1. Some categorical remarks

1.1. **(Co)Limits.** We recall some general notions in ordinary category theory. For a textbook reference see [**?**].

Let $\mathcal{C}$ and $\mathcal{I}$ be categories. For each object $X \in \mathcal{C}$ let $\underline{X} : \mathcal{I} \to \mathcal{C}$ be the functor that sends every object of $\mathcal{I}$ to $X \in \mathcal{C}$ and every morphism to the identity. This construction extends to a functor

$$\underline{(-)} : \mathcal{C} \to \mathsf{Fun}(\mathcal{I}, \mathcal{C}).$$

Definition 1.1. Suppose $\mathcal{I}$ is a small category and let $F : \mathcal{I} \to \mathcal{C}$ be a functor. A **colimit** of F is an object $X \in \mathcal{C}$ together with a natural transformation

$$t : F \to \underline{X}$$

such that for every $Y \in \mathcal{C}$ and every natural transformation $s : F \to \underline{Y}$ there exists a unique map $s' : X \to Y$ making $\underline{s'}t = s$.

Any two colimits are naturally isomorphic. If the colimit of a functor $F : \mathcal{I} \to \mathcal{C}$ exists, we will write it as $\mathrm{colim}_{\mathcal{I}} F \in \mathcal{C}$.

Many familiar categories posses the property that *all* colimits exist. These include $\mathsf{Set}, \mathsf{Top}$, and Vect_k. Moreover, colimits are functorial in the natural way.

Definition 1.2. A category $\mathcal{I}$ is called **filtered** if, for any finite category $\mathcal{J}$ and functor $J : \mathcal{J} \to \mathcal{I}$, there exists an object $i \in \mathcal{I}$ and a natural transformation $F \to \underline{i}$.

If $F : \mathcal{I} \to \mathcal{C}$ is a functor, $\mathcal{I}$ is filtered, and $\mathrm{colim}_{\mathcal{I}} F$ exists, then the colimit is called a *filtered colimit.* A natural example of a filtered category is the poset

$$\mathcal{I} = \mathbb{Z}_+ = \{0 \to 1 \to 2 \to \cdots\}.$$

Resulting colimits are special types of filtered colimits called *sequential colimits.*

1.1.1. The notion of a limit is defined in a dual way. The aforementioned categories also admit all limits.

2. The idea of a model category

Roughly, the theory of model categories was developed to better handle the notion of a "homotopy equivalence". For us, the fundamental example of a homotopy equivalence is a quasi-isomorphism of dg vector spaces.

The first, and perhaps most obvious, attempt to encode homotopy, or *weak*, equivalences in a category is to prescribe some class of morphisms that ware well-behaved with respect to composition. The definition is the following.

Definition 2.1. A **category with weak equivalences** is a category $\mathcal{C}$ together with a set

$$\mathcal{W} \subset \mathrm{Mor}(\mathcal{C})$$

such that

(1) If f is an isomorphism, then $f \in \mathcal{W}$;
(2) if f, g are morphisms such that $f \circ g$ exists then: if two of $f, g, f \circ g$ are in $\mathcal{W}$ then the third is as well ('two out of three").

2.1. **Model categories.**

Definition 2.2. Let $\mathcal{C}$ be a category and $K \subset \mathrm{Mor}(\mathcal{C})$ be a subset of morphisms. A map $f : X \to Y$ has the **left lifting property** (LLP) with respect to K if for any morphism $g : W \to Z$ in K and solid line diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & W \arrow[d, "g"] \\ Y \arrow[r] \arrow[ur, dotted, "\exists h"'] & Z \end{tikzcd}$$

there exists a dotted map $h : Y \to W$ making the total diagram commute. Dually, we say $f : X \to Y$ as the **right lifting property** (RLP) with respect to K if for any morphism $g : W \to Z$ in K and solid line diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Z \arrow[r] \arrow[ur, dotted, "\exists h"'] & Y \end{tikzcd}$$

there exists a dotted map $h : Z \to X$ making the total diagram commute.

3. Localization

4. Adjoints and extensions

5. Examples

6. dgVect, dgLie, etc. Top.