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Deformation theory

Tesi di Laurea Specialistica

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Introduction

Deformation theory is a branch of algebraic geometry whose central problem is the local study of algebraic families of objects. More specifically, it is the study of families of objects X (which can be simply schemes, or more complicated structures, like sheaves, maps, closed subschemes, bundles, and so on) over a scheme S, restricting to a given object X_0 over some point $s_0 \in S$; in this sense these are *deformations* of X_0. This study of families is useful for example in moduli space theory, where it can give informations about the local structure of a moduli space at a fixed point.

The modern study of deformations is usually carried out in several steps. First of all one considers *infinitesimal deformations*, which are roughly obtained by adjoining to the object X_0 we are deforming some nilpotent parameters, in a way such that when these parameters vanish, we get our X_0. More formally, if X_0 is defined over a field k, an infinitesimal deformation will be a family over some local artinian k-algebra A, restricting to X_0 over $\mathrm{Spec}(k) \subseteq \mathrm{Spec}(A)$. A natural tool for this infinitesimal study is given by the so-called functors of Artin rings (or *deformation functors*).

A central question regarding infinitesimal deformations is that of existence and behaviour of liftings: that is, given a surjection $A' \to A$ of local artinian k-algebras and a family over A, does there exist a family over A' restricting to the given one over A? Moreover, how are these liftings related? The answers to these questions are given by obstruction theories and by an action of the tangent space on the set of isomorphism classes of liftings: usually one can find a vector space V and for particular kind of surjections $A' \to A$ as above with a family over A, an element $v \in V$ such that $v = 0$ if and only if there exists a lifting. If this happens, there is another vector space W acting in a free and transitive way on the set of isomorphisms classes of liftings.

The next step is to consider *formal deformations*, which are sequences of compatible infinitesimal deformations over the quotients $R/\mathfrak{m}_R^n$ of a noetherian local complete k-algebra R. There is a natural concept of universality for such formal deformations, and a little less natural notion, called versality, which are related to the prorepresentability of the deformation problem. One of the main results of this part of the theory is a theorem (due to Schlessinger) that gives necessary and sufficient conditions for the existence of such a (uni)versal deformation.

The last step, called the problem of *algebraization*, is to pass from formal deformations to actual ones, and is the analogue of a convergence step in the case of deformations of complex analytic manifolds, which amounts to passing from formal solutions to analytic ones. Whereas in the complex analytic case this problem is usually solvable, in the algebro-geomtric context it is not in general (for example it cannot be solved for surfaces in general). The main tools for this step are an existence theorem of Grothendieck (Theorem 5.4.4), and an approximation theorem of Artin, which allows one to pass to families of finite type.

Historically, the origin of the subject goes back to the work of K. Kodaira and D. C. Spencer on deformations of complex analytic manifolds, in the late 50's. Their methods were actually based more on complex analysis than on algebraic geometry, but some key ideas related to the infinitesimal study and formal deformations were already present.

In the 60's Grothendieck and his school revolutioned the subject (along with the rest of algebraic geometry) by using the language of schemes to translate it in the algebro-geometric context. This language is particularly suited to formalize the study of infinitesimal deformations because of the possibility of having nilpotent elements in the structural sheaf.

Many other people contributed to further developing. Remarkable contributions came in particular from the work of Schlessinger, who in the late 60's shifted the attention on functors of Artin rings, which are a natural way to formalize deformation problems, and the one of Illusie about the so-called cotangent complex. Later, D. S. Rim used (co)fibered categories instead of functors as basic tool to study deformation problems, and M. Artin used this point of view to study openness properties of versal deformations, and to define what today we call an "Artin stack".

The purpose of the present work was to rewrite a part of the classical theory of deformation functors of Schlessinger by using fibered categories instead of functors, following the approach of Rim. This basically amounts to keeping track of automorphisms instead of neglecting them by taking isomorphism classes. This approach seems both more natural (Schlessinger's conditions become more transparent) and can be more useful when one has to study moduli problems that are not represented by schemes.

The treatment is heavily based on a course taught by professor Vistoli in Bologna in 2002-2003 and on his expository paper [Vis]. In the rest of this introduction we summarize the contents of each chapter, and then set up some notations and conventions.

Chapter 1 introduces the subject by analyzing the example of deformations of schemes, and showing how this deformation problem gives rise to a functor of Artin rings. After stating the classical conditions of Schlessinger for a deformation functor and the related theorem of prorepresentability, we shift to fibered categories, and recall definitions and some facts that will be used throughout the rest of the thesis. Then we translate Schlessinger's condition in this alternative language, and end up with the concept of deformation category. Finally we introduce and analyze briefly three important examples, that will give a concrete counterpart to all the abstract constructions we will consider.

In Chapter 2 we start our study of infinitesimal deformations by introducing the concept of tangent space of a deformation category, which basically classifies first order-deformations. After analyzing a canonical action of this space on the set of isomorphism classes of liftings, we calculate it in the three examples introduced in the preceding chapter, and give an application to deformations of smooth hypersurfaces in $\mathbb{P}^n_k$.

Chapter 3 is devoted to the definition of the group of infinitesimal automorphisms, and the analysis of some properties.

Chapter 4 is about obstructions to the existence of liftings. We define obstruction theories, discuss minimal obstruction spaces, and give an useful criterion to recognize unobstructed deformation problems. We describe then an obstruction theory for each one of our guide examples, and give a classical example of a smooth projective variety with nontrivial obstructions.

In Chapter 5 we turn to the study of formal deformations. In particular we see how, thanks to a Yoneda-like result, they lead naturally to a concept of prorepresentability for deformation categories. We study then universal and (mini)versal formal deformations, and give a proof of existence of versal deformation, which will allow us to prove an analogue of Schlessinger's Theorem. After giving some applications to obstruction theories, we briefly examine the problem of algebraization of formal deformations, giving some results.

Finally Chapter 6 applies the results obtained in the preceding ones to the case of deformations of nodal curves. We show how one can get a miniversal deformation of a nodal affine curve from a miniversal deformation of a standard local model of the singularity, and get from this a formal description of any global deformation around a singular point. Finally, we give an algebraization result for projective nodal curves, using the general results of the previous Chapter.

The three appendices gather some results that are used throughout this work, and are more or less standard facts.

Notations and conventions

All rings will be commutative with identity, and noetherian. If A is a local ring, $\mathfrak{m}_A$ will always denote its maximal ideal.

The symbol Λ will always denote a notherian local ring, which is complete with respect to the $\mathfrak{m}_\Lambda$-adic topology, meaning that the natural homomorphism $\Lambda \to \varprojlim \Lambda/\mathfrak{m}_\Lambda^n$ is an isomorphism. By k we denote a field (not necessarily algebraically closed), and it will usually be the residue field $\Lambda/\mathfrak{m}_\Lambda$ of Λ.

We denote by (Art/Λ) the category of local artinian Λ-algebras with residue field k. The *order* of an object $A \in (\mathrm{Art}/\Lambda)$ will be the least n such that $\mathfrak{m}_A^{n+1} = 0$. Similarly (Comp/Λ) will be the category of noetherian local complete Λ-algebras with residue field k.

Notice that all homomorphisms in (Art/Λ) and (Comp/Λ) are automatically local. In general, if A is a local ring with residue field k, we will denote by (Art/A) the category of local artinian A-algebras with residue field k.

When dealing with categories, as customary we will not worry about set-theoretic problems, so in particular the collections of objects and arrows will always treated as sets. A functor $F : \mathcal{A} \to \mathcal{B}$ will always denote a covariant functor from $\mathcal{A}$ to $\mathcal{B}$; a contravariant functor from $\mathcal{A}$ to $\mathcal{B}$ will be considered as a covariant functor from the opposite category, written $F : \mathcal{A}^{op} \to \mathcal{B}$. If $\mathcal{A}$ is a category, $A \in \mathcal{A}$ will mean that A is an object of $\mathcal{A}$.

We denote by (Set) the category of sets, by (Mod/A) (resp. (FMod/A)) the category of (finitely generated) modules over the ring A, and by (Vect/k) (resp. (FVect/k)) the category of (finite-dimensional) k-vector spaces. By *groupoid* we mean a category in which all arrows are invertible. A *trivial groupoid* will be a groupoid in which for any pair of objects there is exactly one arrow from this first to the second.

All schemes we will consider will be noetherian, and if $f : X \to Y$ is a morphism of schemes, $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ will denote the corresponding morphism of sheaves. If X is a scheme, we write $|X|$ for the underlying topological space, and *quasi-coherent $\mathcal{O}_X$-module* as well as *quasi-coherent sheaf* will always mean quasi-coherent sheaf of $\mathcal{O}_X$-modules. Usually, we specify the structure sheaf only when there are different schemes with the same underlying topological space. If $x \in X$ is a point of the scheme X, we will denote by $k(x)$ its residue field $\mathcal{O}_{X,x}/\mathfrak{m}_x$.

If X is a scheme over k, the sheaf of Kähler differentials $\Omega_{X/k}$ on X coming from the morphism $X \to \mathrm{Spec}(k)$ will be denoted simply by Ω_X, and we use the same convention with the tangent sheaf; in the same fashion, the sheaf of continuous differentials $\widehat{\Omega}_{R/k}$ of an object $R \in (\mathrm{Comp}/k)$ will be denoted by $\widehat{\Omega}_R$ (see appendix B). Moreover by *rational point* of X we always mean a k-rational point.

If $X \subseteq Y$ is a closed immersion of schemes over a ring A, with sheaf of ideals I, by *conormal sequence* associated with this immersion we will always mean the exact sequence of $\mathcal{O}_X$-modules

$$I/I^2 \longrightarrow \Omega_{Y/A}|_X \longrightarrow \Omega_{X/A} \longrightarrow 0.$$

If X is a scheme over a ring A and $A \to B$ is a ring homomorphism, we denote by X_B the base change $X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B)$, and we will use the same notation for pullbacks of (quasi-) coherent sheaves.

If $\mathcal{U} = \{U_i\}_{n\in\mathbb{N}}$ is an open cover of a topological space X, we will denote by U_{ij} the double intersection $U_i \cap U_k$, by U_{ijk} the triple intersection $U_i \cap U_j \cap U_k$, and so on.

By *variety* we will always mean an integral separated scheme of finite type over the field k. A *curve* will be a variety of dimension 1.

The symbol 0 will sometimes denote the trivial group, A-module, k-vector space.

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Chapter 1

Deformation categories

In this introductory chapter we will define the basic setting for the study of inifinitesimal deformations, which will be employed throughout the rest of this thesis.

Starting from the example of deformations of schemes, we mention briefly Schlessinger's classical theory of deformation functors, and state Schlessinger's Theorem. After this, we gather some definitions and facts about fibered categories, which are the objects we will use to formalize deformation problems instead of functors. We will then state and examine a basic condition our fibered categories will satisfy, and relate our theory with Schlessinger's one. Finally we will introduce three examples of deformation problems that will be analyzed in detail throughout this work.

1.1 Deformation functors

We start by describing the most basic example of a deformation problem, that considers deformations of schemes.

Let X_0 be a proper scheme over k; we are interested in families having a fiber over a rational point isomorphic to X_0.

Definition 1.1.1. *A **deformation** of X_0 is a cartesian diagram of schemes over k*

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spec}(k) \arrow[r] & S \end{tikzcd}$$

where $f : X \to S$ is a flat and proper morphism.

Sometimes X is called the **total scheme** of the deformation, S the **base scheme**.

Notice that to give a deformation we can equivalently give a flat and proper morphism $f : X \to S$ and an isomorphism of the fiber of f over a rational point $s_0 \in S$ with X_0. We will usually refer to a deformation simply as the morphism of schemes, leaving the rational point of S and the isomorphism with the fiber understood.

Remark 1.1.2. The properness condition on the morphism f is often assumed when dealing with global deformations, to avoid situations one typically does not want to consider. We will drop this hypothesis as soon as we focus on infinitesimal deformations, which do not have this kind of problems.

Definition 1.1.3. *An* ***isomorphism*** *between two deformations $f : X \to S$ and $g : Y \to S$ of X_0 is an isomorphism $F : X \to Y$ inducing the identity on (the fiber isomorphic to) X_0.*

Example 1.1.4. Every X_0 has a **trivial deformation** over any scheme S over k, given by the projection $X_0 \times_{\mathrm{Spec}(k)} S \to S$, and we can take as distinguished fiber any fiber over a rational point of S, since they are all isomorphic to X_0.

A deformation of X_0 over S is called **trivial** if it is isomorphic to this trivial deformation.

Deformations over a fixed S with isomorphisms form a category, which is a groupoid by definition. We call this category $\mathcal{D}ef_{X_0}(S)$.

Remark 1.1.5. This construction is functorial in the base space: given a morphism $\psi : R \to S$ and a deformation $f : X \to S$ of X_0, we can form the fibered product and consider the projection $R \times_S X \to R$, which is a deformation of X_0 over R. Moreover, if we have two isomorphic deformations over S, say $f : X \to S$ and $g : Y \to S$ with an isomorphism $F : X \to Y$, F induces an isomorphism $\mathrm{id} \times_S F : R \times_S X \to R \times_S Y$, and this association gives a **pullback functor** $\psi^* : \mathcal{D}ef_{X_0}(S) \to \mathcal{D}ef_{X_0}(R)$.

As we have already mentioned, the fist step in studying deformations is considering infinitesimal ones.

Definition 1.1.6. *A deformation is called* ***infinitesimal*** *if $S = \mathrm{Spec}(A)$, where $A \in (\mathrm{Art}/k)$, and* ***first-order*** *if $S = \mathrm{Spec}(k[\varepsilon])$.*

Here and from now on, $k[\varepsilon]$ denotes the k-algebra $k[t]/(t^2) \cong k \oplus kt$ (the **ring of dual numbers** of k), so that ε is an "indeterminate" with $\varepsilon^2 = 0$.

In the case of infinitesimal deformations X and X_0 have the same underlying topological space, and what changes is only the structure sheaf. This is because the sheaf of ideals of X_0 in X is nilpotent, being the pullback of the sheaf of ideals on $\mathrm{Spec}(A)$ corresponding to the maximal ideal of A.

A scheme whose first-order deformations are all trivial is said to be **rigid**.

Definition 1.1.7. *The **deformation functor** defined by X_0 is the functor* $\mathrm{Def}_{X_0} : (\mathrm{Art}/k) \to (\mathrm{Set})$ *defined on objects as*

$$\mathrm{Def}_{X_0}(A) = \{\textit{isomorphism classes of deformations of } X_0 \textit{ over } \mathrm{Spec}(A)\}$$

and sending a homomorphism $\varphi : A \to B$ to the pullback $\varphi_* : \mathrm{Def}_{X_0}(A) \to \mathrm{Def}_{X_0}(B)$.

Remark 1.1.8. Notice that we introduced a covariant construction $\varphi \mapsto \varphi_*$ and still called it pullback, and not pushforward. This is because we always want to consider $(\mathrm{Art}/k)^{op}$ as a subcategory of (Sch/k), and from this point of view the function φ_* is the pullback $(\widetilde{\varphi})^*$ induced by the map $\widetilde{\varphi} : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ corresponding to φ.

To avoid this confusion, every time we will have a homomorphism $\varphi : A \to B$ that induces a pullback function in some way, we will still denote it by φ_*, keeping in mind that it is the pullback induced by the associated map on the spectra.

We stress the fact that $\mathrm{Def}_{X_0}(A)$ is the set of isomorphism classes of the category $\mathcal{D}ef_{X_0}(\mathrm{Spec}(A))$, and the function φ_* is just the one induced by the pullback functor we defined before, along the morphism of schemes $\psi : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$.

So the study of infinitesimal deformations of a fixed scheme X_0 is basically the study of a functor $(\mathrm{Art}/k) \to (\mathrm{Set})$.

Definition 1.1.9. *A **predeformation functor** is a functor $F : (\mathrm{Art}/k) \to (\mathrm{Set})$, such that $F(k)$ is a set with one element.*

The idea is of course that the element of $F(k)$ is the object that is getting deformed, and the elements of $F(A)$ are its deformations on $\mathrm{Spec}(A)$. All the cases mentioned in the introduction can be formalized in this setting, and we will see some examples later on.

The theory of these functors has been developed first by Schlessinger, in [Schl] (another exposition can be found in Chapter 2 of [Ser]). Since we will review most of it using fibered categories, there is no point in describing it in detail here. An exception is the so-called Schlessinger's Theorem (which is the central result of Schlessinger's paper), which will provide a basic condition for the fibered categories we will consider. To state the theorem we need a couple of definitions.

Definition 1.1.10. *A predeformation functor is **prorepresentable** if it is isomorphic to a functor of the form $\mathrm{Hom}_k(R, -)$ for some $R \in (\mathrm{Comp}/k)$.*

Prorepresentability corresponds to the existence of what is called a universal formal deformation, and is clearly a good thing to have, but it is also quite restrictive. A substitute when prorepresentability fails is the existence of a hull, which is a formal deformation having a weaker universality property. Again, we will not go into details here, because we will discuss all of this later in a more general context.

Definition 1.1.11. *A **small extension** is a surjective homomorphism $\varphi : A' \to A$ in $(\mathrm{Art}\,/k)$, such that $\ker(\varphi)$ in annihilated by $\mathfrak{m}_{A'}$, so that it is naturally a k-vector space. A small extension is called **tiny** if $\ker(\varphi)$ is also principal and nonzero, or equivalently if $\ker(\varphi) \cong k$ as a k-vector space.*

Definition 1.1.12. *The **tangent space** of a predeformation functor F is $TF = F(k[\varepsilon])$.*

Remark 1.1.13. This is of course only a set in general, but it has a canonical structure of k-vector space if F satisfies condition (H2) below (see [Schl]).

Let F be a predeformation functor, and suppose we are given two homomorphisms $A' \to A$ and $A'' \to A$ in $(\mathrm{Art}\,/k)$. Then we can consider the fibered product $A' \times_A A''$ (notice that this is still an object of $(\mathrm{Art}\,/k)$, see Lemma 1.3.5), and we have a natural map $f : F(A' \times_A A'') \to F(A') \times_{F(A)} F(A'')$ given by the universal property of the target. Schlessinger's condition are as follows:

(H1) f is surjective when $A' \to A$ is a tiny extension.
(H2) f is bijective when $A' = k[\varepsilon]$ and $A = k$.
(H3) The tangent space TF is finite-dimensional.
(H4) f is bijective when $A' = A''$ and $A' \to A$ is a tiny extension.

Theorem 1.1.14 (Schlessinger)**.** *A predeformation functor F has a hull if and only if it satisfies (H1),(H2),(H3) above, and it is prorepresentable if and only if it satisfies also (H4).*

Remark 1.1.15. Conditions (H1) and (H2) are usually satisfied when dealing with functors coming from geometric deformation problems. Because of this, a predeformation functor satisfying (H1)+(H2) is called by some authors a **deformation functor**. Schlessinger's terminology is a bit different, since with "deformation functor" he means our predeformation ones.

1.2 Categories fibered in groupoids

As we have seen, the deformation functor of a scheme X_0 is formed taking isomorphism classes in a certain groupoid. This actually happens systematically when a geometric deformation problem is translated into a functor, and sometimes, for example when using deformation theory to study moduli spaces, it is useful to keep track of isomorphisms and automorphisms.

This leads us to using categories fibered in groupoids instead of functors while developing our theory. For this purpose we recall here the definitions and some basic facts about fibered categories. All the proofs and more about the subject can be found in Chapter 3 of [FGA].

First definitions

In what follows we consider two categories $\mathcal{F}$ and $\mathcal{C}$ with a functor $p_{\mathcal{F}} : \mathcal{F} \to \mathcal{C}$. In this context, the notation $\xi \mapsto T$ where $\xi \in \mathcal{F}$ and $T \in \mathcal{C}$ will mean $p_{\mathcal{F}}(\xi) = T$ (and we will sometimes say that ξ is over T). Moreover we will call a diagram like this

$$\begin{tikzcd} \xi \arrow[r, "f"] \arrow[d, mapsto] & \eta \arrow[d, mapsto] \\ T \arrow[r, "\varphi"] & S \end{tikzcd}$$

commutative if $p_{\mathcal{F}}(f) = \varphi$ (and we will sometimes say that f is over φ).

Definition 1.2.1. *An arrow $f : \xi \to \eta$ of $\mathcal{F}$ is **cartesian** if the following universal property holds: every commutative diagram*

$$\begin{tikzcd} \nu \arrow[dr, dashed, "h"'] \arrow[drr, bend left, "g"] \arrow[dd, mapsto] & & \\ & \xi \arrow[r, "f"'] \arrow[d, mapsto] & \eta \arrow[d, mapsto] \\ U \arrow[r, "\varphi"] & T \arrow[r] & S \end{tikzcd}$$

can always be filled with a dotted arrow, in a unique way.

In other words, given any two arrows $g : \nu \to \eta$ in $\mathcal{F}$ and $\varphi : p_{\mathcal{F}}(\nu) \to p_{\mathcal{F}}(\xi)$ in $\mathcal{C}$ such that $p_{\mathcal{F}}(f) \circ \varphi = p_{\mathcal{F}}(g)$, there exists exactly one arrow $h : \nu \to \xi$ over φ such that $f \circ h = g$.

Remark 1.2.2. It is very easy to see that if we have two cartesian arrows $\xi \to \eta$ and $\nu \to \eta$ in $\mathcal{F}$ over the same arrow of $\mathcal{C}$, then there is a canonically defined isomorphism $\xi \cong \nu$, coming from the universal property, and compatible with the two arrows.

Definition 1.2.3. *$\mathcal{F} \to \mathcal{C}$ is a **fibered category** if for every object η of $\mathcal{F}$ and every arrow $\varphi : T \to p_{\mathcal{F}}(\eta)$ there exists a cartesian arrow $f : \xi \to \eta$ of $\mathcal{F}$ over φ.*

Sometimes we will also say that $\mathcal{F}$ is a fibered category over $\mathcal{C}$.

In the situation above we say that ξ is a **pullback** of η to T along the arrow φ. So fibered categories are basically categories in which we can always find pullbacks along arrows of $\mathcal{C}$. The existence of some sort of pullback is a very common feature when dealing with geometric problems, so it seems convenient to use the formalism of fibered categories in this context.

By the preceding remark, pullbacks are unique, up to a unique isomorphism.

Definition 1.2.4. *If T is an object of $\mathcal{C}$, we can define a **fiber category** that we denote by $\mathcal{F}(T)$: its objects are objects ξ of $\mathcal{F}$ such that $p_{\mathcal{F}}(\xi) = T$, and its arrows are arrows $f : \xi \to \eta$ of $\mathcal{F}$ such that $p_{\mathcal{F}}(f) = \mathrm{id}_T$.*

*A fibered category $\mathcal{F} \to \mathcal{C}$ is a **category fibered in groupoids** if for every object T of $\mathcal{C}$ the category $\mathcal{F}(T)$ is a groupoid, i.e. every arrow of $\mathcal{F}(T)$ is an isomorphism.*

In the following we will always use categories fibered in groupoids (mostly because we will be interested in classifying things, and so the only morphisms we want to have between objects are isomorphisms).

We have the following criterion to decide when a functor $\mathcal{F} \to \mathcal{C}$ gives a category fibered in groupoids.

Proposition 1.2.5. *Let $\mathcal{F} \to \mathcal{C}$ be a functor. $\mathcal{F}$ is a category fibered in groupoids over $\mathcal{C}$ if and only if the following conditions hold:*

(i) *Every arrow of $\mathcal{F}$ is cartesian.*

(ii) *Given an arrow $T \to S$ of $\mathcal{C}$ and an object $\xi \in \mathcal{F}(S)$, there exists an arrow of $\mathcal{F}$ over $T \to S$ and with target ξ.*

So a fibered category is fibered in groupoids if and only if every arrow gives a pullback.

The ambiguity in the choice of a pullback is sometimes annoying when defining things that seem to depend on it. However, in these cases the constructions one ends up with are independent of the choice in some way (the construction of the pullback functors we will see shortly is an example). To avoid this annoyance, we make the choice of a pullback of any object along any arrow once and for all.

Definition 1.2.6. *A **cleavage** for a fibered category $\mathcal{F} \to \mathcal{C}$ is a collection of cartesian arrows of $\mathcal{F}$, such that for every object ξ of $\mathcal{F}$ and every arrow $T \to S$ in $\mathcal{C}$, such that $\xi \in \mathcal{F}(S)$, there is exactly one arrow in the cleavage with target ξ and over $T \to S$.*

We can use some appropriate version of the axiom of choice to see that every fibered category has a cleavage. Fixing a cleavage in a fibered category is somewhat like choosing a basis for a vector space: sometimes it is useful because it makes things clearer and more concrete, but usually one would like to have constructions that are independent of it.

Remark 1.2.7. In what follows we will always assume that we have a fixed cleavage when we are dealing with fibered categories. If we have an arrow $\varphi : T \to S$ of $\mathcal{C}$ and an object $\xi \in \mathcal{F}(S)$, we will denote the pullback given by the cleavage by $\varphi^*(\xi)$, or $\xi|_T$ when no confusion is possible.

Suppose now we have $\varphi : T \to S$ an arrow of $\mathcal{C}$. We can define a **pullback functor** $\varphi^* : \mathcal{F}(S) \to \mathcal{F}(T)$ in the following way: an object ξ goes to $\varphi^*(\xi)$, the pullback along φ, and an arrow $f : \xi \to \eta$ in $\mathcal{F}(S)$ goes to the unique arrow that fills the commutative diagram

$$\begin{tikzcd}
\varphi^*(\xi) \arrow[rr] \arrow[dr, dashed, "\varphi^*(f)"] \arrow[dd, mapsto] & & \xi \arrow[dr, "f"] \arrow[dd, mapsto] & \\
& \varphi^*(\eta) \arrow[rr] \arrow[dd, mapsto] & & \eta \arrow[dd, mapsto] \\
T \arrow[rr] \arrow[dr, equal] & & S \arrow[dr, equal] & \\
& T \arrow[rr] & & S.
\end{tikzcd}$$

As with objects, when no confusion is possible we write $f|_T$ instead of $\varphi^*(f)$.

It is very easy to see that a choice of a different cleavage will give another pullback functor, but the two will be naturally isomorphic. From now on we will leave this type of comment understood when doing constructions that use a cleavage.

Remark 1.2.8. The association that sends on object T of $\mathcal{C}$ to the category $\mathcal{F}(T)$, and an arrow $\varphi : T \to S$ to the pullback functor $\varphi^* : \mathcal{F}(S) \to \mathcal{F}(T)$ seems to give a contravariant functor from $\mathcal{C}$ to the category of categories. This is not quite correct, because it could well happen that, if $\psi : S \to U$ is another arrow in $\mathcal{C}$, we do not have $\varphi^* \circ \psi^* = (\psi \circ \varphi)^*$. It is possible that we only have a natural isomorphism between these two functors, and the association above will then be what is called a **pseudo-functor**.

Taking isomorphism classes in the fiber categories clearly fixes this problem: given a category fibered in groupoids $\mathcal{F} \to \mathcal{C}$ we have a functor $F : \mathcal{C}^{op} \to (\mathrm{Set})$ that sends an object T of $\mathcal{C}$ to the set of isomorphism classes in the category $\mathcal{F}(T)$, and an arrow $\varphi : T \to S$ to the obvious pullback function $\varphi^* : F(S) \to F(T)$.

Definition 1.2.9. *We will call F the **associated functor** of $\mathcal{F}$.*

In general we cannot recover a category fibered in groupoids from its associated functor. This will be true, at least up to equivalence, only for categories fibered in equivalence relations (see later in this section).

Example 1.2.10. One can show that the categories $\mathcal{D}ef_{X_0}(S)$ introduced in Section 1.1 can be put together as fiber categories of a category fibered in groupoids $\mathcal{D}ef_{X_0} \to (\mathrm{Sch}/k)$ (we will do this in detail later, but only for infinitesimal deformations). The deformation functor $\mathrm{Def}_{X_0} : (\mathrm{Art}/k) \to (\mathrm{Set})$ is then precisely the associated functor of the restriction of $\mathcal{D}ef_{X_0} \to (\mathrm{Sch}/k)$ to the full subcategory $(\mathrm{Art}/k)^{op} \subseteq (\mathrm{Sch}/k)$.

Morphisms and equivalence

Suppose $p_{\mathcal{F}} : \mathcal{F} \to \mathcal{C}$ and $p_{\mathcal{G}} : \mathcal{G} \to \mathcal{C}$ are two categories fibered in groupoids.

Definition 1.2.11. *A **morphism** of categories fibered in groupoids from $\mathcal{F}$ to $\mathcal{G}$ is a functor $F : \mathcal{F} \to \mathcal{G}$ which is base-preserving, i.e. such that $p_{\mathcal{G}} \circ F = p_{\mathcal{F}}$.*

Remark 1.2.12. If T is an object of $\mathcal{C}$, F will clearly induce a functor $\mathcal{F}(T) \to \mathcal{G}(T)$ which we denote by F_T. In particular F will induce a natural transformation between the associated functors of $\mathcal{F}$ and $\mathcal{G}$.

With this notion of morphism comes a notion of isomorphism between fibered categories, but as it often happens when dealing with categories, this notion is too strict.

Definition 1.2.13. *Given two morphisms $F, G : \mathcal{F} \to \mathcal{G}$, a natural transformation $\alpha : F \to G$ is said to be **base-preserving** if for every object ξ of $\mathcal{F}$ the arrow $\alpha_\xi : F(\xi) \to G(\xi)$ is in $\mathcal{G}(T)$, where $T = p_{\mathcal{F}}(\xi)$.*

*An **isomorphism** between F and G is a base-preserving natural equivalence.*

Definition 1.2.14. *Two categories fibered in groupoids $\mathcal{F} \to \mathcal{C}$ and $\mathcal{G} \to \mathcal{C}$ are said to be **equivalent** if there exist two morphisms $F : \mathcal{F} \to \mathcal{G}$ and $G : \mathcal{G} \to \mathcal{F}$, with an isomorphism of $F \circ G$ with the identity functor of $\mathcal{G}$ and of $G \circ F$ with the one of $\mathcal{F}$.*

In this case we will say that F is an **equivalence** between $\mathcal{F}$ and $\mathcal{G}$, and that F and G are **quasi-inverse** to each other.

We have a handy criterion to decide whether a morphism of fibered categories is an equivalence.

Proposition 1.2.15. *A morphism of categories fibered in groupoids $F : \mathcal{F} \to \mathcal{G}$ is an equivalence if and only if $F_T : \mathcal{F}(T) \to \mathcal{G}(T)$ is an equivalence for every object T of $\mathcal{C}$.*

Categories fibered in sets

A particularly simple type of fibered categories is that of categories fibered in sets.

Definition 1.2.16. *A **categories fibered in sets** is a fibered category $\mathcal{F} \to \mathcal{C}$ such that $\mathcal{F}(T)$ is a set for any object T of $\mathcal{C}$.*

Here we see a set as a category whose only arrows are the identities. In a category fibered in sets pullbacks are strictly unique, and this feature characterizes them.

Proposition 1.2.17. *Let $\mathcal{F} \to \mathcal{C}$ be a functor. $\mathcal{F}$ is a category fibered in sets over $\mathcal{C}$ if and only if for every arrow $T \to S$ of $\mathcal{C}$ and every object ξ of $\mathcal{F}(S)$ there exists a unique arrow in $\mathcal{F}$ over $T \to \mathcal{C}$ and with target ξ.*

Because of this uniqueness, when $\mathcal{F} \to \mathcal{C}$ is fibered in sets the associated pseudo-functor is actually already a functor, which we denote by $\Phi_{\mathcal{F}} : \mathcal{C}^{op} \to (\mathrm{Set})$. Moreover any morphism $F : \mathcal{F} \to \mathcal{G}$ of categories fibered in sets over $\mathcal{C}$ will give a natural transformation $\varphi_F : \Phi_{\mathcal{F}} \to \Phi_{\mathcal{G}}$, as in Remark 1.2.12. This association gives a functor from the category of categories fibered in sets over $\mathcal{C}$ and the category of functors $\mathcal{C}^{op} \to (\mathrm{Set})$.

Proposition 1.2.18. *The functor defined above is an equivalence of categories.*

We sketch briefly the inverse construction. Suppose we have a functor $F : \mathcal{C}^{op} \to (\mathrm{Set})$, and consider the following category, which we call $\mathcal{F}_F$: as objects take pairs (T, ξ), where T is an object of $\mathcal{C}$ and $\xi \in F(T)$, and an arrow $f : (T, \xi) \to (S, \eta)$ will be an arrow $f : T \to S$ such that $F(f)(\eta) = \xi$. Then $\mathcal{F}_F$ is a category fibered in sets.

Given a natural transformation $\alpha : F \to G$ between two functors $\mathcal{C}^{op} \to (\mathrm{Set})$, we construct a functor $H_\alpha : \mathcal{F}_F \to \mathcal{F}_G$, as follows: an object (T, ξ) of $\mathcal{F}_F$ goes to the object $(T, \alpha(T)(\xi))$ of $\mathcal{F}_G$, and an arrow $f : (T, \xi) \to (S, \eta)$ simply goes to itself (as an arrow $f : T \to S$ of $\mathcal{C}$). It can be shown that this gives a functor, which is a quasi-inverse to the one considered above.

Another class of simple fibered categories are the ones fibered in equivalence relations. We say that a groupoid is an **equivalence relation** if for any pair of objects there is at most one arrow from the first one to the second. Another way to say this is that the only arrow from any object to itself is the identity.

Definition 1.2.19. *A fibered category $\mathcal{F} \to \mathcal{C}$ is said to be* ***fibered in equivalence relations*** *if for every object T of $\mathcal{C}$ the fiber category $\mathcal{F}(T)$ is an equivalence relation.*

Remark 1.2.20. The name "equivalence relation" comes from the fact that if a groupoid $\mathcal{F}$ is an equivalence relation, and we call A and O its sets of arrows and objects respectively, the map $A \to O \times O$ that sends an arrow to the pair (source, target) is injective, and gives an equivalence relation on the set O.

We have the following proposition, which characterizes categories fibered in equivalence relations.

Proposition 1.2.21. *A fibered category $\mathcal{F} \to \mathcal{C}$ is fibered in equivalence relations if and only if it is equivalent to a category fibered in sets.*

Because of this, sometimes categories fibered in equivalence relations are called **quasi-functors**.

Suppose now that T in an object of $\mathcal{C}$, and consider the **comma category** $(\mathcal{C}/T)$, defined as follows: its objects are arrows $S \to T$ of $\mathcal{C}$ with target T, and an arrow from $f : S \to T$ to $g : U \to T$ is an arrow $h : S \to U$ of $\mathcal{C}$, such that $g \circ h = f$. We have a functor $(\mathcal{C}/T) \to \mathcal{C}$ that sends $S \to T$ to S, and an arrow as above to the arrow $h : S \to U$ of $\mathcal{C}$.

$(\mathcal{C}/T) \to \mathcal{C}$ is a category fibered in sets: given an arrow $S \to U$ of $\mathcal{C}$ and an object over U, that is, an arrow $U \to T$, the only possible pullback to S is the composite $S \to U \to T$. It is also easy to see that this category fibered in sets is the one associated with the functor $h_T : \mathcal{C}^{op} \to (\mathrm{Set})$ represented by T (up to equivalence of course).

Definition 1.2.22. *A category fibered in groupoids $\mathcal{F} \to \mathcal{C}$ is called **representable** if it is equivalent to a category fibered in groupoids of the form $(\mathcal{C}/T)$.*

Clearly if $\mathcal{F} \to \mathcal{C}$ is representable, then it is fibered in equivalence relations.

We have a version of Yoneda's Lemma for categories fibered in groupoids. We will not need it, but we state it for completeness' sake. Let $\mathcal{F} \to \mathcal{C}$ be a category fibered in groupoids and T an object of $\mathcal{C}$, and consider the category $\mathrm{Hom}((\mathcal{C}/T), \mathcal{F})$ of morphisms of categories fibered in groupoids $\mathcal{C}/T \to \mathcal{F}$, with base-preserving natural transformations as arrows. We have a functor

$$\mathrm{Hom}((\mathcal{C}/T), \mathcal{F}) \longrightarrow \mathcal{F}(T)$$

that associates to a morphism $\Phi : (\mathcal{C}/T) \to \mathcal{F}$ the object $\Phi(\mathrm{id}_T) \in \mathcal{F}(T)$, and to a base-preserving natural transformation $\alpha : \Phi \to \Psi$ the arrow $\alpha(\mathrm{id}_T) : \Phi(\mathrm{id}_T) \to \Psi(\mathrm{id}_T)$.

Proposition 1.2.23. *The functor defined above is an equivalence of categories.*

Example 1.2.24. In particular if X is a scheme over S, we can see it as a functor $h_X : (\mathrm{Sch}/S)^{op} \to (\mathrm{Set})$ (by the classical Yoneda's Lemma), and also as a category fibered in groupoids $((\mathrm{Sch}/S)/X) \to (\mathrm{Sch}/S)$ (by the preceding proposition). To avoid this cumbersome notation we will write X for h_X and also for $((\mathrm{Sch}/S)/X)$.

1.3 Fibered categories as deformation problems

Now suppose $\mathcal{F} \to (\mathrm{Sch}/S)$ is a category fibered in groupoids coming from a geometric deformation problem, where $S = \mathrm{Spec}(k)$ or some other base scheme (we will see how this happens in practice in some examples below), and that we want to study it.

The idea is of course that objects of $\mathcal{F}(\mathrm{Spec}(k(s_0)))$ where $s_0 \in S$ are things we are deforming, and an object $\xi \in \mathcal{F}(X)$ that restricts to $\xi_0 \in F(\mathrm{Spec}(k(s_0)))$ is a deformation of ξ_0 over X.

As with functors, the first step will be to restrict $\mathcal{F}$ to the full subcategory of (Sch/S) consisting of spectra of artinian local $k(s_0)$-algebras, for some fixed (possibly closed) point $s_0 \in S$. Actually it is sometimes useful to study $\mathcal{F}$ over $(\mathrm{Art}/\Lambda)^{op}$, where Λ is a complete noetherian local ring and we denote by k its residue field. Here are some motivations.

Example 1.3.1. Suppose we want to study the infinitesimal deformations of a given $\xi_0 \in \mathcal{F}(\mathrm{Spec}(k(s_0)))$, that is, we are interested in deformations of ξ_0 over artinian algebras over S (with image of $\mathrm{Spec}(A) \to S$ the fixed point s_0). We notice that any such morphism factors through $\mathrm{Spec}(\widehat{\mathcal{O}}_{S,s_0})$, where $\widehat{\mathcal{O}}_{S,s_0}$ is the usual completion of the local ring of S at s_0 with respect to the maximal ideal $\mathfrak{m}_{s_0}$.

This is simply because every homomorphism $\mathcal{O}_{S,s_0} \to A$ factors through $\mathcal{O}_{S,s_0}/\mathfrak{m}_{s_0}^n$ for some n (because $\mathfrak{m}_A^n = 0$ for some n, and the homomorphism is local), and consequently factors through $\widehat{\mathcal{O}}_{S,s_0}$. So the algebras we are interested in are actually Λ-algebras, where $\Lambda = \widehat{\mathcal{O}}_{S,s_0}$ in this case.

Example 1.3.2. Suppose we have a moduli space M over k parametrizing objects of some kind, and that we want to study its structure at a point $m_0 \in M$. Some properties of M at m_0 (smoothness, for example) can be inferred by studying morphisms $\mathrm{Spec}(A) \to M$ with image m_0, where $A \in (\mathrm{Art}/k)$.

Exactly as before, any such A is actually an object of (Art/Λ) where $\Lambda = \widehat{\mathcal{O}}_{M,m_0}$, and from the properties of the moduli space morphisms $\mathrm{Spec}(A) \to M$ correspond to families over $\mathrm{Spec}(A)$, so for the purpose of understanding M we are led to study $\mathcal{F} \to (\mathrm{Art}/\Lambda)^{op}$, where $\mathcal{F}$ is the category fibered in groupoids that comes from the deformation problem associated with M.

Example 1.3.3. When working with varieties on a perfect field k of positive characteristic p, it is sometimes useful to consider deformations over the ring of Witt vectors $W(k)$ of k (this is related to the problem of liftings from characteristic p to characteristic zero). In these cases our formalism can be applied, with $\Lambda = W(k)$.

From now on we will then study categories fibered in groupoids $\mathcal{F} \to (\mathrm{Art}/\Lambda)^{op}$, where Λ is a complete noetherian local ring with residue field k. We will turn back to "global" deformations only occasionally.

Remark 1.3.4. We stress once again that we will always identify $(\mathrm{Art}/\Lambda)^{op}$ with the corresponding full subcategory of (Sch/Λ), and in this fashion if $\mathcal{F} \to (\mathrm{Art}/\Lambda)^{op}$ is a category fibered in groupoids, $\varphi : A' \to A$ a homomorphism in (Art/Λ), and $f : \xi \to \eta$ an arrow of $\mathcal{F}$ with $\xi \in \mathcal{F}(A)$ and $\eta \in F(A')$, we will say that f is over φ if its image in $(\mathrm{Art}/\Lambda)^{op}$ is the morphism $\widetilde{\varphi} : \mathrm{Spec}(A) \to \mathrm{Spec}(A')$ corresponding to φ.

We will also draw some unpleasant commutative diagrams like this

$$\begin{tikzcd} \xi \arrow[r, "f"] \arrow[d, mapsto] & \eta \arrow[d, mapsto] \\ A & A' \arrow[l, "\varphi"'] \end{tikzcd}$$

which should of course be read as

$$\begin{tikzcd} \xi \arrow[rr, "f"] \arrow[d, mapsto] & & \eta \arrow[d, mapsto] \\ \operatorname{Spec}(A) \arrow[rr, "\widetilde{\varphi}"] & & \operatorname{Spec}(A'). \end{tikzcd}$$

The Rim-Schlessinger condition

We now state a basic condition we will impose on our $\mathcal{F}$: suppose we have a category fibered in groupoids $\mathcal{F} \to (\operatorname{Art}/\Lambda)^{op}$, and two homomorphisms $\pi' : A' \to A$, $\pi'' : A'' \to A$ in $(\operatorname{Art}/\Lambda)$, the second one being surjective. We consider then the fibered product $A' \times_A A''$.

Lemma 1.3.5. *$A' \times_A A''$ is still an object of* $(\operatorname{Art}/\Lambda)$.

Proof. We have to check that $B = A' \times_A A''$ is a local artinian Λ-algebra with residue field k. Call $p_1 : B \to A'$ and $p_2 : B \to A''$ the two projections, and notice that p_1 is surjective.

First, B is a Λ algebra via the homomorphism $\Lambda \to B$ induced by the two structure homomorphisms of A' and A'', and it is artinian because it is of finite length as a Λ-module, being a submodule of the product $A' \times A''$, which is of finite length because the two factors are.

Next, consider the (proper) ideal

$$I = \mathfrak{m}_{A'} \times_{\mathfrak{m}_A} \mathfrak{m}_{A''} = \{(x_1, x_2) \in A' \times_A A'' : x_1 \in \mathfrak{m}_{A'} \text{ and } x_2 \in \mathfrak{m}_{A''}\}$$

of B. We show that every element of $B \setminus I$ is a unit, so that I is the only maximal ideal of B. Take $(x_1, x_2) \in B \setminus I$, and suppose that $x_1 \notin \mathfrak{m}_{A'}$ (the other case is carried out the same way). Then since $\pi''(x_2) = \pi'(x_1) \notin \mathfrak{m}_A$ and π'' is local, we have also $x_2 \notin \mathfrak{m}_{A''}$. Then we have two elements $y_1 \in A'$ and $y_2 \in A''$, inverses to x_1 and x_2 respectively. Since $\pi'(x_1)\pi'(y_1) = 1 = \pi''(x_2)\pi''(y_2) = \pi'(x_1)\pi''(y_2)$ we get then $\pi'(y_1) = \pi''(y_2)$, so that (y_1, y_2) is an element of B, and it is an inverse to (x_1, x_2).

Finally, the composite $B \xrightarrow{p_1} A' \to A'/\mathfrak{m}_{A'} \cong k$ is surjective, and its kernel can only be the maximal ideal I, so we have

$$A' \times_A A''/(\mathfrak{m}_{A'} \times_{\mathfrak{m}_A} \mathfrak{m}_{A''}) \cong k. \qquad \square$$

We have then two pullback functors $\mathcal{F}(A' \times_A A'') \to \mathcal{F}(A')$ and $\mathcal{F}(A' \times_A A'') \to \mathcal{F}(A'')$, such that the composites

$$\mathcal{F}(A' \times_A A'') \longrightarrow \mathcal{F}(A') \longrightarrow \mathcal{F}(A)$$

and

$$\mathcal{F}(A' \times_A A'') \longrightarrow \mathcal{F}(A'') \longrightarrow \mathcal{F}(A)$$

with the pullback functors to A are isomorphic. We get then an induced functor

$$\Phi : \mathcal{F}(A' \times_A A'') \to \mathcal{F}(A') \times_{\mathcal{F}(A)} \mathcal{F}(A'')$$

(see appendix C for the definition of fibered products of categories).

More explicitly, Φ sends an object ξ to $(\xi|_{A'}, \xi|_{A''}, \theta)$ where $\theta : (\xi|_{A'})|_A \to (\xi|_{A''})|_A$ is the canonical isomorphism identifying the pullbacks of $\xi|_{A'}$ and $\xi|_{A''}$ to A as pullbacks of ξ, and an arrow $f : \xi \to \eta$ is mapped to the pair $(f|_{A'}, f|_{A''})$ of induced arrows on the pullbacks.

Definition 1.3.6. *A category fibered in groupoids $\mathcal{F} \to (\mathrm{Art}/\Lambda)^{op}$ satisfies the* ***Rim-Schlessinger condition*** *([RS] from now on) if Φ is an equivalence of categories for every $A, A', A'' \in (\mathrm{Art}/\Lambda)$ and maps as above.*

This condition, which was first formulated by D. S. Rim in [Rim], resembles very much Schlessinger's ones, and actually implies (H1) and (H2) for the associated functor, as is very easy to see ((H4) is a little more subtle, see Proposition 2.1.12 of [Oss], which is essentially Proposition 1.3.13 below).

Despite the fact that [RS] is somewhat stronger than (H1)+(H2), when one proves that a given category fibered in groupoids (or rather its associated functor) satisfies the latter ones, he usually proves that the category satisfies [RS] (or could do so with little extra effort). Moreover all categories fibered in groupoids coming from reasonable geometric deformation problems seem to have the stated property, so we will take it as starting point.

Definition 1.3.7. *A* ***deformation category*** *is a category fibered in groupoids $\mathcal{F} \to (\mathrm{Art}/\Lambda)^{op}$ that satisfies [RS].*

Deformation categories are called "homogeneous groupoids" by Rim in [Rim], and "deformation stacks" by Osserman in [Oss].

Remark 1.3.8. From now on when we have a deformation category $\mathcal{F} \to (\mathrm{Art}/\Lambda)^{op}$ with A, A', A'' artinian algebras as above, and objects $\xi' \in \mathcal{F}(A')$ and $\xi'' \in F(A'')$ with a fixed isomorphism of the pullbacks to A, we denote by $\{\xi', \xi''\}$ an induced object over the fibered product $A' \times_A A''$. When the isomorphism over A or the choice of such an object is relevant, we will be more specific.

Example 1.3.9. As a very simple example, we consider the category fibered in groupoids $X \to (\mathrm{Art}/\Lambda)^{op}$ given by a scheme X over $\mathrm{Spec}(\Lambda)$.

If $X = \mathrm{Spec}(R)$ is affine, for every $B \in (\mathrm{Art}/\Lambda)$ we have a natural bijection $X(B) \cong \mathrm{Hom}_\Lambda(R, B)$, and if we take $A, A', A'' \in (\mathrm{Art}/\Lambda)$ and maps as above, the map $X(A' \times_A A'') \to X(A') \times_{X(A)} X(A'')$ is a bijection because of the properties of the fibered product. When X is not affine one reduces to the affine case by noticing that the image of the morphisms involved is a point of X, and taking an affine neighborhood.

Given a deformation category $\mathcal{F} \to (\mathrm{Art}/\Lambda)^{op}$ and an object ξ_0 over $\mathrm{Spec}(k)$, we can construct another deformation category $\mathcal{F}_{\xi_0}$ that contains only objects of $\mathcal{F}$ that restrict to ξ_0 over $\mathrm{Spec}(k)$ (and in this sense are deformations of ξ_0), taking the (dual) comma category:

Objects: arrows $f : \xi_0 \to \xi$ of $\mathcal{F}$, or equivalently pairs (ξ, φ) where ξ is an object of $\mathcal{F}$ and φ is an arrow in $\mathcal{F}(k)$ between ξ_0 and the pullback of ξ to $\mathrm{Spec}(k)$.

Arrows: from $f : \xi_0 \to \xi$ to $g : \xi_0 \to \eta$ are arrows $h : \xi \to \eta$ of $\mathcal{F}$ such that $h \circ f = g$, or equivalently the arrow $\xi_0 \to \xi_0$ induced by h is the identity.

We have also an obvious functor $\mathcal{F}_{\xi_0} \to (\mathrm{Art}/\Lambda)^{op}$, induced by $\mathcal{F} \to (\mathrm{Art}/\Lambda)^{op}$. The following proposition will be useful when we have to consider deformations of a fixed object over k.

We recall here that a functor is an equivalence of categories if and only if it is fully faithful and essentially surjective.

Proposition 1.3.10. *If $\mathcal{F} \to (\mathrm{Art}/\Lambda)^{op}$ is a deformation category and $\xi_0 \in \mathcal{F}(k)$, then $\mathcal{F}_{\xi_0} \to (\mathrm{Art}/\Lambda)^{op}$ is also a deformation category.*

Proof. It is clear that $\mathcal{F}_{\xi_0}$ is fibered in groupoids: given a homomorphism $\varphi : A \to B$ in (Art/Λ) and an object $\xi_0 \to \xi$ of $\mathcal{F}_{\xi_0}(A)$, we take the pullback $\xi|_B \to \xi$, which is an arrow over φ, and by the fact that this is cartesian in $\mathcal{F}$ we have an induced arrow $\xi_0 \to \xi|_B$, which is then an element of $\mathcal{F}_{\xi_0}(B)$. It is easy to check that this gives a cartesian arrow of $\mathcal{F}_{\xi_0}$ over φ with target ξ. Moreover for every object A of (Art/Λ), every arrow of $\mathcal{F}_{\xi_0}(A)$ will be invertible as an arrow of $\mathcal{F}(A)$, and it is easy to check that the inverse arrow will also be in $\mathcal{F}_{\xi_0}(A)$, so that $\mathcal{F}_{\xi_0}(A)$ is a groupoid.

We turn then to [RS]. Let $A, A', A'' \in (\mathrm{Art}/\Lambda)$, and $\pi' : A' \to A$, $\pi'' : A'' \to A$ be two homorphisms, with π'' surjective, and call

$$\Phi : \mathcal{F}(A' \times_A A'') \to \mathcal{F}(A') \times_{\mathcal{F}(A)} \mathcal{F}(A'')$$

and

$$\Phi_{\xi_0} : \mathcal{F}_{\xi_0}(A' \times_A A'') \to \mathcal{F}_{\xi_0}(A') \times_{\mathcal{F}_{\xi_0}(A)} \mathcal{F}_{\xi_0}(A'')$$

the natural functors. We show that Φ_{ξ_0} is fully faithful and essentially surjective, knowing that Φ is.

First, consider any two objects $\xi_0 \to \xi$ and $\xi_0 \to \eta$ of $\mathcal{F}_{\xi_0}$, and take the induced function

$$F : \mathrm{Hom}((\xi_0 \to \xi), (\xi_0 \to \eta)) \longrightarrow \mathrm{Hom}(\Phi_{\xi_0}(\xi_0 \to \xi), \Phi_{\xi_0}(\xi_0 \to \eta)).$$

From the fact that arrows of $\mathcal{F}_{\xi_0}$ are just arrows of $\mathcal{F}$ with a compatibility condition, faithfulness (that is, injectivity of F) is immediate. Next take an element of the right-hand side, which will be a pair (f, g), where $f : \xi|_{A'} \to \eta|_{A'}$ and $g : \xi|_{A''} \to \eta|_{A''}$ are two arrows commuting with the arrows from ξ_0, and such that the following diagram

$$\begin{tikzcd}
(\xi|_{A'})|_A \arrow[rr, "f|_A"] \arrow[dd, "\theta"'] & & (\eta|_{A'})|_A \arrow[dd, "\nu"] \\
& \xi_0 \arrow[ul] \arrow[ur] \arrow[dl] \arrow[dr] & \\
(\xi|_{A''})|_A \arrow[rr, "g|_A"'] & & (\eta|_{A''})|_A
\end{tikzcd}$$

(where θ and ν are the canonical isomorphisms) is commutative.

Because of the bijectivity of

$$G : \mathrm{Hom}(\xi, \eta) \longrightarrow \mathrm{Hom}(\Phi(\xi), \Phi(\eta))$$

we have an arrow $h : \xi \to \eta$ such that $G(h) = (f, g)$. We have to check that h commutes with the arrows from ξ_0, i.e. if we call $a : \xi_0 \to \xi$ and $b : \xi_0 \to \eta$, then $h \circ a = b$.

Taking the pullbacks of ξ and η to k in $\mathcal{F}$, we get a diagram

$$\begin{tikzcd}
& \xi|_k \arrow[r] \arrow[dd, "h|_k"] & \xi \arrow[dd, "h"] \\
\xi_0 \arrow[ur] \arrow[dr] & & \\
& \eta|_k \arrow[r] & \eta
\end{tikzcd}$$

where the right square commutes, and the composites $\xi_0 \to \xi|_k \to \xi$ and $\xi_0 \to \eta|_k \to \eta$ are just a and b. So it suffices to show that the left triangle commutes. But now we know that the diagram

$$\begin{tikzcd}
& \xi|_{A'} \arrow[r] \arrow[dd, "f"] & \xi \arrow[dd, "h"] \\
\xi_0 \arrow[ur] \arrow[dr] & & \\
& \eta|_{A'} \arrow[r] & \eta
\end{tikzcd}$$

commutes, and then by pullback to k we get that

$$\begin{tikzcd} & (\xi|_{A'})|_k \arrow[r] \arrow[d, "f|_k"] & \xi|_k \arrow[d, "h|_k"] \\ \xi_0 \arrow[ur] \arrow[dr] & & \\ & (\eta|_{A'})|_k \arrow[r] & \eta|_k \end{tikzcd}$$

commutes too, and we are done.

Finally, we show that Φ_{ξ_0} is essentially surjective. Take an object of $\mathcal{F}_{\xi_0}(A') \times_{\mathcal{F}_{\xi_0}(A)} \mathcal{F}_{\xi_0}(A'')$, that is, a triplet $(\xi_0 \to \xi', \xi_0 \to \xi'', \theta)$ where $\xi' \in \mathcal{F}(A')$, $\xi'' \in \mathcal{F}(A'')$ and $\theta : \xi'|_A \to \xi''|_A$ is an isomorphism compatible with the arrows from ξ_0.

Since Φ is essentially surjective we have an object $\xi \in \mathcal{F}(A' \times_A A'')$ such that $\Phi(\xi)$ is isomorphic to (ξ', ξ'', θ): this means in particular that we have two arrows $\xi' \to \xi$ and $\xi'' \to \xi$ in $\mathcal{F}$ (identifying ξ' and ξ'' with the pullbacks of ξ to A' and A'').

It is easy to see that the two composites $\xi_0 \to \xi' \to \xi$ and $\xi_0 \to \xi'' \to \xi$ are the same arrow, and that the image of the resulting object $\xi_0 \to \xi$ of $\mathcal{F}_{\xi_0}(A' \times_A A'')$ in $\mathcal{F}_{\xi_0}(A') \times_{\mathcal{F}_{\xi_0}(A)} \mathcal{F}_{\xi_0}(A'')$ is isomorphic to $(\xi_0 \to \xi', \xi_0 \to \xi'', \theta)$, so that Φ_{ξ_0} is essentially surjective. $\square$

A **morphism** of deformation categories will simply be a morphism of categories fibered in groupoids.

Remark 1.3.11. Using Yoneda's Lemma 1.2.23, and considering $\mathrm{Spec}(A)$ (where $A \in (\mathrm{Art}/\Lambda)$) as a category fibered in groupoids over $(\mathrm{Art}/\Lambda)^{op}$ as explained in Section 1.2, we have that an object of $\mathcal{F}(A)$ corresponds to a morphism $\mathrm{Spec}(A) \to \mathcal{F}$. In this fashion, the objects of $\mathcal{F}_{\xi_0}(A)$ are exactly the morphisms $\mathrm{Spec}(A) \to \mathcal{F}$ such that the composite

$$\mathrm{Spec}(k) \to \mathrm{Spec}(A) \to \mathcal{F}$$

(where $\mathrm{Spec}(k) \to \mathrm{Spec}(A)$ is induced by the quotient map) corresponds to the object $\xi_0 \in \mathcal{F}(k)$.

Of course if $\mathcal{F}$ is the category fibered in groupoids coming from the functor of points of a scheme X, then $\xi_0 \in X(k)$ is simply a rational point, and $X_{\xi_0}(A)$ are the morphism of schemes $\mathrm{Spec}(A) \to X$ with image the point ξ_0.

Remark 1.3.12. If $\mathcal{F} \to (\mathrm{Art}/\Lambda)^{op}$ and $\mathcal{G} \to (\mathrm{Art}/\Lambda)^{op}$ are two deformation categories, a morphism $F : \mathcal{F} \to \mathcal{G}$ will give an induced one $F_{\xi_0} : \mathcal{F}_{\xi_0} \to \mathcal{G}_{F(\xi_0)}$, in the obvious way: an object $\xi_0 \to \xi$ goes to $F(\xi_0) \to F(\xi)$, and an arrow $\xi \to \eta$ to $F(\xi) \to F(\eta)$.

Relation with the classical theory

Now we spend some words about the relation between the point of view of deformation categories and that of deformation functors. If $\mathcal{F} \to (\mathrm{Art}/\Lambda)^{op}$ is a deformation category coming from a geometric deformation problem, we can consider its associated functor, which we recall to be the functor $F : (\mathrm{Art}/\Lambda) \to (\mathrm{Set})$ defined on objects by

$$F(A) = \{\text{isomorphism classes of objects in } \mathcal{F}(A)\}$$

and sending a homomorphism $A \to B$ in (Art/Λ) to the pullback function $F(A) \to F(B)$. Suppose also that $\mathcal{F}(k)$ is a trivial groupoid, so that $F(k)$ will be a singleton, and F a deformation functor.

We can see F as a category fibered in sets over $(\mathrm{Art}/\Lambda)^{op}$ (as explained in Section 1.2), and we have an obvious "quotient morphism" $\mathcal{F} \to F$ of categories fibered in groupoids, sending an object of $\mathcal{F}$ to its isomorphism class. To carry out the study of $F \to (\mathrm{Art}/\Lambda)^{op}$ (and ultimately of the deformation functor F) using the theory we will develop, we need to know that $F \to (\mathrm{Art}/\Lambda)^{op}$ satisfies [RS]. Unfortunately, this is not always true.

The reason is the following: suppose we have $A, A', A'' \in (\mathrm{Art}/\Lambda)$, and two homomorphisms $\pi' : A' \to A$ and $\pi'' : A'' \to A$, the second one being surjective, and consider the induced function

$$f : F(A' \times_A A'') \to F(A') \times_{F(A)} F(A'').$$

We want to know if this is a bijection, knowing that the functor

$$\Phi : \mathcal{F}(A' \times_A A'') \to \mathcal{F}(A') \times_{\mathcal{F}(A)} \mathcal{F}(A'')$$

is an equivalence.

Surjectivity is not a problem: if we have an element $(a,b) \in F(A') \times_{F(A)} F(A'')$, where a and b are isomorphism classes of objects in $\mathcal{F}(A')$ and $\mathcal{F}(A'')$ whose pullbacks to A are isomorphic, we choose representatives $\xi' \in \mathcal{F}(A')$ for a and $\xi'' \in \mathcal{F}(A'')$ for b, and an isomorphism $\theta : \xi'|_A \cong \xi''|_A$ in $\mathcal{F}(A)$, obtaining thus an object $(\xi', \xi'', \theta) \in \mathcal{F}(A') \times_{\mathcal{F}(A)} \mathcal{F}(A'')$. Since Φ is an equivalence we have an object $\xi \in \mathcal{F}(A' \times_A A'')$ such that $\Phi(\xi)$ is isomorphic to (ξ', ξ'', θ), and then its isomorphism class $c \in F(A' \times_A A'')$ is such that $f(c) = (a,b)$.

On the other hand injectivity is not always assured, basically because the datum of the isomorphism between the pullbacks on A in an object of $\mathcal{F}(A') \times_{\mathcal{F}(A)} \mathcal{F}(A'')$ is lost when we consider the corresponding element of $F(A') \times_{F(A)} F(A'')$. Precisely, we have the following proposition.

Proposition 1.3.13. *$F \to (\mathrm{Art}/\Lambda)^{op}$ satisfies [RS] if and only if for every surjection $A' \to A$ in (Art/Λ) and $\xi' \in \mathcal{F}(A')$, the homomorphism $\mathrm{Aut}_{A'}(\xi') \to \mathrm{Aut}_A(\xi'|_A)$ is surjective.*

Here if $A \in (\mathrm{Art}/\Lambda)$ and $\xi \in \mathcal{F}(A)$, we denote by $\mathrm{Aut}_A(\xi)$ the group of automorphisms of the object ξ in the category $\mathcal{F}(A)$. The homomorphism $\mathrm{Aut}_{A'}(\xi') \to \mathrm{Aut}_A(\xi'|_A)$ is defined then by pullback of arrows, as in Section 1.2 (it will be studied further in Chapter 3).

Proof. Suppose first that the second condition of the statement holds. By the discussion above (and with the same notation), we only need to show that f is injective. Take $a, b \in F(A' \times_A A'')$ such that $f(a) = f(b)$, representatives $\xi, \eta \in \mathcal{F}(A' \times_A A'')$ for a and b, and write $\Phi(\xi) = (\xi', \xi'', \theta), \Phi(\eta) = (\eta', \eta'', \nu)$.

Since $f(a) = f(b)$ we have two isomorphisms $f' : \xi' \to \eta'$ and $f'' : \xi'' \to \eta''$ in $\mathcal{F}(A')$ and $\mathcal{F}(A'')$ respectively. Consider then the diagram

$$\begin{tikzcd} \xi'|_A \arrow[r, "\theta"] \arrow[d, "f'|_A"'] & \xi''|_A \arrow[d, "f''|_A"] \\ \eta'|_A \arrow[r, "\nu"'] & \eta''|_A. \end{tikzcd} \tag{1.1}$$

This need not be commutative, but if it is, then (f', f'') will be an isomorphism between $\Phi(\xi)$ and $\Phi(\eta)$ in $\mathcal{F}(A') \times_{\mathcal{F}(A)} \mathcal{F}(A'')$, and from this will follow that $\xi \cong \eta$, and $a = b$.

Notice now that we can modify the isomorphism f', by composing it with an automorphism of ξ' on the right. Let us consider then the composite

$$G(\alpha) = \theta^{-1} \circ (f''|_A)^{-1} \circ \nu \circ (f' \circ \alpha)|_A \in \mathrm{Aut}_A(\xi'|_A)$$

where $\alpha \in \mathrm{Aut}_{A'}(\xi')$ is an automorphism of ξ'.

Diagram 1.1 with $f' \circ \alpha$ in place of f' will be commutative if and only if $G(\alpha) = \mathrm{id}$. This can be rewritten as

$$\alpha|_A = (f'|_A)^{-1} \circ \nu^{-1} \circ f''|_A \circ \theta \in \mathrm{Aut}_A(\xi'|_A)$$

and since $\mathrm{Aut}_{A'}(\xi') \to \mathrm{Aut}_A(\xi'|_A)$ is surjective, we can find an α that satisfies the last equality, and we can conclude by the argument above.

Conversely, suppose that we have a surjection $A' \to A$ and a $\xi' \in \mathcal{F}(A')$ such that $\mathrm{Aut}_{A'}(\xi') \to \mathrm{Aut}_A(\xi'|_A)$ is not surjective, and take $f \in \mathrm{Aut}_A(\xi'|_A)$ that is not in the image.

Then the two objects $(\xi', \xi', \mathrm{id})$ and (ξ', ξ', f) of $\mathcal{F}(A') \times_{\mathcal{F}(A)} \mathcal{F}(A')$ correspond to the same element of $F(A') \times_{F(A)} F(A')$, but they are not isomorphic in $\mathcal{F}(A') \times_{\mathcal{F}(A)} \mathcal{F}(A')$, because if (g', g'') was an isomorphism between them, we would have a commutative diagram

$$\begin{tikzcd} \xi'|_A \arrow[r, "g'|_A"] \arrow[d, equal] & \xi'|_A \arrow[d, "f"] \\ \xi'|_A \arrow[r, "g''|_A"] & \xi'|_A. \end{tikzcd}$$

from which it would follow that $f = (g'' \circ g'^{-1})|_A$ is in the image of $\mathrm{Aut}_{A'}(\xi') \to \mathrm{Aut}_A(\xi'|_A)$.

In conclusion if we take $\eta', \eta'' \in \mathcal{F}(A' \times_A A')$ corresponding to $(\xi', \xi', \mathrm{id})$ and (ξ', ξ', f), and we denote by $a, b \in F(A' \times_A A')$ their isomorphism classes, we have $a \neq b$ but $f(a) = f(b)$, and f is not injective. □

1.4 Examples

We introduce here three examples of deformation problems that will show up systematically in the following, providing concrete examples to our abstract constructions. In each of these example some additional hypotheses may be required (on the ambient scheme over Λ, for example) to make things work out sometimes. We will specify these hypotheses case by case.

Each of these examples has also a classical associated deformation functor, which can be simply obtained by taking the associated functor of the deformation category we will introduce for the problem. Moreover, each of the deformation categories we will consider comes actually from a (and possibly more than one) category fibered in groupoids over (Sch/Λ), which is defined in a similar way. We will not consider these "global" deformations until Section 5.4, where we will briefly discuss the problem of algebraization of formal deformations.

1.4.1 Schemes

The simplest example is the one already introduced, which considers deformations of schemes, without additional structure.

Let us consider the following category fibered in groupoids, which we will denote by $\mathcal{D}ef$:

Objects: flat morphisms of schemes $X \to \mathrm{Spec}(A)$, where $A \in (\mathrm{Art}/\Lambda)$.

Arrows: from $X \to \mathrm{Spec}(A)$ to $Y \to \mathrm{Spec}(B)$ are pairs (φ, f) where $\varphi : B \to A$ is a homomorphism and $f : X \cong Y_A$ is an isomorphism of schemes (recall that Y_A denotes the base change $Y \times_{\mathrm{Spec}(B)} \mathrm{Spec}(A)$).

Given two arrows (φ, f) from $X \to \mathrm{Spec}(A)$ to $Y \to \mathrm{Spec}(B)$ and (ψ, g) from $Y \to \mathrm{Spec}(B)$ to $Z \to \mathrm{Spec}(C)$ the composite $(\psi, g) \circ (\varphi, f)$ is (ρ, h) where $\rho : C \to A$ is simply $\varphi \circ \psi$, and if we call $g_A : Y_A \cong (Z_B)_A$ the isomorphism induced by $g : Y \cong Z_B$ by base change, $h : X \cong Z_A$ is given by the composite

$$X \xrightarrow{f} Y_A \xrightarrow{g_A} (Z_B)_A \cong Z_A$$

where the last isomorphism is the canonical one.

We have a natural forgetful functor $\mathcal{D}ef \to (\mathrm{Art}/\Lambda)^{op}$, and by the properties of the fibered product and the way we defined arrows we see that the conditions of Proposition 1.2.5 are satisfied, so that $\mathcal{D}ef \to (\mathrm{Art}/\Lambda)^{op}$ is a category fibered in groupoids. Notice that if $X_0 \in \mathcal{D}ef(k)$ and $A \in (\mathrm{Art}/\Lambda)$, the category $\mathcal{D}ef_{X_0}(A)$ is (apart from the properness hypothesis) the one we defined before, of flat schemes over $\mathrm{Spec}(A)$ with an isomorphism of the closed fiber with X_0.

We have also a full subcategory $\widetilde{\mathcal{D}ef}$ of $\mathcal{D}ef$, whose objects are flat schemes of finite type, and the restriction $\widetilde{\mathcal{D}ef} \to (\mathrm{Art}/\Lambda)^{op}$ still gives a category fibered in groupoids. Sometimes we will need this additional hypothesis, and will have to restrict our attention to this subcategory.

Proposition 1.4.1. *The categories fibered in groupoids $\mathcal{D}ef \to (\mathrm{Art}/\Lambda)^{op}$ and $\widetilde{\mathcal{D}ef} \to (\mathrm{Art}/\Lambda)^{op}$ are deformation categories.*

For the proof of this and of the other similar propositions below, we state a result (which is basically the affine case) whose proof can be found in Section 8 of [Vis], with a minor modification.

Consider the category fibered in groupoids $\mathrm{Mod} \to (\mathrm{Art}/\Lambda)^{op}$ defined as follows:

Objects: pairs (A, M) where $A \in (\mathrm{Art}/\Lambda)$ and M is a flat A-module.

Arrows: from (A, M) to (B, N) are pairs (φ, f) where $\varphi : B \to A$ is a homomorphism and $f : M \cong N \otimes_B A$ is an isomorphism of A-modules.

The composite of $(\varphi, f) : (A, M) \to (B, N)$ and $(\psi, g) : (B, N) \to (C, P)$ is (ρ, h), where ρ is simply the composite $\varphi \circ \psi : C \to A$, and $h : M \to P \otimes_C A$ is defined as the composite

$$M \cong N \otimes_B A \cong (P \otimes_C B) \otimes_B A \cong P \otimes_C A.$$

We have a natural forgetful functor $\mathrm{Mod} \to (\mathrm{Art}/\Lambda)^{op}$, and by Proposition 1.2.5 again we see that $\mathrm{Mod} \to (\mathrm{Art}/\Lambda)^{op}$ is a category fibered in groupoids.

Proposition 1.4.2. *The category fibered in groupoids $\mathrm{Mod} \to (\mathrm{Art}/\Lambda)^{op}$ is a deformation category.*

In particular, if $A, A', A'' \in (\mathrm{Art}/\Lambda)$ and $\pi' : A' \to A, \pi'' : A'' \to A$ are two homomorphisms, an object

$$(M', M'', \theta) \in \mathrm{Mod}(A') \times_{\mathrm{Mod}(A)} \mathrm{Mod}(A'')$$

can also be seen as a quintuple $(M, M', M'', \alpha', \alpha'')$, where M is a flat A-module, $\alpha' : M' \to M$ is a homomorphism of A'-modules inducing an isomorphism $M' \otimes_{A'} A \cong M$, and similarity for α''. Then a module over $A' \times_A A''$ whose image in $\mathrm{Mod}(A') \times_{\mathrm{Mod}(A)} \mathrm{Mod}(A'')$ is isomorphic to (M', M'', θ)

is given simply by the fibered product $M' \times_M M''$. Most of the proof is devoted to showing that this is a flat module over $A' \times_A A''$.

From the proof of this proposition it is also easy to deduce (as it is done in [Vis] as well) that

$$\Phi : \mathcal{D}ef(A' \times_A A'') \to \mathcal{D}ef(A') \times_{\mathcal{D}ef(A)} \mathcal{D}ef(A'')$$

is an equivalence if restricted to affine schemes, and the same holds also for $\widetilde{\mathcal{D}ef}$.

Proof of 1.4.1. Let us extend the definition of the quasi-inverse

$$\Psi : \mathcal{D}ef(A') \times_{\mathcal{D}ef(A)} \mathcal{D}ef(A'') \to \mathcal{D}ef(A' \times_A A'')$$

of Φ that we described above to non-necessarily affine schemes. Take then an object

$$(X', X'', \theta) \in \mathcal{D}ef(A') \times_{\mathcal{D}ef(A)} \mathcal{D}ef(A'')$$

that is, a pair of flat schemes $X' \to \operatorname{Spec}(A')$ and $X'' \to \operatorname{Spec}(A'')$ with an isomorphism $\theta : X''|_{\operatorname{Spec}(A)} \cong X'|_{\operatorname{Spec}(A)}$ of the pullbacks to A. We can see this object as the following diagram

$$\begin{tikzcd}
X' \arrow[dd, mapsto] & & X'' \arrow[dd, mapsto] \\
& X = X'|_{\operatorname{Spec}(A)} \arrow[ul] \arrow[ur] \arrow[dd, mapsto] & \\
A' \arrow[dr] & & A'' \arrow[dl] \\
& A &
\end{tikzcd}$$

where the morphism $X'|_{\operatorname{Spec}(A)} \to X''$ is the composite of the inverse of θ and of the closed immersion $X''|_{\operatorname{Spec}(A)} \to X''$.

We consider then the sheaf of $A' \times_A A''$-algebras $\mathcal{O}_{X'} \times_{\mathcal{O}_X} \mathcal{O}_{X''}$ on the topological space X. The locally ringed space $\widetilde{X} = (X, \mathcal{O}_{X'} \times_{\mathcal{O}_X} \mathcal{O}_{X''})$ is a scheme by the affine case, and moreover it is flat over $A' \times_A A''$, since flatness is a local property. We set then $\Psi(X', X'', \theta) = \widetilde{X}$.

By the universal property of fibered products one can easily see that an arrow $(X', X'', \theta) \to (Y', Y'', \nu)$ gives a morphism $\widetilde{X} \to \widetilde{Y}$, and routine verifications show that Φ and Ψ are quasi-inverse to each other.

This shows that $\mathcal{D}ef \to (\operatorname{Art}/\Lambda)^{op}$ is a deformation category. The same construction also works for $\widetilde{\mathcal{D}ef} \to (\operatorname{Art}/\Lambda)^{op}$, because of X' and X'' are of finite type over A' and A'', then $\widetilde{X}$ is of finite type over $A' \times_A A''$. $\square$

1.4.2 Closed subschemes

For our second example we want to consider, given a closed immersion of schemes $Y_0 \subseteq X$ over k, families of subschemes of X including the given Y_0 as a fiber over a rational point.

In our setting, given a scheme X over $\mathrm{Spec}(\Lambda)$, we consider the following category, which we will denote by $\mathcal{H}ilb^X$:

Objects: pairs (A, Y) where $A \in (\mathrm{Art}/\Lambda)$ and Y is a closed subscheme of X_A, flat over A.

Arrows: from (A, Y) to (B, Z) are homomorphisms $B \to A$, such that the induced closed subscheme $Z_A \subseteq (X_B)_A$ corresponds to $Y \subseteq X_A$ under the canonical isomorphism $(X_B)_A \cong X_A$.

Composition is given by the usual composition of ring homomorphisms, and it is easily checked that this is well defined: that is, if we have $\varphi : (A, Y) \to (B, Z)$ and $\psi : (B, Z) \to (C, W)$ arrows as above, then the composite $\varphi \circ \psi : C \to A$ is still an arrow in our category, i.e. the induced closed subscheme $W_A \subseteq (X_C)_A$ corresponds to $Y \subseteq X_A$ with respect to the canonical isomorphism $(X_C)_A \cong X_A$.

We have a natural forgetful functor $\mathcal{H}ilb^X \to (\mathrm{Art}/\Lambda)^{op}$, and again by the properties of fibered products and definition of the arrows we easily see that we can apply Proposition 1.2.5, so that $\mathcal{H}ilb^X \to (\mathrm{Art}/\Lambda)^{op}$ is a category fibered in groupoids.

Remark 1.4.3. There is an important difference between this example and the previous one, namely the fact that in $\mathcal{H}ilb^X$ arrows are uniquely determined by their image in (Art/Λ). This "rigidity" phenomenon is strictly related to the associated pseudo-functor of $\mathcal{H}ilb^X \to (\mathrm{Art}/\Lambda)^{op}$ being actually a functor, and our fibered category being fibered in sets.

We will see later on that this is equivalent to saying that our deformation problem has no infinitesimal automorphisms (see Proposition 3.1.8).

Proposition 1.4.4. *The category fibered in groupoids $\mathcal{H}ilb^X \to (\mathrm{Art}/\Lambda)^{op}$ is a deformation category.*

Proof. Let $A, A', A'' \in (\mathrm{Art}/\Lambda)$, $\pi' : A' \to A$, $\pi'' : A'' \to A$ be two homomorphisms, and

$$\Phi : \mathcal{H}ilb^X(A' \times_A A'') \to \mathcal{H}ilb^X(A') \times_{\mathcal{H}ilb^X(A)} \mathcal{H}ilb^X(A'')$$

be the natural functor.

We sketch the definition of a quasi-inverse Ψ of Φ. Take on object

$$(Y', Y'') \in \mathcal{H}ilb^X(A') \times_{\mathcal{H}ilb^X(A)} \mathcal{H}ilb^X(A'')$$

(notice that in this case the isomorphism between the pullbacks to A is irrelevant, because it can only be the identity); in other words Y' is a closed subscheme of $X_{A'}$, and Y'' of $X_{A''}$, such that $Y'|_{\mathrm{Spec}(A)} = Y''|_{\mathrm{Spec}(A)}$ as closed subschemes of X_A. We call this last closed subscheme $Y \subseteq X_A$.

We define $\widetilde{Y}$ as the locally ringed space $(Y, \mathcal{O}_{Y'} \times_{\mathcal{O}_Y} \mathcal{O}_{Y''})$ over $A' \times_A A''$. From the proof of 1.4.1, we know that $\widetilde{Y}$ is actually a flat scheme over $A' \times_A A''$, inducing Y' and Y'' on A' and A''. Moreover we have a commutative diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & Y' \arrow[d] \\ Y'' \arrow[r] & X_{A' \times_A A''} \end{tikzcd}$$

(where $Y' \to X_{A' \times_A A''}$ is the composite $Y' \subseteq X_{A'} \to X_{A' \times_A A''}$ and similarly for Y''), which by the properties of the fibered product induces a morphism $\widetilde{Y} \to X_{A' \times_A A''}$.

Since the pullback of this morphism to A' is a closed immersion, and the projection $A' \times_A A'' \to A'$ is surjective (for $\pi'' : A'' \to A$ is) and has nilpotent kernel, one can easily verify that $\widetilde{Y} \to X_{A' \times_A A''}$ is a closed immersion as well. We set then $\Psi(Y', Y'') = \widetilde{Y}$. Completing the definition on arrows (which is trivial, since $\mathcal{H}ilb^X$ is fibered in sets), one can readily check that Ψ is a quasi-inverse to Φ. $\square$

The name $\mathcal{H}ilb$ comes from the fact that the deformation category is related to the Hilbert functor, if the ambient scheme X is projective and of finite type over Λ.

1.4.3 Quasi-coherent sheaves

For our last example, suppose we are given a quasi-coherent sheaf $\mathcal{E}_0$ on a scheme X over k, and we want to consider families of quasi-coherent sheaves on X having a fiber over a rational point isomorphic to $\mathcal{E}_0$.

Once again, we formulate the problem in terms of fibered categories. Given a scheme X over Λ, we construct the category $\mathcal{QC}oh^X$ as follows:

Objects: pairs $(A, \mathcal{E})$, where $A \in (\mathrm{Art}/\Lambda)$ and $\mathcal{E}$ is a quasi-coherent sheaf on X_A, flat over A.

Arrows: from $(A, \mathcal{E})$ to $(B, \mathcal{F})$ are pairs (φ, f), with $\varphi : B \to A$ a homomorphism and $f : \mathcal{E} \cong \mathcal{F}_A$ an isomorphism of quasi-coherent sheaves on X_A, where $\mathcal{F}_A$ is the pullback of $\mathcal{F}$ along the natural morphism $X_A \to X_B$.

Composition is defined as in the first example: given $(\varphi, f) : (A, \mathcal{E}) \to (B, \mathcal{F})$ and $(\psi, g) : (B, \mathcal{F}) \to (C, \mathcal{G})$, their composite $(\psi, g) \circ (\varphi, f)$ is (ρ, h), where $\rho : C \to A$ is the usual composite $\varphi \circ \psi$, and if we denote by $g_A : \mathcal{F}_A \cong (\mathcal{G}_B)_A$ the isomorphism induced by $g : \mathcal{F} \cong \mathcal{G}_B$ by base change, $h : \mathcal{E} \cong \mathcal{G}_A$ is given by

$$\mathcal{E} \xrightarrow{f} \mathcal{F}_A \xrightarrow{g_A} (\mathcal{G}_B)_A \cong \mathcal{G}_A$$

where the last isomorphism is the canonical one.

As before we have a forgetful functor $\mathcal{QC}oh^X \to (\mathrm{Art}/\Lambda)^{op}$, and by our definition of arrows and properties of the pullback of quasi-coherent sheaves, we can use Proposition 1.2.5, and so $\mathcal{QC}oh^X \to (\mathrm{Art}/\Lambda)^{op}$ is a category fibered in groupoids.

Proposition 1.4.5. *The category fibered in groupoids $\mathcal{QC}oh^X \to (\mathrm{Art}/\Lambda)^{op}$ is a deformation category.*

Proof. Let $A, A', A'' \in (\mathrm{Art}/\Lambda)$ and $\pi' : A' \to A, \pi'' : A'' \to A$ be two homomorphisms; as usual, let

$$\Phi : \mathcal{QC}oh^X(A' \times_A A'') \to \mathcal{QC}oh^X(A') \times_{\mathcal{QC}oh^X(A)} \mathcal{QC}oh^X(A')$$

be the natural functor. We extend the definition of the quasi-inverse Ψ of Φ that we already have in the local case, from the proof of Proposition 1.4.2.

Suppose we have an object

$$(\mathcal{E}', \mathcal{E}'', \theta) \in \mathcal{QC}oh^X(A') \times_{\mathcal{QC}oh^X(A)} \mathcal{QC}oh^X(A')$$

which can also be seen as a diagram

$$\begin{tikzcd}
\mathcal{E}' \arrow[d, mapsto] & & \mathcal{E}'' \arrow[d, mapsto] \\
& \mathcal{E} = \mathcal{E}'|_{\mathrm{Spec}(A)} \arrow[ul] \arrow[ur] \arrow[d, mapsto] & \\
A' \arrow[r] & A & A'' \arrow[l]
\end{tikzcd}$$

where the arrows $\mathcal{E} \to \mathcal{E}'$ and $\mathcal{E} \to \mathcal{E}''$ are respectively the cartesian arrows in $\mathcal{QC}oh^X$ defining $\mathcal{E} = \mathcal{E}'|_{\mathrm{Spec}(A)}$ as the pullback of $\mathcal{E}'$, and the analogue arrow $\mathcal{E}''|_{\mathrm{Spec}(A)} \to \mathcal{E}''$, composed with the isomorphism $\theta : \mathcal{E}'|_{\mathrm{Spec}(A)} \cong \mathcal{E}''|_{\mathrm{Spec}(A)}$ (which is a genuine isomorphism of sheaves of A-modules) respectively.

We consider then the sheaf of $\mathcal{O}_{X_{A' \times_A A''}}$-modules $\widetilde{\mathcal{E}} = \mathcal{E}' \times_{\mathcal{E}} \mathcal{E}''$ on the scheme $X_{A' \times_A A''}$; with $\mathcal{E}' \times_{\mathcal{E}} \mathcal{E}''$ here we mean the sheaf defined over an open subset U of X by

$$\widetilde{\mathcal{E}}(U) = \{(s', s'') : s' \in \mathcal{E}'(U), s'' \in \mathcal{E}''(U) \text{ such that } s' \otimes 1 = s'' \otimes 1 \in \mathcal{E}(U)\}$$

where the equality must be interpreted as: $s' \otimes 1 \in \mathcal{E}'|_{\mathrm{Spec}(A)}$ and $s'' \otimes 1 \in \mathcal{E}''|_{\mathrm{Spec}(A)}$ correspond to each other under the isomorphism θ.

Since $\mathcal{E}'$ and $\mathcal{E}''$ are quasi coherent, $\widetilde{\mathcal{E}}$ is as well, and by the local construction (Proposition 1.4.2) we have that it is flat over $A' \times_A A''$. In conclusion we can set $\Psi(\mathcal{E}', \mathcal{E}'', \theta) = \widetilde{\mathcal{E}}$.

It is easy to see that an arrow $(\mathcal{E}', \mathcal{E}'', \theta) \to (\mathcal{G}', \mathcal{G}'', \nu)$ will yield a homomorphism $\widetilde{\mathcal{E}} \to \widetilde{\mathcal{G}}$, and one can easily check that Ψ is a quasi-inverse of Φ, and so we are done. $\square$

Chapter 2

Tangent space

This chapter is devoted to the introduction and study of the tangent space of a deformation category. This concept generalizes the corresponding ones for schemes and deformation functors, and basically parametrizes first-order deformations.

After defining the tangent space and discussing its action on isomorphism classes of liftings, we will calculate it in our three main examples, and give an application to deformations of smooth hypersurfaces in $\mathbb{P}^n_k$.

Section 2.2 recalls some definitions and facts about extensions of algebras that will be fundamental in the rest of this work.

2.1 Definition

Let $\mathcal{F} \to (\mathrm{Art}/\Lambda)^{op}$ be a deformation category and suppose $\xi_0 \in \mathcal{F}(k)$. We start by defining the tangent space as a set.

Definition 2.1.1. *The* ***tangent space*** *of $\mathcal{F}$ at ξ_0 is the set*

$$T_{\xi_0}\mathcal{F} = \{\textit{isomorphism classes of objects in } \mathcal{F}_{\xi_0}(k[\varepsilon])\}.$$

Remark 2.1.2. Recall that if x_0 is a point of a scheme X over k, there is a bijection between elements of the tangent space $T_{x_0}X$, where x_0 is a rational point of X, and morphisms $\mathrm{Spec}(k[\varepsilon]) \to X$ such that the restriction to $\mathrm{Spec}(k) \to \mathrm{Spec}(k[\varepsilon])$ is the point x_0.

Using the point of view of Remark 1.3.11 we see then that (at least before taking isomorphism classes) there is an analogy between the tangent space just defined, and the classical one of a scheme.

Next we want to justify the name of tangent space, showing that there is a canonical structure of k-vector space on $T_{\xi_0}\mathcal{F}$. To do so, we consider the functor $F : (\mathrm{FVect}/k) \to (\mathrm{Set})$ defined as follows: given a $V \in (\mathrm{FVect}/k)$, we take the ring $k[V]$ of dual numbers of V (which is the k-algebra $k \oplus V$, with product defined by $(x, v) \cdot (y, w) = (xy, xw + yv)$), and associate to V the set

$$F(V) = \{\text{isomorphism classes of object in } \mathcal{F}_{\xi_0}(k[V])\}.$$

If $V \to W$ is a k-linear map, we get a homomorphism $k[V] \to k[W]$, and by pullback (in the fibered category $\mathcal{F}_{\xi_0}$) an arrow $F(V) \to F(W)$. Clearly $F(k) = T_{\xi_0}\mathcal{F}$.

We will show now that F has a lifting $\widetilde{F} : (\mathrm{FVect}\,/k) \to (\mathrm{Vect}\,/k)$ to the category of k-vector spaces, so that every $F(V)$ (in particular $F(k) = T_{\xi_0}\mathcal{F}$) will have a natural structure of k-vector space. As shown in appendix A, to do this it suffices to check that F preserves finite products.

Recall that this means the following: given $V, W \in (\mathrm{FVect}\,/k)$ the two projections $V \oplus W \to V$ and $V \oplus W \to W$ induce functions $F(V \oplus W) \to F(V)$ and $F(V \oplus W) \to F(W)$, which in turn give a function $F(V \oplus W) \to F(V) \times F(W)$. F is said to preserve finite products if the last map is a bijection for every V, W.

Proposition 2.1.3. *The functor F defined above preserves finite products.*

Proof. This follows directly from the fact that $\mathcal{F}_{\xi_0} \to (\mathrm{Art}\,/\Lambda)^{op}$ satisfies [RS], as was shown in Proposition 1.3.10. Take $V, W \in (\mathrm{FVect}\,/k)$, and put $A' = k[V]$, $A'' = k[W]$ with the projections $\pi' : k[V] \to k$, $\pi'' : k[W] \to k$. Then the fibered product $A' \times_k A''$ is just $k[V \oplus W]$, and [RS] gives us an equivalence of categories

$$\Phi : \mathcal{F}_{\xi_0}(k[V \oplus W]) \to \mathcal{F}_{\xi_0}(k[V]) \times_{\mathcal{F}_{\xi_0}(k)} \mathcal{F}_{\xi_0}(k[W]).$$

The induced function on the sets of isomorphism classes of objects coincides with the one $F(V \oplus W) \to F(V) \times F(W)$ induced by the projections as above, which is then a bijection, because Φ is an equivalence of categories. $\square$

For completeness' sake we describe briefly this structure: first of all $F(0)$ has exactly one element, which is simply the isomorphism class of the identity $\xi_0 \to \xi_0$ in $\mathcal{F}_{\xi_0}(k)$. Moreover every $V \in (\mathrm{FVect}\,/k)$ has a natural map $0 \to V$ that induces $F(0) \to F(V)$; the zero element of $F(V)$ is then the image of this map. In our particular case this corresponds to the isomorphism class of the "trivial" pullback of ξ_0 along the inclusion homomorphism $k \to k[V]$.

Addition is defined by the composite

$$F(V) \times F(V) \cong F(V \oplus V) \xrightarrow{F(+)} F(V)$$

where $+ : V \oplus V \to V$ is the addition of V. Similarly multiplication by $a \in k$ is simply $F(\mu_a) : F(V) \to F(V)$, where $\mu_a : V \to V$ is multiplication by a.

From now on we will consider F as a functor $(\mathrm{FVect}/k) \to (\mathrm{Vect}/k)$.

Remark 2.1.4. Suppose we have another object $\eta_0 \in \mathcal{F}(k)$, such that there is an arrow $f : \xi_0 \to \eta_0$ (which is an isomorphism). It is clear then that f will induce a bijection $T_{\xi_0}\mathcal{F} \to T_{\eta_0}\mathcal{F}$, which is actually an isomorphism of k-vector spaces.

So isomorphic objects over k will have isomorphic tangent spaces.

Remark 2.1.5. As discussed in appendix A this canonical lifting $(\mathrm{FVect}/k) \to (\mathrm{Vect}/k)$ is a k-linear functor, so we can apply Proposition A.6 and conclude that for every $V \in (\mathrm{FVect}/k)$ we have

$$F(V) \cong V \otimes_k F(k) = V \otimes_k T_{\xi_0}\mathcal{F}.$$

Remark 2.1.6. If F is the deformation functor associated with $\mathcal{F}_{\xi_0}$, it follows immediately from the definition that $TF = T_{\xi_0}\mathcal{F}$ as k-vector spaces, so that the given definition of tangent space generalizes the standard definition for deformation functors (and for schemes, as noticed in Remark 2.1.2).

In particular if we have a moduli space M representing a certain functor $F : (\mathrm{Sch}/S)^{op} \to (\mathrm{Set})$ (and suppose that the corresponding deformation category satisfies [RS]), we can get informations on the tangent space of M by studying that of the deformation category associated with F.

As expected, along with the concept of tangent space comes the one of differential of a morphism.

Let $H : \mathcal{F} \to \mathcal{G}$ be a morphism of deformation categories, and suppose $\xi_0 \in \mathcal{F}(k)$. Then as in Remark 1.3.12 we have an induced morphism $H_{\xi_0} : \mathcal{F}_{\xi_0} \to \mathcal{G}_{H(\xi_0)}$. If we call $F, G : (\mathrm{FVect}/k) \to (\mathrm{Set})$ the two functors involved in the construction of the tangent spaces of $\mathcal{F}$ at ξ_0 and $\mathcal{G}$ at $H(\xi_0)$ respectively, H_{ξ_0} will induce a natural transformation $\varphi : F \to G$.

Precisely, given a $V \in (\mathrm{FVect}/k)$ we have a functor

$$H_{\xi_0}(k[V]) : \mathcal{F}_{\xi_0}(k[V]) \to \mathcal{G}_{H(\xi_0)}(k[V])$$

and taking isomorphism classes we obtain a function $\varphi(V) : F(V) \to G(V)$. The naturality property follows directly from the fact that $H_{\xi_0} : \mathcal{F}_{\xi_0} \to \mathcal{G}_{H(\xi_0)}$ is a functor.

Since F and G are k-linear functors, from Proposition A.5 we see that φ is automatically k-linear. In particular $\varphi(k) : F(k) \longrightarrow G(k)$ will be a k-linear map.

Definition 2.1.7. *The **differential** of H at ξ_0 is the k-linear map*

$$d_{\xi_0}H = \varphi(k) : T_{\xi_0}\mathcal{F} \to T_{H(\xi_0)}\mathcal{G}.$$

Concretely, given $a \in T_{\xi_0}\mathcal{F}$ and an object $\xi \in \mathcal{F}_{\xi_0}(k[\varepsilon])$ in the isomorphism class a, the image $d_{\xi_0}H(a)$ is just the isomorphism class of $H(\xi) \in \mathcal{G}_{H(\xi_0)}(k[\varepsilon])$.

Remark 2.1.8. As one expects the differential of the composite of two morphisms of deformation categories is the composite of the differentials, as is very easy to see. Moreover if a morphism $H : \mathcal{F} \to \mathcal{F}$ is isomorphic to the identity, then the differential $d_{\xi_0}H : T_{\xi_0}\mathcal{F} \to T_{H(\xi_0)}\mathcal{F}$ is an isomorphism.

If in particular $H : \mathcal{F} \to \mathcal{G}$ is an equivalence, then $d_{\xi_0}H : T_{\xi_0}\mathcal{F} \to T_{H(\xi_0)}\mathcal{G}$ is an isomorphism too. This is because in this case H has a quasi-inverse $K : \mathcal{G} \to \mathcal{F}$, and the composites $H \circ K$ and $K \circ H$ are isomorphic to the identities; this implies that

$$d_{H(\xi_0)}K \circ d_{\xi_0}H : T_{\xi_0}\mathcal{F} \to T_{\xi_0}\mathcal{F}$$

and

$$d_{\xi_0}H \circ d_{H(\xi_0)}K : T_{H(\xi_0)}\mathcal{G} \to T_{H(\xi_0)}\mathcal{G}$$

are isomorphisms, and so $d_{\xi_0}H$ will be too. Here actually $K(H(\xi_0))$ needs only to be isomorphic to ξ_0, so we use the isomorphism of Remark 2.1.4 to identify $T_{K(H(\xi_0))}\mathcal{F}$ and $T_{\xi_0}\mathcal{F}$ in the composites above.

2.2 Extensions of algebras and liftings

In this section we define and state some standard facts about extensions of algebras that will be used very frequently from now on. Let R be a ring, and A be an R-algebra.

Definition 2.2.1. *An* ***extension*** *of A is a surjection $A' \to A$ of R-algebras with square-zero kernel $I = \ker(\varphi) \subseteq A'$. We also say that $A' \to A$ is an extension of A by I.*

An extension as above is usually pictured as the exact sequence or R-modules

$$0 \longrightarrow I \longrightarrow A' \longrightarrow A \longrightarrow 0.$$

It is a standard fact that in this situation I is an A-module in a natural way: given $a \in A$ and $i \in I$ we just take an element $a' \in A'$ in the preimage of a and define $a \cdot i$ as $a'i \in I$. This is well defined because $I^2 = (0)$.

Example 2.2.2. If M is an A-module, then there is a trivial extension of A by M, that we obtain by considering $A \oplus M$ as an R-algebra in the natural way (defining the product by $(a, m) \cdot (a', m') = (aa', am' + a'm)$, so that in particular $M^2 = (0)$), and the projection $A \oplus M \to M$. This is called the **trivial extension** of A by M.

The R-algebra $A \oplus M$ defined above is called the **ring of dual numbers** of M, and we will denote it by $A[M]$. In particular if $A = k$ is a field and

$M \cong k$, we obtain the k-algebra $k[t]/(t^2) \cong k \oplus kt$, which is the usual ring of dual numbers $k[\varepsilon]$ (where $\varepsilon^2 = 0$).

The following is also a standard fact, that will be used later on.

Proposition 2.2.3. *Let $A' \to A$ be an extensions of R-algebras with kernel I, B an R-algebra, and $f, g : B \to A'$ two homomorphisms of R-algebras such that the composites with $A' \to A$ coincide. Then the difference $f - g : B \to I$ is an R-derivation.*

Conversely, if $f : B \to A'$ is a homomorphism of R-algebras and $d : B \to I$ is an R-derivation, then the map $f + d : B \to A'$ is a homomorphism of R-algebras such that the composite with $A' \to A$ coincides with $B \xrightarrow{f} A' \to A$.

Suppose now we have two extensions of R-algebras $A' \to A$ and $B' \to B$, with kernels I and J respectively, and an homomorphism of R-algebras $\varphi : A' \to B'$, such that $\varphi(I) \subseteq J$. Then φ will induce $\overline{\varphi} : A \to B$ and $\varphi|_I : I \to J$, which fit together with φ in a commutative diagram.

Definition 2.2.4. *A **morphism** between two extensions of R-algebras $A' \to A$ with kernel I and J respectively is a triplet of homomorphisms (f, g, h), where $f : I \to J, g : A' \to B', h : A \to B$, such that the diagram*

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d, "f"] & A' \arrow[r] \arrow[d, "g"] & A \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & J \arrow[r] & B' \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

is commutative.

So a homomorphism φ as above induces a morphism $(\varphi|_I, \varphi, \overline{\varphi})$ between the two extensions.

Definition 2.2.5. *A **splitting** of an extension of R-algebras $A' \to A$ is a homomorphism of R-algebras $\varphi : A \to A'$ such that the composite $A \xrightarrow{\varphi} A' \to A$ is the identity.*

Standard arguments show that an extension admits a splitting if and only if it is isomorphic to a trivial extension.

Now we restrict our attention to extensions of algebras in (Art/Λ). The following type of extensions will play a particularly important role.

Definition 2.2.6. *An extension $A' \to A$ in (Art/Λ) is said to be **small** if the kernel I is annihilated by the maximal ideal $\mathfrak{m}_{A'}$, so that it is naturally a k-vector space.*

*A small extension is called **tiny** if $I \cong k$ as a k-vector space, or equivalently if I is principal and nonzero.*

From now on when we write "$A' \to A$ is a small (tiny) extension", we mean also that $A', A \in (\mathrm{Art}/\Lambda)$ (and recall that the homomorphism is automatically local).

The following proposition will allow us to consider small extensions in most of the questions we will face.

Proposition 2.2.7. *Let $A' \to A$ be a surjection in (Art/Λ). Then it can be factored as a composite of tiny extensions*

$$A' = A_0 \to A_1 \to \cdots \to A_n = A.$$

Proof. Let I be the kernel of $A' \to A$, a proper ideal of A'. Since $I \subseteq \mathfrak{m}_{A'}$ and $\mathfrak{m}_{A'}$ is nilpotent, say $\mathfrak{m}_{A'}^n = (0)$ and $\mathfrak{m}_{A'}^{n-1} \neq (0)$, then $I\mathfrak{m}_{A'}^{n-1} = (0)$. Moreover we have a chain of ideals

$$(0) = I\mathfrak{m}_{A'}^{n-1} \subseteq I\mathfrak{m}_{A'}^{n-2} \subseteq \cdots \subseteq I$$

which gives a compostion of surjections

$$A' = A'/I\mathfrak{m}_{A'}^{n-1} \to A'/I\mathfrak{m}_{A'}^{n-2} \to \cdots \to A'/I \cong A$$

which are easily seen to be small extensions.

Finally, since the kernel of each of the homomorphisms $A'/I\mathfrak{m}_{A'}^{i} \to A'/I\mathfrak{m}_{A'}^{i-1}$ is a finite-dimensional k-vector space we can take a basis and consider the successive quotients by elements of this basis, thus factoring the projection $A'/I\mathfrak{m}_{A'}^{i} \to A'/I\mathfrak{m}_{A'}^{i-1}$ into a composite of tiny extensions. □

Now we come to liftings of objects of a deformation category. The idea is that if we want to study the deformations over $A \in (\mathrm{Art}/\Lambda)$ of a given object over k, we should do this inductively using the factorization of the surjection $A \to k$ given by the preceding proposition to reduce to the case of small extensions.

Definition 2.2.8. *Let $\mathcal{F} \to (\mathrm{Art}/\Lambda)^{op}$ be a deformation category, $\varphi : A' \to A$ a surjection in (Art/Λ), and $\xi \in \mathcal{F}(A)$. A **lifting** of ξ to A' is an arrow $\xi \to \xi'$ over φ.*

Equivalently, a lifting of ξ to A' is an object $\xi' \in \mathcal{F}(A')$ together with an isomorphism of its pullback $\varphi_*(\xi')$ with ξ in $\mathcal{F}(A)$. Sometimes we will refer to a lifting only by means of the object ξ' over A', leaving the arrow from ξ understood.

Generalizing the construction of the category $\mathcal{F}_{\xi_0}$, it is easy to see that given φ and ξ as above, the liftings of ξ to A' are the objects of a category $\mathcal{L}if(\xi, A')$, in which arrows from $f : \xi \to \xi'$ to $g : \xi \to \xi''$ are arrows $h : \xi' \to \xi''$ of $\mathcal{F}(A)$ such that $h \circ f = g$. We will call $\mathrm{Lif}(\xi, A')$ the set of isomorphism classes of liftings of ξ to A'.

Both $\mathcal{L}if(\xi, A')$ and $\mathrm{Lif}(\xi, A')$ clearly depend also on the homomorphism $A' \to A$, but we will not specify it in the notation, since it will always be clear from the context which homomorphism we are considering.

Remark 2.2.9. In the following we will make some constructions starting with an isomorphism class $[\xi']$ of a lifting and possibly pick one of its elements in the process, without mentioning that the final result will not depend on this choice (because we will be often taking isomorphism classes again in the end).

In particular if we have an element $a \in I \otimes_k T_{\xi_0}\mathcal{F}$, we will also write a for an object of $\mathcal{F}_{\xi_0}(k[I])$ belonging to the isomorphism class a.

2.3 Actions on liftings

Part of the usefulness of the tangent space is the fact that it gives some control on the liftings of objects of $\mathcal{F}$ along small extensions, as the following theorem shows.

Theorem 2.3.1. *Let $\mathcal{F} \to (\mathrm{Art}/\Lambda)^{op}$ be a deformation category, $A' \to A$ a small extension with kernel I, and take $\xi_0 \in \mathcal{F}(k)$, $\xi \in \mathcal{F}_{\xi_0}(A)$. Then $\mathrm{Lif}(\xi, A')$ is either empty, or there is a free and transitive action of $I \otimes_k T_{\xi_0}\mathcal{F}$ on it.*

Proof. Let $\xi \to \xi_1'$ and $\xi \to \xi_2'$ be two liftings of ξ to A', and notice that together they give an object of the category $\mathcal{F}(A') \times_{\mathcal{F}(A)} \mathcal{F}(A')$. By [RS], they give rise to a lifting $\xi \to \{\xi_1', \xi_2'\}$ of ξ to the fibered product $A' \times_A A'$, and exactly as in the proof of Proposition 1.3.10 one can see that this construction gives a bijective correspondence between pairs of isomorphism classes of liftings of ξ to A' and liftings of ξ to $A' \times_A A'$.

We have an isomorphism of rings $f : A' \times_A A' \cong A' \oplus I = A'[I]$, given by $f(a_1, a_2) = (a_1, a_2 - a_1)$, commuting with the projections on the first factor A'. It is clear that it is an additive isomorphism. Moreover using the fact that $I^2 = (0)$ we see that f is a ring homomorphism: we have

$$f((a_1, a_2)(b_1, b_2)) = f(a_1b_1, a_2b_2) = (a_1b_1, a_2b_2 - a_1b_1)$$

and on the other hand

$$f(a_1, a_2)f(b_1, b_2) = (a_1, a_2 - a_1)(b_1, b_2 - b_1) = (a_1b_1, a_1(b_2 - b_1) + b_1(a_2 - a_1)).$$

But now

$$a_1(b_2 - b_1) + b_1(a_2 - a_1) = a_2b_2 - a_1b_1 - (a_2 - a_1)(b_2 - b_1) = a_2b_2 - a_1b_1$$

because $(a_2 - a_1)(b_2 - b_1) \in I^2$.

Moreover, if we call $\pi : A \to k$ the quotient map, there is an isomorphism $A'[I] \cong A' \times_k k[I]$, defined by $(a, v) \mapsto (a, \pi(a) \oplus v)$, which also commutes with the projections on the first factor, and so as before we have a bijection between the isomorphism classes of liftings of ξ to $A'[I]$ and pairs of isomorphism classes of liftings of ξ to A' and of ξ_0 to $k[I]$.

$$\begin{tikzcd}
 & \xi'_2 \arrow[rr] \arrow[dd, mapsto] & & \{\xi'_1, \xi'_2\} \arrow[dd, mapsto] & & \eta' \arrow[ll] \arrow[rr] \arrow[dd, mapsto] & & \eta'' \arrow[dd, mapsto] & & \eta|''_{k[I]} \arrow[ll] \arrow[dd, mapsto] \\
\xi \arrow[ur] \arrow[rr] \arrow[dd, mapsto] & & \xi'_1 \arrow[ur] \arrow[rrrr, equal] \arrow[dd, mapsto] & & & & \xi'_1 \arrow[ur] \arrow[dd, mapsto] & & \xi_0 \arrow[ll] \arrow[ur] \arrow[dd, mapsto] & \\
 & A' \arrow[dl] & & A' \times_A A' \arrow[ll] \arrow[rr, "\sim"] \arrow[dl] & & A' \oplus I \arrow[dl] & & A' \times_k k[I] \arrow[ll, "\sim"'] \arrow[rr] \arrow[dl] & & k[I] \arrow[dl] \\
A & & A' \arrow[ll] \arrow[rr, equal] & & A' \arrow[rr, equal] & & A' \arrow[rr] & & k &
\end{tikzcd}$$

In conclusion we have a bijection Φ given by

$$\operatorname{Lif}(\xi, A') \times \operatorname{Lif}(\xi, A') \longrightarrow \operatorname{Lif}(\xi, A') \times \operatorname{Lif}(\xi_0, k[I]) \cong \operatorname{Lif}(\xi, A') \times (I \otimes_k T_{\xi_0}\mathcal{F}).$$

By construction if $\pi_1 : \operatorname{Lif}(\xi, A') \times \operatorname{Lif}(\xi, A') \to \operatorname{Lif}(\xi, A')$ is the projection to the first factor, then $\pi_1 \circ \Phi^{-1}$ is also the projection to the first factor $\operatorname{Lif}(\xi, A') \times (I \otimes_k T_{\xi_0}\mathcal{F}) \to \operatorname{Lif}(\xi, A')$. Let us consider now

$$\mu = \pi_2 \circ \Phi^{-1} : \operatorname{Lif}(\xi, A') \times (I \otimes_k T_{\xi_0}\mathcal{F}) \to \operatorname{Lif}(\xi, A')$$

where $\pi_2 : \operatorname{Lif}(\xi, A') \times \operatorname{Lif}(\xi, A') \to \operatorname{Lif}(\xi, A')$ is the projection on the second factor, and let us show that it gives an action of $I \otimes_k T_{\xi_0}\mathcal{F}$ on $\operatorname{Lif}(\xi, A')$. Once we have done this, the action will automatically be free and transitive because of the bijectivity of Φ.

We have to show:

- $\mu([\xi'], 0) = [\xi']$ for every $[\xi'] \in \operatorname{Lif}(\xi, A')$,
- $\mu([\xi'], a + b) = \mu(\mu([\xi'], a), b)$ for every $[\xi'] \in \operatorname{Lif}(\xi, A')$ and $a, b \in I \otimes_k T_{\xi_0}\mathcal{F}$.

Let us start with the first statement; we show that $\Phi([\xi'], [\xi']) = ([\xi'], 0)$. Consider the diagonal map $d : A' \to A' \times_A A'$ given by $d(a) = (a, a)$, and notice that the pullback of ξ' along this map is isomorphic to $\{\xi', \xi'\}$. Moreover the composite $f \circ d : A' \to A' \oplus I$ is $(f \circ d)(a) = (a, 0)$, so the following diagram commutes.

$$\begin{tikzcd}
A' \arrow[r, "f \circ d"] \arrow[dr] & A' \oplus I \arrow[r, "\sim"] & A' \times_k k[I] \arrow[dd] \\
 & k \arrow[dr] & \\
 & & k[I]
\end{tikzcd}$$

Since the second component of $\Phi([\xi'], [\xi'])$ can be obtained by pulling back ξ' along the top homomorphism $A' \to k[I]$ of the above diagram, it is precisely the element we obtain by pulling back ξ' along the "trivial" homomorphism $A' \to k \to k[I]$, and so it is the zero element of $I \otimes_k T_{\xi_0}\mathcal{F}$.

For the second statement, we consider an element $[\xi'] \in \mathrm{Lif}(\xi, A')$ and $a, b \in I \otimes_k T_{\xi_0}\mathcal{F}$. With arguments similar to the ones used before, we see that the triple fiber product $A' \times_A A' \times_A A'$ is isomorphic to $A' \oplus I \oplus I$ by means of the map $(a_1, a_2, a_3) \mapsto (a_1, a_2 - a_1, a_3 - a_2)$, and isomorphism classes of liftings of ξ to $A' \times_A A' \times_A A'$ are in correspondence with triplets of liftings of ξ to A', and similarly for $A' \oplus I \oplus I \cong A' \times_k k[I] \times_k k[I]$. So we have a bijection

$$\mathrm{Lif}(\xi, A') \times \mathrm{Lif}(\xi, A') \times \mathrm{Lif}(\xi, A') \cong \mathrm{Lif}(\xi, A') \times (I \otimes_k T_{\xi_0}\mathcal{F}) \times (I \otimes_k T_{\xi_0}\mathcal{F})$$

such that the triplet $([\xi'], \mu([\xi'], a), \mu(\mu([\xi'], a), b))$ corresponds to $([\xi'], a, b)$.

In particular $\mu(\mu([\xi'], a), b)$ is the isomorphism class of the pullback of the object $\{\xi', a, b\}$ on $A' \times_k k[I] \times_k k[I] \cong A' \oplus I \oplus I$ along the induced projection on the third factor

$$\pi_3' : A' \oplus I \oplus I \cong A' \times_A A' \times_A A' \to A'$$

given by $(a, v, w) \mapsto a + v + w$.

On the other hand we have a homomorphism $A' \oplus I \oplus I \to A' \oplus I \cong A' \times_k k[I]$ induced by addition on I (explicitly given by $(a, v, w) \mapsto (a, v+w)$), and by definition of addition in $I \otimes_k T_{\xi_0}\mathcal{F}$ the pullback of $\{\xi', a, b\}$ to $A' \times_k k[I]$ is just $\{\xi', a + b\}$. We can now pullback further along

$$A' \times_k k[I] \cong A' \times_A A' \xrightarrow{\pi_2} A'$$

and we obtain exactly $\mu([\xi'], a + b) \in \mathrm{Lif}(\xi, A')$. In conclusion we have a commutative diagram

$$\begin{tikzcd}
A' \oplus I \oplus I \arrow[r, "\mathrm{id} \oplus +"] \arrow[d, "\pi_3'"'] & A' \oplus I \arrow[r, "\sim"] & A' \times_k k[I] \arrow[r, "\sim"] & A' \times_A A' \arrow[llld, bend left, "\pi_2"] \\
A' & & &
\end{tikzcd}$$

that lets us conclude that $\mu(\mu([\xi'], a), b) = \mu([\xi'], a + b)$. $\square$

From now on we will drop the notation $\mu : \mathrm{Lif}(\xi, A') \times (I \otimes_k T_{\xi_0}\mathcal{F}) \to \mathrm{Lif}(\xi, A')$ for the action, and we will simply write it as a multiplication on the right.

The following corollary is a straightforward application of Proposition 2.3.1.

Corollary 2.3.2. *Let $\mathcal{F} \to (\mathrm{Art}/\Lambda)^{op}$ be a deformation category, and $\xi_0 \in \mathcal{F}(k)$. If $T_{\xi_0}\mathcal{F} = 0$, then there is at most one isomorphism class in $\mathcal{F}_{\xi_0}(A)$, for every $A \in (\mathrm{Art}/\Lambda)$.*

Proof. Take $A \in (\mathrm{Art}\,/\Lambda)$, and consider the quotient map $\pi : A \to k$. Using Proposition 2.2.7 we can factor π as a composite of small extensions

$$A = A_0 \to A_1 \to \ldots \to A_n = k.$$

Call $n(A)$ the least n with such a factorization. We will use an induction on $n(A)$.

If $n(A) = 0$ we have $A = k$, and the result is clear. So suppose we know the conclusion up to $n(A) - 1$, and consider the small extension $A \to A_1$ (with the above notation), with kernel I.

By inductive hypothesis $\mathcal{F}_{\xi_0}(A_1)$ is either empty or all its objects are isomorphic. If it is empty, $\mathcal{F}_{\xi_0}(A)$ will be too (because there is a pullback functor $\mathcal{F}_{\xi_0}(A) \to \mathcal{F}_{\xi_0}(A_1)$), and in this case we are done; if it is not empty, consider two objects $\xi_0 \to \xi$ and $\xi_0 \to \xi'$ of $\mathcal{F}_{\xi_0}(A)$, if there are any.

We have that the pullbacks $\xi_0 \to \xi|_{A_1}$ and $\xi_0 \to \xi'|_{A_1}$ are isomorphic in $\mathcal{F}_{\xi_0}(A_1)$, and so $[\xi]$ and $[\xi']$ are both elements of $\mathrm{Lif}(\xi|_{A_1}, A)$. Since this set has a transitive action of $I \otimes_k T_{\xi_0}\mathcal{F} = 0$, we have that $[\xi] = [\xi']$, and then $\xi \cong \xi'$ (also as objects of $\mathcal{F}_{\xi_0}(A')$). □

Remark 2.3.3. We will sometimes use the following notation: when $[\xi'], [\xi'']$ are two isomorphism classes of liftings of $\xi \in \mathcal{F}_{\xi_0}(A)$ to A', where $A' \to A$ is a small extension with kernel I, we will denote by $g([\xi'], [\xi''])$ the element $g \in I \otimes_k T_{\xi_0}\mathcal{F}$ such that $[\xi''] \cdot g = [\xi']$.

This action has two natural functoriality properties, that we now discuss. The first one is a functoriality with respect to the small extension. Let $\mathcal{F} \to (\mathrm{Art}\,/\Lambda)^{op}$ be a deformation category, and $A' \to A$, $B' \to B$ two small extensions, with kernels $I \subseteq A'$ and $J \subseteq B'$. Suppose we also have a homomorphism $\varphi : A' \to B'$ such that $\varphi(I) \subseteq J$, and thus inducing $\overline{\varphi} : A \to B$ and $\varphi|_I : I \to J$. In other words, we have a morphism of extensions

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r] \arrow[d, "\varphi|_I"] & A' \arrow[r] \arrow[d, "\varphi"] & A \arrow[r] \arrow[d, "\overline{\varphi}"] & 0 \\
0 \arrow[r] & J \arrow[r] & B' \arrow[r] & B \arrow[r] & 0.
\end{tikzcd}$$

Let us also have $\xi_0 \in \mathcal{F}(k)$, $\xi \in \mathcal{F}_{\xi_0}(A)$ and assume $\mathrm{Lif}(\xi, A')$ is nonempty (so that $\mathrm{Lif}(\overline{\varphi}_*(\xi), B')$ will also be nonempty). We have a k-linear map

$$\varphi|_I \otimes \mathrm{id} : I \otimes_k T_{\xi_0}\mathcal{F} \to J \otimes_k T_{\xi_0}\mathcal{F}$$

(which by naturality of the isomorphism of Remark 2.1.5 corresponds to the pullback function $\mathrm{Lif}(\xi_0, k[I]) \to \mathrm{Lif}(\xi_0, k[J])$ induced by $\mathrm{id} \oplus \varphi|_I$), and a pullback function on isomorphism classes of liftings

$$\varphi_* : \mathrm{Lif}(\xi, A') \to \mathrm{Lif}(\overline{\varphi}_*(\xi), B').$$

Proposition 2.3.4. *We have*

$$\varphi_*([\xi'] \cdot a) = \varphi_*([\xi']) \cdot (\varphi|_I \otimes \mathrm{id})(a)$$

for every $a \in I \otimes_k T_{\xi_0}\mathcal{F}$ *and* $[\xi'] \in \mathrm{Lif}(\xi, A')$.

Proof. Call

$$\mu_A : \mathrm{Lif}(\xi, A') \times (I \otimes_k T_{\xi_0}\mathcal{F}) \to \mathrm{Lif}(\xi, A')$$

and

$$\mu_B : \mathrm{Lif}(\overline{\varphi}_*(\xi), B') \times (J \otimes_k T_{\xi_0}\mathcal{F}) \to \mathrm{Lif}(\overline{\varphi}_*(\xi), B')$$

the two maps giving the actions, as in the proof of Theorem 2.3.1. We have to prove that the following diagram commutes.

$$\begin{tikzcd}
\mathrm{Lif}(\xi, A') \times (I \otimes_k T_{\xi_0}\mathcal{F}) \arrow[r, "\mu_A"] \arrow[d, "(\varphi_*, \varphi|_I \otimes \mathrm{id})"'] & \mathrm{Lif}(\xi, A') \arrow[d, "\varphi_*"] \\
\mathrm{Lif}(\overline{\varphi}_*(\xi), B') \times (J \otimes_k T_{\xi_0}\mathcal{F}) \arrow[r, "\mu_B"] & \mathrm{Lif}(\overline{\varphi}_*(\xi), B')
\end{tikzcd}$$

Fix $[\xi'] \in \mathrm{Lif}(\xi, A')$ and $a \in I \otimes_k T_{\xi_0}\mathcal{F} \cong \mathrm{Lif}(\xi_0, k[I])$, and recall that $\mu_A([\xi'], a)$ is defined as the (isomorphism class of the) pullback of the object $\{\xi', a\}$ over $A' \times_k k[I]$ along the composite

$$A' \times_k k[I] \cong A' \times_A A' \xrightarrow{\pi_2} A'.$$

To get $\varphi_*(\mu_A([\xi'], a))$ we have to pullback further along $\varphi : A' \to B'$. On the other hand we obtain $\mu_B(\varphi_*([\xi']), (\varphi|_I \otimes \mathrm{id})(a))$ by taking first the pullback of $\{\xi', a\}$ along the induced homomorphism

$$(\varphi, \mathrm{id} \oplus \varphi|_I) : A' \times_k k[I] \to B' \times_k k[J]$$

and then pulling back again along $B' \times_k k[J] \cong B' \times_B B' \xrightarrow{\pi_2} B'$.

But we have a commutative diagram

$$\begin{tikzcd}
A' \times_k k[I] \arrow[r, "\sim"] \arrow[d, "(\varphi, \mathrm{id} \oplus \varphi|_I)"'] & A' \times_A A' \arrow[r, "\pi_2"] & A' \arrow[d, "\varphi"] \\
B' \times_k k[J] \arrow[r, "\sim"] & B' \times_B B' \arrow[r, "\pi_2"] & B'
\end{tikzcd}$$

so the pullbacks of $\{\xi', a\}$ along the two homomorphisms are isomorphic, and we are done. $\square$

Remark 2.3.5. Using the notation of Remark 2.3.3, we can equivalently say that if $[\xi'], [\xi''] \in \mathrm{Lif}(\xi, A')$ we have

$$g([\varphi_*(\xi')], [\varphi_*(\xi'')]) = (\varphi|_I \otimes \mathrm{id})(g([\xi'], [\xi''])).$$

The second one is functoriality with respect to the deformation category. Let $\mathcal{F} \to (\mathrm{Art}/\Lambda)^{op}$ and $\mathcal{G} \to (\mathrm{Art}/\Lambda)^{op}$ be two deformation categories with a morphism $F : \mathcal{F} \to \mathcal{G}$, $A' \to A$ a small extension with kernel I, and let $\xi_0 \in \mathcal{F}(k)$, $\xi \in \mathcal{F}_{\xi_0}(A)$. Assume also that $\mathrm{Lif}(\xi, A')$ is nonempty (so that $\mathrm{Lif}(F_{\xi_0}(\xi), A')$ will also be nonempty).

There is a k-linear map

$$\mathrm{id} \otimes d_{\xi_0} F : I \otimes_k T_{\xi_0}\mathcal{F} \to I \otimes_k T_{F(\xi_0)}\mathcal{G}$$

induced by the differential $d_{\xi_0} F : T_{\xi_0}\mathcal{F} \to T_{F(\xi_0)}\mathcal{G}$, and we still denote by

$$F : \mathrm{Lif}(\xi, A') \to \mathrm{Lif}(F(\xi), A')$$

the induced function on isomorphism classes of liftings.

Proposition 2.3.6. *We have*

$$F([\xi'] \cdot a) = F([\xi']) \cdot (\mathrm{id} \otimes d_{\xi_0} F)(a)$$

for every $a \in I \otimes_k T_{\xi_0}\mathcal{F}$ and $[\xi'] \in \mathrm{Lif}(\xi, A')$.

Proof. Consider $[\xi'] \in \mathrm{Lif}(\xi, A')$ and $a \in I \otimes_k T_{\xi_0}\mathcal{F}$; as in the preceding proof, we recall that $[\xi'] \cdot a$ is defined as the isomorphism class of the pullback along $A' \times_k k[I] \cong A' \times_A A' \xrightarrow{\pi_2} A'$ of the object $\{\xi', a\}$ over $A' \times_k k[I]$, so that there is a diagram

$$\begin{tikzcd}
\{\xi', a\} \arrow[d, mapsto] & \{\xi', \xi''\} \arrow[l] \arrow[d, mapsto] & \xi'' \arrow[l] \arrow[d, mapsto] \\
A' \times_k k[I] \arrow[r, "\sim"] & A' \times_A A' \arrow[r, "\pi_2"] & A'
\end{tikzcd}$$

where $\xi'' \in \mathcal{F}_{\xi_0}(A')$ is such that $[\xi''] = [\xi'] \cdot a$.

If we apply the base-preserving functor F to this diagram, we get a similar one with top row

$$\{F(\xi'), F(a)\} \longleftarrow \{F(\xi'), F(\xi'')\} \longleftarrow F(\xi'')$$

so that $F(\xi'')$ is (isomorphic to) the pullback of $\{F(\xi'), F(a)\}$ along $A' \times_k k[I] \cong A' \times_A A' \xrightarrow{\pi_2} A'$, whose isomorphism class is $F([\xi']) \cdot F(a)$ by definition of the action of $I \otimes_k T_{F(\xi_0)}\mathcal{G}$.

But by definition of differential $F(a)$ is precisely $(\mathrm{id} \otimes d_{\xi_0} F)(a)$, and so

$$F([\xi'] \cdot a) = F([\xi'']) = [F(\xi'')] = F([\xi']) \cdot (\mathrm{id} \otimes d_{\xi_0} F)(a). \qquad \square$$

Remark 2.3.7. As before we can reformulate this result using the notation of Remark 2.3.3, and obtain

$$g([F(\xi')], [F(\xi'')]) = (\mathrm{id} \otimes d_{\xi_0} F)(g([\xi'], [\xi'']))$$

for every $[\xi'], [\xi''] \in \mathrm{Lif}(\xi, A')$.

There is a generalization of the previous constructions, that we will need later to state a theorem about vanishing of obstructions (see Theorem 4.1.9). Given $A \in (\mathrm{Art}/\Lambda)$, $\xi \in \mathcal{F}(A)$, we consider the liftings of ξ to Λ-algebras of the form $A[M]$ where $M \in (\mathrm{FMod}/A)$ (and the homomorphism $A[M] \to A$ is the quotient map).

We have a functor $F_\xi : (\mathrm{FMod}/A) \to (\mathrm{Set})$ defined on objects by

$$F_\xi(M) = \{\text{isomorphism classes of liftings of } \xi \text{ to } A[M]\}$$

and sending an A-linear map $M \to N$ to the pullback function $F_\xi(M) \to F_\xi(N)$.

Since $\mathcal{F} \to (\mathrm{Art}/\Lambda)^{op}$ satisfies [RS], one can readily show (as in the construction of the tangent space) that the functor F_ξ preserves finite products, and so by Proposition A.3 it has a canonical lifting $(\mathrm{FMod}/A) \to (\mathrm{Mod}/A)$, which we still call F_ξ. Notice that in opposition to the case $A = k$, the functor F_ξ need not be exact. Nevertheless, one can easily prove using [RS] that it is half-exact, that is, if

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is an exact sequence of finitely generated A-modules, then the sequence

$$F_\xi(M') \longrightarrow F_\xi(M) \longrightarrow F_\xi(M'')$$

is exact.

The following proposition can be proved in the exact same way as Theorem 2.3.1.

Proposition 2.3.8. *If $A' \to A$ is a surjection in (Art/Λ) with kernel I such that $I^2 = (0)$ (so that I is an A-module), and $\xi \in \mathcal{F}(A)$, then $\mathrm{Lif}(\xi, A')$ is either empty, or there is a free and transitive action of $F_\xi(I)$ on it.*

2.4 Examples

We now calculate the tangent space in each of the examples introduced in Section 1.4, and give an application to infinitesimal deformations of smooth hypersurfaces in $\mathbb{P}^n_k$.

2.4.1 Schemes

We first consider the deformation category $\widetilde{\mathcal{D}ef} \to (\mathrm{Art}/\Lambda)^{op}$ corresponding to deformations of schemes of finite type.

Theorem 2.4.1. *Let X_0 be a reduced and generically smooth scheme of finite type over k. There is an isomorphism (sometimes called the* ***Kodaira-Spencer*** *correspondence)*

$$T_{X_0}\widetilde{\mathcal{D}ef} \cong \mathrm{Ext}^1_{\mathcal{O}_{X_0}}(\Omega_{X_0}, \mathcal{O}_{X_0}).$$

Proof. Call $F : (\mathrm{FVect}\,/k) \to (\mathrm{Set})$ the functor defined on objects by

$$F(V) = \{\text{isomorphism classes of objects in } \widetilde{\mathcal{D}ef}_{X_0}(k[V])\}$$

and that sends a k-linear map $V \to W$ to the pullback function $F(V) \to F(W)$. Our aim is to construct a functorial bijection

$$F(V) \cong V \otimes_k \mathrm{Ext}^1_{\mathcal{O}_{X_0}}(\Omega_{X_0}, \mathcal{O}_{X_0})$$

that will give a k-linear natural transformation between the functors F and $- \otimes_k \mathrm{Ext}^1_{\mathcal{O}_{X_0}}(\Omega_{X_0}, \mathcal{O}_{X_0})$, and in particular we will get an isomorphism

$$T_{X_0}\widetilde{\mathcal{D}ef} = F(k) \cong \mathrm{Ext}^1_{\mathcal{O}_{X_0}}(\Omega_{X_0}, \mathcal{O}_{X_0}).$$

We will proceed in several steps.

Step 1. We start by constructing a function

$$\varphi_V : F(V) \to V \otimes_k \mathrm{Ext}^1_{\mathcal{O}_{X_0}}(\Omega_{X_0}, \mathcal{O}_{X_0}).$$

Take an object $X \in \widetilde{\mathcal{D}ef}_{X_0}(k[V])$, which is a flat scheme of finite type over $k[V]$ with an isomorphism

$$X \times_{\mathrm{Spec}(k[V])} \mathrm{Spec}(k) \cong X_0$$

(in particular $\mathcal{O}_X \otimes_{k[V]} k \cong \mathcal{O}_{X_0}$).

We see first that the sheaf of ideals I of X_0 in X can be identified with $V \otimes_k \mathcal{O}_{X_0}$: tensoring the exact sequence of $k[V]$-modules

$$0 \longrightarrow V \longrightarrow k[V] \longrightarrow k \longrightarrow 0 \tag{2.1}$$

with $\mathcal{O}_X$, by flatness of X over $k[V]$ we get an exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow V \otimes_{k[V]} \mathcal{O}_X \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{X_0} \longrightarrow 0$$

where the second map is the canonical projection. So we have

$$I \cong V \otimes_{k[V]} \mathcal{O}_X \cong V \otimes_k (k \otimes_{k[V]} \mathcal{O}_X) \cong V \otimes_k \mathcal{O}_{X_0}.$$

In particular $I^2 = (0)$, and $I/I^2 = I \cong V \otimes_k \mathcal{O}_{X_0}$.

Now consider the conormal sequence of $X_0 \subseteq X$

$$V \otimes_k \mathcal{O}_{X_0} \xrightarrow{d} \Omega_X|_{X_0} \longrightarrow \Omega_{X_0} \longrightarrow 0$$

where d is the homomorphism induced by the universal derivation $\mathcal{O}_X \to \Omega_X$. From Proposition C.9 we see that in our case d is injective, and so we have an exact sequence of $\mathcal{O}_{X_0}$-modules

$$0 \longrightarrow V \otimes_k \mathcal{O}_{X_0} \longrightarrow \Omega_X|_{X_0} \longrightarrow \Omega_{X_0} \longrightarrow 0$$

whose isomorphism class in an element of $\mathrm{Ext}^1_{\mathcal{O}_{X_0}}(\Omega_{X_0}, V \otimes_k \mathcal{O}_{X_0}) \cong V \otimes_k \mathrm{Ext}^1_{\mathcal{O}_{X_0}}(\Omega_{X_0}, \mathcal{O}_{X_0})$. It is also clear that isomorphic objects of $\widetilde{\mathcal{D}ef}_{X_0}(k[V])$ will give isomorphic extensions, and so we have our function

$$\varphi_V : F(V) \to V \otimes_k \mathrm{Ext}^1_{\mathcal{O}_{X_0}}(\Omega_{X_0}, \mathcal{O}_{X_0}).$$

Step 2. We construct a function

$$\psi_V : V \otimes_k \mathrm{Ext}^1_{\mathcal{O}_{X_0}}(\Omega_{X_0}, \mathcal{O}_{X_0}) \to F(V)$$

in the other direction. We start then with an element of $\mathrm{Ext}^1_{\mathcal{O}_{X_0}}(\Omega_{X_0}, V \otimes_k \mathcal{O}_{X_0})$, represented by an extension

$$0 \longrightarrow V \otimes_k \mathcal{O}_{X_0} \longrightarrow E \xrightarrow{f} \Omega_{X_0} \longrightarrow 0$$

of $\mathcal{O}_{X_0}$-modules. We define then a sheaf of k-vector spaces $\mathcal{O}(E)$ by

$$\mathcal{O}(E) = \mathcal{O}_{X_0} \times_{\Omega_{X_0}} E \subseteq \mathcal{O}_{X_0} \oplus E$$

where the morphism $\mathcal{O}_{X_0} \to \Omega_{X_0}$ is the universal derivation d_0. The sheaf $\mathcal{O}(E)$ fits in the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & V \otimes_k \mathcal{O}_{X_0} \arrow[r] \arrow[d, equal] & \mathcal{O}(E) \arrow[r] \arrow[d] & \mathcal{O}_{X_0} \arrow[r] \arrow[d, "d_0"] & 0 \\
0 \arrow[r] & V \otimes_k \mathcal{O}_{X_0} \arrow[r] & E \arrow[r, "f"] & \Omega_{X_0} \arrow[r] & 0.
\end{tikzcd}$$

From Proposition C.11 we see that $\mathcal{O}(E)$ has a natural structure of sheaf of (flat) $k[V]$-modules (coming from the first row).

We check now that it is a sheaf of subrings of $\mathcal{O}_{X_0} \oplus E$, where the product here is defined by $(s_1, e_1)(s_2, e_2) = (s_1 s_2, s_1 e_2 + s_2 e_1)$. Recall that $\mathcal{O}(E) = \mathcal{O}_{X_0} \times_{\Omega_{X_0}} E \subseteq \mathcal{O}_{X_0} \oplus E$ is by definition the submodule of the elements $s + e$ with s is a section of $\mathcal{O}_{X_0}$ and e one of E, such that $d_0(s) = f(e)$ as sections of Ω_{X_0}.

First, the identity $1 + 0$ of $\mathcal{O}_{X_0} \oplus E$ is in $\mathcal{O}(E)$, because $d_0(1) = 0 = f(0)$. Moreover if $s_1 + e_1, s_2 + e_2$ are sections of $\mathcal{O}(E)$, then $(s_1 + e_1)(s_2 + e_2) = s_1 s_2 + s_1 e_2 + s_2 e_2$ and

$$d(s_1 s_2) = s_1 d(s_2) + s_2 d(s_1) = s_1 f(e_2) + s_2 f(e_1) = f(s_1 e_2 + s_2 f_1)$$

because d is a derivation, $d(s_1) = f(e_1)$, $d(s_2) = f(e_2)$, and f is $\mathcal{O}_{X_0}$-linear respectively. So $(s_1 + e_1)(s_2 + e_2)$ is a section of $\mathcal{O}(E)$ as well. Stability under sum and multiplication by elements of k is clear. It is also immediate to check that the defined product is compatible with the structure of $k[V]$-module.

Then $\mathcal{O}(E)$ is a sheaf of flat $k[V]$-algebras on the topological space $|X_0|$. It is easy to see that its stalks are local rings, so that $X(E) = (|X_0|, \mathcal{O}(E))$ is a locally ringed space, and moreover it is a scheme (flat over $k[V]$). This is simply because if $U = \mathrm{Spec}(A)$ is an open affine subscheme of X_0, then $(U, \mathcal{O}(E)|_U)$ is isomorphic to $\mathrm{Spec}(A \times_{\Omega_A} E(U))$.

Since $\mathcal{O}(E) \otimes_{k[V]} k \cong \mathcal{O}_{X_0}$ we see also that $X(E) \times_{\mathrm{Spec}(k[V])} \mathrm{Spec}(k) \cong X_0$. Furthermore, $X(E)$ is quasi compact because X_0 is, and from the exact sequence

$$0 \longrightarrow V \otimes_k \mathcal{O}_{X_0} \longrightarrow \mathcal{O}(E) \longrightarrow \mathcal{O}_{X_0} \longrightarrow 0$$

we see that $X(E)$ is of finite type over $k[V]$.

Suppose $U = \mathrm{Spec}(A)$ is an open affine subscheme of X_0, such that A is a finitely-generated k-algebra; call $x_1, \ldots, x_n$ a set of generators. Then taking cohomology (and observing that $H^1(U, V \otimes_k \mathcal{O}_{X_0}) = 0$ since U is affine and $V \otimes_k \mathcal{O}_{X_0}$ is quasi-coherent) we get the exact sequence

$$0 \longrightarrow \Gamma(U, V \otimes_k \mathcal{O}_{X_0}) \xrightarrow{i} \Gamma(U, \mathcal{O}(E)) \xrightarrow{g} \Gamma(U, \mathcal{O}_{X_0}) = A \longrightarrow 0.$$

Take liftings $y_1, \ldots, y_n \in \Gamma(U, \mathcal{O}(E))$ of $x_1, \ldots, x_n$, and the $k[V]$-subalgebra B they generate; we see that B is the whole $\Gamma(U, \mathcal{O}(E))$. If $x \in \Gamma(U, \mathcal{O}(E))$, then $g(x) \in A$ is $p(x_1, \ldots, x_n)$ for a polynomial $p \in k[z_1, \ldots, z_n]$. Then $a - p(y_1, \ldots, y_n) \in \ker(g)$, and so we have polynomials $p_1, \ldots, p_r \in k[z_1, \ldots, z_n]$ (where r is the dimension of V) such that

$$a - p(y_1, \ldots, y_n) = i(v_1 \otimes p_1(x_1, \ldots, x_n) + \cdots + v_r \otimes p_r(x_1, \cdots, x_n))$$

where $v_1, \ldots, v_r$ is a basis of V. Since

$$i(v_i \otimes p_i(x_1, \ldots, x_n)) = v_i \cdot p_i(y_1, \ldots, y_n) \in \Gamma(U, \mathcal{O}(E))$$

we have written a as a polynomial $P(y_1, \ldots, y_n)$ with coefficients in $k[V]$, and this proves that a is an element of B, and our claim.

Finally, noticing that this construction is independent (up to isomorphism) of the representative chosen for the element of $\mathrm{Ext}^1_{\mathcal{O}_{X_0}}(\Omega_{X_0}, V \otimes_k \mathcal{O}_{X_0})$, we get a function

$$\psi_V : V \otimes_k \mathrm{Ext}^1_{\mathcal{O}_{X_0}}(\Omega_{X_0}, \mathcal{O}_{X_0}) \to F(V).$$

Step 3. We show that φ_V and ψ_V are inverse to each other. Given an object $X \in \widetilde{\mathcal{D}ef}_{X_0}(k[V])$, we have the associated extension

$$0 \longrightarrow V \otimes_k \mathcal{O}_{X_0} \longrightarrow \Omega_X|_{X_0} \longrightarrow \Omega_{X_0} \longrightarrow 0$$

of $\mathcal{O}_{X_0}$-modules, and we have to show that the scheme we get from this one is isomorphic to X over $k[V]$.

We have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & V \otimes_k \mathcal{O}_{X_0} \arrow[r] \arrow[d, equal] & \mathcal{O}_X \arrow[r] \arrow[d, "d"] & \mathcal{O}_{X_0} \arrow[r] \arrow[d, "d_0"] & 0 \\
0 \arrow[r] & V \otimes_k \mathcal{O}_{X_0} \arrow[r] & \Omega_X|_{X_0} \arrow[r, "f"] & \Omega_{X_0} \arrow[r] & 0
\end{tikzcd}$$

where d and d_0 are induced by the universal derivations, and the top row is the exact sequence associated with the closed immersion $X_0 \subseteq X$. The induced map $\mathcal{O}_X \to \mathcal{O}(\Omega_X|_{X_0}) = \mathcal{O}_{X_0} \times_{\Omega_{X_0}} \Omega_X|_{X_0}$ is an isomorphism of sheaves of k-vector spaces, and again by Proposition C.11, also of sheaves of $k[V]$-vector spaces. Furthermore it is immediate to see that it is actually an isomorphism of sheaves of $k[V]$-algebras, and this shows that $X(\Omega_X|_{X_0})$ is isomorphic to X over $k[V]$.

Conversely, if we start from an extension of $\mathcal{O}_{X_0}$-modules

$$0 \longrightarrow V \otimes_k \mathcal{O}_{X_0} \longrightarrow E \longrightarrow \Omega_{X_0} \longrightarrow 0$$

we have to show that the conormal extension

$$0 \longrightarrow V \otimes_k \mathcal{O}_{X_0} \longrightarrow \Omega_{X(E)}|_{X_0} \longrightarrow \Omega_{X_0} \longrightarrow 0$$

is isomorphic to the one above. Consider the second projection $\pi_2 : \mathcal{O}(E) \to E$; it is a k-derivation, because if a is a section of $\mathcal{O}_{X_0}$ coming from k clearly $\pi_2(a) = 0$ (since a corresponds to the section $(a, 0)$ of $\mathcal{O}_{X_0} \oplus E$), additivity is obvious, and if $x_1 + e_1$ and $x_2 + e_2$ are sections of $\mathcal{O}(E)$, then

$$\begin{aligned}
\pi_2((x_1+e_1)(x_2+e_2)) &= x_1e_2 + x_2e_1 = (x_1+e_1)e_2 + (x_2+e_2)e_1 \\
&= (x_1+e_1)\pi_2(x_2+e_2) + (x_2+e_2)\pi_2(x_1+e_1).
\end{aligned}$$

Then we have an induced $\mathcal{O}_{X(E)}$-linear homomorphism $\Omega_{X(E)} \to E$ such that a section of the form $d(x+e)$ of $\Omega_{X(E)}$ goes to the section e of E. This in turn gives an $\mathcal{O}_{X_0}$-linear homomorphism $f : \Omega_{X(E)}|_{X_0} \to E$ that fits into a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & V \otimes_k \mathcal{O}_{X_0} \arrow[r] \arrow[d, equal] & \Omega_{X(E)}|_{X_0} \arrow[r] \arrow[d, "f"] & \Omega_{X_0} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & V \otimes_k \mathcal{O}_{X_0} \arrow[r] & E \arrow[r] & \Omega_{X_0} \arrow[r] & 0
\end{tikzcd}$$

and this (by the five Lemma) gives an isomorphism of extensions, as claimed.

Step 4. We show that φ_V is functorial in V. In other words, given a k-linear map $f : V \to W$, the diagram

$$\begin{tikzcd}
F(V) \arrow[r, "\varphi_V"] \arrow[d, "(\mathrm{id}\oplus f)_*"'] & V \otimes_k \operatorname{Ext}^1_{\mathcal{O}_{X_0}}(\Omega_{X_0}, \mathcal{O}_{X_0}) \arrow[d, "f\otimes\mathrm{id}"] \\
F(W) \arrow[r, "\varphi_W"] & W \otimes_k \operatorname{Ext}^1_{\mathcal{O}_{X_0}}(\Omega_{X_0}, \mathcal{O}_{X_0})
\end{tikzcd}$$

is commutative.

This is almost immediate from the functoriality of the conormal exact sequence: if X is an object of $\widetilde{\mathcal{D}ef}_{X_0}(k[V])$, and X' is the pullback of X to $k[W]$, we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & V \otimes_k \mathcal{O}_{X_0} \arrow[r] \arrow[d, "f\otimes\mathrm{id}"'] & \Omega_X|_{X_0} \arrow[r] \arrow[d] & \Omega_{X_0} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & W \otimes_k \mathcal{O}_{X_0} \arrow[r] & \Omega_{X'}|_{X_0} \arrow[r] & \Omega_{X_0} \arrow[r] & 0 \end{tikzcd} \tag{2.2}$$

where the map $\Omega_X|_{X_0} \to \Omega_{X'}|_{X_0}$ is induced by the natural morphism $X' \to X$.

On the other hand the image of the extension

$$0 \longrightarrow V \otimes_k \mathcal{O}_{X_0} \longrightarrow \Omega_X|_{X_0} \longrightarrow \Omega_{X_0} \longrightarrow 0$$

in $W \otimes_k \mathrm{Ext}^1_{\mathcal{O}_{X_0}}(\Omega_{X_0}, \mathcal{O}_{X_0})$ is the second row of

$$\begin{tikzcd} 0 \arrow[r] & V \otimes_k \mathcal{O}_{X_0} \arrow[r] \arrow[d, "f\otimes\mathrm{id}"'] & \Omega_X|_{X_0} \arrow[r] \arrow[d] & \Omega_{X_0} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & W \otimes_k \mathcal{O}_{X_0} \arrow[r] & E \arrow[r] & \Omega_{X_0} \arrow[r] & 0 \end{tikzcd}$$

where E is the pushout of the following diagram.

$$\begin{tikzcd} V \otimes_k \mathcal{O}_{X_0} \arrow[r] \arrow[d, "f\otimes\mathrm{id}"'] & \Omega_X|_{X_0} \\ W \otimes_k \mathcal{O}_{X_0} & \end{tikzcd}$$

But 2.2 implies (by the five Lemma, as usual) that this "pushout extension" is isomorphic to the one associated with the deformation X' over $k[W]$

$$0 \longrightarrow W \otimes_k \mathcal{O}_{X_0} \longrightarrow \Omega_{X'}|_{X_0} \longrightarrow \Omega_{X_0} \longrightarrow 0$$

so we have the functoriality of φ_V, and this concludes our proof. □

Remark 2.4.2. If X_0 is of finite type over k, it is easy to see that every deformation of X_0 over $A \in (\mathrm{Art}/\Lambda)$, say X, will also be of finite type over A. This follows from the same arguments used to show that $X(E)$ was of finite type in the preceding proof, starting from the exact sequence

$$0 \longrightarrow \mathfrak{m}_A \otimes_k \mathcal{O}_{X_0} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{X_0} \longrightarrow 0$$

where $\mathfrak{m}_A \otimes_k \mathcal{O}_{X_0}$ is seen as ideal sheaf of X_0 in X. In particular if X_0 is of finite type over k the deformation categories $\mathcal{D}ef_{X_0}$ and $\widetilde{\mathcal{D}ef}_{X_0}$ are the same, so from now on we will only consider $\mathcal{D}ef_{X_0}$ in this case.

From this we see that if X_0 is reduced, generically smooth and of finite type over k, then we also have

$$T_{X_0}\mathcal{D}ef \cong \mathrm{Ext}^1_{\mathcal{O}_{X_0}}(\Omega_{X_0}, \mathcal{O}_{X_0}).$$

Given $\xi \in \mathcal{F}_{\xi_0}(k[\varepsilon])$, the element of $\mathrm{Ext}^1_{\mathcal{O}_{X_0}}(\Omega_{X_0}, \mathcal{O}_{X_0})$ associated with ξ is sometimes called its **Kodaira-Spencer class**, from the names of the two mathematicians who first studied this type of problems in the case of complex varieties.

Remark 2.4.3. If X_0 is also smooth over k, then the tangent space $T_{X_0}\mathcal{D}ef$ is isomorphic to $\mathrm{Ext}^1_{\mathcal{O}_{X_0}}(\Omega_{X_0}, \mathcal{O}_{X_0}) \cong H^1(X_0, T_{X_0})$, where $T_{X_0} = \Omega^\vee_{X_0} = \mathcal{H}om(\Omega_{X_0}, \mathcal{O}_{X_0})$ is the tangent sheaf of X_0.

In particular we see that every first-order deformation of a smooth and affine variety X_0 is trivial, because in this case $H^1(X_0, T_{X_0})$ vanishes. So smooth affine varieties are rigid.

Remark 2.4.4. In the general case, in which X_0 is not necessarily reduced and generically smooth, one has to resort to the cotangent complex $L_{X_0/k}$ associated with the structure morphism $X_0 \to \mathrm{Spec}(k)$; the general result, which can be found in [Ill] (III, 2.1.7), states that there is a canonical isomorphism

$$T_{X_0}\mathcal{D}ef \cong \mathrm{Ext}^1_{\mathcal{O}_{X_0}}(L_{X_0/k}, \mathcal{O}_{X_0})$$

and implies Theorem 2.4.1, since if X_0 is reduced and generically smooth the cotangent complex is just the sheaf Ω_{X_0}.

2.4.2 Smooth varieties

Now suppose X_0 is a smooth variety over k. We describe the bijection

$$T_{X_0}\mathcal{D}ef \cong H^1(X_0, T_{X_0})$$

more explicitly in this case, using Čech cohomology. This will be useful in Chapter 4, where we will use this description to give an example of an obstructed variety.

Consider an object $X \in \mathcal{D}ef_{X_0}(k[\varepsilon])$ and take an open affine cover $\mathcal{U} = \{U_i\}_{i\in I}$ of X_0; notice that since X_0 is separated, every finite intersection of elements of this cover will be affine again. Because of Remark 2.4.3 the induced deformation $X|_{U_i}$ of U_i is trivial for every index i, and from this we get a collection $\{\theta_i\}_{i\in I}$ of isomorphisms of deformations

$$\theta_i : U_i \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k[\varepsilon]) \to X|_{U_i}.$$

Now put $\theta_{ij} = \theta_i^{-1}\theta_j$; these are automorphisms of the trivial deformations $U_{ij} \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k[\varepsilon])$ that restrict to the identity on the closed fiber U_{ij}.

It is an easy consequence of Corollary 3.1.4 and Proposition 3.2.1 below that there is an isomorphism between the group of automorphisms of the deformation $U_{ij} \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k[\varepsilon])$ that induce the identity on the closed fiber, and the group $\mathrm{Der}_k(B_{ij}, B_{ij}) = \Gamma(U_{ij}, T_{X_0})$, where $U_{ij} = \mathrm{Spec}(B_{ij})$.

For each θ_{ij} we get then an associated element $d_{ij} \in \Gamma(U_{ij}, T_{X_0})$. Furthermore, on U_{ijk} we have for each triplet of indices the cocycle condition

$$\theta_{ij} \circ \theta_{jk} = \theta_{ik}$$

on automorphisms, which translates into the relation $d_{ij} + d_{jk} - d_{ik} = 0$ (here and from now on when we write relations of this kind, the restriction on the triple intersection is understood). This in turn says that the family $\{d_{ij}\}_{i,j\in I}$ is a Čech 1-cocycle for T_{X_0}, and so defines an element $[\{d_{ij}\}_{i,j\in I}]$ of $\check{H}^1(\mathcal{U}, T_{X_0}) \cong H^1(X_0, T_{X_0})$.

This element does not depend on the cover $\mathcal{U} = \{U_i\}_{i\in I}$. To see this, given another affine cover $\mathcal{V} = \{V_j\}_{j\in J}$ of X_0, it suffices to consider a common affine refinement (for example $\{U_i \cap V_j\}_{(i,j)\in I\times J}$), and restrict the cocycles relative to the two covers to cocycles relative to the common refinement to see that they represent the same class in $H^1(X_0, T_{X_0})$.

Let us check that this construction is invariant under isomorphism. Suppose $Y \in \mathcal{D}ef_{X_0}(k[\varepsilon])$ is another deformation of X_0, with an isomorphism of deformations $F : X \to Y$. Then, writing ν_i and ν_{ij} for the analogues of the θ_i and θ_{ij} relative to the new deformation, the following composite

$$U_i \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k[\varepsilon]) \xrightarrow{\theta_i} X|_{U_i} \xrightarrow{F|_{U_i}} Y|_{U_i} \xrightarrow{\nu_i^{-1}} U_i \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k[\varepsilon])$$

is an automorphism $\alpha_i = \nu_i^{-1} \circ F|_{U_i} \circ \theta_i$ of $U_i \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k[\varepsilon])])$ inducing the identity on the closed fiber, so it defines an element $a_i \in \Gamma(U_i, T_{X_0})$. Since by definition $\nu_i \circ \alpha_i = F|_{U_i} \circ \theta_i$, we get

$$\alpha_i^{-1} \circ \nu_{ij} \circ \alpha_j = (\nu_i \circ \alpha_i)^{-1} \circ (\nu_j \circ \alpha_j) = \theta_i^{-1} \circ F|_{U_{ij}}^{-1} \circ F|_{U_{ij}} \circ \theta_j = \theta_{ij}$$

and in turn this implies $\delta_{ij} + a_j - a_i = d_{ij}$, where δ_{ij} are the elements of $\Gamma(U_{ij}, T_{X_0})$ associated with the automorphisms ν_{ij}. Then $\{d_{ij}\}_{i,j\in I}$ and $\{\delta_{ij}\}_{i,j\in I}$ are cohomologous, and so their class is the same. This gives us a well-defined function

$$T_{X_0}\mathcal{D}ef \to H^1(X_0, T_{X_0})$$

that can be seen to correspond to the one we constructed in the proof of Theorem 2.4.1.

The inverse function is as follows: given an element of $H^1(X_0, T_{X_0})$, we can represent it as a 1-cocycle $\{d_{ij}\}_{i,j\in I}$ for some open affine cover

$\mathcal{U} = \{U_i\}_{i\in I}$ of X_0. The d_{ij} correspond to automorphisms of the trivial deformation $U_{ij} \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k[\varepsilon])$, and the cocycle condition says exactly that these automorphisms can be used to glue the schemes $U_i \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k[\varepsilon])$ along the subschemes $U_{ij} \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k[\varepsilon])$, to get a flat scheme X over $k[\varepsilon]$. It is easy to see that this construction does not depend (up to isomorphism) on the affine cover, and on the cocycle we choose in the cohomology class. Finally it is clear that the two constructions are inverse to each other, so we have the bijection above.

2.4.3 Closed subschemes

Next we consider the case of deformations of closed subschemes. Given an object of $\mathcal{H}ilb^X(k)$, i.e. a closed subscheme $Z_0 \subseteq X_0 = X \times_{\mathrm{Spec}(\Lambda)} \mathrm{Spec}(k)$, call I_0 the ideal sheaf of Z_0 in X_0, and consider the normal sheaf $\mathcal{N}_0 = \mathcal{H}om(I_0/I_0^2, \mathcal{O}_{Z_0})$.

Theorem 2.4.5. *There is an isomorphism*

$$T_{Z_0}\mathcal{H}ilb^X \cong H^0(Z_0, \mathcal{N}_0) \cong \mathrm{Hom}_{\mathcal{O}_{Z_0}}(I_0/I_0^2, \mathcal{O}_{Z_0}).$$

Proof. We consider the functor $F : (\mathrm{FVect}\,/k) \to (\mathrm{Set})$ defined on objects by

$$F(V) = \{\text{objects in } \mathcal{H}ilb^X_{Z_0}(k[V])\}$$

and sending a k-linear map $V \to W$ to the associated pullback function $F(V) \to F(W)$. We will construct a functorial bijection

$$F(V) \cong V \otimes_k \mathrm{Hom}_{\mathcal{O}_{X_0}}(I_0, \mathcal{O}_{Z_0})$$

that will give a k-linear natural transformation, and in particular an isomorphism

$$T_{Z_0}\mathcal{H}ilb^X = F(k) \cong \mathrm{Hom}_{\mathcal{O}_{X_0}}(I_0, \mathcal{O}_{Z_0})$$

(notice that $\mathrm{Hom}_{\mathcal{O}_{X_0}}(I_0, \mathcal{O}_{Z_0}) \cong \mathrm{Hom}_{\mathcal{O}_{Z_0}}(I_0/I_0^2, \mathcal{O}_{Z_0})$). We divide the proof in steps.

Step 1. We define a function

$$\varphi_V : F(V) \to V \otimes_k \mathrm{Hom}_{\mathcal{O}_{X_0}}(I_0, \mathcal{O}_{Z_0}).$$

Take an object $Z \in \mathcal{H}ilb^X_{Z_0}(k[V])$, that is, a closed subscheme $Z \subseteq X_V$, where $X_V = X_0 \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k[V])$ is the trivial deformation of X_0 over $k[V]$, and restricting to Z_0 over k; call $I \subseteq \mathcal{O}_{X_V}$ its sheaf of ideals.

Starting as usual from the exact sequence of $k[V]$-modules

$$0 \longrightarrow V \longrightarrow k[V] \longrightarrow k \longrightarrow 0$$

and tensoring it with $\mathcal{O}_Z$ and $\mathcal{O}_{X_V}$, using flatness we get two exact sequences

$$0 \longrightarrow V \otimes_k \mathcal{O}_{X_0} \longrightarrow \mathcal{O}_{X_V} \longrightarrow \mathcal{O}_{X_0} \longrightarrow 0$$

and

$$0 \longrightarrow V \otimes_k \mathcal{O}_{Z_0} \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{O}_{Z_0} \longrightarrow 0$$

of $\mathcal{O}_{X_V}$-modules and $\mathcal{O}_Z$-modules respectively. Moreover, tensoring

$$0 \longrightarrow I \longrightarrow \mathcal{O}_{X_V} \longrightarrow \mathcal{O}_Z \longrightarrow 0$$

with k, by flatness of Z we get

$$0 \longrightarrow I \otimes_{k[V]} k \longrightarrow \mathcal{O}_{X_0} \longrightarrow \mathcal{O}_{Z_0} \longrightarrow 0$$

and from this we see that $I \otimes_{k[V]} k$ can be identified with I_0, the sheaf of ideals of $Z_0 \subseteq X_0$.

These four exact sequences fit into the following commutative diagram of $\mathcal{O}_{X_V}$-modules.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & I \arrow[r] \arrow[d] & I_0 \arrow[r] \arrow[d, "i"] & 0 \\
0 \arrow[r] & V \otimes_k \mathcal{O}_{X_0} \arrow[r] \arrow[d] & \mathcal{O}_{X_V} \arrow[r, "p_X"'] \arrow[d, "q"'] & \mathcal{O}_{X_0} \arrow[r] \arrow[d, "q_0"] \arrow[l, bend right, "\sigma"'] & 0 \\
0 \arrow[r] & V \otimes_k \mathcal{O}_{Z_0} \arrow[r] & \mathcal{O}_Z \arrow[r, "p_Z"'] \arrow[d] & \mathcal{O}_{Z_0} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd} \tag{2.3}$$

Since

$$O_{X_V} \cong \mathcal{O}_{X_0} \otimes_k k[V] \cong \mathcal{O}_{X_0} \oplus (V \otimes_k \mathcal{O}_{X_0})$$

as an $\mathcal{O}_{X_0}$-module, the map p_X has an $\mathcal{O}_{X_0}$-linear section, which we call σ, simply defined by $\sigma(s) = (s, 0)$, where s is a section of $\mathcal{O}_{X_0}$.

The composite

$$f : I_0 \xrightarrow{i} \mathcal{O}_{X_0} \xrightarrow{\sigma} \mathcal{O}_{X_V} \xrightarrow{q} \mathcal{O}_Z$$

factors through $V \otimes_k \mathcal{O}_{Z_0} \to \mathcal{O}_Z$, because

$$p_Z \circ f = (p_Z \circ q) \circ \sigma \circ i = q_0 \circ (p_X \circ \sigma) \circ i = q_0 \circ i = 0.$$

So we have an $\mathcal{O}_{X_0}$-linear morphism $I_0 \to V \otimes_k \mathcal{O}_{Z_0}$, which is then an element of $\operatorname{Hom}_{\mathcal{O}_{X_0}}(I_0, V \otimes_k \mathcal{O}_{Z_0}) \cong V \otimes_k \operatorname{Hom}_{\mathcal{O}_{X_0}}(I_0, \mathcal{O}_{Z_0})$, and this gives us a function

$$F(V) \to V \otimes_k \operatorname{Hom}_{\mathcal{O}_{X_0}}(I_0, \mathcal{O}_{Z_0})$$

that we call φ_V.

Step 2. We construct a function in the other direction. Take a homomorphism of $\mathcal{O}_{X_0}$-modules $f : I_0 \to V \otimes_k \mathcal{O}_{Z_0}$, and consider the subsheaf I_f of $\mathcal{O}_{X_V} \cong \mathcal{O}_{X_0} \oplus (V \otimes_k \mathcal{O}_{X_0})$ given on an open set U of $|X_V|$ by

$$I_f(U) = \{(s,t) \in \mathcal{O}_{X_V}(U) : s \in I_0(U) \subseteq \mathcal{O}_{X_0}(U) \text{ and } f(s)+(\mathrm{id} \otimes q_0)(t) = 0\}.$$

where $q_0 : \mathcal{O}_{X_0} \to \mathcal{O}_{Z_0}$ is the quotient map.

An easy verification shows that I_f is a sheaf of ideals of $\mathcal{O}_{X_V}$, and it is clearly coherent, begin the kernel of the homomorphism

$$f \circ \pi_1 + (\mathrm{id} \otimes q_0) \circ \pi_2 : \mathcal{O}_{X_V} \to V \otimes_k \mathcal{O}_{Z_0}.$$

between two quasi-coherent sheaves. So I_f defines a closed subscheme of X_V that we call $Z_f \subseteq X_V$.

We see that

$$Z_f \times_{\mathrm{Spec}(k[V])} \mathrm{Spec}(k) \subseteq X_V \times_{\mathrm{Spec}(k[V])} \mathrm{Spec}(k) \cong X_0$$

is the closed subscheme Z_0: this follows simply from the fact that the sheaf of ideals of $Z_f \times_{\mathrm{Spec}(k[V])} \mathrm{Spec}(k)$ in X_0 is $I_f \otimes_{k[V]} k \cong I_0$.

Furthermore, we see that Z_f is flat over $k[V]$. Using the local flatness criterion, we have to check that $\mathrm{Tor}_1^{k[V]}(\mathcal{O}_Z, k) = 0$. We have an exact sequence of $\mathcal{O}_{X_V}$-modules

$$0 \longrightarrow I_f \longrightarrow \mathcal{O}_{X_V} \longrightarrow \mathcal{O}_Z \longrightarrow 0$$

from which, taking the Tor exact sequence (tensoring with k), we get

$$\mathrm{Tor}_1^{k[V]}(\mathcal{O}_{X_V}, k) \longrightarrow \mathrm{Tor}_1^{k[V]}(\mathcal{O}_Z, k) \longrightarrow I_f \otimes_{k[V]} k \longrightarrow \mathcal{O}_{X_0} \longrightarrow \mathcal{O}_{Z_0} \longrightarrow 0.$$

Since X_V is flat over $k[V]$ we have $\mathrm{Tor}_1^{k[V]}(\mathcal{O}_{X_V}, k) = 0$, so we only need to show that the map $I_f \otimes_{k[V]} k \to I_0 \subseteq \mathcal{O}_{X_0}$ is injective, and this is clear (it is the isomorphism already used above).

This gives us a function

$$V \otimes_k \mathrm{Hom}_{\mathcal{O}_{X_0}}(I_0, \mathcal{O}_{Z_0}) \to F(V)$$

that we call ψ_V.

Step 3. We show that φ_V and ψ_V are inverse to each other. Starting from an object $Z \in \mathcal{H}ilb^X_{Z_0}(k[V])$, with sheaf of ideals I in X_V, we have the $\mathcal{O}_{X_0}$-linear homomorphism $f = \varphi_V(Z) : I_0 \to V \otimes_k \mathcal{O}_{Z_0}$, and we have to show that $I_f = I$.

If s is a section of $I \subseteq \mathcal{O}_{X_V} \cong \mathcal{O}_{X_0} \oplus (V \otimes_k \mathcal{O}_{X_0})$, write $s = s_0 + t$ in this decomposition, where s_0 is a section of I_0 and t one of $V \otimes_k \mathcal{O}_{X_0}$. Then $f(s_0) = q(\sigma(s_0))$, and

$$0 = q(s) = q(s_0 + t) = q(\sigma(s_0)) + q(t) = f(s_0) + (\mathrm{id} \otimes q_0)(t).$$

Then $s_0 + t$ is a section of I_f, and $I \subseteq I_f$.

Conversely, take a section (s,t) of I_f, which we see also as the section $s+t$ of $\mathcal{O}_{X_V}$. Then

$$q(s+t) = q(\sigma(s)) + q(t) = f(s) + (\mathrm{id} \otimes q_0)(t) = 0$$

and this implies that $s+t$ is a section of I, so $I_f \subseteq I$. In conclusion $I = I_f$, and so $Z = Z_f$.

The other half of the claim follows from straightforward diagram chasing, using 2.3.

Step 4. Finally we see that φ_V is functorial in V. This is also immediate: given a k-linear map $f : V \to W$, it amounts to the commutativity of the diagram

$$\begin{tikzcd}
\mathcal{H}ilb^X_{Z_0}(k[V]) \arrow[r, "\varphi_V"] \arrow[d, "(\mathrm{id}\oplus f)_*"'] & V \otimes_k \mathrm{Hom}_{\mathcal{O}_{X_0}}(I_0, \mathcal{O}_{Z_0}) \arrow[d, "f\otimes\mathrm{id}"] \\
\mathcal{H}ilb^X_{Z_0}(k[W]) \arrow[r, "\varphi_V"] & W \otimes_k \mathrm{Hom}_{\mathcal{O}_{X_0}}(I_0, \mathcal{O}_{Z_0}).
\end{tikzcd}$$

The map $\mathrm{Hom}_{\mathcal{O}_{X_0}}(I_0, V \otimes_k \mathcal{O}_{Z_0}) \to \mathrm{Hom}_{\mathcal{O}_{X_0}}(I_0, W \otimes_k \mathcal{O}_{Z_0})$ corresponding to $f \otimes \mathrm{id}$ above is given by composite with $f \otimes \mathrm{id} : V \otimes_k \mathcal{O}_{Z_0} \to W \otimes_k \mathcal{O}_{Z_0}$ on the left, and it is easy to see (for example adding the rows corresponding to W to diagram 2.3) that, given $Z \in \mathcal{H}ilb^X_{Z_0}(k[V])$, taking the homomorphism $\varphi_V(Z) : I_0 \to V \otimes_k \mathcal{O}_{Z_0}$ and composing with $V \otimes_k \mathcal{O}_{Z_0} \to W \otimes_k \mathcal{O}_{Z_0}$ will precisely give the homomorphism $I_0 \to W \otimes_k \mathcal{O}_{Z_0}$ associated with the pullback of Z to $k[W]$. So we have the functoriality of φ_V, and this concludes our proof. □

In the following discussion we suppose also that X is of finite type over Λ (and then any of its closed subschemes is as well). We then have a natural forgetful functor $F : \mathcal{H}ilb^X \to \widetilde{\mathcal{D}ef}$ that sends an object $Y \subseteq X_A$ of $\mathcal{H}ilb^X(A)$ to the flat morphism $Y \to \mathrm{Spec}(A)$, which is an object of $\widetilde{\mathcal{D}ef}(A)$, and acts on the arrows in the obvious way.

The functor F is a morphism of fibered categories, and we are interested in its differential at a closed subscheme $Z_0 \subseteq X_0 = X \times_{\mathrm{Spec}(\Lambda)} \mathrm{Spec}(k)$, with Z_0 reduced and generically smooth.

Since $T_{Z_0}\mathcal{H}ilb^X = \mathrm{Hom}_{\mathcal{O}_{Z_0}}(I_0/I_0^2, \mathcal{O}_{Z_0})$ and $T_{Z_0}\widetilde{\mathcal{D}ef} = \mathrm{Ext}^1_{\mathcal{O}_{Z_0}}(\Omega_{Z_0}, \mathcal{O}_{Z_0})$, the differential of F will correspond to a k-linear map

$$d_{Z_0}F : \mathrm{Hom}_{\mathcal{O}_{Z_0}}(I_0/I_0^2, \mathcal{O}_{Z_0}) \to \mathrm{Ext}^1_{\mathcal{O}_{Z_0}}(\Omega_{Z_0}, \mathcal{O}_{Z_0})$$

that we still call $d_{Z_0}F$.

Lemma 2.4.6. *If Z_0 and X_0 are as above, the conormal exact sequence of the closed immersion $Z_0 \subseteq X_0$*

$$I_0/I_0^2 \xrightarrow{d} \Omega_{X_0}|_{Z_0} \longrightarrow \Omega_{Z_0} \longrightarrow 0$$

gives a coboundary map $\delta : \mathrm{Hom}_{\mathcal{O}_{Z_0}}(I_0/I_0^2, \mathcal{O}_{Z_0}) \to \mathrm{Ext}^1_{\mathcal{O}_{Z_0}}(\Omega_{Z_0}, \mathcal{O}_{Z_0})$.

Proof. First, it is well known that d is injective where Z_0 is smooth. Since it is generically smooth, if we let $\mathcal{K} = \ker(d)$, then $\mathrm{supp}(\mathcal{K})$ can't contain any irreducible component of Z_0. It follows that $\mathcal{H}om_{\mathcal{O}_{Z_0}}(\mathcal{K}, \mathcal{O}_{Z_0}) = 0$, and from this we get

$$\mathrm{Hom}_{\mathcal{O}_{Z_0}}(I_0/I_0^2, \mathcal{O}_{Z_0}) \cong \mathrm{Hom}_{\mathcal{O}_{Z_0}}((I_0/I_0^2)/\mathcal{K}, \mathcal{O}_{Z_0}).$$

Then from this isomorphism and the induced exact sequence

$$0 \longrightarrow (I_0/I_0^2)/\mathcal{K} \longrightarrow \Omega_{X_0}|_{Z_0} \longrightarrow \Omega_{Z_0} \longrightarrow 0$$

taking the long Ext exact sequence we get our coboundary

$$\delta : \mathrm{Hom}_{\mathcal{O}_{Z_0}}(I_0/I_0^2, \mathcal{O}_{Z_0}) \cong \mathrm{Hom}_{\mathcal{O}_{Z_0}}((I_0/I_0^2)/\mathcal{K}, \mathcal{O}_{Z_0}) \to \mathrm{Ext}^1_{\mathcal{O}_{Z_0}}(\Omega_{Z_0}, \mathcal{O}_{Z_0}).$$

□

Proposition 2.4.7. *The differential $d_{Z_0}F$ of the forgetful functor F coincides with the homomorphism δ of Lemma 2.4.6.*

Proof. We have to show that the following square (where the vertical functions are the isomorphisms we described in the proofs of Theorems 2.4.1 and 2.4.5) is commutative.

$$\begin{tikzcd}
T_{Z_0}\mathcal{H}ilb^X \arrow[r, "d_{Z_0}F"] \arrow[d, "\wr"] & T_{Z_0}\widetilde{\mathcal{D}ef} \arrow[d, "\wr"] \\
\mathrm{Hom}_{\mathcal{O}_{Z_0}}(I_0/I_0^2, \mathcal{O}_{Z_0}) \arrow[r, "\delta"] & \mathrm{Ext}^1_{\mathcal{O}_{Z_0}}(\Omega_{Z_0}, \mathcal{O}_{Z_0}).
\end{tikzcd}$$

By possibly replacing I_0/I_0^2 with the quotient $(I_0/I_0^2)/\mathcal{K}$, we can suppose that the conormal sequence of $Z_0 \subseteq X_0$

$$0 \longrightarrow I_0/I_0^2 \longrightarrow \Omega_{X_0}|_{Z_0} \longrightarrow \Omega_{Z_0} \longrightarrow 0$$

is exact. Then the coboundary δ sends a homomorphism $f : I_0/I_0^2 \to (\varepsilon) \otimes_k \mathcal{O}_{Z_0}$ to the "pushout extension", the bottom row of the diagram

$$\begin{tikzcd}
0 \arrow[r] & I_0/I_0^2 \arrow[r] \arrow[d, "f"] & \Omega_{X_0}|_{Z_0} \arrow[r] \arrow[d] & \Omega_{Z_0} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & (\varepsilon) \otimes_k \mathcal{O}_{Z_0} \arrow[r] & E \arrow[r] & \Omega_{Z_0} \arrow[r] & 0
\end{tikzcd}$$

where E is the pushout of

$$\begin{tikzcd} I_0/I_0^2 \arrow[r] \arrow[d, "f"'] & \Omega_{X_0}|_{Z_0} \\ (\varepsilon)\otimes_k \mathcal{O}_{Z_0}. & \end{tikzcd}$$

We have to show that the extension we get by taking $f = f_Z$, the homomorphism associated with an object $Z \in \mathcal{H}ilb^X_{Z_0}(k[\varepsilon])$, is (isomorphic to) the extension

$$\begin{tikzcd} 0 \arrow[r] & (\varepsilon)\otimes_k \mathcal{O}_{Z_0} \arrow[r] & \Omega_Z|_{Z_0} \arrow[r] & \Omega_{Z_0} \arrow[r] & 0 \end{tikzcd}$$

corresponding to $Z \in \widetilde{\mathcal{D}ef}_{Z_0}(k[\varepsilon])$.

We now notice that the section $\mathcal{O}_{X_0} \to \mathcal{O}_{X_\varepsilon} \cong \mathcal{O}_{X_0} \oplus ((\varepsilon)\otimes_k \mathcal{O}_{X_0})$ (where X_ε is the trivial deformation $X_0 \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k[\varepsilon])$) used in the proof of Theorem 2.4.5 induces a section $\Omega_{X_0} \to \Omega_{X_\varepsilon}|_{X_0}$ of the homomorphism of $\mathcal{O}_{X_0}$-modules $\Omega_{X_\varepsilon}|_{X_0} \to \Omega_{X_0}$, which is part of the conormal sequence of $X_0 \subseteq X_\varepsilon$. This section induces by pullback an $\mathcal{O}_{Z_0}$-linear $g : \Omega_{X_0}|_{Z_0} \to \Omega_{X_\varepsilon}|_{Z_0}$.

Moreover the inclusion $Z \subseteq X_\varepsilon$ gives $\Omega_{X_\varepsilon}|_Z \to \Omega_Z$, which we can pullback to Z_0, and get another $\mathcal{O}_{Z_0}$-linear homomorphism $h : \Omega_{X_\varepsilon}|_{Z_0} \to \Omega_Z|_{Z_0}$.

The composite $h \circ g$ is an $\mathcal{O}_{Z_0}$-linear homomorphism $\Omega_{X_0}|_{Z_0} \to \Omega_Z|_{Z_0}$ that fits into a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & I_0/I_0^2 \arrow[r] \arrow[d, "f"'] & \Omega_{X_0}|_{Z_0} \arrow[r] \arrow[d, "h\circ g"] & \Omega_{Z_0} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & (\varepsilon)\otimes_k \mathcal{O}_{Z_0} \arrow[r] & \Omega_Z|_{Z_0} \arrow[r] & \Omega_{Z_0} \arrow[r] & 0. \end{tikzcd} \tag{2.4}$$

Commutativity of the right square follows at once from the one of the following square

$$\begin{tikzcd} \Omega_{X_\varepsilon}|_{X_0} \arrow[r] \arrow[d] & \Omega_{X_0} \arrow[d] \arrow[l, bend right] \\ \Omega_Z|_{Z_0} \arrow[r] & \Omega_{Z_0} \end{tikzcd}$$

where the horizontal maps are induced by the closed immersions $X_0 \subseteq X_\varepsilon$ and $Z_0 \subseteq Z$, the vertical ones by $Z \subseteq X_\varepsilon$ and $Z_0 \subseteq X_0$, and the map $\Omega_{X_0} \to \Omega_{X_\varepsilon}|_{X_0}$ is the section mentioned above.

Commutativity of the left square follows from the fact that $f = f_Z$ was defined (in the proof of Theorem 2.4.5) using the section $\mathcal{O}_{X_0} \to \mathcal{O}_{X_\varepsilon}$, which we used also to define $h \circ g$ (by taking the one induced on the shaves of differentials).

Finally we notice that 2.4 implies that the pushout extension above is isomorphic to

$$0 \longrightarrow (\varepsilon) \otimes_k \mathcal{O}_{Z_0} \longrightarrow \Omega_Z|_{Z_0} \longrightarrow \Omega_{Z_0} \longrightarrow 0$$

and this concludes the proof. □

Remark 2.4.8. If X_0 is smooth and Z_0 is a generically smooth local complete intersection in X_0, we have that the conormal sequence

$$0 \longrightarrow I_0/I_0^2 \xrightarrow{d} \Omega_{X_0}|_{Z_0} \longrightarrow \Omega_{Z_0} \longrightarrow 0$$

is also exact on the left, and all the terms are locally free $\mathcal{O}_{Z_0}$-modules. This is because, if we put $\mathcal{K} = \ker(d)$, since Z_0 is generically smooth $\mathcal{K}$ will be concentrated on a nowhere dense closed subset of Z_0 (because d is injective where Z_0 is smooth); from the facts that I_0/I_0^2 is locally free on Z_0 and that Z_0 is a local complete intersection, so it cannot have embedded points, it follows then that $\mathcal{K} = 0$.

If Z_0 is also smooth, dualizing we get another exact sequence

$$0 \longrightarrow T_{Z_0} \longrightarrow T_{X_0}|_{Z_0} \longrightarrow \mathcal{N}_0 \longrightarrow 0$$

that induces a coboundary map $H^0(Z_0, \mathcal{N}_0) \to H^1(Z_0, T_{Z_0})$. This map corresponds to δ when we identify $H^0(Z_0, \mathcal{N}_0)$ with $\mathrm{Hom}_{\mathcal{O}_{Z_0}}(I_0/I_0^2, \mathcal{O}_{Z_0})$ and $H^1(Z_0, T_{Z_0})$ with $\mathrm{Ext}^1_{\mathcal{O}_{Z_0}}(\Omega_{Z_0}, \mathcal{O}_{Z_0})$.

From now on we will write δ also for the coboundary map $H^0(Z_0, \mathcal{N}_0) \to H^1(Z_0, T_{Z_0})$, when Z_0 and X_0 are smooth.

2.4.4 Hypersurfaces in $\mathbb{A}^n_k$

We study now the case of deformations of hypersurfaces in $\mathbb{A}^n_k$; in particular our aim is to describe explicitly the Kodaira-Spencer correspondence $T_{X_0}\mathcal{D}ef \to \mathrm{Ext}^1_{\mathcal{O}_{X_0}}(\Omega_{X_0}, \mathcal{O}_{X_0})$ of a reduced and generically smooth hypersurface $X_0 \subseteq \mathbb{A}^n_k$.

Suppose $X_0 = \operatorname{Spec} A$ where $A = k[x_0, \dots, x_n]/(f)$, and put $I_0 = (f)$. First of all we calculate $\mathrm{Ext}^1_{\mathcal{O}_{X_0}}(\Omega_{X_0}, \mathcal{O}_{X_0}) \cong \mathrm{Ext}^1_A(\Omega_A, A)$. Consider the conormal exact sequence of the immersion $X_0 \subseteq \mathbb{A}^n_k$

$$0 \longrightarrow I_0/I_0^2 \longrightarrow \Omega_{k[x_1,\dots,x_n]} \otimes_k A \longrightarrow \Omega_A \longrightarrow 0$$

which is also exact on the left, by Remark 2.4.8.

We recall that $\Omega_{k[x_1,\dots,x_n]}$ is a free $k[x_1, \dots, x_n]$-module on n generators that we denote by $dx_1, \dots, dx_n$, and the map $I_0/I_0^2 \to \Omega_{k[x_1,\dots,x_n]} \otimes_k A$ is defined by

$$f \mapsto df = \frac{\partial f}{\partial x_1} dx_1 \otimes 1 + \cdots + \frac{\partial f}{\partial x_n} dx_n \otimes 1$$

and extended by linearity.

Let us apply the functor $\mathrm{Hom}_A(-, A)$ to the sequence above, and take the Ext exact sequence. We get

$$\mathrm{Hom}_A(\Omega_{k[x_1,\dots,x_n]} \otimes_k A, A) \xrightarrow{G} \mathrm{Hom}_A(I_0/I_0^2, A) \longrightarrow \mathrm{Ext}^1_A(\Omega_A, A) \longrightarrow 0$$

where the map $\mathrm{Hom}_A(I_0/I_0^2, A) \to \mathrm{Ext}^1_A(\Omega_A, A)$ is the differential of the forgetful morphism $F : \mathcal{H}ilb^{\mathbb{A}^n_\Lambda} \to \widetilde{\mathcal{D}ef}$ at the object $X_0 \in \mathcal{H}ilb^{\mathbb{A}^n_\Lambda}(k)$. In particular we see that this differential is surjective, or in other words, every deformation of X_0 over algebras of the type $k[V]$ is affine as well.

Noticing that $I_0/I_0^2 \cong A$ and $\Omega_{k[x_1,\dots,x_n]} \otimes_k A \cong A^n$, the map G will correspond to an A-linear function $A^n \to A$ that is given by scalar multiplication by the vector $(\partial f/\partial x_1, \dots, \partial f/\partial x_n)$. The image of G corresponds then to the Jacobian ideal $J = (\partial f/\partial x_1, \dots, \partial f/\partial x_n) \subseteq A$ of X_0, and we have

$$\mathrm{Ext}^1_A(\Omega_A, A) \cong A/J \cong k[x_1, \dots, x_n]/(f, \partial f/\partial x_1, \dots, \partial f/\partial x_n).$$

From this we see that $T_{X_0}\mathcal{D}ef$ is finite-dimensional if and only if X_0 has isolated singularities (for the singular locus is exactly defined by the ideal J in X_0, and $k[x_1, \dots, x_n]/(f, \partial f/\partial x_1, \dots, \partial f/\partial x_n)$ will be finite-dimensional exactly when $V(J)$ is zero-dimensional).

The dimension of $k[x_1, \dots, x_n]/(f, \partial f/\partial x_1, \dots, \partial f/\partial x_1)$ as a k-vector space is called the **Tyurina number** of X_0, and since this is also the dimension of $T_{X_0}\mathcal{D}ef$, we see in particular that it is independent of the immersion of X_0 in the affine space.

Suppose now we have a first order deformation $X \in \mathcal{D}ef_{X_0}(k[\varepsilon])$ of X_0; by the remark about the forgetful morphism above we have a closed immersion $X \subseteq \mathbb{A}^n_{k[\varepsilon]}$ that extends $X_0 \subseteq \mathbb{A}^n_k$. Taking a lifting $f + \varepsilon g \in k[\varepsilon][x_1, \dots, x_n]$ of f along the projection $k[\varepsilon][x_1, \dots, x_n] \to k[x_1, \dots, x_n]$ (where g is some element of $k[x_1, \dots, x_n]$) we see easily that

$$X = \mathrm{Spec}(k[\varepsilon][x_1, \dots, x_n]/(f + \varepsilon g)) \subseteq \mathbb{A}^n_{k[\varepsilon]}.$$

Then the class of X in $\mathrm{Ext}^1_{\mathcal{O}_{X_0}}(\Omega_{X_0}, \mathcal{O}_{X_0})$ will be the image along the differential $\mathrm{Hom}_A(I_0/I_0^2, A) \to \mathrm{Ext}^1_A(\Omega_A, A)$ of the homomorphism $I_0/I_0^2 \to A$ corresponding to the object $X \subseteq \mathbb{A}^n_{k[\varepsilon]} \in \mathcal{H}ilb^{\mathbb{A}^n_\Lambda}(k[\varepsilon])$, as in the proof of Theorem 2.4.5.

Using diagram 2.3 in this particular case, one can easily check that the morphism $I_0/I_0^2 \to A$ we are looking for is the one that sends f (a generator of I_0/I_0^2) to the class of g in A. In conclusion we have proved the following:

Proposition 2.4.9. *If $X = \mathrm{Spec}(k[\varepsilon][x_1, \dots, x_n]/(f + \varepsilon g))$ is a first-order deformation of X_0, then the corresponding element of $\mathrm{Ext}^1_{\mathcal{O}_{X_0}}(\Omega_{X_0}, \mathcal{O}_{X_0})$ is*

$$[g] \in k[x_1, \dots, x_n]/(f, \partial f/\partial x_1, \dots, \partial f/\partial x_n).$$

In particular X is a trivial deformation if and only if $g \in (f, \partial f/\partial x_1, \dots, \partial f/\partial x_n)$.

2.4.5 Smooth hypersurfaces in $\mathbb{P}^n_k$

We give an application of the previous constructions to deformations of smooth hypersurfaces of $\mathbb{P}^n_k$. Take $\Lambda = k$, and suppose we have a smooth hypersurface $Z_0 \subseteq \mathbb{P}^n_k$ of degree d, with $n \geq 2$, $d \geq 1$.

We can ask the following question: given a deformation Z of Z_0 over $\mathrm{Spec}(k[V])$, where $V \in (\mathrm{FVect}/k)$, can we find a closed immersion $Z \subseteq \mathbb{P}^n_{k[V]}$ that extends $Z_0 \subseteq \mathbb{P}^n_k$? The existence of such an immersion for every such Z is equivalent to the surjectivity of the differential of the forgetful morphism at Z_0

$$d_{Z_0}F = \delta : H^0(Z_0, \mathcal{N}_0) \to H^1(Z_0, T_{Z_0}).$$

Proposition 2.4.10. *The map δ is surjective exactly in the following cases:*

- $n = 2$, $d \leq 4$.
- $n = 3$, $d \neq 4$.
- $n \geq 4$, *any* d.

Proof. We start with a piece of the cohomology exact sequence

$$H^0(Z_0, \mathcal{N}_0) \xrightarrow{\delta} H^1(Z_0, T_{Z_0}) \longrightarrow H^1(Z_0, T_{\mathbb{P}^n_k}|_{Z_0}) \longrightarrow H^1(Z_0, \mathcal{N}_0) \qquad (2.5)$$

induced by the dual of the conormal sequence of $Z_0 \subseteq \mathbb{P}^n_k$, as in Remark 2.4.8.

The first step is to prove

Lemma 2.4.11. $\mathrm{coker}(\delta) \cong H^2(\mathbb{P}^n_k, T_{\mathbb{P}^n_k}(-d))$.

Proof. First, we notice that $H^1(Z_0, \mathcal{N}_0) = 0$. This is because $\mathcal{N}_0 \cong \mathcal{O}_{Z_0}(d)$ (since $I_0 \cong \mathcal{O}_{\mathbb{P}^n_k}(-d)$), and $H^1(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(d)) = H^2(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}) = 0$, so from the cohomology exact sequence induced by

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n_k} \xrightarrow{f\cdot} \mathcal{O}_{\mathbb{P}^n_k}(d) \longrightarrow \mathcal{O}_{Z_0}(d) \longrightarrow 0$$

where f is an equation for Z_0, we get $H^1(Z_0, \mathcal{N}_0) = H^1(\mathbb{P}^n_k, \mathcal{N}_0) = 0$ (because $\mathcal{N}_0$ has support contained in Z_0).

From 2.5 we deduce then that $\mathrm{coker}(\delta) \cong H^1(Z_0, T_{\mathbb{P}^n_k}|_{Z_0})$, which is the same as $H^1(\mathbb{P}^n_k, T_{\mathbb{P}^n_k}|_{Z_0})$, again because $T_{\mathbb{P}^n_k}|_{Z_0}$ has support contained in Z_0.

Tensoring the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n_k}(-d) \xrightarrow{f\cdot} \mathcal{O}_{\mathbb{P}^n_k} \longrightarrow \mathcal{O}_{Z_0} \longrightarrow 0$$

with $T_{\mathbb{P}^n_k}$ we get

$$0 \longrightarrow T_{\mathbb{P}^n_k}(-d) \longrightarrow T_{\mathbb{P}^n_k} \longrightarrow T_{\mathbb{P}^n_k}|_{Z_0} \longrightarrow 0. \qquad (2.6)$$

Now we notice that $H^i(\mathbb{P}^n_k, T_{\mathbb{P}^n_k}) = 0$ for $i \geq 1$: this follows from $H^i(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}) = H^i(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(1)) = 0$, using the cohomology exact sequence coming from the dual of the Euler sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n_k} \longrightarrow \mathcal{O}_{\mathbb{P}^n_k}(1)^{\oplus(n+1)} \longrightarrow T_{\mathbb{P}^n_k} \longrightarrow 0.$$

From 2.6 we get then an isomorphism

$$H^1(\mathbb{P}^n_k, T_{\mathbb{P}^n_k}|_{Z_0}) \cong H^2(\mathbb{P}^n_k, T_{\mathbb{P}^n_k}(-d)). \qquad \square$$

To understand $H^2(\mathbb{P}^n_k, T_{\mathbb{P}^n_k}(-d))$, we consider the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n_k}(-d) \longrightarrow \mathcal{O}_{\mathbb{P}^n_k}(1-d)^{\oplus(n+1)} \longrightarrow T_{\mathbb{P}^n_k}(-d) \longrightarrow 0$$

obtained by twisting the dual of the Euler sequence by $\mathcal{O}_{\mathbb{P}^n_k}(-d)$, and the following piece of its cohomology exact sequence

$$\begin{tikzcd} H^2(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(-d)) \arrow[r] & H^2(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(1-d))^{n+1} \arrow[r] & H^2(\mathbb{P}^n_k, T_{\mathbb{P}^n_k}(-d)) \arrow[dll] \\ H^3(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(-d)) \arrow[r] & H^3(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(1-d))^{n+1}. & \end{tikzcd} \tag{2.7}$$

Suppose now $n \geq 4$. In this case we have

$$H^2(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(1-d))^{n+1} = H^3(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(-d)) = 0$$

and then from 2.7 we obtain $\mathrm{coker}(\delta) \cong H^2(\mathbb{P}^n_k, T_{\mathbb{P}^n_k}(-d)) = 0$, so that δ is surjective.

Now take $n = 2$. We have then $H^3(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(-d)) = 0$ and so again from 2.7 we get

$$H^2(\mathbb{P}^2_k, T_{\mathbb{P}^2_k}(-d)) \cong \mathrm{coker}\left(H^2(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(-d)) \xrightarrow{\varphi} H^2(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(1-d))^3\right)$$

where the map φ is the one induced by

$$\begin{array}{ccc} \mathcal{O}_{\mathbb{P}^2_k}(-d) & \longrightarrow & \mathcal{O}_{\mathbb{P}^2_k}(1-d)^{\oplus 3} \\ f & \longmapsto & f\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} \end{array}$$

where the x_i's are homogeneous coordinates on $\mathbb{P}^2_k$ (seen as sections of the sheaf $\mathcal{O}_{\mathbb{P}^2_k}(1)$ of course).

By Serre's duality we have $H^2(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(-d)) \cong H^0(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(d-3))^\vee$ and $H^2(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(1-d)) \cong H^0(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(d-4))^\vee$, and the adjoint map

$$H^0(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(d-4))^3 \xrightarrow{\varphi^\vee} H^0(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(d-3))$$

is given by scalar multiplication by the vector (x_0, x_1, x_2).

Now $\operatorname{coker}(\varphi) \cong \ker(\varphi^\vee)$. If $d \leq 3$ the source of $\varphi^\vee$ is trivial, so certainly $\ker(\varphi^\vee) = 0$. If $d = 4$, we have $H^0(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(d-4)) \cong k$, and the map $\varphi^\vee$ is injective, because the sections x_0, x_1, x_2 are linearly independent over $\mathcal{O}_{\mathbb{P}^2_k}$, so that δ is surjective. On the other hand when $d \geq 5$ clearly $\varphi^\vee$ is it not injective anymore, and so δ will not be surjective.

Suppose now that $n = 3$. Then $H^2(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(-d)) = 0$, and using again 2.7, we get

$$H^2(\mathbb{P}^n_k, T_{\mathbb{P}^2_k}(-d)) \cong \ker\left(H^3(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3_k}(-d)) \xrightarrow{\varphi} H^3(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3_k}(1-d))^4\right)$$

where φ is the analogue of the one we had in the preceding case. Again using Serre's duality we have to study

$$\operatorname{coker}\left(H^0(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3_k}(d-5))^4 \xrightarrow{\varphi^\vee} H^0(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3_k}(d-4))\right).$$

If $d \leq 3$ the target is trivial, so that certainly $\operatorname{coker}(\varphi^\vee) = 0$, and if $d \geq 5$ the map $\varphi^\vee$ is surjective, because every homogeneous polynomial of positive degree in variables x_0, x_1, x_2, x_3 can be written as a linear combination of the variables x_i's, with homogeneous polynomials of one degree less as coefficients. In these cases then δ will be surjective.

The only case in which $\varphi^\vee$ is not surjective (and so δ will not be too) is $d = 4$, when the source is trivial and the target is not. □

We will examine the case $n = 3, d = 4$ further in Section 5.4, where it will give a counterexample about algebraizability of deformations of surfaces.

We now state a more general result that follows from what we have shown here, and from the following fact, which will be proved in Section 4.2.4.

Proposition 2.4.12. *If Z_0 is a smooth hypersurface of $\mathbb{P}^n_k$ of degree d, with $n \geq 1$ and $d \geq 1$, any object $Z \subseteq \mathbb{P}^n_A$ of $\mathcal{H}ilb^{\mathbb{P}^n_k}_{Z_0}(A)$ can be lifted along any small extension $A' \to A$.*

Proposition 2.4.13. *Let $Z \to \operatorname{Spec}(A)$ where $A \in (\mathrm{Art}/k)$ be a flat morphism of schemes over k, and $Z_0 = Z \times_{\operatorname{Spec}(A)} \operatorname{Spec}(k) \subseteq \mathbb{P}^n_k$ be a closed immersion, making Z_0 a smooth hypersurface of degree d in $\mathbb{P}^n_k$, with $n \geq 2$ and $d \geq 1$.*

Then there is a closed immersion $Z \subseteq \mathbb{P}^n_A$ inducing $\mathbb{Z}_0 \subseteq \mathbb{P}^n_k$ in all cases except $n = 2, d \geq 5$ and $n = 3, d = 4$.

Proof. We already know from the preceding discussion that in cases $n = 2, d \geq 5$ and $n = 3, d = 4$ there are counterexamples.

Suppose then that we are not in one of the cases above, and take the given $Z \in \mathcal{D}ef_{Z_0}(A)$. We consider a factorization of the homomorphism $A \to k$ as a composite of small extensions

$$A = A_0 \to A_1 \to \ldots \to A_n = k.$$

and proceed by induction on $n(A)$, the least n with such a factorization.

If $n(A) = 0$ there is nothing to prove. Suppose we know the result for $n(A) - 1$, and consider the extension $A \to A_1$ with kernel I. The pullback $Z|_{A_1} \in \mathcal{D}ef_{Z_0}(A_1)$ of Z to A_1 admits then a closed immersion $Z|_{A_1} \subseteq \mathbb{P}^n_{A_1}$ because of the induction hypothesis.

From the discussion above we also know that the differential of the forgetful morphism $d_{Z_0}F : T_{Z_0}\mathcal{H}ilb^{\mathbb{P}^n_k} \to T_{Z_0}\mathcal{D}ef$ is surjective, and in particular

$$\mathrm{id} \otimes d_{Z_0}F : I \otimes_k T_{Z_0}\mathcal{H}ilb^{\mathbb{P}^n_k} \to I \otimes_k T_{Z_0}\mathcal{D}ef$$

will be surjective too.

Because of Proposition 2.4.12 we can find a lifting $Z' \subseteq \mathbb{P}^n_A$ of $Z|_{A_1} \subseteq \mathbb{P}^n_{A_1}$ to A; both $\mathrm{Lif}(Z|_{A_1}, A)$ and $\mathrm{Lif}(Z_{A_1} \subseteq \mathbb{P}^n_{A_1}, A)$ will then be nonempty, and by Theorem 2.3.1 we have free and transitive actions on them, respectively of $I \otimes_k T_{Z_0}\mathcal{D}ef$ and $I \otimes_k T_{Z_0}\mathcal{H}ilb^{\mathbb{P}^n_k}$.

The object $Z' \in \mathcal{D}ef_{Z_0}(A)$ is a lifting of $Z|_{A_1}$, as is Z, so by transitivity of the action we have an element $g \in I \otimes_k T_{Z_0}\mathcal{D}ef$ such that $[Z'] \cdot g = [Z]$; take then $h \in I \otimes_k T_{Z_0}\mathcal{H}ilb^{\mathbb{P}^n_k}$ such that $(\mathrm{id} \otimes d_{Z_0}F)(h) = g$.

Then using Proposition 2.3.6 we have

$$F((Z' \subseteq \mathbb{P}^n_A) \cdot h) = [Z'] \cdot (\mathrm{id} \otimes d_{Z_0}F)(h) = [Z'] \cdot g = [Z].$$

In other words the object $(Z' \subseteq \mathbb{P}^n_A) \cdot h$ is (after possibly composing with an isomorphism of schemes over $\mathrm{Spec}(A)$) a closed immersion $Z \subseteq \mathbb{P}^n_A$ that induces $Z_0 \subseteq \mathbb{P}^n_k$ on the closed fiber, which is what we were looking for. □

The only things we really used in this proof were surjectivity of the differential and existence of liftings in the source deformation category. Everytime these two facts hold in an abstract setting we can repeat the same argument to deduce that every object of the target deformation category is isomorphic to the image of an object of the source.

2.4.6 Quasi-coherent sheaves

Consider now a scheme X over $\mathrm{Spec}(\Lambda)$, and the deformation category $\mathcal{QC}oh^X \to (\mathrm{Art}/\Lambda)^{op}$ of deformations of quasi-coherent sheaves on X. Let $\mathcal{E}_0 \in \mathcal{QC}oh^X(k)$.

Proposition 2.4.14. *There is an isomorphism*

$$T_{\mathcal{E}_0}\mathcal{QC}oh^X \cong \mathrm{Ext}^1_{\mathcal{O}_{X_0}}(\mathcal{E}_0, \mathcal{E}_0).$$

Proof. Consider the functor $F : (\mathrm{FVect}\,/k) \to (\mathrm{Set})$ defined on objects by

$$F(V) = \{\text{isomorphism classes of objects in } \mathcal{QC}oh^X_{\mathcal{E}_0}(k[V])\}$$

and sending a k-linear map $f : V \to W$ to the corresponding pullback function $F(V) \to F(W)$. We show that there is a functorial bijection

$$F(V) \cong V \otimes_k \mathrm{Ext}^1_{\mathcal{O}_{X_0}}(\mathcal{E}_0, \mathcal{E}_0)$$

that will give as usual a k-linear natural transformation, and in particular an isomorphism

$$T_{\mathcal{E}_0}\mathcal{QC}oh^X = F(k) \cong \mathrm{Ext}^1_{\mathcal{O}_{X_0}}(\mathcal{E}_0, \mathcal{E}_0).$$

It is an easy consequence of Proposition C.12 that the category $\mathcal{QC}oh^X_{\mathcal{E}_0}(k[V])$ of quasi-coherent $\mathcal{O}_{X_V}$-modules $\mathcal{E}$ on $X_V = X_0 \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k[V])$ with an isomorphism $\mathcal{E} \otimes_{k[V]} k \cong \mathcal{E}_0$, is equivalent to the category whose objects are extensions of quasi-coherent $\mathcal{O}_{X_0}$-modules

$$0 \longrightarrow V \otimes_k \mathcal{E}_0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_0 \longrightarrow 0$$

and arrows defined in the obvious way. This automatically gives us the bijection φ_V we wanted, taking isomorphism classes.

So we only have to check functoriality. Suppose $f : V \to W$ is a k-linear map; we show that the diagram

$$\begin{tikzcd}
F(V) \arrow[r, "\varphi_V"] \arrow[d, "(\mathrm{id} \oplus f)_*"'] & V \otimes_k \mathrm{Ext}^1_{\mathcal{O}_{X_0}}(\mathcal{E}_0, \mathcal{E}_0) \arrow[d, "f \otimes \mathrm{id}"] \\
F(W) \arrow[r, "\varphi_W"] & W \otimes_k \mathrm{Ext}^1_{\mathcal{O}_{X_0}}(\mathcal{E}_0, \mathcal{E}_0)
\end{tikzcd}$$

is commutative.

Starting with an object $\mathcal{E}$ of $\mathcal{QC}oh^X_{\mathcal{E}_0}(k[V])$, we have the associated extension

$$0 \longrightarrow V \otimes_k \mathcal{E}_0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_0 \longrightarrow 0$$

that gets mapped by $f \otimes \mathrm{id}$ to the "pushout extension", the bottom row of

$$\begin{tikzcd}
0 \arrow[r] & V \otimes_k \mathcal{E}_0 \arrow[r] \arrow[d, "f \otimes \mathrm{id}"'] & \mathcal{E} \arrow[r] \arrow[d] & \mathcal{E}_0 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & W \otimes_k \mathcal{E}_0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{E}_0 \arrow[r] & 0
\end{tikzcd}$$

where $\mathcal{F}$ is the pushout of the following diagram.

$$\begin{tikzcd}
V \otimes_k \mathcal{E}_0 \arrow[r] \arrow[d, "f \otimes \mathrm{id}"'] & \mathcal{E} \\
W \otimes_k \mathcal{E}_0 &
\end{tikzcd}$$

But on the other hand we have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & V \otimes_k \mathcal{E}_0 \arrow[r] \arrow[d, "f\otimes\mathrm{id}"'] & \mathcal{E} \arrow[r] \arrow[d, "\varphi"] & \mathcal{E}_0 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & W \otimes_k \mathcal{E}_0 \arrow[r] & \mathcal{E}' \arrow[r] & \mathcal{E}_0 \arrow[r] & 0
\end{tikzcd}$$

where $\mathcal{E}'$ is the pullback of $\mathcal{E}$ to $k[W]$, coming from the fact that $\mathcal{E}'$ is just $\mathcal{E} \otimes_{\mathcal{O}_{X_V}} \mathcal{O}_{X_W}$ in this case. This gives an isomorphism between the bottom row of the last diagram (which is the extension associated with $\mathcal{E}'$) and the "pushout extension", providing the functoriality we needed and ending our proof. $\square$

Remark 2.4.15. If $\mathcal{E}_0$ is locally free, than we have

$$T_{\mathcal{E}_0}\mathcal{QC}oh^X \cong \mathrm{Ext}^1_{\mathcal{O}_{X_0}}(\mathcal{E}_0, \mathcal{E}_0) \cong H^1(X_0, \mathcal{E}nd_{\mathcal{O}_{X_0}}(\mathcal{E}_0))$$

and moreover if $\mathcal{E}_0$ is invertible, then $\mathcal{E}nd_{\mathcal{O}_{X_0}}(\mathcal{E}_0) \cong \mathcal{E}_0 \otimes_{\mathcal{O}_{X_0}} (\mathcal{E}_0)^\vee \cong \mathcal{O}_{X_0}$, so that

$$T_{\mathcal{E}_0}\mathcal{QC}oh^X \cong H^1(X_0, \mathcal{O}_{X_0})$$

which does not depend on $\mathcal{E}_0$.

Chapter 3

Infinitesimal automorphisms

The purpose of this chapter is to introduce and discuss the so-called group (or space) of infinitesimal automorphisms of a deformation category at an object $\xi_0 \in \mathcal{F}(k)$.

We will see that this space gives a measure of the "rigidity" of a deformation problem, and tell us how far our deformation category is from its corresponding deformation functor. After the definition, we will examine some of its properties, and finally calculate it in some examples.

3.1 The group of infinitesimal automorphisms

Suppose $\mathcal{F} \to (\mathrm{Art}/\Lambda)^{op}$ is a deformation category, and $\varphi : A' \to A$ is a small extension. Fix $\xi \in \mathcal{F}(A)$, and let $\xi' \in \mathcal{F}(A')$ be a lifting of ξ to A'. We are interested in automorphisms of ξ' that induce the identity on ξ.

Definition 3.1.1. *If $A \in (\mathrm{Art}/\Lambda)$ and $\xi \in \mathcal{F}(A)$, we denote by $\mathrm{Aut}_A(\xi)$ the set of automorphisms of the object ξ in the category $\mathcal{F}(A)$.*

Recall from Section 1.2 that φ induces a pullback functor $\varphi_* : \mathcal{F}(A') \to \mathcal{F}(A)$. In particular we have a "restriction" function $\mathrm{Aut}_{A'}(\xi') \to \mathrm{Aut}_A(\xi)$ (given by the composite $\mathrm{Aut}_{A'}(\xi') \to \mathrm{Aut}_A(\xi'|_A) \cong \mathrm{Aut}_A(\xi)$, where the last map comes from the canonical isomorphism $\xi'|_A \cong \xi$), which is a homomorphism of groups. The automorphisms inducing the identity on ξ, which we call **infinitesimal automorphisms** of ξ' (with respect to ξ), are the ones in the kernel of this homomorphism.

In this chapter we will see that the subgroup of infinitesimal automorphisms of ξ' depends only on $\ker(\varphi)$ and on the pullback of ξ to $\mathrm{Spec}(k)$.

We start by defining the group of infinitesimal automorphisms of $\xi_0 \in \mathcal{F}(k)$. Notice that if A is a k-algebra we have a trivial deformation of ξ_0 over A, which we denote by $\xi_0|_A$, given by the pullback of ξ_0 along the structure homomorphism $k \to A$.

Definition 3.1.2. *The **group of infinitesimal automorphisms** of ξ_0 is the subgroup of* $\mathrm{Aut}_{k[\varepsilon]}(\xi_0|_{k[\varepsilon]})$

$$\mathrm{Inf}(\xi_0) = \ker\left(\mathrm{Aut}_{k[\varepsilon]}(\xi_0|_{k[\varepsilon]}) \to \mathrm{Aut}_k(\xi_0)\right).$$

When we need to specify the category $\mathcal{F}$ in the notation, we will write $\mathrm{Inf}_{\xi_0}(\mathcal{F})$ instead of $\mathrm{Inf}(\xi_0)$.

The group of infinitesimal automorphisms has also a canonical k-vector space structure, coming from the fact that it is the tangent space of deformation category.

Consider the functor $\mathrm{Aut}(\xi_0) : (\mathrm{Art}\,/k) \to (\mathrm{Set})$ that sends an object $A \in (\mathrm{Art}\,/k)$ to $\mathrm{Aut}_A(\xi_0|_A)$, and an arrow $A' \to A$ to the function $\mathrm{Aut}_{A'}(\xi_0|_{A'}) \to \mathrm{Aut}_A(\xi_0|_A)$ introduced above. This functor gives a category fibered in sets over $(\mathrm{Art}\,/k)^{op}$, and from the fact that $\mathcal{F}$ satisfies [RS] (precisely from the "fully faithful" part), we get that $\mathrm{Aut}(\xi_0)$ does too.

We can then consider the tangent space $T_{\mathrm{id}_{\xi_0}}\mathrm{Aut}(\xi_0)$, which is easily seen to be as a set exactly $\mathrm{Inf}(\xi_0)$ defined above.

Remark 3.1.3. We see that the addition coming from the definition of the tangent space and the group operation given by composition coincide (so that $\mathrm{Inf}(\xi_0)$ will always be an abelian group).

Recall that addition is defined on $T_{\mathrm{id}_{\xi_0}}\mathrm{Aut}(\xi_0) = \mathrm{Aut}(\xi_0)_{\mathrm{id}_{\xi_0}}(k[\varepsilon])$ by the following diagram

$$\begin{tikzcd}
\mathrm{Aut}(\xi_0)_{\mathrm{id}_{\xi_0}}(k[\varepsilon_1,\varepsilon_2]) \arrow[r, "\sim"] \arrow[dr, "\varphi_*"'] & \mathrm{Aut}(\xi_0)_{\mathrm{id}_{\xi_0}}(k[\varepsilon]) \times \mathrm{Aut}(\xi_0)_{\mathrm{id}_{\xi_0}}(k[\varepsilon]) \arrow[d, "+"] \\
 & \mathrm{Aut}(\xi_0)_{\mathrm{id}_{\xi_0}}(k[\varepsilon])
\end{tikzcd}$$

where we see $k[\varepsilon_1,\varepsilon_2]$ as $k[\varepsilon]\times_k k[\varepsilon]$ (and of course $\varepsilon_1^2 = \varepsilon_2^2 = \varepsilon_1\varepsilon_2 = 0$), the horizontal bijection is given by [RS], and the map φ_* is the pullback map induced by $\varphi : k[\varepsilon_1,\varepsilon_2] \to k[\varepsilon]$ that sends both $\varepsilon_1,\varepsilon_2$ to ε.

The horizontal map and φ_* are group homomorphisms (with componentwise composition as group operation on the product), so that the addition $+$ is a homomorphism too. In other words for every $f, f', g, g' \in T_{\mathrm{id}_{\xi_0}}\mathrm{Aut}(\xi_0)$ we have

$$(f \circ f') + (g \circ g') = (f + g) \circ (f' + g').$$

Taking $f' = g = \mathrm{id}_{\xi_0|_{k[\varepsilon]}}$ we get $f + g' = f \circ g'$.

From the fact that $\mathrm{Inf}(\xi_0)$ is the tangent space of a deformation category, using Theorem 2.3.1 we deduce the following corollary.

Corollary 3.1.4. *Let $\mathcal{F} \to (\mathrm{Art}/\Lambda)^{op}$ be a deformation category, $A' \to A$ a small extension with kernel I, and $f \in \mathrm{Aut}(\xi_0)_{\mathrm{id}_{\xi_0}}(A)$. If $\mathrm{Lif}(f, A')$ is not empty, then there is a free and transitive action of $I \otimes_k \mathrm{Inf}(\xi_0)$ on it.*

We now prove the initial assertion about the infinitesimal automorphisms of a lifting.

Proposition 3.1.5. *Let $\mathcal{F} \to (\mathrm{Art}/\Lambda)^{op}$ be a deformation category, $A' \to A$ a small extension with kernel I, $\xi_0 \in \mathcal{F}(k)$, $\xi \in \mathcal{F}_{\xi_0}(A)$ and ξ' a lifting of ξ to A'. Then we have an isomorphism*

$$\ker\big(\mathrm{Aut}_{A'}(\xi') \to \mathrm{Aut}_A(\xi)\big) \cong I \otimes_k \mathrm{Inf}(\xi_0).$$

Proof. We generalize the construction of the functor $\mathrm{Aut}(\xi_0)$.

Consider the functor $\mathrm{Aut}(\xi') : (\mathrm{Art}/A') \to (\mathrm{Set})$ that sends a A'-algebra B to the set $\mathrm{Aut}_B(\xi'|_B)$ (where $\xi'|_B$ is the trivial pullback of ξ' along the structure homomorphism $A' \to B$), and an arrow $B' \to B$ to the induced function $\mathrm{Aut}_{B'}(\xi'|_{B'}) \to \mathrm{Aut}_B(\xi'|_B)$.

The functor $\mathrm{Aut}(\xi')$ gives a category fibered in groupoids that satisfies [RS] (over (Art/A')), so we have a tangent space $T_{\mathrm{id}_{\xi_0}} \mathrm{Aut}(\xi')$ at $\mathrm{id}_{\xi_0} \in \mathrm{Aut}(\xi')(k)$. Up to isomorphism we can also assume that $\xi'|_A = \xi$ (that is, in the following we leave the isomorphism $\mathrm{Aut}_A(\xi) \cong \mathrm{Aut}_A(\xi'|_A)$ understood).

Now notice that $K = \ker(\mathrm{Aut}_{A'}(\xi') \to \mathrm{Aut}_A(\xi))$ coincides with the set of liftings of id_ξ to A', in the category $\mathrm{Aut}(\xi')$. Since this set of liftings is nonempty (we have at least $\mathrm{id}_{\xi'}$), by Theorem 2.3.1 we have a free and transitive action of $I \otimes_k T_{\mathrm{id}_{\xi_0}} \mathrm{Aut}(\xi')$ on K. Moreover using the fact that sum and composition coincide in $T_{\mathrm{id}_{\xi_0}} \mathrm{Aut}(\xi')$ (as is easily shown with the same argument of Remark 3.1.3) it is easy to see that the bijection $I \otimes_k T_{\mathrm{id}_{\xi_0}} \mathrm{Aut}(\xi') \to K$ defined by $a \mapsto \mathrm{id}_{\xi'} \cdot a$ is an isomorphism of groups.

To conclude it suffices to notice that $T_{\mathrm{id}_{\xi_0}} \mathrm{Aut}(\xi') \cong T_{\mathrm{id}_{\xi_0}} \mathrm{Aut}(\xi_0) = \mathrm{Inf}(\xi_0)$, because every trivial lifting of ξ' to a k-algebra of the form $k[V]$ is in particular (up to isomorphism) a trivial lifting of ξ_0 (the structure homomorphism $A' \to k[V]$ is defined as the composite $A' \to k \to k[V]$). $\square$

Remark 3.1.6. Suppose we have two liftings of ξ to A', say $\xi_1, \xi_2 \in \mathcal{F}_{\xi_0}(A')$, and an isomorphism of liftings $f : \xi_1 \to \xi_2$. Take an infinitesimal automorphism $g_1 \in \mathrm{Aut}_{A'}(\xi_1)$ of ξ_1, and consider $g_2 = f \circ g_1 \circ f^{-1} \in \mathrm{Aut}_{A'}(\xi_2)$, which is an infinitesimal automorphism of ξ_2. Then it is clear from the preceding proof that the elements of $I \otimes_k \mathrm{Inf}(\xi_0)$ corresponding to g_1 and g_2 with respect to the isomorphism constructed above are the same.

From Proposition 3.1.5 we immediately get the following corollary.

Corollary 3.1.7. *Let $\mathcal{F} \to (\mathrm{Art}/\Lambda)^{op}$ be a deformation category, and $\xi_0 \in \mathcal{F}(k)$. If $\mathrm{Inf}(\xi_0) = 0$, then for every $A \in (\mathrm{Art}/\Lambda)$ and $\xi \in \mathcal{F}_{\xi_0}(A)$ the homomorphism $\mathrm{Aut}_A(\xi) \to \mathrm{Aut}_k(\xi_0)$ is injective.*

Proof. Fix $A \in (\mathrm{Art}/\Lambda)$ and $\xi \in \mathcal{F}_{\xi_0}(A)$, and factor the homomorphism $A \to k$ as a composite of small extensions

$$A = A_0 \to A_1 \to \ldots \to A_n = k.$$

As in Corollary 2.3.2, call $n(A)$ the least n with such a factorization, and proceed by induction on $n(A)$.

If $n(A) = 0$, then $A = k$ and the conclusion is trivial. Suppose now that we have our claim for $n(A) - 1$. Then the homomorphism $\mathrm{Aut}_{A_1}(\xi|_{A_1}) \to \mathrm{Aut}_k(\xi_0)$ is injective by inductive hypothesis, and Proposition 3.1.5 applied to the small extension $A \to A_1$ gives that $\ker(\mathrm{Aut}_A(\xi) \to \mathrm{Aut}_{A_1}(\xi|_{A_1}) = I \otimes_k \mathrm{Inf}(\xi_0) = 0$ (where I is the kernel of $A \to A_1$).

Then $\mathrm{Aut}_A(\xi) \to \mathrm{Aut}_{A_1}(\xi|_{A_1})$ is also injective, and so is the composite $\mathrm{Aut}_A(\xi) \to \mathrm{Aut}_{A_1}(\xi|_{A_1}) \to \mathrm{Aut}_k(\xi_0)$. □

Further, we see that the group of infinitesimal automorphisms gives a measure of the "rigidity" of our deformation problem.

Proposition 3.1.8. *Let $\mathcal{F} \to (\mathrm{Art}/\Lambda)^{op}$ be a deformation category and $\xi_0 \in \mathcal{F}(k)$. Then $\mathrm{Inf}(\xi_0) = 0$ if and only if $\mathcal{F}_{\xi_0} \to (\mathrm{Art}/\Lambda)^{op}$ is a category fibered in equivalence relations.*

Proof. Recall that a groupoid is an equivalence relation if and only if the only automorphisms are the identities.

Suppose that $\mathrm{Inf}(\xi_0) = 0$, and consider the category $\mathcal{F}_{\xi_0}$. By Corollary 3.1.7 we have that for every $A \in (\mathrm{Art}/\Lambda)$ and object $\xi_0 \to \xi \in \mathcal{F}_{\xi_0}(A)$, the induced homomorphism $\mathrm{Aut}_A(\xi) \to \mathrm{Aut}_k(\xi_0)$ is injective, and in particular $\mathrm{Aut}_A(\xi_0 \to \xi)$ (which is the preimage of id_{ξ_0}) has at most one element (it will have exactly one, namely id_ξ).

It follows that $\mathcal{F}_{\xi_0}(A)$ is an equivalence relation for every $A \in (\mathrm{Art}/\Lambda)$, and so $\mathcal{F}_{\xi_0} \to (\mathrm{Art}/\Lambda)^{op}$ is fibered in equivalence relations. The converse is trivial. □

In other words if a deformation problem does not have any nontrivial infinitesimal automorphism, we do not lose anything by studying its deformation functor instead of the deformation category.

3.2 Examples

We now analyze the group of infinitesimal automorphisms in our three examples.

3.2.1 Schemes

Consider the deformation category $\mathcal{D}ef \to (\mathrm{Art}/\Lambda)^{op}$ of flat schemes, and $X_0 \in \mathcal{D}ef(k)$.

Proposition 3.2.1. *We have an isomorphism*

$$\mathrm{Inf}_{X_0}(\mathcal{D}ef) \cong \mathrm{Der}_k(\mathcal{O}_{X_0}, \mathcal{O}_{X_0}) \cong \mathrm{Hom}_{\mathcal{O}_{X_0}}(\Omega_{X_0}, \mathcal{O}_{X_0}).$$

Proof. We have to understand the functor $F : (\mathrm{FVect}\,/k) \to (\mathrm{Set})$, that takes $V \in (\mathrm{FVect}\,/k)$ to

$$F(V) = \ker\left(\mathrm{Aut}_{k[V]}(X_0|_{k[V]}) \to \mathrm{Aut}_k(X_0)\right)$$

where $X_0|_{k[V]}$ is the trivial deformation $X_V = X_0 \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k[V])$. In particular as topological spaces $|X_V| = |X_0|$, and on the structure sheaves we have

$$\mathcal{O}_{X_V} = \mathcal{O}_{X_0} \otimes_k k[V] \cong \mathcal{O}_{X_0} \oplus (V \otimes_k \mathcal{O}_{X_0}).$$

Take an element $\varphi \in \mathrm{Aut}(X_0)(V)$. Then φ will clearly be the identity as a map between topological spaces, so we turn to the morphism $\varphi^\sharp : \mathcal{O}_{X_V} \to \mathcal{O}_{X_V}$ on the structure sheaf, which is an automorphism of sheaves of $k[V]$-algebras such that the diagram

$$\begin{tikzcd}
\mathcal{O}_{X_0} \oplus (V \otimes_k \mathcal{O}_{X_0}) \arrow[r, "\varphi^\sharp"] \arrow[d, "\pi_1"'] & \mathcal{O}_{X_0} \oplus (V \otimes_k \mathcal{O}_{X_0}) \arrow[d, "\pi_1"] \\
\mathcal{O}_{X_0} \arrow[r, "\mathrm{id}"] & O_{X_0}
\end{tikzcd}$$

is commutative.

Using the analogue of Proposition 2.2.3 for extensions of sheaves, with respect to the extension

$$0 \longrightarrow V \otimes_k \mathcal{O}_{X_0} \longrightarrow \mathcal{O}_{X_V} \longrightarrow \mathcal{O}_{X_0} \longrightarrow 0$$

we see that $\varphi^\sharp$ differs from the identity of $\mathcal{O}_{X_V}$ by a derivation

$$D_\varphi \in \mathrm{Der}_k(\mathcal{O}_{X_0}, V \otimes_k \mathcal{O}_{X_0}).$$

Conversely every φ as above can be obtained in this way, and so for each $V \in (\mathrm{FVect}\,/k)$ we get a bijection

$$F(V) \cong \mathrm{Der}_k(\mathcal{O}_{X_0}, V \otimes_k \mathcal{O}_{X_0}) \cong V \otimes_k \mathrm{Der}_k(\mathcal{O}_{X_0}, \mathcal{O}_{X_0}).$$

These maps are also functorial in V (as is readily checked), so the corresponding natural transformation is k-linear, and in particular we have an isomorphism

$$\mathrm{Inf}_{X_0}(\mathcal{D}ef) = F(k) \cong \mathrm{Der}_k(\mathcal{O}_{X_0}, \mathcal{O}_{X_0}). \qquad \square$$

Remark 3.2.2. Notice that if X_0 is of finite type over k, then (as we have already remarked) every trivial deformation X_V will be of finite type over $k[V]$, so we will also have

$$\mathrm{Inf}_{X_0}(\widetilde{\mathcal{D}ef}) \cong \mathrm{Der}_k(\mathcal{O}_{X_0}, \mathcal{O}_{X_0}) \cong \mathrm{Hom}_{\mathcal{O}_{X_0}}(\Omega_{X_0}, \mathcal{O}_{X_0}).$$

Remark 3.2.3. In particular if X_0 is smooth $\mathrm{Hom}_{\mathcal{O}_{X_0}}(\Omega_{X_0}, \mathcal{O}_{X_0})$ coincides with $H^0(X_0, T_{X_0})$, so that infinitesimal automorphisms correspond to sections of the tangent sheaf, or "vector fields", which is an old intuitive idea from differential geometry.

3.2.2 Closed subschemes

Now we turn to deformations of closed subschemes. It was already mentioned that in this case the space of infinitesimal automorphisms is trivial.

Proposition 3.2.4. $\mathrm{Inf}_{Z_0}(\mathcal{H}ilb^X)$ *is trivial for every* $Z_0 \in \mathcal{H}ilb^X(k)$.

Proof. This is immediate from the fact that, for source and target fixed, the arrows in $\mathcal{H}ilb^X$ are uniquely determined by their image in $(\mathrm{Art}/\Lambda)^{op}$. In particular an object of $\mathcal{H}ilb^X(k[\varepsilon])$ can only have one automorphism (because they map to the identity of $k[\varepsilon]$ in $(\mathrm{Art}/\Lambda)^{op}$), which is the identity. □

3.2.3 Quasi-coherent sheaves

Finally let us consider the infinitesimal automorphisms of $\mathcal{E}_0 \in \mathcal{QC}oh^X(k)$ in the deformation category $\mathcal{QC}oh^X \to (\mathrm{Art}/\Lambda)^{op}$.

Proposition 3.2.5. *We have an isomorphism*

$$\mathrm{Inf}_{\mathcal{E}_0}(\mathcal{QC}oh) \cong \mathrm{Hom}_{\mathcal{O}_{X_0}}(\mathcal{E}_0, \mathcal{E}_0)$$

Proof. We have to study the functor $F : (\mathrm{FVect}/k) \to (\mathrm{Set})$ defined by

$$F(V) = \ker\left(\mathrm{Aut}_{k[V]}(\mathcal{E}_V) \to \mathrm{Aut}_k(\mathcal{E}_0)\right)$$

where, if $V \in (\mathrm{FVect}/k)$, the sheaf $\mathcal{E}_V$ is the trivial lifting

$$\mathcal{E}_V = \pi_V^*(\mathcal{E}_0) \cong \mathcal{E}_0 \otimes_k k[V] \cong \mathcal{E}_0 \oplus (V \otimes_k \mathcal{E}_0)$$

(where $\pi_V : X_0 \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k[V]) \to X_0$ is the projection).

Consider an automorphism $\varphi : \mathcal{E}_0 \oplus (V \otimes_k \mathcal{E}_0) \to \mathcal{E}_0 \oplus (V \otimes_k \mathcal{E}_0)$ of $\mathcal{O}_{X_V}$-modules that induces the identity on $\mathcal{E}_0$. Using $k[V]$-linearity and $V^2 = (0)$, we see as in Proposition 3.2.1 that φ restricts to the identity on $V \otimes_k \mathcal{E}_0$, and if we write $\varphi(f) = f + G_\varphi(f)$ for a section f of the summand $\mathcal{E}_0 \subseteq \mathcal{E}_V$, then $G_\varphi : \mathcal{E}_0 \to V \otimes_k \mathcal{E}_0$ is a homomorphism of $\mathcal{O}_{X_0}$-modules, and determines φ completely.

Conversely, given an $\mathcal{O}_{X_0}$-module homomorphism $G \in \mathrm{Hom}_{\mathcal{O}_{X_0}}(\mathcal{E}_0, V \otimes_k \mathcal{E}_0)$, we can define a homomorphism of $\mathcal{O}_{X_V}$-modules

$$\varphi_G : \mathcal{E}_0 \oplus (V \otimes_k \mathcal{E}_0) \to \mathcal{E}_0 \oplus (V \otimes_k \mathcal{E}_0)$$

by $\varphi_G(f+\alpha) = f+G(f)+\alpha$, where f is a section of $\mathcal{E}_0$ and α one of $V \otimes_k \mathcal{E}_0$. Moreover φ_G will be an automorphism, with inverse φ_{-G}.

These two correspondences are inverse to each other, so that for each $V \in (\mathrm{FVect}\,/k)$ we have a bijection

$$F(V) \cong \mathrm{Hom}_{\mathcal{O}_{X_0}}(\mathcal{E}_0, V \otimes_k \mathcal{E}_0) \cong V \otimes_k \mathrm{Hom}_{\mathcal{O}_{X_0}}(\mathcal{E}_0, \mathcal{E}_0).$$

These maps are easily seen to be functorial in V, so the resulting natural transformation will be k-linear, and we have an isomorphism

$$\mathrm{Inf}_{\mathcal{E}_0}(\mathcal{QC}oh^X) \cong \mathrm{Hom}_{\mathcal{O}_{X_0}}(\mathcal{E}_0, \mathcal{E}_0). \qquad \square$$

Chapter 4

Obstructions

The present chapter is about obstruction theories, which tell us whether we can lift a given object along a small extension or not. In opposition to tangent spaces and groups of infinitesimal automorphisms, which are canonically defined, there can well be more than one obstruction theory for a given problem, and the choice of a particular one is important in some cases.

After the definition, we will concentrate on minimal obstruction spaces and their properties, and we will state a theorem on the vanishing of obstructions that will be proved in Chapter 5. We will then present a particular obstruction theory for each one of our examples, and give a classical example of a variety over $\mathbb{C}$ with nontrivial obstructions.

4.1 Obstruction theories

We focus now on the problem of existence of liftings. Given a deformation category $\mathcal{F} \to (\mathrm{Art}/\Lambda)^{op}$ and a small extension $A' \to A$, with an object $\xi \in \mathcal{F}(A)$, we would like to have a procedure to decide whether there is a lifting of ξ to A'.

Definition 4.1.1. *An* ***obstruction theory*** *for $\xi_0 \in \mathcal{F}(k)$ is a pair (V_ω, ω), where V_ω is a k-vector space and ω is a function that assigns to every small extension $A' \to A$ with kernel I and $\xi \in \mathcal{F}_{\xi_0}(A)$ an element*

$$\omega(\xi, A') \in I \otimes_k V_\omega$$

called the ***obstruction*** *to lifting ξ to A', in a way such that:*

- *$\omega(\xi, A') = 0$ if and only if there exists a lifting of ξ to A'.*
- *We have the following functoriality property: if $B' \to B$ is another small extension with kernel J, $\varphi : A' \to B'$ is a homomorphism such that $\varphi(I) \subseteq$*

J, and $\overline{\varphi} : A \to B$, $\varphi|_I : I \to J$ are the induced homomorphisms, then

$$(\varphi|_I \otimes \mathrm{id})(\omega(\xi, A')) = \omega(\overline{\varphi}_*(\xi), B') \in J \otimes_k V_\omega.$$

The space V_ω called an **obstruction space** for ξ_0. If the association ω is identically zero (that is, every object can be lifted along any small extension), we say that ξ_0 (or the deformation problem associated with $\mathcal{F}_{\xi_0}$) is **unobstructed**; otherwise, we say it is **obstructed**.

Example 4.1.2. If $\xi_0 \in \mathcal{F}(k)$ has the property that any object of $\mathcal{F}$ restricting to ξ_0 on k can be lifted along any small extension, then it obviously admits a "trivial" obstruction theory, with $V_\omega = 0$ and ω the only possible function. In this case we will also say that ξ_0 is unobstructed.

Remark 4.1.3. Notice that the functoriality property implies in particular that if $\omega(\xi, A') = 0$ (i.e. ξ admits a lifting to A'), then surely $\omega(\overline{\varphi}_*(\xi), B') = 0$ (i.e. $\overline{\varphi}_*(\xi)$ admits a lifting to B'). But this is clear, because the pullback along φ of a lifting of ξ to A' will be a lifting of $\overline{\varphi}_*(\xi)$ to B'.

When dealing with concrete problems, it is usually possible to construct an obstruction theory, and sometimes the obstruction space is a cohomology group of a quasi-coherent sheaf on a certain noetherian scheme (usually one degree higher than the one representing the tangent space of the deformation problem we are considering). We will see some examples of this later. In these cases in particular the obstruction will always vanish locally (at least on affine open subschemes).

If we stick to the abstract setting, that is, if we consider an arbitrary deformation category $\mathcal{F} \to (\mathrm{Art}\,/\Lambda)^{op}$ and an object $\xi_0 \in \mathcal{F}(k)$, it is possible to construct "abstract" obstruction theories for ξ_0. In [Fan] the authors define a more general notion of obstruction theory (for morphisms of deformation functors) using pointed sets, and among other results they show that, with mild hypotheses, one can always find an obstruction theory for a deformation functor (and also a universal one, in some sense).

Nevertheless notice that obstruction spaces are something that is intrinsically "not-canonical", and moreover the choice of the obstruction theory one considers is very important in some cases (for example, in Gromov-Witten theory).

4.1.1 Minimal obstruction spaces

An undesirable thing that can happen (see Proposition 4.2.7), is that, if (V_ω, ω) is an obstruction theory for some $\xi_0 \in \mathcal{F}(k)$, the vector space V_ω is not zero, but nevertheless the map ω is. To try to avoid this type of behavior, we eliminate from the vector space V_ω all the unnecessary elements.

Definition 4.1.4. *Let (V_ω, ω) be an obstruction theory for $\xi_0 \in \mathcal{F}(k)$. The* ***minimal obstruction space*** *Ω_ω of the given obstruction theory is the subspace of V_ω of elements $v \in V_\omega$ that correspond to obstructions along tiny extensions, in the following sense: there exists a tiny extension $A' \to A$, with a fixed isomorphism $I \cong k$, and $\xi \in \mathcal{F}_{\xi_0}(A)$, such that v is the image of the obstruction $\omega(\xi, A') \in I \otimes_k V_\omega$ under the induced isomorphism $I \otimes_k V_\omega \cong k \otimes_k V_\omega \cong V_\omega$.*

For this definition to make sense, we have to check that Ω_ω is a vector subspace of V_ω.

Proposition 4.1.5. *$\Omega_\omega \subseteq V_\omega$ is a vector subspace.*

Proof. First of all notice that $0 \in \Omega_\omega$. For example, we can take the tiny extension $k[\varepsilon] \to k$, and the object $\xi_0 \in \mathcal{F}(k)$. Then we have at least the trivial lifting, obtained by pulling ξ_0 back along $k \to k[\varepsilon]$, so $\omega(\xi_0, k[\varepsilon]) = 0 \in (\varepsilon) \otimes_k V_\omega$, which corresponds to 0 in V_ω.

Next, we check that Ω_ω is closed under scalar multiplication. Suppose we have an element $v \in \Omega_\omega$ corresponding to $\omega(\xi, A')$ for a tiny extension $A' \to A$ with kernel I, an isomorphism $f : I \cong k$, and $\xi \in \mathcal{F}_{\xi_0}(A)$; take also $x \in k$, and suppose $x \neq 0$, since we already know that 0 is in Ω_ω. Then we can consider the same tiny extension $A' \to A$ with the same object $\xi \in \mathcal{F}_{\xi_0}(A)$, but take the isomorphism $x \cdot f : I \cong k$. The element of V_ω that we get this way will clearly be $x \cdot v$.

Now take two elements $v, w \in \Omega_\omega$, corresponding respectively to $\omega(\xi, A')$ and $\omega(\eta, B')$, with $A' \to A$ and $B' \to B$ two tiny extensions with kernels I and J, fixed isomorphisms $f : I \cong k$, $g : J \cong k$, and objects $\xi \in \mathcal{F}_{\xi_0}(A)$, $\eta \in \mathcal{F}_{\xi_0}(B)$.

Then we take the fibered product $A \times_k B$, and notice that by [RS] ξ and η induce an object $\{\xi, \eta\}$ of $\mathcal{F}_{\xi_0}(A \times_k B)$ (since they restrict to ξ_0 over k). The map $A' \times_k B' \to A \times_k B$ gives a small extension, with kernel $f \oplus g : I \oplus J \cong k \oplus k$; we have then an obstruction

$$\omega(\{\xi, \eta\}, A' \times_k B') \in (I \oplus J) \otimes_k V_\omega \cong (k \oplus k) \otimes_k V_\omega \cong V_\omega \oplus V_\omega$$

that corresponds to the pair (u, v).

In fact, the first projection $\pi_1 : A' \times_k B' \to A'$ induces a morphism of extensions

$$\begin{tikzcd}
0 \arrow[r] & I \oplus J \arrow[r] \arrow[d, "\pi_1|_{I \oplus J}"] & A' \times_k B' \arrow[r] \arrow[d, "\pi_1"] & A \times_k B \arrow[r] \arrow[d, "\overline{\pi}_1"] & 0 \\
0 \arrow[r] & I \arrow[r] & A' \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

so by functoriality of the obstruction (and $(\overline{\pi}_1)_*(\{\xi, \eta\}) \cong \xi$) we have

$$(\pi_1|_{I \oplus J} \otimes \mathrm{id})(\omega(\{\xi, \eta\}, A' \times_k B')) = \omega((\overline{\pi}_1)_*(\{\xi, \eta\}), A') = \omega(\xi, A') = u.$$

But $(\pi_1|_{I\oplus J} \otimes \mathrm{id})(\omega(\{\xi,\eta\}, A' \times_k B'))$ is the first component of the corresponding element in $V_\omega \oplus V_\omega$, because $\pi_1|_{I\oplus J} \otimes \mathrm{id} : (I \oplus J) \otimes_k V_\omega \to I \otimes_k V_\omega$ corresponds to the first projection $V_\omega \oplus V_\omega \to V_\omega$. The same goes for v.

We take then the sum $s : I \oplus J \cong k \oplus k \to k$, defined by $s(i,j) = f(i) + g(j)$, and consider $K = \ker(s) \subseteq I \oplus J \subseteq A' \times_k B'$, an ideal. Since s is surjective we have an isomorphism $h : (I \oplus J)/K \cong k$.

Put now $C' = (A' \times_k B')/K$. We have a tiny extension $C' \to A \times_k B$ with kernel $(I \oplus J)/K \cong k$ (which is a sort of "sum extension" of the given ones), and the projection $\pi : A' \times_k B' \to C'$ induces a morphism of extensions

$$\begin{tikzcd}
0 \arrow[r] & I \oplus J \arrow[r] \arrow[d, "\overline{s}"] & A' \times_k B' \arrow[r] \arrow[d, "\pi"] & A \times_k B \arrow[r] \arrow[d, "\overline{\pi}"] & 0 \\
0 \arrow[r] & (I \oplus J)/K \arrow[r] & C' \arrow[r] & A \times_k B \arrow[r] & 0
\end{tikzcd}$$

where the map $\overline{s}$ is the projection to the quotient, and corresponds to the addition $+ : k \oplus k \to k$ under the isomorphisms above.

By functoriality of the obstruction we have then

$$\omega(\overline{\pi}_*(\{\xi,\eta\}), C') = (\overline{s} \otimes \mathrm{id})(\omega(\{\xi,\eta\}, A' \times_k B'))$$

which corresponds to $u + v \in V_\omega$, because the diagram

$$\begin{tikzcd}
(I \oplus J) \otimes_k V_\omega \arrow[r, "\sim"] \arrow[d, "\overline{s}\otimes\mathrm{id}"] & (k \oplus k) \otimes_k V_\omega \arrow[r, "\sim"] \arrow[d, "+\otimes\mathrm{id}"] & V_\omega \oplus V_\omega \arrow[d, "+"] \\
((I \oplus J)/K) \otimes_k V_\omega \arrow[r, "\sim"] & k \otimes_k V_\omega \arrow[r, "\sim"] & V_\omega
\end{tikzcd}$$

(where the horizontal isomorphisms are the one considered before) is commutative. So we also have $u + v \in \Omega_\omega$, and this concludes the proof. □

Next, we see that (Ω_ω, ω) is an obstruction theory.

Proposition 4.1.6. *Given a small (not necessarily tiny) extension $A' \to A$ with kernel I, and $\xi \in \mathcal{F}_{\xi_0}(A)$, we have*

$$\omega(\xi, A') \in I \otimes_k \Omega_\omega \subseteq I \otimes_k V_\omega.$$

In particular (Ω_ω, ω) is an obstruction theory for ξ_0.

Proof. Let $v_1, \ldots, v_n$ be a basis of I as a k-vector space, and write the obstruction $\omega(\xi, A') \in I \otimes_k V_\omega$ as a sum

$$\omega(\xi, A') = v_1 \otimes w_1 + \cdots + v_n \otimes w_n$$

where $w_1, \ldots, w_n \in V_\omega$. We have to show that $w_1, \ldots, w_n$ are elements of Ω_ω.

Fix $1 \leq i \leq n$, and let

$$K_i = \ker(v_i^*) = \{v \in I : \text{if we write } v = a_1 v_1 + \cdots + a_n v_n, \text{ then } a_i = 0\} \subseteq I$$

where $v_i^* : I \to k$ is the dual element of $v_i \in I$.

K_i is an ideal of A', so put $B' = A'/K_i$. We have a tiny extension $B' \to A$ with kernel $I/K_i \cong k$ (where the isomorphism is induced by v_i^*), and the projection $\pi : A' \to B'$ induces a morphism of extensions

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r] \arrow[d, "\pi|_I"] & A' \arrow[r] \arrow[d, "\pi"] & A \arrow[r] \arrow[d, "\overline{\pi}"] & 0 \\
0 \arrow[r] & I/K_i \arrow[r] & B' \arrow[r] & A \arrow[r] & 0.
\end{tikzcd}$$

By functoriality of the obstruction we get then

$$\omega(\overline{\pi}_*(\xi), B') = (\pi|_I \otimes \mathrm{id})(\omega(\xi, A'))$$

which corresponds to the i-th component w_i of $\omega(\xi, A')$ under the isomorphism $I \otimes_k V_\omega \cong k^n \otimes_k V_\omega \cong V_\omega^n$ given by the basis $v_1, \ldots, v_n$, because the diagram

$$\begin{tikzcd}
I \otimes_k V_\omega \arrow[r, "\sim"] \arrow[d, "\pi|_I \otimes \mathrm{id}"] & k^n \otimes_k V_\omega \arrow[r, "\sim"] \arrow[d, "\pi_i \otimes \mathrm{id}"] & V_\omega^n \arrow[d, "\pi_i"] \\
(I/K_i) \otimes_k V_\omega \arrow[r, "\sim"] & k \otimes_k V_\omega \arrow[r, "\sim"] & V_\omega
\end{tikzcd}$$

(where the horizontal isomorphisms are the one already considered) is commutative.

Finally notice that $\omega(\overline{\pi}_*(\xi), B')$ is the obstruction associated with a tiny extension (since $I/K_i \cong k$), so that $w_i \in \Omega_\omega$, and we are done. $\square$

After the study of miniversal deformations in Chapter 5, we will see that we can obtain a formula for the dimension of Ω_ω from a miniversal deformation of ξ_0 (provided it exists). In particular $\dim_k(\Omega_\omega)$ does not depend on the starting obstruction theory (V_ω, ω).

This also follows from the next result, which says that minimal obstruction spaces are canonical.

Proposition 4.1.7. *Let $\mathcal{F} \to (\mathrm{Art}\,/\Lambda)^{op}$ be a deformation category, (V_1, ω_1) and (V_2, ω_2) be two obstruction theories for $\xi_0 \in \mathcal{F}(k)$, and denote by Ω_1 and Ω_2 the corresponding minimal obstruction spaces. Then there is a canonical isomorphism $\varphi : \Omega_1 \cong \Omega_2$ that preserves obstructions.*

With "preserves obstructions" we mean that if $A' \to A$ is a small extension with kernel I, and $\xi \in \mathcal{F}(A)$, then

$$(\mathrm{id} \otimes \varphi)(\omega_1(\xi, A')) = \omega_2(\xi, A') \in I \otimes_k \Omega_2.$$

Proof. We define a function $\varphi : \Omega_1 \to \Omega_2$: take a vector $v \in \Omega_1$ and a tiny extension $A' \to A$, with kernel I and an isomorphism $f : I \cong k$, and an object $\xi \in \mathcal{F}(A)$, such that the image of $\omega(\xi, A') \in I \otimes_k \Omega_1$ in Ω_1 is v. We define $\varphi(v) \in \Omega_2$ to be the image of $\omega_2(\xi, A') \in I \otimes_k \Omega_2$ in Ω_2 (using the same isomorphism $f : I \cong k$).

The main point is to check that this association is well-defined. Suppose that $B' \to B$ is another tiny extension, with kernel J and an isomorphism $g : J \cong k$, and take $\eta \in \mathcal{F}(B)$, such that the image of $\omega(\eta, B') \in J \otimes_k \Omega_1$ in Ω_1 is v again. If we define $\psi(v) \in \Omega_2$ as the element corresponding to $\omega_2(\eta, B') \in J \otimes_k \Omega_2$ in Ω_2 using the isomorphism g, we have to show that $\varphi(v) = \psi(v)$.

We consider the "difference extension", defined similarly as the "sum extension" in the proof of 4.1.5: the small extension

$$0 \longrightarrow I \oplus J \longrightarrow A' \times_k B' \longrightarrow A \times_k B \longrightarrow 0$$

leads to an obstruction

$$\omega_i(\{\xi, \eta\}, A' \times_k B') \in (I \oplus J) \otimes_k \Omega_i \cong (k \oplus k) \otimes_k \Omega_i \cong \Omega_i \oplus \Omega_i$$

which in the case $i = 1$ corresponds to the pair (v, v), and if $i = 2$ to $(\varphi(v), \psi(u))$ (as in the proof above).

There is a difference homomorphism $d : I \oplus J \cong k \oplus k \to k$ defined by $d(i, j) = f(i) - g(j)$, with kernel $K = \ker(d) \subseteq I \oplus J \subseteq A' \times_k B'$. d induces an isomorphism $\overline{d} : (I \oplus J)/K \cong k$, and considering the quotient $C' = (A' \times_k B')/K$ and the projection $\pi : A' \times_k B' \to C'$ we get a morphism of small extensions

$$\begin{tikzcd}
0 \arrow[r] & I \oplus J \arrow[r] \arrow[d, "\overline{d}"] & A' \times_k B' \arrow[r] \arrow[d, "\pi"] & A \times_k B \arrow[r] \arrow[d, "\overline{\pi}"] & 0 \\
0 \arrow[r] & (I \oplus J)/K \arrow[r] & C' \arrow[r] & A \times_k B \arrow[r] & 0
\end{tikzcd}$$

(with $\overline{d}$ corresponding to the difference $- : k \oplus k \to k$) and an element

$$\omega_i(\overline{\pi}_*(\{\xi, \eta\}), C') \in (I \oplus J)/K \otimes_k \Omega_i \cong k \otimes_k \Omega_i \cong \Omega_i$$

which corresponds for $i = 1$ to $v - v = 0$, and for $i = 2$ to $\varphi(v) - \psi(v)$.

But now $\overline{\pi}_*(\{\xi, \eta\})$ will lift to C' (since its obstruction with respect to the first obstruction theory is zero), so we must also have $\omega_2(\overline{\pi}_*(\{\xi, \eta\}), C') = 0$, which implies $\varphi(v) - \psi(v) = 0$.

So we have a well-defined function $\varphi : \Omega_1 \to \Omega_2$, preserving obstructions along tiny extensions. In the same way we define $\psi : \Omega_2 \to \Omega_1$. It is clear that φ and ψ are inverse to each other, so both of them are bijective. Moreover using the fact that, if $v, w \in \Omega_i$ correspond to the obstructions of (two objects with respect to) two tiny extensions, the sum $v+w$ corresponds to the obstruction of the (induced object on the) "sum extension" as in the proof of 4.1.5, we easily see that φ is additive. k-linearity is checked in the same way, and finally we conclude that φ is an isomorphism of k-vector spaces.

By construction φ preserves the obstruction on tiny extensions. With a reasoning similar to that of the proof of 4.1.6 one can readily check that it preserves obstructions in general, and this concludes the proof. □

Remark 4.1.8. Even though the minimal obstruction space seems a good thing to have, in practice it is (in general) very hard to calculate. Because of this, in most application it suffices to have an obstruction theory that is possibly easier to calculate and more naturally defined, as in the examples we will see later on.

4.1.2 A result of unobstructedness

The following theorem (which was first stated and proved in [Kaw]) can be applied in some cases to conclude that a deformation problem is unobstructed. In this section we assume $\Lambda = k$.

Theorem 4.1.9 (Ran-Kawamata)**.** *Let $\mathcal{F} \to (\mathrm{Art}/k)^{op}$ be a deformation category, and take $\xi_0 \in \mathcal{F}(k)$. Assume that:*

- *$T_{\xi_0}\mathcal{F}$ is finite-dimensional.*
- $\mathrm{char}(k) = 0$.
- *If $A \in (\mathrm{Art}/k)$ and $\xi \in \mathcal{F}_{\xi_0}(A)$, then the functor $F_\xi : (\mathrm{FMod}/A) \to (\mathrm{Mod}/A)$ described at the end of Section 2.3 is right-exact (that is, carries surjections to surjections).*

Then ξ_0 is unobstructed.

We postpone the proof until Section 5.3.1.

Example 4.1.10. Let X be a scheme over k and consider an invertible sheaf $\mathcal{L}_0 \in \mathcal{QC}oh^X(k)$ on X. Suppose also that $\mathrm{char}(k) = 0$ and $H^1(X, \mathcal{O}_X)$ is finite-dimensional. We want to show that in this case $\mathcal{L}_0$ is unobstructed in $\mathcal{QC}oh^X$, using the Ran-Kawamata Theorem.

To do this, we consider $A \in (\mathrm{Art}/k)$ and $\mathcal{L} \in \mathcal{QC}oh^X_{\mathcal{L}_0}(A)$, and we want to understand the functor $F_{\mathcal{L}} : (\mathrm{FMod}/A) \to (\mathrm{Mod}/A)$; recall that this is defined by

$$F_{\mathcal{L}}(M) = \{\text{isomorphism classes of liftings of } \mathcal{L} \text{ to } A[M]\}.$$

Notice that if $\mathcal{L}_M$ denotes the trivial pullback of $\mathcal{L}_0$ to $A[M]$ for any $M \in (\mathrm{FMod}/A)$ (the one along the inclusion $A \to A[M]$), then there is a natural equivalence of functors $\varphi : F_{\mathcal{O}_{X_A}} \cong F_{\mathcal{L}}$ (where $X_A = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(A)$ is the trivial deformation, as usual); if $M \in (\mathrm{FMod}/A)$, the function $\varphi_M : F_{\mathcal{O}_{X_A}}(M) \to F_{\mathcal{L}}(M)$ is defined by

$$\varphi_M([\mathcal{E}]) = [\mathcal{E} \otimes_{\mathcal{O}_{X_{A[M]}}} \mathcal{L}_M].$$

Moreover one can show that there is a functorial isomorphism

$$F_{\mathcal{O}_{X_A}}(M) \cong H^1(X, M \otimes_A \mathcal{O}_{X_A}).$$

Now $M \otimes_A \mathcal{O}_{X_A} \cong M \otimes_A (A \otimes_k \mathcal{O}_X) \cong M \otimes_k \mathcal{O}_X$, and the functor $- \otimes_k \mathcal{O}_X$ is exact. Consequently $H^1(X, - \otimes_k \mathcal{O}_X) \cong H^1(X, - \otimes_A \mathcal{O}_{X_A}) \cong F_{\mathcal{O}_{X_A}}$ is exact too, and since $\mathrm{char}(k) = 0$ and

$$T_{\mathcal{L}_0} \mathcal{QC}oh^X \cong \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{L}_0, \mathcal{L}_0) \cong H^1(X, \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{L}_0)) \cong H^1(X, \mathcal{O}_X)$$

is finite-dimensional, we can apply Theorem 4.1.9, and conclude that $\mathcal{L}_0$ is unobstructed.

We have the following corollary, which is useful for example when considering deformations of abelian varieties, Calabi-Yau manifolds, K3 surfaces, etc.

Corollary 4.1.11 (Ran). *Let X_0 be a smooth and projective scheme over k (with $\mathrm{char}(k) = 0$), whose canonical sheaf ω_{X_0} is trivial (i.e. isomorphic to $\mathcal{O}_{X_0}$). Then X_0 is unobstructed.*

Proof. Since $\mathrm{char}(k) = 0$ and the tangent space $T_{X_0}\mathcal{D}ef \cong H^1(X_0, T_{X_0})$ (see Remark 2.4.3) is finite dimensional, to apply the Ran-Kawamata Theorem we only need to show, given $A \in (\mathrm{Art}/k)$ and an object $X \in \mathcal{D}ef_{X_0}(A)$, that the functor $F_X : (\mathrm{FMod}/A) \to (\mathrm{Mod}/A)$ defined by

$$F_X(M) = \{\text{isomorphism classes of liftings of } X \text{ to } A[M]\}$$

is right-exact.

If $M \in (\mathrm{FMod}/A)$, and we call $f : X \to \mathrm{Spec}(A)$ the structure morphism, then one can show (using the same techniques we used to calculate the tangent space of $\mathcal{D}ef$, in section 2.4.2) that there is a functorial isomorphism

$$F_X(M) \cong H^1(X, f^*(M) \otimes_{\mathcal{O}_X} T_{X/A}) \cong H^1(X, M \otimes_A T_{X/A}).$$

Using the results about base change of appendix C we will show that there is also a functorial isomorphism

$$H^1(X, M \otimes_A T_{X/A}) \cong M \otimes_A H^1(X, T_{X/A}) \tag{4.1}$$

which shows that the functor F_X is isomorphic to $- \otimes_A H^1(X, T_{X/A})$, and so it is right-exact.

Set $n = \dim(X_0) = \dim(X)$; since $\omega_{X_0} = \Omega^n_{X_0} \cong \mathcal{O}_{X_0}$ (recall that $\Omega^i_{X_0}$ denotes $\bigwedge^i \Omega_{X_0}$), we have a global nowhere vanishing section s of $\Omega^n_{X_0}$, which is an element of $H^0(X_0, \Omega^n_{X_0})$. By Deligne's Theorem (Theorem C.8) the natural map

$$k \otimes_A H^0(X, \Omega^n_{X/A}) \to H^0(X_0, \Omega^n_{X_0})$$

is an isomorphism, and s corresponds to a global section of $\Omega^n_{X/A}$ that is nowhere vanishing as well, since $|X| = |X_0|$.

From the existence of this section we get that $\Omega^n_{X/A} \cong \mathcal{O}_X$. Moreover for each $j \leq n$ we have a bilinear nondegenerate pairing

$$\Omega^j_{X/A} \times \Omega^{n-j}_{X/A} \to \Omega^n_{X/A} \cong \mathcal{O}_X$$

that induces then an isomorphism $\Omega^{n-j}_{X/A} \cong (\Omega^j_{X/A})^\vee$.

This implies in particular that $T_{X/A} = (\Omega^1_{X/A})^\vee \cong \Omega^{n-1}_{X/A}$, which by Deligne's Theorem again satisfies base change, and then we have our functorial isomorphism 4.1. This concludes the proof, as we already noticed. $\square$

4.2 Examples

We describe now an obstruction theory for each of our main examples, and give a classical example of a variety with nontrivial obstructions.

4.2.1 Schemes

We consider the category of deformations of schemes $\mathcal{D}ef \to (\mathrm{Art}/\Lambda)^{op}$, and $X_0 \in \mathcal{D}ef(k)$ a local complete intersection, generically smooth scheme of finite type over k.

Theorem 4.2.1. *With the hypotheses above, there is an obstruction theory (V_ω, ω) for X_0 with vector space*

$$V_\omega = \mathrm{Ext}^2_{\mathcal{O}_{X_0}}(\Omega_{X_0}, \mathcal{O}_{X_0}).$$

A proof of this theorem can be found in [Vis].

Remark 4.2.2. If X_0 is also affine, then $\mathrm{Ext}^2_{\mathcal{O}_{X_0}}(\Omega_{X_0}, \mathcal{O}_{X_0}) = 0$. In particular deformations of an affine X_0 with the hypotheses above are unobstructed.

Let $\mathrm{Spec}(A) = X_0 \subseteq \mathbb{A}^n_k$ be a closed immersion, with sheaf of ideals I_0. Then as in Remark 2.4.8 the conormal sequence

$$0 \longrightarrow I_0/I_0^2 \xrightarrow{d} \Omega_{\mathbb{A}^n_k}|_{X_0} \longrightarrow \Omega_{X_0} \longrightarrow 0 \qquad (4.2)$$

is exact also on the left. Notice that I_0/I_0^2 and $\Omega_{\mathbb{A}^n_k}|_{X_0}$ are locally free in this case, hence projective, and so sequence 4.2 is a projective resolution of Ω_{X_0}. This implies that $\mathrm{Ext}^2_{\mathcal{O}_{X_0}}(\Omega_{X_0}, \mathcal{O}_{X_0})$ is trivial.

Remark 4.2.3. As with the tangent space, in the general case one can still find an obstruction theory for X_0, by using the cotangent complex. In general X_0 has an obstruction theory with obstruction space

$$V_\omega = \mathrm{Ext}^2_{\mathcal{O}_{X_0}}(L_{X_0/k}, \mathcal{O}_{X_0})$$

(see Théorème 2.1.7 of Chapter III in [Ill]).

4.2.2 Smooth varieties

We give a proof of the theorem above only in the case of smooth varieties, which can be studied using Čech cohomology. Consider the deformation category $\mathcal{D}ef \to (\mathrm{Art}/\Lambda)^{op}$.

Theorem 4.2.4. *Every smooth variety $X_0 \in \mathcal{D}ef(k)$ has an obstruction theory (V_ω, ω) with obstruction space*

$$V_\omega = H^2(X_0, T_{X_0}).$$

Proof. Let $A' \to A$ be a small extension with kernel I, and $X \in \mathcal{D}ef_{X_0}(A)$ be a deformation of X_0 over $\mathrm{Spec}(A)$. We show how to construct the element $\omega(X, A') \in I \otimes_k H^2(X_0, T_{X_0})$.

Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open affine cover of X_0, and denote by $X|_{U_i}$ the induced deformation of U_i over A, obtained just by considering $U_i \subseteq X_0$ as an open subscheme of X (recall that $|X| = |X_0|$). By Remark 4.2.2 we have that U_i is unobstructed, and so we can find deformations $Y_i \in \mathcal{D}ef_{U_i}(A')$ such that the restriction of each Y_i to A is $X|_{U_i}$.

Now notice that, since U_{ij} is affine as well, the restrictions $Y_i|_{U_{ij}}$ and $Y_j|_{U_{ij}}$ are isomorphic deformations of U_{ij} over A', by Remark 2.4.3 and Corollary 2.3.2. For each pair of indices there exists then an isomorphism of deformations

$$\theta_{ij} : Y_j|_{U_{ij}} \to Y_i|_{U_{ij}}$$

which restricts to the identity of $X|_{U_{ij}}$ on the pullback to A, and for each triplet of indices we can consider the composite

$$\theta_{ijk} = \theta_{ij} \circ \theta_{jk} \circ \theta_{ik}^{-1}.$$

Each θ_{ijk} is an automorphism of the deformation $Y_i|_{U_{ijk}}$ of U_{ijk} over A' that restricts to the identity on the pullback to A, and so by Propositions 3.1.5 and 3.2.1 it corresponds to an element

$$d_{ijk} \in I \otimes_k \mathrm{Inf}(U_{ijk}) \cong I \otimes_k \mathrm{Der}_k(A_{ijk}, A_{ijk}) \cong \Gamma(U_{ijk}, I \otimes_k T_{X_0})$$

where $\mathrm{Spec}(A_{ijk}) = U_{ijk}$.

The family $\{d_{ijk}\}_{i,j,k\in I}$ is a Čech 2-cocycle for the sheaf $I \otimes_k T_{X_0}$, with respect to the cover $\mathcal{U}$: we have to show that for every quadruple of indices $i, j, k, l \in I$ we have

$$d_{jkl} - d_{ikl} + d_{ijl} - d_{ijk} = 0$$

as elements of $\Gamma(U_{ijkl}, I \otimes_k T_{X_0}) \cong I \otimes_k \mathrm{Inf}(U_{ijkl})$. We rewrite this as

$$d_{ijl} - d_{ikl} - d_{ijk} = -d_{jkl} \tag{4.3}$$

and notice that the left-hand side corresponds (under the isomorphism of Proposition 3.1.5) to the infinitesimal automorphism

$$\theta_{ijl} \circ \theta_{ikl}^{-1} \circ \theta_{ijk}^{-1} = \theta_{ij} \circ \theta_{jkl}^{-1} \circ \theta_{ij}^{-1}$$

of the deformation $Y_i|_{U_{ijkl}}$ of U_{ijkl}, and the right-hand side to the infinitesimal automorphism θ_{jkl}^{-1} of the deformation $Y_j|_{U_{ijkl}}$. Moreover the restriction of θ_{ij} is an isomorphism between these two deformations, and so 4.3 follows from Remark 3.1.6.

Then we have an element $[\{d_{ijk}\}_{i,j,k\in I}]$ of

$$\check{H}^2(\mathcal{U}, I \otimes_k T_{X_0}) \cong H^2(X_0, I \otimes_k T_{X_0}) \cong I \otimes_k H^2(X_0, T_{X_0})$$

that we call $\omega(X, A')$. We check that it is independent of the choice of the θ_{ij}'s: let $\{\nu_{ij}\}_{i,j\in I}$ be another collection of isomorphisms as above. Then for any pair of indices $\nu_{ij} \circ \theta_{ij}^{-1}$ is an infinitesimal automorphism of the deformation $Y_i|_{U_{ij}}$ of U_{ij}, so that (again by Proposition 3.2.1) it corresponds to an element

$$e_{ij} \in I \otimes_k \mathrm{Der}_k(B_{ij}, B_{ij}) \cong \Gamma(U_{ij}, I \otimes_k T_{X_0})$$

(where $\mathrm{Spec}(B_{ij}) = U_{ij}$). Moreover, if $f_{ijk} \in \Gamma(U_{ijk}, I \otimes_k T_{X_0})$ are the sections corresponding to the automorphisms $\nu_{ijk} = \nu_{ij} \circ \nu_{jk} \circ \nu_{ik}^{-1}$, one easily checks that

$$f_{ijk} = d_{ijk} + (e_{ij} + e_{jk} - e_{ik})$$

which says exactly that $\{f_{ijk}\}_{i,j,k\in I}$ and $\{d_{ijk}\}_{i,j,k\in I}$ are cohomologous, and so define the same element in $\check{H}^2(\mathcal{U}, I \otimes_k T_{X_0})$.

It is also clear that each cocycle in this cohomology class corresponds to a family of isomorphisms, just by reversing this construction. Finally one can check as in Section 2.4.2 that the $\omega(X, A')$ defined does not depend on the open affine cover $\mathcal{U}$ of X_0.

Notice now that the element $\omega(X, A') \in I \otimes_k H^2(X_0, T_{X_0})$ vanishes if and only if a lifting of X to A' exists: $d_{ijk} = 0$ above corresponds to $\theta_{ij} \circ \theta_{jk} = \theta_{ik}$, which is exactly the condition that lets us construct a lifting of X to A' by patching the local liftings Y_i along the restrictions to the intersections U_{ij}. On the other hand if $X' \in \mathcal{D}ef_{X_0}(A')$ is a lifting of X, then (by the arguments already used above) the restriction $X'|_{U_i}$ will be isomorphic to Y_i, and this implies that there is a choice of the θ_{ij}'s that satisfies the cocycle condition, and so $\omega(X, A') = 0$.

The functoriality property is an easy consequence of the functoriality of the isomorphism we constructed in the proof of Proposition 3.2.1. □

Remark 4.2.5. In particular if X_0 is a smooth curve, then $H^2(X_0, T_{X_0}) = 0$ and so X_0 is unobstructed.

Remark 4.2.6. The preceding proof shows a typical pattern that can be used in other cases to construct obstructions. Here is the (rather vague) idea: if our deformation problem has an underlying scheme X (X_0 in the case above), and it localizes naturally on this scheme, in the sense that every deformation over X induces one over any of its open subschemes (just by restriction, in the case above), and moreover:

- Infinitesimal automorphisms form a sheaf $\mathcal{I}$ on X.
- Locally we always have liftings.
- Two liftings of the same deformation are always locally isomorphic.
- We can reconstruct our deformations from local compatible data.

Then we can mimic the preceding proof to construct an obstruction theory, with space $H^2(X, \mathcal{I})$ (see [Oss]).

We show an example of a smooth variety X_0 such that $H^2(X_0, T_{X_0}) \neq 0$, but nevertheless X_0 is unobstructed, so the map ω must be zero.

Proposition 4.2.7. *Let $Z_0 \subseteq \mathbb{P}^3_k$ be a smooth surface of degree $d \geq 6$. Then $H^2(Z_0, T_{Z_0}) \neq 0$, but Z_0 is unobstructed.*

Proof. The fact that Z_0 is unobstructed is immediate from Propositions 2.4.12 and 2.4.13: given an object $Z \in \mathcal{D}ef_{Z_0}(A)$ and a small extension $A' \to A$, because of 2.4.13 we have a closed immersion $Z \subseteq \mathbb{P}^3_A$, which by 2.4.12 lifts to some $Z' \subseteq \mathbb{P}^3_{A'}$ over A', and forgetting the immersion this gives a lifting $Z' \in \mathcal{D}ef_{Z_0}(A')$ of Z to A'.

The fact that $H^2(Z_0, T_{Z_0}) \neq 0$ is proved by the following calculation, similar to the ones we used in the proof of 2.4.10. From the dual of the conormal sequence of $Z_0 \subseteq \mathbb{P}^3_k$

$$0 \longrightarrow T_{Z_0} \longrightarrow T_{\mathbb{P}^3_k}|_{Z_0} \longrightarrow \mathcal{O}_{Z_0}(d) \longrightarrow 0$$

we see that it suffices to show that $H^2(Z_0, T_{\mathbb{P}^3_k}|_{Z_0}) \neq 0$. By Serre's duality we have

$$H^2(Z_0, T_{\mathbb{P}^3_k}|_{Z_0}) \cong H^0(Z_0, \Omega_{\mathbb{P}^3_k}(d-4)|_{Z_0})^\vee$$

and by the twisted and restricted Euler sequence

$$0 \longrightarrow \Omega_{\mathbb{P}^3_k}(d-4)|_{Z_0} \longrightarrow \mathcal{O}_{Z_0}(d-5)^{\oplus 4} \longrightarrow \mathcal{O}_{Z_0}(d-4) \longrightarrow 0$$

it is sufficient to check that

$$\dim_k(H^0(Z_0, \mathcal{O}_{Z_0}(d-5))^4) > \dim_k(H^0(Z_0, \mathcal{O}_{Z_0}(d-4)))$$

or, using known formulas for the dimensions above, that

$$4\binom{d-2}{3} > \binom{d-1}{3}$$

for $d \geq 6$, which is easy to check. $\square$

4.2.3 An obstructed variety

Using the calculations done with Čech cohomology up to this point, we now describe a classical example of Kodaira of a variety over $\mathbb{C}$ with non-trivial obstructions.

Let X be a smooth variety over k. Recall that the tangent sheaf $T_X = \mathcal{H}om(\Omega_X, \mathcal{O}_X)$ has a natural structure of sheaf of Lie algebras over k: on an open affine subset $U = \mathrm{Spec}(A)$, we have $T_X(U) = \mathrm{Der}_k(A, A)$, and given two derivations $D, E : A \to A$ we can define $[D, E] : A \to A$ by

$$[D, E](x) = D(E(x)) - E(D(x)).$$

It is immediate to check that $[D, E]$ is still a k-derivation, and that this product gives a structure of Lie algebra over k to $T_X(U)$. Moreover this local construction gives a global Lie product $[\cdot, \cdot] : T_X \times T_X \to T_X$, which, being k-bilinear, induces $T_X \otimes_k T_X \to T_X$, which we still denote by $[\cdot, \cdot]$.

Moreover if we fix an open affine cover of X, say $\mathcal{U} = \{U_i\}_{i \in I}$, then there is a product

$$\check{H}^1(\mathcal{U}, T_X) \times \check{H}^1(\mathcal{U}, T_X) \to \check{H}^2(\mathcal{U}, T_X \otimes_k T_X)$$

in Čech cohomology, induced by the tensor product (see for example II, § 6 of [God]).

From this product and its properties we get a quadratic form $\check{H}^1(\mathcal{U}, T_X) \to \check{H}^2(\mathcal{U}, T_X)$, defined as the composite

$$\check{H}^1(\mathcal{U}, T_X) \to \check{H}^1(\mathcal{U}, T_X) \times \check{H}^1(\mathcal{U}, T_X) \to \check{H}^2(\mathcal{U}, T_X \otimes_k T_X) \to \check{H}^2(\mathcal{U}, T_X)$$

where the first map is just the diagonal $v \mapsto (v, v)$, the second is induced by the tensor product, and the last one by the Lie product. If we represent an element $v \in \check{H}^1(\mathcal{U}, T_X)$ as a 1-cocycle $\{a_{ij}\}_{i,j \in I}$, then the image of v

along the above map, which we denote by $[v, v]$, is given by the class of the 2-cocycle $\{[a_{ij}, a_{jk}]\}_{i,j,k\in I}$.

We remark that there is an analogous quadratic form $H^1(X, T_X) \to H^2(X, T_X)$ induced in a similar same way, and that is compatible with the preceding one and the canonical isomorphisms $\check{H}^1(\mathcal{U}, T_X) \cong H^1(X, T_X)$ and $\check{H}^2(\mathcal{U}, T_X) \cong H^2(X, T_X)$.

Finally recall that if X_0 is a smooth variety, then $T_{X_0}\mathcal{D}ef \cong H^1(X_0, T_{X_0})$ and we have an obstruction theory with vector space $H^2(X_0, T_{X_0})$.

Proposition 4.2.8. *Let X_0 be a smooth variety over k, with $\mathrm{char}(k) \neq 2$, $\mathcal{U} = \{U_i\}_{i\in I}$ an open affine cover of X_0. Then the map*

$$\Phi : \check{H}^1(\mathcal{U}, T_{X_0}) \to \check{H}^2(\mathcal{U}, T_{X_0})$$

defined by $\Phi(v) = \frac{1}{2}[v, v]$ has the property that $\Phi(v) = 0$ if and only if a first-order deformation X_v associated with v can be lifted to $k[t]/(t^3)$.

Proof. With the notation of Section 2.4.2, we take an element v, a first-order deformation X_v corresponding to v, and the associated 1-cocycle $\{d_{ij}\}_{i,j\in I}$. Recall that d_{ij} is the derivation associated with the infinitesimal automorphism θ_{ij} of $U_{ij} \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k[t]/(t^2))$; in other words we have

$$\theta_{ij}^\sharp(f + tg) = f + t(d_{ij}(f) + g)$$

where we see $\mathcal{O}_{X_v}|_{U_{ij}}$ as $\mathcal{O}_{U_{ij}} \oplus ((t) \otimes_k \mathcal{O}_{U_{ij}})$ and f, g are sections of $\mathcal{O}_{U_{ij}}$ (see proposition 3.2.1).

Now, by the proof of Theorem 4.2.4, X_v will lift to $k[t]/(t^3)$ if and only if the θ_{ij}'s lift to automorphisms ν_{ij} of $U_{ij} \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k[t]/(t^3))$, satisfying the cocycle condition $\nu_{ij} \circ \nu_{jk} = \nu_{ik}$. This in turn is equivalent to the existence of automorphism of sheaves of $k[t]/(t^3)$-algebras

$$\varphi_{ij} : \mathcal{O}_{U_{ij}} \otimes_k k[t]/(t^3) \to \mathcal{O}_{U_{ij}} \otimes_k k[t]/(t^3)$$

that restrict to the $\theta_{ij}^\sharp$'s on $\mathcal{O}_{U_{ij}} \oplus ((t) \otimes_k \mathcal{O}_{U_{ij}})$, and satisfy an analogous cocycle relation.

Let us try to construct such automorphisms: since they must extend the $\theta_{ij}^\sharp$'s, if f is a section of $\mathcal{O}_{U_{ij}}$ we must have

$$\varphi_{ij}(f) = f + td_{ij}(f) + t^2 e_{ij}(f)$$

where we see $\mathcal{O}_{U_{ij}} \otimes_k k[t]/(t^3)$ as $\mathcal{O}_{U_{ij}} \oplus ((t) \otimes_k \mathcal{O}_{U_{ij}}) \oplus ((t^2) \otimes_k \mathcal{O}_{U_{ij}})$, and e_{ij} is a function $\mathcal{O}_{U_{ij}} \to \mathcal{O}_{U_{ij}}$. Because of the $k[t]/(t^3)$-linearity of φ_{ij}, e_{ij} will actually be a k-linear homomorphism, and conversely a k-linear homomorphism $e_{ij} : \mathcal{O}_{U_{ij}} \to \mathcal{O}_{U_{ij}}$ will give a $k[t]/(t^3)$-linear φ_{ij}.

Moreover (using linearity) we see that φ_{ij} is completely determined by d_{ij} and e_{ij}, by the formula

$$\varphi_{ij}(f + tg + t^2h) = f + t(d_{ij}(f) + g) + t^2(d_{ij}(g) + e_{ij}(f) + h)$$

where f, g, h are sections of $\mathcal{O}_{U_{ij}}$.

Now we turn to the conditions on e_{ij} that correspond to φ_{ij} being a homomorphism of algebras. If f, g are sections of $\mathcal{O}_{U_{ij}}$ we have

$$\begin{aligned}\varphi_{ij}(f)\varphi_{ij}(g) &= (f + td_{ij}(f) + t^2 e_{ij}(f))(g + td_{ij}(g) + t^2 e_{ij}(g)) \\ &= fg + t(fd_{ij}(g) + gd_{ij}(g)) \\ &\quad + t^2(fe_{ij}(g) + ge_{ij}(f) + d_{ij}(f)d_{ij}(g))\end{aligned}$$

and on the other hand

$$\varphi_{ij}(fg) = fg + td_{ij}(fg) + t^2 e_{ij}(fg).$$

From these formulas (and recalling that d_{ij} is a derivation) we obtain that φ_{ij} is an homomorphism of algebras if and only if

$$e_{ij}(fg) = fe_{ij}(g) + ge_{ij}(f) + d_{ij}(f)d_{ij}(g) \tag{4.4}$$

for f, g sections of $\mathcal{O}_{U_{ij}}$ (for the "if" part notice that the sections of $\mathcal{O}_{U_{ij}}$ generate the whole $\mathcal{O}_{U_{ij}} \otimes_k k[t]/(t^3)$ as a $k[t]/(t^3)$-algebra).

We consider now $d_{ij}^2 = d_{ij} \circ d_{ij}$, which satisfies

$$d_{ij}^2(fg) = d_{ij}(fd_{ij}(g) + gd_{ij}(f)) = fd_{ij}^2(g) + gd_{ij}^2(f) + 2d_{ij}(f)d_{ij}(g)$$

where f, g are sections of $\mathcal{O}_{U_{ij}}$, so if we set

$$h_{ij} = e_{ij} - \frac{1}{2}d_{ij}^2$$

then h_{ij} is a derivation (i.e. an element of $\Gamma(U_{ij}, T_{X_0})$) if and only if 4.4 holds.

In other words, the automorphisms of $\mathcal{O}_{U_{ij}} \otimes_k k[t]/(t^3)$ as a $k[t]/(t^3)$-algebras that extend the θ_{ij}'s are of the form

$$\varphi_{ij}(f) = f + td_{ij}(f) + t^2\left(h_{ij}(f) + \frac{1}{2}d_{ij}^2(f)\right)$$

for some $h_{ij} \in \Gamma(U_{ij}, T_{X_0})$ and all sections f of $\mathcal{O}_{U_{ij}}$ (the inverse is obtained taking $-d_{ij}$ and $-h_{ij}$).

Finally we examine the cocycle condition $\varphi_{ij} \circ \varphi_{jk} = \varphi_{ik}$: we have

$$\begin{aligned}\varphi_{ij}(\varphi_{jk}(f)) &= \varphi_{ij}(f + td_{jk}(f) + t^2 h_{jk}(f)) \\ &= f + t(d_{ij}(f) + d_{jk}(f)) \\ &\quad + t^2\left(h_{ij}(f) + h_{jk}(f) + \tfrac{1}{2}d_{ij}^2(f) + \tfrac{1}{2}d_{jk}^2(f) + d_{ij}(d_{jk}(f))\right)\end{aligned}$$

whereas

$$\varphi_{ik}(f) = f + td_{ik}(f) + t^2\left(h_{ik}(f) + \frac{1}{2}d_{ik}^2(f)\right).$$

Equating the coefficients of t^2 (and using the cocycle condition $d_{ij} + d_{jk} = d_{ik}$ for d_{ij} in the last formula) we get

$$h_{ij} + h_{jk} - h_{ik} = \frac{1}{2}\left(d_{jk}(d_{ij}(f)) - d_{ij}(d_{jk}(f))\right) = -\frac{1}{2}[d_{ij}, d_{jk}].$$

In conclusion, the cocycle condition holds for the φ_{ij}'s (and so a lifting to $k[t]/(t^3)$ exists) if and only if there is some cochain $\{h_{ij}\}_{i,j\in I}$ of T_{X_0} such that

$$\left\{\frac{1}{2}[d_{ij}, d_{jk}]\right\}_{i,j,k\in I} = -\partial\{h_{ij}\}_{i,j\in I}$$

or in other words if and only if $\frac{1}{2}[v, v] = 0$. □

Example 4.2.9. Because of the proposition just proved, to give an example of an obstructed variety it suffices to find one such that the map Φ above is nonzero, or equivalently such that the quadratic for given by the Lie product is not identically zero.

We take $X_0 = \mathbb{P}^1_{\mathbb{C}} \times_{\mathrm{Spec}(\mathbb{C})} Y$ where Y is an abelian variety over $\mathbb{C}$ of dimension at least 2. Then the product

$$H^1(Y, \mathcal{O}_Y) \otimes_{\mathbb{C}} H^1(Y, \mathcal{O}_Y) \to H^2(Y, \mathcal{O}_Y)$$

induced by the tensor product is not identically zero, because in this case it coincides with the wedge product of the graded algebra $\bigwedge H^1(Y, \mathcal{O}_Y)$ (this can be found for example in Chapter 1 of [Mum]); moreover this is also true for the product

$$H^0(\mathbb{P}^1_{\mathbb{C}}, T_{\mathbb{P}^1_{\mathbb{C}}}) \otimes_{\mathbb{C}} H^0(\mathbb{P}^1_{\mathbb{C}}, T_{\mathbb{P}^1_{\mathbb{C}}}) \to H^0(\mathbb{P}^1_{\mathbb{C}}, T_{\mathbb{P}^1_{\mathbb{C}}})$$

induced by the Lie product. In fact if z is a coordinate on $\mathbb{A}^1_{\mathbb{C}} = \mathbb{P}^1_{\mathbb{C}} \setminus \{\infty\}$, then $\frac{\partial}{\partial z}$ and $z\frac{\partial}{\partial z}$ are two global sections of $T_{\mathbb{P}^1_{\mathbb{C}}}$, and we have

$$\left[\frac{\partial}{\partial z}, z\frac{\partial}{\partial z}\right] = \frac{\partial}{\partial z} \neq 0.$$

Take $a, b \in H^1(Y, \mathcal{O}_Y)$ such that $a \otimes b \neq 0$ and $D, E \in H^0(\mathbb{P}^1_{\mathbb{C}}, T_{\mathbb{P}^1_{\mathbb{C}}})$ such that $[D, E] \neq 0$, and call $\pi_1 : X_0 \to \mathbb{P}^1_{\mathbb{C}}$ and $\pi_2 : X_0 \to Y$ the two projections; we will denote by π_i^* the pullback maps induced on sections of $\mathcal{O}_Y$ and $T_{\mathbb{P}^1_{\mathbb{C}}}$.

We have then two induced elements aD and bE of $H^1(X_0, T_{X_0})$ that we describe using Čech cohomology. We choose an open affine cover $\mathcal{U} = \{U_i\}_{i\in I}$ of Y and one $\mathcal{V} = \{V_\alpha\}_{\alpha\in A}$ of $\mathbb{P}^1_{\mathbb{C}}$, and suppose that a is represented by a 1-cocycle $\{a_{ij}\}_{i,j\in I}$, and D by a 0-cocycle $\{D_\alpha\}_{\alpha\in A}$. We have then a product affine cover $\{U_i \times_{\mathrm{Spec}(\mathbb{C})} V_\alpha\}_{(i,\alpha)\in I\times A}$ of X_0 that we denote by

$\mathcal{U} \times \mathcal{V}$. The element aD is defined by the composite

$$
\begin{array}{ccc}
\check{H}^1(\mathcal{U}, \mathcal{O}_Y) \otimes_{\mathbb{C}} \check{H}^0(\mathcal{V}, T_{\mathbb{P}^1_{\mathbb{C}}}) & & [\{a_{ij}\}] \otimes [\{D_\alpha\}] \\
\downarrow & & \downarrow \\
\check{H}^1(\mathcal{U} \times \mathcal{V}, \mathcal{O}_{X_0}) \otimes_{\mathbb{C}} \check{H}^0(\mathcal{U} \times \mathcal{V}, T_{X_0}) & & [\{\pi_2^*(a_{ij})\}] \otimes [\{\pi_1^*(D_\alpha)\}] \\
\downarrow & & \downarrow \\
\check{H}^1(\mathcal{U} \times \mathcal{V}, \mathcal{O}_{X_0} \otimes_{\mathbb{C}} T_{X_0}) & & [\{\pi_2^*(a_{ij}) \otimes \pi_1^*(D_\alpha)\}] \\
\downarrow & & \downarrow \\
\check{H}^1(\mathcal{U} \times \mathcal{V}, T_{X_0}) & & [\{\pi_2^*(a_{ij})\pi_1^*(D_\alpha)\}]
\end{array}
$$

and bE is defined in the same way. Here the first map from the top is induced by the two pullbacks, the second by the tensor product, and the last one by the homomorphism $\mathcal{O}_{X_0} \otimes_{\mathbb{C}} T_{X_0} \to T_{X_0}$ that corresponds to scalar multiplication of sections of T_{X_0} by sections of $\mathcal{O}_{X_0}$; moreover the various π_1^* and π_2^* on the right are actually the pullbacks along the projections from $U_{ij} \times_{\mathrm{Spec}(\mathbb{C})} V_\alpha$, but to avoid introducing a heavy notation we will not include these indices.

We consider then the Lie product $[aD, bE] \in H^2(X_0, T_{X_0})$, and claim that this is nonzero. This will show that the quadratic form induced by the Lie product is not trivial, and that X_0 is obstructed.

Writing $b = [\{b_{ij}\}_{i,j \in I}]$ and $E = [\{E_\alpha\}_{\alpha \in A}]$ in the same way, as we recalled at the beginning of this section the Lie product $[aD, bE]$ will be represented by the 2-cocycle

$$\{[\pi_2^*(a_{ij})\pi_1^*(D_\alpha), \pi_2^*(b_{jk})\pi_1^*(E_\alpha)]\}_{i,j,k \in I, \alpha \in A}.$$

Fixing the indices $i, j, k \in I$ and $\alpha \in A$, we have that

$$
\begin{aligned}
[\pi_2^*(a_{ij})\pi_1^*(D_\alpha), \pi_2^*(b_{jk})\pi_1^*(E_\alpha)] \quad = \quad & \pi_2^*(a_{ij})\pi_1^*(D_\alpha)(\pi_2^*(b_{jk})\pi_1^*(E_\alpha)) \\
& -\pi_2^*(b_{jk})\pi_1^*(E_\alpha)(\pi_2^*(a_{ij})\pi_1^*(D_\alpha)).
\end{aligned}
$$

But now $\pi_1^*(D_\alpha)(\pi_2^*(b_{jk}))$ and $\pi_1^*(E_\alpha)(\pi_2^*(a_{ij}))$ are zero: this is because we have compatible isomorphisms

$$
\begin{aligned}
\mathcal{O}_{X_0} &\cong \pi_1^*(\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}) \oplus \pi_2^*(\mathcal{O}_Y) \\
T_{X_0} &\cong \pi_1^*(T_{\mathbb{P}^1_{\mathbb{C}}}) \oplus \pi_2^*(T_Y)
\end{aligned}
$$

and if f is a section of $\mathcal{O}_{X_0}$ coming from $\mathcal{O}_Y$, and D is a section of T_{X_0} coming from $T_{\mathbb{P}^1_{\mathbb{C}}}$, then $D(f) = 0$ (taking local coordinates z for $\mathbb{P}^1_{\mathbb{C}}$, and $x_1, \ldots, x_n$ for Y, D will be a linear combination of $\frac{\partial}{\partial z}, z\frac{\partial}{\partial z}, z^2\frac{\partial}{\partial z}$ and f will be a function of $x_1, \ldots, x_n$).

So we have

$$\begin{aligned}
[\pi_2^*(a_{ij})\pi_1^*(D_\alpha), \pi_2^*(b_{jk})\pi_1^*(E_\alpha)] &= \pi_2^*(a_{ij})\pi_2^*(b_{jk})\pi_1^*(D_\alpha)(\pi_1^*(E_\alpha)) \\
&\quad -\pi_2^*(a_{ij})\pi_2^*(b_{jk})\pi_1^*(E_\alpha)(\pi_1^*(D_\alpha)) \\
&= \pi_2^*(a_{ij}b_{jk})[\pi_1^*(D_\alpha), \pi_1^*(E_\alpha)] \\
&= \pi_2^*(a_{ij}b_{jk})(\pi_1^*([D_\alpha, E_\alpha]))
\end{aligned}$$

which is a (component of a) cocycle representing the element $(ab)[D,E] \in H^2(X_0, T_{X_0})$, defined as the preceding ones by the composite

$$\begin{array}{ccc}
\check{H}^2(\mathcal{U}, \mathcal{O}_Y) \otimes_{\mathbb{C}} \check{H}^0(\mathcal{V}, T_{\mathbb{P}^1_{\mathbb{C}}}) & & [\{a_{ij}b_{jk}\}] \otimes [\{[D_\alpha, E_\alpha]\}] \\
\downarrow & & \downarrow \\
\check{H}^2(\mathcal{U} \times \mathcal{V}, \mathcal{O}_{X_0}) \otimes_{\mathbb{C}} \check{H}^0(\mathcal{U} \times \mathcal{V}, T_{X_0}) & & [\{\pi_2^*(a_{ij}b_{jk})\}] \otimes [\{\pi_1^*([D_\alpha, E_\alpha])\}] \\
\downarrow & & \downarrow \\
\check{H}^2(\mathcal{U} \times \mathcal{V}, T_{X_0}) & & [\{\pi_2^*(a_{ij}b_{jk})(\pi_1^*([D_\alpha, E_\alpha]))\}]
\end{array}$$

Finally we deduce that $(ab)[D,E] \in H^2(X_0, T_{X_0})$ is not zero, using the Künneth formula: in our particular case (for more on the Künneth formula see for example VI, § 8 of [Mil]) it implies that there is a canonical isomorphism

$$\bigoplus_{i+j=2} H^i(Y, \mathcal{O}_Y) \otimes_{\mathbb{C}} H^j(\mathbb{P}^1_{\mathbb{C}}, T_{\mathbb{P}^1_{\mathbb{C}}}) \xrightarrow{\sim} H^2(X_0, T_{X_0})$$

and from its definition it is immediate that the restriction

$$H^2(Y, \mathcal{O}_Y) \otimes_{\mathbb{C}} H^0(\mathbb{P}^1_{\mathbb{C}}, T_{\mathbb{P}^1_{\mathbb{C}}}) \to H^2(X_0, T_{X_0})$$

(which is then injective) carries $ab \otimes [D,E]$ to the product $(ab)[DE]$ defined above. Since by hypothesis $ab = a \otimes b \neq 0$ and $[D,E] \neq 0$, we get that $(ab)[D,E] \in H^2(X_0, T_{X_0})$ is also not zero, and since it coincides with $[aD, bE]$ we are done.

4.2.4 Closed subschemes

Now we consider the deformation category $\mathcal{H}ilb^X \to (\mathrm{Art}\,/\Lambda)^{op}$, with X separated. Take $Z_0 \in \mathcal{H}ilb^X(k)$ (which is separated as well), call I_0 its sheaf of ideals, and $\mathcal{N}_0$ the normal sheaf $\mathcal{N}_0 = \mathcal{H}om(I_0/I_0^2, \mathcal{O}_{Z_0})$.

Here we add the hypothesis that Z_0 is a local complete intersection in $X_0 = X \times_{\mathrm{Spec}(\Lambda)} \mathrm{Spec}(k)$. In this case one can show (see [Vis] for the proof) that for any $A \in (\mathrm{Art}\,/\Lambda)$, any $Z \in \mathcal{H}ilb^X_{Z_0}(A)$ is a local complete intersection in X_A.

Theorem 4.2.10. *There is an obstruction theory (V_ω, ω) for Z_0, with obstruction space*

$$V_\omega = H^1(Z_0, \mathcal{N}_0).$$

Proof. Take a small extension $A' \to A$, $Z \in \mathcal{H}ilb^X_{Z_0}(A)$ and call $I \subseteq \mathcal{O}_{X_A}$ its sheaf of ideals, where $X_A = X \times_{\mathrm{Spec}(\Lambda)} \mathrm{Spec}(A)$. By the remark above, Z is a local complete intersection in X_A. Let us show first that liftings always exist locally.

Take an open affine subscheme $U_0 = \mathrm{Spec}(R)$ of X, and the corresponding ones $U = \mathrm{Spec}(R \otimes_\Lambda A)$ and $U' = \mathrm{Spec}(R \otimes_\Lambda A')$ of X_A and X'_A, such that Z is a complete intersection in U. This means that the ideal sheaf I of $Z \cap U$ is generated by a regular sequence $x_1, \ldots, x_n \in R \otimes_\Lambda A$; consider liftings $y_1, \ldots, y_n \in R \otimes_\Lambda A'$. We define $I' = (y_1, \ldots, y_n)$ and $S = (R \otimes_\Lambda A')/I'$, and check that the closed subscheme $Z' = \mathrm{Spec}(S) \subseteq U'$ is a lifting of $Z \cap U \subseteq U$ to A'.

It is clear that the restriction of Z' to X_A will be Z, so the only thing to check is that Z' is flat over A'. For this we use the local flatness criterion: starting from the exact sequence

$$0 \longrightarrow I' \longrightarrow R \otimes_\Lambda A' \longrightarrow S \longrightarrow 0$$

and taking the Tor long exact sequence (tensoring with A over A') we get

$$\mathrm{Tor}_1^{A'}(S, A) \longrightarrow I' \otimes_{A'} A \longrightarrow R \otimes_\Lambda A \longrightarrow (R \otimes_\Lambda A)/I \longrightarrow 0$$

where the next term $\mathrm{Tor}_1^{A'}(R \otimes_\Lambda A', A)$ is zero, because $X_{A'}$ is flat over A'. Then to show that $\mathrm{Tor}_1^{A'}(S, A) = 0$ (and conclude that Z' is flat over A') it suffices to show that the natural map $I' \otimes'_A A = I'/II' \to I$ is an isomorphism.

Since it is clearly surjective, we show that the kernel of $I' \to I$ is II': take an element $a_1 y_1 + \cdots + a_n y_n \in I'$ such that its image $b_1 x_1 + \cdots + b_n x_n$ in I is zero. Since $x_1, \ldots, x_n$ is a regular sequence, we have that $(b_1, \ldots, b_n) \in (R \otimes_\Lambda A)^n$ is a linear combination of standard relations of the form

$$x_{ij} = (0, \ldots, x_i, \ldots, -x_j, \ldots, 0)$$

where x_i is in the j-th place, and $-x_j$ in the i-th one; say $(b_1, \ldots, b_n) = c_1 x_{i_1 j_1} + \cdots + c_r x_{i_r j_r}$.

These standard relations lift to analogous ones y_{ij} among the y_i's, and the difference

$$(a_1, \ldots, a_n) - (c_1 y_{i_1 j_1} + \cdots + c_r y_{i_r j_r})$$

is an element $(d_1, \ldots, d_n) \in (I(R \otimes_\Lambda A'))^n$. In conclusion we have $a_1 y_1 + \cdots + a_n y_n = d_1 y_1 + \cdots + d_n y_n \in II'$, and this gives the flatness of Z'.

Now take an open affine cover $\mathcal{U} = \{U_i\}_{i \in I}$ of X such that Z is a complete intersection in each of the corresponding open subschemes $V_i = U_i \times_{\mathrm{Spec}(\Lambda)} \mathrm{Spec}(A)$ of X_A, and for each index i take a lifting $Z'_i \subseteq W_i = U_i \times_{\mathrm{Spec}(\Lambda)} \mathrm{Spec}(A')$ of $Z \cap V_i \subseteq V_i$. As before, every finite intersection of the U_i's (and the V_i's, and the W_i's) will be affine, because of the separatedness of X.

For each pair of indexes i, j the restrictions $Z'_i \cap W_{ij}$ and $Z'_j \cap W_{ij}$ are both liftings of $Z \cap V_{ij} \subseteq V_{ij}$, so by Theorem 2.3.1 we have a unique element

$$h_{ij} \in I \otimes_k T_{Z_0 \cap U_{ij}} \mathcal{H}ilb^{U_{ij}} \cong \Gamma(Z_0 \cap U_{ij}, I \otimes_k \mathcal{N}_0)$$

such that

$$(Z'_i \cap W_{ij} \subseteq W_{ij}) \cdot h_{ij} = Z'_j \cap W_{ij} \subseteq W_{ij}.$$

From the fact that the action of $\Gamma(Z_0 \cap U_{ij}, I \otimes_k \mathcal{N}_0)$ on the liftings is free (and compatible with restriction to open subsets, as can be easily checked), for every triplet of indices i, j, k we have

$$h_{ij} + h_{jk} = h_{ik}$$

so that $\{h_{ij}\}_{i,j \in I}$ is a Čech 1-cocycle for the sheaf $I \otimes_k \mathcal{N}_0$ (notice that $\mathcal{U} \cap Z_0 = \{Z_0 \cap U_i\}_{i \in I}$ is an open affine cover of Z_0).

Let us check that its cohomology class is independent of the choice of the liftings Z'_i. Suppose for every index i we have another lifting $Z''_i \subseteq W_i$ of $Z \cap V_i \subseteq V_i$, and call $\{k_{ij}\}_{i,j \in I}$ the corresponding cocycle. Then again by Theorem 2.3.1 we have sections $l_i \in \Gamma(Z_0 \cap U_i, I \otimes_k \mathcal{N}_0)$ such that

$$Z'_i \subseteq W_i = (Z''_i \subseteq W_i) \cdot l_i.$$

Restricting to U_{ij} we have

$$Z'_j \subseteq W_{ij} = (Z'_i \subseteq W_{ij}) \cdot h_{ij} = (Z''_i \subseteq W_{ij}) \cdot (l_i + h_{ij})$$

and on the other hand

$$Z'_j \subseteq W_{ij} = (Z''_j \subseteq W_{ij}) \cdot l_j = (Z''_i \subseteq W_{ij}) \cdot (k_{ij} + l_j).$$

Again by freeness of the action we must have

$$k_{ij} = h_{ij} + l_i - l_j$$

and this says that $\{h_{ij}\}_{i,j \in I}$ and $\{k_{ij}\}_{i,j \in I}$ are cohomologous, and so define the same cohomology class in

$$\check{H}^1(\mathcal{U} \cap Z_0, I \otimes_k \mathcal{N}_0) \cong H^1(Z_0, I \otimes_k \mathcal{N}_0) \cong I \otimes_k H^1(Z_0, \mathcal{N}_0)$$

that we call $\omega(Z \subseteq X_A, A')$.

As in the preceding cases it is also easy to see that this class does not depend on the choice of the open cover $\{U_i\}_{i \in I}$, and that we have a lifting $Z' \subseteq X_{A'}$ if and only if $\omega(Z \subseteq X_A, A') = 0$. In fact this corresponds exactly to the situation in which the restrictions of the liftings Z'_i on the intersections W_{ij} are compatible, and can be used to define a global lifting (notice that we do not have infinitesimal automorphisms, so in this case these restrictions are equal, and not only isomorphic).

Finally the functoriality property is immediate from the one of the action of the tangent space. □

Remark 4.2.11. Proposition 2.4.12 is an immediate consequence of this theorem: $Z_0 \subseteq \mathbb{P}^n_k$ has an obstruction theory with space $H^1(Z_0, \mathcal{N}_0)$, which in this case is trivial, as we already saw in the proof of proposition 2.4.10. In other words Z_0 is unobstructed, so any $Z \in \mathcal{H}ilb^{\mathbb{P}^n_k}_{Z_0}(A)$ can be lifted along any small extension $A' \to A$.

4.2.5 Quasi-coherent sheaves

We turn now to the case of deformations of quasi-coherent sheaves. Take $\Lambda = k$ and the deformation category $\mathcal{QC}oh^X \to (\mathrm{Art}/k)^{op}$, and consider a quasi-coherent sheaf $\mathcal{E}_0 \in \mathcal{QC}oh^X(k)$ (notice that $X_0 = X$ in this case).

Theorem 4.2.12. *There is an obstruction theory (V_ω, ω) for $\mathcal{E}_0$, with obstruction space*

$$V_\omega = \mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{E}_0, \mathcal{E}_0).$$

Proof. Take a small extension $A' \to A$ with kernel I, and an object $\mathcal{E} \in \mathcal{QC}oh^X_{\mathcal{E}_0}(A)$, which is a quasi-coherent sheaf on $X_A = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(A)$ with an isomorphism $\mathcal{E} \otimes_A k \cong \mathcal{E}_0$. We construct the obstruction $\omega(\mathcal{E}, A')$.

Take the exact sequence of A'-modules

$$0 \longrightarrow I \longrightarrow \mathfrak{m}_{A'} \longrightarrow \mathfrak{m}_A \longrightarrow 0$$

and notice that, since $\mathfrak{m}_{A'} \cdot I = 0$, $\mathfrak{m}_{A'}$ is also an A-module (and I is too because $I^2 = (0)$), so that the sequence above is also an exact sequence of A-modules. We tensor it with $\mathcal{E}$ to get (by flatness)

$$0 \longrightarrow I \otimes_k \mathcal{E}_0 \longrightarrow \mathfrak{m}_{A'} \otimes_A \mathcal{E} \longrightarrow \mathfrak{m}_A \otimes_A \mathcal{E} \longrightarrow 0$$

(since $I \otimes_A \mathcal{E} \cong I \otimes_k (k \otimes_A \mathcal{E}) \cong I \otimes_k \mathcal{E}_0$) which is an element $e \in \mathrm{Ext}^1_{\mathcal{O}_X}(\mathfrak{m}_A \otimes_A \mathcal{E}, I \otimes_k \mathcal{E}_0)$.

We consider then the exact sequence of A-modules

$$0 \longrightarrow \mathfrak{m}_A \longrightarrow A \longrightarrow k \longrightarrow 0$$

and tensor it with $\mathcal{E}$, getting (by flatness again)

$$0 \longrightarrow \mathfrak{m}_A \otimes_A \mathcal{E} \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_0 \longrightarrow 0.$$

This induces a long Ext exact sequence (taking $\mathrm{Hom}_{\mathcal{O}_X}(-, I \otimes_k \mathcal{E}_0)$) that contains in particular the following piece

$$\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{E}, I \otimes_k \mathcal{E}_0) \xrightarrow{\gamma} \mathrm{Ext}^1_{\mathcal{O}_X}(\mathfrak{m}_A \otimes_A \mathcal{E}, I \otimes_k \mathcal{E}_0) \xrightarrow{\delta} \mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{E}_0, I \otimes_k \mathcal{E}_0).$$

We take as $\omega(\mathcal{E}, A')$ the element $\delta(e) \in \mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{E}_0, I \otimes_k \mathcal{E}_0) \cong I \otimes_k \mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{E}_0, \mathcal{E}_0)$.

Now we have to verify that $\mathcal{E}$ has a lifting to A' if and only if $\delta(e) = 0$. Suppose first that $\mathcal{E}$ has a lifting $\mathcal{E}' \in \mathcal{QC}oh^X_{\mathcal{E}_0}(A')$. Then notice that

$\mathfrak{m}_{A'} \otimes_{A'} \mathcal{E}' \cong (\mathfrak{m}_{A'} \otimes_{A'} A) \otimes_A \mathcal{E} \cong \mathfrak{m}_{A'} \otimes_A \mathcal{E}$ (because $\mathfrak{m}_{A'} \otimes_{A'} A \cong \mathfrak{m}_{A'}$, since $\mathfrak{m}_{A'}$ is already an A-module). Tensoring the diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d, equal] & \mathfrak{m}_{A'} \arrow[r] \arrow[d] & \mathfrak{m}_A \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & I \arrow[r] & A' \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

with $\mathcal{E}'$, we get

$$\begin{tikzcd} 0 \arrow[r] & I \otimes_k \mathcal{E}_0 \arrow[r] \arrow[d, equal] & \mathfrak{m}_{A'} \otimes_A \mathcal{E} \arrow[r] \arrow[d] & \mathfrak{m}_A \otimes_A \mathcal{E} \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & I \otimes_k \mathcal{E}_0 \arrow[r] & \mathcal{E}' \arrow[r] & \mathcal{E} \arrow[r] & 0 \end{tikzcd}$$

where the top row is the extension e obtained before.

But this diagram implies that e is obtained by pullback from an extension in $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{E}, I \otimes_k \mathcal{E}_0)$ (the bottom row), so that it is in the image of the map γ; then we have $\delta(e) = 0$.

Conversely, suppose that $\delta(e) = 0$. Then by exactness of the Ext long exact sequence above, e is in the image of the map γ. In other words we have a commutative diagram of $\mathcal{O}_X$-modules with exact rows

$$\begin{tikzcd} 0 \arrow[r] & I \otimes_k \mathcal{E}_0 \arrow[r] \arrow[d, equal] & \mathfrak{m}_{A'} \otimes_A \mathcal{E} \arrow[r] \arrow[d, "g"] & \mathfrak{m}_A \otimes_A \mathcal{E} \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & I \otimes_k \mathcal{E}_0 \arrow[r] & \mathcal{F} \arrow[r, "\pi"] & \mathcal{E} \arrow[r] & 0 \end{tikzcd} \tag{4.5}$$

where $\mathcal{F}$ is an $\mathcal{O}_X$-module.

We define a structure of $\mathcal{O}_{X_{A'}}$-module on $\mathcal{F}$ in the following way: since

$$\mathcal{O}_{X_{A'}} \cong \mathcal{O}_X \otimes_k A' \cong \mathcal{O}_X \oplus (\mathfrak{m}_{A'} \otimes_k \mathcal{O}_X)$$

(because $A' \cong k \oplus \mathfrak{m}_{A'}$ as a k-vector space) we only need to define $x \cdot s$ where x is a section of $\mathfrak{m}_{A'} \otimes_k \mathcal{O}_X$ and s one of $\mathcal{F}$. Given two such sections $x = a' \otimes t$ and s, we define then

$$(a' \otimes t) \cdot s = g(a' \otimes \pi(ts)) \in \mathcal{F}.$$

(notice that g is injective, since f is by flatness of $\mathcal{E}$). It is readily checked that this gives a structure of $\mathcal{O}_{X_{A'}}$-module to $\mathcal{F}$. Moreover $\mathcal{F}$ is quasi coherent, because it is an extension of two quasi-coherent sheaves.

Finally, we notice that the natural homomorphism $\mathcal{F} \otimes_{A'} A \to \mathcal{E}$ induced by $\mathcal{F} \to \mathcal{E}$ above is an isomorphism (it suffices to tensor the second row of 4.5 with A over A'), and from the local flatness criterion we have that $\mathcal{F}$ is flat over A'. Precisely, tensoring

$$0 \longrightarrow \mathfrak{m}_{A'} \longrightarrow A' \longrightarrow k \longrightarrow 0$$

with $\mathcal{F}$ we get

$$\mathfrak{m}_{A'} \otimes_{A'} \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow \mathcal{E}_0 \longrightarrow 0.$$

But now

$$\mathfrak{m}_{A'} \otimes_{A'} \mathcal{F} \cong \mathfrak{m}_{A'} \otimes_A (A \otimes_{A'} \mathcal{F}) \cong \mathfrak{m}_{A'} \otimes_A \mathcal{E}$$

and using this isomorphism the map $\mathfrak{m}_{A'} \otimes_{A'} \mathcal{F} \to \mathcal{F}$ corresponds to g of 4.5, which is injective. So $\mathrm{Tor}_1^{A'}(\mathcal{F}, k) = 0$, and $\mathcal{F}$ is flat over A'.

In conclusion, $\mathcal{F}$ is a lifting of $\mathcal{E}$ to A'. Functoriality of the obstruction defined is immediate from the construction. □

Remark 4.2.13. If $\mathcal{E}_0$ is locally free, than we have

$$\mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{E}_0, \mathcal{E}_0) \cong H^2(X, \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}_0))$$

and in this case (with the additional hypothesis that X is separated) Theorem 4.2.12 can be proved using Čech cohomology, in the same way as we did for Theorem 4.2.4.

In particular if X is affine, or of dimension at most 1, then every locally free sheaf is unobstructed.

Example 4.2.14. We describe a simple example of a quasi-coherent sheaf with nontrivial obstructions. Take the affine curve $X_0 \subseteq \mathbb{A}^2_k$ over k defined by the equation

$$y^2 = x(x-1)$$

so that the origin $p = (0,0)$ is a singular point of X_0, and put $\mathcal{E}_0 = \mathcal{O}_p$, the pushforward of the structure sheaf of the point $p = \mathrm{Spec}(k)$ along the morphism $\mathrm{Spec}(k) \to X_0$ with image p. Consider the tangent cone C_pX_0 of X_0 at p, which is a union of two lines contained properly in the tangent space T_pX_0, which is two-dimensional, and take a tangent vector $v \in T_pX_0 \setminus C_pX_0$.

We see v as a morphism $v : \mathrm{Spec}(k[\varepsilon]) \to X_0$ in the usual way, and notice that it gives a section $\mathrm{Spec}(k[\varepsilon]) \to X_\varepsilon$ of the structure morphism $X_\varepsilon \to \mathrm{Spec}(k[\varepsilon])$ (where as usual X_ε is the trivial deformation of X_0 over $k[\varepsilon]$).

$$\begin{tikzcd}
\mathrm{Spec}(k[\varepsilon]) \arrow[dr, dashed] \arrow[drr, bend left, "v"] \arrow[ddr, bend right, "\mathrm{id}"'] & & \\
& X_\varepsilon \arrow[r] \arrow[d] & X_0 \arrow[d] \\
& \mathrm{Spec}(k[\varepsilon]) \arrow[r] & \mathrm{Spec}(k)
\end{tikzcd}$$

Moreover the image of this section, call it $Z_1 \subseteq X_\varepsilon$, is closed, because $X_\varepsilon \to \mathrm{Spec}(k[\varepsilon])$ is a separated morphism.

Now put $\mathcal{E}_1 = \mathcal{O}_{Z_1} \in \mathcal{QC}oh^{X_0}_{\mathcal{E}_0}(k[\varepsilon])$, which is again the pushforward on X_ε of the structure sheaf of $\mathrm{Spec}(k[\varepsilon])$. $\mathcal{E}_1$ is flat over $k[\varepsilon]$, being free of rank 1; we claim that there exists $n \geq 2$ such $\mathcal{E}_1$ cannot be lifted to $R_n = k[t]/(t^{n+1})$, so that $\mathcal{E}_0$ must be obstructed.

Suppose that the claim is false, and lift $\mathcal{E}_1$ inductively to every R_n, obtaining a sequence of quasi-coherent flat sheaves $\mathcal{E}_n \in \mathcal{QC}oh^{X_0}_{\mathcal{E}_0}(R_n)$, which will all have support only in the origin p. We also lift successively the surjective homomorphism $\mathcal{O}_{X_\varepsilon} \to \mathcal{E}_1$ together with the generator of $\mathcal{E}_1$, obtaining a sequence of surjective homomorphisms $\mathcal{O}_{X_n} \to \mathcal{E}_n$ (where X_n is the trivial deformation of X_0 over R_n) and generators of $\mathcal{E}_n$, in the following way: suppose we already lifted it for n and that the section e_n (which has sopport only in the origin) is a generator of $\mathcal{E}_n$ over $\mathcal{O}_{X_n}$, and the image of the unit section of $\mathcal{O}_{X_n}$ in $\mathcal{E}_n$. Consider the diagram

$$\begin{tikzcd}
\mathcal{O}_{X_{n+1}} \arrow[r] \arrow[d, dashed] & \mathcal{O}_{X_n} \arrow[d] \\
\mathcal{E}_{n+1} \arrow[r] & \mathcal{E}_n
\end{tikzcd}$$

where the top arrow is the surjection corresponding to the closed immersion $X_n \subseteq X_{n+1}$, the bottom one is obtained tensoring $R_{n+1} \to R_n \to 0$ by $\mathcal{E}_{n+1}$ (notice that $\mathcal{E}_{n+1} \otimes_{R_{n+1}} R_n \cong \mathcal{E}_n$) and in particular it is surjective, and the vertical one is surjective by inductive hypothesis.

If we take an arbitrary section e_{n+1} of $\mathcal{E}_{n+1}$ lifting e_n, this will be a generator for $\mathcal{E}_{n+1}$ (because the kernel of $R_{n+1} \to R_n$ is nilpotent), and so it suffices to define $\mathcal{O}_{X_{n+1}} \to \mathcal{E}_{n+1}$ by sending the unit section of $\mathcal{O}_{X_{n+1}}$ to e_{n+1}.

The kernels of these surjections will define a sequence of compatible closed subschemes $Z_n \subseteq X_n$ (which topologically are just the origin p), and for each n the structure sheaf $\mathcal{E}_n$ of is free of rank 1 over it. The structure morphism $Z_n \to \operatorname{Spec} R_n$ is then an isomorphism, and we have a sequence of compatible sections $\operatorname{Spec}(R_n) \to Z_n \subseteq X_n$ of the structure morphisms $X_n \to \operatorname{Spec}(R_n)$.

In particular we have a system of compatible morphisms $f_n : \operatorname{Spec}(R_n) \to \operatorname{Spec}(\mathcal{O}_{X_0,p}) \to X$ that correspond to homomorphisms of k-algebras $\varphi_n : \mathcal{O}_{X_0,p} \to R_n$. These together induce a homomorphism $\varphi : \mathcal{O}_{X_0,p} \to \varprojlim R_i = k[[t]]$ that in turn corresponds to a morphism $f : \operatorname{Spec}(k[[t]]) \to X_0$ that sends the maximal ideal to p.

Moreover if $i : \operatorname{Spec}(k[\varepsilon]) \to \operatorname{Spec}(k[[t]])$ is the inclusion, we have that $f \circ i = v$, and since f will carry the tangent cone of $\operatorname{Spec}(k[[t]])$ (which is $\mathbb{A}^1_k$) to the one of X_0, we conclude that the vector v must be in the tangent cone of X_0, which is a contradiction with the initial assumption.

Chapter 5

Formal deformations

After developing tools to study infinitesimal deformations, in this chapter we go one step further and try to put together infinitesimal deformations that are successive liftings of a fixed $\xi_0 \in \mathcal{F}(k)$ at higher orders. A collection of such liftings is said to be a *formal deformation*.

After defining precisely these objects and organizing them in a fibered category, we will consider universal and versal formal deformations, whose existence is related to prorepresentability of our deformation category. Using the properties of these particular deformations, we will state and prove an analogue of Schlessinger's Theorem for deformation categories. Finally, we will give some applications to obstruction theories, and consider briefly the problem of algebraization of formal deformations.

Throughout this chapter we will use some notation and results about noetherian local complete Λ-algebras that can be found in appendix B.

5.1 Formal objects

Let $\mathcal{F} \to (\mathrm{Art}/\Lambda)^{op}$ be a deformation category, and $R \in (\mathrm{Comp}/\Lambda)$ (recall that this denotes the category of noetherian local complete Λ-algebras with residue field k). We want to consider sequences of compatible deformations on the quotients $R_n = R/\mathfrak{m}_R^{n+1}$: the idea is that $\mathrm{Spec}(R)$ should be a little piece of the base scheme S of a deformation we are trying to construct or study: for example it could be the spectrum of the completion of the local ring $\mathcal{O}_{S,s_0}$ of that base scheme at a point s_0, and we consider then sequences of compatible deformations on all the "thickenings" $\mathrm{Spec}(\mathcal{O}_{S,s_0}/\mathfrak{m}_{s_0}^n)$ of the point $s_0 = \mathrm{Spec}(k(s_0))$, hoping to get an actual deformation over $\mathrm{Spec}(\widehat{\mathcal{O}}_{S,s_0})$.

Definition 5.1.1. *A* ***formal object*** *of $\mathcal{F}$ over R is a collection $\xi = \{\xi_n, f_n\}_{n\in\mathbb{N}}$, where ξ_n is an object of $\mathcal{F}(R_n)$ and $f_n : \xi_n \to \xi_{n+1}$ is an arrow of $\mathcal{F}$ over the* *canonical projection $R_{n+1} \to R_n$.*

Sometimes we will call ξ_n the **term of order** n of ξ, and say that R is the **base ring** of the formal object ξ. If we need to specify it in the notation, we will denote a formal object as above by (R, ξ).

Remark 5.1.2. The condition of having fixed arrows $f_n : \xi_n \to \xi_{n+1}$ reflects the fact that the objects ξ_n are compatible, in the sense that if $n \geq m$, then the pullback of ξ_n to R_m along the projection $R_n \to R_m$ is isomorphic to ξ_m, and moreover we have a canonical isomorphism, coming from the composite $\xi_m \to \xi_{m+1} \to \cdots \to \xi_n$ of the given arrows.

We also remark explicitly that a formal deformation is known (up to isomorphism) if we know ξ_n for n arbitrarily large. This is because ξ_n determines all the ξ_i's with $i \leq n$, by taking pullbacks along the projections $R_n \to R_i$.

A formal object as above should be thought of as an "inverse limit" object of the sequence $\xi_n \in \mathcal{F}(R_n)$, with respect to the given arrows f_n. We will see that formal objects do actually have some properties similar to those of inverse limits, see for example Remark 5.1.9 below.

Definition 5.1.3. *A **morphism** $\alpha : \xi \to \eta$ of formal objects over R, where $\xi = \{\xi_n, f_n\}_{n\in\mathbb{N}}$ and $\eta = \{\eta_n, g_n\}_{n\in\mathbb{N}}$ is a collection $\alpha = \{\alpha_n\}_{n\in\mathbb{N}}$ of arrows $\alpha_n : \xi_n \to \eta_n$ of $\mathcal{F}(R_n)$, such that for every n the diagram*

$$\begin{tikzcd} \xi_n \arrow[r, "f_n"] \arrow[d, "\alpha_n"'] & \xi_{n+1} \arrow[d, "\alpha_{n+1}"] \\ \eta_n \arrow[r, "g_n"'] & \eta_{n+1} \end{tikzcd}$$

commutes.

As with objects, α_n will sometimes be called the **term of order** n of α.

Formal objects over a fixed R with morphisms form a category (composition of arrows is defined as composition at each order), that we call the **category of formal objects** over R and denote by $\widehat{\mathcal{F}}(R)$.

Here we used the canonical filtration $\{\mathfrak{m}_R^n\}_{n\in\mathbb{N}}$, but to define a formal object we can use any filtration that defines the right topology on R. Let $\mathcal{A} = \{I_n\}_{n\in\mathbb{N}}$ be a filtration of R, that is, a sequence of ideals I_n such that $I_n \subseteq I_m$ whenever $n \geq m$, and inducing on R the same topology as the canonical filtration.

This is equivalent to saying that the filtrations $\mathcal{A}$ and $\{\mathfrak{m}_R^n\}_{n\in\mathbb{N}}$ are cofinal (for every n there exists m such that $\mathfrak{m}_R^m \subseteq I_n$, and conversely), or we can say that R with the $\mathfrak{m}_R$-adic topology is complete with respect to the topology induced by the filtration, or that the canonical homomorphism

$$R \to \varprojlim R/I_i$$

is an isomorphism of topological rings.

We can consider then a category $\widehat{\mathcal{F}}_{\mathcal{A}}(R)$, whose objects are collections $\xi = \{\xi_n, f_n\}_{n\in\mathbb{N}}$ that we still call formal objects of $\mathcal{F}$ over R, where $\xi_n \in \mathcal{F}(R/I_n)$ and $f_n : \xi_n \to \xi_{n+1}$ is an arrow of $\mathcal{F}$ over the projection $R/I_{n+1} \to R/I_n$, and an arrow $\alpha = \{\alpha_n\}_{n\in\mathbb{N}} : \xi \to \eta$ (where $\eta = \{\eta_n, g_n\}_{n\in\mathbb{N}}$) is a collections of arrows $\alpha_n : \xi_n \to \eta_n$ of $\mathcal{F}(R/I_n)$ such that for every n the diagram

$$\begin{tikzcd}
\xi_n \arrow[r, "f_n"] \arrow[d, "\alpha_n"'] & \xi_{n+1} \arrow[d, "\alpha_{n+1}"] \\
\eta_n \arrow[r, "g_n"'] & \eta_{n+1}
\end{tikzcd}$$

commutes.

Proposition 5.1.4. *For any R and filtration $\mathcal{A} = \{I_n\}_{n\in\mathbb{N}}$ that defines the $\mathfrak{m}_R$-adic topology on R, the categories $\widehat{\mathcal{F}}_{\mathcal{A}}(R)$ and $\widehat{\mathcal{F}}(R)$ are equivalent.*

Proof. We define a functor $F : \widehat{\mathcal{F}}_{\mathcal{A}}(R) \to \widehat{\mathcal{F}}(R)$. Given a formal object $\xi = \{\xi_n, f_n\}_{n\in\mathbb{N}}$ in $\widehat{\mathcal{F}}_{\mathcal{A}}(R)$, we define $F(\xi) = \{\eta_n, g_n\}_{n\in\mathbb{N}} \in \widehat{\mathcal{F}}(R)$ in the following way: for every fixed n, there exists an m such that $I_m \subseteq \mathfrak{m}_R^{n+1}$, so that the projection $R \to R_n$ will factor as $R \to R/I_m \to R_n$. We take the least m with such a factorization (we denote it by $m(n)$ when we want to stress its dependence on n), and define η_n to be the pullback of $\xi_m \in \mathcal{F}(R/I_m)$ to R_n.

Since clearly $m(n+1) \geq m(n)$, for every n we get an arrow $g_n : \eta_n \to \eta_{n+1}$ over the projection $R_{n+1} \to R_n$ taking the pullback of $f_{m(n)}$, as in the diagram

$$\begin{tikzcd}
\xi_{m(n+1)} \arrow[dd, mapsto] & & \eta_{n+1} \arrow[ll] \arrow[dd, mapsto] & \\
& \xi_{m(n)} \arrow[ul, "f_{m(n)}"] \arrow[dd, mapsto] & & \eta_n \arrow[ll] \arrow[ul, dashed, "g_n"'] \arrow[dd, mapsto] \\
R/I_{m(n+1)} \arrow[rr] \arrow[dr] & & R_{n+1} \arrow[dr] & \\
& R/I_{m(n)} \arrow[rr] & & R_n .
\end{tikzcd}$$

This defines an object $F(\xi) = \{\eta_n, g_n\}_{n\in\mathbb{N}} \in \widehat{\mathcal{F}}(R)$.

Given an arrow $\alpha = \{\alpha_n\}_{n\in\mathbb{N}} : \xi \to \rho$ where $\rho = \{\rho_n, h_n\}_{n\in\mathbb{N}}$, we get an arrow $F(\alpha) : F(\xi) \to F(\rho)$, where $F(\rho) = \{\nu_n, l_n\}_{n\in\mathbb{N}}$, by taking the sequence $\beta_n : \eta_n \to \nu_n$ of arrows of $\mathcal{F}(R_n)$ obtained by pullback, as in the diagram

$$
\begin{tikzcd}
\rho_{m(n)} \arrow[d, mapsto, "\alpha_{m(n)}"] & & \nu_n \arrow[ll] \arrow[d, mapsto] & \\
& \xi_{m(n)} \arrow[ul] \arrow[d, mapsto] & & \eta_n \arrow[ll] \arrow[ul, dashed, "\beta_n"'] \arrow[d, mapsto] \\
R/I_{m(n)} \arrow[rr] \arrow[dr, equal] & & R_n \arrow[dr, equal] & \\
& R/I_{m(n)} \arrow[rr] & & R_n .
\end{tikzcd}
$$

The F just defined gives a functor $\widehat{\mathcal{F}}_{\mathcal{A}}(R) \to \widehat{\mathcal{F}}(R)$. In the same exact way one can define a functor in the other direction $G : \widehat{\mathcal{F}}(R) \to \widehat{\mathcal{F}}_{\mathcal{A}}(R)$, which will be a quasi-inverse to F. Indeed, starting from an object $\xi = \{\xi_n, f_n\}_{n\in\mathbb{N}}$ of $\widehat{\mathcal{F}}_{\mathcal{A}}$, put $F(\xi) = \{\eta_n, g_n\}_{n\in\mathbb{N}}$ and $G(F(\xi)) = \{\xi'_n, f'_n\}_{n\in\mathbb{N}}$. Then for every n we have a diagram

$$
\begin{tikzcd}
\xi_l \arrow[d, mapsto] & \eta_m \arrow[l] \arrow[d, mapsto] & \xi'_n \arrow[l] \arrow[d, mapsto] \\
R/I_l \arrow[r] & R_m \arrow[r] & R/I_n
\end{tikzcd}
$$

(coming from the definition of η_m and ξ'_n as pullbacks) that gives a canonical isomorphism $\xi'_n \to \xi_n$ identifying the two as pullbacks of ξ_l to R/I_n. Straightforward arguments (using the universal property of pullbacks) show that the collection of these isomorphisms is an arrow $G(F(\xi)) \to \xi$, and that these arrows give a natural equivalence $G \circ F \cong \mathrm{id}$. The same goes for the other composite $F \circ G$, and so we have our equivalence. $\square$

To define a formal object of $\mathcal{F}$ over R we can use then any filtration with the hypotheses above, and not just the canonical one.

The notation $\widehat{\mathcal{F}}(R)$ suggests that we want to consider a fibered category $\widehat{\mathcal{F}} \to (\mathrm{Comp}\,/\Lambda)^{op}$, which is indeed the case.

Definition 5.1.5. *A* ***morphism*** $\alpha : (R, \xi) \to (S, \eta)$ *of formal objects of* $\mathcal{F}$*, where* $\xi = \{\xi_n, f_n\}_{n\in\mathbb{N}}$ *and* $\eta = \{\eta_n, g_n\}_{n\in\mathbb{N}}$*, is a pair* (α, φ)*, where* $\varphi : S \to R$ *is a homomorphism, and* $\alpha = \{\alpha_n\}_{n\in\mathbb{N}}$ *is a collection of arrows* $\alpha_n : \xi_n \to \eta_n$ *of* $\mathcal{F}$ *over* $\varphi_n : S_n \to R_n$*, such that for every* n *the diagram*

$$
\begin{tikzcd}
\xi_n \arrow[r, "f_n"] \arrow[d, "\alpha_n"'] & \xi_{n+1} \arrow[d, "\alpha_{n+1}"] \\
\eta_n \arrow[r, "g_n"'] & \eta_{n+1}
\end{tikzcd}
$$

commutes.

Again, sometimes we will call α_n the **term of order** n of (α, φ).

We define a category $\widehat{\mathcal{F}}$, and call it the **category of formal objects** of $\mathcal{F}$: its objects are formal objects (R, ξ), and an arrow $(R, \xi) \to (S, \eta)$ is a morphism of formal objects. We have a functor $\widehat{\mathcal{F}} \to (\mathrm{Comp}/\Lambda)^{op}$ that takes (R, ξ) to R and an arrow $(\alpha, \varphi) : (R, \xi) \to (S, \eta)$ to the homomorphism φ, which is an arrow $R \to S$ in $(\mathrm{Comp}/\Lambda)^{op}$.

Proposition 5.1.6. *$\widehat{\mathcal{F}} \to (\mathrm{Comp}/\Lambda)^{op}$ is a category fibered in groupoids.*

Proof. First of all we check that $\widehat{\mathcal{F}} \to (\mathrm{Comp}/\Lambda)^{op}$ is a fibered category. Suppose we have a homomorphism $\varphi : R \to S$ in (Comp/Λ), and a formal object $\xi = \{\xi_n, f_n\}_{n \in \mathbb{N}}$ over R; we want to define an object $\eta = \{\eta_n, g_n\}_{n \in \mathbb{N}}$ over S and a cartesian arrow $\eta \to \xi$ over φ.

For each n, we consider the homomorphism $\varphi_n : R_n \to S_n$ induced by φ, and take as $\eta_n \in \mathcal{F}(S_n)$ the pullback of ξ_n to S_n; further, call $\alpha_n : \eta_n \to \xi_n$ the cartesian arrow defining the pullback. For every n we have a diagram

$$\begin{tikzcd}
\xi_{n+1} \arrow[dd, mapsto] & & \eta_{n+1} \arrow[ll, "\alpha_{n+1}"'] \arrow[dd, mapsto] & \\
& \xi_n \arrow[ul, "f_n"] \arrow[dd, mapsto] & & \eta_n \arrow[ll, "\alpha_n"'] \arrow[ul, dashed, "g_n"'] \arrow[dd, mapsto] \\
R_{n+1} \arrow[rr, "\varphi_{n+1}"] \arrow[dr] & & S_{n+1} \arrow[dr] & \\
& R_n \arrow[rr, "\varphi_n"] & & S_n .
\end{tikzcd}$$

that defines an arrow $g_n : \eta_n \to \eta_{n+1}$ by pullback, and shows that $\alpha = \{\alpha_n\}_{n \in \mathbb{N}}$ together with φ gives a morphism of formal objects $\alpha : (S, \eta) \to (R, \xi)$, where $\eta = \{\eta_n, g_n\}_{n \in \mathbb{N}}$.

We check that α is a cartesian arrow in $\widehat{\mathcal{F}}$. Suppose we have another formal object (T, ρ), where $\rho = \{\rho_n, h_n\}_{n \in \mathbb{N}}$, and a diagram

$$\begin{tikzcd}
\rho \arrow[dr, dashed, "\beta"] \arrow[drr, bend left] \arrow[dd, mapsto] & & \\
& \eta \arrow[r, "\alpha"] \arrow[dd, mapsto] & \xi \arrow[dd, mapsto] \\
T & & \\
& S \arrow[ul, "\psi"] & R \arrow[l, "\varphi"']
\end{tikzcd} \tag{5.1}$$

where the morphism ψ and the top arrow are given, and we want to construct a dotted arrow $\beta = \{\beta_n\}_{n \in \mathbb{N}}$ over ψ.

We consider the part of order n of diagram 5.1

$$\begin{tikzcd}
\rho_n \arrow[dr, dashed, "\beta_n"] \arrow[drr, bend left] \arrow[dd, mapsto] & & \\
& \eta_n \arrow[r, "\alpha_n"] \arrow[dd, mapsto] & \xi_n \arrow[dd, mapsto] \\
T_n & & \\
& S_n \arrow[ul, "\psi_n"] & R_n \arrow[l, "\varphi_n"']
\end{tikzcd}$$

where we have a unique dotted arrow β_n that fits in, because $\alpha_n : \eta_n \to \xi_n$ is cartesian. Drawing the diagrams relative to orders n and $n+1$ together, straightforward verifications show that $\{\beta_n\}_{n\in\mathbb{N}}$ gives a morphism of formal objects, that fits in 5.1. Uniqueness is trivial, since the term of order n of an arrow $\rho \to \eta$ fitting in 5.1 will fit in the last diagram, and so it is uniquely determined.

Finally, we check that $\widehat{\mathcal{F}}(R)$ is a groupoid. Suppose then we have two formal objects $\xi = \{\xi_n, f_n\}_{n\in\mathbb{N}}$ and $\eta = \{\eta_n, g_n\}_{n\in\mathbb{N}}$ over R, with a morphism $\alpha = \{\alpha_n\}_{n\in\mathbb{N}} : \xi \to \eta$ over the identity of R. Then for each n we have an inverse $\beta_n : \eta_n \to \xi_n$ of α_n in $\mathcal{F}(R_n)$, and the commutativity of diagram

$$\begin{tikzcd}
\xi_n \arrow[r, "f_n"] \arrow[d, "\alpha_n"'] & \xi_{n+1} \arrow[d, "\alpha_{n+1}"] \\
\eta_n \arrow[r, "g_n"'] & \eta_{n+1}
\end{tikzcd}$$

implies immediately that

$$\begin{tikzcd}
\xi_n \arrow[r, "f_n"] & \xi_{n+1} \\
\eta_n \arrow[r, "g_n"'] \arrow[u, "\beta_n"] & \eta_{n+1} \arrow[u, "\beta_{n+1}"']
\end{tikzcd}$$

is commutative as well.

So $\beta = \{\beta_n\}_{n\in\mathbb{N}}$ is a morphism of formal objects, and is an inverse for α in $\widehat{\mathcal{F}}(R)$, which is then a groupoid. □

Remark 5.1.7. Suppose we have two deformation categories $\mathcal{F} \to (\mathrm{Art}/\Lambda)^{op}$ and $\mathcal{G} \to (\mathrm{Art}/\Lambda)^{op}$, and a morphism $F : \mathcal{F} \to \mathcal{G}$. Then there is a natural induced morphism $\widehat{F} : \widehat{\mathcal{F}} \to \widehat{\mathcal{G}}$ of categories fibered in groupoids: a formal object $\xi = \{\xi_n, f_n\}_{n\in\mathbb{N}}$ of $\mathcal{F}$ goes to the formal object $\widehat{F}(\xi) = \{F(\xi_n), F(f_n)\}_{n\in\mathbb{N}}$ of $\mathcal{G}$, and an arrow $\alpha = \{\alpha_n\}_{n\in\mathbb{N}}$ goes to the arrow $\widehat{F}(\alpha) = \{F(\alpha_n)\}_{n\in\mathbb{N}}$. It is immediate to check that this is well-defined, and gives a morphism of categories fibered in groupoids.

Now we show that $\mathcal{F}$ is a subcategory of $\widehat{\mathcal{F}}$. First notice that if $A \in (\mathrm{Art}/\Lambda)$, then in particular $A \in (\mathrm{Comp}/\Lambda)$, so we can consider the fiber category $\widehat{\mathcal{F}}(A)$.

Proposition 5.1.8. *We have an equivalence $\widehat{\mathcal{F}}(A) \cong \mathcal{F}(A)$. Moreover these equivalences give rise to a morphism of categories fibered in groupoids $F : \mathcal{F} \to \widehat{\mathcal{F}}|_{(\mathrm{Art}/\Lambda)^{op}}$ that is an equivalence, so that $\mathcal{F}$ can be regarded as a full subcategory of $\widehat{\mathcal{F}}$.*

Proof. In the above statement with $\widehat{\mathcal{F}}|_{(\mathrm{Art}/\Lambda)^{op}}$ we mean the full subcategory of $\widehat{\mathcal{F}}$ whose objects are formal objects (A, ξ) of $\mathcal{F}$ with $A \in (\mathrm{Art}/\Lambda)$.

The idea of the proof is quite clear: since A is artinian its maximal ideal $\mathfrak{m}_A$ is nilpotent, so that there exists an m such that $\mathfrak{m}_A^{m+1} = (0)$, and then $A_i = A$ for all $i \geq m$; because of this, a formal deformation will be completely determined (up to isomorphism) by its term of order m.

Formally, there is an obvious functor $F : \widehat{\mathcal{F}}(A) \to \mathcal{F}(A)$ that carries a formal object $\{\xi_n, f_n\}_{n\in\mathbb{N}}$ to the object $\xi_m \in \mathcal{F}(A)$, and an arrow $\alpha = \{\alpha_n\} : \{\xi_n, f_n\}_{n\in\mathbb{N}} \to \{\eta_n, g_n\}_{n\in\mathbb{N}}$ to $\alpha_m : \xi_m \to \eta_m$.

We construct a quasi inverse $G : \mathcal{F}(A) \to \widehat{\mathcal{F}}(A)$ as follows: given an object $\xi \in \mathcal{F}(A)$, for $i \leq m-1$ we can consider the pullbacks ξ_i of ξ along the quotient maps $A \to A/\mathfrak{m}_A^{i+1} = A_i$, and the canonical arrows $f_i : \xi_i \to \xi_{i+1}$, identifying ξ_i as a pullback of ξ_{i+1}, and for $i \geq m$, we put $\xi_i = \xi \in \mathcal{F}(A)$, and $f_i = \mathrm{id} : \xi \to \xi$. Then $G(\xi) = \{\xi_n, f_n\}_{n\in\mathbb{N}}$ is an object of $\widehat{\mathcal{F}}(A)$. Moreover, if η is another object of $\mathcal{F}(A)$, with $G(\eta) = \{\eta_n, g_n\}_{n\in\mathbb{N}}$, and $\alpha : \xi \to \eta$ is an arrow in $\mathcal{F}(A)$, we define an arrow $G(\alpha) = \{\alpha_n\}_{n\in\mathbb{N}} : G(\xi) \to G(\eta)$, taking for $i \leq m-1$ the arrow $\alpha_i : \xi_i \to \eta_i$ that is the pullback of $\alpha : \xi \to \eta$ to A_i, and for $i \geq m$, since $\xi_i = \xi$ and $\eta_i = \eta$, we take simply $\alpha : \xi \to \eta$.

It is clear that the functor $F \circ G$ is the identity, and also that for every formal deformation $\xi = \{\xi_n, f_n\}_{n\in\mathbb{N}}$ the object $G(F(\xi))$ will be isomorphic to ξ: in fact as we already remarked, every formal deformation is determined (up to isomorphism) by knowing terms of arbitrarily high order, and in our case for $i \geq m$ the terms of order i of both $G(F(\xi))$ and ξ are isomorphic to ξ_m. The usual arguments involving the universal property of pullbacks show that these isomorphisms give a natural equivalence $G \circ F \cong \mathrm{id}$.

So for every $A \in (\mathrm{Art}/\Lambda)$ we have an equivalence $F_A : \widehat{\mathcal{F}}(A) \to \mathcal{F}(A)$. We put these together in a morphism $F : \widehat{\mathcal{F}}|_{(\mathrm{Art}/\Lambda)^{op}} \to \mathcal{F}$ that takes a formal object (A, ξ) to $\xi_m \in \mathcal{F}$, where $\xi = \{\xi_n, f_n\}_{n\in\mathbb{N}}$ and m is the order of A, and an arrow $\alpha = \{\alpha_n\}_{n\in\mathbb{N}} : (A, \xi) \to (A', \xi')$ to $\alpha_m : \xi_m \to \xi'_m$, where m is the maximum of the orders of A and A'. Thanks to Proposition 1.2.15 the morphism F is an equivalence of categories fibered in groupoids, and this concludes the proof. $\square$

Because of the preceding proposition we can then talk about arrows between an object ξ over $A \in (\mathrm{Art}/\Lambda)$ and a formal object $\eta = \{\eta_n, g_n\}_{n\in\mathbb{N}}$ over $R \in (\mathrm{Comp}/\Lambda)$, using the above identification.

In particular giving an arrow $\xi \to \eta$ corresponds to giving a homomorphism $\varphi : R \to A$ of Λ-algebras, and an isomorphism $\xi \cong (\varphi_m)_*(\eta_m)$ in $\mathcal{F}(A)$, where m is the order of A, so that φ_m is a homomorphism $\varphi_m : R_m \to A$. A pullback of η to A is $(\varphi_m)_*(\eta_m) \in \mathcal{F}(A)$, where m is as above.

Finally, if $A \in (\mathrm{Art}/\Lambda)$ and $\xi \in \mathcal{F}(A)$, we will sometimes write (A, ξ) for the corresponding object in $\widehat{\mathcal{F}}(A)$ defined in the proof above.

Remark 5.1.9. We point out here (using again the identification given by Proposition 5.1.8) that giving a morphism of formal objects is equivalent to giving a sequence of morphisms from the artinian quotients of the source, compatible with the projections.

Let $(R, \xi), (S, \eta)$ be two formal objects of $\mathcal{F}$, where $\xi = \{\xi_n, f_n\}_{n\in\mathbb{N}}$ and $\eta = \{\eta_n, g_n\}_{n\in\mathbb{N}}$; call A the set of arrows $(R, \xi) \to (S, \eta)$ in $\widehat{\mathcal{F}}$, and B the set of sequences $\{h_n\}_{n\in\mathbb{N}}$ of morphisms of formal objects $h_n : (R_n, \xi_n) \to (S, \eta)$ such that for every n the composite

$$(R_n, \xi_n) \to (R_{n+1}, \xi_{n+1}) \xrightarrow{h_{n+1}} (S, \eta)$$

coincides with h_n.

There is a natural map $A \to B$ sending a morphism $(R, \xi) \to (S, \eta)$ to the sequence of composites $(R_n, \xi_n) \to (R, \xi) \to (S, \eta)$. Conversely given a sequence h_n as above, the arrow $h_n : (R_n, \xi_n) \to (S, \eta)$ of $\widehat{\mathcal{F}}$ corresponds to an isomorphism between ξ_n and the pullback of $\eta_n \in \mathcal{F}(S_n)$ to R_n, which gives an arrow $\alpha_n : \xi_n \to \eta_n$ of $\mathcal{F}$. The fact that $\{h_n\}_{n\in\mathbb{N}}$ has the compatibility property above ensures that $\alpha = \{\alpha_n\}_{n\in\mathbb{N}}$ gives an arrow of formal objects $\alpha : (R, \xi) \to (S, \eta)$, and this gives a map $B \to A$ that is clearly inverse to the previous one.

5.1.1 Formal objects as morphisms

Now we change point of view, and describe formal objects as morphisms of categories fibered in groupoids. For an object $R \in (\mathrm{Comp}/\Lambda)$, consider the opposite category $(\mathrm{Art}/R)^{op}$ of the category of local artinian R-algebras with residue field k, or equivalently objects $A \in (\mathrm{Art}/\Lambda)$ with a homomorphism of Λ-algebras $R \to A$.

There is an obvious functor $(\mathrm{Art}/R)^{op} \to (\mathrm{Art}/\Lambda)^{op}$ that sends an object $R \to A$ of $(\mathrm{Art}/R)^{op}$ to the Λ-algebra A defined by the composite $\Lambda \to R \to A$, and a homomorphism $A \to B$ in $(\mathrm{Art}/R)^{op}$ to itself, as a homomorphism of Λ-algebras.

Proposition 5.1.10. *$(\mathrm{Art}/R)^{op} \to (\mathrm{Art}/\Lambda)^{op}$ is a category fibered in sets that satisfies [RS], and so it is a deformation category. Moreover its tangent space*

$T_{R\to k}(\mathrm{Art}/R)^{op}$ at the unique object $R \to k$ over k is isomorphic to the vertical tangent space of R

$$T_{R\to k}(\mathrm{Art}/R)^{op} \cong T_\Lambda R = (\mathfrak{m}_R/(\mathfrak{m}_\Lambda R + \mathfrak{m}_R^2))^\vee.$$

Proof. We check first that $(\mathrm{Art}/R)^{op} \to (\mathrm{Art}/\Lambda)^{op}$ is a category fibered in groupoids: if $R \to A$ is an object of $(\mathrm{Art}/R)^{op}$ over $A \in (\mathrm{Art}/\Lambda)$, and $\varphi : A \to B$ is a homomorphism in (Art/Λ), then the only possible pullback $\varphi_*(R \to A)$ is simply the composite $R \to A \to B$, and the cartesian arrow from $R \to B$ to $R \to A$ is φ itself, as a homomorphism of R-algebras.

Moreover it is fibered in sets, because if we have $A \in (\mathrm{Art}/\Lambda)$, and $f : R \to A$, $g : R \to A$ are objects of $(\mathrm{Art}/R)^{op}(A)$, then the only morphism of R-algebras $A \to A$ that induces the identity $\mathrm{id} : A \to A$ as Λ-algebras is the identity itself, so in particular we must have $f = g$. This shows that in $(\mathrm{Art}/R)^{op}(A)$ there are no arrows other than the identities, and so it is a set (precisely the set $\mathrm{Hom}_\Lambda(R, A)$).

Now we turn to [RS]. If A, A', A'' are objects of (Art/Λ), and $\pi' : A' \to A$ and $\pi'' : A'' \to A$ are two homomorphisms with π'' surjective, then we have

$$(\mathrm{Art}/R)^{op}(A' \times_A A'') = \mathrm{Hom}_\Lambda(R, A' \times_A A'').$$

On the other hand $(\mathrm{Art}/R)^{op}(A') \times_{(\mathrm{Art}/R)^{op}(A)} (\mathrm{Art}/R)^{op}(A'')$ is by definition $\mathrm{Hom}_\Lambda(R, A') \times_{\mathrm{Hom}_\Lambda(R,A)} \mathrm{Hom}_\Lambda(R, A'')$ and the properties of the fibered product imply that the natural function

$$\mathrm{Hom}_\Lambda(R, A' \times_A A'') \to \mathrm{Hom}_\Lambda(R, A') \times_{\mathrm{Hom}_\Lambda(R,A)} \mathrm{Hom}_\Lambda(R, A'')$$

is a bijection.

Finally, we calculate the tangent space. Notice first that $(\mathrm{Art}/R)^{op}(k) = \mathrm{Hom}_\Lambda(R, k)$ has precisely one element, which is the quotient map $R \to k$. To find the tangent space, we consider the functor $F : (\mathrm{FVect}/k) \to (\mathrm{Set})$ that associates to $V \in (\mathrm{FVect}/k)$ the set $F(V) = \mathrm{Hom}_\Lambda(R, k[V])$, and acts on arrows by pullback.

We will show that there is a functorial bijection

$$F(V) \cong V \otimes_k T_\Lambda R$$

where $T_\Lambda R$ is the vertical tangent space of R

$$T_\Lambda R = (\mathfrak{m}_R/(\mathfrak{m}_\Lambda R + \mathfrak{m}_R^2))^\vee.$$

This will give an isomorphism $T_{R\to k}(\mathrm{Art}/R)^{op} \cong T_\Lambda R$.

We construct a function $F(V) \to V \otimes_k T_\Lambda R$. Take a homomorphism of Λ-algebras $\varphi : R \to k \oplus V$. We can restrict φ to the maximal ideal $\mathfrak{m}_R$ of R to get a function $\overline{\varphi} : \mathfrak{m}_R \to V$, and since $\varphi(\mathfrak{m}_\Lambda R + \mathfrak{m}_R^2) = 0$ (for $\varphi(\mathfrak{m}_\Lambda R) = 0$

and $\varphi(\mathfrak{m}_R^2) = 0$, respectively because of Λ-linearity of φ and $V^2 = (0)$), we can pass $\overline{\varphi}$ to the quotient to get a k-linear function

$$f_\varphi : \mathfrak{m}_R/(\mathfrak{m}_\Lambda R + \mathfrak{m}_R^2) \to V$$

which is an element of

$$\mathrm{Hom}_k(\mathfrak{m}_R/(\mathfrak{m}_\Lambda R + \mathfrak{m}_R^2), V) \cong V \otimes_k (\mathfrak{m}_R/(\mathfrak{m}_\Lambda R + \mathfrak{m}_R^2))^\vee.$$

Conversely, suppose we have an element of $V \otimes_k (\mathfrak{m}_R/(\mathfrak{m}_\Lambda R + \mathfrak{m}_R^2))^\vee$ that corresponds then to a k-linear function

$$f : \mathfrak{m}_R/(\mathfrak{m}_\Lambda R + \mathfrak{m}_R^2) \to V.$$

Since $\mathfrak{m}_R/(\mathfrak{m}_\Lambda R + \mathfrak{m}_R^2) \cong \mathfrak{m}_{\overline{R}}/\mathfrak{m}_{\overline{R}}^2$ (where $\overline{R} = R/\mathfrak{m}_\Lambda R$, see appendix B), we can consider the composite $g : \mathfrak{m}_{\overline{R}} \to \mathfrak{m}_R/(\mathfrak{m}_\Lambda R + \mathfrak{m}_R^2) \to V$, and define $\varphi_f : R \to k[V]$ as

$$\varphi_f(r) = \pi(r) + g(\pi'(r) - \pi(r)).$$

where $\pi : R \to k$ and $\pi' : R \to R/\mathfrak{m}_\Lambda R = \overline{R}$ are the quotient maps (we are using the fact that $\overline{R}$ is a k-algebra, so $\pi(r) \in \overline{R}$).

It can readily be checked that these two functions are inverse to each other, and we have our bijection. Functoriality is immediate. $\square$

Consider now a morphism $\xi : (\mathrm{Art}/R)^{op} \to \mathcal{F}$ of deformation categories. From ξ we get a formal object of $\mathcal{F}$ over R, taking $\xi_n = \xi(R_n)$, and $f_n = \xi(R_{n+1} \to R_n)$, and if we have a base-preserving natural transformation $\alpha : \xi \to \eta$ between two morphisms $(\mathrm{Art}/R)^{op} \to \mathcal{F}$ we get an arrow $\alpha : \{\xi_n, f_n\}_{n\in\mathbb{N}} \to \{\eta_n, g_n\}_{n\in\mathbb{N}}$ taking $\alpha_n = \alpha(R_n) : \xi_n \to \eta_n$.

This association gives a functor $\Phi : \mathrm{Hom}((\mathrm{Art}/R)^{op}, \mathcal{F}) \to \widehat{\mathcal{F}}(R)$. We have the following "Yoneda-like" proposition.

Proposition 5.1.11. *The functor Φ is an equivalence of categories.*

Proof. We construct a quasi-inverse $\Psi : \widehat{\mathcal{F}}(R) \to \mathrm{Hom}((\mathrm{Art}/R)^{op}, \mathcal{F})$ to Φ. Given a formal object $\xi = \{\xi_n, f_n\}_{n\in\mathbb{N}}$, we get a morphism $F_\xi : (\mathrm{Art}/R)^{op} \to \mathcal{F}$ in the following way: if $A \in (\mathrm{Art}/R)^{op}$ we associate to A the pullback $\xi_A = (\varphi_m)_*(\xi_m) \in \mathcal{F}$, where m is the order of A and $\varphi_m : R_m \to A$ is the homomorphism induced by $R \to A$. On arrows, if we have a homomorphism $\varphi : A \to B$ in (Art/R), the commutative diagram

$$\begin{tikzcd}
\xi_m \arrow[dd, mapsto] & & \xi_A \arrow[ll] \arrow[dd, mapsto] & \\
& \xi_n \arrow[ul] \arrow[dd, mapsto] & & \xi_B \arrow[ll] \arrow[ul, dashed, "\xi_\varphi"'] \arrow[dd, mapsto] \\
R_m \arrow[rr] \arrow[dr] & & A \arrow[dr, "\varphi"] & \\
& R_n \arrow[rr] & & B.
\end{tikzcd}$$

gives (by pullback in $\mathcal{F}$) an arrow $\xi_\varphi : \xi_B \to \xi_A$ of $\mathcal{F}$ over φ (as an arrow in (Art/Λ)).

This defines a morphism $F_\xi : (\mathrm{Art}/R)^{op} \to \mathcal{F}$. To an arrow $\alpha = \{\alpha_n\}_{n\in\mathbb{N}} : \xi \to \eta$ between two formal objects over R, where $\eta = \{\eta_n, g_n\}_{n\in\mathbb{N}}$, we associate a natural transformation $F_\alpha : F_\xi \to F_\eta$. Given an object $A \in (\mathrm{Art}/R)^{op}$ of order m, we define an arrow $F_\alpha(A) : F_\xi(A) \to F_\eta(A)$ simply as the pullback of $\alpha_m : \xi_m \to \eta_m$, along the homomorphism $R_m \to A$. Standard arguments show that this gives a natural transformation, and this completes the definition of Ψ.

Routine verifications using the universal property of pullbacks prove that Ψ and Φ are quasi-inverse to each other, and then our result. □

So giving a formal object of $\mathcal{F}$ over R is equivalent to giving a morphism of deformation categories $(\mathrm{Art}/R)^{op} \to \mathcal{F}$. From now on we will use both these points of view.

In particular we will use the same symbol for a formal object and for the associated morphism, and if $\xi : (\mathrm{Art}/R)^{op} \to \mathcal{F}$ is a formal object and $\varphi : R \to A$ a homomorphism of Λ-algebras, we will denote by ξ_φ (or simply $\xi_{R\to A}$ when there is no possibility of confusion) the object $\xi(R \to A)$ of $\mathcal{F}(A)$.

We get the following corollary (which is an analogue of the "weak" Yoneda's Lemma), simply by taking $\mathcal{F} = (\mathrm{Art}/R')$.

Corollary 5.1.12. *There is a natural bijection* $\mathrm{Hom}((\mathrm{Art}/R)^{op}, (\mathrm{Art}/R')^{op}) \cong \mathrm{Hom}_\Lambda(R', R)$ *that respects composition.*

In particular $(\mathrm{Art}/R)^{op}$ *and* $(\mathrm{Art}/R')^{op}$ *are isomorphic if and only if R ad R' are isomorphic.*

By "respects composition" above we mean that if R'' is another object of (Comp/Λ), and $F : (\mathrm{Art}/R)^{op} \to (\mathrm{Art}/R')^{op}$, $G : (\mathrm{Art}/R')^{op} \to (\mathrm{Art}/R'')^{op}$ are two morphisms corresponding to $\varphi : R' \to R$ and $\psi : R'' \to R'$ respectively, then $G \circ F \in \mathrm{Hom}((\mathrm{Art}/R)^{op}, (\mathrm{Art}/R'')^{op})$ corresponds to $\varphi \circ \psi \in \mathrm{Hom}_\Lambda(R'', R)$.

Notice also that $\mathrm{Hom}((\mathrm{Art}/R)^{op}, (\mathrm{Art}/R')^{op})$ is a set, since $(\mathrm{Art}/R')^{op}$ is fibered in sets. As to the proof, bijectivity is immediate from 5.1.11, and the part about respecting composition is very easy.

Remark 5.1.13. From this description of formal objects we get another interpretation of the pullback: if (R, ξ) is a formal object of $\mathcal{F}$, and $\varphi : R \to S$ is a homomorphism in (Comp/Λ), then from the last corollary we have an associated morphism $\overline{\varphi} : (\mathrm{Art}/S)^{op} \to (\mathrm{Art}/R)^{op}$, and we can consider the composite

$$(\mathrm{Art}/S)^{op} \xrightarrow{\overline{\varphi}} (\mathrm{Art}/R)^{op} \xrightarrow{\xi} \mathcal{F}$$

which is then a formal object of $\mathcal{F}$ over S. One can easily see that this formal object is (up to isomorphism) precisely the pullback of ξ to S along φ.

5.1.2 The Kodaira-Spencer map

Given a formal object (R, ξ) of $\mathcal{F}$, we can consider the differential at the only object over k of $(\mathrm{Art}\,/R)^{op}$ of the corresponding morphism $\xi : (\mathrm{Art}\,/R)^{op} \to \mathcal{F}$.

Definition 5.1.14. *The k-linear function $d_{R\to k}\xi : T_{R\to k}(\mathrm{Art}\,/R)^{op} \to T_{\xi_0}\mathcal{F}$ is called the **Kodaira-Spencer map** of the formal object (R, ξ). We will usually denote it by $\kappa_\xi : T_\Lambda R \to T_{\xi_0}\mathcal{F}$.*

Remark 5.1.15. More explicitly, the Kodaira-Spencer map can be described in the following way: if $\varphi : R \to k[\varepsilon]$ is an element of $T_{R\to k}(\mathrm{Art}\,/R)^{op}$ (we do not need to take isomorphism classes here, for $(\mathrm{Art}\,/R)^{op}$ is fibered in sets), the image $\kappa_\xi(\varphi)$ is the isomorphism class of the pullback of the formal object ξ along the map φ.

Notice that, since $\varepsilon^2 = 0$ and by Λ-linearity, φ will factor through the quotient map $R \to \overline{R}_1$, and $\kappa_\xi(\varphi)$ can be described as the isomorphism class of the pullback of $\overline{\xi}_1 \in \mathcal{F}(\overline{R}_1)$ along the induced map $\overline{R}_1 \to k[\varepsilon]$, where $\overline{\xi}_1$ is the pullback of the formal object ξ along the quotient map above.

There is another natural k-linear map $T_\Lambda R \to T_{\xi_0}\mathcal{F}$ associated with (R, ξ), defined in the following way: consider the object $\overline{\xi}_1 \in \mathcal{F}(\overline{R}_1)$ defined in the previous remark. This is a lifting of ξ_0 to $\overline{R}_1$, and since $\overline{R}_1$ is a k-algebra, we can compare it to the trivial lifting $\xi_0|_{\overline{R}_1}$ of ξ_0 to $\overline{R}_1$.

Since these objects are liftings of ξ_0 to $\overline{R}_1$, and

$$0 \longrightarrow \mathfrak{m}_{\overline{R}}/\mathfrak{m}_{\overline{R}}^2 \longrightarrow \overline{R}_1 \longrightarrow k \longrightarrow 0$$

is a small extension, we get an element

$$g([\overline{\xi}_1], [\xi_0|_{\overline{R}_1}]) \in (\mathfrak{m}_{\overline{R}}/\mathfrak{m}_{\overline{R}}^2) \otimes_k T_{\xi_0}\mathcal{F}$$

(using the notation introduced in Remark 2.3.3) that we call the **Kodaira-Spencer class** of ξ, and denote by k_ξ.

This element corresponds to a k-linear function

$$T_\Lambda R \cong (\mathfrak{m}_{\overline{R}}/\mathfrak{m}_{\overline{R}}^2)^\vee \to T_{\xi_0}\mathcal{F}.$$

Proposition 5.1.16. *The last map coincides with the Kodaira-Spencer map κ_ξ of ξ.*

Proof. Notice first that if $F : (\mathrm{FVect}\,/k) \to (\mathrm{Set})$ is the functor defined on objects by $F(V) = \{$isomorphism classes of objects of $\mathcal{F}_{\xi_0}(k[V])\}$, then since $\mathfrak{m}_{\overline{R}_1} = \mathfrak{m}_{\overline{R}}/\mathfrak{m}_{\overline{R}}^2$ is a square-zero ideal we have

$$\overline{R}_1 \cong k \oplus \mathfrak{m}_{\overline{R}}/\mathfrak{m}_{\overline{R}}^2 = k[\mathfrak{m}_{\overline{R}}/\mathfrak{m}_{\overline{R}}^2]$$

so that $F(\mathfrak{m}_{\overline{R}}/\mathfrak{m}_{\overline{R}}^2) = \{\text{isomorphism classes of objects of } \mathcal{F}_{\xi_0}(\overline{R}_1)\}$.

By definition of the action of Theorem 2.3.1, the element $k_\xi = g([\overline{\xi}_1], [\xi_0|_{\overline{R}_1}])$ corresponds then to the isomorphism class of $\overline{\xi}_1$ in

$$F(\mathfrak{m}_{\overline{R}}/\mathfrak{m}_{\overline{R}}^2) \cong (\mathfrak{m}_{\overline{R}}/\mathfrak{m}_{\overline{R}}^2) \otimes_k T_{\xi_0}\mathcal{F}.$$

To conclude it suffices to recall that the isomorphism

$$F(k[V]) \cong V \otimes_k T_{\xi_0}\mathcal{F} \cong \mathrm{Hom}_k(V^\vee, T_{\xi_0}\mathcal{F})$$

of Remark 2.1.5 takes an element $[\xi_V]$ of $F(k[V])$ to the k-linear function $V^\vee \to T_{\xi_0}\mathcal{F}$ that carries a functional $V \to k$ to the (isomorphism class of the) pullback of ξ_V to $k[\varepsilon]$, along the induced $k[V] \to k \oplus k \cong k[\varepsilon]$.

From this description, taking $V = \mathfrak{m}_{\overline{R}}/\mathfrak{m}_{\overline{R}}^2$ and $[\xi_V] = [\overline{\xi}_1]$, and from Remark 5.1.15 we see that the k-linear map corresponding to the element k_ξ is exactly the Kodaira-Spencer map κ_ξ. $\square$

This proposition shows in particular that κ_ξ is completely determined by the first-order term $\xi_1 \in \mathcal{F}(R_1)$ of ξ, because this determines $\overline{\xi}_1 \in \mathcal{F}(\overline{R}_1)$ up to isomorphism. Conversely, by freeness of the action, $\overline{\xi}_1$ is determined (up to isomorphism) once we know κ_ξ.

The following functoriality property is an immediate consequence of Proposition 2.3.4. Take two objects R, S of (Comp/Λ), a homomorphism $\varphi : R \to S$, and $\xi = \{\xi_n, f_n\}_{n\in\mathbb{N}}$ a formal object of $\mathcal{F}$ over R.

Call $\overline{\varphi}_1 : \overline{R}_1 \to \overline{S}_1$ the induced homomorphism, which in turn induces a morphism of small extensions

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{m}_{\overline{R}}/\mathfrak{m}_{\overline{R}}^2 \arrow[r] \arrow[d, "\psi"] & \overline{R}_1 \arrow[r] \arrow[d, "\overline{\varphi}_1"] & k \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathfrak{m}_{\overline{S}}/\mathfrak{m}_{\overline{S}}^2 \arrow[r] & \overline{S}_1 \arrow[r] & k \arrow[r] & 0.
\end{tikzcd}$$

Recall also that φ induces a differential $d\varphi : T_\Lambda S \to T_\Lambda R$ that is the adjoint of the codifferential $\psi : \mathfrak{m}_{\overline{R}}/\mathfrak{m}_{\overline{R}}^2 \to \mathfrak{m}_{\overline{S}}/\mathfrak{m}_{\overline{S}}^2$ (see appendix B).

Proposition 5.1.17. *We have the following relations between the Kodaira-Spencer maps and classes of $\xi \in \widehat{\mathcal{F}}(R)$ and of the pullback $\varphi_*(\xi) \in \widehat{\mathcal{F}}(S)$.*

- $k_{\varphi_*(\xi)} = (\psi \otimes \mathrm{id})(k_\xi) \in \mathfrak{m}_{\overline{S}}/\mathfrak{m}_{\overline{S}}^2 \otimes_k T_{\xi_0}\mathcal{F}.$
- $\kappa_{\varphi_*(\xi)} = \kappa_\xi \circ d\varphi : T_\Lambda S = (\mathfrak{m}_{\overline{S}}/\mathfrak{m}_{\overline{S}}^2)^\vee \to T_{\xi_0}\mathcal{F}.$

Proof. The first property follows immediately from Proposition 2.3.4, and we get the second one by using Remark 5.1.13, and noticing that the differential of the morphism $(\mathrm{Art}/S)^{op} \to (\mathrm{Art}/R)^{op}$ corresponding to φ is precisely $d\varphi : T_\Lambda S \to T_\Lambda R$.

Alternatively the second part follows from the first, the fact that $d\varphi$ is adjoint to ψ, and Proposition 5.1.16. $\square$

5.2 Universal and versal formal deformations

As the classical Yoneda's Lemma, Proposition 5.1.11 lets us speak of "universal formal objects" for a deformation category $\mathcal{F}$.

Definition 5.2.1. *A **universal formal object** over $R \in (\mathrm{Comp}/\Lambda)$ for $\mathcal{F}$ is a formal object $\xi \in \widehat{\mathcal{F}}(R)$, such that the corresponding $\xi : (\mathrm{Art}/R)^{op} \to \mathcal{F}$ is an equivalence.*

Thanks to Proposition 1.2.15, ξ is a universal formal object if and only if $\xi_A : (\mathrm{Art}/R)^{op}(A) \to \mathcal{F}(A)$ is an equivalence for every $A \in (\mathrm{Art}/\Lambda)$, or equivalently if and only if for every $A \in (\mathrm{Art}/\Lambda)$ and $\eta \in \mathcal{F}(A)$ there exist a unique homomorphism of Λ-algebras $R \to A$ and a unique isomorphism $\xi_{R\to A} \cong \eta$ in $\mathcal{F}(A)$. This can also be restated by saying that for every $A \in (\mathrm{Art}/\Lambda)$ and $\eta \in \mathcal{F}(A)$ there is a unique arrow $(A, \eta) \to (R, \xi)$ of formal objects in $\widehat{\mathcal{F}}$.

Using Remark 5.1.9 we see that the above universal property can be strengthened to: for every formal object (S, η) of $\mathcal{F}$ there exists a unique arrow $(S, \eta) \to (R, \xi)$. That is, every formal object can be obtained as pullback of (R, ξ), in a unique way. This can easily be checked by considering the sequence of arrows $h_n : (S_n, \eta_n) \to (R, \xi)$ coming from the "weak" universal property above, and noticing that they are necessarily compatible because of uniqueness.

Using this last universal property it is easy to check that two universal deformations are canonically isomorphic.

Definition 5.2.2. *We say that a deformation category $\mathcal{F} \to (\mathrm{Art}/\Lambda)^{op}$ is **prorepresentable** if it is equivalent to a deformation category of the form $(\mathrm{Art}/R)^{op}$ for some $R \in (\mathrm{Comp}/\Lambda)$, or equivalently if $\mathcal{F}$ has a universal formal object (R, ξ).*

Since $(\mathrm{Art}/R)^{op}$ is a category fibered in sets, a necessary condition for a deformation category $\mathcal{F} \to (\mathrm{Art}/\Lambda)^{op}$ to be prorepresentable is that $\mathcal{F}$ should be fibered in equivalence relations. Other necessary conditions are that $\mathcal{F}(k)$ should be a trivial groupoid, because it will be equivalent to a singleton, and $\mathcal{F}$ should have finite-dimensional tangent space $T_{\xi_0}\mathcal{F}$ at any (actually it suffices that this holds for one, given the former condition) object $\xi_0 \in \mathcal{F}(k)$, because $\dim_k(T_\Lambda R)$ is finite.

The main result of this chapter is that the converse also holds.

Theorem 5.2.3 (Schlessinger)**.** *Let $\mathcal{F} \to (\mathrm{Art}/\Lambda)^{op}$ be a deformation category. Then $\mathcal{F} \to (\mathrm{Art}/\Lambda)^{op}$ is prorepresentable if and only if:*

- *$\mathcal{F}(k)$ is a trivial groupoid.*
- *$T_{\xi_0}\mathcal{F}$ is finite-dimensional for any $\xi_0 \in \mathcal{F}(k)$.*
- *$\mathrm{Inf}(\xi_0)$ is trivial for any $\xi_0 \in \mathcal{F}(k)$.*

This is an analogue of Schlessinger's Theorem 1.1.14 for deformation categories, even though there are no direct implications between the two (see the discussion at the end of Section 1.3). We will prove the theorem in Section 5.3, after discussing miniversal deformations.

Example 5.2.4. As a simple example, suppose that X is a projective scheme of finite type over k, set $\Lambda = k$, and consider the deformation category $\mathcal{H}ilb^X \to (\mathrm{Art}/k)^{op}$.

Take Z_0 a closed subscheme of X, and notice that the deformation category $\mathcal{H}ilb^X_{Z_0} \to (\mathrm{Art}/k)^{op}$ of objects restricting to Z_0 over k meets all hypotheses of Theorem 5.2.3: we have already seen that $\mathrm{Inf}_{Z_0}(\mathcal{H}ilb^X) = 0$, clearly the only object over k is Z_0 itself, and the tangent space $T_{Z_0}\mathcal{H}ilb^X \cong H^0(Z_0, \mathcal{N}_0)$ is finite-dimensional over k. Then we can conclude that the deformation category $\mathcal{H}ilb^X_{Z_0} \to (\mathrm{Art}/k)^{op}$ is prorepresentable.

We can see this in a more concrete way: the deformation category (which is fibered in sets) $\mathcal{H}ilb^X \to (\mathrm{Art}/k)^{op}$ comes from a functor, called the **Hilbert functor** of X, and denoted by $Hilb^X : (\mathrm{Sch}/k) \to (\mathrm{Set})$; a theorem of Grothendieck (see for example Chapter 5 of [FGA]) states that with the hypotheses above this functor is represented by a scheme, called the **Hilbert scheme** of X, that we still denote by $Hilb^X \in (\mathrm{Sch}/k)$.

The closed subscheme Z_0 corresponds then to a point in the Hilbert scheme, $Z_0 \in Hilb^X$. Since $Hilb^X$ represents the Hilbert functor, every object $Z \in \mathcal{H}ilb^X_{Z_0}(A)$ corresponds to a morphism $\mathrm{Spec}(A) \to Hilb^X$ with image Z_0, that factors through $\mathrm{Spec}(\widehat{\mathcal{O}}_{Hilb^X, Z_0})$, by the usual argument. In particular the resulting homomorphism $\widehat{\mathcal{O}}_{Hilb^X, Z_0} \to A$ gives an object of $(\mathrm{Art}/\widehat{\mathcal{O}}_{Hilb^X, Z_0})^{op}$.

This gives a morphism

$$\mathcal{H}ilb^X_{Z_0} \to (\mathrm{Art}/\widehat{\mathcal{O}}_{Hilb^X, Z_0})^{op}$$

of deformation categories that is easily seen to be an equivalence. Then $\mathcal{H}ilb^X_{Z_0}$ is prorepresentable, as we already knew from Theorem 5.2.3. The universal formal object (R, Z) in this case has $R = \widehat{\mathcal{O}}_{Hilb^X, Z_0}$, and the term of order n of the formal deformation $Z = \{Z_n, f_n\}_{n \in \mathbb{N}}$ over R is the pullback to $\mathrm{Spec}(\mathcal{O}_{Hilb^X, Z_0}/\mathfrak{m}^n_{Hilb^X, Z_0})$ of the universal closed subscheme of $Hilb^X$.

5.2.1 Versal objects

The condition of not having infinitesimal automorphisms prevents many deformation categories from being prorepresentable. Because of this, we try to weaken the condition of universality on formal objects, to end up with a more useful notion. The right definition is the following.

Definition 5.2.5. *Let $\mathcal{F} \to (\mathrm{Art}/\Lambda)^{op}$ be a deformation category. A formal object (R, ρ) of $\mathcal{F}$ is called **versal** if the following lifting property holds: for every small extension $\varphi : A' \to A$, every diagram of formal objects*

$$\begin{tikzcd} & (A', \xi') \arrow[dl, dashed, bend right] \\ (R, \rho) & (A, \xi) \arrow[l] \arrow[u] \end{tikzcd}$$

can be filled with a dotted arrow.

Remark 5.2.6. It is easy to check that a formal object (R, ρ) is universal if and only if for any diagram as above there exists a unique dotted arrow making it commutative. So universal deformations are versal.

Proposition 5.2.7. *Let $\mathcal{F} \to (\mathrm{Art}/\Lambda)^{op}$ be a deformation category, and (R, ρ) a versal formal object. Then the lifting property of the definition above holds also when $A' \to A$ is a surjection in (Comp/Λ).*

Proof. First of all, it is easy to see that the lifting property will hold when $A' \to A$ is a surjection in (Art/Λ), as usual by factoring $A' \to A$ into a composite of small extensions and lifting the morphism successively.

Let $A' \to A$ be a surjection in (Comp/Λ), and we write $\xi = \{\xi_n, f_n\}$ and $\xi' = \{\xi'_n, f'_n\}_{n\in\mathbb{N}}$. Let us show inductively that for each n we can find a morphism of formal objects $g_n : (A'_n, \xi'_n) \to (R, \rho)$ such that for all n the composite

$$(A'_n, \xi'_n) \to (A'_{n+1}, \xi'_{n+1}) \xrightarrow{g_{n+1}} (R, \rho)$$

coincides with g_n, and the diagram

$$\begin{tikzcd} & (A'_n, \xi'_n) \arrow[dl, bend right, "g_n"'] \\ (R, \rho) & (A_n, \xi_n) \arrow[l] \arrow[u] \end{tikzcd}$$

commutes.

Suppose we have constructed g_n, and consider the diagram

$$\begin{tikzcd} R \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & B \arrow[r] \arrow[d] & A'_n \arrow[d] \\ & A_{n+1} \arrow[r] & A_n \end{tikzcd}$$

where the maps from R are the homomorphism $R \to A'_n$ coming from g_n and the one $R \to A_{n+1}$ associated with $(A,\xi) \to (R,\rho)$, and B is the fibered product. Taking the pullback of ρ to B along the dotted homomorphism above we get an object $\eta \in \mathcal{F}(B)$ restricting to ξ'_n on A'_n and on ξ_{n+1} on A_{n+1}.

Notice now that there is a homomorphism $A'_{n+1} \to B$ induced by the quotient map $A'_{n+1} \to A'_n$ and the map $A'_{n+1} \to A_{n+1}$ coming from $A' \to A$, and which gives a morphism of formal objects $(B,\eta) \to (A'_{n+1},\xi'_{n+1})$, fitting in the commutative diagram

$$
\begin{tikzcd}
(A'_{n+1},\xi'_{n+1}) & & \\
& (B,\eta) \arrow[ul] & (A'_n,\xi'_n) \arrow[l] \arrow[ull, bend right] \\
& (A_{n+1},\xi_{n+1}) \arrow[u] \arrow[uul, bend left] & (A_n,\xi_n). \arrow[l] \arrow[u]
\end{tikzcd}
$$

Moreover from the fact that $A'_{n+1} \to B$ is a surjection in (Art/Λ) (as is readily checked, using the surjectivity of $A'_{n+1} \to A_{n+1}$), and from the diagram

$$
\begin{tikzcd}
& (A'_{n+1},\xi'_{n+1}) \arrow[dl, dashed, "g_{n+1}"'] \\
(R,\rho) & (B,\eta) \arrow[l] \arrow[u]
\end{tikzcd}
$$

by versality of (R,ρ) we get the dotted morphism $g_{n+1} : (A'_{n+1},\xi'_{n+1}) \to (R,\rho)$ that has the desired properties.

Finally, notice that by Remark 5.1.9 the sequence $\{g_n\}_{n\in\mathbb{N}}$ of compatible morphisms induces a morphism of formal objects $(A',\xi') \to (R,\rho)$ that gives the desired lifting. $\square$

Notice that the dotted arrow in the diagram of definition 5.2.5 will give a lifting $R \to A'$ of the map $R \to A$, and conversely the existence of such a lifting implies at least that the deformation ξ will lift to A' (just by taking the pullback of ρ). In other words in presence of a versal deformations the problem of lifting objects becomes a problem of lifting maps of Λ-algebras. From this remark we will get a criterion to decide wether a deformation problem is obstructed or not, knowing a versal deformation (see Proposition 5.2.12).

As in the case of deformation functors, the property of being versal can be restated as a smoothness condition.

Definition 5.2.8. *Let $\mathcal{F} \to (\mathrm{Art}/\Lambda)^{op}$ and $\mathcal{G} \to (\mathrm{Art}/\Lambda)^{op}$ be two deformation categories, and $F : \mathcal{F} \to \mathcal{G}$ be a morphism. We say that F is **formally smooth** if for every surjection $A' \to A$ in (Art/Λ) the functor $\mathcal{F}(A') \to \mathcal{F}(A) \times_{\mathcal{G}(A)} \mathcal{G}(A')$ induced by the diagram*

$$\begin{tikzcd} \mathcal{F}(A') \arrow[r] \arrow[d, "F_{A'}"'] & \mathcal{F}(A) \arrow[d, "F_A"] \\ \mathcal{G}(A') \arrow[r] & \mathcal{G}(A) \end{tikzcd}$$

is essentially surjective.

Remark 5.2.9. The term “smooth” comes from the fact that if $\mathcal{F}$ and $\mathcal{G}$ are deformation categories corresponding to the functors of points of two schemes X and Y, then a morphism $X \to Y$ locally of finite type is smooth if and only if the corresponding morphism $\mathcal{F} \to \mathcal{G}$ is formally smooth. This is the so-called “infinitesimal smoothness criterion” of Grothendieck (see Théorème 3.1 of Exposé III in [SGA1]).

Proposition 5.2.10. *Let $\mathcal{F} \to (\mathrm{Art}/\Lambda)^{op}$ be a deformation category. A formal object (R, ρ) of $\mathcal{F}$ is versal if and only if the corresponding morphism $\rho : (\mathrm{Art}/R)^{op} \to \mathcal{F}$ is formally smooth.*

Proof. This is immediate from the definitions: if $A' \to A$ is a surjection in (Art/Λ), the natural functor

$$(\mathrm{Art}/R)^{op}(A') \to (\mathrm{Art}/R)^{op}(A) \times_{\mathcal{F}(A)} \mathcal{F}(A')$$

sends a homomorphism $R \to A'$ to the object $(R \to A, \rho_{R\to A'}, \theta)$, where $R \to A$ is the composite $R \to A' \to A$, and $\theta : \rho_{R\to A'}|_A \cong \rho_{R\to A}$ is the canonical isomorphism between the pullbacks of ρ.

From this it is evident that an object $(R \to A, \xi', \theta) \in (\mathrm{Art}/R)^{op}(A) \times_{\mathcal{F}(A)} \mathcal{F}(A')$, which corresponds to a diagram of formal deformations

$$\begin{tikzcd} & (A', \xi') \arrow[dl, dashed] \\ (R, \rho) & (A, \rho_{R\to A}) \arrow[l] \arrow[u] \end{tikzcd}$$

will come from a homomorphism $R \to A'$ exactly when there exists a lifting $(A', \xi') \to (R, \rho)$ of the morphism $(A, \rho_{R\to A}) \to (R, \rho)$. $\square$

Here are two immediate properties of versal deformations.

Proposition 5.2.11. *Let $\mathcal{F} \to (\mathrm{Art}/\Lambda)^{op}$ be a deformation category, and (R, ρ) a versal formal object of $\mathcal{F}$. Then:*

(i) *For every formal object (S, ξ) restricting to ρ_0 on k there is a morphism $(S, \xi) \to (R, \rho)$ (in particular this also holds if $S \in (\mathrm{Art}/\Lambda)$).*

(ii) *The Kodaira-Spencer map $\kappa_\rho : T_\Lambda R \to T_{\rho_0}\mathcal{F}$ is surjective.*

Proof. The first part of the statement is immediate from Proposition 5.2.7, where we consider as surjection the quotient map $S \to k$, and the diagram

$$\begin{tikzcd} & (S,\xi) \arrow[dl, dashed] \\ (R,\rho) & (k,\rho_0) \arrow[l] \arrow[u] \end{tikzcd}$$

that identifies ρ_0 as the pullback of ρ and ξ over k.

Now we prove **(ii)**: take a vector $v \in T_{\rho_0}\mathcal{F}$, the usual ring of dual numbers $k[\varepsilon]$, and consider the element of $(\varepsilon) \otimes_k T_{\rho_0}\mathcal{F}$ corresponding to v. We can then find an object $\xi \in \mathcal{F}_{\rho_0}(k[\varepsilon])$ such that

$$g([\xi], [\rho_0|_{k[\varepsilon]}]) = v \in (\varepsilon) \otimes_k T_{\rho_0}\mathcal{F}$$

(simply by taking a representative of $[\rho_0|_{k[\varepsilon]}] \cdot v$, where this is the usual action of Theorem 2.3.1) which is the same as saying that the Kodaira-Spencer map $\kappa_\xi : k \cong (\varepsilon)^\vee \to T_{\rho_0}\mathcal{F}$ of the formal object $(k[\varepsilon], \xi)$ sends 1 to v.

By the first part of the proposition we get a morphism of formal objects $(k[\varepsilon], \xi) \to (R, \rho)$ (and in particular a homomorphism $\varphi : R \to k[\varepsilon]$) that identifies ξ as a pullback of ρ, and from the second part of Proposition 5.1.17 we get

$$v = \kappa_\xi(1) = \kappa_\rho(d\varphi(1))$$

where $d\varphi : k \to T_\Lambda R$ is the differential induced by φ. From this we see that v is in the image of κ_ρ, and thus this map is surjective. $\square$

In particular if $\mathcal{F}$ admits a versal object (R, ρ), then the tangent space $T_{\rho_0}\mathcal{F}$ is finite-dimensional.

We can now state and prove the anticipated criterion to recognize unobstructed objects.

Proposition 5.2.12. *Let $\mathcal{F} \to (\mathrm{Art}/\Lambda)^{op}$ be a deformation category, and (R, ρ) a versal formal object of $\mathcal{F}$. Then ρ_0 is unobstructed if and only if R is a power series ring over Λ.*

Proof. We first recall the smoothness criterion B.9: $R \in (\mathrm{Comp}/\Lambda)$ is a power series ring over Λ if and only if for any homomorphism $\varphi : R \to A$ with $A \in (\mathrm{Art}/\Lambda)$, and small extension $\psi : A' \to A$, there exists a lifting $\lambda : R \to A'$, that is, a homomorphism λ such that $\psi \circ \lambda = \varphi$.

Suppose ρ_0 is unobstructed, and take a homomorphism $\varphi : R \to A$ and a small extension $A' \to A$. Then considering the pullback $\xi = \varphi^*(\rho)$ of ρ

to A we get a morphism of formal deformations $f : (A, \xi) \to (R, \rho)$, and since ρ_0 is unobstructed (and clearly the pullback of ξ to k will still be ρ_0), we have a lifting $\xi' \in \mathcal{F}(A')$ of ξ, which gives a morphism of formal objects $(A, \xi) \to (A', \xi')$. By versality we have then a morphism $(A', \xi') \to (R, \rho)$ that lifts f, and in particular we get a homomorphism $R \to A'$ lifting φ. By criterion B.9, we conclude that R is a power series ring.

Conversely, suppose R is a power series ring, and take a small extension $\varphi : A' \to A$ with $\xi \in \mathcal{F}(A)$, such that the pullback to k is ρ_0. From the versality of (R, ρ) (and Proposition 5.2.11) we get a morphism of formal objects $(A, \xi) \to (R, \rho)$, that is, a homomorphism $\psi : R \to A$ with an isomorphism $\psi_*(\rho) \cong \xi$. Since R is a power series ring, φ will lift to $\lambda : R \to A'$, and we can take the pullback $\xi' = \lambda_*(\rho) \in \mathcal{F}(A')$. Since $\varphi \circ \lambda = \psi$, the pullbacks $\varphi_*(\xi')$ and $\psi_*(\rho) \cong \xi$ will be isomorphic; in other words we have an arrow $\xi \to \psi_*(\rho) \to \varphi_*(\xi') \to \xi'$ of $\mathcal{F}$ over φ that makes ξ' into a lifting of ξ to A'. This shows that ρ_0 is unobstructed. $\square$

5.2.2 Miniversal objects

The second part of Proposition 5.2.11 suggests us to consider versal deformations where R is as "small" as possible, and leads us to the following definition.

Definition 5.2.13. *A versal formal object (R, ρ) of $\mathcal{F}$ is called **minimal** if the Kodaira-Spencer map $\kappa_\rho : T_\Lambda R \to T_{\rho_0}\mathcal{F}$ is an isomorphism.*

A versal minimal formal object is shortly called **miniversal**; Schlessinger calls the corresponding concept for deformation functors a **hull**. Sometimes we will also say that (R, ρ) is a **miniversal deformation** of $\rho_0 \in \mathcal{F}(k)$.

We now show that all universal deformations are miniversal, and that miniversal deformations are all isomorphic, in a non-canonical way.

Proposition 5.2.14. *Let $\mathcal{F} \to (\mathrm{Art}/\Lambda)^{op}$ be a deformation category. Then:*

(i) *Any universal formal object of $\mathcal{F}$ is miniversal.*

(ii) *Any two miniversal formal objects of $\mathcal{F}$ with the same pullback to k are non-canonically isomorphic.*

Proof. We start by proving **(i)**: it is clear that a universal object is in particular versal (and moreover the lifting morphism in the versality property will be unique), so we only have to prove that if (R, ρ) is a universal formal object of $\mathcal{F}$, then the Kodaira-Spencer map $\kappa_\rho : T_\Lambda R \to \mathcal{F}$ is an isomorphism. But this follows from Remark 2.1.8, since κ_ρ is the differential of $\rho : (\mathrm{Art}/R)^{op} \to \mathcal{F}$, which is an equivalence by definition.

For the second statement, take two miniversal objects (R, ρ) and (S, ν) such that ρ_0 and ν_0 are isomorphic. By Proposition 5.2.11 we have two morphisms of formal objects $(R,\rho) \to (S,\nu)$ and $(S,\nu) \to (R,\rho)$, and we call $\varphi : S \to R$ and $\psi : R \to S$ the corresponding homomorphisms.

By functoriality of the Kodaira-Spencer map and minimality of (R,ρ) and (S,ν), the two differentials $d\varphi : T_\Lambda R \to T_\Lambda S$ and $d\psi : T_\Lambda S \to T_\Lambda R$ will be isomorphisms (so the codifferentials are also), and from Corollary B.4 we get that ψ and φ are isomorphisms. In conclusion (R,ρ) and (S,ν) are isomorphic formal objects. □

Next, we see that all versal formal objects can be described in term of a miniversal one (provided we have one).

Proposition 5.2.15. *Let $\mathcal{F} \to (\mathrm{Art}/\Lambda)^{op}$ be a deformation category, (R,ρ) a miniversal formal object of $\mathcal{F}$, and consider the power series algebra on n indeterminates $S = R[[x_1,\dots,x_n]] \in (\mathrm{Comp}/\Lambda)$, with the inclusion $i : R \to S$. Then the formal object $(S, i_*(\rho))$ obtained by pullback is versal.*

Conversely if (P,ξ) is a versal formal object of $\mathcal{F}$ restricting to ρ_0 on k, and the kernel of $\kappa_\xi : T_\Lambda P \to T_{\xi_0}\mathcal{F}$ has dimension n, then (P,ξ) is isomorphic to the formal object $(S, i_*(\rho))$ above.

Proof. Suppose we have a diagram

$$\begin{tikzcd}
 & (A',\xi') \arrow[dl, dashed, bend right] \\
(S, i_*(\rho)) & (A,\xi) \arrow[l] \arrow[u]
\end{tikzcd}$$

and we want to show that the dotted lifting exists. By versality of (R,ρ), the composite $(A,\xi) \to (S, i_*(\rho)) \to (R,\rho)$ will lift to a morphism of formal objects $(A',\xi') \to (R,\rho)$ that corresponds in particular to a homomorphism $\varphi : R \to A'$. Call also $\psi : S \to A$ and $\lambda : A' \to A$ the two homomorphisms corresponding to the morphisms above.

The following diagram, in which the labels of the arrows are the corresponding homomorphisms (going then in the inverse direction), sums up the situation.

$$\begin{tikzcd}
 & & (A',\xi') \arrow[dll, bend right, "\varphi"'] \arrow[dl, dashed, bend right] \\
(R,\rho) & (S, i_*(\rho)) \arrow[l, "i"'] & (A,\xi) \arrow[l, "\psi"'] \arrow[u, "\lambda"']
\end{tikzcd}$$

We now lift $\varphi : R \to A'$ to $\nu : S \to A'$: if we choose for each i an $a_i \in A'$ such that $\psi(x_i) = \lambda(a_i)$, then by the properties of power series rings there exist a unique homomorphism $\nu : S \to A'$ such that $\nu \circ i = \varphi$ and $\nu(x_i) = a_i$ for all i (this is proved in the same way as Proposition B.6). This homomorphism also satisfies $\lambda \circ \nu = \psi$ by construction.

Since $\nu \circ i = \varphi$, the pullback $\nu_*(i_*(\rho))$ of $i_*(\rho)$ to A' will be isomorphic to ξ', and this isomorphism together with ν will give an arrow $(A', \xi') \to (S, i_*(\rho))$ that fits in the above diagram, showing that $(S, i_*(\rho))$ is versal.

Conversely, suppose (P, ξ) is a versal formal object of $\mathcal{F}$ restricting to ρ_0 over k. Then by Proposition 5.2.11 and versality of (R, ρ) we have a morphism of formal objects $(P, \xi) \to (R, \rho)$ that corresponds to a homomorphism $\varphi : R \to P$. Moreover by 5.1.17 and surjectivity of κ_ξ (by 5.2.11 again), we get that $d\varphi : T_\Lambda P \to T_\Lambda R$ is surjective, so that in turn its adjoint map $\psi : \mathfrak{m}_{\overline{R}}/\mathfrak{m}_{\overline{R}}^2 \to \mathfrak{m}_{\overline{P}}/\mathfrak{m}_{\overline{P}}^2$ is injective, and the codimension of the image is exactly n, since κ_ρ is an isomorphism.

Take then elements $a_1, \dots, a_n \in \mathfrak{m}_P$ such that their image in $\mathfrak{m}_{\overline{P}}/\mathfrak{m}_{\overline{P}}^2$ is a basis for a complement of the image of ψ, and define $\nu : S \to P$ (where $S = R[[x_1, \dots, x_n]]$ as above) by imposing that $\nu \circ i = \varphi$, and $\nu(x_i) = a_i$ for all i. The pullback $\nu_*(i_*(\rho))$ will then be isomorphic to ξ, and this gives us a morphism $(P, \xi) \to (S, i_*(\rho))$, which we now prove to be an isomorphism.

Since $\nu_1 : \overline{S}_1 \to \overline{P}_1$ is an isomorphism by construction, we have an inverse $\lambda_1 : \overline{P}_1 \to \overline{S}_1$ that gives in particular an isomorphism of formal objects $(\overline{S}_1, \overline{\rho}_1) \to (\overline{P}_1, \overline{i_*(\rho)}_1)$, inverse to the restriction of $(P, \xi) \to (S, i_*(\rho))$ to the first-order terms. By versality of (P, ξ), the composite

$$(\overline{S}_1, \overline{i_*(\rho)}_1) \to (\overline{P}_1, \overline{\xi}_1) \to (P, \xi)$$

can be lifted to a morphism $(S, i_*(\rho)) \to (P, \xi)$, which corresponds to a homomorphism $\lambda : P \to S$.

Notice now that the composites $\lambda \circ \nu$ and $\nu \circ \lambda$ induce the identity on $\mathfrak{m}_{\overline{S}}/\mathfrak{m}_{\overline{S}}^2$ and $\mathfrak{m}_{\overline{P}}/\mathfrak{m}_{\overline{P}}^2$, so using Corollary B.4 we conclude that they are isomorphisms, and in particular ν is as well. This shows that $(P, \xi) \to (S, i_*(\rho))$ is an isomorphism. $\square$

5.3 Existence of miniversal deformations

Now we give an analogue of the "existence of hulls" part of Schlessinger's Theorem, in the context of deformations categories.

Theorem 5.3.1. *Let $\mathcal{F} \to (\mathrm{Art}/\Lambda)^{op}$ be a deformation category, and $\xi_0 \in \mathcal{F}(k)$, such that the tangent space $T_{\xi_0}\mathcal{F}$ is finite-dimensional. Then $\mathcal{F}$ admits a miniversal formal object (R, ρ), with $\rho_0 \cong \xi_0$.*

Moreover if n is the dimension of $T_{\xi_0}\mathcal{F}$, then R will be a quotient P/I of the power series ring $P = \Lambda[[x_1, \dots, x_n]]$ on n indeterminates, with $I \subseteq \mathfrak{m}_\Lambda P + \mathfrak{m}_P^2$.

Proof. The proof will be in two steps. First of all, we show that it is sufficient to find a formal object (R, ρ) such that the Kodaira-Spencer map $\kappa_\rho : T_\Lambda R \to T_{\xi_0}\mathcal{F}$ is an isomorphism, and for every small extension $A' \to A$

with a diagram

$$\begin{tikzcd} & (A', \xi') \\ (R, \rho) & (A, \xi) \arrow[l] \arrow[u] \end{tikzcd}$$

of formal objects, the homorphism $R \to A$ lifts to $R \to A'$.

Proof. We show that this weaker lifting property implies versality, if κ_ρ is an isomorphism.

Write $\varphi : A' \to A$ and $\psi : R \to A$ for the homomorphisms associated with the arrows above, and suppose we have a lifting $\nu : R \to A'$ of ψ (which is a lifting of the object $\psi \in (\mathrm{Art}/R)^{op}(A)$ to A'); we consider then the pullback $\rho_{R \to A'}$ of ρ to A'. Since $\varphi \circ \nu = \psi$, the pullback $\varphi_*(\rho_{R \to A'})$ of $\rho_{R \to A'}$ to A will be isomorphic to ξ, and this makes $\rho_{R \to A'}$ into a lifting of ξ to A'.

Since ξ' is a lifting of ξ too, by Theorem 2.3.1 we can find an element $g \in I \otimes_k T_{\xi_0}\mathcal{F}$ such that $[\rho_{R \to A'}] \cdot g = [\xi']$. But now $\kappa_\rho : T_\Lambda R \to T_{\xi_0}\mathcal{F}$ is an isomorphism, so that $\mathrm{id} \otimes \kappa_\rho : I \otimes_k T_\Lambda R \to I \otimes_k T_{\xi_0}\mathcal{F}$ is as well, and in particular we can find an element $h \in I \otimes_k T_\Lambda R$ such that $(\mathrm{id} \otimes \kappa_\rho)(h) = g$.

Recalling that κ_ρ is the differential of the morphism $\rho : (\mathrm{Art}/R)^{op} \to \mathcal{F}$ and that the action on the liftings is functorial with respect to the deformation category (Proposition 2.3.6), we get that

$$[\rho_{(R \to A') \cdot h}] = [\rho_{R \to A'}] \cdot (\mathrm{id} \otimes \kappa_\rho)(h) = [\rho_{R \to A'}] \cdot g = [\xi'].$$

So $(R \to A') \cdot h$ is a homomorphism $R \to A'$ such that the pullback of ρ along it is isomorphic to ξ', and this gives a lifting $(A', \xi') \to (R, \rho)$ of the given $(A, \xi) \to (R, \rho)$, showing that (R, ρ) is versal. $\square$

Now we will construct a formal object (R, ρ) with the two properties above.

Let $E = T^\vee_{\xi_0}\mathcal{F}$, the dual of the tangent space $T_{\xi_0}\mathcal{F}$, and $x_1, \dots, x_n$ be a basis of E as a k-vector space. Put $P = \Lambda[[x_1, \dots, x_n]]$. Then we have $\overline{P}_1 \cong k \oplus E$ and we can consider a lifting $\overline{\rho}_1 \in \mathcal{F}(\overline{P}_1)$ of ξ_0, such that the Kodaira-Spencer map $\kappa_{\overline{\rho}_1} : E^\vee \to T_{\xi_0}\mathcal{F} = E^\vee$ of the formal deformation $(\overline{P}_1, \overline{\rho}_1)$ is the identity.

There exists precisely one such lifting (up to isomorphism), and it is obtained applying to the trivial lifting $\xi_0|_{\overline{P}_1}$ the element of $E \otimes_k T_{\xi_0}\mathcal{F}$ corresponding to the identity (see the comments after the proof of Proposition 5.1.16). One could easily see that this object has the versality property with respect to artinian Λ-algebras of the form $k \oplus V$.

Now we will progressively extend $\overline{\rho}_1$ to a formal object on some bigger quotient of P. We first define inductively a sequence of ideals $I_i \subseteq P$ and objects $\rho_i \in \mathcal{F}(P/I_i)$ (it is easy to check that all the quotients will be actually artinian) starting with $I_1 = \mathfrak{m}_\Lambda P + \mathfrak{m}_P^2$ and $\rho_1 = \overline{\rho}_1$.

Suppose we have I_{n-1} and $\rho_{n-1} \in \mathcal{F}(P/I_{n-1})$. Consider the set A of ideals $I \subseteq P$ such that $\mathfrak{m}_P I_{n-1} \subseteq I \subseteq I_{n-1}$ and there exists a lifting $\rho \in \mathcal{F}(P/I)$ of ρ_{n-1}, and take I_n to be the minimum element of A with respect to inclusion, that is, every element of the set A contains I_n.

To show that such an element exists, we show that A is closed under intersection (it is clearly nonempty, since I_{n-1} satisfies the conditions). Noticing that ideals $\mathfrak{m}_P I_{n-1} \subseteq I \subseteq I_{n-1}$ correspond to subspaces of the finite-dimensional k-vector space $I_{n-1}/\mathfrak{m}_P I_{n-1}$, we only have to show that A is closed under finite (or pairwise) intersection.

So suppose $I, J \in A$, with $\eta \in \mathcal{F}(P/I)$ and $\sigma \in \mathcal{F}(P/J)$; working in the k-vector space $I_{n-1}/\mathfrak{m}_P I_{n-1}$ we can find an ideal J' of P such that $J \subseteq J' \subseteq I_{n-1}$, $I \cap J = I \cap J'$ and $I + J' = I_{n-1}$. Then we have that

$$P/(I \cap J') \cong P/I \times_{P/I_{n-1}} P/J'$$

and using [RS] we get a deformation over $P/(I \cap J') = P/(I \cap J)$ lifting ρ_{n-1}, corresponding to the objects η on P/I, and the pullback of σ along the projection $P/J \to P/J'$, on P/J'. Thus $I \cap J$ is in A as well.

Now set $I = \bigcap_i I_i$, and $R = P/I$. Notice that R is still complete in the $\mathfrak{m}_R$-adic topology, and we have also $R \cong \varprojlim(P/I_i) \cong \varprojlim(R/(I_i/I))$. This is because, since $\mathfrak{m}_P^i \subseteq I_i$ for every i, we have exact sequences

$$0 \longrightarrow I_i/\mathfrak{m}_P^i \longrightarrow P/\mathfrak{m}_P^i \longrightarrow P/I_i \longrightarrow 0$$

for every i, which together give an exact sequence of projective systems. Moreover, since $P/\mathfrak{m}_P^i$ is artinian, the projective system $\{I_i/\mathfrak{m}_P^i\}_{i\in\mathbb{N}}$ with the natural maps satisfies the Mittag-Leffler condition, and so the induced map $P \cong \varprojlim(P/\mathfrak{m}_P^i) \to \varprojlim(P/I_i)$ is surjective, and it is clear that its kernel is precisely $I = \bigcap_i I_i$.

This shows that the filtration $\{I_n/I\}_{n\in\mathbb{N}}$ of R defines the same topology as its canonical one, and so (see Proposition 5.1.4) we can define a formal object ρ on R as $\{\rho_n, f_n\}_{n\in\mathbb{N}}$, where $\rho_i \in \mathcal{F}(R/(I_i/I))$ are the ones defined above, and $f_n : \rho_n \to \rho_{n+1}$ are the arrows defining ρ_{n+1} as a lifting of ρ_n.

Let us show that the formal object (R, ρ) satisfies the two properties above: clearly the Kodaira-Spencer map $\kappa_\rho : T_\Lambda R \cong E^\vee \to T_{\xi_0}\mathcal{F} = E^\vee$ is an isomorphism, since it is nothing else than $\kappa_{\overline{\rho}_1}$.

Now for the lifting property: suppose $A' \to A$ is a small extension, and that we have a diagram of formal objects as above. We want to show that $R \to A$ lifts to $R \to A'$. We can clearly assume that $A' \to A$ is a tiny extension, because if we prove it in this case, we can lift the homomorphism form R successively, using the fact that every small extension is a composite of tiny extensions.

Notice that the homomorphism $R \to A$ factors through some P/I_i, say $P/I_n \to A$. Let us consider the fibered product $R' = (P/I_i) \times_A A'$, and take a lifting of the homomorphism $P \to P/I_n \to A$ to $P \to A'$. These homomorphisms together induce $P \to R'$, such that the following diagram is commutative.

$$\begin{tikzcd} & R' \arrow[r] \arrow[d] & A' \arrow[d] \\ P \arrow[r] \arrow[ur, dashed] & P/I_n \arrow[r] & A \end{tikzcd}$$

Call $J = \ker(P \to R')$, and notice that $J \subseteq I_n$. If $J = I_n$ we are done, because the projection $R' \to P/I_n$ will have a section that we can use to define our lifting as the composite $R \to P/I_n \to R' \to A'$.

So suppose that J is properly contained in I_n. Identifying P/J with its image in R', we have that $I_n/J \subseteq \ker(R' \to P/I_n)$, which is isomorphic to $\ker(A' \to A) \cong k$, so that necessarily $I_n/J = \ker(R' \to P/I_n)$. Looking at the diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & I_n/J \arrow[r] \arrow[d, equal] & P/J \arrow[r] \arrow[d] & P/I_n \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \ker(R' \to P/I_n) \arrow[r] & R' \arrow[r] & P/I_n \arrow[r] & 0 \end{tikzcd}$$

we get that $R' \cong P/J$. It is also easy to check that $\mathfrak{m}_P I_n \subseteq J$ (and we had already that $J \subseteq I_n$), and by [RS] we have a lifting $\overline{\rho} \in \mathcal{F}(R') = \mathcal{F}(P/J)$ of ρ_n. By definition of I_{n+1} as the minimal ideal of P with these properties, we have that $I_{n+1} \subseteq J$, so that the homomorphism $P \to R'$ factors through P/I_{n+1}. Now it is clear that the composite $R \to P/I_{n+1} \to R' \to A'$ is a lifting of the given $R \to A$, so we are done. □

Now we turn to the proof of Schlessinger's Theorem 5.2.3. The key point is the following proposition.

Proposition 5.3.2. *Let $\mathcal{F} \to (\mathrm{Art}/\Lambda)^{op}$ be a deformation category, and (R, ρ) a miniversal formal object of $\mathcal{F}$. If $\mathrm{Inf}(\rho_0) = 0$ and $\mathcal{F}(k)$ is a trivial groupoid, then (R, ρ) is a universal formal object of $\mathcal{F}$.*

To prove it we need a lemma.

Let $\varphi : A' \to A$ be a small extension with kernel I, and $B \in (\mathrm{Art}/\Lambda)$. Suppose we have two homomorphisms $f, g : B \to A'$ such that the composites $h = \varphi \circ f = \varphi \circ g : B \to A$ coincide. Then the difference $f - g : B \to I$ is a Λ-derivation (see Proposition 2.2.3), so easy calculations, which use also the fact that $A' \to A$ is a small extension, show that $(f - g)(\mathfrak{m}_B^2) = 0$ and $(f - g)(\mathfrak{m}_\Lambda B) = 0$; we can consider then the induced k-linear function

$$\Delta(f, g) : \mathfrak{m}_B/(\mathfrak{m}_\Lambda B + \mathfrak{m}_B^2) \cong \mathfrak{m}_{\overline{B}}/\mathfrak{m}_{\overline{B}}^2 \longrightarrow I.$$

Notice that by Λ-linearity of f and g, and the fact that B is generated by $\mathfrak{m}_B$ and Λ as a ring, we have $f = g$ if and only if $\Delta(f,g) = 0$.

Take now an object $\xi \in \mathcal{F}(B)$, and consider the pullbacks $f_*(\xi), g_*(\xi) \in \mathcal{F}(A')$, which are liftings of $h_*(\xi)$. In particular by Theorem 2.3.1 we have an action of $I \otimes_k T_{\xi_0}\mathcal{F}$ on $\mathrm{Lif}(h_*(\xi), A')$, where ξ_0 is the pullback of ξ to k, and recall also that the formal deformation (B,ξ) has an associated Kodaira-Spencer class $k_\xi \in \mathfrak{m}_{\overline{B}}/\mathfrak{m}_{\overline{B}}^2 \otimes_k T_{\xi_0}\mathcal{F}$.

Lemma 5.3.3. *With the notation of Remark 2.3.3, we have*

$$g([f_*(\xi)],[g_*(\xi)]) = (\Delta(f,g)\otimes \mathrm{id})(k_\xi) \in I \otimes_k T_{\xi_0}\mathcal{F}.$$

Proof. Set $V = \mathfrak{m}_{\overline{B}}/\mathfrak{m}_{\overline{B}}^2$, and consider $B' = B \oplus V$ with the obvious Λ-algebra structure, and the trivial small extension

$$0 \longrightarrow V \longrightarrow B' \longrightarrow B \longrightarrow 0.$$

If $\pi : B \to k$ and $\pi' : B \to \overline{B} = B/\mathfrak{m}_\Lambda B$ are the quotient maps, there is a derivation $D : B \to V$ that sends $b \in B$ to the class of $\pi'(b) - \pi(b)$ in $\mathfrak{m}_{\overline{B}}/\mathfrak{m}_{\overline{B}}^2$.

We consider the two homomorphisms $i, u : B \to B'$, defined by $i(b) = (b,0)$ and $u(b) = (b, D(b))$, and the one $F : B' \to A'$ given by

$$F(b,x) = g(b) + \Delta(f,g)(x).$$

One can easily check that $F \circ i = g$ and $F \circ u = f$, and using Proposition 2.3.4 (with $\varphi = F : B' \to A'$) we get

$$g([f_*(\xi)],[g_*(\xi)]) = (\Delta(f,g)\otimes \mathrm{id})(g([u_*(\xi)],[i_*(\xi)]))$$

(since $F|_V = \Delta(f,g)$).

We now consider $\overline{B}_1 = \overline{B}/\mathfrak{m}_{\overline{B}}^2 \cong k \oplus V$, and the homomorphism $h : B' \to \overline{B}_1$ defined by $h(b,x) = \pi(b) + x$. If we call $\pi'' : B \to \overline{B}_1$ the quotient map, we have $h \circ u = \pi''$, and $(h \circ i)(a) = \pi(a) \in \overline{B}_1$.

From this we get that $h_*(u_*(\xi)) \cong \overline{\xi}_1$ and $h_*(i_*(\xi)) \cong \xi_0|_{\overline{B}_1}$; noticing that $h|_V$ is the identity, using Proposition 2.3.4 again we infer that

$$g([u_*(\xi)],[i_*(\xi)]) = g([\overline{\xi}_1],[\xi_0|_{\overline{B}_1}]).$$

But now $g([\overline{\xi}_1],[\xi_0|_{\overline{B}_1}]) = k_\xi$ by definition, and this concludes the proof. □

Now we can prove 5.3.2.

Proof of 5.3.2. By Proposition 5.2.11 we already know that for any formal object (S,ξ) of $\mathcal{F}$ there exists a morphism $(S,\xi) \to (R,\rho)$ (ξ_0 will be necessarily isomorphic to ρ_0, for $\mathcal{F}(k)$ is a trivial groupoid).

As for uniqueness, we have to show that any two morphisms of formal objects $f, g : (S, \xi) \to (R, \rho)$ are the same. Using Proposition 3.1.8 we see that, since $\mathrm{Inf}(\rho_0)$ is trivial, we only need to show that the two homomorphisms $\varphi, \psi : R \to S$ associated with f and g are equal.

Moreover it is sufficient to show that $\varphi_n, \psi_n : R_n \to S_n$ are equal for every n, and we do this inductively. Obviously $\varphi_0 = \psi_0$, so suppose $\varphi_{n-1} = \psi_{n-1}$. In this case $\varphi_n, \psi_n : R_n \to S_n$ are the same map when composed with $S_n \to S_{n-1}$, so we can consider

$$\Delta(\varphi_n, \psi_n) : \mathfrak{m}_{\overline{R}_n}/\mathfrak{m}_{\overline{R}_n}^2 \cong \mathfrak{m}_{\overline{R}}/\mathfrak{m}_{\overline{R}}^2 \longrightarrow \mathfrak{m}_S^n/\mathfrak{m}_S^{n+1}.$$

Since by assumption $(\varphi_n)_*(\rho_n)$ and $(\psi_n)_*(\rho_n)$ are isomorphic as liftings of $(\varphi_{n-1})_*(\rho_{n-1}) = (\psi_{n-1})_*(\rho_{n-1})$, by the preceding lemma we conclude that

$$(\Delta(\varphi_n, \psi_n) \otimes \mathrm{id})(k_\rho) = 0 \in \mathfrak{m}_S^n/\mathfrak{m}_S^{n+1} \otimes_k T_{\rho_0}\mathcal{F}$$

where $k_\rho \in \mathfrak{m}_{\overline{R}}/\mathfrak{m}_{\overline{R}}^2 \otimes_k T_{\rho_0}\mathcal{F}$ is the Kodaira-Spencer class of ρ.

This means that if we compose the adjoint map

$$\Delta(\varphi_n, \psi_n)^\vee : (\mathfrak{m}_S^n/\mathfrak{m}_S^{n+1})^\vee \to (\mathfrak{m}_{\overline{R}}/\mathfrak{m}_{\overline{R}}^2)^\vee$$

with the Kodaira-Spencer map $\kappa_\rho : T_\Lambda R \to T_{\rho_0}\mathcal{F}$ of ρ we get the zero map. But now κ_ρ is an isomorphism, so we conclude that $\Delta(\varphi_n, \psi_n) = 0$, from which follows that $\varphi_n = \psi_n$, as we wanted to show. □

Schlessinger's Theorem is now an easy corollary of 5.3.1 and 5.3.2.

Proof of 5.2.3. We already remarked that if a deformation category is prorepresentable, than it has the properties of the statement. So suppose conversely that we have a deformation category $\mathcal{F}$ satisfying the hypotheses.

Pick $\rho_0 \in \mathcal{F}(k)$. Since $T_{\rho_0}\mathcal{F}$ is finite-dimensional, there exists a miniversal object (R, ρ) of $\mathcal{F}$ restricting to ρ_0 over k. Moreover since $\mathrm{Inf}(\rho_0) = 0$ and $\mathcal{F}(k)$ is a trivial groupoid, by Proposition 5.3.2 we conclude that (R, ρ) is a universal formal object for $\mathcal{F}$, which is then prorepresentable. □

The following proposition gives a useful criterion that will be used later to show that some formal deformations are miniversal.

Proposition 5.3.4. *Let $\mathcal{F} \to (\mathrm{Art}/\Lambda)^{op}$ be a deformation category, and suppose that (R, ρ) is a formal object of $\mathcal{F}$ such that:*

- *R is a power series ring over Λ.*
- *The Kodaira-Spencer map $\kappa_\rho : T_\Lambda R \to T_{\rho_0}\mathcal{F}$ is an isomorphism.*

Then (R, ρ) is a miniversal formal object, and in particular ρ_0 is unobstructed (see 5.2.12).

Proof. Since κ_ρ is an isomorphism, we have that $T_{\rho_0}\mathcal{F}$ is a finite-dimensional k-vector space; by Theorem 5.3.1 we can then find a miniversal object (S, ξ) restricting to ρ_0 over k. By Proposition 5.2.11 and versality of (S, ξ) we have a morphism of formal objects $(R, \rho) \to (S, \xi)$, and since both of the Kodaira-Spencer maps κ_ρ and κ_ξ are isomorphisms, the k-linear map $T^\vee_\Lambda S \to T^\vee_\Lambda R$ induced on the cotangent spaces is an isomorphism too (by Proposition 5.1.17.

Since R is a power series ring over Λ, this implies that the homomorphism $S \to R$ is an isomorphism (see Corollary B.7), and then the morphism $(R, \rho) \to (S, \xi)$ is an isomorphism too, so (R, ρ) is miniversal. □

5.3.1 Applications to obstruction theories

Now that we have proved the existence of miniversal deformations, we can give a proof of the Ran-Kawamata Theorem (Theorem 4.1.9) and the anticipated formula for the dimension of the minimal obstruction space associated with an obstruction theory.

Proof of 4.1.9. Let (R, ρ) be a miniversal deformation of ξ_0 coming from Theorem 5.3.1. In particular R is a quotient P/I, where $P = k[[x_1, \dots, x_n]]$ and $n = \dim_k(T_{\xi_0}\mathcal{F})$, and $I \subseteq \mathfrak{m}_P^2$. We want to show that $I = (0)$, so that R is a power series ring, and by Proposition 5.2.12 ξ_0 will be unobstructed.

The first step is to prove that the module of continuous differentials $\Omega = \widehat{\Omega}_R$ (see appendix B) is a free R-module. Since R is local we can equivalently show that Ω is a projective R-module, and to do this it suffices to check that for every surjection $M' \to M$ of R-modules of finite length the induced homomorphism $\mathrm{Hom}_R(\Omega, M') \to \mathrm{Hom}_R(\Omega, M)$ is surjective.

In fact, since R is noetherian and Ω is finitely generated, to show that Ω is projective it suffices to show that $\mathrm{Ext}^i_R(\Omega, N) = 0$ for all i and finitely generated R-modules N; for then, if N is not finitely generated, we can write $N \cong \varinjlim N_\alpha$ where the N_α's are finitely generated, and

$$\mathrm{Ext}^i_R(\Omega, N) \cong \varinjlim \mathrm{Ext}^i_R(\Omega, N_\alpha) = 0.$$

Now fix a finitely generated R-module N; in particular the quotient modules $N/\mathfrak{m}_R^n N$ have finite lenght and N is separated in the $\mathfrak{m}_R$-adic topology, that is, $\varprojlim N/\mathfrak{m}_R^n N \cong N$. Taking a projective resolution $P_\bullet$ of Ω whose terms are finitely generated R-modules, we have

$$\mathrm{Ext}^i_R(\Omega, N) \cong H^i(\mathrm{Hom}_R(P_\bullet, N)) \cong H^i\left(\varprojlim \mathrm{Hom}_R(P_\bullet, N/\mathfrak{m}_R^n N)\right)$$

where $\mathrm{Hom}_R(P_\bullet, N)$ denotes the complex obtained by applying the functor $\mathrm{Hom}_R(-, N)$ to the complex $P_\bullet$. Finally one can use a "Mittag-Leffler"-like argument to show that the last module is isomorphic to

$$\varprojlim H^i(\mathrm{Hom}_R(P_\bullet, N/\mathfrak{m}_R^n N)) = \varprojlim \mathrm{Ext}^i_R(\Omega, N/\mathfrak{m}_R^n N).$$

But now the condition that for every surjection $M' \to M$ of R-modules of finite lenght the induced homomorphism $\mathrm{Hom}_R(\Omega, M') \to \mathrm{Hom}_R(\Omega, M)$ is surjective implies that $\mathrm{Ext}^i_R(\Omega, Q) = 0$ for every R-module of finite length Q, and from this (and the above isomorphisms) we get that $\mathrm{Ext}^i_R(\Omega, N) = 0$ for every finitely generated R-module N. In conclusion, this condition about surjections implies that Ω is projective.

Let us take then a surjection $M' \to M$ of R-modules of finite length, and n large enough for M' and M to be $R/\mathfrak{m}_R^{n+1}$-modules. Set $A = R_n$, and consider a homomorphism $\varphi \in \mathrm{Hom}_R(\Omega, M)$. This will correspond to a k-derivation $R \to M$, which in turn is the same as a homomorphism of R-modules $R \to A[M]$ (this is a standard fact) that is moreover compatible with the two quotient maps to A. In other words the diagram

$$\begin{tikzcd} R \arrow[r] \arrow[dr] & A[M] \arrow[d] \\ & A \end{tikzcd}$$

is commutative.

Take then the pullbacks $\xi \in \mathcal{F}_{\xi_0}(A)$ and $\xi' \in \mathcal{F}_{\xi_0}(A[M])$ of the miniversal deformation ρ along the two homomorphisms above. The class of ξ' is an element $[\xi'] \in F_\xi(M)$, and so by right-exactness of F_ξ we can find a $[\xi''] \in F_\xi(M')$ that maps to $[\xi']$ via the canonical function $F_\xi(M') \to F_\xi(M)$. In other words $\xi'' \in \mathcal{F}_{\xi_0}(A[M'])$ is an object whose pullback to $A[M]$ is isomorphic to ξ'.

By versality of (R, ρ) the homomorphism $R \to A[M]$ can then be lifted to $R \to A[M']$

$$\begin{tikzcd} & (A[M'], \xi'') \arrow[dl, dashed] \\ (R, \rho) & (A[M], \xi') \arrow[l] \arrow[u] \end{tikzcd}$$

and this lifting corresponds to a k-derivation $R \to M'$, which in turn is the same as a homomorphism $\psi \in \mathrm{Hom}_R(\Omega, M')$. This homomorphism ψ will then be in the preimage of the chosen $\varphi \in \mathrm{Hom}_R(\Omega, M)$, and this proves that $\mathrm{Hom}_R(\Omega, M') \to \mathrm{Hom}_R(\Omega, M)$ is surjective. In conclusion $\Omega = \widehat{\Omega}_R$ is a free R-module.

Now we deduce that $I = (0)$, and this will conclude the proof, as we already noticed. We consider the conormal sequence

$$I/I^2 \xrightarrow{d} \widehat{\Omega}_P \otimes_P R \longrightarrow \widehat{\Omega}_R \longrightarrow 0 \tag{5.2}$$

(see Proposition B.16) and notice that, since P is a power series ring on n indeterminates, the R-module $\widehat{\Omega}_P \otimes_P R$ is free of rank n. Moreover if m is the rank of $\widehat{\Omega}_R$, tensoring 5.2 with k we obtain an isomorphism

$$(\widehat{\Omega}_P \otimes_P R) \otimes_R k \cong \widehat{\Omega}_R \otimes_R k$$

(for the homomorphism d becomes the zero map), and this tells us that $m = n$.

Therefore the surjective homomorphism $\widehat{\Omega}_P \otimes_P R \to \widehat{\Omega}_R$ of 5.2 has to be an isomorphism, and so $d : I/I^2 \to \widehat{\Omega}_P \otimes_P R$ is the zero map. This means that the image of I along the universal derivation $d : P \to \widehat{\Omega}_P$ is contained in the ideal $I\widehat{\Omega}_P$, and this implies that for any $f \in I$ and $i = 1, \dots, n$, the partial derivative $\partial f/\partial x_i$ is an element of I.

Since $\mathrm{char}(k) = 0$, it is easy to see that this implies $I = (0)$ (for example considering an element of I of minimal degree and recalling that $I \subseteq \mathfrak{m}_P^2$), and so we are done. □

Now consider a deformation category $\mathcal{F} \to (\mathrm{Art}/\Lambda)^{op}$, an object $\xi_0 \in \mathcal{F}(k)$ such that $T_{\xi_0}\mathcal{F}$ is finite-dimensional, and an obstruction theory (V_ω, ω) for ξ_0. By Theorem 5.3.1 ξ_0 has a miniversal deformation (R, ρ) where R is a quotient P/I, with $P = \Lambda[[x_1, \dots, x_n]]$, $n = \dim_k(T_{\xi_0}\mathcal{F})$ and $I \subseteq \mathfrak{m}_\Lambda P + \mathfrak{m}_P^2$.

We denote by $\mu(I)$ the minimal number of generators of the ideal $I \subseteq P$, which by Nakayama's Lemma is the same as $\dim_k(I/\mathfrak{m}_P I)$. Finally let Ω_ω denote the minimal obstruction space associated with (V_ω, ω), as in Section 4.1.1.

Proposition 5.3.5. *The dimension of Ω_ω as a k-vector space coincides with $\mu(I) = \dim_k(I/\mathfrak{m}_P I)$.*

Proof. We will show that there is an isomorphism of k-vector spaces $\Omega_\omega \cong (I/\mathfrak{m}_p I)^\vee$, and this will imply the result. Put $\widetilde{R} = P/\mathfrak{m}_P I$ (which is an object of (Comp/Λ) as well), so that we have an exact sequence of P-modules

$$0 \longrightarrow I/\mathfrak{m}_P I \longrightarrow \widetilde{R} \longrightarrow R \longrightarrow 0.$$

Tensoring this with $P_n = P/\mathfrak{m}_P^{n+1}$, we obtain

$$I/\mathfrak{m}_P I \xrightarrow{\alpha_n} \widetilde{R}_n \longrightarrow R_n \longrightarrow 0$$

and by the Artin-Rees Lemma we see that $\ker(\alpha_n) = (I/\mathfrak{m}_P I) \cap \mathfrak{m}_{\widetilde{R}}^{n+1} = (0)$ for n large enough.

For every such n then the sequence

$$0 \longrightarrow I/\mathfrak{m}_P I \xrightarrow{\alpha_n} \widetilde{R}_n \longrightarrow R_n \longrightarrow 0$$

is a small extension, and we have an object $\rho_n \in \mathcal{F}(R_n)$, coming from the versal deformation (R, ρ). We can consider then the obstruction

$$\omega_n = \omega(\rho_n, \widetilde{R}_n) \in I/\mathfrak{m}_P I \otimes_k \Omega_\omega \cong \mathrm{Hom}_k((I/\mathfrak{m}_P I)^\vee, \Omega_\omega).$$

Notice that this element does not depend on n (large enough): this follows immediately from functoriality of the obstruction, and the fact that for every n large enough we have a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & I/\mathfrak{m}_P I \arrow[r] \arrow[d, equal] & \widetilde{R}_{n+1} \arrow[r] \arrow[d] & R_{n+1} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & I/\mathfrak{m}_P I \arrow[r] & \widetilde{R}_n \arrow[r] & R_n \arrow[r] & 0. \end{tikzcd}$$

From this we get a well-defined element $f \in \mathrm{Hom}_k((I/\mathfrak{m}_P I)^\vee, \Omega_\omega)$, that is, a k-linear map $f : (I/\mathfrak{m}_P I)^\vee \to \Omega_\omega$. We show now that f is bijective.

First we show that it is injective. Take a nonzero $u \in (I/\mathfrak{m}_P I)^\vee$, which is a surjective k-linear function $u : I/\mathfrak{m}_P I \to k$, and put $K = \ker(u)$, which is an ideal of $\widetilde{R}_n$ (for n large enough). We consider then $(I/\mathfrak{m}_P I)/K \cong k$ and $S_n = \widetilde{R}_n/K$, and the following commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & I/\mathfrak{m}_P I \arrow[r] \arrow[d, "u"] & \widetilde{R}_n \arrow[r] \arrow[d] & R_n \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & k \arrow[r] & S_n \arrow[r] & R_n \arrow[r] & 0. \end{tikzcd}$$

(where the vertical arrows are the quotient maps) that gives a morphism between the two small extensions.

By definition of the isomorphism $I/\mathfrak{m}_P I \otimes_k \Omega_\omega \cong \mathrm{Hom}_k((I/\mathfrak{m}_P I)^\vee, \Omega_\omega)$ and by functoriality of the obstruction ω, we have that

$$f(u) = \omega_n(u) = (u \otimes \mathrm{id})(\omega_n) = \omega(\rho_n, S_n) \in k \otimes_k \Omega_\omega \cong \Omega_\omega.$$

Suppose that $f(u) = 0$. Then there is a lifting $\eta_n \in \mathcal{F}(S_n)$ of $\rho_n \in \mathcal{F}(R_n)$, and by versality of (R, ρ) the homomorphism $R \to R_n$ will lift to $R \to S_n$. On the other hand since $\mathfrak{m}_{S_n}^{n+1} = (0)$ this last map will factor through R_n, and give then a splitting $R_n \to S_n$ of the small extension above.

Finally notice that this splitting (as well as $S_n \to R_n$) will induce an isomorphism between cotangent spaces of R_n and S_n, and then (by part **(ii)** of Corollary B.4) the map $S_n \to R_n$ is an isomorphism. But this is a contradiction, because the kernel of this map is isomorphic to k. In conclusion this shows that $f(u) \neq 0$, and so f is injective.

We show that it is surjective. Take a vector $v \in \Omega_\omega$, and suppose it corresponds to the obstruction $\omega(\xi, A')$ associated with a small extension $A' \to A$ with kernel J and an isomorphism $g : J \cong k$, and an object $\xi \in \mathcal{F}_{\xi_0}(A)$.

By versality of (R, ρ) and Proposition 5.2.11 we have an arrow of formal objects $(A, \xi) \to (R, \rho)$, and since A is artinian the homomorphism $R \to A$ will factor through R_n for n large enough (and the pullback of ρ_n to A is isomorphic to ξ). Moreover if we lift the homomorphism $P \to R \to A$ to $\varphi : P \to A'$ using the fact that P is a power series ring over Λ, then $\varphi(I)$ will be contained in J, and consequently $\varphi(\mathfrak{m}_P I) = (0)$, so φ will factor through $\widetilde{R}$.

Taking n large enough we get then a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & I/\mathfrak{m}_P I \arrow[r] \arrow[d, "u"] & \widetilde{R}_n \arrow[r] \arrow[d] & R_n \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & J \arrow[r] & A' \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

where $u : I/\mathfrak{m}_P I \to J \cong k$ can be seen as an element of $(I/\mathfrak{m}_P I)^\vee$. By functoriality of the obstruction (and the other arguments used above) we get

$$f(u) = \omega_n(u) = (u \otimes \mathrm{id})(\omega_n) = \omega(\xi, A') \in J \otimes_k \Omega_\omega \cong \Omega_\omega$$

which corresponds to v. This shows that f is surjective, and concludes the proof. $\square$

Remark 5.3.6. Using this we get immediately another proof of Proposition 5.2.12: R is a power series ring if and only if $I = (0)$, and this happens exactly when $\dim_k(\Omega_\omega) = 0$, and ξ_0 is unobstructed.

The last proposition has the following corollaries.

Corollary 5.3.7. *Let $\mathcal{F} \to (\mathrm{Art}/\Lambda)^{op}$ be a deformation category, $\xi_0 \in \mathcal{F}(k)$, and (V_ω, ω) be an obstruction theory for ξ_0. If $T_{\xi_0}\mathcal{F}$ is finite-dimensional, then Ω_ω is as well.*

Corollary 5.3.8. *Let $\mathcal{F} \to (\mathrm{Art}/\Lambda)^{op}$ be a deformation category, $\xi_0 \in \mathcal{F}(k)$ such that $T_{\xi_0}\mathcal{F}$ is finite-dimensional, and (V_ω, ω) be an obstruction theory for ξ_0. Moreover let (R, ρ) be a miniversal deformation of ξ_0. Then*

$$\dim(R) \geq \dim_k(T_{\xi_0}\mathcal{F}) - \dim_k(\Omega_\omega) \geq \dim_k(T_{\xi_0}\mathcal{F}) - \dim_k(V_\omega).$$

Proof. The right inequality is clear, so we prove only the left one, by showing that $\dim(R) \geq \dim_k(T_{\xi_0}\mathcal{F}) - \mu(I)$ for the miniversal deformation (R, ρ) considered above, and using the preceding proposition. Set $n = \dim_k(T_{\xi_0}\mathcal{F})$.

Notice that we can reduce to the case $\Lambda = k$ by considering the canonical homomorphism $P = \Lambda[[x_1, \dots, x_n]] \to k[[x_1, \dots, x_n]]$, and the induced surjection $R = P/I \to k[[x_1, \dots, x_n]]/J$ where J is the extension of I. Indeed if we know that $\dim(k[[x_1, \dots, x_n]]/J) \geq n - \mu(J)$, then we have

$$\dim(R) \geq \dim(k[[x_1, \dots, x_n]]/J) \geq n - \mu(J) \geq n - \mu(I).$$

So we can assume that $\Lambda = k$. By Krull's Hauptidealsatz we have that $\dim_k(I/\mathfrak{m}_P I) = \mu(I) \geq \mathrm{ht}(I)$; because of this inequality and the fact that $P = k[[x_1, \dots, x_n]]$ is catenary, we get

$$\dim(R) = \dim(P/I) = \dim(P) - \mathrm{ht}(I) \geq \dim(P) - \mu(I) = n - \mu(I). \quad \square$$

This result can be applied to find a lower bound on the dimension of the base ring R of a miniversal deformation.

Example 5.3.9. Let $Z_0 \subseteq \mathbb{P}^3_k$ be a smooth curve of genus g and degree d, and (R, ρ) a universal deformation of Z_0 in $\mathcal{H}ilb^{\mathbb{P}^3_k}$ (we have a miniversal one since $\dim_k T_{Z_0}\mathcal{H}ilb^{\mathbb{P}^3_k} = \dim_k H^0(Z_0, \mathcal{N}_0)$ is finite, and it is universal because $\mathcal{H}ilb^{\mathbb{P}^3_k}$ is fibered in sets). Recall that $\mathcal{H}ilb^{\mathbb{P}^3_k}$ comes from a representable functor, so if we denote by $Hilb^{\mathbb{P}^3_k}$ the Hilbert scheme of $\mathbb{P}^3_k$, the dimension of R in this case is the same as $\dim_{Z_0} Hilb^{\mathbb{P}^3_k}$.

By the preceding corollary we get then

$$\begin{aligned} \dim_{Z_0} Hilb^{\mathbb{P}^3_k} &\geq \dim_k(T_{Z_0}\mathcal{H}ilb^{\mathbb{P}^3_k}) - \dim_k(V_\omega) \\ &= \dim_k(H^0(Z_0, \mathcal{N}_0)) - \dim_k(H^1(Z_0, \mathcal{N}_0)) \\ &= \chi(\mathcal{N}_0) \end{aligned}$$

(here we are considering the obstruction theory described in Section 4.2.4) where χ is the Euler characteristic and $\mathcal{N}_0$ is the normal sheaf of Z_0 in $\mathbb{P}^3_k$.

Now from the dual of the conormal sequence of $Z_0 \subseteq \mathbb{P}^3_k$

$$0 \longrightarrow T_{Z_0} \longrightarrow T_{\mathbb{P}^3_k}|_{Z_0} \longrightarrow \mathcal{N}_0 \longrightarrow 0$$

we get $\chi(\mathcal{N}_0) = \chi(T_{\mathbb{P}^3_k}|_{Z_0}) - \chi(T_{Z_0})$, and from the restriction of the dual of the Euler sequence

$$0 \longrightarrow \mathcal{O}_{Z_0} \longrightarrow \mathcal{O}_{Z_0}(1)^{\oplus 4} \longrightarrow T_{\mathbb{P}^3_k}|_{Z_0} \longrightarrow 0$$

we obtain further that $\chi(\mathcal{N}_0) = 4\chi(\mathcal{O}_{Z_0}(1)) - \chi(\mathcal{O}_{Z_0}) - \chi(T_{Z_0})$. Using the Riemann-Roch Theorem to calculate explicitly the three terms in the last expression, we get

$$\dim_{Z_0} Hilb^{\mathbb{P}^3_k} \geq \chi(\mathcal{N}_0) = (4d + 4 - 4g) - (1 - g) - (2 - 2g + 1 - g) = 4d$$

which gives a lower bound on $\dim_{Z_0} Hilb^{\mathbb{P}^3_k}$ independent of the genus g.

Example 5.3.10. Consider now a smooth projective curve X_0 over k. Since $T_{X_0}\mathcal{D}ef \cong H^1(X_0, T_{X_0})$ is finite-dimensional, X_0 has a miniversal deformation (R, ρ), and since $H^2(X_0, T_{X_0}) = 0$ we see that X_0 is unobstructed (by Theorem 4.2.4), and so R is a power series ring, and $\dim(R) = \dim_k(T_{X_0}\mathcal{D}ef)$.

We can calculate this dimension explitictly: if g is the genus of X_0, then T_{X_0} has degree $2 - 2g$, and by the Riemann-Roch Theorem we get

$$\chi(T_{X_0}) = \dim_k(H^0(X_0, T_{X_0})) - \dim_k(H^1(X_0, T_{X_0})) = 2 - 2g + 1 - g = 3 - 3g.$$

Now if $g \geq 2$, then T_{X_0} has negative degree, so $\dim_k(H^0(X_0, T_{X_0})) = 0$ and

$$\dim(R) = \dim_k(H^1(X_0, T_{X_0})) = 3g - 3.$$

On the other hand if $g = 1$ we find $\dim_k(H^1(X_0, T_{X_0})) = 1$, and in the case $g = 0$ we have $\dim_k(H^1(X_0, T_{X_0})) = 0$.

These values give the minimum number of parameters that are necessary to describe a versal deformation of X_0 for a given genus.

5.3.2 Hypersurfaces in $\mathbb{A}^n_k$

As an example (which will be useful in the next chapter), we calculate a miniversal deformation of a reduced and generically smooth hypersurface $X_0 \subseteq \mathbb{A}^n_k$, using the facts already proved in Section 2.4.4. In particular, since we showed that $T_{X_0}\mathcal{D}ef$ is finite-dimensional if and only if X_0 has isolated singularities, we have to restrict to this case.

Suppose then that $X_0 \subseteq \mathbb{A}^n_k$ is a hypersurface as above, with equation $f \in k[x_1, \dots, x_n]$, and so defined by the ideal $I = (f)$ and with coordinate ring $A = k[x_1, \dots, x_n]/I$. Recall from Section 2.4.4 that

$$T_{X_0}\mathcal{D}ef \cong k[x_1, \dots, x_n]/(f, \partial f/\partial x_1, \dots, \partial f/\partial x_n).$$

Let $m = \dim_k(T_{X_0}\mathcal{D}ef)$ (which is finite because X_0 has isolated singularities), and choose elements $g_1, \dots, g_m \in \Lambda[x_1, \dots, x_n]$ such that their images in $k[x_1, \dots, x_n]/(f, \partial f/\partial x_1, \dots, \partial f/\partial x_n)$ form a basis.

We consider then the power series ring $R = \Lambda[[t_1, \dots, t_m]]$, and the closed subscheme

$$X = V(f' + t_1 g_1 + \cdots + t_m g_m) \subseteq \mathbb{A}^n_R$$

where $f' \in \Lambda[x_1, \dots, x_n]$ is a lifting of f. X induces a formal deformation $\widehat{X} = \{X_i, f_i\}_{i \in \mathbb{N}}$ of X_0 over R, by taking X_i to be the pullback of X to $R_i = R/\mathfrak{m}_R^{i+1}$ along the quotient map $R \to R_i$, and as arrows $f_i : X_i \to X_{i+1}$ the natural closed immersions.

Proposition 5.3.11. *The formal object $(R, \widehat{X})$ of the deformation category $\mathcal{D}ef \to (\mathrm{Art}/\Lambda)^{op}$ is miniversal.*

Proof. We use the criterion given by Proposition 5.3.4: R is a power series ring, so we only have to check that the Kodaira-Spencer map

$$\kappa_{\widehat{X}} : T_\Lambda R \to T_{X_0}\mathcal{D}ef$$

is an isomorphism. Recall that this map is the same as $\kappa_{\overline{X}_1}$, where $\overline{X}_1$ is the pullback of X to $\overline{R}_1 \cong k \oplus \mathfrak{m}_{\overline{R}}/\mathfrak{m}_{\overline{R}}^2$ along the projection $R \to \overline{R}_1$.

In this particular case we have

$$\overline{X}_1 = \mathrm{Spec}(\overline{R}_1[x_1, \dots, x_n]/(f + t_1\overline{g}_1 + \cdots + t_m\overline{g}_m))$$

where $\overline{g}_i$ is the image of g_i in $k[x_1, \dots, x_n]$ (and we still write t_i for the class of t_i in $\mathfrak{m}_{\overline{R}}/\mathfrak{m}_{\overline{R}}^2$). Since the images of $t_1, \dots, t_m$ in $T^\vee_\Lambda R = \mathfrak{m}_{\overline{R}}/\mathfrak{m}_{\overline{R}}^2$ form a basis of the cotangent space, we can consider the dual basis $s_1, \dots, s_m \in T_\Lambda R$. The Kodaira-Spencer map

$$\kappa_{\overline{X}_1} : T_\Lambda R \to T_{X_0}\mathcal{D}ef \cong k[x_1, \dots, x_n]/(f, \partial f/\partial x_1, \dots, \partial f/\partial x_n)$$

sends then s_i to the class of g_i.

This is because the k-linear function $s_i : T_\Lambda^\vee R \to k$ corresponds to a homomorphism $\varphi_i : \overline{R}_1 \to k[\varepsilon]$, and by definition of the Kodaira-Spencer map the element $\kappa_{\overline{X}_1}(s_i)$ will be the isomorphism class of the pullback of $\overline{X}_1$ to $k[\varepsilon]$ along φ_i. This pullback is seen to be given by the closed subscheme $V(f + \varepsilon \overline{g}_i)$ in $\mathbb{A}^n_{k[\varepsilon]}$, and by Proposition 2.4.9, the corresponding element in $k[x_1, \dots, x_n]/(f, \partial f/\partial x_1, \dots, \partial f/\partial x_n)$ is exactly the class $[\overline{g}_i]$.

By the choice of the g_i's the map $\kappa_{\overline{X}_1}$ is then an isomorphism (since the two spaces have the same dimension, and a basis goes to a basis), and this concludes the proof. □

Example 5.3.12. Consider the union of the two axes

$$X_0 = V(xy) \subseteq \mathbb{A}^2_k = \operatorname{Spec}(k[x, y]).$$

In this case the Jacobian ideal is $J = (x, y) \subseteq k[x, y]/(xy)$, and a basis of $T_{X_0}\mathcal{D}ef \cong k[x, y]/(x, y)$ is given by the class of -1. A miniversal deformation of X_0 is then for example the one induced by $X = V(xy - t) \subseteq \mathbb{A}^2_{\Lambda[[t]]}$.

Example 5.3.13. Assume $\operatorname{char}(k) \neq 2, 3$, and consider the cuspidal curve

$$X_0 = V(y^2 - x^3) \subseteq \mathbb{A}^2_k = \operatorname{Spec}(k[x, y]).$$

In this case we have $J = (2y, 3x^2) \subseteq k[x, y]/(y^2 - x^3)$, and a basis of $T_{X_0}\mathcal{D}ef \cong k[x, y]/(y, x^2)$ is given by the classes of 1 and x. The formal object induced by the closed subscheme $X = V(y^2 - x^3 + t_1 + t_2 x) \subseteq \mathbb{A}^2_{\Lambda[[t_1, t_2]]}$ is then a miniversal deformation.

5.4 Algebraization

The next step in constructing (or studying) deformations, is to pass from formal ones to "actual" ones (over notherian complete local rings). In other words given a formal deformation, which is a sequence of compatible deformations over the artinian quotients of the base ring, we ask if there is a "true" deformation over the base ring that restricts to the given ones over these quotients.

Formally, suppose $\mathcal{F} \to (\mathrm{Art}/\Lambda)^{op}$ is a deformation category coming (by restriction) from a category fibered in groupoids $\overline{\mathcal{F}} \to (\mathrm{Sch}/\Lambda)$, which is associated with some deformation problem we are trying to study. This is the case for the three deformation categories $\mathcal{D}ef, \mathcal{H}ilb^X, \mathcal{QC}oh^X$ we have studied up to this point.

Definition 5.4.1. *A formal object (R, ξ) of $\mathcal{F}$, where $\xi = \{\xi_n, f_n\}_{n \in \mathbb{N}}$, is called* ***algebraizable*** *if there exists an object $\widetilde{\xi} \in \overline{\mathcal{F}}(\operatorname{Spec}(R))$ with a collection $\{g_n\}_{n \in \mathbb{N}}$*

of arrows $g_n : \xi_n \to \widetilde{\xi}$ of $\overline{\mathcal{F}}$ over the immersion $\mathrm{Spec}(R_n) \to \mathrm{Spec}(R)$, such that for every n the following diagram (in $\overline{\mathcal{F}}$) is commutative.

$$\begin{tikzcd}
\xi_n \arrow[d, "f_n"'] \arrow[dr, bend left, "g_n"] & \\
\xi_{n+1} \arrow[r, "g_{n+1}"] & \widetilde{\xi}
\end{tikzcd}$$

*We call $\widetilde{\xi}$ an **algebraization** of ξ.*

The idea is that $\widetilde{\xi}$ is an actual deformation of ξ_0 over R, whose approximations to the various orders correspond to the terms of the formal object ξ.

Example 5.4.2. The miniversal deformation we constructed in the previous section for a hypersurface of $\mathbb{A}^n_k$ with isolated singularities is algebraizable (if we take $\overline{\mathcal{F}}$ to be the category of flat morphisms of schemes), since we constructed it by taking pullbacks from an actual deformation over an object $R \in (\mathrm{Comp}/\Lambda)$.

Remark 5.4.3. Actually (as we already remarked) when dealing with global deformations of schemes one assumes other additional hypotheses, a typical example being properness of the morphism defining the deformation. So the last example is formally correct, but not so meaningful.

From now on when we say that a formal deformation $(R, \widehat{X})$ of a scheme $X_0 \in \mathcal{D}ef(k)$ is **algebraizable** we will usually mean that there exists a scheme X that is flat and proper over R, and that induces the formal deformation $\widehat{X}$ by pullback.

The problem of algebraization is not solvable in general. The main result when dealing with it in concrete cases is the following theorem, due to Grothendieck.

Let Λ be as usual, and X a scheme over Λ; set $X_n = X|_{\mathrm{Spec}(\Lambda_n)}$. Together with the obvious morphisms, the sequence $\{X_n, f_n\}_{n\in\mathbb{N}}$ gives a formal deformation $\widehat{X}$ of X_0 over Λ.

We denote by $\mathcal{C}oh(X)$ the category of coherent sheaves on X, and by $\mathcal{C}oh(\widehat{X})$ the category of formal coherent sheaves on $\widehat{X}$: its objects are collections $\{\mathcal{E}_n, g_n\}_{n\in\mathbb{N}}$ of coherent sheaves $\mathcal{E}_n$ on X_n, with isomorphisms $g_n : \mathcal{E}_n \cong \mathcal{E}_{n+1}|_{X_n}$ (where this pullback is along the immersion $f_n : X_n \to X_{n+1}$), and an arrow $\{\mathcal{E}_n, g_n\}_{n\in\mathbb{N}} \to \{\mathcal{G}_n, h_n\}_{n\in\mathbb{N}}$ is a sequence $\{F_n\}_{n\in\mathbb{N}}$ of homomorphisms $F_n : \mathcal{E}_n \to \mathcal{G}_n$ of coherent sheaves on X_n, compatible with the isomorphisms g_n, h_n. This is an abelian category, even though in a not completely trivial way.

There is a natural functor $\Phi : \mathcal{C}oh(X) \to \mathcal{C}oh(\widehat{X})$, sending a coherent sheaf $\mathcal{E}$ on X to the formal coherent sheaf $\{\mathcal{E}|_{X_n}, f_n\}_{n\in\mathbb{N}}$, where f_n are the obvious isomorphisms identifying the pullback of $\mathcal{E}|_{X_{n+1}}$ to X_n with the one of $\mathcal{E}$, and a homomorphism $F : \mathcal{E} \to \mathcal{G}$ goes to the sequence $\{F_n\}_{n\in\mathbb{N}}$ of homomorphisms induced on the pullbacks.

Theorem 5.4.4 (Grothendieck's existence Theorem)**.** *If X is proper over Λ, the functor Φ is an equivalence of abelian categories.*

For a discussion about this theorem, see for example Chapter 8 of [FGA].

From this theorem we get an algebraization result that will be used in the next chapter. First of all, we have the following corollary about embedded formal deformations.

Corollary 5.4.5. *Let X be a proper scheme over Λ, and consider the formal deformation $\widehat{X} = \{X_n, f_n\}_{n\in\mathbb{N}}$ of X_0 as above. Consider a sequence $\{Y_n\}_{n\in\mathbb{N}}$ of closed subschemes $Y_n \subseteq X_n$, such that for every n we have $Y_{n+1} \cap X_n = Y_n$ (where we see $X_n \subseteq X_{n+1}$ by means of the closed immersion f_n). Then there exists a closed subscheme $Y \subseteq X$ such that $Y_n = Y \cap X_n$ for any n.*

Proof. We use Grothendieck's Theorem: consider the formal coherent sheaf $\{\mathcal{O}_{Y_n}, f_n\}_{n\in\mathbb{N}}$, where f_n are the obvious isomorphisms. By the theorem we have a coherent sheaf $\mathcal{E}$ on X, and a sequence of isomorphisms $\varphi_n : \mathcal{E}|_{X_n} = \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_n} \cong \mathcal{O}_{Y_n}$ compatible with the projections $\mathcal{O}_{X_{n+1}} \to \mathcal{O}_{X_n}$ and $\mathcal{O}_{Y_{n+1}} \to \mathcal{O}_{Y_n}$.

Moreover we have an arrow $\{\mathcal{O}_{X_n}, g_n\}_{n\in\mathbb{N}} \to \{\mathcal{O}_{Y_n}, f_n\}_{n\in\mathbb{N}}$ of formal sheaves on $\widehat{X}$, given by the surjections $\mathcal{O}_{X_n} \to \mathcal{O}_{Y_n}$ defining the closed subschemes Y_n. This arrow corresponds (by the theorem again) to a homomorphism $\psi : \mathcal{O}_X \to \mathcal{E}$ of coherent sheaves on X, such that for every n the diagram

$$\begin{tikzcd}
\mathcal{O}_X|_{X_n} \arrow[r, "\psi|_{X_n}"] \arrow[d, equal] & \mathcal{E}|_{X_n} \arrow[d, "\varphi_n"] \\
\mathcal{O}_{X_n} \arrow[r] & \mathcal{O}_{Y_n}
\end{tikzcd}$$

is commutative.

Notice finally that since the functor Φ of Grothendieck's Theorem is an equivalence of abelian categories, and $\{\mathcal{O}_{X_n}, g_n\}_{n\in\mathbb{N}} \to \{\mathcal{O}_{Y_n}, f_n\}_{n\in\mathbb{N}}$ has trivial cokernel in $\mathcal{C}oh(\widehat{X})$, we get that ψ is surjective. The kernel of $\psi : \mathcal{O}_X \to \mathcal{E}$ defines then a closed subscheme $Y \subseteq X$ with structure sheaf $\mathcal{E}$, such that $Y \cap X_n = Y_n$ for every n. $\square$

Now we go further, and consider abstract deformations.

Proposition 5.4.6. *Let X_0 be a projective scheme over k such that $H^2(X_0, \mathcal{O}_{X_0}) = 0$, and suppose $\widehat{X} = \{X_n, f_n\}_{n\in\mathbb{N}}$ is a formal deformation of X_0 over Λ (that is, a formal object of the category $\mathcal{D}ef$ restricting to X_0 over k).*

Then $\widehat{X}$ is algebraizable, i.e. there exists a flat and projective scheme X over Λ such that X_n is isomorphic to the pullback of X to Λ_n along the projection $\Lambda \to \Lambda_n$, and the isomorphisms are compatible with the arrows f_n.

For the proof we need the following lemma.

Lemma 5.4.7. *Let Z be a scheme and $Z_0 \subseteq Z$ a closed subscheme with square-zero sheaf of ideals $I \subseteq \mathcal{O}_Z$. Then there is an exact sequence of sheaves of groups*

$$0 \longrightarrow I \xrightarrow{\alpha} \mathcal{O}_Z^* \longrightarrow \mathcal{O}_{Z_0}^* \longrightarrow 0$$

where α is defined by $\alpha(u) = 1 + u$.

Proof. Consider the exact sequence

$$0 \longrightarrow I \longrightarrow \mathcal{O}_Z \xrightarrow{g} \mathcal{O}_{Z_0} \longrightarrow 0$$

coming from the closed immersion $Z_0 \subseteq Z$. We first show that if U is any open subset of $|Z| = |Z_0|$, then $f \in \mathcal{O}_Z(U)$ is a unit if and only if $g(f) \in \mathcal{O}_{Z_0}(U)$ is.

It is clear that if f is a unit then $g(f)$ is, so suppose conversely that we know that $g(f)$ is a unit in $\mathcal{O}_{Z_0}(U)$. Then there exists $h \in \mathcal{O}_{Z_0}(U)$ such that $g(f)h = 1$, and by surjectivity of g there exists $k \in \mathcal{O}_Z(U)$ such that $g(k) = h$. So $g(f)g(k) = 1$, or equivalently $g(fk - 1) = 0$, and hence $fk - 1 \in I$. This implies that $fk = 1 + u$ with $u \in I(U)$, which is invertible in $\mathcal{O}_Z(U)$, since $(1+u)(1-u) = 1$ (because $I^2 = (0)$), so that f is invertible too.

The argument we used above shows actually that the induced homomorphism $\mathcal{O}_Z^* \to \mathcal{O}_{Z_0}^*$ is surjective, and that its kernel is isomorphic to I as a sheaf of groups, by means of the homomorphism $I \to \mathcal{O}_Z^*$ defined by $u \mapsto 1 + u$, where u is a section of I. This concludes our proof. ☐

In particular since $H^1(Z_0, \mathcal{O}_{Z_0}^*) \cong \mathrm{Pic}(Z_0)$ and $H^1(Z_0, \mathcal{O}_Z^*) \cong \mathrm{Pic}(Z)$, taking the cohomology exact sequence we get

$$\cdots \longrightarrow H^1(Z_0, I) \longrightarrow \mathrm{Pic}(Z) \longrightarrow \mathrm{Pic}(Z_0) \longrightarrow H^2(Z_0, I) \longrightarrow \cdots$$

where the map $\mathrm{Pic}(Z) \to \mathrm{Pic}(Z_0)$ is just the natural pullback homomorphism.

Proof of 5.4.6. We start by showing that the natural restriction homomorphism $\mathrm{Pic}(X_n) \to \mathrm{Pic}(X_{n-1})$ is surjective. By the lemma, for a fixed n we have an exact sequence of groups

$$0 \longrightarrow \mathfrak{m}_\Lambda^n / \mathfrak{m}_\Lambda^{n+1} \otimes_k \mathcal{O}_{X_0} \longrightarrow \mathcal{O}_{X_n}^* \longrightarrow \mathcal{O}_{X_{n-1}}^* \longrightarrow 0.$$

Taking the cohomology long exact sequence and recalling that by hypothesis
$$H^2(X_0, \mathfrak{m}_\Lambda^n/\mathfrak{m}_\Lambda^{n+1} \otimes_k \mathcal{O}_{X_0}) \cong \mathfrak{m}_\Lambda^n/\mathfrak{m}_\Lambda^{n+1} \otimes_k H^2(X_0, \mathcal{O}_{X_0})$$
is trivial, we see that the homomorphism $H^1(X_0, \mathcal{O}^*_{X_n}) \to H^1(X_0, \mathcal{O}^*_{X_{n-1}})$ corresponding to the restriction $\mathrm{Pic}(X_n) \to \mathrm{Pic}(X_{n-1})$ is surjective.

Take now a very ample invertible sheaf $\mathcal{L}_0$ on X_0, such that $H^1(X_0, \mathcal{L}_0) = 0$, and let $s_0, \dots, s_m$ be a basis of $H^0(X_0, \mathcal{L}_0)$ as a k-vector space, defining the closed immersion $X_0 \to \mathbb{P}^m_k$. By surjectivity of $\mathrm{Pic}(X_n) \to \mathrm{Pic}(X_{n-1})$ we can lift $\mathcal{L}_0$ successively to X_n, obtaining thus a sequence $\{\mathcal{L}_n\}_{n\in\mathbb{N}}$ of compatible invertible sheaves on the formal deformation $\widehat{X}$; moreover we can also lift the basis above at each step.

In fact tensoring the exact sequence
$$0 \longrightarrow \mathfrak{m}_\Lambda^n/\mathfrak{m}_\Lambda^{n+1} \otimes_k \mathcal{O}_{X_0} \longrightarrow \mathcal{O}_{X_n} \longrightarrow \mathcal{O}_{X_{n-1}} \longrightarrow 0$$
with $\mathcal{L}_n$, we get
$$0 \longrightarrow \mathfrak{m}_\Lambda^n/\mathfrak{m}_\Lambda^{n+1} \otimes_k \mathcal{L}_0 \longrightarrow \mathcal{L}_n \longrightarrow \mathcal{L}_{n-1} \longrightarrow 0.$$
Noticing that
$$H^1(X_0, \mathfrak{m}_\Lambda^n/\mathfrak{m}_\Lambda^{n+1} \otimes_k \mathcal{L}_0) \cong \mathfrak{m}_\Lambda^n/\mathfrak{m}_\Lambda^{n+1} \otimes_k H^1(X_0, \mathcal{L}_0)$$
is trivial, and taking the cohomology long exact sequence of the last short one, we see that the restriction homomorphism $H^0(X_0, \mathcal{L}_n) \to H^0(X_0, \mathcal{L}_{n-1})$ is surjective, and so we can surely lift inductively $s_0, \dots, s_m$ to elements $s_0^n, \dots, s_m^n \in H^0(X_0, \mathcal{L}_n)$.

Moreover the sections $(s_0^n, \dots, s_m^n)$ will not have base points (because if they did, these points would also be base points of $(s_0, \dots, s_m)$), and then for every n we have an induced morphism $\varphi_n : X_n \to \mathbb{P}^m_{\Lambda_n}$; since $\varphi_0 : X_0 \to \mathbb{P}^m_k$ is a closed immersion, every φ_n will be as well.

This makes the sequence $\{X_n\}_{n\in\mathbb{N}}$ into a sequence of closed subschemes $X_n \subseteq \mathbb{P}^m_{\Lambda_n}$ compatible with the immersions $\mathbb{P}^m_{\Lambda_n} \subseteq \mathbb{P}^m_{\Lambda_{n+1}}$. Corollary 5.4.5 gives then a closed subscheme $X \subseteq \mathbb{P}^m_\Lambda$ restricting to X_n over Λ_n. If we show that X is flat over Λ, then it will be an algebraization of $\widehat{X}$.

By generic flatness, the locus of points at which X is flat over Λ is an open subset of X; consider its complement Z, a closed subset of X. Since $X \to \mathrm{Spec}(\Lambda)$ is proper, we have that $Z \cap X_0$ is nonempty (because the image of Z will contain the maximal ideal $\mathfrak{m}_\Lambda$). Now if we take a point $p \in Z \cap X_0$, from the fact that $\mathcal{O}_{X_n,p} \cong \mathcal{O}_{X,p}/\mathfrak{m}_\Lambda^{n+1}\mathcal{O}_{X,p}$ is flat over Λ_n for every n and from the local flatness criterion, it follows that $X \to \mathrm{Spec}(\Lambda)$ is flat at p, which is a contradiction.

Then $Z = \emptyset$, and X is flat over Λ. $\square$

Example 5.4.8. We give here an example of a formal deformation of a scheme that is not algebraizable. To do this, we will take as X_0 a smooth quartic surface in $\mathbb{P}^3_{\mathbb{C}}$, such that the Picard group $\mathrm{Pic}(X_0)$ is cyclic, generated by the invertible sheaf $\mathcal{O}_{X_0}(1)$. One can check that in this case $H^2(X_0, \mathcal{O}_{X_0}) \cong \mathbb{C}$, so that the hypotheses of the last theorem are not satisfied.

To know that such a surface exists, we need the following theorems.

Theorem 5.4.9 (Noether-Lefschetz)**.** *Let $d \geq 4$, and $\mathbb{P}^N_{\mathbb{C}}$ be the projective space of surfaces of degree d in $\mathbb{P}^3_{\mathbb{C}}$. Then there exists countably many hypersurfaces*

$$H_1, H_2, \ldots, H_n, \ldots \subseteq \mathbb{P}^N_{\mathbb{C}}$$

such that if $X_0 \in \mathbb{P}^N_{\mathbb{C}} \setminus \bigcup_i H_i$, then $\mathrm{Pic}(X_0)$ is cyclic and generated by $\mathcal{O}_{X_0}(1)$.

For a discussion about this theorem, see for example [Griff].

Theorem 5.4.10 (Baire)**.** *In a locally compact and Hausdorff topological space, a countable intersection of open dense subsets is itself dense.*

Combining these two theorems, we get a quartic surface $X_0 \subseteq \mathbb{P}^3_{\mathbb{C}}$, such that $\mathrm{Pic}(X_0)$ is cyclic generated by $\mathcal{O}_{X_0}(1)$.

Proposition 5.4.11. *We have that $H^2(X_0, T_{X_0}) = 0$. In particular, by Theorem 4.2.4, X_0 is unobstructed.*

Proof. We start from the exact sequence

$$0 \longrightarrow T_{X_0} \longrightarrow T_{\mathbb{P}^3_{\mathbb{C}}}|_{X_0} \longrightarrow \mathcal{O}_{X_0}(4) \longrightarrow 0$$

that we obtain by dualizing the conormal sequence of $X_0 \subseteq \mathbb{P}^3_{\mathbb{C}}$. Taking the cohomology exact sequence we get

$$\cdots \longrightarrow H^1(X_0, \mathcal{O}_{X_0}(4)) \longrightarrow H^2(X_0, T_{X_0}) \longrightarrow H^2(X_0, T_{\mathbb{P}^3_{\mathbb{C}}}|_{X_0}) \longrightarrow \cdots$$

so it is sufficient to show that the other two cohomology groups are trivial.

The exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3_{\mathbb{C}}} \longrightarrow \mathcal{O}_{\mathbb{P}^3_{\mathbb{C}}}(4) \longrightarrow \mathcal{O}_{X_0}(4) \longrightarrow 0$$

shows that $H^1(X_0, \mathcal{O}_{X_0}(4))$ is trivial. To check that $H^2(X_0, T_{\mathbb{P}^3_{\mathbb{C}}}|_{X_0})$ is as well, consider the restriction of the dual of the Euler sequence

$$0 \longrightarrow \mathcal{O}_{X_0} \longrightarrow \mathcal{O}_{X_0}(1)^{\oplus 4} \longrightarrow T_{\mathbb{P}^3_{\mathbb{C}}}|_{X_0} \longrightarrow 0$$

from which we see that it suffices to show that $H^2(X_0, \mathcal{O}_{X_0}(1)) = 0$ (since $H^3(X_0, \mathcal{O}_{X_0}) = 0$, for X_0 is a surface). This last fact follows from the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3_{\mathbb{C}}}(-3) \longrightarrow \mathcal{O}_{\mathbb{P}^3_{\mathbb{C}}}(1) \longrightarrow \mathcal{O}_{X_0}(1) \longrightarrow 0$$

that yields $H^2(X_0, \mathcal{O}_{X_0}(1)) \cong H^2(\mathbb{P}^3_{\mathbb{C}}, \mathcal{O}_{X_0}(1)) \cong H^3(\mathbb{P}^3_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^3_{\mathbb{C}}}(-3))$, and the last group is trivial, as one can readily check using Serre's duality. □

From the results in Section 2.4.5, we know that the differential of the forgetful morphism $F : \mathcal{H}ilb^{\mathbb{P}^3_k} \to \mathcal{D}ef$ at X_0 is not surjective, so we can take a first-order deformation $X_\varepsilon \to \operatorname{Spec}([\varepsilon])$, such that there does not exists a closed immersion $X_\varepsilon \subseteq \mathbb{P}^3_{\mathbb{C}[\varepsilon]}$ extending $X_0 \subseteq \mathbb{P}^3_{\mathbb{C}}$.

Moreover such a deformation has trivial Picard group: the exact sequence of shaves of groups

$$0 \longrightarrow \mathcal{O}_{X_0} \cong \mathcal{O}_{X_0} \otimes_k (\varepsilon) \longrightarrow \mathcal{O}^*_{X_\varepsilon} \longrightarrow \mathcal{O}^*_{X_0} \longrightarrow 0$$

of Lemma 5.4.7 yields

$$0 = H^1(X_0, \mathcal{O}_{X_0}) \longrightarrow \operatorname{Pic}(X_\varepsilon) \longrightarrow \operatorname{Pic}(X_0) \longrightarrow H^2(X_0, \mathcal{O}_{X_0}) \cong \mathbb{C}.$$

Now since $\operatorname{Pic}(X_0)$ is cyclic infinite and $\mathbb{C}$ is torsion-free, we conclude that the map $\operatorname{Pic}(X_\varepsilon) \to \operatorname{Pic}(X_0)$ must be zero, and then $\operatorname{Pic}(X_\varepsilon) = 0$.

From the fact that X_0 is unobstructed, we can find a formal deformation $\widehat{X} = \{X_n, f_n\}_{n\in\mathbb{N}}$ of X_0 over $\mathbb{C}[[t]]$, with term of order one isomorphic to X_ε.

Proposition 5.4.12. *The formal deformation $\widehat{X}$ is not algebraizable, that is, there does not exist a flat and proper scheme X over $\mathbb{C}[[t]]$ inducing the formal deformation $\widehat{X}$.*

Proof. Assume such an X exists, and take an open affine subscheme $U \subseteq X$. If we denote by $D = X \setminus U$ the complement of U, then every irreducible component of D has codimension 1 (see for example Corollaire 21.12.7 of [EGAIV]), and then, with the structure of reduced closed subscheme, it can be seen as a Weil divisor on X.

Now notice that X is smooth over $\mathbb{C}[[t]]$: if Z is the locus where X is not smooth, then Z is a closed subset of X, and (if it is not empty) it must intersect the central fiber X_0 (since $X \to \operatorname{Spec}(\mathbb{C}[[t]])$ is proper, and then the image of Z contains the maximal ideal of $\mathbb{C}[[t]]$). But X_0 is smooth over $\mathbb{C}$, and this is a contradiction that shows that $Z = \emptyset$.

Hence D is also an effective Cartier divisor over X, and since $X_0 \nsubseteq U$, we have that $D \cap X_0$ is an effective Cartier divisor on X_0, and it is not trivial (i.e. the associated invertible sheaf $\mathcal{O}_{X_0}(D \cap X_0)$ is not isomorphic to $\mathcal{O}_{X_0}$). Finally consider the commutative diagram

$$\begin{tikzcd} \operatorname{Pic}(X) \arrow[rr] \arrow[dr] & & \operatorname{Pic}(X_\varepsilon) \arrow[dl] \\ & \operatorname{Pic}(X_0) & \end{tikzcd}$$

with the maps are the natural pullback homomorphisms. We showed above that the function $\operatorname{Pic}(X) \to \operatorname{Pic}(X_0)$ is not zero, since the invertible sheaf $\mathcal{O}_X(D)$ goes to $\mathcal{O}_{X_0}(D \cap X_0)$, which is not trivial, but on the other hand we proved that $\operatorname{Pic}(X_\varepsilon) = 0$, which gives a contradiction. $\square$

Chapter 6

Deformations of nodal curves

In this last chapter we apply the results we obtained in the preceding ones to deformations of affine and projective curves with a finite number of nodes.

By studying this particular case we will show how knowing a miniversal deformation of a local model for a singularity helps in giving a local (formal) description of any global deformation of such a singularity. Finally we will give an algebraization result for projective curves with a finite number of nodes that relies on the general results of the preceding chapter.

6.1 Nodal curves

We start by describing the type of curves we are interested in. Let X be a curve over k.

Definition 6.1.1. *A closed point $p \in X$ is a **rational node** if p is a rational point, and the complete local ring $\widehat{\mathcal{O}}_{X,p}$ is isomorphic to $k[[x,y]]/(xy)$ as a k-algebra.*

We consider then generically smooth curves, having only rational nodes as singularities.

Definition 6.1.2. *By **nodal curve** we mean a curve X over k that is smooth outside of a finite number of closed points $p_1, \dots, p_n$ that are rational nodes.*

We give a criterion to recognize rational nodes, assuming $\mathrm{char}(k) \neq 2$: suppose X is a curve over k, and that the complete local ring of X in p is isomorphic to $k[[x,y]]/(f)$ for some element $f \in k[[x,y]]$. Write f_i for the homogeneous term of degree i of f, and suppose $f_0 = f_1 = 0$, and that f_2 is a quadratic form equivalent to xy over k. Then there is an isomorphism $k[[x,y]]/(f) \cong k[[x,y]]/(xy)$. This basically says that every rational singular point with multiplicity two and with two rational distinct tangent lines is a rational node.

The main ingredient for the proof is Weierstrass' preparation Theorem (see for example IV, § 9 of [Lang]) that we recall here.

Theorem 6.1.3. *Let R be a noetherian complete local ring, and $f = \sum_i f_i x^i$ an element of the power series ring $R[[x]]$. Assume that $f_0, f_1, \dots, f_{r-1} \in \mathfrak{m}_R$ and $f_r \notin \mathfrak{m}_R$. Then there exists a unit $u = \sum_i u_i x^i \in R[[x]]$ and a monic polynomial $p \in R[x]$ of degree r and with coefficients in $\mathfrak{m}_R$, such that $f = p \cdot u$.*

Example 6.1.4. The remark above does not hold if we do not assume that f_2 is equivalent to the quadratic form xy, unless we add some other hypothesis (for example that the field k is algebraically closed).

For instance, if we take $f \in \mathbb{R}[[x, y]]$ to be $f(x, y) = x^2 + y^2 + x^3$, then the origin is a singular point with multiplicity 2, but the tangent lines (with equations $x + iy = 0$ and $x - iy = 0$) are not rational, and in fact one can check that $\mathbb{R}[[x, y]]/(x^2 + y^2)$ is not isomorphic to $\mathbb{R}[[x, y]]/(xy)$ as an $\mathbb{R}$-algebra.

6.2 Affine curves with one node

The first case we consider is the one of an affine nodal curve X over k with only one rational node p. Since the complete local ring should give some control on the local structure of a scheme at the corresponding point, and by definition we have an isomorphism of the complete local ring $\widehat{\mathcal{O}}_{X,p}$ with $k[[x, y]]/(xy)$, which is the complete local ring at the origin of $V(xy) \subseteq \mathbb{A}^2_k$, one could hope to link the deformations of X and the ones of $V(xy)$ using this isomorphism, which is what we will do now.

The starting point is the following theorem of Michael Artin (see [Art]).

Theorem 6.2.1. *Suppose X, Y are schemes of finite type over a base scheme S, also of finite type over a field k. Let $p \in X$ and $q \in Y$ be two points with a fixed isomorphism $f : k(p) \cong k(q)$ over S, and call s the image of p and q in S. Then f extends to an isomorphism $\widehat{\mathcal{O}}_{X,p} \cong \widehat{\mathcal{O}}_{Y,q}$ of $\widehat{\mathcal{O}}_{S,s}$-algebras if and only if there exists a scheme Z over S with two étale morphisms $Z \to X$ and $Z \to Y$, fitting in the commutative diagram below.*

$$
\begin{tikzcd}
\mathrm{Spec}(k(p)) \arrow[dr] \arrow[dd, bend right] & & \mathrm{Spec}(k(q)) \arrow[ll, "\sim"'] \arrow[dl] \arrow[dd, bend left] \\
& Z \arrow[dl] \arrow[dr] & \\
X \arrow[dr] & & Y \arrow[dl] \\
& S &
\end{tikzcd}
$$

Back to the curve with a single node, we apply the preceding theorem with $S = \mathrm{Spec}(k)$, X our curve, $Y = V(xy) \subseteq \mathbb{A}^2_k$, p the rational node of X, and q the origin of $\mathbb{A}^2_k$.

Since we have an isomorphism $\widehat{\mathcal{O}}_{X,p} \cong k[[x,y]]/(xy) = \widehat{\mathcal{O}}_{y,q}$ extending the identity $k \cong k$ on the residue fields, we conclude that there exist a scheme Z over k with two étale morphisms $Z \to X, Z \to Y$, and a rational point z of Z that gets mapped to p and q respectively, and fitting in a commutative diagram as above.

We will use these two étale maps to link the deformations of X with the ones of the "standard" nodal "curve" Y (quotation marks since Y is not irreducible).

6.2.1 Pullback functor induced by an étale morphism

Suppose we have two schemes X_0, Y_0 over k, with an étale morphism $f : X_0 \to Y_0$. We will show in this section that such an f induces a pullback functor $f^* : \mathcal{D}ef_{Y_0} \to \mathcal{D}ef_{X_0}$ (which is a morphism of deformation categories), and then we will analyze its properties in a particular case. The natural thing to do is, given an infinitesimal deformation Y of Y_0, to take as $f^*(Y)$ a scheme that fits in a cartesian diagram of the form

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d, "f"'] & f^*(Y) \arrow[d] \\ Y_0 \arrow[r] & Y. \end{tikzcd}$$

To show that we can find such a scheme, we start from the following theorem of Grothendieck. We consider a scheme Z', and a closed subscheme $Z \subseteq Z'$, whose sheaf of ideals is nilpotent. We have two categories, which we denote by $\acute{\mathrm{E}}\mathrm{t}(Z), \acute{\mathrm{E}}\mathrm{t}(Z')$, of étale morphisms of schemes $T \to Z$ (respectively $T' \to Z'$), with the obvious arrows.

There is also a natural restriction functor $\Phi : \acute{\mathrm{E}}\mathrm{t}(Z') \to \acute{\mathrm{E}}\mathrm{t}(Z)$, defined on objects by $\Phi(T' \to Z') = Z \times_{Z'} T' \to Z$ (the projection on the first factor of the fibered product), and on arrows in the obvious way.

Theorem 6.2.2. *The functor Φ is an equivalence of categories.*

A proof can be found for example in [Mil] (Theorem 3.23).

Remark 6.2.3. More concretely, the fact that Φ is essentially surjective is equivalent to the statement that if $T \to Z$ is étale, then we can find an étale morphism $T' \to Z'$ (which is unique up to isomorphism) such that the following diagram is cartesian

$$\begin{tikzcd} T \arrow[r] \arrow[d] & T' \arrow[d] \\ Z \arrow[r] & Z'. \end{tikzcd}$$

Φ being fully faithful on the other hand implies that if $T' \to Z', S' \to Z', U \to Z$ are all étale morphisms, and we have two cartesian diagrams

$$\begin{tikzcd} U \arrow[r] \arrow[d] & T' \arrow[d] \\ Z \arrow[r] & Z' \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r] \arrow[d] & S' \arrow[d] \\ Z \arrow[r] & Z' \end{tikzcd}$$

then there exists a unique morphism $T' \to S'$ that fits in the commutative diagram

$$\begin{tikzcd} & & T' \arrow[dl, dashed] \arrow[ddl] \\ U \arrow[urr] \arrow[r] \arrow[d] & S' \arrow[d] & \\ Z \arrow[r] & Z'. & \end{tikzcd}$$

This second property has the following consequence.

Corollary 6.2.4. *Let $Z \to Z'$ and $Z \to Z''$ be two closed immersions with nilpotent sheaf of ideals, and $T \to Z, T' \to Z', T'' \to Z''$ three étale morphisms. Assume also that we have two cartesian diagrams*

$$\begin{tikzcd} T \arrow[r] \arrow[d] & T' \arrow[d] \\ Z \arrow[r] & Z' \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r] \arrow[d] & T'' \arrow[d] \\ Z \arrow[r] & Z'' \end{tikzcd}$$

and also a morphism $Z' \to Z''$, compatible with the closed immersions $Z \to Z'$ and $Z \to Z''$.

Then there exists a unique morphism $T' \to T''$ fitting in the commutative diagram

$$\begin{tikzcd} & T' \arrow[dr, dashed] \arrow[dd] & \\ T \arrow[ur] \arrow[rr] \arrow[ddd] & & T'' \arrow[ddd] \\ & Z' \arrow[ddr] & \\ & & \\ Z \arrow[uur] \arrow[rr] & & Z''. \end{tikzcd}$$

Proof. This follows directly from the second part of the last remark, after noticing that giving a morphism $T' \to T''$ over $Z' \to Z''$ is equivalent to giving a morphism $T' \to Z' \times_{Z''} T''$ compatible with the two morphisms

$T' \to Z'$ and $Z' \times_{Z''} T'' \to Z'$ (which are étale morphisms).

$$\begin{tikzcd}
 & T' \arrow[r, dashed] \arrow[dd] & Z' \times_{Z''} T'' \arrow[d] \arrow[ddl] \\
T \arrow[ur] \arrow[rr] \arrow[dd] & & T'' \arrow[dd] \\
 & Z' \arrow[dr] & \\
Z \arrow[ur] \arrow[rr] & & Z''
\end{tikzcd}$$

□

Now we can construct the functor $f^* : \mathcal{D}ef_{Y_0} \to \mathcal{D}ef_{X_0}$, for an étale morphism $f : X_0 \to Y_0$. Take an object $Y \in \mathcal{D}ef_{Y_0}(A)$, which makes Y_0 into a closed subscheme of Y with a nilpotent sheaf of ideals. The functor $\Phi : \acute{\mathrm{E}}\mathrm{t}(Y) \to \acute{\mathrm{E}}\mathrm{t}(Y_0)$ is then an equivalence (Theorem 6.2.2) and X_0 is an object of $\acute{\mathrm{E}}\mathrm{t}(Y_0)$, so we have an object $X \in \acute{\mathrm{E}}\mathrm{t}(Y)$ (unique up to isomorphism), fitting in the cartesian diagram

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d, "f"'] & X \arrow[d] \\
Y_0 \arrow[r] & Y.
\end{tikzcd}$$

From this it follows that the induced diagram

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d] & X \arrow[d] \\
\mathrm{Spec}(k) \arrow[r] & \mathrm{Spec}(A)
\end{tikzcd}$$

is cartesian too, and so X is an object of $\mathcal{D}ef_{X_0}(A)$ that we denote by $f^*(Y)$. We choose arbitrarily such an object for each $Y \in \mathcal{D}ef_{Y_0}$.

Next, suppose we have a morphism $Y \to Y'$ in $\mathcal{D}ef_{Y_0}$, where $Y \in \mathcal{D}ef_{Y_0}(A)$ and $Y' \in \mathcal{D}ef(A')$. Using Corollary 6.2.4 we get then a morphism $f^*(Y) \to f^*(Y')$, which is the unique one fitting in the diagram

$$\begin{tikzcd}
 & f^*(Y) \arrow[dr, dashed] \arrow[dd] & \\
X_0 \arrow[ur] \arrow[rr] \arrow[dd] & & f^*(Y') \arrow[dd] \\
 & Y \arrow[dr] & \\
Y_0 \arrow[ur] \arrow[rr] & & Y'.
\end{tikzcd}$$

This makes f^* into a functor that we call the **pullback functor** induced by f. It is immediate to check that changing our choice of $f^*(Y)$ for some of the infinitesimal deformations Y of Y_0 gives naturally equivalent functors.

6.2.2 Quasi-equivalences

Next, we want to show that with some additional hypotheses, the pullback functor induced by an étale morphism has some nice properties. Namely, it is what we call a quasi-equivalence.

Definition 6.2.5. *Let $\mathcal{F} \to (\mathrm{Art}/\Lambda)^{op}$ and $\mathcal{G} \to (\mathrm{Art}/\Lambda)^{op}$ be two deformation categories, and $\xi_0 \in \mathcal{F}(k)$. A morphism $F : \mathcal{F} \to \mathcal{G}$ is a **quasi-equivalence** at ξ_0 if:*

- *The differential $d_{\xi_0}F : T_{\xi_0}\mathcal{F} \to T_{F(\xi_0)}\mathcal{G}$ is an isomorphism.*
- *If $A' \to A$ is a small extension and $\xi \in \mathcal{F}_{\xi_0}(A)$, then ξ has a lifting to A' if and only if $F(\xi) \in \mathcal{G}(A)$ does.*

Remark 6.2.6. As usual, if F is a quasi-equivalence the second property will hold for any surjection $A' \to A$ in (Art/Λ), as one sees by factoring it as a composite of small extensions. In the proof, one uses also the fact that the differential of F is an isomorphism, and that it is compatible with the actions on the isomorphism classes of liftings.

Example 6.2.7. If $\xi \in \widehat{\mathcal{F}}(R)$ is a formal object of $\mathcal{F}$, then the corresponding morphism $\xi : (\mathrm{Art}/R)^{op} \to \mathcal{F}$ is a quasi-equivalence (at the only object over k of the source) if and only if ξ is miniversal. This is because the second property of the definition corresponds exactly to the lifting property of versal objects, as one easily checks.

Remark 6.2.8. Another property that is easy to check is that a composite of two quasi-equivalences is still a quasi-equivalence. More precisely, if $F : \mathcal{F} \to \mathcal{G}$ is a quasi-equivalence at $\xi_0 \in \mathcal{F}(k)$, and $G : \mathcal{G} \to \mathcal{H}$ is a quasi-equivalence at $F(\xi_0) \in \mathcal{G}(k)$, then the composite morphism $G \circ F : \mathcal{F} \to \mathcal{H}$ is a quasi-equivalence at ξ_0.

Now suppose we have a quasi-equivalence $F : \mathcal{F} \to \mathcal{G}$ at $\xi_0 \in \mathcal{F}(k)$, and let $\xi \in \widehat{\mathcal{F}}(R)$ be a miniversal formal object restricting to ξ_0 over k, corresponding to $\xi : (\mathrm{Art}/R)^{op} \to \mathcal{F}$; then by the above remarks the composite $F \circ \xi : (\mathrm{Art}/R)^{op} \to \mathcal{F} \to \mathcal{G}$ is a quasi-equivalence. The formal object corresponding to this composite, which is the image of $\xi \in \widehat{\mathcal{F}}$ along the induced morphism $\widehat{F} : \widehat{\mathcal{F}} \to \widehat{\mathcal{G}}$, is then a miniversal formal object of $\mathcal{G}$, restricting to $F(\xi_0)$ over k.

In particular we want to apply the above discussion to the pullback morphism $f^* : \mathcal{D}ef_{Y_0} \to \mathcal{D}ef_{X_0}$, obtaining thus a way to get a miniversal deformation of X_0 from one of Y_0.

Proposition 6.2.9. *Let X_0, Y_0 be affine, reduced and generically smooth local complete intersection schemes of finite type over k, and $f : X_0 \to Y_0$ be an étale morphism. Assume also that Y_0 has isolated singularities, and that for any* *singular point $q \in Y_0$ there exists a unique $p \in X_0$ such that $f(p) = q$, and moreover $k(p) \cong k(q)$. Then the pullback functor $f^* : \mathcal{D}ef_{Y_0} \to \mathcal{D}ef_{X_0}$ is a quasi-equivalence.*

Proof. Let $X_0 = \mathrm{Spec}(A), Y_0 = \mathrm{Spec}(B)$. Since f is étale, we have $\Omega_{X_0} \cong f^*(\Omega_{Y_0})$, or in other words $\Omega_A \cong \Omega_B \otimes_B A$. We start with some preliminary remarks about canonical homomorphisms between the Ext modules of Ω_B and Ω_A.

Thanks to flatness of f, we have canonical isomorphisms

$$\mathrm{Ext}^i_B(\Omega_B, B) \otimes_B A \cong \mathrm{Ext}^i_A(\Omega_B \otimes_B A, A) \cong Ext^i_A(\Omega_A, A)$$

for any i. Moreover we can compose these isomorphisms with the natural maps

$$\varphi_i : \mathrm{Ext}^i_B(\Omega_B, B) \to \mathrm{Ext}^i_B(\Omega_B, B) \otimes_B A$$

given by $x \mapsto x \otimes 1$. We show that the homomorphisms φ_i are actually isomorphisms, so that the composites $\mathrm{Ext}^i_B(\Omega_B, B) \to Ext^i_A(\Omega_A, A)$ will be as well.

Set $M = \mathrm{Ext}^i_B(\Omega_B, B)$. To show that φ_i is an isomorphism, we prove that the localization

$$(\varphi_i)_q : M_q \to (M \otimes_B A)_q \cong M_q \otimes_{B_q} A_q$$

is an isomorphism for any $q \in Y_0 = \mathrm{Spec}(B)$. If q is not a singular point of Y_0, then $\Omega_{B,q}$ is locally free, and

$$M_q = \mathrm{Ext}^i_B(\Omega_B, B)_q \cong \mathrm{Ext}^i_{B_q}(\Omega_{B,q}, B_q)$$

is trivial (as well as $(M \otimes_B A)_q$), so φ_q is an isomorphism.

Take then q to be one of the singular points of Y_0. Notice that by hypothesis there is a unique point $p \in X_0$ over q, so that in this case $A_q \cong A_p$. Then $\varphi_q : M_q \to M_q \otimes_{B_q} A_p$, and moreover we have an étale homomorphism of rings $B_q \to A_p$, induced by f.

The following lemma lets us conclude that φ_q is an isomorphism.

Lemma 6.2.10. *Let $R \to S$ be an étale homomorphism of local rings with isomorphic residue fields, and M be an R-module of finite length. Then the natural map*

$$M \to M \otimes_R S$$

is an isomorphism.

Proof. The hypotheses above imply that the induced homomorphism $f_n : R_n \to S_n$ is an isomorphism for any $n \geq 0$ (where as usual $R_n = R/\mathfrak{m}_R^{n+1}$, and the same for S).

Since $\mathfrak{m}_R^{n+1}M = 0$ for n large enough, we have $M \otimes_R R_n \cong M$ (and the same holds for $M \otimes_R S$); moreover there is a commutative diagram

$$\begin{tikzcd} M \arrow[r] \arrow[d, "\wr"] & M \otimes_R S \arrow[d, "\wr"] \\ M \otimes_R R_n \arrow[r] & (M \otimes_R S) \otimes_R R_n. \end{tikzcd}$$

But now the bottom homomorphism $M \otimes_R R_n \to (M \otimes_R S) \otimes_R R_n$ is an isomorphism too, because $(M \otimes_R S) \otimes_R R_n \cong (M \otimes_R R_n) \otimes_{R_n} S_n$, and $R_n \cong S_n$ for any n.

In conclusion $M \to M \otimes_R S$ is also an isomorphism, and this concludes the proof. $\square$

Now we turn to the pullback functor f^*. First, we show that the differential $d_{Y_0}f^* : T_{Y_0}\mathcal{D}ef \to T_{X_0}\mathcal{D}ef$ corresponds to the homomorphism $\mathrm{Ext}^1_B(\Omega_B, B) \to \mathrm{Ext}^1_A(\Omega_A, A)$ considered above, and then in particular it is an isomorphism.

Notice first that $\mathrm{Ext}^1_B(\Omega_B, B) \to \mathrm{Ext}^1_A(\Omega_A, A)$ can be described as the function that takes an (isomorphism class of an) extension of B-modules

$$0 \longrightarrow B \longrightarrow M \longrightarrow \Omega_B \longrightarrow 0$$

to the (isomorphism class of the) one obtained by tensoring with A

$$0 \longrightarrow A \longrightarrow M \otimes_B A \longrightarrow \Omega_A \longrightarrow 0.$$

Suppose then that $Y \in \mathcal{D}ef_{Y_0}(k[\varepsilon])$, put $X = f^*(Y) \in \mathcal{D}ef_{X_0}(k[\varepsilon])$, and recall from the proof of Theorem 2.4.1 that the class of Y in $\mathrm{Ext}^1_B(\Omega_B, B)$ is given by the conormal sequence

$$0 \longrightarrow B \longrightarrow \Omega_Y|_{Y_0} \longrightarrow \Omega_B \longrightarrow 0$$

and the same holds for X.

Consider now the morphism $g : X \to Y$ given by the diagram

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X = f^*(Y) \arrow[d, "g"] \\ Y_0 \arrow[r] & Y. \end{tikzcd}$$

This induces a homomorphism $\Omega_Y|_{Y_0} \otimes_B A \to \Omega_X|_{X_0}$, fitting in the commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & \Omega_Y|_{Y_0} \otimes_B A \arrow[r] \arrow[d] & \Omega_A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A \arrow[r] & \Omega_X|_{X_0} \arrow[r] & \Omega_A \arrow[r] & 0 \end{tikzcd}$$

that shows that the image of the extension in $\operatorname{Ext}^1_A(\Omega_A, A)$ corresponding to Y is isomorphic to the extension corresponding to X, as we wanted.

The property about liftings is trivially satisfied, since X_0 and Y_0 are both unobstructed (see Remark 4.2.2). □

6.2.3 Deformations of affine curves with one node

Finally we turn to deformations of curves. Let X_0 be an affine nodal curve with exactly one node $p \in X_0$, let $Y_0 = V(xy) \subseteq \mathbb{A}^2_k$, and $q \in Y_0$ be the origin.

Then (by Theorem 6.2.1) we have a scheme Z over k and two étale morphisms $f : Z \to X_0, g : Z \to Y_0$, which induce two pullback functors $f^* : \mathcal{D}ef_{X_0} \to \mathcal{D}ef_Z$ and $g^* : \mathcal{D}ef_{Y_0} \to \mathcal{D}ef_Z$.

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X_0 & & Y_0 \end{tikzcd}$$

By Theorem 6.2.9, these two morphisms are quasi-equivalences.

Then we can get a miniversal deformation of X_0 over $\Lambda[[t]]$, from the "standard" one of Y_0. Set $R = \Lambda[[t]]$ and let $Y \subseteq \mathbb{A}^2_R$ be the closed subscheme $Y = V(xy - t)$. Recall from Section 5.3.2 that the formal deformation $\widehat{Y} = \{Y_n, f_n\}_{n\in\mathbb{N}}$ over R obtained by taking the pullbacks $Y_n = Y|_{R_n}$ and the induced morphisms is a miniversal deformation of Y_0.

Applying the functor g^* we get then a miniversal deformation $\widehat{Z} = \{Z_n, g_n\}$ of Z: here Z_n can be defined inductively as a scheme over k with an étale morphism $Z_n \to Y_n$ that fits in the cartesian diagram

$$\begin{tikzcd} Z_{n-1} \arrow[r, "g_{n-1}"] \arrow[d] & Z_n \arrow[d] \\ Y_{n-1} \arrow[r] & Y_n \end{tikzcd}$$

(we are using the fact that the restriction functor $\text{Ét}(Y_n) \to \text{Ét}(Y_{n-1})$ is an equivalence).

Since X_0 has isolated singularities we can also consider a miniversal deformation $\widehat{X} = \{X_n, h_n\}$ of X_0, say over $S \in (\text{Comp}/\Lambda)$, and apply the functor f^*. This way we get another miniversal deformation $\widehat{Z}' = \{Z'_n, g'_n\}_{n\in\mathbb{N}}$ of Z, defined the same way as the one induced by $\widehat{Y}$.

Since two miniversal deformations of the same scheme over k are isomorphic (Proposition 5.2.14), we have an isomorphism $(S, \widehat{Z}') \to (R, \widehat{Z})$, which consists of an isomorphism of Λ-algebras $\varphi : R \to S$, together with isomorphisms $\alpha_n : Z'_n \to Z_n$, fitting in the diagrams

$$\begin{tikzcd} Z'_n \arrow[r, "\alpha_n"] \arrow[d] & Z_n \arrow[d] \\ \mathrm{Spec}(S_n) \arrow[r] & \mathrm{Spec}(R_n) \end{tikzcd}$$

and that are also compatible with the immersions g_n and g'_n.

Now we can consider the inverse $\psi : S \to R$ of φ, and the pullback X'_n of X_n along $S_n \to R_n$, as in the diagram

$$\begin{tikzcd} X'_n \arrow[r] \arrow[d] & X_n \arrow[d] \\ \mathrm{Spec}(R_n) \arrow[r] & \mathrm{Spec}(S_n). \end{tikzcd}$$

This, together with the induced arrows $h'_n : X'_n \to X'_{n+1}$, gives another formal deformation $\widehat{X}' = \{X'_n, h'_n\}_{n\in\mathbb{N}}$ of X_0 over R that is easily seen to be miniversal too. Moreover the morphisms $Z'_n \to X_n$ and $Z'_n \to \mathrm{Spec}(S_n) \to \mathrm{Spec}(R_n)$ induce an étale morphism $Z'_n \to X'_n$ over $\mathrm{Spec}(R_n)$.

Putting everything together, for every n we have a commutative diagram

$$\begin{tikzcd} & Z'_n \arrow[rr, "\sim", "\alpha_n"'] \arrow[dl] \arrow[ddr] & & Z_n \arrow[dr] \arrow[ddl] & \\ X'_n \arrow[drr] & & & & Y_n \arrow[dll] \\ & & \mathrm{Spec}(R_n) & & \end{tikzcd}$$

where $Z'_n \to X'_n$ and $Z_n \to Y_n$ are étale, and moreover the morphisms in this diagram are compatible with the closed immersions h'_n, g_n, g'_n, f_n and $\mathrm{Spec}(R_n) \to \mathrm{Spec}(R_{n+1})$.

This gives us (by Theorem 6.2.1) a sequence of compatible isomorphisms

$$\lambda_n : \widehat{\mathcal{O}}_{X'_n,p} \to \widehat{\mathcal{O}}_{Y_n,q} \cong R_n[[x,y]]/(xy-t)$$

in the sense that for every n the diagram

$$\begin{tikzcd} \widehat{\mathcal{O}}_{X'_{n+1},p} \arrow[r, "\lambda_{n+1}"] \arrow[d] & \widehat{\mathcal{O}}_{Y_{n+1},q} \arrow[d] \\ \widehat{\mathcal{O}}_{X'_n,p} \arrow[r, "\lambda_n"] & \widehat{\mathcal{O}}_{Y_n,q} \end{tikzcd}$$

commutes, where the vertical maps are the projections.

From this discussion we get the following result, which gives a description of the complete local ring of a global deformation of a curve around a rational node (which is in some sense a "formal" description of the deformation around the node).

Proposition 6.2.11. *Suppose that $f : X \to S$ is a flat morphism of finite type, the fiber $X_0 = f^{-1}(s_0)$ over a point $s_0 \in S$ is a curve over $k(s_0)$ with isolated singularities, and $p \in X_0$ is a rational node. Then there exists $u \in \widehat{\mathcal{O}}_{S,s_0}$ and an isomorphism of $\widehat{\mathcal{O}}_{S,s_0}$-algebras*

$$\widehat{\mathcal{O}}_{X,p} \cong \widehat{\mathcal{O}}_{S,s_0}[[x,y]]/(xy-u).$$

Proof. Since the statement is local, we can assume that p is the unique singular point of X_0. We take $\Lambda = \widehat{\mathcal{O}}_{S,s_0}$, and consider the formal deformation $\widehat{X} = \{X_n, f_n\}_{n\in\mathbb{N}}$ over Λ defined by $X_n = f^{-1}(\operatorname{Spec}(\Lambda_n))$, where we see $\operatorname{Spec}(\Lambda_n) \to \operatorname{Spec}(\Lambda) \to S$ as the n-th infinitesimal neighborhood of s_0; the morphisms f_n are the induced closed immersions.

Since $(\Lambda, \widehat{X})$ is a formal deformation, we have a morphism of formal objects $(\Lambda, \widehat{X}) \to (\Lambda[[t]], \widehat{X}')$ (where $\widehat{X}'$ is the miniversal deformation of X_0 constructed above), that is, a homomorphism of Λ-algebras $\Lambda[[t]] \to \Lambda$ and closed immersions $X_n \to X'_n$ such that for every n the diagram

$$\begin{tikzcd} X_n \arrow[r] \arrow[d] & X'_n \arrow[d] \\ \operatorname{Spec}(\Lambda) \arrow[r] & \operatorname{Spec}(\Lambda[[t]]) \end{tikzcd}$$

is cartesian.

We call u the image of t along the homomorphism $\Lambda[[t]] \to \Lambda$, which we can see as the quotient map $\Lambda[[t]] \to \Lambda[[t]]/(t-u) \cong \Lambda$. Using the preceding discussion we get then a sequence of compatible isomorphisms of Λ-algebras

$$\widehat{\mathcal{O}}_{X_n,p} \cong \Lambda[[t]]_n[[x,y]]/(xy-t, t-u).$$

Finally, passing to the projective limit, this sequence induces an isomorphism

$$\widehat{\mathcal{O}}_{X,p} \cong \Lambda[[t]][[x,y]]/(xy-t, t-u) \cong \Lambda[[x,y]]/(xy-u)$$

which is what we wanted. □

Proposition 6.2.11 can be generalized to local complete intersections with isolated singularities.

Example 6.2.12. If instead of $xy = 0$ we take $y^2 - x^3 = 0$ as standard singularity (and we assume that $\operatorname{char}(k) \neq 2, 3$), then by Example 5.3.13 we know that a miniversal deformation of $V(y^2 - x^3) \subseteq \mathbb{A}^2_k$ is given by the pullbacks $X_n = X|_{R_n}$ of $X = V(y^2 - x^3 + t + ux) \subseteq \mathbb{A}^2_R$ to the quotients of $R = \Lambda[[t, u]]$, together with the induced immersions $X_n \to X_{n+1}$.

Then in the same way one can prove: if $f : X \to S$ is a flat morphism of finite type such that $X_0 = f^{-1}(s_0)$ (with $\mathrm{char}(k(s_0)) \neq 2, 3$) is a curve with isolated singularities over $k(s_0)$, and $p \in X_0$ is a rational point such that $\widehat{\mathcal{O}}_{X_0,p} \cong k(s_0)[[x, y]]/(y^2 - x^3)$, then there exist $v, w \in \widehat{\mathcal{O}}_{S,s_0}$ and an isomorphism of $\widehat{\mathcal{O}}_{S,s_0}$-algebras

$$\widehat{\mathcal{O}}_{X,p} \cong \widehat{\mathcal{O}}_{S,s_0}[[x, y]]/(y^2 - x^3 + v + wx).$$

6.3 Affine curves with a finite number of nodes

Now we analyze the more general case of affine curves with more that one node. We would like to repeat the argument we used in the previous case, but now we cannot use $Y_0 = V(xy) \subseteq \mathbb{A}^2_k$ anymore, since we have more than one node. On the other hand we can do the following: call X_0 our nodal curve, and suppose the nodes are $p_1, \dots, p_r \in X_0$. For each i we have a diagram

$$\begin{tikzcd} & Z_i \arrow[dl, "f_i"'] \arrow[dr, "g_i"] & \\ X_0 & & Y_0 \end{tikzcd}$$

where there is a point $z_i \in Z_i$ that goes to p_i in X_0 and to the origin q in Y_0.

Taking the disjoint unions $Z = \coprod_i Z_i$, $Y = \coprod_i Y_0$, and the induced étale morphisms $f : Z \to X_0$ and $g : Z \to Y$, we get a diagram

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X_0 & & Y \end{tikzcd}$$

that we can use to construct a particular miniversal deformation of X_0, as in the preceding section.

To do this, we first have to describe a miniversal deformation of the disjoint union $Y = \coprod_i Y_0$, starting from the description of the one of Y_0 we already used above. Suppose for example that we have only two copies of Y_0, that is, $Y = Y_0' \amalg Y_0''$ (the general case can be treated inductively starting from this one), and let $Z \in \mathcal{D}ef_Y(A)$ be a deformation of Y over $A \in (\mathrm{Art}/\Lambda)$. Since as topological spaces $|Z| = |Y|$, we can decompose Z as a disjoint union $Z = Z' \amalg Z''$, where $Z' \in \mathcal{D}ef_{Y_0'}(A)$ and $Z'' \in \mathcal{D}ef_{Y_0''}(A)$.

This gives a morphism of fibered categories

$$\mathcal{D}ef_Y \to \mathcal{D}ef_{Y_0'} \times_{(\mathrm{Art}/\Lambda)^{op}} \mathcal{D}ef_{Y_0''}$$

that is clearly an equivalence. So we are led to study products of deformation categories, and in particular the relations between miniversal deformations of the product and the ones of the factors.

6.3.1 Products of deformation categories

Let $\mathcal{F} \to (\mathrm{Art}/\Lambda)^{op}$ and $\mathcal{G} \to (\mathrm{Art}/\Lambda)^{op}$ be two deformation categories.

Definition 6.3.1. *The* ***product category*** *of $\mathcal{F}$ and $\mathcal{G}$ (as deformation categories) denoted by $\mathcal{F} \times \mathcal{G}$ is the fibered product $\mathcal{F} \times_{(\mathrm{Art}/\Lambda)^{op}} \mathcal{G}$, equipped with the natural functor $\mathcal{F} \times_{(\mathrm{Art}/\Lambda)^{op}} \mathcal{G} \to (\mathrm{Art}/\Lambda)^{op}$.*

If ξ and η are objects of $\mathcal{F}$ and $\mathcal{G}$ over $A \in (\mathrm{Art}/\Lambda)$ respectively, we will denote by (ξ, η) the corresponding object of $(\mathcal{F} \times \mathcal{G})(A)$ (the isomorphism $A \to A$ will be the identity, so we omit it in the notation).

In the following proposition we collect a couple of facts, whose proof is very easy.

Proposition 6.3.2. *Let $\mathcal{F} \to (\mathrm{Art}/\Lambda)^{op}$ and $\mathcal{G} \to (\mathrm{Art}/\Lambda)^{op}$ be two deformation categories, $\xi_0 \in \mathcal{F}(k)$ and $\eta_0 \in \mathcal{G}(k)$. Then:*

- *The product category $\mathcal{F} \times \mathcal{G} \to (\mathrm{Art}/\Lambda)^{op}$ is a deformation category.*
- *The two projections $\mathcal{F} \times \mathcal{G} \to \mathcal{F}$ and $\mathcal{F} \times \mathcal{G} \to \mathcal{G}$ are morphisms of deformation categories, and the map*

$$T_{(\xi_0,\eta_0)}(\mathcal{F} \times \mathcal{G}) \to T_{\xi_0}\mathcal{F} \oplus T_{\eta_0}\mathcal{G}$$

induced by the differentials of the projections is an isomorphism.

Now assume that $T_{\xi_0}\mathcal{F}$ and $T_{\eta_0}\mathcal{G}$ are both finite-dimensional; by Theorem 5.3.1 we have then two miniversal deformations (R, ξ) of ξ_0 and (S, η) of η_0. We consider the coproduct $R \widehat{\otimes} S \in (\mathrm{Comp}/\Lambda)$ (as defined in appendix B), and the two pullbacks $\overline{\xi} \in \widehat{\mathcal{F}}(R \widehat{\otimes} S)$ and $\overline{\eta} \in \widehat{\mathcal{G}}(R \widehat{\otimes} S)$ of ξ and η along the two inclusions $R \to R \widehat{\otimes} S$ and $S \to R \widehat{\otimes} S$.

Together $\overline{\xi}$ and $\overline{\eta}$ give a formal object $(\overline{\xi}, \overline{\eta})$ of $\mathcal{F} \times \mathcal{G}$ over $R \widehat{\otimes} S$.

Proposition 6.3.3. *The formal object $(R \widehat{\otimes} S, (\overline{\xi}, \overline{\eta}))$ is a miniversal formal object of $\mathcal{F} \times \mathcal{G}$.*

Proof. First of all, recall from Proposition B.13 that $T_\Lambda(R \widehat{\otimes} S) \cong T_\Lambda R \oplus T_\Lambda S$. It is easy then to check that the Kodaira-Spencer map

$$\kappa_{(\overline{\xi},\overline{\eta})} : T_\Lambda R \oplus T_\Lambda S \to T_{(\xi_0,\eta_0)}(\mathcal{F} \times \mathcal{G}) \cong T_{\xi_0}\mathcal{F} \oplus T_{\eta_0}\mathcal{G}$$

is just the direct sum of $\kappa_\xi : T_\Lambda R \to T_{\xi_0}\mathcal{F}$ and $\kappa_\eta : T_\Lambda S \to T_{\eta_0}\mathcal{G}$, and so it is an isomorphism.

As for the lifting property, suppose we have a diagram of formal objects of $\mathcal{F} \times \mathcal{G}$

$$\begin{tikzcd} & (A', (\rho', \nu')) \\ (R\widehat{\otimes} S, (\overline{\xi}, \overline{\eta})) & (A, (\rho, \nu)). \arrow[l] \arrow[u] \end{tikzcd}$$

By definition this diagram corresponds to two similar diagrams, one relative to $\mathcal{F}$ and one to $\mathcal{G}$.

$$\begin{tikzcd} & (A', \rho') \\ (R\widehat{\otimes} S, \overline{\xi}) & (A, \rho) \arrow[l] \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} & (A', \nu') \\ (R\widehat{\otimes} S, \overline{\eta}) & (A, \nu) \arrow[l] \arrow[u] \end{tikzcd}$$

If we compose the morphism $(A, \rho) \to (R\widehat{\otimes} S, \overline{\xi})$ of the first diagram with $(R\widehat{\otimes} S, \overline{\xi}) \to (R, \xi)$, by versality of (R, ξ) in $\mathcal{F}$ we see that there exists a lifting $(A', \rho') \to (R, \xi)$ of the composite $(A, \rho) \to (R\widehat{\otimes} S, \overline{\xi}) \to (R, \xi)$. This gives in particular a homomorphism of Λ-algebras $R \to A'$.

Repeating the argument for the second diagram, we get another homomorphism $S \to A'$, which, together with the previous one, gives a homomorphism $R\widehat{\otimes} S \to A'$. Moreover we easily see that the two arrows $\rho' \to \xi$ and $\nu' \to \eta$ give (by cartesianity of $\overline{\xi} \to \xi$ and $\overline{\eta} \to \eta$) two other arrows $\rho' \to \overline{\xi}$ and $\nu' \to \overline{\eta}$ over the constructed $R\widehat{\otimes} S \to A'$.

Finally, these two arrows in turn induce a morphism of formal objects

$$(A, (\rho', \nu')) \to (R\widehat{\otimes} S, (\overline{\xi}, \overline{\eta}))$$

of $\mathcal{F} \times \mathcal{G}$ that gives a lifting in the initial diagram, proving the versality of $(\overline{\xi}, \overline{\eta})$. □

Let us consider now the disjoint union $Y = Y_0' \amalg Y_0''$ of two copies of $Y_0 = V(xy) \subseteq \mathbb{A}^2_k$. Call $\{Y_n', f_n'\}_{n\in\mathbb{N}}$ and $\{Y_n'', f_n''\}_{n\in\mathbb{N}}$ the two miniversal deformations of Y_0' and Y_0'' induced by $V(xy - t) \subseteq \mathbb{A}^2_{\Lambda[[t]]}$ and $V(xy - u) \subseteq \mathbb{A}^2_{\Lambda[[u]]}$ respectively. The miniversal deformation of Y given by the proposition above is then $(\Lambda[[t, u]], \widehat{Y})$, where $\widehat{Y} = \{Y_n' \amalg Y_n'', f_n' \amalg f_n''\}_{n\in\mathbb{N}}$.

Going on by induction it is possible to find a similar miniversal deformation, for any disjoint union $Y = \coprod_i Y_0$ of a finite number of copies of Y_0.

6.3.2 Deformations of affine curves with a finite number of nodes

Let us now go back to the affine nodal curve X_0 with a finite number of nodes $p_1, \ldots, p_r \in X_0$. Starting from the last example and proceding exactly as in Section 6.2.3, using the diagram with étale morphisms

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X_0 & & Y \end{tikzcd}$$

constructed above, one can prove the following proposition (which is the analogue of the description we gave in the case with only one node).

Proposition 6.3.4. *There is a miniversal deformation $\widehat{X} = \{X_n, f_n\}_{n\in\mathbb{N}}$ of X_0 over $\Lambda[[t_1, \dots, t_r]]$, with compatible isomorphisms of Λ-algebras*

$$\widehat{\mathcal{O}}_{X_n,p_i} \cong \Lambda[[t_1, \dots, t_r]]_n[[x, y]]/(xy - t_i)$$

for every n and i.

The following result is a straightforward consequence of Proposition 6.2.11.

Proposition 6.3.5. *Let $f : X \to S$ be a flat morphism of finite type, and suppose that the fiber $X_0 = f^{-1}(s_0)$ over a point $s_0 \in S$ is a nodal curve over $k(s_0)$ with r nodes $p_1, \dots, p_r \in X_0$. Then there exist $u_1, \dots, u_r \in \Lambda$ and isomorphisms of $\mathcal{O}_{S,s_0}$-algebras*

$$\widehat{\mathcal{O}}_{X,p_i} \cong \widehat{\mathcal{O}}_{S,s_0}[[xy]]/(xy - u_i).$$

6.4 Projective curves with a finite number of nodes

We turn now to the case of projective nodal curves. After proving that their deformations are unobstructed, we will try to fall back to the case of affine curves, by taking an affine neighborhood of the nodes.

Let X_0 be a projective nodal curve over k, and call $p_1, \dots, p_r \in X_0$ its rational nodes

Proposition 6.4.1. *An X_0 as above is always unobstructed.*

Proof. Using thoerem 4.2.1, it suffices to show that $\mathrm{Ext}^2_{\mathcal{O}_{X_0}}(\Omega_{X_0}, \mathcal{O}_{X_0}) = 0$. In order to do this, we first show that $\Omega_{X_0,p}$ has projective dimension at most 1 over $\mathcal{O}_{X_0,p}$, for every point $p \in X_0$.

If p is a smooth point of X_0, then $\Omega_{X_0,p}$ is free of rank 1 over $= \mathcal{O}_{X_0,p}$, and then in particular it is projective. So take one of the nodes, $p = p_k \in X_0$, and a module M over $R = \mathcal{O}_{X_0,p}$. We want to show that $\mathrm{Ext}^i_R(\Omega_{X_0,p}, M)$ is trivial for any $i \geq 2$.

Since the complete local ring $\widehat{R} = \widehat{\mathcal{O}}_{X_0,p}$ is faithfully flat over R, we can instead consider

$$\mathrm{Ext}^i_R(\Omega_{X_0,p}, M) \otimes_R \widehat{R} \cong \mathrm{Ext}^i_{\widehat{R}}(\Omega_{X_0,p} \otimes_R \widehat{R}, M \otimes_R \widehat{R}).$$

As shown in Proposition B.17, the $\widehat{R}$-module $\Omega_{X_0,p} \otimes_R \widehat{R}$ is isomorphic to the module of continuous Kähler differentials $\widehat{\Omega}_{\widehat{R}}$; in our case since p is a rational node we have

$$\widehat{R} = \widehat{\mathcal{O}}_{X_0,p} \cong k[[x,y]]/(xy)$$

and then $\widehat{\Omega}_{\widehat{R}}$ is an $\widehat{R}$-module with two generators dx and dy, and the only relation $ydx + xdy = 0$.

This module has a projective resolution of the form

$$0 \longrightarrow \widehat{R} \xrightarrow{\alpha} \widehat{R} \oplus \widehat{R} \xrightarrow{\beta} \widehat{\Omega}_{\widehat{R}} \longrightarrow 0$$

where α is defined by $\alpha(1) = (y,x)$, and β by $\beta(f,g) = fdx + gdy$. The existence of this projective resolution implies that $\mathrm{Ext}^i_{\widehat{R}}(\widehat{\Omega}_{\widehat{R}}, M \otimes_R \widehat{R}) = 0$ for $i \geq 2$, which is what we wanted to show.

Moreover from the same resolution we get that $\mathrm{Ext}^1_R(\Omega_{X_0,p}, R) \otimes_R \widehat{R} \cong k$, and so also $\mathrm{Ext}^1_R(\Omega_{X_0,p}, R) \cong k$.

Now we use Grothendieck's spectral sequence for a composite of derived functors (see for example Section 5.8 of [Weib]): there is a spectral sequence $\{E_n^{p,q}\}_{n\in\mathbb{N}}$ such that

$$E_2^{p,q} = H^p(X_0, \mathcal{E}xt^q_{\mathcal{O}_{X_0}}(\Omega_{X_0}, \mathcal{O}_{X_0})) \Longrightarrow \mathrm{Ext}^{p+q}_{\mathcal{O}_{X_0}}(\Omega_{X_0}, \mathcal{O}_{X_0}).$$

Using this and the preceding discussion, and the fact that $\mathcal{E}xt^1_{\mathcal{O}_{X_0}}(\Omega_{X_0}, \mathcal{O}_{X_0})$ has support only on the nodes $p_1, \dots, p_r$, so that $H^p(X_0, \mathcal{E}xt^1_{\mathcal{O}_{X_0}}(\Omega_{X_0}, \mathcal{O}_{X_0}))$ is trivial if $p \geq 1$, we get that $E_2^{p,q} = 0$ if $p + q \geq 2$.

Consequently $\mathrm{Ext}^i_{\mathcal{O}_{X_0}}(\Omega_{X_0}, \mathcal{O}_{X_0})$ is trivial if $i \geq 2$, and in particular $\mathrm{Ext}^2_{\mathcal{O}_{X_0}}(\Omega_{X_0}, \mathcal{O}_{X_0}) = 0$, so X_0 is unobstructed. □

Remark 6.4.2. As by-product, the above proof gives that

$$\mathcal{E}xt^1_{\mathcal{O}_{X_0}}(\Omega_{X_0}, \mathcal{O}_{X_0}) \cong \bigoplus_i k_{p_i}$$

where k_{p_i} is the sheaf on X_0 with support in p_i, and the stalk in p_i is k.

Remark 6.4.3. We will not use this fact, but it is worthwile to notice that the tangent space $T_{X_0}\mathcal{D}ef \cong \mathrm{Ext}^1_{\mathcal{O}_{X_0}}(\Omega_{X_0}, \mathcal{O}_{X_0})$ is finite-dimensional over k (because X_0 is projective), so X_0 will have a miniversal deformation, defined over a power series ring over Λ (because of the last proposition).

Now take an open affine subscheme $U_0 \subseteq X_0$, containing all the nodes $p_1, \dots, p_r$. Since the open immersion $i : U_0 \to X_0$ is étale, we have an induced restriction functor $i^* : \mathcal{D}ef_{X_0} \to \mathcal{D}ef_{U_0}$, which in this case is a "true" restriction (in the sense that $i^*(X)$ is just the open subscheme of X with underlying topological space U_0). We recall also that there are canonical isomorphisms $T_{X_0}\mathcal{D}ef \cong \mathrm{Ext}^1_{\mathcal{O}_{X_0}}(\Omega_{X_0}, \mathcal{O}_{X_0})$ and $T_{U_0}\mathcal{D}ef \cong \mathrm{Ext}^1_{\mathcal{O}_{U_0}}(\Omega_{U_0}, \mathcal{O}_{U_0})$ (see Theorem 2.4.1).

Proposition 6.4.4. *In the situation above the homomorphism*

$$\mathrm{Ext}^1_{\mathcal{O}_{X_0}}(\Omega_{X_0}, \mathcal{O}_{X_0}) \to \mathrm{Ext}^1_{\mathcal{O}_{U_0}}(\Omega_{U_0}, \mathcal{O}_{U_0})$$

corresponding to the differential $d_{X_0} i^* : T_{X_0}\mathcal{D}ef \to T_{U_0}\mathcal{D}ef$ *is surjective.*

Proof. As in the proof of 6.2.9, the differential $d_{X_0} i^* : T_{X_0}\mathcal{D}ef \to T_{U_0}\mathcal{D}ef$ corresponds to the canonical homomorphism

$$\alpha : \mathrm{Ext}^1_{\mathcal{O}_{X_0}}(\Omega_{X_0}, \mathcal{O}_{X_0}) \to \mathrm{Ext}^1_{\mathcal{O}_{U_0}}(\Omega_{U_0}, \mathcal{O}_{U_0})$$

that carries an (isomorphism class of an) extension of $\mathcal{O}_{X_0}$-modules

$$0 \longrightarrow \mathcal{O}_{X_0} \longrightarrow \mathcal{E} \longrightarrow \Omega_{X_0} \longrightarrow 0$$

to the (isomorphism class of the) one obtained by tensoring with $\mathcal{O}_{U_0}$

$$0 \longrightarrow \mathcal{O}_{U_0} \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_{X_0}} \mathcal{O}_{U_0} \longrightarrow \Omega_{U_0} \longrightarrow 0$$

(where $\Omega_{X_0} \otimes_{\mathcal{O}_{X_0}} \mathcal{O}_{U_0} \cong \Omega_{U_0}$ because the open immersion $i : U_0 \to X_0$ is étale).

We have a commutative diagram

$$\begin{tikzcd}
\mathrm{Ext}^1_{\mathcal{O}_{X_0}}(\Omega_{X_0}, \mathcal{O}_{X_0}) \arrow[r, "\alpha"] \arrow[d] & \mathrm{Ext}^1_{\mathcal{O}_{U_0}}(\Omega_{U_0}, \mathcal{O}_{U_0}) \arrow[d, "\wr"] \\
H^0(X_0, \mathcal{E}xt^1_{\mathcal{O}_{X_0}}(\Omega_{X_0}, \mathcal{O}_{X_0})) \arrow[r, "\sim"] & H^0(U_0, \mathcal{E}xt^1_{\mathcal{O}_{U_0}}(\Omega_{U_0}, \mathcal{O}_{U_0}))
\end{tikzcd}$$

where the left vertical map is surjective (as one sees using again the spectral sequence of the proof of 6.4.1), the right vertical one is an isomorphism (because U_0 is affine), and the bottom horizontal one is also an isomorphism (because $\mathcal{E}xt^1_{\mathcal{O}_{X_0}}(\Omega_{X_0}, \mathcal{O}_{X_0})$ has support contained in U_0).

From the diagram we see then that the homomorphism α is surjective, as we wanted to show. □

From the fact that the differential of i^* is surjective and that X_0 is unobstructed, we deduce the following proposition.

Proposition 6.4.5. *Every formal deformation of* U_0 *is (isomorphic to) the restriction of one of* X_0.

Proof. The proof is very similar to the one of Proposition 2.4.13. Suppose we have a formal deformation $\widehat{U} = \{U_n, f_n\}_{n\in\mathbb{N}}$ of U_0 over $R \in (\mathrm{Comp}\,/\Lambda)$, and consider the small extension

$$0 \longrightarrow \mathfrak{m}_R/\mathfrak{m}_R^2 \longrightarrow R_1 \longrightarrow R_0 \cong k \longrightarrow 0.$$

Since X_0 is unobstructed, we can find $X_1' \in \mathcal{D}ef_{X_0}(R_1)$ that is a lifting of X_0. Now the two objects $U_1, i^*(X_1') \in \mathcal{D}ef_{U_0}(R_1)$ are both liftings of U_0 to R_1, and by Theorem 2.3.1 we can find an element $g \in \mathfrak{m}_R/\mathfrak{m}_R^2 \otimes_k T_{U_0}\mathcal{D}ef$ such that $[i^*(X_1')] \cdot g = [U_1]$.

By surjectivity of

$$\mathrm{id} \otimes_k d_{X_0} i^* : \mathfrak{m}_R/\mathfrak{m}_R^2 \otimes_k T_{X_0}\mathcal{D}ef \to \mathfrak{m}_R/\mathfrak{m}_R^2 \otimes_k T_{U_0}\mathcal{D}ef$$

we have an $h \in \mathfrak{m}_R/\mathfrak{m}_R^2 \otimes_k T_{X_0}\mathcal{D}ef$ such that $(\mathrm{id} \otimes_k d_{X_0} i^*)(h) = g$, and then by functoriality of the action with respect to the deformation category (Proposition 2.3.6) we get

$$i^*([X_1'] \cdot h) = [i^*(X_1')] \cdot (\mathrm{id} \otimes_k d_{X_0} i^*)(h) = [i^*(X_1')] \cdot g = [U_0].$$

If we take a representative X_1 for $[X_1'] \cdot h$, then $i^*(X_1)$ and U_1 are isomorphic liftings of U_0.

Repeating this argument inductively we find a formal deformation $\widehat{X} = \{X_n, g_n\}_{n \in \mathbb{N}}$ of X_0 over R such that $\widehat{i^*}(\widehat{X})$ is isomorphic to $\widehat{U}$ (where $\widehat{i^*} : \widehat{\mathcal{D}ef_{X_0}} \to \widehat{\mathcal{D}ef_{U_0}}$ is the induced morphism), as we wanted. □

Using the last proposition, and the results of Section 5.4, we can easily prove an algebraization result for deformations of projective nodal curves.

Proposition 6.4.6. *Let X_0 be a projective nodal curve over k, with rational nodes $p_1, \dots, p_r \in X_0$, and $u_1, \dots, u_r \in \Lambda$ be arbitrary elements. Then there exists a flat and projective scheme X over Λ, having closed fiber isomorphic to X_0, and such that*

$$\widehat{\mathcal{O}}_{X,p_i} \cong \Lambda[[x,y]]/(xy - u_i).$$

for every node $p_i \in X_0$.

Proof. Let $\Lambda[[t_1, \dots, t_r]]$ be the base ring of the miniversal deformation of U_0 of Proposition 6.3.4, and consider the homomorphism of Λ-algebras $\Lambda[[t_1, \dots, t_r]] \to \Lambda$ defined by $t_i \mapsto u_i$. This induces by pullback a formal deformation $\widehat{U} = \{U_n, f_n\}_{n \in \mathbb{N}}$ of U_0 over Λ.

By Proposition 6.4.5 we can find a formal deformation $\widehat{X} = \{X_n, g_n\}_{n \in \mathbb{N}}$ such that the restriction $\widehat{i^*}(\widehat{X})$ is isomorphic to $\widehat{U}$. Since X_0 is projective and $H^2(X_0, \mathcal{O}_{X_0}) = 0$ (since X_0 is a curve), by Theorem 5.4.6 the formal deformation $\widehat{X}$ is algebraizable, that is, we can find a flat and projective scheme $X \to \mathrm{Spec}(\Lambda)$ inducing $\widehat{X}$.

In particular X has closed fiber isomorphic to X_0, and since it restricts to a formal deformation isomorphic to $\widehat{U}$ constructed above, by Proposition 6.3.5 we have

$$\widehat{\mathcal{O}}_{X,p_i} \cong \Lambda[[x,y]]/(xy - u_i).$$

for every node p_i. □

In particular we deduce the following corollary, that shows that if Λ is one-dimensional, we can always deform X_0 in a smooth way.

Corollary 6.4.7. *Let X_0 be a projective nodal curve as in the preceding proposition, and suppose that $\dim(\Lambda) = 1$ (for example $\Lambda = k[[t]]$). Then there exists a flat and projective morphism $X \xrightarrow{\pi} \mathrm{Spec}(\Lambda)$ such that the closed fiber is isomorphic to X_0, and $X \setminus X_0 \to \mathrm{Spec}(\Lambda) \setminus \{\mathfrak{m}_\Lambda\}$ is smooth.*

Proof. Let $u \in \mathfrak{m}_\Lambda$ be a system of parameters for Λ, and take the deformation $X \to \mathrm{Spec}(\Lambda)$ of X_0 given by Proposition 6.4.6, with $u_i = u$ for every i.

Let U be the open subset of X on which the coherent sheaf $\Omega_{X/\Lambda}$ is locally free of rank 1 (or equivalently where X is smooth over Λ). We want to show that $U = X \setminus \{p_1, \dots, p_r\}$.

Consider an irreducible component V of $X \setminus U$, with generic point $p \in V \subseteq X$. Since V is closed in X and $X \to \mathrm{Spec}(\Lambda)$ is proper, we must have $V \cap X_0 \neq \emptyset$ (because the image of $X \to \mathrm{Spec}(\Lambda)$ contains the maximal ideal $\mathfrak{m}_\Lambda$). Since X_0 is smooth outside $p_1, \dots, p_r$, there exists an i such that $p_i \in V \cap X_0$; we will show that $V = \{p_i\}$, and this will conclude the proof.

We consider the complete local ring $R = \widehat{\mathcal{O}}_{X,p_i} \cong \Lambda[[x,y]]/(xy-u)$, and the module of continuous Kähler differentials $\widehat{\Omega}_{R/\Lambda} \cong$ (see appendix B), which is an R-module with two generators dx, dy, and the relation $ydx + xdy = 0$. This can also be seen as the cokernel of the homomorphism $R \to R \oplus R$ given by multiplication by the vector (y, x).

If $p \in \mathrm{Spec}(R)$ does not contain the ideal (x,y), then one of x, y is invertible in p, and then $\widehat{\Omega}_{R/\Lambda}$ is locally free of rank 1 over R_p. Since the radical of (x,y) is $\mathfrak{m}_R$, we conclude that $\widehat{\Omega}_{R/\Lambda}|_{\mathrm{Spec}(R)\setminus\{\mathfrak{m}_R\}}$ is locally free of rank 1.

Now the natural morphism $\mathcal{O}_{X,p_i} \to \widehat{\mathcal{O}}_{X,p_i} = R$ is faithfully flat, and then $\mathrm{Spec}(R) \to \mathrm{Spec}(\mathcal{O}_{X,p_i})$ is flat and surjective. Moreover the inverse image of the closed point $\mathfrak{m}_{p_i}$ is $\{\mathfrak{m}_R\}$, and so we can restrict the morphism above to

$$\mathrm{Spec}(R) \setminus \{\mathfrak{m}_R\} \to \mathrm{Spec}(\mathcal{O}_{X,p_i}) \setminus \{\mathfrak{m}_{p_i}\}$$

that is flat and surjective too. From this, and the fact that the pullback of $\Omega_{X/\Lambda}$ to $\mathrm{Spec}(R) \setminus \{\mathfrak{m}_R\}$ is locally free of rank 1 (see Proposition B.17), we get that its pullback to $\mathrm{Spec}(\mathcal{O}_{X,p_i}) \setminus \{\mathfrak{m}_{p_i}\}$ along the morphism

$$\mathrm{Spec}(\mathcal{O}_{X,p_i}) \setminus \{\mathfrak{m}_{p_i}\} \to X$$

is also locally free of rank 1.

Finally, the generic point p of V is in the image of the morphism above (since this image it is the set of the generic points of irreducible components of X containing p_i), but the stalk $\Omega_{X/\Lambda,p}$ is not free of rank 1 by hypothesis. From this we get that the maximal ideal $\mathfrak{m}_{p_i}$ goes to p, or in other words $p = p_i$, and $V = \{p_i\}$ (since p_i is closed), as we claimed. $\square$

Appendix **A**

Linear functors

In this appendix we give some results about functors from categories of modules (or vector spaces) that preserve finite products. Throughout this appendix A will be a noetherian ring (commutative and with identity, as usual).

Let $F : (\mathrm{FMod}\,/A) \to (\mathrm{Set})$ be a functor. If $M, N \in (\mathrm{FMod}\,/A)$, the two projections $M \oplus N \to M$ and $M \oplus N \to N$ induce two functions $F(M \oplus N) \to F(M)$ and $F(M \oplus N) \to F(N)$, and in turn these induce $\varphi_{M,N} : F(M \oplus N) \to F(M) \times F(N)$.

Definition A.1. *A functor $F : (\mathrm{FMod}\,/A) \to (\mathrm{Set})$ is said to* ***preserve finite products*** *if the function $\varphi_{M,N}$ is bijective for every $M, N \in (\mathrm{FMod}\,/A)$, and $F(0) \neq \emptyset$.*

Definition A.2. *A functor $F : (\mathrm{FMod}\,/A) \to (\mathrm{Mod}\,/A)$ is said to be* ***A-linear*** *if for every $M, N \in (\mathrm{FMod}\,/A)$ the function*

$$\mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(F(M), F(N))$$

is a homomorphism of A-modules.

It is easy to see that if $F : (\mathrm{FMod}\,/A) \to (\mathrm{Mod}\,/A)$ is A-linear, then the induced functor $(\mathrm{FMod}\,/A) \to (\mathrm{Set})$ preserves finite products, and the bijection $\varphi_{M,N} : F(M \oplus N) \to F(M) \times F(N)$ is actually an isomorphism of A-modules (with the product structure on the target).

The following proposition shows that if a functor $F : (\mathrm{FMod}\,/A) \to (\mathrm{Set})$ preserves finite products, then each $F(M)$ has a canonical structure of A-module.

Proposition A.3. *Let $F : (\mathrm{FMod}\,/A) \to (\mathrm{Set})$ be a functor that preserves finite products. Then there exists a unique A-linear lifting $\widetilde{F} : (\mathrm{FMod}\,/A) \to (\mathrm{Mod}\,/A)$ of F, that is, an A-linear functor such that its composite with the forgetful functor $(\mathrm{Mod}\,/A) \to (\mathrm{Set})$ is F.*

With "unique" above we mean really unique, not only up to isomorphism.

Proof. We define a structure of A-module on every $F(M)$, and call $\widetilde{F}(M)$ the set $F(M)$ with this A-module structure. First, notice that $F(0)$ (where 0 is the zero A-module) has exactly one element. In fact the two projections $0 \oplus 0 \to 0$ are the same function, and then the same is true for the two induced functions $F(0 \oplus 0) \to F(0)$. But now we know that $F(0 \oplus 0) \cong F(0) \times F(0)$, and for the projections on the two factors to be the same function, we must have that $F(0)$ has at most one element. Finally, it has at least one, since $F(0) \neq \emptyset$ by hypothesis.

Now fix $M \in (\mathrm{FMod}\,/A)$, and notice that we have a unique homomorphism $0 \to M$. We define the image of the induced $F(0) \to F(M)$ to be the zero vector of $F(M)$.

Next, we define scalar multiplication: if $a \in A$, we have a homomorphism $\mu_A : M \to M$ given by scalar multiplication by a. We define then scalar multiplication by a in $F(M)$ to be the induced function $F(\mu_A) : F(M) \to F(M)$.

Finally, we define addition. Consider the "sum" homomorphism $+ : M \oplus M \to M$ defined by $(m, n) \mapsto m + n$; this induces a function $F(M \oplus M) \to F(M)$, which, using the bijection $\varphi_{M,M} : F(M \oplus M) \cong F(M) \times F(M)$, gives a function $F(M) \times F(M) \to F(M)$. We define the sum in $F(M)$ by means of the last function.

We should verify that the data defined give a structure of A-module on $F(M)$, and that if $M \to N$ is a homomorphism, then the induced $F(M) \to F(N)$ is a homomorphism too. The method to verify the various axioms (and also the last fact about homomorphisms) is the same: one rewrites everything using of commutative diagrams, and then uses the appropriate functorialities to conclude.

As an example, we verify only associativity of $+$ on $F(M)$. Instead of saying that $v + (w + z) = (v + w) + z$ for every $v, w, z \in F(M)$, it can be restated by saying that the diagram

$$\begin{array}{ccc} F(M) \times F(M) \times F(M) & \xrightarrow{+\times \mathrm{id}} & F(M) \times F(M) \\ {\scriptstyle \mathrm{id}\times +}\downarrow & & \downarrow{\scriptstyle +} \\ F(M) \times F(M) & \xrightarrow[+]{} & F(M) \end{array} \tag{A.1}$$

is commutative.

To prove this it suffices to consider the corresponding diagram for M

$$\begin{array}{ccc} M \oplus M \oplus M & \xrightarrow{+\oplus \mathrm{id}} & M \oplus M \\ {\scriptstyle \mathrm{id}\oplus +}\downarrow & & \downarrow{\scriptstyle +} \\ M \oplus M & \xrightarrow[+]{} & M \end{array}$$

(which is clearly commutative) and apply F. We get

$$\begin{tikzcd} F(M \oplus M \oplus M) \arrow[r, "F(+\oplus \mathrm{id})"] \arrow[d, "F(\mathrm{id}\oplus +)"'] & F(M \oplus M) \arrow[d, "F(+)"] \\ F(M \oplus M) \arrow[r, "F(+)"'] & F(M) \end{tikzcd} \tag{A.2}$$

and after noticing that the bijections $F(M \oplus M \oplus M) \cong F(M) \times F(M) \times F(M)$ and $F(M \oplus M) \cong F(M) \times F(M)$ are compatible with the homomorphisms in diagrams A.1 and A.2 (basically by the definition of addition in $F(M)$), we get that A.1 is commutative.

It is easy to see that the defined structure of A-module on each $F(M)$ is uniquely determined if we want an A-linear functor. For example, since addition on $M \in (\mathrm{FMod}/A)$ is the unique function $M \oplus M \xrightarrow{+} M$ such that the composites $M \to M \oplus M \to M$ with the two inclusions are the identities, applying F we see that the function

$$F(M) \oplus F(M) \cong F(M \oplus M) \xrightarrow{F(+)} F(M)$$

satisfies the same property with respect to the inclusions $F(M) \to F(M) \oplus F(M)$, and so coincides necessarily with the addition of $F(M)$.

Finally, let us check that the lifting $\widetilde{F} : (\mathrm{FMod}/A) \to (\mathrm{Mod}/A)$ defined above is A-linear. We have to prove that if $M, N \in (\mathrm{FMod}/A)$ the induced function

$$\Phi : \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(F(M), F(N)).$$

defined by $f \mapsto F(f)$ is a homomorphism of A-modules. We prove only additivity, since linearity is similar.

Call $\Delta : M \to M \oplus M$ the diagonal homomorphism defined by $\Delta(m) = (m, m)$, then $F(\Delta) : F(M) \to F(M \oplus M)$ corresponds to the diagonal function $\Delta' : F(M) \to F(M) \times F(M)$ if we use the bijection $F(M \oplus M) \cong F(M) \times F(M)$. The additivity of Φ (that is, $\Phi(f+g) = \Phi(f) + \Phi(g)$ for every $f, g \in \mathrm{Hom}_A(M, N)$) amounts to the commutativity of the diagram

$$\begin{tikzcd} F(M) \times F(M) \arrow[r, "\Phi(f)\times\Phi(g)"] & F(N) \times F(N) \arrow[d, "+"] \\ F(M) \arrow[u, "\Delta'"] \arrow[r, "\Phi(f+g)"'] & F(N). \end{tikzcd} \tag{A.3}$$

But by definition of the sum homomorphism $f + g$, we have a commutative diagram

$$\begin{tikzcd} M \oplus M \arrow[r, "f\oplus g"] & N \oplus N \arrow[d, "+"] \\ M \arrow[u, "\Delta"] \arrow[r, "f+g"'] & N \end{tikzcd}$$

from which, applying F and using the bijections $F(M \oplus M) \cong F(M) \times F(M)$ and $F(N \oplus N) \cong F(N) \times F(N)$, we see that A.3 is commutative too. □

Now we turn to natural transformations.

Definition A.4. *Let $F, G : (\mathrm{FMod}\,/A) \to (\mathrm{Mod}\,/A)$ be two functors. A natural transformation $\alpha : F \to G$ is said to be A-linear if for every $M \in (\mathrm{FMod}\,/A)$ the function $\alpha_M : F(M) \to G(M)$ is A-linear.*

The following proposition is useful when one has to prove that some bijections are isomorphisms of modules.

Proposition A.5. *Let $F, G : (\mathrm{FMod}\,/A) \to (\mathrm{Set})$ be two functors that preserve finite products, $\widetilde{F}, \widetilde{G} : (\mathrm{FMod}\,/A) \to (\mathrm{Mod}\,/A)$ the two A-linear liftings coming from the preceding proposition, and $\alpha : F \to G$ a natural transformation. Then for every $M \in (\mathrm{FMod}\,/A)$ the function $\alpha_M : \widetilde{F}(A) \to \widetilde{G}(A)$ is A-linear, and so α induces an A-linear natural transformation $\widetilde{\alpha} : \widetilde{F} \to \widetilde{G}$.*

Proof. Fix $M \in (\mathrm{FMod}\,/A)$. We start with additivity; it amounts to showing that the diagram

$$\begin{tikzcd} F(M) \times F(M) \arrow[r, "\alpha_M \times \alpha_M"] \arrow[d, "+"'] & G(M) \times G(M) \arrow[d, "+"] \\ F(M) \arrow[r, "\alpha_M"'] & G(M) \end{tikzcd} \tag{A.4}$$

commutes.

We consider the sum homomorphism $+ : M \oplus M \to M$. By naturality of α we have then a commutative diagram

$$\begin{tikzcd} F(M \oplus M) \arrow[r, "\alpha_{M \oplus M}"] \arrow[d, "F(+)"'] & G(M \oplus M) \arrow[d, "G(+)"] \\ F(M) \arrow[r, "\alpha_M"'] & G(M) \end{tikzcd}$$

and using once again the bijections $F(M \oplus M) \cong F(M) \times F(M)$ and $G(M \oplus M) \cong G(M) \times G(M)$, and the fact that the function $F(M) \times F(M) \to G(M) \times G(M)$ corresponding to $\alpha_{M \oplus M}$ is $\alpha_M \times \alpha_M$, we get the commutativity of A.4.

Linearity is simple: if $a \in A$ we consider the homomorphism $\mu_a : M \to M$ given by multiplication by a. By naturality of α the diagram

$$\begin{tikzcd} F(M) \arrow[r, "\alpha_M"] \arrow[d, "F(\mu_a)"'] & G(M) \arrow[d, "G(\mu_a)"] \\ F(M) \arrow[r, "\alpha_M"] & G(M) \end{tikzcd}$$

is commutative, and this says exactly that α_M is A-linear. □

Finally, we see that if $A = k$ is a field, then k-linear functors are particularly simple to describe.

Proposition A.6. *Let $G : (\mathrm{FVect}/k) \to (\mathrm{Vect}/k)$ be a k-linear functor. Then for every $V \in (\mathrm{FVect}/k)$ there is a functorial isomorphism*

$$G(V) \cong V \otimes_k G(k).$$

In particular G is an exact functor (carries exact sequences to exact sequences), since the functor $- \otimes_k G(k)$ is.

Proof. We define a natural transformation $\tau : - \otimes_k G(k) \to G$ as follows: for $V \in (\mathrm{FVect}/k)$ we define $\tau(V) : V \otimes_k F(k) \to F(V)$ by

$$\tau(V)(v \otimes \alpha) = F(\varphi_v)(\alpha)$$

where $\varphi_v : k \to V$ is the k-linear function sending 1 to v. It is readily checked that τ is a natural transformation.

We check that each $\tau(V)$ is an isomorphism. First of all if $V = k$, then $\tau_k : k \otimes_k F(k) \to F(k)$ is easily seen to be just the canonical isomorphism defined by $a \otimes \alpha \mapsto a \cdot \alpha$.

If $V = k^n$, then we have a commutative diagram

$$\begin{tikzcd} k^n \otimes_k F(k) \arrow[r, "\tau_{k^n}"] \arrow[d] & F(k^n) \arrow[d] \\ (k \otimes_k F(k))^n \arrow[r, "(\tau_k)^n"] & F(k)^n \end{tikzcd}$$

where the left vertical arrow is the canonical isomorphism, the right vertical one is the isomorphism given by k-linearity of F to $k^n \cong k \oplus \cdots \oplus k$ (applied $n-1$ times), and the bottom one is an isomorphism because τ_k is. It follows that τ_{k^n} is an isomorphism too.

Finally, for a general $V \in (\mathrm{FVect}/k)$, we take an isomorphism $V \cong k^n$ where n is the dimension of V, and reduce this case to the preceding one. □

Appendix B

Noetherian complete local rings

In this appendix we gather some definitions and facts about notherian complete local algebras over Λ (which is as usual a noetherian complete local ring) with residue field k, that are applied in Chapters 5 and 6. We denote the category of such rings by $(\mathrm{Comp}\,/\Lambda)$, where as usual we consider only local homomorphisms (which are also precisely the continuous ones with respect to the natural topologies).

Vertical tangent and cotangent spaces

First of all we set up some notation. Let $R \in (\mathrm{Comp}\,/\Lambda)$, and $\mathfrak{m}_R \subseteq R$ be the maximal ideal as usual; there is another important ideal of R, the extension $\mathfrak{m}_\Lambda R \subseteq \mathfrak{m}_R \subseteq R$ of the maximal ideal of Λ. In this situation, we denote by R_n the quotient $R/\mathfrak{m}_R^{n+1}$, which is an object of $(\mathrm{Art}\,/\Lambda)$, and by $\overline{R}$ the quotient $R/\mathfrak{m}_\Lambda R$, an object of $(\mathrm{Comp}\,/k)$. If $\varphi : R \to S$ is a homomorphism in $(\mathrm{Comp}\,/\Lambda)$, we will denote by $\varphi_n : R_n \to S_n$ and $\overline{\varphi} : \overline{R} \to \overline{S}$ the induced ones.

So $\overline{R}_n \in (\mathrm{Art}\,/k)$ will be the quotient $\overline{R}/\mathfrak{m}_{\overline{R}}^{n+1} \cong R_n/\mathfrak{m}_\Lambda R_n \cong R_n \otimes_\Lambda k$, and in particular we have

$$\overline{R}_1 \cong k \oplus \mathfrak{m}_{\overline{R}_1} \cong k \oplus \mathfrak{m}_{\overline{R}}/\mathfrak{m}_{\overline{R}}^2$$

because $\mathfrak{m}_{\overline{R}_1}^2 = (0)$.

Definition B.1. *The **vertical cotangent space** of R is the finite-dimensional k-vector space*

$$T_\Lambda^\vee R = \mathfrak{m}_R/(\mathfrak{m}_\Lambda R + \mathfrak{m}_R^2).$$

Its dual

$$T_\Lambda R = (\mathfrak{m}_R/(\mathfrak{m}_\Lambda R + \mathfrak{m}_R^2))^\vee$$

*is called the **vertical tangent space** of R.*

Remark B.2. The name "vertical tangent space" comes from the fact that $T_\Lambda R$ is the tangent space of the fiber of the morphism $\mathrm{Spec}(R) \to \mathrm{Spec}(\Lambda)$ over the maximal ideal (which is just $\mathrm{Spec}(\overline{R})$), at the only closed point. In fact one easily checks that there is a canonical isomorphism

$$T_\Lambda R = (\mathfrak{m}_R/(\mathfrak{m}_\Lambda R + \mathfrak{m}_R^2))^\vee \cong (\mathfrak{m}_{\overline{R}}/\mathfrak{m}_{\overline{R}}^2)^\vee$$

As one expects, there is a related notion of differential of a homomorphism $\varphi : R \to S$ in (Comp/Λ). This comes from the fact that $\varphi(\mathfrak{m}_R) \subseteq \mathfrak{m}_S$ and $\varphi(\mathfrak{m}_\Lambda R + \mathfrak{m}_R^2) \subseteq \mathfrak{m}_\Lambda S + \mathfrak{m}_S^2$, so φ induces a k-linear map

$$\varphi_* : \mathfrak{m}_R/(\mathfrak{m}_\Lambda R + \mathfrak{m}_R^2) \to \mathfrak{m}_S/(\mathfrak{m}_\Lambda S + \mathfrak{m}_S^2)$$

between the cotangent spaces, that we call the **codifferential** of φ.

Dualizing, we get another k-linear map

$$d\varphi : (\mathfrak{m}_S/(\mathfrak{m}_\Lambda S + \mathfrak{m}_S^2))^\vee \to (\mathfrak{m}_R/(\mathfrak{m}_\Lambda R + \mathfrak{m}_R^2))^\vee$$

that we call the differential of φ, and is just the differential of the morphism induced by φ between the closed fibers $\mathrm{Spec}(\overline{S}) \to \mathrm{Spec}(\overline{R})$.

These constructions are clearly functorial, in the sense that differential and codifferential of a composite coincides with the composites of the differentials and codifferentials respectively.

We have the following important proposition.

Proposition B.3. *Let $R, S \in (\mathrm{Comp}/\Lambda)$, and $\varphi : R \to S$ be a homomorphism. If the codifferential $\varphi_* : T_\Lambda^\vee R \to T_\Lambda^\vee S$ is surjective, than φ itself is surjective.*

Proof. Let us consider first the homomorphisms of k-algebras $\overline{\varphi}_n : \overline{R}_n \to \overline{S}_n$ induced by φ. We show inductively that $\overline{\varphi}_n$ is surjective for every n.

To do this, notice that surjectivity of the codifferential $\varphi_* : \mathfrak{m}_{\overline{R}}/\mathfrak{m}_{\overline{R}}^2 \to \mathfrak{m}_{\overline{S}}/\mathfrak{m}_{\overline{S}}^2$ implies that of the map

$$f_n : \mathfrak{m}_{\overline{R}}^n/\mathfrak{m}_{\overline{R}}^{n+1} \to \mathfrak{m}_{\overline{S}}^n/\mathfrak{m}_{\overline{S}}^{n+1}$$

induced by φ_n, for any n (as is easily checked). Now we come to $\overline{\varphi}_n$: if $n = 1$, we have that $\overline{\varphi}_1 : k \oplus T_\Lambda^\vee R \to k \oplus T_\Lambda^\vee S$ is surjective because the codifferential φ_* is, by hypothesis. Suppose that we know that $\overline{\varphi}_{n-1}$ is surjective; we have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{m}_{\overline{R}}^n/\mathfrak{m}_{\overline{R}}^{n+1} \arrow[r] \arrow[d, "f_n"] & \overline{R}_n \arrow[r] \arrow[d, "\overline{\varphi}_n"] & \overline{R}_{n-1} \arrow[r] \arrow[d, "\overline{\varphi}_{n-1}"] & 0 \\
0 \arrow[r] & \mathfrak{m}_{\overline{S}}^n/\mathfrak{m}_{\overline{S}}^{n+1} \arrow[r] & \overline{S}_n \arrow[r] & \overline{S}_{n-1} \arrow[r] & 0
\end{tikzcd}$$

and by diagram chasing the surjectivity of $\overline{\varphi}_{n-1}$ and f_n implies that of $\overline{\varphi}_n$.

Now consider $\varphi_n : R_n \to S_n$; we show that all these homomorphisms are surjective as well. Notice that R_n and S_n are finite as Λ-modules, because they have a finite filtration (given by the powers of the maximal ideal), such that successive quotients are finite-dimensional k-vector spaces.

Recall also that $\overline{R}_n \cong R_n \otimes_\Lambda k$ and $\overline{S}_n \cong S_n \otimes_\Lambda k$, and $\overline{\varphi}_n$ is the homomorphism induced by φ_n. Since $\overline{\varphi}_n$ is surjective, we can apply Nakayama's Lemma and deduce that φ_n is surjective too.

Finally, we pass to $\varphi : R \to S$, which is the projective limit of the homomorphisms φ_n. If we set $K_n = \ker(R_n \to S_n)$, we have for every n an exact sequence

$$0 \longrightarrow K_n \longrightarrow R_n \longrightarrow S_n \longrightarrow 0$$

that together give an exact sequence of projective systems. Since in our case R_n is artinian (and so K_n is as well), the Mittag-Leffler condition (for every n the image of $K_{n+k} \to K_n$ is the same for all k's large enough) is certainly satisfied, and then the induced homomorphism

$$\varprojlim \varphi_n = \varphi : \varprojlim R_n \cong R \longrightarrow \varprojlim S_n \cong S$$

is surjective. □

From the last proposition we get the following corollary.

Corollary B.4. *Let $R, S \in (\mathrm{Comp}\,/\Lambda)$, and $\varphi : R \to S$ be a homomorphism such that the codifferential $\varphi_* : T_\Lambda^\vee R \to T_\Lambda^\vee S$ is surjective. Then:*

(i) *If $\ell(R_n) = \ell(S_n)$ for all n (where $\ell(-)$ denotes the length as a Λ-module), then φ is an isomorphism.*

(ii) *If there exists a homomorphism $\psi : S \to R$ such that the codifferential $\psi_* : T_\Lambda^\vee S \to T_\Lambda^\vee R$ is surjective, then φ is an isomorphism.*

(iii) *If R and S are isomorphic, then φ is an isomorphism.*

Proof. The first assertion follows from the fact that $\ell(R_n) = \ell(S_n)$ implies $\ell(K_n) = 0$ (with the notation of the preceding proof), and consequently that each $\varphi_n : R_n \to S_n$ is an isomorphism. In conclusion $\varphi = \varprojlim \varphi_n$ is an isomorphism as well.

For the second statement, if $\psi : S \to R$ is a homomorphism with surjective codifferential $\psi_* : T_\Lambda^\vee S \to T_\Lambda^\vee R$, by the proof of the preceding proposition we deduce that $\psi_n : S_n \to R_n$ is surjective for every n, and this, together with the fact that $\varphi_n : R_n \to S_n$ is surjective as well, implies that $\ell(R_n) = \ell(S_n)$, so we can apply the first part of the corollary.

This last argument clearly proves **(iii)** as well (because if $\psi : S \to R$ is an isomorphism, then in particular the codifferential will be surjective), and this concludes the proof. □

Remark B.5. Notice that it is not sufficient to have a surjective map $T^\vee_\Lambda S \to T^\vee_\Lambda R$ to conclude that φ above is an isomorphism, but we must have a homomorphism $S \to R$ with surjective codifferential.

In particular the fact that φ_* is an isomorphism does not imply that φ itself is.

Power series rings

Now we turn to power series rings over Λ. For any n, the power series ring on n indeterminates $R = \Lambda[[x_1, \ldots, x_n]]$ is an object of (Comp/Λ). Since the ideal $\mathfrak{m}_\Lambda R \subseteq R$ coincides with the kernel of the natural homomorphism $\Lambda[[x_1, \ldots, x_n]] \to k[[x_1, \ldots, x_n]]$ (as one easily checks, using noetherianity of Λ), we get that $\overline{R} \cong k[[x_1, \ldots, x_n]]$. In particular

$$T^\vee_\Lambda R \cong \mathfrak{m}_{k[[x_1,\ldots,x_n]]}/\mathfrak{m}^2_{k[[x_1,\ldots,x_n]]}$$

is a k-vector space of dimension n, with basis $[x_1], \ldots, [x_n]$.

The next proposition shows that power series rings have properties similar to those of polynomial rings, with respect to complete algebras.

Proposition B.6. *Let $R \in (\mathrm{Comp}/\Lambda)$, and $a_1, \ldots, a_n \in \mathfrak{m}_R$. Then there exists a unique homomorphism $\varphi : \Lambda[[x_1, \ldots, x_n]] \to R$ such that $\varphi(x_i) = a_i$.*

Proof. By the properties of polynomial rings, for every k we have a unique homomorphism

$$\varphi_k : \Lambda[[x_1, \ldots, x_n]]_k \cong \Lambda[x_1, \ldots, x_n]/\mathfrak{m}^{k+1}_{\Lambda[x_1,\ldots,x_n]} \longrightarrow R_k$$

sending $[x_i]$ to $[a_i]$. By completeness we get then a homomorphism

$$\varphi = \varprojlim \varphi_k : \varprojlim \Lambda[[x_1, \ldots, x_n]]_k \cong \Lambda[[x_1, \ldots, x_n]] \longrightarrow \varprojlim R_k \cong R$$

such that $\varphi(x_i) = a_i$.

Moreover if $\psi : \Lambda[[x_1, \ldots, x_n]] \to R$ is a homomorphism with this property, then for every k the induced homomorphism

$$\psi_k : \Lambda[x_1, \ldots, x_n]/\mathfrak{m}^{k+1}_{\Lambda[x_1,\ldots,x_n]} \to R_k$$

sends $[x_i]$ to $[a_i]$, and so coincides with φ_k above. This implies $\psi = \varprojlim \psi_k = \varprojlim \varphi_k = \varphi$ and concludes the proof. $\square$

The following is an immediate consequence of Proposition B.6 and part **(ii)** of Corollary B.4.

Corollary B.7. *Let $R \in (\mathrm{Comp}/\Lambda)$, and assume we have a homomorphism $\varphi : R \to \Lambda[[x_1, \ldots, x_n]]$ such that the codifferential $\varphi_* : T^\vee_\Lambda R \to T^\vee_\Lambda \Lambda[[x_1, \ldots, x_n]]$ is an isomorphism. Then φ is an isomorphism.*

Proof. Let us take elements $a_1, \dots, a_n \in \mathfrak{m}_R$ such that $\varphi_*([a_i]) = [x_i]$, and $[a_1], \dots, [a_n]$ form a basis of $T^\vee_\Lambda R$. By Proposition B.6 we can find then a homomorphism $\psi : \Lambda[[x_1, \dots, x_n]] \to R$ such that $\psi(x_i) = a_i$; its codifferential will then be surjective, and part **(ii)** of B.4 lets us conclude that φ is an isomorphism. $\square$

We get now a description of noetherian complete local rings as quotients of power series rings.

Corollary B.8. *Every $R \in (\mathrm{Comp}/\Lambda)$ is a quotient of the power series ring $\Lambda[[x_1, \dots, x_n]]$ for some n. Moreover, the minimum such n is the dimension $\dim_k(T^\vee_\Lambda R)$ of the vertical cotangent space of R.*

Proof. Set $n = \dim_k(T^\vee_\Lambda R)$, and consider elements $a_1, \dots, a_n \in \mathfrak{m}_R$ such that $[a_1], \dots, [a_n]$ is a basis of $T^\vee_\Lambda R$. By Proposition B.6 we can define a homomorphism $\varphi : \Lambda[[x_1, \dots, x_n]] \to R$ such that $\varphi(x_i) = a_i$; its codifferential will then be surjective, and by Proposition B.3 φ will be surjective too. In other words, R is a quotient of $\Lambda[[x_1, \dots, x_n]]$.

On the other hand if $\varphi : \Lambda[[x_1, \dots, x_r]] \to R$ is surjective then the codifferential $\varphi_* : T^\vee_\Lambda \Lambda[[x_1, \dots, x_r]] \to T^\vee_\Lambda R$ is surjective too, and this implies that $r \geq n$. $\square$

Finally we prove a criterion that characterizes power series rings as formally smooth algebras in (Comp/Λ).

Theorem B.9. *Let $R \in (\mathrm{Comp}/\Lambda)$. Then R is a power series ring if and only if for every surjection $A' \to A$ in (Art/Λ) and every homomorphism $R \to A$, we can find a lifting $R \to A'$.*

Proof. If R is a power series ring, then Proposition B.6 implies that we can lift homomorphisms along small extensions.

Conversely, suppose that the lifting property holds, and take a homomorphism $\varphi : \Lambda[[x_1, \dots, x_n]] \to R$ that induces an isomorphism on cotangent spaces, $\varphi_* : T^\vee_\Lambda \Lambda[[x_1, \dots, x_n]] \to T^\vee_\Lambda R$ (using the last corollary, for example).

Now notice that the quotient map $\Lambda[[x_1, \dots, x_n]]_1 \to T^\vee_\Lambda \Lambda[[x_1, \dots, x_n]]$ is a surjection in (Art/Λ), and then by hypothesis we can lift the homomorphism $R \to T^\vee_\Lambda R \xrightarrow{(\varphi_*)^{-1}} T^\vee_\Lambda \Lambda[[x_1, \dots, x_n]]$ to $R \to \Lambda[[x_1, \dots, x_n]]_1$

$$\begin{tikzcd} & R \arrow[dl, dashed] \arrow[d] \\ \Lambda[[x_1, \dots, x_n]]_1 \arrow[r] & T^\vee_\Lambda \Lambda[[x_1, \dots, x_n]]. \end{tikzcd}$$

Likewise, since the quotient map $\Lambda[[x_1, \dots, x_n]]_k \to \Lambda[[x_1, \dots, x_n]]_{k-1}$ is a surjection in (Art/Λ), we can lift inductively the homomorphism $R \to \Lambda[[x_1, \dots, x_n]]_{k-1}$ to a homomorphism $R \to \Lambda[[x_1, \dots, x_n]]_k$.

Finally, taking the projective limit of the sequence of compatible homomorphisms above, we obtain a homomorphism $\psi : R \to \Lambda[[x_1, \dots, x_n]]$ such that the codifferential $\psi_* : T_\Lambda^\vee R \to T_\Lambda^\vee \Lambda[[x_1, \dots, x_n]]$ is an isomorphism (the inverse of φ_*), and by Proposition B.7 this implies that φ is an isomorphism, so R is a power series ring. □

Remark B.10. Actually the criterion can be strengthened by replacing "surjection" $A' \to A$ by "small extension". To see this it suffices to factor a surjection as a composite of small extensions and lift the homomorphism successively, as usual.

Coproducts in $(\mathrm{Comp}\,/\Lambda)$

The following discussion is applied in Section 6.3.1, to find miniversal deformations of product deformation categories.

Let R, S be objects of $(\mathrm{Comp}\,/\Lambda)$. We ask if there is a coproduct of R and S in the category $(\mathrm{Comp}\,/\Lambda)$, that is, an object $T \in (\mathrm{Comp}\,/\Lambda)$ with two homomorphisms $i_R : R \to T, i_S : S \to T$, such that given any other object $U \in (\mathrm{Comp}\,/\Lambda)$ with two homomorphisms $\varphi_R : R \to U, \varphi_S : S \to U$ there is a unique homomorphism $\psi : T \to U$ such that $\psi \circ i_R = \varphi_R$ and $\psi \circ i_S = \varphi_S$.

$$\begin{tikzcd} R \arrow[r, "i_R"] \arrow[dr, "\varphi_R"'] & T \arrow[d, dashed, "\psi"] & S \arrow[l, "i_S"'] \arrow[dl, "\varphi_S"] \\ & U & \end{tikzcd}$$

This universal property implies (using the usual argument) that any two coproducts will be canonically isomorphic. To prove existence, the natural thing to do is to try to take the tensor product $R \otimes_\Lambda S$, which is a coproduct in the category of Λ-algebras. Unfortunately does not give the "right" thing.

Example B.11. Take $R = k[[x]], S = k[[y]] \in (\mathrm{Comp}\,/k)$. One can easily verify (using Proposition B.6) that $k[[x, y]]$ with the two natural inclusions $k[[x]] \to k[[x, y]]$ and $k[[y]] \to k[[x, y]]$ is a coproduct of R and S in $(\mathrm{Comp}\,/\Lambda)$. On the other hand $k[[x, y]]$ is not isomorphic to the tensor product $k[[x]] \otimes_k k[[y]]$.

There is an injective map $k[[x]] \otimes_k k[[y]] \to k[[x, y]]$, defined by $f(x) \otimes g(y) \mapsto f(x)g(y)$ and extended by linearity, but this is not surjective, because for example one can see that the series $h(x, y) = \sum_i x^i y^i$ is not in the image.

Instead of taking the tensor product, we do the following: write R and S as quotients of power series rings (Corollary B.8),

$$R \cong \Lambda[[x_1, \dots, x_n]]/I, \quad S \cong \Lambda[[y_1, \dots, y_m]]/J$$

and put

$$R\widehat{\otimes}S = \Lambda[[x_1, \dots, x_n, y_1, \dots, y_m]]/(I\Lambda[[y_1, \dots, y_n]] + J\Lambda[[x_1, \dots, x_n]])$$

which is clearly an object of $(\mathrm{Comp}\,/\Lambda)$. Moreover we have two natural homomorphisms $i_R : R \to R\widehat{\otimes}S$ and $i_S : S \to R\widehat{\otimes}S$.

Proposition B.12. *The object $R\widehat{\otimes}S \in (\mathrm{Comp}\,/\Lambda)$ with the two homomorphisms i_R and i_S is a coproduct of R and S in $(\mathrm{Comp}\,/\Lambda)$.*

Proof. Assume that we have an object $U \in (\mathrm{Comp}\,/\Lambda)$, with two homomorphisms $\varphi_R : R \to U$ and $\psi_S : S \to U$. If $a_i \in U$ is the image of $[x_i] \in R$ and $b_i \in U$ the one of $[y_i] \in S$, then if $f_R : \Lambda[[x_1, \dots, x_n]] \to U$ and $f_S : \Lambda[[y_1, \dots, y_m]] \to U$ are the homomorphisms (Proposition B.6) sending x_i to a_i and y_i to b_i, we have that $I \subseteq \ker(f_R)$, $J \subseteq \ker(f_S)$, and φ_R, φ_S are the induced homomorphisms.

Now we define $f : \Lambda[[x_1, \dots, x_n, y_1, \dots, y_m]] \to U$ by sending x_i to a_i and y_i to b_i (using B.6 again). Because of the above inclusions we have $I\Lambda[[y_1, \dots, y_m]] + J\Lambda[[x_1, \dots, x_n]] \subseteq \ker(f)$, and so f induces $\psi : R\widehat{\otimes}S \to U$ that satisfies the desired property. Uniqueness is easy. $\square$

The following proposition relates the tangent space of $R\widehat{\otimes}S$ to the ones of R and S.

Proposition B.13. *There is an isomorphism $T_\Lambda(R\widehat{\otimes}S) \cong T_\Lambda R \oplus T_\Lambda S$ induced by the two homomorphisms i_R and i_S.*

Proof. We prove the analogous statement for cotangent spaces, and our result will follow by duality. The two homomorphisms $i_R : R \to R\widehat{\otimes}S$ and $i_S : S \to R\widehat{\otimes}S$ define two k-linear maps $(i_R)_* : T_\Lambda^\vee R \to T_\Lambda^\vee(R\widehat{\otimes}S)$ and $(i_S)_* : T_\Lambda^\vee S \to T_\Lambda^\vee(R\widehat{\otimes}S)$, which together induce a k-linear

$$\Phi : T_\Lambda^\vee R \oplus T_\Lambda^\vee S \to T_\Lambda^\vee(R\widehat{\otimes}S).$$

To prove that this is an isomorphism, we use the following property of the cotangent space, which is part of Proposition 5.1.10: if $R \in (\mathrm{Comp}\,/\Lambda)$, then for every $V \in (\mathrm{FVect}\,/k)$ there is a bijection (which is functorial in V)

$$\mathrm{Hom}_\Lambda(R, k[V]) \cong \mathrm{Hom}_k(T_\Lambda^\vee R, V)$$

obtained by sending a homomorphism $R \to k[V]$ to the induced k-linear function $\mathfrak{m}_R/(\mathfrak{m}_\Lambda R + \mathfrak{m}_R^2) \to V$.

For a fixed V, the map Φ induces then a homomorphism

$$\mathrm{Hom}_k(T_\Lambda^\vee(R\widehat{\otimes}S),V) \to \mathrm{Hom}_k(T_\Lambda^\vee R \oplus T_\Lambda^\vee S, V).$$

This is actually an isomorphism, being the composite of the natural isomorphisms

$$\begin{aligned}\mathrm{Hom}_k(T_\Lambda^\vee(R\widehat{\otimes}S),V) &\cong \mathrm{Hom}_\Lambda(R\widehat{\otimes}S, k[V])\\ &\cong \mathrm{Hom}_\Lambda(R,k[V]) \times \mathrm{Hom}_\Lambda(S,k[V])\\ &\cong \mathrm{Hom}_k(T_\Lambda^\vee R, V) \times \mathrm{Hom}_k(T_\Lambda^\vee S, V)\\ &\cong \mathrm{Hom}_k(T_\Lambda^\vee R \oplus T_\Lambda^\vee S, V)\end{aligned}$$

(where the second isomorphism comes from the universal property of $R\widehat{\otimes}S$), and this (together with functoriality of all these isomorphisms) implies that Φ is an isomorphism too. □

Continuous Kähler differentials

In this section we introduce a module of differentials for objects of (Comp/Λ) that is much more useful than the standard one.

Let R be an object of (Comp/Λ). We have then the usual module of Kähler differentials $\Omega_{R/\Lambda}$ with the universal Λ-derivation $d : R \to \Omega_{R/\Lambda}$, which has the following universal property: if $D : R \to M$ is a Λ-derivation then there is a unique homomorphism of R-modules $f : \Omega_{R/\Lambda} \to M$ such that $D = f \circ d$.

For some applications this module is too large: for example, one can show that $\Omega_{k[[x]]/k}$ is not finitely generated over $k[[x]]$, since the field of fractions $k((x))$ of $k[[x]]$ has infinite transcendence degree over k, and

$$\Omega_{k[[x]]/k} \otimes_{k[[x]]} k((x)) = \Omega_{k((x))/k}$$

is not finitely generated over $k((x))$.

Because of this we define another module of differentials that is better behaved. We consider derivations $D : R \to M$ where M is a module that is separated with respect to the $\mathfrak{m}_R$-adic topology, that is, the intersection of the submodules $\{\mathfrak{m}_R^i M\}_{i\in\mathbb{N}}$ is the zero submodule. For example, by one of Krull's Theorems, finitely generated R-modules are separated.

We want then a finitely generated R-module $\widehat{\Omega}_{R/\Lambda}$ with a derivation $d : R \to \widehat{\Omega}_{R/\Lambda}$, such that for every derivation $D : R \to M$, where M is a separated R-module, there exists a homomorphism $f : \widehat{\Omega}_{R/\Lambda} \to M$ such that $D = f \circ d$.

Write R as a quotient of a power series ring (Corollary B.8)

$$R \cong \Lambda[[x_1, \ldots, x_n]]/I$$

and suppose that $I = (f_1, \dots, f_k)$. We consider the free R-module on n elements $dx_1, \dots, dx_n$, and its submodule J generated by the elements

$$df_i = \left[\frac{\partial f_i}{\partial x_1}\right] dx_1 + \cdots + \left[\frac{\partial f_i}{\partial x_n}\right] dx_n$$

for $i = 1, \dots, k$; we define then

$$\widehat{\Omega}_{R/\Lambda} = (Rdx_1 \oplus \cdots \oplus Rdx_n)/J.$$

Moreover we have a derivation $d : R \to \widehat{\Omega}_{R/\Lambda}$ given by

$$d([g]) = \left[\left[\frac{\partial g}{\partial x_1}\right] dx_1 + \cdots + \left[\frac{\partial g}{\partial x_n}\right] dx_n\right]$$

for $[g] \in R$, that is easily seen to be well-defined.

Proposition B.14. *The R-module $\widehat{\Omega}_{R/\Lambda}$ and the derivation $d : R \to \widehat{\Omega}_{R/\Lambda}$ have the universal property above.*

Proof. We sketch the idea of the proof, without going into details. Let $D : R \to M$ be a Λ-derivation of R into a separated R-module M.

We start by defining $Rdx_1 \oplus \cdots \oplus Rdx_n \to M$ by saying that dx_i goes to $D([x_i])$, and then extending by linearity. To see that this induces a homomorphism on the quotient $\widehat{\Omega}_{A/\Lambda}$, the key point is to see that the derivation D is completely determined by $D([x_i])$ for $i = 1, \dots, n$.

This is clearly true for $D([p])$ where p is a polynomial, just by using the leibnitz rule repeteadly. The fact that D is uniquely determined on power series follows from the fact that derivations are continuous with respect to the $\mathfrak{m}_R$-adic topology, and from separatedness of M. $\square$

Definition B.15. *The R-module $\widehat{\Omega}_{R/\Lambda}$ is called the* ***module of continuous Kähler differentials*** *of R, and d is the* ***universal continuous derivation****.*

The proposition above implies in particular that changing the presentation of R as a quotient of a power series ring we get isomorphic modules of continuous differentials.

Suppose now that $R, S \in (\mathrm{Comp}/\Lambda)$, and that $\varphi : R \to S$ is a surjection with kernel $I \subseteq R$. Because of the universal property of $\widehat{\Omega}_{R/\Lambda}$ and the fact that the composite $R \to S \to \widehat{\Omega}_{S/\Lambda}$ is a Λ-derivation, we get a homomorphism of R-modules $\widehat{\Omega}_{R/\Lambda} \to \widehat{\Omega}_{S/\Lambda}$, which tensoring by S induces a homomorphism of S-modules $f : \widehat{\Omega}_{R/\Lambda} \otimes_R S \to \widehat{\Omega}_{S/\Lambda}$.

Moreover the universal derivation $d : R \to \widehat{\Omega}_{R/\Lambda}$ induces as usual a homomorphism of S-modules $I/I^2 \to \widehat{\Omega}_{R/\Lambda} \otimes_R S$ that we still denote by d. The following proposition is proved in the same way as its analogue for the standard modules of differentials.

Proposition B.16. *If $\varphi : R \to S$ is a surjection in* (Comp/Λ)*, then the sequence of S-modules*

$$I/I^2 \xrightarrow{d} \widehat{\Omega}_{R/\Lambda} \otimes_R S \xrightarrow{f} \widehat{\Omega}_{S/\Lambda} \longrightarrow 0$$

is exact.

The sequence above is called the **conormal sequence** associated with the homomorphism φ.

The following proposition will be applied in Chapter 6.

Proposition B.17. *Let X be a scheme of finite type over Λ, and $p \in X$ be a rational point. Then there is a natural isomorphism*

$$\Omega_{X/\Lambda,p} \otimes_{\mathcal{O}_{X,p}} \widehat{\mathcal{O}}_{X,p} \cong \widehat{\Omega}_{\widehat{\mathcal{O}}_{X,p}/\Lambda}.$$

Proof. Since this is a local problem, we can assume that $X \subseteq \mathbb{A}^n_\Lambda$ is a closed subscheme with ideal $I = (f_1, \ldots, f_k) \subseteq k[x_1, \ldots, x_n]$, and moreover that $p \in X$ is the origin of $\mathbb{A}^n_\Lambda$.

So we have

$$\Omega_{X/\Lambda} \cong (\mathcal{O}_X dx_1 \oplus \cdots \oplus \mathcal{O}_X dx_n)/(df_1, \ldots, df_n)$$

and

$$\widehat{\mathcal{O}}_{X,p} \cong k[[x_1, \ldots, x_n]]/(f_1, \ldots, f_k) = R.$$

Consequently, using the properties of localization and tensor product, we get

$$\Omega_{X/\Lambda,p} \otimes_{\mathcal{O}_{X,p}} \widehat{\mathcal{O}}_{X,p} \cong (Rdx_1 \oplus \cdots \oplus Rdx_n)/(d[f_1], \ldots, d[f_n]) = \widehat{\Omega}_{R/\Lambda}$$

where $d[f_i] = [\partial f_i/\partial x_1]dx_1 + \cdots + [\partial f_i/\partial x_n]dx_n$. This concludes the proof. □

Appendix C

Some other facts and constructions

In this appendix we gather some other miscellaneous standard results and constructions that are used throughout this work.

Fibered products of categories

Let $\mathcal{F}, \mathcal{G}, \mathcal{H}$ be three categories, with two functors $F : \mathcal{F} \to \mathcal{H}$ and $G : \mathcal{G} \to \mathcal{H}$. We want to define a "fibered product" category $\mathcal{F} \times_{\mathcal{H}} \mathcal{G}$ with two functors $\pi_{\mathcal{F}} : \mathcal{F} \times_{\mathcal{H}} \mathcal{G} \to \mathcal{F}$ and $\pi_{\mathcal{G}} : \mathcal{F} \times_{\mathcal{H}} \mathcal{G} \to \mathcal{G}$, such that the composites $F \circ \pi_{\mathcal{F}}$ and $G \circ \pi_{\mathcal{G}}$ are isomorphic as functors $\mathcal{F} \times_{\mathcal{H}} \mathcal{G} \to \mathcal{H}$, and such that for any other category $\mathcal{C}$ with two functors $\varphi_{\mathcal{F}} : \mathcal{C} \to \mathcal{F}$ and $\varphi_{\mathcal{G}} : \mathcal{C} \to \mathcal{G}$ and a fixed isomorphism of functors $F \circ \varphi_{\mathcal{F}} \cong G \circ \varphi_{\mathcal{G}}$ there exists a dotted functor as in the diagram below

$$\begin{tikzcd}
\mathcal{C} \arrow[dr, dashed, "\psi"] \arrow[drr, bend left, "\varphi_{\mathcal{F}}"] \arrow[ddr, bend right, "\varphi_{\mathcal{G}}"'] & & \\
& \mathcal{F} \times_{\mathcal{H}} \mathcal{G} \arrow[r, "\pi_{\mathcal{F}}"] \arrow[d, "\pi_{\mathcal{G}}"'] & \mathcal{F} \arrow[d, "F"] \\
& \mathcal{G} \arrow[r, "G"'] & \mathcal{H}
\end{tikzcd}$$

such that $\pi_{\mathcal{F}} \circ \psi = \varphi_{\mathcal{F}}$ and $\pi_{\mathcal{G}} \circ \psi = \varphi_{\mathcal{G}}$ (which are actual equalities, and not merely isomorphisms of functors).

We define such a category $\mathcal{F} \times_{\mathcal{H}} \mathcal{G}$ as follows:

Objects: are triplets (X, Y, f) where $X \in \mathcal{F}, Y \in \mathcal{G}$ and $f : F(X) \to G(X)$ is an isomorphism in the category $\mathcal{H}$.

Arrows: from (X, Y, f) to (Z, W, g) are pairs (h, k) of arrows $h : X \to Z$ of $\mathcal{F}$ and $k : Y \to W$ of $\mathcal{G}$, such that the diagram

$$\begin{tikzcd} F(X) \arrow[r, "F(h)"] \arrow[d, "f"'] & F(Z) \arrow[d, "g"] \\ G(Y) \arrow[r, "F(k)"'] & G(W) \end{tikzcd}$$

is commutative.

Composition of arrows is defined in the obvious way, as well as the two functors $\pi_{\mathcal{F}}, \pi_{\mathcal{G}}$; for example $\pi_{\mathcal{F}} : \mathcal{F} \times_{\mathcal{H}} \mathcal{G} \to \mathcal{F}$ sends an object (X, Y, f) to $X \in \mathcal{F}$, and an arrow (h, k) to h.

Moreover notice that the composites $F \circ \pi_{\mathcal{F}}$ and $G \circ \pi_{\mathcal{G}}$ are clearly isomorphic: starting from $(X, Y, f) \in \mathcal{F} \times_{\mathcal{H}} \mathcal{G}$ we have $(F \circ \pi_{\mathcal{F}})(X, Y, f) = F(X)$ and $(G \circ \pi_{\mathcal{G}})(X, Y, f) = G(Y)$, so $f : F(X) \to G(Y)$ gives the desired isomorphism. The compatibility property on arrows ensures that these isomorphisms altogether give a natural transformation.

Proposition C.1. *The category $\mathcal{F} \times_{\mathcal{H}} \mathcal{G}$ with the functors $\pi_{\mathcal{F}}, \pi_{\mathcal{G}}$ has the property stated above.*

Proof. Suppose we have a category $\mathcal{C}$ with two functors $\varphi_{\mathcal{F}} : \mathcal{C} \to \mathcal{F}$ and $\varphi_{\mathcal{G}} : \mathcal{C} \to \mathcal{G}$, and a fixed isomorphism of functors $\alpha : F \circ \varphi_{\mathcal{F}} \cong G \circ \varphi_{\mathcal{G}}$. We define a functor $\psi : \mathcal{C} \to \mathcal{F} \times_{\mathcal{H}} \mathcal{G}$ as follows: if $X \in \mathcal{C}$, we put $\psi(X) = (\varphi_{\mathcal{F}}(X), \varphi_{\mathcal{G}}(X), \alpha(X))$, and an arrow $f : X \to Y$ of $\mathcal{C}$ goes to the arrow $(\varphi_{\mathcal{F}}(f), \varphi_{\mathcal{G}}(f))$ of $\mathcal{F} \times_{\mathcal{H}} \mathcal{G}$.

It is immediate to check that ψ is well-defined, and that $\pi_{\mathcal{F}} \circ \psi = \varphi_{\mathcal{F}}$ and $\pi_{\mathcal{G}} \circ \psi = \varphi_{\mathcal{G}}$. $\square$

Definition C.2. *The category $\mathcal{F} \times_{\mathcal{H}} \mathcal{G}$ is called the* ***fibered product*** *of $\mathcal{F}$ and $\mathcal{G}$ over $\mathcal{H}$.*

Remark C.3. The property that we used as starting point to define the fibered product looks much like a universal property (which should define it up to equivalence), apart from the fact that there is no uniqueness required on the functor ψ. On the other hand we defined the fibered product explicitly, and we will not need this "uniqueness" part.

Nevertheless, we remark that it is possible to give a universal property that identifies the fibered product up to equivalence, but the natural setting in which this property is stated is that of 2-categories.

The local flatness criterion

The following theorem gives an important flatness criterion.

Theorem C.4 (Local flatness criterion)**.** *Let A be a ring, $I \subseteq A$ a proper ideal, and M an A-module. If either*

(i) *I is nilpotent, or*

(ii) *A is a noetherian local ring and M is a finitely generated B-module, where B is a notherian local ring with a local homomorphism $\varphi : A \to B$ and the two structures of module on M are compatible with φ*

then the following conditions are equivalent:

- *M is a flat A-module.*
- *M/IM is flat over A/I and $\mathrm{Tor}_1^A(M, A/I) = 0$.*
- *$M/I^n M$ is flat over A/I^n for every $n \geq 1$.*

A discussion about this can be found in § 22 of [Mat].

If in particular condition **(ii)** is satisfied, and $I = \mathfrak{m}_A$ is the maximal ideal of A, then $M/\mathfrak{m}_A M$ is certainly flat over $A/\mathfrak{m}_A$, which is a field, and so we get the following corollary.

Corollary C.5. *Let $\varphi : A \to B$ be a local homomorphism of noetherian local rings, and M be a finitely generated B-module. Then M is flat over A if and only if $\mathrm{Tor}_1^A(M, k) = 0$.*

A base change theorem

Let X be a scheme over a noetherian ring A, and $\mathcal{E}$ be a coherent sheaf on X. We want to understand the relation between the A-modules $H^i(X, M \otimes_A \mathcal{E})$ and $M \otimes_A H^i(X, \mathcal{E})$ (this is a particular case of the "base change problem").

There is a natural homomorphism

$$\varphi_M^i : M \otimes_A H^i(X, \mathcal{E}) \to H^i(X, M \otimes_A \mathcal{E})$$

that is defined as follows. An element $m \in M$ corresponds to a homomorphism of A-modules $m : A \to M$, defined by $a \mapsto a \cdot m$. We can consider then the homomorphism $m \otimes \mathrm{id} : \mathcal{E} \cong A \otimes_A \mathcal{E} \to M \otimes_A \mathcal{E}$, which induces a homomorphism in cohomology $(m \otimes \mathrm{id})_* : H^i(X, \mathcal{E}) \to H^i(X, M \otimes_A \mathcal{E})$.

We define then a function $F : M \times H^i(X, \mathcal{E}) \to H^i(X, M \otimes_A \mathcal{E})$ by $F(m, \alpha) = (m \otimes \mathrm{id})_*(\alpha)$; one can check that this function is A-bilinear in both variables, and so it induces a homomorphism of A-modules $\varphi_M^i : M \otimes_A H^i(X, \mathcal{E}) \to H^i(X, M \otimes_A \mathcal{E})$.

We have the following classical result.

Theorem C.6. *Let X be a proper scheme over A, $\mathcal{E}$ a coherent sheaf on X, flat over A, and assume that for every closed point $p \in \operatorname{Spec}(A)$ and a fixed i the homomorphism*

$$\varphi^i_{k(p)} : k(p) \otimes_A H^i(X, \mathcal{E}) \to H^i(X, k(p) \otimes_A \mathcal{E})$$

is an isomorphism. Then for every A-module M the homomorphism φ^i_M is an isomorphism.

Definition C.7. *If the concusion of this theorem holds for a coherent sheaf $\mathcal{E}$ and a natural number i, we say that the cohomology group $H^i(X, \mathcal{E})$* ***satisfies base change****.*

For a discussion about base change and the theorem above, see for example Sections 7.7 and 7.8 of [EGAIII], or III, 12 of [Har].

The following theorem tells us that sheaves of differentials satisfy base change in a particular case (see Theorem 5.5 (i) of [Del]).

Theorem C.8 (Deligne)**.** *Let X be a proper and smooth scheme over a noetherian $\mathbb{Q}$-algebra A, and consider the coherent sheaf of Kähler differentials $\Omega_{X/A}$, and its exterior powers $\Omega^i_{X/A} = \bigwedge^i \Omega_{X/A}$. Then all the cohomology groups $H^i(X, \Omega^j_{X/A})$ satisfy base change.*

Left exactness of the conormal sequence in a particular case

Let X_0 be a scheme of finite type over k, $V \in (\mathrm{FVect}/k)$, and take a deformation $X \in \mathcal{D}ef_{X_0}(k[V])$ of X_0 over the ring of dual numbers $k[V]$. Since the sheaf of ideals of X_0 in X can be identified with $V \otimes_k \mathcal{O}_{X_0}$ (see the proof of Theorem 2.4.1), and in particular its square is zero, we can consider the conormal sequence associated with the closed immersion $X_0 \subseteq X$

$$V \otimes_k \mathcal{O}_{X_0} \xrightarrow{d} \Omega_X|_{X_0} \longrightarrow \Omega_{X_0} \longrightarrow 0.$$

Proposition C.9. *If X_0 is reduced and generically smooth, then d is injective.*

For the proof we will need the following theorem (for a discussion about it, see Exposé II of [SGA1]).

Theorem C.10. *Let X, Y be schemes, with Y noetherian, and $f : X \to Y$ be a morphism of finite type. Then f is smooth if and only if there exists an open cover $\{X_i\}_{i \in I}$ of X and étale morphisms $f_i : X_i \to \mathbb{A}^{n_i}_Y$, such that the diagram*

$$\begin{tikzcd} X_i \arrow[rr, "f_i"] \arrow[dr, "f|_{X_i}"'] & & \mathbb{A}^{n_i}_Y \arrow[dl] \\ & Y & \end{tikzcd}$$

is commutative for all i.

Proof. We proceed by steps, starting from the simplest case.

Case 1. Suppose $X = \mathbb{A}^n_{k[V]} = \operatorname{Spec}(k[V][x_1,\dots,x_n])$ (and then $X_0 \cong \mathbb{A}^n_k$). Take a basis $v_1,\dots,v_r$ of V, and put $R = k[v_1,\dots v_r, x_1,\dots,x_n]$, so that we see X as a closed subscheme $X = \operatorname{Spec}(R/J) \subseteq \mathbb{A}^{n+r}_k$, where J is the ideal generated by all the products v_iv_j.

From the conormal exact sequence of this closed immersion

$$J/J^2 \xrightarrow{d_X} \Omega_{\mathbb{A}^{n+r}_k}|_X \longrightarrow \Omega_X \longrightarrow 0$$

and the fact that $\Omega_{\mathbb{A}^{n+r}_k}$ is a free R-module generated by the $n+r$ elements $dv_1,\dots,dv_r,dx_1,\dots,dx_n$, we see that Ω_X is a $k[V][x_1,\dots,x_n]$-module with generators $dv_1,\dots,dv_r,dx_1,\dots,dx_n$ and relations $d_X(v_iv_j) = v_idv_j+v_jdv_i = 0$.

We conclude from this that $\Omega_X|_{X_0} = \Omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_{X_0}$ is a free $\mathcal{O}_{X_0}$-module, generated by $dv_1|_{X_0},\dots,dv_r|_{X_0}, dx_1|_{X_0},\dots,dx_n|_{X_0}$ (the relations become trivial when forcing $v_i = 0$).

Now consider the conormal sequence of $X_0 \subseteq X$

$$V \otimes_k \mathcal{O}_{X_0} \xrightarrow{d} \Omega_X|_{X_0} \longrightarrow \Omega_{X_0} \longrightarrow 0$$

and notice that d is defined on the generators v_i of $V \otimes_k \mathcal{O}_{X_0}$ by $v_i \mapsto dv_i|_{X_0}$. Since now we know that the images are linearly independent in $\Omega_X|_{X_0}$, we see that d is injective in this case.

Case 2. Suppose X_0 is smooth over k, which is equivalent to saying that X is smooth over $k[V]$. Since our result is a local question, using Theorem C.10 we can assume that we have an étale morphism $f : X \to \mathbb{A}^n_{k[V]}$, such that

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & \mathbb{A}^n_{k[V]} \arrow[dl] \\ & \operatorname{Spec}(k[V]) & \end{tikzcd}$$

commutes. From the properties of étale morphisms we have then that the natural morphism of $\mathcal{O}_X$-modules $f^*(\Omega_{\mathbb{A}^n_{k[V]}}) \to \Omega_X$ is an isomorphism.

Consider now the closed immersion $\mathbb{A}^n_k \subseteq \mathbb{A}^n_{k[V]}$. By the preceding case, its conormal sequence

$$0 \longrightarrow V \otimes_k \mathcal{O}_{\mathbb{A}^n_k} \longrightarrow \Omega_{\mathbb{A}^n_{k[V]}}|_{\mathbb{A}^n_k} \longrightarrow \mathcal{O}_{\mathbb{A}^n_k} \longrightarrow 0$$

is exact. Since f is flat, exactness is preserved if we apply f^*, and doing so we get

$$0 \longrightarrow V \otimes_k \mathcal{O}_{X_0} \xrightarrow{d} \Omega_X|_{X_0} \longrightarrow \mathcal{O}_{X_0} \longrightarrow 0$$

because $f^*(\Omega_{\mathbb{A}^n_{k[V]}}|_{\mathbb{A}^n_k}) \cong \Omega_X|_{X_0}$. So d is injective in this case too.

Case 3. We turn to the general case. Put $\mathcal{K} = \ker(d)$, and call $\mathcal{K}_i$ the image of $\mathcal{K}$ under the i-th projection $\pi_i : V \otimes_k \mathcal{O}_{X_0} \cong \mathcal{O}^n_{X_0} \to \mathcal{O}_{X_0}$; these are $\mathcal{O}_{X_0}$-modules on X_0.

By the preceding case we see that for each i the support of $\mathcal{K}_i$ has to be contained in the singular locus of X_0, which does not contain any component of X_0, since it is generically smooth. In particular $\operatorname{supp}(\mathcal{K}_i)$ can not contain any irreducible component of X_0.

But if we suppose $\mathcal{K}_i \neq 0$, then $\operatorname{supp}(\mathcal{K}_i) \neq \emptyset$, and its irreducible components would correspond to embedded points of X_0, which cannot exist since X_0 is reduced. Then $\mathcal{K}_i = 0$ for every i, and so also $\mathcal{K} = 0$, and d is injective. □

Two equivalences of categories of sheaves

Let X be a topological space, $V \in (\mathrm{FVect}\,/k)$, and consider the following categories:

- The category $\mathcal{A}$ of sheaves of flat $k[V]$-modules on X.
- The category $\mathcal{B}$, whose objects are pairs (E, F) of sheaves of k-vector spaces on X, with an extension

  $$0 \longrightarrow V \otimes_k E \xrightarrow{t} F \xrightarrow{u} E \longrightarrow 0$$

  and arrows from (E, F) to (E', F') are pairs of k-linear homomorphisms $\alpha : E \to E'$, $\beta : F \to F'$, fitting in a commutative diagram

  $$\begin{tikzcd} 0 \arrow[r] & V \otimes_k E \arrow[r] \arrow[d, "\mathrm{id} \otimes \alpha"] & F \arrow[r] \arrow[d, "\beta"] & E \arrow[r] \arrow[d, "\alpha"] & 0 \\ 0 \arrow[r] & V \otimes_k E' \arrow[r] & F' \arrow[r] & E' \arrow[r] & 0 \end{tikzcd}$$

  with the respective extensions.

Consider the functor $\Phi : \mathcal{A} \to \mathcal{B}$ that takes a flat $k[V]$-module F to the pair (F_0, F), where $F_0 = F \otimes_{k[V]} k$, with the extension

$$0 \longrightarrow V \otimes_k F_0 \longrightarrow F \longrightarrow F_0 \longrightarrow 0$$

obtained by tensoring with F (and using its flatness over $k[V]$) the exact sequence of $k[V]$-modules

$$0 \longrightarrow V \longrightarrow k[V] \longrightarrow k \longrightarrow 0$$

(notice that $V \otimes_{k[V]} F \cong V \otimes_k (F \otimes_{k[V]} k)$), and acting on arrows in the obvious way.

Proposition C.11. *The functor $\Phi : \mathcal{A} \to \mathcal{B}$ is an equivalence of categories.*

Proof. We define a quasi-inverse $\Psi : \mathcal{B} \to \mathcal{A}$. Starting from a pair (E, F) with an extension

$$0 \longrightarrow V \otimes_k E \xrightarrow{t} F \xrightarrow{u} E \longrightarrow 0 \tag{C.1}$$

we see that F has a natural structure of $k[V]$-module: given a section s of F, and $x + v \in k[V] = k \oplus V$, we define

$$(x + v) \cdot s = xs + t(v \otimes u(s))$$

It is easy to check that this gives a $k[V]$-module structure.

Moreover, F is flat over $k[V]$, because of the local flatness criterion: if we put $F_0 = F \otimes_{k[V]} k$, we have an isomorphism $F_0 = F \otimes_{k[V]} k \cong E$ (that we get by tensoring C.1 with k over $k[V]$), that induces $V \otimes_k E \cong V \otimes_k F_0$, and these two isomorphisms fit in a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & V \otimes_k E \arrow[r] \arrow[d, "\wr"] & F \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, "\wr"] & 0 \\
 & V \otimes_k F_0 \arrow[r] & F \arrow[r] & F_0 \arrow[r] & 0
\end{tikzcd}$$

where the top row is the extension above, and the bottom is the exact sequence of $k[V]$-modules

$$0 \longrightarrow V \longrightarrow k[V] \longrightarrow k \longrightarrow 0$$

tensored with F. From the diagram we see that also the bottom left map is injective, so we have $\mathrm{Tor}_1^{k[V]}(F, k) = \ker(V \otimes_k F_0 \to F) = 0$, and this implies that F is flat over $k[V]$.

Ψ sends then the pair (E, F) with the extension above to the $k[V]$-module F, and an arrow (α, β) to the $k[V]$-linear homomorphism $\beta : F \to F'$. Straightforward verifications show that the funtors Φ and Ψ are quasi-inverse to each other, and then give an equivalence of categories. □

We will also need a similar result, in which we consider quasi-coherent sheaves instead of simply sheaves of $k[V]$-modules. Take a scheme X over k and $V \in (\mathrm{FVect}/k)$. We denote by X_V the trivial deformation over $k[V]$,

$$X_V = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k[V])$$

and recall that $|X_V| = |X|$, so we can speak indifferently of sheaves over X or over X_V.

Consider the following categories:

- The category $\mathcal{C}$ of quasi-coherent $\mathcal{O}_{X_V}$-modules on X_V that are flat over $k[V]$.

- The category $\mathcal{D}$, whose objects are pairs $(\mathcal{E}, \mathcal{F})$ of quasi-coherent $\mathcal{O}_X$-modules on X, with an extension

  $$0 \longrightarrow V \otimes_k \mathcal{E} \xrightarrow{t} \mathcal{F} \xrightarrow{u} \mathcal{E} \longrightarrow 0$$

  and arrows from $(\mathcal{E}, \mathcal{F})$ to $(\mathcal{E}', \mathcal{F}')$ are pairs of homomorphisms $\alpha : \mathcal{E} \to \mathcal{E}'$, $\beta : \mathcal{F} \to \mathcal{F}'$ of $\mathcal{O}_X$-modules, fitting in a commutative diagram

  $$\begin{tikzcd}
0 \arrow[r] & V \otimes_k \mathcal{E} \arrow[r] \arrow[d, "\mathrm{id} \otimes \alpha"] & \mathcal{F} \arrow[r] \arrow[d, "\beta"] & \mathcal{E} \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & V \otimes_k \mathcal{E}' \arrow[r] & \mathcal{F}' \arrow[r] & \mathcal{E}' \arrow[r] & 0
\end{tikzcd}$$

  with the respective extensions.

We have a functor $\Phi : \mathcal{C} \to \mathcal{D}$ that takes a quasi-coherent O_{X_V}-module $\mathcal{F}$ flat over $k[V]$ to the pair of quasi-coherent $\mathcal{O}_X$-modules $(\mathcal{F}_0, \mathcal{F})$, where $\mathcal{F}_0 = \mathcal{F} \otimes_{k[V]} k$, with the extension

$$0 \longrightarrow V \otimes_k \mathcal{F}_0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_0 \longrightarrow 0$$

that we get tensoring with $\mathcal{F}$ (and using its flatness over $k[V]$) the exact sequence of $k[V]$-modules

$$0 \longrightarrow V \longrightarrow k[V] \longrightarrow k \longrightarrow 0$$

(again, notice that $\mathcal{F} \otimes_{k[V]} k \cong \mathcal{F} \otimes_k (\mathcal{F} \otimes_{k[V]} k)$), and acts on arrows in the obvious way.

Proposition C.12. *The functor $\Phi : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories.*

Proof. The proof is very similar to the preceding one. We construct a quasi-inverse $\Psi : \mathcal{D} \to \mathcal{C}$ as follows: given a pair of quasi-coherent $\mathcal{O}_X$-modules $(\mathcal{E}, \mathcal{F})$ with an extension

$$0 \longrightarrow V \otimes_k \mathcal{E} \xrightarrow{t} \mathcal{F} \xrightarrow{u} \mathcal{E} \longrightarrow 0$$

we define a structure of a quasi-coherent $\mathcal{O}_{X_V}$-module on $\mathcal{F}$. Given sections f of $\mathcal{F}$, and $s + v \otimes s'$ of $\mathcal{O}_{X_V}$, that we see as $\mathcal{O}_X \oplus (V \otimes_k \mathcal{O}_X)$, we define

$$(s + v \otimes s')f = sf + t(v \otimes u(s'f)).$$

It is straightforward to check that this gives indeed a structure of quasi-coherent $\mathcal{O}_{X_V}$-module to $\mathcal{F}$. Exactly as in the preceding proof one can show that $\mathcal{F}$ is flat over $k[V]$, using the local flatness criterion.

Ψ sends then the pair $(\mathcal{E}, \mathcal{F})$ with the extension above to $\mathcal{F}$, and an arrow (α, β) to the homomorphism of $\mathcal{O}_{X_V}$-modules $\beta : \mathcal{F} \to \mathcal{F}'$. Easy verifications show that Φ and Ψ are quasi-inverse to each other, and so they give an equivalence of categories. $\square$

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