilde{L} \oplus \tilde{L}^*$.

One can then associate a GpK structure to the bi-para-Hermitian data (η, K_B, K):

$$\mathcal{K}_+ \begin{pmatrix} K + B & \beta \\ b & -(K+B)^* \end{pmatrix}, \quad \mathcal{K}_- \begin{pmatrix} B & \omega^{-1} + \beta \\ \omega + b & -B^* \end{pmatrix},$$

where $\beta = \eta^{-1} b \eta^{-1}$.

3.2 Generalized Chiral Structures

In this section we explore more in-depth the commuting pair $(\mathcal{G},\mathcal{J})$ giving the generalized chiral structure.

Definition 3.10. A **Generalized Chiral structure** (GCh) is a commuting pair $(\mathcal{G},\mathcal{J}_+)$ of a generalized metric $\mathcal{G}=\mathcal{G}(g,b)$ and a GP structure $\mathcal{J}_+$.

Note that for the commuting pair $(\mathcal{G},\mathcal{J}_+)$, $\mathcal{J}_- \coloneqq \mathcal{G}\mathcal{J}_+$ is another GP structure. For GCh there does not exist a good notion of integrability similar to GK or GpK geometry, although in the following section we will introduce a notion of *weak integrability* for these structures. We now describe the canonical example of a GCh structure:

Example 3.11 (Chiral geometry). Let (η, J) be a chiral structure and $\mathcal{H}=\eta J$ the corresponding Riemannian metric. Define

$$\mathcal{G}(\eta)=\begin{pmatrix}0 & \eta^{-1}\\ \eta & 0\end{pmatrix},\quad \mathcal{G}(\mathcal{H})=\begin{pmatrix}0 & \mathcal{H}^{-1}\\ \mathcal{H} & 0\end{pmatrix},\quad \mathcal{J}_+=\begin{pmatrix}J & 0\\ 0 & J^*\end{pmatrix}.$$

Then both $(\mathcal{G}(\eta),\mathcal{J}_+)$ and $(\mathcal{G}(\mathcal{H}),\mathcal{J}_+)$ define CCh structure such that $\mathcal{J}_-=\mathcal{G}(\eta)\mathcal{J}_+=\mathcal{G}(\mathcal{H})$.

We also have

Lemma 3.12. *A b-field transformation of a GCh structure is again a GCh structure.*

For a general GCh structure $(\mathcal{G}=\mathcal{G}(\eta,b),\mathcal{J}_+)$ we again denote the eigenbundles of $\mathcal{G}$ by $C_\pm$. All facts about the tangent bundle structures corresponding to commuting pairs outlined in section 3 holds true. Denoting isomorphisms (17) associated to $\mathcal{G}$ by $\pi_\pm$, we obtain a pair of product structures $J_\pm$ on the tangent bundle by

$$J_+=\pi_+\mathcal{J}_\pm\pi_+^{-1},\quad J_-=\pm\pi_-\mathcal{J}_\pm\pi_-^{-1},$$

such that $(\eta, J_\pm)$ is a pair of chiral structures on the tangent bundle. The formula that recovers the GCh data from a pair of chiral structures and an arbitrary b-field is given by

$$\mathcal{J}_\pm=\pi_+^{-1}J_+\pi_+P_+^C\pm\pi_-^{-1}J_-\pi_-P_-^C, \tag{31}$$

where $P_\pm^C$ denotes the projections onto $C_\pm$, $P_\pm^C=\frac{1}{2}(\mathbb{1}\pm\mathcal{G})$. One also easily recovers the usual expressions in the matrix form

$$\mathcal{J}_\pm=\frac{1}{2}\begin{pmatrix}\mathbb{1} & 0\\ b & \mathbb{1}\end{pmatrix}\begin{pmatrix}J_+\pm J_- & \mathcal{H}_+^{-1}\mp\mathcal{H}_-^{-1}\\ \mathcal{H}_+\mp\mathcal{H}_- & J_+^*\pm J_-^*\end{pmatrix}\begin{pmatrix}\mathbb{1} & 0\\ -b & \mathbb{1}\end{pmatrix}, \tag{32}$$

where $\mathcal{H}_\pm$ denotes the two metrics corresponding to $(\eta, J_\pm)$.

3.3 Generalized Bismut Connections and Integrability

A generalized Bismut connection associated to a generalized metric $\mathcal{G}$ was introduced in [] as a Courant algebroid connection that preserves $\mathcal{G}$ and has useful properties, in particular, it can be used to formulate integrability of generalized complex structures that commute with $\mathcal{G}$. The idea of this section is to extend this observation to any commuting pair from Section 3 and hence define a notion of integrability of such commuting pairs. For generalized structures with isotropic eigenbundles (GC and GpC structures), this notion of integrability is weaker then the Courant integrability condition; in particular, it does not require the corresponding tangent bundle data to be Frobenius integrable. The advantage of this integrability condition is that it can be used for generalized structures with no isotropic eigenbundles as well.

The definition of a generalized Bismut connection can be straightforwardly extended to indefinite metrics as well:

Definition 3.13. Let $\mathcal{G} = \mathcal{G}(\eta, b) \in \mathrm{End}(T \oplus T^*)$ be a generalized indefinite metric and denote $C_\pm$ its eigenbundles. We split the sections $u \in \Gamma(T \oplus T^*)$ accordingly, $u = u_+ + u_-$. Then the following expression defines a generalized connection parallelizing $\mathcal{G}$:

$$D^H_u v = [u_-, v_+]_{H+} + [u_+, v_-]_{H-} + [Cu_-, v_-]_{H-} + [Cu_+, v_+]_{H+}. \tag{33}$$

Here $u, v \in \Gamma(T \oplus T^*)$, $[\ ,\ ]_H$ is the twisted Dorfman bracket and C is the generalized almost para-Complex structure

$$C = \begin{pmatrix} \mathbb{1} & 0 \\ 2b & -\mathbb{1} \end{pmatrix} = e^b \left(\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \right) \in \mathrm{End}(T \oplus T^*)$$

The generalized Bismut connection of $\mathcal{G}$ has the property that it is related via $\pi_\pm$ to two usual connections on T with a fully skew torsion:

$$\begin{aligned} D_u v &= \pi_+^{-1} \nabla^+_{\pi(u)} \pi_+ v_+ + \pi_-^{-1} \nabla^-_{\pi(u)} \pi_- v_-, \\ \nabla^\pm &= \mathring{\nabla} \pm \frac{1}{2}(H + \mathrm{d}b), \end{aligned} \tag{34}$$

where $\mathring{\nabla}$ is the Levi-Civita connection of η in $\mathcal{G}(\eta, b)$ and b is the two-form. It follows that the **generalized torsion** defined as

$$T^D(u, v, w) = \langle D_u v - D_v u - [u, v], w \rangle + \langle D_w u, v \rangle$$

is given by [9] $T^D = \pi_+^*(H + \mathrm{d}b) + \pi_-^*(H + \mathrm{d}b)$.

The generalized Bismut connection is related to integrability of GC structures in the following way:

Theorem 3.14 ([9])**.** *Let $(\mathcal{G}, \mathcal{J})$ be a commuting pair consisting of a generalized metric $\mathcal{G}$ and a generalized almost complex structure $\mathcal{J}$ and let D be the* *generalized Bismut connection of $\mathcal{G}$. Then $(\mathcal{G},\mathcal{J})$ is a GK structure and in particular both $\mathcal{J}$ and $\mathcal{J}'=\mathcal{G}\mathcal{J}$ are Courant integrable iff $D\mathcal{J}=0$ and T^D is of type $(2,1)+(1,2)$ with respect to $\mathcal{J}$.*

The relationship between the integrability of GK structures and the generalized Bismut connection can be summed up as follows

$$(\mathcal{G},\mathcal{J})\text{ Generalized Kähler} \Longleftrightarrow D\mathcal{J}=0,\ T^D \text{ type } (2,1)+(1,2).$$

We therefore introduce the notion of weak integrability:

Definition 3.15. Let $(\mathcal{G},\mathcal{J})$ be a commuting pair consisting of an indefinite generalized metric $\mathcal{G}$ and arbitrary generalized structure $\mathcal{J}$ and let D be the generalized Bismut connection of $\mathcal{G}$. We say $\mathcal{J}$ is **weakly integrable** if $D\mathcal{J}=0$.

Note that it follows that in the above definition when $\mathcal{J}$ is weakly integrable then also $\mathcal{J}'=\mathcal{G}\mathcal{J}$ is weakly integrable. We will now analyse what the condition $D\mathcal{J}=0$ means in terms of the tangent bundle data corresponding to $(\mathcal{G},\mathcal{J})$. As we have seen previously, we get a pair of tangent bundle endomorphisms for any commuting pair $(\mathcal{G},\mathcal{J})$ whenever $\mathcal{G}$ is an (indefinite) generalized metric via the formula

$$J_\pm = \pm\pi_\pm\mathcal{J}\pi_\pm^{-1}.$$

This can be inverted into a formula for $\mathcal{J}$ in terms of $J_\pm$:

$$\mathcal{J}=\pi_+^{-1}J_+\pi_+P_+ + \pi_-^{-1}J_-\pi_-P_-, \tag{35}$$

where $P_\pm=\frac{1}{2}(\mathbb{1}\pm\mathcal{G})$ projects from $T\oplus T^*$ to $C_\pm$. Using (35) and (34) We can no rephrase the equation $D\mathcal{J}=0$ in terms of $\nabla^\pm$ and $J_\pm$:

Lemma 3.16. *Let $(\mathcal{G},\mathcal{J})$ be a commuting pair with $\mathcal{G}$ a (indefinite) generalized metric and D the generalized Bismut connection of $\mathcal{G}$ given by (34). Then $D\mathcal{J}=0$ if and only if $\nabla^\pm J_\pm=0$.*

Proof. From $(D_u\mathcal{J})v=D_u(\mathcal{J}v)-\mathcal{J}(D_uv)$ we get

$$\begin{aligned}(D_u\mathcal{J})v &= \pi_+^{-1}\nabla^+_{\pi(u)}(J_+\pi_+v_+)+\pi_-^{-1}\nabla^-_{\pi(u)}J_-(\pi_-v_-)\\ &\quad -\pi_+^{-1}J_+(\nabla^+_{\pi(u)}\pi_+v_+)-\pi_-^{-1}J_-(\nabla^-_{\pi(u)}\pi_-v_-),\end{aligned}$$

and combining the terms that take values in $C_\pm$ yields the result. ∎

We now show what the condition $\nabla^\pm J_\pm=0$ means for the endomorphisms $J_\pm$. For this we establish several technical lemmas.

Lemma 3.17. *Let (g,J) be an almost Hermitian structure and $\mathring{\nabla}$ the Levi-Civita connection of g. Then $\nabla^hJ=0$, where $\nabla^h=\mathring{\nabla}+\frac{1}{2}g^{-1}h$ is equivalent to the following set of equations*

$$\begin{aligned} i(\mathrm{d}\omega^{(3,0)}-\mathrm{d}\omega^{(0,3)}) &= 3h^{(3,0)+(0,3)}=\frac{3}{4}N\\ \mathrm{d}\omega^{(2,1)+(1,2)} &= -i(h^{(2,1)}-h^{(1,2)}),\end{aligned}$$

where $\omega = gJ$ is the fundamental form. Analogous statement holds for a para-Hermitian structure (η, K) and its fundamental form ω.

$$\begin{aligned}
\mathrm{d}\omega^{(3,0)} &= -3h^{(3,0)} = \frac{3}{4}N_+ \\
\mathrm{d}\omega^{(0,3)} &= 3h^{(0,3)} = -\frac{3}{4}N_- \\
\mathrm{d}\omega^{(2,1)} &= -h^{(2,1)} \\
\mathrm{d}\omega^{(1,2)} &= h^{(1,2)},
\end{aligned} \tag{36}$$

Proof. We prove the statement for the para-Hermitian case, the Hermitian case follows analogously. Alternative proof of the Hermitian case has been also presented in [10].

Let $\nabla K = 0$. This means that ∇ preserves the eigenbundles $L/\tilde{L}$ of K, which in particular means $\eta(\nabla_X y, z) = 0$ for any $x, y \in \Gamma(L)$ and arbitrary vector field X. We therefore have

$$0 = \eta(\nabla_X y, z) = \eta(\mathring{\nabla}_X y, z) + \frac{1}{2}h(X, y, z),$$

and since $\eta(\mathring{\nabla}_X y, z) = \eta(\mathring{\nabla}_X (Py), z) = \frac{1}{2}\mathring{\nabla}_X \omega(y, z)$, we conclude that

$$\mathring{\nabla}_X \omega(y, z) = -h(X, y, z),$$

and analogously $\mathring{\nabla}_X \omega(\tilde{y}, \tilde{z}) = h(X, \tilde{y}, \tilde{z})$ for any $\tilde{y}, \tilde{z} \in \Gamma(L)$. Using the formula

$$\mathrm{d}\omega(X, Y, Z) = \sum_{Cycl. X,Y,Z} \nabla_X \omega(Y, Z),$$

together with the property $\mathring{\nabla}_X \omega(KY, KZ)$, implying that the mixed components $\mathring{\nabla}_X \omega(y, \tilde{z}) = 0$ vanish, gives the relationships between $\mathrm{d}\omega$ and h in (36). The relationship with $N_\pm$ follows from the definitions of $N_\pm$ and the fact that ∇ preserves $L/\tilde{L}$:

$$N_+(x, y, z) = 4\eta([x, y], z) = 4\eta(\nabla_x y - \nabla_y x - T(x, y), z) = -4h(x, y, z),$$

and similarly for N_-.

Conversely, assume the relationships (36) are satisfied. Because N is fully skew, we have $N_+ = 4\eta([x, y], z) = -4\eta([x, z], y)$ for any $x, y, z \in \Gamma(L)$, implying $\mathring{\nabla}_x \omega(y, z) = -\mathring{\nabla}_y \omega(x, z)$, which in turn means $\mathrm{d}\omega^{3,0} = 3\mathring{\nabla}_x \omega(y, z)$, so that $h^{3,0} = -\mathring{\nabla}_x \omega(y, z)$.

We now expand ∇K in terms of $\mathring{\nabla} K = \eta^{-1}\mathring{\nabla}\omega$:

$$\eta((\nabla_X K)Y, Z) = \mathring{\nabla}_X \omega(Y, Z) + \frac{1}{2}h(X, KY, Z) + \frac{1}{2}h(X, Y, KZ).$$

Plugging in expressions for components of h in terms of ω and using once again $\mathring{\nabla}_X \omega(y, \tilde{z}) = \mathring{\nabla}_X \omega(\tilde{y}, z) = 0$ yields $\nabla K = 0$. $\square$

We now turn to chiral structures. Since the fundamental tensor $F(X,Y,Z) = \eta((\mathring{\nabla}_X J)Y, Z)$ is not fully skew and additionally is a tensor of type $(2,1)+(1,2)$ (with respect to J), one cannot expect a statement fully analogous to Lemma 3.17. However, we can still relate properties of the chiral structures, such as its Nijenhuis tensor, to the three-form h. Before we state the lemma, we recall the following property of F [**?**]:

$$F(X,Y,Z) = F(X,Z,Y) = -F(X,JY,JZ).$$

Lemma 3.18. *Let (η, J) be an almost chiral structure and $\mathring{\nabla}$ the Levi-Civita connection of η. Then $\nabla^h J = 0$, where $\nabla^h = \mathring{\nabla} + \frac{1}{2}g^{-1}h$ is equivalent to the following equation*

$$F(X, y_+, z_-) = -h(X, y_+, z_-), \tag{37}$$

for any $y_+ \in \Gamma(T_+), z_- \in \Gamma(T_-)$ and arbitrary vector field X. In particular, (η, J) is then of type $\mathcal{W}_3$, which means its characteristic tensors satisfy (7)

$$\sum_{Cycl.X,Y,Z} F(X,Y,Z) = 0, \quad \Phi(X,Y) = -\Phi(JX,JY).$$

Additionally, the Nijenhuis tensor is related to h in the following way

$$\sum_{Cycl.X,Y,Z} N(X,Y,Z) = -h^{(2,1)+(1,2)}.$$

Proof. Let $\nabla J = 0$. This implies that ∇ preserves the eigenbundles $T_\pm$ of J, in particular $\eta(\nabla_X y_+, z_-) = 0$. We therefore have

$$0 = \eta(\nabla_X y_+, z_-) = \eta(\mathring{\nabla}_X y_+, z_-) + \frac{1}{2}h(X, y_+, z_-),$$

implying $h(X, y_+, z_-) = -F(X, y_+, z_-)$ which determines all mixed components of h in terms of F.

Conversely, assume (37) is satisfied. We first compute ∇J in terms of F and h:

$$\eta((\nabla_X J)Y, Z) = \nabla_X \mathcal{H}(Y,Z) = F(X,Y,Z) - \frac{1}{2}h(X,Y,JZ) - \frac{1}{2}h(X,Z,JY).$$

The right hand side vanishes identically whenever Y and Z belong to the same eigenbundle since $F(X,JY,JZ) = -F(X,Y,Z)$. It therefore suffices to show that $\eta((\nabla_X J)Y, Z) = 0$ for $Y = y_+ \in T_+$ and $Z = z_- \in T_-$:

$$\eta((\nabla_{x_+} J)y_+, z_-) = F(x_+, y_+, z_-) + h(x_+, y_+, z_-) = 0,$$

proving the first part of the proof. To show that (η, J) is of type $\mathcal{W}_3$ is a straightforward application of (37) and the relationship [**?**]

$$\Phi(X,Y,Z) = \frac{1}{2}(F(X,Y,JZ) + F(Y,JZ,X) - F(JZ,X,Y)).$$

To show the relationship between the Nijenhuis tensor and h, one simply recalls the definition

$$N(X,Y) = (\mathring{\nabla}_X J)JY - (\mathring{\nabla}_Y J)JX + (\mathring{\nabla}_{JX} J)Y - (\mathring{\nabla}_{JY} J)X,$$

which can be rewritten in terms of F as

$$\begin{aligned} N(X,Y,Z) &\coloneqq \eta(N(X,Y),Z) \\ &= F(X,JY,Z) - F(Y,JX,Z) + F(JX,Y,Z) - F(JY,X,Z), \end{aligned} \tag{38}$$

and the result is again derived by a simple application of (37). □

It is interesting to note that in contrast to the (para-)Hermitian geometry, in the chiral case the condition ∇J only determines the $(2,1)+(1,2)$ components of h.

We are now ready to state the main results of this section. In the following, we use the notation for commuting pairs from Section 3 and the one introduced in this section: $(\mathcal{G},\mathcal{J})$ denotes a general commuting pair with $\mathcal{G} = \mathcal{G}(g,b)$ being an (indefinite) generalized metric, D^H its generalized Bismut connection and $\pi_\pm$ are the isomorphisms (17) between the tangent bundle and the eigenbundles $C_\pm$ of $\mathcal{G}$. The tangent bundle data corresponding to the commuting pair are denoted by $(g, J_\pm)$:

$$\begin{aligned} g(X,Y) &= \pm\frac{1}{2}\langle \pi_\pm^{-1}X, \pi_\pm^{-1}Y\rangle \\ J_\pm &= \pi_\pm \mathcal{J} \pi_\pm^{-1}. \end{aligned}$$

Additionally, the subscript $\pm$ refers to quantities and bigradings associated with $J_\pm$.

Theorem 3.19. *When $(\mathcal{G},\mathcal{J})$ defines an* **almost GK** *structure, $D^H\mathcal{J}$ is equivalent to the following set of equations:*

$$\begin{aligned} i(\mathrm{d}\omega_\pm^{(3,0)_\pm} - \mathrm{d}\omega_\pm^{(0,3)_\pm}) = 3(H+\mathrm{d}b)^{(3,0)_\pm+(0,3)_\pm} &= \frac{3}{4}N_\pm \\ \mathrm{d}\omega_\pm^{(2,1)_\pm+(1,2)_\pm} &= -i((H+\mathrm{d}b)^{(2,1)_\pm} - (H+\mathrm{d}b)^{(1,2)_\pm}). \end{aligned} \tag{39}$$

Theorem 3.20. *When $(\mathcal{G},\mathcal{J})$ defines an* **almost GpK** *structure, $D^H\mathcal{J}$ is equivalent to the following set of equations:*

$$\begin{aligned} \mathrm{d}\omega_\pm^{(3,0)_\pm} &= -3(H+\mathrm{d}b)^{(3,0)_\pm} = \frac{3}{4}N^\pm_{(+)} \\ \mathrm{d}\omega_\pm^{(0,3)_\pm} &= 3(H+\mathrm{d}b)^{(0,3)_\pm} = -\frac{3}{4}N^\pm_{(-)} \\ \mathrm{d}\omega_\pm^{(2,1)_\pm} &= -(H+\mathrm{d}b)^{(2,1)_\pm} \\ \mathrm{d}\omega_\pm^{(1,2)_\pm} &= (H+\mathrm{d}b)^{(1,2)_\pm}. \end{aligned} \tag{40}$$

Theorem 3.21. *When $(\mathcal{G}, \mathcal{J})$ defines an* **almost generalized chiral** *structure, $D^H\mathcal{J}$ is equivalent to the following equation:*

$$F_{\pm}(X, y_+, z_-) = -(H + \mathrm{d}b)(X, y_+, z_-), \tag{41}$$

for all $y_+ \in \Gamma(T_+^{\pm})$ and $z_- \in \Gamma(T_-^{\pm})$, $T_+^{\pm}$ ($T_-^{\pm}$) being the $+1$ (-1) eigenbundle of $J_\pm$. In particular, both $(g, J_\pm)$ are of type $\mathcal{W}_3$ almost product pseudo-Riemannian structures whose Nijenhuis tensor is related to $H + \mathrm{d}b$ by

$$\sum_{Cycl.X,Y,Z} N_{(\pm)}(X, Y, Z) = -(H + \mathrm{d}b)^{(2,1)_\pm + (1,2)_\pm}.$$

change notation for Nijenhuis to $N_{(+)}$ in the rest of file

Proof. All the above statements are direct consequences of definitions and Lemmas 3.16-3.18. □

It is easy to see from Theorem 3.19 why Theorem 3.14 holds true; starting with equations (39) and additionally imposing that the torsion of D^H is of type $(2,1) + (1,2)$ with respect to $\mathcal{J}$ simply means that $(H + \mathrm{d}b)$ is of type $(2,1) + (1,2)$ with respect to both $J_\pm$, i.e. the Nijenuis tensor of both $J_\pm$ has to vanish, which recovers the usual integrability conditions of GK structures, in particular we have

$$\mathrm{d}^c_\pm \omega_\pm = \pm(H + \mathrm{d}b).$$

Analogous statement is true for the GpK geometry; the additional requirement of $(H+\mathrm{d}b)$ being of type $(2,1)+(1,2)$ with respect to both para-Hermitian structures $J_\pm$ in equations (40) again implies vanishing of the Nijenhuis tensor and the integrability condition from Theorem 3.6

$$\mathrm{d}^p_\pm \omega_\pm = \mp(H + \mathrm{d}b).$$

We now explain how the weak integrability relates to the almost D-bracket from Proposition 2.13. We first recall from [4, Prop. 4.8] that given an almost para-Hermitian structure (η, K), there is a unique associated D-bracket given by

$$\begin{aligned}\eta([\![X, Y]\!], Z) &= \eta(\mathring{\nabla}_X Y - \mathring{\nabla}_Y X, Z) + \eta(\mathring{\nabla}_Z X, Y) \\ &- \frac{1}{2}[\mathrm{d}\omega^{(3,0)}(X, Y, Z) + \mathrm{d}\omega^{(2,1)}(X, Y, Z) - \mathrm{d}\omega^{(1,2)}(X, Y, Z) - \mathrm{d}\omega^{(0,3)}(X, Y, Z)],\end{aligned} \tag{42}$$

where $\mathring{\nabla}$ is the Levi-Civita connection of η.

Proposition 3.22. *Let $(\mathcal{G}, \mathcal{K})$ be a weakly integrable almost GpK structure with $(\eta, \mathcal{K}_\pm)$ the associated tangent bundle geometry and $\pi_\pm$ the isomorphisms* (17) *associated to $\mathcal{G}$. Then $(\mathcal{G},\mathcal{K})$ is an integrable GpK structure if and only if the almost D-bracket associated to $\mathcal{G}=\mathcal{G}(\eta,b)$*

$$\eta([\![X,Y]\!]^{\pm}_{H+\mathrm{d}b},Z)=\pm\frac{1}{2}\eta([\pi_{\pm}^{-1}X,\pi_{\pm}^{-1}Y]_H,\pi_{\pm}^{-1}Z)$$

are the D-brackets associated to $(\eta,\mathcal{K}_{\pm})$.

Proof. We recall from Proposition 2.13

$$\begin{aligned}\eta([\![X,Y]\!]^{\pm}_{H+\mathrm{d}b},Z)=\eta(\mathring{\nabla}_XY-\mathring{\nabla}_YX,Z)+\eta(\mathring{\nabla}_ZX,Y)\\ \pm\frac{1}{2}(H(X,Y,Z)+\mathrm{d}b(X,Y,Z)).\end{aligned}$$

Let $(\mathcal{G},\mathcal{K})$ be integrable. Then the D-brackets (42) for $(\eta,\omega_{\pm})$ take the form

$$\begin{aligned}\eta([\![X,Y]\!]^{\pm},Z)=\eta(\mathring{\nabla}_XY-\mathring{\nabla}_YX,Z)+\eta(\mathring{\nabla}_ZX,Y)\\ -\frac{1}{2}(\mathrm{d}^p_{\pm}\omega_{\pm}(X,Y,Z)),\end{aligned}$$

since the $(3,0)+(0,3)$ components of $\mathrm{d}\omega_{\pm}$ vanish due to $K_{\pm}$ being integrable, which is a consequence of Theorem 3.6. Another consequence is that $\mathrm{d}^p_{\pm}\omega_{\pm}=\mp(H+\mathrm{d}b)$, showing that $[\![\ ,\ ]\!]^{\pm}_{H+\mathrm{d}b}=[\![\ ,\ ]\!]^{\pm}$.

Conversely, let $(\mathcal{G},\mathcal{K})$ be only weakly integrable and let $[\![\ ,\ ]\!]_{H+\mathrm{d}b}=[\![\ ,\ ]\!]^{\pm}$. The weak integrability implies by Theorem 3.20 that $\mathrm{d}\omega^{(3,0)_{\pm}}=-3(H+\mathrm{d}b)^{(3,0)_{\pm}}$ but $[\![\ ,\ ]\!]_{H+\mathrm{d}b}=[\![\ ,\ ]\!]^{\pm}$ implies $\mathrm{d}\omega^{(3,0)_{\pm}}=-(H+\mathrm{d}b)^{(3,0)_{\pm}}$, which means $\mathrm{d}\omega^{(3,0)_{\pm}}=(H+\mathrm{d}b)^{(3,0)_{\pm}}=0$ and similarly for the $(0,3)$ components. This in turn means $K_{\pm}$ are integrable and from the assumptions it additionally follows $\mathrm{d}^p_{\pm}\omega_{\pm}=\mp(H+\mathrm{d}b)$, implying $(\mathcal{G},\mathcal{K})$ is integrable and completing the proof. □

4 Generalized Geometry for Born Geometry

In this section we describe how Born geometry fits in the GG framework, defining the notion of generalized Born Geometry. Just as the usual Born geometry described in the Section 1.4, which can be seen as a para-Hermitian structure (η,K) together with a Chiral structure (η,J) such that $\{K,J\}=0$, we will define the generalized Born Geometry as a GpK structure $(\mathcal{G},\mathcal{K})$ together with a GCh structure $(\mathcal{G},\mathcal{J})$ such that $\{\mathcal{K},\mathcal{J}\}=0$. The corresponding tangent bundle data then gives a pair of Born structures, as expected. We will also show that a statement analogous to Corollary 1.12 holds in the case of generalized Born geometry.

Definition 4.1. A **Generalized Born Geometry (GBG)** is an (almost-) GpK structure $(\mathcal{G},\mathcal{K}_+)$ and a GCh structure $(\mathcal{G},\mathcal{J}_+)$ such that $\{\mathcal{K}_+,\mathcal{J}_+\}=0$.

A few remarks are in order:

- Denoting $\mathcal{K}_- = \mathcal{G}\mathcal{K}_+$ and $\mathcal{J}_- = \mathcal{G}\mathcal{J}_+$, it is easy to see that $\{\mathcal{K}_-, \mathcal{J}_-\} = 0$ as well as $\{\mathcal{K}_\pm, \mathcal{J}_\mp\} = 0$.
- Due to the integrability of GCh structure being of different nature than integrability of the GpK structure and we will discuss the overall integrability of a GBG separately.

We now continue by describing the corresponding tangent bundle data. By itself, the almost GpK and GCh each define a pair of almost Hermitian (η, K_+, K_-) and Chiral (η, J_+, J_-) structures, respectively. Because $\{\mathcal{K}_+, \mathcal{K}_-\} = 0$,

$$\{K_+, J_+\} = K_+J_+ + J_+K_+ = \pi_+\{\mathcal{K}_+, \mathcal{J}_+\}\pi_+^{-1} = 0,$$

and similarly $\{K_-, J_-\} = 0$. However, in general $\{K_+, J_-\} \neq 0$.

Therefore, in addition to real structures $J_\pm, K_\pm$ we get a pair of almost complex structures $I_\pm = J_\pm K_\pm$. A GBG therefore defines a pair of Born geometries $(\eta, I_\pm, J_\pm, K_\pm)$. Conversely, it is easy to see that whenever there is such pair $(\eta, I_\pm, J_\pm, K_\pm)$ and a B-field b, one can construct a generalized metric $\mathcal{G}(\eta, b)$ and use it to define a GBG via the usual formulas (22), (31).

Lemma 4.2. *A GBG is equivalent to the data of two Born geometries sharing the same split signature metric η, $(\eta, I_\pm, J_\pm, K_\pm)$, and a B-field b.*

We now proceed to showing that GBG is equivalent a choice of a Riemannian structure on an eigenbundle of a GpK structure, hence showing a complete analogy with statement in Corollary 1.12 in the setting of generalized geometry.

Proposition 4.3. *A GBG is equivalent to the data of an almost GpK structure $(\mathcal{K}_+, \mathcal{K}_-)$ and a choice of a positive-definite metric on a $+1$ (or -1) eigenbundle of the generalized para-complex structure $\mathcal{K}_+$ (or $\mathcal{K}_-$).*

Proof. First we show that the GBG indeed defines a positive-definite metric on all of the eigenbundles $L_\pm, \tilde{L}_\pm$ associated to $\mathcal{K}_\pm$. We show this for L_+, the $+1$ eigenbundle of $\mathcal{K}_+$ and the generalization to the remaining bundles is obvious. Note that due to L_+ and $\tilde{L}_+$ being maximally isotropic with respect to $G = \langle \mathcal{G}\cdot, \cdot\rangle$ and $T \oplus T^* = L_+ \oplus \tilde{L}_+$, we have $\tilde{L}_+ \simeq L_+^*$ by G. Moreover, $\mathcal{J}_+$ maps $L_+ \to \tilde{L}_+ \simeq L_+^*$ because $\{\mathcal{K}_+, \mathcal{J}_+\} = 0$ and this map is an isomorphism since $\mathcal{J}_+$ is invertible ($\mathcal{J}_+^2 = \mathbb{1}$). We therefore define a metric h on L_+ by

$$h(u, v) \coloneqq \langle \mathcal{G}\mathcal{J}_+u, v\rangle, \quad u, v \in \Gamma(L_+).$$

This defines a metric because $h(u, v) = h(v, u)$ and both G and $\mathcal{J}_+$ have inverses. The definiteness follows from L_+ and $\tilde{L}_+$ being maximally isotropic with respect to G, therefore $G(u, v) \neq 0$ for $u \in \Gamma(L_+)$ and $v \in \Gamma(\tilde{L}_+)$, in particular when $v = \mathcal{J}_+u$. Then if h defined above is negative-definite, one just takes $-h$.

For the converse, let h be a positive-definite metric on L_+. Then simply define $\mathcal{J}_+$ on L_+ by

$$h(u, v) = G(\mathcal{J}_+u, v), \quad u, v \in \Gamma(L_+), \tag{43}$$

and on $\tilde{L}_+$ by

$$G(u,v) = h(\mathcal{J}_+ u, v), \quad u \in \Gamma(\tilde{L}_+), v \in \Gamma(L_+).$$

Then $\mathcal{J}_+^2 = \mathbb{1}$ and $\mathcal{J}_+$ maps L_+ to $\tilde{L}_+$ and vice versa, which implies $\{\mathcal{J}_+, \mathcal{K}_+\} = 0$. Let now $u, v \in \Gamma(\ell_+ = L_+ \cap C_+)$. From (43) we see that $\mathcal{J}_+ u \in \Gamma(C_+)$, otherwise $h(u,v)$ would vanish and ℓ_+ would be isotropic with respect to h, which contradicts the assumption that h is definite; therefore, we see that $\mathcal{J}_+$ preserves C_+. Similar argument can be carried out for C_- and we see that $\mathcal{J}_+$ commutes with $\mathcal{G}$, which completes the proof. □

5 Examples

6 Physical Interpretation and Relationship to Supersymmetry

In this section we briefly explain how the geometries introduced in this paper (mostly GpK and generalized chiral geometries) appear in physics in the context of supersymmetry (SUSY). This direction will be in detail elaborated on in forthcoming work [].

In this section we mostly follow the discussion in []. The original result for bi-Hermitian geometry is found in []. This section is meant mostly for physicists and therefore we use physical terminology and introduce a lot of new objects customarily used in physics without explaining their exact meanings and definitions. For basics of SUSY and details the reader can consult for example [] or a nice thesis containing many useful details and calculations [].

6.1 Twisted (2,2) SUSY and GpK Geometry

In this subsection we explain how GpK geometry naturally appears in $2D$ $(2,2)$ supersymmetric sigma models, more concretely in **twisted** supersymmetric models. We compare this with the well-known story about how GK geometry appears in the usual $(2,2)$ supersymmetry.

We first consider the general $(1,1)$ SUSY sigma model given by the action

$$S_{(1,1)}(\phi) = \int_{\hat{\Sigma}} [g(\phi) + b(\phi)]_{ij} D_+ \phi^i D_- \phi^j, \tag{44}$$

where $\phi = (\phi_i)_{i=1\cdots n}$ are *fields*, i.e. maps $\phi : \hat{\Sigma} \to (M^n, g)$, where Σ is a super-Riemann surface, (M, g) is (for now) arbitrary pseudo-Riemannian manifold and b denotes a local two-form. $(1,1)$ supersymmetry means that this action is invariant under transformations generated by the two *supercharges* $Q_\pm$, which are vector fields on $T[1]\Sigma$ and in particular obey the supercommutation relations

$$\{Q_\pm, Q_\pm\} = P_\pm, \tag{45}$$

where $P_\pm$ are generators of translations.

The idea now is to study under which conditions the action $S_{(1,1)}$ admits additional supersymmetries. It turns out that this puts severe restrictions on the required geometry of M. In particular, if we are to extend the supersymmetry to $(2,2)$, (M,g) necessarily needs to be a GK manifold. If the $(1,1)$ supersymmetry is to be extended to a twisted $(2,2)$ supersymmetry, the target manifold needs to be a GpK manifold.

It turns out that the two additional supersymmetries necessarily transform the fields by

$$\delta\phi^i = \epsilon_+(T_+)^i_j D_+\phi^j + \epsilon_-(T_-)^i_j D_-\phi^j,$$

for some tensors $T_\pm$. The requirement that the action (44) is invariant under this transformation forces the compatibility between $(g+b)$ and T:

$$\begin{aligned} g(T_\pm\cdot,\cdot) + g(\cdot,T_\pm\cdot) &= 0 \\ b(T_\pm\cdot,\cdot) + b(\cdot,T_\pm\cdot) &= 0, \end{aligned}$$

along with the condition

$$\nabla^\pm T_\pm = 0,$$

where the connections $\nabla^\pm$ (34)

$$\nabla^\pm = \mathring{\nabla} \pm \frac{1}{2}H,$$

naturally appear. We recall that $\mathring{\nabla}$ is the Levi-Civita connection of g and H is a closed global three-form, such that b is locally its potential, $\mathrm{d}b = H$. The additional requirement that the corresponding supercharges $Q'_\pm$ generate supersymmetry, i.e. square to translations via (45), translates into the additional conditions

$$T_\pm^2 = -\mathbb{1} \quad N_\pm = 0,$$

where $N^\pm$ are the Nijenhuis tensors of $T_\pm$. This implies that $(g, T_\pm, b)$ defines a bi-Hermitian data, or equivalently forces M to be generalized Kähler. Alternatively, we can require that the supercharges $Q'_\pm$ generate twisted supersymmetry, i.e.

$$\{Q_\pm, Q_\pm\} = -P_\pm,$$

which forces

$$T_\pm^2 = \mathbb{1} \quad N_\pm = 0,$$

rendering $(g, T_\pm, b)$ a bi-para-Hermitian data, or equivalently, M to be a GpK manifold.

When we require that the theory is parity-symmetric, we find that the b-field term in (44) has to vanish and additionally $T_+ = T_- = T$, which gives the (para-)Kähler limit of the geometry. Additionally, one might require additional supersymmetry, which requires additional (para-)complex structure that anti-commutes with T, which therefore desribes the (para-)hyper-Kähler limit of GK/GpK geometry. Various other heterotic $(p, 1)$ supersymmetries can be realized as well, all as special cases of the GK/GpK geometry.

We conclude this section by two remarks – first, the signature of g is a priori not fixed, therefore the above discussion generalizes to GK (GpK) geometries with different signature metrics and we recover genuine GK or GpK geometry only when we require that g is Riemannian or split, respectively. Second, the integrability of $T_\pm$ can be relaxed [], giving the GK or GpK geometries which are only integrable in the weaker sense introduced in Section 3.3.

6.2 (1,1) Superconformal Algebra and Generalized Chiral Geometry

In [], it has been demonstrated that when the target space of the $(1, 1)$ model (44) admits two (almost-)project structures $J_\pm$ orthogonal w.r.t. g, i.e. a bi-chiral geometry, one can introduce additional symmetries $\delta_{P_\pm}$ and $\delta_{Q_\pm}$ associated to the $+1$ and -1 projectors $P_\pm$ and $Q_\pm$, respectively, corresponding to $J_\pm$. The chiral structures $J_\pm$ are additionally required to be covariantly constant with respect to the connections $\nabla^\pm$ (34), i.e. the data (J_+, J_-, g, b) defines a generalized chiral geometry, which is integrable in the weaker sense described in Section 3.3. The symmetries $\delta_{P_\pm}$ and $\delta_{Q_\pm}$ then form copies of the $(1, 1)$ superconformal algebra even when $J_\pm$ are not integrable; this shows that from the physics point of view, the integrability of the chiral structures behaves differently and it is natural to relax it. The notion of integrability of the corresponding gen. chiral structure introduced in Section 3.3 is therefore necessary and sufficient for this physics application.

Because the superconformal algebra factorizes even without integrability of $J_\pm$, the factorization lacks a spacetime description in terms of Riemannian submanifolds of M. The author of [] then relates this fact to the existence of non-geometric string backgrounds.

References

[1] L. Freidel, F. J. Rudolph and D. Svoboda, *A Unique Connection for Born Geometry*, `1806.05992`.

[2] J. Merker, *On almost hyper-para-kähler manifolds*, *ISRN Geometry* **2012** (2012) .

[3] N. Hitchin, *Hypersymplectic quotients*, *Acta Acad. Sci. Tauriensis* **124** (1990) 169–180.

[4] D. Svoboda, *Algebroid Structures on Para-Hermitian Manifolds*, *J. Math. Phys.* **59** (2018) 122302, [1802.08180].

[5] A. Wade, *Dirac structures and paracomplex manifolds*, *Comptes Rendus Mathematique* **338** (2004) 889–894.

[6] Z.-J. Liu, A. Weinstein and P. Xu, *Manin Triples for Lie Bialgebroids*, *J. Diff. Geom.* **45** (1997) 547–574, [dg-ga/9508013].

[7] I. Vaisman, *Generalized para-kähler manifolds*, *Differential Geometry and its Applications* **42** (2015) 84–103.

[8] L. Freidel, F. J. Rudolph and D. Svoboda, *Generalised Kinematics for Double Field Theory*, *JHEP* **11** (2017) 175, [1706.07089].

[9] M. Gualtieri, *Branes on Poisson varieties*, 2007. 0710.2719. DOI.

[10] M. Gualtieri, *Generalized complex geometry*, Ph.D. thesis, Oxford U., 2003. math/0401221.

[11] M. Gualtieri, *Generalized Kahler geometry*, 1007.3485.

[12] H. Kamada et al., *Neutral hyperkahler structures on primary kodaira surfaces*, *Tsukuba journal of mathematics* **23** (1999) 321–332.