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GENERALIZED COMPLEX STRUCTURES ON COURANT ALGEBROIDS

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Abstract

In this thesis we study generalized complex structures defined on Lie bialgebroids, and arbitrary Courant algebroids. This thesis consists of two parts: the first deals with the generalized complex structures on Courant algebroids, while the second discusses generalized complex submanifolds.

The basic examples of generalized complex structures are given, and certain classes of Poisson-Nijenhuis manifolds are described using generalized complex structures. The Poisson structure arising from a generalized complex structure is also defined explicitly. Generalized complex structures on arbitrary Courant algebroids are also described using generating operators and spinors. A generating operator for the Courant algebroid of a Lie bialgebroid is also given.

In the second part we introduce the notion of twisted generalized complex submanifolds and describe an equivalent characterization in terms of Poisson-Dirac submanifolds. Our characterization recovers a result of Vaisman [38]. An equivalent characterization is also given in terms of spinors. As a consequence, we show that the fixed locus of an involution preserving a twisted generalized complex structure is a twisted generalized complex submanifold. Lastly, we also discuss generalized Kähler submanifolds.

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Chapter 1

Introduction

Generalized complex structures were introduced by Nigel Hitchen [15, 16], and studied by Marco Gualtieri [14]. The main idea is to replace the usual tangent bundle by the generalized tangent bundle, which is the direct sum of the tangent bundle and the cotangent bundle. One important structure on the generalized tangent bundle is the Courant bracket, which was defined as

$$[X+\xi, Y+\eta] = [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \tfrac{1}{2}d\big(\eta(X) - \xi(Y)\big)$$

for all $X, Y \in \mathfrak{X}(M)$ and $\xi, \eta \in \Omega^1(M)$ [7]. The other important structure on the generalized tangent bundle is the following symmetric bilinear form

$$\langle X+\xi, Y+\eta\rangle = \tfrac{1}{2}\big(\eta(X) + \xi(Y)\big).$$

There are three ways to realize generalized complex structures. The first is as a bundle map $J : TM \oplus T^*M \to TM \oplus T^*M$ such that $J^2 = -\mathrm{Id}$, $\langle J(X+\xi), J(Y+\eta)\rangle = \langle X+\xi, Y+\eta\rangle$, and

$$\begin{aligned} 0 = [J(X+\xi), J(Y+\eta)] + J^2\,[X+\xi, Y+\eta] \\ - J\big([J(X+\xi), Y+\eta] + [X+\xi, J(Y+\eta)]\big). \end{aligned} \tag{1.1}$$

A generalized complex structure can also be realized as Dirac structure on the generalized tangent bundle. Lastly a generalized complex structure can be defined as a spinor line bundle satisfying some integrability conditions.

Hitchin's definition of generalized complex structures was motivated by the mirror symmetry relation between complex and symplectic structures from Topological String Theory [41]. It was hoped that, because generalized complex structures encompass complex and symplectic structures as examples, generalized complex structures would give a natural setting in which to study mirror symmetry. In fact generalized complex structures have been used in flux compactifications [13, 18] (see also [12] and the references therein).

Generalized complex structures have also arisen in the study of 2-dimensional field theories. In that a particular type of supersymmetry exists if, and only if, the manifold is generalized complex [44]. Other notions from generalized complex structures have also arisen in the study of these field theories. In his thesis Gualtieri proved that a generalized Kähler structure is equivalent to a Bi-Hermitian structure. Applications of generalized complex structures are examined in [45] and the references cited.

Also of importance are twisted generalized complex structures. These structures arise when the Courant bracket is twisted by a closed 3-form Ω

$$[X+\xi, Y+\eta]_\Omega = [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \tfrac{1}{2}d\big(\eta(X) - \xi(Y)\big) + i_Y i_X \Omega.$$

The untwisted and twisted Courant brackets are both examples of a more general structure: Courant algebroids. Courant algebroids were defined by Liu et al [25] as a way to systematize the properties of both Courant brackets. In particular a Courant algebroid is a vector bundle over a manifold with a skew-symmetric bracket and a field of symmetric bilinear form of signature (n,n) that satisfy

1. $\mathrm{Jac}(e_1,e_2,e_3) = \frac{1}{3}\mathcal{D}\Big(\big\langle\, [e_1,e_2]\,, e_3\big\rangle + \big\langle\, [e_2,e_3]\,, e_1\big\rangle + \big\langle\, [e_3,e_1]\,, e_2\big\rangle\Big).$
2. $[e_1, fe_2] = f\,[e_1,e_2] + \big(\rho(e_1)\cdot f\big)e_2 - \langle e_1, e_2\rangle\, \mathcal{D}f.$
3. $\langle \mathcal{D}f, \mathcal{D}g\rangle = 0.$
4. $\rho(e_1)\cdot\langle e_2, e_3\rangle = \big\langle [e_1,e_2] + \mathcal{D}\big(\,\langle e_1,e_2\rangle\,\big), e_3\big\rangle + \big\langle e_2, [e_1,e_3] + \mathcal{D}\big(\,\langle e_1,e_3\rangle\,\big)\big\rangle.$

Here $e_i \in \Gamma(E)$, $f,g \in C^\infty(M)$ and $\mathcal{D}: C^\infty(M) \to \Gamma(\mathcal{E})$ is the map defined by

$$\langle \mathcal{D}f, e\rangle = \tfrac{1}{2}\rho(e)\cdot f, \tag{1.2}$$

for all $f \in C^\infty(M)$.

This thesis consists of two parts. The first part is the study of generalized complex structures on arbitrary Courant algebroids. Generalized complex structures of this sort have been recently used in a paper of Stiénon [33]. Generalized complex structures on Courant algebroids were also defined by Grabowski as *complex Courant structures* [11]. In particular we study how the symplectic and complex examples manifest themselves in this case. We also realize Complex-Nijenhuis structures as generalized complex structures on a particular Courant algebroid. In this section we also prove the following theorem.

Theorem. *A generalized complex structure on a Courant algebroid gives rise to a Poisson structure on the underlying base manifold.*

We also characterize generalized complex structures on Courant algebroids using pure spinor line bundles. To do this we use the Clifford algebra representation, and generating operators of Courant algebroids. We will consider the case of the Courant algebroid associated to a Lie bialgebroid. In this case the Clifford representation is given by

$$(X+\xi)\cdot\omega = i_X\omega + \xi\wedge\omega$$

for all $X + \xi \in \Gamma(\mathcal{A} \oplus \mathcal{A}^*)$ and $\omega \in \Gamma(\wedge^\bullet \mathcal{A}^*)$, and a generating operator is a differential operator $\mathcal{D} : \Gamma(\wedge^\bullet \mathcal{A}^*) \to \Gamma(\wedge^\bullet \mathcal{A}^*)$ of odd degree. Given a generating operator, the Courant bracket is defined by

$$[A, B]_d \cdot \omega = \frac{1}{2}\Big(\llbracket \llbracket \mathcal{D}, A \rrbracket, B \rrbracket - \llbracket \llbracket \mathcal{D}, B \rrbracket, A \rrbracket \Big) \cdot \omega,$$

for all $\omega \in \Gamma(\wedge^\bullet \mathcal{A}^*)$ and $A, B \in \Gamma(\mathcal{A} \oplus \mathcal{A}^*)$. Here $\llbracket .,. \rrbracket$ denotes the commutator bracket of operators. While the anchor map is defined by

$$\tfrac{1}{2}\rho(A) \cdot f = \llbracket \mathcal{D}, f \rrbracket,$$

for all $f \in C^\infty(M)$. We give a direct proof of the following theorem proved by Alekseev and Xu:

Theorem. *[2] The operator $\mathcal{D} = d_{\mathcal{A}} + \partial$ is a generating operator.*

Here, given a nowhere vanishing volume form $\nu \in \Gamma(\wedge^n \mathcal{A}^*)$, $\partial : \Gamma(\wedge^k \mathcal{A}^*) \to \Gamma(\wedge^{k-1} \mathcal{A}^*)$ is defined by

$$\partial\omega = \partial(i_X \nu) = (-1)^{n-k} i_{d_* X} \nu.$$

When the Courant algebroid has a generating operator we also prove the following theorem.

Theorem. *Let $\mathcal{E}$ be a Courant algebroid with generating operator $\mathcal{D}$. Let $L \subseteq \mathcal{E}_{\mathbb{C}}$ be an almost generalized complex structure, and let Λ be the pure spinor line bundle associated to L. Then L is a generalized complex structure if, and only if, each nonvanishing local section $\lambda \in \Gamma(\Lambda)$ is projectively closed.*

By projectively closed we mean that $\mathcal{D}\lambda = A \cdot \lambda$ for some $A \in \Gamma(\mathcal{A} \oplus \mathcal{A}^*)$. This result is a generalization of Gualtieri's result [14], and also gives a different interpretation. Gualtieri did not use derived brackets, nor generating operators.

In the second part we introduce the notion of twisted generalized complex submanifolds. This work was done jointly with Mathieu Stiénon [3]. A *twisted generalized complex submanifold* N is a submanifold of a twisted generalized complex submanifold (M, Ω, J) such that the pullback of the $+i$-eigenbundle of J can be realized as the $+i$-eigenbundle of a twisted generalized complex structure $J' : TN \oplus T^*N \to TN \oplus T^*N$. As well as finding the form of J' we also find the following characterization for when a submanifold is twisted generalized complex.

Theorem. *Let (M, Ω, J) be a twisted generalized complex manifold with $J = \begin{bmatrix} \phi & \pi^\sharp \\ \sigma_\flat & -\phi^* \end{bmatrix}$. A twisted submanifold N of M inherits a twisted generalized complex structure J', making it a twisted generalized complex submanifold, if and only the following conditions hold:*

1. *N is a Poisson-Dirac submanifold of (M, π),*
2. *$\phi(TN) \subseteq TN + \pi^\sharp(TN^o)$,*
3. *$\mathrm{pr} \circ \phi : TN \to TN$ is smooth.*

This result was obtained independently by Vaisman for the untwisted case [38]. As a consequence we prove the following theorem

Theorem. *The fixed locus of a twisted generalized complex involution is a generalized complex submanifold.*

We apply this result to holomorphic Poisson manifolds. Such a manifold can be realized as a generalized complex structure. Using the above characterization we show

that a submanifold of a holomorphic Poisson manifold will also be a holomorphic Poisson manifold if it is both a complex submanifold and a Poisson-Dirac submanifold.

We also show that the spinor associated to the induced generalized complex structure $J' : TN \oplus T^*N \to TN \oplus T^*N$ is the pull-back of the spinor associated to the original generalized complex structure on M. This allows us to express the submanifold condition for spinors.

Theorem. *Let M be a twisted generalized complex submanifold, with associated spinor line bundle Λ. A twisted submanifold $h : N \to M$ is a twisted generalized complex submanifold if, and only if, $h^*\Lambda$ is a pure spinor line bundle and $\left(h^*\mu, h^*\overline{\mu}\right)_{\mathrm{muk}} \neq 0$ for all $\mu \in \Lambda$.*

Lastly, we consider submanifolds of generalized Kähler structures. A *generalized Kähler structure* is a pair of generalized complex structures J_1 and J_2 such that $J_1 J_2 = J_2 J_1$ and $G = \langle ., J_1 J_2 . \rangle$ is a positive definite metric. We show that a submanifold inherits a generalized Kähler structure if, and only if, the submanifold is a generalized complex submanifold with respect to both generalized complex structures.

This thesis is organized as follows. The second chapter recalls the definitions of many of the structures that will be used throughout this paper. In particular it includes a description of complex structures on Lie algebroids. The third chapter defines the notion of a generalized complex structure for a Courant algebroid. Also included in this chapter are examples of such structures and the proof of the existence of the Poisson bivector field. The fourth chapter deals with generating operators for the Courant bracket on the double of a Lie bialgebroid. A generating operator for such a Courant bracket is found, and this operator is used to give the integrability condition for almost generalized complex structures. The final chapter is devoted to the study of twisted generalized complex submanifolds.

Chapter 2

Lie and Courant Algebroids

Let M be a smooth manifold. The Courant bracket is a skew-symmetric bracket defined on the sections of $TM \oplus T^*M$. This bracket was introduced in [7], and was used to define Dirac structures, which unify Poisson and presymplectic geometry. Liu et al [25] axiomatized the properties of the Courant bracket and defined Courant algebroids.

In this chapter we will define Courant algebroids and the various structures that will be used throughout this thesis. We start with Lie algebroids, which are a generalization of the tangent bundle. Lie algebroids provide the necessary framework to define the other structures we will use: in particular Poisson bivectors and Nijenhuis tensors.

We then introduce Lie bialgebroids, and Courant algebroids. Lie bialgebroids take the place of Lie bialgebras which take the place of Lie bialgebras for algebroids.

Throughout this chapter M will denote a smooth manifold. For a point $p \in M$ the fibre of a vector bundle $\mathcal{A}$ at p will be denoted by $\mathcal{A}_p$. The smooth sections of a vector bundle $\mathcal{A}$ will be denoted by $\Gamma(\mathcal{A})$, the smooth k-sections by $\Gamma(\wedge^k \mathcal{A})$, and the algebra of all smooth multi-sections by $\Gamma(\wedge^\bullet \mathcal{A})$. An exception to this notation is smooth vector fields and forms on M: $\mathfrak{X}(M)$ and $\Omega^1(M)$ will be used instead.

2.1 Lie Algebroids

This section is a review of some basic facts for Lie algebroids, and some of the relevant structures that can be defined for them. The details of these structures can be found in [26], and in fact many of them are due to Mackenzie [28]

A *Lie algebroid* is a triple $(\mathcal{A}, [.,.], \rho)$, consisting of a real vector bundle $\mathcal{A} \to M$, a Lie bracket on $\Gamma(\mathcal{A})$, and a bundle map $\rho : \mathcal{A} \to TM$ called the *anchor*, such that the following conditions hold.

1. The anchor induces a Lie algebra morphism from $\Gamma(\mathcal{A})$ to $\mathfrak{X}(M)$, where the Lie bracket on $\mathfrak{X}(M)$ is the usual commutator of vector fields.
2. For all $f \in C^\infty(M)$ and $X, Y \in \Gamma(\mathcal{A})$,

$$[X, fY] = f\,[X, Y] + \big(\rho(X) \cdot f\big)Y. \tag{2.1}$$

A *representation* of a Lie algebroid $\mathcal{A}$ is a vector bundle $\mathcal{V}$ over M and an $\mathbb{R}$-bilinear map $\Gamma(\mathcal{A}) \otimes \Gamma(\mathcal{V}) \to \Gamma(\mathcal{V})$:$A \otimes v \mapsto D_A v$ such that for all $A, B \in \mathcal{A}$, $v \in \mathcal{V}$ and $f \in C^\infty(M)$,

1. $D_{fA}v = fD_Av$;
2. $D_A(fv) = fD_Av + \big(\rho(A) \cdot f\big)v$;
3. $D_A(D_Bv) - D_B(D_Av) = D_{[A,B]}v$.

The *differential* of a Lie algebroid is the map $d_{\mathcal{A}} : \Gamma(\wedge^k \mathcal{A}^*) \to \Gamma(\wedge^{k+1}\mathcal{A}^*)$ defined for $\omega \in \Gamma(\wedge^k \mathcal{A}^*)$ by

$$d_{\mathcal{A}}\omega(X_1, \ldots, X_{k+1}) = \sum_{i=1}^{k+1}(-1)^{i+1}\rho(X_i) \cdot \omega\left(X_1, \ldots, \widehat{X_i}, \ldots, X_{k+1}\right)$$
$$+ \sum_{i<j}(-1)^{i+j}\omega\left(\left[X_i, X_j\right]), X_1, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_{k+1}\right). \quad (2.2)$$

Here $X_i \in \Gamma(\mathcal{A})$, and $\widehat{X}$ denotes the omission of X from the arguments. If $f \in C^\infty(M)$ then $d_{\mathcal{A}}f(X) = \rho(X) \cdot f$. This differential is similar to the usual differential on forms, and has the property that $d_{\mathcal{A}}^2 = 0$. The cohomology of $(\Gamma(\wedge^\bullet \mathcal{A}^*), d_{\mathcal{A}})$ is called the *Lie algebroid cohomology of $\mathcal{A}$ (with trivial coefficients)*, and is denoted by $H(\mathcal{A})$.

Example 2.1. A Lie algebra is a Lie algebroid over a point. Its representations are the usual Lie algebra representations.

Example 2.2. The tangent bundle of a manifold, with the regular Lie bracket and the identity map, is a Lie algebroid. The Lie algebroid cohomology of this Lie algebroid is the usual deRham cohomology of M.

Definition 2.3. Let $(\mathcal{A}, \rho, [.,.])$ and $(\mathcal{B}, \rho', [.,.]')$ be two Lie algebroids over M. A smooth bundle map $\phi : \mathcal{A} \to \mathcal{B}$ is a *Lie algebroid morphism* if $\rho' \circ \phi = \rho$ and

$$\phi\,[X, Y] = [\phi(X), \phi(Y)]' \text{ for all } X, Y \in \Gamma(\mathcal{A}).$$

With this notion, the condition on the anchor map can be restated: the anchor must be a Lie algebroid morphism from $\mathcal{A}$ to TM.

With the differential, the Lie derivative of a form ω with respect to a vector field $X \in \Gamma(\mathcal{A})$ can be defined using the Cartan formula:

$$\mathcal{L}_X\omega = i_X d_{\mathcal{A}}\omega + d_{\mathcal{A}} i_X\omega.$$

In this formula $i_X\omega$ is the element of $\Omega^{k-1}(M)$ defined by substituting X for the first argument of ω. This operation is often referred to as the *interior product* of ω by X, and it can be defined for $X \in \Gamma(\wedge^k \mathcal{A})$.

The last definition is simply restating the definition of Lie algebroids for complex bundles.

Definition 2.4. A *complex Lie algebroid* is a complex vector bundle $A \to M$ with a $\mathbb{C}$-linear anchor map $\rho : A \to T_{\mathbb{C}}M$ and a Lie bracket on $\Gamma\mathcal{A}$. The anchor and the bracket must obey the same properties as for a real Lie algebroid.

2.2 The Schouten Bracket and Poisson Bivectors

It is well known that a Lie algebroid bracket can be extended from $\Gamma(\mathcal{A})$ to $\Gamma(\wedge^\bullet \mathcal{A})$. This extended bracket is called the Schouten bracket, and it will be the subject of this section.

Definition 2.5. Let $X = X_1 \wedge \cdots \wedge X_x$ and $Y = Y_1 \wedge \cdots \wedge Y_y$, where $X_i, Y_j \in \Gamma(\mathcal{A})$. The *Schouten bracket* of X and Y is denoted by $[X, Y]$ and defined as

$$\sum_{i,j}(-1)^{i+j}\left[X_i, Y_j\right] \wedge X_1 \wedge \cdots \widehat{X_i} \cdots \wedge X_x \wedge Y_1 \wedge \cdots \widehat{Y_j} \cdots \wedge Y_y.$$

On elements of $\Gamma(\wedge^0 \mathcal{A}) = C^\infty(M)$ the Schouten bracket is defined as

$$[f, g] = 0 \text{ for all } f, g \in C^\infty(M), \tag{2.3}$$

$$[X, f] = \rho(X) \cdot f \text{ for all } f \in C^\infty(M) \text{ and } X \in \Gamma(\mathcal{A}). \tag{2.4}$$

The Schouten bracket can now be bilinearly extended to all of $\Gamma(\wedge^\bullet \mathcal{A})$.

Lengthy, but elementary, calculations give the following proposition.

Proposition 2.6. *The Schouten bracket satisfies*

$$[X, Y] = -\,(-1)^{(x-1)(y-1)}\,[Y, X]\,, \tag{2.5}$$

$$[X, Y \wedge Z] = [X, Y] \wedge Z + (-1)^{(x-1)y} Y \wedge [X, Z]\,, \tag{2.6}$$

$$(-1)^{(x-1)(z-1)}\big[X, [Y, Z]\big] + (-1)^{(x-1)(y-1)}\big[Y, [Z, X]\big] + (-1)^{(y-1)(z-1)}\big[Z, [X, Y]\big] = 0. \tag{2.7}$$

for all $X \in \Gamma(\wedge^x \mathcal{A})$, $Y \in \Gamma(\wedge^y \mathcal{A})$ and $Z \in \Gamma(\wedge^z \mathcal{A})$. Equivalently, (2.5) *states that the Schouten bracket is graded skew-symmetric. Also* (2.6) *is the graded Leibniz identity, and* (2.7) *the graded Jacobi identity.*

It is also well known that the Schouten bracket is the only bracket that extends the Lie algebroid bracket and satisfies (2.3), (2.4), (2.5), (2.6) and (2.7). The Schouten bracket may also be defined without the decomposition used in Definition 2.5. Let $X \in \Gamma(\wedge^x \mathcal{A})$ and $Y \in \Gamma(\wedge^y \mathcal{A})$. The Schouten bracket of X and Y is the unique element $[X, Y] \in \Gamma(\wedge^{x+y-1} \mathcal{A})$ such that

$$\omega\big([X, Y] \big) = (-1)^{(x+1)y} i_X d i_Y \omega + (-1)^x i_Y d i_X \omega - i_{X \wedge Y} d\omega, \tag{2.8}$$

for all $\omega \in \Gamma(\wedge^{x+y-1} \mathcal{A}^*)$. See [39] for the proof that this definition and Definition 2.5 are equivalent. We can also use the Schouten bracket to extend the Lie derivative from $\Gamma(\mathcal{A})$ to $\Gamma(\wedge^\bullet \mathcal{A})$. Take $X \in \Gamma(\mathcal{A})$ and define

$$\mathcal{L}_X Y = [X, Y], \tag{2.9}$$

for all $Y \in \Gamma(\wedge^\bullet \mathcal{A})$. The previous proposition ensures that this expression gives a derivation.

One application of the Schouten bracket is to give an alternate definition of Poisson manifolds.

Definition 2.7. A Poisson manifold is a pair of a smooth manifold M and a bivector $\pi \in \mathfrak{X}^2(M)$ such that $[\pi, \pi] = 0$.

Recall that a Poisson manifold is usually defined as a smooth manifold M with a skew-symmetric bracket $\{.,.\}$, defined on $C^\infty(M)$, which satisfies the Leibniz and Jacobi identities. The two definitions are related by $\{f, g\} = \pi(df, dg)$. In this thesis Poisson manifolds will normally be defined in terms of a bivector. This bivector also leads to a Lie algebroid structure on T^*M.

Example 2.8. Let M be a smooth manifold, and let π be a Poisson bivector. Define the following bracket:

$$[\xi, \eta]_\pi = \mathcal{L}_{\pi^\sharp \xi} \eta - \mathcal{L}_{\pi^\sharp \eta} \xi - d\pi(\xi, \eta) \text{ for all } \xi, \eta \in \Omega^1(M). \tag{2.10}$$

The cotangent bundle with this bracket and anchor map $\pi^\sharp$ is a Lie algebroid. This Lie algebroid will be denoted by $(T^*M)_\pi$. The formula for $[.,.]_\pi$ can also be rewritten as

$$[\xi, \eta]_\pi = i_{\pi^\sharp \xi} d\eta - i_{\pi^\sharp \eta} d\xi + d\pi(\xi, \eta). \tag{2.11}$$

The previous proposition shows that the Schouten bracket gives rise to a Gerstenhaber algebra.

Definition 2.9. Let A be a graded set and denote the elements of degree a as $\mathbf{A}_a$. A Gerstenhaber algebra is a triple $(\mathbf{A}, *, [.,.])$, where $\mathbf{A}$ is a graded set, $*$ is a graded product of $\mathbf{A}$ and $[.,.]$ is an odd graded product of $\mathbf{A}$. So $A * B \in \mathbf{A}_{a+b}$ and $[A,B] \in \mathcal{A}_{a+b-1}$. The two products are also required to obey the following for all $A \in \mathbf{A}_a$, $B \in \mathbf{A}_b$, and $C \in \mathbf{A}$:

$$\begin{aligned} A * B &= (-1)^{ab} B * A, \\ [A,B] &= -(-1)^{(a+1)(b+1)} [B,A], \\ \big[A,[B,C]\big] &= \big[[A,B],C\big] + (-1)^{ab}\big[B,[A,C]\big], \\ [A, B * C] &= [A,B] * C + (-1)^{(a+1)b} B * [A,C]. \end{aligned}$$

Example 2.10. If $\mathcal{A}$ is a Lie algebroid then $(\Gamma(\wedge^\bullet \mathcal{A}), \wedge, [.,.])$ is a Gerstenhaber algebra.

2.3 Complex and Nijenhuis Structures

This section states the relevant facts that transfer from complex and Nijenhuis structures on manifolds to Lie algebroids. In this section $\mathcal{A}$ will denote a Lie algebroid. Before defining complex structures, we will briefly mention Nijenhuis structures.

Definition 2.11. Let $\phi : \mathcal{A} \to \mathcal{A}$ be a bundle map. The *Nijenhuis torsion* of ϕ is the (2,1)-tensor $\mathcal{N}_\phi$ on $\mathcal{A}$, defined by

$$\mathcal{N}_\phi(X,Y) = [\phi X, \phi Y] + \phi^2 [X,Y] - \phi\big([\phi X, Y] + [X, \phi Y]\big), \tag{2.12}$$

for all $X, Y \in \Gamma(\mathcal{A})$. A map ϕ is called a *Nijenhuis tensor* if $\mathcal{N}_\phi = 0$.

A Nijenhuis structure on TM gives rise to a new Lie algebroid structure on the tangent bundle.

Example 2.12. [21] Let M be a manifold and $\phi : TM \to TM$ a Nijenhuis tensor. Define the following bracket:

$$[X,Y]_\phi = [\phi X, Y] + [X, \phi Y] - \phi [X,Y] \text{ for all } X, Y \in \mathfrak{X}(M). \tag{2.13}$$

TM with this bracket, and the anchor map ϕ is a Lie algebroid. This Lie algebroid will be denoted by $(TM)_\phi$.

Definition 2.13. An *almost complex structure* on $\mathcal{A}$ is a smooth bundle map $J : \mathcal{A} \to \mathcal{A}$ such that $J^2 = -\mathrm{Id}$.

This is the analogue of an almost complex manifold; indeed, if $\mathcal{A} = TM$ then (M, J) is an almost complex manifold.

Returning now to the general case, just as for a complex manifold the map J leads to a splitting of $\mathcal{A}_{\mathbb{C}}$ into $+i$ and $-i$ eigenbundles. These bundles will be denoted by $\mathcal{A}_{1,0}$ and $\mathcal{A}_{0,1}$, respectively. There is also a splitting $\mathcal{A}^*_{\mathbb{C}} = \mathcal{A}^*_{1,0} \oplus \mathcal{A}^*_{0,1}$. Continuing with this notation $\mathcal{A}_{n,0} = \bigwedge^n \mathcal{A}_{1,0}$ and $\mathcal{A}_{n,m} = \mathcal{A}_{n,0} \bigwedge \mathcal{A}_{0,m}$. Denote the projections by $\pi^*_{n,m} : \bigwedge^\bullet \mathcal{A}^*_{\mathbb{C}} \to \mathcal{A}^*_{n,m}$. Also define the maps $\partial_{\mathcal{A}} : \Gamma(\mathcal{A}^*_{n,m}) \to \Gamma(\mathcal{A}^*_{n+1,m})$, and $\overline{\partial}_{\mathcal{A}} : \Gamma(\mathcal{A}^*_{n,m}) \to \Gamma(\mathcal{A}^*_{n,m+1})$ by

$$\partial_{\mathcal{A}} = \pi^*_{n+1,m} \circ d_{\mathcal{A}} \tag{2.14}$$

$$\overline{\partial}_{\mathcal{A}} = \pi^*_{n,m+1} \circ d_{\mathcal{A}} \tag{2.15}$$

The following proposition is similar to one for the almost complex structure on a manifold.

Proposition 2.14. *The following statements are all equivalent.*

1. *The $+i$-eigenbundle of J is involutive.*
2. *The $-i$-eigenbundle of J is involutive.*
3. *The Nijenhuis torsion of J is zero.*
4. $d_{\mathcal{A}} = \partial_{\mathcal{A}} + \overline{\partial}_{\mathcal{A}}$.

Proof. The conjugation map allows us to easily see that 1 and 2 are equivalent.

1,2 $\Longrightarrow$ 3: Let X_i denote elements of $\Gamma(\mathcal{A}_{1,0})$ and let Y_i denote elements of $\Gamma(\mathcal{A}_{0,1})$.

$$\begin{aligned}
\mathcal{N}_J(X_1, X_2) &= [JX_1, JX_2] - [X_1, X_2] - J\,[JX_1, X_2] - J\,[X_1, JX_2] \\
&= [iX_1, iX_2] - [X_1, X_2] - J\,[iX_1, X_2] - J\,[X_1, iX_2] \\
&= -2\,[X_1, X_2] - 2iJ\,[X_1, X_2]
\end{aligned}$$

By assumption $[X_1, X_2] \in \Gamma(\mathcal{A}_{1,0})$ and this is zero. Similarly

$$\mathcal{N}_J(Y_1, Y_2) = -2\,[Y_1, Y_2] + 2iJ\,[Y_1, Y_2]$$

Once again by assumption $[Y_1, Y_2] \in \Gamma(\mathcal{A}_{0,1})$ and this is also zero. Finally we check

$$\begin{aligned}\mathcal{N}_J(X,Y) &= [JX, JY] - [X,Y] - J\,[JX,Y] - J\,[X,JY] \\ &= [iX, -iY] - [X,Y] - J\big([iX,Y] + [X,-iY]\big) \\ &= [X,Y] - [X,Y] - J\big(i\,[X,Y] - i\,[X,Y]\big) = 0\end{aligned}$$

Since $\Gamma(\mathcal{A}_{\mathbb{C}}) = \Gamma(\mathcal{A}_{1,0}) \oplus \Gamma(\mathcal{A}_{0,1})$ the conclusion follows.

3 $\Longrightarrow$ 1: From the previous part for any two elements of $\Gamma(\mathcal{A}_{1,0})$ the following holds.

$$0 = \mathcal{N}_J(X_1, X_2) = -2\,[X_1, X_2] - 2iJ\,[X_1, X_2]$$

This can only be true if $J\,[X_1, X_2] = i\,[X_1, X_2]$, i.e. $[X_1, X_2] \in \Gamma(\mathcal{A}_{1,0})$. As this holds for any elements, $\Gamma(\mathcal{A}_{1,0})$ is involutive.

4 $\Longrightarrow$ 1: Take $X_1, X_2 \in \Gamma(\mathcal{A}_{1,0})$. For all $\xi \in \Gamma(\mathcal{A}_{0,1})$

$$\begin{aligned}\xi([X_1, X_2]) &= \rho(X_1)\cdot\xi(X_2) - \rho(X_2)\cdot\xi(X_1) - i_{X_2} i_{X_1} d_{\mathcal{A}}\xi \\ &= -i_{X_2} i_{X_1} \partial_{\mathcal{A}}\xi - i_{X_2} i_{X_1} \overline{\partial}_{\mathcal{A}}\xi = 0.\end{aligned}$$

Thus $[X_1, X_2] \in \Gamma(\mathcal{A}_{1,0})$, and $\mathcal{A}_{1,0}$ is an involutive subspace.

1,2 $\Longrightarrow$ 4: We will prove the following claim, which is equivalent to this implication:

Claim: If $\mathcal{A}_{1,0}$ and $\mathcal{A}_{0,1}$ are involutive subspaces then $d_{\mathcal{A}}\omega \in \Gamma(\mathcal{A}^{k,l} \oplus \mathcal{A}^{k-1,l+1})$ for all $\omega \in \Gamma(\mathcal{A}^{k-1,l})$.

Proof. Because $\omega \in \Gamma(\mathcal{A}^{k-1,l})$ it is true that $i_X\omega = 0$ for all $X \in \Gamma(\mathcal{A}_{k,0})$. Similarly, it is also true that $i_Y\omega = 0$ for all $Y \in \Gamma(\mathcal{A}_{0,l+1})$.

The claim will be true if, and only if $i_{X_{k+1}} \dots i_{X_1} d_{\mathcal{A}}\omega = 0$ for all $X_i \in \Gamma(\mathcal{A}_{1,0})$ and $i_{Y_{l+2}} \dots i_{Y_1} d_{\mathcal{A}}\omega = 0$ for all $Y_i \in \Gamma(\mathcal{A}_{0,1})$. Let the Z_i be any elements of $\Gamma(\mathcal{A}_{\mathbb{C}})$.

$$
\begin{aligned}
d_{\mathcal{A}}\omega(X_1, \dots, X_{k+1}, Z_1, \dots, Z_{l-1}) = \\
&\sum_{i=1}^{k+1} (-1)^{i-1} \rho(X_i) \cdot \omega(X_1, \dots \widehat{X_i} \dots, X_{k+1}, Z_1, \dots, Z_{l-1}) \\
&+ \sum_{i=1}^{l-1} (-1)^{i} \rho(Z_l) \cdot \omega(X_1, \dots, X_{k+1}, Z_1, \dots \widehat{Z_i} \dots, Z_{l-1}) \\
&+ \sum_{j=2}^{k+1} \sum_{i=1}^{j-1} (-1)^{i+j} \omega\left(\left[X_i, X_j\right], X_1, \dots \widehat{X_i} \dots \widehat{X_j}, \dots, X_{k+1}, Z_1, \dots, Z_{l-1}\right) \\
&+ \sum_{j=1}^{l-1} \sum_{i=1}^{k+1} (-1)^{i+j+k+1} \omega\left(\left[X_i, Z_j\right], X_1, \dots \widehat{X_i} \dots, X_{k+1}, Z_1, \dots \widehat{Z_j} \dots, Z_{l-1}\right) \\
&+ \sum_{j=2}^{l-1} \sum_{i=1}^{j-1} (-1)^{i+j} \omega\left(\left[Z_i, Z_j\right], X_1, \dots, X_{k+1}, Z_1, \dots \widehat{Z_i} \dots \widehat{Z_j}, \dots, Z_{l-1}\right) \\
&= \sum_{j=2}^{k+1} \sum_{i=1}^{j-1} (-1)^{i+j} \omega\left(\left[X_i, X_j\right], X_1, \dots \widehat{X_i} \dots \widehat{X_j}, \dots, X_{k+1}, Z_1, \dots, Z_{l-1}\right) = 0
\end{aligned}
$$

Where the last equality is true because $\mathcal{A}_{1,0}$ is an involutive subspace under the Lie bracket.

Similarly $d_{\mathcal{A}}(Y_1, \dots, Y_{l+2}, Z_1, \dots, Z_{k-2}) = 0$. □

□

Definition 2.15. An almost complex structure is *integrable* if one of the conditions of Proposition 2.14 is satisfied.

2.4 Lie Bialgebroids

Lie bialgebroids were introduced in [27] as the infinitesimal objects associated to Poisson groupoids. They are a generalization of Lie bialgebras.

Definition 2.16. A *Lie bialgebroid* is a Lie algebroid $(\mathcal{A}, \rho, [.,.])$ with a Lie algebroid structure on $\mathcal{A}^*$ such that

$$
d_{\mathcal{A}^*}[X, Y] = \left[X, d_{\mathcal{A}^*}Y\right] + \left[d_{\mathcal{A}^*}X, Y\right] \qquad \text{for all} \qquad X, Y \in \Gamma(\mathcal{A}).
$$

Here, $d_{\mathcal{A}^*}$ is the differential on $\Gamma(\wedge^\bullet \mathcal{A})$ induced by $[.,.]_*$.

Example 2.17. A Lie bialgebra is a Lie bialgebroid over a point.

Example 2.18. Any Lie algebroid with the zero bracket on its dual and zero anchor map is a Lie bialgebroid. In particular take (TM, T^*M) with the usual Lie bracket on $\mathfrak{X}(M)$.

Example 2.19. [27] Let (M, π) be a Poisson manifold. The pair $\big(TM, (T^*M)_\pi\big)$ is another Lie bialgebroid structure on TM.

Example 2.20. [21] If (M, π, ϕ) is a Poisson Nijenhuis manifold then $\big((TM)_\phi, (T^*M)_\pi\big)$ is a Lie bialgebroid. See Example 2.12 and Example 2.7 for details of the Lie algebroids.

The differential $d_{\mathcal{A}^*}$ can be used to give an alternate definition of a Lie bialgebroid.

Proposition 2.21. *A Lie bialgebroid is equivalent to a Lie algebroid* $(\mathcal{A}, [.,.], \rho)$ *and a degree one differential* δ *of the Gerstenhaber algebra* $(\Gamma(\wedge^\bullet \mathcal{A}), \wedge, [.,.])$ *such that* $\delta^2 = 0$.

More explicitly, $\delta : \Gamma(\wedge^k \mathcal{A}) \to \Gamma(\wedge^{k+1} \mathcal{A})$ is required to satisfy

1. $\delta^2 = 0$,
2. $\delta(X \wedge Y) = \delta X \wedge Y + (-1)^x X \wedge \delta Y$ for all $X \in \Gamma(\wedge^x \mathcal{A})$ and $Y \in \Gamma(\wedge^y \mathcal{A})$,
3. $\delta\,[X, Y] = [\delta X, Y] - (-1)^x\,[X, \delta Y]$ for all $X \in \Gamma(\wedge^x \mathcal{A})$ and $Y \in \Gamma(\wedge^y \mathcal{A})$.

The first definition implies the second by simply taking $\delta = d_{\mathcal{A}^*}$. Now start from the second definition and let δ be the differential. Define the anchor anchor $\rho_* : \mathcal{A}^* \to TM$ of $\xi \in \mathcal{A}^*$ as

$$\rho_*(\xi).f = \xi(\delta f) \text{ for all } f \in C^\infty(M).$$

The bracket $[.,.]_*$ is defined, for $\xi, \eta \in \Gamma(\mathcal{A}^*)$, as

$$[\xi, \eta]_*\,(X) = \rho_*(\xi).\eta(X) - \rho_*(\eta).\xi(X) - (\xi \wedge \eta)(\delta X) \text{ for all } X \in \Gamma(\mathcal{A}).$$

2.5 Courant Algebroids

Courant algebroids were defined in [25] as a generalization of the Courant bracket. The Courant bracket was defined in [7] as

$$[X + \xi, Y + \eta] = [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \tfrac{1}{2} d\big(\eta(X) - \xi(Y)\big) \tag{2.16}$$

for all $X, Y \in \mathfrak{X}(M)$ and $\xi, \eta \in \Omega^1(M)$.

There also exist smoothly varying nondegenerate bilinear forms on each fibre of $TM \oplus T^*M$. These forms are defined as

$$\langle X + \xi, Y + \eta \rangle = \tfrac{1}{2}\big(\eta(X) + \xi(Y)\big) \tag{2.17}$$

for all $X, Y \in T_mM$, $\xi, \eta \in T^*_mM$ and $m \in M$. Courant algebroids axiomatize the properties of (2.16) and (2.17).

If [.,.] is a skew-symmetric bracket then the *Jacobiator* of three elements is defined as

$$\mathrm{Jac}(e_1, e_2, e_3) = \big[\, [e_1, e_2]\, , e_3\big] + \big[\, [e_2, e_3]\, , e_1\big] + \big[\, [e_3, e_1]\, , e_2\big].$$

Definition 2.22. A *real Courant algebroid* is a real vector bundle $\mathcal{E} \to M$ with a bilinear skew-symmetric bracket on its sections [.,.], smoothly varying nondegenerate symmetric bilinear forms on each of its fibres $\langle .,. \rangle$, and a linear map $\rho : \mathcal{E} \to TM$ called the *anchor*. These three structures are required to obey the following relations.

1. $\mathrm{Jac}(e_1, e_2, e_3) = \frac{1}{3}\mathcal{D}\Big(\big\langle\, [e_1, e_2]\, , e_3\big\rangle + \big\langle\, [e_2, e_3]\, , e_1\big\rangle + \big\langle\, [e_3, e_1]\, , e_2\big\rangle\Big).$
2. $[e_1, fe_2] = f\,[e_1, e_2] + \big(\rho(e_1) \cdot f\big)e_2 - \langle e_1, e_2\rangle\, \mathcal{D}f.$
3. $\langle \mathcal{D}f, \mathcal{D}g\rangle = 0.$
4. $\rho(e_1) \cdot \langle e_2, e_3\rangle = \big\langle [e_1, e_2] + \mathcal{D}\big(\langle e_1, e_2\rangle\big), e_3\big\rangle + \big\langle e_2, [e_1, e_3] + \mathcal{D}\big(\langle e_1, e_3\rangle\big)\big\rangle.$

Here $e_i \in \Gamma(E)$, $f, g \in C^\infty(M)$ and $\mathcal{D} : C^\infty(M) \to \Gamma(\mathcal{E})$ is the map defined by

$$\langle \mathcal{D}f, e\rangle = \tfrac{1}{2}\rho(e) \cdot f, \tag{2.18}$$

for all $f \in C^\infty(M)$.

The first example of a Courant algebroid is given by the Courant bracket.

Example 2.23. Consider the vector bundle $TM \oplus T^*M$ over M with the bracket (2.16) and the form (2.17). Also, define the anchor map as the natural projection $TM \oplus T^*M \to TM$. This quadruple is a Courant algebroid.

There are quite a few Courant Algebroid structures on $TM \oplus T^*M$. Another is given by the twisted Courant bracket which was originally defined in [32].

Example 2.24. Consider $TM \oplus T^*M$ with the same form and anchor map as Example 2.23. Take a 3-form Ω. The *Ω-twisted Courant bracket* is defined as

$$[X + \xi, Y + \eta]_\Omega = [X, Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \tfrac{1}{2}d\big(\eta(X) - \xi(Y)\big) + i_Y i_X\Omega. \tag{2.19}$$

Now, $TM \oplus T^*M$ with the Ω-twisted bracket is a Courant algebroid if, and only if, Ω is closed.

Example 2.25. [25] Let $(\mathcal{A}, \mathcal{A}^*)$ be a Lie bialgebroid, and consider the vector bundle $\mathcal{A} \oplus \mathcal{A}^*$. The metric (2.17) can be defined on $\mathcal{A} \oplus \mathcal{A}^*$ by the same formula. Define the anchor as

$\rho = \rho + \rho_*$, and the bracket by

$$\begin{aligned}[X+\xi, Y+\eta] = [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \tfrac{1}{2}d_{\mathcal{A}}\big(\eta(X) - \xi(Y)\big) \\ + [\xi,\eta]_* + \mathcal{L}^*_\xi Y - \mathcal{L}^*_\eta X + \tfrac{1}{2}d_{\mathcal{A}^*}\big(\eta(X) - \xi(Y)\big).\end{aligned} \tag{2.20}$$

Liu et al. showed that $(\mathcal{A} \oplus \mathcal{A}^*, [.,.], \langle .,. \rangle, \rho')$ is a Courant algebroid [25].

Example 2.26. Consider the trivial Lie bialgebroid from Example 2.18. The Courant algebroid arising from this Lie bialgebroid has the same formula as the usual Courant bracket (2.16).

We will finish this section by briefly discussing Courant algebroids on complex vector bundles.

Definition 2.27. A *complex Courant algebroid* is merely a Courant algebroid that is a complex vector bundle, rather than a real one. The bracket and nondegenerate form are required to obey the same properties, except they must be $\mathbb{C}$-linear, rather than $\mathbb{R}$-linear. The anchor is a $\mathbb{C}$-linear map $\rho : \mathcal{E} \to T_{\mathbb{C}}M$. The underlying manifold of a complex Courant algebroid will still be a real manifold.

A real Courant algebroid can be complexified. The complexification of $\mathcal{E}$ will be denoted by $\mathcal{E}_{\mathbb{C}}$, and the bundle is defined as $\mathcal{E}_{\mathbb{C}} = \mathcal{E} \otimes_{\mathbb{R}} \mathbb{C}$. The bracket is extended by defining

$$[e_1 \otimes \alpha_1, e_2 \otimes \alpha_2] = [e_1, e_2] \otimes \alpha_1\alpha_2,$$

where $e_i \in \Gamma(\mathcal{E})$ and $\alpha_i \in \mathbb{C}$. The form and anchor map are extended by defining

$$\langle e_1 \otimes \alpha_1, e_2 \otimes \alpha_2 \rangle = \alpha_1\alpha_2 \langle e_1, e_2 \rangle \qquad \text{and} \qquad \rho(e_1 \otimes \alpha_1) = \alpha_1 \rho(e_1).$$

2.6 Alternative Description of Courant Algebroids

In this short section we give an alternate description of Courant algebroids. Instead of a skew-symmetric bracket this definition uses a bilinear form that is not skew-symmetric. One of the advantages of this approach is that the anomalous terms in the original expression disappear. This approach is also more natural when the Courant algebroid is described using derived brackets. Derived brackets will be studied in Chapter 4.

The possibility of simplifying the expressions was noted in [25], where Courant algebroids were originally defined. In his thesis, Roytenberg showed that this leads to an equivalent definition of Courant algebroids [31].

Definition 2.28. A *Courant algebroid* is a vector bundle $E \to M$, with a nondegenerate symmetric bilinear form $\langle .,. \rangle$, a bilinear operation on $\Gamma(\mathcal{E})$, and a map $\rho : \mathcal{E} \to TM$ such that the following hold for all $A, B, C \in \Gamma(\mathcal{E})$ and $f \in C^{\infty}(M)$:

$$(1) \quad A \circ (B \circ C) = (A \circ B) \circ C + B \circ (A \circ C), \tag{2.21}$$

$$(2) \quad \rho(A \circ B) = [\rho(A), \rho(B)]\,, \tag{2.22}$$

$$(2) \quad A \circ (fB) = f(A \circ B) + (\rho(A) \cdot f)B, \tag{2.23}$$

$$(4) \quad A \circ A = \mathcal{D}\,\langle A, A\rangle\,, \tag{2.24}$$

$$(5) \quad \rho(A) \cdot \langle B, C\rangle = \langle A \circ B, C\rangle + \langle B, A \circ C\rangle\,. \tag{2.25}$$

In the fourth of these expressions the $\mathcal{D}$ is the same one as defined in (2.18).

The details of the equivalence can be found in the second chapter of [31]. Given the bilinear operation from above, the skew-symmetric bracket is defined as

$$[A, B] = \tfrac{1}{2}\big(A \circ B - B \circ A\big).$$

Whereas, given the skew-symmetric bracket the bilinear operation is defined as

$$A \circ B = [A, B] + \mathcal{D}\,\langle A, B\rangle\,.$$

This definition can in fact be simplified.

Remark 2.29. [37] Conditions (2), (3) and the defining relation for $\mathcal{D}$ follow from the remaining axioms in Definition 2.28 and the following Leibniz identity:

$$\mathcal{D}(fg) = f\mathcal{D}g + g\mathcal{D}f, \qquad \text{for all} \qquad f, g]in]snfM.$$

A similar simplification can also be made to the usual definition of Courant algebroids from the previous section.

Complex Courant algebroids may also be considered in the same manner.

2.7 Dirac Structures

In the previous section we showed that a Lie bialgebroid gives rise to a Courant algebroid. It is natural to ask when the converse is also true? I.e. when does a Courant algebroid arise from a Lie bialgebroid? Dirac structures are crucial in answering this question. In this section $(\mathcal{E}, \rho, [.,.]\,, \langle .,. \rangle)$ will denote a Courant algebroid.

Definition 2.30. Let L be a smooth subbundle of $\mathcal{E}$. The subbundle L is *involutive* if $\Gamma(L)$ is closed under the Courant bracket. If L is maximally isotropic and involutive then it is called a *Dirac structure*.

Let L be a Dirac structure. Because L is involutive $[e_1, e_2] \in \Gamma(L)$ for all $e_i \in \Gamma(L)$, and since L is isotropic, $\mathrm{Jac}(e_1, e_2, e_3) = 0$ for all $e_i \in \Gamma(L)$. Thus $(L, [.,.], \rho)$ is a Lie algebroid. As a result if $\mathcal{E}$ can be written as the direct sum of two Dirac structures one might hope it is the double of a Lie bialgebroid. This fact was proven in [25].

Theorem 2.31 ([25]). *Let $(\mathcal{E}, \rho, [.,.], \langle .,. \rangle)$ be a Courant algebroid. If L_1 and L_2 are two Dirac structures of $\mathcal{E}$ such that $\mathcal{E} = L_1 \oplus L_2$ then (L_1, L_2) is a Lie bialgebroid. The nondegenerate form on $\mathcal{E}$ induces the isomorphism between L_2 and L_1^*.*

Liu et. al. noted that an immediate consequence of this theorem is the duality property of Lie bialgebroids, first proven in [27].

Proposition 2.32 ([19], [25] and [27]). *If $(\mathcal{A}, \mathcal{A}^*)$ is a Lie bialgebroid then so is $(\mathcal{A}^*, \mathcal{A})$.*

We now give a couple of examples of Lie bialgebroids and their associated Courant algebroids.

Example 2.33. As mentioned previously a Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ is a Lie bialgebroid over a single point. The Courant algebroid associated to this is just the usual double of a Lie bialgebra, and the triple $(\mathfrak{g}, \mathfrak{g}^*, \mathfrak{g} \oplus \mathfrak{g}^*)$ is a Manin triple.

Example 2.34. Consider the Lie bialgebroid (TM, T^*M) with the usual Lie bracket structure on TM and the zero structure on T^*M. The Courant algebroid associated to this Lie bialgebroid is just $TM \oplus T^*M$ with the usual Courant bracket (2.16).

When the Courant algebroid can be decomposed as $\mathcal{E} = \mathcal{A} \oplus \mathcal{A}^*$ Dirac structure can be described with a subspace of $\mathcal{A}$ and a two-form on this subspace. Similarly a Dirac structure can also be describes with a subspace of $\mathcal{A}^*$ and a two-form on this subspace.

First consider the case $\mathcal{E} = V \oplus V^*$, where V is a finite dimensional vector space. Let q_1 denote the projection of $V \oplus V^*$ onto V, and let q_2 denote the projection onto V^*. Also, let L be a Dirac structure on $\mathcal{E}$.

Because L is a maximal isotropic there exists a skew-symmetric bilinear form Λ on L defined by

$$\Lambda(X + \xi, Y + \eta) = \xi(Y) = -\eta(X) \text{ for all } X + \xi, Y + \eta \in L.$$

It is easy to see that

$$\Lambda(X + \xi_1, Y + \eta_1) = \Lambda(X + \xi_2, Y + \eta_2) \text{ for all } X + \xi_{1,2}, Y + \eta_{1,2} \in L$$

and

$$\Lambda(X_1 + \xi, Y_1 + \eta) = \Lambda(X_2 + \xi, Y_2 + \eta) \text{ for all } X_{1,2} + \xi, Y_{1,2} + \eta \in L.$$

Hence, there exists a two form ω on $q_1(L)$ defined by

$$\omega(X, Y) = \Lambda(X + \xi, Y + \eta) \text{ for all } X + \xi, Y + \eta \in L,$$

and a two form π on $q_2(L)$ defined by

$$\pi(\xi, \eta) = -\Lambda(X + \xi, Y + \eta) \text{ for all } X + \xi, Y + \eta \in L.$$

If $X \in q_1(L)$ then there exists some $\xi \in L^*$ with $X + \xi \in L$, and furthermore $\omega(X, Y) = \xi(Y)$ for all $Y \in q_1(L)$. Thus $i_X\omega = \xi|_{q_1(L)}$, and

$$X + \xi \in L \quad \Longleftrightarrow \quad X \in q_1(L) \text{ and } i_X\omega = \xi|_{q_1(L)}.$$

Thus knowing the maximal isotropic L is exactly the same as knowing the subspace $q_1(L)$ and the two form ω. Similarly, L is equivalent to the pair $(q_2(L), \pi)$. Therefore, we have the converse: any subspace $R \subseteq V$ endowed with a two form ω on R defines a maximal isotropic $L(R, \omega)$ by

$$L(R, \omega) = \{X + \xi \in R \oplus V : i_X\omega = \xi|_R\},$$

and any subspace $S \subseteq V^*$ endowed with a two form π on S defines a maximal isotropic $L(S, \pi)$ by

$$L(S, \pi) = \{X + \xi \in V \oplus S : \pi(\xi, \eta) = -\eta(X) \text{ for all } \eta \in S\}.$$

Now let L be a maximal isotropic subbundle of $\mathcal{B} \oplus \mathcal{B}^*$. This construction can be used to define, point-wise, a generalized distribution (in the sense of [36]) R of $\mathcal{B}$ and a not necessarily smooth 2-form $\omega \in \Gamma(\wedge^2 R^*)$. The maximal isotropic also defines a generalized distribution S of $\mathcal{B}^*$ and a, not necessarily smooth, 2-form $\omega \in \Gamma(\wedge^2 S^*)$. Details of these constructions can be found in [7].

Thus we have the following proposition.

Proposition 2.35. *A Dirac structure on $\mathcal{E} = \mathcal{B} \oplus \mathcal{B}^*$ is defined by one of the following sets of data:*

1. *A maximally isotropic involutive distribution $L \subseteq \mathcal{B} \oplus \mathcal{B}^*$.*

2. *A generalized distribution $R \subseteq \mathcal{B}$ and a 2-form $\omega \in \Gamma(\wedge^2 R^*)$ such that $L(R, \omega)$ is a smooth involutive subbundle.*

3. *A generalized distribution $S \subseteq \mathcal{B}^*$ and a 2-form $\pi \in \Gamma(\wedge^2 S^*)$ such that $L(S, \pi)$ is a smooth involutive subbundle.*

This proposition gives us some examples of Dirac structures. Of particular interest is the following example.

Example 2.36. [7] Consider the Courant algebroid $\mathcal{E} = TM \oplus T^*M$ with the usual Courant bracket and metric. Also, let $\omega \in \Omega^2(M)$. Now

$$L(TM, \omega) = \left\{ X + \omega_\flat(X) : X \in TM \right\},$$

which we call the *graph* of the 2-form ω, is involutive if and only if ω is closed. Thus presymplectic manifolds give an example of a Dirac structure.

Another example arises from Poisson geometry.

Example 2.37. [7] Consider the same Courant algebroid as in the previous example and let $\pi \in \mathfrak{X}^2(M)$. Then

$$L(T^*M, \pi) = \left\{ \pi^\sharp(\xi) + \xi : \xi \in T^*M \right\},$$

is a Dirac structure if, and only if, (M, π) is a Poisson manifold.

Chapter 3

Generalized Complex Structures on Courant Algebroids

Courant algebroids were introduced in [25] to systematize the properties of the Courant bracket defined by T. Courant [7]. Twisted Courant brackets, and Lie bialgebroids are also examples of Courant algebroids. Hitchin and Gualtieri use the usual Courant bracket on $TM \oplus T^*M$ to define generalized complex structures [14, 15, 16]. The aim of this chapter is to pose the definition of generalized complex structures for arbitrary Courant algebroids, and to also examine how some of the common examples behave on Courant algebroids arising from Lie bialgebroids.

In this chapter $(\mathcal{E}, \rho, [.,.], \langle .,. \rangle)$ will denote an arbitrary real Courant algebroid over a smooth manifold M. The complexification of this Courant algebroid will be denoted by $\mathcal{E}_{\mathbb{C}}$, and the fibre at a point $m \in M$ will be denoted by $\mathcal{E}_m$.

3.1 Definition and basic properties

Definition 3.1. An *almost generalized complex structure* on $\mathcal{E}$ is a smooth bundle map $J : \mathcal{E} \to \mathcal{E}$ over the identity such that $J^2 = -\mathrm{Id}$, and $\langle J_m(e), J_m(f) \rangle = \langle e, f \rangle$. An almost complex structure is *integrable*, or a *generalized complex structure* if the Nijenhuis torsion, (2.12), of J is zero.

Remark 3.2. [8] If $J : \mathcal{E} \to \mathcal{E}$ has $J^2 = -\mathrm{Id}$ and zero Nijenhuis torsion then it automatically has $\langle J_m(e), J_m(f) \rangle = \langle e, f \rangle$.

Generalized complex structures can also be described in other ways. One, involving spinors will be left to the next section. Another description, that uses subbundles, is given below.

Proposition 3.3. *An almost generalized complex structure is a equivalent to a maximal isotropic subbundle $L \subseteq \mathcal{E}_{\mathbb{C}}$ such that $L \cap \overline{L} = \{0\}$. An almost generalized complex structure is integrable if L is a Dirac structure.*

This equivalence arises by considering the eigenbundles of J. If $J : \mathcal{E} \to \mathcal{E}$ is a generalized complex map then J has two eigenbundles in $\mathcal{E}_{\mathbb{C}}$: the $+i$ and $-i$-eigenbundles. Let L be the $+i$-eigenbundle. This bundle has the same rank as the real vector bundle $\mathcal{E}$. The fact that J is orthogonal implies that L is isotropic. Because J is integrable the $+i$-eigenbundle is involutive and L is a Dirac structure. Alternately if L is a Dirac structure then defining $J(x) = ix$ for all $x \in L$ and $J(y) = -iy$ for all $y \in \overline{L}$ gives a generalized complex map. While this map is defined on $\mathcal{E}_{\mathbb{C}}$ it maps real vectors to real vectors, and so we can view this as a generalized complex map $J : \mathcal{E} \to \mathcal{E}$.

Consider the case that $\mathcal{E} = \mathcal{A} \oplus \mathcal{A}^*$, where $(\mathcal{A}, \mathcal{A}^*)$ is a Lie bialgebroid.

In this case, on every fibre, the almost generalized complex structure J_m can be divided into four parts:

$$J_m(X \oplus \xi) = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X \\ \xi \end{bmatrix}, \text{ for all } X \in \mathcal{A}_m \text{ and } \xi \in \mathcal{A}^*_m.$$

Here $A : \mathcal{A}_m \to \mathcal{A}_m$, $B : \mathcal{A}^*_m \to \mathcal{A}_m$, $C : \mathcal{A}_m \to \mathcal{A}^*_m$ and $D : \mathcal{A}^*_m \to \mathcal{A}^*_m$. Because J_m is skew-symmetric

$$\begin{bmatrix} D^* & B^* \\ C^* & A^* \end{bmatrix} = \begin{bmatrix} -A & -B \\ -C & -D \end{bmatrix}.$$

Thus B and C are both skew-symmetric, and can be written in terms of a bivector and a two form, respectively. Also D is just $-A^*$. The fact that the J_m vary smoothly means that each of the four components also vary smoothly. Combining these facts allows us to write J as follows:

$$J_m = \begin{bmatrix} \phi & \pi^\sharp \\ \sigma_\flat & -\phi^* \end{bmatrix}. \tag{3.1}$$

Here ϕ is an endomorphism of $\mathcal{A}$, $\pi^\sharp : \mathcal{A}^* \to \mathcal{A}$ is the bundle map induced by a section $\pi \in \Gamma(\wedge^2 \mathcal{A})$, and $\sigma_\flat : \mathcal{A} \to \mathcal{A}^*$ is the bundle map induced by a two form $\sigma \in \Gamma(\wedge^2 \mathcal{A}^*)$. The fact that $(J_m)^2 = -\mathrm{Id}$ also leads to the following formulas

$$\phi^2 + \pi^\sharp \sigma_\flat = -\mathrm{Id}, \quad \phi \pi^\sharp = \pi^\sharp \phi^*, \quad \text{and} \quad \phi^* \sigma_\flat = \sigma_\flat \phi. \tag{3.2}$$

These facts, and relations describing the integrability condition in the case of the usual Courant bracket, were first noted in [8].

3.2 Basic Examples

We start this section by citing two examples from [14]. These examples are formulated for the usual Courant bracket. The first example arises from complex geometry, and motivates the term "generalized complex".

Example 3.4. Let (M, j) be an almost complex manifold; here j denotes the automorphism of TM. Define an endomorphism of $TM \oplus T^*M$ by

$$J_j = \begin{bmatrix} -j & 0 \\ 0 & j^* \end{bmatrix}. \tag{3.3}$$

This map is an almost generalized complex structure, and is integrable if and only if j is integrable – or (M, j) is a complex manifold. The Dirac structure associated to J_j is $T_{0,1}M \oplus T^*_{1,0}M$.

Example 3.5. Let $\omega \in \Omega^2(M)$ be a nondegenerate 2-form. So there exists an inverse to the map $\omega_\flat$. Here $\omega_\flat$ denotes the map from $TM \to T^*M$ defined by $\omega^\sharp(X) = i_X\omega$ for all $X \in TM$.

Define a map on $TM \oplus T^*M$ by

$$J_\omega = \begin{bmatrix} 0 & -\omega_\flat^{-1} \\ \omega_\flat & 0 \end{bmatrix}. \tag{3.4}$$

The map J_ω is an almost generalized complex structure, and it is integrable if and only if $d\omega = 0$. I.e. J_ω is a generalized complex structure if, and only if (M, ω) is a symplectic manifold. The Dirac structure associated to J_ω is $\{X - i\omega_\flat(X) : X \in T_{\mathbb{C}}M\}$.

Now let $(\mathcal{A}, \mathcal{A}^*)$ be a Lie bialgebroid and consider the Courant algebroid $\mathcal{E} = \mathcal{A} \oplus \mathcal{A}^*$ as in Example 2.25. We consider the same two examples.

Let $j : \mathcal{A} \to \mathcal{A}$ be a complex map. Because $\mathcal{A}^*$ is a Lie algebroid it is reasonable to talk about the integrability of j^*. Define an almost generalized complex structure as in (3.3). This map has $+i$-eigenbundle $L_j = \mathcal{A}_{0,1} \oplus \mathcal{A}^*_{1,0}$. The following lemma will be needed to show when L_j is integrable.

Lemma 3.6. *Let j and j^* be integrable. For every $X \in \Gamma(\mathcal{A}_{0,1})$ and $\xi \in \Gamma(\mathcal{A}^*_{1,0})$*

$$\mathcal{L}_X\xi = i_X\overline{\partial}\xi, \qquad \textit{and} \tag{3.5}$$

$$\mathcal{L}^*_\xi X = i_\xi\partial^* X. \tag{3.6}$$

Here $\overline{\partial}$ and ∂ are defined as in (2.14) *and* (2.15).

Proof. Start with (3.5); because j is integrable d splits and by definition

$$\mathcal{L}_X\xi = i_Xd\xi + di_X\xi = i_Xd\xi = i_X\partial\xi + i_X\overline{\partial}\xi = i_X\overline{\partial}\xi.$$

The last equality holds as by definition $\partial\xi \in \Gamma(\mathcal{A}^*_{2,0})$ and taking the inner product with an element of $\Gamma(\mathcal{A}_{0,1})$ will give zero. A similar calculation, using the splitting of d_*, yields (3.6). $\square$

Proposition 3.7. *J_j is a generalized complex structure if, and only if, j and j^* are complex structures on $\mathcal{A}$ and $\mathcal{A}^*$ respectively.*

Proof. First, assume that J_j is a generalized complex structure.

For all $X, Y \in \Gamma(\mathcal{A}_{\mathbb{C}})$ $X + 0, Y + 0 \in \Gamma(L_j)$. Since L_j is involutive, it follows that $[X, Y] \in \Gamma(\mathcal{A}_{\mathbb{C}})$. Similarly, $\Gamma(\mathcal{A}^*_{1,0})$ is also involutive.

Now, assume that j and j^* are both integrable and take $X + \xi, Y + \eta \in L_j$. So $X, Y \in \Gamma(\mathcal{A}_{0,1})$ and $\xi, \eta \in \Gamma(\mathcal{A}^*_{1,0})$. The $\Gamma(\mathcal{A}^*_{\mathbb{C}})$ component of $[X + \xi, Y + \eta]$ is

$$[\xi, \eta]_* + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d(i_X\eta - i_Y\xi) = [\xi, \eta]_* + i_X\overline{\partial}\eta - i_Y\overline{\partial}\xi.$$

The first term is clearly in $\Gamma(\mathcal{A}^*_{1,0})$. Now $\overline{\partial}\eta \in \Gamma(\mathcal{A}^*_{1,1})$, and when contracted with X the result will be in $\Gamma(\mathcal{A}^*_{1,0})$. Similarly the last term is in $\Gamma(\mathcal{A}^*_{1,0})$, and the entire $\Gamma(\mathcal{A}^*_{\mathbb{C}})$ component is in $\Gamma(\mathcal{A}^*_{1,0})$. A similar argument shows that the $\Gamma(\mathcal{A}_{\mathbb{C}})$ component of the bracket is in $\Gamma(\mathcal{A}_{0,1})$. Thus $\Gamma(\mathcal{A}_{0,1} \oplus \mathcal{A}^*_{1,0})$ is involutive, and J_j is integrable. $\square$

The next example is based on the previous symplectic example. Let $\omega \in \Gamma(\wedge^2\mathcal{A}^*)$ and define J_ω as in (3.4). The $+i$-eigenbundle of this map is

$$L_\omega = \left\{X - i\omega_\flat X : X \in \mathcal{A}_{\mathbb{C}}\right\}.$$

The integrability of such a Dirac structure was studied in [25], where ω is called a *Hamiltonian operator*. The following result is a corollary of Theorem 6.1 [25].

Proposition 3.8. *J_ω is a generalized complex structure if, and only if,*

$$d\omega = 0 \qquad \textit{and} \qquad [\omega, \omega]_* = 0.$$

Proof. Theorem 6.1 [25] states that if $\alpha \in \Gamma(\wedge^2\mathcal{A}^*_{\mathbb{C}})$ then the graph of $-\alpha_\flat$ is a Dirac structure iff

$$d\alpha + \tfrac{1}{2}\left[\alpha, \alpha\right]_* = 0.$$

Now L_ω is the graph of $-i\omega_\flat$. Thus L_ω is a Dirac structure iff

$$d(i\omega) + \tfrac{1}{2}\left[i\omega, i\omega\right]_* = 0.$$

Equating real and complex parts gives the proposition. $\square$

As the next example shows, structures satisfying the conditions of Proposition 3.8 have been considered previously.

Example 3.9. Let (M, π) be a Poisson manifold. Vaisman defined a *complementary* 2-form to be an $\omega \in \Omega^2(M)$ such that $[\omega, \omega]_\pi = 0$ [40]. Furthermore Vaisman showed that if ω was symplectic then this lead to a Poisson-Nijenhuis manifold (see the next section for more on Poisson-Nijenhuis manifolds). The Nijenhuis structure is defined by taking $\pi^\sharp \circ \omega_\flat$. This situation is precisely the one considered above when the Lie bialgebroid is $\big(TM, (T^*M)_\pi\big)$.

3.3 Poisson Structure of a Generalized Complex Structure

For the usual Courant bracket it is well known that the existence of a generalized complex structure leads to a Poisson bivector field [1, 8, 24, 17]. The Poisson bivector field is the bivector field appearing in (3.1). In this section we show that this is true for generalized complex structures on arbitrary Courant algebroids. This fact was first used by Stiénon and myself for a twisted generalized complex structure [3]. The main result of this section is the following theorem.

Theorem 3.10. *A generalized complex structure on a Courant algebroid gives rise to a Poisson structure on the underlying base manifold.*

As stated previously, a generalized complex structure J on a Courant algebroid leads to a splitting $\mathcal{E}_\mathbb{C} = L_+ \oplus L_-$, where L_+ and L_- are the $+i$ and $-i$ eigenbundles of J. Because both of these bundles are Dirac structures, the Lie bracket can be restricted to L_+ and L_- to give rise to two Lie algebroids. These Lie algebroids will be denoted by $(L_+, \rho_+, [.,.])$ and $(L_-, \rho_-, [.,.])$. The nondegenerate form gives an isomorphism $\Xi : L_- \to L_+^*$. It is well known [25] that transverse Dirac structures in a Courant algebroid give a Lie bialgebroid. This theorem was originally proven for real Courant algebroids, but it is still true for complex Courant algebroids. Thus (L_+, L_-) is a complex Lie bialgebroid.

Let $\Phi = \rho_- \circ \Xi^{-1} \circ \rho_+^*$ and $\Psi = \rho_+ \circ \Xi^{-1} \circ \rho_-^*$. Both of these maps take $T^*_\mathbb{C}M$ to $T_\mathbb{C}M$. The following result from [27] is for a real Courant algebroid, but will still hold – with a slight modification – to the complex case.

Proposition 3.11 (Proposition 3.6 [27])**.** *The maps Φ and Ψ are both skew-symmetric and define complex Poisson bivector fields on M. I.e. both give rise to elements of $\mathfrak{X}^2_\mathbb{C}(M)$. Moreover, $\Phi = -\Psi$.*

The only modification for the complex case is that the maps are defined on the complexified cotangent bundle and map to the complexified tangent bundle. Thus they give rise to sections in $\Gamma(\wedge^2 T_{\mathbb{C}}M)$, whereas a Poisson bivector field is usually a section in $\Gamma(\wedge^2 TM)$. The bivector field Schouten bracket with itself will still be zero. In fact, as will be shown below, this bivector still gives rise to a usual Poisson structure on M.

The anchor map of $\mathcal{E}_{\mathbb{C}}$ is an extension of a real map, thus $\rho \circ z = z \circ \rho$. Here $z : \mathcal{E}_{\mathbb{C}} \to \mathcal{E}_{\mathbb{C}}$ denotes the conjugation automorphism. The anchor maps of the Lie bialgebroid are restrictions of this anchor, and z is a bijection between the two algebroids. The following identities are a result of these observations.

$$z \circ \rho_{\pm} = \rho_{\mp} \circ z \qquad \text{and} \qquad z \circ \rho^*_{\pm} = \rho^*_{\mp} \circ z$$

When these are applied to Φ and Ψ, we find that $\overline{\Phi(\xi)} = -\Psi(\bar{\xi}) = -\Phi(\bar{\xi})$. Thus $\Phi = i\Pi^\sharp$, where Π is a real bivector field on M. This bivector field Π will still be Poisson and (M, Π) is a Poisson manifold. Thus we have proved Theorem 3.10; the following is an example for the usual Courant bracket.

Example 3.12. Let $\mathcal{E} = TM \oplus T^*M$ with the regular Courant bracket (2.16). By splitting vectors and covectors, a generalized complex structure can be written as

$$J = \begin{bmatrix} \phi & \pi^\sharp \\ \sigma_\flat & -\phi^* \end{bmatrix}. \tag{3.7}$$

Here ϕ is an endomorphism of TM, $\pi^\sharp : T^*M \to TM$ is the bundle map induced by a bivector field π, and $\sigma_\flat : TM \to T^*M$ is the bundle map induced by a two form σ. See [8] and [35] for more on the relations between ϕ, π and σ.

Extending to the complexified bundles $\rho^*_+(\xi) = i\pi^\sharp(\xi) + \xi - \phi^*(\xi)$, so $\rho_- \circ \rho^*_+ = i\pi^\sharp$ and π is Poisson. This is the same Poisson structure as was found in [8].

Lemma 3.13. *The Poisson bivector field can also be expressed as*

$$\pi^\sharp = \tfrac{1}{2}\rho \circ J \circ \Xi^{-1} \circ \rho^*$$

Proof. By definition $\rho = \rho_- + \rho_+$, and so $\rho^* = \rho^*_- + \rho^*_+$. Now

$$\begin{aligned} \rho \circ J \circ \Xi^{-1} \circ \rho^* &= \rho_- \circ J \circ \Xi^{-1} \circ \rho^*_- + \rho_+ \circ J \circ \Xi^{-1} \circ \rho^*_+ \\ &= i\rho + - \circ \Xi^{-1} \circ \rho^*_- - i\rho_- \circ \Xi^{-1} \circ \rho^*_+ \\ &= 2\pi^\sharp. \end{aligned}$$

□

This expression for π can now be used to describe the generalized complex foliation. The generalized complex foliation was defined in [14]

Proposition 3.14. $\rho(L_+) \cap \rho(L_-) = \Delta \otimes \mathbb{C}$ *with* $\Delta = \rho(J \ker \rho)$

Proof. If $v \in \rho(L_+) \cap \rho(L_-)$, then $\overline{v} \in \rho(L_-) \cap \rho(L_+)$. Hence there exists a subbundle Δ of TM such that $\rho(L_+) \cap \rho(L_-) = \Delta \otimes \mathbb{C}$. For all $k \in \ker \rho$, one has $\rho(L_+) \ni \rho(\frac{1+iJ}{2}k) = \frac{i}{2}\rho(Jk)$. Therefore, $\rho(J \ker \rho) \subset \rho(L_+)$. Since J is a real, $\rho(J \ker \rho) \subset \rho(L_+) \cap \rho(E) = \Delta$. It remains to prove the converse inclusion: $\Delta \subset \rho(J \ker \rho)$. Since $\Delta = \rho(L_+) \cap \rho(E)$, given $\delta \in \Delta$, there exists $l_+ \in L_+$ such that $\rho(l_+) = \delta = \rho(\overline{l_+})$. Thus $\delta = \rho(\frac{l_+ + \overline{l_+}}{2}) = \rho\big(J(\frac{l_+ - \overline{l_+}}{2i})\big)$ with $\frac{l_+ - \overline{l_+}}{2i} \in \ker \rho$. $\square$

Remark 3.15. If $\mathcal{E} = TM \oplus T^*M$ is the standard Courant algebroid of Example 2.23 then $(\ker \rho)^\perp = T^*M = \ker \rho$. Therefore, in this particular case, $\pi^\sharp(T^*_{\mathbb{C}}M) = \rho(L_+) \cap \rho(L_-)$, recovering Gualtieri's result [14].

It would be interesting to explore when the symplectic foliation $\pi^\sharp(T^*_{\mathbb{C}}M)$ coincides with $\rho(L_+) \cap \rho(L_-)$ for arbitrary Courant algebroids.

3.4 Poisson-Nijenhuis Structures

In this section we will examine the generalized complex structure that arises from a Poisson-Nijenhuis manifold. Such manifolds were defined while studying biHamiltonian Systems [29, 23]. A Poisson-Nijenhuis manifold is a manifold endowed with both a Poisson bivector field $\pi \in \mathfrak{X}^2(M)$ and a Nijenhuis tensor $\phi : TN \to TN$. These two structures are required to satisfy the following compatibility conditions for all $\xi, \eta \in \Omega^1(M)$:

$$\pi(\phi^*\xi, \eta) = \pi(\xi, \phi^*\eta), \tag{3.8}$$

$$C_{\phi,\pi}(\xi, \eta) = [\xi, \eta]_{\pi_\phi} - \Big[\phi^*\xi, \eta\Big]_\pi - \Big[\xi, \phi^*\eta\Big]_\pi + \phi^* [\xi, \eta]_\pi = 0. \tag{3.9}$$

Before discussing the notation appearing in (3.9) we will examine (3.8). Define a bundle map $\pi^\sharp_\phi : T^*M \to TM$ by $\pi^\sharp_\phi = \pi^\sharp \circ \phi^*$. Equation (3.8) ensures that π_ϕ is skew-symmetric, and so π_ϕ is a bivector field. Equation (3.8) is also often expressed as

$$\pi^\sharp \circ \phi^* = \phi \circ \pi^\sharp. \tag{3.10}$$

In the second condition the subscripts on the brackets refer to the Lie algebroid brackets induced by π and π_ϕ. See (2.10) for the definition of these brackets. It is known that π_ϕ is a Poisson bivector field, and $\left[\pi, \pi_\phi\right] = 0$. Such a pair of Poisson bivectors is called a *bi-Hamiltonian system*.

The concept of a Poisson-Nijenhuis manifold can also be applied to a Lie algebroid. If $\pi \in \Gamma(\wedge^2 \mathcal{A})$ satisfies $[\pi, \pi =] \, 0$, and $\phi : \mathcal{A} \to \mathcal{A}$ is a bundle map with zero Nijenhuis torsion then the same compatibility conditions as above can be defined. We will call a Lie algebroid with such a structure a *Poisson Nijenhuis structure*. The vector bundle $\mathcal{A}^*$ with the bracket $[.,.]_\pi$ is also a Lie algebroid, which we will denote by $\mathcal{A}^*_\pi$ to distinguish it from other Lie algebroid structures on $\mathcal{A}^*$. In fact, $(\mathcal{A}, \mathcal{A}^*_\pi)$ are Lie bialgebroids [27].

Proposition 3.16. *The following are equivalent:*

1. $(\mathcal{A}, \pi, \phi)$ *is a Poisson-Nijenhuis structure with* $\phi^2 = -\mathrm{Id}$.
2. $J_\phi = \begin{bmatrix} \phi & 0 \\ 0 & -\phi^* \end{bmatrix}$ *is a generalized complex structure on* $(\mathcal{A}, \mathcal{A}^*_\pi)$, *with* $\phi \circ \pi^\sharp = \pi^\sharp \circ \phi^*$.

Here J_ϕ *is the generalized complex structure defined in Example 3.3.*

Proof. We first show that 1 implies 2. By definition ϕ is integrable. By Proposition 3.7 J_ϕ will be an generalized complex structure if, and only if, ϕ^* is also integrable. The following calculation of the bracket $[.,.]_{\pi_\phi}$ will be used, where $\xi, \eta \in \Gamma(\mathcal{A}^*_{1,0})$.

$$\begin{aligned}
[\xi, \eta]_{\pi_\phi} &= i_{\phi \circ \pi^\sharp(\xi)} d_{\mathcal{A}} \eta - i_{\phi \circ \pi^\sharp(\eta)} d_{\mathcal{A}} \xi + d_{\mathcal{A}} \left\langle \phi \circ \pi^\sharp(\xi), \eta \right\rangle \\
&= i_{\pi^\sharp \circ \phi^*(\xi)} d_{\mathcal{A}} \eta - i_{\pi^\sharp \circ \phi^*(\eta)} d_{\mathcal{A}} \xi + d_{\mathcal{A}} \left\langle \pi^\sharp \circ \phi^*(\xi), \eta \right\rangle \\
&= i_{\pi^\sharp(i\xi)} d_{\mathcal{A}} \eta - i_{\pi^\sharp(i\eta)} d_{\mathcal{A}} \xi + d_{\mathcal{A}} \left\langle \pi^\sharp(i\xi), \eta \right\rangle \\
&= i \left(i_{\pi^\sharp(\xi)} d_{\mathcal{A}} \eta - i_{\pi^\sharp(\eta)} d_{\mathcal{A}} \xi + d_{\mathcal{A}} \left\langle \pi^\sharp \xi, \eta \right\rangle \right) \\
&= i \, [\xi, \eta]_\pi
\end{aligned}$$

Hence

$$\begin{aligned}
\phi^* \, [\xi, \eta]_\pi &= C_{\phi,\pi}(\xi, \eta) - [\xi, \eta]_{\pi_\phi} + \left[\phi^* \xi, \eta\right]_\pi + [\xi, \phi * \eta]_\pi \\
&= C_{\phi,\pi}(\xi, \eta) - i \, [\xi, \eta]_\pi + 2i \, [\xi, \eta]_\pi \\
&= i \, [\xi, \eta]_\pi + C_{\phi,\pi}(\xi, \eta)
\end{aligned}$$

As we have a Poisson-Nijenhuis manifold $C_{\phi,\pi} = 0$, thus it follows that $[\xi,\eta]_\pi \in \Gamma(\mathcal{A}^*_{1,0})$, which proves that ϕ^* is integrable.

Now, let 2 be true. In order for J_ϕ to be a generalized complex structure $\phi^2 = -\mathrm{Id}$. Now if (3.8) is true then the previous calculation shows that (3.9) is true for all $\xi, \eta \in \Gamma(\mathcal{A}_{1,0})$. A similar calculation also shows that (3.9) is true for all $\xi, \eta \in \Gamma(\mathcal{A}_{0,1})$. Now, take $\xi \in \mathcal{A}_{1,0}$ and $\eta \in \mathcal{A}_{0,1}$. So

$$\pi(\phi^*\xi, \eta) = i\pi(\xi, \eta),$$

and

$$\pi(\xi, \phi^*\eta) = -i\pi(\xi, \eta).$$

Thus, from (3.8) it follows that $\pi(\xi, \eta) = 0$. This fact also implies that $\pi^\sharp(\mathcal{A}_{1,0}) \subseteq \mathcal{A}_{1,0}$, and $\pi^\sharp(\mathcal{A}_{0,1}) \subseteq \mathcal{A}_{0,1}$. Now for all $\xi \in \Gamma(\mathcal{A}_{1,0})$ and $\eta \in \Gamma(\mathcal{A}_{0,1})$

$$\begin{aligned}
\phi^* [\xi,\eta]_\pi &= \phi^* i_{\pi^\sharp \xi} d_{\mathcal{A}}\eta - \phi^* i_{\pi^\sharp \eta} d_{\mathcal{A}}\xi + \phi^*\pi(\xi,\eta) \\
&= -i_{\pi^\sharp \circ \phi(\xi)} \phi^* d_{\mathcal{A}}\eta + i_{\pi^\sharp \circ \phi(\eta)} \phi^* d_{\mathcal{A}}\xi \\
&= -ii_{\pi^\sharp \xi} \phi^* \partial_{\mathcal{A}}\eta - ii_{\pi^\sharp \xi} \phi^* \overline{\partial}_{\mathcal{A}}\eta - ii_{\pi^\sharp \eta} \phi^* \partial_{\mathcal{A}}\xi - ii_{\pi^\sharp \eta} \phi^* \overline{\partial}_{\mathcal{A}}\xi \\
&= -ii_{\pi^\sharp \xi} \partial_{\mathcal{A}}\eta + ii_{\pi^\sharp \xi} \overline{\partial}_{\mathcal{A}}\eta + ii_{\pi^\sharp \eta} \partial_{\mathcal{A}}\xi - ii_{\pi^\sharp \eta} \overline{\partial}_{\mathcal{A}}\xi \\
&= -ii_{\pi^\sharp \xi} \partial_{\mathcal{A}}\eta - ii_{\pi^\sharp \eta} \overline{\partial}_{\mathcal{A}}\xi.
\end{aligned}$$

The next calculation also uses these observations on π:

$$\begin{aligned}
[\xi,\eta]_{\pi_\phi} &= i_{\pi^\sharp \circ \phi^*(\xi)} d_{\mathcal{A}}\eta - \phi^* i_{\pi^\sharp \circ \phi(\eta)} d_{\mathcal{A}}\xi + \pi(\phi^*\xi, \eta) \\
&= ii_{\pi^\sharp \xi} d_{\mathcal{A}}\eta + ii_{\pi^\sharp \eta} d_{\mathcal{A}}\xi + i\pi(\xi,\eta) \\
&= ii_{\pi^\sharp \xi} \partial_{\mathcal{A}}\eta + ii_{\pi^\sharp \eta} \overline{\partial}_{\mathcal{A}}\xi.
\end{aligned}$$

We are now ready to calculate (3.9) on the mixed terms:

$$\begin{aligned}
C_{\pi,\phi}(\xi,\eta) &= [\xi,\eta]_{\pi_\phi} - \left[\phi^*\xi, \eta\right]_\pi - \left[\xi, \phi^*\eta\right]_\pi + \phi^* [\xi,\eta]_\pi \\
&= i(i_{\pi^\sharp \xi} \partial_{\mathcal{A}}\eta + i_{\pi^\sharp \eta} \overline{\partial}_{\mathcal{A}}\xi) - i [\xi,\eta]_\pi + i [\xi,\eta]_\pi - i(i_{\pi^\sharp \xi} \partial_{\mathcal{A}}\eta + i_{\pi^\sharp \eta} \overline{\partial}_{\mathcal{A}}\xi) \\
&= 0.
\end{aligned}$$

Thus (3.9) is satisfied for all $\xi, \eta \in \Gamma(\mathcal{A}^*_{\mathbb{C}})$ and the proposition is proven. $\square$

Example 3.17. Let $(\mathcal{A}, \pi, \phi)$ be a Poisson Nijenhuis structure with $\phi^2 = -\mathrm{Id}$. The Poisson bivector field arising from the generalized complex structure J_ϕ is given by

$$\begin{aligned}
\pi(\xi, \eta) &= \frac{1}{2}\left\langle J \circ \rho^*(\xi), \rho^*(\eta)\right\rangle \\
&= \frac{1}{2}\left\langle \phi\big(\rho^*_{\mathcal{A}}(\xi)\big) - \phi^*\big(\rho_{\mathcal{A}}(\xi)\big), \rho^*_{\mathcal{A}}(\eta) + \rho_{\mathcal{A}}(\eta)\right\rangle \\
&= \frac{1}{2}\left\langle \phi \circ \pi^\sharp(\xi) - \phi^*(\xi), \pi^\sharp(\eta) + \eta\right\rangle \\
&= \frac{1}{2}\big(\pi(\xi, \phi^*\eta) - \pi(\eta, \phi^*\xi)\big) \\
&= \pi_\phi(\xi, \eta).
\end{aligned}$$

Chapter 4

Derived Brackets and Spinors

The aim of this chapter is to characterize generalized complex structures using spinors. Gualtieri showed that a usual generalized complex structure is equivalent to a pure spinor line bundle $\Lambda \subseteq \Omega^{\bullet}(M)$ which satisfied additional conditions. In this chapter we will do the same thing for generalized complex structures on the Courant algebroid of a Lie bialgebroid. See Example 2.25 for the definition of this Courant algebroid.

The main aim of this chapter is, using spinors, to characterize when an almost generalized complex structure is in fact a generalized complex structure. This result is a generalization of a result of Gualtieri's [14]. We also use a different approach, namely generating operators and derived brackets. Derived brackets were motivated by an example from [5]: the Courant bracket. This example is detailed in Section 4.5. The definition of derived brackets appears in [20] and many examples and applications of derived brackets appear in [22], and the papers cited there.

This chapter starts with some introductory material on graded algebra, Clifford algebras, and spinors. We then describe almost generalized complex structures using spinors. After these preliminary ideas we provide the definitions and some examples of derived brackets and generating operators. In particular we find the generating operator for the Courant algebroid of a Lie bialgebroid. The chapter finishes with the application of generating operators to answer our main question: when is an almost generalized complex structure a generalized complex structure.

4.1 Graded algebras and derivations

This section is a brief diversion to define some of the graded structures we will use when discussing derived brackets and generating operators.

Definition 4.1. An algebra, $\mathscr{A}$ is *$\mathbb{Z}$-graded* if there exists a collections of subsets $\mathscr{A}_i \subseteq \mathscr{A}$, where $i \in \mathbb{Z}$, such that $\mathscr{A} = \bigoplus \mathscr{A}_i$, each $\mathscr{A}_i$ is a linear subspace of $\mathscr{A}$, and if $A \in \mathscr{A}_a$ and $B \in \mathscr{A}_b$ then $AB \in \mathscr{A}_{a+b}$. The elements of $\bigcup \mathscr{A}_i$ are called *homogeneous* and this set is denoted by $\mathscr{A}_{\mathrm{hom}}$.

We will often consider algebras that are graded by $\mathbb{Z}_2$. The definition is identical to the usual $\mathbb{Z}$ grading. If using a $\mathbb{Z}_2$ grading we will refer to elements as being even (if they are contained in $\mathscr{A}_0$), and odd (if they are contained in $\mathscr{A}_1$). In fact any $\mathbb{Z}$-graded algebra can be viewed as a $\mathbb{Z}_2$-graded by taking the even elements to be $\bigcup \mathscr{A}_{2i}$ and the odd elements to be $\bigcup \mathscr{A}_{2i-1}$. For much of what follows this $\mathbb{Z}_2$ grading is sufficient. In fact when the Clifford module is introduced this grading is the one that is necessary.

There are many examples of graded algebras. For a smooth manifold M, the spaces $\Omega^\bullet(M)$ and $\mathfrak{X}^\bullet(M)$ are both graded algebras. The product in both these examples is the wedge. For a Lie algebroid $\mathcal{A}$, the spaces $\Gamma(\wedge^\bullet \mathcal{A})$ and $\Gamma(\wedge^\bullet \mathcal{A}^*)$ are also graded algebras.

Definition 4.2. An *operator of degree* k on a graded algebra is a collection of linear maps $\phi_i : \mathscr{A}_i \to \mathscr{A}_{i+k}$.

This collection of maps can be extended linearly to give a map $\phi : \mathscr{A} \to \mathscr{A}$. We will often just use ϕ to denote such an operator. The exterior differentials on $\Omega^\bullet(M)$ and $\Gamma(\wedge^\bullet \mathcal{A}^*)$ are examples of operators of degree 1. The space of operators of all degrees on a graded algebra $\mathscr{A}$ will be denoted by $\operatorname{op} \mathscr{A}$. This space has a natural grading.

Let $\mathscr{A}$ be a graded algebra, and let ϕ and ψ be two operators on $\mathscr{A}$, of degrees k and l. The *graded commutator* is defined as

$$[\![\phi, \psi]\!] = \phi \circ \psi - (-1)^{kl} \psi \circ \phi, \tag{4.1}$$

and denoted by $[\![\phi, \psi]\!]$. This bracket will give an operator of degree $k + l$. Thus we have a graded algebra structure on $\operatorname{op} \mathscr{A}$. The graded commutator is graded skew-symmetric; i.e.

$$[\phi, \psi] = (-1)^{kl} [\psi, \phi],$$

where ϕ and ψ are of order k and l. It also satisfies the graded Jacobi identity; i.e.

$$\big[\phi, [\psi, \delta]\big] = \big[[\phi, \psi], \delta\big] + (-1)^{kl} \big[\psi, [\phi, \delta]\big]. \tag{4.2}$$

Definition 4.3. Let ϕ be an operator on $\mathscr{A}$. This operator of degree k is called a *derivation* if

$$\phi(A \wedge B) = \phi(A) \wedge B + (-1)^{ka} A \wedge \phi(B)$$

for all $A \in \mathscr{A}_a$ and $B \in \mathscr{A}$. A *Lie derivative* is a derivation of degree 0.

The differentials on $\Omega^\bullet(M)$ and $\Gamma(\wedge^\bullet \mathcal{A}^*)$ are examples of derivations. While the usual Lie derivative of a manifold is an example Lie derivative. The following proposition will be used later when derivatives are combined.

Proposition 4.4. *Let d_1 be a derivative of degree k, and let d_2 be a derivative with degree l. Then their commutator,*

$$[\![d_1, d_2]\!] = d_1 \circ d_2 - (-1)^{kl} d_2 \circ d_1,$$

is a derivative of order $k + l$.

Proof. By definition $[\![d_1, d_2]\!]$ will be an operator of order $k+l$. Now if $A \in \mathscr{A}_a$ and $B \in \mathscr{A}_b$ then

$$\begin{aligned}
[\![d_1, d_2]\!]\,(A \wedge B) &= d_1(d_2A \wedge B + (-1)^{la} A \wedge d_oB) \\
&\quad - (-1)^{kl} d_2(d_1A \wedge B + (-1)^{ka} A \wedge d_1B) \\
&= d_1d_2A \wedge B + (-1)^{k(a+l)} d_2A \wedge d_1B + (-1)^{la} d_1A \wedge d_2B \\
&\quad + (-1)^{la+ka} A \wedge d_1d_2B - (-1)^{kl} d_2d_1A \wedge B - (-1)^{kl+l(k+a)} d_1A \wedge d_2B \\
&\quad - (-1)^{kl+ka} d_2A \wedge d_1B - (-1)^{kl+la+ka} A \wedge d_2d_1B \\
&= d_1d_2A \wedge B + (-1)^{la+ka} A \wedge d_1d_2B \\
&\quad - (-1)^{kl} d_2d_1A \wedge B - (-1)^{kl+la+ka} A \wedge d_2d_1B \\
&= [\![d_1, d_2]\!]\,(A) \wedge B + (-1)^{(k+l)a} A \wedge [\![d_1, d_2]\!]\,(B)
\end{aligned}$$

Thus $[\![d_1, d_2]\!]$ is a derivative of order $k+l$. □

4.2 Clifford Algebras

In this section we will define Clifford algebras, and give – as an example – the Clifford algebra that will appear throughout this chapter. We will start with vector spaces and then discuss vector bundles.

Let W be a vector space over $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. A *quadratic form* on W is a map $Q : W \to \mathbb{F}$ such that

1. $Q(\alpha w) = \alpha^2 Q(w)$ for all $w \in W$ and $\alpha \in \mathbb{F}$, and
2. the form defined by $(v, w) \mapsto Q(v+w) - Q(v) - Q(w)$ is bilinear.

Let $\mathcal{T}\,(W)$ denote the tensor algebra of W, and consider the ideal I generated by elements of the form $v \otimes v - Q(v)$. The *Clifford algebra* of W is denoted by $Cl\,(W)$ and defined as

$$Cl\,(W) = \mathcal{T}\,(W)\,/I.$$

Bilinear forms may also be used to define Clifford algebras. If $B : W \otimes W \to \mathbb{F}$ is a bilinear form then $Q(v) = B(v, v)$ is a quadratic form. This quadratic form can then be used to define a Clifford algebra. The bilinear form (2.17) gives an example of a Clifford algebra. Let

$$Q(w + \xi) = \xi(w), \text{ for all } w \in V \text{ and } \xi \in V^*.$$

Now, Q is a quadratic form on $V \oplus V^*$. When $V = T_mM$ this quadratic form is the same as the one induced by the bilinear form (2.17).

The tensor algebra $\mathcal{T}(W)$ has a natural $\mathbb{Z}$-grading, and the quotient we just defined does not respect this grading. However it only changes the order of an element by an even number. Thus the Clifford algebra $Cl(W)$ will inherit a $\mathbb{Z}_2$-grading from the grading on $\mathcal{T}(W)$.

For the remainder of this section we will consider $W = V \oplus V^*$ with the bilinear form (2.17), and the Clifford algebra $Cl\left(V \oplus V^*\right)$. The vector space $V \oplus V^*$ has a representation on $\wedge^\bullet(V^*)$ given by

$$(x + \xi) \cdot \lambda = i_x \omega + \xi \wedge \lambda,$$

for all $x \in V$ and $\xi \in V^*$. This representation is then extended to $\mathcal{T}\left(V \oplus V^*\right)$ preserving the algebra structure. This representation respects the ideal from above because

$$\begin{aligned}
\big((x + \xi) \otimes (x + \xi)\big) \cdot \lambda &= (x + \xi) \cdot \big((x + \xi) \cdot \lambda\big) \\
&= (x + \xi) \cdot (i_x \lambda + \xi \wedge \lambda) \\
&= i_x i_x \lambda + \xi \wedge i_x \lambda + i_x(\xi \wedge \lambda) + \xi \wedge \xi \wedge \lambda \\
&= \xi \wedge i_x \lambda + \xi(x) \wedge \lambda - \xi \wedge i_x \lambda \\
&= \langle x + \xi, x + \xi \rangle \cdot \lambda.
\end{aligned}$$

Thus we have a representation of $Cl\left(V \oplus V^*\right)$ on $\wedge^\bullet(V^*)$. The elements of $\wedge^\bullet(V^*)$ are called *spinors*, and the representation of $Cl\left(V \oplus V^*\right)$ is called the *spin representation*. The spin representation also respects the $\mathbb{Z}_2$-grading of $\wedge^\bullet(V^*)$. In that, if $c \in Cl\left(V \oplus V^*\right)$ is of order i, and $\omega \in \wedge^\bullet(V^*)$ of order j then $c \cdot \omega$ is of order $i + j$. Here $i, j \in \{0, 1\}$ and the addition is modulo 2.

The spin representation maps $Cl\left(V \oplus V^*\right)$ into the space of operators on $\wedge^\bullet(V^*)$. This map is injective, and we will often identify the Clifford algebra element with its image in the space of operators.

To each spinor $\lambda \in \wedge^\bullet(V^*)$ there is an associated subspace of $V \oplus V^*$, called the *null space*. It is defined as

$$L_\lambda = \left\{ A \in V \oplus V^* : A \cdot \lambda = 0 \right\}.$$

This subspace is always an isotropic subspace. However it is not necessarily maximal. If L_λ is maximal then the spinor λ is called a *pure spinor*. There is also a converse to this statement,which will be addressed later in this chapter.

There is one final notion from the theory of Clifford algebras that we will use when describing generalized complex structures. Let ι denote the anti-isomorphism of $\wedge^\bullet(V^*)$

given by $\iota(\xi_1 \wedge \xi_2 \wedge \dots \xi_n) = \xi_n \wedge \dots \xi_2 \wedge \xi_1$ for $\xi_i \in V^*$. The *Mukai pairing* of two spinors $\lambda, \mu \in \wedge^\bullet(V^*)$ is defined as

$$(\lambda, \mu)_{\text{muk}} = (\iota(\lambda) \wedge \mu)_{\text{top}}.$$

Here $(.)_{\text{top}}$ denotes taking the top dimensional component of the form inside the brackets. This pairing is used in the following theorem to classify when the isotropic subspaces associated to pure spinors intersect. The following theorem of [6] relates the Mukai pairing to the intersections of isotropic subspaces.

Theorem 4.5. *Let $\lambda, \mu \in \wedge^\bullet(V^*)$ be two spinors with associated isotropic subspaces $L_\lambda, L_\mu \subseteq V \oplus V^*$. The intersection of L_λ and L_μ is trivial if and only if $(\lambda, \mu)_{\text{muk}} \neq 0$.*

All the definitions in this section can be extended to vector bundles by using the vector space definition on each fibre. If $\mathcal{B}$ is a vector bundle then we will call $Cl\left(\mathcal{B} \oplus \mathcal{B}^*\right))$ the *Clifford bundle*. The Clifford bundle now acts on $\wedge^\bullet(\mathcal{B}^*)$. This action is a smooth action, and by definition preserves the fibres. Thus we will often consider smooth sections of $Cl\left(\mathcal{B} \oplus \mathcal{B}^*\right)$ acting on smooth sections of $\wedge^\bullet(\mathcal{B}^*)$. In the vector bundle context a spinor will be taken to be an element of $\Gamma(\wedge^\bullet\mathcal{B}^*)$. Now, each spinor $\lambda \in \Gamma(\wedge^\bullet\mathcal{B}^*)$ gives a possibly singular isotropic subbundle $L_\lambda \subseteq \mathcal{B} \oplus \mathcal{B}^*$, and the spinor is called *pure* if each fibre is maximal.

4.3 Spinors and Almost Generalized Complex Structures

In this section we will relate the previous section to generalized complex structures. We will start by letting $(\mathcal{A}, \mathcal{A}^*)$ be a Lie bialgebroid. The following proposition is just Theorem 4.5 in this context.

Proposition 4.6. *Consider a spinor $\lambda \in \Gamma(\wedge^\bullet\mathcal{A}^*_{\mathbb{C}})$. Its null space $L_\lambda \subseteq \mathcal{A}_{\mathbb{C}} \oplus \mathcal{A}^*_{\mathbb{C}}$ is an almost generalized complex structure if λ is pure, and $\left(\lambda, \overline{\lambda}\right)_{\text{muk}} \neq 0$.*

The main issue remaining is the integrability condition. The rest of this chapter will be devoted to formulating this condition. Before discussing this issue, however we will obtain a form for the spinor and use this to prove the converse to the previous proposition. Once again we will first consider a vector space V, and then generalize to vector bundles.

Let $L \subseteq V_{\mathbb{C}} \oplus V^*_{\mathbb{C}}$ be a maximal isotropic subspace with $L \cap \overline{L} = \{0\}$. This maximal isotropic subspace is equivalent to a pair (E, ε), where $E \subseteq V_{\mathbb{C}}$ is a subspace and $\varepsilon \in \wedge^2(E^*)$. See Section 2.7 for the details of this construction. Given this notation, Chevally showed [6] that the spinor associated to L is

$$\lambda = c \det(E^o) \wedge \exp(\varepsilon). \tag{4.3}$$

Here c is any nonzero constant, and E^o denotes the subspace of $V_{\mathbb{C}}^*$ consisting of all elements which have $\xi(e) = 0$ for all $e \in E$. Now, $\det(E^o)$ denotes a volume element of E^o. One issue of the above should be noted: $\exp(\varepsilon)$ is an element of $\wedge^\bullet(E^*)$, not $\wedge^\bullet(V_{\mathbb{C}}^*)$ as required. This problem can be addressed by taking $\exp(\tilde{\varepsilon})$, where $\tilde{\varepsilon} \in \wedge^2(V_{\mathbb{C}}^*)$ such that $\tilde{\varepsilon}|_E = \varepsilon$. The choice of extension does not change λ, because of the wedge with $\det(E^o)$. The definition of λ is unique, up to the choice of c. The converse is also true: any pure spinor must be of the form (4.3), and $L_\lambda \cap \overline{L_\lambda} = \{0\}$ if and only if $\left(\lambda, \overline{\lambda}\right)_{\text{muk}} \neq 0$.

Now, consider the vector bundle case, and let $L \subseteq \mathcal{A}_{\mathbb{C}} \oplus \mathcal{A}_{\mathbb{C}}^*$ be an almost generalized complex structure. So L is a maximal isotropic subbundle and $L \cap \overline{L} = \{0\}$. Using the notation of the previous paragraph $E \subseteq \mathcal{A}_{\mathbb{C}}$, and $\varepsilon \in \Gamma(\wedge^2 E^*)$. Once again there is the same issue regarding ε, but this can be solved by considering an extension $\tilde{\varepsilon} \in \Gamma(\wedge^2 \mathcal{A}_{\mathbb{C}}^*)$. This extension, like before, does not change λ. The expression for λ is unique, up to multiplication by a nonzero function $f \in C^\infty(M, \mathbb{C})$. Thus the almost generalized complex structure is uniquely determined by a line subbundle of $\wedge^\bullet(\mathcal{A}_{\mathbb{C}}^*)$, called the *spinor line bundle*. These results are summarized in the following proposition.

Proposition 4.7. *Let $L \subseteq \mathcal{A}_{\mathbb{C}} \oplus \mathcal{A}_{\mathbb{C}}^*$ be an almost generalized complex structure with $L = L(E, \varepsilon)$, where $E \subseteq \mathcal{A}_{\mathbb{C}}$ and $\varepsilon \in \Gamma(\wedge^2 E^*)$. This almost generalized complex structure is equivalent to the spinor line bundle*

$$\Lambda = \left\{ c \det(E^o) \wedge \exp(\tilde{\varepsilon}) : c \in \mathbb{C} \right\}.$$

We will finish this section by summarizing the three different ways of defining almost generalized complex structures on Lie bialgebroids.

Definition 4.8. Let $(\mathcal{A}, \mathcal{A}^*)$ be Lie bialgebroids. An *almost generalized complex structure* on $\mathcal{A} \oplus \mathcal{A}^*$ is given by any of the following equivalent sets of data.

1. A smooth bundle map $J : \mathcal{A} \oplus \mathcal{A}^* \to \mathcal{A} \oplus \mathcal{A}^*$ such that $J^2 = -\mathrm{Id}$, and $J^* J = \mathrm{Id}$.
2. A maximally isotropic subbundle $L \subseteq \mathcal{A}_{\mathbb{C}} \oplus \mathcal{A}_{\mathbb{C}}^*$ such that $L \cap \overline{L} = \{0\}$.
3. A pure spinor line bundle $\Lambda \subseteq \Gamma(\wedge^\bullet \mathcal{A}_{\mathbb{C}}^*)$ such that if $\lambda \in \Lambda_m$ is nonzero then $\left(\lambda, \overline{\lambda}\right)_{\text{muk}} \neq 0$. Moreover Λ can be written as

$$\Lambda = \left\{ c \det(E^o) \wedge \exp(\tilde{\varepsilon}) : c \in \mathbb{C} \right\}, \tag{4.4}$$

4.4 Generating Operators and Derived Brackets

Let $\mathcal{A}$ be a vector bundle and let $\mathcal{E} = \mathcal{A} \oplus \mathcal{A}^*$. In this section, and the remaining sections, the Courant algebroid and its image within the Clifford algebra will not be distinguished. We will also identify the previous two spaces with their image within the space of operators on $\Gamma(\wedge^\bullet \mathcal{A}^*)$

Definition 4.9. [2] A *generating operator* is a differential operator $\mathcal{D} : \Gamma(\wedge^\bullet \mathcal{A}^*) \to \Gamma(\wedge^\bullet \mathcal{A}^*)$ with odd degree such that the following hold:

1. $[\![\mathcal{D}, f]\!] \in \Gamma(\mathcal{E})$ for all $f \in C^\infty(M)$.
2. $[\![\, [\![\mathcal{D}, A]\!] \,, B]\!] \in \Gamma(\mathcal{E})$ for all $A, B \in \Gamma(\mathcal{E})$.
3. $\left[\!\!\left[\mathcal{D}^2, A\right]\!\!\right] \in \Gamma(\mathcal{E})$ for all $A \in \Gamma(\mathcal{E})$.

Recall the commutator bracket (4.1).

We will now prove that a generating operator gives a Courant algebroid structure.

Theorem 4.10. *[2] A generating operator $\mathcal{D} : \Gamma(\wedge^\bullet \mathcal{A}^*) \to \Gamma(\wedge^\bullet \mathcal{A}^*)$ leads to a Courant algebroid structure on $\mathcal{E} = \mathcal{A} \oplus \mathcal{A}^*$.*

Proof. We will use the alternate description of Courant algebroids described in Section 2.6. The bilinear operator on $\Gamma(\mathcal{E})$ is defined as

$$A \circ B = [\![\, [\![\mathcal{D}, A]\!] \,, B]\!] \tag{4.5}$$

for all $A, B \in \Gamma(\mathcal{E})$. While the anchor map, $\rho : \Gamma(\mathcal{E}) \to TM$ is defined by

$$\rho(A) \cdot f = \langle [\![\mathcal{D}, f]\!] \,, A \rangle = [\![\, [\![\mathcal{D}, f]\!] \,, A]\!] \tag{4.6}$$

for all $f \in C^\infty(M)$ and $A \in \Gamma(\mathcal{E})$. The graded Jacobi identity gives

$$[\![\, [\![\mathcal{D}, f]\!] \,, A]\!] = [\![\mathcal{D}, [\![f, A]\!] \,]\!] - [\![f, [\![\mathcal{D}, A]\!] \,]\!] = [\![\, [\![\mathcal{D}, A]\!] \,, f]\!] .$$

Thus the following can also be used to define the anchor map,

$$\rho(A) \cdot f = [\![\, [\![\mathcal{D}, A]\!] \,, f]\!] . \tag{4.7}$$

We first need to show that $\rho(A) \in TM$. It suffices to show that $\rho(A)$ satisfies the Leibniz rule. Consider two functions $f, g \in C^\infty(M)$:

$$\begin{aligned}
\rho(A) \cdot (fg) &= [\![\, [\![\mathcal{D}, A]\!] \,, fg]\!] = [\![\mathcal{D}, A]\!] \, fg - fg \, [\![\mathcal{D}, A]\!] \\
&= [\![\mathcal{D}, A]\!] \, fg - f \, [\![\mathcal{D}, A]\!] \, g + f \, [\![\mathcal{D}, A]\!] \, g - fg \, [\![\mathcal{A}, A]\!] \\
&= [\![\, [\![\mathcal{D}, A]\!] \,, f]\!] g + f [\![\, [\![\mathcal{D}, A]\!] \,, g]\!] \\
&= \big(\rho(A) \cdot f\big) g + f \big(\rho(A) \cdot g\big).
\end{aligned}$$

The three terms from (2.21) are

$$
\begin{aligned}
A \circ (B \circ C) &= \llbracket \llbracket \mathcal{D}, A \rrbracket , (B \circ C) \rrbracket \\
&= \Big\llbracket \llbracket \mathcal{D}, A \rrbracket , \llbracket \llbracket \mathcal{D}, B \rrbracket , C \rrbracket \Big\rrbracket , \\
(A \circ B) \circ C &= \llbracket \llbracket \mathcal{D}, (A \circ B) \rrbracket , C \rrbracket \\
&= \Big\llbracket \Big\llbracket \mathcal{D}, \llbracket \llbracket \mathcal{D}, A \rrbracket , B \rrbracket \Big\rrbracket , C \Big\rrbracket \\
&= \Big\llbracket \Big\llbracket \llbracket \mathcal{D}, \llbracket \mathcal{D}, A \rrbracket \rrbracket , B \Big\rrbracket , C \Big\rrbracket + \Big\llbracket \llbracket \llbracket \mathcal{D}, A \rrbracket , \llbracket \mathcal{D}, B \rrbracket \rrbracket , C \Big\rrbracket \\
&= \Big\llbracket \llbracket \llbracket \mathcal{D}, A \rrbracket , \llbracket \mathcal{D}, B \rrbracket \rrbracket , C \Big\rrbracket , \\
B \circ (A \circ C) &= \llbracket \llbracket \mathcal{D}, B \rrbracket , (A \circ C) \rrbracket \\
&= \Big\llbracket \llbracket \mathcal{D}, B \rrbracket , \llbracket \llbracket \mathcal{D}, A \rrbracket , C \rrbracket \Big\rrbracket .
\end{aligned}
$$

For the second of these we have used the graded Jacobi identity. With these expressions (2.21) is now just the graded Jacobi identity with $\llbracket \mathcal{D}, A \rrbracket$, $\llbracket \mathcal{D}, B \rrbracket$ and C.

Now, for (2.22) we start with the following calculations:

$$
\begin{aligned}
\rho(A) \cdot \big(\rho(B) \cdot f\big) &= \rho(A) \cdot \llbracket \llbracket \mathcal{D}, f \rrbracket , B \rrbracket \\
&= \Big\llbracket \llbracket \mathcal{D}, A \rrbracket , \llbracket \llbracket \mathcal{D}, f \rrbracket , B \rrbracket \Big\rrbracket \\
\rho(B) \cdot \big(\rho(A) \cdot f\big) &= \rho(B) \cdot \llbracket \llbracket \mathcal{D}, f \rrbracket , A \rrbracket \\
&= \Big\llbracket \Big\llbracket \mathcal{D}, \llbracket \llbracket \mathcal{D}, f \rrbracket , A \rrbracket \Big\rrbracket , B \Big\rrbracket \\
&= \Big\llbracket \Big\llbracket \llbracket \mathcal{D}, \llbracket \mathcal{D}, f \rrbracket \rrbracket , A \Big\rrbracket , B \Big\rrbracket - \Big\llbracket \llbracket \llbracket \mathcal{D}, f \rrbracket , \llbracket \mathcal{D}, A \rrbracket \rrbracket , B \Big\rrbracket \\
&= -\Big\llbracket \llbracket \llbracket \mathcal{D}, f \rrbracket , \llbracket \mathcal{D}, A \rrbracket \rrbracket , B \Big\rrbracket
\end{aligned}
$$

Combining these equations gives

$$
\begin{aligned}
[\rho(A), \rho(B)] \cdot f &= \rho(A) \cdot \big(\rho(B) \cdot f\big) - \rho(B) \cdot \big(\rho(A) \cdot f\big) \\
&= \Big\llbracket \llbracket \mathcal{D}, A \rrbracket , \llbracket \llbracket \mathcal{D}, f \rrbracket , B \rrbracket \Big\rrbracket + \Big\llbracket \llbracket \llbracket \mathcal{D}, f \rrbracket , \llbracket \mathcal{D}, A \rrbracket \rrbracket , B \Big\rrbracket \\
&= \Big\llbracket \llbracket \mathcal{D}, f \rrbracket , \llbracket \llbracket \mathcal{D}, A \rrbracket , B \rrbracket \Big\rrbracket \\
&= \rho(A \circ B) \cdot f
\end{aligned}
$$

Now, for all $\omega \in \Gamma(\wedge^\bullet \mathcal{A}^*)$:

$$
\begin{aligned}
A \circ (fB)(\omega) = & \left[\!\left[[\![\mathcal{D}, A]\!], fB \right]\!\right](\omega) \\
& = [\![\mathcal{D}, A]\!] \left(fB \cdot \omega - fB \cdot [\![\mathcal{D}, A]\!] (\omega) \right. \\
& = \mathcal{D}(A \cdot fB \cdot \omega) + A \cdot \mathcal{D}(fB \cdot \omega) - fB \cdot [\![\mathcal{D}, A]\!] (\omega) \\
& = \mathcal{D}(fA \cdot B \cdot \omega) + A \cdot \mathcal{D}(fB \cdot \omega) - fB \cdot [\![\mathcal{D}, A]\!] (\omega) \\
& = \mathcal{D}f \wedge A \cdot B \cdot \omega + f\mathcal{D}(A \cdot B \cdot \omega) + A \cdot \mathcal{D}f \wedge B \cdot \omega \\
& \qquad + fA \cdot \mathcal{D}(B \cdot \omega) - fB \cdot [\![\mathcal{D}, A]\!] (\omega) \\
& = [\![\mathcal{D}f, A]\!] (B \cdot \omega) + f [\![\mathcal{D}, A]\!] (B \cdot \omega) - fB \cdot [\![\mathcal{D}, A]\!] (\omega) \\
& = [\![\mathcal{D}f, A]\!] B \cdot \omega + f \left[\!\left[[\![\mathcal{D}, A]\!], B \right]\!\right](\omega)
\end{aligned}
$$

To show (2.24) is true, consider the following application of the Jacobi identity:

$$
\left[\!\left[\mathcal{D}, [\![A, A]\!] \right]\!\right] = \left[\!\left[[\![\mathcal{D}, A]\!], A \right]\!\right] - \left[\!\left[A, [\![\mathcal{D}, A]\!] \right]\!\right] = 2 \left[\!\left[[\![\mathcal{D}, A]\!], A \right]\!\right].
$$

Now, by definition

$$
A \circ A = \left[\!\left[[\![\mathcal{D}, A]\!], A \right]\!\right] = \tfrac{1}{2} \left[\!\left[\mathcal{D}, [\![A, A]\!] \right]\!\right] = [\![\mathcal{D}, \langle A, A \rangle]\!] .
$$

Finally we will calculate (2.25) using the graded Jacobi identity:

$$
\begin{aligned}
\rho(A) \cdot \langle B, C \rangle = & \left[\!\left[[\![\mathcal{D}, A]\!], \langle B, C \rangle \right]\!\right] \\
& = \tfrac{1}{2} \left[\!\left[[\![\mathcal{D}, A]\!], [\![B, C]\!] \right]\!\right] \\
& = \tfrac{1}{2} \left(\left[\!\left[\left[\!\left[[\![\mathcal{D}, A]\!], B \right]\!\right], C \right]\!\right] + \left[\!\left[B, \left[\!\left[[\![\mathcal{D}, A]\!], C \right]\!\right] \right]\!\right] \right) \\
& = \tfrac{1}{2} [\![(A \circ B), C]\!] + \tfrac{1}{2} [\![B, (A \circ C)]\!] \\
& = \langle (A \circ B), C \rangle + \langle B, (A \circ C) \rangle .
\end{aligned}
$$

□

A generating operator defines a skewsymmetric bracket on $\Gamma(\mathcal{E})$ by defining

$$
[A, B]_d \cdot \omega = \frac{1}{2} \Big(\left[\!\left[[\![\mathcal{D}, A]\!], B \right]\!\right] - \left[\!\left[[\![\mathcal{D}, B]\!], A \right]\!\right] \Big) \cdot \omega \tag{4.8}
$$

for all $\omega \in \Gamma(\wedge^\bullet \mathcal{A}^*)$. Brackets which admit a generating operator are called *derived brackets*. A generating operator that defines a Courant algebroid, where the bracket is a derived bracket and the anchor is defined by (4.6), will be called a *derived Courant algebroid*. Some examples of derived brackets and Courant algebroids will appear in the next section. Several realizations of brackets appearing in differential geometry can be found in Kosmann-Schwarzbach's paper [22]. Before giving these examples we will prove the following simple proposition.

Proposition 4.11. *Let* $\mathcal{D}_1 : \Gamma(\wedge^\bullet \mathcal{A}^*) \to \Gamma(\wedge^\bullet \mathcal{A}^*)$ *and* $\mathcal{D}_2 : \Gamma(\wedge^\bullet \mathcal{A}^*) \to \Gamma(\wedge^\bullet \mathcal{A}^*)$ *be two generating operators. If* $[\![\mathcal{D}_1 \circ \mathcal{D}_2 + \mathcal{D}_2 \circ \mathcal{D}_1, A]\!] \in \Gamma(\mathcal{E})$ *for all* $A \in \Gamma(\mathcal{E})$ *then* $\mathcal{D}_1 + \mathcal{D}_2$ *is also a generating operator. Also* $\mathcal{D}_1 - \mathcal{D}_2$ *will be a generating operator.*

Proof. Because the commutator is linear $\mathcal{D}_1 + \mathcal{D}_2$ will satisfy conditions (1) and (2) of Definition 4.9. Lastly, because $(\mathcal{D}_1 + \mathcal{D}_2)^2 = \mathcal{D}_1^2 + (\mathcal{D}_1 \circ \mathcal{D}_2 + \mathcal{D}_2 \circ \mathcal{D}_1) + \mathcal{D}_2^2$, the assumption implies condition (3) of Definition 4.9. The fact that $\mathcal{D}_1 - \mathcal{D}_2$ is also a generating operator is proved identically. □

4.5 The Courant bracket as a derived bracket

As an example of the derived bracket construction we will construct the usual Courant bracket on $\mathcal{A} \oplus \mathcal{A}^*$, where $\mathcal{A}$ is a Lie bialgebroid and $\mathcal{A}^*$ is the zero Lie algebroid. See Example 2.26. While the Courant bracket is usually defined on $TM \oplus T^*M$, we will consider the Lie algebroid picture.

Let $\mathcal{A}$ be a Lie algebroid and consider the zero Lie algebroid structure on $\mathcal{A}^*$. As noted previously, this gives an example of a Lie bialgebroid. We will consider the vector bundle $\mathcal{A} \oplus \mathcal{A}^*$.

Now consider the space of all forms on $\mathcal{A}$, denoted by $\Gamma(\wedge^\bullet \mathcal{A}^*)$.

Consider the action of $\Gamma(\mathcal{A} \oplus \mathcal{A}^*)$ on $\Gamma(\wedge^\bullet \mathcal{A}^*)$ defined by

$$(X + \xi) \cdot \omega = i_X \omega + \xi \wedge \omega$$

for all $X \in \Gamma(\mathcal{A})$, $\xi \in \Gamma(\mathcal{A}^*)$, $\omega \in \Gamma(\wedge^\bullet \mathcal{A}^*)$. This action is linear, and it is very similar to the one defined in Section 4.2. In what follows we will identify $\Gamma(\mathcal{A} \oplus \mathcal{A}^*)$ with its image in the operators on $\Gamma(\wedge^\bullet \mathcal{A}^*)$.

Now, let $d_{\mathcal{A}} : \Gamma(\wedge^k \mathcal{A}^*) \to \Gamma(\wedge^{k+1} \mathcal{A}^*)$ denote the differential induced by the Lie algebroid structure on $\mathcal{A}$ (see Definition 2.2). This map is also a linear operator on $\Gamma(\wedge^\bullet \mathcal{A}^*)$. Now, the Leibniz identity for $d_{\mathcal{A}}$ implies that

$$[\![d_{\mathcal{A}}, \xi]\!]\,(\omega) = d_{\mathcal{A}}\xi \wedge \omega,$$

for all $\xi \in \Gamma(\mathcal{A}^*)$ and $\omega \in \Gamma(\wedge^\bullet \mathcal{A}^*)$. Thus $[\![d_\mathcal{A}, \xi]\!] = d_\mathcal{A}\xi$. Also, Cartan's formula gives

$$[\![d_\mathcal{A}, X]\!]\,(\omega) = \mathcal{L}_X\omega,$$

for all $X \in \Gamma(\mathcal{A})$ and $\omega \in \Gamma(\wedge^\bullet \mathcal{A}^*)$. So, as operators, $[\![d_\mathcal{A}, X]\!] = \mathcal{L}_X$.

Proposition 4.12. *If $X, Y \in \Gamma(\mathcal{A})$ and $\xi, \eta \in \Gamma(\mathcal{A}^*)$ then the following hold.*

$$[\![\,[\![d_\mathcal{A}, X]\!]\,, Y]\!] = [X, Y] \qquad [\![\,[\![d_\mathcal{A}, X]\!]\,, \xi]\!] = \mathcal{L}_X\xi$$

$$[\![\,[\![d_\mathcal{A}, \xi]\!]\,, X]\!] = -\mathcal{L}_X\xi + d\big(\xi(X)\big) \qquad [\![\,[\![d_\mathcal{A}, \xi]\!]\,, \eta]\!] = 0$$

Proof. These four identities follow from the observations above. The first one also uses the formula

$$i_{[X,Y]} = [\![\mathcal{L}_X, i_Y]\!]\,.$$

□

Proposition 4.13. *The differential $d_\mathcal{A}$ is a generating operator, and its derived bracket has the same formula as the usual Courant bracket* (2.16)*.*

Proof. We first show that $d_\mathcal{A}$ is a generating operator. By definition

$$[\![d_\mathcal{A}, f]\!]\,(\omega) = d_\mathcal{A}(f\omega) - f d_\mathcal{A}\omega = d_\mathcal{A}f \wedge \omega,$$

and thus $[\![d_\mathcal{A}, f]\!] = d_\mathcal{A}f \in \Gamma(\mathcal{A}^*)$. The previous proposition shows that

$$[\![\,[\![d_\mathcal{A}, X+\xi]\!]\,, Y+\eta]\!] \in \Gamma(\mathcal{A} \oplus \mathcal{A}^*),$$

for all $X, Y \in \Gamma(\mathcal{A})$ and $\xi, \eta \in \Gamma(\mathcal{A}^*)$. Because $d_\mathcal{A}^2 = 0$, $\left[\!\!\left[d_\mathcal{A}^2, X+\xi\right]\!\!\right] = 0$ for all $X + \xi \in \Gamma(M \oplus M^*)$. Thus d is a generating operator. Lastly, it is a very simple calculation to show that the usual Courant bracket (2.16) arises when the equations of Proposition 4.12 are substituted into (4.8). □

Example 4.14. A closely related example mirrors the *twisted Courant bracket* Example 2.24. Consider a closed form $\Omega \in \Gamma(\wedge^3 \mathcal{A}^*)$. It follows that

$$d_\Omega = d + \Omega$$

is a generating operator, and the bracket it defines has the same formula as the twisted Courant bracket (2.19).

4.6 The Operator ∂

Now return to the case of an arbitrary Lie bialgebroid. We first assume that there exists some nowhere vanishing element $\nu \in \Gamma(\wedge^n \mathcal{A}^*)$, where n is the rank of $\mathcal{A}$. This elements is called a *volume form*. With this every $\omega \in \Gamma(\wedge^k \mathcal{A}^*)$ can be written as $\omega = i_X \nu$ for some $X \in \Gamma(\wedge^{n-k} \mathcal{A})$.

Define an operator $\partial : \Gamma(\wedge^k \mathcal{A}^*) \to \Gamma(\wedge^{k-1} \mathcal{A}^*)$ by taking

$$\partial\omega = \partial i_X \nu = (-1)^{n-k} i_{d_* X} \nu.$$

This operator is not quite a derivation. The following proposition shows how it fails.

Proposition 4.15. *[42, 30] If $\omega \in \Gamma(\wedge^a \mathcal{A}^*)$ and $\varepsilon \in \Gamma(\wedge^b \mathcal{A}^*)$ then the following formula holds*

$$[\omega, \varepsilon] = (-1)^a \partial(\omega \wedge \varepsilon) - (-1)^a \partial\omega \wedge \varepsilon - \omega \wedge \partial\varepsilon.$$

Thus $\partial : \Gamma(\wedge^k \mathcal{A}^*) \to \Gamma(\wedge^{k-1} \mathcal{A}^*)$ is a BV generating operator of the Gerstenhaber algebra $(\Gamma(\wedge^\bullet \mathcal{A}^*), [.,.], \wedge)$.

Proof. The following two lemmas will be used in the proof of this proposition.

Lemma 4.16. *If $\omega \in \Gamma(\wedge^a \mathcal{A}^*)$, $\varepsilon \in \Gamma(\wedge^b \mathcal{A}^*)$ and $X \in \Gamma(\wedge^{a+b} \mathcal{A})$ then the following formula holds.*

$$i_{i_\omega X} \varepsilon = i_X(\omega \wedge \varepsilon)$$

Lemma 4.17. *If $\omega \in \Gamma(\wedge^a \mathcal{A}^*)$, $\xi \in \Gamma(\mathcal{A}^*)$, and $X \in \Gamma(\wedge^2 \mathcal{A})$ then the following holds:*

$$i_X(\xi \wedge \omega) = i_{i_\xi X} \omega + \xi \wedge i_X \omega.$$

Proof. Check this for $X = Y \wedge Z$, where $Y, Z \in \Gamma(\mathcal{A})$. The left hand side is

$$i_Z i_Y(\xi \wedge \omega) = i_Z\big(\xi(Y)\omega - \xi \wedge i_Y \omega\big) = \xi(Y) i_Z \omega - \xi(Z) i_Y \omega + \xi \wedge i_{Y \wedge Z} \omega.$$

The first term on the right hand side is

$$i_{i_\xi X} \omega = i_{\xi(Y)Z - \xi(Z)Y} \omega = \xi(Y) i_Z \omega - \xi(Z) i_Y \omega.$$

These two terms with the last term from the proposition is the left hand side. $\square$

Let $X \in \Gamma(\wedge^{a+b-1}\mathcal{A})$, and evaluate each of the terms on the right hand side:

$$\begin{aligned}
i_X\Big((-1)^a\partial(\omega\wedge\varepsilon)\Big) &= (-1)^a\Big((-1)^{a+b-1}\partial i_X(\omega\wedge\varepsilon) + i_{d_*X}(\omega\wedge\varepsilon)\Big)\\
&= -(-1)^b\partial i_X(\omega\wedge\varepsilon) + (-1)^a i_{d_*X}(\omega\wedge\varepsilon)\\
i_X(-\omega\wedge\partial\varepsilon) &= -i_{i_\omega X}\partial\varepsilon\\
&= -(-1)^{a+b-1-a}\partial i_{i_\omega X}\varepsilon - i_{d_* i_\omega X}\varepsilon\\
&= (-1)^b\partial i_{i_\omega X}\varepsilon - i_{d_* i_\omega X}\varepsilon\\
i_X\Big(-(-1)^a\partial\omega\wedge\varepsilon\Big) &= -(-1)^a\Big((-1)^{(a-1)b} i_X(\varepsilon\wedge\partial\omega)\Big)\\
&= (-1)^{a-b+ab+1} i_{i_\varepsilon X}\partial\omega\\
&= (-1)^{a-b+ab+1}\Big((-1)^{a+b-1-b}\partial i_{i_\varepsilon X}\omega + i_{d_* i_\varepsilon X}\omega\Big)\\
&= (-1)^{-b+ab}\partial i_{i_\varepsilon X}\omega - (-1)^{a+ab-b} i_{d_* i_\varepsilon X}\omega
\end{aligned}$$

Combining these three together gives the right hand side of the proposition:

$$\begin{aligned}
-(-1)^b\partial i_X(\omega\wedge\varepsilon)+&(-1)^a i_{d_*X}(\omega\wedge\varepsilon) + (-1)^{-b+ab}\partial i_{i_\varepsilon X}\omega\\
&- (-1)^{a+ab-b} i_{d_* i_\varepsilon X}\omega + (-1)^b\partial i_{i_\omega X}\varepsilon - i_{d_* i_\omega X}\varepsilon\\
=&(-1)^b\Big(-\partial i_X(\omega\wedge\varepsilon) + (-1)^{ab}\partial i_{i_\varepsilon X}\omega + \partial i_{i_\omega X}\varepsilon\Big)\\
&+ (-1)^a i_{d_*X}(\omega\wedge\varepsilon) - i_{d_* i_\omega X}\varepsilon - (-1)^{a+ab-b} i_{d_* i_\varepsilon X}\omega\\
=&(-1)^a i_{d_*X}(\omega\wedge\varepsilon) - i_{d_* i_\omega X}\varepsilon - (-1)^{a+ab-b} i_{d_* i_\varepsilon X}\omega.
\end{aligned}$$

Now, using (2.8), the left hand side of the proposition can be expressed as

$$i_X\,[\omega,\varepsilon] = (-1)^{ab-a-b+1} i_{d_* i_\varepsilon X}\omega - i_{d_* i_\omega X}\varepsilon + (-1)^a i_{d_*X}(\omega\wedge\varepsilon),$$

which is exactly what was found for the right hand side. □

4.7 ∂ as a generating operator

In this section we will show that ∂ is a generating operator and describe the Courant algebroid it generates.

If $f \in C^{\infty}(M)$ then $\partial f = \partial(i_X \nu) = i_{d_* X}\nu$, but $X \in \Gamma(\wedge^n \mathcal{A})$. Thus $d_* X = 0$, and $\partial f = 0$. Now, by Proposition 4.15

$$[\![\partial, f]\!] \cdot \omega = \partial(f\omega) - f\partial\omega = (\partial f)\omega + f\partial\omega + [f, \omega] - f\partial\omega = [f, \omega] \, .$$

This fact can now be used to show that ∂ satisfies the first condition of Definition 4.9:

$$[\![\partial, f]\!] \cdot \omega = \partial(f\omega) - f\partial\omega = (\partial f)\omega + f\partial\omega - f\partial\omega = (\partial f)\omega = 0.$$

The second condition is more involved and will examined later in this section, where we calculate $[\![[\![\partial, A]\!] , B]\!]$ on the basis elements of $\Gamma(\mathcal{E})$. Take an arbitrary element $\omega \in \Gamma(\wedge^\bullet \mathcal{A}^*)$ with $\omega = i_X \nu$. Then

$$\partial^2 \omega = \partial(i_{d_* X}\nu) = i_{(d_*)^2 X}\nu = 0.$$

Thus $\partial^2 = 0$, and ∂ satisfies the final condition of Definition 4.9.

Before calculating the $[\![[\![\partial, A]\!] , B]\!]$ a few calculations will be needed.

Lemma 4.18. *If $\xi \in \Gamma(\mathcal{A}^*)$ then $[\![\partial, \xi]\!] = -\mathcal{L}^*_\xi + \partial\xi$.*

Proof. Take $\omega \in \Gamma(\wedge^\bullet \mathcal{A}^*)$. Now,

$$\begin{aligned} [\![\partial, \xi]\!] (\omega) &= \partial(\xi \wedge \omega) + \xi \wedge \partial\omega \\ &= -[\xi, \omega] + \partial\xi \wedge \omega - \xi \wedge \partial\omega + \xi \wedge \partial\omega. \end{aligned}$$

The first term is just the Lie derivative (2.9), and the result follows. □

Lemma 4.19. *If $X \in \Gamma(\wedge^l \mathcal{A})$ then $[\![\partial, X]\!] = d_* X$.*

Proof. Let ω be an arbitrary element of $\Gamma(\wedge^k \mathcal{A}^*)$. By definition

$$[\![\partial, X]\!] \cdot \omega = \partial i_X \omega - (-1)^l i_X \partial\omega.$$

Let $\omega = i_Z \nu$, where $Z \in \Gamma(\wedge^{n-l} \mathcal{A})$. The first term on the right hand side is

$$\begin{aligned} \partial i_X \omega &= \partial i_X i_Z \nu = \partial i_{Z\wedge X}\nu \\ &= -(-1)^{n-l+k} i_{d_*(Z\wedge X)}\nu \\ &= -(-1)^{n-l+k} \left(i_{d_* Z\wedge X}\nu + (-1)^{n-l} i_{Z\wedge d_* X}\nu \right) \\ &= (-1)^l i_X \partial\omega + i_{d_* X} i_Z \nu \\ &= (-1)^l i_X \partial\omega + i_{d_* X}\omega. \end{aligned}$$

□

Proposition 4.20. *For all $X, Y \in \Gamma(\mathcal{A})$ and $\xi, \eta \in \Gamma(\mathcal{A}^*)$ the following formulas hold.*

$$\big[[\partial, X], Y\big] = 0 \qquad\qquad \big[[\partial, X], \xi\big] = \mathcal{L}^*_\xi X - d_*\xi(X)$$

$$\big[[\partial, \xi], X\big] = -\mathcal{L}^*_\xi X \qquad\qquad \big[[\partial, \xi], \eta\big] = -[\xi, \eta]_*$$

Proof. All of these formulas are calculations from the previous lemmas.

$$\llbracket \llbracket \partial, X\rrbracket, Y\rrbracket \cdot \omega = \llbracket d_* X, Y\rrbracket \cdot \omega = i_{d_*X} i_Y \omega - i_Y i_{d_*X}\omega = 0 \cdot \omega$$

$$\begin{aligned}
\llbracket \llbracket \partial, X\rrbracket, \xi\rrbracket \cdot \omega = & \llbracket d_* X, \xi\rrbracket \cdot \omega \\
= & i_{d_*X}(\xi \wedge \omega) - \xi \wedge i_{d_*X}\omega \\
= & i_{i_\xi d_* X}\omega + \xi \wedge i_{d_*X}\omega - \xi \wedge i_{d_*X}\omega \\
= & i_{i_\xi d_* X}\omega = \Big(\mathcal{L}^*_\xi X - d_*\big(\xi(X)\big)\Big) \cdot \omega
\end{aligned}$$

$$\begin{aligned}
\llbracket \llbracket \partial, \xi\rrbracket, X\rrbracket \cdot \omega = & \left\llbracket -\mathcal{L}^*_\xi + \partial\xi, X\right\rrbracket \cdot \omega = -\left\llbracket \mathcal{L}^*_\xi, X\right\rrbracket \cdot \omega \\
= & -\Big(\mathcal{L}^*_\xi X\Big) \cdot \omega
\end{aligned}$$

$$\begin{aligned}
\llbracket \llbracket \partial, \xi\rrbracket, \eta\rrbracket \cdot \omega = & \left\llbracket -\mathcal{L}^*_\xi + \partial\xi, \eta\right\rrbracket \cdot \omega \\
= & -\Big(\mathcal{L}^*_\xi \eta\Big) \cdot \omega = -[\xi, \eta]_* \cdot \omega
\end{aligned}$$

□

Summarizing we have the following proposition.

Proposition 4.21. *Both ∂ and $d_\mathcal{A}$ are generating operators.*

4.8 The generating operator for the Courant algebroid of $\mathcal{A} \oplus \mathcal{A}^*$

In this section we will show that the Courant bracket arising from a pair of Lie bialgebroids – see (2.20) – is a derived bracket. Consider the operator $\mathcal{D} : \Gamma(\wedge^\bullet \mathcal{A}^*) \to$

$\Gamma(\wedge^{\bullet}\mathcal{A}^*)$ defined by

$$\mathcal{D}\omega = d_{\mathcal{A}}\omega - \partial\omega. \tag{4.9}$$

Because both $d_{\mathcal{A}}$ and ∂ have odd degree $\mathcal{D}$ will also have odd degree.

As noted in Proposition 4.11, whether $\mathcal{D}$ is a generating operator will depend on $d_{\mathcal{A}} \circ \partial + \partial \circ d_{\mathcal{A}}$. This operator is of degree zero, and it will be denoted by $\mathcal{L}$.

Proposition 4.22. *The operator $\mathcal{L}$ is a Lie derivative.*

Proof. If $\omega \in \Gamma(\wedge^k\mathcal{A}^*)$ and $\varepsilon \in \Gamma(\wedge^{\bullet}\mathcal{A}^*)$ are arbitrary then

$$\begin{aligned}
\mathcal{L}(\omega \wedge \varepsilon) &= d_{\mathcal{A}}\partial(\omega \wedge \varepsilon) + \partial d_{\mathcal{A}}(\omega \wedge \varepsilon) \\
&= d_{\mathcal{A}}\big(\partial\omega \wedge \varepsilon + (-1)^k\omega \wedge \partial\varepsilon + (-1)^k\,[\omega,\varepsilon]_*\big) + \partial\big(d_{\mathcal{A}}\omega \wedge \varepsilon + (-1)^k\omega \wedge d_{\mathcal{A}}\varepsilon\big) \\
&= d_{\mathcal{A}}\partial\omega \wedge \varepsilon + (-1)^{k-1}\partial\omega \wedge d_{\mathcal{A}}\varepsilon + (-1)^k d_{\mathcal{A}}\omega \wedge \partial\varepsilon + \omega \wedge d_{\mathcal{A}}\partial\varepsilon \\
&\quad + (-1)^k d_{\mathcal{A}}\,[\omega,\varepsilon]_* + \partial d_{\mathcal{A}}\omega \wedge \varepsilon + (-1)^{k+1} d_{\mathcal{A}}\omega \wedge \partial\varepsilon + (-1)^{k+1}\,[d_{\mathcal{A}}\omega,\varepsilon] \\
&\quad + (-1)^k\partial\omega \wedge d_{\mathcal{A}}\varepsilon + \omega \wedge d_{\mathcal{A}}\partial\varepsilon + [\omega, d_{\mathcal{A}}\varepsilon] \\
&= d_{\mathcal{A}}\partial\omega \wedge \varepsilon + \partial d_{\mathcal{A}}\omega \wedge \varepsilon + \omega \wedge d_{\mathcal{A}}\partial\varepsilon + (-1)^{k+1} d_{\mathcal{A}}\omega \wedge \partial\varepsilon \\
&\quad + (-1)^k\big(d_{\mathcal{A}}\,[\omega,\varepsilon]_* - [d_{\mathcal{A}}\omega,\varepsilon] + (-1)^k\,[\omega, d_{\mathcal{A}}\varepsilon]\big) \\
&= \mathcal{L}\omega \wedge \varepsilon + \omega \wedge \mathcal{L}\varepsilon.
\end{aligned}$$

□

Proposition 4.23. *$\mathcal{D}$ is a generating operator.*

Proof. We have already shown that d and ∂ are both generating operators. By Proposition 4.11 the only fact that remains to check is that $[\![\mathcal{L}, A]\!] \in \Gamma(\mathcal{E})$ for all $A \in \Gamma(\mathcal{E})$.

First take $\xi \in \Gamma(\mathcal{A})$, by Proposition 4.22

$$[\![\mathcal{L}, \xi]\!]\,(\omega) = \mathcal{L}(\xi \wedge \omega) - \xi \wedge \mathcal{L}\omega = (\mathcal{L}\xi) \wedge \omega.$$

Now, $\mathcal{L}\xi$ is clearly an element of $\Gamma(\mathcal{E})$.

Now consider $X \in \Gamma(\mathcal{A})$. By Proposition 4.4 $[\![\mathcal{L}, X]\!]$ will be a derivation of degree -1. Thus it suffices to confirm the required property on $\Gamma(\mathcal{A}^*)$. Also,

$$[\![\mathcal{L}, X]\!]\,(f\omega) = [\![\mathcal{L}, X]\!]\,(f)\omega + f\,[\![\mathcal{L}, X]\!]\,\omega$$

for all $f \in C^\infty(M)$. However $[\![\mathcal{L}, X]\!]\,(f) = 0$, and thus

$$[\![\mathcal{L}, X]\!]\,(f\omega) = f\,[\![\mathcal{L}, X]\!]\,(\omega),$$

or $[\![\mathcal{L}, X]\!]$ is $C^\infty(M)$ linear. Thus $[\![\mathcal{L}, X]\!]$ acts tensorially and there must be some vector field $Y \in \Gamma(\mathcal{A})$ such that $i_Y \xi = [\![\mathcal{L}, X]\!]\,(\xi)$ for all $\xi \in \Gamma(\mathcal{A}^*)$. $\square$

Theorem 4.24. *Let $A = X + \xi$ and $B = Y + \eta$ where $X, Y \in \Gamma(\mathcal{A})$ and $\xi, \eta \in \Gamma(\mathcal{A}^*)$. Then, when considered as operators on $\Gamma(\wedge^\bullet \mathcal{A}^*)$, the following formulas hold*

1. $[A, B] = \frac{1}{2}\Big([\![\, [\![\mathcal{D}, A]\!]\,, B]\!] - [\![\, [\![\mathcal{D}, B]\!]\,, A]\!] \Big)$,
2. $\frac{1}{2}\rho(A) \cdot f = [\![\, [\![\mathcal{D}, f]\!]\,, A]\!]$.

Proof. Using Propositions 4.12 & 4.20 we find

$$\begin{aligned}
\big[\,[\mathcal{D}, A]\,, B\big] =& \big[\,[d - \partial, X + \xi]\,, Y + \eta\big] \\
=& \big[\,[d, X]\,, Y\big] + \big[\,[d, X]\,, \eta\big] + \big[\,[d, \xi]\,, Y\big] + \big[\,[d, \xi]\,, \eta\big] \\
& - \big[\,[\partial, X]\,, Y\big] - \big[\,[\partial, X]\,, \eta\big] - \big[\,[\partial, \xi]\,, Y\big] - \big[\,[\partial, \xi]\,, \eta\big] \\
=& [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi + d\xi(Y) + 0 \\
& - 0 - \mathcal{L}^*_\eta X + d_*\eta(X) + \mathcal{L}^*_\xi Y + [\xi, \eta]_*
\end{aligned}$$

Similarly for the second term we get the following formula.

$$\begin{aligned}
\big[\,[\mathcal{D}, B]\,, A\big] =& [Y, X] + \mathcal{L}_Y \xi - \mathcal{L}_X \eta + d\eta(X) + 0 \\
& - 0 - \mathcal{L}^*_\xi Y + d_*\xi(Y) + \mathcal{L}^*_\eta X + [\eta, \xi]_*
\end{aligned}$$

Combining both these gives the theorem.

$$\begin{aligned}
&\frac{1}{2}\Big(\big[\,[\mathcal{D}, A]\,, B\big] - \big[\,[\mathcal{D}, B]\,, A\big]\Big) \\
&\quad =\frac{1}{2}\Big([X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi + d\xi(Y) - \mathcal{L}^*_\eta X + d_*\eta(X) \\
&\qquad + \mathcal{L}^*_\xi Y + [\xi, \eta]_* - [Y, X] - \mathcal{L}_Y \xi + \mathcal{L}_X \eta - d\eta(X) \\
&\qquad + \mathcal{L}^*_\xi Y - d_*\xi(Y) - \mathcal{L}^*_\eta X - [\eta, \xi]_* \Big) \\
&\quad = [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2} d\big(\eta(X) - \xi(Y)\big) \\
&\qquad + [\xi, \eta]_* + \mathcal{L}^*_\xi Y - \mathcal{L}^*_\eta X + \frac{1}{2} d_*\big(\eta(X) - \xi(Y)\big) \\
&\quad = [A, B]
\end{aligned}$$

For the second formula it suffices to show that $[\![\partial, f]\!] = d_{\mathcal{A}^*}(f)$. The required property for $d_{\mathcal{A}}$ was shown in Proposition 4.13. For all $\omega \in \Gamma(\wedge^\bullet \mathcal{A}^*)$

$$\begin{aligned} [\![\partial, f]\!]\,(\omega) &= \partial(f\omega) - f\partial\omega \\ &= (\partial f)\omega + f\partial\omega + [f, \omega]_* - f\partial\omega \\ &= [f, \omega]_* = (d_{\mathcal{A}^*} f)\omega. \end{aligned}$$

□

We summarize this section in the following theorem.

Theorem 4.25. *The operator $\mathcal{D} = d - \partial$ is a generating operator and it's derived Courant algebroid coincides with the Courant bracket for the double of a Lie bialgebroid.*

4.9 Integrability Condition for Spinors

In this chapter we will use the generating operator to describe the integrability condition for spinors

Let $\Lambda \subseteq \wedge^\bullet \mathcal{A}^*$ be a spinor line bundle and L_Λ its null space. See Section 4.2 for details of this construction. Also, in Section 4.2 we obtained conditions on the spinor in order for L_Λ to be a generalized complex structure.

Definition 4.26. [2] A section $\lambda \in \Gamma(\mathcal{A}^*)$ is *projectively closed* if there exists some section $A \in \Gamma(\mathcal{E})$ such that

$$\mathcal{D}(\lambda) = A \cdot \lambda.$$

The following theorem is due to Alekseev and Xu [2], however due to the unpublished nature of this work the proof is reproduced here.

Theorem 4.27 ([2]). *Let $\mathcal{E} = \mathcal{A} \oplus \mathcal{A}^*$ be a Courant algebroid with generating operator $\mathcal{D}$: $\Gamma(\wedge^\bullet \mathcal{A}^*) \to \Gamma(\wedge^\bullet \mathcal{A}^*)$. Also, let $\lambda \in \Gamma(\wedge^\bullet \mathcal{A}^*)$ be a pure spinor. If λ is projectively closed then its null space L_λ is a Dirac structure on $\mathcal{E}$. Conversely, let $L \subseteq \mathcal{E}$ be a Dirac structure and $\Lambda \subseteq \wedge^\bullet \mathcal{A}^*$ its spinor line bundle. Now, if $\lambda \in \Gamma(\Lambda)$ is locally nonvanishing then it must be projectively closed.*

Proof. We will start with the first implication. Take any two sections $A, B \in \Gamma(L_\lambda)$; thus $A \cdot \lambda = B \cdot \lambda = 0$. Now,

$$\begin{aligned} (A \circ B) \cdot \lambda &= [\![\, [\![\mathcal{D}, A]\!]\,, B]\!] \cdot \lambda \\ &= \mathcal{D}(A \cdot B \cdot \lambda) + A \cdot \mathcal{D}(B \cdot \lambda) - B \cdot \mathcal{D}(A \cdot \lambda) - B \cdot A \cdot \mathcal{D}(\lambda) \\ &= -B \cdot A \cdot \mathcal{D}(\lambda) \end{aligned}$$

Now, because λ is projectively closed there is some $F \in \Gamma(\mathcal{E})$ such that $\mathcal{D}(\lambda) = F \cdot \lambda$. So

$$\begin{aligned}(A \circ B) \cdot \lambda &= -B \cdot A \cdot F \cdot \lambda = -(BAF) \cdot \lambda \\ &= -(BFA - B\langle A, F\rangle) \cdot \lambda \\ &= \langle A, F\rangle B \cdot \lambda - B \cdot F \cdot A \cdot \lambda \\ &= 0\end{aligned}$$

Thus $A \circ B \in \Gamma(L_\lambda)$, and L_λ is a Dirac structure.

Now, for the converse statement. Take any locally nonvanishing $\lambda \in \Gamma(\Lambda)$ – remember that Λ is the line bundle associated to the Dirac structure L. Now,

$$(A \circ B) \cdot \lambda = [A, B] \cdot \lambda + \tfrac{1}{2}\mathcal{D}\big(\langle A, B\rangle\big) \cdot \lambda.$$

And, because L is a Dirac structure both $[A, B]$ and $\langle A, B\rangle$ are zero. Thus $(A \circ B) \cdot \lambda = 0$ for all $A, B \in \Gamma(L)$. By definition $A \cdot \lambda = B \cdot \lambda = 0$ too. These facts, combined with the definition of $\circ$, imply that $B \cdot [\![\mathcal{D}, A]\!](\lambda) = 0$. This fact must hold for any $B \in \Gamma(L)$, and thus $[\![\mathcal{D}, A]\!](\lambda) \in \Gamma(\Lambda)$. Because λ is locally nonvanishing there is some function $f_A \in C^\infty(M)$ such that $[\![\mathcal{D}, A]\!](\lambda) = f_A\lambda$.

Now, by definition

$$[\![\mathcal{D}, A]\!](\lambda) = \mathcal{D}(A \cdot \lambda) + A \cdot \mathcal{D}(\lambda) = A \cdot \mathcal{D}(\lambda).$$

Because the Clifford representation is linear, f_A depends on linearly on A. Thus there is some $F \in \Gamma(\mathcal{E})$, determined up to a section of L, such that $f_A = 2\langle F, A\rangle$. Now $f_A\lambda = (FA + AF) \cdot \lambda = (AF) \cdot \lambda$. Thus

$$(AF)\lambda = f_A\lambda = A \cdot \mathcal{D}(\lambda),$$

or $A \cdot (\mathcal{D}(\lambda) - F \cdot \lambda) = 0$. Once again, this expression holds for all $A \in \Gamma(L)$, and thus $\mathcal{D}(\lambda) - F \cdot \lambda = g\lambda$ for some $g \in C^\infty(M)$. Now $\mathcal{D}$ and F are both odd operators, but g is an even operator. Also from (4.4), it is easy to see that λ is a homogeneous element. Thus, the only way for both sides to have the same order is if the operators are zero. Thus $\mathcal{D}(\lambda) = F \cdot \lambda$, and λ is projectively closed. $\square$

Finally we will apply this to generalized complex structures. As a reminder an almost generalized complex structure can be given by a pure spinor line bundle $\Lambda \subseteq \Gamma(\wedge^\bullet \mathcal{A}^*)$ such that $(\lambda, \overline{\mu})_{\mathrm{muk}} \neq 0$ for all $\lambda, \mu \in \Gamma(\Lambda)$. The following corollary gives the condition for the above to be integrable to a generalized complex structure.

Corollary 4.28. *Let $\mathcal{E}$ be a Courant algebroid with generating operator $\mathcal{D}$. A generalized complex structure is equivalent to a pure spinor line bundle $\Phi \subseteq \wedge^{\bullet}\mathcal{A}^*$ such that for all $\varphi \in \Gamma(\Phi)$ nonzero local sections*

- $(\varphi, \overline{\varphi})_{\mathrm{muk}} \neq 0.$
- *φ is projectively closed.*

In the case of the usual Courant bracket on $TM \oplus T^*M$ this result coincides with Gualtieri's [14]. Using our this technique we have extended this result for all Courant algebroids with a generating operator. In particular the twisted Courant bracket, which has generating operator $\mathcal{D}_{\Omega} = d + \Omega$.

Chapter 5

Generalized Complex Submanifolds

The primary objects of study in this chapter are twisted manifolds. A manifold M endowed with a closed 3-form Ω will be called *twisted*. Given a twisted manifold we can define the twisted Courant bracket as in (2.19).

A *twisted generalized complex structure* is a smooth map $J : TM \oplus T^*M \to TM \oplus T^*M$ such that $J^2 = -\mathrm{Id}$, $JJ^* = \mathrm{Id}$ and the $+i$-eigenbundle of J is involutive with respect to (2.19), rather than (2.16). The triple (M, Ω, J) will be called a *twisted generalized complex manifold*.

The aim of this chapter is to characterize when a submanifold of a twisted generalized complex manifold is also a twisted generalized complex manifold. In the untwisted case, several notions of generalized complex submanifolds have been recently introduced. The notion defined here is similar to the one in [4] and [38]. A different notion of generalized complex submanifolds appears in [14].

Definition 5.1. A *twisted immersion*, from one twisted manifold (N, Υ) to another twisted manifold (M, Ω), is defined as a smooth immersion $h : N \to M$ with $\Upsilon = h^*\Omega$. A *twisted generalized complex immersion* from (N, Υ, J') to (M, Ω, J) is a twisted immersion $h : (N, \Upsilon) \to (M, \Omega)$ such that the pullback of the $+i$-eigenbundle of J is the $+i$-eigenbundle of J'. In this case, N is called a *twisted generalized complex submanifold* of M.

As noted in (3.1), a twisted generalized complex structure can be written as

$$J = \begin{bmatrix} \phi & \pi^\sharp \\ \sigma_\flat & -\phi^* \end{bmatrix}, \tag{5.1}$$

where

$$\phi^2 + \pi^\sharp \sigma_\flat = -\mathrm{Id}, \quad \phi\pi^\sharp = \pi^\sharp\phi^*, \quad \text{and} \quad \phi^*\sigma_\flat = \sigma_\flat\phi. \tag{5.2}$$

These facts, and others, were first noted in [8]. These results were also described using Poisson quasi-Nijenhuis manifolds in [35]. The conditions for a submanifold to be twisted generalized complex will be expressed in terms of this splitting. In Chapter 3 we showed that π from the above splitting is a Poisson bivector field.

This work was inspired by [34], where the reduction of generalized complex structures is studied. The main result, in the untwisted case, was also independently obtained by Vaisman [38].

This chapter is organized as follows. In Section 1, we recall induced Dirac structures and prove a lemma that will be used to find when they are involutive. In Section 2, we define the induced generalized complex structure, and characterize when it has the required properties. In Section 3, we prove the main theorem of this paper, and provide examples. Twisted generalized complex involutions are also introduced in this section. In Section 4, we determine when a submanifold of a holomorphic Poisson manifold is itself endowed with an induced holomorphic Poisson structure. Section 5 is a restatement of our main result in terms of spinors. The last section discusses generalized Kähler submanifolds.

5.1 Induced Dirac structures

In this section we will recall the definition of the pull back of a Dirac structure from Section 2.7.This section makes heavy use of the notation from the first chapter.

Let N be a smooth submanifold of M with injection $\varphi : N \to M$. Recall, from Section 2.7, that φ_* pulls back a Dirac structure $L \subseteq TM \oplus T^*M$ to a maximal isotropic $B_\varphi(L) \subseteq TN \oplus T^*N$, given by

$$B_\varphi(L) = \left\{ X + \varphi^*\xi : X \in TN, \xi \in T^*M \text{ such that } \varphi_* X + \xi \in L \right\},$$

The last lemma of this section will be used to characterize when the pullback bundle is involutive. Let (M, Ω) and (N, Υ) be two twisted manifolds with an immersion $\varphi : N \to M$. Two sections $\sigma_N = X + \xi \in \mathfrak{X}(N) \oplus \Omega^1(N)$ and $\sigma_M = Y + \eta \in \mathfrak{X}(M) \oplus \Omega^1(M)$ are said to be *φ-related*, denoted by $\sigma_N \overset{\varphi}{\rightsquigarrow} \sigma_M$, if $Y = \varphi_* X$ and $\xi = \varphi^*\eta$. The following lemma is an extension of Lemma 2.2 from [34]. This lemma is also true for complex sections, which is when it will be applied.

Lemma 5.2. *Assume that $\sigma^i_N \in \Gamma(TN \oplus T^*N)$ and $\sigma^i_M \in \Gamma(TM \oplus T^*M)$ satisfy $\sigma^i_N \overset{\varphi}{\rightsquigarrow} \sigma^i_M$, for $i = 1, 2$. Then, if φ is a twisted immersion,*

$$\left[\sigma^1_N, \sigma^2_N\right]_\Upsilon \overset{\varphi}{\rightsquigarrow} \left[\sigma^1_M, \sigma^2_M\right]_\Omega .$$

Proof. Write $\sigma^i_N = X^i + \xi^i$ and $\sigma^i_M = Y^i + \eta^i$, where $X^i + \xi^i \in \mathfrak{X}(N) \oplus \Omega^1(N)$ and $Y^i + \eta^i \in \mathfrak{X}(M) \oplus \Omega^1(M)$, for $i = 1, 2$. Since $\sigma^i_N \overset{\varphi}{\rightsquigarrow} \sigma^i_M$, for $i = 1, 2$, then $\varphi_* X^i = Y^i$ and $\varphi^*\eta^i = \xi^i$. By definition

$$\left[\sigma^1_N, \sigma^2_N\right]_\Upsilon = \left[X^1, X^2\right] + \mathcal{L}_{X^1}\xi^2 - \mathcal{L}_{X^2}\xi^1 + \tfrac{1}{2}d\big(\xi^1(X^2) - \xi^2(X^1)\big) + i_{X^2} i_{X^1}\Upsilon,$$

and

$$\left[\sigma^1_M, \sigma^2_M\right]_\Omega = \left[Y^1, Y^2\right] + \mathcal{L}_{Y^1}\eta^2 - \mathcal{L}_{Y^2}\eta^1 + \tfrac{1}{2}d\big(\eta^1(Y^2) - \eta^2(Y^1)\big) + i_{Y^2}i_{Y^1}\Omega.$$

Now

$$\varphi_*\left[X^1, X^2\right] = \left[\varphi_* X^1, \varphi_* X^2\right] = \left[Y^1, Y^2\right],$$

and

$$\begin{aligned}\varphi^*(\mathcal{L}_{Y^1}\eta^2) =& \varphi^*(i_{Y^1}d\eta^2 + di_{Y^1}\eta^2) = \varphi^* i_{\varphi_* X^1}d\eta^2 + \varphi^* d\big(\eta^2(Y^1)\big)\\ =& i_{X^1}\varphi^* d\eta^2 + d\big(\xi^2(X^1)\big) = i_{X^1}d\xi^2 + di_{X^1}\xi^2 = \mathcal{L}_{X^1}\xi^2.\end{aligned}$$

Similarly,

$$\varphi^*(\mathcal{L}_{Y^2}\eta^1) = \mathcal{L}_{X^2}\xi^1.$$

The second last term becomes

$$\varphi^* d\big(\eta^1(Y^2) - \eta^2(Y^1)\big) = d\big(\xi^1(X^2) - \xi^2(X^1)\big),$$

since

$$\varphi^*\big(\eta^1(Y^2)\big) = \varphi^*\big(\eta^1(\varphi_* X^2)\big) = (\varphi^*\eta^1)(X^2) = \xi^1(X^2).$$

Finally, because φ is a twisted immersion, the following holds.

$$\varphi^* i_{Y^2}i_{Y^1}\Upsilon = \varphi^* i_{\varphi_* X^2}i_{\varphi_* X^1}\Upsilon = i_{X^2}i_{X^1}\varphi^*\Upsilon = i_{X^2}i_{X^1}\Omega.$$

□

5.2 The induced generalized complex structure

Consider two twisted manifolds (M, Ω) and (N, Υ) with an immersion $h : N \to M$. Also, assume that there is a generalized complex structure J on M with eigenbundles L_+ and L_-. The goal of this section is to characterize when the pull backs of L_+ and L_- give a generalized complex structure on N. The pull backs of L_+ and L_- will be called the *induced bundles*, and are given by

$$L'_\pm = \mathcal{B}_h(L_\pm) = \left\{X + h^*\xi : X \in T_{\mathbb{C}}N, \xi \in T^*_{\mathbb{C}}M \text{ such that } h_* X + \xi \in L_\pm\right\},$$

By definition, both L'_+ and L'_- are maximal isotropics, but they need not be smooth or involutive subbundles. The bundles may also have nontrivial intersection. The rest of this section is devoted to characterizing when the induced bundles have the desired properties. The first of these properties to be addressed will be the intersection property.

Because L'_+ and L'_- are both maximal isotropics, it suffices to check that they span $T_{\mathbb{C}}N \oplus T^*_{\mathbb{C}}N$. Consider the subbundle $B = TN \oplus T^*M|_N$ of $TM \oplus T^*M$. Its orthogonal, $B^{\perp} = TN^o$, is the kernel of the natural projection $s : B \to TN \oplus T^*N$, which maps $X + \xi \mapsto X + h^*\xi$. The following proposition relates s with the decomposition of $T_{\mathbb{C}}N \oplus T^*_{\mathbb{C}}N$.

Proposition 5.3. *Let $B = j_*TN \oplus T^*M|_N$ and $s : B \to T_{\mathbb{C}}N \oplus T^*_{\mathbb{C}}N$ be the map which takes $j_*X + \xi \mapsto X + j^*\xi$. With this notation,*

$$s\big((B \cap JB)_{\mathbb{C}}\big) = E'_+ + E'_- .$$

Proof. First take $X + \xi \in E'_+$. Thus $X \in T_{\mathbb{C}}N$ and $\xi = j^*\eta$ for some $\eta \in T^*_{\mathbb{C}}M|_N$ with $j_*X + \eta \in E_+$. Clearly $j_*X + \eta \in B_{\mathbb{C}}$ and, because $j_*X + \eta = i(-ij_*X - i\eta)$, it is also true that $j_*X + \eta \in (JB)_{\mathbb{C}}$. Similarly $E'_- \subseteq s((B \cap JB)_{\mathbb{C}})$.

Now take $a \in (B \cap JB)_{\mathbb{C}}$, so $a \in B$ and $Ja \in B$. Consider elements $a_+ = \frac{1}{2}(a - iJa)$ and $a_- = \frac{1}{2}(a + iJa)$. Clearly $a_+ \in E_+$, $a_- \in E_-$ and $a = a_+ + a_-$. Finally $s(a_+) \in E'_+$ and $s(a_-) \in E'_-$; so $s((B\cap)_{\mathbb{C}}) \subseteq E'_+ + E'_-$. $\square$

Thus the decomposition, $T_{\mathbb{C}}N \oplus T^*_{\mathbb{C}}N = E'_+ \oplus E'_-$, holds iff $B = B \cap JB + B^{\perp}$. The latter is equivalent to $JB \subseteq B + JB^{\perp}$, which is also equivalent to $JB^{\perp} \cap B \subseteq B^{\perp}$. The preceding can be summarized in the following proposition.

Proposition 5.4. *The following assertions are equivalent.*

1. *The subbundle L'_+ is the $+i$-eigenbundle of a – not necessarily smooth – automorphism J' of $TN \oplus T^*N$ such that $J'^2 = -\mathrm{Id}$ and $J'J'^* = \mathrm{Id}$.*

2. $B = B \cap JB + B^{\perp}$.

3. $JB \subseteq B + JB^{\perp}$.

4. $JB^{\perp} \cap B \subseteq B^{\perp}$.

Conditions (3) and (4) follow from elementary calculations. In the sequel we will assume that the assertions of Proposition 5.4 are satisfied. Consider the restriction of J and s to the J-invariant subspace $B \cap JB$; the latter map will be denoted by s'. The kernel of s' is $B^{\perp} \cap JB$. Under J, this kernel is mapped to $JB^{\perp} \cap B$. This must be in $JB \cap B$ and also, by Proposition 5.4, in $B^{\perp}$, however $B^{\perp} \subseteq B$. So the image of the kernel of s' is in $B^{\perp} \cap JB \cap B = B^{\perp} \cap JB$. Thus the kernel of s' is J-invariant and $J|_{B \cap JB}$ induces an automorphism of $TN \oplus T^*N$:

$$\begin{array}{ccc} B \cap JB & \xrightarrow{J} & B \cap JB \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s} \\ TN \oplus T^*N & \xrightarrow[J']{} & TN \oplus T^*N. \end{array} \tag{5.3}$$

The induced automorphism is nothing but J' from Proposition 5.4. Indeed, the complexification of the above commutative diagram gives

$$\begin{array}{ccc} (L_+ \cap B_{\mathbb{C}}) \oplus (L_- \cap B_{\mathbb{C}}) & \xrightarrow{(+i)\mathrm{Id} \oplus (-i)\mathrm{Id}} & (L_+ \cap B_{\mathbb{C}}) \oplus (L_- \cap B_{\mathbb{C}}) \\ \downarrow & & \downarrow \\ L'_+ \oplus L'_- & \xrightarrow[(+i)\mathrm{Id} \oplus (-i)\mathrm{Id}]{} & L'_+ \oplus L'_-. \end{array}$$

The following lemma relates condition (4) of Proposition 5.4 to the splitting of J:

$$J = \begin{bmatrix} \phi & \pi^{\sharp} \\ \sigma_{\flat} & -\phi^* \end{bmatrix}.$$

Lemma 5.5. *The following assertions are equivalent.*

1. $JB^{\perp} \cap B \subseteq B^{\perp}$.

2. $TN \cap \pi^{\sharp}(TN^o) = 0$ *and* $\phi(TN) \subseteq TN + \pi^{\sharp}(TN^o)$.

Proof. The inclusion

$$J(TN^o) \cap (TN \oplus T^*M|_N) \subseteq TN^o$$

is true if, and only if,

$$\left\{\begin{array}{l}\xi \in TN^o \\ J\xi \in TN \oplus T^*M|_N\end{array}\right\} \implies J\xi \in TN^o$$

if, and only if,

$$\left\{\begin{array}{l}\xi \in TN^o \\ \pi^\sharp\xi \in TN\end{array}\right\} \implies \left\{\begin{array}{l}\pi^\sharp\xi = 0 \\ \phi^*\xi \in TN^o\end{array}\right\}$$

if, and only if,

$$\xi \in TN^o \cap (\pi^\sharp)^{-1}(TN) \implies \left\{\begin{array}{l}\pi^\sharp\xi = 0 \\ \xi \in (\phi(TN))^o\end{array}\right\}$$

if, and only if,

$$\pi^\sharp(TN^o) \cap TN = 0 \qquad \text{and} \qquad TN^o \cap (\pi^\sharp)^{-1}(TN) \subseteq (\phi(TN))^o.$$

Since π is skew-symmetric $(\pi^\sharp)^{-1}(TN) = (\pi^\sharp(TN^o))^o$, and

$$TN^o \cap (\pi^\sharp(TN^o))^o \subseteq (\phi(TN))^o$$

$$TN + \pi^\sharp(TN^o) \supseteq \phi(TN).$$

□

According to this lemma, the sum $TN + \pi^\sharp(TN^o)$ must be direct. In the sequel pr will denote the projection $TN \oplus \pi^\sharp(TN^o) \to TN$. If π is degenerate then neither the bundle $TN \oplus \pi^\sharp(TN^o)$, nor the map pr are necessarily smooth.

For any $\xi \in T^*N$ we claim that if $\eta, \eta' \in B \cap JB$ such that $\xi = h^*\eta = h^*\eta'$ then $\pi^\sharp\eta = \pi^\sharp\eta'$. Because η and η' are preimages of ξ they differ by some element of TN^o, and as $B \cap JB$ is J-stable both $\pi^\sharp\eta$ and $\pi^\sharp\eta'$ are in TN. However $TN \cap \pi^\sharp(TN^o) = \{0\}$, and the difference of the two preimages is zero. Thus the assignment $\xi \mapsto \pi^\sharp\eta$ defines a skew-symmetric vector bundle map from T^*N to TN. Its associated bivector field on N will be denoted by π'.

The following technical lemmas will be used to show when J' is smooth.

Lemma 5.6. *Let $X \in TN$ and $\xi \in TN^o$. If $\phi X + \pi^\sharp\xi \in TN$ then $\phi X + \pi^\sharp\xi = (\mathrm{pr} \circ \phi)X$.*

Proof. For any $X \in TN$ the second assertion of Lemma 5.5 gives $\phi X = Y + \pi^\sharp \eta$, where $Y \in TN$ and $\eta \in TN^o$. By definition $Y = (\mathrm{pr} \circ \phi)X$, and $\phi X + \pi^\sharp \xi = Y + \pi^\sharp(\eta + \xi)$. Both Y and $\phi X + \pi^\sharp \xi$ are elements of TN; thus $\pi^\sharp(\eta + \xi)$ is also an element of TN. But $TN \cap \pi^\sharp(TN^o) = \{0\}$, and $\eta + \xi \in TN^o$. Thus $\pi^\sharp(\eta + \xi) = 0$. □

Lemma 5.7. *Let $p_1 : TN \oplus T^*N \to TN$ and $p_2 : TN \oplus T^*N \to T^*N$ be the projections. If $X \in TN$, then*

$$(p_1 J')X = (\mathrm{pr} \circ \phi)X = \phi X + \pi^\sharp \zeta \tag{5.4}$$

and

$$(p_2 J')X = h^*(\sigma_\flat X - \phi^* \zeta), \tag{5.5}$$

where ζ is some element of TN^o such that $X + \zeta \in B \cap JB$.

If $\xi \in T^*N$, then

$$(p_1 J')\xi = \pi'^\sharp \xi = \pi^\sharp \eta \tag{5.6}$$

and

$$(p_2 J')\xi = -(h^* \phi^*)\eta, \tag{5.7}$$

where η is some element of $T^*M|_N \cap B \cap JB$ such that $h^*\eta = \xi$.

Proof. Consider $X \in TN$. Since s is surjective there exists some $\zeta \in TN^o$ such that $X + \zeta \in B \cap JB$ and $s(X + \zeta) = X$. Now $J(X + \zeta) = (\phi X + \pi^\sharp \zeta) + (\sigma_\flat X - \phi^* \zeta) \in B$. Therefore $\phi X + \pi^\sharp \zeta \in TN$ and, by Lemma 5.6, $\phi X + \pi^\sharp \zeta = (\mathrm{pr} \circ \phi)X$. Both (5.4) and (5.5) follow from (5.3).

Now take $\xi \in T^*N$. Again, since s is surjective there exists some $\eta \in T^*M$ such that $\eta \in B \cap JB$ and $s(\eta) = \xi$. Now $J(\xi) = \pi^\sharp \eta - \phi^* \eta = \pi'^\sharp \xi - \phi^* \eta$, which is in B. Both (5.6) and (5.7) follow from (5.3). □

For the remainder of this section, if L is a smooth vector bundle then $\Gamma(L)$ will denote the space of all – not necessarily smooth – sections of L, and $\Gamma^\infty(L)$ the subspace of smooth sections.

Lemma 5.8. *Let $\xi \in \Gamma(TN^o)$. If $\pi^\sharp\xi \in \Gamma^\infty(TM|_N)$, then $(h^*\phi^*)\xi \in \Gamma^\infty(T^*N)$.*

Proof. As noted previously, if $X \in \Gamma^\infty(TN)$ then $\phi X = Y + \pi^\sharp\eta$, where $Y \in \Gamma(TN)$ and $\eta \in \Gamma^\infty(TN^o)$. Now $(\phi^*\xi)(X) = \xi(\phi X) = \xi(Y) + \xi(\pi^\sharp\eta) = \xi(\pi^\sharp\eta) = -\eta(\pi^\sharp\xi)$. This function and its restriction to TN are smooth because $\pi^\sharp\xi$ is. $\square$

Lemma 5.9. *Assume $\mathrm{pr} \circ\phi$ is a smooth map and $\eta \in \Gamma(T^*M|_N)$. If $h^*\eta \in \Gamma^\infty(TN)$ and $\pi^\sharp\eta \in \Gamma^\infty(TN)$ then $(h^*\phi^*)\eta \in \Gamma^\infty(T^*N)$.*

Proof. Once again, if $Y \in \Gamma^\infty(TN)$ then $\phi Y = (\mathrm{pr} \circ\phi)Y + \pi^\sharp\zeta$ for some $\zeta \in \Gamma^\infty(TN^o)$. Now

$$\begin{aligned}
&\big((h^*\phi^*)\eta\big)(Y) \\
&\qquad = (h^*\eta)(\phi Y) \\
&\qquad = (h^*\eta)\big((\mathrm{pr} \circ\phi)Y\big) + (h^*\eta)(\pi^\sharp\zeta) \\
&\qquad = (h^*\eta)\big((\mathrm{pr} \circ\phi)Y\big) - \zeta\big(\pi^\sharp(h^*\eta)\big).
\end{aligned}$$

Thus $((h^*\phi^*)\eta)(Y)$ is a smooth function, and the lemma follows. $\square$

We are now ready to give the conditions J' must satisfy in order to be smooth.

Proposition 5.10. *The vector bundle automorphism J' of $TN \oplus T^*N$ is smooth if, and only if, $\mathrm{pr} \circ\phi : TN \to TN$ is smooth and π' is a smooth bivector field on N.*

Proof. First assume that J' is smooth. Thus $(p_1 J')X \in \Gamma^\infty(TN)$ for all $X \in \Gamma^\infty(TN)$. It follows from (5.4) that $(\mathrm{pr} \circ\phi)$ must be smooth. Also $(p_1 J')\xi \in \Gamma^\infty(T^*N)$ for all $\xi \in \Gamma^\infty(T^*N)$, and (5.6) shows that $\pi'^\sharp$ is smooth.

Now for the other implication. For every $X \in \Gamma^\infty(T^*N)$ there is some $\zeta \in \Gamma(TN^o)$ such that (5.4) and (5.5) are satisfied. As J is smooth both $\sigma_\flat$ and ϕ are smooth. The smoothness of $\mathrm{pr} \circ\phi$ and (5.4) show that $\pi^\sharp\zeta \in \Gamma^\infty(TM|_N)$. Thus, according to Lemma 5.8, $(h^*\phi^*)\zeta \in \Gamma^\infty(T^*N)$, and the right hand sides of (5.4) and (5.5) are smooth. Finally $J'X = (p_1 J')X + (p_2 J')X \in \Gamma^\infty(TN \oplus T^*N)$.

Now take $\xi \in \Gamma^\infty(T^*N)$. There must exist $\eta \in \Gamma(T^*M|_N)$ such that (5.6) holds, (5.7) holds, and $h^*\eta = \xi$. The smoothness of π' and (5.6) show that $\pi^\sharp\eta \in \Gamma^\infty(TN)$. Now Lemma 5.9 gives $(h^*\phi^*)\eta \in \Gamma^\infty(T^*N)$, and the right hand sides of (5.6) and (5.7) are smooth. Finally, $J'\xi = (p_1 J')\xi + (p_2 J')\xi \in \Gamma^\infty(TN \oplus T^*N)$. $\square$

We finish this section by using Lemma 5.2 to show when J' is integrable.

Proposition 5.11. *If J' is smooth then it is integrable.*

Proof. First, observe that the vector bundles $L_{\pm} \cap B_{\mathbb{C}} = (I \mp iJ)B_{\mathbb{C}}$ are smooth. Since J' is smooth, its eigenbundles $L'_{\pm}$ are also smooth. It is not hard to check that any smooth section of L'_{+} is h-related to a smooth section of $L_{+} \cap B_{\mathbb{C}}$.

Hence for any $\sigma'_1, \sigma'_2 \in \Gamma^{\infty}(L'_{+})$ there exists $\sigma_1, \sigma_2 \in \Gamma^{\infty}(L_{+} \cap B_{\mathbb{C}})$ such that $\sigma_1 \overset{h}{\rightsquigarrow} \sigma'_1$ and $\sigma_2 \overset{h}{\rightsquigarrow} \sigma'_2$. Since L_{+} is integrable $\left[\sigma_1, \sigma_2\right]_{\Omega} \in \Gamma^{\infty}(L_{+})$, and it follows from Lemma 5.2 that $\left[\sigma_1, \sigma_2\right]_{\Omega} \overset{h}{\rightsquigarrow} \left[\sigma'_1, \sigma'_2\right]_{\Upsilon}$. Thus $\left[\sigma'_1, \sigma'_2\right]_{\Upsilon} \in \Gamma^{\infty}(L'_{+})$, and L'_{+} is involutive with respect to the Υ-twisted bracket. □

5.3 Main theorem

The following definition will be used to characterize when a twisted submanifold is also generalized complex; see [9] for the motivation of this definition.

Definition 5.12. Let (M, π) be a Poisson manifold. A smooth submanifold N of M is a *Poisson-Dirac submanifold* of M if $TN \cap \pi^{\sharp}(TN)^{o} = \{0\}$, and the induced Poisson tensor π' on N is smooth.

The next theorem is the main result of this paper. The untwisted version of this result was obtained independently, using a different method, by Vaisman [38].

Theorem 5.13. *Let (M, Ω, J) be a twisted generalized complex manifold with $J = \begin{bmatrix} \phi & \pi^{\sharp} \\ \sigma_{\flat} & -\phi^{*} \end{bmatrix}$. A twisted submanifold N of M inherits a twisted generalized complex structure J', making it a twisted generalized complex submanifold, if and only the following conditions hold:*

1. *N is a Poisson-Dirac submanifold of (M, π),*
2. *$\phi(TN) \subseteq TN + \pi^{\sharp}(TN^{o})$,*
3. *$\mathrm{pr} \circ \phi : TN \to TN$ is smooth.*

The generalized complex structure J' on N is given by

$$J' = \begin{bmatrix} \phi' & (\pi')^{\sharp} \\ \sigma'_{\flat} & -(\phi')^{*} \end{bmatrix}.$$

Here $\phi' = \mathrm{pr} \circ \phi|_{TN'}$, π' is the induced Poisson tensor, and

$$\sigma'_\flat(X) = h^*(\sigma_\flat X - \phi^*\zeta), \tag{5.8}$$

where $\zeta \in (TN)^o$ such that $X + \zeta \in B \cap JB$, as in Lemma 5.7

Proof. This theorem is the construction and confirmation of the properties of J'. Proposition 5.4 combined with Lemma 5.5 shows that $J'^2 = -\mathrm{Id}$ and $J'^* J' = \mathrm{Id}$. The smoothness of J' follows from Proposition 5.10, and the integrability of its $+i$-eigenbundle follows from Proposition 5.11. The form of the generalized complex structure follows from Lemma 5.7. □

For the following examples let $\Omega = 0$.

Example 5.14. Let (M, j) be a complex manifold, and let N be a smooth submanifold of M. There is a generalized complex structure on M given by $\phi = j$, $\sigma = 0$ and $\pi = 0$. Because the Poisson structure is zero, N is automatically a Poisson-Dirac submanifold. Condition (b) of Theorem 5.13 becomes $j(TN) \subseteq TN$, which is exactly the requirement for N to be an immersed complex submanifold of M. Now $\mathrm{pr} \circ j = j|_{TN}$, which is a smooth map. Thus N is a generalized complex submanifold if, and only if, it is an immersed complex submanifold.

Example 5.15. Let (M, ω) be a symplectic manifold and N a smooth submanifold of M. The generalized complex structure on M arising from ω is given by $\phi = 0$, $\sigma_\flat = \omega_\flat$ and $\pi^\sharp = -\omega_\flat^{-1}$. Because $\phi = 0$, conditions (b) and (c) of Theorem 5.13 are automatically satisfied. Now N will be a Poisson-Dirac submanifold of M if, and only if, N is a symplectic submanifold of M. Thus N is a generalized complex submanifold of M if, and only if, it is a symplectic submanifold.

The last result of this section is an application of Theorem 5.13 to the stable locus of a twisted generalized complex involution. This result is similar to one for Poisson involutions [10, 43]. Let (M, Ω, J) be a twisted generalized complex manifold. A *twisted generalized complex involution* is a diffeomorphism $\Psi : M \to M$ such that $\Psi^2 = \mathrm{Id}$, $\Psi^*\Omega = \Omega$ and

$$\Psi^*_* \circ J = J \circ \Psi^*_*. \tag{5.9}$$

Here Ψ^*_* is the map from $TM \oplus T^*M$ to $TM \oplus T^*M$ defined by $\Psi^*_*(X + \xi) = \Psi_* X + \Psi^*\xi$.

Corollary 5.16. *Let (M, Ω, J) be a twisted generalized complex manifold and let Ψ be a twisted generalized complex involution of J. The fixed locus, N, of Ψ is a twisted generalized complex submanifold of M.*

Proof. Let ξ be an arbitrary element of T^*M. Equation (5.9) implies that $(\Psi_* \pi^\sharp \Psi^*)\xi = \pi^\sharp \xi$. Hence $\Psi_* \pi = \pi$, and Ψ_* is a Poisson involution. Because Ψ_* is a Poisson involution, Proposition 4.1 of [43] implies that N is a Dirac submanifold. Thus N is a Poisson-Dirac submanifold, and condition (a) of Theorem 5.13 is satisfied.

Take $X \in TN$. Equation (5.9) implies that $\Psi_*(\phi X) + \Psi^*(\sigma_\flat X) = \phi X + \sigma_\flat X$. The vector field component of this equality proves that $\phi(TN) \subseteq TN$, and condition (b) of Theorem 5.13 is satisfied. Thus $\mathrm{pr} \circ \phi = \phi|_{TN}$, which is a smooth map. Hence condition (c) of Theorem 5.13 is satisfied. $\square$

5.4 Holomorphic Poisson submanifolds

Let (M, j, π) be a Poisson Nijenhuis manifold such that $j : TM \to TM$ is an integrable almost complex structure. Such a structure is equivalent to a holomorphic Poisson structure The holomorphic Poisson tensor is given by $\Pi = \pi_j + i\pi$, where $\pi_j^\sharp = \pi^\sharp \circ j^*$.

A generalized complex structure on M is given by, [8, 35],

$$J = \begin{bmatrix} j & \pi^\sharp \\ 0 & -j^* \end{bmatrix}. \tag{5.10}$$

In general, if N is a generalized complex submanifold then the induced generalized complex need not have $\sigma' = 0$.

Recall that $TN \cap \pi^\sharp(TN^o) = \{0\}$ and $\phi(TN) \subseteq TN + \pi^\sharp(TN^o)$. Thus, we can define the composition

$$TN \xrightarrow{\phi} TN \oplus \pi^\sharp(TN^o) \xrightarrow{\mathrm{pr}_2} \pi^\sharp(TN^o). \tag{5.11}$$

Proposition 5.17. *Consider the generalized complex structure* (5.10)*, and let N be a generalized complex submanifold of M. Now, $\sigma' = 0$ if, and only if,*

$$\phi(TN) \subseteq \pi^\sharp\big((\mathrm{pr}_2 \circ \phi(TN))^o\big). \tag{5.12}$$

Proof. Take $X \in TN$. Then $\begin{bmatrix} X \\ 0 \end{bmatrix} \in TN \oplus T^*M|_N = B$. Since $B = B \cap JB + B^\perp$ and $B^\perp = TN^o$ there exists a $\zeta \in TN^o$ such that $\begin{bmatrix} X \\ \zeta \end{bmatrix} \in B \cap JB$. But then $J \begin{bmatrix} X \\ \zeta \end{bmatrix} \in B$ too, and so $\phi X + \pi^\sharp \zeta \in TN$.

In other words, given $X \in TN$ there exists $\zeta \in TN^o$ such that $\phi X + \pi^\sharp \zeta \in TN$. Since $\phi(TN) \subseteq TN \oplus \pi^\sharp(TN^o)$, this is equivalent to $\mathrm{pr}_2 \circ \phi(X) = -\pi^\sharp \zeta$. Recall that $\sigma'_\flat(X) = h^*(\sigma_\flat(X) - \phi^* \zeta)$.

Now, assume $\sigma = 0$. Then $\sigma' = 0$ if, and only if $\sigma'_\flat(X) = 0$ for all $X \in TN$. From the previous discussion, this will be true if, and only if $h^*(\phi^*\zeta) = 0$ for all $\zeta \in TN^o$ such that $\pi^\sharp\zeta \in \mathrm{pr}_2 \circ\phi(TN)$. Which is equivalent to

$$TN^o \cap (\pi^\sharp)^{-1}(\mathrm{pr}_2 \circ\phi(TN)) \subseteq (\phi^*)^{-1}(TN^o). \tag{5.13}$$

Let $A = \mathrm{pr}_2 \circ\phi(TN)$. Now,

$$(\pi^\sharp)^{-1}(A) = \big((\pi^\sharp)^*\big)^{-1}(A) = \big(\pi^\sharp(A^o)\big)^o,$$

and

$$(\phi^*)^{-1}(TN^o) = \big(\phi(TN)\big)^o.$$

Hence, (5.13) becomes

$$\big(TN + \pi^\sharp(A^o)\big)^o \subseteq \big(\phi(TN)\big)^o,$$

which is equivalent to

$$\phi(TN) \subseteq TN + \pi^\sharp(A^o), \tag{5.14}$$

So a generalized complex submanifold N, of a generalized complex manifold M with generalized complex structure (5.10), will have a generalized complex structure of the same form as (5.10) if and only if $\phi(TN) \subseteq TN + \pi^\sharp(A^o)$. Now, consider the following series of equivalent statements:

$$\begin{aligned} A &\subseteq \pi^\sharp(TN^o) \\ A^o &\supseteq \big(\pi^\sharp(TN^o)\big)^o \\ A^o &\supseteq (\pi^\sharp)^{-1}(TN) \\ \pi^\sharp(A^o) &\supseteq \pi^\sharp\big((\pi^\sharp)^{-1}(TN)\big) = TN. \end{aligned}$$

Thus (5.14) becomes

$$\phi(TN) \subseteq \pi^\sharp(A^o). \tag{5.15}$$

□

If N is both a complex submanifold of (M, j), and a Poisson-Dirac submanifold of (M, π) then this condition will automatically be satisfied and there will be a generalized complex structure of the form (5.10) on N. Thus N will also be a holomorphic Poisson manifold.

5.5 Spinors and generalized complex submanifolds

As discussed in Chapter 4, generalized complex structures may also be realized using Clifford algebras and spinors. The aim of this section is to prove that generalized complex submanifolds can also be realized using spinors.

Recall that a spinor, in this context, is a differential form $\lambda \in \Omega^\bullet(M)$. To each spinor is associated a null space L_λ given by

$$(L_\lambda)_m = \left\{ X_m + \xi_m \in (T_m M \oplus T^*_m M)_\mathbb{C} : (X_m + \xi_m) \cdot \lambda_m = 0 \right\}.$$

This subbundle is isotropic, and if it is also maximal isotropic then the spinor is called *pure*. Given a maximal isotropic $L = L(R, \varepsilon)$, see Section 2.7 for how E and ε are defined, then its pure spinor bundle is given by

$$\Lambda = \left\{ c(\det(R^o)) \wedge \exp(\varepsilon) : c \in \mathbb{C} \right\}. \tag{5.16}$$

See Sections 4.2i & 4.3 for the details of these constructions.

Let $h : N \to M$ be a twisted submanifold with a generalized complex structure defined by a spinor line bundle $\mathcal{L} \subseteq \wedge^\bullet T^*_\mathbb{C} M$. This spinor line bundle naturally induces a line bundle in $\Omega^\bullet_\mathbb{C}(N)$ given by $h^*\mathcal{L}$. This induced line bundle could potentially give a generalized complex structure on N. We will show that the maximal isotropic defined by $h^*\mathcal{L}$ is in fact $\mathcal{B}_h(L_\mathcal{L})$.

Proposition 5.18. *Let (M, J, Ω) be a twisted generalized complex manifold and let θ be the pure spinor line that also gives J. If $h : N \to M$ is a twisted generalized complex submanifold of M, with generalized complex structure J', then the spinor associated to J' is $h^*\theta$.*

Proof. Let L denote the Dirac structure associated to J. The spinor line bundle associated to L is given by

$$\mathcal{L} = \left\{ c(\det(R^o)) \wedge \exp(\varepsilon) : c \in \mathbb{C} \right\}.$$

Now,

$$h^*\big(c \det(R^o) \wedge \exp(\varepsilon)\big) = c \det((h^{-1}R)^o) \wedge \exp(h^*\varepsilon).$$

This line bundle is the same as the line bundle associated to $\mathcal{B}_h(L)$. □

With this proposition it is now a simple matter to give the conditions for a twisted generalized complex submanifold in terms of spinors. The involutivity is guaranteed by Lemma 5.2.

Corollary 5.19. *Let M be a twisted generalized complex submanifold, with associated spinor line bundle $\mathcal{L}$. A twisted submanifold $h : N \to M$ is a twisted generalized complex submanifold if, and only if, $h^*\mathcal{L}$ is a pure spinor line bundle and $\left(h^*\mu, h^*\overline{\mu}\right)_{\text{muk}} \neq 0$ for all $\mu \in \mathcal{L}$.*

5.6 Generalized Kähler submanifolds

Finally we will consider submanifolds of generalized Kähler structures. A *twisted generalized Kähler structure* on M is a pair of twisted generalized complex structures $J_1, J_2 : TM \oplus T^*M \to TM \oplus T^*M$ such that:

1. J_1 and J_2 commute,
2. $\left\langle X + \xi, J_1 J_2(Y + \eta)\right\rangle$, is a positive definite metric.

The first proposition of this section gives a condition, in terms of the eigenbundles, for when two complex maps will commute.

Proposition 5.20. *Let W be a real vector space with two maps $\psi_1, \psi_2 : W \to W$ such that $\psi_1^2 = \psi_2^2 = -\mathrm{Id}$. Also, let L^k_+ denote the $+i$-eigenbundles of these maps, and L^k_- denote the $-i$-eigenbundles . Using this notation, ψ_1 and ψ_2 commute if, and only if,*

$$W_{\mathbb{C}} = (L^1_+ \cap L^2_+) \oplus (L^1_+ \cap L^2_-) \oplus (L^1_- \cap L^2_+) \oplus (L^1_- \cap L^2_-).$$

Proof. First assume the two maps commute. Because of this fact, every $w \in W_{\mathbb{C}}$ can be written as

$$\begin{aligned} w =& \tfrac{1}{4}(\mathrm{Id} - iJ_1)(\mathrm{Id} - iJ_2)(w) + \tfrac{1}{4}(\mathrm{Id} - iJ_1)(\mathrm{Id} + iJ_2)(w) \\ &+ \tfrac{1}{4}(\mathrm{Id} + iJ_1)(\mathrm{Id} - iJ_2)(w) + \tfrac{1}{4}(\mathrm{Id} + iJ_1)(\mathrm{Id} + iJ_2)(w) \\ :=& w^+_+ + w^+_- + w^-_+ + w^-_-. \end{aligned}$$

It is clear that $w^\pm_\bullet \in L^1_\pm$, and $w^\bullet_\pm \in L^2_\pm$. Now assume every $w \in W_{\mathbb{C}}$ can be written as $w = w^+_+ + w^+_- + w^-_+ + w^-_-$, where $w^\pm_\bullet \in L^1_\pm$ and $w^\bullet_\pm \in L^2_\pm$. Now $(\psi_2 \circ \psi_1)(w) = \psi_2(iw^+_+ + iw^+_- - iw^-_+ - iw^-_+) = -w^+_- + w^+_- + w^-_+ - w^-_-$, and $(\psi_1 \circ \psi_2)(w) = -w^+_+ + w^+_- + w^-_+ - w^-_-$. $\square$

Let $J_1, J_2 : TM \oplus T^*M \to TM \oplus T^*M$ be two commuting bundle maps such that $J_1^2 = J_2^2 = -\mathrm{Id}$. Also, let $L^k_\pm$ denote the $+i$-eigenbundles and $-i$-eigenbundles of J_k. We also use the notation of Section 5.2. In that $B = TN \oplus T^*M|_N$, and $s : B \to TN \oplus T^*N$. The next two lemmas relate the condition above to our conditions.

Lemma 5.21. *The following are equivalent*

1. $T_\mathbb{C}N \oplus T^*_\mathbb{C}N = s\big((L^1_+ \cap L^2_+) \cap B_\mathbb{C}\big) + s\big((L^1_+ \cap L^2_-) \cap B_\mathbb{C}\big) + s\big((L^1_- \cap L^2_+) \cap B_\mathbb{C}\big) + s\big((L^1_- \cap L^2_-) \cap B_\mathbb{C}\big)$.
2. $TN \oplus T^*N = s(B \cap J_1B \cap J_2B \cap J_1J_2B)$.
3. $B = B \cap J_1B \cap J_2B \cap J_1J_2B + B^\perp$.
4. $J_1B^\perp \cap B \subseteq B^\perp$, $J_2B^\perp \cap B \subseteq B^\perp$, *and* $B \cap J_1J_2B^\perp \subseteq B^\perp$.

Proof.

(1) $\Longrightarrow$ (2): Every $v \in T_\mathbb{C}N \oplus T^*_\mathbb{C}N$ can be written as $v = s(\tilde{v}^+_+) + s(\tilde{v}^+_-) + s(\tilde{v}^-_+) + s(\tilde{v}^-_-)$, where $\tilde{v}^+_\pm \in L^1_+ \cap L^2_\pm \cap B_\mathbb{C}$ and $\tilde{v}^-_\pm \in L^1_- \cap L^2_\pm \cap B_\mathbb{C}$. Now let $\tilde{v} = \tilde{v}^+_+ + \tilde{v}^+_- + \tilde{v}^-_+ + \tilde{v}^-_-$, and so $v = s(\tilde{v})$ and $\tilde{v} \in B_\mathbb{C}$. Now $J_1(\tilde{v}) \in B_\mathbb{C}$, $J_2(\tilde{v}) \in B_\mathbb{C}$, and $J_1J_2(\tilde{v}) \in B_\mathbb{C}$. Thus $\tilde{v} \in J_1(B_\mathbb{C})$, $\tilde{v} \in J_2(B_\mathbb{C})$, and $\tilde{v} \in J_1J_2(B_\mathbb{C})$. Finally, taking the real parts of each of these gives (2).

(2) $\Longrightarrow$ (1): Every $v \in TN \oplus T^*N$ can be written as $v = s(\tilde{v})$ for some $\tilde{v} \in B \cap J_1B \cap J_2B \cap J_1J_2B$. Alternately $v = s(\tilde{v})$ for some $\tilde{v} \in B$ such that $J_1(\tilde{v}) \in B$, $J_2(\tilde{v}) \in B$, and $J_1J_2(\tilde{v}) \in B$. Now we can write

$$\tilde{v} = \tfrac{1}{4}\big((\mathrm{Id} - iJ_1) \circ (\mathrm{Id} - iJ_2)(\tilde{v}) + (\mathrm{Id} - iJ_1) \circ (\mathrm{Id} + iJ_2)(\tilde{v}) \\ + (\mathrm{Id} + iJ_1) \circ (\mathrm{Id} - J_2)(\tilde{v}) + (\mathrm{Id} + iJ_1) \circ (\mathrm{Id} + iJ_2)(\tilde{v})\big).$$

By definition each of these components is in the intersection of the eigenbundles, and the previous discussion shows that each of these terms is also in $B_\mathbb{C}$.

(2) $\Longleftrightarrow$ (3): We know $s(B) = T_\mathbb{C}N \oplus T^*_\mathbb{C}N$, and $\ker(s) = B^\perp$. Thus these two conditions are equivalent.

(3) $\iff$ *(4):* This equivalence is a fairly straightforward calculation

$$B = B^{\perp} + B \cap J_1 B \cap J_2 B \cap J_1 J_2 B$$

if, and only if,

$$B \subseteq B^{\perp} + J_1 B \cap J_2 B \cap J_1 J_2 B$$

if, and only if,

$$B \cap (J_1 B^{\perp} + J_2 B^{\perp} + J_1 J_2 B^{\perp}) \subseteq B^{\perp}$$

if, and only if,

$$B \cap J_1 B^{\perp} + B \cap J_2 B^{\perp} + B \cap J_1 J_2 B^{\perp} \subseteq B^{\perp}$$

if, and only if,

$$B \cap J_1 B^{\perp} \subseteq B^{\perp},\ B \cap J_2 B^{\perp} \subseteq B^{\perp},\ \text{and } B \cap J_1 J_2 B^{\perp} \subseteq B^{\perp}.$$

□

This last lemma strengthens the conclusions of the first statement in the previous lemma.

Lemma 5.22. *If N is a twisted generalized complex submanifold of (M, J_1) and (M, J_2) then the sums in expression (1), of the previous proposition, are direct. Also, each of the components in this expression can be rewritten as*

$$s\big((L^1_{\pm} \cap L^2_{\pm}) \cap B_{\mathbb{C}}\big) = F^1_{\pm} \cap F^2_{\pm},$$

where $F^k_{\pm} = \mathcal{B}_i(L^k_{\pm})$.

Proof. The fact that J_1 and J_2 descend to generalized complex structures on N implies that $F^k_{+} \cap F^k_{-} = \{0\}$, and the sums must be direct. Now, by definition $s(L^k_{\pm} \cap B_{\mathbb{C}}) = F^k_{\pm}$ and it is obvious that $s\big((L^1_{\pm} \cap L^2_{\pm}) \cap B_{\mathbb{C}}\big) \subseteq F^1_{\pm} \cap F^2_{\pm}$. To see the other inclusion, consider $F^1_{+} \cap F^2_{+}$. This subset will have zero intersection with F^1_{-} and F^2_{-}, and so it will not intersect with any of the other components. However, $F^1_{+} \cap F^2_{+} \subseteq T_{\mathbb{C}} N \oplus T^*_{\mathbb{C}} N$, and the fact that $T_{\mathbb{C}} N \oplus T^*_{\mathbb{C}} N$ is made up of these four components implies that $F^1_{+} \cap F^2_{+} \subseteq s\big((L^1_{\pm} \cap L^2_{\pm}) \cap B_{\mathbb{C}}\big)$. □

We are now ready to prove our last theorem, namely that these conditions are guaranteed to be satisfied by a generalized Kähler structure and so our notion of generalized complex submanifold preserves generalized Kähler structures.

Theorem 5.23. *Let N be a twisted submanifold of a generalized Káhler manifold (M, J_1, J_2). If N is a twisted generalized complex submanifold of (M, J_1) and (M, J_2), then (N, J_1', J_2') is a twisted generalized Kähler manifold.*

Proof. All that we need to show is $J_1' J_2' = J_2' J_1'$, and the metric induced by $G' = J_1' J_2'$ is positive definite. We start with the commutativity. By Lemma 5.22 and Proposition 5.20, if one of the equivalent conditions in Lemma 5.21 is true then J_1 and J_2 will commute. Consider condition (4) of this lemma. By assumption $J_1 B^\perp \cap B \subseteq B^\perp$ and $J_2 B^\perp \cap B \subseteq B^\perp$. All that remains is to show $B \cap J_1 J_2 B^\perp \subseteq B^\perp$. Take $v \in B \cap J_1 J_2 B^\perp$, so $v \in B^\perp$ and $J_1 J_2 v \in B$. Thus $\left\langle v, J_1 J_2 v \right\rangle = 0$. However, by assumption this metric is positive definite and so $v = 0$. Thus $B \cap J_1 J_2 B^\perp \subseteq \{0\}$, and $B \cap J_1 J_2 B^\perp \subseteq B^\perp$ is always true. It remains to show that $J_1' J_2'$ defines a positive definite metric. Take $v \in TM \oplus T^*M$ and $\tilde{v} \in B \cap J_1 B \cap J_2 B \cap J_1 J_2 B$ such that $s(\tilde{v}) = v$. Because s does not change the inner product

$$\left\langle v, J_1' J_2'(v) \right\rangle = \left\langle s(\tilde{v}), J_1' J_2' s(\tilde{v}) \right\rangle = \left\langle s(\tilde{v}), s J_1 J_2(\tilde{v}) \right\rangle = \left\langle \tilde{v}, J_1 J_2(\tilde{v}) \right\rangle,$$

and the positive definiteness of $J_1 J_2$ implies the positive definiteness of $J_1' J_2'$. □

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