HIGHER KAC-MOODY ALGEBRAS AND SYMMETRIES OF HOLOMORPHIC FIELD THEORIES

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INTRODUCTION

The loop algebra $L\mathfrak{g} = \mathfrak{g}[z, z^{-1}]$, consisting of Laurent polynomials valued in a Lie algebra $\mathfrak{g}$, admits a non-trivial central extension $\widehat{\mathfrak{g}}$ for each choice of invariant pairing on $\mathfrak{g}$. This affine Lie algebra and its cousin, the Kac-Moody vertex algebra, are foundational objects in representation theory and conformal field theory. A natural question then arises: do there exist~~s~~ multivariable, or higher dimensional, generalizations of the affine Lie algebra and Kac-Moody vertex algebra?

In this work, we pursue two independent yet related goals:

(1) Use factorization algebras to study the (co)sheaf of Lie algebra~~-~~valued currents on complex manifolds, and their relationship to higher affine algebras;
(2) Develop tools for understanding symmetries of *holomorphic field theory* in any dimension~~,~~ that provide a systematic generalization of methods used in chiral conformal field theory on Riemann surfaces.

Concretely, for every complex dimension d and to every Lie algebra, we define a factorization algebra defined on all d-dimensional complex manifolds. There is also a version of this construction that works for an arbitrary principal bundle. When $d = 1$, it is shown in [CG17], that this factorization algebra recovers the ordinary affine algebra by restricting the factorization algebra to the punctured complex line C^*. When $d > 1$, part of our main result is to show how the factorization algebra on $\mathbb{C}^d \setminus \{0\}$ recovers a higher~~-~~dimensional central extensions of $\mathfrak{g}$-valued functions on the punctured plane. A model for these "higher affine algebras" has recently appeared in work of Faonte-Hennion-Kapranov [FHK], and we will give a systematic study of the relationship between our approaches.

By a standard procedure, there is a way of enhancing the affine algebra to a vertex algebra. The so-called Kac-Moody vertex algebra, as developed in [Fre85, Kac98, Bor86], is important in its own right ~~to~~ in representation theory and conformal field theory. In [CG17] it is also shown how the holomorphic factorization algebra associated to a Lie algebra recovers this vertex algebra. The key point is that the OPE is encoded by the factorization product between disks embedded in $\mathbb{C}$. Our proposed factorization algebra, then, provides a higher dimensional enhancement of this vertex algebra through the factorization product of balls or polydisks in $\mathbb{C}^d$. This structure can be thought of as a holomorphic analog of an algebra over the operad of little d-disks.

It is the general philosophy of [CG17, CG] that every quantum field theory defines a factorization algebra of observables. This perspective allows us to realize the higher Kac-Moody algebra inside of familiar higher-dimensional field theories. In particular, this philosophy leads to higher-dimensional analogs of free field realization via a quantum field theory called the $\beta\gamma$ system, which is defined on any complex manifold.

In complex dimension one, a vertex algebra is a gadget associated to any conformal field theory that completely determines the algebra of local operators. More recently, vertex algebras have been extracted from higher dimensional field theories, such as 4-dimensional gauge theories [BPRvR15, BLL+15]. A future direction, which we do not undertake here, would be to use these higher dimensional vertex algebras as a more refined invariant of the quantum field theory.

etc.

abbr. Before embarking on our main results, we take some time to motivate higher-dimensional current algebras from two different perspectives.

A view from geometry. There is an embedding $\mathfrak{g}[z, z^{-1}] \hookrightarrow C^\infty(S^1) \otimes \mathfrak{g} = \mathrm{Map}(S^1, \mathfrak{g})$, induced by the embedding of algebraic functions on the punctured affine line inside of smooth functions on S^1. Thus, a natural starting point for d-dimensional affine algebras is the "sphere algebra"

$$\mathrm{Map}(S^{2d-1}, \mathfrak{g}), \tag{1}$$

where we view $S^{2d-1} \subset \mathbb{C}^d \setminus \{0\}$.

When $d = 1$, affine algebras are given by extensions of $L\mathfrak{g}$ prescribed by a 2-cocycle involving the algebraic residue pairing. Note that this cocycle is *not* pulled back from any cocycle on $\mathcal{O}_{\text{alg}}(\mathbb{A}^1) \otimes \mathfrak{g} = \mathfrak{g}[z]$.

When $d > 1$, Hartog's theorem implies that the space of holomorphic functions on punctured affine space is the same as the space of holomorphic functions on affine space. The same holds for algebraic functions, so that $\mathcal{O}_{\text{alg}}(\mathring{\mathbb{A}}^d) \otimes \mathfrak{g} = \mathcal{O}_{\text{alg}}(\mathbb{A}^d) \otimes \mathfrak{g}$. In particular, the naive algebraic replacement $\mathcal{O}_{\text{alg}}(\mathring{\mathbb{A}}^d) \otimes \mathfrak{g}$ of (1) has no interesting central extensions. However, as opposed to the punctured line, the punctured affine space $\mathring{\mathbb{A}}^s$ has interesting higher cohomology.

The key idea is to replace the commutative algebra $\mathcal{O}_{\text{alg}}(\mathring{\mathbb{A}}^d)$ by the derived space of functions $\mathbb{R}\Gamma(\mathring{\mathbb{A}}^d, \mathcal{O}_{\text{alg}})$. This complex has interesting cohomology and leads to nontrivial extensions of the Lie algebra object $\mathbb{R}\Gamma(\mathring{\mathbb{A}}^d, \mathcal{O}) \otimes \mathfrak{g}$, as well as its Dolbeault model $\Omega^{0,*}(\mathring{\mathbb{A}}^d) \otimes \mathfrak{g}$. Faonte-Hennion-Kapranov [FHK] have provided a systematic exploration of this situation.

Our starting point is to step back, and consider the full sheaf of $\mathfrak{g}$-valued Dolbeault forms $\Omega^{0,*}(X, \mathfrak{g})$ defined on any complex manifold X. We deem this sheaf of dg Lie algebras, or rather its cosheaf version $\mathcal{G}_X := \Omega^{0,*}_c(X, \mathfrak{g})$, the *holomorphic* $\mathfrak{g}$*-valued currents* on X. We devote Section 2.1.5 to relating our construction to that in [FHK]. The Lie algebra homology, $C^{\text{Lie}}_* \mathcal{G}_X$, of this cosheaf determines the structure of a *factorization algebra* on the manifold X. It serves as a higher dimensional analog of the chiral enveloping algebra of $\mathfrak{g}$ introduced by Beilinson and Drinfeld [BD04], and will [illegible] ~~or~~ model for the higher dimensional Kac-Moody algebra. We will see that there exist~~s~~ cocycles on this sheaf of dg Lie algebras which give rise to interesting extensions of the factorization algebra $C_*^{\mathrm{Lie}}\mathcal{G}_X$.

A view from physics. In conformal field theory, the Kac-Moody algebra generically appears as the symmetry of a system with an action by a group. This appears in Kac-Moody uniformization, for instance, in (?) where~~by~~ the affine algebra describes infinitesimal symmetries of a principal G-bundle inside the moduli space of all G-bundles. The higher Kac-Moody algebra we propose arises naturally as the symmetries of higher dimensional quantum field theories that have a *holomorphic* flavor.

Throughout this paper, we use ideas and techniques from the Batalin-Vilkovisky formalism, as articulated by Costello, and factorization algebras, following [CG17, CG]. In this introduction, however, we will try to explain the key objects and constructions with a light touch, in a way that does not require familiarity with that formalism, merely comfort with basic complex geometry and ideas of quantum field theory.

A running example is the following version of the $\beta\gamma$ system, to which we will return ~~to~~ in detail in Section 3.

Let X be a complex d-dimensional manifold. Let G be a complex algebraic group, such as $GL_n(\mathbb{C})$, and let $P \to X$ be a holomorphic principal G-bundle. Fix a finite-dimensional G-representation V and let $V^\vee$ denote the dual vector space with the natural induced G-action. Let $\mathcal{V} \to X$ denote the holomorphic associated bundle $P \times^G V$, and let $\mathcal{V}^! \to X$ denote the holomorphic bundle $K_X \otimes \mathcal{V}^\vee$, where $\mathcal{V}^* \to X$ is the holomorphic associated bundle $P \times^G V^*$. Note that there is a natural fiberwise pairing

supposed to be $\mathcal{V}^\vee$?

$$\langle -,- \rangle : \mathcal{V} \otimes \mathcal{V}^! \to K_X{}^1$$

arising from the evaluation pairing between V and $V^\vee$.

The field theory involves fields γ~~, for a~~ smooth section of $\mathcal{V}$~~,~~ and β~~, for~~ a smooth section of $\Omega^{0,d-1} \otimes \mathcal{V}^\vee$. Here, $\mathcal{V}^\vee$ denotes the dual bundle. The action functional is

$$S(\beta,\gamma) = \int_X \langle \beta, \overline{\partial}\gamma \rangle,$$

are these supposed to be different fonts?

so that the equations of motion are

$$\overline{\partial}\gamma = 0 = \overline{\partial}\beta.$$

Thus, the classical theory is manifestly holomorphic: it picks out holomorphic sections of $\mathcal{V}$ and $\mathcal{V}^!$ as solutions.

The theory also enjoys a natural symmetry with respect to G, arising from the G-action on $\mathcal{V}$ and $\mathcal{V}^!$. For instance, if $\overline{\partial}\gamma = 0$ and $g \in G$, then the section $g\gamma$ is also holomorphic. In fact, there is a local symmetry as well. Let $\mathrm{ad}(P) \to X$ denote the Lie algebra-valued

[1] The shriek denotes the Serre dual, $\mathcal{V}^! = K_X \otimes V^\vee$.

bundle $P \times^G \mathfrak{g} \to X$ arising from the adjoint representation $\mathrm{ad}(G)$. Then a holomorphic section f of $\mathrm{ad}(P)$ acts on a holomorphic section γ of $\mathcal{V}$, and

$$\overline{\partial}(f\gamma) = (\overline{\partial} f)\gamma + f\overline{\partial}\gamma = 0,$$

so that the sheaf of holomorphic sections of $\mathrm{ad}(P)$ encodes a class of local symmetries of this classical theory.

If one takes a BV/BRST approach to field theory, as we will in this paper, then one works with a cohomological version of fields and symmetries. For instance, it is natural to view the classical fields as consisting of the graded vector space of Dolbeault forms

$$\gamma \in \Omega^{0,*}(X, \mathcal{V}) \quad \text{and} \quad \beta \in \Omega^{0,*}(X, \mathcal{V}^!) \cong \Omega^{d,*}(X, \mathcal{V}^*),$$

but using the same action functional, extended in the natural way. As we are working with a free theory and hence have only a quadratic action, the equations of motion are linear and can be viewed as arising from equipping the fields with the differential $\overline{\partial}$. In this sense, the sheaf $\mathcal{E}$ of solutions to the equations of motion can be identified with the elliptic complex that assigns to an open set $U \subset X$, the complex

$$\mathcal{E}(U) = \Omega^{0,*}(U, \mathcal{V}) \oplus \Omega^{0,*}(U, \mathcal{V}^!),$$

with $\overline{\partial}$ as the differential. This dg approach is certainly appealing from the perspective of complex geometry, where one routinely works with the Dolbeault complex of a holomorphic bundle.

It is natural then to encode the local symmetries in the same way. Let $\mathcal{A}d(P)$ denote the Dolbeault complex of $\mathrm{ad}(P)$ viewed as a sheaf. That is, it assigns to the open set $U \subset X$, the dg Lie algebra

$$\mathcal{A}d(P)(U) = \Omega^{0,*}(U, \mathrm{ad}(P))$$

with differential $\overline{\partial}$ for this bundle. By construction, $\mathcal{A}d(P)$ acts on $\mathcal{E}$. In words, $\mathcal{E}$ is a sheaf of dg modules for the sheaf of dg Lie algebras $\mathcal{A}d(P)$.

So far, we have simply lifted the usual discussion of symmetries to a dg setting, using standard tools of complex geometry. We now introduce a novel maneuver that is characteristic of the BV/factorization package of [CG17, CG].

The idea is to work with *compactly supported* sections of $\mathcal{A}d(P)$, i.e., to work with the precosheaf $\mathcal{A}d(P)_c$ of dg Lie algebras that assigns to an open U, the dg Lie algebra

$$\mathcal{A}d(P)_c(U) = \Omega_c^{0,*}(U, \mathrm{ad}(P)).$$

The terminology *precosheaf* encodes the fact that there is natural way to extend a section supported in U to a larger open $V \supset U$ (namely, extend by zero), and so one has a functor $\mathcal{A}d(P)\colon \mathrm{Opens}(X) \to \mathrm{Alg}_{\mathrm{Lie}}$.

There are several related reasons to consider compact support. First, it is common in physics to consider compactly-supported modifications of a field. Recall that in the variational calculus, ~~where~~ one extracts the equations of motion by working with precisely such first-order perturbations. Hence, it is natural to focus on such symmetries as well. Second, one could ask how such compactly supported actions of $\mathcal{A}d(P)$ affect observables. More specifically, one can ask about the charges of the theory with respect to this local symmetry.[2] Third—and this reason will become clearer in a moment—the anomaly that appears when trying to quantize this symmetry ~~are~~ is naturally local in X, and hence it is encoded by a kind of Lagrangian density L on sections of $\mathcal{A}d(P)$. Such a density only defines a functional on compactly supported sections, since when evaluated on a noncompactly supported section f, the density $L(f)$ may be non-integrable. Thus L determines a central extension of $\mathcal{A}d(P)_c$ as a precosheaf of dg Lie algebras, but not as a sheaf.[3]

Let us sketch how to make these reasons explicit. The first step is to understand how $\mathcal{A}d(P)_c$ acts on the observables of this theory.

Modulo functional analytic issues, we say that the observables of this classical theory are the commutative dg algebra

$$(\mathrm{Sym}(\Omega^{0,*}(X,\mathcal{V})^* \oplus \Omega^{0,*}(X,\mathcal{V}^!)^*), \bar{\partial}),$$

i.e., the polynomial functions on $\mathcal{E}(X)$. More accurately, we work with a commutative dg algebra essentially generated by the continuous linear functionals on $\mathcal{E}(X)$, which are compactly supported distributional sections of certain Dolbeault complexes (*aka* Dolbeault currents). We could replace X by any open set $U \subset X$, in which case the observables with support in U arise from such distributions supported in U. We denote this commutative dg algebra by $\mathrm{Obs}^{cl}(U)$. Since observables on an open U extend to observables on a larger open $V \supset U$, we recognize that Obs^{cl} forms a precosheaf.

Manifestly, $\mathcal{A}d(P)_c(U)$ acts on $\mathrm{Obs}^{cl}(U)$, by precomposing with its action on fields. Moreover, these actions are compatible with the extension maps of the precosheaves, so that Obs^{cl} is a module for $\mathcal{A}d(P)_c$ in precosheaves of cochain complexes. This relationship already exhibits why one might choose to focus on $\mathcal{A}d(P)_c$, as it naturally intertwines with the structure of the observables.

But Noether's theorem provides a further reason, when understood in the context of the BV formalism. The idea is that Obs^{cl} has a Poisson bracket $\{-,-\}$ of degree 1 (although there are some issues with distributions here that we ~~suppress~~ ignore for the moment). Hence one can ask to realize the action of $\mathcal{A}d(P)_c$ via the Poisson bracket. In other words, we ask to find a map of (precosheaves of) dg Lie algebras

$$J\colon \mathcal{A}d(P)_c \to \mathrm{Obs}^{cl}[-1]$$

[2]We remark that it is precisely this relationship with traditional physical terminology of currents and charges that led de Rham to use *current* to mean a distributional section of the de Rham complex.

[3]We remark that to stick with sheaves, one must turn to quite sophisticated tools [Wit88, Get88, BMS87] that can be tricky to interpret, much less generalize to higher dimension, whereas the cosheaf-theoretic version is quite mundane and easy to generalize, as we'll see.

such that for any $f \in \mathcal{A}d(P)_c(U)$ and $F \in \mathrm{Obs}^{cl}(U)$, we have

$$f \cdot F = \{J(f), F\}.$$

Such a map would realize every symmetry as given by an observable, much as in Hamiltonian mechanics.

In this case, there is such a map:

$$J(f)(\gamma, \beta) = \int_U \langle \beta, f\gamma \rangle.$$

This functional is local, and it is natural to view it as describing the "minimal coupling" between our free $\beta\gamma$ system and a kind of gauge field implicit in $\mathcal{A}d(P)$. This construction thus shows again that it is natural to work with compactly supported sections of $\mathcal{A}d(P)$, since it allows one to encode the Noether map in a natural way. We call $\mathcal{A}d(P)_c$ the Lie algebra of *classical currents* because ~~as we have explained how~~, via J, we have realize these symmetries as classical observables.

there *Remark* 0.1. We remark that it is not always possible to produce such a Noether map, but is an obstruction which ~~the obstruction always~~ determines a central extension of $\mathcal{A}d(P)_c$ as a precosheaf of dg Lie algebras, and one can then produce such a map out of the central extension to the classical observables.

In the BV formalism, quantization amounts to a deformation of the differential on Obs^{cl}, where the deformation is required to satisfy certain properties. Two conditions are preeminent:

- the differential satisfies a *quantum master equation*, which ensures that $\mathrm{Obs}^q(U)[-1]$ is still a dg Lie algebra via the bracket,[4] and
- it respects support of observables so that Obs^q is still a precosheaf.

The first condition is more or less what BV quantization means, whereas the second is a version of the locality of field theory.

We can now ask whether the Noether map J determines a map of precosheaves of dg Lie algebras from $\mathcal{A}d(P)_c$ to $\mathrm{Obs}^q[-1]$. Since the Lie bracket has not changed on the observables, the only question is where J is a cochain map for the new differential d^q If we write $\mathrm{d}^q = \mathrm{d}^{cl} + \hbar\Delta$,[5] then

$$[\mathrm{d}, J] = \hbar\Delta \circ J.$$

Naively—i.e., ignoring renormalization issues—this term is the functional ob on $\mathcal{A}d(P)_c$ given

$$ob(f) = \int \langle f K_\Delta \rangle,$$

[4]Again, we are suppressing—for the moment ~~important,~~ issues about renormalization, which will play a key role when we get to the real work.

[5]By working with smeared observables, one really can work with the naive BV Laplacian Δ. Otherwise, one must take a little more care.

where K_Δ is the integral kernel for the identity with respect to the pairing $\langle -, - \rangle$. (It encodes a version of the trace of f over $\mathcal{E}$.) This obstruction should resemble standard anomalies.

This functional *ob* is a cocycle in Lie algebra cohomology for $\mathcal{A}d(P)$ and hence determines a central extension $\widehat{\mathcal{A}d(P)}_c$ as precoshea~~ves~~ of dg Lie algebras. It is the Lie algebra of *quantum* currents, as there is a lift of J to a map J^q out of this extension to the quantum observables. This cocycle will be precisely the one corresponding to the central extensions for the higher dimensional affine algebras.

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CONTENTS

1. Current algebras on complex manifolds

This paper takes general definitions and constructions from [CG17] and specializes them to the context of complex manifolds. In this subsection we will review some of the key ideas but refer to [CG17] for foundational results.

Remark 1.1. It might help to bear in mind the one-dimensional case that we wish to extend. Let Σ be a Riemann surface, and let $\mathfrak{g}$ be a simple Lie algebra with Killing form κ. Consider the local Lie algebra $\mathcal{G}_\Sigma = \Omega_c^{0,*}(\Sigma) \otimes \mathfrak{g}$ on Σ. There is a natural cocycle depending precisely on two inputs:

$$\theta(\alpha \otimes M, \beta \otimes N) = \kappa(M,N) \int_\Sigma \alpha \wedge \partial\beta,$$

where $\alpha, \beta \in \Omega_c^{0,*}(\Sigma)$ and $M, N \in \mathfrak{g}$. In Chapter 5 of [CG17] it is shown how the twisted enveloping factorization algebra of $\mathcal{G}_\Sigma$ for this cocycle recovers the Kac-Moody vertex algebra associated to the affine algebra extending $L\mathfrak{g} = \mathfrak{g}[z, z^{-1}]$.

1.1. Local Lie algebras.

A key notion for us is a sheaf of Lie algebras on a smooth manifold. These often appear as infinitesimal automorphisms of geometric objects, and hence as symmetries in classical field theories.

Definition 1.2. A *local Lie algebra* on a smooth manifold X is

(i) a $\mathbb{Z}$-graded vector bundle L on X of finite total rank;
(ii) a degree 1 operator $\ell_1 : \mathcal{L}^{sh} \to \mathcal{L}^{sh}$ on the sheaf $\mathcal{L}^{sh}$ of smooth sections of L, and
(iii) a degree 0 bilinear operator

$$\ell_2 : \mathcal{L}^{sh} \times \mathcal{L}^{sh} \to \mathcal{L}^{sh}$$

such that $\ell_1^2 = 0$, ℓ_1 is a differential operator, and ℓ_2 is a bidifferential operator, and

$$\ell_1(\ell_2(x,y)) = \ell_2(\ell_1(x), y) + (-1)^{|x|}\ell_2(x, \ell_1(y))$$

for any sections x, y of $\mathcal{L}^{sh}$. We call ℓ_1 the *differential* and ℓ_2 the *bracket*.

In other words, a local Lie algebra is a sheaf of dg Lie algebras where the underlying sections are smooth sections of a vector bundle and where the operations are local in the sense of not enlarging support of sections. (As we will see, such Lie algebras often appear by acting naturally on the local functionals from physics, namely functionals determined by Lagrangian densities.)

Remark 1.3. For a local Lie algebra, we reserve the more succinct notation $\mathcal{L}$ to denote the precosheaf of *compactly supported* sections of L, which assigns a dg Lie algebra to each open set $U \subset X$, since the differential and bracket respect support. At times we will abusively refer to $\mathcal{L}$ to mean the data determining the local Lie algebra, when the support of the sections is not relevant to the discussion at hand.

The key examples for this paper all arise from studying the symmetries of holomorphic principal bundles. We begin with the specific and then examine a modest generalization.

Let $\pi : P \to X$ be a holomorphic principal G-bundle over a complex manifold. We use $\mathrm{ad}(P) \to X$ to denote the associated *adjoint bundle* $P \times^G \mathfrak{g} \to X$, where the Borel construction uses adjoint action of G on $\mathfrak{g}$ from the left. The complex structure defines a $(0,1)$-connection $\overline{\partial}_P : \Omega^{0,q}(X; \mathrm{ad}(P)) \to \Omega^{0,q+1}(X; \mathrm{ad}(P))$ on the Dolbeault forms with values in the adjoint bundle, and this connection satisfies $\overline{\partial}_P^2 = 0$. Note that the Lie bracket on $\mathfrak{g}$ induces a pointwise bracket on smooth sections of $\mathrm{ad}(P)$ by

$$[s,t](x) = [s(x), t(x)]$$

where s,t are sections and x is a point in X. This bracket naturally extends to Dolbeault forms with values in the adjoint bundle, as the Dolbeault forms are a graded-commutative algebra.

Definition 1.4. For $\pi : P \to X$ a holomorphic principal G-bundle, let $\mathcal{A}d(P)^{sh}$ denote the local Lie algebra whose sections are $\Omega^{0,*}(X, \mathrm{ad}(P))$, whose differential is $\overline{\partial}_P$, and whose bracket is the pointwise operation just defined above.

This dg Lie algebra $\mathcal{A}d(P)^{sh}(X)$ controls formal deformations of the holomorphic principal G-bundle P. (For a proof see Section 5 of [FHK].) Indeed, given a Maurer-Cartan element $\alpha \in \mathcal{A}d(P)^{sh}(X)^1 = \Omega^{0,1}(X, \mathrm{ad}(P))$ one considers the new complex structure defined by the connection $\overline{\partial}_P + \alpha$. The Maurer-Cartan condition is equivalent to $(\overline{\partial}_P + \alpha)^2 = 0$.

This construction admits important variations. For example, we can move from working over a fixed manifold X to working over a site. Let Hol_d denote the category whose objects are complex d-folds and whose morphisms are local biholomorphisms,[6] This category admits a natural Grothendieck topology where a cover $\{\phi_i : U_i \to X\}$ means a collection of morphisms into X such that union of the images is all of X. It then makes sense to talk about a local Lie algebra on the site Hol_d. Here is a particularly simple example that appears throughout the paper.

[6] A biholomorphism is a bijective map $\phi : X \to Y$ such that both ϕ and ϕ^{-1} are holomorphic. A *local* biholomorphism means a map $\phi : X \to Y$ such that every point $x \in X$ has a neighborhood on which ϕ is a biholomorphism.

Definition 1.5. Let G be a complex Lie group and let $\mathfrak{g}$ denote its ordinary Lie algebra. There is a natural functor

$$\begin{array}{cccc} \mathcal{G}^{sh}: & \mathrm{Hol}_d^{\mathrm{op}} & \to & \mathrm{dgLie} \\ & X & \mapsto & \Omega^{0,*}(X) \otimes \mathfrak{g}, \end{array}$$

which defines a sheaf of dg Lie algebras. Restricted to each slice $\mathrm{Hol}_{d/X}$, it determines the local Lie algebra for the trivial principal bundle $G \times X \to X$, in the sense described above. We use $\mathcal{G}_d$ to denote the cosheaf of compactly supported sections $\Omega_c^{0,*} \otimes \mathfrak{g}$ on this site.

Can do a site where objects have P G-bdls; then can assign $\Omega^{0,*}(X; ad(P))$

Remark 1.6. It is not necessary to start with a complex Lie group: the construction makes sense for a dg Lie algebra over $\mathbb{C}$ of finite total dimension. We lose, however, the interpretation in terms of infinitesimal symmetries of the principal bundle.

Remark 1.7. For any complex manifold X we can restrict the functor $\mathcal{G}^{sh}$ to the overcategory of opens in X, that we denote by $\mathcal{G}_X^{sh}$. In this case, $\mathcal{G}_X^{sh}$, or its compactly supported version $\mathcal{G}_X$, comes from the local Lie algebra of Definition 1.4 in the case of the trivial G-bundle on X. In the case that $X = \mathbb{C}^d$ we will denote the sheaves and cosheaves of the local Lie algebra by $\mathcal{G}_d^{sh}, \mathcal{G}_d$ respectively.

1.2. **Current algebras as enveloping factorization algebras of local Lie algebras.** Local Lie algebras often appear as symmetries of classical field theories. For instance, as we will show in Section 3, each finite-dimensional complex representation V of a Lie algebra $\mathfrak{g}$ determines a charged $\beta\gamma$-type system on a complex d-fold X with choice of holomorphic principal bundle $\pi : P \to X$. Namely, the on-shell γ fields are holomorphic sections for the associated bundle $P \times^G V \to X$, and the on-shell β fields are holomorphic d-forms with values in the associated bundle $P \times^G V^* \to X$. It should be plausible that $\mathcal{A}d(P)^{sh}$ acts as symmetries of this classical field theory, since holomorphic sections of the adjoint bundle manifestly send on-shell fields to on-shell fields.

Such a symmetry determines currents, which we interpret as observables of the classical theory. Note, however, a mismatch: while fields are contravariant in space(time) because fields pull back along inclusions of open sets, observables are covariant because an observable on a smaller region extends to any larger region containing it. The currents, as observables, thus do not form a sheaf but a precosheaf. We introduce the following terminology.

maybe this is better to introduce this when dealing

Definition 1.8. For a local Lie algebra $(L \to X, \ell_1, \ell_2)$, its precosheaf $\mathcal{L}[1]$ of *linear currents* is given by taking compactly supported sections of L.

There are a number of features of this definition that may seem peculiar on first acquaintance. First, we work with $\mathcal{L}[1]$ rather than $\mathcal{L}$. This shift is due to the Batalin-Vilkovisky formalism. In that formalism the observables in the classical field theory possesses a 1-shifted Poisson bracket $\{-,-\}$ (also known as the antibracket), and so if the current $J(s)$ associated to a section $s \in \mathcal{L}$ encodes the action of s on the observables, i.e.,

$$\{J(s), F\} = s \cdot F,$$

then we need the cohomological degree of $J(s)$ to be 1 less than the degree of s. In short, we want a map of dg Lie algebras $J : \mathcal{L} \to \mathrm{Obs}^{\mathrm{cl}}[-1]$, or equivalently a map of 1-shifted dg Lie algebras $J : \mathcal{L}[1] \to \mathrm{Obs}^{\mathrm{cl}}$, where $\mathrm{Obs}^{\mathrm{cl}}$ denotes the algebra of classical observables.

Second, we use the term "linear" here because the product of two such currents is not in $\mathcal{L}[1]$ itself, although such a product will exist in the larger precosheaf $\mathrm{Obs}^{\mathrm{cl}}$ of observables. In other words, if we have a Noether map of dg Lie algebras $J : \mathcal{L} \to \mathrm{Obs}^{\mathrm{cl}}[-1]$, it extends to a map of 1-shifted Poisson algebras

$$J : \mathrm{Sym}(\mathcal{L}[1]) \to \mathrm{Obs}^{\mathrm{cl}}$$

as $\mathrm{Sym}(\mathcal{L}[1])$ is the 1-shifted Poisson algebra freely generated by the 1-shifted dg Lie algebra $\mathcal{L}[1]$. We hence call $\mathrm{Sym}(\mathfrak{g}[1])$ the *enveloping 1-shifted Poisson algebra* of a dg Lie algebra $\mathfrak{g}$.

Say what this means?

For any particular field theory, the currents generated by the symmetry for *that* theory are given by the image of this map J of 1-shifted Poisson algebras. To study the general structure of such currents, without respect to a particular theory, it is natural to study this enveloping algebra by itself.

Definition 1.9. For a local Lie algebra $(L \to X, \ell_1, \ell_2)$, its *classical currents* $\mathrm{Cur}^{\mathrm{cl}}(\mathcal{L})$ is the precosheaf $\mathrm{Sym}(\mathcal{L}[1])$ on X given by taking the enveloping 1-shifted Poisson algebra of the compactly supported sections of L. It assigns

$$\mathrm{Cur}^{\mathrm{cl}}(\mathcal{L})(U) = \mathrm{Sym}(\mathcal{L}(U)[1])$$

to an open subset $U \subset X$.

We emphasize here that by $\mathrm{Sym}(\mathcal{L}(U)[1])$ we do *not* mean the symmetric algebra in the purely algebraic sense, but rather a construction that takes into account the extra structures on sections of vector bundles (e.g., the topological vector space structure). Explicitly, the nth symmetric power $\mathrm{Sym}^n(\mathcal{L}(U)[1])$ means the smooth, compactly supported, and S_n-invariant sections of the graded vector bundle

$$L[1]^{\boxtimes n} \to U^n.$$

For further discussion of functional analytic aspects (which play no tricky role in our work here), see [CG17], notably the appendices. Appendices B–D

A key result of [CG17], namely Theorem 5.6.0.1, is that this precosheaf of currents forms a factorization algebra. From hereon we refer to $\mathrm{Cur}^{\mathrm{cl}}(\mathcal{L})$ as the *factorization algebra of classical currents*. If the local Lie algebra acts as symmetries on some classical field theory, we obtain a map of factorization algebras $J : \mathrm{Cur}^{\mathrm{cl}}(\mathcal{L}) \to \mathrm{Obs}^{\mathrm{cl}}$ that encodes each current as a classical observable.

action vs. inner action?

There is a quantum counterpart to this construction, in the Batalin-Vilkovisky formalism. The idea is that for a dg Lie algebra $\mathfrak{g}$, the enveloping 1-shifted Poisson algebra $\mathrm{Sym}(\mathfrak{g}[1])$ admits a natural BV quantization via the Chevalley-Eilenberg chains $C_*(\mathfrak{g})$. This assertion is transparent by examining the Chevalley-Eilenberg differential:

$$\mathrm{d}_{CE}(xy) = \mathrm{d}_{\mathfrak{g}}(x)y \pm x\,\mathrm{d}_{\mathfrak{g}}(y) + [x,y]$$

for x, y elements of $\mathfrak{g}[1]$. The first two terms behave like a derivation of $\mathrm{Sym}(\mathfrak{g}[1])$, and the last term agrees with the shifted Poisson bracket. More accurately, to keep track of the $\hbar$-dependency in quantization, we introduce a kind of Rees construction.

Definition 1.10. The *enveloping BD algebra* $U^{BD}(\mathfrak{g})$ of a dg Lie algebra $\mathfrak{g}$ is given by the graded-commutative algebra in $\mathbb{C}[\hbar]$-modules

$$\mathrm{Sym}(\mathfrak{g}[1])[\hbar] \cong \mathrm{Sym}_{\mathbb{C}[\hbar]}(\mathfrak{g}[\hbar][1]),$$

but the differential is defined as a coderivation with respect to the natural graded-cocommutative coalgebra structure, by the condition

$$\mathrm{d}(xy) = \mathrm{d}_{\mathfrak{g}}(x)y \pm x\,\mathrm{d}_{\mathfrak{g}}(y) + \hbar[x,y].$$

This construction determines a BV quantization of the enveloping 1-shifted Poisson algebra, as can be verified directly from the definitions. (For further discussion see [GH18] and [CG].) It is also straightforward to extend this construction to "quantize" the factorization algebra of classical currents.

Definition 1.11. For a local Lie algebra $(L \to X, \ell_1, \ell_2)$, its *factorization algebra of quantum currents* $\mathrm{Cur}^{\mathrm{q}}(\mathcal{L})$ is given by taking the enveloping BD algebra of the compactly supported sections of L. It assigns

$$\mathrm{Cur}^{\mathrm{q}}(\mathcal{L})(U) = U^{BD}(\mathcal{L}(U))$$

to an open subset $U \subset X$.

As mentioned just after the definition of the classical currents, the symmetric powers here mean the construction involving sections of the external tensor product. Specializing $\hbar = 1$, we recover the following construction.

Definition 1.12. For a local Lie algebra $(L \to X, \ell_1, \ell_2)$, its *enveloping factorization algebra* $\mathbb{U}(\mathcal{L})$ is given by taking the Chevalley-Eilenberg chains $C_*^{\mathrm{Lie}}(\mathcal{L})$ of the compactly supported sections of L.

Here the symmetric powers use sections of the external tensor powers, just as with the classical or quantum currents.

When a local Lie algebra acts as symmetries of a classical field theory, it sometimes also lifts to symmetries of a BV quantization. In that case the map $J : \mathrm{Sym}(\mathcal{L}[1]) \to \mathrm{Obs}^{\mathrm{cl}}$

of 1-shifted Poisson algebras lifts to a cochain map $J^{\mathrm{q}} : \mathrm{Cur}^{\mathrm{q}}(\mathcal{L}) \to \mathrm{Obs}^{\mathrm{q}}$ realizing quantum currents as quantum observables. Sometimes, however, the classical symmetries do not lift directly to quantum symmetries. We turn to discussing the natural home for the obstructions to such lifts after a brief detour to offer a structural perspective on the enveloping construction.

1.2.1. *A digression on the enveloping E_n algebras.* This construction $\mathbb{U}(\mathcal{L})$ has a special feature when the local Lie algebra is obtained by taking the de Rham forms with values in a dg Lie algebra $\mathfrak{g}$, i.e., when $\mathcal{L} = \Omega^*_c \otimes \mathfrak{g}$. In that case the enveloping factorization algebra is locally constant and, on the d-dimensional real manifold $\mathbb{R}^d$, determines an E_d algebra, also known as an algebra over the little d-disks operad, by a result of Lurie (see Theorem 5.5.4.10 of [Lur]). This construction satisfies a universal property: it is the d-dimensional generalization of the universal enveloping algebra of a Lie algebra.

To state this result of Knudsen precisely, we need to be in the context of ∞-categories.

what result of Knudsen?

Theorem 1.13 ([Knu]). *Let $\mathcal{C}$ be a stable, $\mathbb{C}$-linear, presentable, symmetric monoidal ∞-category. There is an adjunction*

$$U^{E^d} : \mathrm{LieAlg}(\mathcal{C}) \leftrightarrows E_d\mathrm{Alg}(\mathcal{C}) : F$$

between Lie algebra objects in $\mathcal{C}$ and E_d algebra objects in $\mathcal{C}$. This adjunction intertwines with the free-forget adjunctions from Lie/E_d algebras in $\mathcal{C}$ to $\mathcal{C}$ so that

$$\mathrm{Free}_{E_d}(X) \simeq U^{E_d}\mathrm{Free}_{Lie}(\Sigma^{d-1}X)$$

for any object $X \in \mathcal{C}$.

When $\mathcal{C}$ is the ∞-category of chain complexes over a field of characteristic zero, the E_d algebra $U^{E_d}\mathfrak{g}$ is modeled by the locally constant factorization algebra $\mathbb{U}(\Omega^*_c \otimes \mathfrak{g})$ on $\mathbb{R}^d$.

This theorem is highly suggestive for us: our main class of examples is $\mathcal{G}_d$ and $\mathbb{U}\mathcal{G}_d$, which replaces the de Rham complex with the Dolbeault complex. In other words, we anticipate that $\mathbb{U}\mathcal{G}_d$ should behave like a holomorphic version of an E_d algebra and that it should be the canonical such algebra determined by a dg Lie algebra. We do not pursue this structural result in this paper, but it provides some intuition behind our constructions.

1.3. **Local cocycles and shifted extensions.** Some basic questions about a dg Lie algebra $\mathfrak{g}$, such as the classification of extensions and derivations, are encoded cohomologically, typically as cocycles in the Chevalley-Eilenberg cochains $C^*_{\mathrm{Lie}}(\mathfrak{g}, V)$ with coefficients in some $\mathfrak{g}$-representation V. When working with local Lie algebras, it is natural to focus on cocycles that are also local in the appropriate sense. (Explicitly, we want to restrict to cocycles that are built out of polydifferential operators.) After introducing the relevant construction, we turn to studying how such cocycles determine modified current algebras.

1.3.1. *Local cochains of a local Lie algebra.* We refer the reader to Section 4.4 of [CG] for the definition of the "local cohomology" of a local Lie algebra. We briefly recall it here. The basic idea is that a local cochain is a Lagrangian density: it takes in a section of the local Lie algebra and produces a smooth density on the manifold. Such a cocycle determines a functional by integrating the density. As usual with Lagrangian densities, we wish to work with them up to total derivatives, i.e., we identify Lagrangian densities related using integration by parts and hence ignore boundary terms.

In a bit more detail, for L is a graded vector bundle, let JL denote the corresponding ∞-jet bundle, which has a canonical flat connection. In other words, it is a left D_X-module, where D_X denotes the sheaf of smooth differential operators on X. For a local Lie algebra, this JL obtains the structure of a dg Lie algebra in left D_X-modules. Thus, we may consider its reduced Chevalley-Eilenberg cochain complex $C^*_{\mathrm{Lie}}(JL)$ in the category of left D_X-modules. By taking the de Rham complex of this left D_X-module, we obtain the local cochains. For a variety of reasons, it is useful to ignore the "constants" term and work with the reduced cochains. Hence we have the following definition.

Definition 1.14. Let $\mathcal{L}$ be a local Lie algebra on X. The *local Chevalley-Eilenberg cochains* of $\mathcal{L}$ is

$$C^*_{\mathrm{loc}}(\mathcal{L}) = \Omega^{*,*}_X[2d] \otimes_{D_X} C^*_{\mathrm{Lie,red}}(JL).$$

This sheaf of cochain complexes on X has global sections that we denote by $C^*_{\mathrm{loc}}(\mathcal{L}(X))$.

Remark 1.15. This construction $C^*_{\mathrm{loc}}(\mathcal{L})$ is just a version of diagonal Gelfand-Fuks cohomology [Fuk86, Los98], where the adjective "diagonal" indicates that we are interested in continuous cochains whose integral kernels are supported on the small diagonals.

1.3.2. *Shifted extensions.* For an ordinary Lie algebra $\mathfrak{g}$, central extensions are parametrized by 2-cocycles on $\mathfrak{g}$ valued in the trivial module $\mathbb{C}$. It is possible to interpret arbitrary cocycles as determining as determining *shifted* central extensions as L_∞ *algebras.* Explicitly, a k-cocycle Θ of degree n on a dg Lie algebra $\mathfrak{g}$ determines an L_∞ algebra structure on the direct sum $\mathfrak{g} \oplus \mathbb{C}[n-k]$ with the following brackets $\{\widehat{\ell}_m\}_{m \geq 1}$: $\widehat{\ell}_1$ is simply the differential on $\mathfrak{g}$, $\widehat{\ell}_2$ is the bracket on $\mathfrak{g}$, $\widehat{\ell}_m = 0$ for $m > 2$ except

$$\widehat{\ell}_k(x_1 + a_1, \ldots, x_k + a_k) = 0 + \Theta(x_1, x_2, \ldots, x_k).$$

(See [KS] for further discussion. Note that $n = 2$ for $k = 2$ with ordinary Lie algebras.) ?
Similarly, local cocycles provide shifted central extensions of local Lie algebras.

Definition 1.16. For a local Lie algebra (L, ℓ_1, ℓ_2), a cocycle Θ of degree $2+k$ in $C^*_{\mathrm{loc}}(\mathcal{L})$ determines a *k-shifted central extension*

$$0 \to \mathbb{C}[k] \to \widehat{\mathcal{L}}_\Theta \to \mathcal{L} \to 0 \tag{2}$$

of precosheaves of L_∞ algebras, where the L_∞ structure maps are defined by

$$\widehat{\ell}_n(x_1, \ldots, x_n) = (\ell_n(x_1, \ldots, x_n), \int \Theta(x_1, \ldots, x_n)).$$

Here we set $\ell_n = 0$ for $n > 2$.

As usual, cohomologous cocycles determine quasi-isomorphic extensions. Much of the rest of the section is devoted to constructing and analyzing various cocycles and the resulting extensions.

1.3.3. *Twists of the current algebras.* Local cocycles give a direct way of deforming the various current algebras a local Lie algebra. For example, we have the following construction.

Definition 1.17. Let Θ be a degree 1 local cocycle for a local Lie algebra $(L \to X, \ell_1, \ell_2)$. Let K denote a degree zero parameter so that $\mathbb{C}[K]$ is a polynomial algebra concentrated in degree zero. The *twisted enveloping factorization algebra* $\mathbb{U}_\Theta(\mathcal{L})$ assigns to an open $U \subset X$, the cochain complex

$$\begin{aligned} \mathbb{U}_\Theta(\mathcal{L})(U) &= (\mathrm{Sym}(\mathcal{L}(U)[1] \oplus \mathbb{C} \cdot K), \mathrm{d}_{\mathcal{L}} + K \cdot \Theta) \\ &= (\mathrm{Sym}(\mathcal{L}(U)[1])[K], \mathrm{d}_{\mathcal{L}} + K \cdot \Theta), \end{aligned}$$

isn't this $\oplus \hat{\ell}_{\Theta}$?

where $\mathrm{d}_{\mathcal{L}}$ denotes the differential on the untwisted enveloping factorization algebra and Θ is the operator extending the cocycle $\Theta : \mathrm{Sym}(\mathcal{L}(U)[1]) \to \mathbb{C} \cdot K$ to the symmetric coalgebra as a graded coderivation. This twisted enveloping factorization algebra is a module for the commutative ring $\mathbb{C}[K]$, and so specializing the value of K determines nontrivial modifications of $\mathbb{U}(\mathcal{L})$.

An analogous construction applies to the quantum currents, which we will denote $\mathrm{Cur}^{\mathfrak{q}}_{\Theta}(\mathcal{L})$.

1.3.4. *A special class of cocycles: the $\mathfrak{j}$ functional.* There is a particular family of local cocycles that has special importance in studying symmetries of higher dimensional holomorphic field theories.

Consider

$$\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}},$$

so that θ is a $\mathfrak{g}$-invariant polynomial on $\mathfrak{g}$ of homogenous degree $d+1$. This data determines a local functional for $\mathcal{G} = \Omega^{0,*} \otimes \mathfrak{g}$ on any complex d-fold as follows.

Definition 1.18. For any complex d-fold X, extend θ to a functional $\mathfrak{J}_X(\theta)$ on $\mathcal{G}_X = \Omega^{0,*}_c(X) \otimes \mathfrak{g}$ by the formula

$$\mathfrak{J}_X(\theta)(\alpha_0, \ldots, \alpha_d) = \int_X \theta(\alpha_0, \partial\alpha_1, \ldots, \partial\alpha_d), \tag{3}$$

where ∂ denotes the holomorphic de Rham differential. In this formula, we define the integral to be zero whenever the integrand is not a (d,d)-form.

To make this formula as clear as possible, suppose the α_i are pure tensors of the form $\omega_i \otimes y_i$ with $\omega_i \in \Omega_c^{0,*}(X)$ and $y_i \in \mathfrak{g}$. Then

$$\mathfrak{J}_X(\theta)(\omega_0 \otimes y_0, \dots, \omega_d \otimes y_d) = \theta(y_0, \dots, y_d) \int_X \omega_0 \wedge \partial\omega_1 \cdots \wedge \partial\omega_d. \tag{4}$$

Note that we use d copies of the holomorphic derivative $\partial : \Omega^{0,*} \to \Omega^{1,*}$ to obtain an element of $\Omega_c^{d,*}$ in the integrand and hence something that can be integrated.

This formula manifestly makes sense for any complex d-fold X, and since integration is local on X, it intertwines nicely with the structure maps of $\mathcal{G}_X$.

Definition 1.19. For any complex d-fold X and any $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$, let $\mathfrak{j}_X(\theta)$ denote the local cochain in $C^*_{\mathrm{loc}}(\mathcal{G}_X)$ defined by

$$\mathfrak{j}_X(\theta)(\alpha_0, \dots, \alpha_d) = \theta(\alpha_0, \partial\alpha_1, \dots, \partial\alpha_d).$$

Hence $\mathfrak{J}_X(\theta) = \int_X \mathfrak{j}_X(\theta)$.

This integrand $\mathfrak{j}_X(\theta)$ is in fact a local cocycle, and in a moment we will use it to produce an important shifted central extension of $\mathcal{G}_X$.

Proposition 1.20. *The assignment*

$$\begin{array}{rccc} \mathfrak{j}_X : & \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}[-1] & \to & C^*_{\mathrm{loc}}(\mathcal{G}_X) \\ & \theta & \mapsto & \mathfrak{j}_X(\theta) \end{array}$$

is an cochain map.

Proof. The element $\mathfrak{j}_X(\theta)$ is local as it is expressed as a density produced by polydifferential operators. We need to show that $\mathfrak{j}_X(\theta)$ is closed for the differential on $C^*_{\mathrm{loc}}(\mathcal{G}_X)$. Note that $\mathcal{G}_X$ is the tensor product of the dg commutative algebra $\Omega_X^{0,*}$ and the Lie algebra $\mathfrak{g}$. Hence the differential on the local cochains of $\mathcal{G}_X$ splits as a sum $\bar\partial + d_{\mathfrak{g}}$ where $\bar\partial$ denotes the differential on local cochains induced from the $\bar\partial$ differential on the Dolbeault forms and $d_{\mathfrak{g}}$ denotes the differential induced from the Lie bracket on $\mathfrak{g}$. We now analyze each term separately.

Observe that for any collection of $\alpha_i \in \mathcal{G}$, we have

$$\begin{aligned} \bar\partial(\theta(\alpha_0, \partial\alpha_1, \dots, \partial\alpha_d)) &= \theta(\bar\partial\alpha_0, \partial\alpha_1, \dots, \partial\alpha_d) \pm \theta(\alpha_0, \bar\partial\partial\alpha_1, \dots, \partial\alpha_d) \pm \cdots \pm \theta(\alpha_0, \partial\alpha_1, \dots, \bar\partial\partial\alpha_d) \\ &= \sum_{i=0}^{d} \pm\theta(\alpha_0, \partial\alpha_1, \dots, \partial\alpha_d) = \theta(\bar\partial(\alpha_0, \partial\alpha_1, \dots, \partial\alpha_d)) \end{aligned}$$

because $\bar\partial$ is a derivation and θ wedges the form components. (It is easy to see this assertion when one works with inputs like in equation (4).) Hence viewing $\mathfrak{j}_X(\theta)$ as a map from $\mathcal{G}$ to the Dolbeault complex, it commutes with the differential $\bar\partial$. This fact is equivalent to $\bar\partial\mathfrak{j}_X(\theta) = 0$ in local cochains.

Similarly, observe that for any collection of $\alpha_i \in \mathcal{G}$, we have

$$\begin{aligned}(\mathrm{d}_{\mathfrak{g}} \mathrm{j}_X(\theta))(\alpha_0, \alpha_1, \ldots, \alpha_d) &= (\mathrm{d}_{\mathfrak{g}}\theta)(\alpha_0, \partial\alpha_1, \ldots, \partial\alpha_d)) \\ &= 0\end{aligned}$$

since θ is closed in $C^*_{\mathrm{Lie}}(\mathfrak{g})$. □

As should be clear from the construction, everything here works over the site Hol_d of complex d-folds, and hence we use $\mathrm{j}(\theta)$ to denote the local cocycle for the local Lie algebra $\mathcal{G}$ on Hol_d.

This construction works nicely for an arbitrary holomorphic G-bundle P on X, because the cocycle is expressed in a coordinate-free fashion. To be explicit, on a coordinate patch $U_i \subset X$ with a choice of trivialization of the adjoint bundle $\mathrm{ad}(P)$, the formula for $\mathrm{j}_X(\theta)$ makes sense. On an overlap $U_i \cap U_j$, the cocycles patch because $\mathrm{j}_X(\theta)$ is independent of the choice of coordinates. Hence we can glue over any sufficiently refined cover to obtain a global cocycle. Thus, we have a cochain map

I wouldn't say this explanation is necessary.

$$\mathrm{j}_X^P : \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}[-1] \to C^*_{\mathrm{loc}}(\mathcal{A}d(P)(X))$$

given by the same formula as in (3).

1.3.5. *Another special class: the LMNS extensions.* Much of this paper focuses on local cocycles of type $\mathrm{j}_X(\theta)$, where $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$. But there is another class of local cocycles that appear naturally when studying symmetries of holomorphic theories. Unlike the cocycle $\mathrm{j}_X(\theta)$, which only depends on the manifold X through its dimension, this class of cocycles depends on the geometry.

In complex dimension two, this class of cocycles has appeared in the work of Losev-Moore-Nekrasov-Shatashvili (LMNS) [LMNS96, LMNS97, LMNS98] in their construction of a higher analog of the "chiral WZW theory". Though our approach~~es~~ differs from theirs we share their ambition to formulate a higher analogs of constructions and ideas in chiral CFT.

Let X be a complex manifold of dimension d with a choice of (k,k)-form η. Choose a form $\theta_{d+1-k} \in \mathrm{Sym}^{d+1-k?}(\mathfrak{g}^*)^{\mathfrak{g}}$. This data determines a local cochain on $\mathcal{G}_X$ whose local functional is:

$$\begin{array}{cccc}\phi_{\theta,\eta} : & \mathcal{G}(X)^{\otimes d+1-k} & \to & \mathbb{C} \\ & \alpha_0 \otimes \cdots \otimes \alpha_{d-k} & \mapsto & \displaystyle\int_X \eta \wedge \theta_{d+1-k}(\alpha_0, \partial\alpha_1, \ldots, \partial\alpha_{d-k})\end{array}.$$

Such a cochain is a cocycle only if $\bar{\partial}\eta = 0$, because η does not interact with the Lie structure.

Note that a Kähler manifold always produces natural choices of η by taking $\eta = \omega^k$, where ω is the symplectic form. In this way, Kähler geometry determines an important class of extensions. It would be interesting to explore what aspects of the geometry are reflected by these associated current algebras. ¶ The following is a direct calculation.

Transition

Lemma 1.21. *Fix $\theta \in \mathrm{Sym}^{d+1-k}(\mathfrak{g}^*)^{\mathfrak{g}}$. If a form $\eta \in \Omega^{k,k}(X)$ satisfies $\bar{\partial}\eta = 0$ and $\partial\eta = 0$, then the local cohomology class $[\phi_{\theta,\omega}] \in H^1_{\mathrm{loc}}(\mathcal{G}_X)$ depends only on the cohomology class $[\omega] \in H^k(X, \Omega^k_{cl})$.* $\eta \propto \omega$?

When $\eta = 1$, it trivially satisfies the conditions of the lemma. In this case $\phi_{\theta,1} = \mathrm{j}_X(\theta)$ in the notation of the last section.

1.4. **The higher Kac-Moody factorization algebra.**

Finally, we can introduce the central object of this paper.

Definition 1.22. Let X be a complex manifold of complex dimension d equipped with a holomorphic principal G-bundle P. Let Θ be a degree 1 cocycle in $C^*_{\mathrm{loc}}(\mathcal{A}\mathrm{d}(P))$, which determines a 1-shifted central extension $\mathcal{A}\mathrm{d}(P)_\Theta$. The *Kac-Moody factorization algebra on X of type* Θ is the twisted enveloping factorization algebra $\mathbb{U}_\Theta(\mathcal{A}\mathrm{d}(P))$ that assigns

$$\left(\mathrm{Sym}\left(\Omega^{0,*}_c(U, \mathrm{ad}(P))[1]\right)[K], \bar{\partial} + \mathrm{d}_{CE} + \Theta\right)$$

to an open set $U \subset X$.

Remark 1.23. As in the definition of twisted enveloping factorization algebras, the factorization algebras $\mathbb{U}_\Theta(\mathcal{A}\mathrm{d}(P))$ are modules for the ring $\mathbb{C}[K]$. In keeping with conventions above, when P is the trivial bundle on X, we will denote the Kac-Moody factorization algebra by $\mathbb{U}_\Theta(\mathcal{G}_X)$.

The most important class of such higher Kac-Moody algebras makes sense over the site Hol_d of all complex d-folds.

Definition 1.24. Let $\mathfrak{g}$ be an ordinary Lie algebra and let $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$. Let $\mathcal{G}_{d,\theta}$ denote the 1-shifted central extension of $\mathcal{G}_d$ determined by the local cocycle $\mathrm{j}(\theta)$. Let $\mathbb{U}_\theta(\mathcal{G})$ denote the *θ-twisted enveloping factorization algebra* $\mathbb{U}_{\mathrm{j}(\theta)}(\mathcal{G})$ for the local Lie algebra $\mathcal{G} = \Omega^{0,*}_c \otimes \mathfrak{g}$ on the site Hol_d of complex d-folds.

In the case $d = 1$ the definition above agrees with the Kac-Moody factorization algebra on Riemann surfaces given in [CG17]. There, it is shown that this factorization algebra, restricted to the complex manifold $\mathbb{C}$, recovers a vertex algebra isomorphic to that of the ordinary Kac-Moody vertex algebra. (See Section 5 of Chapter 5.) Thus, we think of the object $\mathbb{U}_\Theta(\mathcal{A}\mathrm{d}(P))$ as a higher dimensional version of the Kac-Moody vertex algebra.

1.4.1. *Holomorphic translation invariance and higher dimensional vertex algebras.*

To put some teeth into the previous paragraph, we note that [CG17] introduces a family of colored operads $\mathrm{PDiscs}_{\mathrm{d}}$, the little d-dimensional polydiscs operads, that provide a holomorphic analog of the little d-disks operads E_d. Concretely, this operad $\mathrm{PDiscs}_{\mathrm{d}}$ encodes the idea of the operator product expansion, where one now understands observables supported in small disks mapping into observables in large disks, rather than point-like observables.

In the case $d = 1$, Theorem 5.3.3 of [CG17] shows that a PDiscs_1-algebra $\mathcal{A}$ determines a vertex algebra $\mathbb{V}(\mathcal{A})$ so long as $\mathcal{A}$ is suitably equivariant under rotation . This construction $\mathbb{V}$ is functorial. As shown in [CG17], many vertex algebras appear this way, ~~and any~~ vertex algebras that arise from physics should, in light of the main results of [CG17, CG]. appear this way.

For this reason, one can interpret PDiscs_d-algebras, particularly when suitably equivariant under rotation, as providing a systematic and operadic generalization of vertex algebras to higher dimensions. Proposition 5.2.2 of [CG17] provides a useful mechanism for producing PDiscs_d-algebras: it says that if a factorization algebra is equivariant under translation in a holomorphic manner, then it determines such an algebra.

Hence it is interesting to identify when the higher Kac-Moody factorization algebras are invariant in the sense needed to produce PDiscs_d-algebras. We now address this question.

First, note that on the complex d-fold $X = \mathbb{C}^d$, the local Lie algebra $\mathcal{G}_d$ is manifestly equivariant under translation.

It is important to recognize that this translation action is holomorphic in the sense that the infinitesimal action of the (complexified) vector fields $\partial/\partial\overline{z}_i$ is homotopically trivial. Explicitly, consider the operator $\eta_i = \iota_{\partial/\partial\overline{z}_i}$ on Dolbeault forms (and which hence extends to $\mathcal{G}_{\mathbb{C}^d}$), and note that

$$[\overline{\partial}, \eta_i] = \partial/\partial\overline{z}_i.$$

Both the infinitesimal actions and this homotopical trivialization extend canonically to the Chevalley-Eilenberg chains of $\mathcal{G}_{\mathbb{C}^d}$ and hence to the enveloping factorization algebra and the current algebras. (For more discussion of these ideas see [Wilb] and Chapter 10 of [CG].)

A succinct way to express this feature is to introduce a dg Lie algebra

$$\mathbb{C}^d_{\mathrm{hol}} = \mathrm{span}_{\mathbb{C}}\{\partial/\partial z_1, \ldots, \partial/\partial z_d, \partial/\partial\overline{z}_1, \ldots, \partial/\partial\overline{z}_d, \eta_1, \ldots, \eta_d\}$$

where the partial derivatives have degree 0 and the η_i have degree -1, where the brackets are all trivial, and where the differential behaves like $\overline{\partial}$ in the sense that the differential of η_i is $\partial/\partial\overline{z}_i$. We just argued in the preceding paragraph that $\mathcal{G}_{\mathbb{C}^d}$ and its current algebras are all strictly $\mathbb{C}^d_{\mathrm{hol}}$-invariant.

When studying shifted extensions of $\mathcal{G}_{\mathbb{C}^d}$, it then makes sense to consider local cocycles that are also translation invariant in this sense. Explicitly, we ask to work with cocycles in

$$C^*_{\mathrm{loc}}(\mathcal{G}_d)^{\mathbb{C}^d_{\mathrm{hol}}} \subset C^*_{\mathrm{loc}}(\mathcal{G}_d).$$

Local cocycles here determine higher Kac-Moody algebras that are holomorphically translation invariant and hence yield PDiscs_d-algebras.

The following result indicates ~~tells~~ us that we have already encountered all the relevant cocycles so long as we also impose rotation invariance, which is a natural condition.

Proposition 1.25. *The map* $\mathrm{j}_{\mathbb{C}^d} : \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}[-1] \to \mathrm{C}^*_{\mathrm{loc}}(\mathcal{G}_d)$ *factors through the subcomplex of local cochains that are rotationally and holomorphically translation invariant. Moreover, it determines an isomorphism on* H^1

$$H^1(\mathrm{j}_{\mathbb{C}^d}) : \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}} \xrightarrow{\cong} H^1\left(\mathrm{C}^*_{\mathrm{loc}}(\mathcal{G}_d))^{\mathbb{C}^d_{\mathrm{hol}}}\right)^{U(d)}.$$

As the proof is rather lengthy, we provide it in Appendix A.

2. Local aspects of the higher Kac-Moody factorization algebras

A factorization algebra encodes an enormous amount of information, and hence it is important to extract aspects that are simpler to understand. In this section we will take two approaches:

(1) by compactifying along a sphere of real dimension $2d-1$, we obtain an algebra (more precisely, a homotopy-coherent associative algebra) that encodes the higher dimensional version of "radial ordering" of operators from two-dimensional conformal field theory, and
(2) by compactifying along a torus $(S^1)^d$, we obtain an algebra over the little d-disks operad.

In both cases these algebras behave like enveloping algebras of homotopy-coherent Lie algebras (in a sense we will spell out in detail below), which allows for simpler descriptions of some phenomena. It is important to be aware, however, that these algebras do not encode the full algebraic structure produced by the compactification; instead, they sit as dense subalgebras. We will elaborate on this subtlety below. inside the full alg. structure

For factorization algebras, compactification is accomplished by the pushforward operation. Given a map $f : X \to Y$ of manifolds and a factorization algebra $\mathcal{F}$ on X, its *pushforward* $f_*\mathcal{F}$ is the factorization algebra on Y where

$$f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$$

for any open $U \subset Y$. The first example we treat arises from the radial projection map

$$r : \mathbb{C}^d \setminus \{0\} \to (0,\infty)$$

sending z to its length $|z|$. The preimage of a point is simply a $2d-1$-sphere, so one can interpret the pushforward Kac-Moody factorization algebra $r_*\mathbb{U}_\theta\mathcal{G}_d$ as compactification along these spheres. Our first main result is that there is a locally constant factorization algebra $\mathcal{A}$ along $(0,\infty)$ with a natural map

$$\phi : \mathcal{A} \to r_*\mathbb{U}_\theta\mathcal{G}_d$$

that is dense from the point of view of the topological vector space structure. By a theorem of Lurie, locally constant factorization algebras on $\mathbb{R}$ correspond to homotopy-coherent associative algebras, so that we can interpret ϕ as saying that the pushforward is approximated by an associative algebra, in this derived sense. We will show explicitly that this algebra is the A_∞ algebra arising as the enveloping algebra of an L_∞ algebra already introduced by Faonte-Hennion-Kapranov.

For the physically-minded reader, this process should be understood as a version of radial ordering. Recall from the two-dimensional setting that it can be helpful to view the punctured plane as a cylinder, and to use the radius as a kind of time parameter. Time ordering of operators is then replaced by radial ordering. Many computations can be nicely organized in this manner, because a natural class of operators arises by using a Cauchy integral around the circle of a local operator. The same technique works in higher dimensions where one now computes residues along the $2d-1$-spheres. From this perspective, the natural Hilbert space is associated to the origin in the plane (more accurately to an arbitrarily small disk around the origin), and this picture also extends to higher dimensions. Hence we obtain a kind of vacuum module for this higher dimensional generalization of the Kac-Moody algebras.

Our second cluster of results uses compactification along the projection map

$$\begin{array}{ccc} \mathbb{C}^d \setminus \{\text{coordinate hyperplanes}\} & \to & (0,\infty)^d \\ (z_1,\ldots,z_d) & \mapsto & (|z_1|,\ldots,|z_d|). \end{array}$$

is there a physical interpretation?

We construct a locally constant factorization algebra on $(0,\infty)^d$ that maps densely into the pushforward of the higher Kac-Moody algebra. Lurie's theorem shows that locally constant factorization algebras on $\mathbb{R}^d$ correspond to E_d algebras, so we obtain a higher-dimensional analog of the spherical result.

2.1. **Compactifying the higher Kac-Moody algebras along spheres.** Our approach is modeled on the construction of the affine Kac-Moody Lie algebras and their associated vertex algebras from Section 5.5 of [CG17] and [Gwi12], so we review the main ideas to orient the reader.

isn't this just always true on a cpct. mfld?

On the punctured plane $\mathbb{C}^*$, the sheaf $\mathcal{G}_1^{sh} = \Omega^{0,*} \otimes \mathfrak{g}$ is quasi-isomorphic to the sheaf $\mathcal{O} \otimes \mathfrak{g}$. The restriction maps of this sheaf tell us that for any open set U, there is a map of Lie algebras

$$\mathcal{O}(\mathbb{C}^*) \otimes \mathfrak{g} \to \mathcal{O}(U) \otimes \mathfrak{g},$$

so that we get a map of Lie algebras

$$\mathcal{O}_{\text{alg}}(\mathbb{C}^*) \otimes \mathfrak{g} = \mathfrak{g}[z,z^{-1}] \to \mathcal{O}(U) \otimes \mathfrak{g}$$

because Laurent polynomials $\mathbb{C}[z,z^{-1}] = \mathcal{O}_{\text{alg}}(\mathbb{C}^*)$ are well-defined on any open subset of the punctured plane. This *loop algebra* $L\mathfrak{g} = \mathfrak{g}[z,z^{-1}]$ admits interesting central extensions, known as the affine Kac-Moody Lie algebras. These extensions are labeled by elements of $\text{Sym}^2(\mathfrak{g}^*)^{\mathfrak{g}}$, which is compatible with our work in Section 1.3.4.

To apply radial ordering to this sheaf—or rather, its associated current algebras—it is convenient to study the pushforward along the radial projection map $r(z) = |z|$. Note that the preimage of an interval (a, b) is an annulus, so

$$r_* \mathcal{G}_1^{sh}((a,b)) = \mathcal{G}_1^{sh}(\{a < |z| < b\})$$

and hence we have a canonical map of Lie algebras

$$\mathfrak{g}[z, z^{-1}] \to \mathcal{O}(\{a < |z| < b\}) \otimes \mathfrak{g} \hookrightarrow r_* \mathcal{G}_1^{sh}((a,b)).$$

We can refine this situation by replacing the left hand side with the locally constant sheaf $\underline{\mathfrak{g}[z,z^{-1}]}$ to produce a map of sheaves $\underline{\mathfrak{g}[z,z^{-1}]} \to r_* \mathcal{G}_1^{sh}((a,b))$. The Poincaré lemma tells us that Ω^* is quasi-isomorphic to the locally constant sheaf $\underline{\mathbb{C}}$, and so we can introduce a sheaf

$$\mathrm{L}\mathfrak{g}^{sh} = \Omega^* \otimes \mathfrak{g}[z, z^{-1}]$$

that is a soft resolution of $\underline{\mathfrak{g}[z,z^{-1}]}$. There is then a map of sheaves of dg Lie algebras

$$\mathrm{L}\mathfrak{g}^{sh} \to r_* \mathcal{G}_1^{sh} \tag{5}$$

that sends $\alpha \otimes x z^n$ to $[r^*\alpha]_{0,*} \cdot z^n \otimes x$, with $x \in \mathfrak{g}$, α a differential form on $(0, \infty)$, and $[r^*\alpha]_{0,*}$ the $(0,*)$-component of the pulled back form. This map restricts nicely to compactly support sections $\mathrm{L}\mathfrak{g} \to r_* \mathcal{G}_1$. By taking Chevalley-Eilenberg chains on both sides, we obtain a map of factorization algebras

$$\mathbb{U}\mathrm{L}\mathfrak{g} = \mathrm{C}_*^{\mathrm{Lie}}(\mathrm{L}\mathfrak{g}) \to \mathrm{C}_*^{\mathrm{Lie}}(r_* \mathcal{G}_1) = r_* \mathbb{U} \mathcal{G}_1. \tag{6}$$

The left hand side $\mathbb{U}\mathrm{L}\mathfrak{g}$ encodes the associative algebra $U(L\mathfrak{g})$, the enveloping algebra of $L\mathfrak{g}$, as can be seen by direct computation (see section 3.4 of [CG17]) or by a general result of Knudsen [Knu]. The right hand side contains operators encoded by Cauchy integrals, and it is possible to identify such as operator, up to exact terms, as the limit of a sequence of elements from $U(L\mathfrak{g})$.

We extend this argument to the affine Kac-Moody Lie algebras by working with suitable extensions on $\mathrm{L}\mathfrak{g}$. It is a deformation-theoretic argument, ~~as~~ Since we can view the extensions as deforming the bracket.

We wish to replace the punctured plane $\mathbb{C}^*$ by the punctured d-dimensional affine space

$$\mathring{\mathbb{A}}^d = \mathbb{C}^d \setminus \{0\},$$

the current algebras of $\mathcal{G}_1$ by the current algebras of $\mathcal{G}_d$, and, of course, the extensions depending on $\mathrm{Sym}^2(\mathfrak{g}^*)^{\mathfrak{g}}$ by other local cocycles. There are two nontrivial steps to this generalization:

(1) finding a suitable replacement for the Laurent polynomials, so that we can ~~reca-~~ generalize ~~pitulate~~ (without any issues) the construction of the maps (5) and (6), and
(2) deforming this construction to encompass the extensions of $\mathcal{G}_d$ and hence the twisted enveloping factorization algebras $\mathbb{U}_\theta \mathcal{G}_d$.

We undertake the steps in order.

2.1.1. *Derived functions on punctured affine space.* When $d = 1$, we note that

$$\mathbb{C}[z, z^{-1}] \subset \mathcal{O}(\mathbb{C}^*) \xrightarrow{\simeq} \Omega^{0,*}(\mathbb{C}^*),$$

and so the Laurent polynomials are a dense subalgebra of the Dolbeault complex. When $d > 1$, Hartog's lemma tells us that every holomorphic function on punctured d-dimensional space extends through the origin:

$$\mathcal{O}(\mathring{\mathbb{A}}^d) = \mathcal{O}(\mathbb{A}^d).$$

This result might suggest that $\mathring{\mathbb{A}}^d$ is an unnatural place to seek a generalization of the loop algebra, but such pessimism is misplaced because $\mathring{\mathbb{A}}^d$ is not affine and so its *derived* algebra of functions, given by the derived global sections $\mathbb{R}\Gamma(\mathring{\mathbb{A}}^d, \mathcal{O})$, is more interesting than the underived global sections $\mathcal{O}(\mathring{\mathbb{A}}^d)$.

Indeed, a straightforward computation in algebraic geometry shows

$$H^*(\mathring{\mathbb{A}}^d, \mathcal{O}_{\text{alg}}) = \begin{cases} 0, & * \neq 0, d-1 \\ \mathbb{C}[z_1, \ldots, z_d], & * = 0 \\ \mathbb{C}[z_1^{-1}, \ldots, z_d^{-1}]\frac{1}{z_1 \cdots z_d}, & * = d-1 \end{cases}.$$

(For instance, use the cover by the affine opens of the form $\mathbb{A}^d \setminus \{z_i = 0\}$.) When $d = 1$, this computation recovers the Laurent polynomials, so we should view the cohomology in degree $d - 1$ as providing the derived replacement of the polar part of the Laurent polynomials. A similar result holds in analytic geometry, of course, so that we have a natural map

$$\mathbb{R}\Gamma(\mathring{\mathbb{A}}^d, \mathcal{O}_{\text{alg}}) \to \mathbb{R}\Gamma(\mathring{\mathbb{A}}^d, \mathcal{O}_{\text{an}}) \simeq \Omega^{0,*}(\mathring{\mathbb{A}}^d)$$

that replaces our inclusion of Laurent polynomials into the Dolbeault complex on $\mathring{\mathbb{A}}^d$.

For explicit constructions, it is convenient to have an explicit dg commutative algebra that models the derived global sections. It should be no surprise that we like to work with the Dolbeault complex, but there is also an explicit dg model A_d for the algebraic version of derived global sections due to Faonte-Hennion-Kapranov [FHK] and based on the Jouanolou method for resolving singularities. In fact, they provide a model for the algebraic p-forms as well.

weird notation to have an algebra be represented by lowercase letter.

Definition 2.1. Let a_d denote the algebra

$$\mathbb{C}[z_1, \ldots, z_d, z_1^*, \ldots, z_d^*][(zz^*)^{-1}]$$

defined by localizing the polynomial algebra with respect to $zz^* = \sum_i z_i z_i^*$. View this algebra a_d as concentrated in bidegree $(0,0)$, and consider the bigraded-commutative algebra $R_d^{*,*}$ over a_d that is freely generated in bidegree $(1,0)$ by elements

$$\mathrm{d}z_1, \ldots, \mathrm{d}z_d,$$

and in bidegree $(0,1)$ by

$$\mathrm{d}z_1^*, \ldots, \mathrm{d}z_d^*.$$

We care about the subalgebra $A_d^{*,*}$ where $A_d^{p,m}$ consists consisting of elements $\omega \in R_d^{p,m}$ such that

(i) the coefficient of $\mathrm{d}z_{i_1}^* \cdots \mathrm{d}z_{i_m}^*$ has degree $-m$ with respect to the z_k^* variables, and

(ii) the contraction $\iota_\xi \omega$ with the Euler vector field $\xi = \sum_i z_i^* \partial_{z_i^*}$ vanishes. Not the same thing?

This bigraded algebra admits natural differentials in both directions:

(1) define a map $\overline{\partial} : A_d^{p,q} \to A_d^{p,q+1}$ of bidegree $(0,1)$ by

$$\overline{\partial} = \sum_i \mathrm{d}z_i^* \frac{\partial}{\partial z_i^*},$$

(2) define a a map of bidegree $(1,0)$ by

$$\partial = \sum_i \mathrm{d}z_i \frac{\partial}{\partial z_i}.$$

These differentials commute $\overline{\partial}\partial = \partial\overline{\partial}$, and each squares to zero.

We denote the subcomplex with $p = 0$ by

$$(A_d, \overline{\partial}) = \left(\bigoplus_{q=0}^{d} A_d^q[-q], \overline{\partial}\right),$$

and it has the structure of a dg commutative algebra. For $p > 0$, the complex $A_d^{p,*} = (\oplus_q A^{p,q}[-q], \overline{\partial})$ is a dg module for $(A_d, \overline{\partial})$.

From the definition, one can guess that the variables z_i should be understood as the usual holomorphic coordinates on affine space $\mathbb{C}^d$ and the variables z_i^* should be understood as the antiholomorphic coordinates $\overline{z}_i$. The following proposition confirms that guess; it also summarizes key properties of the dg algebra A_d and its dg modules $A_d^{p,*}$, by aggregating several results of [FHK].

Proposition 2.2 ([FHK], Section 1).

(1) The dg commutative algebra $(A_d, \overline{\partial})$ is a model for $\mathbb{R}\Gamma(A^{d\times}, \mathcal{O}^{alg})$:

$$A_d \simeq \mathbb{R}\Gamma(\mathbb{A}^{d\times}, \mathcal{O}^{alg}).$$

Similarly, $(A_d^{p,*}, \overline{\partial}) \simeq \mathbb{R}\Gamma(\mathbb{A}^{d\times}, \Omega^{p,alg})$.

(2) There is a dense map of commutative bigraded algebras

$$\mathrm{j} : A_d^{*,*} \to \Omega^{*,*}(\mathbb{C}^d \setminus \{0\})$$

sending z_i to z_i, z_i^* to $\overline{z}_i$, and $\mathrm{d}z_i^*$ to $\mathrm{d}\overline{z}_i$, and the map intertwines with the $\overline{\partial}$ and ∂ differentials on both sides.

(3) There is a unique GL_n*-equivariant residue map*

$$\mathrm{Res}_{z=0} : A_d^{d,d-1} \to \mathbb{C}$$

that satisfies

$$\mathrm{Res}_{z=0}\left(f(z)\omega_{BM}^{alg}(z,z^*)\mathrm{d}z_1\cdots\mathrm{d}z_d\right) = f(0)$$

for any $f(z) \in \mathbb{C}[z_1,\ldots,z_d]$*. In particular, for any* $\omega \in A_d^{d,d-1}$*,*

$$\mathrm{Res}_{z=0}(\omega) = \oint_{S^{2d-1}} \mathrm{j}(\omega)$$

where S^{2d-1} *is any sphere centered at the origin in* $\mathbb{C}^d$*.*

It is a straightforward to verify that the formula for the $\boxed{\text{Bochner-Martinelli kernel}}$ makes sense in the algebra A_d. That is, we define

did you introduce this earlier?

$$\omega_{BM}^{alg}(z,z^*) = \frac{(d-1)!}{(2\pi i)^d}\frac{1}{(zz^*)^d}\sum_{i=1}^{d}(-1)^{i-1}z_i^*\mathrm{d}z_1^*\wedge\cdots\wedge\widehat{\mathrm{d}z_i^*}\wedge\cdots\wedge\mathrm{d}z_d^*,$$

which is an element of $A_d^{0,d-1}$.

2.1.2. *The sphere algebra of* $\mathfrak{g}$. The loop algebra $L\mathfrak{g} = \mathfrak{g}[z,z^{-1}]$ arises as an algebraic model of the mapping space $\mathrm{Map}(S^1,\mathfrak{g})$, which obtains a natural Lie algebra structure from the target space $\mathfrak{g}$. For a topologist, a natural generalization is to replace the circle S^1, which is equal to the unit vectors in $\mathbb{C}$, by the sphere S^{2d-1}, which is equal to the unit vectors in $\mathbb{C}^d$. That is, consider the "sphere algebra" ~~of~~ $\mathrm{Map}(S^{2d-1},\mathfrak{g})$. An algebro-geometric ~~sphere~~ replacement of this sphere is the punctured affine d-space $\mathring{\mathbb{A}}^d$ or a punctured formal d-disk, and so we introduce an algebraic model for the sphere algebra.

Definition 2.3. For a Lie algebra $\mathfrak{g}$, the *sphere algebra* in complex dimension d is the dg Lie algebra $A_d \otimes \mathfrak{g}$. Following [FHK] we denote it by $\mathfrak{g}_d^{\bullet}$.

?

There are natural central extensions of this sphere algebra as em L_∞ algebras, in parallel with our discussion of extensions of the local Lie algebras. For any $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$, Faonte-Hennion-Kapranov define the cocycle

$$\begin{array}{rccc} \theta_{FHK}: & (A_d\otimes\mathfrak{g})^{\otimes(d+1)} & \to & \mathbb{C} \\ & a_0\otimes\cdots\otimes a_d & \mapsto & \mathrm{Res}_{z=0}\theta(a_0,\partial a_1,\ldots,\partial a_d) \end{array}.$$

This cocycle has cohomological degree 2 and so determines an unshifted central extension as L_∞ algebras of $A_d\otimes\mathfrak{g}$:

$$\mathbb{C}\cdot K \to \widetilde{\mathfrak{g}}_{d,\theta}^{\bullet} \to A_d\otimes\mathfrak{g}. \tag{7}$$

Our aim is now to show how the Kac-Moody factorization algebra $\mathbb{U}_\theta\mathcal{G}_d$ is related to this L_∞ algebra, which is a higher-dimensional version of the affine Kac-Moody Lie algebras.

2.1.3. *The case of zero level.* Here we will consider the higher Kac-Moody factorization algebra on $\mathbb{C}^d \setminus \{0\}$ "at level zero," namely the factorization algebra $\mathbb{U}(\mathcal{G}_{\mathbb{C}^d\setminus\{0\}})$. In this section we will omit $\mathbb{C}^d \setminus \{0\}$ from the notation, and simply refer to the factorization algebra by $\mathbb{U}(\mathcal{G}_d)$. Our construction will follow the model case outlined in the introduction to this section. Recall that $r : \mathring{\mathbb{A}}^d \to (0, \infty)$ is the radial projection map that sends $(z_1, \ldots, z_d)$ to its length $\sqrt{z_1\overline{z}_1 + \cdots z_d\overline{z}_d}$.

Lemma 2.4. *There is a map of sheaves of dg commutative algebras on* $\mathbb{R}_{>0}$

$$\pi : \Omega^* \to r_*\Omega^{0,*}$$

sending a form α *to the* $(0,*)$*-component of its pullback* $r^*\alpha$.

This result is straightforward since the pullback r^* denotes a map of dg algebras to $r_*\Omega^{*,*}$ and we are simply postcomposing with the canonical quotient map of dg algebras $\Omega^{*,*} \to \Omega^{0,*}$.

We also have a map of dg commutative algebras $A_d \to \Omega^{0,*}(U)$ for any open set $U \subset \mathring{\mathbb{A}}^d$, by postcomposing the map j of proposition 2.2 with the restriction map. We abusively denote the composite by j as well. Thus we obtain a natural map of dg commutative algebras

Zariski open?

could be condensed

$$\pi_A : \Omega^* \otimes A_d \to r_*\Omega^{0,*}$$

sending $\alpha \otimes \omega$ to $\pi(\alpha) \wedge \mathrm{j}(\omega)$. By tensoring with $\mathfrak{g}$, we obtain the following.

Corollary 2.5. *There is a map of sheaves of dg Lie algebras on* $\mathbb{R}_{>0}$

$$\pi_{\mathfrak{g},d} : \Omega^* \otimes \mathfrak{g}_d^\bullet \to r_*(\Omega^{0,*} \otimes \mathfrak{g}) = r_*(\mathcal{G}_d^{sh})$$

sending $\alpha \otimes x$ *to* $\pi(\alpha) \otimes x$.

Note that $\Omega^* \otimes \mathfrak{g}_d^\bullet = \Omega^* \otimes A_d \otimes \mathfrak{g}$, so $\pi_{\mathfrak{g},d}$ is simply $\pi_A \otimes \mathrm{id}_{\mathfrak{g}}$.

This map preserves support and hence restricts to compactly-supported sections. In other words, we have a map between the associated cosheaves of complexes (and precosheaves of dg Lie algebras). In summary, we have shown our key result.

Proposition 2.6. *The map*

$$\pi_{\mathfrak{g},d} : \Omega^*_{\mathbb{R}_{>0},c} \otimes \mathfrak{g}_d^\bullet \to r_*\mathcal{G}_d$$

is a map of precosheaves of dg Lie algebras. It determines a map of factorization algebras

$$C_*^{\mathrm{Lie}}(\pi_{\mathfrak{g},d}) : \mathbb{U}\left(\Omega^*_{\mathbb{R}_{>0}} \otimes \mathfrak{g}_d^\bullet\right) \to r_*\left(\mathbb{U}\mathcal{G}_d\right).$$

The map of factorization algebras follows from applying the functor $C_*^{\mathrm{Lie}}(-)$ to the map $\pi_{\mathfrak{g},d}$; this construction commutes with push-forward by inspection.

Both maps are dense in every cohomological degree with respect to the natural topologies on these vector spaces, leading to the following observation.

Corollary 2.7. *By Theorem 1.13 of Knudsen, the enveloping* E_1 *algebra of the sphere algebra* $\mathfrak{g}_d^\bullet$ *is dense inside the pushforward factorization algebra* $r_*\left(\mathbb{U}\mathcal{G}_d\right)$.

2.1.4. *The case of non-zero level.* Pick a $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$. This choice determines a higher Kac-Moody factorization algebra $\mathbb{U}_\theta \mathcal{G}_d$, and we would like to produce maps akin to those of Proposition 2.6.

The simplest modification of the level zero situation is to introduce a central extension of the precosheaf

$$\mathtt{G}_d = \Omega^*_{\mathbb{R}_{>0},c} \otimes \mathfrak{g}_d^\bullet$$

as a precosheaf of L_∞ algebras on $\mathbb{R}_{>0}$, with the condition that this extension intertwines with the extension $r_* \mathcal{G}_{d,\theta}$ of $r_* \mathcal{G}_d$. In other words, we need a map

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{C} \cdot K[-1] \arrow[r] \arrow[d, "="] & \mathtt{G}_{d,\Theta'} \arrow[r] \arrow[d, "\widehat{\pi}_{\mathfrak{g},d}"] & \mathtt{G}_d \arrow[r] \arrow[d, "\pi_{\mathfrak{g},d}"] & 0 \\
0 \arrow[r] & \mathbb{C} \cdot K[-1] \arrow[r] & r_* \mathcal{G}_{d,\theta} \arrow[r] & r_* \mathcal{G}_d \arrow[r] & 0.
\end{tikzcd}$$

of central extensions of L_∞ algebras. This condition fixes the problem completely, because we simply pull back the extension defining $r_* \mathcal{G}_{d,\theta}$. Let us extract an explicit description, which will be useful later. On an open $U \subset \mathbb{R}_{>0}$, the extension for $r_* \mathcal{G}_{d,\theta}$ is given by an integral

$$\int_{r^{-1}(U)} \theta(\alpha_0, \partial\alpha_1, \ldots, \partial\alpha_d) = \int_U \int_{S^{2d-1}} \theta(\alpha_0, \partial\alpha_1, \ldots, \partial\alpha_d)$$

that can be factored into a double integral. This formula indicates that Θ' must be given by the cocycle whose value on elements $\phi_i \otimes a_i \in \Omega^*_c \otimes \mathfrak{g}^\bullet_d$ is

$$\Theta'(\phi_0 \otimes a_0, \ldots, \phi_d \otimes a_d) = \int_U \int_{S^{2d-1}} \theta(\pi(\phi_0) \wedge \mathtt{j}(a_0), \partial(\pi(\phi_1) \wedge \mathtt{j}(a_1)), \ldots, \partial(\pi(\phi_d) \wedge \mathtt{j}(a_d)))$$

We thus obtain the following result.

Lemma 2.8. *For $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$, let $\mathtt{G}_{d,\theta}$ denote the precosheaf of L_∞ algebras obtained by extending $\mathtt{G}_d$ by the cocycle*

$$(\phi_0 \otimes a_0, \ldots, \phi_d \otimes a_d) \mapsto \int_U \int_{S^{2d-1}} \theta(\pi(\phi_0) \wedge \mathtt{j}(a_0), \partial(\pi(\phi_1) \wedge \mathtt{j}(a_1)), \ldots, \partial(\pi(\phi_d) \wedge \mathtt{j}(a_d))).$$

By construction, there is a canonical map

$$\pi_{\mathfrak{g},d,\theta} : \mathtt{G}_{d,\theta} \to r_* \mathcal{G}_{d,\theta}$$

of precosheaves of L_∞ algebras on $\mathbb{R}_{>0}$, and hence there is a map of factorization algebras

$$\mathbb{U}(\pi_{\mathfrak{g},d,\theta}) : \mathbb{U}_\theta \mathtt{G}_d \to r_* \mathbb{U}_\theta \mathcal{G}_d.$$

The maps remain degreewise dense, but now we are working with a twisted enveloping factorization algebra, which is slightly different in flavor than Knudsen's construction. The central parameter K parametrizes, in fact, a family of E_1 algebras that specializes at $K = 0$ to the enveloping E_1 algebra of the sphere algebra $\mathfrak{g}^\bullet_d$.

Corollary 2.9. *There is a family of E_1 algebras over the affine line* $\mathrm{Spec}(\mathbb{C}[K])$ *with the enveloping E_1 algebra of the sphere algebra $\mathfrak{g}_d^\bullet$ at the origin. This family is dense within the pushforward* $r_*(\mathbb{U}_\theta \mathcal{G}_d)$.

2.1.5. *A comparison with the work of Faonte-Hennion-Kapranov.* There is a variant of the preceding result that is particularly appealing in light of [FHK], which is to provide a map of factorization algebras on the positive reals

$$\mathbb{U}(\widetilde{\pi}_{\mathfrak{g},d,\theta}) : \mathbb{U}(\Omega_c^* \otimes \widetilde{\mathfrak{g}}_{d,\theta}^\bullet) \to r_* \mathcal{G}_{d,\theta},$$

~~I thought you used [illegible]?~~

where the source is the factorization algebra encoding the enveloping E_1 algebra of $\widetilde{\mathfrak{g}}_{d,\theta}^\bullet$. Specializing the central parameters to zero on both sides must recover the map $\mathbb{U}\pi_{\mathfrak{g},d}$ of Proposition 2.6. Such a map has two connected consequences:

(1) It shows that the higher current Lie algebras $\widetilde{\mathfrak{g}}_{d,\theta}^\bullet$ of [FHK] "control" our twisted current factorization algebras $\mathcal{G}_{d,\theta}$ in the same way that the affine Kac-Moody Lie algebras control their vertex algebras.
(2) It shows that our factorization algebras $\mathcal{G}_{d,\theta}$ know the information encoded by the Lie algebras $\widetilde{\mathfrak{g}}_{d,\theta}^\bullet$ introduced in [FHK].

In short this map provides a conduit for transferring insights between derived algebraic geometry (as represented by the [FHK] approach) and quantum field theory (as represented by ours).

Remark 2.10. Before embarking on the construction of the map, we remark that it was a pleasant surprise to come upon [FHK] and to find that they had explored terrain that we had approached from the direction exposed in this paper, i.e., the higher dimensional generalization of results from [CG17]. Their Jouanolou model A_d gave a more explicit and more tractable analogue to Laurent polynomials and hence allowed us to sharpen our results into something more punchy, and their discussion of the global derived geometry verified natural guesses, which were beyond our technical powers. Although we had found the same extensions, our explanations were based on finding an explicit generalization of the $d = 1$ formula, with confirmation arising from Feynman diagram computations. By contrast, [FHK] gave a beautiful structural explanation via cyclic homology, which resonates with our physical view of large N limits. We thank Faonte, Hennion, and Kapranov for inspiring and enlightening conversations and correspondence on these subjects.

underlined things haven't been discussed yet, so it ~~[illegible]~~ as it doesn't seem too fruitful to bring up here.

Constructing the map requires overcoming two issues. First, note that

$$\widetilde{\mathsf{G}}_{d,\theta} = \Omega_c^* \otimes \widetilde{\mathfrak{g}}_{d,\theta}^\bullet$$

can be viewed as an extension

$$\Omega_c^* \otimes \mathbb{C} \to \widetilde{\mathsf{G}}_{d,\theta} \to \mathsf{G}_d$$

of precosheaves of L_∞ algebras on $\mathbb{R}_{>0}$. By contrast, $r_* \mathcal{G}_{d,\theta}$ is an extension by the constant precosheaf $\mathbb{C}K[-1]$. There is, however, a natural map of precosheaves

$$\int : \Omega^*_c \to \mathbb{C}[-1]$$

to employ, since integration is well-defined on compactly-supported forms. This map indicates the shape of the underlying map of short exact sequences.

The second issue looks more serious: the two cocycles at play seem different at first glance. The pushforward $r_* \mathcal{G}_{d,\theta}$ uses a cocycle whose behavior on the image under $\pi_{\mathfrak{g},d}$ is given by

$$\begin{aligned} &\Theta_{push}(\phi_0 \otimes a_0, \ldots, \phi_d \otimes a_d) \\ &\quad = \int_U \int_{S^{2d-1}} \theta(\pi(\phi_0) \wedge \mathrm{j}(a_0), \partial(\pi(\phi_1) \wedge \mathrm{j}(a_1)), \ldots, \partial(\pi(\phi_d) \wedge \mathrm{j}(a_d))), \end{aligned}$$

where we use elements of the form $\phi_i \otimes a_i \in \Omega^*_c(U) \otimes \mathfrak{g}^\bullet_d$ with U an open subset of $\mathbb{R}_{>0}$. On the other hand, on those same elements, the FHK extension is given by

$$\begin{aligned} &\Theta_{FHK}(\phi_0 \otimes a_0, \ldots, \phi_d \otimes a_d) \\ &\qquad = (\phi_0 \wedge \cdots \wedge \phi_d) \int_{S^{2d-1}} \theta(\mathrm{j}(a_0), \partial(\mathrm{j}(a_1)), \ldots, \partial(\mathrm{j}(a_d))). \end{aligned}$$

(Note that in the FHK case, we do not integrate over U because we extend by Ω^*_c.) The key difference here is that the FHK extension does not involve applying ∂ to the $(0,*)$-components of the pulled back forms $r^*\phi_i$. It separates the ϕ_i and a_i contributions, whereas the other cocycle mixes them. The tension is resolved by showing these cocycles are cohomologous.

Lemma 2.11. *There is a cochain η for G_d such that*

$$\Theta_{push} = \int \Theta_{FHK} + \mathrm{d}\eta,$$

where d *here denotes the differential on the Lie algebra cochains of* G_d.

Proof. We note that the Lie algebra $\mathfrak{g}$ and the invariant polynomial θ play no substantive role in the problem. The issue here is about calculus. Hence it suffices to consider the case that $\mathfrak{g}$ is the one-dimensional abelian Lie algebra and θ is the unique-up-to-scale monomial of degree $d+1$ (i.e., "x^{d+1}").

Let

$$E = r\frac{\partial}{\partial r}$$

denote the Euler vector field on $\mathbb{R}_{>0}$, and let

$$\mathrm{d}\vartheta = \sum_i \frac{\mathrm{d}z_i}{z_i}$$

is $d\vartheta$ $\neq$ d applied to some form?

denote a $(1,0)$-form on $\mathring{\mathbb{A}}^d = \mathbb{C}^d \setminus 0$.

For concision we express the element $\varphi_i \otimes a_i$ in $\Omega^*_c(U) \otimes A_d$ by $\varphi_i a_i$. We now define

$$\eta(\varphi_0 a_0, \dots, \varphi_d a_d) = \sum_{i=1}^{d} \left(\int_U \varphi_0 \left(\iota_E \varphi_i\right) \varphi_1 \cdots \widehat{\varphi_i} \cdots \varphi_d \right) \left(\oint \left(a_0 a_i \mathrm{d}\vartheta\right) \partial a_1 \cdots \widehat{\partial a_i} \cdots \partial a_d \right).$$

is this the notation for d-dim residue pairing?

It is a straightforward exercise in integration by parts and the bigrading of Dolbeault forms to verify that η cobounds the difference of the cocycles. □

With this explicit cochain η in hand, we can produce the desired map.

Proposition 2.12. *There is an L_∞ map of L_∞ algebras*

$$\widetilde{\pi}_{\mathfrak{g},d,\theta} : \Omega^*_c \otimes \widetilde{\mathfrak{g}}^{\bullet}_{d,\theta} \rightsquigarrow r_* \mathcal{G}_{d,\theta},$$

by which we mean there is a sequence of multilinear maps

$$\widetilde{\pi}_{\mathfrak{g},d,\theta}\langle n \rangle : \prod_{i=1}^{n} \Omega^*_c \otimes \widetilde{\mathfrak{g}}^{\bullet}_{d,\theta} \to r_* \mathcal{G}_{d,\theta},$$

that have degree $2 - n$ and are skew-symmetric and intertwine with the L_∞ brackets on both sides (cf. [KS, LV12]). The terms $\widetilde{\pi}_{\mathfrak{g},d,\theta}\langle n \rangle$ vanish for $n \neq 1, d+1$. The $n = 1$ map fits into the commuting diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \Omega^*_c \cdot K[-1] \arrow[r] \arrow[d, "\int"] & \Omega^*_c \otimes \widetilde{\mathfrak{g}}^{\bullet}_{d,\theta} \arrow[r] \arrow[d, "\widetilde{\pi}_{\mathfrak{g},d,\theta}\langle n \rangle"] & \mathsf{G}_d \arrow[r] \arrow[d, "\pi_{\mathfrak{g},d}"] & 0 \\
0 \arrow[r] & \mathbb{C} \cdot K[-1] \arrow[r] & r_* \mathcal{G}_{d,\theta} \arrow[r] & r_* \mathcal{G}_d \arrow[r] & 0.
\end{tikzcd}$$

The $n = d+1$ map sends the $d+1$-tuple $(\phi_0 \otimes a_0, \dots, \phi_d \otimes a_d)$ to

$$\eta(\phi_0 \otimes a_0, \dots, \phi_d \otimes a_d).$$

This L_∞ map is equivalent to giving a map of dg conilpotent cocommutative coalgebras on the Chevalley-Eilenberg chains of these L_∞ algebras, which in fact provides a map

$$\mathbb{U}(\widetilde{\pi}_{\mathfrak{g},d,\theta}) : \mathbb{U}(\Omega^*_c \otimes \widetilde{\mathfrak{g}}^{\bullet}_{d,\theta}) \to r_* \mathbb{U}_\theta \mathcal{G}_d$$

of factorization algebras.

Proof. Note that for our L_∞ algebras, the only nontrivial brackets are ℓ_1, ℓ_2, and ℓ_{d+1}. We already know that the $n = 1$ map intertwines with ℓ_1 and ℓ_2 brackets, as it does modulo the central extensions. We can thus set the maps for $n = 2, \dots, d$ to zero. The first nontrivial issue arises at $n = d+1$, as the $n = 1$ map does not intertwine the ℓ_{d+1} brackets. The defining property of η, however, ensures that $\widetilde{\pi}_{\mathfrak{g},d,\theta}\langle d+1 \rangle$ corrects the failure. Hence we may set the maps for $n > d+1$ to zero as well. □

Corollary 2.13. *The enveloping E_1 algebra of $\mathfrak{g}^{\bullet}_{d,\theta}$ is dense inside the pushforward $r_* \mathbb{U}_\theta \mathcal{G}_d$.*

2.2. **Compactifying along tori.** There is another direction that one may look to extend the notion of affine algebras to higher dimensions. The affine algebra is a central extension of the loop algebra on $\mathfrak{g}$. Instead of looking at higher dimensional sphere algebras, one can consider higher *torus* algebras, i.e., iterated loop algebras:

$$L^d\mathfrak{g} = \mathbb{C}[z_1^{\pm}, \cdots, z_d^{\pm}] \otimes \mathfrak{g}.$$

These iterated loop algebras are algebraic versions of the torus mapping space

$$\mathrm{Map}(S^1 \times \cdots \times S^1, \mathfrak{g}).$$

We now explore what information the Kac-Moody factorization algebras encode about extensions of such iterated loop algebras.

To do this, we study the Kac-Moody factorization algebras on the complex manifold $(\mathbb{C}^\times)^d$, which is an algebro-geometric version of the torus $(S^1)^d$. As with the punctured affine space $\mathring{\mathbb{A}}^d$, we compactify by pushing forward to $(\mathbb{R}_{>}0)^d$ along a radial projection map

$$\begin{array}{cccc} \vec{r}: & (\mathbb{C}^\times)^d & \to & (\mathbb{R}_{>0})^d \\ & (z_1, \ldots, z_d) & \mapsto & (|z_1|, \cdots, |z_d|) \end{array}.$$

The preimage of a point $(r_1, \ldots, r_d)$ is a d-fold product of circles, and the preimage of an open d-cube is a polyannulus—a d-fold product of annuli. Observe that on a polyannulus U, the underived and derived algebras of functions coincide,

$$\Gamma(U, \mathcal{O}) \xhookrightarrow{\simeq} \mathbb{R}\Gamma(U, \mathcal{O}),$$

as U is a Stein manifold because it is a product of Stein manifolds. Similarly, the scheme $(\mathbb{A}^1 \setminus \{0\})^d$ is affine and so its structure sheaf has no higher cohomology:

$$\mathbb{R}\Gamma((\mathbb{A}^1 \setminus \{0\})^d, \mathcal{O}) \simeq \mathbb{C}[z_1, z_1^{-1}, \ldots, z_d, z_d^{-1}].$$

Note that the iterated loop algebras $L^d\mathfrak{g}$ appear precisely by tensoring $\mathfrak{g}$ with functions on this product of punctured affine lines. Thus, in contrast to $\mathring{\mathbb{A}}^d$, we seem to be able to work in an underived setting.

This impression is misleading, however, in the sense that it ignores some additional algebraic structure that naturally appears at the level of current algebras: there is an E_d algebra that sits densely inside the pushforward $\vec{r}_*\mathcal{G}_d$.

Lemma 2.14. *There is a map*

$$\rho_d : \Omega^* \to \vec{r}_*\Omega^{0,*}$$

of sheaves of dg commutative algebras on $(\mathbb{R}_{>0})^d$ *sending a form* α *to the projection of the pulled back* $\vec{r}^*\alpha$ *onto its* $(0,*)$*-components.*

form

As algebraic functions sit inside holomorphic functions and hence inside the Dolbeault complex, there is a map of dg commutative algebras

$$\mathbb{C}[z_1, z_1^{-1}, \ldots, z_d, z_d^{-1}] \to \Omega^{0,*}(U)$$

for any open $U \subset (\mathbb{C} \setminus \{0\})^d$. There is thus a map

$$\rho'_d : \Omega^* \otimes \mathbb{C}[z_1, z_1^{-1}, \ldots, z_d, z_d^{-1}] \to \vec{r}_* \Omega^{0,*}$$

of dg commutative algebras. We tensor with $\mathfrak{g}$ to obtain the following result.

Lemma 2.15. *There is a map*

$$\rho_{d,\mathfrak{g}} : \Omega^* \otimes L^d\mathfrak{g} \to \vec{r}_* \mathcal{G}_d^{sh}$$?

of sheaves of dg Lie algebras on $(\mathbb{R}_{>0})^d$ sending an element $\alpha \otimes x$ to $\rho_d(\alpha) \otimes x$.s As this map preserves support, it restricts to a map

$$\rho_{d,\mathfrak{g}} : \Omega_c^* \otimes L^d\mathfrak{g} \to \vec{r}_* \mathcal{G}_d$$

of precosheaves of dg Lie algebras on $(\mathbb{R}_{>0})^d$.

By taking Chevalley-Eilenberg chains, we obtain a statement at the level of factorization algebras.

Corollary 2.16. *There is a map* L^d ?

$$\mathbb{U}(\rho_{d,\mathfrak{g}}) : \mathbb{U}(\Omega_c^* \otimes \mathfrak{g}) \to \vec{r}_* \mathbb{U}\mathcal{G}_d$$

of factorization algebras on $(\mathbb{R}_{>0})^d$. As the source is locally constant, it corresponds to an E_d algebra, which is the enveloping E_d algebra of $L^d\mathfrak{g}$, by Knudsen's theorem.

This map has dense image in each degree, and so we see that the enveloping E_d algebra of the iterated loop algebra $L^d\mathfrak{g}$ "controls" the pushforward $\vec{r}_* \mathbb{U}\mathcal{G}_d$ in this sense.

Remark 2.17. When $d = 1$ one can understand the radially ordered products of operators ? by evaluating these current factorization algebras on nested annuli. For $d > 1$ one can read likewise understand interesting phenomena about operator products by evaluating these current factorization algebras on ~~these~~ polyannuli. In particular, the connection with E_d-algebras indicates that there is a (possibly nontrivial) $(1-d)$-shifted Poisson bracket between operators, even at the level of cohomology.

In the case of the centrally-extended Lie algebras $\mathcal{G}_{d,\theta}$, we note that one can pull back the extension along the map $\rho_{d,\mathfrak{g}}$ to determine an extension of $\Omega_c^* \otimes L^d\mathfrak{g}$ as a precosheaf of L_∞ algebras. One can view this extension as an extension of $L^d\mathfrak{g}$ as an L_∞ algebra:

$$\mathbb{C}[d-1] \to \widehat{L^d\mathfrak{g}_\theta} \to L^d\mathfrak{g},$$

It is essentially immediate from the definitions that the cocycle is

$$L^d\theta(f_0 \otimes x_0) \otimes \cdots \otimes (f_d \otimes x_d) = \theta(x_0, \ldots, x_d) \oint_{|z_1|=1} \cdots \oint_{|z_d|=1} f_0 \mathrm{d}f_1 \cdots \mathrm{d}f_d$$

where $f_i \in \mathbb{C}[z_1^{\pm 1}, \ldots, z_d^{\pm 1}]$ and $x_i \in \mathfrak{g}$. This formula is just an iterated version of the usual residue pairing.

This extension then determines a twist of the enveloping E_d algebra, as well. By techniques analogous to what we did in comparing with [FHK], one can show the following.

Proposition 2.18. *For $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$, there is a map of factorization algebras*

$$\rho_{d,\mathfrak{g},\theta} : \mathbb{U}(\Omega^*_c \otimes \widehat{L^d\mathfrak{g}_\theta}) \to \rho_* \mathbb{U}_\theta \mathcal{G}_d$$

that has dense image in each degree.

In this sense the enveloping E_d algebra of $\widehat{L^d\mathfrak{g}_\theta}$ controls the twisted enveloping factorization algebra.

3. The holomorphic charge anomaly

In this section, we switch gears a bit to propose a natural occurrence of the Kac-Moody factorization algebra as a symmetry of a simple class of higher dimensional quantum field theories. This situation is analogous to the free field realization of the affine Kac-Moody algebra as a subalgebra of differential operators on the loop space.

maybe say a bit more about which concepts you use, with some ~~[illegible]~~ pointers to specific places.

Our approach is through the general machinery of perturbative quantum field theory developed by Costello [Cos11] and Costello-Gwilliam [CG17, CG]. We study the quantization of a particular *free* field theory, which makes sense in any complex dimension. Classically, the theory depends on the data of a G-representation, and the holomorphic nature of the theory allows us to lift this to a symmetry for the classical current algebra $\mathrm{Cur}^{\mathrm{cl}}(\mathcal{G}_X)$ at "zero level". We find that upon quantization, the symmetry is broken, but in a way that we can measure by an explicit anomaly, or cocycle. This leads to a symmetry of the theory via the quantum current algebra $\mathrm{Cur}^{\mathrm{q}}(\mathcal{G}_X)$ twisted by this cocycle.

3.1. **Charged holomorphic matter.** We introduce a classical field theory in the BV formalism that one may think of as higher dimensional "matter" in a holomorphic setting. When the complex dimension is $d = 1$, this returns the chiral $\beta\gamma$ system from ordinary conformal field theory. In dimensions 2 and 4, this theory is ~~equal~~ a minimal twist of supersymmetric matter.

equivalent to?
equal to?

To start, we fix a finite dimensional $\mathfrak{g}$-module V and an integer $d > 0$. The classical fields of the complex d-dimensional theory consist of a map

$$\gamma : \mathbb{C}^d \to V \tag{8}$$

and a differential form of Hodge type $(d, d-1)$, $\beta \in \Omega^{d,d-1}(\mathbb{C}^d, V^\vee)$ valued in the dual space $V^\vee$. The action functional describing the classical field theory is of the usual form

$$S(\gamma, \beta) = \int \langle \beta, \bar\partial \gamma \rangle_V \tag{9}$$

where $\langle -, - \rangle_V$ denotes the natural pairing between V and its dual. The classical equations of motion of this theory consist of those (γ, β) that are holomorphic, namely $\bar\partial\beta = \bar\partial\gamma = 0$.

The symmetry we consider comes from the $\mathfrak{g}$-action on V. This extends, in a natural way, to an action of the gauge Lie algebra $C^\infty(\mathbb{C}^d, \mathfrak{g})$ on the fields (8): an element $X(z, \bar z) \in C^\infty(X, \mathfrak{g})$ acts simply by $X(z, \bar z) \cdot \gamma(z, \bar z)$ where the dot indicates product of functions and the $\mathfrak{g}$-module structure on V. The condition that this action be compatible with the action functional (9) says precisely that $X(z,\bar{z})$ must be holomorphic: $\bar{\partial}X(z,\bar{z}) = 0$.

Notice that the original action functional (9) has an internal symmetry via the gauge transformation

$$\beta \mapsto \beta + \bar{\partial}\beta'$$

where $\beta' \in \Omega^{d,d-2}(X, V^*)$. Thus, it is natural to include the space $\Omega^{d,d-2}(X, V^\vee)$ as the ghosts of the BRST formulation of this theory. Moreover, there are ghosts for ghosts $\beta'' \in \Omega^{d,d-3}(X, V^\vee)$, and so on. Together with all of the antifields and antighosts, the full theory comprises of two copies of the Dolbeualt complex. The precise definition is the following.

did you motivate/ discuss ~~of~~ anti-fields

Definition 3.1. The *classical $\beta\gamma$ system* on the complex manifold X, in the BV formalism, has space of fields

$$\mathcal{E}_V = \Omega^{0,*}(X, V) \oplus \Omega^{d,*}(X, V^*)[d-1],$$

with the linear BRST operator given by $Q = \bar{\partial}$. We will write fields as (γ, β) to match with the notation above. The (-1)-shifted symplectic pairing is given by integration along X combined with the evaluation pairing between V and its dual: $(\gamma, \beta) \mapsto \int_X \langle \gamma, \beta \rangle_V$. The action functional for this free theory is thus

$$S_V(\beta, \gamma) = \int_X \langle \beta, \bar{\partial}\gamma \rangle_V.$$

Remark 3.2. As usual, the notation $[d-1]$ means we shift that copy of the fields down by $d-1$. Note that the elements in degree zero, where the physical fields live, are precisely maps $\gamma : \mathbb{C} \to V$ and sections $\beta \in \Omega^{d,d-1}(X; V^\vee)$, just as in the description above. In this flat case the section β has no dependence on γ. The gauge symmetry $\beta \to \beta + \bar{\partial}\beta'$, where $\beta' \in \Omega^{d,d-2}(X, V^\vee)$ has naturally been incorporated into our BRST complex (which only consists of a linear operator since the theory is free).

?

By our discussion above, once we include the full BV complex, the symmetry by the Lie algebra $C^\infty(X, \mathfrak{g})$ now extends to a symmetry by the *dg Lie algebra* $\mathcal{G}_X^{sh} = \Omega^{0,*}(X, \mathfrak{g})$. The action by $\mathcal{G}_X^{sh}$ extends to a natural action on the fields of the $\beta\gamma$ system in such a way that the shifted symplectic pairing is preserved. We encode the action by an element $\alpha \in \mathcal{G}_X^{sh}$ by a *local* Hamiltonian functional $I_\alpha^{\mathcal{G}} \in \mathcal{O}_{\text{loc}}(\mathcal{E}_V)$.

Numbering?

Definition/Lemma 1. The $\mathcal{G}_X$-equivariant $\beta\gamma$ system on X with values in V is defined by the ~~local functional~~

$$I^{\mathcal{G}}(\alpha, \beta, \gamma) = \int \langle \beta, \alpha \cdot \gamma \rangle_V \in \mathcal{O}_{\text{loc}}(\mathcal{E}_V \oplus \mathcal{G}_X[1]).$$

This functional satisfies the $\mathcal{G}_X$-equivariant classical master equation

$$(\bar{\partial} + \mathrm{d}_{\mathcal{G}}) I^{\mathcal{G}} + \frac{1}{2}\{I^{\mathcal{G}}, I^{\mathcal{G}}\} = 0.$$

The classical master equation simply says that the Hamiltonian function $I^{\mathcal{G}}$ defines a dg Lie algebra action on the theory $\mathcal{E}_V$. In particular, $I^{\mathcal{G}}$ determines a map of sheaves of dg Lie algebras

$$I^{\mathcal{G}} : \mathcal{G}_X^{sh} \to \mathcal{O}_{\mathrm{loc}}(\mathcal{E}_V)[-1],$$

where the Lie bracket on the right hand side is defined by the BV bracket $\{-,-\}$.

For the remainder of the section we will restrict ourselves to the complex d-fold $X = \mathbb{C}^d$.

3.1.1. *The $\beta\gamma$ factorization algebra.* It is the general philosophy of [CG17, CG] that the observables of a quantum field theory form a factorization algebra on the underlying space-time.

For any theory, the factorization algebra of classical observables assigns to every open set U the cochain complex of algebraic functions on the fields supported on U. For the example of the $\beta\gamma$ system, the differential is just given by the $\bar\partial$ operator. Concretely, the complex of classical observables[7] assigned to an open set $U \subset \mathbb{C}^d$ is

$$\mathrm{Obs}_V^{\mathrm{cl}}(U) = \left(\mathrm{Sym}\left(\Omega^{0,*}(U)^\vee \otimes V^\vee \oplus \Omega^{d,*}(U)^\vee \otimes V[-d+1]\right), \bar\partial\right).$$

As usual, we use the completed tensor product when defining the symmetric products. It follows from the general results of Chapter 6 of [CG] that this assignment defines a factorization algebra on $\mathbb{C}^d$.

Upon taking global sections, the functional $I^{\mathcal{G}}$ defines a map of dg Lie algebras $I^{\mathcal{G}} : \mathcal{G}_d(\mathbb{C}^d) \to \mathrm{Obs}_V^{\mathrm{cl}}(\mathbb{C}^d)$. We have seen that $\mathrm{Obs}_V^{\mathrm{cl}}$ is a factorization algebra equipped with a P_0-structure. In the introduction, we also discussed how a local Lie algebra determines a P_0-factorization algebra via its classical current algebra. The classical Noether's theorem, as ~~proved~~ stated in Theorem 11.0.1.1 of [CG], says that $I^{\mathcal{G}}$ determines a map between these factorization algebras.

Proposition 3.3 ([CG]). *(Classical Noether's Theorem) The assignment that sends an element $\alpha \in \Omega_c^{0,*}(U, \mathfrak{g})$ to the observable*

$$J^{cl}_\alpha(\gamma \otimes \beta) = \cancel{\gamma \otimes \beta \in \Omega^{0,*}(U, V) \otimes \Omega^{d,*}(U, V^*) \mapsto} \int_U \langle \beta, \alpha \cdot \gamma \rangle_V$$

determines a map of P_0-factorization algebras on $\mathbb{C}^d$

$$J^{\mathrm{cl}} : \mathrm{Cur}^{\mathrm{cl}}(\mathcal{G}_d) \to \mathrm{Obs}_V^{\mathrm{cl}}.$$

Remark 3.4. The formula for J^{cl} is identical to that of the local functional $I^{\mathfrak{g}}(\alpha)$ defining the action of $\mathcal{G}_d$ on the $\beta\gamma$ system. Ordinarily, a local functional does not determine an observable on an open set since the integral may not exist. However, since α is compactly supported on U, it makes sense to restrict $I^{\mathcal{G}}(\alpha)$ to an observable on U. This is precisely the observable $J^{\mathrm{cl}}(\alpha)$.

[7] There is also the completed version where one uses $\prod$ in place of $\oplus$ when defining the symmetric algebra, but we will not use it here.

As usual, the statement we are after pertains to the quantum situation. Being a free field theory, the $\beta\gamma$ system admits a unique quantization and hence a factorization algebra $\mathrm{Obs}_V^{\mathrm{q}}$ of quantum observables (whose definition we recall below). The natural question arises whether the symmetry by the dg Lie algebra $\mathcal{G}_d$ persists upon quantization. We are asking if we can lift J^{cl} to a "quantum current" $J^{\mathrm{q}} : \mathrm{Cur}^{\mathrm{q}}(\mathcal{G}_d) \to \mathrm{Obs}_V^{\mathrm{q}}$, where $\mathrm{Cur}^{\mathrm{q}}(\mathcal{G}_d)$ is the factorization algebras of quantum currents defined in the introduction. The existence of this map of factorization algebras is controlled by the equivariant quantum master equation, which we now turn to.

3.2. **The equivariant quantization.** The approach to quantum field theory we use follows Costello's theory of renormalization and the Batalin-Vilkovisky formalism developed in [Cos11]. The formalism dictates that in order to define a quantization, it suffices to define the theory at each energy (or length) scale and to ask that these descriptions be compatible as we vary the scale. Concretely, this compatibility is through the *renormalization group (RG) flow* and is encoded by an operator $W(P_{\epsilon<L}, -)$ acting on the space of functionals. The functional $W(P_{\epsilon<L}, -)$ is defined as a sum over ~~weights of~~ graphs~~, which~~ is how Feynman diagrams appear in Costello's formalism. A theory that is compatible with the RG flow is called a "prequantization". In order to obtain a quantization, one must solve the quantum master equation (QME). For us, the quantum master equation encodes the failure of lifting the classical $\mathcal{G}_d$-symmetry to one ~~on~~ of the prequantization.

of diagrammatic weight, associated to those graphs This

The quantization we work with follows Costello's general approach very closely, with the slight caveat that we are working equivariantly, and so some of the fields are actually background fields. The two main ingredients to construct the weight are the propagator $P_{\epsilon<L}$ and the classical interaction $I^{\mathcal{G}}$. The propagator only depends on the underlying free theory, ~~that is, the~~ higher dimensional $\beta\gamma$ system. For the general definition, we refer the reader to can be ~~found~~ in Section 10 of [CG]. As above, the interaction describes ~~how~~ the linear currents $\mathcal{G}_d$ act on the free theory.

The construction of $P_{\epsilon<L}$, which makes sense for a wide class theories of this holomorphic flavor, can be found in Section 3.2 of [Wilb]. For us, it is important to know that $P_{\epsilon<L}$ satisfies the following properties:

(1) For $0 < \epsilon < L < \infty$ the propagator is a symmetric element

$$P_{\epsilon<L} \in \mathcal{E}_V \widehat{\otimes} \mathcal{E}_V.$$

Moreover, $P_{0<\infty} = \lim_{\epsilon\to 0}\lim_{L\to\infty}$ $P_{\epsilon<L}$ is a symmetric element of the distributional completion $\overline{\mathcal{E}}_V \widehat{\otimes} \overline{\mathcal{E}}_V$.

(2) The propagator lies in the subspace

$$\Omega^{d,*}(\mathbb{C}^d \times \mathbb{C}^d, V \otimes V^*) \oplus \Omega^{d,*}(\mathbb{C}^d \times \mathbb{C}^d, V^* \otimes V) \subset \mathcal{E}_V \widehat{\otimes} \mathcal{E}_V.$$

If we ~~coordinatize~~ $(z, w) \in \mathbb{C}^d \times \mathbb{C}^d$, the propagator has the form

choose coordinates

$$P_{\epsilon<L} = P^{an}_{\epsilon<L}(z, w) \otimes (\mathrm{id}_V + \mathrm{id}_{V^*}) \tag{10}$$

where $\mathrm{id}_V, \mathrm{id}_{V^*}$ are the elements representing the identity maps which lie in $V \otimes V^*, V^* \otimes V$ respectively. Moreover, $P^{an}_{0<\infty}(z,w)$ is the Green's function for the Hodge Laplacian $\triangle_{\mathrm{Hodge}}$ on $\mathbb{C}^d$:

$$\triangle_{\mathrm{Hodge}} P^{an}_{0<\infty}(z,w) = \delta(z-w).$$

(3) Let $K_t \in C^\infty((0,\infty)_t) \otimes \mathcal{E}_V \widehat{\otimes} \mathcal{E}_V$ be the heat kernel for the Hodge Laplacian

$$\triangle_{\mathrm{Hodge}} K_t + \frac{\partial}{\partial t} K_t = 0.$$

Then, $P_{\epsilon<L}$ provides a $\bar{\partial}$-homotopy between K_ϵ and K_L:

$$\bar{\partial} P_{\epsilon<L} = K_{t=L} - K_{t=\epsilon}.$$

To define the quantization, we must recall the definition of a weight of a Feynman diagram adjusted to this equivariant context. To simplify our discussion, we introduce the notation $\mathcal{O}(\mathcal{G}_d[1])$ to mean the underlying graded vector space of $C^*_{\mathrm{Lie}}(\mathcal{G}_d)$, ~~so~~ i.e. the symmetric algebra on the dual of $\mathcal{G}_d$.

For the free $\beta\gamma$ system, the homotopy RG flow from scale $L > 0$ to $L' > 0$ is an invertible linear map

$$W(P_{L<L'}, -) : \mathcal{O}(\mathcal{E})[[\hbar]] \to \mathcal{O}(\mathcal{E})[[\hbar]] \tag{11}$$

defined as a sum over weights of graphs $W(P_{L<L'}, I) = \sum_\Gamma W_\Gamma(P_{L<L'}, I)$. Here, Γ denotes a graph, and the weight W_Γ associated to Γ is defined as follows. One labels the vertices of valence k by the kth homogenous component of the functional I. The edges of the graph are labeled by the propagator $P_{L<L'}$. The total weight is given by iterative contractions of the homogenous components of the interaction with the propagator. Formally, we can write the weight as

$$e^{W(P_{\epsilon<L}, I)} = e^{\hbar \partial_{P_{\epsilon<L}}} e^{I/\hbar}$$

where ∂_P denotes contraction with P. For a more precise definition see Chapter 2 of [Cos11].

To define the equivariant version, we extend (11) to a $\mathcal{O}(\mathcal{G}_d[1])$-linear map

$$W^{\mathcal{G}}(P_{L<L'}, -) : \mathcal{O}(\mathcal{E} \oplus \mathcal{G}_d[1])[[\hbar]] \to \mathcal{O}(\mathcal{E} \oplus \mathcal{G}_d[1])[[\hbar]].$$

Definition/Lemma 2. The *prequantization* of the $\mathcal{G}_d$-equivariant $\beta\gamma$ system on $\mathbb{C}^d$ is defined by the family of functionals $\{I^{\mathcal{G}}[L]\}_{L>0}$, where

$$I^{\mathcal{G}}[L] = \lim_{\epsilon \to 0} W^{\mathcal{G}}(P_{\epsilon<L}, I^{\mathcal{G}}). \tag{12}$$

This family satisfies homotopy RG flow and solves the quantum master equation modulo $C^*_{\mathrm{loc}}(\mathcal{G}_d)$.

Proof. The non-trivial claim to justify here is why the $\epsilon \to 0$ limit of $W^{\mathcal{G}}(P_{\epsilon<L}, I^{\mathcal{G}})$ exists. This follows from the following two claims:

(1) Only one-loop graphs appear in the weight expansion $W^{\mathcal{G}}(P_{\epsilon<L}, I^{\mathcal{G}})$.
(2) Let Γ be a one-loop graph. Then

$$\lim_{\epsilon\to 0} W^{\mathcal{G}}_{\Gamma}(P_{\epsilon<L}, I^{\mathcal{G}})$$

exists.

Claim (1) follows from direct combinatorial observation. Recall that the weight is defined as a sum over *connected* graphs. The inner edges that we can label by the propagator $P_{\epsilon<L}$ only depend on the fields $\beta\gamma$. Since there are only trivalent ~~wheels,~~ vertices of the form $\int \beta[\alpha,\gamma]$, where α is a section in $\mathcal{G}_d$, the connected diagrams must be wheels with trees sticking out of them. Even stronger: wheels with only external edges, or trees.

Now, Theorem 3.4 of [Wilb] applies to show that the $\epsilon \to 0$ limit of the weights is finite. This proves claim (2). □

As an immediate consequence of the proof, we see that only polynomial values of $\hbar$ occur in the expansion of $I^{\mathcal{G}}[L]$. This fact will be used later on when we make sense of the "free field realization" of the Kac-Moody granted by this equivariant quantization.

Corollary 3.5. *For each $L > 0$, the functional $I^{\mathcal{G}}[L]$ lies in the subspace $\mathcal{O}(\mathcal{E} \oplus \mathcal{G}_d[1])[\hbar] \subset \mathcal{O}(\mathcal{E} \oplus \mathcal{G}_d[1])[[\hbar]]$.*

To define the quantum master equation, we must introduce the BV Laplacian Δ_L and the scale L BV bracket $\{-,-\}_L$. For $L > 0$, the operator $\Delta_L : \mathcal{O}(\mathcal{E}_V) \to \mathcal{O}(\mathcal{E}_V)$ is defined by contraction with the heat kernel K_L defined above. Similarly, $\{-,-\}_L$ is a bilinear operator on $\mathcal{O}(\mathcal{E}_V)$ defined by

$$\{I, J\}_L = \Delta_L(IJ) - (\Delta_L I)J - (-1)^{|I|} I \Delta_L J.$$

There are equivariant versions of each of these operators given by extending via $\mathcal{O}(\mathcal{G}_d[1])$-linearity. For instance, the BV Laplacian is a degree one operator of the form

$$\Delta_L : \mathcal{O}(\mathcal{E} \oplus \mathcal{G}_d[1]) \to \mathcal{O}(\mathcal{E} \oplus \mathcal{G}_d[1]).$$

We say a functional $J \in \mathcal{O}(\mathcal{E}_V \oplus \mathcal{G}_d[1])$ satisfies the scale L, *$\mathcal{G}_d$-equivariant quantum master equation* if

$$(\overline{\partial} + \mathrm{d}_{\mathcal{G}})J + \frac{1}{2}\{J, J\}_L + \hbar\Delta_L J = 0.$$

The main object of study in this section is the failure for the quantization $I^{\mathcal{G}}[L]$ to satisfy this equivariant QME.

Definition 3.6. The scale L, *$\mathcal{G}_d$-equivariant charge anomaly* is

$$\hbar\Theta_V[L] = (\overline{\partial} + \mathrm{d}_{\mathcal{G}})I^{\mathcal{G}}[L] + \frac{1}{2}\{I^{\mathcal{G}}[L], I^{\mathcal{G}}\}_L + \hbar\Delta I^{\mathcal{G}}[L].$$

As above, $\mathrm{d}_{\mathcal{G}}$ denotes the Chevalley-Eilenberg differential $\mathrm{C}^*_{\mathrm{Lie}}(\mathcal{G}_d) = (\mathcal{O}(\mathcal{G}_d[1]), \mathrm{d}_{\mathcal{G}})$.

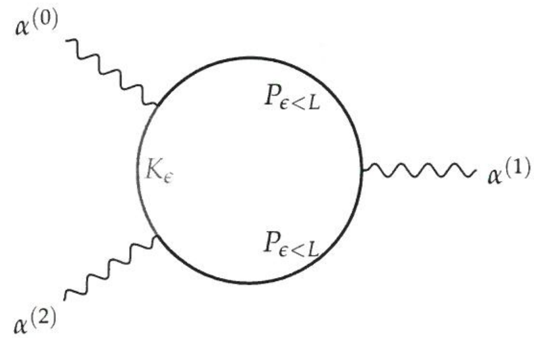

FIGURE 1. The diagram representing the weight $W_{\Gamma,e}(P_{\epsilon<L}, K_\epsilon, I^{\mathfrak{g}})$ in the case $d = 2$. On the black internal edges are we place the propagator $P_{\epsilon<L}$ of the $\beta\gamma$ system. On the red edge labeled by e we place the heat kernel K_ϵ. The external edges are labeled by elements $\alpha^{(i)} \in \Omega^{0,*}_c(\mathbb{C}^2)$.

3.3. **The charge anomaly for $\beta\gamma$.** To calculate the anomaly to solving the $\mathcal{G}_d$-equivariant master equation we utilize a general result about the quantum master equation for holomorphic field theories formulated in [Wilb]. In general, since the effective field theory defining the prequantization $\{I^{\mathcal{G}}[L]\}$ is given by a Feynman diagram expansion, the anomaly to solving the quantum master equation is also given by a potentially complicated sum of diagrams. As an immediate corollary of Proposition 4.4 of [Wilb] for holomorphic theories on $\mathbb{C}^d$, we find that only a simple class of diagrams appear in the anomaly.

Lemma 3.7. *Let $\Theta_V[L]$ be the $\mathcal{G}_d$-equivariant* charge anomaly *for the $\beta\gamma$ system with values in V. Then, $\Theta_V := \lim_{L\to 0}\Theta_V[L]$ exists, and is a local cocycle*

$$\Theta_V \in C^*_{\rm loc}(\mathcal{G}_d).$$

Furthermore, Θ is computed by the following limit

$$\hbar\Theta_V = \frac{1}{2}\lim_{L\to 0}\lim_{\epsilon\to 0}\sum_{\Gamma\in \mathrm{Wheel}_{d+1},e} W_{\Gamma,e}(P_{\epsilon<L}, K_\epsilon, I^{\mathcal{G}}),$$

where the sum is over all wheels of valency $(d+1)$ and choices of *internal edges e.*

Is there a heuristic explanation for why only the (d+1)-leg terms contribute.

Remark 3.8. If Γ is a graph with a distinguished edge e and A, B are elements of the tensor square of fields $A, B \in \mathcal{E}\otimes\mathcal{E}$, we let $W_{\Gamma,e}(A,B,I)$ denotes the weight of the graph where we place B at the internal edge labeled e and A on the remaining internal edges.

In the remainder of this section we will characterize this anomaly algebraically, using the identification of Proposition ~~1.25. Actually, it~~ is immediate based on symmetry arguments ~~what~~ the obstruction ~~is~~ up to a scalar multiple.

First, we note that the element $\Theta_V \in C^*_{\rm loc}(\mathcal{G}_d)$ lifts to the invariant subspace of $U(d)$-invariant, holomorphic translation invariant local cocycles. This follows from the fact that both the functional $I^{\mathcal{G}}$ and propagator $P_{\epsilon<L}$ are $U(d)$-invariant, holomorphic translation invariant. By Proposition 1.25 we see that Θ_V must be cohomological to a cocycle of the form

$$(\alpha_0, \ldots, \alpha_d) \mapsto \int_{\mathbb{C}^d} \theta(\alpha_0 \wedge \partial\alpha_1 \wedge \cdots \wedge \partial\alpha_d)$$

where θ is some element of $\mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$. In the notation of Section 1, this is the cocycle $\mathfrak{J}_d(\theta)$. This cocycle factors in the following way:

(13)

$$\left(\Omega_c^{0,*}(\mathbb{C}^d) \otimes \mathfrak{g}\right)^{\otimes(d+1)} \xrightarrow{\mathfrak{an}} \left(\Omega_c^{0,*}(\mathbb{C}^d) \otimes \mathfrak{g}\right) \otimes \left(\Omega_c^{1,*}(\mathbb{C}^d) \otimes \mathfrak{g}\right)^{\otimes d} \xrightarrow{\theta} \Omega_c^{d,*}(\mathbb{C}^d) \xrightarrow{\int} \mathbb{C}.$$

The first map is $\mathfrak{an} : \alpha_0 \otimes \cdots \otimes \alpha_d \mapsto \alpha_0 \otimes \partial\alpha_1 \otimes \cdots \otimes \partial\alpha_d$. The second map is given by extending the Lie algebraic functional $\theta : \mathfrak{g}^{\otimes(d+1)} \to \mathbb{C}$ to the Dolbeault complex in the obvious way:

$$\theta\left(\alpha_0 \otimes (\alpha_1 \otimes \cdots \otimes \alpha_d)\right) = \alpha_0 \wedge \theta(\alpha_1 \wedge \cdots \wedge \alpha_d) \in \Omega_c^{d,*}(\mathbb{C}^d).$$

Lemma 3.7 implies that the obstruction is given by the sum over Feynman weights associated to graphs of wheels of valency $(d+1)$. We can identify the algebraic component, corresponding to θ in the above composition (13), directly from the shape of this graph. The propagator and heat kernel $P_{\epsilon<L}, K_\epsilon$ labeling the edges factor as

$$\begin{aligned} P_{\epsilon<L} &= P^{an}_{\epsilon<L} \otimes (\mathrm{id}_V + \mathrm{id}_{V^*}), \\ K_\epsilon &= K^{an}_\epsilon \otimes (\mathrm{id}_V + \mathrm{id}_{V^*}) \end{aligned}$$

where $\mathrm{id}_V, \mathrm{id}_{V^*}$ are the elements representing the identity maps which lie in $V \otimes V^*$, $V^* \otimes V$ respectively. The analytic factors $P^{an}_{\epsilon<L}, K^{an}_\epsilon$ only depend on the dimension d.

Each trivalent vertex of the wheel is also labeled by both an analytic factor and Lie algebraic factor. The Lie algebraic part of each vertex can be thought of as the defining map of the representation $\rho : \mathfrak{g} \to \mathrm{End}(V)$. The diagrammatics of the wheel amounts to taking the trace of the symmetric $(d+1)$st power of this Lie algebra factor. Thus, the Lie algebraic factor of the weight of the wheel is the $(d+1)$st component of the character of the representation

$$\mathrm{ch}^{\mathfrak{g}}_{d+1}(V) = \frac{1}{(d+1)!} \mathrm{Tr}\left(\rho(X)^{d+1}\right) \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*).$$

By these symmetry arguments, we know that the anomaly will be of the form $\Theta = A\mathfrak{j}(\mathrm{ch}^{\mathfrak{g}}_{d+1}(V))$ for some number $A \in \mathbb{C}$. In Appendix B, we perform an explicit calculation of this constant A, which depend on the specific form of the analytic propagator and heat kernel. We arrive at the following.

Proposition 3.9. *The charge anomaly for quantizing the $\mathcal{G}_d$-equivariant $\beta\gamma$ system on $\mathbb{C}^d$ is equal to*

$$\Theta_V = \frac{1}{(4\pi)^d} \mathfrak{j}(\mathrm{ch}^{\mathfrak{g}}_{d+1}(V)),$$

where $\jmath$ is the isomorphism from Proposition 1.25.

3.4. **The quantum observables of the $\beta\gamma$ system.** Before deducing the main consequence of the anomaly calculation, we introduce the quantum observables of the $\beta\gamma$ system. The quantum observables $\mathrm{Obs}^{\mathrm{q}}_V$ define a quantization of the classical observables in the sense that there is an isomorphism

$$\mathrm{Obs}^{\mathrm{cl}}_V \cong \mathrm{Obs}^{\mathrm{q}}_V \otimes_{\mathbb{C}[\hbar]} \mathbb{C}_{\hbar=0}.$$

In practice, the BV formalism suggests that the quantum observables arise by

(a) tensoring the underlying graded vector space of $\mathrm{Obs}^{\mathrm{cl}}_n$ with $\mathbb{C}[[\hbar]]$ and
(b) deforming the differential to $\overline{\partial} + \hbar\Delta_L$, where Δ_L is the BV Laplacian.

This actually defines a family of quantum observables, one for each length scale L. A main idea of [CG] says that by considering the collection of functionals at all length scales L, the observables $\mathrm{Obs}^{\mathrm{q}}_V$ still define a factorization algebra.

The fact that this works is quite subtle, since naively the differential Δ_L seems to have support on all of $\mathbb{C}^d$, so it's not obvious how to define the corestriction maps of the factorization algebra. In the case of free theories, such as the $\beta\gamma$ system, there is a way to circumvent this difficulty. One can work with a smaller class of observables — such as those arising from smooth functionals, not distributional ones. Then, one can make sense of the limit $\Delta = \lim_{L\to 0}\Delta_L$, and just use this single BV Laplacian. This approach is developed in detail for the free $\beta\gamma$ system on $\mathbb{C}$ in Chapter 5, Section 3 of [CG17]. The case for $\mathbb{C}^d$ is similar.

A classical result of Atiyah-Bott, Proposition 6.1 in [AB67], implies that for any complex manifold U the subcomplex

homotopy equiv. even!

$$\Omega^{p,*}_c(U) \subset \overline{\Omega}^{p,*}(U)$$

is quasi-isomorphic to the full complex of distributional forms. This follows from ellipticity of the Dolbeault complex. Consequently we can introduce the qausi-isomorphic subcomplex

just keep this part.

$$\begin{tikzcd}
\left(\mathrm{Sym}(\Omega^{d,*}_c(U,V^*)[d] \oplus \Omega^{0,*}_c(U,V)[1]), \overline{\partial}\right) \arrow[r, hook, "\simeq"] \arrow[d, equal] & \left(\mathrm{Sym}(\overline{\Omega}^{d,*}_c(U,V^*)[d] \oplus \overline{\Omega}^{0,*}_c(U,V)[1]), \overline{\partial}\right) \arrow[d, equal] \\
\widetilde{\mathrm{Obs}}^{\mathrm{cl}}_V(U) \arrow[r, hook, "\simeq"] & \mathrm{Obs}^{\mathrm{cl}}_V(U).
\end{tikzcd}$$

The assignment $U \mapsto \widetilde{\mathrm{Obs}}^{\mathrm{cl}}_V(U)$ still defines a factorization algebra on $\mathbb{C}^d$, and we have a resulting quasi-isomorphism of factorization algebras $\widetilde{\mathrm{Obs}}^{\mathrm{cl}}_V \xrightarrow{\simeq} \mathrm{Obs}^{\mathrm{cl}}_V$.

Definition 3.10. The quantum observables supported on $U \subset \mathbb{C}^d$ is the cochain complex

$$\widetilde{\mathrm{Obs}}^{\mathrm{q}}_V(U) = \left(\mathrm{Sym}(\Omega^{d,*}_c(U,V^*)[d] \oplus \Omega^{0,*}_c(U,V)[1]), \overline{\partial} + \hbar\Delta\right).$$

By Theorem 5.3.10 of [Gwi12] the assignment $U \mapsto \mathrm{Obs}^{\mathrm{q}}_V(U)$ defines a factorization algebra on $\mathbb{C}^d$. Just as in the classical case, there is an induced quasi-isomorphism of factorization algebras $\widetilde{\mathrm{Obs}}^{\mathrm{q}}_V \xrightarrow{\simeq} \mathrm{Obs}^{\mathrm{q}}_V$. This is proved in Lemma 11.24 of [GGW].

3.5. **Free field realization.** We now appeal to a general result about lifting the classical Noether map from the current algebra $\mathrm{Cur}^{\mathrm{cl}}(\mathcal{G}_d)$ to the factorization algebra of quantum observables. This factorization enhancement of the quantum Noether theorem is Theorem 12.1.0.2 [CG]. The general situation for which the result is stated is for an action of a local Lie algebra $\mathcal{L}$ on a classical theory. The factorization enhancement of the quantum Noether theorem says that if Θ is the obstruction to solving the the $\mathcal{L}$-equivariant quantum master equation, then we have a map from the *twisted* quantum current algebra $\mathrm{Cur}^{\mathrm{q}}_{\Theta}(\mathcal{L})$ to the observables of the quantum theory. Thus, applied to our situation, we have the following consequence of our Feynman diagram calculation above.

Proposition 3.11. *Let $\hbar\Theta_V$ be the obstruction to satisfying the $\mathcal{G}_d$-equivariant quantum master equation. There is a map of factorization algebras on $\mathbb{C}^d$ from the twisted quantum current algebra to the quantum observables*

$$J^{\mathrm{q}} : \mathrm{Cur}^{\mathrm{q}}_{\hbar\Theta_V}(\mathcal{G}_d) \to \mathrm{Obs}^{\mathrm{q}}_V \tag{14}$$

that fits into the diagram of factorization algebras

$$\begin{tikzcd}
\mathrm{Cur}^{\mathrm{q}}_{\hbar\Theta_V}(\mathcal{G}_d) \arrow[r, "J^{\mathrm{q}}"] \arrow[d, "\hbar\to 0"'] & \mathrm{Obs}^{\mathrm{q}}_V \arrow[d, "\hbar\to 0"] \\
\mathrm{Cur}^{\mathrm{cl}}(\mathcal{G}_d) \arrow[r, "J^{\mathrm{cl}}"] & \mathrm{Obs}^{\mathrm{cl}}_V .
\end{tikzcd}$$

The quantum current algebra $\mathrm{Cur}^{\mathrm{q}}_{\hbar\Theta_V}$ is a factorization algebra on $\mathbb{C}^d$ taking values in $\mathbb{C}[\hbar]$. It therefore makes sense to specialize the value of $\hbar$. The convention we take is to specialize the value of $\hbar$ to be

$$\hbar = (4\pi)^d.$$

From our calculation of the charge anomaly Θ_V above, once we specialize $\hbar$ we can realize the current algebra as an enveloping factorization algebra

$$\mathrm{Cur}^{\mathrm{q}}_{\hbar\Theta_V}(\mathcal{G}_d)\Big|_{\hbar=(2\pi i)^d} \cong \mathbb{U}_{\mathrm{ch}^{\mathfrak{g}}_{d+1}(V)}(\mathcal{G}_d).$$

is it 4π?

Thus, as an immediate corollary of the above proposition, J^{q} specializes to a map of factorization algebras

$$J^{\mathrm{q}} : \mathbb{U}_{\mathrm{ch}^{\mathfrak{g}}_{d+1}(V)}(\mathcal{G}_d) \to \mathrm{Obs}^{\mathrm{q}}_V|_{\hbar=(4\pi)^d} . \tag{15}$$

We interpret this as a *free field realization* of the higher Kac-Moody factorization algebra. Indeed, this is an embedding of the Kac-Moody algebra into the factorization algebra of observables of the free theory described classically by the $\beta\gamma$ system.

We obtain a more concrete result once we specialize to the sphere operators. To state this, we introduce a dg associative algebra closely related to the $\beta\gamma$ system. Recall, the algebra A_d which provides a dg model for functions on punctured affine space $\mathbb{A}^d$. Consider the dg vector space

$$A_d \otimes (V \oplus V^*[d-1])$$

where V is our $\mathfrak{g}$-representation. The dual pairing between V and V^* combined with the higher residue defines a symplectic structure ω_V on this dg vector space via

(−1)-shifted?

$$\omega_V(\alpha \otimes v, \beta \otimes v^*) = \langle v, v^* \rangle_V \oint_{S^{2d-1}} \alpha \wedge \beta \mathrm{d}^d z.$$

From this symplectic dg vector space, we define the dg Lie algebra $\mathcal{H}_V$ as a central extension

$$\mathbb{C} \to \mathcal{H}_V \to A_d \otimes (V \oplus V^*[d-1]).$$

The 2-cocycle defining this extension is simply ω_V. Thus, $\mathcal{H}_V$ is the Heisenberg Lie algebra associated to the symplectic dg vector space.

If we restrict the factorization algebra $\mathrm{Obs}_V^{\mathfrak{q}}$ to spheres S^{2d-1} × intervals, we obtain a locally constant factorization algebra, analogous to the case of the spherical algebra of the Kac-Moody factorization algebras. This locally constant factorization algebra is equivalent, as E_1-algebras, to the dg associative algebra obtained as the enveloping algebra $U(\mathcal{H}_V)$ of the Heisenberg Lie algebra.

Upon compactification of the quantum Noether map along the sphere S^{2d-1} we obtain the following.

Corollary 3.12. *The map (15) determines a map of E_1-algebras*

$$\oint_{S^{2d-1}} J^{\mathfrak{q}} : U\left(\widehat{\mathfrak{g}}_{d,\mathrm{ch}^{\mathfrak{g}}_{d+1}(V)}\right) \to U(\mathcal{H}_V). \tag{16}$$

We denote the restriction by $\oint_{S^{2d-1}}$ since the value of the pushforward factorization algebra along radial projection on open intervals coincides with the "sphere observables".

Proof. Let $r : \mathbb{C}^d \setminus 0 \to \mathbb{R}_{>0}$ be the radial projection. Consider the induced map

$$r_* J^{\mathfrak{q}} : r_* \mathbb{U}_{\mathrm{ch}^{\mathfrak{g}}_{d+1}(V)}(\mathcal{G}_d) \to r_* \mathrm{Obs}_V^{\mathfrak{q}}|_{\hbar=(4\pi)^d}.$$

The classical $\beta\gamma$ system on $\mathbb{C}^d$ (and $\mathbb{C}^d \setminus \{0\}$) is manifestly equivariant for the group $U(d)$ given by rotating the plane. This symmetry persists upon quantization, since the BV Laplacian is also compatible with this action. Thus, we can consider the subfactorization algebra of $r_* \mathrm{Obs}_V^{\mathfrak{q}}|_{\hbar=(4\pi)^d}$ consisting of the direct sum of the $U(d)$-eigenspaces $U(d)$-eigenspaces.

Similarly, the Kac-Moody factorization algebra $\mathbb{U}_{\mathrm{ch}^{\mathfrak{g}}_{d+1}(V)}(\mathcal{G}_d)$ is $U(d)$-equivariant. The subfactorization algebra given by the $U(d)$-invariants is precisely $\mathbb{U}_{\mathrm{ch}^{\mathfrak{g}}_{d+1}(V)}\mathtt{G}_d$, where $\mathtt{G}_d$ is as in Section 2.1.

By an argument completely analogous to the Kac-Moody case, one checks that the subfactorization algebra of $r_* \operatorname{Obs}^{\mathrm{q}}_V|_{\hbar=(4\pi)^d}$ consisting of $U(d)$-eigenspaces is locally constant. To an interval $I \subset \mathbb{R}_{>0}$, the subfactorization algebra assigns a dg vector space isomorphic to $U(\mathcal{H}_V)$. It is shown in Chapter 3 of [Wila] that this locally constant factorization algebra is equivalent to $U(\mathcal{H}_V)$ as an E_1 ~~algebras~~ algebra.

Finally, note that the family of functionals $\{I^{\mathfrak{g}}[L]\}$ defining the Noether map are all $U(d)$-invariant. Thus, J^{q} preserves the subfactorization algebras of $U(d)$-eigenspaces, and the result follows. $\square$

Should say more about how J^q is defined in order for this argument to make sense.

4. Some global aspects of the higher Kac-Moody factorization algebras

A compelling aspect of factorization algebras is that they are local-to-global objects, and hence the global sections—the factorization homology—can be quite interesting. We focus here on a class of complex manifolds called *Hopf manifolds*, whose underlying smooth manifold has the form $S^1 \times S^{2d-1}$. We choose to focus on these because the factorization homology answer admits a concise description in terms of Hochschild homology, and for the application for studying the equivariance of the partition function of the higher dimensional $\beta\gamma$-systems. After this, we return to the LMNS variants of the twisted Kac-Moody factorization algebra that exist on complex d-folds.

w.r.t. what?

4.1. Hopf manifolds and twisted indices.

We focus on a family of complex manifolds defined by Hopf in [Hop48] defined in every complex dimension d.

Definition 4.1. Fix an integer $d \geq 1$. Let $f : \mathbb{C}^d \to \mathbb{C}^d$ be a polynomial map such that $f(0) = 0$ such that its Jacobian at zero $Jac(f)(0)$ is invertible with eigenvalues $\{\lambda_i\}$ all satisfying $|\lambda_i| < 1$. Define the *Hopf manifold associated to f* to be the d-dimensional complex manifold

$$X_f := \left(\mathbb{C}^d \setminus \{0\}\right) \Big/ (x \sim f(x)).$$

Note that X_f is compact for any f. In the case $d = 1$ all Hopf surfaces are equivalent to elliptic curves.

Lemma 4.2. *For any f there is a diffeomorphism $X_f \cong S^{2d-1} \times S^1$.*

This implies that when $d > 1$, the cohomology $H^2_{dR}(X_f) = 0$ for any f. In particular, X_f is *not* Kähler when $d > 1$. For $1 \leq i \leq d$ let $q_i \in D(0,1)^\times$ be a nonzero complex number of modulus $|q_i| < 1$. The d-dimensional *Hopf manifold of type* $\mathbf{q} = (q_1, \ldots, q_d)$ is the following quotient of punctured affine space $\mathbb{C}^d \setminus \{0\}$ by the discrete group $\mathbb{Z}^d$:

$$X_{\mathbf{q}} = \left(\mathbb{C}^d \setminus \{0\}\right) \Big/ \left((z_1, \ldots, z_d) \sim (q_1^{2\pi i \mathbb{Z}} z_1, \ldots, q_d^{2\pi i \mathbb{Z}} z_d)\right).$$

Note that in the case $d = 1$ we recover the usual description of an elliptic curve $X_{\mathbf{q}} = E_q = \mathbb{C}^\times / q^{2\pi i \mathbb{Z}}$. We will denote by $p_{\mathbf{q}} : \mathbb{C}^d \setminus \{0\} \to X_{\mathbf{q}}$ the quotient map.

For any d and tuple $(q_1, \ldots, q_d)$ as above, we see that as a smooth manifold there is a diffeomorphism $X_{\mathbf{q}} \cong S^{2d-1} \times S^1$. Indeed, the radial projection map $\mathbb{C}^d \setminus \{0\} \to \mathbb{R}_{>0}$ defines a smooth S^{2d-1}-fibration over $\mathbb{R}_{>0}$. Passing to the quotient, we obtain an S^{2d-1}-fibration

$$X_{\mathbf{q}} \to \mathbb{R}_{>0} / \left(r \sim \lambda^{\mathbb{Z}} \cdot r \right) \cong S^1.$$

Here, $\lambda = (|q_1|^2 + \cdots + |q_d|^2)^{1/2} > 0$. Since there are no non-trivial S^{2d-1} fibrations over S^1 we obtain $X_{\mathbf{q}} = S^{2d-1} \times S^1$ as smooth manifolds.

Proposition 4.3. *Let X be a Hopf manifold and suppose $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$ is any $\mathfrak{g}$-invariant polynomial of degree $(d+1)$. Then, there is a quasi-isomorphism of $\mathbb{C}[K]$-modules*

$$\int_X \mathbb{U}_\theta(\mathcal{G}_X) \simeq \mathrm{Hoch}_*(U\mathfrak{g})[K].$$

Proof. Let's first consider the untwisted case where the statement reduces to $\int_X \mathbb{U}(\mathcal{G}_X) \simeq \mathrm{Hoch}_*(U\mathfrak{g})$. The factorization homology on the left hand side is computed by

$$\int_X \mathbb{U}(\mathcal{G}_X) = \mathrm{C}^{\mathrm{Lie}}_*(\Omega^{0,*}(X) \otimes \mathfrak{g}).$$

would have been nice to mention this earlier.

Now, since every ~~Hopf manifold is~~ Dolbeault formal there is a quasi-isomorphism of commutative dg algebras

$$(H^{0,*}(X), 0) \overset{?}{\simeq} \left(\Omega^{0,*}(X), \bar\partial\right).$$

where?

In fact, we have written down a preferred presentation for the cohomology ring of X given by $H^{0,*}(X) = \mathbb{C}[\delta]$ where $|\delta| = 1$. (δ^2) A particular Dolbeault representative for δ given by λ is

$$\bar\partial(\log |z|^2) = \sum_i \frac{z_i \mathrm{d}\bar z_i}{|z|^2}$$

where $z = (z_1, \ldots, z_d)$ is the coordinate on $\mathbb{C}^d \setminus \{0\}$.

Applied to the global sections of the Kac-Moody factorization algebra we see that there is a quasi-isomorphism

$$\int_X \mathbb{U}(\mathcal{G}_X) \simeq \mathrm{C}^{\mathrm{Lie}}_*(\mathbb{C}[\delta] \otimes \mathfrak{g}).$$

$\mathbb{C}[\delta] \otimes \mathfrak{g} = \mathfrak{g} \oplus \mathfrak{g}[-1] \oplus \mathfrak{g}[-2] \oplus \cdots$

Now, note that $\mathrm{C}^{\mathrm{Lie}}_*(\mathbb{C}[\delta] \otimes \mathfrak{g}) \neq \mathrm{C}^{\mathrm{Lie}}_*(\mathfrak{g} \oplus \mathfrak{g}[-1]) = \mathrm{C}^{\mathrm{Lie}}_*(\mathfrak{g}, \mathrm{Sym}(\mathfrak{g}))$, where $\mathrm{Sym}(\mathfrak{g})$ is the symmetric product of the adjoint action of $\mathfrak{g}$ on itself. By Poincaré-Birkhoff-Witt ~~there is~~ an isomorphism of vector spaces $\mathrm{Sym}(\mathfrak{g}) = U\mathfrak{g}$, so we can write this as $\mathrm{C}^{\mathrm{Lie}}_*(\mathfrak{g}, \mathrm{Sym}(\mathfrak{g}))$. Og

Now, any $U(\mathfrak{g})$-bimodule M is automatically a module for the Lie algebra $\mathfrak{g}$ by the formula $x \cdot m = xm - mx$ where $x \in \mathfrak{g}$ and $m \in M$. Moreover, for any such bimodule there is a quasi-isomorphism of cochain complexes

$$\mathrm{C}^{\mathrm{Lie}}_*(\mathfrak{g}, M) \simeq \mathrm{Hoch}_*(U\mathfrak{g}, M).$$

This is proved, for instance, in Section 2.3 of [CR11]. Applied to the bimodule $M = U\mathfrak{g}$ itself we obtain a quasi-isomorphism $\mathrm{C}^{\mathrm{Lie}}_*(\mathfrak{g}, U\mathfrak{g}) \simeq \mathrm{Hoch}(U\mathfrak{g})$.

The twisted case is similar. Let θ be as in the statement. Then, the factorization homology is equal to

$$\int_X \mathbb{U}_\theta(\mathcal{G}_X) = \left(\mathrm{Sym}(\Omega^{0,*}(X) \otimes \mathfrak{g})[K], \overline{\partial} + \mathrm{d}_{CE} + \mathrm{d}_\theta\right).$$

Applying Dolbeault formality again we see that this is quasi-isomorphic to the cochain complex

$$(\mathrm{Sym}(\mathfrak{g}[\delta])[K], \mathrm{d}_{CE} + \mathrm{d}_\theta). \tag{17}$$

We note that d_θ is identically zero on $\mathrm{Sym}(\mathfrak{g}[\delta])$. Indeed, for degree reasons, at least one of the inputs must be from $\mathfrak{g} \hookrightarrow \mathfrak{g}[\delta] = \mathfrak{g} \oplus \mathfrak{g}[-1]$, which consists of constant functions on X with values in the Lie algebra $\mathfrak{g}$. In the formula for the local cocycle from Proposition 1.20 associated to θ it is clear that if any one of the inputs is constant the cocycle vanishes. Indeed, one can integrate by parts to put it in the form $\int \partial\alpha, \cdots \partial\alpha$, which is the integral of a total derivative, hence zero since X has no boundary. Thus, (17) just becomes the Chevalley-Eilenberg complex with values in the trivial module $\mathbb{C}[K]$. By the same argument as in the untwisted case, we conclude that in this case the factorization homology is quasi-isomorphic to $\mathrm{Hoch}_*(U\mathfrak{g})[K]$, as desired. □

There is an interesting consequence of this calculation ~~to~~ for the Hochschild homology for the A_∞ algebra $U(\widehat{\mathfrak{g}}_{d,\theta})$. It is easiest to state this when X is a Hopf manifold of the form $(\mathbb{C}^d \setminus \{0\})/q^{\mathbb{Z}}$ for a single $q = q_1 = \cdots = q_d \in D(0,1)^\times$ where the quotient is by the relation $(z_1, \ldots, z_d) \simeq (q^{\mathbb{Z}} z_1, \ldots, q^{\mathbb{Z}})$. Let $p_q : \mathbb{C}^d \setminus \{0\} \to X$ be the quotient map. Consider the following diagram

$$\begin{tikzcd}
\mathbb{C}^d \setminus \{0\} \arrow[r, "p_q"] \arrow[d, "\rho"] & X \arrow[d, "\overline{\rho}"] \\
\mathbb{R}_{>0} \arrow[r, "\overline{p}_q"] & S^1
\end{tikzcd}$$

Here, ρ is the radial projection map and $\overline{\rho}$ is the induced map defined by the quotient. The action of $\mathbb{Z}$ on $\mathbb{C}^d \setminus \{0\}$ gives $\mathcal{G}_{\mathbb{C}^d \setminus \{0\}}$ the structure of a $\mathbb{Z}$-equivariant factorization algebra. In turn, this determines an action of $\mathbb{Z}$ on pushforward factorization algebra $\rho_* \mathcal{G}_{\mathbb{C}^d \setminus \{0\}}$. We have seen that there is a dense locally constant subfactorization algebra on $\mathbb{R}_{>0}$ of the pushforward that is equivalent as an E_1 algebra to $U(\widehat{\mathfrak{g}}_{d,\theta})$. A consequence of excision for factorization homology, see Lemma 3.18 [AF15], implies that there is a quasi-isomorphism

left-hand?

$$\mathrm{Hoch}_*(U(\widehat{\mathfrak{g}}_{d,\theta}), q) \simeq \int_{S^1} \overline{\rho}_* \mathbb{U}_\theta(\mathcal{G}_X),$$

where the right-hand side is the Hochschild homology of the algebra $U\widehat{\mathfrak{g}}_{d,\theta}$ with coefficients in the bimodule $U\widehat{\mathfrak{g}}_{d,\theta}$ with the ordinary left-module structure and right-module structure given by twisting the ordinary action by the automorphism corresponding to the element $1 \in \mathbb{Z}$ on the algebra.

Moreover, by the push-forward for factorization homology, Proposition 3.23 [AF15], there is an equivalence

$$\int_{S^1} \overline{p}_* \mathbb{U}_\alpha(\mathcal{G}_X) \xrightarrow{\simeq} \int_X \mathbb{U}_\alpha(\mathcal{G}_X).$$

We have just shown that the factorization homology of $\mathcal{G}_X$ is equal to the Hochschild homology of $U\mathfrak{g}$ so that

$$\mathrm{Hoch}_*(U(\widehat{\mathfrak{g}}_{d,\theta}), q) \simeq \mathrm{Hoch}_*(U\mathfrak{g})[K].$$

This statement is purely algebraic as the dependence on the manifold for which the Kac-Moody lives has dropped out. It may be easiest to understand in the case $d = 1$ and $\theta = 0$. Then $\mathfrak{g}_{d,\theta}$ is the loop algebra $L\mathfrak{g} = g[z, z^{-1}]$. The action of $\mathbb{Z}$ on $L\mathfrak{g}$ rotates the loop parameter: for $z^n \otimes \mathfrak{g} \in L\mathfrak{g} = \mathbb{C}[z, z^{-1}] \otimes \mathfrak{g}$ the action of $1 \in \mathbb{Z}$ is $1 \cdot (z^n \otimes \mathfrak{g}) = q^n z^n \otimes \mathfrak{g}$. In turn, the bimodule structure of $U(\mathfrak{g}[z, z^{-1}])$ on itself, which we denote $U(\mathfrak{g}[z, z^{-1}])_q$ is the ordinary one on the left and on the right is given by twisting by the automorphism corresponding to $1 \in \mathbb{Z}$. The complex $\mathrm{Hoch}_*(U(g[z, z^{-1}]), q)$ is the Hochschild homology of $U(\mathfrak{g}[z, z^{-1}])$ with values in this bimodule. Thus, the statement implies that there is a quasi-isomorphism

Is this well-known?

$$\mathrm{Hoch}_*\left(U(\mathfrak{g}[z, z^{-1}]), U(\mathfrak{g}[z, z^{-1}])_q\right) \simeq \mathrm{Hoch}(U\mathfrak{g}).$$

4.2. **The Kac-Moody vertex algebra and compactification.** We turn briefly to the variant of the Kac-Moody factorization algebra associated to the cocycles from Section 1.3.5. This class of cocycles is related to the ordinary Kac-Moody vertex algebra on Riemann surfaces through dimensional reduction, as we will now show.

Consider the complex manifold $X = \Sigma \times \mathbb{P}^{d-1}$ where Σ is a Riemann surface and $\mathbb{P}^{d-1}$ is $(d-1)$-dimensional complex projective space. Suppose that $\omega \in \Omega^{d-1,d-1}(\mathbb{P}^{d-1})$ is the natural volume form; this clearly satisfies the conditions of Lemma 1.21 and so determines a degree one cocycle $\phi_{\kappa,\omega} \in \mathrm{C}^*_{\mathrm{loc}}(\mathcal{G}_{\Sigma\times\mathbb{P}^{d-1}})$ where κ is some $\mathfrak{g}$-invariant bilinear form $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$. We can then consider the twisted enveloping factorization algebra of $\mathcal{G}_{\Sigma\times\mathbb{P}^{d-1}}$ by the cocycle $\phi_{\kappa,\omega}$.

Recall that if $p : X \to Y$ and $\mathcal{F}$ is a factorization algebra on X, then the pushforward $p_*\mathcal{F}$ on Y is defined on opens by $p_*\mathcal{F} : U \subset Y \mapsto \mathcal{F}(p^{-1}U)$.

you've used this construction a lot elsewhere and if you haven't defined it before, you should move this def$^{\underline{n}}$ to where it first arises.

Proposition 4.4. *Let $\pi : \Sigma \times \mathbb{P}^{d-1} \to \Sigma$ be the projection. Then, there is a quasi-isomorphism between the following two factorization algebras on Σ:*

(1) $\pi_*\mathbb{U}_{\phi_{\kappa,\theta}}(\mathcal{G}_{\Sigma\times\mathbb{P}^{d-1}})$, *the pushforward along π of the Kac-Moody factorization algebra on $\Sigma \times \mathbb{P}^{d-1}$ of type $\phi_{\kappa,\omega}$;* level ?

(2) $\mathbb{U}_{\mathrm{vol}(\omega)\kappa}(\mathcal{G}_\Sigma)$, *the Kac-Moody factorization algebra on Σ associated to the invariant pairing $\mathrm{vol}(\omega) \cdot \kappa$.*

The twisted enveloping factorization on the right-hand side is the familiar Kac-Moody factorization alegbra on Riemann surfaces associated to a multiple of the pairing κ. The twisting $\mathrm{vol}(\omega)\kappa$ corresponds to a cocycle of the type in the previous section more precisely reference?

$$J(\mathrm{vol}(\omega)\kappa) = \mathrm{vol}(\omega) \int_{\Sigma} \kappa(\alpha, \partial\beta)$$

where $\mathrm{vol}(\omega) = \int_{\mathbb{P}^{d-1}} \omega$.

Proof. Suppose that $U \subset \Sigma$ is open. Then, the factorization algebra $\pi_* \mathbb{U}_{\phi_{\kappa,\theta}}(\mathcal{G}_{\Sigma \times \mathbb{P}^{d-1}})$ assigns to U the cochain complex

$$\left(\mathrm{Sym}\left(\Omega^{0,*}(U \times \mathbb{P}^{d-1})\right)[1][K], \bar{\partial} + K\phi_{\kappa,\omega}|_{U \times \mathbb{P}^{d-1}}\right), \tag{18}$$

where $\phi_{\kappa,\omega}|_{U\times\mathbb{P}^{d-1}}$ is the restriction of the cocycle to the open set $U \times \mathbb{P}^{d-1}$. Since projective space is Dolbeault formal its Dolbeault complex is quasi-isomorphic to its cohomology. Thus, we have address subtleties?

$$\Omega^{0,*}(U \times \mathbb{P}^{d-1}) = \Omega^{0,*}(U) \otimes \Omega^{0,*}(\mathbb{P}^{d-1}) \simeq \Omega^{0,*}(U) \otimes H^*(\mathbb{P}^{d-1}, \mathcal{O}) \cong \Omega^{0,*}(U).$$

Under this quasi-isomorphism, the restricted cocycle has the form

$$\phi_{\kappa,\omega}|_{U\times\mathbb{P}^{d-1}}(\alpha \otimes 1, \beta \otimes 1) = \int_U \kappa(\alpha, \partial\beta) \int_{\mathbb{P}^{n-1}} \omega$$

where $\alpha, \beta \in \Omega^{0,*}(U)$ and 1 denotes the unit constant function on $\mathbb{P}^{d-1}$. This is precisely the value of the local functional $\mathrm{vol}(\omega)J_\Sigma(\kappa)$ on the open set $U \subset \Sigma$. Thus, the cochain complex (18) is quasi-isomorphic to

$$\left(\mathrm{Sym}\left(\Omega^{0,*}(U)\right)[1][K], \bar{\partial} + K\mathrm{vol}(\omega)J_\Sigma(\kappa)\right). \tag{19}$$

We recognize this as the value of the Kac-Moody factorization algebra on Σ of type $\mathrm{vol}(\omega)J_\Sigma(\kappa)$. It is immediate to see that identifications above are natural with respect to maps of opens, so that the factorization structure maps are ~~the desired~~ ones intertwined by the quasi-isos. This completes the proof. $\square$

Now, suppose Σ_1, Σ_2 are Riemann surfaces and let ω_1, ω_2 be the Kähler forms. Then, we can consider the two projections

$$\begin{array}{ccc} & \Sigma_1 \times \Sigma_2 & \\ \swarrow^{\pi_1} & & \searrow^{\pi_2} \\ \Sigma_1 & & \Sigma_2 \end{array}$$

Consider the following closed $(1,1)$ form $\omega = \pi_1^*\omega_1 + \pi_2^*\omega_2 \in \Omega^{1,1}(\Sigma_1 \times \Sigma_2)$. According to the proposition above, for any symmetric invariant pairing $\kappa \in \mathrm{Sym}^2(\mathfrak{g}^*)^{\mathfrak{g}}$ this form determines a bilinear local functional

$$\phi_{\kappa,\omega}(\alpha) = \int_{\Sigma_1\times\Sigma_2} \omega \wedge \kappa(\alpha, \partial\alpha)$$

on the local Lie algebra $\mathcal{G}_{\Sigma_1\times\Sigma_2}$. A similar calculation as in the previous example implies that the pushforward factorization algebra $(\pi_i)_* \mathbb{U}_{\phi_{\kappa,\omega}}\mathcal{G}$, $i = 1, 2$, is isomorphic to the two-dimensional Kac-Moody factorization algebra on the Riemann surface Σ_i with level equal

Let $\partial : \Omega^k(B\mathfrak{g}) \to \Omega^{k+1}(B\mathfrak{g})$ be the de Rham operator for $B\mathfrak{g}$. The space of closed k-forms is

$$\Omega^k_{cl}(B\mathfrak{g}) = \left(\Omega^k(B\mathfrak{g}) \xrightarrow{\partial} \Omega^{k+1}(B\mathfrak{g})[-1] \to \cdots\right).$$

Proposition A.1. *Let $\mathfrak{g}$ be an ordinary Lie algebra. The map $\mathrm{j} : \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}[-1] \to \mathrm{C}^*_{\mathrm{loc}}(\mathcal{G}_d)$ factors through the holomorphically translation invariant deformation complex:*

what do you mean here? it both lifts to a map $\Omega^{d+1}_{cl} \to C^*_{loc}$ and the image of j is in the

(21) $$\mathrm{j} : \Omega^{d+1}_{cl}(B\mathfrak{g})[d] \to (\mathrm{C}^*_{\mathrm{loc}}(\mathcal{G}_d))^{\mathbb{C}^d_{\mathrm{hol}}}.$$

Furthermore, J defines a quasi-isomorphism into the $U(d)$-invariant subcomplex of the right-hand side. In particular, on $H^1(-)$, we obtain an isomorphism

$$H^1(\mathrm{j}) : \mathrm{Sym}^{d+1}(\mathfrak{g}^\vee)^{\mathfrak{g}} \xrightarrow{\cong} H^1\left(\mathrm{C}^*_{\mathrm{loc}}(\mathcal{G}_d)\right)^{U(d),\mathbb{C}^d_{\mathrm{hol}}}.$$

Proof. To compute the translation invariant deformation complex we will invoke Corollary 2.29 from [Wilb]. Note that the deformation complex is simply the (reduced) local cochains on the local Lie algebra $\Omega^{0,*}_{\mathbb{C}^d} \otimes \mathfrak{g}$. Thus, we see that the translation invariant local cochain complex is quasi-isomorphic to the following

(22) $$(\mathrm{C}^*_{\mathrm{loc}}(\mathcal{G}_d))^{\mathbb{C}^d_{\mathrm{hol}}} \simeq \mathbb{C} \cdot \mathrm{d}^d z \otimes^{\mathbb{L}}_{\mathbb{C}\left[\frac{\partial}{\partial z_i}\right]} \mathrm{C}^*_{\mathrm{Lie,red}}(\mathfrak{g}[[z_1, \ldots, z_d]])[d].$$

We'd like to recast the right-hand side in a more geometric way.

Note that the the algebra $\mathbb{C}\left[\frac{\partial}{\partial z_i}\right]$ is the enveloping algebra of the abelian Lie algebra $\mathbb{C}^d = \mathbb{C}\left\{\frac{\partial}{\partial z_i}\right\}$. Thus, the complex we are computing is of the form

(23) $$\mathbb{C} \cdot \mathrm{d}^d z \otimes^{\mathbb{L}}_{U(\mathbb{C}^d)} \mathrm{C}^*_{\mathrm{Lie,red}}(\mathfrak{g}[[z_1, \ldots, z_d]])[d].$$

Since $\mathbb{C} \cdot \mathrm{d}^d z$ is the trivial module, this is precisely the Chevalley-Eilenberg cochain complex computing Lie algebra homology of $\mathbb{C}^d$ with values in the module $\mathrm{C}^*_{\mathrm{Lie,red}}(\mathfrak{g}[[z_1, \ldots, z_d]])$:

(24) $$\mathrm{C}^{\mathrm{Lie}}_*\left(\mathbb{C}^d; \mathrm{C}^*_{\mathrm{Lie,red}}(\mathfrak{g}[[z_1, \ldots, z_d]])\mathrm{d}^d z\right)[d].$$

We will keep $\mathrm{d}^d z$ in the notation since below we are interested in computing the $U(d)$-invariants, and it has non-trivial weight under the action of this group.

To compute the cohomology of this complex, we will first describe the differential explicitly. There are two components to the differential. The first is the "internal" differential coming from the Lie algebra cohomology of $\mathfrak{g}[[z_1, \ldots, z_d]]$, we will write this as $\mathrm{d}_{\mathfrak{g}}$. The second comes from the $\mathbb{C}^d$-module structure on $\mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g}[[z_1, \ldots, z_n]])$ and is the differential computing the Lie algebra homology, which we denote $\mathrm{d}_{\mathbb{C}^d}$. We will employ a spectral sequence whose first term turns on the $\mathrm{d}_{\mathfrak{g}}$ differential. The next term turns on the differential $\mathrm{d}_{\mathbb{C}^d}$.

As a graded vector space, the cochain complex we are trying to compute has the form

(25) $$\mathrm{Sym}(\mathbb{C}^d[1]) \otimes \mathrm{C}^*_{\mathrm{Lie,red}}\left(\mathfrak{g}[[z_1, \ldots, z_d]]\right)) \, \mathrm{d}^d z[d].$$

to the Euler characteristic $\chi(\Sigma_j)$, where $j \neq i$. This result was alluded to in the work of Johansen in [Joh94] where he showed that there exists a copy of the Kac-Moody chiral algebra inside the operators of a twist of the $\mathcal{N} = 1$ supersymmetric multiplet (both the gauge and matter multiplets, in fact) on the Kähler manifold $\Sigma_1 \times \Sigma_2$. In ~~the~~ Section 3 we saw how the $d = 2$ Kac-Moody factorization algebra embeds inside the operators of a free holomorphic theory on a complex surface. This holomorphic theory, the $\beta\gamma$ system, is the minimal twist of the $\mathcal{N} = 1$ chiral multiplet. Thus, we obtain an enhancement of Johansen's result to a two-dimensional current algebra. in the operators ...

obtain an embedding of

Appendix A. Computing the deformation complex

Some words about goal of Appendix

There is a succinct way of expressing holomorphic translation invariance as the Lie algebra invariants of a certain *dg Lie algebra*. Denote by $\mathbb{C}^d[1]$ the abelian d-dimensional graded Lie algebra in concentrated in degree -1 by the elements $\{\overline{\eta}_i\}$. We want to consider deformations that are invariant for the action by the total *dg* Lie algebra $\mathbb{C}^d_{\text{hol}} = \mathbb{C}^{2d} \oplus \mathbb{C}^d[1]$. The differential sends $\eta_i \mapsto \frac{\partial}{\partial \overline{z}_i}$. The space of holomorphically translation invariant local functionals are denoted by $\mathcal{O}_{\text{loc}}(\mathcal{E}_V)^{\mathbb{C}^d_{\text{hol}}}$. The enveloping algebra of $\mathbb{C}^d_{\text{hol}}$ is of the form

$$U(\mathbb{C}^d_{\text{hol}}) = \mathbb{C}\left[\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z}_i}, \eta_i\right] \tag{20}$$

with differential induced from that in $\mathbb{C}^d_{\text{hol}}$. Note that this algebra is quasi-isomorphic to the algebra of constant coefficient holomorphic differential operators $\mathbb{C}[\partial/\partial z_i] \xrightarrow{\simeq} U(\mathbb{C}^d_{\text{hol}})$.

In this section we specialize the functional J to the space $Y = \mathbb{C}^d$ and use it to completely characterize the $U(d)$-invariant, holomorphically translation invariant local functionals on $\mathcal{G}_d$.

In this appendix, we identity the subcomplex of translation-invariant local cochains with purely Lie algebraic data. To state the result, we introduce some notation. The functions on a formal space $B\mathfrak{g}$ are given by the Chevalley-Eilenberg complex $\mathcal{O}(B\mathfrak{g}) = C^*_{\text{Lie}}(\mathfrak{g})$. By definition, the k-forms on $B\mathfrak{g}$ are defined by

$$\Omega^k(B\mathfrak{g}) := C^*_{\text{Lie}}(\mathfrak{g}; \text{Sym}^k \mathfrak{g}^\vee[-k])$$

where $\mathfrak{g}^\vee$ denotes the coadjoint module of $\mathfrak{g}$.

As a simple check, note that in the case $\mathfrak{g} = \mathbb{C}^n[-1]$ the above complex reduces to

$$\Omega^k(B\mathfrak{g}) = \mathbb{C}[t_1, \ldots, t_n] \otimes \wedge^k(t_1^\vee, \cdots, t_n^\vee),$$

where $t_i^\vee$ denotes the dual coordinate. Everything is in cohomological degree zero. If we identify $t_i^\vee \leftrightarrow \mathrm{d}t_i$, this is the usual definition of the algebraic de Rham forms.

The spectral sequence is induced by the increasing filtration of $\mathrm{Sym}(\mathbb{C}^d[1])$ by symmetric powers

$$F^k = \mathrm{Sym}^{\leq k}(\mathbb{C}^d[1]) \otimes C^*_{\mathrm{Lie,red}}\left(\mathfrak{g}[[z_1, \ldots, z_d]]\right) \mathrm{d}^d z[d]. \tag{26}$$

Remark A.2. In the examples we are most interested in (namely $\mathfrak{g} = \mathbb{C}^n[-1]$ and $\mathfrak{g} = \mathfrak{g}_{X_{\bar{\partial}}}$) we can understand the spectral sequence we are using as a version of the Hodge-to-de Rham spectral sequence.

Have you used this notation elsewhere?

As above, we write the generators of $\mathbb{C}^d$ by $\frac{\partial}{\partial z_i}$. Also, note that the reduced Chevalley-Eilenberg complex has the form

$$C^*_{\mathrm{Lie,red}}(\mathfrak{g}[[z_1, \ldots, z_n]]) = \left(\mathrm{Sym}^{\geq 1}\left(\mathfrak{g}^\vee[z_1^\vee, \ldots, z_d^\vee][-1]\right), \mathrm{d}_{\mathfrak{g}}\right), \tag{27}$$

where $z_i^\vee$ is the dual variable to z_i.

Recall, we are only interested in the $U(d)$-invariant subcomplex of this deformation complex. Sitting inside of $U(d)$ we have $S^1 \subset U(d)$ as multiples of the identity. This induces an overall weight grading to the complex. The group $U(d)$ acts in the standard way on $\mathbb{C}^d$. Thus, z_i has weight $(+1)$ and both $z_i^\vee$ and $\frac{\partial}{\partial z_i}$ have S^1-weight (-1). Moreover, the volume element $\mathrm{d}^d z$ has S^1 weight d. It follows that in order to have total S^1-weight 0, ~~that~~ the total number of $\frac{\partial}{\partial z_i}$ and $z_i^\vee$ must add up to d. Thus, as a graded vector space the invariant subcomplex has the following decomposition

$$\bigoplus_k \mathrm{Sym}^k(\mathbb{C}^d[1]) \otimes \left(\bigoplus_{\substack{i \leq d-k \\ i \geq 0}} \mathrm{Sym}^i\left(\mathfrak{g}^\vee[z_1^\vee, \ldots, z_d^\vee][-1]\right)\right) \mathrm{d}^d z[d]. \tag{28}$$

then?

Not all such guys are inv't, right?

It follows from Schur-Weyl that the space of $U(d)$ invariants of the dth tensor power of the fundamental representation $\mathbb{C}^d$ is one-dimensional, spanned by the top exterior power. Thus, when we pass to the $U(d)$-invariants, only the unique totally antisymmetric tensor involving $\frac{\partial}{\partial z_i}$ and $z_i^\vee$ survives. Thus, for each k, we have

$$\left(\mathrm{Sym}^k(\mathbb{C}^d[1]) \otimes \left(\bigoplus_{i \leq d-k} \mathrm{Sym}^i\left(\mathfrak{g}^\vee[z_1^\vee, \ldots, z_d^\vee][-1]\right)\right) \mathrm{d}^d z\right)^{U(d)?} \cong \tag{29}$$

$$\wedge^k \left(\frac{\partial}{\partial z_i}\right) \wedge \wedge^{d-k}(z_i^\vee)\, C^*_{\mathrm{Lie}}\left(\mathfrak{g}, \mathrm{Sym}^{d-k}(\mathfrak{g}^\vee)\right) \mathrm{d}^d z. \tag{30}$$

What abt $\wedge^k\left(\frac{\partial}{\partial z_i}\right) \wedge \mathrm{Sym}^{d-k}(z_i^\vee) \otimes C^*_{\mathrm{Lie}}(\mathfrak{g}, \mathfrak{g}^\vee)\, d^d z$?

Here, $\wedge^k\left(\frac{\partial}{\partial z_i}\right) \wedge \wedge^{d-k}(z_i^\vee)$ is just a copy of the determinant $U(d)$-representation, but we'd like to keep track of the appearances of the partial derivatives and $z_i^\vee$. Note that for degree reasons, we must have $k \leq d$. When $k = 0$ this complex is the (shifted) space of functions modulo constants on the formal moduli space $B\mathfrak{g}$, $\mathcal{O}_{red}(B\mathfrak{g})[d]$. When $k \geq 1$ this is the (shifted) space of k-forms on the formal moduli space $B\mathfrak{g}$, which we write as $\Omega^k(B\mathfrak{g})[d+k]$. Thus, we see that before turning on the differential on the next page, our complex looks like

$$\begin{array}{cccc} \underline{-2d} & \cdots & \underline{-d-1} & \underline{-d} \\ \\ \mathcal{O}_{red}(B\mathfrak{g}) & \cdots & \Omega^{d-1}(B\mathfrak{g}) & \Omega^{d}(B\mathfrak{g}). \end{array}$$

We've omitted the extra factors for simplicity.

We now turn on the differential $\mathrm{d}_{\mathbb{C}^d}$ coming from the Lie algebra homology of $\mathbb{C}^d = \mathbb{C}\left\{\frac{\partial}{\partial z_i}\right\}$ with values in the above module. Since this Lie algebra is abelian the differential is completely determined by how the operators $\frac{\partial}{\partial z_i}$ act. We can understand this action explicitly as follows. Note that $\frac{\partial}{\partial z_i} z_j = \delta_{ij}$, ~~thus~~ so we may as well think of $z_i^\vee$ as the element $\frac{\partial}{\partial z_i}$. Consider the subspace corresponding to $k = d$ in Equation (29):

(31)
$$\frac{\partial}{\partial z_1} \cdots \frac{\partial}{\partial z_d} \mathrm{C}^*_{\mathrm{Lie,red}}(\mathfrak{g}) \mathrm{d}^d z.$$

Then, if $x \in \mathfrak{g}^\vee[-1] \subset \mathrm{C}^*_{\mathrm{Lie,red}}(\mathfrak{g})$ we observe that

does f appear here?

(32)
$$\mathrm{d}_{\mathbb{C}^d}\left(\frac{\partial}{\partial z_1} \cdots \frac{\partial}{\partial z_d} \otimes f \otimes \mathrm{d}^d z\right) = \det(\partial_i, z_j^\vee) \otimes 1 \otimes x \otimes \mathrm{d}^d z \in \wedge^{d-1}\left(\frac{\partial}{\partial z_i}\right) \wedge \mathbb{C}\{z_i^\vee\} \mathrm{C}^*_{\mathrm{Lie}}\left(\mathfrak{g}, \mathfrak{g}^\vee\right) \mathrm{d}^d z.$$

This follows from the fact that the action of $\frac{\partial}{\partial z_i}$ on $x = x \otimes 1 \in \mathfrak{g}^\vee \otimes \mathbb{C}[z_i^\vee]$ is given by

(33)
$$\frac{\partial}{\partial z_i} \cdot (x \otimes 1) = 1 \otimes x \otimes z_i^\vee \in \mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g}, \mathfrak{g}^\vee) z_i^\vee.$$

By the Leibniz rule we can extend this to get the formula for general elements $f \in \mathrm{C}^*_{\mathrm{Lie,red}}(\mathfrak{g})$. We find that getting rid of all the factors of z_i we recover precisely the de Rham differential

(34)
$$\begin{tikzcd} \mathrm{C}^*_{\mathrm{Lie,red}}(\mathfrak{g})[2d] \arrow[r, "\mathrm{d}_{\mathbb{C}^d}"] \arrow[d, equal] & \mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g}, \mathfrak{g}^\vee)[2d-1] \arrow[d, equal] \\ \mathcal{O}_{red}(B\mathfrak{g}) \arrow[r, "\overset{?\,d}{\partial}"] & \Omega^1(B\mathfrak{g}). \end{tikzcd}$$

A similar argument shows that $\mathrm{d}_{\mathbb{C}^d}$ agrees with the de Rham differential on each $\Omega^k(B\mathfrak{g})$.

We conclude that the E_2 page of this spectral sequence is quasi-isomorphic to the following truncated de Rham complex.

$$\begin{array}{ccccc} \underline{-2d} & \underline{-2d+1} & \cdots & \underline{-d-1} & \underline{-d} \end{array}$$

$$\mathcal{O}_{red}(B\mathfrak{g}) \xrightarrow{\partial} \Omega^1(B\mathfrak{g}) \longrightarrow \cdots \longrightarrow \Omega^{d-1}(B\mathfrak{g}) \xrightarrow{\partial} \Omega^d(B\mathfrak{g}).$$

This complex is quasi-isomorphic, through the de Rham differential, to $\Omega^{d+1}_{cl}[d]$. This completes the proof. □

APPENDIX B. NORMALIZING THE CHARGE ANOMALY

In this section we conclude the proof of Proposition 3.9 by an explicit calculation of the Feynman diagrams controlling the charge anomaly for the $\beta\gamma$ system on $\mathbb{C}^d$. We have already identified the algebraic piece of the anomaly with the $(d+1)$st component of the Chern character of the representation. The only thing left to compute is the analytic factor. We can therefore assume that we have an abelian Lie algebra, and simply compute the weight of the wheel Γ with $(d+1)$-vertices where the external edges are labeled by elements $\alpha \in \Omega_c^{0,*}(\mathbb{C}^d)$. After choosing a numeration of the internal edges $e = 0, \ldots d$, we can label the edges $e = 0, \ldots, d-1$ by the analytic propagator by $P^{an}_{\epsilon<L}$ and the label the edge $e = d$ by the analytic heat kernel K^{an}_ϵ. We recall the precise form of these kernels in the proof below. The vertices are labeled by the trivalent functional $I^{an}(\alpha, \beta, \gamma) = \int \alpha \wedge \beta \wedge \gamma$ (there is no Lie bracket since the algebra is abelian). Denote the resulting weight, which is a functional on the space $\Omega_c^{0,*}(\mathbb{C}^d)$, by

$$W^{an}_\Gamma(P_{\epsilon<L}, K_\epsilon, I^{an}).$$

The main computation left to do is the $\epsilon \to 0, L \to 0$ limit of this weight.

Lemma B.1. *As a functional on the abelian dg Lie algebra $\Omega_c^{0,*}(\mathbb{C}^d)$, one has*

$$\lim_{L\to 0}\lim_{\epsilon \to 0} W^{an}_\Gamma(P_{\epsilon<L}, K_\epsilon, I^{an})(\alpha^{(0)}, \ldots, \alpha^{(d)}) = \frac{1}{(4\pi)^d}\frac{1}{(d+1)!}\int \alpha^{(0)}\partial\alpha^{(1)}\cdots\partial\alpha^{(d)}.$$

Should introduce this notation (a) before statement.

Proof. We enumerate the vertices by integers $a = 0, \ldots, d$. Label the coordinate at the ith vertex by $z^{(a)} = (z_1^{(a)}, \ldots, z_d^{(a)})$. The incoming edges of the wheel will be denoted by homogeneous Dolbeault forms

$$\alpha^{(a)} = \sum_J A_J^{(a)} \mathrm{d}\bar{z}_J^{(a)} \in \Omega_c^{0,*}(\mathbb{C}^d).$$

where the sum is over the multiindex $J = (j_1, \ldots, j_k)$ where $j_a = 1, \ldots, d$ and $(0,k)$ is the homogenous Dolbeault form type. For instance, if α is a $(0,2)$ form we would write

$$\alpha = \sum_{j_1<j_2} A_{(j_1,j_2)}\mathrm{d}\bar{z}_{j_1}\mathrm{d}\bar{z}_{j_2}.$$

Denote by W^{an}_L weight $\epsilon \to 0$ limit of the analytic weight of the wheel with $(d+1)$ vertices. The $L \to 0$ limit of W^{an}_L is the local functional representing the one-loop anomaly Θ.

The weight has the form

$$W^{an}_L(\alpha^{(0)}, \ldots, \alpha^{(d)}) = \lim_{\epsilon\to 0}\int_{\mathbb{C}^{d(d+1)}}\left(\alpha^{(0)}(z^{(0)})\cdots\alpha^{(d)}(z^{(d)})\right)K_\epsilon(z^{(0)}, z^{(d)})\prod_{a=1}^{d} P_{\epsilon,L}(z^{(a-1)}, z^{(a)}).$$

We introduce coordinates

$$w^{(0)} = z^{(0)}$$
$$w^{(a)} = z^{(a)} - z^{(a-1)} \quad 1 \le a \le d.$$

The heat kernel and propagator part of the integral is of the form

$$
\begin{aligned}
K_\epsilon(w^{(0)}, w^{(d)}) \prod_{a=1}^{d} P_{\epsilon,L}(w^{(a-1)}, w^{(a)}) &= \frac{1}{(4\pi\epsilon)^d} \int_{t_1,\ldots,t_d=\epsilon}^{L} \frac{\mathrm{d}t_1 \cdots \mathrm{d}t_d}{(4\pi t_1)^d \cdots (4\pi t_d)^d} \frac{1}{t_1 \cdots t_d} \\
&\times \mathrm{d}^d w^{(0)} \prod_{i=1}^{d} (\mathrm{d}\overline{w}_i^{(1)} + \cdots + \mathrm{d}\overline{w}_i^{(d)}) \prod_{a=1}^{d} \mathrm{d}^d w^{(a)} \left(\sum_{i=1}^{d} \overline{w}_i^{(a)} \prod_{j \neq i} \mathrm{d}\overline{w}_j^{(a)} \right) \\
&\times e^{-\sum_{a,b=1}^{d} M_{ab} w^{(a)} \cdot \overline{w}^{(b)}}.
\end{aligned}
$$

L

$= K_\epsilon(z^{(0)} - z^{(d)} + z^{(d-1)})$

$\neq K_\epsilon(z^{(0)} - z^{(d)})$

Here, M_{ab} is the $d \times d$ square matrix satisfying

$$
\sum_{a,b=1}^{d} M_{ab} w^{(a)} \cdot \overline{w}^{(b)} = |\sum_{a=1}^{d} w^{(a)}|^2/\epsilon + \sum_{a=1}^{d} |w^{(a)}|^2/t_a.
$$

supposed to be diff?

Note that

$$
\prod_{i=1}^{d} (\mathrm{d}\overline{w}_i^{(1)} + \cdots + \mathrm{d}\overline{w}_i^{(d)}) \prod_{a=1}^{d} \left(\sum_{i=1}^{d} \overline{w}_i^{(a)} \prod_{j \neq i} \mathrm{d}\overline{w}_j^{(a)} \right) = \left(\sum_{i_1,\ldots i_d} \epsilon_{i_1 \cdots i_d} \prod_{a=1}^{d} \overline{w}_{i_a}^{(a)} \right) \prod_{a=1}^{d} \mathrm{d}^d \overline{w}^{(a)}.
$$

In particular, only the $\mathrm{d}w_i^{(0)}$ components of $\alpha^{(0)} \cdots \alpha^{(d)}$ can contribute to the weight.

For some compactly supported function Φ we can write the weight as

$$
\begin{aligned}
W(\alpha^{(0)}, \ldots, \alpha^{(d)}) &= \lim_{\epsilon \to 0} \int_{\mathbb{C}^{d(d+1)}} \left(\prod_{a=0}^{d} \mathrm{d}^d w^{(a)} \mathrm{d}^d \overline{w}^{(a)} \right) \Phi \\
&\times \frac{1}{(4\pi\epsilon)^d} \int_{t_1,\ldots,t_d=\epsilon}^{L} \frac{\mathrm{d}t_1 \cdots \mathrm{d}t_d}{(4\pi t_1)^d \cdots (4\pi t_d)^d} \frac{1}{t_1 \cdots t_d} \sum_{i_1,\ldots,i_d} \epsilon_{i_1 \cdots i_d} \overline{w}_{i_1}^{(1)} \cdots \overline{w}_{i_d}^{(d)} e^{-\sum_{a,b=1}^{d} M_{ab} w^{(a)} \cdot \overline{w}^{(b)}}
\end{aligned}
$$

Applying Wick's lemma in the variables $w^{(1)}, \ldots, w^{(d)}$, together with some elementary analytic bounds, we find that the weight above becomes to the following integral over $\mathbb{C}^d$

$$
f(L) \int_{w^{(0)} \in \mathbb{C}^d} \mathrm{d}^d w^{(0)} \mathrm{d}^d \overline{w}^{(0)} \sum_{i_1,\ldots,i_d} \epsilon_{i_1 \cdots i_d} \left(\frac{\partial}{\partial w_{i_1}^{(1)}} \cdots \frac{\partial}{\partial w_{i_d}^{(d)}} \Phi \right) \Bigg|_{w^{(1)} = \cdots = w^{(d)} = 0}
$$

where

$$
f(L) = \frac{1}{(4\pi)^2} \lim_{\epsilon \to 0} \int_{t_1,\ldots,t_d=\epsilon}^{L} \frac{\epsilon}{(\epsilon + t_1 + \cdots + t_d)^{d+1}} \mathrm{d}^d t.
$$

In fact, $f(L)$ is independent of L and is equal to $\frac{1}{(d+1)!}$ after direct computation. Finally, plugging in the forms $\alpha^{(0)}, \ldots, \alpha^{(d)}$, we observe that the integral over $w^{(0)} \in \mathbb{C}^d$ simplifies to

$$
\frac{1}{(4\pi)^2} \frac{1}{(d+1)!} \int_{\mathbb{C}^d} \alpha^{(0)} \partial\alpha^{(1)} \cdots \partial\alpha^{(d)}
$$

as desired. $\square$