LECTURE 4: DG MODULES FOR DG LIE ALGEBRAS

The enveloping algebra of a Lie algebra recognizes quasi-isomorphisms of dg Lie algebras in the following way.

Lemma 0.1. *A map of dg Lie algebras $f : \mathfrak{g} \to \mathfrak{h}$ is a quasi-isomorphism if and only if the induced map*

$$U(f) : U(\mathfrak{g}) \to U(\mathfrak{h})$$

is a quasi-isomorphism of associative dg algebras.

Proof. We consider the spectral sequences induced by the natural filtrations on the enveloping algebras of $\mathfrak{g}$ and $\mathfrak{h}$. The map $U(f)$ clearly preserves the natural filtration on the enveloping algebra and so it induces a map of spectral sequences. In particular, it induces a map at the E_1 page, which is simply the associated graded algebras

$$\mathrm{Gr}\, U(f) : \mathrm{Sym}^*(\mathfrak{g}) \to \mathrm{Sym}^*(\mathfrak{h}).$$

The map of spectral sequence converges to the map $U(f) : U(\mathfrak{g}) \to U(\mathfrak{h})$.

Now, it is an easy exercise to show that $\mathfrak{g} \to \mathfrak{h}$ is a quasi-isomorphism if and only if it induces a quasi-isomorphism of free commutative dg algebras $\mathrm{Sym}^*(\mathfrak{g}) \to \mathrm{Sym}^*(\mathfrak{h})$. Thus, in one direction, if $\mathfrak{g} \to \mathfrak{h}$ is a quasi-isomorphism, we obtain a quasi-isomorphism at the E_1-page, and the result follows.

In the other direction, if $U(\mathfrak{g}) \to U(\mathfrak{h})$ is a quasi-isomorphism, then taking associated gradeds we obtain a quasi-isomorphism $\mathrm{Sym}^*(\mathfrak{g}) \to \mathrm{Sym}^*(\mathfrak{h})$ and hence $\mathfrak{g} \to \mathfrak{h}$ is a quasi-isomorphism by the above remark. □

1. Modules and (co)homology

1.1. **(dg) Modules.** Recall that to every dg vector space V we can associated an associative dg algebra $\mathrm{End}_k(V)$ of endomorphisms.

Definition 1.1. Let $\mathfrak{g}$ be a dg Lie algebra. A **dg $\mathfrak{g}$-module** is a dg vector space M together with a map of dg Lie algebras

$$\rho : \mathfrak{g} \to \mathrm{End}_k(M).$$

A map of dg modules is defined in the obvious way. Let $\mathsf{Mod}_{\mathfrak{g}}^{\mathsf{dg}}$ denote the category of dg $\mathfrak{g}$-modules.

Remark 1.2. The notion of a (left) dg module for an associative dg algebra is defined in the natural way. By the universal property of the enveloping algebra, we see that a dg module for $\mathfrak{g}$ is equivalent to a (left) dg module for the associative dg algebra $U(\mathfrak{g})$.

Unlike the category of dg Lie algebras, the category of dg $\mathfrak{g}$-modules is an abelian category in the obvious way.

As a matter of notation, when $M, N \in \mathsf{Mod}_{\mathfrak{g}}^{\mathsf{dg}}$, we will write

$$M \otimes_k N \in \mathsf{Vect}_k^{\mathsf{dg}}$$

to be the tensor product as underlying dg vector spaces. While it is true that this tensor product still has the structure of a dg Lie algebra, we will not use it in what immediately follows. [1]

1.2. **(Co)Homology.** In what follows we only recount facets of Lie algebra homology/cohomology that we will use later on. For a more complete survey of these ideas we recommend the textbook reference [FF00].

As for ordinary modules, we have a pair of functors

$$\begin{array}{rccl} (-)^{\mathfrak{g}}: & \mathsf{Mod}_{\mathfrak{g}}^{\mathsf{dg}} & \to & \mathsf{Vect}_k^{\mathsf{dg}} \\ & M & \mapsto & M^{\mathfrak{g}} = \{m \in M \mid x \cdot m = m \ , \ \forall x \in \mathfrak{g}\} \\ (-)_{\mathfrak{g}}: & \mathsf{Mod}_{\mathfrak{g}}^{\mathsf{dg}} & \to & \mathsf{Vect}_k^{\mathsf{dg}} \\ & M & \mapsto & M/\mathfrak{g} \cdot M \end{array}$$

called the invariants/coinvariants respectively.

Remark 1.3. Note that

$$M^{\mathfrak{g}} = \mathrm{Hom}_{U\mathfrak{g}}(k, M)$$

and

$$M_{\mathfrak{g}} = k \otimes_{U\mathfrak{g}} M.$$

Here, we regard M as a left dg $U(\mathfrak{g})$-module and k as a trivial (left and right) dg $U(\mathfrak{g})$-module.

Lemma 1.4. *Consider the functor*

$$\mathrm{triv}_{\mathfrak{g}} : \mathsf{Vect}_k^{\mathsf{dg}} \to \mathsf{Mod}_{\mathfrak{g}}^{\mathsf{dg}}$$

that sends a dg vector space to the trivial $\mathfrak{g}$-module. The functor $M \mapsto M^{\mathfrak{g}}$ is left adjoint to $\mathrm{triv}_{\mathfrak{g}}$*. The functor $M \mapsto M_{\mathfrak{g}}$ is right adjoint to* $\mathrm{triv}_{\mathfrak{g}}$*.*

As a consequence, taking invariants/coinvariants is left/right exact respectively. This motivates the following definition.

Definition 1.5. Let $\mathfrak{g}$ be a dg Lie algebra. The **Lie algebra homology** of $\mathfrak{g}$ is the left derived functor of coinvariants

$$\begin{array}{rccc} H_*(\mathfrak{g}; -): & \mathsf{Mod}_{\mathfrak{g}}^{\mathsf{dg}} & \to & \mathsf{Vect}_k \\ & M & \mapsto & \mathbb{L}_*(-)_{\mathfrak{g}}(M) \end{array}$$

.

[1] In fact, the category $\mathsf{Mod}_{\mathfrak{g}}^{\mathsf{dg}}$ is symmetric monoidal.

Similarly, the **Lie algebra cohomology** of $\mathfrak{g}$ is the right derived functor of invariants

$$\begin{array}{rccc} H^*(\mathfrak{g};-): & \mathsf{Mod}^{\mathsf{dg}}_{\mathfrak{g}} & \to & \mathsf{Vect}_k \\ & M & \mapsto & \mathbb{R}_*(-)^{\mathfrak{g}}(M) \end{array}$$

.

Remark 1.6. Using the Tor and Ext notation, we can write

$$H_*^{\mathrm{Lie}}(\mathfrak{g};M) = \mathrm{Tor}_*^{U\mathfrak{g}}(k,M)$$

and

$$H^*_{\mathrm{Lie}}(\mathfrak{g};M) = \mathrm{Ext}^*_{U\mathfrak{g}}(k,M).$$

To compute the Lie algebra homology, for instance, one uses the usual trick for derived functors. By first finding a projective resolution, tensoring, then computing the cohomology. We proceed by finding a projective resolution of $U\mathfrak{g}$

1.2.1. First, we sketch the following general construction for dg Lie algebras. Given a dg Lie algebra $\mathfrak{g}$ define its *cone* to be the dg Lie algebra $\mathrm{Cone}(\mathfrak{g})$ to be

$$\mathrm{Cone}(\mathfrak{g})^n = \mathfrak{g}^{n+1} \oplus \mathfrak{g}^n$$

with differential

$$\mathrm{d}_n = \begin{pmatrix} \mathrm{d}_{\mathfrak{g},n+1} & 0 \\ \mathrm{id}_{\mathfrak{g}^{n+1}} & \mathrm{d}_{\mathfrak{g},n} \end{pmatrix} : \mathrm{Cone}(\mathfrak{g})^n = \mathfrak{g}^{n+1} \oplus \mathfrak{g}^n \to \mathfrak{g}^{n+2} \oplus \mathfrak{g}^{n+1} = \mathrm{Cone}(\mathfrak{g})^{n+1}$$

and bracket

$$[(x,y),(x',y')] = \left([x,y]_{\mathfrak{g}}, [x,y'] + [y,x']\right).$$

Lemma 1.7. *There is a natural map of dg Lie algebras*

$$\mathfrak{g} \hookrightarrow \mathrm{Cone}(\mathfrak{g}).$$

Furthermore, $\mathrm{Cone}(\mathfrak{g})$ *is acyclic.*

Proof. The construction of the cone above is a special case of a general construction. Given any map of dg vector spaces

$$f : V \to W$$

we can construct the *cone* of f, $\mathrm{Cone}(f)$, as follows. For a textbook reference on these sorts of things, see [Wei94]. The complex $\mathrm{Cone}(\mathfrak{g})$ is the special case where $V = W = \mathfrak{g}$ and $f = \mathrm{id}$.

As an underlying vector space

$$\mathrm{Cone}(f) = V[1] \oplus W$$

(thus $\mathrm{Cone}(f)^n = V^{n+1} \oplus W^n$). The differential is

$$\mathrm{d}_{\mathrm{Cone}} = \begin{pmatrix} \mathrm{d}_{V[1]} & 0 \\ f[1] & \mathrm{d}_W \end{pmatrix}.$$

Observe that the cone of the identity morphism is acyclic. An explicit nullhomotopy is given by the identity map itself. □

Remark 1.8. The reason for the terminology comes from topology. If $f : X \to Y$ is a map of topological spaces then it induces a map of singular cochains $f^* : \mathrm{C}^*_{\mathrm{sing}}(Y) \to \mathrm{C}^*_{\mathrm{sing}}(Y)$. Then, in the construction above, $\mathrm{Cone}(f^*)$ is equal to singular cochains of the *mapping cone* of f. From this intuition, it is easy to see why the cone of the identity is contractible / acyclic.

By functoriality of the enveloping functor, the associative algebra $U(\mathrm{Cone}(\mathfrak{g}))$ is acyclic and hence a resolution for the trivial $U\mathfrak{g}$-module. Thus, we have

$$H^{\mathrm{Lie}}_*(\mathfrak{g}, M) = H^*\left(U(\mathrm{Cone}(\mathfrak{g})) \otimes_{U\mathfrak{g}} M\right).$$

Definition 1.9. The Chevalley-Eilenberg cochain complex computing Lie algebra *homology* is the dg vector space

$$\mathrm{C}^{\mathrm{Lie}}_*(\mathfrak{g}; M) := U(\mathrm{Cone}(\mathfrak{g})) \otimes_{U\mathfrak{g}} M.$$

Its cohomology is precisely the Lie algebra homology of M.

Remark 1.10. Explicitly, as a graded vector space, the CE complex is of the form

$$\begin{aligned}\mathrm{C}^{\mathrm{Lie}}_*(\mathfrak{g}; M) &= U(\mathrm{Cone}(\mathfrak{g})) \otimes_{U\mathfrak{g}} M \\ &= (\mathrm{Sym}(\mathfrak{g}[1]) \otimes_k U(\mathfrak{g})) \otimes_{U\mathfrak{g}} M \\ &= \mathrm{Sym}(\mathfrak{g}[1]) \otimes_k M.\end{aligned}$$

Tracing through these isomorphisms, one can deduce that the differential is

$$\begin{aligned}\mathrm{d}_{CE}(x_1, \ldots, x_n) = &\sum_{i=1}^{n} (\pm) x_1 \cdots x_{i-1} (\mathrm{d}x_i) x_{i+1} \cdots x_n \\ &+ \sum_{i<j} (\pm) x_1 \cdots \widehat{x_i} \cdots x_{j-1} [x_i, x_j] x_{j+1} \cdots x_n.\end{aligned}$$

Thus, we can write the differential as a sum of two terms

$$\mathrm{d}_{CE} = \mathrm{d}_{\mathfrak{g}} + \mathrm{d}_{\mathrm{Lie}}$$

where $\mathrm{d}_{\mathfrak{g}}$ is the differential induced from that on the dg Lie algebra $\mathfrak{g}$ by the Leibniz rule (this preserves Sym-degree) and $\mathrm{d}_{\mathrm{Lie}}$ encodes the Lie bracket (this decreases Sym-degree by one).

1.2.2. There is a completely analogous construction for Lie algebra cohomology.

Definition 1.11. The Chevalley-Eilenberg cochain complex computing Lie algebra *cohomology* is the dg vector space

$$\mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g}; M) := U(\mathrm{Cone}(\mathfrak{g})) \otimes_{U\mathfrak{g}} M.$$

Its cohomology is precisely the Lie algebra homology of M

$$H^*_{\mathrm{Lie}}(\mathfrak{g}; M) = H^*\left(\mathrm{Hom}_{U\mathfrak{g}}(U(\mathrm{Cone}(\mathfrak{g})), M)\right).$$

Remark 1.12. One can identify $\mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g}; M)$ with a complex of the form

$$\mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g}; M) = \left(\mathrm{Sym}(\mathfrak{g}^\vee[-1]) \otimes_k M, \mathrm{d}^{CE}\right).$$

Note that, when $M = k$ there is an identification

$$\mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g}; k) = \left(\mathrm{C}^{\mathrm{Lie}}_*(\mathfrak{g}; k)\right)^\vee = \mathrm{Hom}_k(\mathrm{C}^{\mathrm{Lie}}_*(\mathfrak{g}; k), k).$$

Warning 1.13. We abusively refer to $\mathrm{C}^{\mathrm{Lie}}_*(\mathfrak{g}; -)$ and $\mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g}; -)$ as the Chevalley-Eilenberg (CE) chain and cochain complexes, respectively. Note that both are really *cochain complexes* (equivalently dg vector spaces) since their differentials increase grading degree. We keep this terminology since they compute Lie algebra homology and cohomology, respectively.

1.2.3. The CE complexes for homology and cohomology are functorial in the module input. They both determine functors

$$\begin{array}{rccc} \mathrm{C}^{\mathrm{Lie}}_*(\mathfrak{g}; -): & \mathsf{Mod}^{\mathsf{dg}}_{\mathfrak{g}} & \to & \mathsf{Vect}^{\mathsf{dg}}_k \\ \mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g}; -): & \mathsf{Mod}^{\mathsf{dg}}_{\mathfrak{g}} & \to & \mathsf{Vect}^{\mathsf{dg}}_k. \end{array}$$

We are interested in a different type of functoriality in the case that $M = k$, the trivial module. In this case, we write $\mathrm{C}^{\mathrm{Lie}}_*(\mathfrak{g}) = \mathrm{C}^{\mathrm{Lie}}(\mathfrak{g}; k)$ and similarly for cohomology. A silly statement is that this trivial module is *universal* in the sense that it is a module for all dg Lie algebras. Thus, we can contemplate the functoriality of homology/cohomology in the Lie algebra factor.

Lemma 1.14. *The CE complex for homology/cohomology determine functors*

$$\begin{array}{rccc} \mathrm{C}^{\mathrm{Lie}}_*(-): & \mathsf{Lie}^{\mathsf{dg}}_k & \to & \mathsf{Vect}^{\mathsf{dg}}_k \\ \mathrm{C}^*_{\mathrm{Lie}}(-): & \mathsf{Lie}^{\mathsf{dg}}_k & \to & \left(\mathsf{Vect}^{\mathsf{dg}}_k\right)^{op}. \end{array}$$

1.2.4. There is a filtration on CE chains/cochains similar to the one we discussed for the enveloping algebra.

For chains, the filtration is increasing

$$k = F^0\mathrm{C}^{\mathrm{Lie}}_*(\mathfrak{g}) \hookrightarrow F^1\mathrm{C}^{\mathrm{Lie}}_*(\mathfrak{g}) \hookrightarrow \cdots$$

defined by

$$F^k\mathrm{C}^{\mathrm{Lie}}_*(\mathfrak{g}) = \left(\mathrm{Sym}^{\leq k}(\mathfrak{g}[1]), \mathrm{d}_{CE}\right)$$

with the obvious inclusions. Note that the successive quotients are

$$F^k/F^{k-1} \cong \left(\mathrm{Sym}^k(\mathfrak{g}), \mathrm{d}_{\mathfrak{g}}\right)$$

where $\mathrm{d}_{\mathfrak{g}}$ denotes the differential on the symmetric algebra induced from that on the dg Lie algebra $\mathfrak{g}$ by the Leibniz rule. The filtration is complete in the sense that $\mathrm{C}_*^{\mathrm{Lie}}(\mathfrak{g}) \cong \mathrm{colim}_k F^k$.

The filtration on cochains is defined similarly.

Lemma 1.15. *Let $\mathfrak{g}, \mathfrak{h}$ be dg Lie algebras. If $f : \mathfrak{g} \to \mathfrak{h}$ is a quasi-isomorphism of dg Lie algebras, then the induced maps*

$$\begin{array}{rccc} \mathrm{C}_*^{\mathrm{Lie}}(f): & \mathrm{C}_*^{\mathrm{Lie}}(\mathfrak{g}) & \to & \mathrm{C}_*^{\mathrm{Lie}}(\mathfrak{h}) \\ \mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g}): & \mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{h}) & \to & \mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g}) \end{array}$$

are quasi-isomorphisms.

Proof. We prove the statement for chains. The proof for cochains is completely analogous. Note that the induced map $\mathrm{C}_*^{\mathrm{Lie}}(f)$ preserves Sym-degree, and in particular, preserves the filtration $\mathrm{C}_*^{\mathrm{Lie}}(f) : F^\bullet \mathrm{C}_*^{\mathrm{Lie}}(\mathfrak{g}) \to F^\bullet \mathrm{C}_*^{\mathrm{Lie}}(\mathfrak{h})$.

We prove the claim by induction. When $n = 0$ it is obvious. In general, consider the map of short exact sequences induced from $F^{n-1} \to F^n \to F^n/F^{n-1}$:

$$\begin{tikzcd} F^{n-1}\mathrm{C}_*^{\mathrm{Lie}}(\mathfrak{g}) \arrow[r] \arrow[d, "f"] & \mathrm{C}_*^{\mathrm{Lie}}(\mathfrak{g}) \arrow[r] \arrow[d, "f"] & \mathrm{Sym}^n(\mathfrak{g}[1]) \arrow[d, "f"] \\ F^{n-1}\mathrm{C}_*^{\mathrm{Lie}}(\mathfrak{h}) \arrow[r] & \mathrm{C}_*^{\mathrm{Lie}}(\mathfrak{h}) \arrow[r] & \mathrm{Sym}^n(\mathfrak{h}[1]). \end{tikzcd}$$

By assumption, the left vertical arrow is a quasi-isomorphism. Also, since k is characteristic zero, the map $\otimes^n f : \otimes^n \mathfrak{g} \to \otimes^n \mathfrak{h}$ is a quasi-isomorphism since f is. Thus, the middle arrow is a quasi-isomorphism. □

References

[FF00] B. L. Feigin and D. B. Fuchs. Cohomologies of Lie groups and Lie algebras [MR0968446 (90k:22014)]. In *Lie groups and Lie algebras, II*, volume 21 of *Encyclopaedia Math. Sci.*, pages 125–223. Springer, Berlin, 2000.

[Wei94] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.