1 The AKSZ formalism and para-geometry

In this section we will show how to associate a class of $2D$ topological field theories to generalized para-complex structures. The construction closely follows the relationship between ordinary (non-para) generalized complex structures [Gua11] and $2D$ topological field theories developed in [CQZ10, Pes07]. For another class of $2D$ boundary theories to the $3D$ Courant algebroid σ-model see [Š16]. BM to DB: Please add any more references you think fit.

For us, the prescription of defining the $2D$ topological theory associated to a generalized para-complex structure starts with a $3D$ topological field theory which is defined for any manifold M. More generally, this $3D$ theory can be defined for any Courant algebroid on M [Roy02], but for most examples we will only consider the standard exact Courant algebroid (or twisted versions) defined by the Dorfman bracket. The key to our construction is to realize the data of a generalized para-complex structure in terms of its eigenbundles, which are Dirac structures in the real Courant algebroid $T \oplus T^*$. More precisely, we obtain a pair of Dirac structures $L, \tilde{L}$ corresponding to the $\pm$ eigenbundles of the generalized para-complex structure.

Via the AKSZ construction [ASZK97], this correspondence translates into the statement that the pair of Dirac structures define a pair of topological boundary theories of the $3D$ topological theory. In turn, we will directly describe the BRST observables of the $2D$ theories in terms of the cohomology of the Lie algebroid L.

By similar reasoning, one can obtain a single $2D$ TFT from the data of a generalized para-complex structure. One considers the para-complexification $(T \oplus T^*) \otimes_{\mathbb{R}} \mathbf{C}$ of the standard Courant algebroid. The real Dirac structures $L, \tilde{L}$ combine to give a para-complex Dirac structure $L \oplus \tilde{L}$. In turn, we obtain a single $2D$ TFT defined over the para-complex numbers.

In Section **??**, we will relate these theories topological twists of $2D$ $(2,2)$ (para-)SUSY sigma models described in Section **??** closely following the methods in [KL07] for ordinary generalized complex geometry.

All of the topological field theories in this section are of σ-model type and arise through the AKSZ formalism [ASZK97]. Throughout, we will use the formalism of dg manifolds and shifted symplectic geometry.

1.1 Courant algebroids, $3D$ TFTs, and boundary conditions

From the point of view of topological field theory, Courant algebroids are important because they provide geometric examples of 2-shifted symplectic spaces. Via the AKSZ construction, 2-shifted symplectic spaces are the natural home for 3-dimensional topological field theories in the BV formalism. We recall the construction of the $3D$ topological field theory from the data of a Courant algebroid. For other references, see [Roy02, CQZ10].

1.1.1 Shifted symplectic geometry

We begin by setting up our model for the theory of derived manifolds. For us, this is a well-behaved class of NQ manifolds which are appropriate for setting up quantization of σ-models, see for instance [**?**]. A dg manifold is a pair $\mathcal{N} = (N, \mathcal{A}^*_{\mathcal{N}})$ where N is a smooth manifold, called the body, and $\mathcal{A}^*_{\mathcal{N}}$ a sheaf of graded commutative algebras over the de Rham complex Ω^*_N. Here, $\mathrm{d}_{\mathcal{N}}$ is a linear differential operator of degree $+1$, and together the data must satisfy the following conditions:

(1) $\mathcal{A}_{\mathcal{N}}$ is concentrated in finitely many degrees;

(2) For each k, $\mathcal{A}^k_{\mathcal{N}}$ is a locally free sheaf of C^∞_N-modules of finite rank;

(3) The differential $\mathrm{d}_N : \mathcal{A}^*_{\mathcal{N}} \to \mathcal{A}^{*+1}_{\mathcal{N}}$ is square zero differential operator making $(\mathcal{A}_{\mathcal{N}}, \mathrm{d}_{\mathcal{N}})$ into a sheaf of commutative dg algebras over the de Rham complex Ω^*_N.

In particular, as a graded algebra $\mathcal{A}^*_{\mathcal{N}}$ is given by functions on the total space of some graded vector bundle $A_{\mathcal{N}}$ on N (which is of finite rank and concentrated in finitely many degrees). In the language of NQ manifolds, the homological vector field defining the Q-structure is $\mathrm{d}_{\mathcal{N}}$.

BM: Symplectic form

A striking result of [Roy02] classifies all 2-shifted symplectic spaces in terms of Courant algebroids.

Theorem 1.1 ([Roy02])**.** *There is an equivalence between isomorphism classes of* 2*-shifted symplectic NQ manifolds with body M and isomorphism classes of Courant algebroids on M.*

Remark 1.2. In [**?**] it was shown that a more general class of 2-shifted symplectic derived spaces, called L_∞ algebroids, are equivalent to *twisted* Courant algebroids. This is similar to a Courant algebroid, where the Jacobi identity only holds up to homotopy given by some closed 4-form.

We briefly recount the equivalence between the data of a 2-shifted symplectic dg manifold and the data of a Courant algebroid.

Let $E \to M$ be a vector bundle equipped with a fiberwise nondegenerate inner product $\langle -, - \rangle$. The body of the dg manifold is the manifold M, and the underlying graded manifold X_E is given by $T^*M[2] \oplus E[1]$.

We will use coordinates $\{x^i\}$ on M, $\{p_i\}$ for the fiber coordinate of T^*, and $\{e^a\}$ for the fiber of E. Note that x^i is of degree zero, η_i is of degree 2, and e^a is of degree 1. Suppose also that $\langle e^a, e^b \rangle = g^{ab}$ and let (g_{ab}) be the inverse to the inner product. Notice, X_E comes equipped with a natural 2-shifted symplectic form, which in coordinates is

$$\omega_E = dx^i dp_i + g_{ab} de^a de^b.$$

Denote by $\{-,-\}$ the shifted Poisson bracket corresponding to this symplectic structure. An arbitrary degree 3 function on the graded manifold X_E has the form

$$\Theta = p_i a_a^i(x) e^a + \frac{1}{6} f_{abc}(x) e^a e^b e^c.$$

Globally, $a = (a_a^i)$ defines a bundle map $a : E \to T$ and through the pairing the collection (f_{abc}) defines a bilinear map $[-,-] : E \times E \to E$. It is a result of [Roy02] that this function satisfies $\{\Theta, \Theta\} = 0$ if and only if the data $(E, \langle -,- \rangle, a, [-,-])$ has the structure of a Courant algebroid. Here a is the anchor map, and $[-,-]$ is the bracket.

1.1.2 The AKSZ construction

Fix the following data:

- an *n-oriented* dg manifold $\mathcal{N} = (N, (\mathcal{A}_N^*, \mathrm{d}_{\mathcal{N}}))$ with body a smooth oriented manifold N and orientation μ.
- An $(n-1)$-shifted symplectic dg manifold (X, ω).

The starting point of the AKSZ construction is the mapping space

$$\mathcal{E}(\mathcal{N}, X) = \mathrm{Map}\,(\mathcal{N}, X) \tag{1}$$

As a graded manifold, the mapping space $\mathcal{E}(N, X)$ is given by the space of smooth maps between the underlying graded manifolds A_N and X. The graded manifold $\mathcal{E}(\mathcal{N}, X)$ is equipped with the homological vector field $\mathrm{d}_{\mathcal{N}} + Q$ where Q is the homological vector field on X.

The AKSZ construction endows the mapping space $\mathcal{E}(N, X)$ in (1) with a (-1)-shifted symplectic symplectic form as follows, compatible with the homological vector field $\mathrm{d}_{\mathcal{N}} + Q$ as follows. The fundamental observation is that the diagram

$$\begin{tikzcd} & N \times \mathcal{E}(N, X) \arrow[dl, "\pi_N"'] \arrow[dr, "\mathrm{ev}"] & \\ N & & X \end{tikzcd}$$

induces a pairing

$$\begin{array}{ccccc} \Omega_N^p & \times & \Omega_X^q & \to & \Omega_{\mathcal{E}(N,X)}^{p+q-n} \\ \alpha & \times & \beta & \mapsto & \displaystyle\int_N \pi_N^* \alpha \wedge \mathrm{ev}^* \beta \end{array}$$

Applied to the element $1 \times \omega_X \in \Omega_N^0 \times \Omega_X^2$ we obtain a (-1)-shifted symplectic form that we denote $\int_N \omega_X$.

Example 1.3. The most important example for us will be the source dg manifold $\mathcal{N} = (N, \Omega_N^*)$, that is $\mathcal{A}_{\mathcal{N}} = \Omega_N^*$ equipped with de Rham differential. We denote this dg manifold by $\mathcal{N} = N_{\mathrm{dR}}$. Take the dg manifold (N, Ω^*_N) where N is any smooth manifold. If N is closed and oriented there exists an integration map

$$\int_N : \Omega^*_N \to \mathbb{R}[n]$$

of degree $-n$ thus equipping the dg manifold N_{dR} with an n-orientation.

Example 1.4. If N has a para-complex structure, $(N, \mathbf{\Omega}^{0,\bullet})$ has the structure of a dg manifold. Here $\mathbf{\Omega}^{0,\bullet}$ is the para-Dolbeault complex from Section **??** with differential given by the para $\overline{\boldsymbol{\partial}}$-operator. We will denote this dg manifold by $N_{\overline{\boldsymbol{\partial}}}$. To equip the dg manifold $(N, \mathbf{\Omega}^{0,\bullet})$ with an orientation, one must fix additional data. One way to do this is to assume that the para-complex manifold N is equipped with a para-holomorphic volume form Ω_N.

BM: Hamiltonians and action functional

1.1.3 The AKSZ theory associated to a Courant algebroid

From here on, to any Courant algebroid E on a manifold M we associate the dg manifold X_E which carries a 2-shifted symplectic structure. In the case that the Courant algebroid is exact, call the associated 2-shifted symplectic space X_H, where H labels the Severa class.

We first review some basic examples of AKSZ theories associated to Courant algebroids. The simplest type of Courant algebroid is one over a point, see Example **??**. The resulting AKSZ theory is a familiar one.

Example 1.5. Any Lie algebra $\mathfrak{g}$ together with a non-degenerate invariant pairing defines a 2-shifted symplectic structure on the graded manifold $X_E = \mathfrak{g}[1]$ living over a point. The dg algebra of functions is the Chevalley-Eilenberg cochain complex computing Lie algebra cohomology $C^*(\mathfrak{g})$. The differential is the Chevalley-Eilenberg differential. All such 2-symplectic dg manifolds over a point are of this form. The resulting AKSZ theory on a 3-manifold M is Chern-Simons theory on M. Indeed, the space of graded maps $N_{dR} \to \mathfrak{g}[1]$ is identified as a graded vector space with $\Omega^*(N, \mathfrak{g})[1]$. The linear part of the BRST operator encodes the de Rham differential on M and the remaining piece encodes the Lie bracket on $\mathfrak{g}$.

Example 1.6. If E is an exact Courant algebroid on M let X_E be the corresponding 2-shifted symplectic manifold. We use the same coordinates as in Section 1.1.1. The fields are the graded maps $M_{dR} \to X_E = T^*[2]T[1]M$ which we can identify in local coordinates with

$$\begin{aligned} x^i &\in C^\infty(N) \\ p_i &\in \Omega^2(N) \\ e^a &\in \Omega^1(N) \end{aligned}$$

where $i = 1, \ldots, \dim(M)$ indexes the local coordinates on M and $a = 1, \ldots, \mathrm{rank}(E)$ the coordinates on the fiber of E. The action functional is

$$S = \int_N p_i \mathrm{d}x^i + \frac{1}{2} g_{ab}(x) e^a \mathrm{d}e^b - a^i_a(x) p_i e^a + \frac{1}{6} f_{abc}(x) e^a e^b e^c.$$

1.1.4 Dirac structures and boundary conditions

The AKSZ formalism is a construction which produces a (-1)-shifted symplectic space from the data of a closed manifold and a target shifted symplectic manifold. There is a generalization of this construction which produces (-1)-shifted symplectic spaces on manifolds with boundary.

Suppose N is a manifold with boundary, and let $\mathcal{E}(N)$ denote the space of fields of a classical theory on M. For us, $\mathcal{E}(N)$ will be a mapping space of the form $\mathrm{Map}(M, X)$ where X is shifted symplectic. Let $\mathcal{E}(\partial N)$ denote the restriction of the fields to the boundary of N. For the mapping space example, this is simply the space of maps $\mathrm{Map}(\partial N, X)$.

In general, when N is not closed, the pairing defined by the AKSZ construction will not endow $\mathcal{E}(N)$ will with (-1)-shifted symplectic structure. However, the space $\mathcal{E}(\partial N)$ does carry a natural ordinary (0-shifted) symplectic structure from the restriction of the pairing defined on $\mathcal{E}(N)$. Moreover, the natural restriction map $\mathcal{E}(N) \to \mathcal{E}(\partial N)$ is a Lagrangian morphism.

A boundary condition is the choice of an additional Lagrangian subspace $\mathcal{L}$ of $\mathcal{E}(\partial N)$. One then forms the intersection $\mathcal{L} \times_{\mathcal{E}(\partial N)} \mathcal{E}(N)$ of the two Lagrangians $\mathcal{L}$ and $\mathcal{E}(\partial N)$ inside of the phase space $\mathcal{E}(\partial N)$. The intersection defines the fields of the theory on ∂N. When the bulk theory is trivial, so there are no degrees of freedom, the resulting theory on ∂N is also equipped with a (-1)-symplectic structure. For proofs of these facts about the Lagrangian intersection we refer to [**?**, **?**].

This formalism can be made explicit in the case of the $3D/2D$ system. BM: finish

Example 1.7. Suppose (M, π) is a Poisson manifold. Then consider the graph of π as a Lagrangian subbundle of $T \oplus T^*$:

$$\mathrm{Graph}(\pi) = \{(\pi \vee \alpha, \alpha) \mid \alpha \in T^*\} \subset T \oplus T^*.$$

By the Jacobi identity, this subbundle is integrable and so determines a Dirac structure. If we place the 3D AKSZ theory with target the Courant algebroid $T \oplus T^*$ on N^3 with $\partial N = \Sigma$, the boundary condition determined by this Dirac structure is equivalent to the Poisson σ-model on Σ, see [KSS05], for instance.

1.2 Two dimensional TFT from generalized para-complex structures

Let M be a manifold. Throughout this section we will only consider the standard Dorfman Courant algebroid X_H on M, possibly twisted by an H-flux. By Theorem **??**, the choice of a generalized para-complex structure $\mathcal{K}$ determines a pair of transversal Dirac structures $L_{\mathcal{K}}$ and $\tilde{L}_{\mathcal{K}}$ on M given by the positive and negative eigenbundles of $\mathcal{K}$. In particular, both $L_{\mathcal{K}}, \tilde{L}_{\mathcal{K}}$determine Lagrangians in the shifted symplectic space X_H. This Lagrangian then defines a pair of boundary conditions for the three-dimensional AKSZ theory. Thus, we see that any generalized para-complex structure $\mathcal{K}$ defines a 2D TFT living at the boundary of the 3D theory.

More generally, if $\mathcal{K}$ is a generalized para-complex structure in an arbitrary Courant algebroid E, then we obtain a pair of 2-shifted Lagrangians inside of the 2-shifted symplectic space X_E. Many of our examples take E to be the standard Courant algebroid, but the following discussion applies in general. We will continue to denote $L_{\mathcal{K}}$ and $\tilde{L}_{\mathcal{K}}$ the $\pm$ eigenbundles in E.

There is a different presentation of the 2-symplectic manifold X_E in terms of $L_{\mathcal{K}}$ and $\tilde{L}_{\mathcal{K}}$ that we will use in our identification of the boundary $2D$ TFT's. Consider the graded manifold $T^*[2]\tilde{L}_{\mathcal{K}}[1]$. In coordinates, we can parametrize this graded manifold by: $\{l^m, \tilde{l}_n, x^i, p_j\}$ where l^m parameterizes the fiber of $\tilde{L}^*[1] \cong L[1]$ and is of degree 1, $\tilde{l}_n$ parametrizes the fiber of $\tilde{L}[1]$ and is of degree 1, and x^i, p_j are as before.

BM: define $\omega_{\mathcal{K}}$

Proposition 1.8. *Suppose $\mathcal{K}$ is a generalized para-complex structure on a Courant algebroid E. Then, there are symplectomorphisms of 2-shifted symplectic dg manifolds*

$$\left(T^*[2]\tilde{L}, \omega_{\mathcal{K}}\right) \cong X_E \cong \left(T^*[2]L, \tilde{\omega}_{\mathcal{K}}\right).$$

Proof. The proof is completely analogous to the proofs of the results in Appendix A of [CQZ10]. □

Equipped with the description of the 2-symplectic manifold X_E in terms of the generalized para-complex structure of E, we can read off the resulting $2D$ boundary TFT as follows. The $3D$ theory is given by the AKSZ formalism as $\mathrm{Map}(N_{dR}, X_E)$ where N_{dR} is a smooth oriented 3-manifold. In the case that $\partial N = \Sigma$, we see that by Proposition 1.8 the phase space can be written as

$$\mathrm{Map}(\Sigma_{dR}, X_E) \cong \mathrm{Map}(\Sigma_{dR}, T^*[2]\tilde{L}_{\mathcal{K}})$$

which is a dg manifold equipped with a natural 0-shifted symplectic structure. On the right-hand side, we use the symplectic structure defined by $\omega_{\mathcal{K}}$.

The $2D$ theory is given as a Lagrangian inside of this symplectic dg manifold. In the coordinates granted by Proposition 1.8 this Lagrangian takes a particularly nice form. The Dirac structure $L_{\mathcal{K}}$ defines the Lagrangian submanifold $L_{\mathcal{K}}[1] \subset T^*[2]\tilde{L}_{\mathcal{K}}$. At the level of the AKSZ theory, this Lagrangian is simply

$$\mathrm{Map}(\Sigma_{dR}, L_{\mathcal{K}}[1]) \hookrightarrow \mathrm{Map}(\Sigma_{dR}, T^*[2]\tilde{L}_{\mathcal{K}}).$$

For a general Courant algebroid E, this graded manifold is equipped with a natural (-1)-shifted Poisson structure. In the case that E is exact, the AKSZ construction equips the σ-model with a (-1)-shifted symplectic structure. We will refer to this boundary condition as the **positive** $2D$ TFT corresponding to the generalized para-complex structure $\mathcal{K}$.

An analogous construction applies to the $(-)$-eigenbundle. There, we present the phase space as $\mathrm{Map}(\Sigma_{dR}, T^*[2]L_{\mathcal{K}})$ equipped with the symplectic structure $\tilde{\omega}_{\mathcal{K}}$. The Lagrangian is $\mathrm{Map}(\Sigma_{dR}, \tilde{L}_{\mathcal{K}}[1])$. Again, when E is exact this is equipped with a (-1)-symplectic structure. We will refer to this boundary condition as the **negative** $2D$ TFT corresponding to the generalized para-complex structure $\mathcal{K}$.

A completely similar construction works at the level of the para-complexification. By Proposition **??** the sum $L_{\mathcal{K}} \oplus \tilde{L}_{\mathcal{K}}$ determines a Dirac structure on $E \otimes_{\mathbb{R}} \mathbf{C}$. The proof of the following is identical to Proposition 1.8.

Proposition 1.9. *Suppose $\mathcal{K}$ is a generalized para-complex structure on a Courant algebroid E, and consider the para-complexified Courant algebroid $E \otimes_{\mathbb{R}} \mathbf{C}$. Then, there is a symplectomorphisms of 2-shifted symplectic dg manifolds*

$$X_{E\otimes\mathbf{C}} \cong \left(T^*[2](L \oplus \tilde{L}), \tilde{\omega}_{\mathcal{K}}\right).$$

By the same reasoning as above, we see that the Lagrangian $(L \oplus \tilde{L})[1] \subset T^*[2](L \oplus \tilde{L})$ determines a the $2D$ TFT with fields

$$\mathrm{Map}\left(\Sigma_{dR}, (L \oplus \tilde{L})[1]\right)$$

lying at the boundary of the para-complexified Courant σ-model. Again, in the case that E is exact, the AKSZ construction endows this $2D$ σ-model with a (-1)-symplectic structure. We will refer to this boundary condition as the **full** $2D$ TFT corresponding to the generalized para-complex structure $\mathcal{K}$.

There is a familiar description of the local BRST operators of the resulting theory. First, we recall the following general construction. Given any Lie algebroid L on M consider the graded manifold $L[1]$ with body M. Functions on this graded manifold are given by sections of the graded vector bundle

$$\mathcal{O}(L[1]) = \bigoplus_{k\geq 0} \wedge^k L[-k].$$

Here, $\wedge^k(L)$ is placed in cohomological degree $+k$. There is square-zero derivation d_L of degree $+1$ on $\mathcal{O}(L[1])$ defined by the following rule. If φ is a section of $\wedge^k L$, then

$$\begin{aligned}(\mathrm{d}_L\varphi)(\ell_0,\ldots,\ell_k) &= \sum_{i=0}^{k}(-1)^i a(\ell_i)\varphi(\ell_0,\ldots,\hat{\ell}_i,\ldots,\ell_k) \\ &+ \sum_{i<j}(-1)^{i+j}\varphi([\ell_i,\ell_j],\ell_0\cdots,\hat{\ell}_i,\ldots,\hat{\ell}_j,\ldots,\ell_k)\end{aligned}$$

Here $a : L \to T_M$ is the anchor and $[-,-]$ is the bracket on L. In other words d_L is a homological vector field on L, which can be written in coordinates:

$$\mathrm{d}_L = a^i_a\theta^a\frac{\partial}{\partial x^i} + f^c_{ab}\theta^a\theta^b\frac{\partial}{\partial\theta^c}$$

where $\{x^i\}$ are the coordinates on M and $\{\theta^a\}$ are the coordinates for the fiber of $L[1]$.

Equipped with this differential, the cohomology

$$H^*(\mathcal{O}(L[1]), \mathrm{d}_L)$$

is called the Lie algebroid cohomology of L.

Proposition 1.10. *Let $\mathcal{K}$ be a generalized para-complex structure on a Courant algebroid E. The following is true about the local observables of the $2D$ TFTs determined by $\mathcal{K}$:*

(1) the BRST cohomology of the positive $2D$ TFT associated to $\mathcal{K}$ is isomorphic to the Lie algebroid cohomology of the real Lie algebroid $L_{\mathcal{K}}$;

(2) the BRST cohomology of the negative $2D$ TFT associated to $\mathcal{K}$ is isomorphic to the Lie algebroid cohomology of the real Lie algebroid $\tilde{L}_{\mathcal{K}}$;

(3) the BRST cohomology of the full $2D$ TFT is isomorphic to the Lie algebroid cohomology of para-complex Lie algebroid $L_{\mathcal{K}} \oplus \tilde{L}_{\mathcal{K}}$.

We now restrict to the case $E = T \oplus T^*$ with its standard Courant algebroid structure and outline important special cases.

Trivial generalized para-complex structures

Let M be a smooth manifold. Consider the case of the trivial generalized para-complex structure on M defined by

$$\mathcal{K}_0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}.$$

The positive eigenbundle is simply $L_{\mathcal{K}} = T$ and the resulting Lie algebroid structure is the standard one. The positive $2D$ theory is the trivial $2D$ theory.

The negative eigenbundle is $L_{\mathcal{K}} = T^*$ with zero Lie algebroid structure. The negative $2D$ theory is given by the AKSZ theory

$$\mathrm{Map}(\Sigma_{dR}, T^*[1]M)$$

where $T^*[1]M$ is equipped with its standard 1-symplectic structure.

Poisson σ-model

Consider the case of the generalized para-complex structure on M defined by

$$\mathcal{K}_0 = \begin{pmatrix} \mathbb{1} & 2\beta \\ 0 & -\mathbb{1} \end{pmatrix}.$$

where $\beta \in \wedge^2 T$. We have seen that this is integrable if and only if the bivector β is a Poisson structure.

The positive eigenbundle is $L_{\mathcal{K}} = T$ with standard Lie algebroid structure. Thus, the negative $2D$ theory is the same as in the previous example.

The negative eigenbundle is $\tilde{L}_{\mathcal{K}} = T^*$ equipped with Lie algebroid structure determined by the Poisson structure β. The $2D$ TFT is the Poisson σ-model with target (M, β).

Para-complex A/B-models

Consider the case of the generalized para-complex structure on M defined by

$$\mathcal{K}_\omega = \begin{pmatrix} 0 & \omega^{-1} \\ \omega & 0 \end{pmatrix}.$$

where $\omega \in \wedge^2 T^*$ is a symplectic form.

The positive and negative eigenbundles are isomorphic to $L = T^*$ with anchor $\omega^{-1} : T^* \to T$. The $2D$ AKSZ theory is equivalent to the usual A-model.

Finally, consider

$$\mathcal{K}_K = \begin{pmatrix} K & 0 \\ 0 & -K^* \end{pmatrix}.$$

where K is an ordinary para-complex structure, see Example **??**.

In this case, the $2D$ theories specialize to the following para-complex versions of the B-model. First, $L_{\mathcal{K}}$ can be identified with $T^{1,0} \oplus T^{*0,1}$ and $\tilde{L}_{\mathcal{K}}$ is $T^{0,1} \oplus T^{*1,0}$.

Definition 1.11. The **para-B-model** with source a closed Riemann surface Σ and target a para-complex manifold X is the AKSZ theory with source the de Rham space Σ_{dR} and target the 1-shifted symplectic space $T^*[1]X_{\overline{\boldsymbol{\partial}}}$:

$$\mathrm{Map}\left(\Sigma_{dR}, T^*[1]X_{\overline{\boldsymbol{\partial}}}\right).$$

The classical BRST cohomology of the para-B-model is isomorphic to the cohomology of para-holomorphic polyvector fields on the target para-complex manifold X.

BM: below needs to be fixed up a bit

The Lie algebroid cohomology $H^*(\mathcal{O}(L_{\mathcal{K}}[1]), \mathrm{d}_{L_{\mathcal{K}}})$ is computed by a differential of the form

$$\mathrm{d}_{L_{\mathcal{K}}} : \Gamma(M, \wedge^k(T^{1,0} \oplus T^{*0,1})) \to \Gamma(M, \wedge^{k+1}(T^{1,0} \oplus T^{*0,1}))$$

Using the splitting $\wedge^*(T^{1,0} \oplus T^{*0,1}) = \wedge^* T^{1,0} \otimes \wedge^*(T^{*0,1})$ we can identify this differential with the Dobleault differential for the bundle of para-holomorphic polyvector fields $\Theta = \wedge^* T^{1,0}$:

$$\mathrm{d}_L = \overline{\boldsymbol{\partial}} : \boldsymbol{\Omega}^{0,q}(M, \Theta) \to \boldsymbol{\Omega}^{0,q+1}(M, \Theta).$$

Thus, the BRST cohomology is precisely the Dolbeault cohomology of para-holomorphic polyvector fields.

BM: discuss anomalies. Para-holomorphic volume form.

BM: Discuss Si's work on perturbative quantization

BM: ***para Kahler example, **example where the paracomplex structures commute but are not equal (this is where people see "twisted chiral"), para version of T-duality, **Poisson σ-model, going back and forth between generalized (para) complex.

BM: Case of para structure on $M \times M$. Two copies of BF theory.

1.3 Para holomorphic variants

BM: My plan is to just turn this into an extended remark

So far, in each of the σ-models we have discussed in the AKSZ formalism, the fields have depended only topologically on the source Riemann surface. There are closely related σ-models which depending *para-holomorphically* on the source Riemann surface that we briefly discuss.

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