SOMETHING ABOUT LEVELS

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1. Vacuum modules

Definition 1.1. Let $d > 1$, and consider the dg algebra A_d. Define the A_d-module of *positive modes*

$$A_{d,+} = H^{d-1}(A_d).$$

Note that there is a natural map of dg A_d-modules $A_d \to A_{d,+}[-d+1]$. Define the dg ideal of *negative modes*

$$A_{d,-} = \ker\left(A_d \to A_{d,+}[-d+1]\right).$$

Remark 1.2. We are modeling our terminology on the usual definition of positive and negative modes for Laurent polynomials in one-variable $A_1 = \mathbb{C}[z, z^{-1}]$ via

$$A_{1,+} = \mathbb{C}[z] \subset \mathbb{C}[z, z^{-1}] \quad \text{and} \quad A_{2,-} = z^{-1}\mathbb{C}[z^{-1}] \subset \mathbb{C}[z, z^{-1}]$$

respectively.

Definition 1.3. Fix an element $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$ and let $k \in \mathbb{C}$. The *vacuum module* $\mathrm{Vac}_{(\theta,k)}$ associated to the pair (θ, k) is the induced $\mathfrak{g}_{d,\theta}$-module

$$\mathrm{Ind}_{U(A_{d,+}\otimes\mathfrak{g})[K]}^{U(\mathfrak{g}_{d,\theta})}(\mathbb{C}_{K=k}) = U(\mathfrak{g}_{d,\theta}) \otimes_{U(A_{d,+}\otimes\mathfrak{g})[K]} \mathbb{C}_{K=k}.$$

When θ is understood, we refer to this as the *level k* vacuum module.

There is a variant of this definition that makes sense for a fixed θ and no specification of k. It is defined by

$$\mathrm{Vac}_\theta = U(\mathfrak{g}_{d,\theta}) \otimes_{U(A_{d,+}\otimes\mathfrak{g})[K]} \mathbb{C}[K].$$

This is a $U(\mathfrak{g}_{d,\theta})$-module in the category of $\mathbb{C}[K]$-modules.

1.1. BW: Let V be the disk module of $\mathbb{U}_\theta(\mathcal{G})$.

Proposition 1.4. *The factorization product endows V_θ with the structure of a $U(\mathfrak{g}_{d,\theta})$-module. Moreover, it is equivalent to the θ-vacuum module $V_\theta \simeq \mathrm{Vac}_\theta$.*