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Kähler–Chern–Simons theory and symmetries of anti-self-dual gauge fields *

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Kähler–Chern–Simons theory, which was proposed as a generalization of ordinary Chern–Simons theory, is explored in more detail. The theory describes anti-self-dual instantons on a four-dimensional Kähler manifold. The phase space is the space of gauge potentials, whose symplectic reduction by the constraints of anti-self-duality leads to the moduli space of instantons. We show that infinitesimal Bäcklund transformations, previously related to "hidden symmetries" of instantons, are canonical transformations generated by the anti-self-duality constraints. The quantum wave functions naturally lead to a generalized Wess–Zumino–Witten action, which in turn has associated chiral current algebras. The dimensional reduction of the anti-self-duality equations leading to integrable two-dimensional theories is briefly discussed in this framework.

1. Introduction

Recently [1] we proposed a five-dimensional theory, referred to as Kähler–Chern–Simons theory (KCST), as a generalization to a (4 + 1)-dimensional setting of many features of three-dimensional Chern–Simons theory (3d CST) [2]. In this paper, we give a more elaborate presentation of the theory as well as several new results. The three-dimensional Chern–Simons theory, it is by now well known, gives an intrinsically three-dimensional quantum field theoretic interpretation of the Jones polynomials for links; the polynomials are essentially the correlation functions of the Wilson loop operators for the links. Three-dimensional CST is also very closely related to two-dimensional conformal field theory. In particular, 3d CST on $\Sigma \times \mathrm{R}$, where Σ is a Riemann surface, is an exactly solvable theory with a finite-dimensional Hilbert space which can be identified as the space of chiral blocks of a rational conformal field theory. The relevant chiral algebra is the current algebra of a Wess–Zumino–Witten (WZW) model, a Kac–Moody algebra, defined on Σ [3]. The construction of link invariants highlights the topological nature of 3d CST. The current algebraic aspects are, however, related to the fact

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that the reduced phase space is the space of flat gauge potentials on Σ modulo gauge transformations. The Narasimhan–Seshadri theorem [4] shows that, for gauge group SU(N), this space is also the moduli space of stable, rank-N holomorphic vector bundles of Chern class zero over Σ; holomorphic gauge transformations and chiral algebras are naturally defined in this case. It is as a generalization of the current algebraic features that we introduced KCST.

The four-dimensional manifold in KCST, the analogue of the Riemann surface Σ in 3d CST, is a Kähler manifold. The analogue of the Narasimhan–Seshadri theorem is Donaldson's theorem [5] which relates the moduli space of holomorphic vector bundles to instanton moduli spaces. The equations of motion of KCST are thus, not surprisingly, the anti-self-duality conditions in four dimensions and the classical solutions are anti-self-dual instantons. Instantons or anti-self-dual (ASD) gauge fields have also been of interest recently for different but related reasons; this is in connection with integrable systems. There are two connections between integrable systems and conformal field theories. First, integrable theories describe a class of perturbations of conformal field theories away from criticality [6]. Second, the Poisson bracket algebras associated with certain integrable systems are classical analogues of the Virasoro algebra and the W_N algebras (the chiral algebras for conformal field theories with higher-spin operators) [7]. The connection of ASD gauge theories with integrable systems is that ASD gauge theories are conjectured [8] to provide a unified description of all two-dimensional integrable systems. Systematic derivations of integrable systems by gauge and dimensional reduction of ASD gauge theories have given strong support to this conjecture [9,10]. Virasoro and W_N symmetries emerge as the residual gauge symmetry of the ansätze for gauge and dimensional reduction. By virtue of this idea that ASD gauge theories are "master" integrable systems, and the known connection of integrable systems and conformal field theory, it is natural to look at ASD gauge theories for an extension of conformal field theoretic ideas to four dimensions. Quite apart from this, there are many hints that a four-dimensional Kähler manifold may be the natural setting for the study of W_N algebras [11]; in particular, the most appropriate description, from a geometric point of view, of W_∞ gravity in two dimensions may be in terms of ASD gravity on a four-dimensional Kähler manifold [12]. Thus it seems appropriate that we find a lagrangian formulation of ASD gauge theory as a theory defined on a Kähler manifold, despite the fact that the notion of anti-self-duality exists on any riemannian manifold.

From all the above we have ample motivation for the study of ASD gauge fields, especially in a lagrangian and symplectic framework. We shall see that some of the symmetries of ASD gauge theories, known at the level of equations of motion, especially Bäcklund transformations, are realized as canonical transformations within KCST.

This paper is organized as follows. In sect. 2, we discuss the action and equations of motion of our theory. The (reduced) phase space is identified as the moduli space of ASD instantons, obtained by reduction of the space of gauge potentials on the Kähler manifold by the conditions of anti-self-duality. We discuss briefly the hamiltonian version of the theory, and identify the symplectic form on the space of gauge potentials defined by the theory. In sect. 3 we discuss some of the properties of this symplectic form, specifically its gauge invariance, that it is an instance of the Donaldson μ-map [13], and its evaluation for specific instanton moduli spaces. We then look at the Poisson bracket algebra of our theory, and particularly the algebra of the anti-self-duality constraints. Classical and quantum symplectic reductions are discussed in sect. 4. We show that infinitesimal Bäcklund transformations are generated as canonical transformations by the anti-self-duality constraints. At the quantum level, the wave functions naturally involve a generalized form of the WZW action with an accompanying Polyakov–Wiegmann-type factorization property. We examine briefly the chiral algebra associated with this action. In sect. 5, we specialize to the case of $\mathbb{R}^4$ as our Kähler manifold; we show that the infinitesimal transformations found in sect. 4 give the known "hidden symmetries" of the $\mathbb{R}^4$ ASD equations. We discuss reduction of the ASD equations to integrable systems such as KdV, putting known results into a unified framework, exploiting a critical insight from our theory. In sect. 6, in addition to some concluding discussion, we mention higher-dimensional analogues of our theory.

2. Action, equations of motion and phase space

We define Kähler–Chern–Simons theory on a space-time of form $\mathrm{M}^4 \times \mathbb{R}$, where M^4 is a four- (real) dimensional Kähler manifold, with a Kähler form denoted by ω. The action is taken to be

$$S = \int_{\mathrm{M}^4 \times \mathbb{R}} \left[-\frac{k}{4\pi} \mathrm{Tr}\big(A \wedge \mathrm{d}A + \tfrac{2}{3} A \wedge A \wedge A\big) \wedge \omega + \mathrm{Tr}\big((\Phi + \bar{\Phi}) \wedge \mathscr{F}\big) \right]. \quad (2.1)$$

A is the gauge potential; it is a locally defined one-form on $\mathrm{M}^4 \times \mathbb{R}$, with values in the Lie algebra of the gauge group G. We take G to be a compact semi-simple Lie group; when needed we use a basis $\{T^i\}$ for the Lie algebra of G, with $\mathrm{Tr}(T^i T^j) = -\frac{1}{2}\delta^{ij}$. $\mathscr{F}$ denotes the field strength, $\mathscr{F} = \mathrm{d}A + A \wedge A$; we use F for the "magnetic field", i.e. that part of $\mathscr{F}$ which is a two-form on M^4. Φ and $\bar{\Phi}$ are, respectively, locally defined, Lie algebra valued (2, 0) and (0, 2) forms on M^4, which are also one-forms on $\mathbb{R}$. Thus, if z^a, $a = 1, 2$, denote local complex coordinates on M^4, and t denotes the coordinate on $\mathbb{R}$, then we can write locally

$$\begin{aligned} \Phi &= \phi \wedge \mathrm{d}t, \qquad \phi = \tfrac{1}{2} \sum_{a,b=1,2} \phi_{ab}\, \mathrm{d}z^a \wedge \mathrm{d}z^b, \qquad \phi_{ab} = -\phi_{ba}, \\ \bar{\Phi} &= \bar{\phi} \wedge \mathrm{d}t, \qquad \bar{\phi} = \tfrac{1}{2} \sum_{\bar{a},\bar{b}=1,2} \bar{\phi}_{\bar{a}\bar{b}}\, \mathrm{d}\bar{z}^{\bar{a}} \wedge \mathrm{d}\bar{z}^{\bar{b}}, \qquad \bar{\phi}_{\bar{a}\bar{b}} = -\bar{\phi}_{\bar{b}\bar{a}}. \end{aligned} \quad (2.2)$$

The behavior of the fields in the theory under gauge transformations is given by

$$A \to A^u = uAu^{-1} - \mathrm{d}u\, u^{-1},$$
$$\Phi \to \Phi^u = u\Phi u^{-1}, \qquad \bar{\Phi} \to \bar{\Phi}^u = u\bar{\Phi}u^{-1}, \tag{2.3}$$

where u is a locally defined G-valued function on $\mathrm{M}^4 \times \mathbb{R}$. By virtue of the fact that ω is closed, the action (2.1) is invariant under gauge transformations on $\mathrm{M}^4 \times \mathbb{R}$ which are homotopic to the identity; in the case where M^4 is non-compact or has a boundary, it may be necessary to impose the vanishing of F at "infinity" or on the boundary (of course, in such a case, to completely define the theory it is necessary to give some boundary conditions, and we require that these should be compatible with gauge invariance of S). Invariance of e^{iS} under homotopically nontrivial gauge transformations can lead to quantization of k; we will discuss this quantization from another point of view later in the paper.

In local complex coordinates we write $\omega = \frac{1}{2}ig_{a\bar{a}}\, \mathrm{d}z^a \wedge \mathrm{d}\bar{z}^{\bar{a}}$, and the Kähler metric is given by $\mathrm{d}s^2 = g_{a\bar{a}}\, \mathrm{d}z^a\, \mathrm{d}\bar{z}^{\bar{a}}$. We remind the reader that the components of the Kähler form can be derived from a Kähler potential K, via $\omega = i\,\partial\,\bar{\partial}K$. On a Kähler four-manifold, the notions of self-dual and anti-self dual two-forms become especially simple: a two-form is anti-self-dual if it has no (2, 0) or (0, 2) part and its (1, 1) part is perpendicular to ω (i.e. it vanishes upon taking the wedge product of it with ω).

The equations of motion of the theory from varying Φ, $\bar{\Phi}$, A_t are

$$F^{(2,0)} = F^{(0,2)} = 0, \qquad F \wedge \omega = 0, \tag{2.4a,b}$$

and express the fact that for all t the two-form F on M^4 is anti-self-dual, i.e. that A_a, $A_{\bar{a}}$ is an instanton potential. The equations of motion from varying the spatial components of A yield, in the gauge $A_t = 0$,

$$\frac{k}{4\pi}\frac{\partial A_a}{\partial t} = -ig^{\bar{a}b}\bar{\nabla}_{\bar{a}}\phi_{ba}, \qquad \frac{k}{4\pi}\frac{\partial A_{\bar{a}}}{\partial t} = +ig^{\bar{b}a}\nabla_a\bar{\phi}_{\bar{b}\bar{a}}. \tag{2.4c,d}$$

Here $g^{\bar{a}a}$ is defined by $g_{b\bar{a}}g^{\bar{a}a} = \delta^a_b$, and we have denoted by ∇ and $\bar{\nabla}$ the gauge-covariant versions of ∂ and $\bar{\partial}$, respectively, so, for instance $\bar{\nabla}_{\bar{a}}\phi_{ba} = \partial_{\bar{a}}\phi_{ba} + [A_{\bar{a}}, \phi_{ba}]$. Since ∇, $\bar{\nabla}$ appear in the above equations acting, respectively, on (0, 2) and (2, 0) forms, we could replace both of them by the gauge-covariant version of the exterior derivative $\mathrm{d} = \partial + \bar{\partial}$. Furthermore, since on a Kähler manifold the Christoffel symbols $\Gamma^a_{b\bar{c}}$ and $\Gamma^{\bar{a}}_{\bar{b}c}$ vanish we can consider ∇, $\bar{\nabla}$ to be covariant with respect to both gauge and coordinate transformations. We note that we can write eqs. (2.4c, d) in coordinate independent form as, for instance

$$\frac{k}{2\pi}\frac{\partial A^{(1,0)}}{\partial t} = -i * \bar{\nabla}\phi, \qquad \frac{k}{2\pi}\frac{\partial A^{(0,1)}}{\partial t} = +i * \nabla\bar{\phi}, \tag{2.4e,f}$$

where $*$ denotes the usual Hodge star operator. Eqs. (2.4c,d) give us the time evolution of the gauge potentials, or, rather, since the gauge potential is an instanton potential, and eqs. (2.4c,d) are gauge invariant under t-independent gauge transformations (only these since we have fixed $A_t = 0$ gauge), eqs. (2.4c,d) give us the time evolution of the moduli of the instanton potential A_a, $A_{\bar{a}}$. We clearly need to check that the time evolution (2.4c,d) keeps us in the space of instanton potentials, and this will give us constraints on ϕ, $\bar{\phi}$. We will do this shortly.

First we make some other observations relevant to the action (2.1). We have not considered so far the equation of motion that could be obtained by varying the Kähler form ω, or, rather, varying the Kähler potential K. This is in line with our comments in sect. 1 that our theory should be viewed as being defined in a background Kähler metric. The equation of motion obtained by variation of K is satisfied automatically if the time derivatives of A_a, $A_{\bar{a}}$ are zero, which, as we shall see shortly, is certainly the case when M^4 is compact without boundary. Thus, although we use a metric explicitly, many of the properties of the theory will be independent of the metric, i.e. "topological" within the class of Kähler metrics.

The case where M^4 is hyper Kähler merits some special comments. We then have, in addition to the Kähler form ω, a closed (2, 0) form ω^+ and a closed (0, 2) form ω^-, related to each other by complex conjugation. Writing $\omega^3 = \omega$, $\omega^1 = \frac{1}{2}(\omega^+ + \omega^-)$, $\omega^2 = (1/2i)(\omega^+ - \omega^-)$, eqs. (2.4a,b) may be written

$$F \wedge \omega^i = 0, \quad i = 1, 2, 3. \tag{2.5}$$

These, along with the equations $\partial A_a/\partial t = \partial A_{\bar{a}}/\partial t = 0$, can be derived by variation of the imaginary quaternionic action

$$S' = -\frac{k}{4\pi}\int_{\mathrm{M}^4\times\mathbb{R}} \mathrm{Tr}\big(A \wedge \mathrm{d}A + \tfrac{2}{3}A \wedge A \wedge A\big) \wedge \omega^i e_i, \tag{2.6}$$

where $e_\mu = (1, e_i)$ are a basis of quaternions. In this action we regard ω^i as fixed. For the case $\mathrm{M}^4 = \mathbb{R}^4$, the action (2.6) has the advantage over the action (2.1) that it does not require one to pick a specific Kähler structure (which breaks SO(4) invariance). But imposing hyper Kähler structure may, in general, be too restrictive. We note that we can also consider the action (2.6) for a riemannian manifold M^4 which is endowed with quaternion Kähler structure [14]; it would then be necessary to use quaternionic quantization techniques [15].

We return now to consider the question of when the time evolution (2.4c,d) keeps us within the space of instanton potentials. Computing the time derivative of eq. (2.4b) using eqs. (2.4c,d) we find no constraint on ϕ or $\bar{\phi}$. But taking the time derivative of $F^{(2,0)} = 0$, we find that ϕ must satisfy

$$\nabla\big(*\bar{\nabla}\phi\big) = 0. \tag{2.7}$$

For M^4 compact, with no boundary, this implies straightaway that $\bar{\nabla}\phi = 0$, since in this case we can write

$$\int_{M^4} \mathrm{Tr}\big(\bar{\phi} \wedge \nabla(* \bar{\nabla}\phi)\big) = \int_{M^4} \mathrm{Tr}\Big(\big(\overline{\bar{\nabla}\phi}\big) \wedge *\big(\bar{\nabla}\phi\big)\Big) \equiv - \| \bar{\nabla}\phi \|^2. \tag{2.8}$$

where $\|\ \ \|$ denotes the standard norm. Thus for M^4 compact with no boundary, the time evolution is trivial and the equations of motion reduce to

$$F^{(2,0)} = F^{(0,2)} = 0, \quad F \wedge \omega = 0, \quad \frac{\partial A_a}{\partial t} = \frac{\partial A_{\bar{a}}}{\partial t} = 0. \tag{2.9}$$

In fact it is possible to say a little more than this; for the case where M^4 is compact, without boundary, and has positive scalar curvature it has been shown by Itoh [16], using a Bochner-type argument, that the dimension of the solution space of (2.7) is zero. This result is of importance in computing the dimension of the instanton moduli spaces for such manifolds, using the index of the twisted Dolbeault complex. The complex dimension of the moduli space (if it is not empty) of q-instantons on such a manifold is [16] $4q - \frac{1}{4}\dim \mathrm{G}(\chi + \tau)$, where χ is the Euler number and τ is the signature of the manifold; furthermore the moduli spaces have Kähler structure [17]. These moduli spaces are exactly the phase spaces for our theory (at least for M^4 compact, without boundary, and with positive scalar curvature).

For more general M^4 one can have nontrivial solutions to eq. (2.7). We shall see below that changes in $A^{(1,0)}$ of form $* \bar{\nabla}\phi$ are generated canonically by the action of $F^{(0,2)}$ (and similarly the action of $F^{(2,0)}$ generates changes in $A^{(0,1)}$ of the form $* \nabla\bar{\phi}$). The equation $F^{(0,2)} = 0$ shows that such transformations may be regarded as a different type of gauge symmetry. The time evolution of A_a and $A_{\bar{a}}$ is the flow along such gauge directions, and is still therefore trivial in a larger sense.

Consider now the hamiltonian version of the theory, treating the coordinate t as time. The action (2.1) immediately gives the following first-class constraints on the canonical momenta:

$$\pi_{A_t} = 0, \qquad \pi_{\phi} = 0, \qquad \pi_{\bar{\phi}} = 0. \tag{2.10}$$

We can eliminate these constraints by choosing the gauge-fixing constraints

$$A_t = 0, \qquad \phi = 0, \qquad \bar{\phi} = 0, \tag{2.11}$$

but we have to futher impose the equations

$$\frac{\partial \pi_{A_t}}{\partial t} = 0, \qquad \frac{\partial \pi_{\phi}}{\partial t} = 0, \qquad \frac{\partial \pi_{\bar{\phi}}}{\partial t} = 0 \tag{2.12}$$

as constraints, which give eqs. (2.4a, b). Time evolution is once again trivial.

One more ingredient is necessary to complete the picture of the classical phase space of the theory. The action (2.1) determines naturally a symplectic two-form on the space $\mathscr{A}$ of all gauge potentials on M^4,

$$\Omega = \frac{k}{4\pi}\int_{\mathrm{M}^4}\mathrm{Tr}(\delta A \wedge \delta A)\wedge\omega. \tag{2.13}$$

Here δ denotes the exterior derivative on $\mathscr{A}$, and we have suppressed the wedge product between forms on $\mathscr{A}$. So our theory essentially describes the symplectic reduction of the space $\mathscr{A}$, endowed with the symplectic form Ω of (2.13), by $F^{(2,0)}$, $F^{(0,2)}$ and $F\wedge\omega$. $F\wedge\omega$ is, with respect to the symplectic form Ω, the generator of usual gauge transformations, as we shall see later, so (2.4b) is the "Gauss law" of the theory.

3. Properties of the symplectic form and the algebra of constraints

Ω, being non-degenerate, cannot be gauge invariant on the full space $\mathscr{A}$, but the general theory of symplectic reductions [18] dictates that there should be gauge invariance on the subset of $\mathscr{A}$ where the moment map arising from gauge transformations vanishes, which is the set of potentials such that $F\wedge\omega = 0$. It is interesting to see this gauge invariance emerge explicitly, without appealing to the general theory of symplectic reductions. Let us take the case where M^4 is compact and without boundary first. The Lie algebra valued one-form A is only defined locally on M^4, so to compute (2.13) we need to introduce a set of patches on M^4, and to sum the contributions to (2.13) from a set of patches that exactly cover M^4. Explicitly, let $\{\mathrm{B}_p\}$ be a (sufficiently large) collection of closed sets that cover M^4, with $\mathrm{B}_p \cap \mathrm{B}_q = \partial\mathrm{B}_p \cap \partial\mathrm{B}_q \equiv \partial\mathrm{B}_{pq}$. Let A_p represent the gauge potential on B_p; we are given a set of G-valued transition functions h_{pq}, one for each pair p, q such that $p \neq q$ and $\partial\mathrm{B}_{pq} \neq 0$ (these satisfy the usual relations). On $\partial\mathrm{B}_{pq}$ we have $A_p = h_{pq}A_q h_{pq}^{-1} - \mathrm{d}h_{pq}\, h_{pq}^{-1}$; gauge transformations act via $A_p \to g_p A_p g_p^{-1} - \mathrm{d}g_p\, g_p^{-1}$, where g_p is a Lie group valued function on B_p and on $\partial\mathrm{B}_{pq}$ we have $g_p = h_{pq}g_q h_{pq}^{-1}$ *. Under gauge transformations we have $\delta A_p \to g_p(\delta A_p + D_{A_p}(g_p^{-1}\delta g_p))g_p^{-1}$, where D_{A_p} denotes the gauge-covariant derivative. With all this

* Gauge transformations are correctly defined as fiber-preserving automorphisms of the underlying principal bundle. The gauge potentials A_p are obtained from the bundle connection in terms of local trivializations. Gauge transformations can then be described in terms of local G-valued functions g_p with the prescribed overlap relation $g_p = h_{pq}g_q h_{pq}^{-1}$.

one finds, after integration by parts and a few other simple manipulations,

$$\sum_p \int_{B_p} \mathrm{Tr}(\delta A_p \wedge \delta A_p) \wedge \omega$$

$$\to \sum_p \int_{B_p} \mathrm{Tr}(\delta A_p \wedge \delta A_p) \wedge \omega + 2\delta\left(\sum_p \int_{B_p} \mathrm{Tr}\Big((g_p^{-1}\delta g_p) F(A_p) \Big) \wedge \omega \right)$$

$$+ \sum_p \int_{\partial B_p} \mathrm{Tr}\Big(2\delta A_p (g_p^{-1}\delta g_p) + (g_p^{-1}\delta g_p) D_{A_p}(g_p^{-1}\delta g_p) \Big) \wedge \omega. \tag{3.1}$$

On the subspace of $\mathscr{A}$ where $F \wedge \omega$ vanishes, the second term in this expression is clearly zero. For the third term there are two contributions from ∂B_{pq}, which is contained in both ∂B_p and ∂B_q, and these can be shown to cancel; thus this term is also zero. This establishes gauge invariance. For non-compact manifolds, or manifolds with boundary, it clearly is necessary to impose certain boundary conditions to avoid contributions to the third term in (3.1).

Despite the fact that Ω is not gauge invariant on the whole space $\mathscr{A}$, it does define a (nontrivial) element of the second cohomology $H^2(\mathscr{A}/\mathscr{G})$, where $\mathscr{G}$ is the group of gauge transformations. This follows from the fact that if we define

$$\Omega^1(A, g) = \int \mathrm{Tr}\big((g^{-1}\delta g) F(A) \big) \wedge \omega \tag{3.2}$$

(Ω^1 is the one-form on $\mathscr{A}$ appearing in the second term in (3.1) above, and is in some sense the obstruction to gauge invariance), then $\Omega^1(A, g)$ obeys a cocycle condition,

$$\Omega^1(A, hg) = \Omega^1(A, g) + g\Omega^1(A^g, h). \tag{3.3}$$

Furthermore, the cohomology class of Ω in $H^2(\mathscr{A}/\mathscr{G})$ depends only on the cohomology class of ω in $H^2(M^4)$. This is easily seen if we write Ω as

$$\Omega = \frac{k}{4\pi} \int \mathrm{Tr}(\tilde{F} \wedge \tilde{F}) \wedge \omega \tag{3.4a}$$

where

$$\tilde{F} = (\mathrm{d} + \delta) A + A \wedge A. \tag{3.4b}$$

Then we have, for $\omega \to \omega + \mathrm{d}\alpha$,

$$\Omega \to \Omega + \delta\left[\frac{k}{4\pi} \int \mathrm{Tr}(\tilde{F} \wedge \tilde{F}) \wedge \alpha \right]. \tag{3.5}$$

Since $\int \mathrm{Tr}(\tilde{F} \wedge \tilde{F}) \wedge \alpha$ is a one-form on $\mathscr{A}/\mathscr{G}$, it follows that the cohomology class of Ω is not changed. Actually, Ω is an example of the Donaldson map from $\mathrm{H}^2(\mathrm{M}^4)$ to $\mathrm{H}^2(\mathscr{A}/\mathscr{G})$. (This is the cohomology version of the Donaldson map, as described in ref. [13]. We have an extra factor $2\pi k$, since our Ω is derived from an action, which is measured in units of 2π.)

With this understanding we can now discuss the quantization of k, for the case where M^4 is compact and without boundary. If k is an integer, then, assuming that ω represents an integer cohomology class of M^4, the requirement that k is an integer is exactly the requirement that $\Omega/2\pi$ represents an integer cohomology class of $\mathscr{A}/\mathscr{G}$. (Being Kähler, M^4 has nontrivial homology two-cycles; this implies that $\pi_1(\mathscr{G})$ is nonzero, which implies that there are nontrivial homology two-cycles in $\mathscr{A}/\mathscr{G}$.) The relevance of this to the quantum theory, is that in a geometric quantization of our theory we construct a line bundle, called the pre-quantum line bundle, on the phase space with curvature Ω; sections of this line bundle satisfying a certain polarization condition are the wave functions. The existence of the pre-quantum line bundle requires that the integral of $\Omega/2\pi$ over any nontrivial homology two-cycle in the phase space should be an integer [19], i.e. $\Omega/2\pi$ must belong to an integral cohomology class, which, as stated above, means k must be an integer. Let us see this explicitly in a specific example, say $\mathrm{M}^4 = \mathrm{S}^2 \times \mathrm{S}^2$. We consider a homology two-cycle in $\mathscr{A}/\mathscr{G}$, parametrized by σ, τ, $0 \leqslant \sigma, \tau \leqslant 1$, given by

$$A(x^1, x^2, \sigma, \tau) = -\tau \, \mathrm{d}g \, g^{-1}, \tag{3.6}$$

where x^1, x^2 are coordinates of one of the component S^2's of M^4, and $g(x^1, x^2, \sigma)$, with $g(x^1, x^2, 0) = g(x^1, x^2, 1) = 1$, is a nontrivial element of $\pi_3(\mathrm{G}) = \mathbb{Z}$. Integration of Ω over this two-cycle gives

$$\int \Omega = 2\pi k Q[g] \int_{\mathrm{S}^2} \omega, \tag{3.7}$$

where $Q[g]$ is the winding number of g, i.e.

$$Q[g] = \frac{1}{24\pi^2} \int_{\mathrm{S}^3} \mathrm{Tr}(\mathrm{d}g \, g^{-1})^3. \tag{3.8}$$

Since $Q[g]$ and $\int \omega$ are integers, k must be an integer. This example we have given here is, since we essentially ignore the second component S^2 in M^4 in construction of the nontrivial two-cycle in $\mathscr{A}/\mathscr{G}$, a simple extension of an argument that can be given for the quantization of k in three-dimensional Chern–Simons theory; for arbitrary M^4 we use the nontrivial homology two-cycles in place of the component S^2's of the example. For non-compact M^4 there is no quantization of k.

Having now established various properties of Ω, we consider briefly the evaluation of Ω on specific instanton moduli spaces. For instanton solutions we have

$F^{(0,2)} = 0$, allowing us to write locally $A^{(0,1)} = -\bar{\partial}U\, U^{-1}$, where U is $\mathrm{G}^{\mathbb{C}}$-valued. It follows that on some patch B we can write $A = UA'U^{-1} - \mathrm{d}U\, U^{-1}$, where A' is a $\mathrm{G}^{\mathbb{C}}$ Lie algebra valued (1, 0) form, $A' = (U^{\dagger}U)^{-1}\, \partial(U^{\dagger}U)$. We have $\delta A = U(\delta A' - D'_A(U^{-1}\ \delta U))U^{-1}$; comparing with (3.1) and using the fact that if A is an instanton potential we must have $F(A') \wedge \omega = 0$, we deduce that the contribution to Ω from the patch B is simply given by

$$\Omega_{\mathrm{B}} = \int_{\partial \mathrm{B}} \mathrm{Tr}\big(2\delta A'(U^{-1}\delta U) + (U^{-1}\delta U) D_{A'}(U^{-1}\delta U)\big) \wedge \omega. \tag{3.9}$$

We see at once that Ω can be calculated on an instanton moduli space by summing contributions from surfaces of patches. Note that there, unlike in the calculation leading to gauge invariance above, the contributions to Ω from boundaries of neighboring patches do not cancel in general; adopting the patching notation from earlier on in this section, it is easy to see that on $\partial \mathrm{B}_{pq}$ we have $U_p = h_{pq} U_q g_{pq}$, where the g_{pq} are some holomorphic $\mathrm{G}^{\mathbb{C}}$ matrices, depending on the moduli of the instanton potential A. There is some freedom in choosing the matrices g_{pq}, arising from the freedom in choice of the matrices U_p, but it is insufficient freedom to set them all to the identity (if we could set them all to the identity the sum of all the contributions of type (3.9) would vanish). More precisely, the matrices g_{pq} define a holomorphic vector bundle on M^4, and the freedom we have in the choice of g_{pq} means that a specific instanton solution determines exactly an isomorphism class of holomorphic vector bundles. This is one part of Donaldson's theorem [5] that states that there is an isomorphism between moduli of (irreducible) SU(N) instanton potentials on M^4 and moduli of (stable) holomorphic rank-N vector bundles on M^4.

We now turn to an examination of the Poisson bracket algebra of our theory. The canonical Poisson brackets following from (2.13) for the components of A (defined by writing $A = (A^i_a\, \mathrm{d}z^a + A^i_{\bar a}\, \mathrm{d}\bar{z}^{\bar a})T^i$) are

$$\big[A^i_a(x),\, A^j_{\bar a}(y)\big] = \frac{2\pi}{ik} g_{a\bar a} \delta^{ij} \frac{\delta^{(4)}(x-y)}{\det(g)}. \tag{3.10}$$

Here $\det(g) = \det(g_{a\bar a})$. The basic structure of the theory is the symplectic reduction of $\mathscr{A}$ by the constraints $F^{(2,0)}$, $F^{(0,2)}$, $F \wedge \omega$, so central to the quantization procedure is the algebra of these functions. We introduce the following generators:

$$\begin{aligned}
E(\bar\varphi) &= -\frac{k}{2\pi}\int \mathrm{Tr}(\bar\varphi \wedge F) = \frac{k}{2\pi}\int \mathrm{d}V\, g^{\bar a a} g^{\bar b b} \bar\varphi^i_{\bar a \bar b} F^i_{ab},\\
\bar E(\varphi) &= -\frac{k}{2\pi}\int \mathrm{Tr}(\varphi \wedge F) = \frac{k}{2\pi}\int \mathrm{d}V\, g^{\bar a a} g^{\bar b b} \varphi^i_{ab} F^i_{\bar a \bar b},\\
G(\theta) &= \frac{-k}{2\pi}\int \mathrm{Tr}(\theta\omega \wedge F) = \frac{-ik}{2\pi}\int \mathrm{d}V\, g^{\bar a a} \theta^i F^i_{a\bar a},
\end{aligned} \tag{3.11}$$

where θ, φ, $\bar{\varphi}$ are, respectively, Lie algebra valued (0, 0), (2, 0) and (0, 2) forms (which essentially serve as parameters for the transformations generated by $F \wedge \omega$, $F^{(0,2)}$, $F^{(2,0)}$); the components of θ, φ, $\bar{\varphi}$ are defined by

$$\theta = \sum \theta^i T^i,$$

$$\varphi = \tfrac{1}{2} \sum \varphi^i_{ab} T^i \, \mathrm{d}z^a \wedge \mathrm{d}z^b, \qquad \varphi^i_{ab} = -\varphi^i_{ba}, \tag{3.12}$$

$$\bar{\varphi} = \tfrac{1}{2} \sum \bar{\varphi}^i_{\bar{a}\bar{b}} T^i \, \mathrm{d}\bar{z}^{\bar{a}} \wedge \mathrm{d}\bar{z}^{\bar{b}}, \qquad \varphi^i_{\bar{a}\bar{b}} = -\bar{\varphi}^i_{\bar{a}\bar{b}}.$$

In eq. (3.11) the volume element is given by $\mathrm{d}V = \frac{1}{4}\mathrm{d}^2 z \; \mathrm{d}^2 \bar{z} \; \det(g)$. We find the following Poisson brackets of these generators with the components of A:

$$\begin{aligned}
&\left[G(\theta),\, A^i_a(x)\right] = -(\nabla\theta)^i_a(x), && \left[G(\theta),\, A^i_{\bar{a}}(x)\right] = -(\bar{\nabla}\theta)^i_{\bar{a}}(x),\\
&\left[E(\bar{\varphi}),\, A^i_a(x)\right] = 0, && \left[E(\bar{\varphi}),\, A^i_{\bar{a}}(x)\right] = i(*\nabla\bar{\varphi})^i_{\bar{a}}(x),\\
&\left[\bar{E}(\varphi),\, A^i_a(x)\right] = -i(*\bar{\nabla}\varphi)^i_a(x), && \left[\bar{E}(\varphi),\, A^i_{\bar{a}}(x)\right] = 0,
\end{aligned} \tag{3.13}$$

These tell us the facts that we have cited previously, that $F \wedge \omega$ is the generator, with respect to the symplectic form (2.13) on $\mathscr{A}$, of gauge transformations, and $F^{(0,2)}$, $F^{(2,0)}$ are, respectively, the generators of transformations of the form

$$A^{(1,0)} \to A^{(1,0)} - i * \bar{\nabla}\varphi, \qquad A^{(0,1)} \to A^{(0,1)} + i * \nabla\bar{\varphi}. \qquad (3.14), (3.15)$$

The symplectic form (2.13) on $\mathscr{A}$ is Lie-invariant with respect to flows on $\mathscr{A}$ generated by both these transformations and gauge transformations. The three constraints of our theory are moment maps corresponding to these three sets of transformations. This result generalizes both the observation of Donaldson [5] that $F \wedge \omega$ is the moment map corresponding to gauge transformations, and also the result [20] that the anti-self-dual equations on $\mathbb{R}^4$ (or for that matter any hyper Kähler manifold) can be obtained via a hyper Kähler reduction.

It remains to write down the algebra of the generators $E(\bar{\varphi})$, $\bar{E}(\varphi)$, $G(\theta)$. We obtain

$$\left[G(\theta),\, G(\theta')\right] = G(\theta \times \theta'), \tag{3.16a}$$

$$\left[G(\theta),\, E(\bar{\varphi})\right] = E(\theta \times \bar{\varphi}), \tag{3.16b}$$

$$\left[G(\theta),\, \bar{E}(\varphi)\right] = \bar{E}(\theta \times \varphi), \tag{3.16c}$$

$$\left[\bar{E}(\varphi),\, E(\bar{\varphi})\right] = \frac{ik}{2\pi} \int \mathrm{Tr}(\bar{\varphi} \wedge \nabla * \bar{\nabla}\varphi). \tag{3.16d}$$

Here $\theta \times \theta' = f^{ijk}\theta^j\theta'^k T^i$, etc. $E(\bar{\varphi})$ and $E(\bar{\varphi}')$, and $\bar{E}(\varphi)$ and $\bar{E}(\varphi')$ evidently commute. We note that on compact M^4 without boundary, we can rewrite the right-hand side of (3.16d) using (2.8). The algebra (3.16) is clearly of central importance in our theory. We should mention that while we obtained the symplectic structure (2.13) from our action (2.1), it is also that natural symplectic form to consider on $\mathscr{A}$ from the point of view of the connection of the moduli space of instantons with the moduli space of holomorphic vector bundles via Donaldson's theorem mentioned above. We thus, independently from the point of view of Kähler–Chern–Simons theory, expect the algebra (3.10), (3.13) and (3.16) to play a significant role in the quantization of instantons, i.e. the construction of line bundles over the moduli spaces of holomorphic vector bundles on M^4.

In the quantum theory, A, E, $\bar{E}$, G become operators, and the Poisson bracket relations (3.10), (3.13) and (3.16) are replaced by commutators. Notice that the right-hand side of eq. (3.16d) contains both the holomorphic and anti-holomorphic components of A, and hence one has to take care with operator ordering in eq. (3.16d).

4. Symplectic reductions, classical and quantum

The $G(\theta)$, $\bar{E}(\varphi)$ operators have a closed Poisson bracket algebra, given by (3.16a, c). We attempt a two-stage (classical) symplectic reduction of $\mathscr{A}$, first setting $\bar{E}(\varphi)$ (or equivalently $F^{(0,2)}$) to zero, and then setting $G(\theta)$ (or equivalently $F \wedge \omega$) to zero. The $\bar{E}(\varphi) = 0$ subspace of $\mathscr{A}$ consists of potentials locally of the form

$$(A_a, A_{\bar{a}}) = (A_a, -\partial_{\bar{a}} U U^{-1}) \tag{4.1}$$

where U is $G^{\mathbb{C}}$-valued. The flow on this subspace generated by $\bar{E}(\varphi)$ is given by

$$U \to U, \qquad A_a \to A'_a = A_a - i(*\bar{\nabla}\varphi)_a. \tag{4.2}$$

We need a gauge-fixing condition that will restrict us to the orbit space of the flow (4.2). We note that under an infinitesimal change of the form (4.2) we have

$$F^{(2,0)} \to F^{(2,0)} - i\nabla * \bar{\nabla}\varphi. \tag{4.3}$$

Thus it follows that provided there are no solutions to eq. (2.7), the condition $F^{(2,0)} = 0$ (or equivalently $E(\bar{\varphi}) = 0$) will be a good gauge-fixing condition. Consistently with this, we note that if there are no solutions of (2.7), then the right-hand side of eq. (3.16d) can be regarded as an invertible inner product on the space of Lie algebra valued (2, 0) forms φ; the invertibility of this inner product is exactly the criterion for $E(\bar{\varphi}) = 0$ to be a good gauge fixing for flows generated by $\bar{E}(\varphi)$.

We are of course, only considering infinitesimal flows at this stage. Note that, even if there are no solutions to eq. (2.7), the gauge fixing $F^{(2,0)}=0$ could in principle suffer from a Gribov ambiguity. The solution to $F^{(2,0)}=0$ satisfying the appropriate reality conditions is

$$A_a=(U^\dagger)^{-1}\,\partial_a U^\dagger. \tag{4.4}$$

The phase space after reduction by $\bar{E}(\varphi)$ is thus given by the space of U's, U being a locally defined $G^{\mathbb{C}}$-valued function. Gauge transformations act on this space via $U\to gU$.

Continuing with discussion of the case where eq. (2.7) has no solutions, let us proceed to the second stage reduction, by setting $G(\theta)$ to zero and dividing out by gauge transformations. The condition $G(\theta)=0$ is easily seen to be

$$g^{\bar{a}a}\,\partial_{\bar{a}}\big(J^{-1}\,\partial_a J\big)=0, \tag{4.5}$$

where $J=U^\dagger U$. J is gauge invariant. Eq. (4.5) gives the instanton equations in the so-called J-formulation. The reduced phase space now (i.e. the solutions of (4.5)) is the moduli space of instantons.

Consider now the case of manifolds where there are nontrivial solutions to eq. (2.7), and impose $\bar{E}(\varphi)=0$. Imposing $E(\bar{\varphi})=0$ will not be a good gauge fixing for the flow generated by the $\bar{E}$'s. More specifically, if A_a solves $E=0$ then so will A'_a defined by (4.2) provided $\nabla * \bar{\nabla}\varphi=0$. Another way to view this is as follows: if $\nabla * \bar{\nabla}\varphi=0$, then locally we can write $*\bar{\nabla}\varphi=\nabla\sigma$ for some Lie algebra valued function σ, and it is straightforward to check that for such φ we have

$$\left[\bar{E}(\varphi),\,E(\bar{\varphi})\right]=-iE(\sigma\times\bar{\varphi}). \tag{4.6}$$

So the flow generated by $\bar{E}(\varphi)$ for these φ leaves the condition $E=0$ invariant, i.e. in the space $\bar{E}=0$, some of the flows generated by $\bar{E}$ are tangential to the subspace $E=0$; in other words, whereas in the compact case $E=0$ was a good gauge fixing for the symmetries generated by $\bar{E}$ on the space $\bar{E}=0$, in the non-compact case there is an infinitesimal Gribov ambiguity.

Now instanton solutions are defined by $E=\bar{E}=0$ and $G=0$. From what we have said above, for the first two conditions we have two solutions, both with $A_{\bar{a}}=-\partial_{\bar{a}}UU^{-1}$. If we take the first to be the "real" solution, i.e. $A_a=(U^\dagger)^{-1}\,\partial_a(U^\dagger)$, then the second is given by

$$\begin{aligned}A'_a&=(U^\dagger)^{-1}\,\partial_a(U^\dagger)-i(*\bar{\nabla}\varphi)_a\\&=(U^\dagger)^{-1}\,\partial_a(U^\dagger)-i(\nabla\sigma)_a\\&\equiv(V^\dagger)^{-1}\,\partial_a(V^\dagger),\end{aligned} \tag{4.7}$$

where $V^\dagger = U^\dagger \mathrm{e}^{-i\sigma}$ (φ and σ are infinitesimal). If $U^\dagger$, U solves $F \wedge \omega = 0$, then so will $V^\dagger$, U. This $\bar{E}$-flow on the $\bar{E} = 0$ subspace may be regarded as generating new instanton solutions from old ones via solutions of $\nabla * \bar{\nabla}\varphi = 0$. This will be made more precise for the case of $\mathbb{R}^4$ in sect. 5. The new solution will clearly not satisfy reality conditions; but we can also consider the analogous E-flow on the space $E = 0$, which will use solutions of $\bar{\nabla} * \nabla\bar{\varphi} = 0$ to generate new instanton solutions. Taking the right combination of these flows we can generate new real solutions from old real solutions. Here we shall just write eq. (4.7) in a more useful form. The quantities relevant for the condition $G = 0$ are $J = U^\dagger U$ and $J' = V^\dagger U$. We find

$$(J^{-1}\,\partial J) - (J'^{-1}\,\partial J') = iU^{-1}(\nabla\sigma)U. \tag{4.8}$$

Note that eq. (4.8) implies that if J satisfies eq. (4.5) so does J' (use the identity $\bar{\partial}(U^{-1}\alpha U) = U^{-1}(\bar{\nabla}\alpha)U$). Thus (4.8) is an infinitesimal Bäcklund transformation for (4.5).

We now turn to the subject of quantum reductions. We can consider quantization of our theory in two ways. We can quantize the whole space $\mathscr{A}$, endowed with the symplectic form Ω, and then impose the constraints (2.4) by restricting to the set of wave functions annihilated by the appropriate operators. Alternatively, we can directly quantize the reduced phase space, i.e. the subset of $\mathscr{A}$ defined by eq. (2.4). We shall consider the first method, and at the end of this section make some brief comments on the second method. For quantizing the space $\mathscr{A}$, a natural choice of polarization for the wave functions is the holomorphic polarization, i.e. the wave functions are functionals of $A^i_{\bar{a}}$ with the action of A^i_a given by

$$A^i_a\Psi(A^i_{\bar{a}}) = \frac{2\pi}{k} g_{a\bar{a}} \frac{\delta\Psi}{\delta A^i_{\bar{a}}}. \tag{4.9}$$

The scalar product is given by

$$\langle\Psi_1 | \Psi_2\rangle = \int \mathrm{e}^{-\tilde{K}}\ \Psi_1^* \Psi_2\ \mathrm{d}\mu(A), \tag{4.10}$$

where $\mathrm{d}\mu(A)$ is the Liouville measure on $\mathscr{A}$ given by Ω, and $\tilde{K}$ is the Kähler potential for Ω, i.e.

$$\tilde{K} = \frac{k}{2\pi}\int \mathrm{d}V\, g^{\bar{a}a} A^i_a A^i_{\bar{a}}. \tag{4.11}$$

We perform a first stage of reduction by requiring

$$\bar{E}(\varphi)\Psi(A^i_{\bar{a}}) = 0. \tag{4.12}$$

This implies that the wave functions have support only on configurations for which $F^{(0,2)}=0$. For the simplest case, this means $A_{\bar{a}} = -\partial_{\bar{a}} U U^{-1}$ for some *global* G-valued function U on M^4 (in this case, of course, the instanton number is zero). We can now consider Ψ's to be functionals of U. The scalar product becomes

$$\langle \Psi_1 | \Psi_2 \rangle = \int [\mathrm{d}U]\, \mathrm{e}^{-\tilde{K}(U^\dagger, U)}\, \Psi_1^*(U) \Psi_2(U), \tag{4.13}$$

where $[\mathrm{d}U]$ is defined as the product over M^4 of the Haar measure on $\mathrm{G}^{\mathbb{C}}$. The second stage of reduction is now performed by imposing gauge invariance, i.e.

$$G(\theta) \Psi(U) = 0. \tag{4.14}$$

Using the definition of $G(\theta)$ from eqs. (3.11) and (4.9), we see that this is equivalent to

$$\Psi(\mathrm{e}^{\theta} U) = \exp\left\{ -\frac{k}{\pi} \int \mathrm{d}V g^{\bar{a}a}\ \mathrm{Tr}\big(\partial_a \theta\, \partial_{\bar{a}} U U^{-1}\big) \right\} \Psi(U) \tag{4.15}$$

for infinitesimal θ. The solution to this is given by $\Psi = \mathrm{e}^{\mathscr{S}(U)}$, where

$$\begin{aligned}\mathscr{S}(U) = {} & \frac{k}{2\pi} \int_{\mathrm{M}^4} \mathrm{d}V g^{\bar{a}a}\ \mathrm{Tr}\big(\partial_a U\, \partial_{\bar{a}} U^{-1}\big) \\ & + \frac{ik}{12\pi} \int_{\mathrm{M}^5} \mathrm{Tr}\big((U^{-1}\,\mathrm{d}U) \wedge (U^{-1}\,\mathrm{d}U) \wedge (U^{-1}\,\mathrm{d}U)\big) \wedge \omega. \end{aligned} \tag{4.16}$$

This is an analogue of the Wess–Zumino–Witten (WZW) action which appears in the wave functions for three-dimensional Chern–Simons theory. In this expression M^5 is taken to be $\mathrm{M}^4 \times [0, 1]$; we identify one boundary component of M^5 (say $\lambda = 1$, where λ is the coordinate on [0, 1]) with our space M^4, and extend U into M^5 in such a way that it tends to some fixed function U_0 on the other component of the boundary ($\lambda = 0$). Depending on what M^4 is, the set of $\mathrm{G}^{\mathbb{C}}$-valued functions on M^4, i.e. the set of U's, may fall into distinct homotopy classes. Then we would need to specify a set of fixed functions U_0 on $\lambda = 0$, one in each homotype class, in order to define $\mathscr{S}(U)$, and we define the extension of U into M^5 so that it tends to the appropriate U_0 on $\lambda = 0$. Note that eq. (4.15) only gives the behavior of the Ψ's under homotopically trivial gauge transformations. The behavior of Ψ's under homotopically non-trivial transformations can involve additional phase factors (the same for all states), in a way analogous to the θ-vacua of QCD (see e.g. ref. [21]).

Before we go on to look at (4.16) as an action, we note that in a more general case than the one we have considered, when $F^{(0,2)} = 0$ cannot be solved in terms of a globally defined function U, but rather we have $A_{\bar{a}} = -\partial_{\bar{a}} U U^{-1}$ where U is only locally defined, we still expect a factor of the form $\mathrm{e}^{\mathscr{S}(U)}$ in the wave functions, where $\mathscr{S}$ is now some refined notion of the functional (4.16) (to construct this we could exploit e.g. ref. [22], where it is shown how to construct the usual two-dimensional WZW functional for a Riemann surface with boundary). In the general case, the gauge invariance condition will not determine a unique wave function either; the residual freedom has to do with the degrees of freedom of the reduced phase space or moduli. The quantization of the latter will complete the identification of the wave functions. We will not consider these issues here (apart from some comments at the end of this section), but content ourselves with some discussion of the functional (4.16) as an action.

Eq. (4.16) clearly can be used as an action for a field theory on any Kähler manifold M^4. The action satisfies a Polyakov–Wiegmann-type formula,

$$\mathscr{S}(U_1U_2) = \mathscr{S}(U_1) + \mathscr{S}(U_2) + \Gamma(U_1, U_2), \tag{4.17a}$$

$$\Gamma(U_1, U_2) = -\frac{k}{\pi}\int_{M^4} \mathrm{d}V g^{\bar{a}a}\ \mathrm{Tr}\big(U_1^{-1}(\partial_a U_1)(\partial_{\bar{a}} U_2)U_2^{-1}\big). \tag{4.17b}$$

By use of this formula we see that the normalization integral in eq. (4.13) will involve $\mathrm{e}^{\mathscr{S}(J)}$. From eq. (4.17) it is clear that the variational equation for $\mathscr{S}(J)$ is eq. (4.5). Thus we may think of $\mathscr{S}(J)$ as an action for anti-self-dual gauge theory in the J-formulation *, something that has been sought in the past [23]. Note that if we choose any parametrization for the group G, the action (4.16) can be expressed as an integral over M^4 [24]; so for instance we can obtain Pohlmeyer's action [25] for Yang's equations [26].

Eq. (4.17) shows that the transformations $U \to hU$, $U \to U\tilde{h}$, where h is anti-holomorphic and $\tilde{h}$ is holomorphic, are gauge symmetries of the action (4.16). (Obviously such symmetries also exist for $\mathscr{S}(J)$; this has been noticed before at the level of the equations of motion.) In the case of the usual WZW action, these are, of course, the Kac–Moody symmetries. One can then consider a hamiltonian quantization using the holomorphic coordinate z as the time coordinate [24]. An analogous quantization can be carried out in our case, using, say, z_1 as the time variable (of course, this procedure is only meaningful for four-manifolds for which one can define a time coordinate consistently, e.g. $M^4 = \Sigma \times \mathbb{R}^2$, where Σ is a Riemann surface). The symplectic two-form, on the space $\mathscr{G}^{(3)}$ of G-valued functions of z_2, $\bar{z}_1$, $\bar{z}_2$, given by (4.17) is then

$$\Omega^{(3)} = \frac{k}{4\pi}\int \mathrm{d}^2\bar{z}\ \mathrm{d}z_2\ \det(g)g^{\bar{a}1}\ \mathrm{Tr}\big(\xi\ \partial_{\bar{a}}\xi + 2\xi\xi(\partial_{\bar{a}}U)U^{-1}\big), \tag{4.18}$$

* J in eq. (4.5), of course, is not globally defined, but on the intersection of patches we have $J_p = g_{pq}^{\dagger}J_q g_{pq}$ where g_{pq} is holomorphic. From eq. (4.17b) we see that $\mathscr{S}(U)$ is unaffected by multiplication on the left(right) by anti-holomorphic(holomorphic) matrices. Thus $\mathscr{S}(J)$ can be defined for all instanton numbers.

where $\xi = \delta U U^{-1}$ is a one-form on $\mathscr{G}^{(3)}$, and we suppress the wedge product between forms on $\mathscr{G}^{(3)}$. The basic current density of interest is

$$I = \frac{k}{4\pi} \det(g) g^{\bar{a}1} (\partial_{\bar{a}} U) U^{-1}. \tag{4.19}$$

The Poisson bracket algebra of these quantities is determined from (4.18) to be

$$\begin{aligned} &\left[I^i(z_2, \bar{z}_1, \bar{z}_2), I^j(z_2', \bar{z}_1', \bar{z}_2') \right] \\ &= f^{ijk} I^k(z_2, \bar{z}_1, \bar{z}_2) \delta^{(3)}(z - z') - \frac{k}{4\pi} \det(g) g^{\bar{a}1}\, \partial_{\bar{a}} \delta^{(3)}(z - z') \delta^{ij}. \end{aligned} \tag{4.20}$$

The anti-holomorphic symmetries $U \to hU$ are generated by $Q(\bar{z}_1, \bar{z}_2) = \int \mathrm{d}z_2\, I$. The algebra of Q's is obtained from (4.20) as

$$\left[Q^i(\bar{z}), Q^j(\bar{z}') \right] = f^{ijk} Q^k(\bar{z}) \delta^{(2)}(\bar{z} - \bar{z}') - \frac{k}{4\pi} \delta^{ij} C(\bar{z}, \bar{z}'), \tag{4.21a}$$

$$C(\bar{z}, \bar{z}') = \int \mathrm{d}z_2 \det(g) g^{\bar{a}1}\, \partial_{\bar{a}} \delta^{(2)}(\bar{z} - \bar{z}'). \tag{4.21b}$$

This algebra is obviously similar to the Kac–Moody algebra. However it is very limited in its utility in solving the theory defined by (4.16); this is because, unlike its two-dimensional analogue, the solution space of eq. (4.5) is not given just by some finite-dimensional space of solutions, up to multiplication on the left and right by anti-holomorphic and holomorphic matrices, respectively.

Finally in this section, we note that the quantization of the reduced phase space can be carried out in a relatively straightforward way by calculating Ω on a specific instanton moduli space. The Hilbert space will be characterized by k and by q, the instanton number. As mentioned in sect. 2, the complex dimension of the moduli space (on a suitable manifold) is $4q - \frac{1}{4}\dim \mathrm{G}(\chi + \tau)$; furthermore the moduli space has finite volume, for compact M^4. Thus the number of states (dimension of the Hilbert space) will be finite. For non-compact M^4 this is not the case.

It would be interesting to provide a specific example of the reduced phase space quantization, but as far as we know, no one has succeeded in writing down all anti-self-dual instantons for any compact Kähler manifold, for any value of q for which instantons are known to exist, even for $\mathrm{G} = \mathrm{SU}(2)$, which is the only case we will consider in this paragraph. For the case of CP^2, where all self-dual one-instantons are known [27], there are no anti-self-dual one-instantons [28], and the complex dimension of the anti-self-dual q-instanton moduli space for $q \geqslant 2$ is $4q - 3$. By virtue of Donaldson's theorem relating moduli spaces of instantons and moduli spaces of holomorphic vector bundles, we can identify at least the topology of the two-instanton moduli space, which must be isomorphic [17] to the complement in CP^5 of the hypersurface

$$z_0 z_1 z_2 + 2 z_3 z_4 z_5 - z_0 z_5^2 - z_2 z_3^2 - z_1 z_4^2 = 0. \tag{4.22}$$

Here $z_0, \dots, z_5$ are homogeneous coordinates for CP^5. For the case of the four-torus $(\mathrm{S}^1)^4$, some anti-self-dual one-instantons are known [29], and the complex dimension of the q-instanton moduli space is $4q$. The one-instanton moduli space is conjectured to be isomorphic to the product of a four-torus and a K3 surface [30]; if this is true, finding all one-instantons on the torus would provide us with a Kähler structure on a K3 surface. The closest we can get, for now, to an explicit calculation, is to consider the evaluation of Ω for the subset of solutions of the $\mathbb{R}^4$ anti-self-dual equations that are S^4 q-instantons. The real dimensions of these moduli spaces are well known to be $8q-3$, which is odd, so Ω is degenerate (which is not in contradiction with anything we have said). The evaluation for one-instantons can be found in ref. [1]; since we have shown above that Ω can be calculated as a sum of surface terms, we suspect it is possible to compute Ω for $q>1$ explicitly, but have not, as of yet, succeeded in doing this.

5. Analysis on $\mathbb{R}^4$

$\mathbb{R}^4$ is an interesting example of M^4, since as mentioned in sect. 1, the study of instantons on $\mathbb{R}^4$ is relevant to the study of integrable systems. We shall look at reductions to integrable systems shortly. Before this, however, we explain how some of the general analysis of KCST can be carried farther in the special case $\mathrm{M}^4 = \mathbb{R}^4$.

Consider the (classical) symplectic reduction of $\mathscr{A}$ by $\bar{E}(\varphi)$ and $G(\theta)$. Eq. (2.7) now has solutions, or, equivalently, we can find Lie algebra valued 0-forms and (2,0) forms σ and φ such that

$$* \bar{\nabla}\varphi = \nabla\sigma \tag{5.1}$$

The flow generated by $\bar{E}(\varphi)$ for such φ will generate new instanton solutions from old. Note that eq. (5.1) implies

$$* \nabla * \bar{\nabla}\varphi = 0, \qquad * \bar{\nabla} * \nabla\sigma = 0. \tag{5.2}$$

Thus both φ and σ satisfy the covariant Laplace equation. On $\mathbb{R}^4$ we have a covariantly constant (2, 0) tensor ϵ_{ab}, and one can write $\varphi_{ab} = f\epsilon_{ab}$; f and σ satisfy the same equation. So we might try choosing f to be a multiple of σ, i.e. we might look for solutions of eq. (5.1) with

$$\varphi_{ab} = \tfrac{1}{2}\lambda\sigma\epsilon_{ab}, \tag{5.3}$$

where λ is a complex constant. (For a general hyper Kähler manifold M^4 we might look for solutions of eq. (5.1) with $\varphi = \lambda\sigma\omega^+$, in the notation of sect. 2.) From the definition of $V^\dagger$ in eq. (4.7) we see that

$$J^{-1}J' = U^{-1}\,e^{-i\sigma}\,U \approx 1 - iU^{-1}\sigma U. \tag{5.4}$$

Using (5.3) and (5.4) we can rewrite (4.8),

$$J^{-1}\,\partial_a J - J'^{-1}\,\partial_a J' = -\lambda g^{\bar{a}b}\epsilon_{ba}\,\partial_{\bar{a}}(J^{-1}J'). \tag{5.5}$$

This is exactly the form of the infinitesimal Bäcklund transformations and the associated infinite-dimensional symmetry of the ASD equations on $\mathbb{R}^4$ (see e.g. ref. [31]). We see these are indeed generated as canonical transformations by $\bar{E}(\varphi)$. One may regard eq. (4.7) as giving the analogue of the Bäcklund transformation on a general Kähler manifold M^4.

Eq. (5.2a) is of course the same as (2.7); it is the variational equation obtained by varying $\|\bar{\nabla}\varphi\|^2$, the functional defined in (2.8). It is clear that the only way we can have a solution to (5.2a) with $\bar{\nabla}\varphi$ non-vanishing, is to have a non-zero surface integral of the form

$$\int_{\partial M^4} \mathrm{Tr}(\bar{\varphi} \wedge *\bar{\nabla}\varphi) \tag{5.6}$$

(here $\bar{\varphi}$ is the hermitian conjugate of φ). The asymptotic behavior of $J^{-1}J'$ on $\mathbb{R}^4$ will thus be nontrivial.

We now turn to the consideration of reductions of the ASD equations on $\mathbb{R}^4$ that give rise to integrable systems. In ref. [9] Mason and Sparling showed that in a certain reduction to two dimensions, the SL(2, $\mathbb{C}$) ASD equations yielded both the KdV and non-linear Schrödinger (NLS) equations, and these were (up to gauge transformations) essentially the only reductions of this kind. In ref. [10], Bakas and Depireux, realised that the Mason–Sparling reduction to KdV could be gauge transformed into a particularly simple form, and by taking an ansatz of this form for larger gauge groups, found many more integrable equations arising as dimensional reductions of the ASD equations. In the context of this paper, it would be appropriate to fully realise these reductions in a symplectic framework, but we do not do this here. We shall, however, reconsider these reductions with a particular insight from our work. In refs. [9,10] it is apparent that the equations $F^{(0,2)} = 0$ and $F \wedge \omega = 0$ play a different role from the equation $F^{(2,0)} = 0$. This distinction is natural, of course, in our symplectic framework, and furthermore, from our viewpoint it is more natural to gauge fix after imposing $F^{(0,2)}=0$ and $F\wedge\omega=0$ (these are moment maps in our presentation). Following this procedure, at least partially, we show that the Bakas–Depireux ansätze are actually *gauge choices*; this increases the significance of their results substantially.

We introduce complex coordinates w, z on $\mathbb{R}^4$. We dimensionally reduce by restricting to potentials that in some gauge are independent of $\bar{w}$. We clearly still have the freedom to do gauge transformations that are independent of $\bar{w}$, under which we have

$$A_{\bar{w}} \to u A_{\bar{w}} u^{-1}, \tag{5.7}$$

where $u(w, z, \bar{z})$ is the gauge transformation matrix. Exploiting this freedom, we can put $A_{\bar{w}}$ into a canonical form. For $\mathrm{G}=\mathrm{SL}(N, \mathbb{C})$, the possible canonical forms are just the possible Jordan normal forms for a traceless $N\times N$ matrix function of the variables w, z, $\bar{z}$. For $\mathrm{SL}(2, \mathbb{C})$ we have two possible forms,

$$A_{\bar{w}}=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} \quad \text{or} \quad \kappa\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \tag{5.8}$$

and for $\mathrm{SL}(3, \mathbb{C})$ we have four possible forms,

$$A_{\bar{w}}=\begin{pmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \quad \begin{pmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}, \quad \begin{pmatrix}\kappa & 0 & 0\\ 1 & \kappa & 0\\ 0 & 0 & -2\kappa\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}\kappa & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & -\kappa-\lambda\end{pmatrix}. \tag{5.9}$$

Here κ, λ are arbitrary functions of w, z, $\bar{z}$. Each canonical form gives rise to a different type of reduction: the two $\mathrm{SL}(2, \mathbb{C})$ choices give KdV- and NLS-type equations, as found in ref. [9]; equations based on the first two $\mathrm{SL}(3, \mathbb{C})$ choices are considered in ref. [10]. Each canonical form also has associated with it a set of residual gauge transformations u, which leave it invariant; for the first $\mathrm{SL}(2, \mathbb{C})$ form we can take

$$u=\begin{pmatrix}1 & 0\\ \gamma & 1\end{pmatrix} \tag{5.10}$$

for some function γ of w, z, $\bar{z}$. Thus for each reduction we will obtain a whole gauge equivalence class of integrable systems, and this is the notion of gauge equivalence described in ref. [32].

To proceed further let us look at a specific example; we will look at the first $\mathrm{SL}(2, \mathbb{C})$ form, i.e. the KdV-type reduction, but it is straightforward to work out any particular example. We parametrize the remaining potentials suitably,

$$A_w=\begin{pmatrix}a & b\\ c & -a\end{pmatrix}, \qquad A_z=\begin{pmatrix}d & e\\ f & -d\end{pmatrix}, \qquad A_{\bar{z}}=\begin{pmatrix}g & h\\ j & -g\end{pmatrix}. \tag{5.11}$$

All the entries in these matrices are functions of w, z, $\bar{z}$. We will *not* exploit the remaining gauge freedom at this juncture. Instead, we impose $F_{\bar{w}\bar{z}} = F_{w\bar{w}} + F_{z\bar{z}} = 0$ in line with our comments above. The potentials then must take the form

$$A_w = \begin{pmatrix} (j_z - f_{\bar{z}} - 2dj)/2 & d_{\bar{z}} - je \\ c & -(j_z - f_{\bar{z}} - 2dj)/2 \end{pmatrix},$$

$$A_z = \begin{pmatrix} d & e \\ f & -d \end{pmatrix}, \quad A_{\bar{z}} = \begin{pmatrix} 0 & 0 \\ j & 0 \end{pmatrix}, \tag{5.12}$$

where $e_{\bar{z}} = 0$. We see we are left with unfixed functions c, d, e, f, j with $e_{\bar{z}} = 0$. e is unchanged by the residual gauge transformations, and it actually emerges that any choice of e will give us an integrable system. The choice $e = 0$ gives a trivial system, so we will look at $e = 1$. Having fixed e, we have four functions left, one of which corresponds to the gauge degree of freedom γ of eq. (5.10), and the remaining three of which will be "almost fixed" by imposing $F_{wz} = 0$ (these equations do not fix e in any way). Under gauge transformation with u given by (5.10), we find

$$\begin{aligned} &c \to c - \gamma_w + \gamma(j_z - f_{\bar{z}} - 2dj) + \gamma^2(j - d_{\bar{z}}), \\ &d \to d - \gamma, \\ &f \to f + 2\gamma d - \gamma^2 - \gamma_z, \\ &j \to j - \gamma_z. \end{aligned} \tag{5.13}$$

We consider three possible gauge choices, $j = 0$ (Mason–Sparling gauge), $d = 0$ (Bakas–Depireux gauge) and $f = 0$ (MKdV gauge). In these three gauges we obtain, respectively, the following equations by imposing $F_{wz} = 0$ (and making suitable choices of integration coefficients):

$$\begin{aligned} (d_z)_w &= \left[\tfrac{1}{4}\partial_z^2 + 2(d_z) + (d_z)_z\, \partial_z^{-1}\right](d_z)_{\bar{z}}, \\ f_w &= \left[\tfrac{1}{4}\partial_z^2 - f - \tfrac{1}{2}f_z\, \partial_z^{-1}\right] f_{\bar{z}}, \\ d_w &= \left[\tfrac{1}{4}\partial_z^2 - d^2 - d_z\, \partial_z^{-1} d\right] d_{\bar{z}}. \end{aligned} \tag{5.14}$$

These are three-dimensional versions [33] of the KdV, KdV and MKdV equations, respectively. Further reduction to two dimensions by imposing $\partial_z = \partial_{\bar{z}}$, as in refs. [9,10], yields the standard equations. For gauge group SL(N, $\mathbb{C}$), for any N, in the case when $A_{\bar{w}}$ is chosen in the canonical form with exactly one non-zero entry, not on the leading diagonal, it is straightforward to define the gauge choice to reproduce the ansätze in ref. [10]. We note that, in addition to clarifying some issues of gauge freedom in reductions of the ASD equations to integrable systems, we also inherit from our work on the ASD equations a full understanding of the hidden symmetries of the integrable systems we obtain (see also ref. [34]).

6. Concluding remarks

The emergence of the anti-self-dual equations in the symplectic reduction of the space of gauge potentials $\mathscr{A}$ by $F^{(0,2)}$ and $F \wedge \omega$ is perhaps the most important feature of Kähler–Chern–Simons theory. The algebra (3.10), (3.13) and (3.16) of the gauge potentials and the generators $F^{(0,2)}$, $F^{(2,0)}$ and $F \wedge \omega$ plays a crucial role in this picture. Within this framework, the previously known "hidden symmetries" of the instanton equations, related to Bäcklund transformations, can be understood as canonical transformations. Our discussion in sect. 5 shows that reduction of $\mathscr{A}$ by $F^{(0,2)}$ and $F \wedge \omega$ is also the most appropriate setting for the gauge and dimensional reductions of ASD gauge fields leading to two-dimensional integrable systems. In this context we expect the study of analogous reductions and subalgebras of (3.10), (3.13) and (3.16) to shed light on how Virasoro and W_N symmetries emerge, and the role they play, in two-dimensional integrable systems.

As an obvious generalization of what we have presented here, we can consider a KCST theory on a Kähler manifold of arbitrary even (real) dimension M^{2d}, $d \geqslant 2$. The natural action to look at is

$$S = \int_{M^{2d} \times \mathbb{R}} \left[-\frac{k}{4\pi} \mathrm{Tr}\big(A \wedge \mathrm{d}A + \tfrac{2}{3} A \wedge A \wedge A\big) \wedge \omega^{d-1} + \mathrm{Tr}\big((\Phi + \overline{\Phi}) \wedge \mathscr{F}\big) \right] \tag{6.1}$$

where now Φ and $\overline{\Phi}$ are, respectively, $(d,\ d-2)$ and $(d-2,\ \mathrm{d})$ forms on M^{2d}, as well as being one-forms on $\mathbb{R}$. The equations of motion (for suitable M^{2d}) are just

$$F^{(2,0)} = F^{(0,2)} = 0, \qquad \omega^{d-1} \wedge F = 0. \tag{6.2a,b}$$

On $\mathbb{R}^{2d}$, these equations were studied, amongst others, as possible candidates for the appropriate higher-dimensional extension of the ASD equations [35]. The main reason we have focused our attention in this paper on the $d = 2$ case, is that for $d > 2$ it seems these equations of motion are not integrable on $\mathbb{R}^{2d}$. They cannot be written as consistency conditions for the integrability of a set of linear equations [36], and in the J-formulation (obtained by solving eq. (6.2a) to write $A_{\bar{a}} = -\partial_{\bar{a}} U U^{-1}$, $A_a = (U^\dagger)^{-1} \partial_a (U^\dagger)$, in which case eq. (6.2b) becomes $g^{\bar{a}a} \partial_{\bar{a}}(J^{-1} \partial_a J) = 0$, where $J = U^\dagger U$), the equation of motion fails the Painlevé test [37]. Nevertheless, the set of solutions to eq. (6.2) on an arbitrary Kähler manifold M^{2d}, might well merit study; we are not aware of any work on the relationship of the moduli spaces of solutions to eq. (6.2) and the moduli spaces of holomorphic vector bundles on M^{2d}. In ref. [9] it is shown that higher-order equations in the KdV hierarchy can be obtained from eq. (6.2) on $\mathbb{R}^{2d}$, for appropriate choices of d. Also we note that much of what we have said in this paper goes through for the action (6.1); particularly, in the procedure of quantization we find the obvious extension to higher dimensions of the WZW functional (4.16).

Certain remarks in the last paragraph also may help clarify the distinction between 3d CST and KCST. The equations (6.2) on $\mathbb{R}^2$ are completely solvable, but on $\mathbb{R}^4$ they are only "integrable". The notion of integrability, for partial differential equations, is a (currently) non-precise notion, which reflects a degree of solvability, falling just short of the notion of complete solvability, which we take to mean the ability to write down explicitly the most general solution. This reinforces to us the possibility that KCST is the appropriate arena to discuss the host of phenomena now known that are generalizations, in one sense or another, of conformal field theory.

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