The Geometry of two dimensional Supersymmetric Nonlinear σ Model

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Abstract

In this report, we investigate the relation between supersymmetry (twisted and untwisted) and geometry for two dimensional σ model with target spaces of arbitrary signature, and Lorentzian or Euclidean world sheets. It can be shown that the number of allowed supersymmetry is constrained purely by the geometric considerations. The possible geometries are classified according to properties of the target space.

1 Introduction

We discuss the supersymmetric nonlinear σ model on 2 dimensional space-time. The quantum field theory of nonlinear σ model is the theory of harmonic maps from the general space-time to the Riemannian manifold. While the supersymmetric version of nonlinear σ model can be viewed as a map which maps the supersymmetric space-time or superspace to the target space. For example, the action of (1,1) supersymmetric σ model can be written as $S_{(1,1)} = \int d^2\sigma d\theta^+ d\theta^- [g_{ij}(\phi) + b_{ij}(\phi)] D_+\phi^i D_-\phi^j$, where the bosonic field ϕ^i are superfield which can be viewed as coordinates on some D-dimensional manifold M with metric g_{ij} and torsion which is determined by the antisymmtric tensor b_{ij}. It is found that there is an interesting relation between the possible SUSY that the σ model respects and the geometry structure of which the σ model maps to. The extended supersymmetry will impose strong restrictions on the target manifold, which in turn would predict strong statement about the vacuum structure and ultra-violet behavior. In this paper, it is meaningful to impose some conditions to the representation of the super algebra we discussed to let it has definite chirality. The (p,q) super algebra means that it contains p supercharges with positive chirality and q supercharges with negative chirality. The reason to study such kind of models is their application to the string theory. For example, consider the superstrings compactified to four dimensions on some space $M_4 \times K$ (M_4 is four dimensional Minkowski space and K is a compact manifold of dimension six). The string propagation on K is described by a (1+1)-dimensional nonlinear σ model with internal manifold K. For effective 4 dimensional theory which has unbroken supersymmetry, the relevant σ model have(at least) supersymmetry of type (2,0). For detained discussion, see [3].

2 Introduction to the (p,q) Super Algebra of dimension 2

2.1 Spinors

Before we discuss the supersymmetric σ model, the knowledge about the general (p,q) type super algebra is need. We introduce the discussion from Tanii[2] here. In general case, let's consider the d dimensional space-time with an metric

$$\eta_{ab} = diag(1, \cdots, 1, -1, \cdots, -1) \tag{1}$$

which has t +1(time-like directions) and s -1(space-like directions) and $d = t+s$. It is invariant under the group SO(t, s). The supercharges are the $2^{[d/2]}$(at least) dimensional complex spinor representation of of $SO(t, s)$ and transforms as:

$$\delta_L \psi = -\frac{1}{4}\lambda^{ab}\gamma_{ab}\psi \tag{2}$$

Where $[x]$ is the largest integer not larger than x. Gamma matrices $\gamma^a, a = 1 \ldots d$, satisfy the clifford algebra

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab}. \tag{3}$$

As we will discuss below, the number of independent components of the spinors can be reduced by imposing Weyl and/or Majorana conditions. These conditions must be consistent with the Lorentz transformation law (3). Spinors satisfying these conditions are called Weyl spinors and Majorana spinors respectively. To discuss supersymmetry it is convinient to use spinors with the smallest number of independent components in each dimension. Weyl spinors are those having a definite chirality in even dimensions. The chirality of spinors is defined as an eigenvalue of the generalized $\bar{\gamma}$ matrix

$$\bar{\gamma} = (-1)^{\frac{1}{4}(s-t)}\gamma^1\gamma^2\cdots\gamma^d. \tag{4}$$

Weyl spinors with positive(negative) chirality are defined by $\bar{\gamma}\psi = +(-)\psi$. It is easy to see in odd dimensions $\bar{\gamma}$ is proportional to the identity and one cannot define Weyl spinors. Majorana spinors are those satisfying a certain kind of reality condition: $\psi^c = \psi$, where the superscript c represents charge conjugation. We should discuss it for d =even and d =odd separately. When d is even, it is defined by the following procedure.

The matrices $\pm(\gamma^a)^*$ also satisfy the same anticommutation realtion as γ^a, Then it can be shown that there exists matrices $B_\pm$ which relate $\pm(\gamma^a)^*$ to γ^a by similarity transformations:

$$(\psi^a)^* = B_+ \gamma^a B_+^{-1} \qquad (\psi^a)^* = -B_- \gamma^a B_-^{-1}.$$

The charge conjugation is defined by using one of these matrices:

$$\gamma^c = B_+^{-1} \psi^* \qquad \text{or} \;\; \gamma^c = B_-^{-1} \psi^*. \tag{5}$$

For the Majorana condition to be consistent, it must be that $(\psi^c)^c = \psi$, which is equivalent to $B_+^* B_+ = 1$ or $B_-^* B_- = 1$. It can be shown that $B_\pm$ satify

$$B_\pm^* B_\pm = \epsilon_\pm(t,s)1 \qquad \epsilon_\pm(t,s) = \sqrt{2}\cos[\frac{\pi}{4}(s-t\pm 1)]$$

We have summarized the values of $\epsilon_\pm$ in Table 1. It is not difficult to see that $s-t$ has a period of 8.

s-t	1	2	3	4	5	6	7	8
ϵ_+	0	-1	-1	-1	0	1	1	1
ϵ_-	1	1	0	-1	-1	-1	0	1

Table 1: Values of $\epsilon_\pm$ as a function of $s-t$. $+1$ corresponds to space-times in which the pseudo-Majorana or Majorana condition may be imposed.

2.2 Superalgebras and Superspaces

Now we can define (pseudo) Majorana-Weyl spinors, which satisfy both of the (pseudo) Majorana condition and Weyl condition. It is possible when ψ and ψ^c have the same chirality. It can be shown that this occurs exactly when $s-t=0(8)$. For Minkowski space-time, i.e. $t=1, s=d-1$, this requirement gives $d=2$ mod 8. While for Euclidian case, we can not impose these 2 conditions simultaneously but separately. So the spinors contains 2 real components. Now specialize to the case 2 dimensional Minkowski spacetime. One can impose both the Majorana and Weyl conditions or both the pseudo-Majorana and Weyl conditions. This allows one to reduce the general spinor with two complex components to a spinor with one real component. Now the (p,q) supersymmetry algebra for $d=2$ can be given. Central charges will not be considered. The supercharges commute with translations and are spinors under Lorentz transformations:

$$[P_a, Q^i] = 0, \qquad [M_{ab}, Q^i] = \frac{1}{2}\gamma_{ab} Q^i \tag{6}$$

.

The form of anticommutators $\{Q,Q\}$ depend on the spinor type of Q^i. Here we only list the result for $d=2$ mod 8.

$$
\begin{aligned}
\{Q_+^i, Q_+^{jT}\} &= \frac{1}{2}(1+\gamma_5)\gamma^a C_- P_a \delta^{ij}, \\
\{Q_-^i, Q_-^{jT}\} &= \frac{1}{2}(1-\gamma_5)\gamma^a C_- P_a \delta^{ij}, \\
\{Q_+^i, Q_-^{jT}\} &= 0.
\end{aligned} \tag{7}
$$

The supercharges are Majorana-Weyl spinors with positive chirality $Q_+^i(i = 1, 2, \cdots, N_+)$ and Majorana-Weyl spinors with negative chirality $Q_-^i(i = 1, 2, \cdots, N_-)$. Both kind of supercharges are real and 1-component spinors. We call kind super algebra as (p,q) super algebra (twisted or untwisted).Specialize to two dimensional Minkowski case, taking $\gamma^1 = \sigma^1, \gamma^2 = i\sigma^2 \Rightarrow \bar{\gamma} = \sigma^3$, and $C_- = \imath\sigma^2$, it is easy to verify that the above reduces to

$$\{Q_+^i, Q_+^j\} = 2\delta^{ij} P_+ \tag{8}$$

$$\{Q_-^{i\prime}, Q_-^{j\prime}\} = 2\delta^{i\prime j\prime} P_- \tag{9}$$

$$\{Q_+^i, Q_-^{j\prime}\} = 0, \tag{10}$$

where $i, j = 1, \cdots, p$ and $i\prime, j\prime = 1, \cdots, q$, and $P_\pm = (P_1 \pm P_2)$ are the momenta associated with light-cone coordinates. This is just the example showed in [3].

The super algebra (8),(9) and (10) can be generalized to the metric with arbitrary signature, or we can rewrite the relations as follows:

$$
\begin{aligned}
\{Q_+^i, Q_+^j\} &= 2\eta^{ij} P_+, \\
\{Q_-^{i\prime}, Q_-^{j\prime}\} &= 2\eta^{i\prime j\prime} P_-, \\
, \qquad \{Q_+^i, Q_-^{j\prime}\} &= 0,
\end{aligned}
$$

This is also called the twisted form of super algebra. If we choose suitable representation, the metric η^{ij} (consider the invertible case) can be brought into the form:

$$\eta^{ij} = \begin{pmatrix} 1_u & 0 \\ 0 & -1_v \end{pmatrix}, \tag{11}$$

where $u + v = p$ and similar remarks apply for $\eta^{i\prime j\prime}$. If $u = 0 or v = 0$, it is no different than the corresponding positive (negative) chirality part of the algebra. When $u and v$ are nonzero, this algebra is referred to as a *twisted* type (p,q) algebra. For twisted super algebra with metric (11), it's easy to see that $(Q_+^i)^2 = -P_+$ for $i = u+1, \cdots, d$, and we refer to the super algebra as being twisted.

It is straightforward to introduce a superspace for general (p,q) super algebra. There are two real bosonic coordinates $\sigma^+ = \sigma^1 + \sigma^2$,$\sigma^- = \sigma^1 - \sigma^2$, p real positive-chirality Fermi coordinate θ_i^+ and q real negative-chirality Fermi coordinate $\theta_{i'}^-$, The supersymmetry generators are

$$Q_+^i = \frac{\partial}{\partial\theta_i^+} - \imath\eta^{ij}\theta_j^+\partial_+, \qquad Q_-^{i'} = \frac{\partial}{\partial\theta_{i'}^-} - \imath\eta^{i'j'}\theta_{j'}^-\partial_- \tag{12}$$

satisfy the super algebra. The corresponding supercovariant derivatives are

$$D_+^i = \frac{\partial}{\partial\theta_i^+} + \imath\eta^{ij}\theta_j^+\partial_+, \qquad D_-^{i'} = \frac{\partial}{\partial\theta_{i'}^-} + \imath\eta^{i'j'}\theta_{j'}^-\partial_-. \tag{13}$$

and satisfy the anticommutators

$$\begin{aligned} \{D_+^i, D_+^j\} &= 2\eta^{ij}\partial_+, \\ \{D_-^{i'}, D_-^{j'}\} &= 2\eta^{i'j'}\partial_-, \\ , \qquad \{D_+^i, D_-^{j'}\} &= 0, \end{aligned}$$

However, we are not going to use such kind of method to construct supersymmetric σ model.

3 Some Definitions and Results from Complex Geometry

In this section, several key concepts such as the holonomy group will be introduced and discussed. The holonomy group associated with a connection, complex structure, and almost product structure are very important to analysis different kinds of geometry of the target space. Most of the discussion comes from chapter 15 of the book *Superstring theory* by C. Hull and E. Witten.

3.1 Holonomy Group

Let (M^n, g) be an n-dimensional Reimannian manifold, where g has signature (t, s), and let ∇ be a connection (covariant derivative), possibly with torsion. The element of holonomy group is associated with the parallel transportation along a closed curve γ. Suppose $p \in M$ is point which stays on γ. Then parallel transportation defines a map, $\rho_\gamma : TM_p \to TM_p$, from tangent vectors at p, to tangent vectors at p which is given by parallelly transported around the γ. The map is linear because parallel transport is defined through a linear differential operator. Furthermore, the map preserves lengths, as parallel transport preserves lengths. Finally, if the M is orientable, the map will also preserve orientation. So, $\rho_\gamma \in SO(t, s)$. Do all these maps form g group? The answer is yes. First, for every given map which is related to a contractible closed curve, one can always traverse it in the other direction (*the inverse*). Second, given two such closed curves γ_1 and γ_2, traversing γ_1 first and then γ_2 second yields a new contractible closed curve. Finally, it has been shown that $\mathcal{H}_\nabla \subset SO(t, s)$, where $\mathcal{H}_\nabla$ is associated with the connection ∇. Similarly, One can also define the parallel transport of tensors around closed curves. Recall that a covariantly constant vector field is one that always returns to its initial value upon parallel transport around a contractible closed loop. It is easy enough to extend this definition to covariantly constant tensors. It can be shown that a covariantly constant tensor of type $(1,1), f^i_j$ commutes with all the elements of $\mathcal{H}_\nabla$:

$$\rho^i_k(\gamma) f^k_j - f^i_k \rho^k_j(\gamma) = 0 \qquad \forall \rho(\gamma) \epsilon \mathcal{H}_\nabla. \tag{14}$$

The existence of covariantly constant tensor fields implies that the holonomy group can not be all of $SO(t,s)$. The action of the holonomy group is said to be reducible (irreducible) according to whether there is (there is not) a non-trivial invariant subspace. And here we restrict ourselves to holonomy groups which are irreducible.

3.2 Complex Structures and Real Structure

Complex structure is a type (1,1) covariant constant tensor J^i_j field defined on the Riemannian manifold which satisfies $J^2 = -\mathbf{1}$. Such kind of tensor field must have an even dimension. To discuss it in detail, let's consider a certain type of (1,1) tensor field which can be changed to the canonical form by choosing local coordinates at each point $p \in M$

$$J = \begin{pmatrix} 0 & 1_m \\ -1_m & 0 \end{pmatrix}. \tag{15}$$

By coordinates transformation, J can be put in the form

$$J = \begin{pmatrix} \imath 1_m & 0 \\ 0 & -\imath 1_m \end{pmatrix}. \tag{16}$$

Such kind of tensor field is called almost complex structure. Furthermore, If around the neighborhood of each point p the almost complex structure J can be put into the canonical form (16), then J is called a complex structure. What is the necessary and sufficient condition to hold this proposition? The answer is that the Nijenhuis tensor of J vanishes $\Longleftrightarrow$ J is complex structure, where the Nijenhuis tensor is defined as

$$N^k_{ij} = J^l_i(\partial_l J^k_j - \partial_j J^k_l) - J^l_j(\partial_l J^k_i - \partial_i J^k_l). \tag{17}$$

Equivalently, $N^k_{ij} = 0$ and J is said to be integrable. An even-dimensional Riemannian manifold with a complex structure is complex manifold. Given any complex manifold M, we can introduce an Hermitian metric g_{ij} on M. From the hermitian metric and the complex structure we can construct a type (0,2) tensor $k_{ij} \equiv J_{ij} = g_{ik} J^k_j$. The tensor k_{ij} can be viewed as the components of a 2-form called Kähler form. If the Kähler form is closed or equivalently the hermitian metric is torsion free, then the manifold is called Kähler manifold. The invariance of the complex structure J under the holonomy group $\mathcal{H}_\nabla$ is of great importance to the structure of $\mathcal{H}_\nabla$. Here we just list the conclusion of the book. In the general case of metric with signature (t,s), the existence of J implies $t = 2m_1, s = 2m_2$, and its invariance under holonomy implies $\mathcal{H}_\nabla \subset U(m_1, m_2) \subset SO(2m_1, 2m_2)$.

Similar to the almost complex structure, if there is a type (1,1) tensor, S, at each $p \in M$, which squares to the identity. Then there are local coordinates around p which put S into the canonical form:

$$S = \begin{pmatrix} 1_u & 0 \\ 0 & -1_v \end{pmatrix}, \tag{18}$$

with $u + v = n = dim(M)$. Such a tensor is referred to as almost real structure. Similarly discussion as before: under what conditions can S be put in the canonical form RealStructure in a neighborhood about any point $p \in M$? The answer is the same as before. The associated Nijenhuis tensor of S must vanish. If this is the case, S is integrable and called a real structure or an almost product structure. Again, if the real structure is covariantly constant with respect to the connection ∇, then the holonomy group is reduced to $\mathcal{H}_\nabla \subset GL(m, \mathcal{R}) \subset SO(m, m)$. Finally, if there is no torsion, so that the connection is the Levi-Chivita connection, then such manifold is referred to as pseudo-K$\ddot{a}$hler.

3.3 Multiple Complex and/or Real Structures

The number of distinct complex or real structure actually can't be arbitrary, which gives a restriction on the possible values of p,q. If the holonomy group of M is irreducible, we can use Schur's Lemma for real representations of the holonomy group. This implies (see [5]) that the matrices which commute with all elements of $\mathcal{H}_\nabla$ form an associative division algebra over the reals. The only possibilities are the reals, the complexes, and the quaternions. The quaternions are 1,i,j,k satisfying:

$$\begin{aligned} &i^2 = j^2 = k^2 = -1, \\ &ij = -ji = k \qquad jk = -kj = i \qquad ki = -ik = j \end{aligned}$$

In particular, i,j,and k satisfy a clifford algebra $\{I^{(a)}, I^{(b)}\} = -2\delta^{ab}$. There are two more associative algebras over the reals, which are not division algebras, that are possible [1]. This is the algebra E $(1, e)$ with $e^2 = 1$, and the algebra of "pseudo-quaternions" $(\tilde{i}, \tilde{j}, \tilde{k})$:

$$\begin{aligned} &\tilde{i}^2 = -1, \qquad \tilde{j}^2 = \tilde{k}^2 = 1, \\ &\tilde{j}\tilde{k} = -\tilde{k}\tilde{k} = -\tilde{i} \qquad \tilde{k}\tilde{i} = -\tilde{i}\tilde{k} = \tilde{j} \qquad \tilde{i}\tilde{j} = -\tilde{j}\tilde{i} = \tilde{k} \end{aligned}$$

.

Here, $(\tilde{i}, \tilde{j}, \tilde{k})$ satisfy the clifford algebra $\{I^{(a)}, I^{(b)}\} = \eta^{ab}, \eta = diag(1, -1, -1)$. The case where the algebra is the reals corresponds simply to the identiy δ^i_j being the only matrix that commutes with all elements of the holonomy group. Clearly, the complex algebra corresponds to the identity and one complex structure. E corresponds to the identity and one real structure. The quaternions correspond to the identity and three complex structures, where the complex structures satisfy the clifford algebra $\{I^{(a)}, I^{(b)}\} = -2\delta^{ab}$. Finally, the pseudo-quaternions correspond to two real structures and one complex structure, satisfying the clifford algebra $\{I^{(a)}, I^{(b)}\} = \eta^{ab}, \eta = diag(1, -1, -1)$. These are *all* of the possibilities for covariantly constant complex and or real structures on M. If the metric has signature (t, s), the quaternionic structure is possible only if t,s are both divisible by 4, $t = 4m_1, s = 4m_2$. The holonomy group satisfies $\mathcal{H}_\nabla \subset USp(2m_1, 2m_2) \subset SO(4m_1, 4m_2)$, where $USp(2m_1, 2m_2)$ is the subgroup of $U(2m_1, 2m_2)$ preserving the symplectic structure. In order for the pseudo-quaternionic structure to be possible, the signature must be (m,m) with m even, $m = 2k$. Then the holonomy group satisfies $\mathcal{H}_\nabla \subset Sp(2k, R) \subset SO(2k, 2k)$ [1]. Finally, in the case where the torsion vanishes, these manifolds are called hyper-Kähler and pseudo-hyper-Kähler respectively.

4 Construcction SUSY σ Model with General Target Space Signature

4.1 (p,q) $p, q > 0$ σ Model

Now we turn to the construction of nonlinear two dimensional σ model with twisted or untwisted (p,q) supersymmetry on target space of arbitrary signature[1]. We have two ways to approach it. The first is a Noether procedure where one writes down a manifestly (1,1) supersymmetric action and finds conditions under which it has other additional $p - 1$ positive chirality and $q - 1$ negative chirality supersymmetries which are not manifest. From the constraints imposed by the (1,1) original supersymmetry and other extra supersymmetries we can determine the geometry structure of the target space. The second method is to introduce a (p,q) superspace directly and construct an action which is manifestly invariant under all the supersymmetries. This section describes the Noether procedure in some generality. Let's consider the (1,1) sypersymmetric σ model with superspace action

$$\mathcal{S}_(1, 1) = \int d^2x d\theta^+ d\theta^- [g_{ij}(\Phi) + b_{ij}(\Phi)] D_+ \Phi^i D_- \Phi^j, \qquad (19)$$

where the Φ^i are chiral superfields which can be viewed as coordinates on a D-dimensional manifold M with metric g_{ij} and a torsion 3-form H given by the curl of the antisymmetric tensor b_{ij} which is included because the kinetic piece $D_+\Phi^i D_-\Phi^j$ need not be symmetric in i, j.

$$H_{ijk} = \frac{3}{2}\partial_{[i}b_{jk]}. \tag{20}$$

The model will be invariant under (1,1) supersymmetry, general coordinate transformation on the target space and antisymmetric tensor gauge transformation

$$\delta b_{ij} = \partial_{[i}\lambda_{j]} \tag{21}$$

No other restrictions placed on the manifold M for $(1,1)$ supersymmetry. This model will also be conformally invariant at 1-loop if there is a function Ψ such that

$$R^{(+)}_{ij} - \nabla_{(i}\nabla_{j)}\Phi - H^k_{ij}\nabla_k\Phi = 0 \tag{22}$$

where $R^{(+)}_{ij}$ is the Ricci tensor for a connection with torsion. We define the torsion as

$$\Gamma^{(\pm)i}_{jk} = C^i_{jk} \pm H^i_{jk} \tag{23}$$

C^i_{jk} are the usual Christoffel connection.

We now seek the conditions on the target space geometry under which the (1,1) superspace action Sigma11 is invariant under extra supersymmetries. If there are $p-1$ right-handed and $q-1$ left-handed extra supersymmetry transformations, then they must be of the form

$$\delta\Phi^i = \varepsilon^r T^i_{(+)rj} D_+\Phi^j + \varepsilon^{r\prime} T^i_{(-)r\prime j} D_-\Phi^j \tag{24}$$

for some tensors $(T_{(+)r})^i_j$, $(T_{(-)r\prime})^i_j$ with $r = 1, \ldots, p-1$ and $r\prime = 1, \ldots, q-1$. Invariance of the action (19) requires that the tensors $T^i_{(+)rj}, T^i_{(-)r\prime j}$ satisfy

$$\begin{aligned} g_{ki}T^k_{(+)rj} + g_{kj}T^k_{(+)ri} &= 0, \\ g_{ki}T^k_{(-)r\prime j} + g_{kj}T^k_{(-)r\prime i} &= 0, \end{aligned} \quad , \tag{25}$$

$$\nabla^+_k T^i_{(+)rj} = \nabla^-_k T^i_{(-)r\prime j} = 0. \tag{26}$$

If the supersymmetry transformations (24) are to satisfy a superalgebra with general signature, then the matrices $T_{r(+)} and T_{r\prime(-)}$ must satisfy the anticommutation relations of the form

$$\{T_{(+)r}, T_{(+)s}\} = -2\eta^{rs}, \qquad \{T_{(-)r\prime}, T_{(-)s\prime}\} = -2\eta^{r\prime s\prime} \tag{27}$$

for some metrics $\eta^{rs} and \eta^{r\prime s\prime}$. In addition, the generalized Nijenhuis concomitants $\mathcal{N}(T^r_+, T^s_+)$ and $\mathcal{N}(T^{r\prime}_-, T^{s\prime}_-)$ must vanish. For any two (1,1) tensors, T_1 and T_2, the generalized Nijenhuis concomitant is defined by[1]

$$\mathcal{N}(T_1,T_2)^i_{jk} = T^l_{1j}\partial_l T^i_{2k} - T^l_{1k}\partial_l T^i_{2j} - T^i_{1l}\partial_j T^l_{2k} - T^i_{1l}\partial_k T^l_{2j} + (1 \longleftrightarrow 2) \quad (28)$$

.

Note that $\frac{1}{4}\mathcal{N}(T,T) \equiv \mathcal{N}(T)$ is the usual Nijenhuis tensor of T,

$$\mathcal{N}^k_{ij}(T) = T^l_i T^k_{[j,l]} - T^l_j T^k_{[i,l]} \quad (29)$$

The condition $\mathcal{N}(T) = 0$ implies that T is integrable, i.e. $T_{(+)r}$ and $T_{(-)r\prime}$ are complex structures and/or real structures of the manifold, depending on the corresponding sign of η^{rs} and $\eta^{r\prime s\prime}$. If all of the above conditions are satisfied, then the supersymmetry transformations (24) together with the manifest (1,1) supersymmetries satisfy the extended super algebra with

$$\eta^{ij} = diag(1,\eta^{rs}), \qquad \eta^{i\prime j\prime} = diag(1,\eta^{r\prime s\prime}). \quad (30)$$

Now, equations (25) and (26) imply the the $T_{(+)r}$ and $T_{(-)r\prime}$ are hermitian and invariant under the local holonomy group. So M is hermitian manifold. If M is torsion free, then it is Kähler or pseudo-K$\ddot{a}$hler depending on if $T_{(+)r}$ and $T_{(-)r\prime}$ are complex structure or real structure. Next, we give an example to analyze the geometry of M according to the number of $T_{(+)r}$ and $T_{(-)r\prime}$. Consider the positive chirality supersymmetries of extended supersymmtry, corresponding to the $T_{(+)r}$, and suppose for the moment that the T_- vanish. All possibilities are (1) no $T_+ \Rightarrow (1,1)$ SUSY; (2) one complex structure $J_+ \Rightarrow (2,1)$ untwisted SUSY; (3) one real structure $S_+ \Rightarrow (2,1)$ twisted SUSY; (4) three complex structures $J_{(+)r} \Rightarrow (4,1)$ untwisted SUSY; (5) one complex structure J_+ and two real structures $S_{(+)r} \Rightarrow (4,1)$ twisted SUSY. The corresponding holonomy groups $\mathcal{H}_{\nabla^+}$ are list in Table 2. The result for the case $p = 1, q > 1$ are similar. The possibilities are again $q = 2$ twisted or untwisted and $q = 4$ twisted or untwisted. These results are summarized in Table 2.

In the case of $p > 1$ and $q > 1$, we discuss the result with no torsion here. And the discussion with non-zero torsion is involved with introduction of a (p,q) superspace which is complicated and we will not discuss these cases here. For the case of torsion free, the connections are the same and equal to the Levi-Chivita connection $\nabla^+ = \nabla^- \equiv \nabla$. The holonomy groups coincide, as do the complex/real structures $T_{(+)r} = T_{(-)r\prime} \equiv T_r$. In particular, $r = r\prime$, or $p = q$, so the number of left and right supercharges is the same. Actually, if $p \neq q$ and $p, q > 0$, the torsion must be non-zero. As the holonomy is now defined with respect to the Levi-Chivita connection, the possible geometries have names: Kahler, pseudo-Kahler, hyper-Kahler, and pseudo-hyper-Kahler. These results are summarized in Table 3.

4.2 $(p, 0)$ $p > 0$ (twisted) σ Models

In the previous section, we discussed the (p,q) σ models with $p, q > 0$. Now we turn to the type (p,0) or equivalently the type (0,q). The (2,0) σ model

Target Sign.	Holo. of ∇	Geometry with torsion	Supersymmetry
(d_1, d_2)	$\mathrm{O}(\mathrm{d}_1, d_2)$	no restriction	$(1,1),(1,0)$
$(2\mathrm{n}_1, 2n_2)$	$\mathrm{U}(\mathrm{n}_1, n_2)$	1 complex Hermitean structure	$(1,1),(1,0)$,
$(4\mathrm{m}_1, 4m_2)$	$\mathrm{USp}(2\mathrm{m}_1, 2m_2)$	3 complex Hermiean str. (quaternionic)	$(4,1),(4,0)$,
(2n,2n)	GL(n,R)	1 real Heritean structure	tw $(2,1),(2,0)$,
(4m,4m)	Sp(2m,R)	2 real and 1 cmplx Hrmtn. str. (pseudo-qtrnc)	tw $(4,1),(4,0)$,

Table 2: The possible $(p,1)$ and $(p,0)$ twisted and untwisted SUSY, where the target manifold has torsion. The first column is the allowable signatures of the target manifold. The second column is the (maximal) local holonomy group for each case, where the holonomy is with respect to the connection with torsion. The third column lists the complex and/or real structures on the manifold. In the case of multiple structures, the algebra given refers to that of the structures plus the identity. The last column are the applicable supersymmetries. Note that it is not possible to have the $(p,1), p>1$ supersymmetries if the manifold does not have torsion.

Target Sign	Holo. of ∇	Geometry without torsion	Supersymmetry
$(2n_1, 2n_2)$	$\mathrm{U}(\mathrm{n}_1, n_2)$	Kähler	$(2,2),(2,0)$
$(4m_1, 4m_2)$	$\mathrm{USp}(2\mathrm{m}_1, 2m_2)$	hyper-Kähler	$(4,4),(4,0)$
$(2n, 2n)$	GL(n,R)	pseudo-Kähler	twisted (2,2),(2,0)
$(4m, 4m)$	Sp(2m,R)	pseudo-hyper-Kähler	twisted (4,4),(4,0)

Table 3: The possible (p,p) and $(p,0)$ twisted and untwisted supersymmetries, and the target manifold is torsion free. Note that the torsion free condition forces, $p=q$ for supersymmetries (p,q) with $p,q>0$.

is especially useful and discussed in detail by C. Hull and E. Witten[3] The general case for (p,0) is much the same as (2,0). Let's start with a manifestly $(1,0)$ supersymmetric action

$$\mathcal{S}_(1,0) = \imath \int d^2x d\theta [g_{ij}(\Phi) + b_{ij}(\Phi)] D\Phi^i \partial_- \Phi^j, \tag{31}$$

where the superscript $^+$ on θ and D has been dropped. This is sometimes referred to as $N = 1/2$ supersymmetry. The superfield Φ has a rather short expansion:

$$\Phi(x^\mu, \theta) = \phi(x^\mu) + \theta\lambda(x^\mu), \tag{32}$$

where $\phi(x^\mu)$ is a real scalar field and $\lambda(x^\mu)$ is a left-handed Majorana-Weyl spinor for Minkowski signature space-time and a left-handed Weyl spinor for Euclidian signature. It is straightforward to show that equation (31) becomes[3]

$$\mathcal{S}_(1,0) = \int d^2x [(g_{ij}(\phi) + b_{ij}(\phi)) \partial_+ \phi^i \partial_- \phi^j + \imath g_{ij}(\phi) \lambda^i (\partial_- \lambda^j + \Gamma^j_{kl} \partial_- \phi^k \lambda^l)], \tag{33}$$

where the connection Γ differs from the Christoffel connection by the torsion, which is related to b_{ij} as before. Then we impose extra supersymmetry with the form

$$\delta\phi^i = \varepsilon^r T^i_{rj} \lambda^j \qquad \delta(T^i_{rj}\lambda^j) = -\imath \varepsilon^r \partial_+ \phi^i. \tag{34}$$

Since the action is invariant under all these supersymmetries, we have

$$g_{ki} T^k_{rj} + g_{kj} T^k_{ri} = 0, \qquad \nabla_k T^i_{rj} = 0. \tag{35}$$

In order to anti-commute with one another and satisfy the supersymmetry algebra TSusyAlgebra they must satisfy

$$\{T_r, T_s\} = -2\eta_{rs}, \qquad \mathcal{N}(T_r, T_s) = 0. \tag{36}$$

Now we have the same conclusions as above about the possible types of tensors T_r. In fact, the $(p,0)$ and $(p,1)$ cases are equivalent. If the torsion vanishes, then the structure of the manifold is (1) *Kähler* for untwisted $(2,0)$ supersymmetry, (2) pseudo-*Kähler* for twisted $(2,0)$ supersymmetry, (3) hyper-*Kähler* for $(4,0)$ untwisted supersymmetry, and (4) pseudo-hyper-*Kähler* for twisted $(4,0)$ supersymmetry. So, the $(p,0)$ geometrical structure is equivalent to the (p,p) geometrical structure. There is no need for us to use another table to summarize the results.

5 Conclusions

In this paper, we discussed the possible geometry of the target manifold of the (p,q) twisted supersymmetric two dimensional σ model. The case where p,q¿0 and the torsion is nonzero is not considered. It is found that the holonomy group and the geometry is constrained by the possible supersymmetry that the action respect. While the possible values for p,q are also constrained by the geometry structure of the target manifold. All possible values for p,q are 0,1,2 and 4.

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