NEW TITLE

CONTENTS

OG: Obviously we'll write a new intro when we know what we want to accomplish in this paper.

In this chapter we investigate the symmetries that generic holomorphic quantum field theories possess. Our overarching goal is to develop tools for understanding such symmetries that provide a systematic generalization of methods used in chiral conformal field theory on Riemann surfaces, especially for the Kac-Moody and Virasoro vertex algebras [**?**, **?**, **?**]. We will utilize the tools of BV quantization and factorization algebras that have already heavily percolated this thesis. The primordial example of a holomorphic theory we consider is the holomorphic σ-model studied in the previous chapter.

We will focus on two main types of symmetries: holomorphic gauge symmetries and symmetries by holomorphic diffeomorphisms (or holomorphic reparametrizations). An ordinary gauge symmetry is characterized as being local on the spacetime manifold. Each of the types of symmetries we consider share this characteristic, but they also enjoy an additional structure: they are holomorphic (up to homotopy) on the spacetime manifold. This means that they are specific to the type of theories we consider. Moreover, they store more information about the geometry of the underlying manifold as compared to the smooth version of such symmetries.

Infinitesimally speaking, a symmetry is encoded by the action of a Lie algebra. For the holomorphic gauge symmetry this will become a sort of current algebra which is equivalent to holomorphic functions on the complex manifold with values in a Lie algebra. For the holomorphic diffeomorphisms this Lie algebra is that of holomorphic vector fields. Locality implies that this actually extends to a symmetry by a sheafy version of a Lie algebra. The precise sheafy version we mean is called a *local Lie algebra*, which we will recall in the main body of the text. To every local Lie algebra we can assign a factorization algebra through the so-called enveloping factorization algebra:

$$\mathbb{U} : \mathrm{Lie}_X \to \mathrm{Fact}_X .$$

Here, Lie_X is the category of local Lie algebras. By this construction, we see that the Lie algebra of symmetries of a theory define a factorization algebra on the manifold where the theory lives.

One compelling reason for constructing a factorization algebra model for Lie algebras encoding the symmetries of a theory is that it allows one to consider universal versions of such objects. There is a variation of the definition of a factorization algebra that lives, in some sense, on the entire category of manifolds (or complex manifolds). Such a perspective has been developed in great generality by Ayala-Francis in [**?**]. In the case of the symmetry by a current algebra on Riemann surfaces a universal version of the Kac-Moody has been studied in [**?**]. For the case of conformal symmetry our work in [**?**] provides a factorization algebra lift of the ordinary Virasoro vertex algebra that exists uniformly on the site of Riemann surfaces. In this chapter, we extend each of these objects to arbitrary complex dimensions. Our formulation lends itself to an explicit computation of the factorization homology along certain complex manifolds, for which we will focus on a class of examples called *Hopf manifolds*.

Studying such local symmetries involves rich geometric input even at the classical level, but the skeptical mathematician may view this as a repackaging of already familiar objects in complex geometry. The main advantage of working with factorization algebra analogs of such symmetries is in their relationship to studying quantizations of field theories. A similar obstruction deformation theory for studying quantizations of classical field theories also allows us to study the problem of *quantizing* the action of a (local) Lie algebra on a theory. Moreover, we already know that factorization algebras describe the operator product expansion of the observables of a QFT. A formulation of Noether's theorem in Chapter 12 of [**?**] makes the relationship between the associated factorization algebra corresponding to a symmetry and the factorization algebra of observables of the theory.

Of course, quantizing a symmetry of a field theory may not always exist. In fact, this failure sheds light into subtle field theoretic phenomena of the underlying system. For example, in the case of conformal symmetries of a conformal field theory, the failure is exactly measured by the *central charge* of the theory. It is well established that the central charge is a very important invariant associated to a conformal field theory. At the Lie theoretic level, this failure is measured by a cocycle which in turn defines a central extension of the Lie algebra. It is this central extension that acts on the theory.

For this reason, an essential aspect of studying the local symmetries of holomorphic field theories we mentioned above is to characterize the possible cocycles that give rise to central extensions. As we have already mentioned, for vector fields in complex dimension one this is related to the central charge and the central extension of the Witt algebra (vector fields on the circle) known as the Virasoro Lie algebra. In the case of a current algebra associated to a Lie algebra, central extensions are related to the *level* and the corresponding central extensions are called affine algebras.

Theorem 0.1. *The following is true about the local Lie algebras associated to holomorphic diffeomorphisms and holomorphic gauge symmetries.*

(1) *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{g}^X$ is associated current algebra defined on any complex manifold X. There is an embedding of the cohomology $H^*_{Lie}(\mathfrak{g}, \mathrm{Sym}^{d+1} g^\vee[-d-1])$ inside of the local cohomology of $\mathfrak{g}^X$.*
(2) *There is an isomorphism between the local cohomology of holomorphic vector fields on any complex manifold X of dimension d and $H^*_{dR}(\mathrm{X}) \otimes H^*_{GF}(\mathrm{W}_d)[2d]$, where $H^*_{GF}(\mathrm{W}_d)$ is the Gelfand-Fuks cohomology of vector fields on the formal disk.*

The central extensions we are interested in come from classes of degree $+1$ of the above local Lie algebras. In the case of holomorphic vector fields the result above implies that all such extensions are parametrized by $H^{2d+1}(\mathrm{W}_d)$. It is a classical result of Fuks **[?]** that this cohomology is isomorphic to $H^{2d+2}(BU(d))$. In complex dimension one this cohomology is one dimensional corresponding to the class c_1^2. In general, we obtain new classes, which are shown to agree with calculations in the physics literature in dimensions four and six.

In general, any of these cohomology classes define factorization algebras by twisting the enveloping factorization algebra. We especially focus on this construction in the case that the complex d-fold is equal to affine space $\mathbb{C}^d$, or some natural open submanifolds thereof. In the case of the current algebra, our result is compatible with recent work of Kapranov et. al. in **[?]** where they study higher dimensional versions of affine algebras, and their relationship to the (derived) moduli space of G-bundles in an analogous way that affine algebras are related to the moduli of bundles on curves via Kac-Moody uniformization. Our second main result shows how to recover these higher affine algebras from our factorization algebra on punctured affine space $\mathbb{C}^d \setminus \{0\}$, see Theorem 2.1.5.

The extensions of part (1) of Theorem 0.1 are related to cohomology classes in the moduli of G-bundles on complex d-folds. We will show how techniques in equivariant BV quantization lead to natural families of QFTs defined over formal neighborhoods in the moduli space of G-bundles. Our techniques allow us to study quantizations of such families, in particular there are anomalies to quantization. An explicit analysis of Feynman diagrams leads to a computation of certain classes in the local cohomology which we relate to Chern classes of natural line bundles on $\mathrm{Bun}_G(X)$. This leads us to our next main result which is to prove a version of the Grothendieck-Riemann-Roch (GRR) theorem using the aforementioned methods of BV quantization, see Theorem **??**.

1. Current algebras on complex manifolds

This paper takes general definitions and constructions from **[?]** and specializes them to the context of complex manifolds. In this subsection we will review some of the key ideas but refer to **[?]** for foundational results.

OG: In the introduction (or somewhere else TBD), we should explain that while the symmetries of the fields and action functional are encoded by a sheaf of Lie algebras, the associated observables/operators (under a Noether-type relationship) for a (pre)cosheaf. This is a simple consequence of the fact that observables are covariant in spacetime while fields are contravariant.

OG: The following appear later but I think it would fit more naturally here or in the intro

Let us recall the familiar complex one-dimensional case that we wish to extend. Let Σ be a Riemann surface, and let $\mathfrak{g}$ be a simple Lie algebra with Killing form κ. Consider the local Lie algebra $\mathcal{G}_\Sigma = \Omega^{0,*}_c(\Sigma) \otimes \mathfrak{g}$ on Σ. There is a natural cocycle depending precisely on two inputs:

$$\theta(\alpha \otimes M, \beta \otimes N) = \kappa(M, N) \int_\Sigma \alpha \wedge \partial \beta,$$

where $\alpha, \beta \in \Omega_c^{0,*}(\Sigma)$ and $M, N \in \mathfrak{g}$. In Chapter 5 of **[?]** it is shown how the twisted enveloping factorization of $\mathcal{G}_X$ via this cocycle recovers the Kac-Moody vertex algebra and the affine algebra extending $L\mathfrak{g} = \mathfrak{g}[z, z^{-1}]$.

1.1. **Local Lie algebras.** BW: fix these ... The factorization algebras we study in this paper all arise from Lie algebras that are sufficiently local on the manifold in an analogous way that associative algebras arise from Lie algebras via the universal enveloping construction. ... We recall some definitions that we will use throughout the paper. The first concept we introduce is that of a *local Lie algebra*. This is the central object needed to discuss symmetries of field theories that are local on the spacetime manifold.

A key notion for us is a sheaf of Lie algebras on a smooth manifold. These often appear as infinitesimal automorphisms of geometric objects, and hence as symmetries in classical field theories.

Definition 1.1. A *local Lie algebra* on a smooth manifold X is

(i) a $\mathbb{Z}$-graded vector bundle L on X of finite total rank;
(ii) a degree 1 operator $\ell_1 : \mathcal{L}^{sh} \to \mathcal{L}^{sh}$ on the sheaf $\mathcal{L}^{sh}$ of smooth sections of L, and
(iii) a degree 0 bilinear operator

$$\ell_2 : \mathcal{L}^{sh} \times \mathcal{L}^{sh} \to \mathcal{L}^{sh}$$

such that $\ell_1^2 = 0$, ℓ_1 is a differential operator, and ℓ_2 is a bidifferential operator, and

$$\ell_1(\ell_2(x, y)) = \ell_2(\ell_1(x), y) + (-1)^{|x|}\ell_2(x, \ell_1(y))$$

for any sections x, y of $\mathcal{L}^{sh}$. We call ℓ_1 the *differential* and ℓ_2 the *bracket*.

In other words, a local Lie algebra is a sheaf of dg Lie algebras where the underlying sections are smooth sections of a vector bundle and where the operations are local in the sense of not enlarging support of sections. (As we will see, such Lie algebras often appear by acting naturally on the local functionals from physics, namely functionals determined by Lagrangian densities.)

Remark 1.2. For a local Lie algebra, we reserve the more succinct notation $\mathcal{L}$ to denote the precosheaf of *compactly supported* sections of L, which assigns a dg Lie algebra to each open set $U \subset X$, since the differential and bracket respect support. At times we will abusively refer to $\mathcal{L}$ to mean the data determining the local Lie algebra, when the support of the sections is not relevant to the discussion at hand.

The key examples for this paper all arise from studying the symmetries of holomorphic principal bundles. We begin with the specific and then examine a modest generalization.

Let $\pi : P \to X$ be a holomorphic principal G-bundle over a complex manifold. We use $\mathrm{ad}(P) \to X$ to denote the associated *adjoint bundle* $P \times^G \mathfrak{g} \to X$, where the Borel construction uses adjoint action of G on $\mathfrak{g}$ from the left. The complex structure defines a $(0,1)$-connection $\overline{\partial}_P : \Omega^{0,q}(X; \mathrm{ad}(P)) \to \Omega^{0,q+1}(X; \mathrm{ad}(P))$ on the Dolbeault forms with values in the adjoint bundle, and this connection satisfies $\overline{\partial}_P^2 = 0$. Note that the Lie bracket on $\mathfrak{g}$ induces a pointwise bracket on smooth sections of $\mathrm{ad}(P)$ by

$$[s,t](x) = [s(x), t(x)]$$

where s,t are sections and x is a point in X. This bracket naturally extends to Dolbeault forms with values in the adjoint bundle, as the Dolbeault forms are a graded-commutative algebra.

Definition 1.3. For $\pi : P \to X$ a holomorphic principal G-bundle, let $\mathcal{A}\mathrm{d}(P)^{sh}$ denote the local Lie algebra whose sections are $\Omega^{0,*}(X, \mathrm{ad}(P))$, whose differential is $\overline{\partial}_P$, and whose bracket is the pointwise operation just defined above.

OG: We should add some remark about Atiyah algebras ... We could also add a comment about the deformation-theoretic content of this dg Lie algebra.

BW: I did a bit in my thesis, but it may be a hack job

This construction admits important variations. For example, we can move from working over a fixed manifold X to working over a site. Let Hol_d denote the category whose objects are complex d-folds and whose morphisms are local biholomorphisms,[1] This category admits a natural Grothendieck topology where a cover $\{\phi_i : U_i \to X\}$ means a collection of morphisms into X such that union of the images is all of X. It then makes sense to talk about a local Lie algebra on the site Hol_d. Here is a particularly simple example that appears throughout the paper.

Definition 1.4. Let G be a complex Lie group and let $\mathfrak{g}$ denote its ordinary Lie algebra. There is a natural functor

$$\begin{array}{rccc} \mathcal{G}^{sh}: & \mathrm{Hol}_d^{\mathrm{op}} & \to & \mathrm{dgLie} \\ & X & \mapsto & \Omega^{0,*}(X) \otimes \mathfrak{g}, \end{array}$$

which defines a sheaf of dg Lie algebras. Restricted to each slice $\mathrm{Hol}_{d/X}$, it determines the local Lie algebra for the trivial principal bundle $G \times X \to X$, in the sense described above. We use $\mathcal{G}$ to denote the cosheaf of compactly supported sections $\Omega_c^{0,*} \otimes \mathfrak{g}$ on this site.

Remark 1.5. It is not necessary to start with a complex Lie group: the construction makes sense for a dg Lie algebra over $\mathbb{C}$ of finite total dimension. We lose, however, the interpretation in terms of infinitesimal symmetries of the principal bundle.

[1] A biholomorphism is a bijective map $\phi : X \to Y$ such that both ϕ and ϕ^{-1} are holomorphic. A *local* biholomorphism means a map $\phi : X \to Y$ such that every point $x \in X$ has a neighborhood on which ϕ is a biholomorphism.

Remark 1.6. For any complex manifold X we can restrict the functor $\mathcal{G}^{sh}$ to the overcategory of opens in X, that we denote by $\mathcal{G}_X^{sh}$. In this case, $\mathcal{G}_X^{sh}$, or its compactly supported version $\mathcal{G}_X$, comes from the local Lie algebra of Definition 1.3 in the case of the trivial G-bundle on X. In the case that $X = \mathbb{C}^d$ we will denote the sheaves and cosheaves of the local Lie algebra by $\mathcal{G}_d^{sh}, \mathcal{G}_d$ respectively.

1.2. **Current algebras as enveloping factorization algebras of local Lie algebras.** Local Lie algebras often appear as symmetries of classical field theories. For instance, as we will show in Section **??**, each finite-dimensional complex representation V of a Lie algebra $\mathfrak{g}$ determines a charged $\beta\gamma$-type system on a complex d-fold X with choice of holomorphic principal bundle $\pi : P \to X$. Namely, the on-shell γ fields are holomorphic sections for the associated bundle $P \times^G V \to X$, and the on-shell β fields are holomorphic d-forms with values in the associated bundle $P \times^G V^* \to X$. It should be plausible that $\mathcal{A}d(P)^{sh}$ acts as symmetries of this classical field theory, since holomorphic sections of the adjoint bundle manifestly send on-shell fields to on-shell fields.

Such a symmetry determines currents, which we interpret as observables of the classical theory. Note, however, a mismatch: while fields are contravariant in space(time) because fields pull back along inclusions of open sets, observables are covariant because an observable on a smaller region extends to any larger region containing it. The currents, as observables, thus do not form a sheaf but a precosheaf. We introduce the following terminology.

Definition 1.7. For a local Lie algebra $(L \to X, \ell_1, \ell_2)$, its precosheaf $\mathcal{L}[1]$ of *linear currents* is given by taking compactly supported sections of L.

There are a number of features of this definition that may seem peculiar on first acquaintance. First, we work with $\mathcal{L}[1]$ rather than $\mathcal{L}$. This shift is due to the Batalin-Vilkovisky formalism. In that formalism the observables in the classical field theory possesses a 1-shifted Poisson bracket $\{-,-\}$ (also known as the antibracket), and so if the current $J(s)$ associated to a section $s \in \mathcal{L}$ encodes the action of s on the observables, i.e.,

$$\{J(s), F\} = s \cdot F,$$

then we need the cohomological degree of $J(s)$ to be 1 less than the degree of s. In short, we want a map of dg Lie algebras $J : \mathcal{L} \to \mathrm{Obs}^{\mathrm{cl}}[-1]$, or equivalently a map of 1-shifted dg Lie algebras $J : \mathcal{L}[1] \to \mathrm{Obs}^{\mathrm{cl}}$, where $\mathrm{Obs}^{\mathrm{cl}}$ denotes the algebra of classical observables.

Second, we use the term "linear" here because the product of two such currents is not in $\mathcal{L}[1]$ itself, although such a product will exist in the larger precosheaf $\mathrm{Obs}^{\mathrm{cl}}$ of observables. In other words, if we have a Noether map of dg Lie algebras $J : \mathcal{L} \to \mathrm{Obs}^{\mathrm{cl}}[-1]$, it extends to a map of 1-shifted Poisson algebras

$$J : \mathrm{Sym}(\mathcal{L}[1]) \to \mathrm{Obs}^{\mathrm{cl}}$$

as $\mathrm{Sym}(\mathcal{L}[1])$ is the 1-shifted Poisson algebra freely generated by the 1-shifted dg Lie algebra $\mathcal{L}[1]$. We hence call $\mathrm{Sym}(\mathfrak{g}[1])$ the *enveloping 1-shifted Poisson algebra* of a dg Lie algebra $\mathfrak{g}$.[2]

For any particular field theory, the currents generated by the symmetry for *that* theory are given by the image of this map J of 1-shifted Poisson algebras. To study the general structure of such currents, without respect to a particular theory, it is natural to study this enveloping algebra by itself.

Definition 1.8. For a local Lie algebra $(L \to X, \ell_1, \ell_2)$, its *classical currents* $\mathrm{Cur}^{\mathrm{cl}}(\mathcal{L})$ is the precosheaf $\mathrm{Sym}(\mathcal{L}[1])$ given by taking the enveloping 1-shifted Poisson algebra of the compactly supported sections of L. It assigns

$$\mathrm{Cur}^{\mathrm{cl}}(\mathcal{L})(U) = \mathrm{Sym}(\mathcal{L}(U)[1])$$

to an open subset $U \subset X$.

We emphasize here that by $\mathrm{Sym}(\mathcal{L}(U)[1])$ we do *not* mean the symmetric algebra in the purely algebraic sense, but rather a construction that takes into account the extra structures on sections of vector bundles (e.g., the topological vector space structure). Explicitly, the nth symmetric power $\mathrm{Sym}^n(\mathcal{L}(U)[1])$ means the smooth, compactly supported, and S_n-invariant sections of the graded vector bundle

$$L[1]^{\boxtimes n} \to U^n.$$

For further discussion of functional analytic aspects (which play no tricky role in our work here), see **[?]**, notably the appendices.

A key result of **[?]**, namely Theorem 5.6.0.1, is that this precosheaf of currents forms a factorization algebra. From hereon we refer to $\mathrm{Cur}^{\mathrm{cl}}(\mathcal{L})$ as the *factorization algebra of classical currents*. If the local Lie algebra acts as symmetries on some classical field theory, we obtain a map of factorization algebras $J : \mathrm{Cur}^{\mathrm{cl}}(\mathcal{L}) \to \mathrm{Obs}^{\mathrm{cl}}$ that encodes each current as a classical observable.

There is a quantum counterpart to this construction, in the Batalin-Vilkovisky formalism. The idea is that for a dg Lie algebra $\mathfrak{g}$, the enveloping 1-shifted Poisson algebra $\mathrm{Sym}(\mathfrak{g}[1])$ admits a natural BV quantization via the Chevalley-Eilenberg chains $C_*(\mathfrak{g})$. This assertion is transparent by examining the Chevalley-Eilenberg differential:

$$\mathrm{d}_{CE}(xy) = \mathrm{d}_{\mathfrak{g}}(x)y \pm x\,\mathrm{d}_{\mathfrak{g}}(y) + [x,y]$$

for x, y elements of $\mathfrak{g}[1]$. The first two terms behave like a derivation of $\mathrm{Sym}(\mathfrak{g}[1])$, and the last term agrees with the shifted Poisson bracket. More accurately, to keep track of the $\hbar$-dependency in quantization, we introduce a kind of Rees construction. OG: cross ref stuff with Rune and the other paper

[2] OG: Add some references?

Definition 1.9. The *enveloping BD algebra* $U^{BD}(\mathfrak{g})$ of a dg Lie algebra $\mathfrak{g}$ is given by the graded-commutative algebra in $\mathbb{C}[\hbar]$-modules

$$\mathrm{Sym}(\mathfrak{g}[1])[\hbar] \cong \mathrm{Sym}_{\mathbb{C}[\hbar]}(\mathfrak{g}[\hbar][1]),$$

but the differential is defined as a coderivation with respect to the natural graded-cocommutative coalgebra structure, by the condition

$$\mathrm{d}(xy) = \mathrm{d}_{\mathfrak{g}}(x)y \pm x\,\mathrm{d}_{\mathfrak{g}}(y) + \hbar[x,y].$$

This construction determines a BV quantization of the enveloping 1-shifted Poisson algebra, as can be verified directly from the definitions. (For further discussion see **[?]** and **[?]**.) It is also straightforward to extend this construction to "quantize" the factorization algebra of classical currents.

Definition 1.10. For a local Lie algebra $(L \to X, \ell_1, \ell_2)$, its *factorization algebra of quantum currents* $\mathrm{Cur}^{\mathrm{q}}(\mathcal{L})$ is given by taking the enveloping *BD* algebra of the compactly supported sections of L. It assigns

$$\mathrm{Cur}^{\mathrm{q}}(\mathcal{L})(U) = U^{BD}(\mathcal{L}(U))$$

to an open subset $U \subset X$.

As mentioned just after the definition of the classical currents, the symmetric powers here mean the construction involving sections of the external tensor product. Specializing $\hbar = 1$, we recover the following construction.

Definition 1.11. For a local Lie algebra $(L \to X, \ell_1, \ell_2)$, its *enveloping factorization algebra* $\mathbb{U}(\mathcal{L})$ is given by taking the Chevalley-Eilenberg chains $\mathrm{C}^{\mathrm{Lie}}_*(\mathcal{L})$ of the compactly supported sections of L.

Here the symmetric powers use sections of the external tensor powers, just as with the classical or quantum currents.

When a local Lie algebra acts as symmetries of a classical field theory, it sometimes also lifts to symmetries of a BV quantization. In that case the map $J : \mathrm{Sym}(\mathcal{L}[1]) \to \mathrm{Obs}^{\mathrm{cl}}$ of 1-shifted Poisson algebras lifts to a cochain map $J^{\mathrm{q}} : \mathrm{Cur}^{\mathrm{q}}(\mathcal{L}) \to \mathrm{Obs}^{\mathrm{q}}$ realizing quantum currents as quantum observables. Sometimes, however, the classical symmetries do not lift directly to quantum symmetries. We turn to discussing the natural home for the obstructions to such lifts after a brief detour to offer a structural perspective on the enveloping construction.

1.2.1. *A digression on the enveloping E_n algebras.* OG: I added this here because it feels natural and then we can refer to it easily later.

This construction $\mathbb{U}(\mathcal{L})$ has a special feature when the local Lie algebra is obtained by taking the de Rham forms with values in a dg Lie algebra $\mathfrak{g}$, i.e., when $\mathcal{L} = \Omega^*_c \otimes \mathfrak{g}$. In that case the enveloping factorization algebra is locally constant and, on the d-dimensional real manifold $\mathbb{R}^d$, determines an E_d algebra, also known as an algebra over the little d-disks operad, by a result of Lurie (see OG: give precise citation). This construction satisfies a universal property: it is the d-dimensional generalization of the universal enveloping algebra of a Lie algebra.

To state this result of Knudsen precisely, we need to be in the context of ∞-categories.

Theorem 1.12 ([?]). *Let $\mathcal{C}$ be a stable, $\mathbb{C}$-linear, presentable, symmetric monoidal ∞-category. There is an adjunction*

$$U^{E^d} : \mathrm{LieAlg}(\mathcal{C}) \leftrightarrows E_d\mathrm{Alg}(\mathcal{C}) : F$$

between Lie algebra objects in $\mathcal{C}$ and E_d algebra objects in $\mathcal{C}$. This adjunction intertwines with the free-forget adjunctions from Lie/E_d algebras in $\mathcal{C}$ to $\mathcal{C}$ so that

$$\mathrm{Free}_{E_d}(X) \simeq U^{E_d}\mathrm{Free}_{Lie}(\Sigma^{d-1}X)$$

for any object $X \in \mathcal{C}$.

When $\mathcal{C}$ is the ∞-category of chain complexes over a field of characteristic zero, the E_d algebra $U^{E_d}\mathfrak{g}$ is modeled by the locally constant factorization algebra $\mathbb{U}(\Omega^*_c \otimes \mathfrak{g})$ on $\mathbb{R}^d$.

This theorem is highly suggestive for us: our main class of examples is $\mathcal{G}_d$ and $\mathbb{U}\mathcal{G}_d$, which replaces the de Rham complex with the Dolbeault complex. In other words, we anticipate that $\mathbb{U}\mathcal{G}_d$ should behave like a holomorphic version of an E_d algebra and that it should be the canonical such algebra determined by a dg Lie algebra. We do not pursue this structural result in this paper, but it provides some intuition behind our constructions.

1.3. **Local cocycles and shifted extensions.** Some basic questions about a dg Lie algebra $\mathfrak{g}$, such as the classification of extensions and derivations, are encoded cohomologically, typically as cocycles in the Chevalley-Eilenberg cochains $C^*_{\mathrm{Lie}}(\mathfrak{g}, V)$ with coefficients in some $\mathfrak{g}$-representation V. When working with local Lie algebras, it is natural to focus on cocycles that are also local in the appropriate sense. (Explicitly, we want to restrict to cocycles that are built out of polydifferential operators.) After introducing the relevant construction, we turn to studying how such cocycles determine modified current algebras.

1.3.1. *Local cochains of a local Lie algebra.* In Appendix A we define the local cochains of a local Lie algebra in some detail, but we briefly recall it here. The basic idea is that a local cochain is a Lagrangian density: it takes in a section of the local Lie algebra and produces a smooth density on the manifold. Such a cocycle determines a functional by integrating the density. As usual with Lagrangian densities, we wish to work with them up to total derivatives, i.e., we identify Lagrangian densities related using integration by parts and hence ignore boundary terms.

In a bit more detail, for L is a graded vector bundle, let JL denote the corresponding ∞-jet bundle, which has a canonical flat connection. In other words, it is a left D_X-module, where D_X denotes the sheaf of smooth differential operators on X. For a local Lie algebra, this JL obtains the structure of a dg Lie algebra in left D_X-modules. Thus, we may consider its reduced Chevalley-Eilenberg cochain complex $\mathrm{C}^*_{\mathrm{Lie}}(JL)$ in the category of left D_X-modules. By taking the de Rham complex of this left D_X-module, we obtain the local cochains. OG: I took the full de Rham complex, but if you prefer, we can just tensor with densities. BW: don't you have to use derived tensor product if you use full de Rham? For a variety of reasons, it is useful to ignore the "constants" term and work with the reduced cochains. Hence we have the following definition.

Definition 1.13. Let $\mathcal{L}$ be a local Lie algebra on X. The *local Chevalley-Eilenberg cochains* of $\mathcal{L}$ is

$$\mathrm{C}^*_{\mathrm{loc}}(\mathcal{L}) = \Omega^{*,*}_X[2d] \otimes_{D_X} \mathrm{C}^*_{\mathrm{Lie,red}}(JL).$$

This sheaf of cochain complexes on X has global sections that we denote by $\mathrm{C}^*_{\mathrm{loc}}(\mathcal{L}(X))$.

Remark 1.14. This construction $\mathrm{C}^*_{\mathrm{loc}}(\mathcal{L})$ is just a version of diagonal Gelfand-Fuks cohomology [**?**, **?**], where the adjective "diagonal" indicates that we are interested in continuous cochains whose integral kernels are supported on the small diagonals.

1.3.2. *Shifted extensions.* For an ordinary Lie algebra $\mathfrak{g}$, central extensions are parametrized by 2-cocycles on $\mathfrak{g}$ valued in the trivial module $\mathbb{C}$. It is possible to interpret arbitrary cocycles as determining as determining *shifted* central extensions as L_∞ *algebras*. Explicitly, a k-cocycle Θ of degree n on a dg Lie algebra $\mathfrak{g}$ determines an L_∞ algebra structure on the direct sum $\mathfrak{g} \oplus \mathbb{C}[n-k]$ with the following brackets $\{\widehat{\ell}_m\}_{m\geq 1}$: $\widehat{\ell}_1$ is simply the differential on $\mathfrak{g}$, $\widehat{\ell}_2$ is the bracket on $\mathfrak{g}$, $\widehat{\ell}_m = 0$ for $m > 2$ except

$$\widehat{\ell}_k(x_1 + a_1, \ldots, x_k + a_k) = 0 + \Theta(x_1, x_2, \ldots, x_k).$$

(See OG: add ref for further discussion. Note that $n = 2$ for $k = 2$ with ordinary Lie algebras.) Similarly, local cocycles provide shifted central extensions of local Lie algebras.

Definition 1.15. For a local Lie algebra (L, ℓ_1, ℓ_2), a cocycle Θ of degree $2+k$ in $\mathrm{C}^*_{\mathrm{loc}}(\mathcal{L})$ determines a *k-shifted central extension*

$$0 \to \mathbb{C}[k] \to \widehat{\mathcal{L}}_\Theta \to \mathcal{L} \to 0 \tag{1}$$

of precosheaves of L_∞ algebras, where the L_∞ structure maps are defined by

$$\widehat{\ell}_n(x_1, \ldots, x_n) = \left(\ell_n(x_1, \ldots, x_n), \int \Theta(x_1, \ldots, x_n)\right).$$

Here we set $\ell_n = 0$ for $n > 2$.

OG: I think we must include the integral sign here!

As usual, cohomologous cocycles determine quasi-isomorphic extensions. Much of the rest of the section is devoted to constructing and analyzing various cocycles and the resulting extensions.

1.3.3. *Twists of the current algebras.* Local cocycles give a direct way of deforming the various current algebras a local Lie algebra. For example, we have the following construction.

Definition 1.16. Let Θ be a degree 1 local cocycle for a local Lie algebra $(L \to X, \ell_1, \ell_2)$. Let K denote a degree zero parameter so that $\mathbb{C}[K]$ is a polynomial algebra concentrated in degree zero. The *twisted enveloping factorization algebra* $\mathbb{U}_\Theta(\mathcal{L})$ assigns to an open $U \subset X$, the cochain complex

$$\begin{aligned} \mathbb{U}_\Theta(\mathcal{L})(U) &= (\mathrm{Sym}(\mathcal{L}(U)[1] \oplus \mathbb{C} \cdot K), \mathrm{d}_\mathcal{L} + K \cdot \Theta) \\ &= (\mathrm{Sym}(\mathcal{L}(U)[1])[K], \mathrm{d}_\mathcal{L} + K \cdot \Theta)\,, \end{aligned}$$

where $\mathrm{d}_\mathcal{L}$ denotes the differential on the untwisted enveloping factorization algebra and Θ is the operator extending the cocycle $\Theta : \mathrm{Sym}(\mathcal{L}(U)[1]) \to \mathbb{C} \cdot K$ to the symmetric coalgebra as a graded coderivation. This twisted enveloping factorization algebra is module for the commutative ring $\mathbb{C}[K]$, and so specializing the value of K determines nontrivial modifications of $\mathbb{U}(\mathcal{L})$.

An analogous construction applies to the quantum currents, which we will denote $\mathrm{Cur}^{\mathfrak{q}}_\Theta(\mathcal{L})$.

1.3.4. *A special class of cocycles: the $\mathfrak{j}$ functional.* There is a particular family of local cocycles that has special importance in studying symmetries of higher dimensional holomorphic field theories.

Consider

$$\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}},$$

so that θ is a $\mathfrak{g}$-invariant polynomial on $\mathfrak{g}$ of homogenous degree $d+1$. This data determines a local functional for $\mathcal{G} = \Omega^{0,*} \otimes \mathfrak{g}$ on any complex d-fold as follows.

Definition 1.17. For any complex d-fold X, extend θ to a functional $\mathfrak{J}_X(\theta)$ on $\mathcal{G}_X = \Omega^{0,*}_c(X) \otimes \mathfrak{g}$ by the formula

$$\mathfrak{J}_X(\theta)(\alpha_0, \ldots, \alpha_d) = \int_X \theta(\alpha_0, \partial\alpha_1, \ldots, \partial\alpha_d), \tag{2}$$

where ∂ denotes the holomorphic de Rham differential. In this formula, we define the integral to be zero whenever the integrand is not a (d,d)-form.

To make this formula as clear as possible, suppose the α_i are pure tensors of the form $\omega_i \otimes y_i$ with $\omega_i \in \Omega^{0,*}_c(X)$ and $y_i \in \mathfrak{g}$. Then

$$\mathfrak{J}_X(\theta)(\omega_0 \otimes y_0, \ldots, \omega_d \otimes y_d) = \theta(y_0, \ldots, y_d) \int_X \omega_0 \wedge \partial\omega_1 \cdots \wedge \partial\omega_d. \tag{3}$$

Note that we use d copies of the holomorphic derivative $\partial : \Omega^{0,*} \to \Omega^{1,*}$ to obtain an element of $\Omega_c^{d,*}$ in the integrand and hence something that can be integrated.

This formula manifestly makes sense for any complex d-fold X, and since integration is local on X, it intertwines nicely with the structure maps of $\mathcal{G}_X$.

Definition 1.18. For any complex d-fold X and any $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$, let $\mathrm{j}_X(\theta)$ denote the local cochain in $\mathrm{C}^*_{\mathrm{loc}}(\mathcal{G}_X)$ defined by

$$\mathrm{j}_X(\theta)(\alpha_0, \ldots, \alpha_d) = \theta(\alpha_0, \partial\alpha_1, \ldots, \partial\alpha_d).$$

Hence $\mathfrak{J}_X(\theta) = \int_X \mathrm{j}_X(\theta)$.

This integrand $\mathrm{j}_X(\theta)$ is in fact a local cocycle, and in a moment we will use it to produce an important shifted central extension of $\mathcal{G}_X$.

Proposition 1.19. *The assignment*

$$\begin{array}{rccc} \mathrm{j}_X : & \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}[-1] & \to & \mathrm{C}^*_{\mathrm{loc}}(\mathcal{G}_X) \\ & \theta & \mapsto & \mathrm{j}_X(\theta) \end{array}$$

is an cochain map.

Proof. The element $\mathrm{j}_X(\theta)$ is local as it is expressed as a density produced by polydifferential operators. We need to show that $\mathrm{j}_X(\theta)$ is closed for the differential on $\mathrm{C}^*_{\mathrm{loc}}(\mathcal{G}_X)$. Note that $\mathcal{G}_X$ is the tensor product of the dg commutative algebra $\Omega_X^{0,*}$ and the Lie algebra $\mathfrak{g}$. Hence the differential on the local cochains of $\mathcal{G}_X$ splits as a sum $\overline{\partial} + \mathrm{d}_{\mathfrak{g}}$ where $\overline{\partial}$ denotes the differential on local cochains induced from the $\overline{\partial}$ differential on the Dolbeault forms and $\mathrm{d}_{\mathfrak{g}}$ denotes the differential induced from the Lie bracket on $\mathfrak{g}$. We now analyze each term separately.

Observe that for any collection of $\alpha_i \in \mathcal{G}$, we have

$$\begin{aligned} \overline{\partial}(\theta(\alpha_0, \partial\alpha_1, \ldots, \partial\alpha_d)) &= \theta(\overline{\partial}\alpha_0, \partial\alpha_1, \ldots, \partial\alpha_d) \pm \theta(\alpha_0, \overline{\partial}\partial\alpha_1, \ldots, \partial\alpha_d) \pm \cdots \pm \theta(\alpha_0, \partial\alpha_1, \ldots, \overline{\partial}\partial\alpha_d) \\ &= \sum_{i=0}^{d} \pm\theta(\alpha_0, \partial\alpha_1, \ldots, \partial\alpha_d) \end{aligned}$$

because $\overline{\partial}$ is a derivation and θ wedges the form components. (It is easy to see this assertion when one works with inputs like in equation (3).) Hence viewing $\mathrm{j}_X(\theta)$ as a map from $\mathcal{G}$ to the Dolbeault complex, it commutes with the differential $\overline{\partial}$. This fact is equivalent to $\overline{\partial}\mathrm{j}_X(\theta) = 0$ in local cochains.

Similarly, observe that for any collection of $\alpha_i \in \mathcal{G}$, we have

$$\begin{aligned} (\mathrm{d}_{\mathfrak{g}}\mathrm{j}_X(\theta))(\alpha_0, \alpha_1, \ldots, \alpha_d) &= (\mathrm{d}_{\mathfrak{g}}\theta)(\alpha_0, \partial\alpha_1, \ldots, \partial\alpha_d)) \\ &= 0 \end{aligned}$$

since θ is closed in $\mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g})$. □

As should be clear from the construction, everything here works over the site Hol_d of complex d-folds, and hence we use $\mathfrak{j}(\theta)$ to denote the local cocycle for the local Lie algebra $\mathcal{G}$ on Hol_d.

This construction works nicely for an arbitrary holomorphic G-bundle P on X, because the cocycle is expressed in a coordinate-free fashion. To be explicit, on a coordinate patch $U_i \subset X$ with a choice of trivialization of the adjoint bundle $\mathrm{ad}(P)$, the formula for $\mathfrak{j}_X(\theta)$ makes sense. On an overlap $U_i \cap U_j$, the cocycles patch because $\mathfrak{j}_X(\theta)$ is independent of the choice of coordinates. Hence we can glue over any sufficiently refined cover to obtain a global cocycle. Thus, we have a cochain map

$$\mathfrak{j}_X^P : \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}[-1] \to \mathrm{C}^*_{\mathrm{loc}}(\mathcal{A}\mathrm{d}(P)(X))$$

given by the same formula as in (2).

1.3.5. *Another special class: the LMNS extensions.* Much of this paper focuses on local cocycles of type $\mathfrak{j}_X(\theta)$, where $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$. But there is another class of local cocycles that appear naturally when studying symmetries of holomorphic theories. Unlike the cocycle $\mathfrak{j}_X(\theta)$, which only depend on the manifold X through its dimension, this class of cocycles depends on the geometry.

In complex dimension two, this class of cocycles has appeared in the work of Losev-Moore-Nekrasov-Shatashvili (LMNS) [**?**, **?**, **?**] in their construction of a higher analog of the "chiral WZW theory". Though our approaches differ, we share their ambition to formulate a higher dimensional version of chiral CFT. BW: this sentence seems out of place.

Let X be a complex manifold of dimension d with a choice of (k,k)-form η. Choose a form $\theta_{d+1-k} \in \mathrm{Sym}(\mathfrak{g}^*)^{\mathfrak{g}}$. This data determines a local cochain on $\mathcal{G}_X$ whose local functional is:

$$\begin{array}{rccc} \phi_{\theta,\eta}: & \mathcal{G}(X)^{\otimes d+1-k} & \to & \mathbb{C} \\ & \alpha_0 \otimes \cdots \otimes \alpha_{d-k} & \mapsto & \displaystyle\int_X \eta \wedge \theta_{d+1-k}(\alpha_0, \partial\alpha_1, \ldots, \partial\alpha_{d-k}) \end{array}.$$

Such a cochain is a cocycle only if $\overline{\partial}\eta = 0$, because η does not interact with the Lie structure.

Note that a Kähler manifold always produces natural choices of η by taking $\eta = \omega^k$, where ω is the symplectic form. In this way, Kähler geometry determines an important class of extensions. It would be interesting to explore what aspects of the geometry are reflected by these associated current algebras.

Lemma 1.20. *Fix $\theta \in \mathrm{Sym}^{d+1-k}(\mathfrak{g}^*)^{\mathfrak{g}}$. If a form $\eta \in \Omega^{k,k}(X)$ satisfies $\overline{\partial}\eta = 0$ and $\partial\eta = 0$, then the local cohomology class $[\phi_{\theta,\omega}] \in H^1_{\mathrm{loc}}(\mathcal{G}_X)$ depends only on the cohomology class $[\omega] \in H^k(X, \Omega^k_{cl})$.*

When $\eta = 1$, it trivially satisfies the conditions of the lemma. In this case $\phi_{\theta,1} = \mathfrak{j}_X(\theta)$ in the notation of the last section.

1.4. **The higher Kac-Moody factorization algebra.** Finally, we can introduce the central object of this paper.

Definition 1.21. Let X be a complex manifold of complex dimension d equipped with a holomorphic principal G-bundle P. Let Θ be a degree 1 cocycle in $C^*_{\mathrm{loc}}(\mathcal{A}\mathrm{d}(P))$, which determines a 1-shifted central extension $\mathcal{A}\mathrm{d}(P)_\Theta$. The *Kac-Moody factorization algebra on X of type* Θ is the twisted enveloping factorization algebra $\mathbb{U}_\Theta(\mathcal{A}\mathrm{d}(P))$ that assigns

$$\left(\mathrm{Sym}\left(\Omega^{0,*}_c(U, \mathrm{ad}(P))[1]\right)[K], \bar{\partial} + \mathrm{d}_{CE} + \Theta\right)$$

to an open set $U \subset X$.

Remark 1.22. As in the definition of twisted enveloping factorization algebras, the factorization algebras $\mathbb{U}_\Theta(\mathcal{A}\mathrm{d}(P))$ are modules for the ring $\mathbb{C}[K]$. In keeping with conventions above, when P is the trivial bundle on X, we will denote the Kac-Moody factorization algebra by $\mathbb{U}_\Theta(\mathcal{G}_X)$.

The most important class of such higher Kac-Moody algebras makes sense over the site Hol_d of all complex d-folds.

Definition 1.23. Let $\mathfrak{g}$ be an ordinary Lie algebra and let $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$. Let $\mathcal{G}_{d,\theta}$ denote the 1-shifted central extension of $\mathcal{G}_d$ determined by the local cocycle $\mathfrak{j}(\theta)$. Let $\mathbb{U}_\theta(\mathcal{G})$ denote the *θ-twisted enveloping factorization algebra* $\mathbb{U}_{\mathfrak{j}(\theta)}(\mathcal{G})$ for the local Lie algebra $\mathcal{G} = \Omega^{0,*}_c \otimes \mathfrak{g}$ on the site Hol_d of complex d-folds.

In the case $d = 1$ the definition above agrees with the Kac-Moody factorization algebra on Riemann surfaces given in **[?]**. There, it is shown that this factorization algebra, restricted to the complex manifold $\mathbb{C}$, recovers a vertex algebra isomorphic to that of the ordinary Kac-Moody vertex algebra. (See Section 5 of Chapter 5.) Thus, we think of the object $\mathbb{U}_\Theta(\mathcal{A}\mathrm{d}(P))$ as a higher dimensional version of the Kac-Moody vertex algebra.

1.4.1. *Holomorphic translation invariance and higher dimensional vertex algebras.* To put some teeth into the previous paragraph, we note that **[?]** introduces a family of colored operads $\mathrm{PDiscs_d}$, the little d-dimensional polydiscs operads, that provide a holomorphic analog of the little d-disks operads E_d. Concretely, this operad $\mathrm{PDiscs_d}$ encodes the idea of the operator product expansion, where one now understands observables supported in small disks mapping into observables in large disks, rather than point-like observables.

OG: Should we hark back to Knudsen's theorem here? I feel like it might be possible to find a nice analog using PDisks.

In the case $d = 1$, Theorem 5.3.3 of [?] shows that a PDiscs_1-algebra $\mathcal{A}$ determines a vertex algebra $\mathbb{V}(\mathcal{A})$ so long as $\mathcal{A}$ is suitably equivariant under rotation . This construction $\mathbb{V}$ is functorial. As shown in [?], many vertex algebras appear this way, and any vertex algebras that arise from physics should, in light of the main results of [?, ?].

For this reason, one can interpret $\mathrm{PDiscs}_\mathrm{d}$-algebras, particularly when suitably equivariant under rotation, as providing a systematic and operadic generalization of vertex algebras to higher dimensions. Proposition 5.2.2 of [?] provides a useful mechanism for producing $\mathrm{PDiscs}_\mathrm{d}$-algebra: it says that if a factorization algebra is equivariant under translation in a holomorphic manner, then it determines such an algebra.

Hence it is interesting to identify when the higher Kac-Moody factorization algebras are invariant in the sense needed to produce $\mathrm{PDiscs}_\mathrm{d}$-algebras. We now address this question.

First, note that on the complex d-fold $X = \mathbb{C}^d$, the local Lie algebra $\mathcal{G}_d$ is manifestly equivariant under translation.

It is important to recognize that this translation action is holomorphic in the sense that the infinitesimal action of the (complexified) vector fields $\partial/\partial\bar{z}_i$ is homotopically trivial. Explicitly, consider the operator $\eta_i = \iota_{\partial/\partial\bar{z}_i}$ on Dolbeault forms (and which hence extends to $\mathcal{G}_{\mathbb{C}^d}$), and note that

$$[\bar{\partial}, \eta_i] = \partial/\partial\bar{z}_i.$$

Both the infinitesimal actions and this homotopical trivialization extend canonically to the Chevalley-Eilenberg chains of $\mathcal{G}_{\mathbb{C}^d}$ and hence to the enveloping factorization algebra and the current algebras. (For more discussion of these ideas see [?] and Chapter 10 of [?].)

A succinct way to express this feature is to introduce a dg Lie algebra

$$\mathbb{C}^d_{\mathrm{hol}} = \mathrm{span}_{\mathbb{C}}\{\partial/\partial z_1, \ldots, \partial/\partial z_d, \partial/\partial\bar{z}_1, \ldots, \partial/\partial\bar{z}_d, \eta_1, \ldots, \eta_d\}$$

where the partial derivatives have degree 0 and the η_i have degree -1, where the brackets are all trivial, and where the differential behaves like $\bar{\partial}$ in the sense that the differential of η_i is $\partial/\partial\bar{z}_i$. We just argued in the preceding paragraph that $\mathcal{G}_{\mathbb{C}^d}$ and its current algebras are all strictly $\mathbb{C}^d_{\mathrm{hol}}$-invariant.

When studying shifted extensions of $\mathcal{G}_{\mathbb{C}^d}$, it then makes sense to consider local cocycles that are also translation invariant in this sense. Explicitly, we ask to work with cocycles in

$$\mathrm{C}^*_{\mathrm{loc}}(\mathcal{G}_d)^{\mathbb{C}^d_{\mathrm{hol}}} \subset \mathrm{C}^*_{\mathrm{loc}}(\mathcal{G}_d).$$

Local cocycles here determine higher Kac-Moody algebras that are holomorphically translation invariant and hence yield PDiscs_d-algebras.

The following result indicates tells us that we have already encountered all the relevant cocycles so long as we also impose rotation invariance, which is a natural condition.

Proposition 1.24. *The map* $\mathrm{j}_{\mathbb{C}^d} : \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}[-1] \to \mathrm{C}^*_{\mathrm{loc}}(\mathcal{G}_d)$ *factors through the subcomplex of local cochains that are rotationally and holomorphically translation invariant. This map*

$$\mathrm{j}_{\mathbb{C}^d} : \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}[-1] \xrightarrow{\simeq} \left(\mathrm{C}^*_{\mathrm{loc}}(\mathcal{G}_d))^{\mathbb{C}^d_{\mathrm{hol}}}\right)^{U(d)}$$

is a quasi-isomorphism.

As the proof is rather lengthy, we provide it in Appendix B.

2. Local aspects of the higher Kac-Moody factorization algebras

A factorization algebra encodes an enormous amount of information, and hence it is important to extract aspects that are simpler to understand. In this section we will take two approaches:

(1) by compactifying along a sphere of real dimension $2d-1$, we obtain an algebra (more precisely, a homotopy-coherent associative algebra) that encodes the higher dimensional version of "radial ordering" of operators from two-dimensional conformal field theory, and
(2) by compactifying along a torus $(S^1)^d$, we obtain an algebra over the little d-disks operad.

In both cases these algebras behave like enveloping algebras of homotopy-coherent Lie algebras (in a sense we will spell out in detail below), which allows for simpler descriptions of some phenomena. It is important to be aware, however, that these algebras do not encode the full algebraic structure produced by the compactification; instead, they sit as dense subalgebras. We will elaborate on this subtlety below.

For factorization algebras, compactification is accomplished by the pushforward operation. Given a map $f : X \to Y$ of manifolds and a factorization algebra $\mathcal{F}$ on X, its *pushforward* $f_*\mathcal{F}$ is the factorization algebra on Y where

$$f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$$

for any open $U \subset Y$. The first example we treat arises from the radial projection map

$$r : \mathbb{C}^d \setminus \{0\} \to (0, \infty)$$

sending z to its length $|z|$. The preimage of a point is simply a $2d-1$-sphere, so one can interpret the pushforward Kac-Moody factorization algebra $r_*\mathbb{U}_\theta\mathcal{G}_d$ as compactification along these spheres. Our first main result is that there is a locally constant factorization algebra $\mathcal{A}$ along $(0,\infty)$ with a natural map

$$\phi : \mathcal{A} \to r_*\mathbb{U}_\theta\mathcal{G}_d$$

that is dense from the point of view of the topological vector space structure. By a theorem of Lurie, locally constant factorization algebras on $\mathbb{R}$ correspond to homotopy-coherent associative algebras, so that we can interpret ϕ as saying that the pushforward is approximated by an associative algebra, in this derived sense. We will show explicitly that this algebra is the A_∞ algebra arising as the enveloping algebra of an L_∞ algebra already introduced by Faonte-Hennion-Kapranov.

For the physically-minded reader, this process should be understood as a version of radial ordering. Recall from the two-dimensional setting that it can be helpful to view the punctured plane as a cylinder, and to use the radius as a kind of time parameter. Time ordering of operators is then replaced by radial ordering. Many computations can be nicely organized in this manner, because a natural class of operators arises by using a Cauchy integral around the circle of a local operator. The same technique works in higher dimensions where one now computes residues along the $2d-1$-spheres. From this perspective, the natural Hilbert space is associated to the origin in the plane (more accurately to an arbitrarily small disk around the origin), and this picture also extends to higher dimensions. Hence we obtain a kind of vacuum module for this higher dimensional generalization of the Kac-Moody algebras.

Our second cluster of results uses compactification along the projection map

$$\begin{array}{ccc} \mathbb{C}^d \setminus \{\text{coordinate hyperplanes}\} & \to & (0,\infty)^d \\ (z_1,\ldots,z_d) & \mapsto & (|z_1|,\ldots,|z_d|). \end{array}$$

We construct a locally constant factorization algebra on $(0,\infty)^d$ that maps densely into the pushforward of the higher Kac-Moody algebra. Lurie's theorem shows that locally constant factorization algebras on $\mathbb{R}^d$ correspond to E_d algebras, so we obtain a higher-dimensional analog of the spherical result.

OG: I want to move this hidden stuff into the next subsections or into the introduction to the paper.

2.1. **Compactifying the higher Kac-Moody algebras along spheres.** Our approach is modeled on the construction of the affine Kac-Moody Lie algebras and their associated vertex algebras from Section 5.5 of [**?**] and [**?**], so we review the main ideas to orient the reader.

On the punctured plane $\mathbb{C}^*$, the sheaf $\mathcal{G}_1^{sh} = \Omega^{0,*} \otimes \mathfrak{g}$ is quasi-isomorphic to the sheaf $\mathcal{O} \otimes \mathfrak{g}$. The restriction maps of this sheaf tell us that for any open set U, there is a map of Lie algebras

$$\mathcal{O}(\mathbb{C}^*) \otimes \mathfrak{g} \to \mathcal{O}(U) \otimes \mathfrak{g},$$

so that we get a map of Lie algebras

$$\mathcal{O}_{\mathrm{alg}}(\mathbb{C}^*) \otimes \mathfrak{g} = \mathfrak{g}[z,z^{-1}] \to \mathcal{O}(U) \otimes \mathfrak{g}$$

because Laurent polynomials $\mathbb{C}[z,z^{-1}] = \mathcal{O}_{\mathrm{alg}}(\mathbb{C}^*)$ are well-defined on any open subset of the punctured plane. This *loop algebra* $L\mathfrak{g} = \mathfrak{g}[z,z^{-1}]$ admits interesting central extensions, known as the affine Kac-Moody Lie algebras. These extensions are labeled by elements of $\mathrm{Sym}^2(\mathfrak{g}^*)^{\mathfrak{g}}$, which is compatible with our work in Section 1.3.4.

To apply radial ordering to this sheaf—or rather, its associated current algebras—it is convenient to study the pushforward along the radial projection map $r(z) = |z|$. Note that the preimage of an interval (a, b) is an annulus, so

$$r_*\mathcal{G}_1^{sh}((a,b)) = \mathcal{G}_1^{sh}(\{a < |z| < b\})$$

and hence we have a canonical map of Lie algebras

$$\mathfrak{g}[z, z^{-1}] \to \mathcal{O}(\{a < |z| < b\}) \otimes \mathfrak{g} \hookrightarrow r_*\mathcal{G}_1^{sh}((a,b)).$$

We can refine this situation by replacing the left hand side with the locally constant sheaf $\underline{\mathfrak{g}[z, z^{-1}]}$ to produce a map of sheaves $\underline{\mathfrak{g}[z, z^{-1}]} \to r_*\mathcal{G}_1^{sh}((a,b))$. The Poincaré lemma tells us that Ω^* is quasi-isomorphic to the locally constant sheaf $\underline{\mathbb{C}}$, and so we can introduce a sheaf

$$\mathtt{L}\mathfrak{g}^{sh} = \Omega^* \otimes \mathfrak{g}[z, z^{-1}]$$

that is a soft resolution of $\underline{\mathfrak{g}[z, z^{-1}]}$. There is then a map of sheaves of dg Lie algebras

$$\mathtt{L}\mathfrak{g}^{sh} \to r_*\mathcal{G}_1^{sh} \tag{4}$$

that sends $\alpha \otimes x z^n$ to $[r^*\alpha]_{0,*} \cdot z^n \otimes x$, with $x \in \mathfrak{g}$, α a differential form on $(0, \infty)$, and $[r^*\alpha]_{0,*}$ the $(0, *)$-component of the pulled back form. This map restricts nicely to compactly support sections $\mathtt{L}\mathfrak{g} \to r_*\mathcal{G}_1$. By taking Chevalley-Eilenberg chains on both sides, we obtain a map of factorization algebras

$$\mathbb{U}\mathtt{L}\mathfrak{g} = C_*^{\mathrm{Lie}}(\mathtt{L}\mathfrak{g}) \to C_*^{\mathrm{Lie}}(r_*\mathcal{G}_1) = r_*\mathbb{U}\mathcal{G}_1. \tag{5}$$

The left hand side $\mathbb{U}\mathtt{L}\mathfrak{g}$ encodes the associative algebra $U(L\mathfrak{g})$, the enveloping algebra of $L\mathfrak{g}$, as can be seen by direct computation (see section 3.4 of **[?]**) or by a general result of Knudsen **[?]**. The right hand side contains operators encoded by Cauchy integrals, and it is possible to identify such as operator, up to exact terms, as the limit of a sequence of elements from $U(L\mathfrak{g})$.

We extend this argument to the affine Kac-Moody Lie algebras by working with suitable extensions on $\mathtt{L}\mathfrak{g}$. It is a deformation-theoretic argument, as we view the extensions as deforming the bracket.

We wish to replace the punctured plane $\mathbb{C}^*$ by the punctured d-dimensional affine space

$$\mathring{\mathbb{A}}^d = \mathbb{C}^d \setminus \{0\},$$

the current algebras of $\mathcal{G}_1$ by the current algebras of $\mathcal{G}_d$, and, of course, the extensions depending on $\mathrm{Sym}^2(\mathfrak{g}^*)^{\mathfrak{g}}$ by other local cocycles. There are two nontrivial steps to this generalization:

(1) finding a suitable replacement for the Laurent polynomials, so that we can recapitulate (without any issues) the construction of the maps (4) and (5), and
(2) deforming this construction to encompass the extensions of $\mathcal{G}_d$ and hence the twisted enveloping factorization algebras $\mathbb{U}_\theta\mathcal{G}_d$.

We undertake the steps in order.

2.1.1. *Derived functions on punctured affine space.* When $d = 1$, we note that

$$\mathbb{C}[z, z^{-1}] \subset \mathcal{O}(\mathbb{C}^*) \xrightarrow{\simeq} \Omega^{0,*}(\mathbb{C}^*),$$

and so the Laurent polynomials are a dense subalgebra of the Dolbeault complex. When $d > 1$, Hartog's lemma tells us that every holomorphic function on punctured d-dimensional space extends through the origin:

$$\mathcal{O}(\mathring{\mathbb{A}}^d) = \mathcal{O}(\mathbb{A}^d).$$

This result might suggest that $\mathring{\mathbb{A}}^d$ is an unnatural place to seek a generalization of the loop algebra, but such pessimism is misplaced because $\mathring{\mathbb{A}}^d$ is not affine and so its *derived* algebra of functions, given by the derived global sections $\mathbb{R}\Gamma(\mathring{\mathbb{A}}^d, \mathcal{O})$, is more interesting than the underived global sections $\mathcal{O}(\mathring{\mathbb{A}}^d)$.

Indeed, a straightforward computation in algebraic geometry shows

$$H^*(\mathring{\mathbb{A}}^d, \mathcal{O}_{\mathrm{alg}}) = \begin{cases} 0, & * \neq 0, d-1 \\ \mathbb{C}[z_1, \ldots, z_d], & * = 0 \\ \mathbb{C}[z_1^{-1}, \ldots, z_d^{-1}]\frac{1}{z_1 \cdots z_d}, & * = d-1 \end{cases}.$$

(For instance, use the cover by the affine opens of the form $\mathbb{A}^d \setminus \{z_i = 0\}$.) When $d = 1$, this computation recovers the Laurent polynomials, so we should view the cohomology in degree $d - 1$ as providing the derived replacement of the polar part of the Laurent polynomials. A similar result holds in analytic geometry, of course, so that we have a natural map

$$\mathbb{R}\Gamma(\mathring{\mathbb{A}}^d, \mathcal{O}_{\mathrm{alg}}) \to \mathbb{R}\Gamma(\mathring{\mathbb{A}}^d, \mathcal{O}) \simeq \Omega^{0,*}(\mathring{\mathbb{A}}^d)$$

that replaces our inclusion of Laurent polynomials into the Dolbeault complex on $\mathring{\mathbb{A}}^d$.

For explicit constructions, it is convenient to have an explicit dg commutative algebra that models the derived global sections. It should be no surprise that we like to work with the Dolbeault complex, but there is also an explicit dg model A_d for the algebraic version derived global sections due to Faonte-Hennion-Kapranov [**?**] and based on the Jouanolou method for resolving singularities. In fact, they provide a model for the algebraic p-forms as well.

Definition 2.1. Let a_d denote the algebra

$$\mathbb{C}[z_1, \ldots, z_d, z_1^*, \ldots, z_d^*][(zz^*)^{-1}]$$

defined by localizing the polynomial algebra with respect to $zz^* = \sum_i z_i z_i^*$. View this algebra a_d as concentrated in bidegree $(0, 0)$, and consider the bigraded-commutative algebra $R_d^{*,*}$ over a_d that is freely generated in bidegree $(1, 0)$ by elements

$$\mathrm{d}z_1, \ldots, \mathrm{d}z_d,$$

and in bidegree $(0,1)$ by

$$\mathrm{d}z_1^*, \dots, \mathrm{d}z_d^*.$$

We care about the subalgebra $A_d^{*,*}$ where $A_d^{p,m}$ consisting of elements $\omega \in R_d^{p,m}$ such that

(i) the coefficient of $\mathrm{d}z_{i_1}^* \cdots \mathrm{d}z_{i_m}^*$ has degree $-m$ with respect to the z_k^* variables, and
(ii) the contraction $\iota_\xi \omega$ with the Euler vector field $\xi = \sum_i z_i^* \partial_{z_i^*}$ vanishes.

This bigraded algebra admits natural differentials in both directions:

(1) define a map $\overline{\partial} : A_d^{p,q} \to A_d^{p,q+1}$ of bidegree $(0,1)$ by

$$\overline{\partial} = \sum_i \mathrm{d}z_i^* \frac{\partial}{\partial z_i^*},$$

(2) define a a map of bidegree $(1,0)$ by

$$\partial = \sum_i \mathrm{d}z_i \frac{\partial}{\partial z_i}.$$

These differentials commute $\overline{\partial}\partial = \partial\overline{\partial}$, and each squares to zero.

We denote the subcomplex with $p = 0$ by

$$(A_d, \overline{\partial}) = (\bigoplus_{q=0}^{d} A_d^q[-q], \overline{\partial}),$$

and it has the structure of a dg commutative algebra. For $p > 0$, the complex $A_d^{p,*} = (\oplus_q A^{p,q}[-q], \overline{\partial})$ is a dg module for $(A_d, \overline{\partial})$.

From the definition, one can guess that the variables z_i should be understood as the usual holomorphic coordinates on affine space $\mathbb{C}^d$ and the variables z_i^* should be understood as the antiholomorphic coordinates $\overline{z}_i$. The following proposition confirms that guess; it also summarizes key properties of the dg algebra A_d and its dg modules $A_d^{p,*}$, by aggregating several results of **[?]**.

Proposition 2.2 ([**?**], Section 1)**.**

(1) The dg commutative algebra $(A_d, \overline{\partial})$ is a model for $\mathbb{R}\Gamma(A^{d\times}, \mathcal{O}^{alg})$:

$$A_d \simeq \mathbb{R}\Gamma(\mathbb{A}^{d\times}, \mathcal{O}^{alg}).$$

Similarly, $(A_d^{p,*}, \overline{\partial}) \simeq \mathbb{R}\Gamma(\mathbb{A}^{d\times}, \Omega^{p,alg})$.

(2) There is a dense map of commutative bigraded algebras

$$\mathrm{j} : A_d^{*,*} \to \Omega^{*,*}(\mathbb{C}^d \setminus \{0\})$$

sending z_i to z_i, z_i^* to $\overline{z}_i$, and $\mathrm{d}z_i^*$ to $\mathrm{d}\overline{z}_i$, and the map intertwines with the $\overline{\partial}$ and ∂ differentials on both sides.

(3) *There is a unique* GL_n*-equivariant residue map*

$$\mathrm{Res}_{z=0} : A_d^{d,d-1} \to \mathbb{C}$$

that satisfies

$$\mathrm{Res}_{z=0}\left(f(z)\omega^{alg}_{BM}(z,z^*)\mathrm{d}z_1 \cdots \mathrm{d}z_d\right) = f(0)$$

for any $f(z) \in \mathbb{C}[z_1, \dots, z_d]$*. In particular, for any* $\omega \in A_d^{d,d-1}$*,*

$$\mathrm{Res}_{z=0}(\omega) = \oint_{S^{2d-1}} \mathrm{j}(\omega)$$

where S^{2d-1} *is any sphere centered at the origin in* $\mathbb{C}^d$*.*

It is a straightforward to verify that the formula for the Bochner-Martinelli kernel makes sense in the algebra A_d. That is, we define

$$\omega^{alg}_{BM}(z,z^*) = \frac{(d-1)!}{(2\pi i)^d}\frac{1}{(zz^*)^d}\sum_{i=1}^{d}(-1)^{i-1}z_i^*\mathrm{d}z_1^* \wedge \cdots \wedge \widehat{\mathrm{d}z_i^*} \wedge \cdots \wedge \mathrm{d}z_d^*,$$

which is an element of $A_d^{0,d-1}$.

2.1.2. *The sphere algebra of* $\mathfrak{g}$. The loop algebra $L\mathfrak{g} = \mathfrak{g}[z, z^{-1}]$ arises as an algebraic model of the mapping space $\mathrm{Map}(S^1, \mathfrak{g})$, which obtains a natural Lie algebra structure from the target space $\mathfrak{g}$. For a topologist, a natural generalization is to replace the circle S^1, which is equal to the unit vectors in $\mathbb{C}$, by the sphere S^{2d-1}, which is equal to the unit vectors in $\mathbb{C}^d$. That is, consider the "sphere algebra" of $\mathrm{Map}(S^{2d-1}, \mathfrak{g})$. An algebro-geometric sphere replacement of this sphere is the punctured affine d-space $\mathring{\mathbb{A}}^d$ or a punctured formal d-disk, and so we introduce an algebraic model for the sphere algebra.

Definition 2.3. For a Lie algebra $\mathfrak{g}$, the *sphere algebra* in complex dimension d is the dg Lie algebra $A_d \otimes \mathfrak{g}$. Following **[?]** we denote it by $\mathfrak{g}_d^{\bullet}$.

There are natural central extensions of this sphere algebra as em L_∞ algebras, in parallel with our discussion of extensions of the local Lie algebras. For any $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$, Faonte-Hennion-Kapranov define the cocycle

$$\begin{array}{rccc} \theta_{FHK} : & (A_d \otimes \mathfrak{g})^{\otimes(d+1)} & \to & \mathbb{C} \\ & a_0 \otimes \cdots \otimes a_d & \mapsto & \mathrm{Res}_{z=0}\theta(a_0, \partial a_1, \dots, \partial a_d) \end{array}.$$

OG: I changed d (for de Rham) with ∂, since I think that's clearer. Tell me what you think. This cocycle has cohomological degree 2 and so determines an unshifted central extension as L_∞ algebras of $A_d \otimes \mathfrak{g}$:

$$\mathbb{C} \cdot K \to \widetilde{\mathfrak{g}}^{\bullet}_{d,\theta} \to A_d \otimes \mathfrak{g}. \tag{6}$$

Our aim is now to show how the Kac-Moody factorization algebra $\mathbb{U}_\theta \mathcal{G}_d$ is related to this L_∞ algebra, which is a higher-dimensional version of the affine Kac-Moody Lie algebras.

2.1.3. *The case of zero level.* Here we will consider the higher Kac-Moody factorization algebra on $\mathbb{C}^d \setminus \{0\}$ "at level zero," namely the factorization algebra $\mathbb{U}(\mathcal{G}_{\mathbb{C}^d\setminus\{0\}})$. In this section we will omit $\mathbb{C}^d \setminus \{0\}$ from the notation, and simply refer to the factorization algebra by $\mathbb{U}(\mathcal{G}_d)$. Our construction will follow the model case outlined in the introduction to this section. Recall that $r : \mathring{\mathbb{A}}^d \to (0,\infty)$ is the radial projection map that sends $(z_1, \ldots, z_d)$ to its length $\sqrt{z_1\bar{z}_1 + \cdots z_d\bar{z}_d}$.

Lemma 2.4. *There is a map of sheaves of dg commutative algebras on* $\mathbb{R}_{>0}$

$$\pi : \Omega^* \to r_*\Omega^{0,*}$$

sending a form α *to the* $(0,*)$*-component of its pullback* $r^*\alpha$*.*

This result is straightforward since the pullback r^* denotes a map of dg algebras to $r_*\Omega^{*,*}$ and we are simply postcomposing with the canonical quotient map of dg algebras $\Omega^{*,*} \to \Omega^{0,*}$.

We also have a map of dg commutative algebras $A_d \to \Omega^{0,*}(U)$ for any open set $U \subset \mathring{\mathbb{A}}^d$, by postcomposing the map j of proposition 2.2 with the restriction map. We abusively denote the composite by j as well. Thus we obtain a natural map of dg commutative algebras

$$\pi_A : \Omega^* \otimes A_d \to r_*\Omega^{0,*}$$

sending $\alpha \otimes \omega$ to $\pi(\alpha) \wedge \mathrm{j}(\omega)$. By tensoring with $\mathfrak{g}$, we obtain the following.

Corollary 2.5. *There is a map of sheaves of dg Lie algebras on* $\mathbb{R}_{>0}$

$$\pi_{\mathfrak{g}_d^\bullet} : \Omega^* \otimes \mathfrak{g}_d^\bullet \to r_*(\Omega^{0,*} \otimes \mathfrak{g}) = r_*(\mathcal{G}_d^{sh})$$

sending $\alpha \otimes x$ *to* $\pi(\alpha) \otimes x$*.*

Note that $\Omega^* \otimes \mathfrak{g}_d^\bullet = \Omega^* \otimes A_d \otimes \mathfrak{g}$, so $\pi_{\mathfrak{g}_d^\bullet}$ is simply $\pi_A \otimes \mathrm{id}_{\mathfrak{g}}$.

This map preserves support and hence restricts to compactly-supported sections. In other words, we have a map between the associated cosheaves of complexes (and precosheaves of dg Lie algebras). In summary, we have shown our key result.

Proposition 2.6. *The map*

$$\pi_{\mathfrak{g}_d^\bullet} : \Omega^*_{\mathbb{R}_{>0},c} \otimes \mathfrak{g}_d^\bullet \to r_*\mathcal{G}_d$$

is a map of precosheaves of dg Lie algebras. It determines a map of factorization algebras

$$C_*^{\mathrm{Lie}}(\pi_{\mathfrak{g}_d^\bullet}) : \mathbb{U}\left(\Omega^*_{\mathbb{R}_{>0}} \otimes \mathfrak{g}_d^\bullet\right) \to r_*\left(\mathbb{U}\mathcal{G}_d\right).$$

The map of factorization algebras follows from applying the functor $C_*^{\mathrm{Lie}}(-)$ to the map $\pi_{\mathfrak{g}_d^\bullet}$; this construction commutes with push-forward by inspection.

Both maps are dense in every cohomological degree with respect to the natural topologies on these vector spaces, leading to the following observation.

OG: This phrasing isn't great. Can you think of something better?

Corollary 2.7. *By Theorem 1.12 of Knudsen, the enveloping E_1 algebra of the sphere algebra $\mathfrak{g}_d^\bullet$ is dense within the pushforward $r_*(\mathbb{U}\mathcal{G}_d)$.*

2.1.4. *The case of non-zero level.* Pick a $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$. This choice determines a higher Kac-Moody factorization algebra $\mathbb{U}_\theta \mathcal{G}_d$, and we would like to produce maps akin to those of Proposition 2.6.

The simplest modification of the level zero situation is to introduce a central extension of the precosheaf

$$\mathtt{G}_d = \Omega^*_{\mathbb{R}_{>0},c} \otimes \mathfrak{g}_d^\bullet$$

as a precosheaf of L_∞ algebras on $\mathbb{R}_{>0}$, with the condition that this extension intertwines with the extension $r_*\mathcal{G}_{d,\theta}$ of $r_*\mathcal{G}_d$. In other words, we need a map

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{C}\cdot K[-1] \arrow[r] \arrow[d, "="] & \mathtt{G}_{d,\Theta'} \arrow[r] \arrow[d, "\widehat{\pi}"] & \mathtt{G}_d \arrow[r] \arrow[d, "\pi_{\mathfrak{g}_d^\bullet}"] & 0 \\ 0 \arrow[r] & \mathbb{C}\cdot K[-1] \arrow[r] & r_*\mathcal{G}_{d,\theta} \arrow[r] & r_*\mathcal{G}_d \arrow[r] & 0. \end{tikzcd}$$

of central extensions of L_∞ algebras. This condition fixes the problem completely, because we simply pull back the extension defining $r_*\mathcal{G}_{d,\theta}$. Let us extract an explicit description, which will be useful later. On an open $U \subset \mathbb{R}_{>0}$, the extension for $r_*\mathcal{G}_{d,\theta}$ is given by an integral

$$\int_{r^{-1}(U)} \theta(\alpha_0, \partial\alpha_1, \ldots, \partial\alpha_d) = \int_U \int_{S^{2d-1}} \theta(\alpha_0, \partial\alpha_1, \ldots, \partial\alpha_d)$$

that can be factored into a double integral. This formula indicates that Θ' must be given by the cocycle whose value on elements $\phi_i \otimes a_i \in \Omega^*_c \otimes \mathfrak{g}_d^\bullet$ is

$$\Theta'(\phi_0 \otimes a_0, \ldots, \phi_d \otimes a_d) = \int_U \int_{S^{2d-1}} \theta(\pi(\phi_0) \wedge \mathtt{j}(a_0), \partial(\pi(\phi_1) \wedge \mathtt{j}(a_1)), \ldots, \partial(\pi(\phi_d) \wedge \mathtt{j}(a_d)))$$

We thus obtain the following result.

Lemma 2.8. *For $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$, let $\mathtt{G}_{d,\theta}$ denote the precosheaf of L_∞ algebras obtained by extending $\mathtt{G}_d$ by the cocycle*

$$(\phi_0 \otimes a_0, \ldots, \phi_d \otimes a_d) \mapsto \int_U \int_{S^{2d-1}} \theta(\pi(\phi_0) \wedge \mathtt{j}(a_0), \partial(\pi(\phi_1) \wedge \mathtt{j}(a_1)), \ldots, \partial(\pi(\phi_d) \wedge \mathtt{j}(a_d))).$$

By construction, there is a canonical map

$$\pi_{\mathfrak{g}_d^\bullet,\theta} : \mathtt{G}_{d,\theta} \to r_*\mathcal{G}_{d,\theta}$$

of precosheaves of L_∞ algebras on $\mathbb{R}_{>0}$, and hence there is a map of factorization algebras

$$\mathbb{U}(\pi_{\mathfrak{g}_d^\bullet,\theta}) : \mathbb{U}_\theta \mathtt{G}_d \to r_* \mathbb{U}_\theta \mathcal{G}_d.$$

The maps remain degreewise dense, but now we are working with a twisted enveloping factorization algebra, which is slightly different in flavor than Knudsen's construction. The central parameter K parametrizes, in fact, a family of E_1 algebras that specializes at $K = 0$ to the enveloping E_1 algebra of the sphere algebra $\mathfrak{g}_d^\bullet$.

Corollary 2.9. *There is a family of E_1 algebras over the affine line* $\mathrm{Spec}(\mathbb{C}[K])$ *with the enveloping E_1 algebra of the sphere algebra $\mathfrak{g}_d^\bullet$ at the origin. This family is dense within the pushforward* $r_*(\mathbb{U}_\theta \mathcal{G}_d)$.

2.1.5. *A comparison with the work of Faonte-Hennion-Kapranov.* There is a variant of the preceding result that is particularly appealing in light of [**?**], which is to provide a map of factorization algebras on the positive reals

$$\mathbb{U}(\Omega_c^* \otimes \widetilde{\mathfrak{g}}_{d,\theta}^\bullet) \to r_* \mathcal{G}_{d,\theta},$$

where the source is the factorization algebra encoding the enveloping E_1 algebra of $\widetilde{\mathfrak{g}}_{d,\theta}^\bullet$. Specializing the central parameters to zero on both sides must recover the map $\pi_{\mathfrak{g}_d^\bullet}$ of Proposition 2.6. Such a map has two connected consequences:

(1) It shows that the higher current Lie algebras $\widetilde{\mathfrak{g}}_{d,\theta}^\bullet$ of [**?**] "control" our twisted current factorization algebras $\mathcal{G}_{d,\theta}$ in the same way that the affine Kac-Moody Lie algebras control their vertex algebras.
(2) It shows that our factorization algebras $\mathcal{G}_{d,\theta}$ know the information encoded by the Lie algebras $\widetilde{\mathfrak{g}}_{d,\theta}^\bullet$ introduced in [**?**].

In short this map provides a conduit for transferring insights between derived algebraic geometry (as represented by the [**?**] approach) and quantum field theory (as represented by ours).

OG: What do you think of the preceding and following?

Remark 2.10. Before embarking on the construction of the map, we remark that it was a pleasant surprise to come upon [**?**] and to find that they had explored terrain that we had approached from the direction exposed in this paper, i.e., the higher dimensional generalization of results from [**?**]. Their Jouanolou model A_d allowed us to sharpen our results into something more tractable and more obviously analogous to Laurent polynomials, and their discussion of the global derived geometry verified our guesses, which were beyond our technical powers. As we will see in Section OG: add cross ref to large N section, their use of cyclic homology to obtain natural extensions resonated beautifully with some of our thoughts and suggested a different vision, rooted in derived geometry, of the role of large N limits. We thank Faonte, Hennion, and Kapranov for inspiring and enlightening conversations and correspondence on these subjects.

Constructing the map requires overcoming two issues. First, note that

$$\widetilde{\mathfrak{G}}_{d,\theta} = \Omega_c^* \otimes \widetilde{\mathfrak{g}}_{d,\theta}^\bullet$$

can be viewed as an extension

$$\Omega_c^* \otimes \mathbb{C}K[-1] \to \widetilde{\mathtt{G}}_{d,\theta} \to \mathtt{G}_d$$

of precosheaves of L_∞ algebras on $\mathbb{R}_{>0}$. By contrast, $r_* \mathcal{G}_{d,\theta}$ is an extension by the constant precosheaf $\mathbb{C}K[-1]$. There is, however, a natural map of precosheaves

$$\int : \Omega_c^* \to \mathbb{C}$$

to employ, since integration is well-defined on compactly-supported forms. The second issue looks more serious: the two cocycles at play seem different at first glance. On an open $U \subset \mathbb{R}_{>0}$, we found that the cocycle for the pushforward is given by

$$\int_U \int_{S^{2d-1}} \theta(\pi(\phi_0) \wedge \mathtt{j}(a_0), \partial(\pi(\phi_1) \wedge \mathtt{j}(a_1)), \dots, \partial(\pi(\phi_d) \wedge \mathtt{j}(a_d))),$$

where we evaluate it on the image under $\pi_{\mathfrak{g}_d^\bullet}$ of elements $\phi_i \otimes a_i \in \Omega_c^* \otimes \mathfrak{g}_d^\bullet$. On the other hand, on these elements the FHK extension is given by

$$(\phi_0 \wedge \cdots \wedge \phi_d) \int_{S^{2d-1}} \theta(\mathtt{j}(a_0), \partial(\mathtt{j}(a_1)), \dots, \partial(\mathtt{j}(a_d))).$$

(Note that in the FHK case, we do not integrate over U because we extend by Ω_c^*.) The key difference here is that the FHK extension does not involve applying ∂ to the $(0,*)$-components of the pulled back forms $r^*\phi_i$. It separates the ϕ_i and a_i contributions, whereas the other cocycle mixes them. The tension is resolved by showing these cocycles are cohomologous.

Lemma 2.11. *There is a local cochain η that cobounds the difference between these cocycles.*

Proof. We note that the Lie algebra $\mathfrak{g}$ and the invariant polynomial θ play no substantive role in the problem. The issue here is about calculus. Hence it suffices to consider the case that $\mathfrak{g}$ is the one-dimensional abelian Lie algebra and θ is the unique-up-to-scale monomial of degree $d+1$ (i.e., "x^{d+1}").

Let

$$E = r\frac{\partial}{\partial r}$$

denote the Euler vector field on $\mathbb{R}_{>0}$, and let

$$\mathrm{d}\vartheta = \sum_i \frac{\mathrm{d}z_i}{z_i}$$

denote a $(1,0)$-form on $\mathring{\mathbb{A}}^d = \mathbb{C}^d \setminus 0$.

For concision we express the element $\varphi_i \otimes a_i$ in $\Omega_c^*(U) \otimes A_d$ by $\varphi_i a_i$. We now define

$$\eta(\varphi_0 a_0, \dots, \varphi_d a_d) = \sum_{i=1}^d \left(\int_U \varphi_0 \, (\iota_E \varphi_i) \, \varphi_1 \cdots \widehat{\varphi}_i \cdots \varphi_d \right) \left(\oint (a_0 a_i \mathrm{d}\vartheta) \, \partial a_1 \cdots \widehat{\partial a_i} \cdots \partial a_d \right).$$

OG: And now your explicit proof peters out. It looks like a direct computation now, but there's probably something short of writing the details that nonetheless explains where this formula comes from. □

OG: Everything below here is the raw material of your original approach.

We wish to produce a map

$$\pi_\theta : \Omega^*_{\mathbb{R}_{>0},c} \otimes \widehat{\mathfrak{g}}_{d,\theta} \to r_* \mathcal{G}_\theta$$

of precosheaves of L_∞ algebras on $\mathbb{R}_{>0}$, generalizing the map $\pi_{\mathfrak{g}^\bullet_d}$ of Proposition 2.6. Specializing the parameter K to zero, we should recover that map. By applying $C^{\mathrm{Lie}}_*(-)$ to π_θ, we will obtain the desired map of factorization algebras.

We now proceed to the proof of Theorem . The dg Lie algebra $\mathfrak{g}_{d,\theta}$ determines a dg associative algebra via its universal enveloping algebra $U(\mathfrak{g}_{d,\theta})$. This dg algebra determines a factorization algebra on the one-manifold $\mathbb{R}_{>0}$ that assigns to every open interval $I \subset \mathbb{R}_{>0}$ the dg vector space $U(A_d \otimes \mathfrak{g})$. The factorization product is uniquely determined by the algebra structure. Henceforth, we denote this factorization algebra by $U(\mathfrak{g}_{d,\theta})^{fact}$.

To prove the theorem we will construct a sequence of maps of factorization Lie algebras on $\mathbb{R}_{>0}$:

$$\begin{tikzcd} & \mathcal{G}_1 \arrow[dr, "\Phi_1"] & & \mathcal{G}_2 \\ \mathcal{G}_0 \arrow[ur, "\simeq", "\Phi_0"'] & & \mathcal{G}_1' \arrow[ur, "\Phi_2"'] & . \end{tikzcd}$$

The enveloping factorization of $\mathcal{G}_0$ is equivalent to the factorization algebra $U(\widehat{\mathfrak{g}}_{d,\theta})^{fact}$. Moreover, the enveloping factorization of $\mathcal{G}_2$ is the push-forward of of the higher Kac-Moody factorization algebra $r_*\mathbb{U}\mathcal{G}$. Hence, the desired map of factorization algebras is produced by applying the enveloping factorization functor to the above composition of factorization Lie algebras.

First, we introduce the factorization Lie algebra $\mathcal{G}_0$. To an open set $I \subset \mathbb{R}$, it assigns the dg Lie algebra $\mathcal{G}_0(I) = \Omega^*_c(I) \otimes \widehat{\mathfrak{g}}_{d,\theta}$, where $\widehat{\mathfrak{g}}_{d,\theta}$ is the central extension from Equation (6). The differential and Lie bracket are determined by the fact that we are tensoring a commutative dg algebra with a dg Lie algebra. A slight variant of Proposition 3.4.0.1 in **[?]**, which shows that the one-dimensional enveloping factorization of an ordinary Lie algebra produces its ordinary universal enveloping algebra, shows that there is a quasi-isomorphism of factorization algebras on $\mathbb{R}$,

$$(U\widehat{\mathfrak{g}}_{d,\theta})^{fact} \xrightarrow{\simeq} C^{\mathrm{Lie}}_*(\mathcal{G}_0).$$

The factorization Lie algebra $\mathcal{G}_0$ is a central extension of the factorization Lie algebra $\Omega^*_{\mathbb{R},c} \otimes (A_d \otimes \mathfrak{g})$ by the trivial module $\Omega^*_c \oplus \mathbb{C} \cdot K$. Indeed, the cocycle determining the central extension is given by

$$\theta_0(\varphi_0\alpha_0, \ldots, \varphi_d\alpha_d) = (\varphi_0 \wedge \cdots \wedge \varphi_d)\theta_{A_d}(\alpha_1, \ldots, \alpha_d).$$

The factorization Lie algebra $\Omega^*_{\mathbb{R},c} \otimes (A_d \otimes \mathfrak{g})$ is the compactly supported sections of the local Lie algebra $\Omega^*_{\mathbb{R}} \otimes (A_d \otimes \mathfrak{g})$ and this cocycle determining the extension is a local cocycle.

Next, we define the factorization dg Lie algebra $\mathcal{G}_1$ on $\mathbb{R}$. This is also obtained as a central extension of the factorization Lie algebra $\Omega^*_{\mathbb{R},c} \otimes (A_d \otimes \mathfrak{g})$:

$$0 \to \mathbb{C} \cdot K[-1] \to \mathcal{G}_1 \to \Omega^*_{\mathbb{R},c} \otimes (A_d \otimes \mathfrak{g}) \to 0$$

determined by the following cocycle. For an open interval I write $\varphi_i \in \Omega^*_c(I)$, $\alpha_i \in A_d \otimes \mathfrak{g}$. The cocycle is defined by

$$\theta_1(\varphi_0\alpha_0, \ldots, \varphi_d\alpha_d) = \left(\int_I \varphi_0 \wedge \cdots \varphi_d\right) \theta_{\mathrm{FHK}}(\alpha_0, \ldots, \alpha_d) \tag{7}$$

where θ_{FHK} was defined in Equation 2.1.2.

The functional θ_1 determines a local cocycle in $\mathrm{C}^*_{\mathrm{loc}}\left(\Omega^*_{\mathbb{R}} \otimes (A_d \otimes \mathfrak{g})\right)$ of degree one.

We now define a map of factorization Lie algebras $\Phi_0 : \mathcal{G}_0 \to \mathcal{G}_1$. On and open set $I \subset \mathbb{R}$, we define the map $\Phi_0(I) : \mathcal{G}_0(I) \to \mathcal{G}_1(I)$ by

$$\Phi_0(I)(\varphi\alpha, \psi K) = \left(\varphi\alpha, \int \psi \cdot K\right).$$

For a fixed open set $I \subset \mathbb{R}$, the map Φ_0 fits into the commutative diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \Omega^*_c(I) \otimes \mathbb{C} \cdot K \arrow[r] \arrow[d, "\int", "\simeq"'] & \mathcal{G}_0(I) \arrow[r] \arrow[d, "\Phi_0(I)"] & \Omega^*_c(I) \otimes (A_d \otimes \mathfrak{g}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{C} \cdot K[-1] \arrow[r] & \mathcal{G}_1(I) \arrow[r] & \Omega^*_c(I) \otimes (A_d \otimes \mathfrak{g}) \arrow[r] & 0.
\end{tikzcd}$$

To see that $\Phi_0(I)$ is a map of dg Lie algebras we simply observe that the cocycles determining the central extensions are related by $\theta_1 = \int \circ\, \theta_0$, where $\int : \Omega^*_c(I) \to \mathbb{C}$ as in the diagram above. Since $\int$ is a quasi-isomorphism, the map $\Phi_0(I)$ is as well. It is clear that as we vary the interval I we obtain a quasi-isomorphism of factorization Lie algebras $\Phi_0 : \mathcal{G}_0 \xrightarrow{\simeq} \mathcal{G}_1$.

We now define the factorization dg Lie algebra $\mathcal{G}_1'$. Like $\mathcal{G}_0$ and $\mathcal{G}_0$, it is a central extension of $\Omega^*_{\mathbb{R},c} \otimes (A_d \otimes \mathfrak{g})$. The cocycle determining the central extension is defined by

$$\theta_1'(\varphi_0 a_0 X_0, \ldots, \ldots, \varphi_d a_d X_d) = \theta_1(\varphi_0 a_0 X_0, \ldots, \ldots, \varphi_d a_d X_d) + \widetilde{\theta}_1(\varphi_0 a_0 X_0, \ldots, \ldots, \varphi_d a_d X_d)$$

where θ_1 was defined in Equation (7). Before writing down the explicit formula for $\widetilde{\theta}_1$ we introduce some notation. Set

$$E = r\frac{\partial}{\partial r},$$
$$\mathrm{d}\vartheta = \sum_i \frac{\mathrm{d}z_i}{z_i}.$$

We view E as a vector field on $\mathbb{R}_{>0}$ and $\mathrm{d}\vartheta$ as a $(1,0)$-form on $\mathbb{C}^d \setminus 0$. Define the functional

$$\widetilde{\theta}_1(\varphi_0 a_0 X_0, \ldots, \varphi_d a_d X_d) = \frac{1}{2}\sum_{i=1}^{d}\left(\int_I \varphi_0 (E \cdot \varphi_i)\varphi_1 \cdots \widehat{\varphi_i} \cdots \varphi_d\right)\left(\oint (a_0 a_i \mathrm{d}\vartheta)\, \partial a_1 \cdots \widehat{\partial a_i} \cdots \partial a_d\right)\theta(X_0, \ldots, X_d).$$

The functional $\widetilde{\theta}$ defines a local functional in $\mathrm{C}^*_{\mathrm{loc}}\left(\Omega^*_{\mathbb{R}_{>0}} \otimes (A_d \otimes \mathfrak{g})\right)$ of cohomological degree one. One immediately checks that it is a cocycle. This completes the definition of the factorization Lie algebra $\mathcal{G}_1'$.

The factorization Lie algebras $\mathcal{G}_1$ and $\mathcal{G}_1'$ are identical as precosheaves of vector spaces. In fact, if we put a filtration on $\mathcal{G}_1$ and $\mathcal{G}_1'$ where the central element K has filtration degree one, then the associated graded factorization Lie algebras $\mathrm{Gr}\ \mathcal{G}_1$ and $\mathrm{Gr}\ \mathcal{G}_1'$ are also identified. The only difference in the Lie algebra structures comes from the deformation of the cocycle determining the extension of $\mathcal{G}_1'$ given by $\widetilde{\theta}_1$.

In fact, we will show that $\widetilde{\theta}_1$ is actually an exact cocycle via the cobounding element $\eta \in \mathrm{C}^*_{\mathrm{loc}}\left(\Omega^*_{\mathbb{R}_{>0}} \otimes (A_d \otimes \mathfrak{g})\right)$ defined by

$$\eta(\varphi_0 a_0 X_0, \ldots, \varphi_d a_d X_d) = \sum_{i=1}^{d}\left(\int_I \varphi_0\, (\iota_E \varphi_i)\, \varphi_1 \cdots \widehat{\varphi_i} \cdots \varphi_d\right)\left(\oint (a_0 a_i \mathrm{d}\vartheta)\, \partial a_1 \cdots \widehat{\partial a_i} \cdots \partial a_d\right)\theta(X_0, \ldots, X_d).$$

Lemma 2.12. *One has $\mathrm{d}\eta = \widetilde{\theta}_1$, where d is the differential for the cochain complex $\mathrm{C}^*_{\mathrm{loc}}(\Omega^*_{\mathbb{R}_{>0}} \otimes (A_d \otimes \mathfrak{g}))$. In particular, the factorization Lie algebras $\mathcal{G}_1$ and $\mathcal{G}_1'$ are quasi-isomorphic (as L_∞ algebras). An explicit quasi-isomorphism is given by the L_∞ map $\Phi_1 : \mathcal{G}_1 \to \mathcal{G}_1'$ that sends the central element K to itself and an element $(\varphi_0 a_0 X_0, \ldots, \varphi_d a_d X_d) \in \mathrm{Sym}^{d+1}(\Omega^*_c \otimes (A_d \otimes \mathfrak{g})$ to*

$$(\varphi_0 a_0 X_0, \ldots, \varphi_d a_d X_d) + \eta(\varphi_0 a_0 X_0, \ldots, \varphi_d a_d X_d) \cdot K \in \mathrm{Sym}^{d+1}(\Omega^*_c \otimes (A_d \otimes \mathfrak{g})) \oplus \mathbb{C} \cdot K.$$

2.2. **An E_d algebra by compactifying along tori.** There is another direction that one may look to extend the notion of affine algebras to higher dimensions. The affine algebra is a central extension of the loop algebra on $\mathfrak{g}$. Instead of looking at higher dimensional sphere algebras, one can consider higher *torus* algebras; or iterated loop algebras:

$$L^d\mathfrak{g} = \mathbb{C}[z_1^{\pm}, \cdots, z_d^{\pm}] \otimes \mathfrak{g}.$$

These iterated loop algebras are algebraic versions of the torus mapping space $\mathrm{Map}(S^1 \times \cdots \times S^1, \mathfrak{g})$. In this section we show what information the Kac-Moody vertex algebra implies about extensions of such iterated loop algebras.

To do this we specialize the Kac-Moody factorization algebra to the complex manifold $(\mathbb{C}^\times)^d$, which is homotopy equivalent to the topologists torus $(S^1)^{\times d}$. We show, in a similar way as above, how to extract the structure of an E_d algebra from considering the nesting of "polyannuli" in $(\mathbb{C}^\times)^d$. These E_d-algebras are related to interesting extensions of the Lie algebra $L^d\mathfrak{g}$.

When $d = 1$, we have seen that the nesting of ordinary annuli give rise to the structure of an associative algebra. For $d > 1$, a polyannulus is a complex submanifold of the form $\mathrm{Ann}_1 \times \cdots \times \mathrm{Ann}_d \subset (\mathbb{C}^\times)^d$ where each $\mathrm{Ann}_i \subset \mathbb{C}^\times$ is an ordinary annulus.

Equivalently, a polyannulus is the complement of a closed polydisk inside of a larger open polydisk. We will see how the nesting of annuli in each component gives rise to the structure of a locally constant factorization algebra in d *real* dimensions, and hence defines an E_d algebra.

A result of Knudsen [**?**], which we recall below, states that every dg Lie algebra determines an E_d-algebra, for any $d > 1$, called the universal E_d enveloping algebra. To state the result precisely we need to be in the context of ∞-categories.

Theorem 2.13 ([**?**]). *Let $\mathcal{C}$ be a stable, $\mathbb{C}$-linear, presentable, symmetric monoidal ∞-category. There is an adjunction*

$$U^{E^d} : \mathrm{LieAlg}(\mathcal{C}) \leftrightarrows E_d\mathrm{Alg}(\mathcal{C}) : F$$

such that for any object $X \in \mathcal{C}$ one has $\mathrm{Free}_{E_d}(X) \simeq U^{E_d}\mathrm{Free}_{Lie}(\Sigma^{d-1}X)$.

We are most interested in the case $\mathcal{C}$ is the category of chain complexes with tensor product $\mathrm{Ch}^{\otimes}$. In this situation, the enveloping algebra U^{E^d} agrees with the ordinary universal enveloping algebra when $d = 1$.

When the twisting cocycle defining the Kac-Moody factorization algebra is zero we will see that the E_d algebra coming from the product of polyannuli is equivalent to $U^{E_d}(L^d\mathfrak{g})$. When we turn on a twisting cocycle we will find the E_d-enveloping algebra of a central extension of the iterated loop algebra.

The Kac-Moody factorization algebra on the d-fold $(\mathbb{C}^\times)^d$ determines a real d-dimensional factorization algebra by considering the radius in each complex direction. This factorization algebra on $(\mathbb{R}_{>0})^d$ is defined by the pushforward $\vec{r}_*(\mathcal{G}_{\mathbb{C}^{\times d}})$, where $\vec{r} : (\mathbb{C}^\times)^d \to (\mathbb{R}_{>0})^d$ is the projection $(z_1, \ldots, z_d) \mapsto (|z_1|, \cdots, |z_d|)$.

On the Lie algebra side, it is an immediate calculation to see that the following formula defines a cocycle on $L^d\mathfrak{g}$ of degree $(d+1)$:

$$\begin{array}{rccl} L^d\theta : & (L^d\mathfrak{g})^{\otimes d+1} & \to & \mathbb{C} \\ & (f_0 \otimes X_0) \otimes \cdots \otimes (f_d \otimes X_d) & \mapsto & \theta(X_0, \ldots, X_d) \oint_{|z_1|=1} \cdots \oint_{|z_d|=1} f_0 \mathrm{d}f_1 \cdots \mathrm{d}f_d. \end{array}$$

Here $f_i \otimes X_i \in L^d\mathfrak{g} = \mathbb{C}[z_1^{\pm}, \cdots, z_d^{\pm}] \otimes \mathfrak{g}$. The above is just an iterated version of the usual residue pairing. This cocycle determines a shifted Lie algebra extension of the iterated loop algebra

$$\mathbb{C}[d-1] \to \widehat{L^d}\mathfrak{g}_\theta \to L^d\mathfrak{g},$$

that appears in the theorem below.

The following can be proved in exact analogy as the above result for sphere algebras and we omit the proof here.

Proposition 2.14. *Fix $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$ and let $\vec{r}_*\mathbb{U}_\theta\mathcal{G}_{(\mathbb{C}^\times)^d}$ be the factorization algebra on $(\mathbb{R}_{>0})^d$ obtained by reducing the Kac-Moody factorization algebra along the d-torus. There exists* *a dense d-dimensional subfactorization algebra $\mathcal{F}^{lc}$ that is locally constant and is equivalent, as E_d-algebras, to*

$$U^{E_d}\left(\widehat{L^d\mathfrak{g}_\theta}\right).$$

3. Higher Kac-Moody factorization algebras as symmetries of field theories

OG: Please insert your revised version here at some point

4. Global aspects of the higher Kac-Moody factorization algebras

In this section we explore global properties of the Kac-Moody factorization algebra on complex manifolds. The first of which is the (shifted) Poisson structure on the "classical limit" of the Kac-Moody factorization algebra. In the world of CFT, many vertex algebras admit classical limits which have the structure of *Poisson vertex algebras*. Roughly, these are vertex algebras with a commutative OPE together with a family of z-dependent brackets which are biderivations for the OPE. The concept of a P_0-factorization algebra specializes to this in the case of complex one-dimensional holomorphic factorization algebras but applies more generally to factorization algebras in any dimension.

Next, we will compute the factorization homology, or global sections, of the Kac-Moody factorization algebra along a class of complex manifolds called *Hopf manifolds*. We choose to focus on these because the answer admits a concise description in terms of classical algebra, and for the application for studying the gauge equivariance for the partition function of the higher dimensional σ-model from Chapter **??**. After this, we discuss variants of the twisted Kac-Moody factorization algebra that exist on complex d-folds. These variants are related to the approach of studying higher dimensional holomorphic gauge symmetries due to Nekrasov, et. al..

4.1. Hopf manifolds and twisted indices.

We focus on a family of complex manifolds defined by Hopf in **[?]** defined in every complex dimension d.

Definition 4.1. Fix an integer $d \geq 1$. Let $f : \mathbb{C}^d \to \mathbb{C}^d$ be a polynomial map such that $f(0) = 0$ such that its Jacobian at zero $Jac(f)(0)$ is invertible with eigenvalues $\{\lambda_i\}$ all satisfying $|\lambda_i| < 1$. Define the *Hopf manifold associated to f* to be the d-dimensional complex manifold

$$X_f := \left(\mathbb{C}^d \setminus \{0\}\right) \big/ (x \sim f(x)).$$

Note that X_f is compact for any f. In the case $d = 1$ all Hopf surfaces are equivalent to elliptic curves.

Lemma 4.2. *For any f there is a diffeomorphism $X_f \cong S^{2d-1} \times S^1$.*

This implies that when $d > 1$, the cohomology $H^2_{dR}(X_f) = 0$ for any f. In particular, X_f is *not* Kähler when $d > 1$. For $1 \leq i \leq d$ let $q_i \in D(0,1)^\times$ be a nonzero complex number of modulus $|q_i| < 1$. The d-dimensional *Hopf manifold of type* $\mathbf{q} = (q_1, \ldots, q_d)$ is the following quotient of punctured affine space $\mathbb{C}^d \setminus \{0\}$ by the discrete group $\mathbb{Z}^d$:

$$X_{\mathbf{q}} = \left(\mathbb{C}^d \setminus \{0\}\right) \Big/ \left((z_1, \ldots, z_d) \sim (q_1^{2\pi i \mathbb{Z}} z_1, \ldots, q_d^{2\pi i \mathbb{Z}} z_d)\right).$$

Note that in the case $d = 1$ we recover the usual description of an elliptic curve $X_{\mathbf{q}} = E_q = \mathbb{C}^\times / q^{2\pi i \mathbb{Z}}$. We will denote the quotient map $p_{\mathbf{q}} : \mathbb{C}^d \setminus \{0\} \to X_{\mathbf{q}}$.

For any d and tuple $(q_1, \ldots, q_d)$ as above, we see that as a smooth manifold there is a diffeomorphism $X_{\mathbf{q}} \cong S^{2d-1} \times S^1$. Indeed, the radial projection map $\mathbb{C}^d \setminus \{0\} \to \mathbb{R}_{>0}$ defines a smooth S^{2d-1}-fibration over $\mathbb{R}_{>0}$. Passing to the quotient, we obtain an S^{2d-1}-fibration

$$X_{\mathbf{q}} \to \mathbb{R}_{>0} / \left(r \sim \lambda^{\mathbb{Z}} \cdot r\right) \cong S^1.$$

Here, $\lambda = (|q_1|^2 + \cdots + |q_d|^2)^{1/2} > 0$. Since there are no non-trivial S^{2d-1} fibrations over S^1 we obtain $X_{\mathbf{q}} = S^{2d-1} \times S^1$ as smooth manifolds.

Proposition 4.3. *Let X be a Hopf manifold and suppose $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$ is any $\mathfrak{g}$-invariant polynomial of degree $(d+1)$. Then, there is a quasi-isomorphism of $\mathbb{C}[K]$-modules*

$$\int_X \mathbb{U}_\theta(\mathcal{G}_X) \simeq \mathrm{Hoch}_*(U\mathfrak{g})[K].$$

Proof. Let's first consider the untwisted case where the statement reduces to $\int_X \mathbb{U}(\mathcal{G}_X) \simeq \mathrm{Hoch}_*(U\mathfrak{g})$. The factorization homology on the left hand side is computed by

$$\int_X \mathbb{U}(\mathcal{G}_X) = \mathrm{C}_*^{\mathrm{Lie}}(\Omega^{0,*}(X) \otimes \mathfrak{g}).$$

Now, since every Hopf manifold is Dolbeault formal there is a quasi-isomorphism of commutative dg algebras

$$\left(H^{0,*}(X), 0\right) \simeq \left(\Omega^{0,*}(X), \overline{\partial}\right).$$

In fact, we have written down a preferred presentation for the cohomology ring of X given by $H^{0,*}(X) = \mathbb{C}[\delta]$ where $|\delta| = 1$. A particular Dolbeault representative for δ given by

$$\overline{\partial}(\log |z|^2) = \sum_i \frac{z_i \mathrm{d}\overline{z}_i}{|z|^2}$$

where $z = (z_1, \ldots, z_d)$ is the coordinate on $\mathbb{C}^d \setminus \{0\}$.

Applied to the global sections of the Kac-Moody we see that there is a quasi-isomorphism

$$\int_X \mathbb{U}(\mathcal{G}_X) \simeq \mathrm{C}_*^{\mathrm{Lie}}(\mathbb{C}[\delta] \otimes \mathfrak{g}).$$

Now, note that $\mathrm{C}_*^{\mathrm{Lie}}(\mathbb{C}[\delta] \otimes \mathfrak{g}) = \mathrm{C}_*^{\mathrm{Lie}}(\mathfrak{g} \oplus \mathfrak{g}[-1]) = \mathrm{C}_*^{\mathrm{Lie}}(\mathfrak{g}, \mathrm{Sym}(\mathfrak{g}))$, where $\mathrm{Sym}(\mathfrak{g})$ is the symmetric product of the adjoint action of $\mathfrak{g}$ on itself. By Poincaré-Birkoff-Witt there is an isomorphism of vector spaces $\mathrm{Sym}(\mathfrak{g}) = U\mathfrak{g}$, so we can write this as $\mathrm{C}_*^{\mathrm{Lie}}(\mathfrak{g}, \mathrm{Sym}(\mathfrak{g}))$.

Now, any $U(\mathfrak{g})$-bimodule M is automatically a module for the Lie algebra $\mathfrak{g}$ by the formula $x \cdot m = xm - mx$ where $x \in \mathfrak{g}$ and $m \in M$. Moreover, for any such bimodule there is a quasi-isomorphism of cochain complexes

$$C_*^{\mathrm{Lie}}(\mathfrak{g}, M) \simeq \mathrm{Hoch}_*(U\mathfrak{g}, M).$$

This is proved, for instance, in Section 2.3 of [**?**]. Applied to the bimodule $M = U\mathfrak{g}$ itself we obtain a quasi-isomorphism $C_*^{\mathrm{Lie}}(\mathfrak{g}, U\mathfrak{g}) \simeq \mathrm{Hoch}(U\mathfrak{g})$.

The twisted case is similar. Let θ be as in the statement. Then, the factorization homology is equal to

$$\int_X \mathbb{U}_\theta(\mathcal{G}_X) = \left(\mathrm{Sym}(\Omega^{0,*}(X) \otimes \mathfrak{g})[K], \overline{\partial} + \mathrm{d}_{CE} + \mathrm{d}_\theta\right).$$

Applying Dolbeault formality again we see that this is quasi-isomorphic to the cochain complex

$$(\mathrm{Sym}(\mathfrak{g}[\delta])[K], \mathrm{d}_{CE} + \mathrm{d}_\theta)\,. \tag{8}$$

We note that d_θ is identically zero on $\mathrm{Sym}(\mathfrak{g}[\delta])$. Indeed, for degree reasons, at least one of the inputs must be from $\mathfrak{g} \hookrightarrow \mathfrak{g}[\delta] = \mathfrak{g} \oplus \mathfrak{g}[-1]$, which consists of constant functions on X with values in the Lie algebra $\mathfrak{g}$. In the formula for the local cocycle from Proposition 1.19 associated to θ it is clear that if any one of the inputs is constant the cocycle vanishes. Indeed, one can integrate by parts to put it in the form $\int \partial\alpha \cdots \partial\alpha$, which is the integral of a total derivative, hence zero since X has no boundary. Thus (8) just becomes the Chevalley-Eilenberg complex with values in the trivial module $\mathbb{C}[K]$. By the same argument as in the untwisted case, we conclude that in this case the factorization homology is quasi-isomorphic to $\mathrm{Hoch}_*(U\mathfrak{g})[K]$ as desired. □

There is an interesting consequence of this calculation to the Hochschild homology for the A_∞ algebra $U(\widehat{\mathfrak{g}}_{d,\theta})$. It is easiest to state this when X is a Hopf manifold of the form $(\mathbb{C}^d \setminus \{0\})/q^{\mathbb{Z}}$ for a single $q = q_1 = \cdots = q_d \in D(0,1)^\times$ where the quotient is by the relation $(z_1, \ldots, z_d) \simeq (q^{\mathbb{Z}} z_1, \ldots, q^{\mathbb{Z}})$. Let $p_q : \mathbb{C}^d \setminus \{0\} \to X$ be the quotient map. Consider the following diagram

$$\begin{tikzcd}
\mathbb{C}^d \setminus \{0\} \arrow[r, "p_q"] \arrow[d, "\rho"] & X \arrow[d, "\overline{\rho}"] \\
\mathbb{R}_{>0} \arrow[r, "\overline{p}_q"] & S^1
\end{tikzcd}$$

Here, ρ is the radial projection map and $\overline{\rho}$ is the induced map defined by the quotient. The action of $\mathbb{Z}$ on $\mathbb{C}^d \setminus \{0\}$ gives $\mathcal{G}_{\mathbb{C}^d\setminus\{0\}}$ the structure of a $\mathbb{Z}$-equivariant factorization algebra. In turn, this determines an action of $\mathbb{Z}$ on pushforward factorization algebra $\rho_*\mathcal{G}_{\mathbb{C}^d\setminus\{0\}}$. We have seen that there is a dense locally constant subfactorization algebra on $\mathbb{R}_{>0}$ of the pushforward that is equivalent as an E_1 algebra to $U(\widehat{\mathfrak{g}}_{d,\theta})$. A consequence of excision for factorization homology, see Lemma 3.18 [**?**], implies that there is a quasi-isomorphism

$$\mathrm{Hoch}_*(U(\widehat{\mathfrak{g}}_{d,\theta}), q) \simeq \int_{S^1} \overline{\rho}_* \mathbb{U}_\alpha(\mathcal{G}_X),$$

where the right-hand side is the Hochschild homology of the algebra $U\widehat{\mathfrak{g}}_{d,\theta}$ with coefficients in the bimodule $U\widehat{\mathfrak{g}}_{d,\theta}$ with the ordinary left-module structure and right-module structure given by twisting the ordinary action by the automorphism corresponding to the element $1 \in \mathbb{Z}$ on the algebra.

Moreover, by the push-forward for factorization homology, Proposition 3.23 [**?**], there is an equivalence

$$\int_{S^1} \overline{\rho}_* \mathbb{U}_\alpha(\mathcal{G}_X) \xrightarrow{\simeq} \int_X \mathbb{U}_\alpha(\mathcal{G}_X).$$

We have just shown that the factorization homology of $\mathcal{G}_X$ is equal to the Hochschild homology of $U\mathfrak{g}$ so that

$$\mathrm{Hoch}_*(U(\widehat{\mathfrak{g}}_{d,\theta}), q) \simeq \mathrm{Hoch}_*(U\mathfrak{g})[K].$$

This statement is purely algebraic as the dependence on the manifold for which the Kac-Moody lives has dropped out. It may be easiest to understand in the case $d = 1$ and $\theta = 0$. Then $\mathfrak{g}_{d,\theta}$ is the loop algebra $L\mathfrak{g} = g[z, z^{-1}]$. The action of $\mathbb{Z}$ on $L\mathfrak{g}$ rotates the loop parameter: for $z^n \otimes \mathfrak{g} \in L\mathfrak{g} = \mathbb{C}[z, z^{-1}] \otimes \mathfrak{g}$ the action of $1 \in \mathbb{Z}$ is $1 \cdot (z^n \otimes \mathfrak{g}) = q^n z^n \otimes \mathfrak{g}$. In turn, the bimodule structure of $U(\mathfrak{g}[z, z^{-1}])$ on itself, which we denote $U(\mathfrak{g}[z, z^{-1}])_q$ is the ordinary one on the left and on the right is given by twisting by the automorphism corresponding to $1 \in \mathbb{Z}$. The complex $\mathrm{Hoch}_*(U(g[z, z^{-1}]), q)$ is the Hochschild homology of $U(\mathfrak{g}[z, z^{-1}])$ with values in this bimodule. Thus, the statement implies that there is a quasi-isomorphism

$$\mathrm{Hoch}_*\left(U(\mathfrak{g}[z, z^{-1}]), U(\mathfrak{g}[z, z^{-1}])_q\right) \simeq \mathrm{Hoch}(U\mathfrak{g}).$$

4.2. **Variants of the higher Kac-Moody factorization algebras.** OG: Find more specific title

So far we have mostly restricted ourselves to studying the Kac-Moody factorization algebra corresponding to local cocycles of type $\mathrm{j}_X(\theta)$ where $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$. There is another class of local cocycles that appear when studying symmetries of holomorphic theories. Unlike the cocycle $\mathrm{j}_X(\theta)$, which in some sense did not depend on the manifold X, this class of cocycles is more dependent on the manifold for which the current algebra lives.

Let X be a complex manifold of dimension d and suppose ω is a (k,k) form on X. Fix, in addition, a form $\theta_{d+1-k} \in \mathrm{Sym}(\mathfrak{g}^*)^{\mathfrak{g}}$. Then, we may consider the cochain on $\mathcal{G}(X)$:

$$\begin{array}{cccc} \phi_{\theta,\omega} : & \mathcal{G}(X)^{\otimes d+1-k} & \to & \mathbb{C} \\ & \alpha_0 \otimes \cdots \otimes \alpha_{d-k} & \mapsto & \int_X \omega \wedge \theta_{d+1-k}(\alpha_0, \partial\alpha_1, \ldots, \partial\alpha_{d-k}) \end{array}$$

It is clear that $\phi_{\theta,\omega}$ is a local cochain on $\mathcal{G}(X)$.

Lemma 4.4. *Let $\theta \in \mathrm{Sym}^{d+1-k}(\mathfrak{g}^*)^{\mathfrak{g}}$ and suppose $\omega \in \Omega^{k,k}(X)$ satisfies $\overline{\partial}\omega = 0$ and $\partial\omega = 0$. Then, $\phi_{\theta,\omega} \in C^*_{\mathrm{loc}}(\mathcal{G}_X)$ is a local cocycle. Moreover, for fixed θ the cohomology class $[\phi_{\theta,\omega}] \in H^1_{\mathrm{loc}}(\mathcal{G}_X)$ only depends on the cohomology class*

$$[\omega] \in H^k(X, \Omega^k_{cl}).$$

Note that when $\omega = 1$ it trivially satisfies the conditions of the lemma. In this case $\phi_{\theta,1} = \mathfrak{j}_X(\theta)$ in the notation of the last section.

OG: I moved everything above to Section 1.3.5.

This class of cocycles is related to the ordinary Kac-Moody factorization and vertex algebra on Riemann surfaces in a natural way. Consider the following two examples.

Example 4.5. We consider the complex manifold $X = \Sigma \times \mathbb{P}^{d-1}$ where Σ is a Riemann surface and $\mathbb{P}^{d-1}$ is $(d-1)$-dimensional complex projective space. Suppose that $\omega \in \Omega^{d-1,d-1}(\mathbb{P}^{d-1})$ is the natural volume form, this clearly satisfies the conditions of Lemma 4.4 and so determines a degree one cocycle $\phi_{\kappa,\omega} \in C^*_{\mathrm{loc}}(\mathcal{G}_{\Sigma\times\mathbb{P}^{d-1}})$ where κ is some $\mathfrak{g}$-invariant bilinear form $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$. We can then consider the twisted enveloping factorization algebra of $\mathcal{G}_{\Sigma\times\mathbb{P}^{d-1}}$ by the cocycle $\phi_{\kappa,\omega}$.

Recall that if $p : X \to Y$ and $\mathcal{F}$ is a factorization algebra on X, then the pushforward $p_*\mathcal{F}$ on Y is defined on opens by $p_*\mathcal{F} : U \subset Y \mapsto \mathcal{F}(p^{-1}U)$.

Proposition 4.6. *Let $\pi : \Sigma \times \mathbb{P}^{d-1} \to \Sigma$ be the projection. Then, there is a quasi-isomorphism between the following two factorization algebras on Σ:*

(1) $\pi_*\mathbb{U}_{\phi_{\kappa,\theta}}(\mathcal{G}_{\Sigma\times\mathbb{P}^{d-1}})$, the pushforward along π of the Kac-Moody factorization algebra on $\Sigma \times \mathbb{P}^{d-1}$ of type $\phi_{\kappa,\omega}$;

(2) $\mathbb{U}_{\mathrm{vol}(\omega)\kappa}(\mathcal{G}_\Sigma)$, the Kac-Moody factorization algebra on Σ associated to the invariant pairing $\mathrm{vol}(\omega)\cdot\kappa$.

The twisted enveloping factorization on the right-hand side is the familiar Kac-Moody factorization alegbra on Riemann surfaces associated to a multiple of the pairing κ. The twisting $\mathrm{vol}(\omega)\kappa$ corresponds to a cocycle of the type in the previous section

$$J(\mathrm{vol}(\omega)\kappa) = \mathrm{vol}(\omega)\int_\Sigma \kappa(\alpha, \partial\beta)$$

where $\mathrm{vol}(\omega) = \int_{\mathbb{P}^{d-1}} \omega$.

Proof. Suppose that $U \subset \Sigma$ is open. Then, the factorization algebra $\pi_*\mathbb{U}_{\phi_{\kappa,\theta}}(\mathcal{G}_{\Sigma\times\mathbb{P}^{d-1}})$ assigns to U the cochain complex

$$\left(\mathrm{Sym}\left(\Omega^{0,*}(U\times\mathbb{P}^{d-1})\right)[1][K], \overline{\partial} + K\phi_{\kappa,\omega}|_{U\times\mathbb{P}^{d-1}}\right), \tag{9}$$

where $\phi_{\kappa,\omega}|_{U\times\mathbb{P}^{d-1}}$ is the restriction of the cocycle to the open set $U \times \mathbb{P}^{d-1}$. Since projective space is Dolbeault formal its Dolbeault complex is quasi-isomorphic to its cohomology. Thus, we have

$$\Omega^{0,*}(U\times\mathbb{P}^{d-1}) = \Omega^{0,*}(U)\otimes\Omega^{0,*}(\mathbb{P}^{d-1}) \simeq \Omega^{0,*}(U)\otimes H^*(\mathbb{P}^{d-1}, \mathcal{O}) \cong \Omega^{0,*}(U).$$

Under this quasi-isomorphism, the restricted cocycle has the form

$$\phi_{\kappa,\omega}|_{U\times\mathbb{P}^{d-1}}(\alpha\otimes 1,\beta\otimes 1)=\int_U \kappa(\alpha,\partial\beta)\int_{\mathbb{P}^{n-1}}\omega$$

where $\alpha,\beta\in\Omega^{0,*}(U)$ and 1 denotes the unit constant function on $\mathbb{P}^{d-1}$. This is precisely the value of the local functional $\mathrm{vol}(\omega)J_\Sigma(\kappa)$ on the open set $U\subset\Sigma$. Thus, the cochain complex (9) is quasi-isomorphic to

$$\left(\mathrm{Sym}\left(\Omega^{0,*}(U)\right)[1][K],\bar\partial+K\mathrm{vol}(\omega)J_\Sigma(\kappa)\right). \tag{10}$$

We recognize this as the value of the Kac-Moody factorization algebra on Σ of type $\mathrm{vol}(\omega)J_\Sigma(\kappa)$. It is immediate to see that identifications above are natural with respect to maps of opens, so that the factorization structure maps are the desired ones. This completes the proof. □

Example 4.7. Fix two Riemann surfaces Σ_1,Σ_2 and let ω_1,ω_2 be the Kähler forms. Then, we can consider the two projections

$$\begin{tikzcd} & \Sigma_1\times\Sigma_2 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \Sigma_1 & & \Sigma_2 \end{tikzcd}$$

Consider the following closed $(1,1)$ form $\omega=\pi_1^*\omega_1+\pi_2^*\omega_2\in\Omega^{1,1}(\Sigma_1\times\Sigma_2)$. According to the proposition above, for any symmetric invariant pairing $\kappa\in\mathrm{Sym}^2(\mathfrak{g}^*)^{\mathfrak{g}}$ this form determines a bilinear local functional

$$\phi_{\kappa,\omega}(\alpha)=\int_{\Sigma_1\times\Sigma_2}\omega\wedge\kappa(\alpha,\partial\alpha)$$

on the local Lie algebra $\mathcal{G}_{\Sigma_1\times\Sigma_2}$. A similar calculation as in the previous example implies that the pushforward factorization algebra $\pi_{i*}\mathbb{U}_{\phi_{\kappa,\omega}}\mathcal{G}$, $i=1,2$, is isomorphic to the two-dimensional Kac-Moody factorization algebra on the Riemann surface Σ_i with level equal to the Euler characteristic $\chi(\Sigma_j)$, where $j\neq i$. This result was alluded to in the work of Johansen in [**?**] where he showed that there exists a copy of the Kac-Moody chiral algebra inside the operators of a twist of the $\mathcal{N}=1$ supersymmetric multiplet (both the gauge and matter multiplets, in fact) on the Kähler manifold $\Sigma_1\times\Sigma_2$. In the next section we will see how the *two-dimensional* Kac-Moody factorization algebra embeds inside the operators of a holomorphic theory on a complex surface. This holomorphic theory is the twist (as we stated in the previous chapter) of the $\mathcal{N}=1$ multiplet. Thus, we obtain an enhancement of Johansen's result to a two-dimensional current algebra.

5. Large N limits

We take a slight detour from the main course of this paper to remark on something special that happens for the case of $\mathfrak{gl}_N$ as N goes to infinity. The observations we make here are borrowed from unpublished work of the first author with Greg Ginot and Mahmoud Zeinalian, but they are closely related to prior work of Costello-Li [] and Movshev-Schwarz [].

The essential fact is the remarkable theorem of Loday-Quillen [**?**] and Tsygan [**?**], which yields a natural map OG: ugly notation so lets find a better one

$$\ell qt(A) : \mathrm{colim}_N\, \mathrm{C}^{\mathrm{Lie}}_*(\mathfrak{gl}_N(A)) \cong \mathrm{C}^{\mathrm{Lie}}_*(\mathfrak{gl}_\infty(A)) \to \mathrm{Sym}(\mathrm{Cyc}_*(A)[1])$$

for any dg algebra A over a field k of characteristic 0. (It works even for A_∞ algebras.) Naturality here means that it works over the category of dg algebras and maps of dg algebras. When restricted to the $\mathfrak{sl}_\infty(k)$-invariants, we obtain a quasi-isomorphism

$$\ell qt(A) : \mathrm{C}^{\mathrm{Lie}}_*(\mathfrak{gl}_\infty(A))^{\mathfrak{sl}_\infty(k)} \xrightarrow{\simeq} \mathrm{Sym}(\mathrm{Cyc}_*(A)[1]),$$

even when A is nonunital. (When A is unital, the $\mathfrak{sl}_\infty(k)$-invariants are quasi-isomorphic to the full Chevalley-Eilenberg chains, making for a very nice relationship. Note that it is potentially problematic to use strict invariants with a particular model for derived coinvariants of a Lie algebra, namely Chevalley-Eilenberg chains.)

By taking A to be the cosheaf $\Omega^{0,*}_c$ on a complex manifold X, we obtain the following, whose proof is deferred to the end of this section.

Proposition 5.1. *Let $\mathcal{G}l_N$ denote the local Lie algebra $\Omega^{0,*} \otimes \mathfrak{gl}_N$. For every N, there is a map of prefactorization algebras*

$$\ell qt_N : \mathbb{U}\mathcal{G}l_N \to \mathrm{Sym}(\mathrm{Cyc}_*(\Omega^{0,*}_c)[1])$$

that factors through a map of prefactorization algebras

$$\ell qt : \mathbb{U}\mathcal{G}l_\infty \to \mathrm{Sym}(\mathrm{Cyc}_*(\Omega^{0,*}_c)[1]).$$

On any complex d-fold X, there is a quasi-isomorphism

$$\ell qt(X) : \mathbb{U}\mathcal{G}l_\infty(X)^{\mathfrak{sl}_\infty(\mathbb{C})} \to \mathrm{Sym}(\mathrm{Cyc}_*(\Omega^{0,*}_c(X))[1]),$$

and on closed X, there is a quasi-isomorphism

$$\ell qt(X) : \mathbb{U}\mathcal{G}l_\infty(X) \to \mathrm{Sym}(\mathrm{Cyc}_*(\Omega^{0,*}_c(X))[1]).$$

Remark 5.2. We note that, as with the definition of the Chevalley-Eilenberg chains of a local Lie algebra, we use here a construction of cyclic chains that plays nicely with the kind of vector spaces relevant to this situation, namely smooth sections of vector bundles. Where the cyclic quotient $A^{\otimes n}/C_n$ would appear for an ordinary algebra in complex vector spaces, we take the $\Omega^{0,*}(X^n)/C_n$ and so on. OG: I need to check that the Sym doesn't lead to issues ... If we must, we can ignore the quasi-isomorphism and focus on the map just to cyclic homology.

BW: Would it also be a good idea to remark on the "local cyclic cohomology"? I think that's even easier to compute in this case, and we could point to the extensions. I can put that in if you'd like.

OG: Yes, we should do that. We could then relate to FHK again, the idea being that an extension of the cyclic jobby determines an extension of the $\mathfrak{gl}_\infty$ jobby, which pulls back along the map to $\mathfrak{g}$ induced by any finite-dimensional representation. We would also obtain an interesting twist of the LQT set-up, I hope.

This result has teeth because it is possible to compute the relevant cyclic homology. For simplicity, consider the case where X is closed, so that we are working with the Dolbeault complex and hence are implicitly computing the cyclic homology of the structure sheaf $\mathcal{O}$ on X. Standard results OG: e.g., Thm 3.4.12 of Loday then imply that

$$H^*(\mathrm{Cyc}_*(\Omega^{0,*}(X))) \cong \bigoplus_{n\geq 0} \left(H^*(X, \Omega^n_{hol}/\partial\Omega^{n-1}_{hol}) \oplus \bigoplus_{k>0} H^{n-2k}_{dR}(X) \right) [-n]$$

In conjunction with the proposition, we see that the large N limit of the enveloping factorization algebras $\mathbb{U}\mathcal{G}l_\infty$ depends primarily on the underlying topology of the complex manifold X, along with a subtle dependence on the complex geometry through the cohomology of the quotient sheaves $\Omega^n_{hol}/\partial\Omega^{n-1}_{hol}$. In the future we hope to pursue the consequences of this observation, as it indicates that there is an important class of currents that can be understand through cyclic methods. In particular, it would be interesting to relate these results to aspects of the large N limits of holomorphic gauge theories.

Remark 5.3. Loday and Procesi proved variants of the Loday-Quillen-Tsygan theorem for the Lie algebras $\mathfrak{o}_n$ and $\mathfrak{sp}_{2n}$, in which cyclic homology of the algebra is replaced by its dihedral homology. As nothing substantive changes in proving analogous versions of our results above, we do not spell out the details here. It would be interesting to pursue the analogues of questions just raised for these Lie algebras.

Proof. The main issue is to show that $\mathrm{Sym}(\mathrm{Cyc}_*(\Omega^{0,*}_c)[1])$ is a prefactorization algebra, since the Loday-Quillen-Tsygan construction then implies the rest of the claim.

As Cyc_* is a functor on the category of dg algebras, we see that $\mathrm{Cyc}_*(\Omega^{0,*}_c)$ is a precosheaf and hence $\mathcal{C} = \mathrm{Sym}(\mathrm{Cyc}_*(\Omega^{0,*}_c)[1])$ is also a precosheaf.

It remains to provide the structure maps of the putative prefactorization algebra $\mathcal{C}$. We note that for two algebras A and B,

$$\mathrm{Cyc}_*(A) \oplus \mathrm{Cyc}_*(B) \simeq \mathrm{Cyc}_*(A \times B)$$

by OG: find convenient reference (use the two idempotents). Hence, for the cosheaf $\Omega^{0,*}_c$ on pairwise disjoint opens $U_1, \ldots, U_n$, the isomorphism of dg algebras

$$\Omega^{0,*}_c(U_1) \times \cdots \times \Omega^{0,*}_c(U_n) \cong \Omega^{0,*}_c(U_1 \sqcup \cdots \sqcup U_n),$$

determines a quasi-isomorphism

$$\operatorname{Cyc}_*(\Omega_c^{0,*}(U_1)) \oplus \cdots \oplus \operatorname{Cyc}_*(\Omega_c^{0,*}(U_n)) \xrightarrow{\simeq} \operatorname{Cyc}_*(\Omega_c^{0,*}(U_1 \sqcup \cdots \sqcup U_n)). \tag{11}$$

Now suppose these pairwise disjoint opens $U_1, \ldots, U_n$ sit inside a larger open V. We need to provide a multilinear structure map

$$\mathcal{C}(U_1) \times \cdots \times \mathcal{C}(U_n) \to \mathcal{C}(V) \tag{12}$$

to describe $\mathcal{C}$ as a prefactorization algebra. The inclusion $U_1 \sqcup \cdots \sqcup U_n \hookrightarrow V$ provides a map

$$\operatorname{Cyc}_*(\Omega_c^{0,*}(U_1 \sqcup \cdots \sqcup U_n)) \to \operatorname{Cyc}_*(V),$$

via the precosheaf $\operatorname{Cyc}_*(\Omega_c^{0,*})$, and so applying Sym gives us

$$\mathcal{C}(U_1 \sqcup \cdots \sqcup U_n) \to \mathcal{C}(V). \tag{13}$$

Likewise, applying Sym to map (11) provides

$$\mathcal{C}(U_1) \times \cdots \times \mathcal{C}(U_n) \to \mathcal{C}(U_1 \sqcup \cdots \sqcup U_n).$$

We thus obtain the desired map (12) as a composite. This construction is automatically associative for nested inclusions of pairwise disjoint opens, and so $\mathcal{C}$ is a prefactorization algebra. □

Appendix A. Local functionals

In our approach, the space of fields will always be equal to the space of smooth sections of a $\mathbb{Z}$-graded vector bundle $E \to X$ on a manifold $\mathcal{E} = \Gamma(X, E)$. The class of functionals $S : \mathcal{E} \to \mathbb{R}$ defining the classical theories we consider are required to be *local*, or given by the integral of a Lagrangian density. We define this concept now.

Let D_X denote the sheaf of differential operators on X. The ∞-jet bundle $\operatorname{Jet}(E)$ of a vector bundle E is the vector bundle whose fiber over $x \in X$ is the space of formal germs at x of sections of E. It is a standard fact that $\operatorname{Jet}(E)$ is equipped with a flat connection giving its space of sections $J(E) = \Gamma(X, \operatorname{Jet}(E))$ the structure of a D_X-module.

Above, we have defined the algebra of functions $\mathcal{O}(\mathcal{E}(X))$ on the space of sections $\mathcal{E}(X)$. Similarly, let $\mathcal{O}_{red}(\mathcal{E}(X)) = \mathcal{O}(\mathcal{E}(X))/\mathbb{R}$ be the quotient by the constant polynomial functions. The space $\mathcal{O}_{red}(J(E))$ inherits a natural D_X-module structure from $J(E)$. We refer to $\mathcal{O}_{red}(J(E))$ as the space of *Lagrangians* on the vector bundle E. Every element $F \in \mathcal{O}_{red}(J(E))$ can be expanded as $F = \sum_n F_n$ where each F_n is an element

$$F_n \in \operatorname{Hom}_{C_X^\infty}(J(E)^{\otimes n}, C_X^\infty)_{S_n} \cong \operatorname{PolyDiff}(\mathcal{E}^{\otimes n}, C^\infty(X))_{S_n} \tag{14}$$

where the right-hand side is the space of polydifferential operators. The proof of the isomorphism on the right-hand side can be found in Chapter 5 of [**?**].

A local functional is given by a Lagrangian densities modulo total derivatives. The mathematical definition is the following.

Definition A.1. Let E be a graded vector bundle on X. Define the sheaf of *local functionals* on X to be

$$\mathcal{O}_{\text{loc}}(\mathcal{E}) = \text{Dens}_X \otimes_{D_X} \mathcal{O}_{red}(J(E)), \tag{15}$$

where we use the natural right D_X-module structure on densities.

Note that we always consider local functionals coming from Lagrangians modulo constants. We will not be concerned with local functions associated to constant Lagrangians.

From the expression for functionals in Lemma **??** we see that integration defines an inclusion of sheaves

$$i : \mathcal{O}_{\text{loc}}(\mathcal{E}) \hookrightarrow \mathcal{O}_{red}(\mathcal{E}_c).$$

Often times when we describe a local functional we will write down its value on test compactly supported sections, then check that it is given by integrating a Lagrangian density, which amounts to lifting the functional along i.

Appendix B. Computing the deformation complex

There is a succinct way of expressing holomorphic translation invariance as the Lie algebra invariants of a certain *dg Lie algebra*. Denote by $\mathbb{C}^d[1]$ the abelian d-dimensional graded Lie algebra in concentrated in degree -1 by the elements $\{\overline{\eta}_i\}$. We want to consider deformations that are invariant for the action by the total *dg* Lie algebra $\mathbb{C}^d_{\text{hol}} = \mathbb{C}^{2d} \oplus \mathbb{C}^d[1]$. The differential sends $\eta_i \mapsto \frac{\partial}{\partial \overline{z}_i}$. The space of holomorphically translation invariant local functionals are denoted by $\mathcal{O}_{\text{loc}}(\mathcal{E}_V)^{\mathbb{C}^d_{\text{hol}}}$. The enveloping algebra of $\mathbb{C}^d_{\text{hol}}$ is of the form

$$U(\mathbb{C}^d_{\text{hol}}) = \mathbb{C}\left[\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z}_i}, \eta_i\right] \tag{16}$$

with differential induced from that in $\mathbb{C}^d_{\text{hol}}$. Note that this algebra is quasi-isomorphic to the algebra of constant coefficient holomorphic differential operators $\mathbb{C}[\partial/\partial z_i] \xrightarrow{\simeq} U(\mathbb{C}^d_{\text{hol}})$.

In this section we specialize the functional J to the space $Y = \mathbb{C}^d$ and use it to completely characterize the $U(d)$-invariant, holomorphically translation invariant deformation complex.

Proposition B.1. *The map $J : \Omega^{d+1}_{cl}(B\mathfrak{g})[d] \to \text{Def}^{\text{cot}}_{\mathbb{C}^d \to B\mathfrak{g}}$ factors through the holomorphically translation invariant deformation complex:*

$$J : \Omega^{d+1}_{cl}(B\mathfrak{g})[d] \to \left(\text{Def}^{\text{cot}}_{\mathbb{C}^d \to B\mathfrak{g}}\right)^{\mathbb{C}^d_{\text{hol}}}. \tag{17}$$

Furthermore, J defines a quasi-isomorphism into the $U(d)$-invariant subcomplex of the right-hand side.

Proof. To compute the translation invariant deformation complex we will invoke Corollary **??** from Section B. Note that the deformation complex is simply the (reduced) local cochains on the local Lie algebra $\Omega^{0,*}_{\mathbb{C}^d} \otimes \mathfrak{g}$. Thus, in the notation of Section **??** the bundle V is simply the trivial bundle $\mathfrak{g}$. Thus, we see that the translation invariant deformation complex is quasi-isomorphic to the following cochain complex

$$\left(\mathrm{Def}^{\mathrm{cot}}_{Y \to B\mathfrak{g}}\right)^{\mathbb{C}^d_{\mathrm{hol}}} \simeq \mathbb{C} \cdot \mathrm{d}^d z \otimes^{\mathbb{L}}_{\mathbb{C}\left[\frac{\partial}{\partial z_i}\right]} \mathrm{C}^*_{\mathrm{Lie,red}}(\mathfrak{g}[[z_1, \ldots, z_d]])[d]. \tag{18}$$

We'd like to recast the right-hand side in a more geometric way.

Note that the the algebra $\mathbb{C}\left[\frac{\partial}{\partial z_i}\right]$ is the enveloping algebra of the abelian Lie algebra $\mathbb{C}^d = \mathbb{C}\left\{\frac{\partial}{\partial z_i}\right\}$. Thus, the complex we are computing is of the form

$$\mathbb{C} \cdot \mathrm{d}^d z \otimes^{\mathbb{L}}_{U(\mathbb{C}^d)} \mathrm{C}^*_{\mathrm{Lie,red}}(\mathfrak{g}[[z_1, \ldots, z_d]])[d]. \tag{19}$$

Since $\mathbb{C} \cdot \mathrm{d}^d z$ is the trivial module, this is precisely the Chevalley-Eilenberg cochain complex computing Lie algebra homology of $\mathbb{C}^d$ with values in the module $\mathrm{C}^*_{\mathrm{Lie,red}}(\mathfrak{g}[[z_1, \ldots, z_d]])$:

$$\left(\mathrm{Def}^{\mathrm{cot}}_{Y \to B\mathfrak{g}}\right)^{\mathbb{C}^d} \simeq \mathrm{C}^{\mathrm{Lie}}_*\left(\mathbb{C}^d; \mathrm{C}^*_{\mathrm{Lie,red}}(\mathfrak{g}[[z_1, \ldots, z_d]])\mathrm{d}^d z\right)[d]. \tag{20}$$

We will keep $\mathrm{d}^d z$ in the notation since below we are interested in computing the $U(d)$-invariants, and it has non-trivial weight under the action of this group.

To compute the cohomology of this complex, we will first describe the differential explicitly. There are two components to the differential. The first is the "internal" differential coming from the Lie algebra cohomology of $\mathfrak{g}[[z_1, \ldots, z_d]]$, we will write this as $\mathrm{d}_{\mathfrak{g}}$. The second comes from the $\mathbb{C}^d$-module structure on $\mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g}[[z_1, \ldots, z_n]])$ and is the differential computing the Lie algebra homology, which we denote $\mathrm{d}_{\mathbb{C}^d}$. We will employ a spectral sequence whose first term turns on the $\mathrm{d}_{\mathfrak{g}}$ differential. The next term turns on the differential $\mathrm{d}_{\mathbb{C}^d}$.

As a graded vector space, the cochain complex we are trying to compute has the form

$$\mathrm{Sym}(\mathbb{C}^d[1]) \otimes \mathrm{C}^*_{\mathrm{Lie,red}}\left(\mathfrak{g}[[z_1, \ldots, z_d]]\right)\mathrm{d}^d z[d]. \tag{21}$$

The spectral sequence is induced by the increasing filtration of $\mathrm{Sym}(\mathbb{C}^d[1])$ by symmetric powers

$$F^k = \mathrm{Sym}^{\leq k}(\mathbb{C}^d[1]) \otimes \mathrm{C}^*_{\mathrm{Lie,red}}\left(\mathfrak{g}[[z_1, \ldots, z_d]]\right)\mathrm{d}^d z[d]. \tag{22}$$

Remark B.2. In the examples we are most interested in (namely $\mathfrak{g} = \mathbb{C}^n[-1]$ and $\mathfrak{g} = \mathfrak{g}_{X_{\bar\partial}}$) we can understand the spectral sequence we are using as a version of the Hodge-to-de Rham spectral sequence.

As above, we write the generators of $\mathbb{C}^d$ by $\frac{\partial}{\partial z_i}$. Also, note that the reduced Chevalley-Eilenberg complex has the form

$$\mathrm{C}^*_{\mathrm{Lie,red}}(\mathfrak{g}[[z_1, \ldots, z_n]]) = \left(\mathrm{Sym}^{\geq 1}\left(\mathfrak{g}^\vee[z_1^\vee, \ldots, z_d^\vee][-1]\right), \mathrm{d}_{\mathfrak{g}}\right), \tag{23}$$

where $z_i^\vee$ is the dual variable to z_i.

Recall, we are only interested in the $U(d)$-invariant subcomplex of this deformation complex. Sitting inside of $U(d)$ we have $S^1 \subset U(d)$ as multiples of the identity. This induces an overall weight grading to the complex. The group $U(d)$ acts in the standard way on $\mathbb{C}^d$. Thus, z_i has weight $(+1)$ and both $z_i^\vee$ and $\frac{\partial}{\partial z_i}$ have S^1-weight (-1). Moreover, the volume element $\mathrm{d}^d z$ has S^1 weight d. It follows that in order to have total S^1-weight that the total number of $\frac{\partial}{\partial z_i}$ and $z_i^\vee$ must add up to d. Thus, as a graded vector space the invariant subcomplex has the following decomposition

$$\bigoplus_k \mathrm{Sym}^k(\mathbb{C}^d[1]) \otimes \left(\bigoplus_{i \leq d-k} \mathrm{Sym}^i \left(\mathfrak{g}^\vee [z_1^\vee, \ldots, z_d^\vee][-1] \right) \right) \mathrm{d}^d z[d]. \tag{24}$$

It follows from Schur-Weyl that the space of $U(d)$ invariants of the dth tensor power of the fundamental representation $\mathbb{C}^d$ is one-dimensional, spanned by the top exterior power. Thus, when we pass to the $U(d)$-invariants, only the unique totally antisymmetric tensor involving $\frac{\partial}{\partial z_i}$ and $z_i^\vee$ survives. Thus, for each k, we have

$$\left(\mathrm{Sym}^k(\mathbb{C}^d[1]) \otimes \left(\bigoplus_{i \leq d-k} \mathrm{Sym}^i \left(\mathfrak{g}^\vee [z_1^\vee, \ldots, z_d^\vee][-1] \right) \right) \mathrm{d}^d z \right) \cong \wedge^k \left(\frac{\partial}{\partial z_i} \right) \wedge \wedge^{d-k} \left(z_i^\vee \right) \mathrm{C}^*_{\mathrm{Lie}} \left(\mathfrak{g}, \mathrm{Sym}^{d-k}(\mathfrak{g}^\vee) \right) \mathrm{d}^d z$$

Here, $\wedge^k \left(\frac{\partial}{\partial z_i} \right) \wedge \wedge^{d-k} \left(z_i^\vee \right)$ is just a copy of the determinant $U(d)$-representation, but we'd like to keep track of the appearances of the partial derivatives and $z_i^\vee$. Note that for degree reasons, we must have $k \leq d$. When $k = 0$ this complex is the (shifted) space of functions modulo constants on the formal moduli space $B\mathfrak{g}$, $\mathcal{O}_{red}(B\mathfrak{g})[d]$. When $k \geq 1$ this the (shifted) space of k-forms on the formal moduli space $B\mathfrak{g}$, which we write as $\Omega^k(B\mathfrak{g})[d+k]$. Thus, we see that before turning on the differential on the next page, our complex looks like

$$\begin{array}{cccc} \underline{-2d} & \cdots & \underline{-d-1} & \underline{-d} \\ & & & \\ \mathcal{O}_{red}(B\mathfrak{g}) & \cdots & \Omega^{d-1}(B\mathfrak{g}) & \Omega^d(B\mathfrak{g}). \end{array}$$

We've omitted the extra factors for simplicity.

We now turn on the differential $\mathrm{d}_{\mathbb{C}^d}$ coming from the Lie algebra homology of $\mathbb{C}^d = \mathbb{C}\left\{ \frac{\partial}{\partial z_i} \right\}$ with values in the above module. Since this Lie algebra is abelian the differential is completely determined by how the operators $\frac{\partial}{\partial z_i}$ act. We can understand this action explicitly as follows. Note that $\frac{\partial}{\partial z_i} z_j = \delta_{ij}$, thus we may as well think of $z_i^\vee$ as the element $\frac{\partial}{\partial z_i}$. Consider the subspace corresponding to $k = d$ in Equation (B):

$$\frac{\partial}{\partial z_1} \cdots \frac{\partial}{\partial z_d} \mathrm{C}^*_{\mathrm{Lie,red}}(\mathfrak{g}) \mathrm{d}^d z. \tag{25}$$

Then, if $x \in \mathfrak{g}^\vee[-1] \subset C^*_{\mathrm{Lie,red}}(\mathfrak{g})$ we observe that

(26)

$$d_{\mathbb{C}^d}\left(\frac{\partial}{\partial z_1}\cdots\frac{\partial}{\partial z_d}\otimes f\otimes d^d z\right) = \det(\partial_i, z_j^\vee)\otimes 1\otimes x\otimes d^d z \in \wedge^{d-1}\left(\frac{\partial}{\partial z_i}\right)\wedge \mathbb{C}\{z_i^\vee\}C^*_{\mathrm{Lie}}\left(\mathfrak{g},\mathfrak{g}^\vee\right) d^d z.$$

This follows from the fact that the action of $\frac{\partial}{\partial z_i}$ on $x = x\otimes 1 \in \mathfrak{g}^\vee \otimes \mathbb{C}[z_i^\vee]$ is given by

$$\frac{\partial}{\partial z_i}\cdot (x\otimes 1) = 1\otimes x\otimes z_i^\vee \in C^*_{\mathrm{Lie}}(\mathfrak{g},\mathfrak{g}^\vee) z_i^\vee. \tag{27}$$

By the Leibniz rule we can extend this to get the formula for general elements $f \in C^*_{\mathrm{Lie,red}}(\mathfrak{g})$. We find that getting rid of all the factors of z_i we recover precisely the de Rham differential

$$\begin{array}{ccc} C^*_{\mathrm{Lie,red}}(\mathfrak{g})[2d] & \xrightarrow{d_{\mathbb{C}^d}} & C^*_{\mathrm{Lie}}(\mathfrak{g},\mathfrak{g}^\vee)[2d-1] \\ \| & & \| \\ \mathcal{O}_{red}(B\mathfrak{g}) & \xrightarrow{\quad\partial\quad} & \Omega^1(B\mathfrak{g}). \end{array} \tag{28}$$

A similar argument shows that $d_{\mathbb{C}^d}$ agrees with the de Rham differential on each $\Omega^k(B\mathfrak{g})$.

We conclude that the E_2 page of this spectral sequence is quasi-isomorphic to the following truncated de Rham complex.

$$\begin{array}{ccccc} \underline{-2d} & \underline{-2d+1} & \cdots & \underline{-d-1} & \underline{-d} \\ & & & & \\ \mathcal{O}_{red}(B\mathfrak{g}) \xrightarrow{\partial} & \Omega^1(B\mathfrak{g}) \longrightarrow & \cdots \longrightarrow & \Omega^{d-1}(B\mathfrak{g}) \xrightarrow{\partial} & \Omega^d(B\mathfrak{g}). \end{array}$$

This is precisely a shifted version of the complex we had in (**??**). We saw that it was quasi-isomorphic, through the de Rham differential, to $\Omega^{d+1}_{cl}[d]$. This completes the proof. $\square$

We can apply this general result to the case $\mathfrak{g} = \mathbb{C}^n[-1]$. Doing this we have the following corollary.

Corollary B.3. *Let* Def_n *be the deformation complex of the formal* $\beta\gamma$ *system with target* $\widehat{D}^n$*. There is a* $(\mathrm{W}_n, \mathrm{GL}_n)$*-equivariant quasi-isomorphism*

$$J : \widehat{\Omega}^{d+1}_{n,cl}[d] \xrightarrow{\simeq} \left(\left(\mathrm{Def}_n^{\mathrm{cot}}\right)^{\mathbb{C}^d_{\mathrm{hol}}}\right)^{U(d)} \subset \mathrm{Def}_n. \tag{29}$$

This induces a quasi-isomorphism into the $(\mathrm{W}_n, \mathrm{GL}_n)$-equivariant deformation complex

$$J^{\mathrm{W}} : C^*_{\mathrm{Lie}}(\mathrm{W}_n, \mathrm{GL}_n; \widehat{\Omega}^{d+1}_{n,cl}) \xrightarrow{\simeq} \left(\left(\mathrm{Def}_n^{\mathrm{W,cot}}\right)^{\mathbb{C}^d_{\mathrm{hol}}}\right)^{U(d)} \subset \mathrm{Def}_n^{\mathrm{W}}.$$

Moreover, upon performing Gelfand-Kazhdan descent, it implies that on any complex manifold X we can use J to identify the deformation complex for the holomorphic σ-model of maps $\mathbb{C}^d \to X$:

$$J^X : \Omega^{d+1}_{X,cl}[d] \xrightarrow{\simeq} \left(\left(\mathrm{Def}^{\mathrm{cot}}_{\mathbb{C}^d \to X}\right)^{\mathbb{C}^d_{\mathrm{hol}}} \right)^{U(d)}. \tag{30}$$