CS IN CHIRAL GAUGE

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Fix a holomorphic volume form on Σ. Typically, when $\Sigma = \mathbb{C}$ we will pick the form $\mathrm{d}z$.

In this section, we will work on flat space $\mathbb{R}^3$ which we will identify with $\mathbb{C} \times \mathbb{R}$. Choose holomorphic coordinates

$$(z, \overline{z}, t) \in \mathbb{C} \times \mathbb{R}$$

The fields of critical Chern-Simons theory are

$$(\gamma, \beta) \in \Omega^{0,*}(\mathbb{C}) \widehat{\otimes} \Omega^*(\mathbb{R}) \otimes \mathfrak{g}[1] \oplus \Omega^{0,*}(\mathbb{C}) \widehat{\otimes} \Omega^*(\mathbb{R}) \otimes \mathfrak{g}.$$

The action is

$$S(\gamma, \beta) = \int_{\mathbb{C}\times\mathbb{R}} \mathrm{d}z \wedge \left(\langle \beta, \mathrm{d}\gamma \rangle + \frac{1}{2} \langle \beta, [\gamma, \gamma] \rangle \right).$$

1. The chiral gauge

We use the gauge fixing condition

$$Q^{GF} = \overline{\partial}^*_{\mathbb{C}} + \mathrm{d}^*_{\mathbb{R}}.$$

BW: might need to adjust factors in this equation so that $[Q, Q^{GF}]$ is the standard flat Laplacian in each degree. In this gauge, we can write the heat kernel as

$$K_T((z, \overline{z}, t), (w, \overline{w}, s)) = k^{an}_T((z, \overline{z}, t), (w, \overline{w}, s))(\mathrm{d}\overline{z} - \mathrm{d}\overline{w})(\mathrm{d}t - \mathrm{d}s) \otimes \mathsf{Cas}_{\mathfrak{g}},$$

where $k^{an}_T \in C^\infty((\mathbb{C} \times \mathbb{R}) \times (\mathbb{C} \times \mathbb{R}))$, $T > 0$, is the heat kernel for the flat Laplacian on $\mathbb{R}^3$. Explicitly

$$k^{an}_T((z, \overline{z}, t), (w, \overline{w}, s)) = \frac{1}{(4\pi T)^{3/2}} e^{-|z-w|^2/4T - |t-s|^2/4T}.$$

2. One-loop quantization

Theorem 2.1. *Critical Chern-Simons admits a one-loop exact and finite quantization on $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$. The space of one-loop quantizations is a torsor for the group $H^3_{\mathrm{Lie}}(\mathfrak{g})$.*

Proposition 2.2. *The following limit*

$$\lim_{\epsilon \to 0} W(P_{\epsilon < L}, I)$$

exists for all $L > 0$. Thus, in the holomorphic gauge, critical Chern-Simons admits a finite prequantization.

Lemma 2.3. *In the holomorphic gauge, only diagrams of at most one-loop are nonzero in the expansion of* $W(P_{\epsilon<L}, I)$*.*

The strategy is to analyze the one-loop diagram by the order of the number of vertices. There is a special diagram that could potentially give rise to issues.

Lemma 2.4. *Let* Γ_2 *be the wheel with two trivalent vertices. Then* $W_{\Gamma_2}(P_{\epsilon<L}, I) = 0$ *identically as a distribution for all* $\epsilon, L > 0$*.*

Proof. Note that the propagator is of the form

$$P_{\epsilon<L}((z,\overline{z},t),(w,\overline{w},s)) = p^{an}_{\epsilon<L} \cdot ((\mathrm{d}\overline{z} - \mathrm{d}\overline{w}) - (\mathrm{d}t - \mathrm{d}s))$$

where $p^{an}_{\epsilon<L}$ is some smooth function on $(\mathbb{C} \times \mathbb{R})^2$. For any fields α_1, α_2, the weight $W_{\Gamma_2}(P_{\epsilon<L}, I)(\alpha_1, \alpha_2)$ is given by some integral over $(\mathbb{C} \times \mathbb{R})^2 = \mathbb{C}_z \times \mathbb{R}_t \times \mathbb{C}_w \times \mathbb{R}_s$, but the integrand is proportional to

$$P_{\epsilon<L}((z,\overline{z},t),(w,\overline{w},s)) \wedge P_{\epsilon<L}((w,\overline{w},s),(z,\overline{z},t)).$$

This expression is identically zero being the wedge product of proportional one-forms on the same space. $\square$

Lemma 2.5. *Let* Γ_k *be the wheel with* k *trivalent vertices. Then, for* $k > 2$ *the limit*

$$\lim_{\epsilon \to 0} W_{\Gamma_k}(P_{\epsilon<L}, I)$$

exists for all $L > 0$*.*

When $\mathfrak{g}$ is semi-simple, one can see in a similar way to the observations made for ordinary Chern-Simons that the obstruction deformation complex guarantees the existence of a quantization of critical Chern-Simons. However, we can see that the anomaly for critical Chern-Simons vanishes directly using the holomorphic gauge.

Lemma 2.6. *As above, let* Γ_k *denote the wheel with* k *trivalent vertices. When* $k = 2$*, one has*

$$W_{\Gamma_k}(P_{\epsilon<L}, K_\epsilon, I) = 0$$

identically as a distribution. When $k > 2$*, then*

$$\lim_{\epsilon \to 0} W_{\Gamma_k}(P_{\epsilon<L}, K_\epsilon, I) = 0.$$

This lemma shows that the one-loop obstruction to solving the quantum master equation of critical Chern-Simons on $\mathbb{C} \times \mathbb{R}$ in the holomorphic gauge vanishes for any Lie algebra $\mathfrak{g}$. Since the quantization is one-loop exact, this is enough to guarantee a full quantization.