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An introduction to algebraic deformation theory

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Abstract

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Algebraic deformation theory is primarily concerned with the interplay between homological algebra and the perturbations of algebraic structures. We here offer a self-contained introduction to the subject, first describing the classical theory of deformations of associative algebras, then passing to the general case of algebras, coalgebras, and bialgebras defined by triples and cotriples.

Introduction

Our purpose in writing these notes is to give a short elementary introduction to the theory of deformations of algebraic structures, with an eye towards elucidating the categorical point of view. Presently the best overview of the subject is the long paper by Gerstenhaber and Schack [18], while Gerstenhaber's original papers remain eminently readable [14–17]. We intend to cast these notes at a more rudimentary level, in the hope that they will make the general theory more accessible to the non-expert.

Algebraic deformation theory was introduced for associative algebras by Gerstenhaber [14], and was extended to Lie algebras by Nijenhuis and Richardson [23, 24]. Their work closely parallels the theory of deformations of complex analytic structures, initiated by Kodaira and Spencer [21]. The fundamental

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results connect deformation theory with the appropriate cohomology groups, and it was assumed that these results would extend to any equationally-defined class of algebras. What was missing was a suitably general cohomology theory. Early attempts to create such a theory were unconvincing [15, 26], and it was not until the discovery of triple cohomology that the program could be completed.

This paper is divided into two chapters. In the first, we give simple examples of deformations of commutative algebras, especially curves, and then present the basic definitions and theorems of the classical theory. In the second chapter, we will show how the classical theory may be extended by using triple cohomology. All of the results contained in this paper have appeared elsewhere, particularly in [13–18] and [6–11].

We assume the reader is familiar with the basic ideas of homological algebra [22]. If A is an associative k-algebra, where k is a commutative ring, recall that a Hochschild n-cochain is just a k-linear map from $A^{\otimes n}$ to A (we will only consider coefficients in A itself). The group of all n-cochains is denoted C^n, and the boundary map $\partial : C^{n-1} \to C^n$ is defined by

$$\begin{aligned}\partial f(a_1, \ldots, a_n) = a_1 f(a_2, \ldots, a_n) &- f(a_1 a_2, a_3, \ldots, a_n) \\ &+ f(a_1, a_2 a_3, \ldots, a_n) - \cdots \\ &+ (-1)^{n+1} f(a_1, \ldots, a_{n-1}) a_n .\end{aligned}$$

The kernel of ∂ in C^n is the group of n-cocycles, and is denoted Z^n. The image of ∂ in C^n is the group of n-coboundaries, and is denoted B^n. The Hochschild cohomology groups $\mathrm{Hoch}^n(A, A)$ (or just $H^n(A, A)$) are defined to be Z^n/B^n [20].

We also assume the reader is familiar with the language of categories and functors. In Chapter II we give a brief introduction to triple cohomology. For further information on triples, the reader should consult [2] and the introduction to that volume.

I. The classical theory

1. Definition and examples of deformations

Let X be an object with structure (algebraic, analytic, topological, ...). Roughly speaking, a deformation of X is a family X_t of objects whose structures are obtained by 'deforming' the structure on X as t varies over a suitable space of parameters in a smooth way. If A is an algebra over a commutative ring k, a one-parameter algebraic deformation of A is a family of algebras $\{A_t\}$ parameterized by k such that $A_0 \cong A$ and the multiplicative structure of A_t varies algebraically with t. Before giving a formal definition, we will look at a simple example (to which the formal definition must apply).

1.1. Example. Let $A = k[x, y]/(y^2 - x^3)$ and $A_t = k[x, y, t]/(y^2 - x^3 - x^2t)$; see Fig. 1.

Geometrically it looks inviting to think of A_t as a 'deformation' of A. The question is, in what sense is this deformation achieved by deforming the algebraic structure of A? The key is noticing that $y \cdot y = x^3$ in A, while $y \cdot y = x^3 + x^2t$ in A_t, i.e. the product $y \cdot y$ varies with t.

We will consider the case where A is an associative k-algebra and the deformation of A is again associative. Let $A[\![t]\!]$ be the $k[\![t]\!]$-module of formal power-series with coefficients in the k-module A, i.e. $A[\![t]\!] = A \otimes_k k[\![t]\!]$ as a module. The algebra A is a submodule of $A[\![t]\!]$, and we could make $A[\![t]\!]$ an algebra by bilinearly extending the multiplication of A, but we may also impose other multiplications on $A[\![t]\!]$ that agree with that of A when we specialize to $t = 0$.

Working backwards, suppose a multiplication $F : A[\![t]\!] \otimes_{k[\![t]\!]} A[\![t]\!] \rightarrow A[\![t]\!]$ is given by a formal power-series of the form

$$F(a, b) = f_0(a, b) + f_1(a, b)t + f_2(a, b)t^2 + \cdots . \tag{1.1}$$

Since we are defining F over $k[\![t]\!]$, it is enough to consider a and b in A, and we further presume that each f_n is a linear map $A \otimes A \rightarrow A$. Since we want the specialization $t = 0$ to give the original multiplication on A, we insist that $f_0(a, b) = ab$ (multiplication in A).

1.2. Definition. A *one-parameter formal deformation* of a k-algebra A is a formal power-series $F = \sum_{n=0}^{\infty} f_n t^n$ with coefficients in $\mathrm{Hom}_k(A \otimes A, A)$ such that $f_0 : A \otimes A \rightarrow A$ is multiplication in A. The deformation is called *associative* if $F(F(a, b), c) = F(a, F(b, c))$ for all a, b, c in A.

We often refer to $A[\![t]\!]$ with the multiplication defined by F as the deformation of A, and we may write this $A[\![t]\!]_F$ or A_F. If F is finite, or at least finite for each pair (a, b) in $A \otimes A$, the multiplication may be defined on $A[t]$ over $k[t]$.

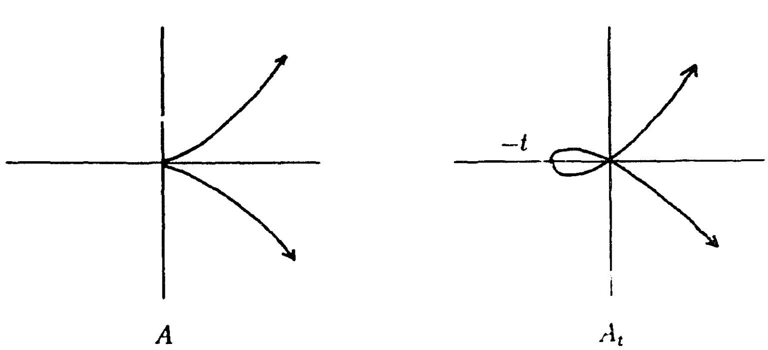

Fig. 1.

Back to Example 1.1, where $A = k[x, y]/(y^2 - x^3)$ and $A_t = k[x, y, t]/(y^2 - x^3 - x^2t)$. Of course $\{x^n, yx^n\}$ is a $k[t]$-basis for $A[t]$, and we define a multiplication F on $A[t]$ by

$$F(x^m, x^n) = X^{m+n}\,, \qquad F(y, y) = x^3 + x^2t\,,$$
$$F(y, x^n) = F(x^n, y) = x^n y\,, \qquad F(yx^m, yx^n) = x^{m+n+3} + x^{m+n+2}t\,.$$

Then $A[t]_F \cong A_t$. Note that $f_1(yx^m, yx^n) = x^{m+n+2}$.

A simpler example, though less geometric, is given by deforming $k[x]/x^2$ to $k[x, t]/(x^2 - t)$. The latter is isomorphic to $(k[x]/x^2)[t]$ with a multiplication given by $F(x, x) = t$, i.e. $f_1(x, x) = 1$. Of course $F(1, x) = F(x, 1) = x$. The next example will explain why we defined deformations in terms of power-series, and not just polynomials, as well as showing that not any old power-series will serve to yield an associative multiplication.

1.3. Example. Let $A = k[x]$ and define $F : A[t] \otimes_{k[t]} A[t] \to A[t]$ by

$$F = f_0 + f_1 t\,, \qquad f_1(x^m, x^n) = mnx^{m+n}\,.$$

This multiplication is associative mod t^2, but not mod t^3, since $F(F(x^2, x), x) = F(x^3 + 2x^3t, x) = x^4 + 5x^4t + 6x^4t^2$ while $F(x^2, F(x, x)) = x^4 + 5x^4t + 4x^4t^2$. This may be patched up by *extending* F to F_1 defined by

$$F_1 = f_0 + f_1t + f_2t^2\,, \qquad f_2(x^m, x^n) = \frac{m^2n^2}{2}\, x^{m+n}\,.$$

Now we find that $F_1(F_1(x^2, x), x) = x^4 + 5x^4t + 25/2x^4t^2 + 15x^4t^3 + 9x^4t^4$ and $F_1(x^2, F_1(x, x)) = x^4 + 5x^4t + 25/2x^4t^2 + 10x^4t^3 + 4x^4t^4$, so F_1 is associative mod t^3 but not mod t^4. Further extensions may be made to patch this up, leading to F_* defined by

$$F_* = \sum_{r=0}^{\infty} f_r t^r\,, \qquad f_r(x^m, x^n) = \frac{m^r n^r}{r!}\, x^{m+n}\,.$$

F_* defines an associative multiplication on $A[\![t]\!]$ which cannot be defined on $A[t]$. If the reader objects to the formal power-series $F_*(a, b)$, we can modify this example so that $F(a, b)$ is finite for each pair of elements of A even though F itself is a formal power-series, as follows:

1.4. Example. Again let $A = k[x]$, and let $Dx^n = nx^{n-1}$. If we begin with $F_1(a, b) = ab + DaDbt$, which is not associative, we are led to $F(a, b) = \sum_{r=0}^{\infty} (D^r aD^r b/r!)t^r$ which is finite for given a and b in $k[x]$, and thus defines a multiplication on $k[x, t]$ (see [16, p. 13]).

Before closing this section, we will give a collection of examples which are particularly satisfying from a geometric point of view.

1.5. Example. Throughout this example let $A = k[x, y]/(y^2)$. We will define several deformations of A over $k[t]$ by giving the multiplication on (y, y) only (we leave the reader to define the multiplication on terms of the form (yx^m, yx^n)). Define F, G, H, J, and K by $F(y, y) = t^2$, $G(y, y) = xt$, $H(y, y) = x^2t^2$, $J(y, y) = x^3t$, and $K(y, y) = (x^2 + x^3)t$ respectively. Then $A_F \cong k[x, y, t]/(y^2 - t^2)$, $A_G \cong k[x, y, t]/(y^2 - xt)$, $A_H \cong k[x, y, t]/(y^2 - x^2t^2)$, $A_J \cong k[x, y, t]/(y^2 - x^3t)$, and $A_K \cong k[x, y, t]/(y^2 - x^2t - x^3t)$. Now we may visualize A itself as a double line (i.e. consider the closed points of Spec A), and our deformations of A may be visualized as shown in Fig. 2.

These graphs represent the deformation for general values of t. The deformation is actually a family of such curves parameterized by t; it may be visualized as a surface in k^3 (see [19, p. 90]).

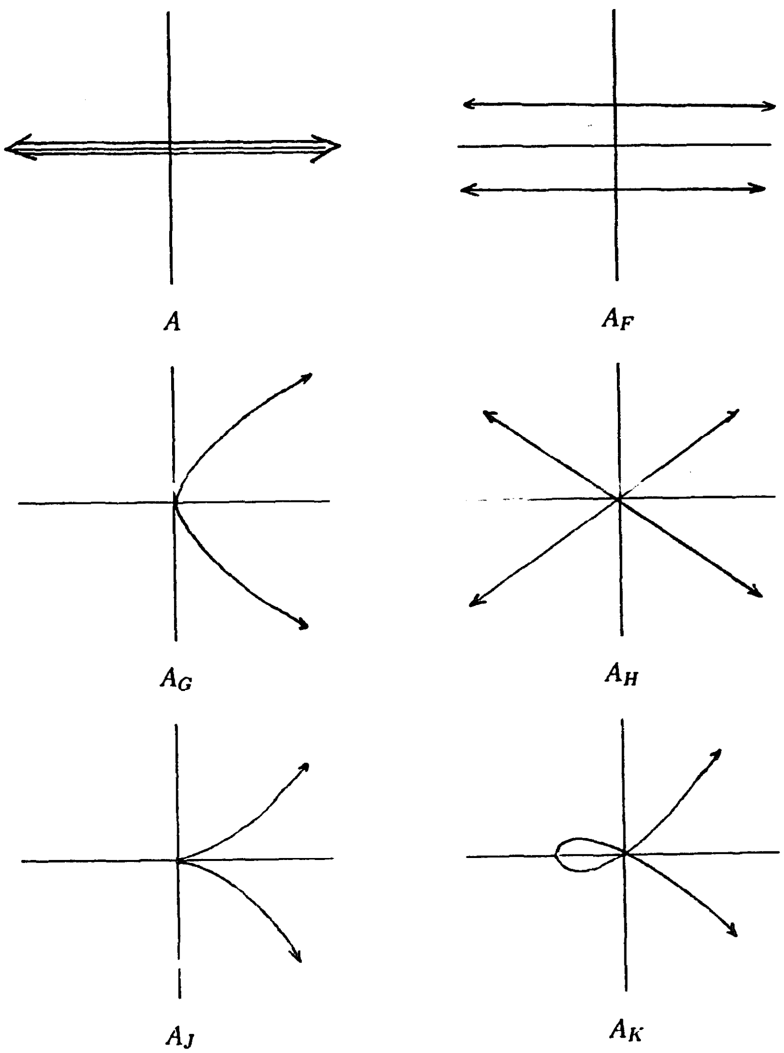

Fig. 2.

2. Infinitesimals and obstructions

As is shown by Example 1.3, not all formal deformations of A will be associative. This condition restricts our choice of coefficient maps $f_n : A \otimes A \to A$. If F is associative we have $F(F(a, b), c) = F(a, F(b, c))$. Expanding both sides of this equation and collecting coefficients of t^n yields

$$\sum_{i=0}^{n} f_i(f_{n-i}(a, b), c) = \sum_{i=0}^{n} f_i(a, f_{n-i}(b, c)), \tag{2.1}$$

which gives necessary and sufficient conditions for the associativity of F. Let f_n be the first non-zero coefficient after f_0 in the expansion $F = \sum f_n t^n$. This f_n is the *infinitesimal* of F, and (2.1) reads $f_0(f_n(a, b), c) + f_n(f_0(a, b), c) = f_0(a, f_n(b, c)) + f_n(a, f_0(b, c))$, or

$$af_n(b, c) - f_n(ab, c) + f_n(a, bc) - f_n(a, b)c = 0. \tag{2.2}$$

The left-hand side of (2.2) is the Hochschild coboundary of f_n, so (2.2) may be written $\partial f_n = 0$, yielding the first connection between deformation theory and cohomology theory.

2.1. Theorem. *If F is an associative deformation of A then the infinitesimal of F is a Hochschild* 2*-cocycle.*

For arbitrary n, (2.1) may be written as

$$\partial f_n(a, b, c) = \sum_{i=1}^{n-1} f_i(f_{n-i}(a, b), c) - f_i(a, f_{n-i}(b, c)). \tag{2.3}$$

If f_n satisfying (2.3) have been given for $0 < n < m$, the right-hand side of (2.3) with $m = n$ is the obstruction to finding f_n extending the deformation. The most important theorem in deformation theory is the following.

2.2. Theorem. *The obstruction is a Hochschild* 3*-cocycle.*

2.3. Corollary. *If* $\mathrm{Hoch}^3(A, A) = 0$ *then every* 2*-cocycle of A may be extended to an associative deformation of A.*

We will give a proof of Theorem 2.2 in Section 9 (Theorem 9.1). The pattern above will be repeated several times in what follows. We will be looking for power-series whose coefficients are n-cochains of A, and it will turn out that the infinitesimal of the series is a cocycle, while the obstructions to extending the infinitesimal to a full power-series will lie in $H^{n+1}(A, A)$.

3. Equivalent and trivial deformations

The next problem to consider is when two deformations are significantly different from one another. Given associative deformations A_F and A_G of A we ask when there is an isomorphism $A_F \to A_G$ which keeps A fixed. By a *formal isomorphism* $\Psi : A_F \to A_G$ we mean a $k[\![t]\!]$-linear map $A[\![t]\!]_F \to A[\![t]\!]_G$ that may be written in the form

$$\Psi(a) = a + \psi_1(a)t + \psi_2(a)t^2 + \psi_3(a)t^3 + \cdots . \tag{3.1}$$

Since Ψ is defined over $k[\![t]\!]$, it is enough to consider a in A, and we further presume that each ψ_n is a k-linear map $A \to A$. We called such a map an 'isomorphism', and not just a 'homomorphism', because it has an inverse of the same form (its leading coefficient, the identity, is invertible). If Ψ is multiplication-preserving, we say it is an *algebraic* isomorphism. The condition that Ψ preserve multiplication means that $G(\Psi(a), \Psi(b)) = \Psi(F(a, b))$ for all a and b in A. Expanding both sides of this equation and collecting the coefficients of t^n yields

$$\sum_{i+j+k=n} g_i(\psi_j(a), \psi_k(b)) = \sum_{i+j=n} \psi_i(f_j(a, b)) \tag{3.2}$$

We say that F and G are *equivalent* if such a Ψ exists, and we write $A_F \cong A_G$ (it is easy to check that this is an equivalence relation on the set of deformations of A). In that case (3.2) yields $\partial\psi_1 = f_1 - g_1$, i.e. the 2-cocycles f_1 and g_1 are in the same cohomology class. Given ψ_1, we may ask when it may be extended to an isomorphism from A_F to A_G.

For general n, (3.2) may be written as

$$\partial\psi_n(a, b) = \sum_{\substack{i+j=n\\ i\neq n}} \psi_i(f_j(a, b)) - \sum_{\substack{i+j+k=n\\ j,k\neq n}} g_i(\psi_j(a), \psi_k(b)) . \tag{3.3}$$

Given a truncated algebraic isomorphism $\Psi = \sum \psi_i t^i$, $i < n$, from A_F to A_G, the right-hand side of (3.3) defines the obstruction to finding ψ_n extending the isomorphism. This obstruction is a Hochschild 2-cocycle, and if its class in $\mathrm{Hoch}^2(A, A)$ is zero then ψ_n exists. If all such obstructions vanish, then $A_F \cong A_G$.

3.1. Theorem. *If* $\mathrm{Hoch}^2(A, A) = 0$, *then all deformations of* A *are isomorphic.*

The most trivial deformation of A is the A-algebra $A[\![t]\!]$, i.e. A_K where $K(a, b) = ab$ for all a and b in A. A deformation F is *trivial* if $A[\![t]\!]_F \cong A[\![t]\!]$. One consequence of Theorem 3.1 is that $\mathrm{Hoch}^2(A, A) = 0$ implies all deformations of A are trivial, i.e. A is *rigid*.

Suppose F is non-trivial, and suppose that its infinitesimal f_n is a coboundary, say $\partial\psi = f_n$. Let Ψ be the formal automorphism of $A[[t]]$ defined by $\Psi = \psi_0 + \psi t^n$ and define $F'(a, b) = \Psi^{-1}F(\Psi(a), \Psi(b))$. Then $F' \cong F$ by construction, and $\Psi^{-1} = \psi_0 - \psi t^n + \psi_2 t^{2n} - \cdots$, so

$$\begin{aligned} f'_n(a, b) &= \sum_n \psi_h^{-1} f_i(\psi_j(a), \psi_k(b)) \\ &= -\psi(ab) + f_n(a, b) + a\psi(b) + \psi(a)b = 0 \, . \end{aligned}$$

Clearly $f'_m = 0$ for $m < n$, so the index of the infinitesimal of F' must be greater than n. We may repeat this process of killing off an infinitesimal that is a coboundary, and the process must stop because F is non-trivial.

3.2. Theorem. *A non-trivial deformation is equivalent to a deformation whose infinitesimal is not a coboundary.*

If F is a trivial deformation, it is not true that its infinitesimal f_n need be a coboundary (unless the infinitesimal is f_1) since (3.2) does not give $\partial\psi_n = f_n$ (see Theorem 4.3).

4. Automorphisms of the deformed algebra

An *algebraic automorphism* of A_F is just an algebraic isomorphism $\Psi : A_F \to A_F$. If ψ_n is the infinitesimal of Ψ (the first non-zero coefficient after ψ_0), then (3.2) gives us $\partial\psi_n = 0$, or

$$\psi_n(ab) = \psi_n(a)b + a\psi_n(b) \, .$$

Hence, the infinitesimal of an algebraic automorphism is a derivation of A, and we may ask when a derivation of A may be extended to an automorphism of A_F. If a derivation ψ_1 has been extended to a truncated automorphism $\Psi = \sum \psi_i t^i$, $i < n$, the obstruction defined by (3.3) may be written as

$$\sum_{\substack{i+j=n \\ i \neq n}} \psi_i(f_j(a, b)) - \sum_{\substack{i+j+k=n \\ j,k \neq n}} f_i(\psi_j(a), \psi_k(b)) \, . \tag{4.1}$$

4.1. Theorem. *The obstruction to extending a truncated automorphism defines an element of* $\mathrm{Hoch}^2(A, A)$ *which must vanish if the truncated automorphism is to be extended.*

4.2. Corollary. *If* $\mathrm{Hoch}^2(A, A) = 0$, *then every derivation of A may be extended to an algebraic automorphism of A_F for any deformation F.*

Suppose G is a trivial deformation whose infinitesimal g_n is not a coboundary, and let $\Psi : A[\![t]\!] \to A[\![t]\!]_G$ be the necessary isomorphism. From (3.3)

$$\partial\psi_n(a, b) + g_n(a, b) = -\sum_{\substack{i+j=n \\ i,j\neq n}} \psi_i(a)\psi_j(b)\,. \tag{4.2}$$

The right-hand side of this equation defines the obstruction to extending $\sum \psi_i t^i$, $i<n$ as an automorphism of the trivial deformation $A[\![t]\!]$, and since g_n is not a coboundary, this obstruction fails to vanish.

4.3. Theorem. *If every derivation of A extends to an algebraic automorphism of $A[\![t]\!]$, then every trivial deformation of A has a trivial infinitesimal.*

4.4. Remarks. We have defined algebraic homomorphisms and deformations to be formal power-series, but this is a bit misleading, since we cannot add them (the leading term would be wrong). Also, since we are not concerned with questions of convergence, the powers of t play no role other than as place-keepers. It would be less deceptive if we defined algebraic automorphisms and deformations to be sequences of elements, in $\mathrm{Hom}(A, A)$ and $\mathrm{Hom}(A\otimes A, A)$ respectively, with composition defined by convolution. The fundamental theorems say that a truncated sequence may be extended if and only if a certain cohomology class (in the next dimension) vanishes.

5. The cohomology of the deformed algebra

In order to measure how much an algebra may vary under deformation, we will compare the cohomology of the deformed algebra with the cohomology of A. A *0-cocycle of A_F* is an element $P = \sum p_n t^n$ of $A[\![t]\!]$ such that $F(P, a) - F(a, P) = 0$, i.e.

$$\sum_{i+j=n} f_i(p_j, a) - f_i(a, p_j) = 0\,. \tag{5.1}$$

For $n=0$ this gives $p_0 a - a p_0 = 0$, so p_0 is a 0-cocycle of A that may be *lifted* to a 0-cocycle of A_F. It is clear from (5.1) that the sum of two 0-cocycles of A_F is again a 0-cocycle, and if P is a 0-cocycle so is Pt^n. Thus the set of cocycles form a $k[\![t]\!]$-module. This module has an obvious filtration obtained by considering the index of the first non-zero coefficient. The associated graded module we denote by $H^0(A_F, A_F)$, and this is generated by the set of liftable 0-cocycles of A. Thus $\dim_{k[\![t]\!]} H^0(A_F, A_F) \le \dim_k H^0(A, A)$, and the inequality is strict if there are cocycles of A that fail to lift to cocycles of A_F. (5.1) may be written as

$$\partial p_n(a) = \sum_{\substack{i+j=n \\ j\neq n}} f_i(a, p_j) - f_i(p_j, a) . \tag{5.2}$$

If $P = \sum_{i<n} p_i t^i$ is a truncated 0-cocycle of A_F, then the right-hand side of (5.2) defines the obstruction to finding p_n extending P. As the reader should expect by now, the obstruction to extending a truncated 0-cocycle of A_F is a 1-cocycle of A. If $\mathrm{Hoch}^1(A, A) = 0$, then all such obstructions vanish, and any 0-cocycle of A may be lifted to a 0-cocycle of A_F.

5.1. Theorem. *If k is a field and* $\mathrm{Hoch}^1(A, A) = 0$, *then* $\dim_{k[[t]]} H^0(A_F, A_F) = \dim_k H^0(A, A)$.

We now turn to $H^1(A_F, A_F)$. By a *1-cochain* of A_F we mean a $k[[t]]$-linear map $\Psi : A[[t]] \to A[[t]]$ that may be written in the form

$$\Psi(a) = \psi_0(a) + \psi_1(a)t + \psi_2(a)t^2 + \cdots . \tag{5.3}$$

It is enough to consider a in A, and we further presume that each ψ_n is a k-linear map $A \to A$. Note that ψ_0 need not be the identity map, as was the case for an algebraic automorphism. Ψ is a *1-cocycle of* A_F if $\partial\Psi = 0$, i.e. if

$$\begin{gathered} F(a, \Psi(b)) - \Psi(F(a, b)) + F(\Psi(a), b) = 0 , \\ \sum_{i+j=n} f_i(a, \psi_j(b)) - \psi_j(f_i(a, b)) + f_i(\psi_j(a), b) = 0 . \end{gathered} \tag{5.4}$$

For $n = 0$ this gives $a\psi_0(b) - \psi_0(ab) + \psi_0(a)b = 0$, i.e. ψ_0 is a 1-cocycle of A that may be lifted to a 1-cocycle of A_F. The set of 1-cocycles of A_F form a $k[[t]]$-module, again with an obvious filtration. The associated graded module is $Z^1(A_F, A_F)$, which is clearly isomorphic to $Z_l^1 \otimes_k k[[t]]$, where Z_l^1 is the k-module of all liftable 1-cocycles of A.

A 1-cochain Ψ of A_F is a 1-coboundary of A_F if there is an element $P = \sum p_n t^n$ of $A[[t]]$ (a 0-cochain of A_F) such that $\partial P = \Psi$, i.e.

$$\begin{gathered} F(P, a) - F(a, P) = \Psi(a) , \\ \sum_{i+j=n} f_i(p_j, a) - f_i(a, p_j) = \psi_n(a) . \end{gathered} \tag{5.5}$$

We let $B^1(A_F, A_F)$ denote the module of all 1-coboundaries of A_F and $H^1(A_F, A_F) = Z^1(A_F, A_F)/B^1(A_F, A_F)$. If Ψ is a 1-coboundary of A_F, then (5.5) yields $p_0 a - a p_0 = \psi_0(a)$, so ψ_0 is a coboundary of A. However, a non-trivial element of $Z^1(A, A)$ may lift to a trivial element of $Z^1(A_F, A_F)$. To see this, suppose that ψ_n is the infinitesimal of a 1-coboundary Ψ of A_F. Then (5.5) does not imply that ψ_n is a 1-coboundary of A. It may happen that $H^1(A_F, A_F)$ has $k[[t]]$-torsion.

5.2. Theorem. $\dim_{k[[t]]} H^1(A_F, A_F) \le \dim_k H^1(A, A)$. *The inequality is strict if there is a non-liftable* 1*-cocycle of* A, *or if there is non-coboundary of* A *that lifts to a coboundary of* A_F.

It thus behooves us to consider the two possibilities mentioned in the theorem above. We first ask under what conditions a 1-cocycle of A may be lifted. (5.4) may be written

$$\partial \psi_n = \sum_{\substack{i+j=n \\ j \ne n}} \psi_j(f_i(a, b)) - f_i(a, \psi_j(b)) - f_i(\psi_j(a), b)\,. \tag{5.6}$$

If $\Psi = \sum \psi_i t^i$, $i < n$, is a truncated 1-cocycle of A_F lifting ψ_0, then the right-hand side of (5.6) defines the obstruction to extending Ψ. The obstruction is a 2-cocycle of A whose class must vanish for ψ_n to exist. If $\mathrm{Hoch}^2(A, A) = 0$, then all obstructions to lifting ψ_0 vanish.

5.3. Theorem. *If* $\mathrm{Hoch}^2(A, A) = 0$, *then every* 1*-cocycle of* A *may be lifted to a* 1*-cocycle of* A_F *for any deformation* F *of* A.

Suppose ψ_n is not a coboundary of A, but is the infinitesimal of a 1-coboundary of A_F. If $\partial P = \Psi$, then (5.6) yields

$$\sum_{i+j=m} f_i(a, p_j) - f_i(p_j, a) = 0 \quad (m < n)\,, \tag{5.7}$$

$$\sum_{\substack{i+j=n \\ j \ne n}} f_i(a, p_j) - f_i(p_j, a) = \psi_n(a) - \partial p_n(a)\,. \tag{5.8}$$

But (5.7) just says that $P' = \sum p_m t^m$, $m < n$, is a truncated 0-cocycle of A_F (cf. (5.1)), while the left-hand side of 5.8 is the obstruction to extending P' (cf. (5.2)). Since ψ_n is not a coboundary, (5.8) shows that this obstruction does not vanish.

5.4. Proposition. *If there is a non-trivial* 1*-cocycle of* A *that lifts to a trivial* 1*-cocycle of* A_F, *then there is an obstructed* 0*-cocycle of* A_F.

The entire analysis of $H^1(A_F, A_F)$ carries over to $H^2(A_F, A_F)$, and we will leave the reader to supply the definitions necessary to make sense out of the following results.

5.5. Theorem. (a) $\dim_{k[[t]]} H^2(A_F, A_F) \le \dim_k H^2(A, A)$.

(b) *The inequality above is strict if there is a non-liftable* 2*-cocycle of* A, *or if a non-trivial* 2*-cocycle of* A *lifts to a trivial* 2*-cocycle of* A_F.

(c) *If* $\mathrm{Hoch}^3(A, A) = 0$, *then every* 2*-cocycle of* A *lifts.*

(d) *If a non-trivial* 2*-cocycle of* A *lifts to a trivial* 2*-cocycle of* A_F, *then there is an obstructed* 1*-cocycle of* A_F.

There is, however, one new wrinkle associated with Theorem 5.5(d). Let g_n be a non-trivial 2-cocycle of A that is the infinitesimal of a 2-coboundary G of A_F, and suppose that $\partial\Psi = G$. Then corresponding to (5.7) and (5.8) we have

$$\sum_{i+j=m} f_i(a, \psi_j(b)) - \psi_j(f_i(a, b)) + f_i(\psi_j(a), b) = 0 \quad (m < n)\,, \tag{5.9}$$

$$\sum_{\substack{i+j=n \\ j\neq n}} \psi_j(f_i(a, b)) - f_i(a, \psi_j(b)) - f_i(\psi_j(a), b) = \partial\psi_n(a, b) - g_n(a, b)\,. \tag{5.10}$$

This last equation looks suspiciously like the obstruction of an algebraic automorphism of A_F (cf. (4.2)), and it is. Define a truncated automorphism Ξ by $\Xi(a) = a - \Psi(a)t^{n+1}$, that is, $\xi_i = 0$ for $0 < i \le n$, $\xi_i = \psi_{i-n-1}$ for $n < i \le 2n+1$. Note that $\xi_{2n+1} = \psi_n$. Then (5.9) shows that $\sum \xi_m t^m$, $m \le 2n$, is indeed a truncated algebraic automorphism of A_F, while (5.10) says that the obstruction to extending it is the class of g_n.

5.6. Theorem. *If a non-trivial* 2*-cocycle of A lifts to a trivial* 2*-cocycle of A_F, then there is an obstructed algebraic automorphism of A_F.*

5.7. Example. Let k be a field of characteristic $\neq 2$, let $A = k[x]/(x^2)$ and $A_t = k[x, t]/(x^2 - t)$. Consider the 2-cocycle g of A given by $g(1, 1) = g(1, x) = g(x, 1) = 0$ and $g(x, x) = 1$. Then g is not a 2-coboundary of A, since if $\partial\phi = g$ for some k-linear map $\phi : A \to A$, we would have $\partial\phi(x, x) = x\phi(x) - \phi(xx) + \phi(x)x = g(x, x)$, hence $2x\phi(x) = 1$, which is impossible. On the other hand, gt is a 2-coboundary of A_t. Let $\psi(1) = 0$ and $\psi(x) = \frac{1}{2}x$ define the $k[t]$-linear map $\Psi : A_t \to A_t$. Then $\partial\Psi = gt$ (the only thing to check is $\partial\psi(x, x) = x\psi(x) - \psi(xx) + \psi(x)x = \frac{1}{2}x^2 - \psi(t) + \frac{1}{2}x^2 = t = (g, x)t$). Now define $\Xi : A_t \to A_t$ by $\Xi(a) = a + \psi(a)t$. Then Ξ is algebraic mod t^2, since $t = \Xi(x^2) \equiv (x + \frac{1}{2}xt)^2 \bmod t^2$. But if we want to make Ξ an algebraic automorphism mod t^3 by extending it to $\Xi_1(a) = a + \psi(a)t + \xi(a)t^2$, we find ourselves trying to solve $1 + 2x\xi(x) = 0$, which is again impossible. Thus the truncated automorphism Ξ is obstructed. It is by no means a coincidence that the 2-cocycle g is the infinitesimal of the deformation, as we will see in the following section.

6. Jump deformations

As explained in Section 1, an algebraic deformation F of A gives rise to a family of algebras parameterized by the elements of the ring k. The algebra A_F is but a generic element of this family. As t varies over k, we may have non-isomorphic specializations of A_F. A jump deformation of A is one such that these specializations are all isomorphic except perhaps, the specialization to $t = 0$, which must be A itself. Of course we ask that the isomorphisms between the specializations of F arise in a generic algebraic manner.

If $F = \sum f_n t^n$ is a deformation of A, let F^ν be the deformation of A_F defined by

$$F^\nu(a, b) = \sum f_n(a, b)(t(1 + u))^n . \tag{6.1}$$

F^ν should be thought of as a generic element for the family of specializations of F. If F^ν is a trivial deformation of F, then those specializations are isomorphic to F itself. Of course, F^ν may be written as a power-series in u with coefficients in $\mathrm{Hom}_{k[[t]]}(A_F \otimes A_F, A_F)$, which yields

$$F^\nu = \sum_{n=0}^{\infty} f^\nu_n u^n , \qquad f^\nu_n = \sum_{m=n}^{\infty} \binom{m}{n} f_m t^m . \tag{6.2}$$

6.1. Definition. If F^ν is a trivial deformation of A_F, then F is a *jump* deformation of A.

From (6.2) we see that $f^\nu_0 = F$, while $f^\nu_1 = f_1 t + 2f_2 t^2 + 3f_3 t^3 + \cdots$. Now suppose that k is a field of characteristic zero (or an algebra over the rationals). Then f^ν_1 is the infinitesimal of F^ν. If F is a jump deformation, then f^ν_1 must be a 2-coboundary of A_F, as seen in Section 3. On the other hand, if F is a non-trivial deformation of A, we may assume that its infinitesimal is not a 2-coboundary of A (Theorem 3.2). But the infinitesimal of F lifts to f^ν_1, yielding the following result due to J.P. Coffee (see [17]):

6.2. Theorem. *If F is a non-trivial jump deformation of A in characteristic zero, then there is a non-trivial 2-cocycle of A that lifts to a trivial 2-cocycle of A_F, namely the infinitesimal of F.*

6.3. Corollary. *If F is a non-trivial jump deformation of A in characteristic zero, then* $\dim_{k[[t]]} H^2(A_F, A_F) < \dim_k H^2(A_F, A_F)$ *and* $\dim_{k[[t]]} H^1(A_F, A_F) < \dim_k H^1(A_F, A_F)$.

This is exactly what happened in Example 5.7. Theorems 3.1 and 3.2 show that A can be non-trivially deformed only if $H^2(A, A)$ is not zero. In particular, A cannot be regular (if A is the local coordinate ring of a curve, there must be a singularity). In as much as the cohomology of A measures the deficiencies of A, such as singularity or incomplete intersection, Corollary 6.3 says that a non-trivial jump deformation must improve A. The reader may look again at the examples given in 1.5, all of which are jump deformations.

The situation in characteristic p is complicated by the fact that infinitesimal of F^ν may not be f^ν_1. This situation is discussed at length in [17] and [18].

We have not touched upon many of the important topics in algebraic deformation theory, must notably the classification of deformations of plane curves [1, 4], multiparameter deformations and spaces of moduli [14], and perturbation theory for operator algebras [27]. As well, we have barely mentioned the algebraic automorphisms of $A[\![t]\!]$, which are particularly important to the study of inseparable extensions. These depend on analyzing automorphisms of the form $\exp(\psi t)$, where ψ is a derivation of A. For more information about deformation theory, the reader should see [18] and the other papers in the same volume. Our hope is that this chapter has given the reader sufficient understanding of the connection between deformation theory and homological algebra to make the following chapter accessible.

II. Deformations using triples

7. Triples and power-series

If we were to follow the classical approach to algebraic deformation theory, we would now replicate the results of Chapter I using commutative algebras and Harrison or André cohomology, then we would repeat the exercise for Lie algebras, using Chevalley–Eilenberg cohomology groups, then we would consider Lie triple systems, using Yamaguti–Harris groups, etc. (see [22] for references), or we could wave our hands and say everything works for 'equationally-defined' classes of algebras. All this is unnecessary, since the categories defined by triples on the category of k-modules include all the interesting examples, and triple cohomology unifies the classical theories.

Let $\mathcal{M}$ denote the category of k-modules, and recall that a *triple* (T, μ, η) on $\mathcal{M}$ is just a functor $T : \mathcal{M} \to \mathcal{M}$ equipped with natural transformations $\eta : I \to T$ and $\mu : T^2 \to T$ satisfying $\mu \cdot T\mu = \mu \cdot \mu T$ and $\mu \cdot T\eta = \mu \cdot \eta T = I$. A *T-algebra* (A, α) is a k-module A equipped with a multiplication map $\alpha : TA \to A$ satisfying $\alpha \cdot \eta = I$ and $\alpha \cdot \mu = \alpha \cdot T\alpha$. A T-algebra map $(A, \alpha) \to (B, \beta)$ is a k-linear map $\psi : A \to B$ satisfying $\beta \cdot T\psi = \psi \cdot \alpha$. We will use $\mathcal{A}$ to denote the category of all T-algebras.

If we are looking at associative algebras, TA is just the tensor algebra generated by the module A, and α multiplies everything in sight. The equations $\alpha \cdot \eta = I$ and $\alpha \cdot \mu = \alpha \cdot T\alpha$ ensure that α is unitary and associative respectively (amongst other things [3, 9]). If we let T be the symmetric algebra functor, we will get commutative algebras, while the free Lie algebra functor yields the category of Lie algebras, etc. Hence any construction carried out in this general setting applies equally well to all these classical special cases.

We will now recast algebraic deformation theory in the setting of triples. Hence, we wish to deform the multiplication $\alpha : TA \to A$ on a T-algebra to a multiplication $F : T(A[\![t]\!]) \to A[\![t]\!]$ on the $k[\![t]\!]$-module $A[\![t]\!]$. Of course $T(A[\![t]\!]) \cong TA \otimes_k k[\![t]\!]$, so it makes sense to consider multiplications given by power-series $F = \sum f_n t^n$ where $f_n : TA \to A$ is a k-linear map and $f_0 = \alpha$. We now consider what conditions associativity

$$F \cdot \mu = F \cdot TF \tag{7.1}$$

places on the coefficient maps. We will temporarily make the gross assumption that T is additive, even though *no triple of interest to us is additive* (we will fix this up later). Expanding (7.1) and collecting coefficients of like power yields

$$f_n \cdot \mu = \sum_{i+j=m} f_i \cdot Tf_j \,. \tag{7.2}$$

For $n = 1$ this reads $f_1 \cdot \mu = f_1 \cdot T\alpha + \alpha \cdot Tf_1$ or

$$f_1 \cdot T\alpha - f_1 \cdot \mu + \alpha \cdot Tf_1 = 0 \tag{7.3}$$

which looks suspiciously like a cocycle condition, which it is. (7.3) says that $f_1 : TA \to A$ is a 1-cocycle of A, just as in Section 2. However, the complex we use to define triple cohomology groups is not the usual one, as in [2], but the 'non-homogeneous' complex defined by Beck [3]. Since this does not appear in the standard category theory texts, we will review it here.

8. Triple cohomology

The usual definition of the cotriple cohomology groups $H^n(A, A)$ is as follows: There is a pair of adjoint functors $F : \mathcal{M} \to \mathcal{A}$ and $U : \mathcal{A} \to \mathcal{M}$, the free algebra and underlying module respectively. More precisely, if $\alpha : TA \to A$ is in $\mathcal{A}$, then $U(A, \alpha) = A$, while if M is in $\mathcal{M}$, then FM is the T-algebra TM with its multiplication $\mu : T^2M \to TM$. These give rise to a cotriple (G, ε, δ) on $\mathcal{A}$, where $G = FU$ and the natural transformation $\varepsilon : GA \to A$ is the multiplication on A. This yields the following cotriple-generated simplicial complex over A:

$$A \leftarrow GA \leftleftarrows G^2A \overset{\leftarrow}{\leftleftarrows} G^3A \overset{\leftleftarrows}{\leftleftarrows} G^4A \cdots \tag{8.1}$$

where the ith face $G^{n+1}A \to G^nA$ is $G^{n-i}\varepsilon G^i$, for $0 \le i \le n$ [2]. In the general theory, one applies a contravariant abelian group-valued functor E to the complex (8.1), yielding a complex of abelian groups

$$EA \to EGA \rightrightarrows EG^2A \overset{\rightarrow}{\rightrightarrows} EG^3A \overset{\rightrightarrows}{\rightrightarrows} \cdots . \tag{8.2}$$

The homology of the associated chain complex defines the cotriple cohomology groups (denoted $H^n(A, E)_G$ in [2]). Note that if E preserves equalizers the term

EA is dropped from (8.2), since $0 \to EA \to EGA \to EG^2A$ would be exact, yielding trivial low level homology groups.

We are only interested in the cohomology of A with coefficients in itself, so the appropriate candidate for E is the functor $\mathrm{Der}_k(-, A)$ [2, 1.3, 1.4]. Thus the cohomology groups $H^n(A, A)$ are defined by the 'homogeneous' chain complex of abelian groups given below:

$$0 \to \mathrm{Der}_k(GA, A) \to \mathrm{Der}_k(G^2A, A) \to \mathrm{Der}_k(G^3A, A) \to \cdots,$$
$$\partial^{n-1} f = \sum_{i=0}^{n} (-1)^i f \cdot G^{n-i} \varepsilon G^i . \tag{8.3}$$

This does not look anything like the coboundary formula (7.3). However, since F is the left adjoint of U, where $G = FU$ and $T = UF$, the group $\mathrm{Der}_k(G^{n+1}A, A) = \mathrm{Der}_k(FUG^nA, A)$ is isomorphic to $\mathrm{Hom}_k(UG^nA, UA) = \mathrm{Hom}_k(T^nUA, UA)$. If the switch from Der_k to Hom_k seems strange, remember that any linear map lifts to a unique derivation from the free algebra. Dropping the superfluous applications of U, we find the complex (8.3) is isomorphic to Beck's 'non-homogeneous' complex

$$0 \to \mathrm{Hom}_k(A, A) \to \mathrm{Hom}_k(TA, A) \to \mathrm{Hom}_k(T^2A, A) \to \cdots,$$
$$\partial^n f = f \cdot T^n \alpha + \sum_{i=1}^{n} (-1)^i f \cdot T^{n-i} \mu T^{i-1} + (-1)^{n-1} \alpha \cdot Tf . \tag{8.4}$$

In particular, $\partial^1 f = f \cdot T\alpha - f \cdot \mu + \alpha \cdot Tf$, which looks like the left-hand side of (7.3), so this is the complex we will work with when doing deformation theory in the context of triples. Note that the additivity of T is implicit in the definition of the boundary operator.

Before returning to deformation theory, we note that there is a product on the cochains of the non-homogeneous complex that makes it a graded ring. If $f \in \mathrm{Hom}_k(T^mB, C)$ and $g \in \mathrm{Hom}_k(T^nA, B)$, then define $f \circ g$ in $\mathrm{Hom}_k(T^{m+n}A, C)$ by

$$f \circ g = f \cdot T^m g . \tag{8.5}$$

It is quite easy to see that this circle product satisfies the Leibnitz formula

$$\partial(f \circ g) = f \circ \partial g + (-1)^n \partial f \circ g . \tag{8.6}$$

Thus the circle product lifts to the level of homology [10]. This replaces both the cup product and the circle product in the classical theory [13], and will be useful in the obstruction theory to follow. Note that the associativity of α may be written $\alpha\mu = \alpha \circ \alpha$.

9. Obstruction theory

It now becomes easy to carry the obstruction theory of Chapter I to our general setting, though we will dispense with power-series and only work with sequences of (coefficient) functions. Thus, a one-parameter algebraic deformation F of a T-algebra $\alpha : TA \to A$ is a sequence of maps $f_n : TA \to A$, $n \geq 0$, such that $f_0 = \alpha$, $F \cdot \mu = F \cdot TF$, and $F \cdot \eta = I$, i.e.

$$f_n \cdot \mu = \sum_{i+j=n} f_i \circ f_j \quad (n \geq 0)\,, \tag{9.1}$$

$$f_n \cdot \eta = 0 \quad (n \geq 1)\,. \tag{9.2}$$

The latter says that the coefficient maps f_n are normalized 1-cochains and will play no further role here. The associativity formula (9.1) corresponds to (2.1) and may be written as

$$\partial f_n = -\sum_{\substack{i+j=n \\ i,j\neq n}} f_i \circ f_j\,. \tag{9.3}$$

An immediate consequence of (9.3) is that the infinitesimal of the deformation must be a 1-cocycle. If $\{f_m\}_{m<n}$ is a truncated deformation of α, the right-hand side of (9.3) is the obstruction to finding f_n extending the sequence. Hence the following is our version of Theorem 2.1.

9.1. Theorem. *The obstruction to extending a truncated deformation is a 2-cocycle.*

Proof. Using (8.6) we have $\partial(-\sum f_i \circ f_j) = \sum \partial f_i \circ f_j - f_i \circ \partial f_j = \sum f_i \circ f_j \circ f_k - f_i \circ f_j \circ f_k = 0$ where the sum of the indices for each $\sum$ is n, with no index equal to 0. □

9.2. Corollary. *If $H^2(A, A) = 0$, then every 1-cocycle may be extended to an algebraic deformation of A.*

Turning to the automorphisms of the deformed algebra, as in Section 4, we define an automorphism of F to be a sequence $\Psi = \{\psi_n\}_{n\geq 1}$ such that $\psi_0 = I$ and $F \cdot T\Psi = \Psi \cdot F$ (Ψ is a T-algebra map $A_F \to A_F$), i.e.

$$\sum f_i \circ \psi_j = \sum \psi_i \circ f_j\,,$$

$$\partial \psi_n = \sum_{\substack{i+j=n \\ i,j\neq n}} f_i \circ \psi_j - \psi_i \circ f_j\,. \tag{9.4}$$

It is immediate that the infinitesimal of the automorphism is a 0-cocycle of A. The right-hand side of (9.4) defines the obstruction to extending a truncated automorphism $\{\psi_m\}_{m<n}$, and another application of the Leibnitz formula for the circle product shows that this obstruction is a 1-cocycle (cf. Theorem 4.1).

Two deformations, F and G, of α are isomorphic if there is an algebraic isomorphism $\Psi : F \to G$, i.e. a sequence $\Psi = \{\psi\}_{n\geq 0}$ of maps $\psi_n : A \to A$ such that $\psi_0 = I$ and $\Psi \cdot F = G \cdot T\Psi$. This leads to an obstruction formula similar to (9.4), and the obstruction is again a 1-cocycle. If $H^1(A, A) = 0$ all such obstructions vanish, and any two deformations of α are isomorphic. In particular, any deformation is isomorphic to the trivial deformation, and $\alpha : TA \to A$ is rigid.

The remainder of the classical theory, as presented in Chapter I, may be carried over in its entirety to the setting of triples, including the results on the cohomology of the deformed algebra and jump deformations [7, 8], thus extending these results to all algebraic categories over $\mathcal{M}$. By using Van Osdol's 'bicohomology theory' [32], we may also develop the deformation theory for coalgebraic categories, bialgebras, sheaves of algebras over a sheaf of rings, etc. [11]. We will outline this approach in Section 11. Note that even in the case of associative algebras the obstruction formulas and computations in the triple-theoretic version of deformation theory are easier than in the classical presentation.

There is a catch to all this—the assumption that T is additive, which is false for any classical category of algebras. In the following section we will explain how we circumvent this problem.

10. Enriched cohomology

Think of A as only a k-module, and let TA be the tensor algebra (free algebra) generated by A. Everybody knows that a linear map $f : A \to A$ may be lifted to a unique algebra homomorphism $f^* : TA \to TA$, defined by $f(ab) = fa \cdot fb$. This is just how the functor T acts on linear maps—$T(f) = f^*$. However, f may also be lifted to a unique derivation $df : TA \to TA$ defined by $df(ab) = fa \cdot b + a \cdot fb$. If we think of f as a derivation to begin with (on the trivial algebra A) it seems perfectly natural that T should take a derivation to a derivation. Should not $T(f)$ be df? The key is that T should act differently on linear maps cum derivations than on linear maps cum algebra homomorphisms. We need two copies of $\mathrm{Hom}_k(A, A)$. On one copy T sends f to f^*, and on the other copy T sends f to df. But this can be carried further. If $g : A \to A$ is another linear map, we may lift g to a second order derivation $g' : TA \to TA$ over df by setting $g'(ab) = ga \cdot b + fa \cdot fb + a \cdot gb$. Any linear map $A \to A$ may be lifted in many different ways to 'multiplication-respecting' (not 'multiplication-preserving') linear maps $TA \to TA$. What we need is a way of indexing our linear maps, a way of saying 'this is how the map would act on products if there were any'. The functor T will then take an indexed map to the same type of map on TA.

Let (A, A) be the cofree coalgebra over $\mathrm{Hom}_k(A, A)$ (all coalgebras in this section will be counitary, coassociative, and cocommutative). The coalgebra structure of (A, A) includes a diagonal map $\Delta : (A, A) \to (A, A) \otimes (A, A)$ which sends each element of (A, A) to a decomposition in terms of other elements of (A, A). Since (A, A) is cofree over $\mathrm{Hom}_k(A, A)$, there is a natural epimorphism $(A, A) \to \mathrm{Hom}_k(A, A)$ and one should think of the elements of (A, A) as linear maps with an assigned decomposition, or factorization, into other maps [9]. For example, suppose $f \in \mathrm{Hom}_k(A, A)$. Then in the fibre over f we will find an element f^* such that $\Delta f^* = f^* \otimes f^*$. This is f in its guise as an algebra map. Also in the fibre over f we will find df such that $\Delta df = df \otimes 1 + 1 \otimes df$, which is f in its guise as a derivation. Rather than acting on $\mathrm{Hom}_k(A, A)$, the functor T will act on (A, A), that is, we will have a coalgebra map $T : (A, A) \to (TA, TA)$ taking algebra maps to algebra maps and derivations to derivations.

Let C be a k-coalgebra with diagonal map $\Delta : C \to C \otimes C$ and counit $\varepsilon : C \to k$. Recall that a *point* in C is an element p such that $\Delta p = p \otimes p$ and $\varepsilon p = 1$. A *primitive over* p is an element f such that $\Delta f = f \otimes p + p \otimes f$ and $\varepsilon f = 0$. An m-deformation over p (sequence of divided powers) is a sequence of elements $\{f_n\}_{n \le m}$ such that $f_0 = p$, $\varepsilon f_n = 0$ for $n > 0$, and

$$\Delta f_n = \sum_{i+j=n} f_i \otimes f_j . \tag{10.1}$$

A deformation of a point is an approximation to another point in C. If $\{f_n\}_{n>0}$ is an ∞-deformation, then the formal sum $F = \sum f_n$ satisfies $\Delta F = F \otimes F$ and $\varepsilon F = 1$.

We will need to apply our generalized functors to Hom sets other than (A, A), so let M and N be k-modules, and let (M, N) be the cofree coalgebra over $\mathrm{Hom}_k(M, N)$. The adjunction $(M, N) \to \mathrm{Hom}_k(M, N)$ yields an evaluation map $(M, N) \otimes M \to N$. This is just another way of saying that the elements of (M, N) represent linear maps $M \to N$. Now let $T : \mathcal{M} \to \mathcal{M}$ be the triple defining our category of algebras A. Unless T is additive there is no linear map $\mathrm{Hom}_k(M, N) \to \mathrm{Hom}_k(TM, TN)$ defining the action of T on maps, but there is always a coalgebra map $\mathbf{T} : (M, N) \to (TM, TN)$ which agrees with T on points [6]. This coalgebra map is the additive enrichment of T.

We will illustrate this construction using the tensor algebra functor $TM = k + M + M \otimes M + M \otimes M \otimes M + \cdots$. Suppose f in (M, N) is an element such that $f = \sum f_{(1)} \otimes f_{(2)}$ (Sweedler's notation [31]). Then $\mathbf{T}f$ in (TM, TN) is defined by $\mathbf{T}f(1) = \varepsilon f$ and

$$\mathbf{T}f(m_1 \otimes m_2 \otimes \cdots \otimes m_i) = \sum f_{(1)}m_1 \otimes f_{(2)}m_2 \cdots \otimes f_{(i)}m_i . \tag{10.2}$$

Note that if f is a point in (M, N) then $f_{(n)} = f$ for all n, so $\mathbf{T}$ acts just like the usual functor T. In particular, if $\{f_n\}$ is an ∞-deformation in (M, N) then $\mathbf{T}$ and T agree on the formal point $\sum f_n$. More generally, if $F = \sum f_n t^n$ is a formal power-series whose coefficients form a sequence of divided powers in (M, N), then $\mathbf{T}F = TF$, though $\mathbf{T}$ is additive, and T is not.

This gets us back to deformation theory. A one-parameter algebraic deformation of a T-algebra $\alpha : TA \to A$ is a sequence of divided powers $F = \{f_n\}$ over α in (TA, A) such that $F \cdot \mu = F \cdot \mathbf{T}F$ and $F \cdot \eta = I$. This leads to obstruction formulas just as in Section 9, except that the complex (8.4) is replaced by the non-homogeneous complex of coalgebras

$$0 \to (A, A) \to (TA, A) \to (T^2A, A) \to \cdots \tag{10.3}$$

where $\mathbf{T}$ replaces T in the boundary formula, and the circle product $\circ : (T^mA, A) \otimes (T^nA, A) \to (T^{m+n}A, A)$ is defined by $f \circ g = f \cdot \mathbf{T}^m g$. The question of extending truncated deformations is more complicated in this setting, so we will work out the analogue of Theorem 2.2 to illustrate how things work.

Suppose $\{f_i\}_{i<n}$ is a truncated deformation of α. Then the obstruction problem is two-fold: We must find f_n satisfying (9.3), and f_n must extend the sequence of divided powers. This may be reduced to a question of finding an appropriate primitive over α. Let g be any element of (TA, A) extending the sequence of divided powers $\{f_i\}$ (there are many since (TA, A) is cofree), and consider

$$\partial g + \sum_{\substack{i+j=n \\ i+j \neq 0}} f_i \circ f_j \,. \tag{10.4}$$

This is the *primitive obstruction* to extending $\{f_i\}_{i<n}$, and as its name implies it is a primitive whose cohomology class must vanish if our deformation is to be extended.

10.1. Lemma. *The primitive obstruction is a primitive over $\alpha \circ \alpha$.*

Proof. Let P denote the sum (10.4). We must show that $\Delta P = P \otimes (\alpha \circ \alpha) + (\alpha \circ \alpha) \otimes P$ (obviously $\varepsilon P = 0$). Remembering that the circle product is a coalgebra map we have

$$\Delta \partial g = \sum_{i+j=n} (f_i \circ \alpha) \otimes (f_j \circ \alpha) - (f_i \circ \mu) \otimes (f_j \circ \mu) + (\alpha \circ f_i)(\alpha \circ f_j) \,,$$

$$\begin{aligned}
\sum_{\substack{i+j=n \\ i+j \neq 0}} & \Delta(f_i \circ f_j) \\
&= \sum_{\substack{i+j+k+m=n \\ i,k \neq 0 \\ j+m \neq 0}} (f_i \circ f_j) \otimes (f_k \circ f_m) \\
&= \sum_{i+j \neq 0} (f_i \circ f_j) \otimes (\alpha \circ \alpha) + (\alpha \circ \alpha) \otimes (f_i \circ f_j) \\
&\qquad + (f_i \circ \alpha) \otimes (\alpha \circ f_j) + (\alpha \circ f_i) \otimes (f_j \circ \alpha)
\end{aligned}$$

$$+\sum_{i+j+k\neq 0}(\alpha\circ f_i)\otimes(f_j\circ f_k)+(f_i\circ\alpha)\otimes(f_j\circ f_k)$$
$$+(f_i\circ f_j)\otimes(\alpha\circ f_k)+(f_i\circ f_j)\otimes(f_k\circ\alpha)$$
$$+\sum_{i+j+k+m\neq 0}(f_i\circ f_j)\otimes(f_k\circ f_m)$$
$$=\sum_{i+j\neq 0}(f_i\circ f_j)\otimes(\alpha\circ\alpha)+(\alpha\circ\alpha)\otimes(f_i\circ f_j)+(f_i\circ\alpha)\otimes(\alpha\circ f_j)$$
$$+(\alpha\circ f_i)\otimes(f_j\circ\alpha)-(\alpha\circ f_i)\otimes(\partial f_j)-(f_i\circ\alpha)\otimes(\partial f_j)$$
$$-(\partial f_i)\otimes(\alpha\circ f_j)-(\partial f_i)\otimes(f_j\circ\alpha)+(\partial f_i)\otimes(\partial f_j)\,.$$

Now write out ∂f_i and add everything up, remembering that $\alpha\circ\mu=\alpha\circ\alpha$. □

10.2. Theorem. *The primitive obstruction P is a 2-cocycle. If there is a primitive h over α in (TA, A) such that $\partial h = P$, then the truncated deformation may be extended.*

Proof. The first assertion follows from Theorem 9.1 and $\partial\partial = 0$. Given h as advertized, $(g-h)$ extends the sequence of divided powers, and setting $f_n = g-h$ we find f_n satisfies (9.3). □

Of course the statement that $\partial h = P$ asserts that P is a 2-coboundary, but not just in the sense of the complex (10.3), since h must be primitive. Consider the subcomplex of (10.3) consisting of the primitives over $I\in(A, A)$, $\alpha\in(TA, A)$, $a\circ\alpha\in(T^2A, A)$, etc. (this is a subcomplex because the boundary maps preserve primitives). Since the primitives exactly represent derivations between the appropriate algebras, the cohomology of this subcomplex yields the traditional cotriple cohomology groups of [2]. Thus the condition for extending a truncated deformation is that the class of the primitive obstruction be 0 in $H^2(A, A)$. Thus Corollary 9.2 remains valid, even though we have added the condition that the deformation be a sequence of divided powers.

We must apologize for the ugly computations in the proof of Lemma 10.1, but they illustrate the interplay between the coalgebra structures and the boundary operators. This could all be done away with if the boundary maps were coalgebra homomorphisms. This points out the basic inadequacy of our coalgebra-enriched theory. Since the category of coalgebras is not additive, we can only take the homology of (10.3) as abelian groups, not as coalgebras. We will return to this in Section 11 (Remarks 11.2).

10.3. Remarks. We mentioned above that the cohomology of the subcomplex of primitives yields the traditional triple cohomology groups of [2]. In the case of associative algebras these are the same as the Hochschild cohomology groups. In

Section 8 we said our circle product 'replaces' the circle product defined by Gerstenhaber in [13]. In fact, on the subcomplex of primitives they are the same, at least in low dimensions [10]. However, the circle product of [13] does not lift to homology. This seems to be a contradiction, but the solution is quite simple: The circle product of two primitives is not a primitive, so our boundary of $f \circ g$ is different from the Hochschild boundary. The reader should note there are other cohomology operations on cohomology groups in classical examples [29, 30].

11. Coalgebras and bialgebras

The cohomology of coalgebras and bialgebras (in a general sense) can also be explicated using triple cohomology [32], and this may be used to study their deformation theories. We will very briefly outline how this may be done (for details see [11]).

The situation for coalgebras is completely dual to that of algebras. Suppose we have a cotriple (S, ε, δ) on $\mathcal{M}$ where $\varepsilon : S \to 1$ and $\delta : S \to S^2$ are natural transformations satisfying well known conditions. Now the category $\mathscr{C}$ of S-coalgebras may be described as the category of modules C equipped with a comultiplication $c : C \to SC$ satisfying the usual identities: $\varepsilon c = 1$ and $\delta c = Sc \cdot c$. If we use the usual cofree coalgebra construction for S [12, 31] we get $\mathscr{C}$ to be the category of coassociative coalgebras. Of course, using the cocommutative variant of S gives cocommutative coalgebras, using the Lie version of S gives Lie coalgebras, etc.

The cohomology of coalgebras is defined using a simplicial complex generated by repeated applications of S, i.e. if C and D are coalgebras, the groups $H^n(C, D)$ are defined by a complex (C, S^*D) whose boundary map d depends on the comultiplications on C and D (see 11.1). In this setting there is also a circle product $\circ : (D, S^mE) \otimes (C, S^nD) \to (C, S^{m+n}E)$ defined by $f \circ g = S^nf \cdot g$ and satisfying the usual Leibnitz formula. The comultiplication c is in (C, SC), and the *coalgebraic* deformations of c are given by deformations G in (C, SC) satisfying $\delta G = G \circ G$. This gives equation (11.1), analogous to (9.1), and leads us into deformation theory,

$$dg_n = \sum_{i+j=n} g_i \circ g_j \,. \tag{11.1}$$

Now suppose that we are also given a triple (T, μ, η) on $\mathcal{M}$ which defines the category of T-algebras. To ensure harmony between T and S, we insist that there be a 'distributive law', i.e. a natural transformation $\lambda : TS \to ST$ satisfying certain conditions which ensure that S may be lifted to the category $\mathscr{A}$ of T-algebras, and T may be lifted to $\mathscr{C}$ (see 11.1). A *T-S-bialgebra* is a module B equipped with two structure maps $\beta : TB \to B$ and $b : B \to SB$, making it a T-algebra and an S-coalgebra. Further, the structure maps β and b must satisfy

$$S\beta \cdot \lambda \cdot Tb = b \cdot \beta . \tag{11.2}$$

This says β is a coalgebra map and b is an algebra map. If A is also a bialgebra, the bialgebra cohomology groups are defined via a double complex (T^*A, S^*B) whose boundaries depend on the structure maps of A and B, as well as λ [11, 32]. Note that $S\beta \cdot \lambda : TSB \to SB$ makes SB a T-algebra, and $\lambda \cdot Tb : TB \to STB$ makes TB an S-coalgebra. With this understanding, the boundaries of the double complex are just the usual boundaries for algebra and coalgebra cohomology. The cohomology groups $H^*(A, B)$ are then defined as the cohomology of the total complex of (T^*A, S^*B).

Now the group of 1-cochains $C^1(B, B)$ in this setting is $(TB, B) \oplus (B, SB)$, and the bialgebra structure given by (β, b) is a point there. The *bialgebraic* deformations of (β, b) are given by sequences of elements in $C^1(B, B)$ which satisfy conditions (9.1) and (11.1), as well as an extra condition coming from (11.2). Once again we find that such a sequence must have an infinitesimal lying in $Z^1(B, B)$, and the obstruction to extending a truncated deformation lies in $H^2(B, B)$.

Note that by using different mixes of T and S this gives a deformation theory for all the usual types of bialgebras—associative and coassociative, Lie bialgebras, ...

In the context of bialgebras there is, once again, a circle product defined at the level of cochains. If $f \in (T^m B, S^n B)$ and $g \in (T^i B, S^j B)$, then $f \circ g \in (T^{i+m}, S^{j+n} B)$ is defined by

$$f \circ g = S^j f \cdot \Lambda \cdot T^m g , \tag{11.3}$$

where Λ is 'the' obvious natural transformation $T^m S^j \to S^j T^m$ built from λ (naturality guarantees all choices are equal). This circle product satisfies a Leibnitz-type formula, so lifts to homology. It also makes for an elegant presentation of the various formulas needed for bicohomology, which we list below. Here $f \in (T^m B, S^n B)$.

11.1. Formulas for bicohomology.

Boundary formulas:

$$\partial f = f \circ \beta + \sum_{i=1}^{m} (-1)^i f \circ \mu T^{i-1} + (-1)^{m+1} \beta \circ f ,$$

$$df = b \circ f + \sum_{i=1}^{n} (-1)^i \delta S^{i-1} \circ f + (-1)^{n+1} f \circ b .$$

Distributive laws:

$$\Lambda \cdot \mu S = S\mu \cdot \Lambda , \qquad \delta \cdot T\Lambda = \Lambda \cdot T\delta ,$$

$$\lambda : \mu S = S\mu \ , \qquad \varepsilon T \cdot \lambda = T\varepsilon \ .$$

Two useful consequences are: $f \circ \mu S \circ g = f \circ \mu \circ g$ and $f \circ \delta T \circ g = f \circ \delta \circ g$.

Bialgebra equations:

$$\beta \circ \mu = \beta \circ \beta \ , \quad \delta \circ b = b \circ b \ , \quad \beta \circ b = b \circ \beta \ .$$

Naturality:

$$f \circ \mu T^{i+m} = \mu T^i S^n \circ f \ , \qquad \delta S^{i+n} \circ f = f \circ \delta S^i T^m \ .$$

11.2. Remarks. As mentioned at the end of Section 10, the great deficiency of the enriched cohomology is that we cannot add coalgebra maps and get coalgebra maps, so the chain complex (8.4) is only a complex of modules. If k is a field, a much more elegant theory results from switching to the category of abelian k-Hopf algebras by taking the free abelian Hopf algebras generated by our coalgebras (T^nA, A) [23, 35]. This yields a chain complex of Hopf algebras, whose homology yields Hopf algebras $H^n[A, A]$ which have the characteristic properties of cohomology groups, including long exact sequences in both variables [33]. Furthermore, the circle product makes the total 'group' $H^*[A, A]$ a Hopf ring, similar to the Hopf ring that appears in the study of complex cobordism [28, 34]. This Hopf-enriched theory will be the subject of another paper.

The enriched cohomology is defined using the category $\mathscr{C}$ of cocommutative, coassociative, counitary coalgebras as a base category over which all other categories of algebras are enriched. This is because we can show that all our categories of interest have such enrichments, and that the triples and cotriples we need can be lifted to this level [6]. However, other variants $\mathscr{C}$ may also be used. For example, using non-associative coalgebras would allow us to consider the subcomplex of (8.4) of skew derivations, leading to a completely different definition of cohomology groups. Or we could consider the enrichment over the category of graded commutative coalgebras, leading to further unexplored ground.

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