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A chiral Borel–Weil–Bott theorem

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Abstract

We compute the cohomology of modules over the algebra of twisted chiral differential operators over the flag manifold. This is applied to (1) finding the character of G-integrable irreducible highest weight modules over the affine Lie algebra at the critical level, and (2) computing a certain elliptic genus of the flag manifold. The main tool is a result that interprets the Drinfeld–Sokolov reduction as a derived functor.

Keywords: Chiral differential operators; Representations at the critical level; Elliptic genus

1. Introduction and the main result

Let G be a simple complex Lie group, $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ its Lie algebra, X the corresponding flag manifold. If $\lambda \in \mathfrak{h}^*$ is an integral weight, denote by $\mathcal{L}_\lambda$ the corresponding invertible G-equivariant sheaf of $\mathcal{O}_X$-modules and by $\mathcal{D}_X^\lambda$ the algebra of twisted differential operators acting on $\mathcal{L}_\lambda$. The action of $\mathfrak{g}$ on $\mathcal{L}_\lambda$ defines a Lie algebra morphism

$$\mathfrak{g} \to \Gamma\left(X, \mathcal{D}_X^\lambda\right) \tag{1.1}$$

and a localization functor [8]

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$$\Delta : \mathfrak{g}\text{-Mod} \to \mathcal{D}_X^\lambda\text{-Mod}, \qquad \Delta(A) = \mathcal{D}_X^\lambda \bigotimes_{\mathfrak{g}} A, \tag{1.2}$$

which has proved of essence in representation theory and served as a template in modern mathematical physics.

From various points of view, it is important to find a reasonable analogue of functor (1.2) in the case of the affine Lie algebra, $\hat{\mathfrak{g}}$, a universal central extension of the loop algebra $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$. Various approaches have been developed, see papers such as [31–33,11,19], all valuable in many respects yet deficient one way or another. We would like to explore yet another proposal, which is based on consistently replacing the notions of Lie or associative algebras with that of a vertex algebra. This approach is not a panacea either, but it does lead to a pleasing result, and it gives answers to a few natural questions arising independently of vertex algebras (characters of irreducible $\hat{\mathfrak{g}}$-modules at the critical level) or even representation theory (elliptic genera attached to flag manifolds).

Constructed in [37] is a sheaf of vertex algebras, $\mathcal{D}_X^{ch}$, known also as an algebra of *chiral differential operators* (CDO); this is an analogue of $\mathcal{D}_X = \mathcal{D}_X^0$.[3] Recently, a sheaf of *twisted chiral differential operators*, $\mathcal{D}_X^{ch,tw}$, was proposed [6]; it is an analogue of not so much $\mathcal{D}_X^\lambda$ as of its universal version, where, roughly speaking, λ becomes a variable. An analogue of (1.1) is a vertex algebra morphism

$$\pi : V_{-h^\vee}(\mathfrak{g}) \to \Gamma\big(X, \mathcal{D}_X^{ch,tw}\big), \tag{1.3}$$

where $V_{-h^\vee}(\mathfrak{g})$ is the vertex algebra attached to $\hat{\mathfrak{g}}$ at the critical level $-h^\vee$.

It is a peculiar property of the critical level that $V_{-h^\vee}(\mathfrak{g})$ acquires a big center $\mathfrak{z}(V_{-h^\vee}(\mathfrak{g})) \subset V_{-h^\vee}(\mathfrak{g})$, a fundamental result of Feigin and Frenkel [15–17]. Analogously, the center of the vertex algebra $\Gamma(X, \mathcal{D}_X^{ch,tw})$ equals H_X, a commutative vertex algebra of differential polynomials on $\mathfrak{h}^*$. Restricting morphism (1.3) to the center one obtains

$$\pi\big(\mathfrak{z}\big(V_{-h^\vee}(\mathfrak{g})\big)\big) \subset H_X. \tag{1.4}$$

It is clear that a 1-dimensional representation of H_X is the same as a Laurent series $\nu(z) \in \mathfrak{h}^*((z))$. Let us introduce $\mathcal{D}_X^{ch,tw}\text{-Mod}_{\nu(z)}$, the category of (sheaves of) $\mathcal{D}_X^{ch,tw}$-modules such that H_X acts according to $\nu(z) \in \mathfrak{h}^*((z))$. This is a reasonable analogue of $\mathcal{D}_X^\lambda$-Mod.

For each $\nu(z) = \nu_0/z + \nu_{-1} + \nu_{-2}z + \cdots$, there is a functor [6]

$$\mathcal{Z}hu_{\nu(z)} : \mathcal{D}_X^{\nu_0}\text{-Mod} \to \mathcal{D}_X^{ch,tw}\text{-Mod}_{\nu(z)}. \tag{1.5}$$

In fact, one can prove [6], if some extra assumptions hold, that $\mathcal{D}_X^{ch,tw}\text{-Mod}_{\nu(z)}$ has more than one, trivial, object if and only if $\nu(z)$ is as demanded, and if so, then the functor $\mathcal{Z}hu_{\nu(z)}$ is an equivalence of categories.

Now form the composition

$$\mathcal{Z}hu_{\nu(z)} \circ \Delta : \mathfrak{g}\text{-Mod} \to \mathcal{D}_X^{ch,tw}\text{-Mod}_{\nu(z)}. \tag{1.6}$$

[3] This sheaf has been further studied in [26,7]. The case of an arbitrary underlying smooth variety is treated in [10,27].

By virtue of (1.3), (1.4), for each $A \in \mathfrak{g}$-Mod, $\mathcal{Z}hu_{\nu(z)} \circ \Delta(A)$ is a sheaf of $V_{-h^\vee}(\mathfrak{g})$-modules, hence $\hat{\mathfrak{g}}$-modules, with central character $\nu(z) \circ \pi$, see (1.4), and one can think of $\mathcal{Z}hu_{\nu(z)} \circ \Delta(A)$ as a localization of $\Gamma(X, \mathcal{Z}hu_{\nu(z)} \circ \Delta(A))$. This is why $\mathcal{Z}hu_{\nu(z)} \circ \Delta$ can be regarded as an affine version of (1.2).

Thus various $A \in \mathfrak{g}$-Mod serve to localize various $\hat{\mathfrak{g}}$-modules. For example, if ν_0 is dominant, and $M^c_{\nu_0}$ is the corresponding contragredient Verma module, then $\mathcal{Z}hu_{\nu(z)} \circ \Delta(M^c_{\nu_0})$ is a localization of the Wakimoto module of critical level with highest weight ν_0 [15–17,41].

The most interesting example of such localization occurs when ν_0 is a regular dominant integral weight and V_{ν_0} is the simple (finite dimensional) $\mathfrak{g}$-module with highest weight ν_0. In this case, $\mathcal{Z}hu_{\nu(z)} \circ \Delta(V_{\nu_0})$ is a G-equivariant sheaf, and so $\Gamma(X, \mathcal{Z}hu_{\nu(z)} \circ \Delta(V_{\nu_0}))$ is an object of $\hat{\mathfrak{g}}\text{-Mod}^{G[\![t]\!]}_{\nu(z)\circ\pi}$, the category of those $\hat{\mathfrak{g}}$-modules at the critical level with central character $\nu(z) \circ \pi$, where the action can be integrated to an action of $G[\![t]\!]$.

In the impressive series of papers [19–21], Frenkel and Gaitsgory prove that $\hat{\mathfrak{g}}\text{-Mod}^{G[\![t]\!]}_{\chi(z)\circ\pi}$ is a semi-simple category with a unique simple object, $\mathbb{V}_{\nu(z)}$, the Weyl module with highest weight ν_0 quotiented out by the central character $\nu(z)\pi$. It follows that the cohomology $H^i(X, \mathcal{Z}hu_{\nu(z)} \circ \Delta(V_{\nu_0}))$ is a direct sum of a number of copies of $\mathbb{V}_{\nu(z)}$. Here is the main result of the paper.

Theorem 1.1. *Denote by $\mathcal{L}^{ch}_{\nu(z)}$ the sheaf $\mathcal{Z}hu_{\nu(z)} \circ \Delta(V_{\nu_0})$ and let*

$$\chi(\mathcal{L}^{ch}_{\nu(z)}) = \sum_{i=0}^{\dim X} (-1)^i \operatorname{ch} H^i(X, \mathcal{L}^{ch}_{\nu(z)}),$$

where ch *stands for the formal character, cf.* (3.12). *Then*

$$\chi(\mathcal{L}^{ch}_{\nu(z)}) = \sum_{w\in W} (-1)^{\ell(w)} e^{w\circ\nu_0} \times \prod_{\alpha\in\hat{\Delta}^{re}_+} (1-e^{-\alpha})^{-1}, \tag{1}$$

$$H^i(X, \mathcal{L}^{ch}_{\nu(z)}) = \bigoplus_{w\in W,\ l(w)=i} \mathbb{V}_{\nu(z)}\big[\langle \nu_0 - w\circ\nu_0, \rho^\vee\rangle\big], \tag{2}$$

where W is the Weyl group of $\mathfrak{g}$, $l(w)$ is the length of $w \in W$, and $\mathbb{V}_{\nu(z)}[m]$ stands for $\mathbb{V}_{\nu(z)}$ as a $\hat{\mathfrak{g}}$-module with conformal filtration shifted by m, cf. Section 3.1.5.

This result is an extension (perhaps one can say "chiralization") of the Borel–Weil–Bott theorem,

$$H^i(X, \mathcal{L}_{\nu_0}) = \begin{cases} V_{\nu_0} & \text{if } i = 0, \\ 0 & \text{otherwise}, \end{cases}$$

where $\mathcal{L}_{\nu_0} = \Delta(V_{\nu_0})$ is the G-equivariant line bundle attached to the highest weight ν_0; note the appearance of the higher cohomology in our situation.

Theorem 1.1(1) considerably simplifies in the limit when $e^\alpha \mapsto 1$, $e^{-\delta} \mapsto q$ (homogeneous grading):

$$\chi(\mathcal{L}^{ch}_{\nu(z)}, q) = \dim V_{\nu_0} \prod_{j=1}^{+\infty} (1-q^j)^{-2\dim X}. \tag{1.7}$$

The assertions of Theorem 1.1 can be interpreted as solutions of problems stated independently of chiral differential operator algebra theory and interesting in their own right.

One such interpretation is the following character formula.

Corollary 1.2.

$$\operatorname{ch}\mathbb{V}_{\nu(z)}=\frac{\sum_{w\in W}(-1)^{\ell(w)}e^{w\circ\nu_0}}{\prod_{\alpha\in\Delta_+}(1-e^{-\langle\nu_0+\rho,\alpha^\vee\rangle\delta})\prod_{\alpha\in\hat{\Delta}_+^{re}}(1-e^{-\alpha})}.$$

Note that the homogeneous grading specialization ($e^{\alpha}\mapsto 1$, $e^{-\delta}\mapsto q$) again makes this formula into an infinite product

$$\dim_q\mathbb{V}_{\nu(z)}=\dim V_{\nu_0}\prod_{j=1}^{+\infty}\left(1-q^j\right)^{-2\dim X}\prod_{\alpha\in\Delta_+}\left(1-q^{\langle\nu_0+\rho,\alpha^\vee\rangle}\right)^{-1}. \tag{1.8}$$

We would like to point out that this character formula is not new: in the case of sl_2 it was worked out in [36]; in the full generality it was first recorded in [4]; and it immediately follows from Theorem 5 and formula (5.3) of [20]. Our point is not so much the formula itself but the fact that it nicely fits in and follows from the proposed geometric framework.

To obtain another interpretation, introduce the following generating function of locally free sheaves over X:

$$\mathcal{E}_\lambda=\mathcal{L}_\lambda\otimes\left(\bigotimes_{n=1}^{\infty}\left(\bigoplus_{m=0}^{\infty}q^{nm}S^m\mathcal{T}_X\right)\right)\otimes\left(\bigotimes_{n=1}^{\infty}\left(\bigoplus_{m=0}^{\infty}q^{nm}S^m\Omega_X\right)\right). \tag{1.9}$$

Formally expanding out we obtain

$$\mathcal{E}_\lambda=\mathcal{L}_\lambda+q\mathcal{E}_{\lambda,1}+q^2\mathcal{E}_{\lambda,2}+\cdots.$$

Now define

$$\chi(\mathcal{E}_\lambda,q)=\chi(\mathcal{L}_\lambda)+q\chi(\mathcal{E}_{\lambda,1})+q^2\chi(\mathcal{E}_{\lambda,2})+\cdots\in\mathbb{Z}[\![q]\!],$$

where $\chi(\mathcal{E}_{\lambda,n})=\sum_i(-1)^i\dim H^i(X,\mathcal{E}_{\lambda,n})$, the Euler characteristic of $\mathcal{E}_{\lambda,n}$.

It is easy to show, following [12], that

$$\chi\left(\mathcal{L}^{ch}_{\nu(z)},q\right)=\chi(\mathcal{E}_{\nu_0},q).$$

On the other hand, it is known [12], see also some explanations in [25], that $\chi(\mathcal{L}^{ch}_{\nu(z)},q)$ is a version of elliptic genus of X. More precisely, an elliptic genus $g_Q(X,q)$ is attached [28] to a formal power series in x, $Q(X)$, that may also depend, as it does in our situation, on q. We have (letting for simplicity $\nu(z)=0$)

$$\chi\left(\mathcal{L}^{ch}_{\nu(z)=0},q\right)=g_Q(X,q),\qquad Q(x)=\frac{x}{1-e^{-x}}\prod_{n=1}^{\infty}\left(1-q^ne^{-x}\right)^{-1}\left(1-q^ne^{x}\right)^{-1}.$$

An alternative physics interpretation is obtained by recalling that Witten [42] has identified the cohomology vertex algebra $\sum_i H^i(X, \mathcal{L}^{ch}_{\nu(z)=0})$ with the chiral algebra of an appropriate $(0,2)$-supersymmetric sigma-model on X. It follows that $\chi(\mathcal{L}^{ch}_{0/z}, q)$ is the index of the corresponding Dirac operator on the loop space $\mathcal{L}X$:

$$\chi\left(\mathcal{L}^{ch}_{0/z}, q\right) = \operatorname{Ind}(\not{D}, \mathcal{L}X).$$

Theorem 1.1(1), or rather its corollary (1.7), is then a computation of either of these three, defined differently if at all but equal to each other, quantities.

Note also that all coefficients of the genus computed in (1.7) happen to be positive. This is a bit mysterious; more examples of this positivity phenomenon can be found in [25].

As an aside, we would like to mention that the problem of computing the cohomology groups $H^i(X, \mathcal{L}^{ch}_{\nu(z)})$, $0 \leqslant i \leqslant \dim X$, has been around since [37], where what is denoted here by $\mathcal{L}^{ch}_{0/z}$ was introduced and the cohomology found in the case of $\mathfrak{g} = sl_2$. The cohomological dimension zero case of the problem was worked out in [6]. Also proved in [6] is an extension of Theorem 1.1 to not necessarily dominant highest weights ν_0 in the case where $\mathfrak{g} = sl_2$; this is based on the earlier work [36].

Our proof of Theorem 1.1 involves the study of the Drinfeld–Sokolov reduction on an appropriate category $\mathcal{O}$ at the critical level. The result we obtain may be of interest in its own right. The Drinfeld–Sokolov reduction is a version of semi-infinite cohomology. The latter has been known since its inception in [13] to be a mixture of homology and cohomology; a refined treatment of this phenomenon can be found in [39]. We find, somewhat unexpectedly, that the Drinfeld–Sokolov reduction, $H_{DS}^{\infty/2+\bullet}(L\mathfrak{n}_+, ?)$, is more like homology (cf. [1,3]):

- the functor $H_{DS}^{\infty/2+i}(L\mathfrak{n}_+, ?) = 0$ if $i > 0$;
- the functor $H_{DS}^{\infty/2+0}(L\mathfrak{n}_+, ?)$ is right exact, and the class of modules with Verma filtration is adapted to this functor; and
- $H_{DS}^{\infty/2-i}(L\mathfrak{n}_+, ?)$, $i > 0$, is isomorphic to the derived functor $L^i H_{DS}^{\infty/2+0}(L\mathfrak{n}_+, ?)$.

This is recorded in the main body of the text as Theorem 3.5.

2. Vertex algebras and chiral differential operators: examples

We will work over $\mathbb{C}$; all vector spaces will actually be vector superspaces; if V is a vector space, $a \in V$, then by $\tilde{a}$ we shall denote the parity of a.

2.1. Examples of vertex algebras

2.1.1. Definition of a vertex algebra

A *field* on a vector space V is a formal series

$$a(z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1} \in (\operatorname{End} V)\left[\!\left[z, z^{-1}\right]\!\right]$$

such that for any $v \in V$ one has $a_{(n)} v = 0$ if $n \gg 0$.

Let *Fields*(V) denote the space of all fields on V.

A *vertex algebra* is a vector space V with the following data:

- a linear map $Y: V \to \mathit{Fields}(V)$, $V \ni a \mapsto a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$,
- an even vector $\mathbf{1} \in V$, called *vacuum vector*,
- a linear operator $\partial: V \to V$, called *translation operator*,

that satisfy the following axioms:

(1) (Translation covariance)

$$(\partial a)(z) = \partial_z a(z).$$

(2) (Vacuum)

$$\mathbf{1}(z) = \mathrm{id};$$
$$a(z)|0\rangle \in V[\![z]\!] \quad \text{and} \quad a_{(-1)}|0\rangle = a.$$

(3) (Borcherds identity)

$$\sum_{j\geqslant 0} \binom{m}{j} (a_{(n+j)}b)_{(m+k-j)} = \sum_{j\geqslant 0} (-1)^j \binom{n}{j} \bigl\{ a_{(m+n-j)} b_{(k+j)} - (-1)^{n+\tilde{a}\tilde{b}} b_{(n+k-j)} a_{(m+j)} \bigr\}. \tag{2.1}$$

A vertex algebra V is *graded* if $V = \bigoplus_{n\geqslant 0} V_n$ and for $a \in V_i$, $b \in V_j$ we have

$$a_{(k)}b \in V_{i+j-k-1}$$

for all $k \in \mathbb{Z}$. (We put $V_i = 0$ for $i < 0$.)

We say that a vector $v \in V_m$ has *conformal weight* m and write $\Delta_v = m$.

If $v \in V_m$ we denote $v_k = v_{(k-m+1)}$, this is the so-called conformal weight notation for operators. One has

$$v_k V_m \subset V_{m-k}.$$

A *morphism* of vertex algebras is a map $f: V \to W$ that preserves vacuum and satisfies $f(v_{(n)}v') = f(v)_{(n)} f(v')$.

2.1.2. Vertex algebra modules

A *module* over a vertex algebra V is a vector space M together with a map

$$Y^M: V \to \mathit{Fields}(M), \qquad a \to Y^M(a,z) = \sum_{n\in\mathbb{Z}} a^M_{(n)} z^{-n-1}, \tag{2.2}$$

that satisfy the following axioms:

(1) $|0\rangle^M(z) = \mathrm{id}_M$.
(2) (Borcherds identity)

$$\sum_{j\geqslant 0}\binom{m}{j}(a_{(n+j)}b)^M_{(m+k-j)}=\sum_{j\geqslant 0}(-1)^j\binom{n}{j}\bigl\{a^M_{(m+n-j)}b^M_{(k+j)}-(-1)^{n+\tilde{a}\tilde{b}}b^M_{(n+k-j)}a^M_{(m+j)}\bigr\}. \tag{2.3}$$

A module M over a graded vertex algebra V is called *graded* if $M=\bigoplus_{n\geqslant 0}M_n$ with $v_kM_l\subset M_{l-k}$ (assuming $M_n=0$ for negative n) for all $v\in V$. Note that we have switched to the conformal weight notation.

An increasing filtration $\{F_nM,\ n\geqslant 0\}$, $\bigcup_n F_nM=M$, $F_nM=\{0\}$ if $n\ll 0$, is called *conformal* if

$$v_kF_lM\subset F_{l-k}M,\quad \forall v\in V. \tag{2.4}$$

Note that if M is graded, then $\{F_nM=\bigoplus_{i=0}^n M_i\}$ is a conformal filtration, but we shall have a chance to encounter conformally filtered V-modules that are not graded.

A *morphism of modules* over a vertex algebra V is a map $f:M\to N$ that satisfies $f(v^M_{(n)}m)=v^N_{(n)}f(m)$ for $v\in V$, $m\in M$.

2.1.3. Commutative vertex algebras

A vertex algebra is said to be *commutative* if $a_{(n)}b=0$ for a,b in V and $n\geqslant 0$. It is known that a commutative vertex algebra is the same as a commutative associative algebra with derivation.

If W is a vector space we denote by H_W the algebra of differential polynomials on W. As an associative algebra it is a polynomial algebra in variables $x_i,\partial x_i,\partial^{(2)}x_i,\dots$ where $\{x_i\}$ is a basis of W^*. A commutative vertex algebra structure on H_W is uniquely determined by attaching the field $x(z)=e^{z\partial}x_i$ to each x_i.

H_W is equipped with grading such that

$$(H_W)_0=\mathbb{C},\qquad (H_W)_1=W^*. \tag{2.5}$$

2.1.4. Beta–gamma system or a CDO over an affine space

Let U be a purely even vector space, $\{x_i\}\subset U^*$ and $\{\partial_i\}\subset U$, $1\leqslant i\leqslant N$, a pair of dual bases. Denote by $\mathcal{D}^{ch}(U)$ the vertex algebra that is generated by the vector space $U\oplus U^*$ and relations

$$x_{i(n)}x_j=\partial_{i(n)}\partial_j=\partial_{i(n+1)}x_j=0\quad \text{if } n\geqslant 0,\qquad \partial_{i(0)}x_j=\delta_{ij}\mathbf{1}. \tag{2.6}$$

The Borcherds identity (2.1) implies the following commutation relations

$$[x_{i(m)},x_{j(n)}]=[\partial_{i(m)},\partial_{j(n)}]=0,\qquad [\partial_{i(m)},x_{j(n)}]=\delta_{ij}\delta_{m,-n+1}. \tag{2.7}$$

This suggests an index shift $\partial_{in}=\partial_{i(n)}$, $x_{in}=x_{i(n-1)}$, which allows to beautify the last relation as follows

$$[\partial_{im},x_{jn}]=\delta_{ij}\delta_{m,-n}. \tag{2.8}$$

As a vector space, $\mathcal{D}^{ch}(U)$ is freely generated from $\mathbf{1}$ by the family of pairwise commuting operators ∂_{in-1}, x_{in} with $n\leqslant 0$. Thus

$$\mathcal{D}^{ch}(U)\xrightarrow{\sim}\mathbb{C}[\partial_{in-1},x_{in};\ n\leqslant 0,\ 1\leqslant i,j\leqslant N],\qquad \mathbf{1}\xrightarrow{\sim}1\in\mathbb{C}. \tag{2.9}$$

This vertex algebra is graded so that the degree of operators ∂_{in}, x_{in} is $(-n)$. In particular,

$$\mathcal{D}^{ch}(U)_0 = \mathbb{C}[x_{10}, \dots, x_{N0}], \qquad \mathcal{D}^{ch}(U)_1 = \bigoplus_{j=1}^{N} \bigl(x_{j,-1}\mathcal{D}^{ch}\bigl(\mathbb{C}^n\bigr)_0 \oplus \partial_{j,-1}\mathcal{D}^{ch}\bigl(\mathbb{C}^n\bigr)_0\bigr). \tag{2.10}$$

We tend to think of $x_{j0}\mathbf{1}$ as the function x_j on U, $\partial_{j,-1}\mathbf{1}$ as the vector field $\partial/\partial x_j$, $x_{j,-1}$ as the differential form dx_j so that $\mathcal{D}^{ch}(U)_0$ is identified with functions on U and $\mathcal{D}^{ch}(\mathbb{C}^n)_1$ becomes $\mathcal{T}_U(U) \oplus \Omega_U(U)$. We shall soon make more sense out of this interpretation; in particular, we shall see that $\mathcal{D}^{ch}(U)$ is the space of global sections of a sheaf of *chiral differential operators* (CDO) over U, $\mathcal{D}^{ch}_U$, and that the latter direct sum is a result of making choices, but the exact sequence

$$0 \to \Omega_U(U) \to \mathcal{D}^{ch}(U)_1 \to \mathcal{T}_U(U) \to 0 \tag{2.11}$$

is natural; here $\Omega_U(U) \to \mathcal{D}^{ch}(U)_1$ is defined by

$$f(x_1, \dots x_N)\, dx_j \mapsto x_{j,-1} f(x_{10}, \dots x_{N0})\mathbf{1},$$

and $\mathcal{D}^{ch}(U)_1 \to \mathcal{T}_U(U)$ is defined by

$$x_{j,-1} f(x_{10}, \dots x_{N0})\mathbf{1} \mapsto 0, \qquad \partial_{j,-1} f(x_{10}, \dots x_{N0})\mathbf{1} \mapsto f(x_{10}, \dots x_{N0})\partial/\partial x_j.$$

It is quite clear that the assignment $U \mapsto \mathcal{D}^{ch}(U)$ defines a functor on the category of affine spaces with affine isomorphisms for morphisms; in other words, a change of variables $x_i \mapsto a_{is}x_s + b_i$ canonically lifts to an isomorphism of $\mathcal{D}^{ch}(U)$.

2.1.5. A super-version: Clifford algebra

The discussion in Section 2.1.4 is easily carried over to the case where the purely even U is replaced with a supervector space of dimension $M|N$. We shall need the example of dimension $0|N$, so we define the Clifford vertex algebra, $Cl(U)$, to be a vertex superalgebra that is generated by the *purely odd* vector space $U \oplus U^*$ and relations

$$\phi^*_{(n)}\phi^* = \phi_{(n)}\phi = \phi_{(n+1)}\phi^* = 0 \quad \text{if } n \geqslant 0, \qquad \phi_{(0)}\phi^* = \langle \phi, \phi^* \rangle, \quad \phi \in U,\ \phi^* \in U^*. \tag{2.12}$$

Upon introducing $\phi_n = \phi_{(n)}$, $\phi^*_n = \phi^*_{(n-1)}$ the last relation becomes

$$\bigl[\phi_m, \phi^*_n\bigr] = \langle \phi, \phi^* \rangle \delta_{m,-n}. \tag{2.13}$$

As a vector space, $Cl(U)$ is freely generated from $\mathbf{1}$ by the family of pairwise commuting operators ϕ_{in-1}, ϕ^*_{in} with $n \leqslant 0$; here $\{\phi_i\}$ and $\{\phi^*_i\}$ is a pair of dual bases. Thus

$$Cl(U) \xrightarrow{\sim} \mathbb{C}\bigl[\phi_{in-1}, \phi^*_{in};\ n \leqslant 0,\ 1 \leqslant i, j \leqslant N\bigr]\mathbf{1},$$

where the polynomial ring is regarded as a superpolynomial ring, all generators being odd.

The functoriality of $U \mapsto Cl(U)$ is obvious.

2.1.6. Affine vertex algebras

Let $\mathfrak{g}$ be a simple Lie algebra and $(.,.) : S^2\mathfrak{g} \to \mathbb{C}$ the *normalized* invariant bilinear form on $\mathfrak{g}$, i.e., the form such that square of the length of the longest root w.r.t. the bilinear form induced on the dual Cartan subalgebra is 2. The *affine Lie algebra* $\hat{\mathfrak{g}}$ associated with $\mathfrak{g}$ and $(.,.)$ is a central extension of $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] + \mathbb{C}K$ with bracket

$$\left[x \otimes t^n, y \otimes t^m\right] = [x, y] \otimes t^{m+n} + \delta_{n+m,0}(x, y).$$

Denote by $V_k(\mathfrak{g})$ the vertex algebra generated by $\mathfrak{g}$ with relations

$$x_{(0)} y = [x, y], \qquad x_{(1)} y = k(x, y). \tag{2.14}$$

Denote by $\hat{\mathfrak{g}}_{\geqslant}$ the subalgebra $\mathfrak{g} \otimes \mathbb{C}[t] \oplus \mathbb{C}k$ and by $\mathbb{C}_k$ its 1-dimensional module, where $\mathfrak{g} \otimes \mathbb{C}[t]$ acts as 0 and $K \mapsto k$. One has

$$V_k(\mathfrak{g}) \xrightarrow{\sim} \operatorname{Ind}_{\hat{\mathfrak{g}}_{\geqslant}}^{\mathfrak{g}} \mathbb{C}_k. \tag{2.15}$$

The field attached to $x \in \mathfrak{g}$ is $x(z) = \sum_{n\in\mathbb{Z}} x_n z^{-n-1}$, where x_n stands for $x \otimes t^n \in \hat{\mathfrak{g}}$ and is regarded as an operator acting on $\operatorname{Ind}_{\hat{\mathfrak{g}}_{\geqslant}}^{\mathfrak{g}} \mathbb{C}_k$.

$V_k(\mathfrak{g})$ is a graded vertex algebra, $V_k(\mathfrak{g}) = \bigoplus_{n\geqslant 0} V_k(\mathfrak{g})_n$, the grading being uniquely determined by the condition that the conformal weight of $\mathbf{1} \stackrel{\text{def}}{=} 1 \in \mathbb{C}_k$ be 0, and the corresponding degree of x_n be $(-n)$.

One can likewise define a vertex algebra associated with any Lie algebra $\mathfrak{a}$ and an invariant bilinear form on it, $(.,.)$. Since in general there is no distinguished such form, not even up to proportionality, we shall use the notation $V_{(.,.)}(\mathfrak{a})$, or $V_0(\mathfrak{a})$ if $(.,.) = 0$.

2.2. Wakimoto modules and algebras of chiral differential operators

Let G be a simple connected complex Lie group, $\mathfrak{g}$ the corresponding Lie algebra, $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ a triangular decomposition, B, B_- resp., the subgroups of G corresponding to $\mathfrak{n}_+ \oplus \mathfrak{h}$, $\mathfrak{n}_- \oplus \mathfrak{h}$ resp., U, U_- the maximal unipotent subgroups of B and B_- resp. We will be interested in the flag manifold of G to be denoted by X and realized as G/B_-.

2.2.1. Feigin–Frenkel–Wakimoto bozonization

The natural action $G \times G/B_- \to G/B_-$ defines a Lie algebra homomorphism

$$\mathfrak{g} \to \Gamma(X, \mathcal{T}_X). \tag{2.16}$$

Denote by Δ_+ the set of positive (relative to the fixed triangular decomposition) roots of $\mathfrak{g}$. Let $w(\mathfrak{n}_+) \subset \mathfrak{g}$ be the maximal nilpotent subalgebra spanned by the root vectors with roots in $w(\Delta_+)$, w being any element of the Weyl group W. The flag manifold has an atlas consisting of U^w-orbits, $U^w(w\overline{B_-})$, where U^w is the maximal unipotent subgroup associated to $w(\mathfrak{n}_+)$. Each such orbit is a U^w-torsor, and in order to simplify the notation, we will identify U^w with $U^w(w\overline{B_-})$ by sending id in the former to $w\overline{B_-}$ in the latter.

Morphism (2.16) defines morphisms

$$\mathfrak{g} \to \Gamma\bigl(U^w, \mathcal{T}_X\bigr), \quad w \in W. \tag{2.17}$$

Enter the vertex algebra $\mathcal{D}^{ch}(U^w)$, Section 2.1.4. Since $V_k(\mathfrak{g})$ is generated by $\mathfrak{g}$, see Section 2.1.6, it is natural to ask whether (2.17) can be lifted to a vertex algebra morphism $V_k(\mathfrak{g}) \to \mathcal{D}^{ch}(U^w)$. Note that (2.11) implies a lifting is determined by (2.17) modulo $\Gamma(U^w, \Omega_{U^w})$.

The answer is 'yes' but only for $k = -h^\vee$, minus the dual Coxeter number. This is the content of an important result of Feigin and Frenkel (building on earlier work of Wakimoto [41]).

Theorem 2.1. *(See [14,17].) There is a unique lift of* (2.17) *to* $\mathfrak{g} \to \mathcal{D}^{ch}(U^w)_1$ *that extends to a vertex algebra morphism*

$$\pi_w : V_{-h^\vee}(\mathfrak{g}) \to \mathcal{D}^{ch}\big(U^w\big). \tag{2.18}$$

The center of a vertex algebra V is defined to be

$$\mathfrak{z}(V) = \{v \in V \text{ s.t. } v_{(n)}V = 0,\ n \geqslant 0\}. \tag{2.19}$$

It is a striking feature of the *critical level* $k = -h^\vee$ that at this level $V_{-h^\vee}(\mathfrak{g})$ acquires a big center, another important result of Feigin and Frenkel.

Theorem 2.2. *(See [15,17].)*

(1) *There are elements* $p_i \in V_{-h^\vee}(\mathfrak{g})_{d_i}$, $1 \leqslant i \leqslant \operatorname{rk}\mathfrak{g}$, *such that* $\mathfrak{z}(V_{-h^\vee}(\mathfrak{g}))$ *is generated by* $\{p_1, \ldots, p_r\}$.
(2) *As a vertex algebra,* $\mathfrak{z}(\mathfrak{g}) \stackrel{\text{def}}{=} \mathfrak{z}(V_{-h^\vee}(\mathfrak{g}))$ *is isomorphic to the algebra of differential polynomials on a* $\operatorname{rk}\mathfrak{g}$*-dimensional space.*

A useful complement to Theorem 2.1, also due to Feigin and Frenkel, is that

$$\pi_w\big(\mathfrak{z}(\mathfrak{g})^*\big) = 0, \tag{2.20}$$

where $\mathfrak{z}(\mathfrak{g})^*$ is the argumentation ideal of $\mathfrak{z}(\mathfrak{g})$.

2.2.2. A CDO over the flag manifold

Since U^w is an affine space, the discussion in Section 2.1.4 allows us to attach to each U^w a graded vertex algebra $\mathcal{D}^{ch}(U^w) = \bigoplus_{j\geqslant 0} \mathcal{D}^{ch}(U^w)_j$. We shall now see how these can be glued together into a sheaf over X.

First of all, we observe [37] that there is a sheaf over U^w of which $\mathcal{D}^{ch}(U^w)$ is the space of global sections. For each polynomial $f \in \mathbb{C}[U^w]$, let U^w_f be the Zariski open subset of U^w obtained by deleting the zero locus $\{f = 0\}$. Define

$$\mathcal{D}^{ch}\big(U^w_f\big) = \mathbb{C}\big[U^w\big]_{(f)} \bigotimes_{\mathbb{C}[U^w]} \mathcal{D}^{ch}\big(U^w\big),$$

which makes sense due to (2.9), where we identify, as promised, $\mathbb{C}[x_{10}, \ldots, x_{N0}]$ with $\mathbb{C}[x_1, \ldots, x_N] = \mathbb{C}[U]$.

The fact of the matter is the remark, proved in [37] and going back to Feigin, that the vertex algebra structure on $\mathcal{D}^{ch}(U^w)$ extends to that on $\mathcal{D}^{ch}(U^w_f)$. It is clear that the assignment $Y \mapsto \mathcal{D}^{ch}(Y)$ is a sheaf of graded vertex algebras on U^w such that its space of global sections is $\mathcal{D}^{ch}(U^w)$. Denote this sheaf $\mathcal{D}^{ch}_{U^w}$; this is a CDO over U^w.

In this setting, (2.11) reads

$$0 \to \Omega_{U^w} \to \left(\mathcal{D}^{ch}_{U^w}\right)_1 \to \mathcal{T}_{U^w} \to 0.$$

In fact, by construction, this extension is split, but this splitting is not natural, as we shall see in a second. In any case, it defines a filtration, $\Omega_{U^w} \subset (\mathcal{D}^{ch}_{U^w})_1$, and the corresponding $\mathrm{gr}(\mathcal{D}^{ch}_{U^w})_1 = \Omega_{U^w} \oplus \mathcal{T}_{U^w}$.

Now recall that $\{U^w\}$ is an atlas of X, and there are transition functions

$$\rho_{wv} : \Omega_{U^w}|_{U^w \cap U^v} \xrightarrow{\sim} \Omega_{U^v}|_{U^w \cap U^v}; \qquad \mathcal{T}_{U^w}|_{U^w \cap U^v} \xrightarrow{\sim} \mathcal{T}_{U^v}|_{U^w \cap U^v}.$$

We would like to lift them to the CDOs $\mathcal{D}^{ch}_{U^w}$, $w \in W$.

Theorem 2.3. *(See [26].)*

(1) *There exist $\mathbb{C}$-isomorphisms*

$$\hat{\rho}_{wv} : \left(\mathcal{D}^{ch}_{U^w}\big|_{U^w \cap U^v}\right)_1 \xrightarrow{\sim} \left(\mathcal{D}^{ch}_{U^v}\big|_{U^w \cap U^v}\right)_1$$

that satisfy

(a) *they uniquely extend to vertex algebra isomorphisms*

$$\hat{\rho}_{wv} : \mathcal{D}^{ch}_{U^w}\big|_{U^w \cap U^v} \xrightarrow{\sim} \mathcal{D}^{ch}_{U^v}\big|_{U^w \cap U^v};$$

(b) *they preserve the filtration and the corresponding graded morphisms equal the classical ρ_{wv};*

(c) *the cocycle condition, $\hat{\rho}_{wu} = \hat{\rho}_{vu} \circ \hat{\rho}_{wv}$, holds on triple intersections $U^w \cap U^v \cap U^u$.*

(2) *Over X, there is a CDO, $\mathcal{D}^{ch}_X$, such that $\Gamma(U^w, \mathcal{D}^{ch}_X) = \mathcal{D}^{ch}(U^w)$ and $\{\hat{\rho}_{wu}\}$ are transition functions.*

(3) *The vertex algebra morphisms* (2.18) *are compatible with $\{\hat{\rho}_{wu}\}$ and define a vertex algebra morphism*

$$\pi : V_{-h^\vee}(\mathfrak{g}) \to \Gamma\left(X, \mathcal{D}^{ch}_X\right), \qquad \pi\left(\mathfrak{z}(\mathfrak{g})^*\right) = 0.$$

(4) *The transition functions $\{\hat{\rho}_{wv}\}$ are not unique but the equivalence class of the corresponding CDO is independent of the choice.*

Note that the transition functions $\{\hat{\rho}_{wu}\}$ are not $\mathcal{O}_X$-linear, and so $\mathcal{D}^{ch}_X$ is not a sheaf of $\mathcal{O}_X$-modules, but the filtration on $(\mathcal{D}^{ch}_X)_1$ extends to the entire $\mathcal{D}^{ch}_X$ so that the corresponding graded object is a sheaf of locally trivial $\mathcal{O}_X$-modules. We have, cf. (1.9),

$$\mathrm{Gr}\,\mathcal{D}^{ch}_X = \left(\bigotimes_{n=1}^{\infty}\left(\bigoplus_{m=0}^{\infty} q^{nm} S^m \mathcal{T}_X\right)\right) \otimes \left(\bigotimes_{n=1}^{\infty}\left(\bigoplus_{m=0}^{\infty} q^{nm} S^m \Omega_X\right)\right). \tag{2.21}$$

2.2.3. Wakimoto modules of critical level

By pull-back, $\mathcal{D}^{ch}(U^{\mathrm{id}})$ is a $V_{-h^\vee}(\mathfrak{g})$-module, hence a $\hat{\mathfrak{g}}$-module. According to (2.20), the center operates on $\mathcal{D}^{ch}(U^{\mathrm{id}})$ as 0; we shall call this module, following [14,17], the *Wakimoto module* of critical level and zero central character and denote it by $\mathbb{W}_{\nu(z)=0}$.

2.2.4. A universal twisted CDO over the flag manifold

The constructions of Section 2.2.2 can be, roughly speaking, deformed. For each integral weight $\lambda \in P \in \mathfrak{h}^*$, denote by $\mathcal{L}_\lambda$ the corresponding G-equivariant invertible sheaf. The action of G determines a Lie algebra morphism

$$\mathfrak{g} \to \Gamma\big(X, \mathcal{D}_X^\lambda\big), \tag{2.22}$$

where $\mathcal{D}_X^\lambda$ is the algebra of (twisted) differential operators acting on $\mathcal{L}_\lambda$. Trivializing $\mathcal{L}_\lambda$ over U^w, the latter morphism becomes

$$\mathfrak{g} \to \Gamma\big(U^w, \mathcal{T}_X\big) \oplus \Gamma\big(U^w, \mathcal{O}_X\big).$$

It is clear that the choices can be made to ensure that this morphism depends on λ polynomially, cf. a similar discussion in [9, Section 2.5]. Thus we obtain a morphism

$$\mathfrak{g} \to \Gamma\big(U^w, \mathcal{T}_X\big) \oplus \Gamma\big(U^w, \mathcal{O}_X\big) \otimes \mathbb{C}\big[\mathfrak{h}^*\big]. \tag{2.23}$$

A vertex algebra version of this is as follows. Let H_X be the commutative vertex algebra of differential polynomials on $\mathfrak{h}^*$, Section 2.1.3; this is an analogue of $\mathbb{C}[\mathfrak{h}^*]$. Feigin and Frenkel proved [14,17] that (2.23) extends, for each w, to a vertex algebra morphism

$$\pi_w : V_{-h^\vee}(\mathfrak{g}) \to \mathcal{D}^{ch}\big(U^w\big) \otimes H_X \quad \text{s.t.} \quad \pi_w|_{\mathfrak{z}(V_{-h^\vee}(\mathfrak{g}))} : \mathfrak{z}\big(V_{-h^\vee}(\mathfrak{g})\big) \hookrightarrow H_X, \tag{2.24}$$

where $\mathcal{D}^{ch}(U^w) \otimes H_X$ is the result of the well-known operation of tensor product of vertex algebras, see e.g. [29]. Note that the first assertion of (2.24) is a reasonably easy consequence of (2.18) [18].

In this case, too, the morphisms π_w can be arranged into a single morphism from $V_{-h^\vee}(\mathfrak{g})$ to a certain sheaf of *twisted chiral differential operators* (TCDO), $\mathcal{D}_X^{ch,tw}$.

Theorem 2.4. *(See [6].) There is a sheaf of graded vertex algebras, $\mathcal{D}_X^{ch,tw}$, over X such that*

(1) *$\Gamma(U^w, \mathcal{D}_X^{ch,tw})$ is isomorphic to $\mathcal{D}^{ch}(U^w) \otimes H_X$;*
(2) *the tautological embeddings $H_X \hookrightarrow \mathcal{D}^{ch}(U^w) \otimes H_X$ define an embedding $H_X \hookrightarrow \mathcal{D}_X^{ch,rw}$ as a constant subsheaf; furthermore, this makes H_X the center of $\mathcal{D}_X^{ch,tw}$;*
(3) *$\mathcal{D}_X^{ch}$ is isomorphic to $\mathcal{D}_X^{ch,tw}$ modulo the ideal generated by H_X;*
(4) *the morphisms* (2.24) *define a graded vertex algebra morphism*

$$\pi : V_{-h^\vee}(\mathfrak{g}) \to \Gamma\big(X, \mathcal{D}_X^{ch,tw}\big) \quad s.t. \quad \pi\big(\mathfrak{z}\big(V_{-h^\vee}(\mathfrak{g})\big)\big) \subset H_X.$$

2.2.5. Wakimoto modules

By pull-back, each $\Gamma(U^w, \mathcal{D}_X^{ch,tw})$ is a $V_{-h^\vee}(\mathfrak{g})$-, hence $\hat{\mathfrak{g}}$-module. Call, following [14,17], $\Gamma(U^{\mathrm{id}}, \mathcal{D}_X^{ch,tw})$ a *Wakimoto module* with highest weight $(0, -h^\vee)$ and denote it by $\mathbb{W}_{0,-h^\vee}$.

Note that $\mathbb{W}_{0,-h^\vee}$ is different from a closely related Wakimoto module *of critical level* and zero central character $\mathbb{W}_{\nu(z)=0}$ introduced in Section 2.2.3: $\mathbb{W}_{0,-h^\vee}$ are bigger than $\mathbb{W}_{\nu(z)=0}$ because they contain H_X and, unlike $\mathbb{W}_{\nu(z)=0}$, can be deformed away from the critical level.

2.2.6. Modules over the twisted CDO

The twisted CDO $\mathcal{D}_X^{ch,tw}$ is a deformation of $\mathcal{D}_X^{ch}$ only morally: the vertex algebra axioms resist letting $p \in H_X$ be a number. The situation changes pleasingly upon passing to $\mathcal{D}_X^{ch,tw}$-modules, where we shall at once get families of modules depending on $\operatorname{rk}\mathfrak{g}$ functional parameters.

We will call a sheaf of vector spaces $\mathcal{M}$ a $\mathcal{D}_X^{ch,tw}$-*module* if

(1) for each open $U \subset X$, $\Gamma(U, M)$ is a $\Gamma(U, \mathcal{D}_X^{ch,tw})$-module;
(2) the restriction morphisms $\Gamma(U, \mathcal{M}) \to \Gamma(V, \mathcal{M})$, $V \subset U$, are $\Gamma(U, \mathcal{D}_X^{ch,tw})$-module morphisms, where the $\Gamma(U, \mathcal{D}_X^{ch,tw})$-module structure on $\Gamma(V, \mathcal{M})$ is that of the pull-back w.r.t. the restriction map $\Gamma(U, \mathcal{D}_X^{tw}) \to \Gamma(V, \mathcal{D}_X^{tw})$;
(3) $\mathcal{M}$ is conformally filtered, cf. (2.4), namely, there is an increasing sequence of subsheaves

$$\{\mathrm{F}_n\mathcal{M},\ n \in \mathbb{Z}\}, \qquad \bigcup_{n=-\infty}^{+\infty} \mathrm{F}_n\mathcal{M}_n = \mathcal{M}, \qquad \mathrm{F}_n\mathcal{M} \subset \mathrm{F}_{n+1}\mathcal{M}, \qquad \mathrm{F}_n\mathcal{M} = \{0\} \quad \text{if } n \ll 0 \tag{2.25}$$

so that

$$\left(\mathcal{D}_X^{ch,tw}\right)_l \mathcal{M}_n \subset \mathcal{M}_{n-l}. \tag{2.26}$$

Denote by $\mathcal{D}_X^{ch,tw}$-Mod the category of $\mathcal{D}_X^{ch,tw}$-modules.

Since the vertex algebra H_X is commutative, its irreducibles are all 1-dimensional and are in 1–1 correspondence with the algebra of Laurent series with values in $\mathfrak{h}^*$. Specifically, if $\nu(z) \in \mathfrak{h}^*((z))$, then the character $\mathbb{C}_{\nu(z)}$ is a 1-dimensional H_X-module defined by

$$\nu : H_X \to \mathit{Fields}(\mathbb{C}_{\nu(z)}), \qquad H_X \ni \lambda \mapsto \lambda^{\mathbb{C}_{\nu(z)}}(z) \stackrel{\text{def}}{=} \lambda\big(\nu(z)\big), \tag{2.27}$$

cf. Section 2.1.2 and recall that H_X is the algebra of differential polynomials on $\mathfrak{h}^*$. For example, if $\lambda \in \mathfrak{h}$, thus λ is a linear function on $\mathfrak{h}^*$, and $\nu(z) = \sum_n \nu_n z^{-n-1}$, then

$$\nu(\lambda)(z) = \sum_{n\in\mathbb{Z}} \lambda(\nu_n) z^{-n-1} \quad \text{or} \quad \nu(\lambda)_{(n)} = \lambda(\nu_n).$$

Denote by $\mathcal{D}_X^{ch,tw}$-$\mathrm{Mod}_{\nu(z)}$ the full subcategory of $\mathcal{D}_X^{ch,tw}$-Mod consisting of those $\mathcal{D}_X^{ch,tw}$-modules, where H_X acts according to the character $\nu(z)$.

We will say that a character $\nu(z) \in \mathfrak{h}^*((z))$ has *regular singularity* if $\nu(z) = \nu_0 z^{-1} + \nu_{-1} + \nu_{(-2)} z + \cdots$.

It is easy to see [6] that if $\nu(z)$ has regular singularity, then for each $\mathcal{M} \in \mathcal{D}_X^{ch,tw}$-$\mathrm{Mod}_{\nu(z)}$,

$$\operatorname{Sing}\mathcal{M} \stackrel{\text{def}}{=} \left\{m \in \mathcal{M} \text{ s.t. } v_n m = 0\ \forall v \in \mathcal{D}_X^{ch,tw},\ n > 0\right\}$$

is naturally a $\mathcal{D}_X^{\nu_0}$-module, hence a functor

$$\operatorname{Sing} : \mathcal{D}_X^{ch,tw}\text{-}\mathrm{Mod}_{\nu(z)} \to \mathcal{D}_X^{\nu_0}\text{-}\mathrm{Mod}, \qquad \mathcal{M} \mapsto \operatorname{Sing}\mathcal{M}. \tag{2.28}$$

This functor has the left adjoint [6]

$$\mathcal{Z}hu_{\nu(z)} : \mathcal{D}_X^{\nu_0}\text{-Mod} \to \mathcal{D}_X^{ch,tw}\text{-Mod}_{\nu(z)}, \tag{2.29}$$

which, as the notation suggests, is closely related to Zhu's work [43]. Its construction is simple enough: define

$$\mathcal{Z}hu_{\nu(z)}(\mathcal{A})\big(U^w\big) = \mathcal{D}^{ch}\big(U^w\big) \otimes_{\mathbb{C}} \Gamma\big(U^w, \mathcal{A}\big) \tag{2.30}$$

and then mimic the proof of Theorem 2.4 to glue these pieces together. In particular, one sees that even though $\mathcal{Z}hu_{\nu(z)}(\mathcal{A})(U^w) = \mathcal{D}^{ch}(U^w) \otimes_{\mathbb{C}} \Gamma(U^w, \mathcal{A})$ appears graded by setting $\mathcal{Z}hu_{\nu(z)}(\mathcal{A})(U^w)_n = \mathcal{D}^{ch}(U^w)_n \otimes_{\mathbb{C}} \Gamma(U^w, \mathcal{A})$, the actual sheaf is only filtered by

$$\Gamma\big(U^w, \mathcal{Z}hu_{\nu(z)}(\mathcal{A})_n\big) = \bigoplus_{j=0}^{n} \mathcal{D}^{ch}\big(U^w\big)_j \otimes_{\mathbb{C}} \Gamma\big(U^w, \mathcal{A}\big). \tag{2.31}$$

Theorem 2.5. *(See [6, Theorem 5.2, Remark 5.4].) The functors*

$$\mathcal{Z}hu_{\nu(z)} : \mathcal{D}_X^{\nu_0}\text{-Mod} \leftrightarrow \mathcal{D}_X^{ch,tw}\text{-Mod}_{\nu(z)} : \mathrm{Sing}$$

are quasiinverses of each other that establish an equivalence of categories.

Note that for each $\mathcal{M} \in \mathcal{D}_X^{ch,tw}\text{-Mod}_{\nu(z)}$, the corresponding graded object, $\mathrm{Gr}_F\,\mathcal{M}$ is an object of the category $\mathcal{M} \in \mathcal{D}_X^{ch,tw}\text{-Mod}_{\nu_0/z}$. (Indeed, the character $\nu(z) = \nu_0/z + \nu_{-1} + \cdots$ is the only source of inhomogeneity, see e.g. (2.30), and the passage to the graded object replaces $\nu(z)$ with ν_0/z, a homogeneous character.)

Having refined the filtration (2.31) further, as in the discussion that led to (2.21), one obtains

$$\mathrm{gr}\,\mathcal{Z}hu_{\nu(z)}(\mathcal{A}) = \mathcal{A} \otimes \left(\bigotimes_{n=1}^{\infty}\left(\bigoplus_{m=0}^{\infty} q^{nm} S^m \mathcal{T}_X\right)\right) \otimes \left(\bigotimes_{n=1}^{\infty}\left(\bigoplus_{m=0}^{\infty} q^{nm} S^m \Omega_X\right)\right), \tag{2.32}$$

where we have used $\mathrm{gr}\,\mathcal{Z}hu(\mathcal{A})$ in place of a more logical but awkward $\mathrm{Gr}\,\mathrm{Gr}_F\,\mathcal{A}$.

We shall use Theorem 2.5 only as a source of examples of modules from $\mathcal{D}_X^{ch,tw}\text{-Mod}_{\nu(z)}$.

2.2.7. Examples of $\mathcal{D}_X^{ch,tw}$-modules

To obtain examples of $\mathcal{D}_X^{ch,tw}$-modules with central character $\nu(z)$, we need a supply of $\mathcal{D}_X^{\nu_0}$-modules. For the purposes of representation theory, the most interesting are those $\mathcal{D}_X^{\nu_0}$-modules that are U-equivariant, hence supported on a union of U-orbits.

Let $X_w \stackrel{\text{def}}{=} Uw\overline{B_-}$, $w \in W$, $i_w : X_w \hookrightarrow X$ the tautological embedding, $i_{w,+} : \mathcal{D}_{X_w}\text{-Mod} \to \mathcal{D}_X\text{-Mod}$ the $\mathcal{D}$-module direct image functor. We obtain a family of $\mathcal{D}_X$-modules, $i_{w,+}\mathcal{O}_{X_w}$, $w \in W$; $i_{w,+}\mathcal{O}_{X_w}$ is often referred to as "the module of distributions supported on X_w."

The space $\Gamma(U^w, i_{w,+}\mathcal{O}_{X_w})$, which is essentially the space of global sections, is easy to describe. Identify U^w with $w(\mathfrak{n}_+)$ be means of the exponential map. Let $\{x_\alpha,\ \alpha \in w(\Delta_+)\}$ be a basis of $\mathfrak{n}_+^*$ dual to the root vector basis of $\mathfrak{n}_+$. In this basis, the submanifold X_w is defined by linear equations:

$$X_w = \{x_\alpha = 0,\ \alpha \in w(\Delta_+) \setminus \Delta_+\}.$$

By definition, the space $\Gamma(U^w, i_{w,+}\mathcal{O}_{X_w})$ is a module over the Weyl algebra (one generated by x_α, $\partial/\partial x_\alpha$, $\alpha \in w(\Delta_+)$) with one generator, $\mathbf{1}_w$, and relations

$$x_\alpha \mathbf{1}_w = \partial/\partial x_\beta \mathbf{1}_w = 0 \quad \text{if } \alpha \in w(\Delta_+) \setminus \Delta_+,\ \beta \in w(\Delta_+) \cap \Delta_+.$$

Similarly, $\Gamma(U^w, \mathcal{Z}hu_0(i_{w,+}\mathcal{O}_{X_w}))$ is a $\mathcal{D}^{ch}(U^w) \otimes H_X$-module with generator $\mathbf{1}_w$ and relations

$$x_{\alpha,n+1}\mathbf{1}_w = \partial_{\alpha,n+1}\mathbf{1}_w = 0 \quad \text{if } \alpha \in w(\Delta_+),\ n \geqslant 0, \tag{2.33}$$

$$x_{\alpha,0}\mathbf{1}_w = \partial/\partial x_{\beta,0}\mathbf{1}_w = 0 \quad \text{if } \alpha \in w(\Delta_+) \setminus \Delta_+,\ \beta \in w(\Delta_+) \cap \Delta_+, \tag{2.34}$$

$$(H_X)_{(m)}\mathbf{1}_w = 0 \quad \text{if } m \in \mathbb{Z}. \tag{2.35}$$

All of this is easy to twist by a character $\nu(z) = \nu_0/z + \nu_{-1} + \cdots$. There still is a functor [9, Section 2.5.5]

$$i_{w,+} : \mathcal{D}_{X_w}^{i_w^*(\mathcal{L}_{\nu_0})}\text{-Mod} \to \mathcal{D}_X^{\nu_0}\text{-Mod}.$$

This gives us a collection of $\mathcal{D}_X^{\nu_0}$-modules, $i_{w,+}i_w^*\mathcal{L}_{\nu_0}$, $w \in W$, if ν_0 is integral. Note that since X_w is affine

$$i_{w,+}i_w^*\mathcal{L}_{\nu_0} \xrightarrow{\sim} i_{w,+}\mathcal{O}_{X_w} \quad \text{as } \mathcal{O}_X\text{-modules}, \tag{2.36}$$

but the actions of $\mathfrak{g}$, on the former induced by (2.22), on the latter by (2.16), are different.

$\Gamma(U^w, \mathcal{Z}hu_{\nu(z)}(i_{w,+}\mathcal{O}_{X_w}))$ is a $\mathcal{D}^{ch}(U^w) \otimes H_X$-module with generator $\mathbf{1}_{w\circ\nu_0}$ and relations (2.33), (2.34), and the following replacement of (2.35)

$$p_{(n)}\mathbf{1}_{w\circ\lambda} = p\big(\nu(z)\big)_{(n)}\mathbf{1}_{w\circ\lambda}, \quad p \in H_X, \tag{2.37}$$

where $p(\nu(z))_{(n)}$ stands for $\mathrm{Res}_{z=0}\, z^n p(\nu(z))$.

By pull-back, each $\Gamma(U^w, \mathcal{Z}hu_{\nu(z)}(i_{w,+}\mathcal{O}_{X_w}))$ is a $V_{-h^\vee}(\mathfrak{g})$ – hence $\hat{\mathfrak{g}}$-module. It is quite clear that $\Gamma(U^{\mathrm{id}}, \mathcal{Z}hu_0(i_*\mathcal{O}_{X_{\mathrm{id}}}))$ is precisely the Wakimoto module of critical level and zero central character, $\mathbb{W}_{\nu(z)=0}$, that was introduced in Section 2.2.3. We can now see how by passing to the twisted CDO we have gained considerable flexibility: denote by

$$\mathbb{W}_{\nu(z)} = \Gamma\big(U^{\mathrm{id}}, \mathcal{Z}hu_{\nu(z)}(i_*\mathcal{O}_{X_{\mathrm{id}}})\big); \tag{2.38}$$

these are Wakimoto modules of critical level and central character $\nu(z)$, by construction [14,17].

Furthermore, as a quick scan of [17, Section 9.5.1] shows,

$$\mathbb{W}_{\nu(z)}^w \stackrel{\text{def}}{=} \Gamma\big(U^w, \mathcal{Z}hu_{\nu(z)}\big(i_{w,+}i_w^*\mathcal{L}_{\nu_0}\big)\big), \quad w \in W, \tag{2.39}$$

are the so-called *w-twisted* Wakimoto modules of critical level and central character $\nu(z)$. (Indeed, used in [17, Section 9.5.1], is the "β–γ-system" $M_{\mathfrak{g}}^{w}$, which is our $\mathcal{D}^{ch}(U^{\mathrm{id}})$ except that the choice of vacuum is different. To give $\mathcal{D}^{ch}(U^{\mathrm{id}})$ a $V_{-h^\vee}(\mathfrak{g})$-module structure, the standard morphism $\pi_{\mathrm{id}}: V_{-h^\vee}(\mathfrak{g}) \to \mathcal{D}^{ch}(U^{\mathrm{id}})$ is twisted in [17, Section 9.5.1], by the Tits lifting of $w \in W \subset \mathrm{Aut}(\mathfrak{h}^*)$ to $\tilde{w} \in \mathrm{Aut}(\mathfrak{g})$. If we identify $\mathcal{D}^{ch}(U^{\mathrm{id}})$ with $\mathcal{D}^{ch}(U^{\mathrm{w}})$ via the same $\tilde{w} \in \mathrm{Aut}(\mathfrak{g})$, see Section 2.2.1, then, by definition, $\pi_w = \pi \circ \tilde{w}^{-1}$, where π_w comes from (2.24). According to Theorem 2.4(4), it is these $\{\pi_w\}$ that 'conspire' to define a $V_{-h^\vee}(\mathfrak{g})$-module structure on $\mathcal{D}_X^{ch,tw}$. Now the obvious observation that our choice of vacuum (2.33)–(2.35) is consistent with the one made in [17, Section 9.5.1], concludes this little bit of translation.)

Lemma 2.6.

$$H^0\big(X, \mathcal{Z}hu_{\nu(z)}\big(i_{w,+}i_w^*\mathcal{L}_{\nu_0}\big)\big) = \Gamma\big(U^w, \mathcal{Z}hu_{\nu(z)}\big(i_{w,+}i_w^*\mathcal{L}_{\nu_0}\big)\big),$$
$$H^i\big(X, \mathcal{Z}hu_{\nu(z)}\big(i_{w,+}i_w^*\mathcal{L}_{\nu_0}\big)\big) = 0 \quad \textit{if } i > 0.$$

Proof. As (2.32) shows, $\mathcal{Z}hu_{\nu(z)}(i_{w,+}i_w^*\mathcal{L}_{\nu_0})$ carries a filtration such that the corresponding graded object, $\mathrm{gr}\,\mathcal{Z}hu_{\nu(z)}(i_{w,+}i_w^*\mathcal{L}_{\nu_0})$, is an $\mathcal{O}_X$-module. Furthermore, this graded object is actually a push-forward, $i_{w,*}\mathcal{E}$, of a locally free sheaf of $\mathcal{O}_{X_w}$-modules: to obtain this sheaf simply replace in (2.32) $\mathcal{T}_X$ with $i_w^*\mathcal{T}_X$, Ω_X with $i_w^*\Omega_X$, $\mathcal{A}$ with $S^\bullet\mathcal{N}_{X_w}$, where $\mathcal{N}_{X_w}$ is the normal bundle to X_w.

The fact that X_w is affine implies that

$$H^0(X, i_{w,*}\mathcal{E}) = \Gamma\big(U^{\mathrm{w}}, i_{w,*}\mathcal{E}\big), \qquad H^i(X, i_{w,*}\mathcal{E}) = 0 \quad \text{if } i > 0.$$

An application of the standard spectral sequence associated with this filtration gives the assertion of Lemma 2.6 at once. □

3. Drinfeld–Sokolov reduction at the critical level

3.1. Categories of $\hat{\mathfrak{g}}$-modules

A triangular decomposition $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ determines a triangular decomposition $\hat{\mathfrak{g}} = \hat{\mathfrak{n}}_+ \oplus \hat{\mathfrak{h}} \oplus \hat{\mathfrak{n}}_-$, cf. Section 2.1.6, where $\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K$, $\hat{\mathfrak{n}}_\pm$ is the preimage of $\mathfrak{n}_\pm$ w.r.t. the evaluation map $\mathfrak{g}[t^{\pm 1}] \to \mathfrak{g}$ defined by letting $t \to 0$ or ∞ respectively.

3.1.1. Definitions of categories

Define $\hat{\mathcal{O}}_k$ to be the category consisting of $\hat{\mathfrak{g}}$-modules M that satisfy the following conditions:

(1) weight space decomposition: if we let $\hat{\mathfrak{h}}_k^*$ be the subspace of $\hat{\mathfrak{h}}^*$ defined by the equation $K = k$, then

$$M = \bigoplus_{\mu \in \hat{\mathfrak{h}}_k^*} M_\mu, \qquad M_\mu = \big\{m \in M\colon\ hm = \mu(h)m,\ h \in \hat{\mathfrak{h}}\big\}; \tag{3.1}$$

(2) local finiteness: for each $m \in M$

$$\dim U(\hat{\mathfrak{n}}_+)m < \infty; \tag{3.2}$$

(3) conformal filtration: there exists a family of subspaces

$$\cdots \subset \mathrm{F}_n M \subset \mathrm{F}_{n+1} M \subset \cdots \subset M, \qquad M = \bigcup_{n\in\mathbb{Z}} \mathrm{F}_n M,$$

compatible with the grading of $\hat{\mathfrak{g}}$ in that

$$\mathfrak{g} \otimes t^m \mathrm{F}_n M \subset \mathrm{F}_{n-m} M, \qquad \mathrm{F}_n M = \{0\} \quad \text{if } n \ll 0 \tag{3.3}$$

and locally finitely generated: for each $n \in \mathbb{Z}$ there are $m_1, \dots, m_s$ such that

$$\mathrm{F}_n M \subset \sum_{j=1}^{s} U(\hat{\mathfrak{g}}) m_j. \tag{3.4}$$

If the *level* $k = -h^\vee$, the case we shall be interested in almost exclusively, then we shall use the notation $\hat{\mathcal{O}}_{crit} = \hat{\mathcal{O}}_{-h^\vee}$.

Condition (3.3) implies that an object of $\hat{\mathcal{O}}_k$ is automatically a $V_k(\mathfrak{g})$-module. Furthermore, if $k = -h^\vee$, then by pull-back, an object of $\hat{\mathcal{O}}_k$ is automatically a $\mathfrak{z}(\mathfrak{g})$-module, see Theorem 2.2. One-dimensional irreducible $\mathfrak{z}(\mathfrak{g})$-modules are nothing but characters, $\chi(z)$, Laurent series with values in the space dual to the linear span of the generating set $\{p_1, \dots, p_r\}$. Given such $\chi(z) = \sum_n \chi_{(n)} z^{-n-1}$, we have a $\mathfrak{z}(\mathfrak{g})$-module $\mathbb{C}_{\chi(z)}$ to be $\mathbb{C}$ as a vector space with action $p_i \mapsto \sum_n \langle \chi_{(n)}, p_i \rangle z^{-n-1}$.

Define $\hat{\mathcal{O}}_{\chi(z)}$ to be a full subcategory of $\hat{\mathcal{O}}_{crit}$ consisting of modules M that satisfy: for each $m \in M$ and i, $1 \leqslant i \leqslant \operatorname{rk} \mathfrak{g}$, there is N such that

$$\bigl(p_{i,(n)} - \langle \chi_{(n)}, p_i \rangle\bigr)^N m = 0. \tag{3.5}$$

3.1.2. Examples of $\hat{\mathfrak{g}}$*-modules*

A rich supply of objects of $\hat{\mathcal{O}}_k$ is obtained by induction: for each $\mathfrak{g}$-module M, finite dimensional or an object of appropriately defined $\mathcal{O}$-category of $\mathfrak{g}$-modules, define

$$\operatorname{Ind}_{\mathfrak{g}[t]\oplus\mathbb{C}K}^{\hat{\mathfrak{g}}} M,$$

where $\mathfrak{g}[t]$ operates on M via the evaluation $t \to 0$ and $K \mapsto k$. In this way we obtain:

the Weyl module $\mathbb{V}_{\lambda,k} = \operatorname{Ind}_{\mathfrak{g}[t]\oplus\mathbb{C}K}^{\hat{\mathfrak{g}}} V_\lambda$, where V_λ is the finite dimensional simple $\mathfrak{g}$-module with highest weight λ; note that $V_k(\mathfrak{g}) = \mathbb{V}_{0,k}$;

the Verma module $\mathbb{M}_{\lambda,k} = \operatorname{Ind}_{\mathfrak{g}[t]\oplus\mathbb{C}K}^{\hat{\mathfrak{g}}} M_\lambda$, where $M_\lambda = \operatorname{Ind}_{\mathfrak{n}_+\oplus\mathfrak{h}}^{\mathfrak{g}} \mathbb{C}_\lambda$, the Verma module over $\mathfrak{g}$.

The Wakimoto module $\mathbb{W}_{0,-h^\vee}$, see Section 2.2.5, belongs to $\hat{\mathcal{O}}_{crit}$; it is obtained not so much by induction as by *semi-infinite induction* [40,18,17].

If $k = -h^\vee$, we can introduce *restricted* versions, those obtained by quotienting out by a central character. For any $M \in \hat{\mathcal{O}}_{crit}$ and a central character $\chi(z)$, define

$$M_{\chi(z)} = M / \operatorname{span}\bigl\{\bigl(p_{i,(n)} - \langle \chi_{(n)}, p_i \rangle\bigr) M\bigr\}.$$

Thus we obtain $\mathbb{V}_{\chi(z)}$ and $\mathbb{M}_{\chi(z)}$; these are objects of $\hat{\mathcal{O}}_{\chi(z)}$.

Twisted Wakimoto modules of critical level, $\mathbb{W}^w_{\nu(z)}$, which were obtained via localization in Section 2.2.7, are objects of $\hat{\mathcal{O}}_{\nu(z)\circ\pi_w}$, where $\nu(z)\circ\pi_w$ stands for the composition of the vertex algebra morphism $\pi_w|_{\mathfrak{z}(\mathfrak{g})}$, see (2.24) and the character $\nu(z): H_X \to \mathit{Fields}(\mathbb{C})$. Note that $\mathbb{W}^w_{\nu(z)}$ is not necessarily a quotient of some bigger module by a central character.

3.1.3. Modules with a Verma flag

We shall say that $M \in \hat{\mathcal{O}}_k$ is filtered by Verma modules if it carries a filtration $G_0M \subset G_1M \subset \cdots$, $\bigcup_i G_iM = M$, such that, for each i, $G_iM/G_{i-1}M$ is a direct sum of Verma modules $\mathbb{M}_{\lambda,k}$, $\lambda \in \mathfrak{h}^*$.

Let $\{v_1, v_2, \ldots\}$ generate M and denote by V the $\hat{\mathfrak{n}}_+ \oplus \hat{\mathfrak{h}}$-submodule of M generated by $\{v_1, v_2, \ldots\}$. Define $P^0 \overset{\text{def}}{=} U(\hat{\mathfrak{g}}) \otimes_{\hat{\mathfrak{n}}_+\oplus\hat{\mathfrak{h}}} V$. It is clear that P^0 has a filtration by Verma modules and projects onto M:

$$P^0 \to M.$$

Continuing in the same vein we obtain, for each $M \in \hat{\mathcal{O}}_k$ a resolution

$$\cdots \to P^{-j} \to \cdots \to P^{-1} \to P^0 \to M \tag{3.6}$$

by modules with a Verma flag.

Now the locally finite generation condition (3.4) implies

Lemma 3.1. *For each $M \in \hat{\mathcal{O}}_k$ and any conformal filtration $\{\mathrm{F}_nM\}$ there exist a resolution* (3.6) *and conformal filtrations $\{\mathrm{F}_nP^{-j}\}$ of all the terms so that*

(1) *the differential is a morphism of filtered modules;*
(2) *for each n, there is N such that $\mathrm{F}_mP^{-j} = \{0\}$ for all $j > N$, $m < n$.*

This is all standard, cf. [38, Section 4], and we omit the details.

3.1.4. Action of the center and a decomposition into blocks

For generators $p_1, \ldots, p_r$ of $\mathfrak{z}(\mathfrak{g})$ chosen as in Theorem 2.2, ‘Fourier coefficients’ $p_{(n)}$ can be regarded as elements of the completed universal enveloping algebra $\tilde{U}(\hat{\mathfrak{g}})$, cf. [18, Section 4.3].

The conformal weight zero subalgebra $\tilde{U}(\hat{\mathfrak{g}})_0$ has an ideal $(\tilde{U}(\hat{\mathfrak{g}})U(\mathfrak{g}\otimes\mathbb{C}[t]t))_0$. We have an obvious isomorphism

$$\tilde{U}(\hat{\mathfrak{g}})_0/\big(\tilde{U}(\hat{\mathfrak{g}})U\big(\mathfrak{g}\otimes\mathbb{C}[t]t\big)\big)_0 \xrightarrow{\sim} U(\mathfrak{g}).$$

Note that $\mathbb{C}[p_1, \ldots, p_r]$ is naturally a commutative associative subalgebra of $\mathfrak{z}(\mathfrak{g})$ w.r.t. multiplication $_{(-1)}$.

Lemma 3.2. *(See [3].) The composite map*

$$\mathbb{C}[p_1, \ldots, p_r] \to \tilde{U}(\hat{\mathfrak{g}})_0 \to U(\mathfrak{g}), \qquad p \mapsto p_0 \mod \big(\tilde{U}(\hat{\mathfrak{g}})U\big(\mathfrak{g}\otimes\mathbb{C}[t]t\big)\big)_0$$

has the center $Z(\mathfrak{g})$ as its image and delivers a commutative associative algebra isomorphism $\mathbb{C}[p_1, \ldots, p_r] \xrightarrow{\sim} Z(\mathfrak{g})$.

Note that p_0 stands for the *conformal weight zero* Fourier coefficient of the field $p(z)$. For example, since p_i has conformal weight d_i+1, we have $p_i(z)=\sum_n p_{i,n}z^{-n-d_i-1}$, which shows that $p_{i,0}$ is the coefficient of z^{-d_i-1}.

The significance of Lemma 3.2 is that it describes the action of the center on Verma modules. To see this, compose the isomorphism $\mathbb{C}[p_1,\dots,p_r]\xrightarrow{\sim} Z(\mathfrak{g})$ with the classical Harish-Chandra isomorphism $\theta: Z(\mathfrak{g})\to\mathbb{C}[\mathfrak{h}^*]^W$ to obtain an isomorphism

$$\theta_{aff}:\mathbb{C}[p_1,\dots,p_r]\xrightarrow{\sim}\mathbb{C}\big[\mathfrak{h}^*\big]^W. \tag{3.7}$$

It follows from the definition of the Verma module $\mathbb{M}_{\lambda,-h^\vee}$ that for each $f\in\mathbb{C}[p_1,\dots,p_r]$, f_n acts on $\mathbb{M}_{\lambda,-h^\vee}$ as 0 if $n>0$ and as multiplication by $\langle\theta_{aff}(f),\lambda\rangle$ if $n=0$.

The same applies to any highest weight module, and, since any object of $\hat{\mathcal{O}}_{crit}$ has a filtration by highest weight modules (alternatively, use Lemma 3.1), one obtains a block decomposition

$$\hat{\mathcal{O}}_{crit}=\bigoplus_{[\lambda]\in\mathfrak{h}^*/W}\hat{\mathcal{O}}_{crit}^{[\lambda]}, \tag{3.8}$$

where $[\lambda]=W\circ\lambda$ and $\hat{\mathcal{O}}_{crit}^{[\lambda]}$ is defined to be the full subcategory of modules such that $(f_0-\theta_{aff}(f)(\lambda))$ acts locally nilpotently for each $f\in\mathbb{C}[p_1,\dots,p_r]$.

Introduce a polynomial algebra

$$\mathcal{Z}_-=\mathbb{C}[p_{i,n},\ 1\leqslant i\leqslant \operatorname{rk}\mathfrak{g},\ n<0]. \tag{3.9}$$

Its is clear that f_n, $f\in\mathfrak{z}(\mathfrak{g})$, $n>0$, acts locally nilpotently on each $M\in\hat{\mathcal{O}}_{crit}$. The action of f_0 being described by (3.8), what one needs to describe the action of the entire $\mathfrak{z}(\mathfrak{g})$ is the action of $\mathcal{Z}_-$. Here is an example.

Theorem 3.3. *(See [17, Theorem 9.5.3].) $\mathbb{M}_{\lambda,-h^\vee}$ is a free $\mathcal{Z}_-$-module. Furthermore, $\operatorname{End}_{\hat{\mathfrak{g}}}(\mathbb{M}_{\lambda,-h^\vee})\xrightarrow{\sim}\mathcal{Z}_-$.*

3.1.5. Gradings and character formulas

A conformal filtration that $M\in\hat{\mathcal{O}}_k$ carries by definition, see Section 3.1.1, is not unique. But if M is a Verma module or its quotient, then there are obvious choices: pick an m, declare $\mathrm{F}_nM=\{0\}$ if $n<m$, let F_mM contain v, a highest weight vector, and define $\mathrm{F}_{n+m}M=U(\hat{\mathfrak{n}}_-)_{\geqslant -n}v$; here $U(\hat{\mathfrak{n}}_-)_{\geqslant -n}$ stands for the subspace of $U(\hat{\mathfrak{n}}_-)$ spanned by $x_1\otimes t^{-n_1}\cdots x_l\otimes t^{-n_l}$ with all $n_j\geqslant 0$ and $\sum_j n_j\leqslant n$.

If M is a highest weight module with filtration, we shall tacitly assume that $\mathrm{F}_nM=\{0\}$ if $n<0$, and that the highest weight belongs to F_0M. We shall denote by $M[m]$ this same M with filtration shifted by m, as in the paragraph above. This explains the assertion (2) of Theorem 1.1.

In any case, given a filtration $\mathrm{F}_\bullet M$ of M, the graded object, $\mathrm{gr}_{\mathrm{F}}M$, is also a $\hat{\mathfrak{g}}$-module, the action being defined to be that on symbols. Furthermore, the canonical grading of $\mathrm{gr}_{\mathrm{F}}M$ makes it into a graded $\hat{\mathfrak{g}}$-module.

This can be made a little more explicit: extend $\hat{\mathfrak{g}}$ to $\hat{\mathfrak{g}}_{ext}$ by adjoining, as usual, a *degree derivation D* so that

$$\big[D,x\otimes t^n\big]=-nx\otimes t^n.$$

Letting D act on $\mathrm{F}_n M/\mathrm{F}_{n-1}M$ as multiplication by n, we make $\operatorname{gr} M$ into a $\hat{\mathfrak{g}}_{ext}$-module. More generally, we shall call $M \in \hat{\mathcal{O}}_k$ *graded* if the action of $\hat{\mathfrak{g}}$ extends to that of $\hat{\mathfrak{g}}_{ext}$ so that the action of D is diagonalizable. We have the weight space decomposition

$$M = \bigoplus_{n \in (\mathbb{C}D)^*} M_n.$$

This is reflected in the following *q-dimension* formula, written down in case of an arbitrary filtered module,

$$\dim_q M = \sum_{n \gg -\infty} q^n \dim \operatorname{gr} M_n. \tag{3.10}$$

Note that it makes sense only if $\dim \operatorname{gr} M_n < \infty$. For example,

$$\dim_q \mathbb{V}_{\lambda,k} = \dim V_\lambda \prod_{j=1}^{\infty} \left(1 - q^j\right)^{-\dim \mathfrak{g}}. \tag{3.11}$$

Invoking the semi-simplicity of the action of $\mathfrak{h}$, we now obtain a weight space decomposition of an arbitrary graded module:

$$M = \bigoplus_{\alpha \in (\hat{\mathfrak{h}} \oplus \mathbb{C}D)^*} M_\alpha,$$

and refine (3.10) by defining the *formal character* as usual

$$\operatorname{ch} M \stackrel{\text{def}}{=} \operatorname{ch}(\operatorname{gr} M) \stackrel{\text{def}}{=} \sum_{\alpha \in (\hat{\mathfrak{h}} \oplus \mathbb{C}D)^*} e^\alpha \dim \operatorname{gr} M_\alpha. \tag{3.12}$$

Note that, as it follows from the definition of $\hat{\mathcal{O}}_k$, Section 3.1.1, $\dim \operatorname{gr} M_\alpha < \infty$.

For example, the Verma module $\mathbb{M}_{\lambda,k}$ is actually graded and

$$\operatorname{ch} \mathbb{M}_{\lambda,k} = e^\lambda \prod_{\alpha \in \hat{\Delta}_+} \left(1 - e^{-\alpha}\right)^{-1}, \tag{3.13}$$

where $\hat{\Delta}_+$ is the set of positive roots of $\hat{\mathfrak{g}}$.

The freeness result, Theorem 3.3, implies

$$\operatorname{ch} \mathbb{M}_{\mu(z)} = e^\lambda \prod_{\alpha \in \Delta_+} \left(1 - e^{-\alpha}\right)^{-1} \times \prod_{n=1}^{+\infty} \prod_{\alpha \in \Delta_+} \left(1 - e^{-\alpha - n\delta}\right)^{-1} \times \prod_{\alpha \in \Delta_-} \left(1 - e^{-\alpha - n\delta}\right)^{-1}, \tag{3.14}$$

where $\mu(z)$ and λ must be compatible, of course: $\mu(z) = \lambda/z + \mu_1 + \mu_2 z + \cdots$.

Some modules are graded by the very construction, for example w-twisted Wakimoto modules with *homogeneous* character $\nu(z) = \nu_0/z$, and so the formal character of $\mathbb{W}^w_{\nu_0/z}$ does not require specification of a filtration. On the other hand, it is quite clear how to define a filtration on an arbitrary w-twisted Wakimoto module so that its formal character coincides with that of $\mathbb{W}^w_{\nu_0/z}$. The result is, cf. [17, formula (9.5.4)],

$$\operatorname{ch}\mathbb{W}^w_{\nu(z)} = e^{w\circ\nu_0}\prod_{\alpha\in\Delta_+}\left(1-e^{-\alpha}\right)^{-1}\times\prod_{n=1}^{+\infty}\prod_{\alpha\in\Delta_+}\left(1-e^{-\alpha-n\delta}\right)^{-1}\times\prod_{\alpha\in\Delta_-}\left(1-e^{-\alpha-n\delta}\right)^{-1}. \tag{3.15}$$

This, of course, coincides with the character of the restricted Verma module (3.14) with highest weight $\lambda = w\circ\nu_0$.

3.1.6. Since $M\in\hat{\mathcal{O}}_k$ being graded implies $\dim \operatorname{gr} M_\alpha < \infty$, we define its dual

$$M^c = \bigoplus_{\alpha\in(\hat{\mathfrak{h}}\oplus\mathbb{C}D)^*} M^*_\alpha,$$

with the action of $\hat{\mathfrak{g}}_{ext}$ determined by $\langle x\phi, m\rangle = \langle\phi, \omega(x)m\rangle$, $x\in\hat{\mathfrak{g}}_{ext}$, $\phi\in M^c$, $m\in M$, and $\omega:\hat{\mathfrak{g}}_{ext}\to\hat{\mathfrak{g}}_{ext}$ is the canonical antiinvolution that sends $\mathfrak{g}_\alpha\otimes t^n$ to $\mathfrak{g}_{-\alpha}\otimes t^{-n}$, $\alpha\in\Delta$ being a root of $\mathfrak{g}$.

It is obvious that the assignment $M\mapsto M^c$ is a contragredient exact functor on the full subcategory of $\hat{\mathcal{O}}_k$ consisting of graded modules.

3.2. The Drinfeld–Sokolov reduction

3.2.1. Definition

Consider the vertex superalgebra $V(\mathfrak{n}_+)\otimes Cl(\mathfrak{n}_+)$, cf. Sections 2.1.5 and 2.1.6. Let $\{e^\alpha,\ \alpha\in\Delta_+\}$ be a root vector basis of $\mathfrak{n}_+$, $\{\phi_\alpha,\ \alpha\in\Delta_+\}$ its copy, albeit with changed parity, that appears inside $Cl(\mathfrak{n}_+)$, $\{\phi^*_\alpha,\ \alpha\in\Delta_+\}\subset\mathfrak{n}^*_+$ the basis dual to the latter.

Let $\{c^{\alpha\beta}_\gamma\}$ be the structure constants so that

$$\left[e^\alpha, e^\beta\right] = \sum_\gamma c^{\alpha\beta}_\gamma e^\gamma.$$

The following elements of $V(\mathfrak{n}_+)\otimes Cl(\mathfrak{n}_+)$ are of importance

$$Q_{st} = \sum_\alpha e^\alpha\otimes\phi^*_\alpha - \frac{1}{2}\sum_{\alpha,\beta,\gamma} c^{\alpha,\beta}_\gamma \phi^*_{\alpha(-1)}\left(\phi^*_{\beta(-1)}\phi_\gamma\right), \qquad \chi = \sum_{i=1}^r \phi^*_{\alpha_i}, \qquad Q_{DS} = Q_{st}+\chi,$$

where $\{\alpha_1,\dots,\alpha_r\}$ is the set of simple roots. Define

$$d_{st} = (Q_{st})_{(0)}, \qquad d = d_{st}+\chi_{(0)}.$$

Since $d^2_{st} = \chi^2_{(0)} = [d_{st}, \chi_{(0)}] = 0$, there arise 3 differential graded vertex algebras, $(V(\mathfrak{n}_+)\otimes Cl(\mathfrak{n}_+), d_{st})$, $(V(\mathfrak{n}_+)\otimes Cl(\mathfrak{n}_+), \chi_{(0)})$, $(V(\mathfrak{n}_+)\otimes Cl(\mathfrak{n}_+), d)$, with grading defined by setting $\deg\phi^*_\alpha = 1$, $\deg\phi_\alpha = -1$.

Furthermore, if M is a $V(\mathfrak{n}_+)$-module, then $(M \otimes Cl(\mathfrak{n}_+), d_{st})$, $(M \otimes Cl(\mathfrak{n}_+), \chi_{(0)})$, $(M \otimes Cl(\mathfrak{n}_+), d)$ are differential graded modules over their respective differential graded vertex algebras. To emphasize the fact that all these are to be treated as complexes, we shall change the notation and write $C^{\infty/2+\bullet}(L\mathfrak{n}_+, M)$ instead of $M \otimes Cl(\mathfrak{n}_+)$. The corresponding cohomology will be denoted as follows: $H^{\infty/2+\bullet}(L\mathfrak{n}_+, M)$, $H_{\chi_{(0)}}^{\infty/2+\bullet}(L\mathfrak{n}_+, M)$, $H_{DS}^{\infty/2+\bullet}(L\mathfrak{n}_+, M)$.

Remark 3.4. $H^{\infty/2+\bullet}(L\mathfrak{n}_+, M)$, and $H_{DS}^{\infty/2+\bullet}(L\mathfrak{n}_+, M)$ can be related as follows. One can regard χ as a (Drinfeld–Sokolov) character, $\chi : L\mathfrak{n}_+ \to \mathbb{C}$, which sends $e^{\alpha_i} \otimes t^{-1}$ to 1 and the rest of $\{e^{\alpha} \otimes t^n\}$ to 0. Denote by $\mathbb{C}_\chi$ the corresponding 1-dimensional $L\mathfrak{n}_+$-module. Then by definition $H_{DS}^{\infty/2+\bullet}(L\mathfrak{n}_+, M) = H^{\infty/2+\bullet}(L\mathfrak{n}_+, M \otimes \mathbb{C}_\chi)$.

If, in addition, $M \in \hat{\mathcal{O}}_{crit}$ and is regarded as an $L\mathfrak{n}_+$-module via pull-back, then each of the three series of cohomology groups above is a $\mathfrak{z}(\mathfrak{g})$-module, because the action of the center commutes with that of $L\mathfrak{n}_+$. Thus we obtain three series of functors

$$H^{\infty/2+\bullet}(L\mathfrak{n}_+, ?),\ H_{\chi_{(0)}}^{\infty/2+\bullet}(L\mathfrak{n}_+, ?),\ H_{DS}^{\infty/2+\bullet}(L\mathfrak{n}_+, ?) : \hat{\mathcal{O}}_{crit} \to \mathfrak{z}(\mathfrak{g})\text{-Mod}. \tag{3.16}$$

Each of these functors makes sense away from the critical level. For example, if we let $\mathcal{W}_k \stackrel{\text{def}}{=} H_{DS}^{\infty/2+\bullet}(L\mathfrak{n}_+, V_k(\mathfrak{g}))$, then we obtain

$$H_{DS}^{\infty/2+\bullet}(L\mathfrak{n}_+, ?) : \hat{\mathcal{O}}_{crit} \to \mathcal{W}_k\text{-Mod}.$$

All of this is well known, of course: $H^{\infty/2+\bullet}(L\mathfrak{n}_+, ?)$ was introduced by Feigin in [13], the entrance point of the BRST business in mathematics, $H_{DS}^{\infty/2+\bullet}(L\mathfrak{n}_+, ?)$ is the Drinfeld–Sokolov reduction functor, proposed by Feigin and Frenkel as a tool to define $\mathcal{W}_k$, the celebrated W-algebra, see [18,17] and references therein. The Drinfeld–Sokolov reduction functor has been studied in [1–3] in a more general setting. A thorough analysis of the functors (3.16) has been carried out recently by Frenkel and Gaitsgory [19–21].

3.2.2. Torus action, grading, twisted grading and filtration

Note that $Cl(\mathfrak{n}_+)$ carries a natural grading determined by the condition that the degree of $\phi_{\alpha,n}$ and $\phi^*_{\alpha,n}$ be $-n$. If $M \in \hat{\mathcal{O}}_{crit}$ is graded, i.e., carries an action of the extended $\hat{\mathfrak{g}}_{ext}$ with diagonalizable D, cf. Section 3.1.5, then the entire complex $C^{\infty/2+\bullet}(L\mathfrak{n}_+, M)$ acquires a grading, that of the tensor products of graded spaces, so that d_{st} has degree 0. Therefore, this grading descends on $H^{\infty/2+\bullet}(L\mathfrak{n}_+, M)$.

This can be refined by noting that we can extend an action not only of D but of the entire $L\mathfrak{h}$ to $C^{\infty/2+\bullet}(L\mathfrak{n}_+, M)$. Consider the linear map

$$\mathfrak{h} \to C^{\infty/2+\bullet}\big(L\mathfrak{n}_+, V_{-h^\vee}(\mathfrak{h})\big), \qquad h \mapsto h \otimes \mathbf{1} - \sum_{\alpha\in\Delta_+} \alpha(h)\mathbf{1} \otimes \phi^*_{\alpha,(-1)}\phi_\alpha. \tag{3.17}$$

It is easy to see that it defines a vertex algebra morphism, cf. the end of Section 2.1.6,

$$V_0(\mathfrak{h}) \to C^{\infty/2+0}\big(L\mathfrak{n}_+, V_{-h^\vee}(\mathfrak{g})\big), \tag{3.18}$$

where the central charge has got shifted: $-h^\vee(.,.)$ has been replaced with 0.

Therefore, $C^{\infty/2+\bullet}(L\mathfrak{n}_+, V_{-h^\vee}(\mathfrak{g}))$, $M \in \hat{\mathcal{O}}_{crit}$, is a $V_0(\mathfrak{h})$-module, and it is easy to see that d_{st} is a $V_0(\mathfrak{h})$-module morphism. In particular, if M is graded and $M = \bigoplus_{\alpha\in\hat{\mathfrak{h}}^*} M_\alpha$, then $H^{\infty/2+\bullet}(L\mathfrak{n}_+, M)$ is also graded:

$$H^{\infty/2+\bullet}(L\mathfrak{n}_+, M) = \bigoplus_{\alpha\in\hat{\mathfrak{h}}^*} H^{\infty/2+\bullet}(L\mathfrak{n}_+, M)_\alpha. \tag{3.19}$$

None of this carries over to the Drinfeld–Sokolov case, because $\chi_{(0)}$ does not preserve either of the gradings introduced. To rescue the situation – partially – denote by $\tilde{D} \in \operatorname{End} C^{\infty/2+\bullet}(L\mathfrak{n}_+, M)$ the operator whose eigenspace decomposition coincides with the grading just discussed and introduce $\hat{\rho}^\vee \in \operatorname{End} C^{\infty/2+\bullet}(L\mathfrak{n}_+, M)$, the operator whose eigenvalues are negative those of the half-sum of positive coroots $\rho^\vee$. Specifically, we demand that

$$[\hat{\rho}^\vee, e_{\alpha,n}] = -\rho^\vee(\alpha)e_{\alpha,n}, \qquad [\hat{\rho}^\vee, \phi_{\alpha,n}] = -\rho^\vee(\alpha)\phi_{\alpha,n}, \qquad [\hat{\rho}^\vee, \phi^*_{\alpha,n}] = \rho^\vee(\alpha)\phi^*_{\alpha,n}.$$

Now notice [18,3] that $\chi_{(0)}$, hence $d = d_{st} + \chi_{(0)}$, commutes with $\tilde{D} + \hat{\rho}^\vee$, and the eigenvalues of the latter provide a *twisted grading* of the Drinfeld–Sokolov reduction $H_{DS}^{\infty/2+\bullet}(L\mathfrak{n}_+, M)$. This allows us to define, cf. (3.11), the notion of *q-dimension*

$$\dim_q H_{DS}^{\infty/2+\bullet}(L\mathfrak{n}_+, M) = \sum_{n\in\mathbb{Z}} q^n \dim H_{DS}^{\infty/2+\bullet}(L\mathfrak{n}_+, M)_n. \tag{3.20}$$

Similarly, if M is filtered, then this filtration extends to $C^{\infty/2+\bullet}(L\mathfrak{n}_+, M)$ and then descends to $H_{DS}^{\infty/2+\bullet}(L\mathfrak{n}_+, M)$ automatically. A spectral sequence arises

$$\{E_r^{\bullet\bullet}, d_r\} \quad \Rightarrow \quad H_{DS}^{\infty/2+\bullet}(L\mathfrak{n}_+, M) \quad \text{s.t.} \quad E_1^{\bullet\bullet}, d_1 = H^{\infty/2+\bullet}(L\mathfrak{n}_+, M). \tag{3.21}$$

3.2.3. Drinfeld–Sokolov reduction as a derived functor

Note that each *singular vector*, i.e., an element of $M^{\hat{\mathfrak{n}}_+}$, is a cocycle:

$$M^{\hat{\mathfrak{n}}_+} \hookrightarrow Z^{\infty/2+0}(L\mathfrak{n}_+, M), \qquad m \mapsto m \otimes \mathbf{1}; \tag{3.22}$$

this is true of any differential, d or d_{st}, and is an obvious consequence of the definition.

Theorem 3.5.

(1) *If $P \in \hat{\mathcal{O}}_k$ is filtered by Verma modules, then $H_{DS}^{\infty/2+i}(L\mathfrak{n}_+, P) = 0$ for all $i \neq 0$* [1]. *Furthermore, if $k = -h^\vee$ and all Verma modules appearing in the composition series of P have regular integral highest weight, then $H_{DS}^{\infty/2+0}(L\mathfrak{n}_+, P)$ is a free $\mathcal{Z}_-$-module (see* (3.9)*) on generators $[v] = \operatorname{class} v$, where v varies over the set of highest weight vectors of Verma modules that appear in the associated graded of P.*
(2) *If $i > 0$, then $H_{DS}^{\infty/2+i}(L\mathfrak{n}_+, M) = 0$ for all $M \in \hat{\mathcal{O}}_k$.*
(3) *The functor $H_{DS}^{\infty/2+0}(L\mathfrak{n}_+, ?)$ is right exact.*
(4) *The class of modules carrying a filtration by Verma modules is adapted to the functor $H_{DS}^{\infty/2+0}(L\mathfrak{n}_+, ?)$.*

Let us discuss some consequences. Items (3) and (4) allow us to define, as usual [24], the derived functors $L^i H_{DS}^{\infty/2+0}(L\mathfrak{n}_+, ?)$.

Corollary 3.6. *The functors $H_{DS}^{\infty/2-i}(L\mathfrak{n}_+, ?)$ and $L^i H_{DS}^{\infty/2+0}(L\mathfrak{n}_+, ?)$ are isomorphic.*

Corollary 3.7. *If μ_0 is regular integral, then*

$$H_{DS}^{\infty/2+i}(L\mathfrak{n}_+, \mathbb{M}_{\mu(z)}) = \begin{cases} \mathbb{C}[v_{\mu_0}] & \text{if } i = 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $[v_{\mu_0}]$ is the cohomology class of a highest weight vector v_{μ_0}.

Proof of Corollary 3.6. Pick a resolution of M

$$P^\bullet : \cdots \to P^{-j} \to \cdots \to P^{-1} \to P^0 \to M$$

by modules with Verma filtration. By definition, $L^\bullet H_{DS}^{\infty/2+0}(L\mathfrak{n}_+, ?)$ is the cohomology of the complex $H_{DS}^{\infty/2+0}(L\mathfrak{n}_+, P^\bullet)$. That the latter complex also computes $H_{DS}^{\infty/2-\bullet}(L\mathfrak{n}_+, M)$ is derived from Theorem 3.5(1) and (3) by a standard argument; it is based on the long exact sequences of various $H_{DS}^{\infty/2-\bullet}(L\mathfrak{n}_+, ?)$ that are associated to the short exact sequences

$$0 \to \mathcal{Z}^0 \to P^0 \to M \to 0,$$
$$0 \to \mathcal{Z}^{-i-1} \to P^{-i-1} \to \mathcal{Z}^{-i} \to 0, \quad i \geqslant 0.$$

It is that argument which allows to compute the cohomology of a sheaf via its $\bar{\partial}$-resolution. We leave the details to the interested reader. □

Proof of Corollary 3.7. For a commutative algebra A, a collection of elements $\underline{a} \subset A$, and an A-module E, denote by $K^\bullet(A, \underline{a}; E)$ the corresponding Koszul complex.

Let now $\underline{z} = \{p_{in} - p_i(\nu_n),\ n < 0,\ 1 \leqslant i \leqslant \operatorname{rk}\mathfrak{g}\} \subset \mathcal{Z}_-$. This collection being regular, Theorem 3.3 implies that $K^\bullet(\mathcal{Z}_-, \underline{z}; \mathbb{M}_{\nu_0, -h^\vee})$ is a resolution of $\mathbb{M}_{\mu(z)}$. Due to Corollary 3.6, the complex $H^{\infty/2+0}(L\mathfrak{n}_+, K^\bullet(\mathcal{Z}_-, \underline{z}; \mathbb{M}_{\nu_0, -h^\vee}))$ computes $H^{\infty/2-\bullet}(L\mathfrak{n}_+, \mathbb{M}_{\mu(z)})$. Theorem 3.5(1) says that the latter complex is nothing but $K^\bullet(\mathcal{Z}_-, \underline{z}; \mathcal{Z}_-)$, the Koszul resolution of $\mathcal{Z}/\langle \underline{z} \rangle = \mathbb{C}$, where by $\langle \underline{z} \rangle$ we have denoted the ideal generated by $\underline{z}$. □

3.2.4. Proof of Theorem 3.5(1)

Except for the freeness assertion in the case where $k = -h^\vee$, this item is proved in [1]. Let us review the details.

First of all [1, Theorem 5.7 and Remark 5.8],

$$\dim_q H_{DS}^i(L\mathfrak{n}_+, \mathbb{M}_{\lambda, -h^\vee}) = \begin{cases} 0 & \text{if } i \neq 0, \\ q^{\langle \lambda, D - \rho^\vee \rangle} \prod_{j=1}^\infty (1 - q^j)^{-\operatorname{rk}\mathfrak{g}} & \text{if } i = 0, \end{cases} \tag{3.23}$$

cf. (3.20); note that our conformal grading convention is different from that in [1]. Next, if P is an arbitrary module with Verma filtration, then there arises the standard spectral sequence associated to this filtration, which is easily seen to converge to $H_{DS}^{\infty/2+\bullet}(L\mathfrak{n}_+, P)$. The vanishing result

(3.23) implies that (a) the spectral sequence collapses in the first term, (b) $H_{DS}^{\infty/2+i}(L\mathfrak{n}_+, P) = 0$ if $i \neq 0$, (c) $H_{DS}^{\infty/2+0}(L\mathfrak{n}_+, P)$ is filtered so that the corresponding graded object is a direct sum of various $H_{DS}^{\infty/2+0}(L\mathfrak{n}_+, \mathbb{M}_{\lambda,-h^\vee})$, one for each Verma module occurring in the Verma composition series of P. This proves

Lemma 3.8. *If P is a module with Verma filtration so that* $\mathrm{gr}\, P = \bigoplus_{\alpha\in A} \mathbb{M}_{\lambda_\alpha,-h^\vee}$, *$A$ being an index set, then*

$$\mathrm{gr}\, H_{DS}^{\infty/2+i}(L\mathfrak{n}_+, P) = \begin{cases} \bigoplus_{\alpha\in A} H_{DS}^{\infty/2+0}(L\mathfrak{n}_+, \mathbb{M}_{\lambda_\alpha,-h^\vee}) & \textit{if } i = 0, \\ 0 & \textit{otherwise.} \end{cases}$$

As to the freeness assertion, let us start off by making an informal remark. If v_λ is a highest weight vector of $\mathbb{M}_{\lambda,k}$, then according to (3.22) it determines a cohomology class, $[v_\lambda]$. Note that its twisted degree, Section 3.2.2, is precisely $\langle \lambda, D - \rho^\vee \rangle$. Since the q-dimension of $\mathcal{Z}_-$ is clearly $\prod_{j=1}^{\infty}(1-q^j)^{-\mathrm{rk}\,\mathfrak{g}}$, the second line of (3.23) is a strong indication that $[v_\lambda]$ freely generates $H_{DS}^0(L\mathfrak{n}_+, \mathbb{M}_{\lambda,-h^\vee})$. Furthermore, if each $H_{DS}^{\infty/2+i}(L\mathfrak{n}_+, \mathbb{M}_{\lambda_\alpha,-h^\vee})$ is a free $\mathcal{Z}_-$-module on one generator, then the following refinement of Lemma 3.8 is valid:

$$H_{DS}^{\infty/2+i}(L\mathfrak{n}_+, P) = \bigoplus_{\alpha\in A} \mathcal{Z}_-,$$

because a filtration by free modules splits. It remains then to prove

Theorem 3.9. *If λ is a regular integral weight, then $H_{DS}^{\infty/2+0}(L\mathfrak{n}_+, \mathbb{M}_{\lambda,-h^\vee})$ is a free $\mathcal{Z}_-$-module on one generator $[v_\lambda]$.*

Since the rest of Theorem 3.5 is independent of this result, we will postpone proving Theorem 3.9 until Section 3.3.

3.2.5. Proof of Theorem 3.5(2)

Given $M \in \hat{\mathcal{O}}_k$, consider a resolution of M by modules with a Verma filtration (3.6). Letting $Z^{-j} = \mathrm{Ker}(P^{-j} \to P^{-j+1})$ we obtain a collection of short exact sequences

$$0 \to Z^0 \to P^0 \to M \to 0, \qquad 0 \to Z^{-j} \to P^{-j} \to Z^{-j+1} \to 0, \quad j > 0.$$

By virtue of the vanishing result in Lemma 3.8, an application of the long exact sequence of cohomology gives, for each $i > 0$ and $n \in \mathbb{Z}$, a chain of isomorphisms

$$\begin{aligned} &\mathrm{F}_n H^{\infty/2+i}(L\mathfrak{n}_+, M) \xrightarrow{\sim} \mathrm{F}_n H^{\infty/2+i+1}\big(L\mathfrak{n}_+, Z^0\big) \\ &\quad \xrightarrow{\sim} \mathrm{F}_n H^{\infty/2+i+2}\big(L\mathfrak{n}_+, Z^{-1}\big) \xrightarrow{\sim} \cdots \xrightarrow{\sim} \mathrm{F}_n H^{\infty/2+j+1}\big(L\mathfrak{n}_+, Z^{-j}\big) \xrightarrow{\sim} \cdots, \end{aligned}$$

where F_n denotes the nth conformal filtration component, see Section 3.1.1.

Making sure that the resolution $P^\bullet$ is one from Lemma 3.1, we see that

$$\mathrm{F}_n H^{\infty/2+j+1}\big(L\mathfrak{n}_+, Z^{-j}\big) = \{0\} \quad \text{if } j \gg 0.$$

3.2.6. Proof of Theorem 3.5(3)

If $0 \to A \to B \to C \to 0$ is exact, we obtain the long exact sequence of cohomology

$$\begin{aligned}\cdots &\to H_{DS}^{\infty/2+0}(L\mathfrak{n}_+, A) \to H_{DS}^{\infty/2+0}(L\mathfrak{n}_+, B) \to H_{DS}^{\infty/2+0}(L\mathfrak{n}_+, C)\\ &\to H_{DS}^{\infty/2+1}(L\mathfrak{n}_+, A) \to \cdots.\end{aligned}$$

The right exactness follows, because by virtue of Theorem 3.5(2), $H_{DS}^{\infty/2+1}(L\mathfrak{n}_+, A) = 0$.

3.2.7. Proof of Theorem 3.5(4)

Being adapted means [24] that (a) each module has a resolution by modules with Verma filtration, (b) if a complex $P^\bullet$ consisting of modules with Verma filtration is exact, then $H_{DS}^{\infty/2+0}(L\mathfrak{n}_+, P^\bullet)$ is also exact. Item (a) is the assertion of Lemma 3.1. Item (b) is a standard consequence of Theorem 3.5(1) and (3): present an exact sequence

$$\cdots \to P^{-n} \to \cdots \to P^0 \to 0$$

as a chain of short exact sequences

$$0 \to Z^{-1} \to P^{-1} \to P^0 \to 0, \qquad 0 \to Z^{-j-1} \to P^{-j-1} \to Z^{-j} \to 0, \quad j > 0.$$

Then an induction on j, using Theorem 3.5(1) and (3), will show that

$$\cdots \to H_{DS}^{\infty/2+0}\big(L\mathfrak{n}_+, P^{-n}\big) \to \cdots \to H_{DS}^{\infty/2+0}\big(L\mathfrak{n}_+, P^0\big) \to 0$$

is the composition of the short exact sequences

$$0 \to H_{DS}^{\infty/2+0}\big(L\mathfrak{n}_+, Z^{-1}\big) \to H_{DS}^{\infty/2+0}\big(L\mathfrak{n}_+, P^{-1}\big) \to H_{DS}^{\infty/2+0}\big(L\mathfrak{n}_+, P^0\big) \to 0,$$
$$0 \to H_{DS}^{\infty/2+0}\big(L\mathfrak{n}_+, Z^{-j-1}\big) \to H_{DS}^{\infty/2+0}\big(L\mathfrak{n}_+, P^{-j-1}\big) \to H_{DS}^{\infty/2+0}\big(L\mathfrak{n}_+, Z^{-j}\big) \to 0, \quad j > 0,$$

and is, therefore, also exact.

3.3. Proof of Theorem 3.9

3.3.1. We begin with a series of 2 reductions, to be followed by 2 "approximations." The first reduction is the content of

Lemma 3.10. *Let λ be a regular integral weight. The following conditions are equivalent:*

(i) $H_{DS}^{\infty/2+0}(L\mathfrak{n}_+, \mathbb{M}_{\lambda,-h^\vee})$ *is a rank 1 free $\mathcal{Z}_-$-module generated by $[v_\lambda]$;*
(ii) $H_{DS}^{\infty/2+0}(L\mathfrak{n}_+, \mathbb{M}_{\lambda,-h^\vee})$ *is a cyclic $\mathcal{Z}_-$-module generated by $[v_\lambda]$;*
(iii) $H_{DS}^{\infty/2+0}(L\mathfrak{n}_+, \mathbb{M}_{\lambda,-h^\vee})$ *is a rank 1 free $\mathcal{Z}_-$-module generated by $[v_\lambda]$ if λ is dominant.*

Proof. Equivalence (i) $\Leftrightarrow$ (ii). The implication (i) $\Rightarrow$ (ii) being clear, it suffices to show (i) $\Leftarrow$ (ii). Item (ii) implies a surjection of $\mathcal{Z}_-$-modules $\mathcal{Z}_- \to H_{DS}^{\infty/2+0}(L\mathfrak{n}_+, \mathbb{M}_{\lambda,-h^\vee})$. On the other hand, formula (3.23) shows that $\dim_q \mathcal{Z}_- = \dim_q H_{DS}^{\infty/2+0}(L\mathfrak{n}_+, \mathbb{M}_{\lambda,-h^\vee})$, cf. a discussion in Section 3.2.4. Hence this surjection is an isomorphism.

Equivalence (i) $\Leftrightarrow$ (iii). We only need the implication (i) $\Leftarrow$ (iii), and it suffices to show that the class $[v_{w\circ\lambda}]$, $w \in W$, is not annihilated by any non-zero element of $\mathcal{Z}_-$. (Consider the map $\mathcal{Z}_- \to H_{DS}^{\infty/2+0}(L\mathfrak{n}_+, \mathbb{M}_{w\circ\lambda,-h^\vee})$, $P \mapsto P[v_{w\circ\lambda}]$, and use the character formula (3.23), as we have just done.)

Consider a reduced expression $w = s_{i_k} s_{i_{k-1}} \cdots s_{i_1}$. It is known that any morphism of Verma modules (over $\mathfrak{g}$) $M_{w\circ\lambda} \to M_\lambda$ factors out in the composition of morphisms $M_{s_{i_j}\circ\lambda_{j-1}} \to M_{\lambda_{j-1}}$ determined by the assignment $v_{s_{i_j}\circ\lambda_{j-1}} \mapsto f_{i_j}^{\langle\lambda,\alpha_{i_j}^\vee\rangle+1} v_{\lambda_{j-1}}$; here $\lambda_{j-1} = (s_{i_{j-1}} \cdots s_{i_1}) \circ \lambda$, f_{i_j} is a Cartan–Serre generator corresponding to the simple root α_{i_j}. Now consider the induced morphisms $\mathbb{M}_{s_{i_j}\circ\lambda_{j-1},-h^\vee} \to \mathbb{M}_{\lambda_{j-1},-h^\vee}$. If the class $[f_{i_1}^{\langle\lambda,\alpha_{i_1}^\vee\rangle+1} v_\lambda] \neq 0$, then thanks to (iii), $[v_{\lambda_1}]$ is not annihilated by any non-zero element of $\mathcal{Z}_-$, hence, as we have just argued, $H_{DS}^{\infty/2+0}(L\mathfrak{n}_+, \mathbb{M}_{\lambda_1,-h^\vee})$ is a free $\mathcal{Z}_-$-module generated by $[v_{\lambda_1}]$. Now an obvious inductive argument shows that it suffices to prove the following:

$$f_i^n v_\mu \text{ is not a coboundary.} \tag{3.24}$$

Proof. Denote by $\mathfrak{m}_\chi$ the Lie subalgebra of $U(\mathfrak{n}_+[t^{-1}]t^{-1})$ obtained by shifting $\mathfrak{n}_+[t^{-1}]t^{-1}$ by the Drinfeld–Sokolov character, see Section 3.2.1, Remark 3.4. Namely, we define $\mathfrak{m}_\chi$ to be generated (as a Lie algebra) by $e^\alpha \otimes t^n$ if $n \leqslant -1$, $\alpha \in \Delta_+$ but not simple, and $e^\alpha \otimes t^{-1} + 1$ if $\alpha \in \Delta_+$ is simple. The definition of the Drinfeld–Sokolov differential implies $f_i^n v_\mu$ is a coboundary only if $f_i^n v_\mu \in \mathfrak{m}_\chi \mathbb{M}_{\mu,-h^\vee}$. We have an $\mathfrak{m}_\chi$-module decomposition $\mathbb{M}_{\mu,-h^\vee} \xrightarrow{\sim} U(\mathfrak{m}_\chi) \otimes U(\mathfrak{h}[t^{-1}]t^{-1}) \otimes U(\mathfrak{n}_-[t^{-1}])$. Upon this identification $f_i^n v_\mu \in 1 \otimes 1 \otimes U(\mathfrak{n}_-[t^{-1}])$; therefore it cannot belong in $\mathfrak{m}_\chi \mathbb{M}_{\mu,-h^\vee}$. This concludes the proof of (3.24) and hence of Lemma 3.10. $\square$

From now on λ will be assumed to be regular, integral, dominant.

3.3.2. Recall that $\mathbb{M}_{\lambda/z}$ denotes the quotient $\mathbb{M}_{\lambda,-h^\vee}/\mathcal{Z}_-\mathbb{M}_{\lambda,-h^\vee}$ and let N_λ be the (unique) maximal proper graded submodule of $\mathbb{M}_{\lambda/z}$. We have a short exact sequence

$$0 \to N_\lambda \to \mathbb{M}_{\lambda/z} \to \mathbb{V}_{\lambda/z} \to 0. \tag{3.25}$$

Here is the 2nd reduction:

Lemma 3.11. *If $H_{DS}^{\infty/2+0}(L\mathfrak{n}_+, N_\lambda) = 0$, then $H_{DS}^{\infty/2+0}(L\mathfrak{n}_+, \mathbb{M}_{\lambda,-h^\vee})$ is a free $\mathcal{Z}_-$-module generated by $[v_\lambda]$.*

Proof. Frenkel and Gaitsgory have computed [21]

$$H_{DS}^{\infty/2+i}(L\mathfrak{n}_+, \mathbb{V}_{\lambda/z}) = \begin{cases} \mathbb{C}[v_\lambda] & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the long exact sequence of cohomology applied to (3.25) gives

$$0 \to H_{DS}^{\infty/2+0}(L\mathfrak{n}_+, N_\lambda) \to H_{DS}^{\infty/2+0}(L\mathfrak{n}_+, \mathbb{M}_{\lambda/z}) \to \mathbb{C} \to 0.$$

Hence the implication

$$H_{DS}^{\infty/2+0}(L\mathfrak{n}_+, N_\lambda) = 0 \quad \Rightarrow \quad H_{DS}^{\infty/2+0}(L\mathfrak{n}_+, \mathbb{M}_{\lambda/z}) = \mathbb{C} = \mathbb{C}[v_\lambda].$$

It remains to prove the implication

$$H_{DS}^{\infty/2+0}(L\mathfrak{n}_+, \mathbb{M}_{\lambda/z}) = \mathbb{C}[v_\lambda] \quad \Rightarrow \quad H_{DS}^{\infty/2+0}(L\mathfrak{n}_+, \mathbb{M}_{\lambda,-h^\vee}) \xrightarrow{\sim} \mathcal{Z}_-. \tag{3.26}$$

We have already had a chance (Theorem 3.3) to cite the Frenkel–Gaitsgory result that $\mathbb{M}_{\lambda,-h^\vee}$ is a free $\mathcal{Z}_-$-module. The (increasing) degree filtration of $\mathcal{Z}_-$ induces a filtration of $\mathbb{M}_{\lambda,-h^\vee}$. Thus we obtain a spectral sequence

$$\{E_r^{pq}\} \quad \Rightarrow \quad H_{DS}^{\infty/2+p+q}(L\mathfrak{n}_+, \mathbb{M}_{\lambda,-h^\vee}).$$

All its terms are $\mathcal{Z}_-$-modules and, in particular,

$$\bigoplus_p E_1^{p,-p} = \mathcal{Z}_- \otimes H_{DS}^{\infty/2+0}(L\mathfrak{n}_+, \mathbb{M}_{\lambda/z}).$$

The left-hand side of (3.26) implies

$$\bigoplus_p E_1^{p,-p} \xrightarrow{\sim} \mathcal{Z}_-.$$

The higher terms of the spectral sequence are subquotients of the latter, but we already know, (3.22), that v_λ determines a cohomology class, which "survives" to the end, e.g. (3.24); hence the higher terms of the spectral sequence are, in fact, quotients of $\mathcal{Z}_-$. The character formula (3.23) implies that all these quotients are isomorphic to $\mathcal{Z}_-$. $\square$

It remains to prove

Proposition 3.12.

$$H_{DS}^{\infty/2+0}(L\mathfrak{n}_+, N_\lambda) = 0.$$

3.3.3. The highest weight λ being dominant, integral, regular, we obtain an exact sequence

$$\bigoplus_{i=1}^{\operatorname{rk}\mathfrak{g}} \mathbb{M}_{s_i\circ\lambda,-h^\vee} \to N_\lambda.$$

Let $N_i = \operatorname{Im}\{\mathbb{M}_{s_i\circ\lambda,-h^\vee} \to N_\lambda\}$. It is a highest weight module generated by v_i, the image of $v_{s_i\circ\lambda}$. Its twisted degree is $\langle s_i \circ \lambda, D - \rho^\vee\rangle$. To unburden the notation we will set $d_\mu = \langle \mu, D - \rho^\vee\rangle$. Here is one approximation to Proposition 3.12.

Proposition 3.13. $H^{\infty/2+0}(L\mathfrak{n}_+, N_i)_{d_{s_i\circ\lambda}} = 0$.

The proof will be given in Section 3.3.5 after some preparatory work in Section 3.3.4.

3.3.4. Let $\mathbb{L}_{\mu,-h^\vee}$ be the irreducible quotient of $\mathbb{M}_{\mu,-h^\vee}$.

Lemma 3.14. *If $M \in \hat{\mathcal{O}}_k$, then*

$$H_{DS}^{\infty/2+0}(L\mathfrak{n}_+, M) = \bigoplus_{d \geqslant d_{\min}} H_{DS}^{\infty/2+0}(L\mathfrak{n}_+, M)_d,$$

where $d_{\min} = \min\{d_\mu\colon [M : \mathbb{L}_{\mu,-h^\vee}] \neq 0\}$.

Proof. This assertion is an immediate consequence of the right exactness of the functor $H_{DS}^{\infty/2+0}(L\mathfrak{n}_+, ?)$ (Theorem 3.5(3)) and the character formula (3.23). □

Lemma 3.15. *Let λ be integral, regular, dominant.*

(i) *If $[\mathbb{M}_{s_i\circ\lambda,-h^\vee} : \mathbb{L}_{\mu,-h^\vee}] \neq 0$ and $d_\mu \leqslant d_{s_i\circ\lambda}$, then $d_\mu = d_{s_i\circ\lambda}$ and either $\mu = s_i \circ \lambda$ or $\mu = t_{\alpha_i^\vee}\circ\lambda$, where $t_{\alpha_i^\vee} = s_{-\alpha_i+\delta}s_{\alpha_i}$.*
(ii) *$t_{\alpha_i^\vee}\circ\lambda$ maximal among weights $\mu \neq s_i \circ \lambda$ such that $[\mathbb{M}_{s_i\circ\lambda,-h^\vee} : \mathbb{L}_{\mu,-h^\vee}] \neq 0$.*
(iii) $[\mathbb{M}_{s_i\circ\lambda,-h^\vee} : \mathbb{L}_{s_i\circ\lambda,-h^\vee}] = [\mathbb{M}_{s_i\circ\lambda,-h^\vee} : \mathbb{L}_{t_{\alpha_i^\vee}\circ\lambda-h^\vee}] = 1$.

Proof. (i) By virtue of the Kac–Kazhdan theorem [30], $[\mathbb{M}_{s_i\circ\lambda,-h^\vee} : \mathbb{L}_{\mu,-h^\vee}] \neq 0$ implies $\mu \in W\circ\lambda - \mathbb{Z}_+\delta$. It is clear that $\langle w\circ\lambda, -\rho^\vee\rangle \geqslant \langle s_i\circ\lambda, -\rho^\vee\rangle$ unless $w = \mathrm{id}$, and so $d_\mu \geqslant d_{s_i\circ\lambda}$ unless $\mu = \lambda \bmod \mathbb{Z}\delta$. Furthermore, $\langle w\circ\lambda, -\rho^\vee\rangle > \langle s_i\circ\lambda, -\rho^\vee\rangle$, hence $d_\mu > d_{s_i\circ\lambda}$, if $l(w) > 1$.

If $\mu = s_j \circ \lambda - n\delta$, then $n \geqslant 1$, hence $d_\mu > d_{s_i\circ\lambda}$ unless $j = i$, $n = 0$, in which case of course $\mu = s_i \circ \lambda$.

Finally, if $\mu = \lambda$, then $\mu = s_{-\alpha_i+\delta} \circ s_i \circ \lambda$, which shows at once that $d_\mu = d_{s_i\circ\lambda}$.

(ii) This important for what follows but technical assertion will be proved in Section 3.3.10.

(iii) The equality $[\mathbb{M}_{s_i\circ\lambda,-h^\vee} : \mathbb{L}_{s_i\circ\lambda,-h^\vee}] = 1$ is obvious. As to the equality $[\mathbb{M}_{s_i\circ\lambda,-h^\vee} : \mathbb{L}_{t_{\alpha_i^\vee}\circ\lambda,-h^\vee}] = 1$, item (ii) implies that the Shapovalov form on $(\mathbb{M}_{s_i\circ\lambda,-h^\vee})_{t_{\alpha_i^\vee}\circ\lambda}$ has a simple zero, and a standard application of the Jantzen filtration gives (iii). □

Lemma 3.16. *If $[N_i : \mathbb{L}_{\mu,-h^\vee}] \neq 0$ and $\mu \neq s_i \circ \lambda$, then $d_\mu > d_{s_i\circ\lambda}$.*

Proof. By Lemma 3.15(i), we have to eliminate the possibility of $\mu = t_{\alpha_i^\vee} \circ \lambda$. Lemma 3.15(iii) implies that the occurrence of $\mathbb{L}_{\mu,-h^\vee}$ in the composition series of N_i is due to the composite morphism

$$\mathbb{M}_{t_{\alpha_i^\vee}\circ\lambda,-h^\vee} \hookrightarrow \mathbb{M}_{s_i\circ\lambda,-h^\vee} \to N_i \subset \mathbb{M}_{\lambda/z} = \mathbb{M}_{\lambda,-h^\vee}/\mathcal{Z}_-\mathbb{M}_{\lambda,-h^\vee}.$$

But the latter lifts to a morphism $\mathbb{M}_{t_{\alpha_i^\vee}\circ\lambda,-h^\vee} \to \mathbb{M}_{\lambda,-h^\vee}$, hence defines an element of $\mathrm{End}_{\hat{\mathfrak{g}}}(\mathbb{M}_{\lambda,-h^\vee})$. (This is because $\mathbb{M}_{t_{\alpha_i^\vee}\circ\lambda,-h^\vee}$ and $\mathbb{M}_{\lambda,-h^\vee}$ are isomorphic as $\hat{\mathfrak{g}}$-modules.) By Theorem 3.3, this element equals an element of the center $\mathcal{Z}_-$, hence the above composite morphism is 0. □

3.3.5. To the proof of Proposition 3.13.

Dualizing (3.25), see Section 3.1.6 for the definition, we obtain the short exact sequence

$$0 \to \mathbb{L}_{\lambda,-h^\vee} \to \mathbb{M}^c_{\lambda/z} \to N^c_\lambda \to 0. \tag{3.27}$$

We have $\mathbb{M}^c_{\lambda/z} \xrightarrow{\sim} \mathbb{W}_{\lambda/z}$ (see Lemma 4.3 below). Since

$$H^{\infty/2+i}_{DS}(L\mathfrak{n}_+, \mathbb{L}_{\lambda,-h^\vee}) = H^{\infty/2+i}_{DS}(L\mathfrak{n}_+, \mathbb{W}_{\lambda/z}) = \begin{cases} \mathbb{C} & \text{if } i = 0, \\ 0 & \text{otherwise,} \end{cases}$$

an application of the long exact cohomology sequence to (3.27) gives

$$H^{\infty/2+\bullet}_{DS}\big(L\mathfrak{n}_+, N^c_\lambda\big) = 0. \tag{3.28}$$

N_i being a submodule of N_λ, we get a surjection $N^c_\lambda \to N^c_i$, hence, by Theorem 3.5(3) and (3.28),

$$H^{\infty/2+\bullet}_{DS}\big(L\mathfrak{n}_+, N^c_i\big) = 0. \tag{3.29}$$

Lemma 3.17.

(i) *The surjection* $N_i \to \mathbb{L}_{s_i\circ\lambda,-h^\vee}$ *induces an isomorphism* $H^{\infty/2+0}_{DS}(L\mathfrak{n}_+, N_i)_{d_{s_i\circ\lambda}} \to H^{\infty/2+0}_{DS}(L\mathfrak{n}_+, \mathbb{L}_{s_i\circ\lambda,-h^\vee})_{d_{s_i\circ\lambda}}$.

(ii) *The embedding* $\mathbb{L}_{s_i\circ\lambda,-h^\vee} \to N^c_i$ *induces an isomorphism* $H^{\infty/2+0}_{DS}(L\mathfrak{n}_+, \mathbb{L}_{s_i\circ\lambda,-h^\vee})_{d_{s_i\circ\lambda}} \to H^{\infty/2+0}_{DS}(L\mathfrak{n}_+, N^c_i)_{d_{s_i\circ\lambda}}$.

Proof. Let V be the kernel of $N_i \to \mathbb{L}_{s_i\circ\lambda,-h^\vee}$. Obviously, $[V : \mathbb{L}_{s_i\circ\lambda,-h^\vee}] = 0$. If so, Lemma 3.16 says that $[V : \mathbb{L}_{\mu,-h^\vee}] = [V^c : \mathbb{L}_{\mu,-h^\vee}] \neq 0$ implies $d_\mu > d_{s_i\circ\lambda}$. Now Lemma 3.14 implies $H^{\infty/2+0}_{DS}(L\mathfrak{n}_+, V)_{d_{s_i\circ\lambda}} = 0$. It remains to use the long cohomology sequence (in case (ii) combined with the inequality $\dim H^{\infty/2+0}_{DS}(L\mathfrak{n}_+, \mathbb{L}_{s_i\circ\lambda,-h^\vee})_{d_{s_i\circ\lambda}} \leqslant 1$, which is an obvious consequence of the right exactness and formula (3.23)). $\square$

Conclusion of the proof of Proposition 3.13. Lemma 3.17 implies the composition $N_i \to \mathbb{L}_{s_i\circ\lambda,-h^\vee} \to N^c_i$ induces an isomorphism $H^{\infty/2+0}_{DS}(L\mathfrak{n}_+, N_i)_{d_{s_i\circ\lambda}} \to H^{\infty/2+0}_{DS}(L\mathfrak{n}_+, N^c_i)_{d_{s_i\circ\lambda}}$. By (3.29), the target of this map is 0, hence so is its source. $\square$

3.3.6. Recall, Remark 3.4, that $H^{\infty/2+\bullet}_{DS}(L\mathfrak{n}_+, ?) = H^{\infty/2+\bullet}(L\mathfrak{n}_+, ? \otimes \mathbb{C}_\chi)$. Here is another approximation to Proposition 3.12, one that replaces the semi-infinite with ordinary Lie algebra homology.

Proposition 3.18. $H_0(\mathfrak{n}_+[t^{-1}]t^{-1}, N_\lambda \otimes \mathbb{C}_\chi) = 0$.

The proof will occupy the rest of this subsection with a proof of one auxiliary assertion spilling over to Section 3.3.7.

Let $\{U_i(\hat{\mathfrak{g}})\}$ be the standard filtration of the universal enveloping algebra. The latter is acted upon by $\rho^\vee$, and we set $U_i(\hat{\mathfrak{g}})[j] = \{u \in U_i(\hat{\mathfrak{g}})\colon\ [\rho^\vee, u] = -j\}$. The collection of subspaces $K_n U_i(\hat{\mathfrak{g}}) = \sum_{i+j\leqslant n} U_i(\hat{\mathfrak{g}})[j]$ defines a filtration of $U(\hat{\mathfrak{g}})$ known as the *Kazhdan filtration*, cf. [23,5]. We have $\mathrm{gr}_K\, U(\hat{\mathfrak{g}}) = S(\hat{\mathfrak{g}})$.

Any Lie subalgebra $\mathfrak{a} \subset \hat{\mathfrak{g}}$ carries an induced filtration $\{K_n\mathfrak{a}\}$ so that $\mathrm{gr}_K\,\mathfrak{a} = S(\mathfrak{a})$, and the embedding $\mathfrak{a} \hookrightarrow \hat{\mathfrak{g}}$ induces the embedding $S(\mathfrak{a}) \hookrightarrow S(\hat{\mathfrak{g}})$.

Denote by v_i a highest weight vector of N_i. Define $K_n(N_\lambda \otimes \mathbb{C}_\chi) = \sum_i (K_n U(\hat{\mathfrak{g}}))(v_i \otimes 1)$; this is a Kazhdan filtration of $N_\lambda \otimes \mathbb{C}_\chi$ compatible with that of $U(\mathfrak{n}_+[t^{-1}]t^{-1})$, and so $\mathrm{gr}_K(N_\lambda \otimes \mathbb{C}_\chi)$ is an $S(\mathfrak{n}_+[t^{-1}]t^{-1})$-module generated by a collection $\{v_i\}$. Note that a similar definition applies to any finitely generated $\hat{\mathfrak{g}}$-module.

This defines a filtration on the standard homology complex, $\{K_n C_\bullet(\mathfrak{n}_+[t^{-1}]t^{-1}, N_\lambda \otimes \mathbb{C}_\chi)\}$. In the arising spectral sequence $\{E^r_{\bullet\bullet}\}$, the term $E^1_{\bullet\bullet}$ is the homology of the Koszul complex $C_\bullet(\{e^\alpha \otimes t^n,\ \alpha \in \Delta_+,\ n < 0\}, \mathrm{gr}_K(N_\lambda \otimes \mathbb{C}_\chi))$. Proposition 3.13 implies (use the definition of the Drinfeld–Sokolov differential) that the symbol of each $v_i \otimes 1$ belongs to $\mathfrak{n}_+[t^{-1}]t^{-1}\,\mathrm{gr}_K(N_\lambda \otimes \mathbb{C}_\chi)$. Therefore

$$\bigoplus_{p=-\infty}^{+\infty} E^1_{-p,p} = 0.$$

Proposition 3.18 will follow once the convergence of the spectral sequence is proved. Since we are only interested in the vanishing of the zeroth homology group, it suffices to prove that for each twisted degree d, Section 3.2.2,

$$H_0\bigl(K_p C_\bullet\bigl(\mathfrak{n}_+\bigl[t^{-1}\bigr]t^{-1}, N_\lambda \otimes \mathbb{C}_\chi\bigr)_d\bigr) = 0 \quad \text{if } p \ll 0. \tag{3.30}$$

3.3.7. Proof of (3.30)

Set $M = \bigoplus_i \mathbb{M}_{s_i \circ \lambda, -h^\vee}$. Consider the standard homology complex $\{C_\bullet(\mathfrak{n}_+[t^{-1}]t^{-1}, M \otimes \mathbb{C}_\chi)\}$ and the Kazhdan filtration $\{K_n C_\bullet(\mathfrak{n}_+[t^{-1}]t^{-1}, N_\lambda \otimes \mathbb{C}_\chi)\}$.

Lemma 3.19. *For each twisted degree d*

$$H_\bullet\bigl(K_p C_\bullet\bigl(\mathfrak{n}_+\bigl[t^{-1}\bigr]t^{-1}, M \otimes \mathbb{C}_\chi\bigr)_d\bigr) = 0 \quad \textit{if } p \ll 0.$$

Proof. For the purpose of this proof let $C_\bullet(M) = C_\bullet(\mathfrak{n}_+[t^{-1}]t^{-1}, M \otimes \mathbb{C}_\chi)$. Consider a short exact sequence of complexes

$$0 \to K_p C_\bullet(M)_d \to C_\bullet(M)_d \to C_\bullet(M)_d / K_p C_\bullet(M)_d \to 0.$$

Note that M being free over $\mathfrak{n}_+[t^{-1}]t^{-1}$, $H_i(\mathrm{gr}_K\, C_\bullet(M)) = 0$ if $i \neq 0$; furthermore, it is clear that $\dim H_0(\mathrm{gr}_K\, C_\bullet(M)_d) = \dim \mathrm{gr}_K\, H_0(C_\bullet(M)_d) < \infty$ for each d. The Kazhdan filtration on $C_\bullet(M)_d / K_p C_\bullet(M)_d$ being regular, $H_0(\mathrm{gr}_K\, C_\bullet(M)_d / K_p C_\bullet(M)_d) = \mathrm{gr}_K\, H_0(C_\bullet(M)_d / K_p C_\bullet(M)_d)$. But $\mathrm{gr}_K\, C_\bullet(M)_d / K_p C_\bullet(M)_d$ is a direct summand of the complex $C_\bullet(M)_d$. The finite dimensionality, $\dim H_0(\mathrm{gr}_K\, C_\bullet(M)_d)_d < \infty$, implies that the rightmost map in the exact sequence above, $C_\bullet(M)_d \to C_\bullet(M)_d / K_p C_\bullet(M)_d$ induces an isomorphism of homology $H_\bullet(C_\bullet(M)_d) \to H_\bullet(C_\bullet(M)_d / K_p C_\bullet(M)_d)$ for all $p \ll 0$. Lemma 3.19 follows. □

Now an obvious surjection $M \to N_\lambda$ induces a surjection of complexes $K_p C_\bullet(\mathfrak{n}_+[t^{-1}]t^{-1}, M \otimes \mathbb{C}_\chi)_d \to K_p C_\bullet(\mathfrak{n}_+[t^{-1}]t^{-1}, N_\lambda \otimes \mathbb{C}_\chi)_d$. Since the homology functor is exact, we obtain a surjection $H_0(K_p C_\bullet(\mathfrak{n}_+[t^{-1}]t^{-1}, M \otimes \mathbb{C}_\chi)_d) \to H_0(K_p C_\bullet(\mathfrak{n}_+[t^{-1}]t^{-1}, N_\lambda \otimes \mathbb{C}_\chi)_d)$. Thus Lemma 3.19 implies (3.30). The proof of Proposition 3.18 is now complete.

3.3.8. We will now undertake a short digression on relative cohomology. For any $M \in \mathcal{O}_k$, consider

$$C^{\infty/2+\bullet}\big(L\mathfrak{n}_+, \mathfrak{n}_+\big[t^{-1}\big]t^{-1}, M \otimes \mathbb{C}_\chi\big) = \frac{C^\bullet(L\mathfrak{n}_+, M \otimes \mathbb{C}_\chi)}{\operatorname{span}\{(\phi_\alpha)_{(-n)}c, [d_{st}, (\phi_\alpha)_{(-n)}]c,\ \alpha \in \Delta_+,\ n > 0,\ c \in C^\bullet(L\mathfrak{n}_+, M \otimes \mathbb{C}_\chi)\}}.$$

It is a quotient complex of the Drinfeld–Sokolov complex $(C^{\infty/2+\bullet}(L\mathfrak{n}_+, M \otimes \mathbb{C}_\chi), d_{st})$, cf. Remark 3.4, known as the *relative semi-infinite cohomology complex* of $L\mathfrak{n}_+$ with coefficients in $M \otimes \mathbb{C}_\chi$. The corresponding relative cohomology groups will be denoted by $H^{\infty/2+i}(L\mathfrak{n}_+, \mathfrak{n}_+[t^{-1}]t^{-1}, M \otimes \mathbb{C}_\chi)$; note that $i \geqslant 0$.

Theorem 3.20. *The functors $H^{\infty/2+0}(L\mathfrak{n}_+, \mathfrak{n}_+[t^{-1}]t^{-1}, ? \otimes \mathbb{C}_\chi)$ and $H_{DS}^{\infty/2+0}(L\mathfrak{n}_+, ?)$ are naturally isomorphic.*

The rest of this subsection will be devoted to the proof of this theorem.

Lemma 3.21. *If P is a module with a Verma filtration, then*

$$H^{\infty/2+\bullet}\big(L\mathfrak{n}_+, \mathfrak{n}_+\big[t^{-1}\big]t^{-1}, P \otimes \mathbb{C}_\chi\big) \xrightarrow{\sim} H_{DS}^{\infty/2+\bullet}(L\mathfrak{n}_+, P).$$

Proof. Consider the Serre–Hochschild spectral sequence [39] $\{E_r^{pq}\}$ with $E_1^{pq} = H_{-p}(\mathfrak{n}_+[t^{-1}]t^{-1}, P \otimes \mathbb{C}_\chi \otimes \Lambda^q \mathfrak{n}_+[t]^*)$. Since P is filtered by Verma modules, it is free over $\mathfrak{n}_+[t^{-1}]t^{-1}$, hence $E_1^{pq} = 0$ unless $p = 0$, in which case it equals $H_0(\mathfrak{n}_+[t^{-1}]t^{-1}, P \otimes \mathbb{C}_\chi \otimes \Lambda^\bullet \mathfrak{n}_+[t]^*)$, and the spectral sequence collapses. In fact, a little more is valid.

Note that although the differential does not preserve conformal grading (this is $\mathbb{C}_\chi$'s doing), it preserves the increasing conformal filtration $\{F_n C_{DS}^{\infty/2+p+q}(L\mathfrak{n}_+, P)\}$, cf. Section 3.1.5. It is easy to see that the aforementioned vanishing takes place upon restriction to each term of this filtration: $(F_n E)_1^{pq} = 0$ unless $p = 0$. Now notice that the filtration that leads to $\{E_r^{pq}\}$ induces a finite filtration of $F_n C_{DS}^{\infty/2+p+q}(L\mathfrak{n}_+, P)$ for each n. A quick diagram chase then shows the convergence of the spectral sequence: $\{E_r^{pq}\} \Rightarrow H_{DS}^{\infty/2+p+q}(L\mathfrak{n}_+, P)$.

Since the spectral sequence collapses, we obtain that the natural map $H_{DS}^{\infty/2+\bullet}(L\mathfrak{n}_+, P) \to H^\bullet(E_{0\bullet}^1)$ defines an isomorphism $H_{DS}^{\infty/2+\bullet}(L\mathfrak{n}_+, P) \xrightarrow{\sim} H^\bullet(E_{0\bullet}^1)$.

On the other hand, by definition, the complexes $E_{0\bullet}^1$ and $(C^\bullet(L\mathfrak{n}_+, \mathfrak{n}_+[t^{-1}]t^{-1}, M \otimes \mathbb{C}_\chi), d_{st})$ are identical. □

Since the Drinfeld–Sokolov functor $H_{DS}^{\infty/2+0}(L\mathfrak{n}_+, ?)$ is right exact and the class of modules with a Verma filtration is adapted, it remains to prove that the functor $H^{\infty/2+0}(L\mathfrak{n}_+, \mathfrak{n}_+[t^{-1}]t^{-1}, ? \otimes \mathbb{C}_\chi)$ is right exact.

Instrumental for proving the right exactness is Proposition 3.7.1 of [5]. This result is but a jet scheme version of [23, Sections 5 and 6], and it deals with the following situation. Let $\mathfrak{b} \subset \mathfrak{a}$ be a pair "Lie algebra, Lie subalgebra," and let a vector space M be acted upon by $S(\mathfrak{a})$ as a commutative algebra, and by $\mathfrak{b}$ as a Lie algebra. Denote by $a \bullet ?$ and $\{b, ?\}$ the respective actions by $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$. We shall say that the 2 structures are compatible if $\{b, a \bullet m\} = [b, a] \bullet m + a \bullet \{b, m\}$.

Lemma 3.22. *(Cf. [5, Proposition 3.7.1].) Let M be acted upon by $S(\hat{\mathfrak{g}}/\mathfrak{g}[t])$ as a commutative algebra and by $\mathfrak{g}[t]$ as a Lie algebra, and let the 2 structures be compatible. If the shifted subalgebra $S(\mathfrak{m}_\chi) = S(\mathrm{span}\{e_\alpha \otimes t^n + \chi(e_\alpha \otimes t^n),\ n < 0,\ \alpha \in \Delta_+\})$ (cf. Remark 3.4) satisfies $S(\mathfrak{m}_\chi)^N M = 0$ for some $N > 0$, then*

$$H^i\big(\mathfrak{n}_+[t], M\big) = \begin{cases} M^{\mathfrak{n}_+[t]} & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

This lemma will be applied via the decreasing Li filtration [34]. Given a vertex algebra V and a V-module M, Li defines $\mathrm{L}_n M$ to be the linear span $x^1_{(-i_1-1)} \cdots x^l_{(-i_l-1)} m$ with $x^j \in V$, $1 \leqslant j \leqslant l$, $m \in M$, and $\sum_j i_j \geqslant n$. Li checks that the graded objects satisfy: $\mathrm{gr}_L V$ is a vertex Poisson algebra, and $\mathrm{gr}_L M$ is a $\mathrm{gr}_L V$-module. Whatever else this means (see also [18] for details), this implies that if $M \in \hat{\mathcal{O}}_k$, then $\mathrm{gr}_L M$ is an $S(\hat{\mathfrak{g}}/\mathfrak{g}[t])$-module and a $\mathfrak{g}[t]$-module, and the 2 structures are compatible.

Corollary 3.23. *For each $M \in \hat{\mathcal{O}}_k$, $H^{\infty/2+i}(L\mathfrak{n}_+, \mathfrak{n}_+[t^{-1}]t^{-1}, M \otimes \mathbb{C}_\chi) = 0$ if $i \neq 0$. Furthermore, $H^{\infty/2+0}(L\mathfrak{n}_+, \mathfrak{n}_+[t^{-1}]t^{-1}, M \otimes \mathbb{C}_\chi) = 0$ is filtered, and $\mathrm{gr}\, H^{\infty/2+0}(L\mathfrak{n}_+, \mathfrak{n}_+[t^{-1}]t^{-1}, M \otimes \mathbb{C}_\chi) = (\mathrm{gr}\, M/\mathfrak{m}_\chi\, \mathrm{gr}\, M)^{\mathfrak{n}_+[t]}$.*

Proof. Consider the decreasing Li [34] filtration of the relative semi-infinite cohomology complex. A spectral sequence arises, and its second term is $H^\bullet(\mathfrak{n}_+[t], \mathrm{gr}_L M/\mathfrak{m}_\chi\, \mathrm{gr}_L M)$. As explained above, this brings us into the situation of Lemma 3.22. An application of Lemma 3.22 concludes the proof. □

The right exactness of the functor $H^{\infty/2+0}(L\mathfrak{n}_+, \mathfrak{n}_+[t^{-1}]t^{-1}, ? \otimes \mathbb{C}_\chi)$ is now easy. Given an exact sequence

$$0 \to K \to M \to N \to 0,$$

we consider its Li graded version (as in the proof above)

$$0 \to \mathrm{gr}_L K \to \mathrm{gr}_L M \to \mathrm{gr}_L N \to 0,$$

obtain an exact sequence of co-invariants (0th Lie algebra homology)

$$\mathrm{gr}_L K/\mathfrak{m}_\chi\, \mathrm{gr}_L K \to \mathrm{gr}_L M/\mathfrak{m}_\chi\, \mathrm{gr}_L M \to \mathrm{gr}_L N/\mathfrak{m}_\chi\, \mathrm{gr}\, N_L \to 0,$$

and letting $U = \mathrm{Ker}\{\mathrm{gr}_L K/\mathfrak{m}_\chi\, \mathrm{gr}_L K \to \mathrm{gr}_L M/\mathfrak{m}_\chi\, \mathrm{gr}_L M\}$, an exact cohomology sequence

$$(\mathrm{gr}_L M/\mathfrak{m}_\chi\, \mathrm{gr}_L M)^{\mathfrak{n}_+[t]} \to (\mathrm{gr}_L N/\mathfrak{m}_\chi\, \mathrm{gr}_L N)^{\mathfrak{n}_+[t]} \to H^1\big(\mathfrak{n}_+[t], U\big) \overset{\text{Lemma 3.22}}{=} 0.$$

By virtue of Corollary 3.23, this shows that the graded functor, $\mathrm{gr}\, H^{\infty/2+0}(L\mathfrak{n}_+, \mathfrak{n}_+[t^{-1}]t^{-1}, ? \otimes \mathbb{C}_\chi)$, is right exact. Hence so is the functor itself. This concludes the proof of Theorem 3.20.

3.3.9. We can now derive Proposition 3.12 from its “2nd approximation,” Proposition 3.18. Recall that we want to show $H_{DS}^{\infty/2+0}(L\mathfrak{n}_+, N_\lambda) = 0$. According to Theorem 3.20, this is equivalent to $H^{\infty/2+0}(L\mathfrak{n}_+, \mathfrak{n}_+[t^{-1}]t^{-1}, N_\lambda \otimes \mathbb{C}_\chi) = 0$. But we have the vanishing already at the level of the complex: indeed, by the definition of the relative semi-infinite cohomology, Section 3.3.8, $C^{\infty/2+0}(L\mathfrak{n}_+, \mathfrak{n}_+[t^{-1}]t^{-1}, N_\lambda \otimes \mathbb{C}_\chi) = N_\lambda/\mathfrak{m}_\chi N_\lambda$, which is 0 thanks to Proposition 3.18. This concludes the proof of Proposition 3.12, hence of Theorem 3.9 – modulo, that is, Lemma 3.15(ii). Its proof is as follows:

3.3.10. We will be freely using the results and notation from [35]. Let $A_w = wA_e$ be the alcove attached to an affine Weyl group element $w \in \hat{W}$. Choose quarters C_v^+ (cf. [35, Section 1.1]) as the translates of the fundamental chamber of $\mathfrak{g}$.

Defined in [35, Section 1.4], is a number, $d(A, B)$, for any two alcoves A, B.

Lemma 3.24. $d(A_w, A_1) = l^{\frac{\infty}{2}}(w)$, *where*

$$l^{\frac{\infty}{2}}(w) = \#\{\alpha \in \hat{\Delta}_+^{re} \cap w^{-1}(\hat{\Delta}_-^{re}),\ \bar{\alpha} \in \Delta_+\} - \#\{\alpha \in \hat{\Delta}_+^{re} \cap w^{-1}(\hat{\Delta}_-^{re}),\ \bar{\alpha} \in \Delta_-\},$$

and $\hat{\mathfrak{h}}^* \ni \lambda \mapsto \bar{\lambda} \in \mathfrak{h}$ *is the restriction map, and* $\hat{\Delta}_\pm^{re}$ *is the set of positive (negative) real roots of* $\hat{\mathfrak{g}}$.

Proof. Use [35, Section 1.4] to proceed by induction on the length $l(w)$. If $l(w) = 1$, the assertion follows from the definition. Let $w = s_i y$ with $l(w) = l(y) + 1$. Then A_y and A_w are next to each other and are separated by the hyperplane $y^{-1}(H_i)$. Hence $d(A_w, A_1) = d(A_y, A_1) \pm 1$. By definition, $d(A_w, A_1) = d(A_y, A_1) + 1$ iff $\overline{y^{-1}(\alpha_i)} \in \Delta_+$. Since $\hat{\Delta}_+^{re} \cap w^{-1}(\hat{\Delta}_-^{re}) = \hat{\Delta}_+^{re} \cap y^{-1}(\hat{\Delta}_-^{re}) \sqcup \{y^{-1}(\alpha_i)\}$, the assertion follows. $\square$

Lemma 3.25. *Let* $\lambda \in \hat{\mathfrak{h}}^*$ *be a weight at the critical level such that* $\bar{\lambda}$ *is dominant regular,* $w \in \hat{W}$, $\alpha \in \hat{\Delta}_+^{re}$. *If there is an embedding* $\mathbb{M}_{s_\alpha w \circ \lambda} \to \mathbb{M}_{w \circ \lambda}$, *then* $l^{\frac{\infty}{2}}(s_\alpha w) > l^{\frac{\infty}{2}}(w)$.

Proof. By [35, (1.4.1)], $l^{\frac{\infty}{2}}(s_\alpha w) = l^{\frac{\infty}{2}}(w) + d(A_{s_\alpha w}, A_w)$. Note that $A_{s_\alpha w} = A_w \sigma_{H_{w^{-1}(\alpha)}}$. By assumption, $\overline{w^{-1}(\alpha)} \in \Delta_+$. Therefore, by [35, Lemma 3.1], $d(A_{s_\alpha w}, A_w) > 0$. $\square$

Now Lemma 3.15(ii) follows from

Lemma 3.26. *For* $i \neq 0$ *(equivalently,* α_i *is a simple root of the finite root system),* $l^{\frac{\infty}{2}}(s_i) = 1$ *and* $l^{\frac{\infty}{2}}(t_{\alpha_i^\vee}) = 2$.

Proof. The first assertion is obvious. The second is also elementary: each element of $\beta \in \Delta_+$ will contribute $\langle \beta, \alpha_i^\vee \rangle$ toward $l^{\frac{\infty}{2}}(t_{\alpha_i^\vee})$; overall we obtain $2\langle \rho, \alpha_i^\vee \rangle$, which is 2. $\square$

The proof of Theorem 3.9 is now complete.

4. Proof of Theorem 1.1

4.1. Resolution

We shall work in the setting of Section 2.2.7.

The Cousin resolution of $\mathcal{L}_{\nu_0}$ w.r.t. the filtration of X by $\{X_w,\ w \in W\}$ reads

$$0 \to \mathcal{L}_{\nu_0} \to i_{\mathrm{id},+} i_{\mathrm{id}}^* \mathcal{L}_{\nu_0} \to \bigoplus_{w \in W^{(1)}} i_{w,+} i_w^* \mathcal{L}_{\nu_0} \to \bigoplus_{w \in W^{(2)}} i_{w,+} i_w^* \mathcal{L}_{\nu_0} \to \cdots. \tag{4.1}$$

Note that

$$\Gamma\big(X, i_{w,+} i_w^* \mathcal{L}_{\nu_0}\big) = M^c_{w \circ \nu_0}, \tag{4.2}$$

the contragredient Verma module.

Let $\mathcal{M}^w_{\nu(z)}$ stand for $\mathcal{Z}hu_{\nu(z)}(i_{w,+} i_w^* \mathcal{L}_{\nu_0})$, see (2.29).

The functor $\mathcal{Z}hu_{\nu(z)}$ is exact [6], and its application to (4.1) gives a resolution in the category of $\mathcal{D}^{ch,tw}_X$-modules

$$0 \to \mathcal{L}^{ch}_{\nu(z)} \to \mathcal{M}^{\mathrm{id}}_{\nu(z)} \to \bigoplus_{w \in W^{(1)}} \mathcal{M}^w_{\nu(z)} \to \bigoplus_{w \in W^{(2)}} \mathcal{M}^w_{\nu(z)} \to \cdots. \tag{4.3}$$

Lemma 2.6 implies

$$H^i\big(X, \mathcal{M}^w_{\nu(z)}\big) = 0 \quad \text{if } i > 0.$$

Since the class of sheaves with vanishing higher cohomology is adapted to the functor of global sections, the complex

$$0 \to \Gamma\big(X, \mathcal{M}^{\mathrm{id}}_{\nu(z)}\big) \to \bigoplus_{w \in W^{(1)}} \Gamma\big(X, \mathcal{M}^w_{\nu(z)}\big) \to \bigoplus_{w \in W^{(2)}} \Gamma\big(X, \mathcal{M}^w_{\nu(z)}\big) \to \cdots \tag{4.4}$$

computes the cohomology $H^i(X, \mathcal{L}^{ch}_{\nu(z)})$.

Now recall (2.39) that $\Gamma(X, \mathcal{M}^w_{\nu(z)})$ is precisely the w-twisted Wakimoto module $\mathbb{W}^w_{\nu(z)}$. We can summarize our discussion as follows.

Lemma 4.1. *The cohomology of the complex*

$$0 \to \mathbb{W}^{\mathrm{id}}_{\nu(z)} \to \bigoplus_{w \in W^{(1)}} \mathbb{W}^w_{\nu(z)} \to \bigoplus_{w \in W^{(2)}} \mathbb{W}^w_{\nu(z)} \to \cdots \tag{4.5}$$

is isomorphic to $H^\bullet(X, \mathcal{L}^{ch}_{\nu(z)})$.

Both items of Theorem 1.1 follow from this lemma, the first easily, the second after some work is done.

4.2. Proof of Theorem 1.1(1) and formula (1.7)

First of all, the definition used in Theorem 1.1(1),

$$\chi\big(\mathcal{L}^{ch}_{\nu(z)}\big) = \sum_{i=0}^{\dim X} (-1)^i \operatorname{ch} H^i\big(X, \mathcal{L}^{ch}_{\nu(z)}\big),$$

has been made sense of: $\chi(\mathcal{L}^{ch}_{\nu(z)})$ being a sheaf of filtered, see (2.31), $\hat{\mathfrak{g}}$-modules, see Theorem 2.4(4), the cohomology groups $H^i(X, \mathcal{L}^{ch}_{\nu(z)})$ are objects of $\hat{\mathcal{O}}_{\nu(z)\circ\pi}$, the full subcategory of $\hat{\mathcal{O}}_{crit}$, cf. Section 3.1.1; their formal characters, $\mathrm{ch}\, H^i(X, \mathcal{L}^{ch}_{\nu(z)})$, are discussed in Section 3.1.5.

Lemma 4.1 implies

$$\chi\big(\mathcal{L}^{ch}_{\nu(z)}\big) = \sum_{w\in W} (-1)^{l(w)}\, \mathrm{ch}\, \mathbb{W}^{w}_{\nu(z)}. \tag{4.6}$$

Since, see (3.15),

$$\mathrm{ch}\, \mathbb{W}^{w}_{\nu(z)} = e^{w\circ\nu_0} \prod_{\alpha\in\Delta_+} \big(1-e^{-\alpha}\big)^{-1} \times \prod_{n=1}^{+\infty} \prod_{\alpha\in\Delta_+} \big(1-e^{-\alpha-n\delta}\big)^{-1} \times \prod_{\alpha\in\Delta_-} \big(1-e^{-\alpha-n\delta}\big)^{-1}$$

(4.6) can be rewritten as follows

$$\begin{aligned}\chi\big(\mathcal{L}^{ch}_{\nu(z)}\big) = {}&\Big(\sum_{w\in W} (-1)^{l(w)} e^{w\circ\nu_0} \prod_{\alpha\in\Delta_+} \big(1-e^{-\alpha}\big)^{-1}\Big)\\ &\times \prod_{n=1}^{+\infty} \prod_{\alpha\in\Delta_+} \big(1-e^{-\alpha-n\delta}\big)^{-1} \prod_{\alpha\in\Delta_-} \big(1-e^{-\alpha-n\delta}\big)^{-1},\end{aligned} \tag{4.7}$$

which is Theorem 1.1(1) in a slightly expanded form. In order to obtain (1.7), we have to let $e^{\alpha} \to 1$, $\alpha \in \Delta$, and set $e^{-\delta} = q$. In the limit, the first factor in (4.7) equals $\dim V_{\nu_0}$ (the Weyl character formula!), and (4.7) becomes

$$\chi_P\big(\mathcal{L}^{ch}_{\nu(z)}\big) = \dim V_{\nu_0} \prod_{n=1}^{+\infty} \big(1-q^n\big)^{-2\dim X},$$

as desired.

4.3. *Proof of Theorem 1.1(2): the case of a homogeneous character*

Since each $\mathcal{L}^{ch}_{\nu(z)}$ is G-equivariant, $H^i(X, \mathcal{L}^{ch}_{\nu(z)}) \in \hat{\mathcal{O}}^{G}_{\mathrm{crit},\nu(z)\circ\pi}$, the full subcategory of $\hat{\mathcal{O}}_{crit}$ that consists of those $\hat{\mathfrak{g}}$-modules where the action of $\mathfrak{g} \subset \hat{\mathfrak{g}}$ integrates to an action of G. This category being semi-simple with a unique simple object $\mathbb{V}_{\nu(z)}$ [21], we obtain

$$H^i\big(X, \mathcal{L}^{ch}_{\nu(z)}\big) = \bigoplus_{j=1}^{m_i} \mathbb{V}_{\nu(z)}[n_{ij}], \tag{4.8}$$

for some nonnegative integers m_i and $\{n_{ij}\}$, which we have to determine. The meaning of $\mathbb{V}_{\nu(z)}[n_{ij}]$ is as follows: the left-hand side of (4.8) is filtered via (2.31); each $\mathbb{V}_{\nu(z)}$ appearing on the right inherits a filtration, and the inherited filtration can be different from the *natural* one, see Section 3.1.5 by a shift; this shift is denoted by $[n_{ij}]$.

It is proved in [21] that the Drinfeld–Sokolov reduction functor extracts these numbers:

$$H_{DS}^{\infty/2+k}\big(L\mathfrak{n}_+, H^i\big(X, \mathcal{L}^{ch}_{\nu(z)}\big)\big) = \begin{cases} \bigoplus_{j=1}^{m_i} \mathbb{C}[n_{ij}] & \text{if } k=0, \\ 0 & \text{otherwise.} \end{cases} \tag{4.9}$$

It is easy to interpret (4.9) is a computation of the cohomology of a certain double complex. Let

$$\mathcal{C}^j = \bigoplus_{w\in W^{(j)}} \mathbb{W}^w_{\nu(z)},$$

and extend (4.5) to a double complex

$$K^{\bullet\bullet} = \bigoplus_{pq} K^{pq}, \qquad K^{pq} = \big(C^{\infty/2+p}\big(L\mathfrak{n}_+, \mathcal{C}^q\big), d_{DS}+d\big), \tag{4.10}$$

where by d we have denoted the differential of complex (4.9).

Either of the two spectral sequences associated with K^{pq} converges to $H^{p+q}_{d_{DS}+d}(K^{\bullet\bullet})$ because complex (4.9) is of finite length. What (4.9) says is that one of these spectral sequences collapses in the second term and

$$H^i_{d_{DS}+d}\big(K^{\bullet\bullet}\big) = \bigoplus_{j=1}^{m_i} \mathbb{C}[n_{ij}]. \tag{4.11}$$

We will compute another spectral sequence (not without a twist) thereby proving the following

Lemma 4.2. *If* $\nu(z) = \nu_0/z$,

$$H^i_{d_{DS}+d}\big(K^{\bullet\bullet}\big) = \bigoplus_{w\in W^{(i)}} \mathbb{C}\big[\langle \nu_0 - w\circ\nu_0, \rho^\vee\rangle\big].$$

This lemma along with (4.8), (4.9), (4.11) implies Theorem 1.1(2).

4.3.1. Proof of Lemma 4.2

Since $\nu(z) = \nu_0/z$, each $\mathcal{C}^j$ is a graded $\hat{\mathfrak{g}}$-module, and we can dualize,[4] see Section 3.1.6, to obtain $(\mathcal{C}^q)^c$ and

$$\tilde{K}^{\bullet\bullet} = \bigoplus_{pq} \tilde{K}^{pq}, \qquad K^{pq} = \big(C^{\infty/2+p}\big(L\mathfrak{n}_+, \big(\mathcal{C}^q\big)^c\big), d_{DS}+d\big).$$

Since $\mathbb{V}_{\nu(z)}$ is irreducible, $(\mathbb{V}_{\nu_0/z})^c = \mathbb{V}_{\nu_0/z}$ and the argument that led above to (4.11) can be repeated to give us

$$H^i_{d_{DS}+d}\big(\tilde{K}^{\bullet\bullet}\big) = \bigoplus_{j=1}^{m_i} \mathbb{C}[n_{ij}] = H^i_{d_{DS}+d}\big(K^{\bullet\bullet}\big). \tag{4.12}$$

[4] This is the only place, where the homogeneity assumption $\nu(z) = \nu_0/z$ is used.

Therefore, it suffices to prove Lemma 4.2 with $K^{\bullet\bullet}$ replaced with $\tilde{K}^{\bullet\bullet}$. In order to do this, consider that spectral sequence $\{E_r^{pq}\}$ where

$$E_1^{pq} = H_{DS}^{\infty/2+q}\big((\mathcal{C}^p)^c\big).$$

Now we wish to compute $\{E_1^{pq}\}$.

Lemma 4.3.

$$\mathbb{W}^w_{\nu_0/z} \tilde{=} \mathbb{M}^c_{w\circ\nu_0/z}.$$

Lemma 4.3 implies

$$\big(\mathbb{W}^w_{\nu_0/z}\big)^c \tilde{=} \mathbb{M}_{w\circ\nu_0/z}.$$

Since $(\mathcal{C}^p)^c = \bigoplus_{w\in W^{(p)}}(\mathbb{W}^w_{\nu_0/z})^c$, now known to be $\bigoplus_{w\in W^{(p)}}\mathbb{M}_{w\circ\nu_0/z}$, Corollary 3.7 gives

$$E_1^{pq} = \begin{cases} \bigoplus_{w\in W^{(i)}}\mathbb{C}[\langle\nu_0 - w\circ\nu_0, \rho^\vee\rangle] & \text{if } q=0,\\ 0 & \text{otherwise,}\end{cases} \tag{4.13}$$

where $\mathbb{C}[\langle\nu_0 - w\circ\nu_0, \rho^\vee\rangle]$ is spanned by the class of the highest weight vector of $\mathbb{M}_{w\circ\nu_0/z}$; note that the grading shift is precisely the one obtained by placing a highest weight vector of $\mathbb{M}_{\nu_0/z}$ in degree 0 component and then using the twisted grading as defined in Section 3.2.2. The following dimensional argument shows that all higher differentials vanish. The cohomology classes recorded in (4.13) are represented by the highest weight vectors $\mathbf{1}_{w\circ\nu_0} \in \mathbb{W}^w_{\nu_0/z}$. The twisted degree, the one that is preserved by the differential of the double complex (4.10), of $\mathbf{1}_{w\circ\nu_0}$ is $\langle w\circ\nu_0, \rho^\vee\rangle$. The differential d_1 that operates on $E_1^{\bullet\bullet}$ is induced by the differential of complex (4.5). The latter (by the construction of the Cousin resolution (4.1)) is a direct sum of the maps

$$\mathbb{W}^w_{\nu_0/z} \to \mathbb{W}^v_{\nu_0/z} \quad \text{with } v > w, \quad l(v) = l(w)+1.$$

Since ν_0 is regular dominant, $\langle w\circ\nu_0, \rho^\vee\rangle < \langle v\circ\nu_0, \rho^\vee\rangle$ provided $w < v$. Hence E_1^{p0} and E_1^{p-10} have different twisted degrees, which makes d_1 be equal to zero. The vanishing of higher differentials is obvious. This concludes the proof of Lemma 4.2.

4.3.2. Proof of Lemma 4.3

This proof is but a version of the argument in [17, Section 9.5.2], just as Lemma 4.3 is a generalization of Proposition 9.5.1 proved there.

The universal property of the Verma module implies the existence of a non-zero morphism

$$\mathbb{M}_{w\circ\nu_0/z} \to \big(\mathbb{W}^w_{\nu_0/z}\big)^c.$$

Dualizing we obtain

$$\mathbb{W}^w_{\nu_0/z} \to \mathbb{M}^c_{w\circ\nu_0/z}.$$

In order to prove that this map is an isomorphism, it suffices to prove that $\mathbb{W}^w_{\nu_0/z}$ has a unique up to proportionality singular vector; in which case it may only belong to $\mathbb{C}\mathbf{1}_{w\circ\nu_0}$. Formally, we want to prove that

$$\big(\mathbb{W}^w_{\nu_0/z}\big)^{\hat{\mathfrak{n}}_+}=\mathbb{C}\mathbf{1}_{w\circ\nu_0}. \tag{4.14}$$

Consider the Lie subalgebra $Lw(\mathfrak{n}_+)\cap\hat{\mathfrak{n}}_+$. By construction, $\mathbb{W}^w_{\nu_0/z}$ is co-free as an $Lw(\mathfrak{n}_+)\cap\hat{\mathfrak{n}}_+$-module; this sort of assertion has been the cornerstone of the Wakimoto module theory since its inception, cf. [17, Section 9.5.2]. Informally speaking, if U^w is the maximal unipotent group that corresponds to $w(\mathfrak{n}_+)$, then $U^w\cap U^{\mathrm{id}}$ acts freely on U^{id}, hence co-freely on $\mathbb{C}[U^{\mathrm{id}}]$. Since $U^w\cap U^{\mathrm{id}}$ and U^{id} are affine spaces, this translates into the co-freeness of the action of the Lie algebra $Lw(\mathfrak{n}_+)\cap\hat{\mathfrak{n}}_+$ on $\mathbb{C}[U^{\mathrm{id}}]$. The passage to the space of loops in U^{id} is straightforward.

$\mathbb{W}^w_{\nu_0/z}$ being co-free as an $Lw(\mathfrak{n}_+)\cap\hat{\mathfrak{n}}_+$-module means that $(\mathbb{W}^w_{\nu_0/z})^c$ is a free module over the "opposite" subalgebra, $Lw(\mathfrak{n}_-)\cap\hat{\mathfrak{n}}_-$. Therefore,

$$\big(\mathbb{W}^w_{\nu_0/z}\big)^c=U\big(Lw(\mathfrak{n}_-)\cap\hat{\mathfrak{n}}_-\big)\otimes_{\mathbb{C}}U \tag{4.15}$$

for some graded subspace $U\subset(\mathbb{W}^w_{\nu_0/z})^c$. The space of co-invariants is

$$\big(\big(\mathbb{W}^w_{\nu_0/z}\big)^c\big)_{Lw(\mathfrak{n}_-)\cap\hat{\mathfrak{n}}_-}=U.$$

Dualizing back one obtains the space of invariants

$$\big(\mathbb{W}^w_{\nu_0/z}\big)^{Lw(\mathfrak{n}_+)\cap\hat{\mathfrak{n}}_+}=U^c$$

or, in terms of formal characters,

$$\big(\operatorname{ch}\mathbb{W}^w_{\nu_0/z}\big)^{Lw(\mathfrak{n}_+)\cap\hat{\mathfrak{n}}_+}=\operatorname{ch}U^c=\frac{\operatorname{ch}\mathbb{W}^w_{\nu_0/z}}{\operatorname{ch}U(Lw(\mathfrak{n}_-)\cap\hat{\mathfrak{n}}_-)}.$$

One has the character formula

$$\operatorname{ch}U\big(Lw(\mathfrak{n}_-)\cap\hat{\mathfrak{n}}_-\big)=\prod_{\substack{\alpha\in w(\Delta_+)\cap\Delta_+\\ n\geqslant 0}}\big(1-e^{-\alpha-n\delta}\big)^{-1}\prod_{\substack{\alpha\in w(\Delta_+)\cap\Delta_-\\ n>0}}\big(1-e^{-\alpha-n\delta}\big)^{-1}.$$

Dividing (3.15) by the latter we obtain

$$\begin{aligned}&\big(\operatorname{ch}\mathbb{W}^w_{\nu_0/z}\big)^{Lw(\mathfrak{n}_+)\cap\hat{\mathfrak{n}}_+}\\&\quad=e^{w\circ\nu_0}\prod_{\substack{\alpha\in w(\Delta_-)\cap\Delta_+\\ n\geqslant 0}}\big(1-e^{-\alpha-n\delta}\big)^{-1}\prod_{\substack{\alpha\in w(\Delta_-)\cap\Delta_-\\ n>0}}\big(1-e^{-\alpha-n\delta}\big)^{-1}.\end{aligned} \tag{4.16}$$

Since $(\mathbb{W}^w_{\nu_0/z})^{\hat{\mathfrak{n}}_+}\subset(\mathbb{W}^w_{\nu_0/z})^{Lw(\mathfrak{n}_+)\cap\hat{\mathfrak{n}}_+}$, it follows that if $v_\mu\in(\mathbb{W}^w_{\nu_0/z})^{\hat{\mathfrak{n}}_+}$ is a singular vector of weight μ, then $\mu\in w\circ\nu_0+w(Q_+)$. Equivalently

$$\mu=w\circ(\nu_0+\alpha)\quad\text{for some }\alpha\in Q_+.$$

On the other hand, the block decomposition (3.8) implies

$$\mu \in W \circ \nu_0.$$

Therefore $\nu_0 + \alpha \in W \circ \nu_0$, but ν_0 being dominant, this may happen only if $\alpha \in Q_-$, which requires that $\alpha \in Q_+ \cap Q_- = \{0\}$, hence $\mu = w \circ \nu_0$. A glance at (4.16) shows that in $(\mathbb{W}^w_{\nu_0/z})^{Lw(\mathfrak{n}_+)\cap\hat{\mathfrak{n}}_+}$ there is only one vector of weight $w \circ \nu_0$, $\mathbf{1}$; (4.14) and Lemma 4.3 follow.

In order to conclude our proof of Lemma 4.2, hence of Theorem 1.1(2) we need to consider an arbitrary character $\nu(z)$.

4.4. Proof of Theorem 1.1(2): the case of an arbitrary character

Let us again denote by $\mathcal{C}^j$ the direct sum $\bigoplus_{w\in W^{(j)}} \mathbb{W}_{\nu(z)}$. According to the key Lemma 4.1, we have to compute the cohomology of the complex

$$0 \to \mathcal{C}^0 \to \mathcal{C}^1 \to \cdots \to \mathcal{C}^{\dim X} \to 0. \tag{4.17}$$

For an arbitrary $\nu(z)$, each $\mathcal{C}^j$ carries an increasing conformal filtration $\{F^i(\mathcal{C}^j),\ n \geqslant 0\}$. A spectral sequence arises (yet another one!), $\{(E_r^{ij}, d_r)\}$, with

$$E_0^{ij} = F^{-i}\mathcal{C}^{i+j}.$$

By definition, $(\bigoplus_{ij} E_0^{ij}, d_0)$ is precisely the complex of Lemma 4.1 with $\nu(z)$ replaced with ν_0/z. This places in the situation of a homogeneous character, the case we have just considered. Hence $\bigoplus_{ij} E_1^{ij}$ is as asserted by Theorem 1.1(2). The higher differentials vanish. Indeed, all higher differentials are morphisms of graded $\hat{\mathfrak{g}}$-modules. The modules in question are direct sums of irreducibles, and morphisms among them are determined by the images of highest weight vectors. But on those the differentials vanish according to the following dimensional argument: by construction,

$$d_r\big(E_r^{ij}\big) \subset E_r^{i+r, j-r+1},$$

thus *decreasing* the conformal weight. On the other hand, the class of the highest weight vector $[\mathbf{1}_{w\circ\nu_0}]$ may be mapped only at a linear combination of classes $[\mathbf{1}_{u\circ\nu_0}]$ with $u > w$ (another use of the construction of the Cousin complex (4.1)), and ν_0 being regular dominant, the conformal weights of the latter are strictly *greater* than that of the former.

That the spectral sequence converges follows easily from the fact that it lies inside a finite width band $\{(i, j),\ 0 < i + j < \dim X,\ i < 0\}$.

4.5. Proof of Corollary 1.2

Since by definition

$$\chi\big(\mathcal{L}^{ch}_{\nu(z)}\big) = \sum_{i=0}^{\dim X} (-1)^i \operatorname{ch} H^i\big(X, \mathcal{L}^{ch}_{\nu(z)}\big),$$

Theorem 1.1(2) gives

$$\chi\left(\mathcal{L}^{ch}_{\nu(z)}\right)=\sum_{w\in W}(-1)^{l(w)}e^{-\langle\nu_0-w\circ\nu_0,\rho^\vee\rangle\delta}\,\mathrm{ch}\,\mathbb{V}_{\nu(z)}.$$

The Euler character, $\chi(\mathcal{L}^{ch}_{\nu(z)})$, is known (Theorem 1.1(1))

$$\chi\left(\mathcal{L}^{ch}_{\nu(z)}\right)=\sum_{w\in W}(-1)^{\ell(w)}e^{w\circ\nu_0}\times\prod_{\alpha\in\hat{\Delta}^{re}_+}\left(1-e^{-\alpha}\right)^{-1}.$$

Therefore

$$\mathrm{ch}\,\mathbb{V}_{\nu(z)}=\sum_{w\in W}(-1)^{\ell(w)}e^{w\circ\nu_0}\times\prod_{\alpha\in\hat{\Delta}^{re}_+}\left(1-e^{-\alpha}\right)^{-1}\times\left(\sum_{w\in W}(-1)^{l(w)}e^{-\langle\nu_0-w\circ\nu_0,\rho^\vee\rangle\delta}\right)^{-1}.$$

The rightmost factor can be further factored out as follows

$$\sum_{w\in W}(-1)^{l(w)}e^{-\langle\nu_0-w\circ\nu_0,\rho^\vee\rangle\delta}=\prod_{\alpha\in\Delta_+}\left(1-e^{-\langle\nu_0+\rho,\alpha^\vee\rangle\delta}\right);$$

this is a well-known identity, cf. [22, p. 399]. Plugging the latter identity into the above expression for $\mathrm{ch}\,\mathbb{V}_{\nu(z)}$ we obtain the desired result.

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