LECTURE 4: MODULES FOR DG LIE ALGEBRAS

The enveloping algebra of a Lie algebra recognizes quasi-isomorphisms of dg Lie algebras in the following way/

Lemma 0.1. *A map of dg Lie algebras $f : \mathfrak{g} \to \mathfrak{h}$ is a quasi-isomorphism if and only if the induced map*

$$U(f) : U(\mathfrak{g}) \to U(\mathfrak{h})$$

is a quasi-isomorphism of associative dg algebras.

Proof. We consider the spectral sequences induced by the natural filtrations on the enveloping algebras of $\mathfrak{g}$ and $\mathfrak{h}$. The map $U(f)$ clearly preserves the natural filtration on the enveloping algebra and so it induces a map of spectral sequences. In particular, it induces a map at the E_1 page, which is simply the associated graded algebras

$$\mathrm{Gr}\, U(f) : \mathrm{Sym}^*(\mathfrak{g}) \to \mathrm{Sym}^*(\mathfrak{h}).$$

The map of spectral sequence converges to the map $U(f) : U(\mathfrak{g}) \to U(\mathfrak{h})$.

Now, it is an easy exercise to show that $\mathfrak{g} \to \mathfrak{h}$ is a quasi-isomorphism if and only if it induces a quasi-isomorphism of free commutative dg algebras $\mathrm{Sym}^*(\mathfrak{g}) \to \mathrm{Sym}^*(\mathfrak{h})$. Thus, in one direction, if $\mathfrak{g} \to \mathfrak{h}$ is a quasi-isomorphism, we obtain a quasi-isomorphism at the E_1-page, and the result follows.

In the other direction, if $U(\mathfrak{g}) \to U(\mathfrak{h})$ is a quasi-isomorphism, then taking associated gradeds we obtain a quasi-isomorphism $\mathrm{Sym}^*(\mathfrak{g}) \to \mathrm{Sym}^*(\mathfrak{h})$ and hence $\mathfrak{g} \to \mathfrak{h}$ is a quasi-isomorphism by the above remark. □

1. Modules and (co)homology

1.1. **(dg) Modules.** Recall that to every dg vector space V we can associated an associative dg algebra $\mathrm{End}_k(V)$ of endomorphisms.

Definition 1.1. Let $\mathfrak{g}$ be a dg Lie algebra. A **dg $\mathfrak{g}$-module** is a dg vector space M together with a map of dg Lie algebras

$$\rho : \mathfrak{g} \to \mathrm{End}_k(M).$$

A map of dg modules is defined in the obvious way. Let $\mathsf{Mod}_{\mathfrak{g}}^{\mathsf{dg}}$ denote the category of dg $\mathfrak{g}$-modules.

Remark 1.2. By the universal property of the enveloping algebra, we see that a dg module for $\mathfrak{g}$ is equivalent to a (left) dg module for the associative dg algebra $U(\mathfrak{g})$.

The category of modules is an abelian category in the obvious way. Also, there is the notion of tensor product.

Definition 1.3. Suppose M, N are two dg $\mathfrak{g}$-modules. Define the tensor product dg $\mathfrak{g}$-module $M \otimes N$ to be the tensor product of underlying dg vector spaces with $\mathfrak{g}$-module structure given by

$$\rho_M \otimes 1 + 1 \otimes \rho_N : \mathfrak{g} \to \mathrm{End}_k(M \otimes_k N) = \mathrm{End}_k(M) \otimes_k \mathrm{End}_k(N).$$

Remark 1.4. If we think of M, N as dg $U\mathfrak{g}$-modules, then

$$M \otimes N = M \otimes_{U\mathfrak{g}} N$$

as $U\mathfrak{g}$-modules.

1.2. **(Co)Homology.** As for ordinary modules, we have a pair of functors

$$\begin{array}{rccl} (-)^{\mathfrak{g}}: & \mathsf{Mod}_{\mathfrak{g}}^{\mathsf{dg}} & \to & \mathsf{Vect}_k^{\mathsf{dg}} \\ & M & \mapsto & M^{\mathfrak{g}} = \{m \in M \mid x \cdot m = m \ , \ \forall x \in \mathfrak{g}\} \\ (-)_{\mathfrak{g}}: & \mathsf{Mod}_{\mathfrak{g}}^{\mathsf{dg}} & \to & \mathsf{Vect}_k^{\mathsf{dg}} \\ & M & \mapsto & M/\mathfrak{g} \cdot M \end{array}$$

called the invariants/coinvariants respectively.

Remark 1.5. Note that

$$M^{\mathfrak{g}} = \mathrm{Hom}_{U\mathfrak{g}}(k, M)$$

and

$$M_{\mathfrak{g}} = k \otimes_{U\mathfrak{g}} M.$$

Lemma 1.6. *Consider the functor*

$$\mathrm{triv}_{\mathfrak{g}} : \mathsf{Vect}_k^{\mathsf{dg}} \to \mathsf{Mod}_{\mathfrak{g}}^{\mathsf{dg}}$$

that sends a dg vector space to the trivial $\mathfrak{g}$-module. The functor $M \mapsto M^{\mathfrak{g}}$ is left adjoint to $\mathrm{triv}_{\mathfrak{g}}$. *The functor $M \mapsto M_{\mathfrak{g}}$ is right adjoint to* $\mathrm{triv}_{\mathfrak{g}}$.

As a consequence, taking invariants/coinvariants is left/right exact respectively. This motivates the following definition.

Definition 1.7. Let $\mathfrak{g}$ be a dg Lie algebra. The **Lie algebra homology** of $\mathfrak{g}$ is the left derived functor of coinvariants

$$\begin{array}{rccc} H_*(\mathfrak{g}; -): & \mathsf{Mod}_{\mathfrak{g}}^{\mathsf{dg}} & \to & \mathsf{Vect}_k \\ & M & \mapsto & \mathbb{L}_*(-)_{\mathfrak{g}}(M) \end{array}.$$

Similarly, the **Lie algebra cohomology** of $\mathfrak{g}$ is the right derived functor of invariants

$$\begin{array}{rccc} H^*(\mathfrak{g}; -): & \mathsf{Mod}_{\mathfrak{g}}^{\mathsf{dg}} & \to & \mathsf{Vect}_k \\ & M & \mapsto & \mathbb{R}_*(-)^{\mathfrak{g}}(M) \end{array}.$$

Remark 1.8. Using the Tor and Ext notation, we can write

$$H_*^{\mathrm{Lie}}(\mathfrak{g}; M) = \mathrm{Tor}_*^{U\mathfrak{g}}(k, M)$$

and

$$H^*_{\mathrm{Lie}}(\mathfrak{g}; M) = \mathrm{Ext}^*_{U\mathfrak{g}}(k, M).$$

To compute the Lie algebra homology, for instance, one uses the usual trick for derived functors. By first finding a projective resolution, tensoring, then computing the cohomology. We proceed by finding a projective resolution of $U\mathfrak{g}$

1.2.1. First, we sketch the following general construction for dg Lie algebras. Given a dg Lie algebra $\mathfrak{g}$ define its *cone* to be the dg Lie algebra $\mathrm{Cone}(\mathfrak{g})$ to be

$$\mathrm{Cone}(\mathfrak{g})_n = \mathfrak{g}_n \oplus \mathfrak{g}_{n-1}$$

with differential

$$\mathrm{d}_n = \begin{pmatrix} \mathrm{d}_{\mathfrak{g},n} & \mathrm{id}_{\mathfrak{g}_n} \\ 0 & \mathrm{d}_{\mathfrak{g},n-1} \end{pmatrix} : \mathrm{Cone}(\mathfrak{g})_n = \mathfrak{g}_n \oplus \mathfrak{g}_{n-1} \to \mathfrak{g}_{n+1} \oplus \mathfrak{g}_n = \mathrm{Cone}(\mathfrak{g})_{n+1}$$

and bracket

$$[(x, y), (x', y')] = \left([x, y]_{\mathfrak{g}}, [x, y'] + [y, x']\right).$$

Lemma 1.9. *There is a natural map of dg Lie algebras*

$$\mathfrak{g} \hookrightarrow \mathrm{Cone}(\mathfrak{g}).$$

Furthermore, $\mathrm{Cone}(\mathfrak{g})$ *is acyclic.*

By functoriality of the enveloping functor, the associative algebra $U(\mathrm{Cone}(\mathfrak{g}))$ is acyclic and hence a resolution for the trivial $U\mathfrak{g}$-module. Thus, we have

$$H_*^{\mathrm{Lie}}(\mathfrak{g}, M) = H^*\left(U(\mathrm{Cone}(\mathfrak{g})) \otimes_{U\mathfrak{g}} M\right).$$

Definition 1.10. The Chevalley-Eilenberg complex computing Lie algebra *homology* is the dg vector space

$$\mathrm{C}_*^{\mathrm{Lie}}(\mathfrak{g}; M) := U(\mathrm{Cone}(\mathfrak{g})) \otimes_{U\mathfrak{g}} M.$$

Its cohomology is precisely the Lie algebra homology of M.

Remark 1.11. Explicitly, as a graded vector space, the CE complex is of the form

$$\begin{aligned} \mathrm{C}_*^{\mathrm{Lie}}(\mathfrak{g}; M) &= U(\mathrm{Cone}(\mathfrak{g})) \otimes_{U\mathfrak{g}} M \\ &= (\mathrm{Sym}(\mathfrak{g}[1]) \otimes_k U(\mathfrak{g})) \otimes_{U\mathfrak{g}} M \\ &= \mathrm{Sym}(\mathfrak{g}[1]) \otimes_k M. \end{aligned}$$

Tracing through these isomorphisms, one can deduce that the differential is

$$\begin{aligned}\mathrm{d}_{CE}(x_1, \ldots, x_n) = &\sum_{i=1}^{n} (\pm) x_1 \cdots x_{i-1} (\mathrm{d}x_i) x_{i+1} \cdots x_n \\ &+ \sum_{i<j} (\pm) x_1 \cdots \widehat{x_i} \cdots x_{j-1} [x_i, x_j] x_{j+1} \cdots x_n.\end{aligned}$$

1.2.2. There is a completely analogous construction for Lie algebra cohomology.

Definition 1.12. The Chevalley-Eilenberg complex computing Lie algebra *cohomology* is the dg vector space

$$\mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g}; M) := U(\mathrm{Cone}(\mathfrak{g})) \otimes_{U\mathfrak{g}} M.$$

Its cohomology is precisely the Lie algebra homology of M

$$H^*_{\mathrm{Lie}}(\mathfrak{g}; M) = H^* \left(\mathrm{Hom}_{U\mathfrak{g}}(U(\mathrm{Cone}(\mathfrak{g})), M) \right).$$

Remark 1.13. One can identify $\mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g}; M)$ with a complex of the form

$$\mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g}; M) = \left(\mathrm{Sym}(\mathfrak{g}^\vee[-1]) \otimes_k M, \mathrm{d}^{CE} \right).$$

Note that, when $M = k$ there is an identification

$$\mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g}; k) = \left(\mathrm{C}^{\mathrm{Lie}}_*(\mathfrak{g}; k) \right)^\vee = \mathrm{Hom}_k(\mathrm{C}^{\mathrm{Lie}}_*(\mathfrak{g}; k), k).$$

1.2.3. The CE complexes for homology and cohomology are functorial in the module input. They both determine functors

$$\begin{array}{rcll} \mathrm{C}^{\mathrm{Lie}}_*(\mathfrak{g}; -) : & \mathsf{Mod}^{\mathsf{dg}}_{\mathfrak{g}} & \to & \mathsf{Vect}^{\mathsf{dg}}_k \\ \mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g}; -) : & \mathsf{Mod}^{\mathsf{dg}}_{\mathfrak{g}} & \to & \mathsf{Vect}^{\mathsf{dg}}_k. \end{array}$$

We are interested in a different type of functoriality in the case that $M = k$, the trivial module. In this case, we write $\mathrm{C}^{\mathrm{Lie}}_*(\mathfrak{g}) = \mathrm{C}^{\mathrm{Lie}}(\mathfrak{g}; k)$ and similarly for cohomology. A silly statement is that this trivial module is *universal* in the sense that it is a module for all dg Lie algebras. Thus, we can contemplate the functoriality of homology/cohomology in the Lie algebra factor.

Lemma 1.14. *The CE complex for homology/cohomology determine functors*

$$\begin{array}{rcll} \mathrm{C}^{\mathrm{Lie}}_*(-) : & \mathsf{Lie}^{\mathsf{dg}}_k & \to & \mathsf{Vect}^{\mathsf{dg}}_k \\ \mathrm{C}^*_{\mathrm{Lie}}(-) : & \mathsf{Lie}^{\mathsf{dg}}_k & \to & \left(\mathsf{Vect}^{\mathsf{dg}}_k\right)^{op}. \end{array}$$

Moreover, if $f : \mathfrak{g} \to \mathfrak{h}$ is a quasi-isomorphism of dg Lie algebras, then the induced maps

$$\begin{array}{rcll} \mathrm{C}^{\mathrm{Lie}}_*(f) : & \mathrm{C}^{\mathrm{Lie}}_*(\mathfrak{g}) & \to & \mathrm{C}^{\mathrm{Lie}}_*(\mathfrak{h}) \\ \mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g}) : & \mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{h}) & \to & \mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g}) \end{array}$$

are quasi-isomorphisms.