HIGHER KAC-MOODY

BW: Everything here is essentially in Kapranov, Hennion, ...

1. The local Lie algebra

Let X be a fixed complex d-fold and let $\mathfrak{g}$ be a Lie algebra. (We assume it is an ordinary Lie algebra, but slight modifications will allow one to handle dg Lie or L_∞ algebras.) For each open set $U \subset X$ define

$$\mathfrak{g}^X(U) = \Omega^{0,*}(U) \otimes \mathfrak{g}.$$

The $\bar{\partial}$ differential for U extended naturally to $\mathfrak{g}^X(U)$ by $\bar{\partial} \otimes 1$. Moreover, $\mathfrak{g}^X(U)$ has a natural Lie bracket defined by the rule

$$[\omega \otimes X, \omega' \otimes X'] = \omega \wedge \omega' \otimes [X, X']_{\mathfrak{g}}$$

where $[-,-]_{\mathfrak{g}}$ is the Lie bracket for $\mathfrak{g}$. Thus, $\mathfrak{g}^X(U)$ has the structure of a dg-Lie algebra.

Lemma 1.1. *The assignment $\mathfrak{g}^X : U \mapsto \mathfrak{g}^X(U)$ defines a local Lie algebra.*

1.1. **dg vs L_∞.** Suppose V is a dg vector space. Then, the symmetric algebra

$$\mathrm{Sym}(V) := \prod_k \mathrm{Sym}^k(V)$$

has the natural structure of a dg cocommutative coalgebra.

Definition 1.2. An L_∞ *algebra* is a dg vector space V together with a coderivation

$$D : \mathrm{Sym}(V) \to \mathrm{Sym}(V).$$

A *morphism* of L_∞ algebras $f : (V, D) \to (V', D')$ is a morphism of dg cocommutative coalgebras

$$f : (\mathrm{Sym}(V), D) \to \left(\mathrm{Sym}(V'), D'\right).$$

Denote the category of L_∞ algebras by $L_\infty\mathrm{Alg}$.

We may a remark about dg Lie algebras and their close relatives, L_∞ algebras.

Theorem 1.3. *BW: Kriz and May? Every L_∞ algebra (V, D) is quasi-isomorphic (in the category $L_\infty\mathrm{Alg}$) to a dg Lie algebra.*

By an L_∞ algebra model for a dg Lie algebra $\mathfrak{g}$, we mean an L_∞ algebra (L, D) together with a quasi-isomorphism $(L, D) \simeq \mathfrak{g}$.

Suppose $\mathfrak{g}$ is a dg Lie algebra. Let $\theta \in C^*_{\mathrm{Lie}}(\mathfrak{g})$ be a cocycle of degree 2, so its cohomology class is an element $[\theta] \in H^2_{\mathrm{Lie}}(\mathfrak{g})$. By BW: ref, we know that θ determines a central extension in the category of dg Lie algebras:

$$0 \to \mathbb{C} \cdot K \to \widehat{\mathfrak{g}} \to \mathfrak{g} \to 0$$

that only depends, up to isomorphism, on the cohomology class of θ.

The explicit dg Lie algebra structure on $\widehat{\mathfrak{g}}$ may be tricky to describe. However, if we are willing to work in the category of L_∞ algebras, there is an explicit model for $\mathfrak{g}$ as an L_∞ algebra. The underlying dg vector space for the L_∞ algebra is the same as that of the dg Lie algebra, $\widehat{\mathfrak{g}} \oplus \mathbb{C} \cdot K$. To equip this with an L_∞ structure we need to provide a coderivation $D = D_1 + D_2 + \cdots$ for the cocommutative coalgebra $\mathrm{Sym}(\mathfrak{g} \oplus \mathbb{C} \cdot K) = \prod_k \mathrm{Sym}^k(\mathfrak{g} \oplus \mathbb{C} \cdot K)$. Indeed, we define

$$\begin{aligned} D_1(X_1) &= \mathrm{d}_{\mathfrak{g}}(X_1) + \theta(X_1) \\ D_2(X_1, X_2) &= [X_1, X_2]_{\mathfrak{g}} + \theta(X_1, X_2) \\ D_k(X_1, \ldots, X_k) &= \theta(X_1, \ldots, X_k) \ , \ \text{for } k \geq 3. \end{aligned}$$

One immediately checks that $(\mathfrak{g} \oplus \mathbb{C}, D)$ is an L_∞ model for $\widehat{\mathfrak{g}}$.

Example 1.4. As an example, consider the following L_∞ model for the dg Lie algebra $\widehat{\mathfrak{g}}_{d,\theta}$. As a dg vector space $\widehat{\mathfrak{g}}_{d,\theta}$ is of the form $A_d \otimes \mathfrak{g} \oplus \mathbb{C} \cdot K$. The only nonzero components of the coderivation determining the L_∞ structure are D_1,D_2, and D_{d+1} and they are determined by $D_1(aX) = (\bar{\partial} a)X$, $D_2(aX, bY) = (a \wedge b)[X,Y]_{\mathfrak{g}}$, and

$$D_{d+1}(a_0 X_0, \ldots, a_d X_d) = \operatorname*{Res}_{z=0} (a_0 \wedge \partial a_1 \wedge \cdots \wedge \partial a_d)\, \theta(X_0, \ldots, X_d) \cdot K.$$

2. Local cocycles from polynomials

Being a local Lie algebra we can consider its local Chevalley-Eilenberg complex. It defined as

$$\mathrm{C}^*_{\mathrm{loc}}(\mathfrak{g}^X) =$$

Recall, a local k-cocycle of a local Lie algebra determines a $(k-2)$-shifted central extension, by the constant sheaf $\underline{\mathbb{C}}$. We are interested in (-1)-shifted central extensions, and hence, local 1-cocycles. For $\mathfrak{g}^X$ we can describe such a family of 1-cocycles.

Let P be an invariant polynomial of $\mathfrak{g}$ of homogenous degree $d+1$. That is, $P \in \mathrm{Sym}^{d+1}(\mathfrak{g}^\vee)^{\mathfrak{g}}$. We can extend P to a functional on $\Omega^{0,*}(X) \otimes \mathfrak{g}$ by the rule

$$\begin{array}{cccc} P^X : & \mathrm{Sym}^{d+1}(\Omega^{0,*}(X) \otimes \mathfrak{g}) & \to & \mathbb{C} \\ & (\omega_1 \otimes X_1, \ldots, \omega_{d+1} \otimes X_{d+1}) & \mapsto & (\omega_1 \wedge \cdots \wedge \omega_{d+1}) P(X_1, \ldots, X_{d+1}) \end{array}$$

Proposition 2.1. *The assignment*

$$J : \mathrm{Sym}^{d+1}(\mathfrak{g}^\vee)^{\mathfrak{g}}[-1] \to \mathrm{C}^*_{\mathrm{loc}}(\mathfrak{g}^X)$$

sending and invariant polynomial P, of homogeneous degree $d+1$, to the local functional

$$(\alpha_1, \ldots, \alpha_{d+1}) \mapsto \int P^X (\alpha_1, \partial\alpha_2, \ldots, \partial\alpha_{d+1})$$

is a cochain map. Moreover, it is injective at the level of cohomology.

Remark 2.2. We extend the operator $\partial : \Omega^{k,l} \to \Omega^{k+1,l}$ to $\Omega^{0,*}(X) \otimes \mathfrak{g} \to \Omega^{1,*}(X) \otimes \mathfrak{g}$ by the operator $\partial \otimes 1$.

3. The factorization algebra

Given any cocycle $\theta \in \mathrm{C}^*_{\mathrm{loc}}(\mathfrak{g}^X)$ of degree one we define a factorization algebra on X.

Definition 3.1. Let θ be a local cocycle of $\mathfrak{g}^X$ of cohomological degree one. Define $\mathrm{KM}^X_{\mathfrak{g},\theta}$ to be the factorization algebra on X that assigns to an open set $U \subset X$ the cochain complex $\mathrm{C}^{\mathrm{Lie},\theta}_* \left(\mathfrak{g}^X(U)\right)$. In other words, $\mathrm{KM}^X_{\mathfrak{g},\theta}$ is the twisted factorization envelope $\mathrm{U}^{\mathrm{fact}}_\theta(\mathfrak{g}^X)$.

Explicitly, on an open set $U \subset X$, the cochain complex $\mathrm{KM}^X_{\mathfrak{g},\theta}(U)$ has as its underlying graded vector space

$$\mathrm{Sym}\left(\mathfrak{g}^X_c(U)[1] \oplus \mathbb{C} \cdot K\right)$$

and the differential is given by $\bar{\partial} + \mathrm{d}_{\mathfrak{g}} + \theta$ where $\mathrm{d}_{\mathfrak{g}}$ is the extension of the Chevalley-Eilenberg differential for $\mathfrak{g}$ to the Dolbeault complex, and where θ is extended to the full symmetric algebra by the rule that it is a (graded) derivation.

Example 3.2. As an example, using the map J of Proposition 2.1, we can construct a factorization algebra on X for any invariant polynomial $P \in \mathrm{Sym}^{d+1}(\mathfrak{g}^\vee)^{\mathfrak{g}}$. Since j is injective, we obtain a unique factorization algebra for every such polynomial, hence it makes sense to denote $\mathrm{KM}^X_{\mathfrak{g},P} := \mathrm{KM}^X_{\mathfrak{g},j(P)}$.

4. Higher loop algebras

In this section we restrict to the complex manifold $X = \mathbb{C}^d$. We will extract from the Kac-Moody factorization algebra on $\mathbb{C}^d$ an associative algebra...

4.1. **A model for the annulus.** Recall, the polydisk centered at $z \in \mathbb{C}^d$ of radius r was defined to be the following open subset

$$\mathrm{PD}^d_r(z) = \{(w_1, \ldots, w_d) \in \mathbb{C}^d \mid |w_i - z_i| < r\} \subset \mathbb{C}^d.$$

For $z \in \mathbb{C}^d$, and $0 < r < R < \infty$ define the following open subset

$$A^d_{r<R}(z) = \mathrm{PD}^d_R(z) \setminus \overline{\mathrm{PD}^d_r(z)}$$

We think of this as a model for the d-dimensional annulus. When $z = 0$ we simply denote this by $A^d_{r<R}$.

We will need a convenient model for the Dolbeault complex $\Omega^{0,*}(A^d_{r<R})$ of this d-dimensional annulus. For $d = 1$ the $\overline{\partial}$-cohomology of $A^d_{r<R}$ is concentrated in degree zero (in fact, any open subset of $\mathbb{C}$ is Stein).

For $d > 1$, the $\overline{\partial}$-cohomology of $A^d_{r<R}$ is concentrated in degrees 0 and $d-1$. In degree zero, of course, $H^0_{\overline{\partial}}(A^d_{r<R})$ is identified with holomorphic functions on $A^d_{r<R}$. In degree $d-1$ ...

There is a natural action of the d-dimensional torus $(S^1)^d = S^1 \times \cdots \times$ on $A_{r<R}$ given by rotating each coordinate:

$$(\lambda_1, \ldots, \lambda_d) \cdot (z_1, \ldots, z_d) = (\lambda_1 z_1, \ldots, \lambda_d z_d).$$

We obtained an induced action of S^1 via the diagonal embedding $S^1 \to S^1 \times \cdots \times S^1$. This induces an action on the Dolbeault complex of $A^d_{r<R}$. Let

$$\left(\Omega^{0,*}(A^d_{r<R})\right)^{(k)} \subset \Omega^{0,*}(A^d_{r<R})$$

denote the weight k subspace.

BW: Recall Jouanolou model, denoted A_d.

There is a map of commutative dg algebras

$$j : A_d \to \Omega^{0,*}(\mathbb{C}^d \setminus 0).$$

If $a \in A_d$ we will denote the resulting element in the Dolbeault complex of $\mathbb{C}^d \setminus 0$ by $a(z) := j(a)$.

4.2. First we will consider the higher Kac-Moody factorization algebra on $\mathbb{C}^d$ "at level zero". That is, the factorization algebra $\mathrm{KM}^{\mathbb{C}^d}_{\mathfrak{g},0}$. There is a natural way to obtain an associative algebra out of this factorization algebra. Namely, we look at its restriction to $\mathbb{C}^d \setminus 0$ and consider the radial projection map

$$\mathrm{rad} : \mathbb{C}^d \setminus 0 \to \mathbb{R}_{>0}$$

sending $z = (z_1, \ldots, z_d)$ to $|z| = \sqrt{|z_1|^2 + \cdots + |z_d|^2}$. We obtain a factorization algebra on $\mathbb{R}_{>0}$ via pushing forward the higher Kac-Moody factorization algebra $\mathrm{rad}_* \mathrm{KM}^{\mathbb{C}^d \setminus 0}_{\mathfrak{g},0}$. Explicitly, to an open set $I \subset \mathbb{R}_{>0}$ it assigns the dg vector space

$$C^{\mathrm{Lie}}_* \left(\Omega^{0,*}_c(\mathrm{rad}^{-1}(I)) \otimes \mathfrak{g}\right).$$

When $I = (-\epsilon, \epsilon) \subset \mathbb{R}_{>0}$ is an interval, the subset $\mathrm{rad}^{-1}(I) \subset \mathbb{C}^d$ is an ϵ-neighborhood of the $2d-1$ sphere. It is homeomorphic to $S^{2d-1} \times (-\epsilon, \epsilon)$.

We wish to compare this one-dimensional factorization algebra to the higher loop Lie algebra we described above, $A_d \otimes \mathfrak{g}$. Consider the associative algebra given by the universal enveloping algebra of this dg Lie algebra $U(A_d \otimes \mathfrak{g})$. BW: define We construct a map of factorization algebras on $\mathbb{R}_{>0}$ from the factorization algebra corresponding to the associative algebra $U(A_d \otimes \mathfrak{g})$ to $\mathrm{rad}_* \mathrm{KM}^{\mathbb{C}^d \setminus 0}_{\mathfrak{g},0}$.

Let $I \subset \mathbb{R}_{>0}$ be an open subset. There is the natural map $\mathrm{rad}^* : \Omega^*_c(I) \to \Omega^*_c(\mathrm{rad}^{-1}(I))$ sending a form φ to the pull-back $\mathrm{rad}^* \varphi$. We can post-compose this with the natural projection $\mathrm{pr}_{\Omega^{0,*}} : \Omega^*_c \to \Omega^{0,*}_c$ to obtain a map of commutative algebras $\mathrm{pr}_{\Omega^{0,*}} \circ \mathrm{rad}^* : \Omega^*_c(I) \to \Omega^{0,*}_c(\mathrm{rad}^{-1}(I))$. Now, consider the map

$$\begin{array}{rccc} \Phi = (\mathrm{pr}_{\Omega^{0,*}} \circ \mathrm{rad}^*) \otimes j : & \Omega^*_c(I) \otimes A_d & \to & \Omega^{0,*}_c\left((\rho^{-1}(I)\right) \\ & \varphi \otimes a & \mapsto & ((\mathrm{pr}_{\Omega^{0,*}} \circ \mathrm{rad}^*)\varphi) \wedge j(a) \end{array}$$

Since this is a map of commutative dg algebras it defines a map of dg Lie algebras

$$\Phi \otimes \mathrm{id}_{\mathfrak{g}} : (\Omega^*_c(I) \otimes A_d) \otimes \mathfrak{g} = \Omega^*_c(I) \otimes (A_d \otimes \mathfrak{g}) \to \Omega^{0,*}(\mathrm{rad}^{-1}(I)) \otimes \mathfrak{g}$$

which maps $\varphi \otimes a \otimes X \mapsto \Phi(\varphi \otimes a) \otimes X$. BW: Explicitly... . We will drop the $\mathrm{id}_{\mathfrak{g}}$ from the notation and will denote this map simply by Φ. Clearly, this map is compatible with inclusions and hence extends to a map of cosheaves of dg Lie algebras

$$\Phi : \Omega^*_{\mathbb{R}_{>0},c} \otimes (A_d \otimes \mathfrak{g}) \to \mathrm{rad}_*(\Omega^{0,*}_{\mathbb{C}^d \setminus 0,c} \otimes \mathfrak{g}).$$

BW: factorization Lie algebras

Proposition 4.1. *The map Φ is a map of factorization Lie algebras on $\mathbb{R}_{>0}$. Hence, it induces a map of factorization algebras*

$$C_*(\Phi) : (U(A_d \otimes \mathfrak{g}))^{fact} \to \mathrm{rad}_* \left(\mathrm{KM}^{\mathbb{C}^d}_{\mathfrak{g},0}|_{\mathbb{C}^d \setminus 0} \right).$$

Theorem 4.2. *There is a map of factorization algebras on $\mathbb{R}_{>0}$*

$$(U\widehat{\mathfrak{g}}_{d,\theta})^{fact} \to \mathrm{rad}_* \left(\mathrm{KM}^{\mathbb{C}^d}_{\mathfrak{g},\theta}|_{\mathbb{C}^d \setminus 0} \right).$$

Moreover, its image is quasi-isomorphic to the subfactorization algebra consisting of the S^1-eigenspaces

$$\mathcal{A}_{d,\mathfrak{g},\theta} := \bigoplus_{k \in \mathbb{Z}} \mathrm{rad}_* \left(\mathrm{KM}^{\mathbb{C}^d}_{\mathfrak{g},\theta}|_{\mathbb{C}^d \setminus 0} \right)^{(k)} \subset \mathrm{rad}_* \left(\mathrm{KM}^{\mathbb{C}^d}_{\mathfrak{g},\theta}|_{\mathbb{C}^d \setminus 0} \right).$$

We will write down a sequence of maps of factorization Lie algebras

$$\begin{tikzcd} & \mathcal{G}_1 & & \mathcal{G}_2 \\ \mathcal{G}_0 \arrow[ur, "\simeq", "\Phi_1"'] & & \mathcal{G}_1' \arrow[ul] \arrow[ur, "\simeq", "\Phi_2"'] & \end{tikzcd}.$$

First, we introduce the factorization Lie algebra $\mathcal{G}_0 := \Omega^*_{\mathbb{R},c} \otimes \widehat{\mathfrak{g}}_{d,\theta}$. To an open set $I \subset \mathbb{R}$, it assigns the dg Lie algebra $\mathcal{G}_0(I) = \Omega^*_c(I) \otimes \widehat{\mathfrak{g}}_{d,\theta}$. The differential and Lie bracket are determined by the fact that we are tensoring a commutative dg algebra with a dg Lie algebra. This factorization Lie algebra is a central extension of the factorization Lie algebra $\Omega^*_{\mathbb{R},c} \otimes (A_d \otimes \mathfrak{g})$ by the trivial module $\Omega^*_c \oplus \mathbb{C} \cdot K$. The cocycle determining the central extension is given by

$$\theta_0(\varphi_0\alpha_0, \ldots, \varphi_d\alpha_d) = (\varphi_0 \wedge \cdots \wedge \varphi_d)\theta_{A_d}(\alpha_1, \ldots, \alpha_d).$$

A slight variant of Proposition 3.4.0.1 in [?] shows that there is a quasi-isomorphism of factorization algebras on $\mathbb{R}$

$$(U\widehat{\mathfrak{g}}_{d,\theta})^{fact} \xrightarrow{\simeq} C^{\mathrm{Lie}}_*(\mathcal{G}_0).$$

We define the factorization dg Lie algebra $\mathcal{G}_1$ on $\mathbb{R}$. First, consider the factorization dg Lie algebra on $\mathbb{R}$ given by $\Omega^*_{\mathbb{R},c} \otimes (A_d \otimes \mathfrak{g})$. This assigns to an open set $I \subset \mathbb{R}$ the dg Lie algebra $\Omega^*_c(I) \otimes (A_d \otimes \mathfrak{g})$. Equivalently, this is the compactly supported sections of the local Lie algebra $\Omega^*_{\mathbb{R}} \otimes (A_d \otimes \mathfrak{g})$.

The factorization dg Lie algebra $\mathcal{G}_1$ is a central extension of $\Omega^*_{\mathbb{R},c} \otimes (A_d \otimes \mathfrak{g})$

$$0 \to \mathbb{C} \cdot K[-1] \to \mathcal{G}_1 \to \Omega^*_{\mathbb{R},c} \otimes (A_d \otimes \mathfrak{g}) \to 0$$

determined by the following cocycle. For an open interval I write $\varphi_i \in \Omega^*_c(I)$, $\alpha_i \in A_d \otimes \mathfrak{g}$. The cocycle is defined by

$$\theta_1(\varphi_0\alpha_0, \ldots, \varphi_d\alpha_d) = \left(\int_I \varphi_0 \wedge \cdots \varphi_d \right) \theta_{A_d}(\alpha_0, \ldots, \alpha_d) \tag{1}$$

Recall, if we write $\alpha_i = a_i X_i$ for $a_i \in A_d$, $X_i \in \mathfrak{g}$ the cocycle θ_{A_d} is given by

$$\theta_{A_d}(a_0X_0, \ldots, \ldots, a_dX_d) = \operatorname*{Res}_{z=0} (a_0 \wedge \partial a_1 \wedge \cdots \wedge \partial a_d)\, \theta(X_0, \ldots, X_d).$$

The functional θ_1 actually determines a local cocycle in $C^*_{\mathrm{loc}}(\Omega^*_{\mathbb{R}} \otimes (A_d \otimes \mathfrak{g}))$ of degree one. As above, if

We define a map of factorization Lie algebras $\Phi_1 : \mathcal{G}_0 \to \mathcal{G}_1$. On and open set $I \subset \mathbb{R}$, we define

$$\Phi_1(\varphi\alpha, \psi K) = \left(\varphi\alpha, \int \psi \cdot K \right)$$

For a fixed open set $I \subset \mathbb{R}$, the map Φ_1 fits into the commutative diagram of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \Omega^*_c(I) \otimes \mathbb{C} \cdot K \arrow[r] \arrow[d, "\int"] & \mathcal{G}_0(I) \arrow[r] \arrow[d, "\Phi_1"] & \Omega^*_c(I) \otimes (A_d \otimes \mathfrak{g}) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathbb{C} \cdot K[-1] \arrow[r] & \mathcal{G}_1(I) \arrow[r] & \Omega^*_c(I) \otimes (A_d \otimes \mathfrak{g}) \arrow[r] & 0. \end{tikzcd}$$

To see that Φ_1 is a map of dg Lie algebras we simply observe that the cocycles determining the central extensions are related by $\theta_1 = \int \circ \theta_0$, where $\int : \Omega^*_c(I) \to \mathbb{C}$ as in the diagram above.

We now define the factorization dg Lie algebra $\mathcal{G}_1'$. Like $\mathcal{G}_1$, it is a central extension of $\Omega^*_{\mathbb{R},c} \otimes (A_d \otimes \mathfrak{g})$. The cocycle determining the central extension is defined by

$$\theta_1'(\varphi_0 a_0 X_0, \ldots, \ldots, \varphi_d a_d X_d) = \theta_1(\varphi_0 a_0 X_0, \ldots, \ldots, \varphi_d a_d X_d) + \widetilde{\theta}_1(\varphi_0 a_0 X_0, \ldots, \ldots, \varphi_d a_d X_d)$$

where θ_1 was defined in Equation (1). Before writing down the explicit formula for $\widetilde{\theta}_1$ we introduce some notation. Set

$$E = r\frac{\partial}{\partial r},$$
$$\mathrm{d}\vartheta = \sum_i \frac{\mathrm{d}z_i}{z_i}.$$

We view E as a vector field on $\mathbb{R}_{>0}$ and $\mathrm{d}\vartheta$ as a $(1,0)$-form on $\mathbb{C}^d \setminus 0$. The functional

$$\widetilde{\theta}_1(\varphi_0 a_0 X_0, \ldots, \varphi_d a_d X_d) = \frac{1}{2}\sum_{i=1}^d \left(\int_I \varphi_0 (E \cdot \varphi_i) \varphi_1 \cdots \widehat{\varphi_i} \cdots \varphi_d\right)\left(\oint (a_0 a_i \mathrm{d}\vartheta)\, \partial a_1 \cdots \widehat{\partial a_i} \cdots \partial a_d\right) \theta(X_0, \ldots, X_d).$$

The functional $\widetilde{\theta}$ defines a local functional in $\mathrm{C}^*_{\mathrm{loc}}\left(\Omega^*_{\mathbb{R}_{>0}} \otimes (A_d \otimes \mathfrak{g})\right)$ of cohomological degree one. One immediately checks that it is a cocycle.

In fact, we will show that $\widetilde{\theta}_1$ is actually an exact cocycle. We will see this by displaying an explicit cobounding functional. Define the local functional

$$\eta(\varphi_0 a_0 X_0, \ldots, \varphi_d a_d X_d) = \sum_{i=1}^d \left(\int_I \varphi_0 \left(\iota_E \varphi_i\right) \varphi_1 \cdots \widehat{\varphi_i} \cdots \varphi_d\right)\left(\oint (a_0 a_i \mathrm{d}\vartheta)\, \partial a_1 \cdots \widehat{\partial a_i} \cdots \partial a_d\right) \theta(X_0, \ldots, X_d).$$

Lemma 4.3. *One has $\mathrm{d}\eta = \widetilde{\theta}_1$, where d is the differential for the cochain complex $\mathrm{C}^*_{\mathrm{loc}}(\Omega^*_{\mathbb{R}_{>0}} \otimes (A_d \otimes \mathfrak{g}))$. In particular, the factorization Lie algebras $\mathcal{G}_1$ and $\mathcal{G}_1'$ are quasi-isomorphic. An explicit quasi-isomorphism is given by BW: ....*

Finally, we define the factorization Lie algebra $\mathcal{G}_2$. We have already seen that the local cocycle $J(\theta) \in \mathrm{C}^*_{\mathrm{loc}}(\mathfrak{g}^{\mathbb{C}^d})$ determines a central extension of factorization Lie algebras

$$0 \to \mathbb{C} \cdot K[-1] \to \mathcal{G}_{J(\theta)} \to \Omega^{0,*}_{\mathbb{C}^d,c} \otimes \mathfrak{g} \to 0.$$

Of course, we can restrict $\mathcal{G}_{J(\theta)}$ to a factorization algebra on $\mathbb{C}^d \setminus 0$. The factorization algebra $\mathcal{G}_2$ is defined as the pushforward of this restriction along the radial projection: $\mathcal{G}_2 := \mathrm{rad}_*\left(\mathcal{G}_{J(\theta)}|_{\mathbb{C}^d \setminus 0}\right)$.

Recall the map $\Phi : \Omega^*_{\mathbb{R}_{>0},c} \otimes (A_d \otimes \mathfrak{g}) \to \mathrm{rad}_*(\Omega^{0,*}_{\mathbb{C}^d \setminus 0,c} \otimes \mathfrak{g})$ defined in BW: ref. On each open set $I \subset \mathbb{R}_{>0}$ we can extend Φ by the identity on the central element to a linear map $\Phi_2 : \mathcal{G}_1'(I) \to \mathcal{G}_2(I)$.

Lemma 4.4. *The map $\Phi_2 : \mathcal{G}_1'(I) \to \mathcal{G}_2(I)$ is a map of dg Lie algebras. Moreover, it extends to a map of factorization Lie algebras $\Phi_2 : \mathcal{G}_1' \to \mathcal{G}_2$.*

Proof. Modulo the central element Φ_2 reduces to the map Φ, which we have already seen is a map of factorization Lie algebras in Proposition BW: ref. Thus, to show that Φ_2 is a map of factorization Lie algebras we need to show that it is compatible with the cocycles determing the respective central extensions. That is, we need to show that

$$\theta_1'(\varphi_0 a_0 X_0, \ldots, \varphi_d a_d X_d) = \theta_2(\Phi(\varphi_0 a_0 X_0), \ldots, \Phi(\varphi_d a_d X_d)) \tag{2}$$

for all $\varphi_i a_i X_i \in \Omega^*_c(I) \otimes (A_d \otimes \mathfrak{g})$. The cocycle θ_1' is only nonzero if one of the φ_i inputs is a 1-form. We evaluate the left-hand side on the $(d+1)$-tuple $(\varphi_0 \mathrm{d}r a_0 X_0, \varphi_1 a_1 X_1, \ldots, \varphi_d a_d X_d)$ where $\varphi_i \in C^\infty_c(I)$, $a_i \in A_d$, $X_i \in \mathfrak{g}$ for $i = 0, \ldots, d$. The result is

$$\left(\int_I \varphi_0 \cdots \varphi_d \mathrm{d}r\right)\left(\oint a_0 \partial a_1 \cdots \partial a_d\right)\theta(X_0, \ldots, X_d) \tag{3}$$

$$+ \quad \frac{1}{2}\sum_{i=1}^d \left(\int_I \varphi_0 (E \cdot \varphi_i) \varphi_1 \cdots \widehat{\varphi_i} \cdots \varphi_d \mathrm{d}r\right)\left(\oint (a_0 a_i \mathrm{d}\vartheta)\, \partial a_1 \cdots \widehat{\partial a_i} \cdots \partial a_d\right)\theta(X_0, \ldots, X_d) \tag{4}$$

We wish to compare this to the right-hand side of Equation (2). Recall that $\Phi(\varphi_0 \mathrm{d}r a_0 X_0) = \varphi(r)\mathrm{d}r a_0(z) X_0$ and $\Phi(\varphi_i a_i X_i) = \varphi(r) a_i(z) X_i$. Plugging this into the explicit formula for the cocycle θ_2 we see the right-hand side of (2) is

$$\left(\int_{\mathrm{rad}^{-1}(I)} \varphi_0(r)\mathrm{d}r a_0(z) \partial(\varphi_1(r) a_1(z)) \cdots \partial(\varphi_d(r) a_d(z))\right)\theta(X_0, \ldots, X_d). \tag{5}$$

We pick out the term in (5) in which the ∂ operators only act on the elements $a_i(z)$, $i = 1, \ldots, d$. This term is of the form

$$\int_{\mathrm{rad}^{-1}(I)} \varphi_0(r) \cdots \varphi_d(r) \mathrm{d}r a_0(z) \partial(a_1(z)) \cdots \partial(a_d(z)) \theta(X_0, \ldots, X_d).$$

Separating variables we find that this is precisely the first term (3) in the expansion of the left-hand side of (2).

Now, note that we can rewrite the d-operator in terms of the radius r as

$$\partial = \sum_{i=1}^d \mathrm{d}z_i \frac{\partial}{\partial z_i} = \sum_{i=1}^d \mathrm{d}z_i \bar{z}_i \frac{\partial}{\partial (r^2)} = \sum_{i=1}^d \mathrm{d}z_i \frac{r^2}{2z_i}\frac{\partial}{\partial r}.$$

The remaining terms in (5) correspond to the expansion of

$$\partial(\varphi_1(r) a_1(z)) \cdots \partial(\varphi_d(r) a_d(z)),$$

using the Leibniz rule, for which the ∂ operators act on at least one of the functions $\varphi_1, \ldots, \varphi_d$. In fact, only terms in which ∂ acts on precisely one of the functions $\varphi_1, \ldots, \varphi_d$ will be nonzero. For instance, consider the term

$$(\partial\varphi_1) a_1(z) (\partial \varphi_2) a_2(z) \partial(\varphi_3(z) a_3(z)) \cdots \partial(\varphi_d(z) a_d(z)). \tag{6}$$

Now, $\partial \varphi_i(r) = \omega \frac{\partial \varphi}{\partial r}$ where ω is the one-form $\sum_i (r^2/2z_i)\mathrm{d}z_i$. Thus, (6) is equal to

$$\left(\omega \frac{\partial \varphi_1}{\partial r}\right) a_1(z) \left(\omega \frac{\partial \varphi_2}{\partial r}\right) a_2(z) \partial(\varphi_3(z) a_3(z)) \cdots \partial(\varphi_d(z) a_d(z),$$

which is clearly zero as ω appears twice.

We observe that terms in the expansion of (5) for which ∂ acts on precisely one of the functions $\varphi_1, \ldots, \varphi_d$ can be written as

$$\sum_{i=1}^d \int_{\mathrm{rad}^{-1}(I)} \varphi_0(r)\left(r \frac{\partial}{\partial r}\varphi_i(r)\right)\varphi_1(r) \cdots \widehat{\varphi_i(r)} \cdots \varphi_d(r) \mathrm{d}r \frac{r}{2z_i}\mathrm{d}z_i a_0(z) a_i(z) \partial a_1(z) \cdots \widehat{\partial a_i(z)} \cdots \partial a_d(z).$$

Finally, notice that the function $z_i/2r$ is independent of the radius r. Thus, separating variables we find the integral can be written as

$$\frac{1}{2}\sum_{i=1}^d \left(\int_I \varphi_0 \left(r\frac{\partial}{\partial r}\varphi_i\right)\varphi_1 \cdots \widehat{\varphi_i} \cdots \varphi_d \mathrm{d}r\right)\left(\oint \frac{\mathrm{d}z_i}{z_i} a_0 a_i \partial a_2 \cdots \widehat{\partial a_i} \cdots \partial a_d\right).$$

This is precisely equal to the second term (4) above. Hence, the cocycles are compatible and the proof is complete.

□

5. MODULI SPACES

6. THE DETERMINANT LINE

Let V be a representation of the Lie group G. There is an associate *determinant line bundle* on $\mathrm{Bun}_G(X)$ constructed as follows. Consider the universal G-bundle $\mathcal{B}\mathrm{un}_G(X)$ on $X \times \mathrm{Bun}_G(X)$. We can form the vector bundle $\mathcal{V}_{\mathcal{B}\mathrm{un}} = \mathcal{B}\mathrm{un}_G(X) \times^G \mathcal{V}$ on $X \times \mathrm{Bun}_G(X)$. Denote the projection $\pi : X \times \mathrm{Bun}_G(X) \to \mathrm{Bun}_G(X)$. The *determinant line bundle* associated to V is the line bundle on $\mathrm{Bun}_G(X)$ defined by

$$\theta_V := \det\left(\mathbb{R}\pi_* \mathcal{V}_{\mathcal{B}\mathrm{un}}\right).$$

For $x \in X$, denote by $\theta_{V,x}$ the pull-back of this line bundle along the map $\mathrm{Bun}_G(X, x) \to \mathrm{Bun}_G(X)$.

Theorem 6.1. *There is an action of the dg Lie algebra $\widehat{\mathfrak{g}}_{\mathrm{ch}_{d+1}(V),x}$ by infinitesimal symmetries on $\mathrm{Tot}(\theta_{V,x})$, where $\mathrm{Tot}(\theta_{V,x})$ is the total space of the determinant line bundle associated to V. Moreover, this action is compatible with the action of $\mathfrak{g}_x$ on $\mathrm{Bun}_G(X, x)$.*

We will obtain a formal version of this result from the point of view of equivariant BV quantization. It will follow from a local version of Grothendieck–Riemann–Roch that we will obtain in terms of explicit Feynman diagrammatics.

To state the result we need to introduce the following BV theory. As above we fix a $\mathfrak{g}$-module V. Consider the following elliptic complex $\Omega^{0,*}(X) \otimes V$ with differential given by $\bar\partial \otimes 1$. One readily observes that the assignment $U \subset X \mapsto \Omega^{0,*}(U) \otimes V$ has the structure of an elliptic moduli problem as defined in **[?]**. Inasmuch, we can defined the associated *cotangent theory* $\mathcal{E}_V^X = T^*[-1](\Omega_X^{0,*} \otimes V)$. This is the classical BV theory; it's action functional can be described explicitly as follows. Label the fields with respect to the obvious decomposition of the shifted cotangent bundle as

$$\begin{aligned}(\gamma, \beta) &\in \left(\Omega^{0,*}(X) \otimes V\right) \oplus \left(\Omega^{0,*}(X) \otimes V\right)^! [-1] \\ &\cong \left(\Omega^{0,*}(X) \otimes V\right) \oplus \left(\Omega^{d,*}(X) \otimes V^\vee\right) [d-1].\end{aligned}$$

The action functional is

$$S^V(\gamma, \beta) = \int \langle \beta, \bar\partial\gamma\rangle_V$$

where $\langle -, -\rangle_V$ is the extension of the dual pairing between V and $V^\vee$ to Dolbeault forms.

Proposition 6.2. *There is map of factorization algebras on X*

$$\Phi^{\mathfrak{q}} : \mathrm{KM}^X_{\mathfrak{g},\mathrm{ch}_{d+1}(V)} \to \mathrm{Obs}_V^{\mathfrak{q}}$$

which, when ...

6.0.1. The action of $\mathfrak{g}$ on V extends to an action of the local Lie algebra $\mathfrak{g}^X$ on the classical theory $\mathcal{E}_V^X$. In Chapter 14 of **[?]** it is shown that such an action is equivalent to a Maurer-Cartan element in the dg Lie algebra $\mathrm{C}^*_{\mathrm{loc}}(\mathfrak{g}^X) \otimes \mathcal{O}_{\mathrm{loc}}(\mathcal{E}_V^X)[-1]$. Here, the dg Lie algebra structure is obtained by tensoring the dg Lie algebra structure on $\mathcal{O}_{\mathrm{loc}}(\mathcal{E}_V^X)[-1]$ given by the BV bracket with the commutative algebra $\mathrm{C}^*_{\mathrm{loc}}(\mathfrak{g}^X)$. Thus, we have an element

$$I^{\mathfrak{g}} \in \mathrm{C}^*_{\mathrm{loc}}(\mathfrak{g}^X) \otimes \mathcal{O}_{\mathrm{loc}}(\mathcal{E}_V^X)[-1]$$

that encodes the action of $\mathfrak{g}^X$ on $\mathcal{E}_V^X$. Explicitly, the Maurer-Cartan equation reads

$$\bar\partial I^{\mathfrak{g}} + \mathrm{d}_{\mathfrak{g}} I^{\mathfrak{g}} + \frac{1}{2}\{I^{\mathfrak{g}}, I^{\mathfrak{g}}\} = 0.$$

The result about factorization algebras above is a consequence of an explicit calculation of the anomaly to having an inner action of $\mathfrak{g}^X$ at the quantum level.

Proposition 6.3. *The anomaly to having a quantum inner action of $\mathfrak{g}^X$ on $\mathcal{E}_V^X$ is a local cocycle $\Theta_V^X \in \mathrm{C}^*_{\mathrm{loc}}(\mathfrak{g}^X)$ of degree one. Moreover, with respect to the map j of Proposition 2.1 we have an identification $\Theta_V^X = aj\left(\mathrm{ch}^{\mathfrak{g}}_{d+1}(V)\right)$ for some nonzero complex number a. BW: need to nail down a.*

First, we reduce this result to a calculation on $X = \mathbb{C}^d$. First, we consider the following general situation.

A result of [**?**] states that the space of BV-theories on a fixed manifold X form a sheaf. In particular, a fixed BV theory $\mathcal{E}^X$ on X can be enhanced to a sheaf of BV theories $\mathcal{E}^X : U \subset X \mapsto \mathcal{E}^U$. Suppose $\mathcal{L}$ is a local Lie algebra on X that acts on $\mathcal{E}^X$. In particular, for each open set $U \subset X$ we have a local Lie algebra $\mathcal{L}^U$ that acts on the BV theory $\mathcal{E}^U$.

Now, suppose we have an inclusion of open sets $i_U^V : U \subset V$ in X. There is an induced map of complexes

$$(i_U^V)^* : \mathrm{C}^*_{\mathrm{loc}}(\mathcal{L}^V) \to \mathrm{C}^*_{\mathrm{loc}}(\mathcal{L}^U).$$

Lemma 6.4. *Let Θ^U be the obstruction to having a quantum inner action of $\mathcal{L}^U$ on $\mathcal{E}^U$. Likewise, define Θ^V. The obstructions satisfy $(i_U^V)^*(\Theta^V) = \Theta^U$.*

7. Higher vertex algebras