THE HIGHER KAC-MOODY FACTORIZATION ALGEBRA

BW: Add intro comparing to Kapranov-Hennion-Faonte.

1. LOCAL LIE ALGEBRAS AND FACTORIZATION

1.1. **A recollection of local Lie algebras.**

Definition 1.1. A *local Lie algebra* (or *local L_∞ algebra*) on X is the following data:
   (i) a $\mathbb{Z}$-graded vector bundle L on X, with sheaf of sections that we denote $\mathcal{L}$;
   (ii) for each $n \in \mathbb{Z}$ a polydifferential operator

$$\ell_n : \mathcal{L}^{\otimes n} \to \mathcal{L}[2-n];$$

such that the collection $\{\ell_n\}$ endow $\mathcal{L}$ with the structure of a sheaf of L_∞ algebras.

We often refer to a local Lie algebra $(L, \{\ell_n\})$ simply by its sheaf of sections $\mathcal{L}$. A local Lie algebra defines the sheaf of complexes $\mathrm{C}_*^{\mathrm{Lie}}(\mathcal{L})$ that sends an open set $U \subset X$ to the complex $\mathrm{C}_*^{\mathrm{Lie}}(\mathcal{L}(U))$. Note that $\mathrm{C}_*^{\mathrm{Lie}}(\mathcal{L})$ is itself the sheaf of sections of a graded vector bundle and that it has the structure of a sheaf of cocommutative coalgebras.

Definition 1.2. A map $f : \mathcal{L} \to \mathcal{L}'$ of local Lie algebras on X is a polydifferential operator

$$f : \mathrm{C}_*^{\mathrm{Lie}}(\mathcal{L}) \to \mathrm{C}_*^{\mathrm{Lie}}(\mathcal{L}')$$

that is, in addition, a map of sheaves of cocommutative coalgebras.

1.2. **Universal objects.** Let CplxMan be the category of complex manifolds with holomorphic maps. There is a fibered category VB of holomorphic vector bundles over CplxMan. Likewise, there is a category of local Lie algebras fibered over CplxMan. Its objects are pairs (X, L) consisting of a complex manifold X together with a local Lie algebra L on X. Maps between $(f, F) : (X, L) \to (X', L')$ is a holomorphic map $f : X \to X'$ together with a map of local Lie algebras on X, $F : L \to f^* L'$. ...

Given a local Lie algebra with underlying $\mathbb{Z}$-graded vector bundle L we can consider both its sheaf of sections $\mathcal{L}$. This has the structure of a sheaf of L_∞ algebras. We can also consider its cosheaf of compactly supported sections, that we denote $\mathcal{L}_c$. The cosheaf of compactly supported sections is not, however, a cosheaf of Lie algebras. It does, however, have a certain "factorization" property that we will exploit to define factorization algebras on the underlying manifold.

Definition 1.3. A *prefactorization Lie algebra* ℓ on a manifold X is the data:
   (i) for each open set $U \subset X$ an L_∞ algebra $\ell(U)$;
   (ii) for each pairwise disjoint collection of open sets $U_1, \ldots, U_n$ contained inside some open set $V \subset X$ a map of L_∞ algebras

$$\ell(U_1) \oplus \cdots \oplus \ell(U_n) \to \ell(V).$$

There is a symmetric monoidal structure on the category of L_∞ algebras $L_\infty\mathrm{Alg}$ given by the direct sum $\oplus$ of underlying chain complexes. Thus, a prefactorization Lie algebra is simply a symmetric monoidal functor

$$\ell : \mathrm{Op}(X)^\sqcup \to L_\infty\mathrm{Alg}^\oplus.$$

In particular, ℓ is a precosheaf of L_∞ algebras.

In the holomorphic setting the above definition makes sense in a wider context, where we consider all complex manifolds of a fixed dimension uniformly.

Definition 1.4. A *universal holomorphic prefactorization Lie algebra* of dimension d is a symmetric monoidal functor

$$\ell : \mathrm{Hol}_d^\sqcup \to L_\infty\mathrm{Alg}^\oplus$$

from the symmetric monoidal category of holomorphic manifolds with embeddings equipped with disjoint union to the category of L_∞ algebras equipped with direct sum.

Just like in the case of factorization algebras, we have the following definition.

Definition 1.5. A *factorization Lie algebra* on X is a prefactorization Lie algebra satisfying descent for Weiss covers on X. Likewise, a *universal holomorphic factorization Lie algebra* is a universal holomorphic prefactorization Lie algebra satisfying descent for Weiss covers in Hol_d.

Local Lie algebras provide a nice class of factorization Lie algebras.

Lemma 1.6. *Suppose L is a local Lie algebra on X. Then the precosheaf of compactly supported sections $\mathcal{L}_c$ is a factorization Lie algebra on X. Similarly, if L is a universal holomorphic local Lie algebra then its functor of compactly supported sections $\mathcal{L}_c$ is a universal holomorphic factorization Lie algebra.*

We briefly elaborate by what we mean by the compactly supported sections of a universal local Lie algebra L. Such an object determines a functor

$$\mathcal{L}_c : \mathrm{Hol}_d \to L_\infty\mathrm{Alg}$$

defined by sending a complex d-fold X to the space of compactly supported sections of the bundle $L(X)$. This has the structure of an L_∞ algebra by definition. Given a holomorphic embedding $f : X \to Y$ one defines the map

$$f_c : \mathcal{L}_c(X) \to \mathcal{L}_c(Y)$$

by BW: finish...

1.2.1. We review how one constructs a factorization algebra from a local Lie algebra or a factorization Lie algebra. This will be the primary way that we will construct factorization algebras in this paper. We show that the construction also works to define, from universal Lie algebras, universal factorization algebras. Much of this section is a recollection of the material in Section 3.6 of [**?**].

Given an ordinary Lie algebra $\mathfrak{g}$ we can define the Chevalley-Eilenberg chain complex $\mathrm{C}_*^{\mathrm{Lie}}(\mathfrak{g})$ computing Lie algebra homology.

Lemma 1.7. *Suppose $\mathcal{G}$ is a factorization Lie algebra on X. Then, the assignment*

$$\mathrm{C}^{\mathrm{Lie}}_*(\mathcal{L}_c) : U \mapsto \mathrm{C}^{\mathrm{Lie}}_*(\mathcal{L}_c(U))$$

defines a factorization algebra on X. Similarly, if L is a universal holomorphic factorization Lie algebra then $\mathrm{C}^{\mathrm{Lie}}_*(\mathcal{L}_c)$ defines a universal holomorphic factorization algebra.

1.3. In this section we introduce the local Lie algebra that will be the main focus of the paper. The local Lie algebra will be defined on any complex manifold and is constructed using the data of a Lie algebra $\mathfrak{g}$. For most of this paper we will assume that we have an ordinary Lie algebra, but a very slight generalization can be used to handle dg Lie or L_∞ algebras.

Fix a complex manifold X of complex dimension d. The complex structure determines a splitting of the tangent bundle $TX = TX^{1,0} \oplus TX^{0,1}$ into its holomorphic and anti-holomorphic subbundles. Likewise, the cotangent bundle splits as $T^*X = TX^{1,0} \oplus TX^{0,1}$. Define the following $\mathbb{Z}$-graded vector bundle on X

$$\mathfrak{g}(X) := \wedge^* T^* X^{0,1} \otimes \underline{\mathfrak{g}} = \oplus_{i=0}^d \wedge^i T^* X^{0,1}[-i]$$

where $\underline{\mathfrak{g}}$ denotes the trivial vector bundle on X with fiber $\mathfrak{g}$. The differential operator $\bar{\partial}$ on X extends to a degree one operator on $\mathfrak{g}(X)$. On the ith graded piece it is defined by

$$\bar{\partial} \otimes \mathrm{id}_{\mathfrak{g}} : \wedge^i T^* X^{0,1} \otimes \underline{\mathfrak{g}} \to \wedge^{i+1} T^* X^{0,1} \otimes \underline{\mathfrak{g}}.$$

The Lie bracket on $[-,-]_{\mathfrak{g}}$ on $\mathfrak{g}$ extends to a polydifferential operator on $\mathfrak{g}(X)$ of degree zero

$$[-,-] := \wedge \otimes [-,-]_{\mathfrak{g}} : \left(\wedge^i T^* X^{0,1} \otimes \underline{\mathfrak{g}} \right) \otimes \left(\wedge^j T^* X^{0,1} \otimes \underline{\mathfrak{g}} \right) = \left(\wedge^i T^* X^{0,1} \otimes \wedge^j T^* X^{0,1} \right) \otimes (\underline{\mathfrak{g}} \otimes \underline{\mathfrak{g}}) \to \wedge^{i+j} T^* X^{0,1} \otimes \underline{\mathfrak{g}}.$$

Here $\wedge$ denotes the wedge product of differential forms. The sheaf of sections of $\wedge^i T^* X^{0,1}$ is denoted $\Omega^{0,*}_X$ and we write the sheaf of sections of $\mathfrak{g}(X)$ as $\mathfrak{g}^X = \Omega^{0,*}_X \otimes \mathfrak{g}$.

Definition/Lemma 1. The $\mathbb{Z}$-graded bundle $\mathfrak{g}(X)$ together with the polydifferential operators $\bar{\partial}, [-,-]$ determine the structure of local Lie algebra on X. We call $\mathfrak{g}(X)$, or its sheaf of sections $\mathfrak{g}^X$, the *holomorphic $\mathfrak{g}$-current algebra* on X.

Proof. It suffices to show that $\mathfrak{g}^X$ is a presheaf of dg Lie algebras. For each open $U \subset X$ the restriction of the polydifferential operators $\bar{\partial}$ and $[-,-]$ to the vector space $\mathfrak{g}^X(U)$ coincides with structure of a dg Lie algebra obtained by tensoring the dg commutative algebra $\Omega^{0,*}(U)$ with the Lie algebra $\mathfrak{g}$. Now, if $U \hookrightarrow V$ is an inclusion of open sets we need to show that the induced map $\Omega^{0,*}(V) \otimes \mathfrak{g} \to \Omega^{0,*}(U) \otimes \mathfrak{g}$ is a map of dg Lie algebras. This follows from the general fact that if $f : A \to B$ to is a map of commutative dg algebras then the induced map $f \otimes \mathrm{id}_{\mathfrak{g}} : A \otimes \mathfrak{g} \to B \otimes \mathfrak{g}$ is a map of dg Lie algebras (where the dg Lie structure on $A \otimes \mathfrak{g}$ and $B \otimes \mathfrak{g}$ is the one mentioned above). $\square$

Given the local Lie algebra $\mathfrak{g}(X)$ we obtain a factorization Lie algebra on X by considering its compactly supported sections $\mathfrak{g}^X_c : U \subset X \mapsto \Omega^{0,*}_c(U) \otimes \mathfrak{g}$.

The local Lie algebra $\mathfrak{g}(X)$ makes sense on any complex manifold and is functorial in the universal sense discussed above. That is, we have a bundle $\mathfrak{g}(-)$ on the category of all complex

dimensional d-folds. Thus, its compactly supported sections restricted to the subcategory Hol_d defines a universal holomorphic factorization Lie algebra. Explicitly, this is the functor

$$\mathfrak{g}_c^d : \text{Hol}_d \to L_\infty\text{Alg}$$

sending $X \to \mathfrak{g}_c^X(X)$.

In fact, there is a certain functoriality in the complex manifold that we now describe

2. CENTRAL EXTENSIONS FROM LOCAL COCYCLES

In this section we describe the extensions of the local Lie algebra $\mathfrak{g}^X$. Let $\underline{C}[k]$ be the local Lie algebra defined on any complex manifold X given by the constant bundle concentrated in cohomological degree $-k$. We wish to describe extensions of a local Lie algebra $\mathcal{L}$ on X by the constant Lie algebra $\mathbb{C}[k]$. This is a local Lie algebra $\widehat{\mathcal{L}}$ that fits into an exact sequence of local Lie algebras

(1) $$0 \to \underline{C}[k] \to \widehat{\mathcal{L}} \to \mathcal{L} \to 0.$$

Every cocycle $\alpha \in C^*_{\text{loc}}(\mathcal{L})(X)$ of total degree $2 + k$ determines a central extension as in (1) as follows. The underlying vector bundle for the extended local Lie algebra is given by $L \oplus \underline{C}[k]$.

Moreover, any two cohomologous cocycles determine quasi-isomorphic extensions.

Lemma 2.1. *The space of k-shifted central extensions as in Equation (1) is a torsor for the abelian group $H^{2+k}(\mathcal{L})(X)$.*

$$C^*_{\text{loc}}(\mathfrak{g}^X) =$$

Recall, a local k-cocycle of a local Lie algebra determines a $(k-2)$-shifted central extension, by the constant sheaf $\underline{C}$. We are interested in (-1)-shifted central extensions, and hence, local 1-cocycles. For $\mathfrak{g}^X$ we can describe such a family of 1-cocycles.

Let P be an invariant polynomial of $\mathfrak{g}$ of homogenous degree $d + 1$. That is, $P \in \text{Sym}^{d+1}(\mathfrak{g}^\vee)^{\mathfrak{g}}$. We can extend P to a functional on $\Omega^{0,*}(X) \otimes \mathfrak{g}$ by the rule

$$
\begin{array}{cccc}
P^X : & \text{Sym}^{d+1}(\Omega^{0,*}(X) \otimes \mathfrak{g}) & \to & \mathbb{C} \\
& (\omega_1 \otimes X_1, \ldots, \omega_{d+1} \otimes X_{d+1}) & \mapsto & (\omega_1 \wedge \cdots \wedge \omega_{d+1})P(X_1, \ldots, X_{d+1})
\end{array}
$$

Proposition 2.2. *The assignment*

$$J : \text{Sym}^{d+1}(\mathfrak{g}^\vee)^{\mathfrak{g}}[-1] \to C^*_{\text{loc}}(\mathfrak{g}^X)$$

sending and invariant polynomial P, of homogeneous degree $d + 1$, to the local functional

$$(\alpha_1, \ldots, \alpha_{d+1}) \mapsto \int P^X(\alpha_1, \partial\alpha_2, \ldots, \partial\alpha_{d+1})$$

is a cochain map. Moreover, it is injective at the level of cohomology.

Remark 2.3. We extend the operator $\partial : \Omega^{k,l} \to \Omega^{k+1,l}$ to $\Omega^{0,*}(X) \otimes \mathfrak{g} \to \Omega^{1,*}(X) \otimes \mathfrak{g}$ by the operator $\partial \otimes 1$.

3. The factorization algebra

Given any cocycle $\theta \in C^*_{\mathrm{loc}}(\mathfrak{g}^X)$ of degree one we define a factorization algebra on X.

Definition 3.1. Let θ be a local cocycle of $\mathfrak{g}^X$ of cohomological degree one. Define $\mathrm{KM}^X_{\mathfrak{g},\theta}$ to be the factorization algebra on X that assigns to an open set $U \subset X$ the cochain complex $C^{\mathrm{Lie},\theta}_* \left(\mathfrak{g}^X(U) \right)$. In other words, $\mathrm{KM}^X_{\mathfrak{g},\theta}$ is the twisted factorization envelope $U^{\mathrm{fact}}_\theta(\mathfrak{g}^X)$.

Explicitly, on an open set $U \subset X$, the cochain complex $\mathrm{KM}^X_{\mathfrak{g},\theta}(U)$ has as its underlying graded vector space

$$\mathrm{Sym}\left(\mathfrak{g}^X_c(U)[1] \oplus \mathbb{C} \cdot K \right)$$

and the differential is given by $\bar{\partial} + d_{\mathfrak{g}} + \theta$ where $d_{\mathfrak{g}}$ is the extension of the Chevalley-Eilenberg differential for $\mathfrak{g}$ to the Dolbeault complex, and where θ is extended to the full symmetric algebra by the rule that it is a (graded) derivation.

Example 3.2. As an example, using the map J of Proposition 2.2, we can construct a factorization algebra on X for any invariant polynomial $P \in \mathrm{Sym}^{d+1}(\mathfrak{g}^\vee)^\mathfrak{g}$. Since j is injective, we obtain a unique factorization algebra for every such polynomial, hence it makes sense to denote $\mathrm{KM}^X_{\mathfrak{g},P} := \mathrm{KM}^X_{\mathfrak{g},j(P)}$.

3.1. **For an arbitrary principal bundle.** There is a local Lie algebra related to $\mathfrak{g}^X$ associated to any principal G bundle. Formally speaking, one can understand $\mathfrak{g}^X$, or rather its global sections $\mathfrak{g}^X(X)$, as being the dg Lie algebra describing the formal neighborhood of the *trivial* G-bundle inside the derived moduli stack of G-bundle on X. Indeed, if triv denotes the trivial bundle then one has

$$\widehat{\mathrm{triv}} = B\mathfrak{g}^X(X)$$

where the hat denotes formal completion. In other words, the (-1)-shifted tangent space of the moduli stack of G-bundles is identified with the dg Lie algebra $\mathfrak{g}^X(X)$. At an arbitrary principal G bundle P, the dg Lie algebra describing the formal completion $\widehat{P}$ is also the global sections of a local Lie algebra that we now.

Let $\mathrm{ad}(P)$ denote the bundle of Lie algebras on X associated to P. We define the local Lie algebra by

$$\mathfrak{g}^{P \to X} := \Omega^{0,*}(X; \mathrm{ad}(P)),$$

i.e. the $(0, *)$-forms on X with coefficients in the bundle $\mathrm{ad}(P)$. The Lie bracket on $\mathrm{ad}(P)$ together with the Dolbeault operator $\bar{\partial}$ define the structure of the local Lie algebra. The global sections of this local Lie algebra describe the formal completion of P in the moduli of G bundles: $\widehat{P} = B\mathfrak{g}^{P \to X}(X)$.

3.2. **A variant on the construction.** The definition of the following flavor of factorization algebras have appeared in Section 3.6 of [**?**], but we wish to further analyze them here. As in the cases above, we work on a complex d-fold X and consider the local Lie algebra $\mathfrak{g}^X = \Omega^{0,*}(X; \mathfrak{g})$. The variant we discuss in this section involves a different (-1)-shifted central extension of this local Lie algebra. In this section, we fix an invariant pairing $\langle -, - \rangle$ on the Lie algebra $\mathfrak{g}$.

Fix a closed $(d-1, d-1)$-form $\omega \in \Omega^{d-1,d-1}(X)$. Define the quadratic functional on $\mathfrak{g}^X$ by

$$\phi_\omega(\alpha, \beta) = \int_X \omega \wedge \langle \alpha, \partial \beta \rangle.$$

Lemma 3.3. *The functional ϕ_ω is a local cocycle of degree one in $C^*_{loc}(\mathfrak{g}^X)$.*

Proof. Clearly ϕ_ω is local and degree one. The differential on $C^*_{loc}(\mathfrak{g}^X)$ is of the form $\bar\partial + d_{\mathfrak{g}}$ where $d_{\mathfrak{g}}$ is the Chevalley-Eilenberg differential on $\mathfrak{g}$ extended to $(0, *)$-forms. Since the pairing is invariant one has $d_{\mathfrak{g}}(\phi_\omega) = 0$. Finally, to see that it is a cocycle we note that

$$\int_X d_{dR}(\omega \wedge \langle \alpha, \partial \beta \rangle) = \int_X \omega \wedge \langle \bar\partial \alpha, \partial \beta \rangle \pm \int_X \omega \wedge \langle \alpha, \bar\partial \partial \beta \rangle$$

using the fact that ω is closed and $\omega \wedge \langle \alpha, \partial \beta \rangle$ is ∂-closed. $\qquad\square$

Definition 3.4. Let X be a complex d-fold and $\omega \in \Omega^{d-1,d-1}(X)$ a closed form. Define the factorization algebra $U^{fact}_\omega(\mathfrak{g}^X)$ on X as the twisted factorization envelope of $\mathfrak{g}^X$ twisted by the cocycle ϕ_ω.

Example 3.5. Suppose that X is a Kähler d-fold and let $\omega \in \Omega^{1,1}(X)$ be the Kähler form. We can then take the $(d-1, d-1)$-form above to be the $(d-1)$st power of the Kähler form ω^{d-1}. We will refer to the factorization algebra

$$\mathrm{KKM}^X := U^{fact}_{\omega^{d-1}}(\mathfrak{g}^X)$$

as the *Kähler-Kac-Moody* factorization algebra on X. In the case that $d = 2$, the factorization algebra is related to the four-dimensional generalization of the Wess-Zumino-Witten model studied by Nair and Schiff in [**?**] and later by Nekrasov et. al. in [**?, ?**]. We will return to this example later to describe its local operators as a consequence of its factorization algebra structure and to give an interpretation of it as a boundary of a certain Chern-Simons–like gauge theory.

4. RELATION TO THE ORDINARY KAC-MOODY ON RIEMANN SURFACES

In this section we pause to discuss a direct relationship of the higher dimensional Kac-Moody factorization algebras discussed above to the familiar Kac-Moody vertex algebras which are defined on one-dimensional complex manifolds.

Throughout this section we fix a Riemann surface Σ and consider a holomorphic family of complex $(d-1)$-folds over it. That is, we have a holomorphic fibration $\pi : X \to \Sigma$ whose fibers $\pi^{-1}(x)$, $x \in \Sigma$ are $(d-1)$-dimensional. For a fixed Lie algebra $\mathfrak{g}$ we put the higher dimensional Kac-Moody on X and consider its pushforward along π to get some factorization algebra on Σ. We will see how this pushforward is related to the one-dimensional Kac-Moody factorization (and vertex) algebra on Σ.

4.1. **The one-dimensional Kac-Moody.**

4.1.1. The affine algebra $\hat{\mathfrak{g}}$ of a Lie algebra $\mathfrak{g}$ together with a invariant pairing $\langle -, - \rangle_{\mathfrak{g}}$ is defined as a Lie algebra central extension of the loop algebra $L\mathfrak{g} = \mathfrak{g}[t, t^{-1}]$ defined by the cocycle $(f, g) \mapsto \mathrm{Res}_0(f \partial g)$. There is a slight generalization of this construction defined for any dg Lie algebra $(\mathfrak{g}, d_{\mathfrak{g}})$. We take as the input data a $\mathfrak{g}$-invariant pairing $\langle -, - \rangle_{\mathfrak{g}}$ that is closed for the differential $d_{\mathfrak{g}}$. This means that for any $X, Y \in \mathfrak{g}$ we have $\langle d_{\mathfrak{g}} X, Y \rangle + (-1)^{|X|} \langle X, d_{\mathfrak{g}} Y \rangle = 0$ where $|X|$ is the cohomological degree of X in $\mathfrak{g}$. Equivalently, $\langle -, - \rangle$...

The loop algebra of a dg Lie algebra $L\mathfrak{g} = \mathfrak{g}[t, t^{-1}]$ is still defined and from the $d_{\mathfrak{g}}$-closed invariant pairing we get a 2-cochain on $L\mathfrak{g}$ defined by the same formula as in the ordinary case.

The fact that it is a cocycle comes from being closed for both the differential $d_{\mathfrak{g}}$ and the Chevalley-Eilenberg differential for $L\mathfrak{g}$ (by invariance). Thus, we obtain a dg Lie algebra central extension $\widehat{\mathfrak{g}}$ of $L\mathfrak{g}$.

From the affine algebra associated to $\mathfrak{g}$ one builds the Kac-Moody vertex algebra by inducing the trivial module for $\widehat{\mathfrak{g}}$ up via the subalgebra of positive loops $L_+\mathfrak{g} \subset L\mathfrak{g}$. It is immediate that the same construction carries over for the case of a dg Lie algebra. One obtains, in this way, a *dg vertex algebra*. That is, a vertex algebra in the category of cochain complexes. We denote the level κ vacuum Kac-Moody dg vertex algebra obtained in this way by $\widehat{\mathfrak{g}}_\kappa$.

4.2. **Level zero.**

4.3.

Corollary 4.1. *Fix a Lie algebra with invariant pairing $\langle -, - \rangle$. Let Σ be an arbitrary Riemann surface and $d > 1$. Consider the volume form $\omega \in \Omega^{d-1,d-1}(\mathbb{P}^{d-1})$. Then, the pushforward of the factorization algebra $\mathrm{KM}_\omega^{\Sigma \times \mathbb{P}^{d-1}}$ along the projection $\pi : \Sigma \times \mathbb{P}^{d-1} \to \Sigma$ is quasi-isomorphic to the ordinary Kac-Moody factorization algebra of central charge $\mathrm{vol}(\omega)$*

$$\pi_* \mathrm{KM}_\omega^{\Sigma \times \mathbb{P}^{d-1}} \simeq \mathrm{KM}_{\mathrm{vol}(\omega)}^\Sigma.$$

APPENDIX A. L_∞ STRUCTURES

BW: this may be an unnecessary section. Want to stress that KHF do not write down an explicit L_∞-model but it will often be convenient for us to use one.

Suppose V is a dg vector space. Then, the symmetric algebra

$$\mathrm{Sym}(V) := \prod_k \mathrm{Sym}^k(V)$$

has the natural structure of a dg cocommutative coalgebra.

Definition A.1. An L_∞ *algebra* is a dg vector space V together with a coderivation

$$D : \mathrm{Sym}(V) \to \mathrm{Sym}(V).$$

A *morphism* of L_∞ algebras $f : (V, D) \to (V', D')$ is a morphism of dg cocommutative coalgebras

$$f : (\mathrm{Sym}(V), D) \to (\mathrm{Sym}(V'), D').$$

Denote the category of L_∞ algebras by $L_\infty\mathrm{Alg}$.

The complex $(\mathrm{Sym}(V), D)$ is the complex of Chevalley-Eilenberg chains of the L_∞ algebra $\mathfrak{g} = (V, D)$. In the case of a dg Lie algebra this is the usual complex of Chevalley-Eilenberg chains. Without loss of generality we denote this complex by $\mathrm{C}^{\mathrm{Lie}}(\mathfrak{g})$ just as in the classical case.

We may a remark about dg Lie algebras and their close relatives, L_∞ algebras.

Theorem A.2. *BW: Kriz and May?* *Every L_∞ algebra (V, D) is quasi-isomorphic (in the category $L_\infty\mathrm{Alg}$) to a dg Lie algebra.*

By an L_∞ algebra model for a dg Lie algebra $\mathfrak{g}$, we mean an L_∞ algebra (L, D) together with a quasi-isomorphism $(L, D) \simeq \mathfrak{g}$.

A.1. **Extensions from cocycles.** Suppose $\mathfrak{g}$ is a dg Lie algebra. Let $\theta \in C^*_{\text{Lie}}(\mathfrak{g})$ be a cocycle of degree 2, so its cohomology class is an element $[\theta] \in H^2_{\text{Lie}}(\mathfrak{g})$. By BW: ref, we know that θ determines a central extension in the category of dg Lie algebras:

$$0 \to \mathbb{C} \cdot K \to \widehat{\mathfrak{g}} \to \mathfrak{g} \to 0$$

that only depends, up to isomorphism, on the cohomology class of θ.

The explicit dg Lie algebra structure on $\widehat{\mathfrak{g}}$ may be tricky to describe. However, if we are willing to work in the category of L_∞ algebras, there is an explicit model for $\mathfrak{g}$ as an L_∞ algebra. The underlying dg vector space for the L_∞ algebra is the same as that of the dg Lie algebra, $\widehat{\mathfrak{g}} \oplus \mathbb{C} \cdot K$. To equip this with an L_∞ structure we need to provide a coderivation $D = D_1 + D_2 + \cdots$ for the cocommutative coalgebra $\text{Sym}(\mathfrak{g} \oplus \mathbb{C} \cdot K) = \prod_k \text{Sym}^k(\mathfrak{g} \oplus \mathbb{C} \cdot K)$. Indeed, we define

$$
\begin{aligned}
D_1(X_1) &= d_{\mathfrak{g}}(X_1) + \theta(X_1) \\
D_2(X_1, X_2) &= [X_1, X_2]_{\mathfrak{g}} + \theta(X_1, X_2) \\
D_k(X_1, \ldots, X_k) &= \theta(X_1, \ldots, X_k) \ , \quad \text{for } k \geq 3.
\end{aligned}
$$

One immediately checks that $(\mathfrak{g} \oplus \mathbb{C}, D)$ is an L_∞ model for $\widehat{\mathfrak{g}}$.

Example A.3. As an example, consider the following L_∞ model for the dg Lie algebra $\widehat{\mathfrak{g}}_{d,\theta}$. As a dg vector space $\widehat{\mathfrak{g}}_{d,\theta}$ is of the form $A_d \otimes \mathfrak{g} \oplus \mathbb{C} \cdot K$. The only nonzero components of the coderivation determining the L_∞ structure are D_1, D_2, and D_{d+1} and they are determined by $D_1(aX) = (\bar{\partial}a)X$, $D_2(aX, bY) = (a \wedge b)[X, Y]_{\mathfrak{g}}$, and

$$D_{d+1}(a_0 X_0, \ldots, a_d X_d) = \underset{z=0}{\text{Res}}\, (a_0 \wedge \partial a_1 \wedge \cdots \wedge \partial a_d)\, \theta(X_0, \ldots, X_d) \cdot K.$$

Lemma A.4. *Suppose $\mathfrak{g}$ is an L_∞ algebra and we are given two central extensions*

$$0 \to \mathbb{C} \cdot K[k] \to \widetilde{\mathfrak{g}}, \widetilde{\mathfrak{g}}' \to \mathfrak{g} \to 0$$

of L_∞ algebras by the trivial module placed in degree $-k$. Suppose that the cocycles determining the central extensions differ by an exact cocycle of the form $d\eta \in C^*_{\text{Lie}}(\mathfrak{g})$ where η is a cochain of degree $k+1$. Then, the map

$$\text{id} + \eta \cdot K : C^{\text{Lie}}_*(\widetilde{\mathfrak{g}}) \to C^{\text{Lie}}_*(\widetilde{\mathfrak{g}}')$$

determines an L_∞-isomorphism $\widetilde{\mathfrak{g}} \cong \widetilde{\mathfrak{g}}'$.

In the lemma above the map $\text{id} + \eta$ sends the element $X_1 \cdots X_n \in \text{Sym}^n(\mathfrak{g})$ to $X_1 \cdots X_n + \eta(X_1, \ldots, X_n) \cdot K$ and is the identity on the subspace generated by the central element K.

A.2. **Homotopy Poisson structures.**