LECTURE ??: DEFORMATIONS OF COMPLEX STRUCTURE

1. Macroscopic families

Definition 1.1. A *holomorphic family of compact complex manifolds* is a proper holomorphic map

$$\pi : X \to B$$

such that: X, B are complex compact manifolds, B is connected, and for each $x \in X$ the derivative $\pi_{*x} : T_x X \to T_{f(x)} B$ is surjective. In other words, π is a proper holomorphic submersion.

We think of B parametrizing a family of complex manifolds $\{X_b = f^{-1}(b)\}_{b \in B}$.

Remark 1.2. There is a weaker notion of a "smooth family" of complex compact manifolds. Here, one does not require that the base B have a complex structure, and one considers smooth surjective submersions $\pi : X \to B$ such that the fibers $X_b \subset X$ have a complex structure. For more details see Chapter 4 of [**?**].

Exercise 1.3. The normal bundle of a submanifold $K \subset M$ is the orthogonal complement of TK inside of $TM|_K$. In other words, it fits into a short exact sequence of bundles on N:

$$0 \to TK \to TM|_K \to N_{K \subset M} \to 0.$$

Suppose that π is a holomorphic family. Show that for any b, $X_b = \pi^{-1}(b) \subset X$ is a closed submanifold and that the normal bundle $N_{X_b \subset X}$ is trivializable.

Example 1.4. *Elliptic curves*

Example 1.5. *Hopf manifolds.* Let $d > 1$ and fix complex numbers $\alpha_1, \ldots, \alpha_d$ such that $|\alpha_i| < 1$ for all $i \leq d$. Consider the action of the infinite cyclic group $\mathbb{Z}$ on punctured affine space $\mathbb{C}^d \setminus \{0\}$ generated by

$$(z_1, \ldots, z_d) \mapsto (\alpha_1 z_1, \ldots, \alpha_d z_d).$$

This action is proper and discontinuous without fixed points. The quotient

$$X = \left(\mathbb{C}^d \setminus 0\right) / \mathbb{Z}$$

is called a *Hopf manifold*. It is a d-dimensional compact complex manifold that is diffeomorphic to $S^{2d-1} \times S^1$.

2. Formal families of complex manifolds

Recall that a local Artinian algebra is a commutative algebra with a unique maximal ideal that is finite dimensional as a $\mathbb{C}$-vector space.

Definition 2.1. Fix a complex manifold X and an Artinian algebra A. A *formal A-deformation* (or simply just an A-deformation) of X is a flat morphism

$$\pi_A : X_A \to \mathrm{Spec}(A)$$

together with a closed embedding $i : X \to X_A$ that induces a biholomorphism $i : X \xrightarrow{\simeq} \pi^{-1}(\star)$, where $\star = \mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(A)$ is the closed point.

A formal deformation fits into a pull back diagram of locally ringed spaces

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d] & X_A \arrow[d, "\pi"] \\ \mathrm{Spec}(\mathbb{C}) \arrow[r] & \mathrm{Spec}(A) \end{tikzcd}$$

and should look familiar to the general deformation picture of algebraic spaces. Just as in the algebraic case, for each complex manifold X we can define the following functor

$$\begin{array}{rccc} \mathrm{Def}_X : & \mathsf{Art}_{\mathbb{C}} & \to & \mathsf{Set} \\ & A & \mapsto & \{A\text{−deformations of } X\} \ / \ \{\text{iso}\} \end{array}$$

It is an exercise that uses very similar arguments as in the previous lectures to show that Def_X defines a deformation functor. We will utilize a more algebraic description of this functor.

Lemma 2.2. *The functor* Def_X *is a deformation functor. Moreover, it is equivalent to the functor*

$$A \mapsto \{\text{sheaves } \mathcal{F} \text{ of flat } A\text{−algebras such that } \mathcal{F} \otimes_A \mathbb{C} \cong \mathcal{O}_X\} \ / \ \{\text{iso}\}.$$

2.1. **The Kodaira-Spencer map.** By definition, a *first order* deformation of X is an A-deformation where $A = \mathbb{C}[\epsilon]/\epsilon^2$ is the ring of dual numbers. The collection of all first order deformations of X is equal to the tangent space of Def_X

$$T_{\mathrm{Def}_X} = \mathrm{Def}_X(\mathbb{C}[\epsilon]/\epsilon^2).$$

As we have seen in the previous lectures, it is a general fact that the tangent space is a $\mathbb{C}$-vector space.

The Kodaira-Spencer map characterizes the tangent space of Def_X, that is all first order deformations of X. It is of the form

$$\mathrm{ks}_X : T_{\mathrm{Def}_X} \to H^1(X, T_X). \tag{1}$$

Its construction is the main goal of this subsection.

To construct the Kodaira-Spencer map we will use a Čech description of the cohomology of the tangent sheaf T_X. A definition using a Dolbeault model will be given later.

First, we have a lemma asserting that any A-deformation of a Stein manifold is trivial. The proof of this fact is completely analogous to the proof in the algebraic context where "Stein" is replaced by affine.

Lemma 2.3. *Any A-deformation of a Stein manifold is isomorphic to a trivial one.*

Now, suppose a first order deformation $X_\epsilon \in T_{\mathrm{Def}_X} = \mathrm{Def}_X(\mathbb{C}[\epsilon]/\epsilon^2$ is given. Pick a Stein open cover $\{U_\alpha\}$ of X such that all double, $U_{\alpha\beta} = U_\alpha \cap U_\beta$, and triple, $U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cap U_\gamma$, intersections are also Stein. By Lemma 2.3, we can pick isomorphisms of deformations

$$\varphi_\alpha : U_\alpha \times \mathrm{Spec}(\mathbb{C}[\epsilon]/\epsilon^2) \to X_\epsilon|_{U_\alpha}$$

for all α. Moreover, for any α, β we obtain an automorphism of the trivial deformation

$$\varphi_{\alpha\beta} \in \mathrm{Aut}\left(U_{\alpha\beta} \times \mathrm{Spec}(\mathbb{C}[\epsilon]/\epsilon^2)\right).$$

Lemma 2.4. *Infinitesimally, on $U_{\alpha\beta}$, the automorphism $\varphi_{\alpha\beta}$ determines a vector field*

$$\theta_{\alpha\beta} \in \Gamma(U_{\alpha\beta}, T_X).$$

Moreover, the collection $\{\theta_{\alpha\beta}\}$ satisfy the Čech cocycle condition

$$\theta_{\alpha\beta} - \theta_{\alpha\gamma} + \theta_{\beta\gamma} = 0.$$

By the lemma, the first order deformation X_ϵ determines an element $\{\theta_{\alpha\beta}\} \in C^1(\{U_\alpha\}, T_X)$. In cohomology, we obtain a class

$$[\{\theta_{\alpha\beta}\}] = H^1(\{U_\alpha\}, T_X) \cong H^1(X, T_X).$$

The second isomorphism is a standard fact about Čech cohomology and Stein covers.

Theorem 2.5. *There is a map of $\mathbb{C}$-vector spaces*

$$\begin{array}{rccc} \mathrm{ks}_X : & \mathrm{Def}_X & \to & H^1(X, T_X) \\ & X_\epsilon & \mapsto & [\{\theta_{\alpha\beta}\}]. \end{array}$$

Moreover, $\mathrm{ks}_X(X_\epsilon) = 0$ if and only if X_ϵ is isomorphic to the trivial deformation.