A FAMILIES INDEX THEOREM

1. A statement of the index theorem

2. The $\beta\gamma$ system

3. Equivariant field theories

3.1. Inner actions and obstructions.

4. Computing the anomaly

Proposition 4.1. *Let V be a $\mathfrak{g}$ module and X a complex d-fold. The classical $\mathfrak{g}^X$-equivariant theory*

$$\mathcal{E}_{X,V} = T^*[-1](\Omega^{0,*}(X;V))$$

admits a canonical $\mathfrak{g}^X$-equivariant quantization. The cohomology class of the obstruction $[\Theta_V] \in H^1(\mathrm{C}^*_{\mathrm{loc}}(\mathfrak{g}^X))$ to lifting this to an inner action by the local Lie algebra $\mathfrak{g}^X$ is identified with the image of

$$\#\mathrm{ch}_{d+1}(V) \in \mathrm{Sym}^{d+1}(\mathfrak{g}^\vee)^{\mathfrak{g}}$$

under the map $J : \mathrm{Sym}^{d+1}(\mathfrak{g}^\vee)^{\mathfrak{g}}[-1] \to \mathrm{C}^*_{\mathrm{loc}}(\mathfrak{g}^X)$.

As a simple corollary we find the anomaly in a slightly more general situation.

Corollary 4.2. *Let P be a principal G bundle on X, and V a G-representation. Then we can consider the $\mathfrak{g}^X_P = \Omega^{0,*}(X; \mathrm{ad}(P))$-equivariant theory*

$$\mathcal{E}_{P\to X,V} = T^*[-1](\Omega^{0,*}(X; P \times^G V)).$$

This theory admits a canonical $\mathfrak{g}^X_P$-equivariant quantization. Moreover, the cohomology class of the obstruction $[\Theta_V]$ to an inner action is also identified with $\#\mathrm{ch}_{d+1}(V)$.

We will prove the proposition in the following steps. First, we argue that it suffices to calculate this obstruction on an arbitrary open set in X. Taking this open set to be a disk we see that it suffices to compute the cocycle in the case that $X = \mathbb{C}^d$. This calculation is done explicitly in terms of one-loop Feynman diagrams.

4.1. By construction, the data of a classical BV theory on X is sheaf-like on the manifold. That is, we have a sheaf of (-1)-shifted elliptic complexes $\mathcal{E}$ on X together with a local functional $I \in \mathcal{O}_{\mathrm{loc}}(\mathcal{E})(X)$. The space of local functionals $\mathcal{O}_{\mathrm{loc}}(\mathcal{E})$ also forms a sheaf on X, so it makes sense to restrict I to any open set $U \subset X$. In this way, for each open we have a (-1)-shifted elliptic complex $\mathcal{E}(U)$ together with a local functional $I|_U \in \mathcal{O}_{\mathrm{loc}}(\mathcal{E})(U)$ – that is, a classical field theory on $U \subset X$. A fancy way of saying this is that the space of classical field theories on X forms a sheaf.

A very slightly refined version of this takes into account an action of a local Lie algebra. If $\mathcal{L}$ is a local Lie algebra on X then the space of $\mathcal{L}$-equivariant classical BV theories also forms a sheaf on X.

Costello has shown in [**?**] that the space of quantum field theories also form a sheaf on X. In a completely analogous way, one can show that the space of $\mathcal{L}$-equivariant quantum field theories forms a sheaf on X.

We have already seen how the obstruction to lifting a quantum field theory with an action of a local Lie algebra $\mathcal{L}$ to an inner action arises as a failure of satisfying the QME. Since an $\mathcal{L}$-equivariant theory satisfies the QME modulo terms in $C^*_{\mathrm{loc}}(\mathcal{L})(X)$, this obstruction $\Theta(X)$is a degree one cocycle in $C^*_{\mathrm{loc}}(\mathcal{L})(X)$. By the remarks above, we can restrict any $\mathcal{L}$-equivariant field theory to an arbitrary open set $U \subset X$. Hence, for each open $U \subset X$ we have an obstruction element Θ^U. The complex $C^*_{\mathrm{loc}}(\mathcal{L})(X)$ also has a refinement to a sheaf of complexes on X and the obstruction Θ^U is an element in $C^*_{\mathrm{loc}}(\mathcal{L})(U)$. We will need the following elementary fact that the obstruction to having an inner action is natural with respect to the restriction of open sets.

Lemma 4.3. *Let $i_U^V : U \hookrightarrow V$ be any inclusion of open sets in X. Then*

$$(i_U^V)^*([\Theta^V]) = [\Theta^U]$$

where $(i_U^V)^* : C^*_{\mathrm{loc}}(\mathcal{L})(V) \to C^*_{\mathrm{loc}}(\mathcal{L})(U)$ is the restriction map and the brackets $[-]$ denotes the cohomology class of the cocycle. In other words, the map that sends a quantum field theory on X with an $\mathcal{L}$-action to its obstruction to having an inner $\mathcal{L}$-action is a map of sheaves.

For any complex d-fold X we have defined the map $J^X : \mathrm{Sym}^{d+1}(\mathfrak{g}^\vee)^{\mathfrak{g}} \to C^*_{\mathrm{loc}}(\mathfrak{g}^X)$. The complex $C^*_{\mathrm{loc}}(\mathfrak{g}^X)$

Lemma 4.4. *The map*

$$J : \underline{\mathrm{Sym}^{d+1}(\mathfrak{g}^\vee)^{\mathfrak{g}}} \to C^*_{\mathrm{loc}}(\mathfrak{g}^X)$$

defined on each open by $J|_U = J^U$ is a map of sheaves. Here, the underline means the constant sheaf.

Lemma 4.5. *For any open sets $i_U^V : U \subset V$ in X the induced map*

$$(i_U^V)^* : H^1\left(V; C^*_{\mathrm{loc}}(\mathfrak{g}^X)\right) \to H^1\left(U; C^*_{\mathrm{loc}}(\mathfrak{g}^X)\right)$$

is injective.

BW: The last key observation is that $(i_U^V)^* J^V = J^U$.