Topological Sigma Models from Generalized Para-Complex Geometry

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Abstract

We study a class of $2d$ topological sigma models that are formulated on generalized para-complex target manifolds via the AKSZ framework. When the target is additionally generalized para-Kähler, these topological theories can be realized as topological twists of $(2,2)$ para-supersymmetric sigma models of Abou-Zeid and Hull. We compare our results to the case of generalized complex and generalized Kähler geometry, where similar results have been described in the context of the usual $(2,2)$ SUSY by Kapustin and Li. We present several examples and show that our construction recovers the A-model and the Poisson sigma model as special cases. Moreover, we also introduce new examples of topological theories inherently tied to the para-complex geometry, the para-B-model and a para-complex version of the Landau-Ginzburg model.

Contents

1 Introduction

Since the work of (..) [**?**] it has been known that on any Kähler manifold (M, I, ω), one can define a $2d$ sigma model admitting a $(2, 2)$ supersymmetry (SUSY) and such model has a simple local description in terms of the Kähler potential. Moreover, Witten has shown [**?**] that such sigma model gives rise to two distinct topological theories – the A- and B-models – via a procedure called topological twisting. These topological theories have an important interpretation in terms of their underlying geometry: the A-model only depends on the symplectic structure (M, ω), while the B-model only depends on the complex structure (M, I). Additionally, there exists an $\mathbb{Z}_2$ automorphism acting on the original $(2, 2)$ supersymmetric theory that exchanges its A- and B-models. This is the simplest way one can observe the phenomenon of mirror symmetry in the context of $2d$ sigma models.

The Kähler geometry does not describe the most general $(2, 2)$ sigma model, as was shown in [**?**]. Instead, the most general geometry is described by a bi-Hermitian geometry, defined by a pair of complex structures $I_\pm$ compatible with the same Riemannian metric g and satisfying the compatibility relation

$$\mathrm{d}^c_\pm \omega_\pm = \pm H \in \Omega^3_{cl.}(M),$$

$\omega_\pm$ being the fundamental forms of $I_\pm$ and $\mathrm{d}^c_\pm$ the associated d^c-operators. For this geometry, the local description is more complicated than in the Kähler case, in particular because the analogue of the Kähler potential is much harder to describe. Consequently, the topological twists are also more complicated.

This problem is nevertheless partially mitigated by an alternative geometric formulation in terms of generalized Kähler (GK) geometry, which has been shown in [**?**] to be equivalent to the above-described bi-Hermitian geometry $(g, I_\pm, H)$. A GK structure on M is given by the data of two complex endomorphisms $\mathcal{J}_\pm$ of the bundle $(T \oplus T^*)M$:

$$\mathcal{J}_\pm \in \mathrm{End}((T \oplus T^*)M), \quad \mathcal{J}_\pm^2 = -\mathbb{1}, \quad \langle \mathcal{J}\cdot, \mathcal{J}\cdot \rangle = \langle \, , \, \rangle,$$

where $\langle \, , \, \rangle$ denotes the natural duality pairing on $(T \oplus T^*)M$:

$$\langle X + \alpha, Y + \beta \rangle = \alpha(Y) + \beta(X), \quad X, Y \in \Gamma(TM), \quad \alpha, \beta \in \Gamma(T^*M).$$

Additionally, $\mathcal{J}_\pm$ are required to commute and their product $\mathcal{G} \coloneqq -\mathcal{J}_+ \mathcal{J}_-$ to define a

1.1 Notations and conventions

BM: Conventions for odd Lie brackets $[-, -]$ and $\{-, -\}$.

2 Background on paracomplex and generalized geometry

DB: would just put the introduction of paracomplex geometry (minus the generalized part) in appendix, we did that even in the other paper which is literally a differential geometry paper

2.1 Paracomplex geometry

In this section we introduce para-complex geometry with emphasis on the analogy with complex geometry. More details can be found in [**?**] or [**?**, Ch. 15], for example. BM: I think you should cite your work here too.

Definition 2.1. An (almost) **product structure** on a smooth manifold $\mathcal{P}$ is an endomorphism $K \in \text{End}(T\mathcal{P})$ which squares to the identity, $K^2 = \mathbb{1}_{T\mathcal{P}}$, $K \neq \mathbb{1}$. An (almost) **para-complex structure** is a product structure such that the $+1$ and -1 eigenbundles of K have the same rank.

A direct consequence of above definition is that any para-complex manifold is of even dimension. From now on, the $+1$ and -1 eigenbundles of an almost product/para-complex structure will be denoted L and $\tilde{L}$, respectively.

Integrability The use of the word *almost* as usual refers to integrability of the endomorphism, i.e. whether its eigenbundles are involutive with respect to the Lie bracket and therefore define a foliation of the underlying manifold. Similarly to the complex case, the integrability is governed by the **Nijenhuis tensor**

$$
\begin{aligned}
N_K(X,Y) &= [X,Y] + [KX,KY] - K([KX,Y] + [X,KY]) \\
&= (\nabla_{KX}K)Y + (\nabla_X K)KY - (\nabla_{KY}K)X - (\nabla_Y K)KX \\
&= 4(\text{P}[\tilde{\text{P}}X, \tilde{\text{P}}Y] + \tilde{\text{P}}[\text{P}X, \text{P}Y]),
\end{aligned}
\tag{1}
$$

where ∇ is any torsionless connection and $\text{P} \coloneqq \frac{1}{2}(\mathbb{1} + K)$ and $\tilde{\text{P}} \coloneqq \frac{1}{2}(\mathbb{1} - K)$ are the projections onto L and $\tilde{L}$, respectively. We say K is integrable and call K a product/para-complex manifold if $N_K = 0$. From (**??**) it is apparent that K is integrable if and only if *both* eigenbundles are simultaneously Frobenius integrable, i.e. involutive distributions in $T\mathcal{P}$. This is one of the main differences between complex and para-complex geometry; one of the eigenbundles can be integrable while the other is not. For this reason, it is useful to introduce the notion of **half-integrability**:

Type decomposition DB: simplify this section The splitting of the tangent bundle $T\mathcal{P} = L \oplus \tilde{L}$ in the real case and $T\mathcal{P}^{\mathbf{C}}$ in the para-complex case gives rise to a decomposition of tensors analogous to the (p,q)-decomposition in complex geometry. For differential forms, we denote the real and para-complex

decompositions as

$$\Lambda^k(T^*\mathcal{P}) = \bigoplus_{k=m+n} \Lambda^{(+m,-n)}(T^*\mathcal{P}), \qquad (2)$$

$$\Lambda^k(T^*\mathcal{P}^{\mathbf{C}}) = \bigoplus_{k=m+n} \Lambda^{(m,n)}(T^*\mathcal{P}), \qquad (3)$$

where $\Lambda^{(+m,-n)}(T^*\mathcal{P}) = \Lambda^m(L^*) \otimes \Lambda^n(\tilde{L}^*)$ and $\Lambda^{(m,n)}(T^*\mathcal{P}) = \Lambda^m(T^{(1,0)*}) \otimes \Lambda^n(T^{(0,1)*})$. The corresponding sections are denoted by $\Omega^{(+m,-n)}(\mathcal{P})$ and $\Omega^{(m,n)}$, respectively. The bigradings (**??**) yield the natural projections

$$\Pi^{(+p,-q)} : \Lambda^k(T^*\mathcal{P}) \to \Lambda^{(+p,-q)}(T^*\mathcal{P}),$$
$$\Pi^{(p,q)} : \Lambda^k(T^*\mathcal{P}^{\mathbf{C}}) \to \Lambda^{(p,q)}(T^*\mathcal{P}),$$

so that when K is integrable, the de-Rham differential splits as $\mathrm{d} = \partial_+ + \partial_-$, in the real case and $\mathrm{d} = \partial + \bar{\partial}$ in the para-complexified case, where

$$\partial_+ := \Pi^{(+p+1,-q)} \circ \mathrm{d}, \quad \partial_- := \Pi^{(+p,-q-1)} \circ \mathrm{d},$$
$$\partial := \Pi^{(p+1,q)} \circ \mathrm{d}, \qquad \bar{\partial} := \Pi^{(p,q+1)} \circ \mathrm{d},$$

are the **para-complex Dolbeault operators**, satisfying

$$\partial_+^2 = \partial_-^2 = \partial_+\partial_- + \partial_-\partial_+ = 0$$
$$\partial^2 = \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0$$

One can also introduce the *twisted differential* $\mathrm{d}^p := (\Lambda^{k+1}K^*) \circ \mathrm{d} \circ (\Lambda^k K^*)$. When K is integrable, it can be simply written as $\mathrm{d}^p = (\partial_+ + \partial_-)$ on real forms and $\mathrm{d}^p = \mathrm{k}(\partial + \bar{\partial})$ on para-complex forms.

2.2 Generalized para-geometry

BM: motivate generalized geometry through courant algebroids
    BM: get rid of repetitions and references to dg / NQ manifolds.
    In this paper, the term *generalized geometry* is used for the study of geometric BM: geometrical is not a word structures on the bundle $TM \oplus T^*M$ over some manifold M. We will typically abbreviate this bundle to $T \oplus T^*$ whenever the base is understood or unimportant for the discussion.
    For now, we record a relationship between generalized paracomplex structures and Courant algebroids.

2.2.1 Background on Courant algebroids

Recall, a Courant algebroid on a manifold M is a vector bundle E, whose sheaf of sections we denote $\mathcal{E}$, together with a nondegenerate symmetric bilinear pairing $\langle -, - \rangle : \mathcal{E} \otimes \mathcal{E} \to C^\infty(M)$, an anchor map $a : E \to TM$, and a bilinear operator

$$[-, -] : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$$

called the Courant-Dorfman bracket. BM: spell out properties, find a good
reference Denote by $a^* : T^*M \to E^*$ the bundle map dual to a.

To every Courant algebroid we can define the following sequence of vector
bundles on M

$$T^*M \xrightarrow{a^*} E \xrightarrow{a} TM.$$

A Courant algebroid is called *exact* if this sequence of bundles is exact.

The simplest type of Courant algebroid is one over a point.

Example 2.2. Any Lie algebra $\mathfrak{g}$ together with a non-degenerate invariant pairing
defines a 2-shifted symplectic structure on the graded manifold $\mathfrak{g}[1]$ living over a
point. The dg algebra of functions is the Chevalley-Eilenberg cochain complex
computing Lie algebra cohomology $C^*(\mathfrak{g})$. All such 2-symplectic NQ manifolds
over a point are of this form.

The 2-shifted symplectic space of an exact Courant Algebroid We
recall the description of the 2-shifted symplectic NQ manifold X_E associated
to an exact Courant algebroid E. Of course, as a vector bundle $E = T \oplus T^*$,
and hence as a graded manifold X_E is of the form $T^*[2]T[1]M$. Concretely, the
coordinates on this manifold are given by $(x^i, v^a, p_i, \mu_a)_{i,a=1\cdots n}$, where x^i are
the usual coordinates on M with degree 0, v^a are the corresponding "velocities",
i.e. the fibre coordinates on $T[1]M$ with a degree 1, and p_i, μ_a are the momenta
corresponding to x^i, v^a with degrees 2, 1, respectively. The canonical symplectic
form

$$\Omega = dx^i dp_i + dv^a d\mu_a,$$

is then of degree 2. $T^*[2]T[1]M$ is the minimal symplectic realization of the
shifted bundle $(T \oplus T^*)[1]M$, which is recovered by setting $p_i = 0$. The sym-
metric part of Ω represents the pairing $\langle \, , \, \rangle$ on $(T \oplus T^*)M$; this is because we
can identify $d\mu_a \leftrightarrow \partial_i$ and $dv^a \leftrightarrow dx^i$

BM: Severa classification

Theorem 2.3. *The set of isomorphism classes of exact Courant algebroids is
a torsor for $H^3(X)$.*

The Poisson Courant algebroid BM: Both real and holomorphic versions

2.2.2 The exact Courant algebroid structure

The bundle $T \oplus T^*$ has a natural Courant algebroid structure given by the
symmetric pairing

$$\langle X + \alpha, Y + \beta \rangle = \alpha(Y) + \beta(Y),$$

the Dorfman bracket,

$$[X + \alpha, Y + \beta] = [X, Y] + \mathcal{L}_X \beta - \imath_Y d\alpha, \tag{4}$$

and the anchor $\pi : X + \alpha \mapsto X$. In the above, $X + \alpha$ denotes a section of $T \oplus T^*$ with the splitting to tangent and cotangent parts given explicitly. The Dorfman bracket can be thought of as an extension of the Lie bracket from T to $T \oplus T^*$ and therefore we opt to use the same notation for both brackets; the expression $[X, Y]$ is always the Lie bracket of vector fields whether we think of $[\ ,\ ]$ as the Lie bracket or the Dorfman bracket and no confusion is therefore possible.

The Courant algebroid on $T \oplus T^*$ is exact In fact, all possible Courant algebroid structures on $T \oplus T^*$ are parametrized by a closed three-form $H \in \Omega^3_{cl}$, which enters the definition of the bracket (??), changing it to a *twisted* Dorfman bracket

$$[X + \alpha, Y + \beta]_H = [X, Y] + \mathcal{L}_X \beta - \imath_Y \mathrm{d}\alpha + \imath_Y \imath_X H.$$

Moreover, any isotropic splitting of (??) $s : T \to T \oplus T^*$ is given by a two-form b, such that $X \overset{s}{\mapsto} X + b(X)$. This is equivalent to an action of a b-field transformation on $T \oplus T^{*1}$

$$e^b = \begin{pmatrix} \mathbb{1} & 0 \\ b & \mathbb{1} \end{pmatrix} \in \mathrm{End}(T \oplus T^*),$$

which consequently changes the bracket as

$$[e^b(X + \alpha), e^b(Y + \beta)]_H = e^b([X + \alpha, Y + \beta]_{H + \mathrm{d}b}),$$

which implies that when H is trivial in cohomology, then a choice of a b-field transformation such that $\mathrm{d}b = -H$ brings the twisted bracket $[\ ,\ ]_H$ into the standard form (??). When H is cohomologically non-trivial this can be done at least locally. This also means that any choice of splitting with a non-trivial b-field can be absorbed in the Courant algebroid bracket in terms of the *flux*[2] $\mathrm{d}b$.

Dirac Structures   An important object in Dirac geometry are (almost) dirac structures, which are subbundles $L \subset T \oplus T^*$ with special properties.

Definition 2.4. An **almost Dirac structure** L is a maximally isotropic subbundle of $T \oplus T^*$, i.e. $\langle u, v \rangle = 0$ for any $u, v \in \Gamma(L)$ and $\mathrm{rank}(L) = \mathrm{rank}(T)$. When L is involutive under the Dorfman bracket, i.e. it satisfies $[L, L] \subset L$, we call L simply a **Dirac structure**.

An important fact we will repeatedly use is the following:

Proposition 2.5 ([?])**.** *Let L be a Dirac structure in the Courant algebroid $T \oplus T^*$ with a flux H. Then the restriction of the Dorfman bracket to L, $[\ ,\ ]|_L$ is skew and gives L a structure of a Lie algebroid compatible with the anchor $\pi|_L$, a restriction of the projection $\pi : T \oplus T^* \to T$ to L.*

---

[1]Here we are using the term b-field transformation more liberally as it is customary to use the term only in the cases when $\mathrm{d}b = 0$ so that e^b is a symmetry of $[\ ,\ ]$.

[2]Flux is a term used mainly in physics, in this context simply meaning the "tensorial contribution to the bracket".

We remark here that all the results in this paper remain valid for any exact courant algebroid E (i.e. E fits in the sequence (??)), which can be always identified with $T \oplus T^*$ by the choice of splitting equivalent to a choice of a representative $H \in \Omega^3_{cl}$. This also amounts to setting $b = 0$ in all formulas since the b-field appears as a difference of two splittings.

We now recall the notion of a generalized para-complex [?, ?, ?] and a generalized para-Kähler [?] structure. We follow the discussion presented in [?].

Definition 2.6. A **(twisted) generalized para-complex** (GpC) structure $\mathcal{K}$ is an endomorphism of $T \oplus T^*$, such that $\mathcal{K}^2 = \mathbb{1}$ and $\langle \mathcal{K}, \mathcal{K} \rangle = -\langle \ , \ \rangle$, whose generalized Nijenhuis tensor vanishes:

$$\mathcal{N}_{\mathcal{K}}(u, v) = [\mathcal{K}u, \mathcal{K}v]_H + \mathcal{K}^2[u, v]_H - \mathcal{K}([\mathcal{K}u, v]_H + [u, \mathcal{K}v]_H) = 0. \qquad (5)$$

It is easy to check that the condition (??) is equivalent to the ± 1 eigenbundles of $\mathcal{K}$ to be involutive under the twisted Dorfman bracket. Since in this paper the flux H will be generically non-zero but all results hold for the special case $H = 0$ as well, we will drop the word "twisted" from our definitions.

Generalized para-complex structures can be understood as the data one needs to specify a splitting of $T \oplus T^*$ to a pair of integrable Dirac structures:

Theorem ([?]). *There is a one-to-one correspondence between generalized para-complex structures on M and pairs of transversal Dirac subbundles of $T \oplus T^*$.*

Combining this result with the well-known result of [?] which states that any pair of transversal Dirac structures $(L, \tilde{L})$ forms a Lie bialgebroid $(L, L^* \simeq \tilde{L})$, one can immediately infer the following

Corollary 2.7. *Generalized para-complex structures on $T \oplus T^*$ are in one-to-one correspondence with Lie bialgebroid pairs (L, L^*) such that $L \oplus L^* = T \oplus T^*$.*

Example 2.8 (The trivial structure and its deformations). Any manifold supports the following GpC structure

$$\mathcal{K}_0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix},$$

that has eigenbundles T and T^* and is always integrable. The following two GpC structures can be seen as deformations of $\mathcal{K}_0$ by either a two-form b or a bi-vector β:

$$\mathcal{K}_b = \begin{pmatrix} \mathbb{1} & 0 \\ 2b & -\mathbb{1} \end{pmatrix}, \quad \mathcal{K}_\beta = \begin{pmatrix} \mathbb{1} & 2\beta \\ 0 & -\mathbb{1} \end{pmatrix}.$$

$\mathcal{K}_b$ is integrable iff $\mathrm{d}b = -H$, and its eigenbundles are $\mathbb{L}_b = \mathrm{graph}(b) = \{X + b(X) \mid X \in \mathfrak{X}\}$ and $\widetilde{\mathbb{L}} = T^*$. Similarly, $\mathcal{K}_\beta$ is integrable iff β is a Poisson structure, (see [?, Lemma 2.13]) and its eigenbundles are $\mathbb{L} = T$ and $\widetilde{\mathbb{L}} = \mathrm{graph}(-\beta) = \{\alpha - \beta(\alpha)\} \mid \alpha \in \Omega\}$.

Example 2.9 (Para-complex structures). A para-complex structure $K \in \mathrm{End}(T)$, defines the diagonal generalized para-complex structure:

$$\mathcal{K}_K = \begin{pmatrix} K & 0 \\ 0 & -K^* \end{pmatrix}.$$

The corresponding Dirac structures are given by $\mathbb{L} = T^{(1,0)} \oplus T^{*(0,1)}$ and $\widetilde{\mathbb{L}} = T^{(0,1)} \oplus T^{*(1,0)}$, where the bigrading is with respect to K. The integrability of $\mathcal{K}_K$ is equivalent to Frobenius integrability of K, i.e. vanishing of the Nijenhuis tensor of K.

Example 2.10 (Symplectic structures). A symplectic form ω defines the anti-diagonal GpC structure

$$\mathcal{K}_\omega = \begin{pmatrix} 0 & \omega^{-1} \\ \omega & 0 \end{pmatrix}.$$

The ± 1 eigenbundles are given by $\mathrm{graph}(\pm\omega) = \{X \pm \omega(X) \mid X \in \mathfrak{X}\}$, and the integrability of $\mathcal{K}_\omega$ is equivalent to $\mathrm{d}\omega = 0$.

BM: Can we import some of the other examples here? Maybe some explicit ones like the toric fibrations you mentioned.

Comparision with generalized complex structures   Example ?? shows that a symplectic manifold (M, ω) is a GpC manifold and it is well-known [?] that (M, ω) is also generalized complex (GC). However, while almost GC structures exist only on almost complex manifolds [?], Example ?? demonstrates that GpC structures exist on any Poisson manifold and in particular on any smooth manifold (with trivial Poisson structure). Another feature of GpC geometry that is not present in GC geometry is that the GpC structures can be half-integrable (similarly to ordinary para-complex structures): see the cases of $\mathcal{K}_b$ and $\mathcal{K}_\beta$ from Example ?? which are always at least half integrable and are fully integrable iff b is closed and β is Poisson, respectively. On the other hand, $\mathcal{K}_\omega$ in Example ?? can only be half-integrable when the H-flux is non-zero.

2.3   Generalized para-Kähler geometry

BM: add buffer

Definition 2.11. A **generalized para-Kähler structure** (GpK) is a commuting pair $(\mathcal{K}_+, \mathcal{K}_-)$ of GpC structures, such that their product $\mathcal{G} = \mathcal{K}_+ \mathcal{K}_-$ defines a split-signature metric G on $T \oplus T^*$ via

$$G(\cdot, \cdot) \coloneqq \langle \mathcal{G}\cdot, \cdot \rangle.$$

Because $\mathcal{K}_\pm$ are GpC structures, they induce the splitting $T \oplus T^* = \mathbb{L}_\pm \oplus \widetilde{\mathbb{L}}_\pm$. Moreover, because they commute, each of $\mathbb{L}_+$ and $\widetilde{\mathbb{L}}_+$ further splits to

eigenbundles of $\mathcal{K}_-$, so that one arrives at the decomposition of $T \oplus T^*$ into four bundles:

$$T \oplus T^* = \ell_+ \oplus \ell_- \oplus \tilde{\ell}_+ \oplus \tilde{\ell}_-, \tag{6}$$

such that the eigenbundles of $\mathcal{K}_+$ are $\mathbb{L}_+ = \ell_+ \oplus \ell_-$ and $\tilde{\mathbb{L}}_+ = \tilde{\ell}_+ \oplus \tilde{\ell}_-$, the eigenbundles of $\mathcal{K}_-$ are $\mathbb{L}_- = \ell_+ \oplus \tilde{\ell}_-$ and $\tilde{\mathbb{L}}_- = \ell_- \oplus \tilde{\ell}_+$ and the eigenbundles of the generalized metric are $C_\pm = \ell_\pm \oplus \tilde{\ell}_\pm$. All $\ell_\pm$ and $\tilde{\ell}_\pm$ are isotropic and integrable under the Dorfman bracket [?] and bundles with opposite chirality are orthogonal.

Similarly to generalized Kähler structures, the GpK structures can be equivalently given in terms of a pair of para-Hermitian structures satisfying a particular integrability condition:

Theorem 2.12 ([?]). *The data of a GpK structure is equivalent to a pair of para-Hermitian structures (η, K_+, K_-), such that $\mathcal{K}_\pm$ are given by*

$$\mathcal{K}_\pm = \frac{1}{2} \begin{pmatrix} \mathbb{1} & 0 \\ b & \mathbb{1} \end{pmatrix} \begin{pmatrix} K_+ \pm K_- & \omega_+^{-1} \mp \omega_-^{-1} \\ \omega_+ \mp \omega_- & -(K_+^* \pm K_-^*) \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ -b & \mathbb{1} \end{pmatrix}, \tag{7}$$

for some 2-form b. The integrability of $\mathcal{K}_\pm$ then translates into $K_\pm$ being integrable and satisfying

$$\nabla^\pm K_\pm = 0,$$

where the connections $\nabla^\pm$ are defined by the Levi-Civita connection $\mathring{\nabla}$ of η and the H-flux:

$$\eta(\nabla_X^\pm Y, Z) = \eta(\mathring{\nabla}_X Y, Z) \pm \frac{1}{2}(H + \mathrm{d}b).$$

The additional integrability condition on $K_\pm$, $\nabla^\pm K_\pm$, can be equivalently expressed using the fundamental forms associated to $K_\pm$, $\omega_\pm = \eta K_\pm$, and a para-complex version of the d^c-operator, which we call d^p:

$$\nabla^\pm K_\pm = 0 \iff \mathrm{d}_\pm^p \omega_\pm = \pm(H + \mathrm{d}b).$$

$\mathrm{d}_\pm^p$ here denote the d^p operators associated to $K_\pm$, $\mathrm{d}_\pm^p := K_\pm^* \circ \mathrm{d} \circ K_\pm^*$. We will call the geometry given by the data $(\eta, K_\pm, H + \mathrm{d}b)$ above a **bi-para-Hermitian** geometry.

2.3.1 Isomorphism between $T \oplus T^*$ and $T \oplus T$

In both the GK and GpK cases, the product of the pair of G(p)C structures defines a generalized (indefinite) metric $\mathcal{G}$, which means that

$$G(\cdot, \cdot) := \langle \mathcal{G} \cdot, \cdot \rangle,$$

defines a genuine (indefinite) metric on the bundle $T \oplus T^*$. It can be shown
(see eg. [?, ?]) that such metric can be always locally expressed in terms of a
two-form b and a metric g

$$\mathcal{G} = \mathcal{G}(g, b) = \begin{pmatrix} \mathbb{1} & 0 \\ b & \mathbb{1} \end{pmatrix} \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ -b & \mathbb{1} \end{pmatrix},$$

and the signature of g then determines the signature of $\mathcal{G}(g, b)$. Therefore, a
generalized metric determines a splitting of $T \oplus T^* \simeq e^b(T) \oplus T^*$ in which the
eigenbundles $C_\pm$ of $\mathcal{G}$ are $C_\pm = graph(\pm g)$ and the flux of this splitting is given
by the closed three-form $H \stackrel{loc.}{=} \mathrm{d}b$. In the usual splitting $T \oplus T^*$, we then have
$C_\pm = graph(b \pm g)$.

Remark 2.13. BM: maybe this should go later? The most general $2D$ $(1, 1)$
sigma model on a pseudo-Riemannian target (M, g) is locally given by the action
functional

$$S_{(1,1)}(\Phi) = \int_{\hat{\Sigma}} [g(\Phi) + b(\Phi)]_{ij} D_+ \Phi^i D_- \Phi^j, \tag{8}$$

where $\mathrm{d}b = H$ is a global closed three-form. Therefore, the data of a generalized
metric $\mathcal{G}$ is equivalent to a specification of the sigma model (??).

An important property of the generalized metric $\mathcal{G}$ is that its eigenbundles
$C_\pm$ are isomorphic to the tangent bundle T via the projection:

$$\begin{aligned} \pi_\pm &: C_\pm \simeq T, \\ X + \alpha &\stackrel{\pi_\pm}{\longmapsto} X, \\ X + (b \pm g)X &\stackrel{\pi_\pm^{-1}}{\longleftarrow} X, \end{aligned} \tag{9}$$

where g and b are the defining data of $\mathcal{G}$. Therefore, since $T \oplus T^* = C \oplus C_-$,
we also have $T \oplus T^* \simeq T \oplus T$:

$$\begin{aligned} (X_+, X_-) &\stackrel{\pi_+^{-1} \oplus \pi_-^{-1}}{\longmapsto} X_+ + (b + g)X_+ + X_- + (b - g)X_- \\ &= \begin{pmatrix} X_+ + X_- \\ g(X_+ - X_-) + b(X_+ + X_-) \end{pmatrix}. \end{aligned} \tag{10}$$

In particular, one can check that $\pi_+ \oplus \pi_-$ maps the eigenbundles of the gener-
alized structures $\mathcal{K}_\pm$ $(\mathcal{J}_\pm)$ to eigenbundles of the corresponding pair of tangent
bundle endomorphisms $\mathcal{K}_\pm$ $(\mathcal{J}_\pm)$. This is because of the following relationship
[?],

$$K_+ = \pi_+ \mathcal{K}_\pm \pi_+^{-1}, \quad K_- = \pm \pi_- \mathcal{K}_\pm \pi_-^{-1},$$

and analogously for $\mathcal{J}_\pm$ and $J_\pm$. From here, it is easy to see that the $+1$
eigenbundle of $\mathcal{K}_+$ $\mathbb{L}_+$, for example, is given by

$$\mathbb{L}_+ = \pi_+^{-1}(T^{(1,0)+}) \oplus \pi_-^{-1}(T^{(1,0)-}),$$

where $T^{(1,0)\pm}$ denotes the $+1$ eigenbundle of $K_\pm$. Analogously, the -1 eigenbundle of $\mathcal{K}_-$, $\widetilde{\mathbb{L}}_-$, is given by

$$\widetilde{\mathbb{L}}_- = \pi_+^{-1}(T^{(0,1)+}) \oplus \pi_-^{-1}(T^{(1,0)-}),$$

and so on. Therefore, one should think about the pair $K_\pm$ corresponding to the GpK structure as each of the $K_\pm$ acting on its own copy of T, which are then mapped via $\pi_\pm$ to $C_\pm$ in $T \oplus T^*$.

We will also notice that $\pi_\pm$ map the the pseudo-Riemannian structures G and g on each other:

$$g(X,Y) = \pm\frac{1}{2}\langle \pi_\pm^{-1}X, \pi_\pm^{-1}Y \rangle = \frac{1}{2}\langle \mathcal{G}\pi_\pm^{-1}X, \pi_\pm^{-1}Y \rangle = \frac{1}{2}G(\pi_\pm^{-1}X, \pi_\pm^{-1}Y). \quad (11)$$

Another important property we will make use of is the fact that the connections $\nabla^\pm$ give rise via $\pi_\pm$ to a generalized connection D on $T \oplus T^*$ called the generalized Bismut connection [**?**]:

$$D_u v = \pi_+^{-1}\nabla_{\pi u}^+ \pi_+ v_+ + \pi_-^{-1}\nabla_{\pi u}^- \pi_- v_-. \quad (12)$$

This connection preserves $\langle\ ,\ \rangle$ and $\mathcal{G}$ and in the case when $\mathcal{G}$ is a generalized metric corresponding to a GpK structure, D also preserves all the eigenbundles in (**??**).

There is a tensorial quantity associated to any generalized connection called generalized torsion, which is defined as [**?**]

$$T^D(u,v,w) = \langle D_u v - D_v u - [u,v]_H, w \rangle + \langle D_w u, v \rangle. \quad (13)$$

For the generalized Bismut connection, this torsion has only pure components in $\Lambda^3 C_\pm$ and satisfies the following formula [**?**, Prop. 2.29]:

$$T^D(\pi_\pm^{-1}X, \pi_\pm^{-1}Y, \pi_\pm^{-1}Z) = 2H_b(X,Y,Z). \quad (14)$$

3   Two-dimensional parasupersymmetry

In this section we introduce the parasupersymmetry algebra in two-dimensions.
BM: add more

3.1   A reminder on $(1,1)$ supersymmetry

The 2-dimensional $(1,1)$ supertranslation algebra is the super Lie algebra

$$\mathfrak{t}_{(1,1)} = \mathbb{R}^{1,1} \oplus \Pi(S_+) \oplus \Pi(S_-)$$

where $S_\pm \cong \mathbb{R}$ are the semi-spin representations of $\mathrm{Spin}(1,1)$ labeled by helicities $\pm\frac{1}{2}$. Note that there are natural $\mathrm{Spin}(1,1)$-equivariant maps

$$\Gamma_\pm : \mathrm{Sym}^2(S_\pm) \to \mathbb{R}^{1,1}$$

which are non-degenerate and whose images hit the subalgebras spanned by $\partial_\pm$, respectively. The only nonvanishing Lie brackets in $\mathfrak{t}_{(1,1)}$ are defined by

$$[Q_+, Q'_+] = \Gamma_+(Q_+ \otimes Q'_+) \ , \ \ [Q_-, Q'_-] = \Gamma_-(Q_- \otimes Q'_-)$$

where $Q_\pm, Q'_\pm$ are generic elements in $S_\pm$.

We will use the lightcone basis $\partial_\pm$ for $\mathbb{R}^{1,1}$ and we also fix a basis $Q^1_\pm$ for $S_\pm$ with a relative normalization so that

$$[Q^1_\pm, Q^1_\pm] = 2\partial_\pm. \tag{15}$$

3.1.1   The $(1,1)$ supersymmetric σ-model

Let $\Sigma^{2|2}$ be a super Riemann surface with two real odd directions. The even part of this supermanifold is a Riemann surface (with coordinates σ, τ). For instance, in the flat case we consider the superspace $\mathbb{R}^{2|2} = \{(x^i, \theta^\pm_1) \mid i = 1, 2\}$, where $\theta^\pm_1$ are odd variables.

The most general $(1,1)$ supersymmetric sigma model into a target pseudo-Riemannian manifold (M, g) is given by the action functional

$$S_{(1,1)}(\Phi) = \int_{\Sigma^{2|2}} [g(\Phi) + b(\Phi)]_{ij} D^1_+ \Phi^i D^1_- \Phi^j, \tag{16}$$

where $\Phi = (\Phi^i)_{i=1\cdots n}$ are local representatives for the $(1,1)$ superfields which are maps $\Phi : \Sigma^{2|2} \to M$. In the action, b denotes a local two-form potential for a closed three-form, $H = \mathrm{d}b$, and $D_\pm$ are the superderivatives

$$D^1_\pm = \frac{\partial}{\partial\theta^\pm_1} - \theta^\pm_1 \partial_\pm, \tag{17}$$

where $\partial_\pm = \frac{\partial}{\partial x_\pm}$ are derivatives along the lightcone coordinates on Σ, $x_\pm = \tau \pm \sigma$.

In superworldsheet coordinates, the superfields decompose as

$$\Phi^i(\sigma, \tau, \theta_+, \theta_-) = \phi^i(\sigma, \tau) + \psi^i_+(\sigma, \tau)\theta^+ + \psi_-(\sigma, \tau)\theta^- + F^i(\sigma, \tau)\theta^+\theta^-.$$

Here, (ϕ^i) are local representatives for a smooth map $\phi : \Sigma \to M$.

The fact that this action carries $(1,1)$ supersymmetry means that the action (??) is invariant under transformations generated by the two supercharges $Q^1_\pm$,

$$Q^1_\pm = \frac{\partial}{\partial\theta^\pm_1} + \theta^\pm_1 \partial_\pm, \tag{18}$$

obeying the supercommutation relations of the $(1,1)$ supersymmetry algebra (??).

3.2 $(2,2)$ para-supersymmetry

DB: Import paragraph on (2,2)

　　BM: point to references who have also discussed versions of (2,2) para susy

Definition 3.1. The **(2,2) para-supersymmetry algebra** has the underlying supervector space

$$\mathfrak{t}_{(2,2)} = \mathbb{R}^{1,1} \oplus \Pi(S_+ \oplus S_+) \oplus \Pi(S_- \oplus S_-).$$

The Lie brackets are described as follows. If we choose a basis $\{Q_\pm^1, Q_\pm^2\}$ for $S_\pm \oplus S_\pm$ then the nontrivial brackets read

$$(2,2) \text{ para} \;\; : \;\; [Q_\pm^1, Q_\pm^1] = 2\partial_\pm \;\; , \;\; [Q_\pm^2, Q_\pm^2] = -2\partial_\pm. \tag{19}$$

Remark 3.2. Note that this differs from ordinary $(2,2)$ supersymmetry by a sign:

$$(2,2) \text{ ordinary} \;\; : \;\; [Q_\pm^1, Q_\pm^1] = 2\partial_\pm \;\; , \;\; [Q_\pm^2, Q_\pm^2] = 2\partial_\pm. \tag{20}$$

　　More generally, suppose $W_\pm$ are vector spaces of dimension $k_\pm$, respectively. In addition, suppose $\eta_\pm$ are symmetric nondegenerate forms on $W_\pm$. Then, we can define a super Lie algebra of the form

$$\mathfrak{t}_W = \mathbb{R}^{1,1} \oplus \Pi(S_+ \otimes W_+) \oplus \Pi(S_- \otimes W_-)$$

with brackets defined by

$$[Q_\pm^a, Q_\pm^b] = 2\eta_\pm^{ab}\partial_\pm.$$

The case of $\mathfrak{t}_{(2,2)}$, which will be of most interested to us, corresponds to taking $k_+ = k_- = 2$ and $\eta_\pm^{11} = 1$, $\eta_\pm^{22} = -1$, $\eta_\pm^{12} = 0$.

　　BM: Can change basis to off diagonal and write in more conventional way.

3.2.1   R-symmetry

DB: Move to mirror symmetry speculation section

　　We have seen above that the $(2,2)$ para-SUSY algebra $\mathfrak{t}_{(2,2)}$ can be diagonalised in terms of the charges[3]

$$\begin{aligned}
[Q_\pm^1, Q_\pm^1] &= 2\partial_\pm \\
[Q_\pm^2, Q_\pm^2] &= -2\partial_\pm.
\end{aligned} \tag{21}$$

We make a change of basis by the formulas $Q_\pm = \frac{1}{\sqrt{2}}(Q_\pm^2 + Q_\pm^1)$ and $\tilde{Q}_\pm = \frac{1}{\sqrt{2}}(Q_\pm^2 - Q_\pm^1)$. Equations (**??**) can be written collectively as

$$[Q_\pm^a, Q_\pm^b] = 2\eta^{ab}\partial_\pm, \tag{22}$$

---

[3]Note that for the usual $(2,2)$ SUSY one cannot diagonalize the algebra over $\mathbb{R}$.

where $\eta^{ab} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. From here it is easy to see that matrices $(M_\pm)^a_b$ that rotate the $\pm$-chirality charges among each other while preserving the commutation relations (**??**),

$$[(M_\pm)^a_c Q^c_\pm, (M_\pm)^b_d Q^d_\pm] = 2\eta^{ab}\partial_\pm$$

have to satisfy

$$(M_\pm)^a_c \eta^{cd} (M_\pm)^a_d = \eta^{ab},$$

i.e. $M_\pm$ have to lie in $SO(1,1)$ for each subscript $\pm$ and the R-symmetry group is therefore $G_R = SO(1,1)_+ \times SO(1,1)_-$. DB: this is only relevant for the (2,2) formalism, will se if we need this or not The R-symmetry acts on the odd coordinates accordingly as

$$\begin{pmatrix} \theta^1_\pm \\ \theta^2_\pm \end{pmatrix} \mapsto \begin{pmatrix} \cosh(\alpha_\pm) & \sinh(\alpha_\pm) \\ \sinh(\alpha_\pm) & \cosh(\alpha_\pm) \end{pmatrix} \begin{pmatrix} \theta^1_\pm \\ \theta^2_\pm \end{pmatrix}$$

or, equivalently, $(\theta^\pm, \tilde\theta^\pm) \mapsto (e^{\alpha_\pm}\theta^\pm, e^{-\alpha_\pm}\tilde\theta^\pm)$. There are therefore two inequivalent embeddings of the Lorentz group $SO(1,1)$ into the R-symmetry group, we label them by V and A:

$$R_V(\alpha) : (\theta^\pm, \tilde\theta^\pm) \mapsto (e^\alpha \theta^\pm, e^{-\alpha}\tilde\theta^\pm)$$
$$R_A(\alpha) : (\theta^\pm, \tilde\theta^\pm) \mapsto (e^{\pm\alpha}\theta^\pm, e^{\mp\alpha}\tilde\theta^\pm),$$

mapping the supercharges as

$$R_V(\alpha) :(Q_\pm, \tilde Q_\pm) \mapsto (e^{-\alpha}Q_\pm, e^\alpha \tilde Q_\pm)$$
$$R_A(\alpha) :(Q_\pm, \tilde Q_\pm) \mapsto (e^{\mp\alpha}Q_\pm, e^{\pm\alpha}\tilde Q_\pm).$$

The generators of these transformations are

$$F_{V/A} = \theta^+ \frac{\partial}{\partial\theta^+} - \tilde\theta^+ \frac{\partial}{\partial\tilde\theta^+} \pm \theta^- \frac{\partial}{\partial\theta^-} \mp \tilde\theta^- \frac{\partial}{\partial\tilde\theta^-}.$$

Because $F_{V/A}$ has same commutators with other operators as M, the Lorentz generator, and additionally all $F_{V/A}$ and M commute among themselves, we can define new "Lorentz" generators in one of the two following ways:

$$M_A = M + F_V, \quad M_B = M + F_A.$$

3.3   The $(2,2)$ parasupersymmetric σ-model

It has been known since [**?**] that the $(1,1)$ supersymmetric σ-model (**??**) admits $(2,2)$ supersymmetry when (M,g) is a Kähler manifold. In general, $(2,2)$ supersymmetry in fact requires the target to be **generalized Kähler** [**?**], which is equivalent to a data of a bi-Hermitian geometry discovered by [**?**]. For $(2,2)$ para-SUSY, the story is analogous. It was shown in [**?**] that in order to extend the $(1,1)$ SUSY of **??** to $(2,2)$ para-SUSY, one needs a para-complex version of the biHermitian geometry. In [**?**] it was then shown that this can be equivalently described by **generalized para-Kähler** geometry.

Expanding the $(1,1)$ **action**   First, we further analyze the $(1,1)$ supersymmetric theory. Recall the $(1,1)$ supersymmetric σ-model is described by the action

$$S_{(1,1)}(\Phi) = \int_{\hat{\Sigma}} [g(\Phi) + b(\Phi)]_{ij} D_+\Phi^i D_-\Phi^j.$$

In components, the superfield has the form

$$\Phi^i = \phi^i + \theta^+\psi_+^i + \theta^-\psi_-^i + \theta^+\theta^- F^i$$

The super covariant derivatives $D_\pm$ applied to Φ^i read

$$D_\pm\Phi^i = \psi_\pm^i \pm \theta^\mp F^i - \theta^\pm\partial_\pm\phi^i \mp \theta^+\theta^-\partial_\pm\psi_\mp^i.$$

Further, we can expand the contributions from the metric and b-field as

$$[g(\Phi) + b(\Phi)]_{ij} = [g(\phi) + b(\phi)]_{ij} + \partial_k[g(\phi) + b(\phi)]_{ij}(\theta^+\psi_+^k + \theta^-\psi_-^k + \theta^+\theta^- F^k)$$
$$+ \frac{1}{2}\partial_k\partial_l[g(\phi) + b(\phi)]_{ij}(\theta^+\psi_+^k + \theta^-\psi_-^k)(\theta^+\psi_+^l + \theta^-\psi_-^l).$$

After performing the odd integration, the only terms that survive are the $\theta^+\theta^-$ coefficients:

$$\theta^+\theta^- \left[(g+b)_{ij}(\psi_+^i\partial_-\psi_+^j + \psi_-^j\partial_-\psi_-^i + F^iF^j + \partial_+\phi^i\partial_-\phi^j) \right.$$
$$+ \partial_k(g+b)_{ij}\left(F^k\psi_+^i\psi_-^j + \psi_+^k(-F^i\psi_-^j - \psi_+^i\partial_-\phi^j) + \psi_-^k(-\partial_+\phi^i\psi_-^j + \psi_+^iF^j) \right)$$
$$\left. + \frac{1}{2}\partial_k\partial_l(g+b)_{ij}(-\psi_+^k\psi_-^l\psi_+^i\psi_-^j + \psi_-^k\psi_+^l\psi_+^i\psi_-^j) \right].$$

The field F is auxiliary (since it enters only algebraically: no derivations of F appear in the action), and therefore can be integrated out. To obtain the equations of motion for F, we use the formula $\partial_k g_{ij} = \Gamma_{ikj} + \Gamma_{jki}$ and collect the terms involving F:

$$g_{ij}F^iF^j + (\Gamma_{ikj} + \Gamma_{jki})(F^k\psi_+^i\psi_-^j - F^i\psi_+^k\psi_-^j + F^j\psi_-^k\psi_+^i)$$
$$+ F^k\psi_+^i\psi_-^j(\partial_k b_{ij} + \partial_i b_{jk} + \partial_j b_{ki}).$$

Using $\Gamma_{ikj} = \Gamma_{kij}$ and $H_{ijk} = (\partial_k b_{ij} + \partial_i b_{jk} + \partial_j b_{ki})$, we can rewrite this as

$$g_{ij}F^iF^j + 2\Gamma_{ikj}F^j\psi_-^k\psi_+^i + F^k\psi_+^i\psi_-^j H_{ijk} = g_{ij}(F^iF^j + 2F^j(\Gamma^-)_{lk}^i\psi_-^k\psi_+^l),$$

where Γ^- are the Christoffel symbols of the connection $\nabla^- = \overset{\circ}{\nabla} - \frac{1}{2}g^{-1}H$. This yields the equation of motion for F,

$$\frac{\delta S_{(1,1)}}{\delta F^i} = 0 \quad\Longleftrightarrow\quad F^i = -(\Gamma^-)_{jk}^i\psi_-^k\psi_+^j \tag{23}$$

Extended supersymmetry The σ-model we have just described has manifest $(1,1)$ supersymmetry. We now propose the a symmetry of the action $S_{(1,1)}$ by the extended $(2,2)$ parasupersymmetry algebra (**??**). In general, this extended symmetry exists only after we make some additional assumptions on the geometric data prescribing the action.

Setting $\theta_1^\pm := \theta^\pm$ for simplicity, we propose the following $(2,2)$ parasupersymmetry in $(1,1)$ superfield notation:

$$Q_\pm^1 = \frac{\partial}{\partial\theta^\pm} + \theta^\pm \partial_\pm \tag{24}$$

$$Q_\pm^2 \Phi^i = (K_\pm)_j^i(\Phi) D_\pm \Phi^j \tag{25}$$

where $D_\pm = \frac{\partial}{\partial\theta^\pm} - \theta^\pm\partial_\pm$. BM: We need to either recall what $K_\pm$ is geometrically, or state that it is a free parameter which we will interpret momentarily. In other words, the infinitesimal action of the element $Q_\pm^1 \in \mathfrak{t}_{(2,2)}$ is:

$$\Phi^i \mapsto \Phi^i + \epsilon_1^\pm \left(\frac{\partial}{\partial\theta^\pm} + \theta^\pm\partial_\pm \right) \Phi^i$$

where $\epsilon_1^\pm$ is an infinitesimal parameter. Similarly, the infinitesimal action by the element $Q_\pm^2 \in \mathfrak{t}_{(2,2)}$ is:

$$\Phi \mapsto \Phi + \epsilon_2^\pm (K_\pm)_j^i(\Phi) D_\pm \Phi^j$$

where $\epsilon_2^\pm$ is an infinitesimal parameter.

Lemma 3.3. *Expanding out in the components of the $(1,1)$ superfield, the infinitesimal action by Q_+^1 and in Equation (**??**) reads:*

$$Q_+^1 : (\phi^i, \psi_+^i, \psi_-^i, F^i) \mapsto (\phi^i, \psi_\pm^i, F^i) + \epsilon_1^+ (\psi_+^i, \partial_+\phi, F^i, \partial_+\psi_-^i).$$

and by Q_-^1:

$$Q_-^1 : (\phi^i, \psi_+^i, \psi_-^i, F^i) \mapsto (\phi^i, \psi_\pm^i, F^i) + \epsilon_1^- (\psi_-^i, \partial_-^i\phi^i, -F^i, -\partial_-\psi_+^i)$$

*Similarly, the infinitesimal action by Q_+^2 in Equation (**??**) reads:*

$$Q_+^2 : (\phi^i, \psi_+^i, \psi_-^i, F^i) \mapsto ((K_+)_j^i\psi_+^j, -(K_+)_j^i\partial_+\phi^j - \partial_k(K_+)_j^i\psi_+^k\psi_+^k, (K_+)_j^iF^j + \partial_k(K_+)_j^i\psi_-^k\psi_+^j, BM: Fterm) \tag{26}$$

and for Q_-^2:

$$Q_-^2 : (\phi^i, \psi_+^i, \psi_-^i, F^i) \mapsto ((K_-)_j^i\psi_-^j, -(K_-)_j^i\partial_-\phi^j - \partial_k(K_-)_j^i\psi_-^k\psi_-^k, (K_-)_j^iF^j + \partial_k(K_-)_j^i\psi_+^k\psi_-^j, BM: Fterm) \tag{27}$$

Proof. Recall, in components the superfield is of the form

$$\Phi^i = \phi^i + \theta^+\psi_+^i + \theta^-\psi_-^i + \theta^+\theta^- F^i.$$

The action of the $Q^1_\pm$ charges is straightforward and can be simply read off from the infinitesimal action. To obtain the action by the charges $Q^2_\pm$ on the individual components we first need to expand $K_\pm(\Phi)$ in terms of the components of the superfield:

$$(K_\pm)^i_j(\Phi) = (K_\pm)^i_j(\phi) + \partial_k(K_\pm)^i_j\theta^+\psi^k_+ + \partial_k(K_\pm)^i_j\theta^-\psi^k_- + \partial_k(K_\pm)\theta^+\theta^- F^k,$$

so that the infinitesimal action by $Q^2_\pm$ on Φ^i is:

$$\Phi^i + \epsilon^\pm_2(K_\pm)^i_j(\Phi)(\psi^j_\pm \pm \theta^\mp F^j - \theta^\pm \partial_\pm\phi^j \mp \theta^+\theta^-\partial_\pm\psi^j_\mp). \tag{28}$$

Reading off the various components this expression yields Equations (**??**) and (**??**).   □

The $(2,2)$ formalism   BM: There is the following superspace model for $(2,2)$ parasupersymmetry. Consider the superspace $\mathbb{R}^{1,1|4}$ BM: maybe leave this as a remark that we are aware of this...but we don't need it.

3.4 $(2,1)$ and $(2,0)$ parasupersymmetry

Many of the constructions above make sense when there is less total supersymmetry, and we briefly mention two important cases.

The first concerns the $(2,1)$ parasupersymmetry algebra. This algebra $\mathfrak{t}_{(2,1)}$ consists of generators $\{Q^1_\pm, Q^2_+\}$ satisfying the same algebra as in (**??**). Starting with the $(1,1)$ supersymmetric σ-model (**??**), and labeling the odd coordinates by $\theta^\pm = \theta^\pm_1$, one can ask for an additional odd symmetry

$$Q^+_2 : \Phi \mapsto \Phi + \epsilon^+_2 K_+(\Phi)^j_i D_+\Phi^i$$

which determines a symmetry by the $(2,1)$ parasupersymmetry algebra.

BM: SKT, Witten-Hull

Proposition 3.4. *The odd vector field Q^+_2 is a symmetry of the $(1,1)$ super-symmetric σ-model provided $(K_+)^j_i$ is a paracomplex structure satisfying* BM: *What conditions here? Thus, the $(1,1)$ supersymmetric σ-model has $(2,1)$ para-supersymmetry provided the target is* BM: *same conditions.*

The next symmetry algebra we wish to consider is the $(2,0)$ parasupersymmetry algebra $\mathfrak{t}_{(2,0)}$. This algebra has the same even generators as the $(2,1)$ and $(2,2)$ algebras, and odd generators $\{Q^1_+, Q^2_+\}$ satisfying the relations

$$[Q^1_+, Q^1_+] = 2\partial_\pm \quad , \quad [Q^2_+, Q^2_+] = -2\partial_\pm$$

BM: The (1,0) action

We propose an extended symmetry of the $(1,0)$ σ-model by the following action by the odd vector field Q^2_+:

$$Q^2_+ : \Phi \mapsto \Phi + \epsilon^+_2 K_+(\Phi)^j_i D_+\Phi^i.$$

Proposition 3.5. *The odd vector field Q_2^+ is a symmetry of the $(1,0)$ supersymmetric σ-model provided $(K_+)_i^j$ is a paracomplex structure. Thus, the $(1,0)$ supersymmetric σ-model has $(2,0)$ parasupersymmetry provided the target is paracomplex.* BM: there is a version for a target vector bundle as well...

4   AKSZ and generailzed para-geometry

In this section we will show how to associate $2D$ topological theories to a generalized (para-)complex structure. Later on, in Section **??**, we relate these theories to topological twists of $2D$ $(2,2)$ para-SUSY sigma models introduced in Section **??**. The key to our construction is to realize the data of a G(p)C structure in terms of its eigenbundles, which are Dirac structures L in the Courant algebroid $T \oplus T^*$. Via the AKSZ construction [**?**], this correspondence translates into the statement that the Dirac structure L defines a topological boundary theory of a $3D$ topological theory defined by the Courant algebroid $T \oplus T^*$ [**?**].

Throughout, we will use symplectic NQ manifold as our model for shifted symplectic geometry. We will recall the general setup of associating a $3D$ topological field theory to the data of a Courant algebroid. For a reference, see [**?**, **?**]. Later we will use the idea of Lagrangian intersections BM: Toen, ButsonYoo, Costello,....

4.1   Courant algebroids, $3D$ TFTs, and boundary conditions

From the point of view of topological field theory, Courant algebroids are important because they provide geometric examples of 2-shifted symplectic spaces. Via the AKSZ construction, 2-shifted symplectic spaces are the natural home for 3-dimensional topological field theories in the BV formalism. BM: give citations here.

In [**?**], it is shown how to associate a 2-shifted symplectic NQ manifold from the data of a Courant algebroid.

Theorem 4.1 ([**?**])**.** *There is an equivalence between isomorphism classes of 2-shifted symplectic NQ manifolds with body M and isomorphism classes of Courant algebroids on M.*

We briefly recount this construction.

Using this sequence, we can define the associated NQ manifold X_E, following [**?**]. The body of the NQ manifold is M, and the underlying graded vector bundle is given by $T^*[2] \oplus E[1]$. Define the degree $+1$ vector field

$$Q : \Gamma(M, \text{Sym} \, (T^*[2] \oplus E[1])^*) \to \Gamma(M, \text{Sym} \, (T^*[2] \oplus E[1])^*)$$

as follows: for $\psi \in \text{Sym}^{k+\ell+2} \, (T^*[2] \oplus E[1])^*)$ and

$$(\xi_0 \otimes \cdots \otimes \xi_k) \otimes (\alpha_0 \otimes \cdots \otimes \alpha_\ell) \in \text{Sym}^{k+1}(T^*[2]) \otimes \text{Sym}^{\ell+1}(E[1])$$

define

$$(Q\psi)((\xi_0 \otimes \cdots \otimes \xi_k) \otimes (\alpha_0 \otimes \cdots \otimes \alpha_\ell)) = BM :finish$$

All that remains is to describe the 2-shifted symplectic structure on the NQ manifold X_E. This is defined via the obvious pairing between $T^*[2]$ and T together with the pairing $\langle -, - \rangle$ on $E[1]$. In local coordinates BM: finish

4.1.1   The AKSZ theory associated to a Courant algebroid

To any Courant algebroid E, we associate an NQ manifold X_E which carries a 2-shifted symplectic structure. In the case that the Courant algebroid is exact, call the associated 2-shifted symplectic space X_H, where H labels the Severa class.

The starting point of the AKSZ construction is the mapping space

$$\mathrm{Map}\left((M, (\mathcal{A}_M^*, \mathrm{d}_M)), X_E\right)$$

where $(M, (\mathcal{A}_M^*, \mathrm{d}_M))$ is a dg manifold with body a smooth oriented manifold M. That is, $\mathcal{A}_M^*$ is a sheaf of graded commutative algebras over the de Rham complex Ω_M^* and d_M is a linear differential operator of degree $+1$ satisfying:

(1) $\mathcal{A}_M$ is concentrated in finitely many degrees;

(2) For each k, $\mathcal{A}_M^k$ is a locally free sheaf of C_M^∞-modules of finite rank;

(3) The differential $\mathrm{d}_M : \mathcal{A}_M^* \to \mathcal{A}_M^{*+1}$ is square zero differential operator making $(\mathcal{A}_M, \mathrm{d}_M)$ into a sheaf of commutative dg algebras over the de Rham complex Ω_M^*

Example 4.2. The most important example for us will be the source dg manifold (M, Ω_M^*), that is $\mathcal{A}_M = \Omega_M^*$ equipped with de Rham differential. We denote this dg manifold by M_{dR}.

Example 4.3. If M has a para-complex structure, then $(M, \Omega_M^{0,*})$ has the structure of a dg manifold with differential given by the para $\bar{\partial}$-operator. BM: David, is this consistent notation? We will denote this dg manifold by $M_{p\bar{\partial}}$.

Example 4.4. If $\mathfrak{g}$ is a Lie algebra equipped with a non-degenerate invariant pairing, and $X_E = \mathfrak{g}[1]$ as in Example ??, then the AKSZ theory (whose target base manifold is a point) is equivalent to Chern-Simons theory on M.

4.1.2   Dirac structures and boundary conditions

The AKSZ formalism is a construction which produces a (-1)-shifted symplectic space from the data of a closed manifold and a target shifted symplectic manifold. There is a generalization of this construction which produces (-1)-shifted symplectic spaces on manifolds with boundary.

Suppose M is a manifold with boundary, and let $\mathcal{E}(M)$ denote the space of fields of a classical theory on M. For instance, $\mathcal{E}(M)$ will be a mapping space of the form $\mathrm{Map}(M, X)$ where X is shifted symplectic. Let $\mathcal{E}(\partial M)$ denote the restriction of the fields to the boundary of M. For the mapping space example, this is simply the space of maps $\mathrm{Map}(\partial M, X)$.

4.2   Two dimensional TFT from GpC structures

BM: Recall that a generalized (para-)complex structure $\mathcal{J}$ defines a Dirac structure in $T \oplus T^*$, and hence a Lagrangian inside of the 2-shifted symplectic space $\mathfrak{X}_H$. This Lagrangian then defines a boundary condition for the three-dimensional AKSZ theory.

Trivial GpC structure

(Real) Poisson σ-model

Para-complex A/B-models

We point out two special cases of the above general construction.

    BM: The A-model is the usual thing.

Definition 4.5. The **para-complex B-model** with source a closed Riemann surface Σ and target a para-complex manifold X is the AKSZ theory with source the de Rham space Σ_{dR} and target the 1-shifted symplectic space $T^*[1]X_{p\bar{\partial}}$:

$$\mathrm{Map}\left(\Sigma_{dR}, T^*[1]X_{p\bar{\partial}}\right).$$

    BM: Discuss Si's work on perturbative quantization

    BM: ***para Kahler example, **example where the paracomplex structures commute but are not equal (this is where people see "twisted chiral"), para version of T-duality, **Poisson σ-model, going back and forth between generalized (para) complex.

    BM: discuss anomalies

4.3   Para holomorphic variants

So far, in each of the σ-models we have discussed in the AKSZ formalism, the fields have depended only topologically on the source Riemann surface. There are closely related σ-models which depending *para-holomorphically* on the source Riemann surface that we briefly discuss.

5   Twisted parasupersymmetry

In this section we will present how para-supersymmetry theories can be twisted, in exact analogy with twisting in ordinary supersymmetry. Applied to the $(2,2)$ parasupersymmetric theories that we have constructed in §**??**, we find that twisting yield topological field field theories. In the next section, we will construct these topological theories geometrically using the AKSZ formalism.

    The procedure for twisting will again follow the analogy between generalized Kähler (GK) and generalized para-Kähler (GpK). geometries, and we will follow the approach presented in [**?**] for the GK case.

BM: geometry, change in gradings DB: how does the embedding of $SO(1,1)$ into G_R reflect on the definition of $\mathcal{Q}$?

For the topological twist, we will use the following nilpotent supercharge

$$\mathcal{Q} = Q_L + Q_R, \quad Q_L = \frac{1}{2}(Q_+^1 + Q_+^2), \quad Q_R = \frac{1}{2}(Q_-^1 + Q_-^2), \qquad (29)$$

and we also introduce the usual notation

$$\chi = \frac{1}{2}(\mathbb{1} + K_+)\psi_+ = P_+\psi_+ = \psi_+^{(1,0)+}, \quad \lambda = \frac{1}{2}(\mathbb{1} + K_-)\psi_- = P_-\psi_- = \psi_-^{(1,0)-}.$$

Proposition 5.1. *The action of $Q_{L/R}$ on the fields χ, λ and ϕ is given by*

$$
\begin{aligned}
&[Q_L, \phi^i] = \chi^i & &[Q_R, \phi^i] = \lambda^i \\
&[Q_L, \chi^i] = 0 & &[Q_R, \lambda^i] = 0 \\
&[Q_L, \lambda^i] = -(\Gamma^-)^i_{jk}\chi^j\lambda^k & &[Q_R, \chi^i] = -(\Gamma^+)^i_{jk}\lambda^j\chi^k.
\end{aligned}
\qquad (30)
$$

Proof. The action of $Q_{L/R}$ on ϕ can be immediately read off (**??**):

$$[Q_L, \phi^i] = \frac{1}{2}[Q_+^1 + Q_+^2, \phi^i] = \chi^i. \qquad (31)$$

We now prove the formulas for the action of $Q_{L/R}$ on χ, the action on λ is entirely analogous. First, we expand

$$[Q_L, \chi^i] = [Q_L, (P_+)^i_j\psi_+^j] = [Q_L, (P_+)^i_j]\psi_+^j + (P_+)^i_j[Q_L, \psi_+^j]. \qquad (32)$$

Or maybe here? I think this is the one Because K_+ is a function of the bosonic coordinates, we also have $P_+ = P_+(\phi)$ and so

$$[Q_L, (P_+)^i_j] = \partial_k(P_+)^i_j[Q_L, \phi^k] = \partial_k(P_+)^i_j\chi^k. \qquad (33)$$

Combining (**??**) and (**??**) and using (**??**), we arrive at

$$
\begin{aligned}
[Q_L, \chi^i] &= \partial_k(P_+)^i_j\chi^k\psi_+^j \overset{?}{\pm} (P_+)^i_j[(\tilde{P}_+)^j_k\partial_+\phi^k + \frac{1}{2}\partial_k(K_+)^j_l\psi_+^k\psi_+^l] \\
&= \partial_k(P_+)^i_j(P_+)^k_l\psi_+^l\psi_+^j - (P_+)^i_j\partial_k(P_+)^j_l\psi_+^k\psi_+^l,
\end{aligned}
$$

where we denoted $\tilde{P}_+ = \frac{1}{2}(\mathbb{1} - K_+)$ and used $P_+\tilde{P}_+ = 0$. We now expand the projectors, $(P_+)^i_j = \frac{1}{2}(\delta^i_j + (K_+)^i_j)$:

$$
\begin{aligned}
[Q_L, \chi^i] &= \partial_k(P_+)^i_j(P_+)^k_l\psi_+^l\psi_+^j - (P_+)^i_j\partial_k(P_+)^j_l\psi_+^k\psi_+^l \\
&= \frac{1}{2}(\partial_j(K_+)^i_l(P_+)^j_k - (P_+)^i_j\partial_k(K_+)^j_l)\psi_+^k\psi_+^l \\
&= \frac{1}{4}(\partial_j(K_+)^i_l(K_+)^j_k - (K_+)^i_j\partial_k(K_+)^j_l)\psi_+^k\psi_+^l \\
&\quad + \frac{1}{4}(\partial_j(K_+)^i_l\psi_+^j\psi_+^l - \partial_k(K_+)^i_l\psi_+^k\psi_+^l) = -\frac{1}{2}(N_{K_+})^i_{kl}\psi_+^k\psi_+^l,
\end{aligned}
$$

This must be the right sign choice where in the last line we rewrote $\psi_+^k \psi_+^l = \frac{1}{2}(\psi^k \psi^l - \psi^l \psi^k)$ to get the expression for the Nijenhuis tensor of K_+:

$$4(N_{K_+})^i_{kl} = (K_+)^j_k \partial_j (K_+)^i_l - (K_+)^j_l \partial_j (K_+)^i_k - (K_+)^i_j (\partial_k (K_+)^j_l - \partial_l (K_+)^j_k).$$

Invoking integrability of K_+, $N_{K_+} =$ and we conclude that $[Q_L, \chi^i] = 0$.

Similarly, $[Q_L, \lambda]$ can be expanded

$$[Q_L, \lambda^i] = [Q_L, (P_-)^i_j \psi_-^j] = [Q_L, (P_-)^i_j] \psi_-^j + (P_-)^i_j [Q_L, \psi_-^j] \tag{34}$$

$$= \partial_k (P_-)^i_j \chi^k \psi_-^j + (P_-)^i_k ((P_+)^k_j F^j + \frac{1}{2} \partial_l (K_+)^k_j \psi_-^l \psi_+^j), \tag{35}$$

where we used

$$[Q_L, \psi_-^i] = (P_+)^i_j F^j + \frac{1}{2} \partial_k (K_+)^i_j \psi_-^k \psi_+^j.$$

Next, we rewrite the derivative in terms of ∇^- and use the property $\nabla^- P_- = 0$:

$$\partial_k (P_-)^i_j = \nabla^-_k (P_-)^i_j - (\Gamma^-)^i_{kl} (P_-)^l_j + (\Gamma^-)^l_{kj} (P_-)^i_l$$
$$= -(\Gamma^-)^i_{kl} (P_-)^l_j + (\Gamma^-)^l_{kj} (P_-)^i_l,$$

so that

$$\partial_k (P_-)^i_j \chi^k \psi_-^j = (-(\Gamma^-)^i_{kl} (P_-)^l_j + (\Gamma^-)^l_{kj} (P_-)^i_l)(P_+)^k_m \psi_+^m \psi_-^j. \tag{36}$$

Similarly,

$$\frac{1}{2}(P_-)^i_k \partial_l (K_+)^k_j \psi_-^l \psi_+^j = (P_-)^i_k \partial_l (P_+)^k_j \psi_-^l \psi_+^j$$
$$= (P_-)^i_k (-(\Gamma^-)^k_{ml} (P_+)^m_j + (\Gamma^-)^m_{jl} (P_+)^k_m) \psi_-^l \psi_+^j, \tag{37}$$

where we used

$$(\Gamma^+)^i_{jk} = g^{im}(\Gamma_{mjk} + \frac{1}{2} H_{jkm}) = g^{im}(\Gamma_{mkj} - \frac{1}{2} H_{kjm}) = (\Gamma^-)^i_{kj}.$$

Plugging (**??**) and (**??**) into (**??**) with use of the equation of motion for F (**??**) then yields

$$[Q_L, \lambda^i] = ((\Gamma^-)^i_{km}(P_-)^m_l - (\Gamma^-)^m_{kl}(P_-)^i_m)(P_+)^k_j \psi_-^l \psi_+^j$$
$$+ (P_-)^i_k (-(\Gamma^-)^k_{ml}(P_+)^m_j + (\Gamma^-)^m_{jl}(P_+)^k_m) \psi_-^l \psi_+^j$$
$$- (P_-)^i_k (P_+)^k_m (\Gamma^-)^m_{jl} \psi_-^l \psi_+^j$$

terms cancel (up to another relative minus sign!) as well as cancelling the term $(P_-)^i_k (P_+)^k_j F^j$ by

$$(P_-)^i_k (\Gamma^+)^m_{lj}(P_+)^k_m \psi_-^l \psi_+^j = (P_-)^i_k (P_+)^k_j (\Gamma^-)^j_{ml} \psi_-^l \psi_+^m = -(P_-)^i_k (P_+)^k_j F^j$$

and we arrive at

$$\{Q_L, \lambda^i\} = -(\Gamma^-)^i_{kl}\chi^k\lambda^l.$$

Now for Q_R, we have

$$\{Q_R, \phi^i\} = \lambda^i$$

$\square$

The observables of the twisted theory are therefore polynomials in the odd variables χ, λ with even coefficients dependent on ϕ:

$$\mathcal{O}_f = f(\phi)_{i_1 \cdots i_a, j_1 \cdots j_b} \chi^{i_1} \cdots \chi^{i_a} \lambda^{j_1} \cdots \lambda^{j_b}.$$

Because $\chi \in \mathfrak{X}^{(1,0)+}$ and $\lambda \in \mathfrak{X}^{(1,0)-}$, we can identify $\mathcal{O}_f$ with a function on the supermanifold $(T^{(1,0)+} \oplus T^{(1,0)-})[1]$, which in turn can be identified with differential form

$$\Omega_f = f(\phi)_{i_1 \cdots i_a, j_1 \cdots j_b} dx_+^{i_1} \cdots dx_+^{i_a} dx_-^{j_1} \cdots dx_-^{j_b},$$

where $dx_\pm$ are the one-forms satisfying $K_\pm dx_\pm = dx_\pm$, i.e. sections of the bundle $T^{*(1,0)\pm}$. Via this identification, the operator $\mathcal{Q} := Q_L + Q_R$ defines a differential $d_\mathcal{Q}$, which acts on the sections of $\bigwedge^k(T^{*(1,0)+} \oplus T^{*(1,0)-})$ and because $\mathcal{Q}$ squares to zero $d_\mathcal{Q}$ also squares to zero. Therefore, $d_\mathcal{Q}$ gives rise to a Chevalley-Eilenberg complex $CE(L)$ of a certain Lie algebroid L which we now describe.

Lie algebroid from bi-para-Hermitian data   The underlying vector bundle of the Lie algebroid is $L = T^{(1,0)+} \oplus T^{(1,0)-}$, and the anchor is the sum of the two inclusions $\imath_\pm : T^{(1,0)\pm} \hookrightarrow T$, i.e.

$$a : L = T^{(1,0)+} \oplus T^{(1,0)-} \to T$$
$$e = (x_+, x_-) \mapsto \imath_+(x_+) + \imath_-(x_-).$$

From (**??**) we can read off the action of $d_\mathcal{Q}$ on degree one elements, i.e. sections $\alpha = (\alpha^+, \alpha^-)$ of $L^* = T^*_{(1,0)_+} \oplus T^*_{(1,0)_-}$:

$$d_\mathcal{Q}\alpha = \partial_k \alpha_i^+ [dx_+^k + dx_-^k] \wedge dx_+^i - \alpha_i^+ (\Gamma^+)^i_{jk} dx_-^j \wedge dx_+^k$$
$$+ \partial_k \alpha_i^- [dx_+^k + dx_-^k] \wedge dx_-^i - \alpha_i^- (\Gamma^-)^i_{jk} dx_+^j \wedge dx_-^k.$$

By contracting in two sections of L, $e_1 = x_+ + x_-$, $e_2 = y_- + y_-$, we get the coordinate-free expression

$$(d_\mathcal{Q}\alpha)(e_1, e_2) = (\partial_+\alpha^+)(x_+, y_+) + \nabla^+_{x_-}\alpha^+(y_+) - \nabla^+_{y_-}\alpha^+(x_+) \\ + (\partial_-\alpha^-)(x_-, y_-) + \nabla^-_{x_+}\alpha^-(y_-) - \nabla^-_{y_+}\alpha^-(x_-), \tag{38}$$

where $\partial_\pm$ denotes the para-complex Dolbeault operators ∂ for $K_\pm$. We can further simplify this expression by writing $\alpha = (\eta(\tilde{z}_+), \eta(\tilde{z}_-))$ for $\tilde{z}_\pm \in \Gamma(T^{(1,0)\pm})$, invoking the formula $\partial_\pm \alpha^\pm(x_\pm, y_\pm) = \mathring{\nabla}_{x_\pm} \alpha^\pm(y_\pm) - \mathring{\nabla}_{y_\pm} \alpha^\pm(x_\pm)$ and recalling $\nabla^\pm = \mathring{\nabla} \pm \frac{1}{2}\eta^{-1}H$:

$$
\begin{aligned}
(\mathrm{d}_{\mathcal{Q}}\alpha)(e_1, e_2) &= \eta((\mathring{\nabla}_{x_+} + \nabla^+_{x_-})\tilde{z}_+, y_+) - \eta((\mathring{\nabla}_{y_+} + \nabla^+_{y_-})\tilde{z}_+, x_+) \\
&\quad + \eta((\mathring{\nabla}_{x_-} + \nabla^-_{x_+})\tilde{z}_-, y_-) - \eta((\mathring{\nabla}_{y_-} + \nabla^-_{y_+})\tilde{z}_-, x_-) \\
&= \eta(\nabla^+_{x_++x_-}\tilde{z}_+, y_+) - \eta(\nabla^+_{y_++y_-}\tilde{z}_+, x_+) - H(x_+, \tilde{z}_+, y_+) \\
&\quad + \eta(\nabla^-_{x_++x_-}\tilde{z}_-, y_-) - \eta(\nabla^-_{y_++y_-}\tilde{z}_-, x_-) + H(x_-, \tilde{z}_-, y_-)
\end{aligned}
\tag{39}
$$

Lie algebroid from generalized para-Kähler data   Recall that a GpK structure defines the decomposition (**??**)

$$
T \oplus T^* = \ell_+ \oplus \ell_- \oplus \tilde{\ell}_+ \oplus \tilde{\ell}_-,
$$

where the the eigenbundles of $\mathcal{K}_+$ are $\mathbb{L}_+ = \ell_+ \oplus \ell_-$ and $\tilde{\mathbb{L}}_+ = \tilde{\ell}_+ \oplus \tilde{\ell}_-$. Because $\mathbb{L}$ is a Dirac structure, it defines a Lie algebroid by restriction of the Courant algebroid on $T \oplus T^*$ to $\mathbb{L}$, i.e. the Lie algebroid bracket and anchor are given by the restriction of the Dorfman bracket and the projection onto tangent bundle, respectively (see Proposition **??**.

Theorem 5.2. *The Lie algebroids* $(L, a, \mathrm{d}_{\mathcal{Q}})$ *and* $(\mathbb{L}_+, \pi_T|_{\mathbb{L}_+}, [\ ,\ ]|_{\mathbb{L}_+})$ *are isomorphic via the map* $\pi = \pi_+ \oplus \pi_-$.

Proof. First, we notice that the bundles themselves are related by π, $\pi(\mathbb{L}_+) = \pi(\ell_+ \oplus \ell_-) = \pi_+(\ell_+) \oplus \pi_-(\ell_-) = T^{(1,0)+} \oplus T^{(1,0)-} = L$ and also π is clearly an isomorphism since both $\pi_\pm$ are. Additionally, the anchors are related by $a \circ \pi = \pi_T|_{\mathbb{L}_+}$. The only non-trivial task is therefore to show that the respective differentials satisfy

$$
\pi^* \circ \mathrm{d}_Q = \mathrm{d}_{\mathbb{L}_+} \circ \pi^*,
\tag{40}
$$

where $\pi^* = \pi^*_+ \oplus \pi^*_-$ and $\pi^*_\pm$ are dual maps to $\pi_\pm : \ell_+ \to T^{(1,0)\pm}$, extended to a map $\pi^* : \Lambda^k(L^*) \to \Lambda^k(\mathbb{L}^*_+)$.

   Because of the chain rule, we only need to check this on degree 0 and degree 1 elements in the respective complexes. Degree 0 follows immediately:

$$
(\mathrm{d}_Q f)(\pi(u)) = a \circ \pi(u)[f] = \pi_T|_{\mathbb{L}_+}(u)[f] = (\mathrm{d}_{\mathbb{L}_+} f)(u),
$$

and we will now prove **??** for degree 1 elements. In the following we will make use of the identifications provided by the metric η and pairing $\langle\ ,\ \rangle$:

$$
\eta : T^*_{(1,0)_\pm} \leftrightarrow T^{(0,1)\pm}
$$
$$
\langle\ ,\ \rangle : \quad \ell^*_\pm \leftrightarrow \tilde{\ell}_\pm,
$$

so that elements in $T^*_{(1,0)_\pm}$ can be written as $\eta(\tilde{z}_\pm)$ with $\tilde{z}_\pm$ vector in $T^{(0,1)\pm}$ and similarly, elements in $\ell^*_\pm$ can be expressed as $\langle \tilde{w}_\pm, \cdot \rangle$ with $\tilde{w}_\pm$ section of $\tilde{\ell}_\pm$.

To declutter notation, we shall also denote the bracket and anchor on $\mathbb{L}_+$ by $[\,,\,]$ and π, respectively.

The differential $\mathrm{d}_{\mathbb{L}_+}$ is defined by the bracket on the Lie algebroid $\mathbb{L}_+$ via the following formula:

$$\mathrm{d}_{\mathbb{L}_+} w^*(u,v) = \pi(u)w^*(v) - \pi(u)w^*(v) - w^*([u,v]), \tag{41}$$

where $w^* \in \Gamma(\mathbb{L}_+^*)$ and $u,v \in \Gamma(\mathbb{L}_+)$. Writing w^* as $\langle \tilde{w}, \cdot \rangle$, with $\tilde{w} \in \Gamma(\tilde{\mathbb{L}}_+)$ we can rewrite this as

$$
\begin{aligned}
\mathrm{d}_{\mathbb{L}_+} w^*(u,v) &= \pi(u)\langle v, \tilde{w}\rangle - \pi(v)\langle u, \tilde{w}\rangle - \langle [u,v], \tilde{w}\rangle \\
&= \langle D_u v, \tilde{w}\rangle + \langle v, D_u \tilde{w}\rangle - \langle D_v u, \tilde{w}\rangle - \langle u, D_v \tilde{w}\rangle - \langle D_u v - D_v u, \tilde{w}\rangle \\
&\quad + T(u,v,\tilde{w}) \\
&= \langle D_u \tilde{w}, v\rangle - \langle D_v \tilde{w}, u\rangle + T(u,v,\tilde{w}) \\
&= \langle D_u \tilde{w}_+, v_+\rangle - \langle D_v \tilde{w}_+, u_+\rangle + T(u_+, v_+, \tilde{w}_+) \\
&\quad + \langle D_u \tilde{w}_-, v_-\rangle - \langle D_v \tilde{w}_-, u_-\rangle + T(u_-, v_-, \tilde{w}_-),
\end{aligned}
\tag{42}
$$

where we used (**??**) to express $[\,,\,]$ in terms of D, the generalized Bismut connection of the generalized metric for the GpK structure. We then used the fact that D preserves $\langle\,,\,\rangle$, $C_\pm$ as well as $\mathbb{L}_+$ and that T only has components in $\Lambda^3 C_+ \oplus \Lambda^3 C_-$ [**?**].

We will now show that $\mathrm{d}_{\mathbb{L}_+}$ is related to $\mathrm{d}_\mathcal{Q}$ via (**??**), so that $\mathrm{d}_\mathcal{Q} = (\pi^{-1})^* \circ \mathrm{d}_{\mathbb{L}_+} \circ \pi^*$, i.e.

$$\mathrm{d}_\mathcal{Q}\alpha(e_1, e_2) = \mathrm{d}_{\mathbb{L}_+}(\pi^*\alpha)(\pi^{-1}e_1, \pi^{-1}e_2).$$

To read this off from (**??**) and compare with (**??**), we need to express $\pi^*\alpha$ as $\pi^*\alpha = w^* = \langle \tilde{w}, \cdot\rangle$ for some $\tilde{w} \in \Gamma(\tilde{\mathbb{L}}_+)$ and $\alpha = (\eta(\tilde{z}_+), \eta(\tilde{z}_-))$:

$$\pi^*\alpha = \pi^*(\eta(\tilde{z}_+), \eta(\tilde{z}_-)) = (\eta(\tilde{z}_+, \pi_+\cdot), \eta(\tilde{z}_-, \pi_-\cdot)) = \frac{1}{2}\langle \pi^{-1}(\tilde{z}_+, -\tilde{z}_-), \cdot\rangle,$$

where $\tilde{z}_\pm \in \Gamma(T^{(1,0)\pm})$ and we used (**??**). Therefore, denoting $e_1 = (x_+, x_-)$ and $e_2 = (y_+, y_-)$ as in (**??**), (**??**) yields (denoting $e_1 = (x_+, x_-)$ and $e_2 = (y_+, y_-)$ as in (**??**))

$$
\begin{aligned}
&\mathrm{d}_{\mathbb{L}_+}(\pi^*\alpha)(\pi^{-1}e_1, \pi^{-1}e_2) \\
&= \eta(\nabla^+_{x_+ + x_-}\tilde{z}_+, y_+) - \eta(\nabla^+_{y_+ + y_-}\tilde{z}_+, x_+) + \frac{1}{2}T(\pi_+^{-1}x_+, \pi_+^{-1}y_+, \pi_+^{-1}\tilde{z}_+) \\
&\quad + \eta(\nabla^-_{x_+ + x_-}\tilde{z}_-, y_-) - \eta(\nabla^-_{y_+ + y_-}\tilde{z}_-, x_-) - \frac{1}{2}T(\pi_-^{-1}x_-, \pi_-^{-1}y_-, \pi_-^{-1}\tilde{z}_-),
\end{aligned}
$$

where we used (**??**) and once again (**??**). Finally, invoking (**??**), we arrive at (**??**), showing that (**??**) holds and completing the proof. $\square$

5.1 Topological twists of the $(2,2)$-model

5.2 The (para)holomorphic twist of the $(2,0)$-model