LECTURE 3: DG LIE ALGEBRAS AND THE MAURER-CARTAN EQUATION

1. Where are we going?

This lecture marks the end of motivation from classical deformation theory and begins the bulk portion of course. Recall, we have defined the theory of formal moduli problems from a functor of points perspective. Precisely, a formal moduli problem over k is a functor

$$F : \mathsf{Art}_k \to \mathsf{Set}$$

satisfying some hypotheses. In many examples, we found that such a formal moduli problem are "controlled" by the cohomology of another algebraic object that was equipped with a Lie bracket of sorts.

In this lecture we begin to formalize this algebraic notion, namely *dg Lie algebras*, as well as connect it back to deformation theory through the Maurer-Cartan equation. After introducing some definitions and foundational context, our goal in the next few lectures is to construct a functor

$$\mathrm{Def} : \mathsf{Lie}_k^{\mathsf{dg}} \to \mathsf{Moduli}_k$$

from the category of dg Lie algebras to the category of formal moduli problems over k. Most of the rest of the course contemplates how far (or close) this functor is from being an equivalence.

2. A introduction to dg Lie algebras

Let R be a commutative ring over k. A derivation of R is a k-linear map $D : R \to R$ such that for all $a, b \in R$ one has

$$D(ab) = D(a)b + aD(b).$$

Let $\mathrm{Der}(R)$ be the vector space of all derivations. Suppose D_1, D_2 are R-derivations and consider the composition $D_1 \circ D_2 : R \to R$. Applied to the product of two elements we compute

$$(D_1 \circ D_2)(ab) = D_1(D_2(a)b + aD_2(b)) = (D_1 \circ D_2)(a)b + D_1(a)D_2(b) + D_2(a)D_1(b) + a(D_1 \circ D_2)(b).$$

In particular, $D_1 \circ D_2$ is not a derivation. However, if we compute the flipped composition

$$(D_2 \circ D_1)(ab) = (D_2 \circ D_1)(a)b + D_2(a)D_1(b) + D_1(a)D_2(b) + a(D_2 \circ D_1)(b)$$

we notice that the *commutator*

$$[D_1, D_2] := D_1 \circ D_2 - D_2 \circ D_1$$

is a derivation. This equips the vector space $\mathrm{Der}(R)$ with the structure of a *Lie algebra*.

Definition 2.1. A **Lie algebra** over k is a vector space V equipped with a bilinear map

$$[-,-] : V \times V \to V$$

called the *bracket* satisfying the conditions:

(1) Skew symmetry. For all $x, y \in V$, one has $[x,y] = -[y,x]$.
(2) Jacobi identity. For all $x, y, z \in V$ one has

$$[x,[y,z]] = [[x,y],z] + [y,[x,z]].$$

A map of Lie algebras is a linear map $f : V \to W$ preserving the brackets.

Remark 2.2. Standard examples of Lie algebras from representation theory include the $n \times n$ matrix Lie algebras $\mathfrak{gl}(n), \mathfrak{sl}(n), \mathfrak{u}(n)$, etc.. More generally, if W is any vector space we let $\mathfrak{gl}(W)$ be the Lie algebra of endomorphisms of W with bracket given by the commutator. Note that for commutative rings R, there is an inclusion of Lie algebras $\mathrm{Der}(R) \hookrightarrow \mathfrak{gl}(R)$.

Remark 2.3. If $(A, \cdot)$ is any associative algebra, the the commutator $[a,b] := a \cdot b - b \cdot a$ endows A with the structure of a Lie algebra. Thus, there is a functor

$$\mathsf{AssAlg}_k \to \mathrm{Lie}_k.$$

Example 2.4. One can associate a natural Lie algebra to any smooth (or complex) manifold M. Let $TM \to M$ be the tangent bundle and T_M its space of smooth sections. The Lie bracket of vector fields

$$[-,-] : T_M \times T_M \to T_M$$

endows T_M with the structure of a Lie algebra. Locally, T_M is generated over $C^\infty(M)$ by symbols $\{\partial/\partial x_1, \ldots, \partial/\partial x_{\dim(M)}\}$. The Lie bracket is defined locally by the formula

$$\left[f(x)\frac{\partial}{\partial x_i}, g(x)\frac{\partial}{\partial x_j}\right] = \frac{\partial f}{\partial x_j}(x) g(x) \frac{\partial}{\partial x_i} - f(x)\frac{\partial g}{\partial x_i}(x)\frac{\partial}{\partial x_j}$$

where $f, g \in C^\infty(M)$.

2.1. **dg Lie algebras.** In this course a more general object than a plain Lie algebra will play a central role. We have already met the notion of a cochain complex as a $\mathbb{Z}$-graded vector space equipped with a *differential* that is square zero and of grading degree one. The notion of a dg Lie algebra marries this concept with that of an ordinary Lie algebra.

Definition 2.5. A **dg Lie algebra** (dgla) over k is:

(i) a $\mathbb{Z}$-graded k-vector space $V = \{V^n\}_{n\in\mathbb{Z}}$;
(ii) a degree $+1$ linear map $\mathrm{d} : V \to V$ called the *differential*. This is equivalent to a collection of linear maps

$$\mathrm{d}^n : V^n \to V^{n+1};$$

(iii) a bilinear map of degree zero $[-,-] : V \times V \to V$ called the *bracket*. This is equivalent to a collection of maps

$$[-,-] : V^m \times V^n \to V^{m+n}.$$

This data must satisfy the conditions:

(1) Differential. $\mathrm{d}^2 = 0$ (Thus (V, d) is a cochain complex);
(2) Graded skew symmetry. For all $x, y \in V$ one has $[x, y] = (-1)^{|x||y|}[y, x]$;
(3) Graded Jacobi. For all $x, y, z \in V$ one has

$$[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}[y, [x, z]];$$

(4) Graded derivation. For all $x, y \in V$ one has

$$\mathrm{d}([x, y]) = [\mathrm{d}x, y] + (-1)^{|x|}[x, \mathrm{d}y].$$

A map of dg Lie algebras is a linear map $f : V \to W$ of grading degree zero that preserves the differentials and Lie brackets. We denote the resulting category by $\mathsf{Lie}_k^{\mathsf{dg}}$.

Remark 2.6. A dg Lie algebra is simply a Lie algebra object in the category of dg vector spaces (cochain complexes).

Remark 2.7. If V is a dg Lie algebra then the bracket restricts to $V^0 \subset V$ and endows V^0 with the structure of an ordinary Lie algebra. This defines a functor of ordinary Lie algebras into the category $\mathsf{Lie}_k^{\mathsf{dg}}$.

Since $\mathrm{d}^2 = 0$, it makes sense to consider the cohomology $H^*(V)$. The cohomology inherits the Lie bracket from V and has the structure of a dg Lie algebra with differential identically zero.

Example 2.8. Let (W, d) be a dg vector space and consider its degree n endomorphisms [1]

$$\mathrm{End}^n(V, W) = \{\varphi : W \to W \mid \varphi \text{ linear },\ \varphi(W^i) \subset W^{i+n}\}.$$

Define the $\mathbb{Z}$-graded vector space

$$\mathrm{End}(W) = \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}^n(V, W)[-n].$$

There is a natural differential on this graded vector space defined by

$$\mathrm{d}_{\mathrm{End}}\varphi = \mathrm{d}_W \circ \varphi - (-1)^{|\varphi|}\varphi \circ \mathrm{d}_V.$$

Note that a *cochain map* is $\varphi : V \to W$ such that $\mathrm{d}_{\mathrm{Hom}}\varphi = 0$.

The complex $(\mathrm{End}(W), \mathrm{d}_{\mathrm{End}})$ has a natural Lie bracket extending the ordinary one in degree zero. If $\varphi, \psi \in \mathrm{End}(W)$, define the graded commutator

$$[\varphi, \psi] = \varphi \circ \psi - (-1)^{|\varphi||\psi|}\psi \circ \varphi.$$

[1] If W is a graded vector space we let $W[n]$ denote the graded vector space obtained by shifting all degrees down by n. That is, $W[n]^i = W^{i+n}$. A linear endomorphism of degree n on W is then a degree zero map $\varphi : W \to W[n]$.

The bracket satisfies the graded version of skew symmetry and the graded version of the Jacobi identity (exercise). In addition, it is compatible with the differential in the sense that $\mathrm{d}_{\mathrm{End}}$ is a graded derivation for it. Thus, the triple $(\mathrm{End}(V), \mathrm{d}_{\mathrm{End}}, [-,-])$ is naturally a dg Lie algebra.

Example 2.9. For a slightly more general example, take any associative dg algebra A. Then, the graded commutator makes A into a dg Lie algebra as well.

Exercise 2.10. Let R be a commutative dg algebra. Define a derivation of degree n to be a linear map $D : R \to R[n]$ satisfying

$$D(ab) = D(a)b + (-1)^{n|a|} a D(b).$$

Define the $\mathbb{Z}$-graded vector space $\mathrm{Der}(R) = \{\mathrm{Der}^n(R)\}_{n\in\mathbb{Z}}$ where $\mathrm{Der}^n(R)$ is the space of derivations of degree n. Show that $\mathrm{Der}(R)$ has the natural structure of a dg Lie algebra in such a way that there is an inclusion of dg Lie algebras $\mathrm{Der}(R) \hookrightarrow \mathrm{End}(R)$.

3. Maurer-Cartan equation and nilpotency

Throughout the rest of this section we let $(\mathfrak{g}, \mathrm{d}, [-,-])$ be a dg Lie algebra which we will simply refer to by $\mathfrak{g}$. We have already noticed that the degree zero part of $\mathfrak{g}$ has the structure of an ordinary Lie algebra. We are actually most interested in what goes on in higher cohomological degrees.

Definition 3.1. A degree one element $x \in \mathfrak{g}^1$ is a **Maurer-Cartan element** of $\mathfrak{g}$ if it satisfies the equation

$$\mathrm{d}x + \frac{1}{2}[x,x] = 0. \tag{1}$$

Let $\mathrm{MC}(\mathfrak{g}) \subset \mathfrak{g}^1$ denote the set of Maurer-Cartan elements.

Equation (1) is called the Maurer-Cartan equation. It cuts out a quadratic algebraic variety inside the linear space $\mathfrak{g}^1$.

Lemma 3.2. *Suppose $\alpha \in \mathrm{MC}(\mathfrak{g})$ and define the new triple*

$$\mathfrak{g}_\alpha = (\mathfrak{g}, \mathrm{d} + [\alpha, -], [-,-]).$$

Then $\mathfrak{g}_\alpha$ has the structure of a dg Lie algebra.

Proof. We need to check that $(\mathrm{d} + [\alpha,-])^2 = 0$. This equation is evidently equivalent to the Maurer-Cartan equation for α. □

3.1. **Gauge transformations.** Given any Lie group G, the tangent space at the unit T_eG has the natural structure of a Lie algebra. Conversely, if $\mathfrak{g}$ is any Lie algebra there is a unique simply connected Lie group whose tangent space at the unit is $\mathfrak{g}$. This means that phenomena in Lie groups can be translated to problems in Lie algebras and vice-versa.

For simplicity, we work over $\mathbb{C}$. Let G is a Lie group with Lie algebra $\mathfrak{g}$. For any $X \in \mathfrak{g}$ there is a map of Lie algebras

$$\tau_X : \mathbb{C} \to \mathfrak{g}$$

sending $t \mapsto tX$. Here, $\mathbb{C}$ is a Lie algebra with trivial bracket. Via the correspondence between Lie groups and Lie algebras, there is then a unique map of Lie groups

$$\gamma_X : \mathbb{C} \to G.$$

We are viewing $\mathbb{C}$ as a Lie group with group operation given by addition. We define the *exponential map* by

$$\exp : \mathfrak{g} \to G \ , \ X \mapsto \gamma_X(1).$$

Given two elements X, Y of $\mathfrak{g}$ we can consider the product $\exp(X)\exp(Y)$ in G. A natural question is whether this product can be expressed in the form $\exp(Z)$ for some $Z \in \mathfrak{g}$. Since exp is a local diffeomorphism, this is possible whenever X, Y are close enough to zero in $\mathfrak{g}$. However, in general, Z may not exist.

Example 3.3. Let $\mathfrak{g} = \mathfrak{gl}(2)$ and consider the elements

$$X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ , \ Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then, there is no matrix Z such that $\exp(Z) = \exp(X)\exp(Y)$.

When such a Z exists, a combinatorial presentation for it in terms of X, Y is given by the *Baker-Campbell-Hausdorff* formula.

3.1.1. Now, let's go back to the case of a dg Lie algebra $(\mathfrak{g}, \mathrm{d}, [-,-])$. Notice that $\mathfrak{g}^0 \subset \mathfrak{g}$ is a sub dg Lie algebra. In fact, $\mathfrak{g}^n$ is a module for the Lie algebra $\mathfrak{g}^0$ through the adjoint action for each $n \in \mathbb{Z}$. In formulas, if $X \in \mathfrak{g}^0$ and $Y \in \mathfrak{g}^n$ then this action is

$$\mathrm{ad}(X)(Y) = [X, Y] \in \mathfrak{g}^n.$$

The object we are interested in is the Maurer-Cartan set $\mathrm{MC}(\mathfrak{g})$ of the dg Lie algebra. Although $\mathfrak{g}^0$ does not act on $\mathrm{MC}(\mathfrak{g})$ in any natural way (in fact, $\mathrm{MC}(\mathfrak{g})$ is rarely even a vector space) we can formally define the action of a group $\exp(\mathfrak{g}^0)$ on $\mathrm{MC}(\mathfrak{g})$. For now, we think of $\exp(\mathfrak{g}^0)$ as the simply connected Lie group corresponding via Lie's theorem to $\mathfrak{g}^0$. In most examples, this is not quite valid since $\mathfrak{g}^0$ may not even by finite dimensional.

Assuming that this exponential exists, we observe the following. Since $\mathfrak{g}^1$ is a $\mathfrak{g}^0$-module, we see that $\mathfrak{g}^1$ is formally a $\exp(\mathfrak{g}^0)$-representation. By formal properties of the exponential, we see that if an element $\alpha \in \mathfrak{g}^1$ solves the Maurer-Cartan equation then so does $\exp(X) \cdot \alpha$ for any $X \in \mathfrak{g}^0$. Thus $\exp(\mathfrak{g}^0)$ acts on the set $\mathrm{MC}(\mathfrak{g})$.

To make this situation well-defined, we need to make some extra assumptions about the Lie algebra $\mathfrak{g}^0$.

Definition 3.4. Let $\mathfrak{h}$ be an ordinary Lie algebra. We say $\mathfrak{h}$ is **nilpotent** if its lower central series terminates at some finite order. That is, there exists $N \in \mathbb{Z}_{\geq 0}$ such that

$$[X_1, [X_2, [\cdots [X_N, Y] \cdots]]] = 0$$

for all $X_1, \ldots, X_N, Y \in \mathfrak{h}$.

If $\mathfrak{h}$ is nilpotent then for any $X, Y \in \mathfrak{h}$ the Baker-Campell-Hausdorff formula for $\exp(X)\exp(Y)$ converges. In fact, one has the following.

Proposition 3.5. *If $\mathfrak{h}$ is a nilpotent Lie algebra with corresponding simply connected Lie group H, then the exponential map*

$$\exp : \mathfrak{h} \to H$$

is a diffeomorphism.

We can now make the above arguments precise. We say a dg Lie algebra $\mathfrak{g}$ is nilpotent if its degree zero sub Lie algebra $\mathfrak{g}^0 \subset \mathfrak{g}$ is nilpotent. Let $\mathsf{Lie}^{\mathsf{dg,nil}}$ be the category of nilpotent dg Lie algebras. Recall that every graded piece $\mathfrak{g}^n$ of $\mathfrak{g}$ is a module for the Lie algebra $\mathfrak{g}^0$.

Lemma 3.6. *Suppose $\mathfrak{g}$ is a dg Lie algebra such that $\mathfrak{g}^0$ is nilpotent. Let G^0 be the corresponding simply connected Lie group. Then, the action of $\mathfrak{g}^0$ on $\mathfrak{g}^n$ exponentiates to an action of G^0 on $\mathfrak{g}^n$ for each n. Moreover, G^0 preserves the solutions to the Maurer-Cartan equation in $\mathfrak{g}^1$.*

There is an explicit formula for this action given by the BCH formula. Given $X \in \mathfrak{g}^0$ and $\alpha \in \mathfrak{g}$, we have

$$\exp(X) \cdot \alpha = \alpha + \sum_{n \geq 0} \frac{\mathrm{ad}(X)^n}{(n+1)!}([X, \alpha] - \mathrm{d}X).$$

Evidently, the right hand side only makes sense if $\mathrm{ad}(X)^n = 0$ for n large enough. This is where nilpotence is essential!

Remark 3.7. Note that if $\mathfrak{g}^0$ is nilpotent, then the action of $\exp(X)$ corresponding to an element in $X \in \mathfrak{g}^0$ on $\mathfrak{g}^1$ makes sense even if $\mathfrak{g}^0$ is infinite dimensional.

Lemma 3.8. *Suppose $\alpha \in \mathrm{MC}(\mathfrak{g})$ and consider the dg Lie algebra $\mathfrak{g}_\alpha$ as in Lemma 3.2. If $\mathfrak{g}^0$ is nilpotent, then for any $X \in \mathfrak{g}^0$ there is an isomorphism of dg Lie algebras*

$$\varphi_X : \mathfrak{g}_\alpha \xrightarrow{\cong} \mathfrak{g}_{\exp(X)\cdot\alpha}$$

defined by $\varphi_X(y) = \exp(X) \cdot y$.