HIGHER DIMENSIONAL KAC-MOODY SYMMETRIES

Contents

Kevin: This is very incomplete, as I've been distracted by writing up the other projects and focusing on applications. The main goals I want to accomplish in this are: 1) construct the universal Kac-Moody in any dimension, 2) show how to recover the sphere and iterated loop algebras and compare the sphere algebra to Faonte-Hennion-Kapranov, 3) prove a version of GRR over the formal moduli of G-bundles by an explicit calculation of the anomaly of higher Kac-Moody acting on beta-gamma with coefficients in a module, 4) present the realization of these higher factorization algebras as the boundary of both 5d and 7d supersymmetric gauge theories.

I believed I've worked out all of these, and my goal is to have this on the arXiv by the end of October, in time for application decisions. (I wanted to include a formula for the OPE in general dimensions, but I think I'll just include that in my thesis and wait until I have a better idea of the full higher vertex algebra structure.)

1. Lie algebras of currents

1.1. **Motivational discussion.** OG: I'm just letting it flow. This paragraph might profitably go elsewhere.

Our focus in this paper is upon field theories that depend upon complex geometry, specifically upon the symmetries they possess. Our overarching goal is to explain tools for understanding such symmetries that provide a systematic generalization of methods used in chiral conformal field theory on Riemann surfaces, notably the Kac-Moody vertex algebras. These tools will use ideas and techniques from the Batalin-Vilkovisky formalism, as articulated by Costello, and factorization algebras, following [**?**, **?**]. In this subsection, however, we will try to explain the key objects and constructions with a light touch, in a way that does not require familiarity with that formalism, merely comfort with basic complex geometry and ideas of quantum field theory.

1.1.1. A running example is the following version of the $\beta\gamma$ system.

Let X be a complex d-dimensional manifold. Let G be a complex algebraic group, such as $GL_n(\mathbb{C})$, and let $P \to X$ be a holomorphic principal G-bundle. Fix a finite-dimensional G-representation V and let V^* denote the dual vector space with the natural induced G-action. Let $\mathcal{V} \to X$ denote the holomorphic associated bundle $P \times^G V$, and let $\mathcal{V}^! \to X$ denote the holomorphic bundle $K_X \otimes \mathcal{V}^*$, where $\mathcal{V}^* \to X$ is the holomorphic associated bundle $P \times^G V^*$. Note that there is a natural fiberwise pairing

$$\langle -, - \rangle : \mathcal{V} \otimes \mathcal{V}^! \to K_X$$

arising from the evaluation pairing between V and V^*.

The field theory involves fields γ, for a smooth section of $\mathcal{V}$, and β, for a smooth section of $\mathcal{V}^!$. OG: I need to adjust where β lives in a way depending on dimension d. The action functional is

$$S(\beta, \gamma) = \int_X \langle \beta, \bar{\partial}\gamma \rangle,$$

so that the equations of motion are

$$\bar{\partial}\gamma = 0 = \bar{\partial}\beta.$$

Thus, the classical theory is manifestly holomorphic: it picks out holomorphic sections of $\mathcal{V}$ and $\mathcal{V}^!$ as solutions.

The theory also enjoys a natural symmetry with respect to G, arising from the G-action on $\mathcal{V}$ and $\mathcal{V}^!$. For instance, if $\bar{\partial}\gamma = 0$ and $g \in G$, then the section $g\gamma$ is also holomorphic. In fact, there is a local symmetry as well. Let $\mathrm{ad}(P) \to X$ denote the Lie algebra-valued bundle $P \times^G \mathfrak{g} \to X$ arising from the adjoint representation $\mathrm{ad}(G)$. Then a holomorphic section f of $\mathrm{ad}(P)$ acts on a holomorphic section γ of $\mathcal{V}$, and

$$\bar{\partial}(f\gamma) = (\bar{\partial}f)\gamma + f\bar{\partial}\gamma = 0,$$

so that the sheaf of holomorphic sections of $\mathrm{ad}(P)$ encodes a class of local symmetries of this classical theory.

1.1.2. If one takes a BV/BRST approach to field theory, as we will in this paper, then one works with a cohomological version of fields and symmetries. For instance, it is natural to view the classical fields as consisting of the graded vector space of Dolbeault forms

$$\gamma \in \Omega^{0,*}(X, \mathcal{V}) \quad \text{and} \quad \beta \in \Omega^{0,*}(X, \mathcal{V}^!) \cong \Omega^{d,*}(X, \mathcal{V}^*),$$

but using the same action functional, extended in the natural way. As we are working with a free theory and hence have only a quadratic action, the equations of motion are linear and can be viewed as equipping the fields with the differential $\bar{\partial}$. In this sense, the sheaf $\mathcal{E}$ of solutions to the equations of motion can be identified with the elliptic complex that assigns to an open set $U \subset X$, the complexe

$$\mathcal{E}(U) = \Omega^{0,*}(U, \mathcal{V}) \oplus \Omega^{0,*}(U, \mathcal{V}^!),$$

with $\bar{\partial}$ as the differential. This dg approach is certainly appealing from the perspective of complex geometry, where one routinely works with the Dolbeault complex of a holomorphic bundle.

It is natural then to encode the local symmetries in the same way. Let $\mathcal{A}\mathrm{d}(P)$ denote the Dolbeault complex of $\mathrm{ad}(P)$ viewed as a sheaf. That is, it assigns to the open set $U \subset X$, the dg Lie algebra

$$\mathcal{A}\mathrm{d}(P)(U) = \Omega^{0,*}(U, \mathrm{ad}(P))$$

with differential $\bar{\partial}$ for this bundle. By construction, $\mathcal{A}\mathrm{d}(P)$ acts on $\mathcal{E}$. In words, $\mathcal{E}$ is a sheaf of dg modules for the sheaf of dg Lie algebra $\mathcal{A}\mathrm{d}(P)$.

1.1.3. So far, we have simply lifted the usual discussion of symmetries to a dg setting, using standard tools of complex geometry. We now introduce a novel maneuver that is characteristic of the BV/factorization package of [**?**, **?**].

The idea is to work with compactly supported sections of $\mathcal{A}\mathrm{d}(P)$, i.e., to work with the precosheaf $\mathcal{A}\mathrm{d}(P)_c$ of dg Lie algebras that assigns to an open U, the dg Lie algebra

$$\mathcal{A}\mathrm{d}(P)_c(U) = \Omega^{0,*}_c(U, \mathrm{ad}(P)).$$

The terminology *precosheaf* encodes the fact that there is natural way to extend a section supported in U to a larger open $V \supset U$ (namely, extend by zero), and so one has a functor $\mathcal{A}\mathrm{d}(P)\colon \mathrm{Opens}(X) \to \mathrm{Alg}_{\mathrm{Lie}}$.

There are several related reasons to consider compact support.[1] First, it is common in physics to consider compactly-supported modifications of a field. Recall the variational calculus, where one extracts the equations of motion by working with precisely such first-order perturbations. Hence, it is natural to focus on such symmetries as well. Second, one could ask how such compactly supported actions of $\mathcal{A}\mathrm{d}(P)$ affect observables. More specifically, one can ask about the charges

[1]In Section 2 we extract factorization algebras from $\mathcal{A}\mathrm{d}(P)_c$, and then extract associative and vertex algebras of well-known interest. We postpone discussions within that framework till that section.

of the theory with respect to this local symmetry.[2] Third—and this reason will become clearer in a moment—the anomaly that appears when trying to quantize this symmetry are naturally local in X, and hence it is encoded by a kind of Lagrangian density L on sections of $\mathcal{A}\mathrm{d}(P)$. Such a density only defines a functional on compactly supported sections, since when evaluated a noncompactly supported section f, the density $L(f)$ may be non-integrable. Thus L determines a central extension of $\mathcal{A}\mathrm{d}(P)_c$ as a precosheaf of dg Lie algebras, but not as a sheaf.[3]

1.1.4. Let us sketch how to make these reasons explicit. The first step is to understand how $\mathcal{A}\mathrm{d}(P)_c$ acts on the observables of this theory.

Modulo functional analytic issues, we say that the observables of this classical theory are the commutative dg algebra

$$(\mathrm{Sym}(\Omega^{0,*}(X,\mathcal{V})^* \oplus \Omega^{0,*}(X,\mathcal{V}^!)^*), \bar{\partial}),$$

i.e., the polynomial functions on $\mathcal{E}(X)$. More accurately, we work with a commutative dg algebra essentially generated by the continuous linear functionals on $\mathcal{E}(X)$, which are compactly supported distributional sections of certain Dolbeault complexes (*aka* Dolbeault currents). We could replace X by any open set $U \subset X$, in which case the observables with support in U arise from such distributions supported in U. We denote this commutative dg algebra by $\mathrm{Obs}^{cl}(U)$. Since observables on an open U extend to observables on a larger open $V \supset U$, we recognize that Obs^{cl} forms a precosheaf.

Manifestly, $\mathcal{A}\mathrm{d}(P)_c(U)$ acts on $\mathrm{Obs}^{cl}(U)$, by precomposing with its action on fields. Moreover, these actions are compatible with the extension maps of the precosheaves, so that Obs^{cl} is a module for $\mathcal{A}\mathrm{d}(P)_c$ in precosheaves of cochain complexes. This relationship already exhibits why one might choose to focus on $\mathcal{A}\mathrm{d}(P)_c$, as it naturally intertwines with the structure of the observables.

But Noether's theorem provides a further reason, when understood in the context of the BV formalism. The idea is that Obs^{cl} has a Poisson bracket $\{-,-\}$ of degree 1 (although there are some issues with distributions here that we suppress for the moment). Hence one can ask to realize the action of $\mathcal{A}\mathrm{d}(P)_c$ via the Poisson bracket. In other words, we ask to find a map of (precosheaves of) dg Lie algebras

$$J \colon \mathcal{A}\mathrm{d}(P)_c \to \mathrm{Obs}^{cl}[-1]$$

such that for any $f \in \mathcal{A}\mathrm{d}(P)_c(U)$ and $F \in \mathrm{Obs}^{cl}(U)$, we have

$$f \cdot F = \{J(f), F\}.$$

Such a map would realize every symmetry as given by an observable, much as in Hamiltonian mechanics.

In this case, there is such a map:

$$J(f)(\gamma,\beta) = \int_U \langle \beta, f\gamma \rangle.$$

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[2] We remark that it is precisely this relationship with traditional physical terminology of currents and charges that led de Rham to use *current* to mean a distributional section of the de Rham complex.

[3] We remark that to stick with sheaves, one must turn to quite sophisticated tools [**?**, **?**, **?**] that can be tricky to interpret, much less generalize to higher dimension, whereas the cosheaf-theoretic version is quite mundane and easy to generalize, as we'll see.

This functional is local, and it is natural to view it as describing the "minimal coupling" between our free $\beta\gamma$ system and a kind of gauge field implicit in $\mathcal{A}d(P)$. OG: This is a little misleading, given the nature of the forms, but I think it is fixable. This construction thus shows again that it is natural to work with compactly supported sections of $\mathcal{A}d(P)$, since it allows one to encode the Noether map in a natural way. We call $\mathcal{A}d(P)_c$ the Lie algebra of *classical currents* as we have explained how, via J, we realize these symmetries as classical observables.

Remark 1.1. We remark that it is not always possible to produce such a Noether map, but the obstruction always determines a central extension of $\mathcal{A}d(P)_c$ as a precosheaf of dg Lie algebras, and one can then produce such a map to the classical observables.

1.1.5. In the BV formalism, quantization amounts to a deformation of the differential on Obs^{cl}, where the deformation is required to satisfy certain properties. Two conditions are preeminent:

- the differential satisfies a *quantum master equation*, which ensures that $\mathrm{Obs}^q(U)[-1]$ is still a dg Lie algebra via the bracket,[4] and
- it respects support of observables so that Obs^q is still a precosheaf.

The first condition is more or less what BV quantization means, whereas the second is a version of the locality of field theory.

We can now ask whether the Noether map J determines a map of precosheaves of dg Lie algebras from $\mathcal{A}d(P)_c$ to $\mathrm{Obs}^q[-1]$. Since the Lie bracket has not changed on the observables, the only question is where J is a cochain map for the new differential d^q If we write $\mathrm{d}^q = \mathrm{d}^{cl} + \hbar\Delta$,[5] then

$$[\mathrm{d}, J] = \hbar\Delta \circ J.$$

Naively—i.e., ignoring renormalization issues—this term is the functional ob on $\mathcal{A}d(P)_c$ given

$$ob(f) = \int \langle f K_\Delta \rangle,$$

where K_Δ is the integral kernel for the identity with respect to the pairing $\langle -, - \rangle$. (It encodes a version of the trace of f over $\mathcal{E}$.) This obstruction should resemble standard anomalies. OG: Is that transition too abrupt? Should we provide an example?

This functional ob is a cocycle in Lie algebra cohomology for $\mathcal{A}d(P)$ and hence determines a central extension $\widehat{\mathcal{A}d(P)}_c$ as precosheaves of dg Lie algebras. It is the Lie algebra of *quantum* currents, as there is a lift of J to a map J^q out of this extension to the quantum observables.

1.2. **Definitions.** We now introduce some definitions that aim to capture the abstract structure of the example just discussed.

[4] Again, we are suppressing—for the moment important—issues about renormalization, which will play a key role when we get to the real work.

[5] By working with smeared observables, one really can work with the naive BV Laplacian Δ. Otherwise, one must take a little more care.

1.2.1. It will be convenient to generalize Lie algebras to L_∞ algebras, which involve multilinear brackets that satisfy higher versions of the Jacobi relation up to homotopy.

OG: Just wanted to mention that your original definition of local Lie algebra was a little misleading, because it used $n \in \mathbb{Z}$ (not just positive integers) and said $\ell_n : \mathcal{L}^{\otimes n} \to \mathcal{L}[2-n]$. This tensor product might mislead people into thinking you mean tensor of sheaves of C^∞-modules, which isn't correct.

Definition 1.2. A *local L_∞ algebra* on X is the following data:

(i) a $\mathbb{Z}$-graded vector bundle L on X, whose sheaf of smooth sections we denote $\mathcal{L}^{sh}$, and

(ii) for each positive integer n, a polydifferential operator in n inputs

$$\ell_n : \underbrace{\mathcal{L}^{sh} \times \cdots \times \mathcal{L}^{sh}}_{n \text{ times}} \to \mathcal{L}[2-n]$$

such that the collection $\{\ell_n\}_{n\in\mathbb{N}}$ satisfy the conditions of an L_∞ algebra. Thus, $\mathcal{L}^{sh}$ is a sheaf of L_∞ algebras.

In practice, we prefer to work with the compactly supported sections of L, as explained in OG: cross ref, for which we reserve the more succinct notation $\mathcal{L}$.

Definition 1.3. Given a local L_∞ algebra $(L, \{\ell_n\})$ on X, let $\mathcal{L}$ denote the precosheaf of L_∞ algebras that assigns compactly supported sections of L to each open of X.

We typically refer to the local L_∞ algebra $(L, \{\ell_n\})$ by $\mathcal{L}$. We will often use local *Lie* algebra, especially if $\mathcal{L}$ is a precosheaf of dg Lie algebras and hence has trivial $\ell_{n\geq 3}$.

Example 1.4. Our favorite example, of course, arises from the adjoint bundle $\mathrm{ad}(P) \to X$ associated to a holomorphic principal G-bundle $P \to X$. We will hereafter use $\mathcal{A}d(P)$ to denote the compactly supported sections of Dolbeault complex of $\mathrm{ad}(P)$. OG: Is that too confusing?

Example 1.5. Another key local Lie algebra makes sense on an arbitrary complex d-fold. Let $\mathfrak{g}$ be an ordinary Lie algebra, such as $\mathfrak{sl}_n$. Let

$$\mathcal{G}^{sh} = \Omega^{0,*} \otimes \mathfrak{g},$$

which is a sheaf of dg Lie algebras on the category of complex d-folds and local biholomorphisms,[6] and $\mathcal{G}$ to denote $\Omega^{0,*}_c \otimes \mathfrak{g}$. We use $\mathcal{G}|_X$ to denote the restriction of $\mathcal{G}$ to a fixed complex d-fold X.

[6] A biholomorphism is a map $\phi : X \to Y$ that is biijective and both ϕ and ϕ^{-1} are holomorphic. A *local* biholomorphism means a map $\phi : X \to Y$ such that for every point $x \in X$ has a neighborhood on which ϕ is a biholomorphism. OG: Not sure what you think of burying this in a footnote, but it seems tangential and not worth elaborating on in the main text.

1.2.2. We are interested in a certain class of central extensions of such an $\mathcal{L}$.

Definition 1.6. A *local functional* on $\mathcal{L}$ of cohomological degree k is OG: yuck

The graded vector space of local functionals is a subcomplex of $C^*_{\mathrm{Lie}}(\mathcal{L})$, the naive Lie algebra cochains of $\mathcal{L}$. Let $C^*_{\mathrm{loc}}(\mathcal{L})$ denote this cochain complex, as explained in detail in OG: give precise citation. The differential is, in essence, just precomposition with the polydifferentials defining the brackets of $\mathcal{L}$.[7]

Definition 1.7. A cocycle θ of degree $2+k$ in $C^*_{\mathrm{loc}}(\mathcal{L})$ determines a *k-shifted central extension*

$$0 \to \mathbb{C}[k] \to \widehat{\mathcal{L}}_\theta \to \mathcal{L} \to 0 \tag{1}$$

of precosheaves of L_∞ algebras, where

$$\widehat{\ell}_n(x_1, \ldots, x_n) = (\ell_n(x_1, \ldots, x_n), \theta(x_1, \ldots, x_n)).$$

Cohomologous cocycles determine quasi-isomorphic extensions.

Example 1.8. Let X be a Riemann surface, i.e., a complex 1-fold, and let $\mathfrak{g}$ be a simple Lie algebra with Killing form κ. Consider the local Lie algebra $\mathcal{G}|_X$. There is a natural cocycle depending precisely on two inputs:

$$\theta(\alpha \otimes x, \beta \otimes y) = \kappa(x, y) \int_X \alpha \wedge \partial\beta,$$

where $\alpha, \beta \in \Omega^{0,*}_c(X)$ and $x, y \in \mathfrak{g}$. As explained in OG: cross ref and Section ??? of [?], this cocycle determines an affine Kac-Moody algebra extending the loop algebra $L\mathfrak{g} = \mathfrak{g}[z, z^{-1}]$.

Much of the rest of the section is devoted to constructing and analyzing various cocycles and extensions.

1.2.3. There is a particular family of local cocycles that has special importance for us and that generalizes the preceding example.

Let p be an invariant polynomial on $\mathfrak{g}$ of homogenous degree $d+1$. That is, p is an element of $\mathrm{Sym}^{d+1}(\mathfrak{g}^\vee)^{\mathfrak{g}}$. We can extend p to a functional $J(p)$ on $\mathcal{G}(X) = \Omega^{0,*}_c(X) \otimes \mathfrak{g}$ by the formula

$$J(p)(\omega_1 \otimes x_1, \ldots, \omega_{d+1} \otimes x_{d+1}) = p(x_1, \ldots, x_{d+1}) \int_X \omega_1 \wedge \partial\omega_2 \cdots \wedge \partial\omega_{d+1}.$$

Note that we use d copies of the holomorphic derivative $\partial : \Omega^{0,*} \to \Omega^{1,*}$ to obtain an element of $\Omega^{d,*}_c$ in the integrand (and hence something that has a chance of being integrated).

This formula clearly makes sense for any complex d-fold X, and since integration is local on X, it intertwines nicely with the structure maps of $\mathcal{G}$.

Proposition 1.9. *The assignment*

$$J : \mathrm{Sym}^{d+1}(\mathfrak{g}^\vee)^{\mathfrak{g}}[-1] \to C^*_{\mathrm{loc}}(\mathcal{G})$$

is a cochain map. Moreover, it is injective at the level of cohomology.

[7] Altogether $C^*_{\mathrm{loc}}(\mathcal{L})$ is just a version of diagonal Gelfand-Fuks cohomology for this kind of Lie algebra.

OG: Need to prove it.

OG: We have another notational challenge here. For a fixed manifold X, one can talk about the local cochains $C^*_{loc}(\mathcal{G}|_X)$, but we don't yet say what we mean by the thing that works on all d-folds simultaneously. If one takes a little care, it is a sheaf on that site, but I'm not sure we should do all that heavy lifting in this section.

1.3. **The sphere Lie algebras, or FHK current algebras.** Our goal in this section is to develop a higher-dimensional analogue of the loop algebra $L\mathfrak{g} = \mathfrak{g}[z, z^{-1}]$ and its central extensions. As the title might suggest, the analogue explored in this section has been developed substantially by Faonte-Hennion-Kapranov [**?**], and our treatment will bear a strong imprint from their approach. For further discussion on their work and its connection with ours, see Remark **??** and Section 1.3.2.

One common approach is to take iterated loops (and we discuss that direction OG: somewhere else), but a different natural generalization is to generalize the circle S^1 by the sphere S^{2d-1}. That is, we work with a "sphere algebra" of maps from S^{2d-1} into $\mathfrak{g}$. For topologists, this direction might seem natural, but it may not seem too natural from the perspective of algebraic geometry. In particular, an algebro-geometric sphere is given by a punctured affine d-space $\mathring{\mathbb{A}}^d = \mathbb{A}^d \setminus \{0\}$ or a punctured formal d-disk, but every map from these spaces to $\mathfrak{g}$ extends to a map from $\mathbb{A}^d$ or the formal d-disk into $\mathfrak{g}$ (essentially, by Hartog's lemma). Thus, this direction seems fruitless. The key to evading this issue is to work with the derived space of maps, since the derived sections of $\mathcal{O}$ on the punctured affine d-space is interesting.

This fact ought not to be too surprising: as a smooth manifold, punctured affine d-space is equivalent to $\mathbb{R}_{>0} \times S^{2d-1}$, and this equivalence manifests itself in the cohomology of the structure sheaf. Explicitly,

$$H^*(\mathring{\mathbb{A}}^d, \mathcal{O}_{alg}) = \begin{cases} 0, & * \neq 0, d-1 \\ \mathbb{C}[z_1, \ldots, z_d], & * = 0 \\ \mathbb{C}[z_1^{-1}, \ldots, z_d^{-1}] \frac{1}{z_1 \cdots z_d}, & * = d-1 \end{cases}$$

as one can show by direct computation (e.g., use the cover by the affine opens of the form $\mathbb{A}^d \setminus \{z_i = 0\}$). When $d = 1$, this recovers the usual Laurent series; and it is natural to view the above as the higher-dimensional analogue of the Laurent series, with the polar part now in degree $d-1$.

Hence, the derived global sections $\mathbb{R}\Gamma(\mathring{\mathbb{A}}^d, \mathcal{O})$ of $\mathcal{O}$ provide a homotopy-commutative algebra, and thus one obtains a homotopy-Lie algebra by tensoring with $\mathfrak{g}$, which we will call the sphere Lie algebra by analogy with the loop Lie algebra. One can then study central extensions of this homotopy-Lie algebra, which are analogous to the affine Kac-Moody Lie algebras. For explicit constructions, it is convenient to have a commutative dg algebra that models the derived global sections. It should be no surprise that we like to work with the Dolbeault complex. We will use this approach to relate the sphere Lie algebra and its extensions to the current algebras that we've already introduced.

Remark 1.10. We strongly encourage the reader to examine the seminal work [**?**], as it puts these constructions into a clear geometric context and offers a systematic generalization in derived algebraic geometry of the rich, fruitful relationship between affine Kac-Moody algebras and the moduli of G-bundles on algebraic curves. Indeed, for us, the lack of such an interpretation was a long-standing conceptual obstacle, as we had found these higher extensions and the action of these current algebras on holomorphic field theories by direct computation; but we did not have the toolkit to understand them in derived geometry. In turn we hope our work here helps derived geometers recognize how such constructions fit into field theory.

1.3.1. *Compactification along the sphere.*

1.3.2. *Comparison with the FHK approach.* We use the punctured algebraic disk $D^{1\times} = \text{Spec}\ \mathbb{C}[z, z^{-1}]$, but the definition also makes sense for the puncture formal disk (formal loops).

Let $D^d = \text{Spec}\ \mathbb{C}[z_1, \ldots, z_d]$ be the d-dimensional algebraic disk. The punctured d-disk is no longer affine, in fact its cohomology is given by

$$H^*(D^{d\times}, \mathcal{O}) =$$

Instead of working with the naive commutative algebra $\Gamma(D^{d\times}, \mathcal{O})$ we will use the dg commutative algebra of *derived* sections $\mathbb{R}\Gamma(D^{d\times}, \mathcal{O})$. An explicit model for this has been written down in **[?]** based on the Jouanolou method for resolving singularities. We recall its definition.

Definition 1.11. Let A_d be the commutative dg algebra generated by elements

$$z_1, \ldots, z_d, z_1^*, \ldots, z_d^*, (z_1 z_1^*)^{-1}, \ldots (z_1 z_d^*)^{-1}$$

in degree zero and

$$\mathrm{d}z_1, \ldots, \mathrm{d}z_d, \mathrm{d}z_1^*, \ldots, \mathrm{d}z_d^*$$

in degree one. Introduce a $*$-weight, so that $z_i^*, \mathrm{d}z_i^*$ have $*$-weight $+1$ and $(z_i^*)^{-1}$ has $*$-weight -1. We require that:

(i) every element is of total $*$-weight zero and

(ii) the contraction of every element with the Euler vector field $\sum_i z_i^* \partial_{z_i^*}$ vanishes.

The key properties of the dg algebra A_d we will utlilize are summarized in the following result of **[?]**.

Proposition 1.12 (**[?]** Proposition 1.3.1)**.** *The commutative dg algebra A_d is a model for $\mathbb{R}\Gamma(D^{d\times}, \mathcal{O})$. Moreover, there is a dense map of commutative dg algebras*

$$j : A_d \to \Omega^{0,*}(\mathbb{C}^d \setminus 0)$$

sending $z_i \mapsto z_i$, $z_i^* \mapsto \bar{z}_i$, and $\mathrm{d}z_i^* \mapsto \mathrm{d}\bar{z}_i$.

We are interested in the dg Lie algebra $A_d \otimes \mathfrak{g}$. In **[?]** they show, via knowledge of the Lie algebra cohomology, that there is a central extension of this BW: not sure what to say

Definition 1.13. Fix an element $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g})^{\mathfrak{g}}$. Let $\widehat{\mathfrak{g}}_{d,\theta}$ be the dg Lie algebra central extension of $A_d \otimes \mathfrak{g}$ determined by the degree two cocycle $\theta_{\mathrm{FHK}} \in \mathrm{C}^*_{\mathrm{Lie}}(A_d \otimes \mathfrak{g})$ defined by

$$\theta_{\mathrm{FHK}}(a_0 \otimes X_0, \ldots, a_d \otimes X_d) = \operatorname*{Res}_{z=0} (a_0 \wedge \mathrm{d}a_1 \wedge \cdots \wedge \mathrm{d}a_d)\, \theta(X_0, \ldots, X_d)$$

where $a_i \otimes X_i \in A_d \otimes \mathfrak{g}$.

1.4. **Dimension d extensions via Gelfand-Kazhdan geometry.** OG: Commented out is the earlier stuff about local Lie algebras, to be cannabalized

2. Factorization algebras of currents

2.1. **The factorization algebra.** Given any cocycle $\theta \in \mathrm{C}^*_{\mathrm{loc}}(\mathfrak{g}^X)$ of degree one we define a factorization algebra on X.

Definition 2.1. Let θ be a local cocycle of $\mathfrak{g}^X$ of cohomological degree one. Define $\mathcal{F}^X_{\mathfrak{g},\theta}$ to be the factorization algebra on X to be the twisted factorization envelope $U^{\mathrm{fact}}_{\theta}(\mathfrak{g}^X)$. Equivalently, this is the factorization envelope of the extended Lie algebra $\widehat{\mathfrak{g}}^X_{\theta}$ determined by θ.

Explicitly, on an open set $U \subset X$, the cochain complex $\mathcal{F}^X_{\mathfrak{g},\theta}(U)$ has as its underlying graded vector space

$$\mathrm{Sym}\left(\mathfrak{g}^X_c(U)[1] \oplus \mathbb{C} \cdot K\right)$$

and the differential is given by $\bar{\partial} + \mathrm{d}_{\mathfrak{g}} + \theta$ where $\mathrm{d}_{\mathfrak{g}}$ is the extension of the Chevalley-Eilenberg differential for $\mathfrak{g}$ to the Dolbeault complex, and where θ is extended to the full symmetric algebra by the rule that it is a (graded) derivation.

Example 2.2. As an example, using the map J of Proposition 1.9, we can construct a factorization algebra on X for any invariant polynomial $P \in \mathrm{Sym}^{d+1}(\mathfrak{g}^\vee)^{\mathfrak{g}}$. Since j is injective, we obtain a unique factorization algebra for every such polynomial, hence it makes sense to denote $\mathcal{F}^X_{\mathfrak{g},P} := \mathcal{F}^X_{\mathfrak{g},j(P)}$.

2.1.1. *Arbitrary principal bundle.* There is a local Lie algebra related to $\mathfrak{g}^X$ associated to any principal G bundle. Formally speaking, one can understand $\mathfrak{g}^X$, or rather its global sections $\mathfrak{g}^X(X)$, as being the dg Lie algebra describing the formal neighborhood of the *trivial* G-bundle inside the derived moduli stack of G-bundle on X. Indeed, if triv denotes the trivial bundle then one has

$$\widehat{\mathrm{triv}} = B\mathfrak{g}^X(X)$$

where the hat denotes formal completion. In other words, the (-1)-shifted tangent space of the moduli stack of G-bundles is identified with the dg Lie algebra $\mathfrak{g}^X(X)$. At an arbitrary principal G bundle P, the dg Lie algebra describing the formal completion $\widehat{P}$ is also the global sections of a local Lie algebra that we now.

Let $\mathrm{ad}(P)$ denote the bundle of Lie algebras on X associated to P. We define the local Lie algebra by

$$\mathfrak{g}^{P \to X} := \Omega^{0,*}(X; \mathrm{ad}(P)),$$

i.e. the $(0,*)$-forms on X with coefficients in the bundle $\mathrm{ad}(P)$. The Lie bracket on $\mathrm{ad}(P)$ together with the Dolbeault operator $\bar{\partial}$ define the structure of the local Lie algebra. The global sections of this local Lie algebra describe the formal completion of P in the moduli of G bundles: $\widehat{P} = B\mathfrak{g}^{P\to X}(X)$.

2.1.2. *A variant on the construction.* The definition of the following flavor of factorization algebras have appeared in Section 3.6 of [**?**], but we wish to further analyze them here. As in the cases above, we work on a complex d-fold X and consider the local Lie algebra $\mathfrak{g}^X = \Omega^{0,*}(X;\mathfrak{g})$. The variant we discuss in this section involves a different (-1)-shifted central extension of this local Lie algebra. In this section, we fix an invariant pairing $\langle -,-\rangle$ on the Lie algebra $\mathfrak{g}$.

Fix a closed $(d-1,d-1)$-form $\omega \in \Omega^{d-1,d-1}(X)$. Define the quadratic functional on $\mathfrak{g}^X$ by

$$\phi_\omega(\alpha,\beta) = \int_X \omega \wedge \langle \alpha, \partial\beta\rangle.$$

Lemma 2.3. *The functional ϕ_ω is a local cocycle of degree one in $\mathrm{C}^*_{\mathrm{loc}}(\mathfrak{g}^X)$.*

Proof. Clearly ϕ_ω is local and degree one. The differential on $\mathrm{C}^*_{\mathrm{loc}}(\mathfrak{g}^X)$ is of the form $\bar{\partial} + \mathrm{d}_{\mathfrak{g}}$ where $\mathrm{d}_{\mathfrak{g}}$ is the Chevalley-Eilenberg differential on $\mathfrak{g}$ extended to $(0,*)$-forms. Since the pairing is invariant one has $\mathrm{d}_{\mathfrak{g}}(\phi_\omega) = 0$. Finally, to see that it is a cocycle we note that

$$\int_X \mathrm{d}_{dR}(\omega\wedge\langle\alpha,\partial\beta\rangle) = \int_X \omega\wedge\langle\bar{\partial}\alpha,\partial\beta\rangle \pm \int_X \omega\wedge\langle\alpha,\bar{\partial}\partial\beta\rangle$$

using the fact that ω is closed and $\omega\wedge\langle\alpha,\partial\beta\rangle$ is ∂-closed. □

Definition 2.4. Let X be a complex d-fold and $\omega \in \Omega^{d-1,d-1}(X)$ a closed form. Define the factorization algebra $U^{fact}_\omega(\mathfrak{g}^X)$ on X as the twisted factorization envelope of $\mathfrak{g}^X$ twisted by the cocycle ϕ_ω.

Example 2.5. Suppose that X is a Kähler d-fold and let $\omega \in \Omega^{1,1}(X)$ be the Kähler form. We can then take the $(d-1,d-1)$-form above to be the $(d-1)$st power of the Kähler form ω^{d-1}. We will refer to the factorization algebra

$$\mathcal{F}^{(X,\omega)} := U^{fact}_{\omega^{d-1}}(\mathfrak{g}^X)$$

as the *Kähler-Kac-Moody* factorization algebra on X. In the case that $d=2$, the factorization algebra is related to the four-dimensional generalization of the Wess-Zumino-Witten model studied by Nair and Schiff in [**?**] and later by Nekrasov et. al. in [**?**, **?**]. We will return to this example later to describe its local operators as a consequence of its factorization algebra structure and to give an interpretation of it as a boundary of a certain Chern-Simons–like gauge theory.

2.2. **Relation to the ordinary Kac-Moody on Riemann surfaces.** In this section we pause to discuss a direct relationship of the higher dimensional Kac-Moody factorization algebras discussed above to the familiar Kac-Moody vertex algebras which are defined on one-dimensional complex manifolds.

Throughout this section we fix a Riemann surface Σ and consider a holomorphic family of complex $(d-1)$-folds over it. That is, we have a holomorphic fibration $\pi : X \to \Sigma$ whose fibers $\pi^{-1}(x)$, $x \in \Sigma$ are $(d-1)$-dimensional. For a fixed Lie algebra $\mathfrak{g}$ we put the higher dimensional

Kac-Moody on X and consider its pushforward along π to get some factorization algebra on Σ. We will see how this pushforward is related to the one-dimensional Kac-Moody factorization (and vertex) algebra on Σ.

2.2.1. *A reminder of the ordinary current algebra.* The affine algebra $\widehat{\mathfrak{g}}$ of a Lie algebra $\mathfrak{g}$ together with a invariant pairing $\langle -,-\rangle_{\mathfrak{g}}$ is defined as a Lie algebra central extension of the loop algebra $L\mathfrak{g} = \mathfrak{g}[t,t^{-1}]$ defined by the cocycle $(f,g) \mapsto \mathrm{Res}_0(f\partial g)$. There is a slight generalization of this construction defined for any dg Lie algebra $(\mathfrak{g}, \mathrm{d}_{\mathfrak{g}})$. We take as the input data a $\mathfrak{g}$-invariant pairing $\langle -,-\rangle_{\mathfrak{g}}$ that is closed for the differential $\mathrm{d}_{\mathfrak{g}}$. This means that for any $X, Y \in \mathfrak{g}$ we have $\langle \mathrm{d}_{\mathfrak{g}} X, Y\rangle + (-1)^{|X|}\langle X, \mathrm{d}_{\mathfrak{g}} Y\rangle = 0$ where $|X|$ is the cohomological degree of X in $\mathfrak{g}$. Equivalently, $\langle -,-\rangle$...

The loop algebra of a dg Lie algebra $L\mathfrak{g} = \mathfrak{g}[t,t^{-1}]$ is still defined and from the $\mathrm{d}_{\mathfrak{g}}$-closed invariant pairing we get a 2-cochain on $L\mathfrak{g}$ defined by the same formula as in the ordinary case. The fact that it is a cocycle comes from being closed for both the differential $\mathrm{d}_{\mathfrak{g}}$ and the Chevalley-Eilenberg differential for $L\mathfrak{g}$ (by invariance). Thus, we obtain a dg Lie algebra central extension $\widehat{\mathfrak{g}}$ of $L\mathfrak{g}$.

From the affine algebra associated to $\mathfrak{g}$ one builds the Kac-Moody vertex algebra by inducing the trivial module for $\widehat{\mathfrak{g}}$ up via the subalgebra of positive loops $L_+\mathfrak{g} \subset L\mathfrak{g}$. It is immediate that the same construction carries over for the case of a dg Lie algebra. One obtains, in this way, a *dg vertex algebra*. That is, a vertex algebra in the category of cochain complexes. We denote the level κ vacuum Kac-Moody dg vertex algebra obtained in this way by $\widehat{\mathfrak{g}}_\kappa$.

2.2.2. *Level zero.*

2.2.3.

Corollary 2.6. *Fix a Lie algebra with invariant pairing $\langle -,-\rangle$. Let Σ be an arbitrary Riemann surface and $d > 1$. Consider the volume form $\omega \in \Omega^{d-1,d-1}(\mathbb{P}^{d-1})$. Then, the pushforward of the factorization algebra $\mathcal{F}_\omega^{\Sigma\times\mathbb{P}^{d-1}}$ along the projection $\pi : \Sigma \times \mathbb{P}^{d-1} \to \Sigma$ is quasi-isomorphic to the ordinary Kac-Moody factorization algebra of central charge* $\mathrm{vol}(\omega)$

$$\pi_*\mathcal{F}_\omega^{\Sigma\times\mathbb{P}^{d-1}} \simeq \mathcal{F}^{\Sigma}_{\mathrm{vol}(\omega)}.$$

3. Sphere and loop algebras

We have defined the Kac–Moody factorization algebra as a universal holomorphic factorization algebra in any dimension. In this section we focus on the restriction of the factorization algebra to two complex manifolds of dimension d, $X = \mathbb{C}^d \setminus \{0\}$ and $X = (\mathbb{C} \setminus \{0\})^d$. In each case we show how the factorization product encodes the structure of a dg Lie algebra. Our main results in this section identify these dg Lie algebras with higher dimensional generalizations of loop and affine algebras.

We first consider a dg Lie algebra $\widehat{\mathfrak{g}}_{d,\theta}$, labeled by the dimension and a parameter $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$, whose zeroeth cohomology is a Lie algebra extension of the $(2d-1)$-sphere algebra

$$\mathrm{Map}(S^{2d-1}, \mathfrak{g}).$$

At the level of cohomology this extension is trivial, but at the level of cochain complexes it is non-trivial.

The dg Lie algebra determines a dg associative algebra via the universal enveloping algebra $U(\widehat{\mathfrak{g}}_{d,\theta})$. Our first main result in this section relates this associative algebra to the Kac–Moody factorization algebra.

Theorem 3.1. *The associative algebra $U(\widehat{\mathfrak{g}}_{d,\theta})$ determines a locally constant factorization algebra on the real one-manifold $\mathbb{R}$ that we denote $U(\widehat{\mathfrak{g}}_{d,\theta})^{fact}$. Moreover, there is an injective dense map of factorization algebras on $\mathbb{R}$:*

$$\Phi^{S^{2d-1}} : (U\widehat{\mathfrak{g}}_{d,\theta})^{fact} \to \rho_* \left(\mathcal{F}_{\mathfrak{g},\theta}^{\mathbb{C}^d \setminus \{0\}} \right).$$

where the right-hand side is the push-forward of the Kac–Moody factorization algebra on $\mathbb{C}^d \setminus \{0\}$ along the radial projection map.

Next, we consider the higher loop Lie algebra

$$L^d \mathfrak{g} = L(\cdots(L\mathfrak{g})\cdots) = \mathrm{Map}(S^1 \times S^1, \mathfrak{g}).$$

We study a class of *shifted* central extension of this Lie algebra, also parametrized by $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$, that we denote by $\widehat{L^d \mathfrak{g}}_\theta$.

A result of Knudsen [**?**], which we recall in Section **??**, states that every dg Lie algebra determines an E_d-algebra, for any $d > 1$, called the universal E_d enveloping algebra. This agrees with the ordinary universal enveloping algebra in the case $d = 1$. For the dg Lie algebra $\widehat{L^d \mathfrak{g}}_\theta$, we denote this E_d algebra by $U^{E_d}(\widehat{L^d \mathfrak{g}}_\theta)$. Its associated locally constant factorization algebra on $\mathbb{R}^d$ is denoted $U^{E_d}(\widehat{L^d \mathfrak{g}}_\theta)^{fact}$.

The Kac–Moody factorization algebra on the d-fold $(\mathbb{C}^\times)^d$ determines a real d-dimensional factorization algebra by considering the radius in each complex direction. We denote this factorization algebra on $\mathbb{R}^d$ by $\vec{\rho}_* \left(\mathcal{F}_{\mathfrak{g},\theta}^{\mathbb{C}^{\times d}} \right)$.

Theorem 3.2. *There is a dense injective map of factorization algebras on $\mathbb{R}^d$:*

$$\Phi^{L^d} : \left(U_{E_d} \left(\widehat{L^d g}_\theta \right) \right)^{fact} \to \vec{\rho}_* \left(\mathcal{F}_{\mathfrak{g},\theta}^{\mathbb{C}^{\times d}} \right).$$

3.1. **The higher sphere algebra.** The affine algebra associated to a Lie algebra $\mathfrak{g}$ together with an invariant pairing is defined as a central extension of the loop algebra of $\mathfrak{g}$

$$\mathbb{C} \to \widehat{\mathfrak{g}} \to Lg$$

where loop algebra is equal to $\mathcal{O}(D^{1\times}) \otimes \mathfrak{g} = \mathfrak{g}[z, z^{-1}]$. The central extension is determined by the cocycle

$$f \otimes X, g \otimes Y \mapsto \oint f \mathrm{d}g \langle X, Y \rangle.$$

We use the punctured algebraic disk $D^{1\times} = \mathrm{Spec}\ \mathbb{C}[z, z^{-1}]$, but the definition also makes sense for the puncture formal disk (formal loops).

Let $D^d = \text{Spec } \mathbb{C}[z_1, \dots, z_d]$ be the d-dimensional algebraic disk. The punctured d-disk is no longer affine, in fact its cohomology is given by

$$H^*(D^{d\times}, \mathcal{O}) =$$

Instead of working with the naive commutative algebra $\Gamma(D^{d\times}, \mathcal{O})$ we will use the dg commutative algebra of *derived* sections $\mathbb{R}\Gamma(D^{d\times}, \mathcal{O})$. An explicit model for this has been written down in [**?**] based on the Jouanolou method for resolving singularities. We recall its definition.

Definition 3.3. Let A_d be the commutative dg algebra generated by elements

$$z_1, \dots, z_d, z_1^*, \dots, z_d^*, (z_1 z_1^*)^{-1}, \dots (z_1 z_d^*)^{-1}$$

in degree zero and

$$\mathrm{d}z_1, \dots, \mathrm{d}z_d, \mathrm{d}z_1^*, \dots, \mathrm{d}z_d^*$$

in degree one. Introduce a $*$-weight, so that $z_i^*, \mathrm{d}z_i^*$ have $*$-weight $+1$ and $(z_i^*)^{-1}$ has $*$-weight -1. We require that:

(i) every element is of total $*$-weight zero and

(ii) the contraction of every element with the Euler vector field $\sum_i z_i^* \partial_{z_i^*}$ vanishes.

The key properties of the dg algebra A_d we will utlilize are summarized in the following result of [**?**].

Proposition 3.4 ([**?**] Proposition 1.3.1)**.** *The commutative dg algebra A_d is a model for $\mathbb{R}\Gamma(D^{d\times}, \mathcal{O})$. Moreover, there is a dense map of commutative dg algebras*

$$j : A_d \to \Omega^{0,*}(\mathbb{C}^d \setminus 0)$$

sending $z_i \mapsto z_i$, $z_i^* \mapsto \bar{z}_i$, and $\mathrm{d}z_i^* \mapsto \mathrm{d}\bar{z}_i$.

We are interested in the dg Lie algebra $A_d \otimes \mathfrak{g}$. In [**?**] they show, via knowledge of the Lie algebra cohomology, that there is a central extension of this BW: not sure what to say

Definition 3.5. Fix an element $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g})^{\mathfrak{g}}$. Let $\widehat{\mathfrak{g}}_{d,\theta}$ be the dg Lie algebra central extension of $A_d \otimes \mathfrak{g}$ determined by the degree two cocycle $\theta_{\mathrm{FHK}} \in \mathrm{C}^*_{\mathrm{Lie}}(A_d \otimes \mathfrak{g})$ defined by

$$\theta_{\mathrm{FHK}}(a_0 \otimes X_0, \dots, a_d \otimes X_d) = \operatorname*{Res}_{z=0} (a_0 \wedge \mathrm{d}a_1 \wedge \cdots \wedge \mathrm{d}a_d)\, \theta(X_0, \dots, X_d)$$

where $a_i \otimes X_i \in A_d \otimes \mathfrak{g}$.

3.2. **The strategy.** We consider the restriction of the factorization algebra $\mathcal{F}_{\mathfrak{g},\theta}$ on $\mathbb{C}^d \setminus \{0\}$ to the collection of open sets diffeomorphic to spherical shells. This restriction has the structure of a one-dimensional factorization algebra corresponding to the iterated nesting of spherical shells. We show that there is a dense subfactorization algebra that is locally constant, hence corresponds to an A_∞ algebra. We conclude by identifying this A_∞ algebra as a the universal enveloping algebra of a certain L_∞ algebra, that agree with the higher dimensional affine algebras of [**?**]

Introduce the radial projection map

$$\rho : \mathbb{C}^d \setminus 0 \to \mathbb{R}_{>0}$$

sending $z = (z_1, \ldots, z_d)$ to $|z| = \sqrt{|z_1|^2 + \cdots + |z_d|^2}$. We will restrict our factorization algebra to spherical shells by pushing forward the factorization algebra along this map. Indeed, the preimage of an open interval is such a spherical shell, and the factorization product on the line is equivalent to the nesting of shells.

3.2.1. *The case of zero level.* First we will consider the higher Kac-Moody factorization algebra on $\mathbb{C}^d \setminus \{0\}$ "at level zero". That is, the factorization algebra $\mathcal{F}_{\mathfrak{g},0}^{\mathbb{C}^d\setminus\{0\}}$. In this section we will omit $\mathbb{C}^d \setminus \{0\}$ from the notation, and simply refer to the factorization algebra by $\mathcal{F}_{\mathfrak{g},0}$.

Let $\rho_*(\mathcal{F}_{\mathfrak{g},0})$ be the factorization algebra on $\mathbb{R}_{>0}$ obtained by pushing forward along the radial projection map. Explicitly, to an open set $I \subset \mathbb{R}_{>0}$ this factorization algebra assigns the dg vector space

$$C_*^{\mathrm{Lie}}\left(\Omega_c^{0,*}(\rho^{-1}(I)) \otimes \mathfrak{g}\right).$$

Let $I \subset \mathbb{R}_{>0}$ be an open subset. There is the natural map $\rho^* : \Omega_c^*(I) \to \Omega_c^*(\rho^{-1}(I))$ given by the pull back of differential forms. We can post compose this with the natural projection $\mathrm{pr}_{\Omega^{0,*}} : \Omega_c^* \to \Omega_c^{0,*}$ to obtain a map of commutative algebras $\mathrm{pr}_{\Omega^{0,*}} \circ \rho^* : \Omega_c^*(I) \to \Omega_c^{0,*}(\rho^{-1}(I))$. The map j from Proposition **??** determines a map of dg commutative algebras $j : A_d \to \Omega^{0,*}(\rho^{-1}(I))$. Thus, we obtain a map

$$\begin{array}{rccc}\Phi(I) = (\mathrm{pr}_{\Omega^{0,*}} \circ \rho^*) \otimes j : & \Omega_c^*(I) \otimes A_d & \to & \Omega_c^{0,*}\left((\rho^{-1}(I)\right) \\ & \varphi \otimes a & \mapsto & ((\mathrm{pr}_{\Omega^{0,*}} \circ \rho^*)\varphi) \wedge j(a)\end{array}$$

Since this is a map of commutative dg algebras it defines a map of dg Lie algebras

$$\Phi(I) \otimes \mathrm{id}_{\mathfrak{g}} : (\Omega_c^*(I) \otimes A_d) \otimes \mathfrak{g} = \Omega_c^*(I) \otimes (A_d \otimes \mathfrak{g}) \to \Omega^{0,*}(\rho^{-1}(I)) \otimes \mathfrak{g}$$

which maps $(\varphi \otimes a) \otimes X \mapsto \Phi(\varphi \otimes a) \otimes X$. We will drop the $\mathrm{id}_{\mathfrak{g}}$ from the notation and will denote this map simply by $\Phi(I)$. Note that $\Phi(I)$ is compatible with inclusions of open sets, hence extends to a map of cosheaves of dg Lie algebras that we will call Φ.

We can summarize the results as follows.

Proposition 3.6. *The map Φ extends to a map of factorization Lie algebras*

$$\Phi : \Omega^*_{\mathbb{R}_{>0},c} \otimes (A_d \otimes \mathfrak{g}) \to \rho_*\left(\Omega^{0,*}_{\mathbb{C}^d\setminus 0,c} \otimes \mathfrak{g}\right).$$

Hence, it defines a map of factorization algebras

$$C_*(\Phi) : U^{fact}\left(\Omega^*_{\mathbb{R}_{>0}} \otimes (A_d \otimes \mathfrak{g})\right) \to \rho_*\left(\mathcal{F}_{\mathfrak{g},0}^{\mathbb{C}^d\setminus 0}\right).$$

The fact that we obtain a map of factorization algebras follows from the universal property of the universal enveloping factorization algebra we discussed in Section **??**.

3.2.2. *The case of non-zero level.* We now proceed to the proof of Theorem . The dg Lie algebra $\mathfrak{g}_{d,\theta}$ determines a dg associative algebra via its universal enveloping algebra $U(\mathfrak{g}_{d,\theta})$. BW: define it? By BW: ref this dg algebra determines a factorization algebra on the one-manifold $\mathbb{R}_{>0}$ that assigns to every open interval $I \subset \mathbb{R}_{>0}$ the dg vector space $U(A_d \otimes \mathfrak{g})$. The factorization product is uniquely determined by the algebra structure. Henceforth, we denote this factorization algebra by $U(\mathfrak{g}_{d,\theta})^{fact}$.

To prove the theorem we will construct a sequence of maps of factorization Lie algebras on $\mathbb{R}_{>0}$:

$$
\begin{tikzcd}
 & \mathcal{G}_1 \arrow[dr, "\Phi_1"] & & \mathcal{G}_2 \\
\mathcal{G}_0 \arrow[ur, "\simeq", "\Phi_0"'] & & \mathcal{G}_1' \arrow[ur, "\Phi_2"'] &
\end{tikzcd}
.
$$

The factorization envelope of $\mathcal{G}_0$ is equivalent to the factorization algebra $U(\widehat{\mathfrak{g}}_{d,\theta})^{fact}$. Moreover, the factorization envelope of $\mathcal{G}_2$ is the push-forward of of the higher Kac–Moody factorization algebra $\rho_* \mathcal{F}_{\mathfrak{g},\theta}$. Hence, the desired map of factorization algebras is produced by applying the factorization envelope functor to the above composition of factorization Lie algebras.

First, we introduce the factorization Lie algebra $\mathcal{G}_0$. To an open set $I \subset \mathbb{R}$, it assigns the dg Lie algebra $\mathcal{G}_0(I) = \Omega^*_c(I) \otimes \widehat{\mathfrak{g}}_{d,\theta}$, where $\widehat{\mathfrak{g}}_{d,\theta}$ is the central extension from BW: ref. The differential and Lie bracket are determined by the fact that we are tensoring a commutative dg algebra with a dg Lie algebra. A slight variant of Proposition 3.4.0.1 in [**?**], which shows that the one-dimensional factorization envelope of an ordinary Lie algebra produces its ordinary universal enveloping algebra, shows that there is a quasi-isomorphism of factorization algebras on $\mathbb{R}$,

$$(U\widehat{\mathfrak{g}}_{d,\theta})^{fact} \xrightarrow{\simeq} \mathrm{C}^{\mathrm{Lie}}_*(\mathcal{G}_0).$$

The factorization Lie algebra $\mathcal{G}_0$ is a central extension of the factorization Lie algebra $\Omega^*_{\mathbb{R},c} \otimes (A_d \otimes \mathfrak{g})$ by the trivial module $\Omega^*_c \oplus \mathbb{C} \cdot K$. Indeed, the cocycle determining the central extension is given by

$$\theta_0(\varphi_0 \alpha_0, \ldots, \varphi_d \alpha_d) = (\varphi_0 \wedge \cdots \wedge \varphi_d) \theta_{A_d}(\alpha_1, \ldots, \alpha_d).$$

The factorization Lie algebra $\Omega^*_{\mathbb{R},c} \otimes (A_d \otimes \mathfrak{g})$ is the compactly supported sections of the local Lie algebra $\Omega^*_{\mathbb{R}} \otimes (A_d \otimes \mathfrak{g})$ and this cocycle determining the extension is a local cocycle.

Next, we define the factorization dg Lie algebra $\mathcal{G}_1$ on $\mathbb{R}$. This is also obtained as a central extension of the factorization Lie algebra $\Omega^*_{\mathbb{R},c} \otimes (A_d \otimes \mathfrak{g})$:

$$0 \to \mathbb{C} \cdot K[-1] \to \mathcal{G}_1 \to \Omega^*_{\mathbb{R},c} \otimes (A_d \otimes \mathfrak{g}) \to 0$$

determined by the following cocycle. For an open interval I write $\varphi_i \in \Omega^*_c(I)$, $\alpha_i \in A_d \otimes \mathfrak{g}$. The cocycle is defined by

$$\theta_1(\varphi_0 \alpha_0, \ldots, \varphi_d \alpha_d) = \left(\int_I \varphi_0 \wedge \cdots \varphi_d \right) \theta_{\mathrm{FHK}}(\alpha_0, \ldots, \alpha_d) \tag{2}$$

where θ_{FHK} was defined in Definition **??**.

The functional θ_1 determines a local cocycle in $\mathrm{C}^*_{\mathrm{loc}}(\Omega^*_{\mathbb{R}} \otimes (A_d \otimes \mathfrak{g}))$ of degree one.

We now define a map of factorization Lie algebras $\Phi_0 : \mathcal{G}_0 \to \mathcal{G}_1$. On and open set $I \subset \mathbb{R}$, we define the map $\Phi_0(I) : \mathcal{G}_0(I) \to \mathcal{G}_1(I)$ by

$$\Phi_0(I)(\varphi\alpha, \psi K) = \left(\varphi\alpha, \int \psi \cdot K\right).$$

For a fixed open set $I \subset \mathbb{R}$, the map Φ_0 fits into the commutative diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \Omega^*_c(I) \otimes \mathbb{C} \cdot K \arrow[r] \arrow[d, "\simeq"', "\int"] & \mathcal{G}_0(I) \arrow[r] \arrow[d, "\Phi_0(I)"] & \Omega^*_c(I) \otimes (A_d \otimes \mathfrak{g}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{C} \cdot K[-1] \arrow[r] & \mathcal{G}_1(I) \arrow[r] & \Omega^*_c(I) \otimes (A_d \otimes \mathfrak{g}) \arrow[r] & 0.
\end{tikzcd}$$

To see that $\Phi_0(I)$ is a map of dg Lie algebras we simply observe that the cocycles determining the central extensions are related by $\theta_1 = \int \circ \theta_0$, where $\int : \Omega^*_c(I) \to \mathbb{C}$ as in the diagram above. Since $\int$ is a quasi-isomorphism, the map $\Phi_0(I)$ is as well. It is clear that as we vary the interval I we obtain a quasi-isomorphism of factorization Lie algebras $\Phi_0 : \mathcal{G}_0 \xrightarrow{\simeq} \mathcal{G}_1$.

We now define the factorization dg Lie algebra $\mathcal{G}_1'$. Like $\mathcal{G}_0$ and $\mathcal{G}_0$, it is a central extension of $\Omega^*_{\mathbb{R},c} \otimes (A_d \otimes \mathfrak{g})$. The cocycle determining the central extension is defined by

$$\theta_1'(\varphi_0 a_0 X_0, \ldots, \ldots, \varphi_d a_d X_d) = \theta_1(\varphi_0 a_0 X_0, \ldots, \ldots, \varphi_d a_d X_d) + \widetilde{\theta}_1(\varphi_0 a_0 X_0, \ldots, \ldots, \varphi_d a_d X_d)$$

where θ_1 was defined in Equation (2). Before writing down the explicit formula for $\widetilde{\theta}_1$ we introduce some notation. Set

$$E = r\frac{\partial}{\partial r},$$
$$\mathrm{d}\vartheta = \sum_i \frac{\mathrm{d}z_i}{z_i}.$$

We view E as a vector field on $\mathbb{R}_{>0}$ and $\mathrm{d}\vartheta$ as a $(1,0)$-form on $\mathbb{C}^d \setminus 0$. Define the functional

$$\widetilde{\theta}_1(\varphi_0 a_0 X_0, \ldots, \varphi_d a_d X_d) = \frac{1}{2}\sum_{i=1}^d \left(\int_I \varphi_0 (E \cdot \varphi_i)\varphi_1 \cdots \widehat{\varphi}_i \cdots \varphi_d\right)\left(\oint (a_0 a_i \mathrm{d}\vartheta)\, \partial a_1 \cdots \widehat{\partial a_i} \cdots \partial a_d\right)\theta(X_0, \ldots, X_d).$$

The functional $\widetilde{\theta}$ defines a local functional in $\mathrm{C}^*_{\mathrm{loc}}\left(\Omega^*_{\mathbb{R}_{>0}} \otimes (A_d \otimes \mathfrak{g})\right)$ of cohomological degree one. One immediately checks that it is a cocycle. This completes the definition of the factorization Lie algebra $\mathcal{G}_1'$.

The factorization Lie algebras $\mathcal{G}_1$ and $\mathcal{G}_1'$ are identical as precosheaves of vector spaces. In fact, if we put a filtration on $\mathcal{G}_1$ and $\mathcal{G}_1'$ where the central element K has filtration degree one, then the associated graded factorization Lie algebras $\mathrm{Gr}\ \mathcal{G}_1$ and $\mathrm{Gr}\ \mathcal{G}_1'$ are also identified. The only difference in the Lie algebra structures comes from the deformation of the cocycle determining the extension of $\mathcal{G}_1'$ given by $\widetilde{\theta}_1$.

In fact, we will show that $\widetilde{\theta}_1$ is actually an exact cocycle via the cobounding element $\eta \in \mathrm{C}^*_{\mathrm{loc}}\left(\Omega^*_{\mathbb{R}_{>0}} \otimes (A_d \otimes \mathfrak{g})\right)$ defined by

$$\eta(\varphi_0 a_0 X_0, \ldots, \varphi_d a_d X_d) = \sum_{i=1}^d \left(\int_I \varphi_0\, (\iota_E \varphi_i)\, \varphi_1 \cdots \widehat{\varphi}_i \cdots \varphi_d\right)\left(\oint (a_0 a_i \mathrm{d}\vartheta)\, \partial a_1 \cdots \widehat{\partial a_i} \cdots \partial a_d\right)\theta(X_0, \ldots, X_d).$$

Lemma 3.7. *One has* $\mathrm{d}\eta = \widetilde{\theta}_1$, *where* d *is the differential for the cochain complex* $C^*_{\mathrm{loc}}(\Omega^*_{\mathbb{R}_{>0}} \otimes (A_d \otimes \mathfrak{g}))$. *In particular, the factorization Lie algebras* $\mathcal{G}_1$ *and* $\mathcal{G}_1'$ *are quasi-isomorphic (as* L_∞ *algebras). An explicit quasi-isomorphism is given by the* L_∞ *map* $\Phi_1 : \mathcal{G}_1 \to \mathcal{G}_1'$ *that sends the central element* K *to itself and an element* $(\varphi_0 a_0 X_0, \ldots, \varphi_d a_d X_d) \in \mathrm{Sym}^{d+1}(\Omega^*_c \otimes (A_d \otimes \mathfrak{g})$ *to*

$$(\varphi_0 a_0 X_0, \ldots, \varphi_d a_d X_d) + \eta(\varphi_0 a_0 X_0, \ldots, \varphi_d a_d X_d) \cdot K \in \mathrm{Sym}^{d+1}(\Omega^*_c \otimes (A_d \otimes \mathfrak{g})) \oplus \mathbb{C} \cdot K.$$

Finally, we define the factorization Lie algebra $\mathcal{G}_2$. We have already seen that the local cocycle $J(\theta) \in C^*_{\mathrm{loc}}(\mathfrak{g}^{\mathbb{C}^d})$ determines a central extension of factorization Lie algebras

$$0 \to \mathbb{C} \cdot K[-1] \to \mathcal{G}_{J(\theta)} \to \Omega^{0,*}_{\mathbb{C}^d,c} \otimes \mathfrak{g} \to 0.$$

Of course, we can restrict $\mathcal{G}_{J(\theta)}$ to a factorization algebra on $\mathbb{C}^d \setminus 0$. The factorization algebra $\mathcal{G}_2$ is defined as the pushforward of this restriction along the radial projection: $\mathcal{G}_2 := \rho_* \left(\mathcal{G}_{J(\theta)}|_{\mathbb{C}^d \setminus 0}\right)$.

Recall the map $\Phi : \Omega^*_{\mathbb{R}_{>0},c} \otimes (A_d \otimes \mathfrak{g}) \to \rho_*(\Omega^{0,*}_{\mathbb{C}^d \setminus 0,c} \otimes \mathfrak{g})$ defined in BW: ref. On each open set $I \subset \mathbb{R}_{>0}$ we can extend Φ by the identity on the central element to a linear map $\Phi_2 : \mathcal{G}_1'(I) \to \mathcal{G}_2(I)$.

Lemma 3.8. *The map* $\Phi_2 : \mathcal{G}_1'(I) \to \mathcal{G}_2(I)$ *is a map of dg Lie algebras. Moreover, it extends to a map of factorization Lie algebras* $\Phi_2 : \mathcal{G}_1' \to \mathcal{G}_2$.

Proof. Modulo the central element Φ_2 reduces to the map Φ, which we have already seen is a map of factorization Lie algebras in Proposition BW: ref. Thus, to show that Φ_2 is a map of factorization Lie algebras we need to show that it is compatible with the cocycles determing the respective central extensions. That is, we need to show that

$$\theta_1'(\varphi_0 a_0 X_0, \ldots, \varphi_d a_d X_d) = \theta_2(\Phi(\varphi_0 a_0 X_0), \ldots, \Phi(\varphi_d a_d X_d)) \tag{3}$$

for all $\varphi_i a_i X_i \in \Omega^*_c(I) \otimes (A_d \otimes \mathfrak{g})$. The cocycle θ_1' is only nonzero if one of the φ_i inputs is a 1-form. We evaluate the left-hand side on the $(d+1)$-tuple $(\varphi_0 \mathrm{d}r a_0 X_0, \varphi_1 a_1 X_1, \ldots, \varphi_d a_d X_d)$ where $\varphi_i \in C^\infty_c(I)$, $a_i \in A_d$, $X_i \in \mathfrak{g}$ for $i = 0, \ldots, d$. The result is

$$\left(\int_I \varphi_0 \cdots \varphi_d \mathrm{d}r\right)\left(\oint a_0 \partial a_1 \cdots \partial a_d\right) \theta(X_0, \ldots, X_d) \tag{4}$$

$$+ \quad \frac{1}{2}\sum_{i=1}^{d} \left(\int_I \varphi_0 (E \cdot \varphi_i) \varphi_1 \cdots \widehat{\varphi_i} \cdots \varphi_d \mathrm{d}r\right)\left(\oint (a_0 a_i \mathrm{d}\vartheta)\, \partial a_1 \cdots \widehat{\partial a_i} \cdots \partial a_d\right) \theta(X_0, \ldots, X_d) \tag{5}$$

We wish to compare this to the right-hand side of Equation (3). Recall that $\Phi(\varphi_0 \mathrm{d}r a_0 X_0) = \varphi(r)\mathrm{d}r a_0(z) X_0$ and $\Phi(\varphi_i a_i X_i) = \varphi(r) a_i(z) X_i$. Plugging this into the explicit formula for the cocycle θ_2 we see the right-hand side of (3) is

$$\left(\int_{\rho^{-1}(I)} \varphi_0(r)\mathrm{d}r a_0(z) \partial(\varphi_1(r) a_1(z)) \cdots \partial(\varphi_d(r) a_d(z))\right) \theta(X_0, \ldots, X_d). \tag{6}$$

We pick out the term in (6) in which the ∂ operators only act on the elements $a_i(z)$, $i = 1, \ldots, d$. This term is of the form

$$\int_{\rho^{-1}(I)} \varphi_0(r) \cdots \varphi_d(r) \mathrm{d}r a_0(z) \partial(a_1(z)) \cdots \partial(a_d(z)) \theta(X_0, \ldots, X_d).$$

Separating variables we find that this is precisely the first term (4) in the expansion of the left-hand side of (3).

Now, note that we can rewrite the ∂-operator in terms of the radius r as

$$\partial = \sum_{i=1}^{d} \mathrm{d}z_i \frac{\partial}{\partial z_i} = \sum_{i=1}^{d} \mathrm{d}z_i \bar{z}_i \frac{\partial}{\partial (r^2)} = \sum_{i=1}^{d} \mathrm{d}z_i \frac{r^2}{2z_i} \frac{\partial}{\partial r}.$$

The remaining terms in (6) correspond to the expansion of

$$\partial(\varphi_1(r)a_1(z)) \cdots \partial(\varphi_d(r)a_d(z)),$$

using the Leibniz rule, for which the ∂ operators act on at least one of the functions $\varphi_1, \ldots, \varphi_d$. In fact, only terms in which ∂ acts on precisely one of the functions $\varphi_1, \ldots, \varphi_d$ will be nonzero. For instance, consider the term

$$(\partial\varphi_1)a_1(z)(\partial\varphi_2)a_2(z)\partial(\varphi_3(z)a_3(z)) \cdots \partial(\varphi_d(z)a_d(z)). \tag{7}$$

Now, $\partial\varphi_i(r) = \omega \frac{\partial \varphi}{\partial r}$ where ω is the one-form $\sum_i (r^2/2z_i)\mathrm{d}z_i$. Thus, (7) is equal to

$$\left(\omega \frac{\partial \varphi_1}{\partial r}\right) a_1(z) \left(\omega \frac{\partial \varphi_2}{\partial r}\right) a_2(z)\partial(\varphi_3(z)a_3(z)) \cdots \partial(\varphi_d(z)a_d(z),$$

which is clearly zero as ω appears twice.

We observe that terms in the expansion of (6) for which ∂ acts on precisely one of the functions $\varphi_1, \ldots, \varphi_d$ can be written as

$$\sum_{i=1}^{d} \int_{\rho^{-1}(I)} \varphi_0(r) \left(r\frac{\partial}{\partial r}\varphi_i(r)\right) \varphi_1(r) \cdots \widehat{\varphi_i(r)} \cdots \varphi_d(r) \mathrm{d}r \frac{r}{2z_i} \mathrm{d}z_i a_0(z) a_i(z) \partial a_1(z) \cdots \widehat{\partial a_i(z)} \cdots \partial a_d(z).$$

Finally, notice that the function $z_i/2r$ is independent of the radius r. Thus, separating variables we find the integral can be written as

$$\frac{1}{2}\sum_{i=1}^{d} \left(\int_I \varphi_0 \left(r\frac{\partial}{\partial r}\varphi_i\right) \varphi_1 \cdots \widehat{\varphi}_i \cdots \varphi_d \mathrm{d}r\right) \left(\oint \frac{\mathrm{d}z_i}{z_i} a_0 a_i \partial a_2 \cdots \widehat{\partial a_i} \cdots \partial a_d\right).$$

This is precisely equal to the second term (5) above. Hence, the cocycles are compatible and the proof is complete.

□

3.3. **Higher loop algebras.** We now put the Kac-Moody factorization algebra on the d-fold $(\mathbb{C}^\times)^d$. Our main result in this section involves extracting the structure of an E_d algebra from considering the nesting of "polyannuli" in $(\mathbb{C}^\times)^d$. When $d = 1$, we have seen that the nesting of ordinary annuli give rise to the structure of an associative algebra. For $d > 1$, a polyannulus is a complex submanifold of the form $\mathbb{A}_1 \times \cdots \times \mathbb{A}_d \subset (\mathbb{C}^\times)^d$ where each $\mathbb{A}_i \subset \mathbb{C}^\times$ is an ordinary annulus. Equivalently, a polyannulus is the complement of a closed polydisk inside of a larger open polydisk. We will see how the nesting of annuli in each component gives rise to the structure of a locally constant factorization algebra in d *real* dimensions, and hence defines an E_d algebra.

3.3.1. Define the commutative algebra

$$B_d = \mathbb{C}[z_1, z_1^{-1}] \otimes \cdots \otimes \mathbb{C}[z_d, z_d^{-1}].$$

If $\mathfrak{g}$ is any Lie algebra we define the Lie algebra $L^d\mathfrak{g} := B_d \otimes \mathfrak{g}$. This is the algebraic version of the d-fold loop space of the Lie algebra $\mathfrak{g}$:

$$L(L(\cdots L(\mathfrak{g})\ldots)) = \mathrm{Map}((S^1)^{\times d}, \mathfrak{g}).$$

We will write elements as $f \otimes X \in B_d \otimes \mathfrak{g}$ for $f = f(z_1, \ldots, f_d) \in B_d$ and $X \in \mathfrak{g}$.

In the commutative algebra B_d there are derivations $\partial/\partial z_1, \ldots, \partial/\partial z_d$. Let $\Omega^1_{B_d} = B_d[\mathrm{d}z_1, \ldots, \mathrm{d}z_d]$ be the vector space of algebraic differentials. Similarly, define $\Omega^k_{B_d}$ by $B_d \otimes \wedge^k \mathbb{C}\{\mathrm{d}z_1, \ldots, \mathrm{d}z_d\}$. There is a universal algebraic differential $\partial : B_d \to \Omega^1_{B_d}$ given in coordinates by $\partial = \sum_i \frac{\partial}{\partial z_i}\mathrm{d}z_i$.

We note that the space of d-forms $\Omega^d_{B_d}$ admits a residue map defined by taking d-fold iterated one-dimensional residues:

$$\oint_{|z_1|=1} \cdots \oint_{|z_d|=1} : \Omega^d_{B_d} \to \mathbb{C}.$$

Explicitly, if $f\mathrm{d}z_1 \cdots \mathrm{d}z_d$ is a top form then

$$\oint_{|z_1|=1} \cdots \oint_{|z_d|=1} f\mathrm{d}z_1 \cdots \mathrm{d}z_d = (2\pi i)^n \times \{\text{coefficient of } z_1^{-1} \cdots z_d^{-1}\}.$$

Given a homogenous degree d invariant polynomial on $\mathfrak{g}$ there is a shifted extension of $L^d\mathfrak{g}$ that is closely related to the extension we discussed in the previous section.

Proposition 3.9. *Given any $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g}^\vee)^{\mathfrak{g}}$ there is $(d-1)$-shifted L_∞-central extension of $L^d\mathfrak{g}$*

$$0 \to \mathbb{C}[d-1] \to \widehat{L^d\mathfrak{g}}_\theta \to L^d\mathfrak{g} \to 0$$

with brackets given by $\ell_2 = [-,-]_{L^d\mathfrak{g}}$ and

$$\ell_{d+1}(f_0 \otimes X_0, \cdots, f_d \otimes X_d) = \theta(X_1, \ldots, X_d) \oint_{|z_1|=1} \cdots \oint_{|z_d|=1} f_0 \partial f_1 \cdots \partial f_d \cdot K$$

and all other brackets zero. Here, K is the generator of the central part of the Lie algebra of degree $-d+1$.

3.3.2. Given any Lie algebra $\mathfrak{h}$ we can define the universal enveloping algebra $U\mathfrak{h}$ which is an associative. In fact, the functor $\mathfrak{h} \mapsto U\mathfrak{h}$ from Lie algebras to associative algebras is left adjoint to the forgetful functor obtained by forming the commutator in the associative algebra. The homotopical generalization of associative algebras are E_1-algebras which are algebras over the operad of little 1-disks.

Theorem 3.10 ([?]). *There is a forgetful functor $F : \mathrm{Alg}_{E_d} \to \mathrm{dgLie}_{\mathbb{C}}$ and it admits a left adjoint*

$$U_{E_d} : \mathrm{dgLie}_{\mathbb{C}} \to \mathrm{Alg}_{E_d}$$

called the E_d-universal enveloping algebra. If $\mathfrak{h}$ is an ordinary Lie algebra the E_d-algebra has underlying graded vector space

$$U_{E_d}(\mathfrak{h}) = \mathrm{Sym}\left(\mathfrak{h}[1-d]\right).$$

There is an equivalence of categories between E_d algebras and locally constant factorization algebras on $\mathbb{R}^d$. If A is an E_d algebra we denote by A^{fact} its associated locally constant factorization algebra on $\mathbb{R}^d$.

Proposition 3.11. *Suppose $\mathfrak{h}$ is a dg Lie algebra. Then, there is a quasi-isomorphism of factorization algebras on $\mathbb{R}^d$:*

$$\left(U_{E_d}\mathfrak{h}\right)^{fact} \simeq C_*^{\mathrm{Lie}}(\Omega^*_{c,\mathbb{R}^d} \otimes \mathfrak{h})$$

We now explain how the higher dimensional Kac-Moody factorization algebra is related to the universal E_d enveloping algebra of the Lie algebra $B_d \otimes \mathfrak{g}$ (and its central extension). We will consider the factorization algebra restricted to the complex manifold $(\mathbb{C}^\times)^d \subset \mathbb{C}^d$. Throughout this section we will denote the factorization algebra $\mathcal{F}^{(\mathbb{C}^\times)^d}_{\mathfrak{g},\theta}$ on $(\mathbb{C}^\times)^d$ simply by $\mathcal{F}_{\mathfrak{g},\theta}$.

Let $\vec{\rho} : (\mathbb{C}^\times)^d \to (\mathbb{R}_{>0})^d$ be the map sending $(z_1, \ldots, z_d) \mapsto (|z_1|, \ldots, |z_d|)$. If $I_1, \ldots, I_d \subset \mathbb{R}_{>0}$ is any collection of intervals we see that $\vec{\rho}^{-1}(I_1 \times \cdots \times I_d) \subset (\mathbb{C}^\times)^d$ is a polyannulus. Thus, to understand the behavior of a factorization algebra $\mathcal{F}$ on $(\mathbb{C}^\times)^d$ with respect to the nesting of polyannuli, as discussed in the beginning of this section, it suffices to understand the factorization product of cubes of the pushforward of the factorization algebra $\vec{\rho}_*\mathcal{F}$ on $(\mathbb{R}_{>0})^d$.

A general factorization algebra $\mathcal{F}$ on $(\mathbb{C}^\times)^d$ does not define a E_d algebra in the way we have just described. Indeed, even in the case of a holomorphic factorization algebra, it is reasonable to expect that the pushforward factorization algebra will be sensitive to the length of the sides of the cubes in $\mathbb{R}_{>0}$. Just as in the case of the previous section, where we considered compactification along the $2d-1$ sphere in $\mathbb{C}^d \setminus 0$, we will show that there is a well-behaved sub-factorization algebra that *is* locally constant and hence does define the structure of an E_d algebra.

There is a holomorphic action of the d-torus $T^d = S^1 \times \cdot \times S^1$ on the complex manifold $(\mathbb{C}^\times)^d$ by rotating component-wise. Hence, there is an induced action of T^d on the Dolbeault complex $\Omega^{0,*}((\mathbb{C}^\times)^d) \cong \Omega^{0,*}(\mathbb{C}^\times)^{\otimes d}$. The action of the torus is induced from a tensor product of S^1 representations with respect to this decomposition. For an integer $n \in \mathbb{Z}$ let $\Omega^{0,*}(\mathbb{C}^\times)^{(n)} \subset \Omega^{0,*}(\mathbb{C}^\times)$ be the dg subspace consisting of all forms with eigenvalue n. Similarly, for each sequence of integers $(n_1, \ldots, n_d)$ we let

$$\Omega^{0,*}\left((\mathbb{C}^\times)^d\right)^{(n_1,\ldots,n_d)} \subset \Omega^{0,*}\left((\mathbb{C}^\times)^d\right)$$

be the tensor product $\Omega^{0,*}(\mathbb{C}^\times)^{(n_1)} \otimes \cdots \otimes \Omega^{0,*}(\mathbb{C}^\times)^{(n_d)}$.

For each open set $U \subset (\mathbb{C}^\times)^d$ we can define, in a completely analogous way, the subspace

$$\mathcal{F}^{(\mathbb{C}^\times)^d}_{\mathfrak{g},\theta}(U)^{(n_1,\ldots,n_d)} \subset \mathcal{F}^{(\mathbb{C}^\times)^d}_{\mathfrak{g},\theta}(U).$$

4. Formal index theorem on the moduli of G-bundles

The main goal of the BV formalism developed in **[?]** is to rigorously construct quantum field theories using a combination of homological methods and a rigorous model for renormalization. A particular nicety of this approach is the ability to study *families* of field theories. In this section we will consider a family of QFT's parametrized by the moduli space of principal G-bundles.

Our main result is to interpret a certain anomaly coming from BV quantization as a families index over $\mathrm{Bun}_G(X)$. This anomaly is computed via an explicit Feynman diagrammatic calculation and is related to a local cocycle of the current algebra discussed in Section BW: ref. An immediate corollary is a formal universal version of the Grothendieck–Riemann–Roch theorem over the moduli space of bundles.

We will arrive at this result in a way that is local-to-global on space-time which we formulate in terms of factorization algebras. In [?, ?] it is shown how the observables of a QFT determine a factorization algebra. We study the associated family of factorization algebras associated to the family of QFT's over the moduli space of G-bundles mentioned in the preceding paragraph. We recollect a formulation of Noether's theorem for symmetries of a theory in terms of factorization algebras developed in Chapter ?? of [?]. The central object in this discussion is a "local index" which describes how the Kac–Moody factorization algebra acts on the observables of the QFT. Locally on space-time we see how Noether's theorem provides a *free field realization* of the Kac–Moody factorization algebra generalizing that of the Kac–Moody vertex algebra in chiral conformal field theory [?].

4.1. **The families index.** Fix a complex d-fold X. Let $\mathrm{Bun}_G(X)$ denote the moduli space of G-bundles on the complex d-fold X. For $d > 1$ [?] have constructed a global smooth derived realization of this space, but its full structure will not be used in this discussion.

Suppose $\mathcal{P} \to X \times B$ is a holomorphic family of principal G-bundles on X. For each point b in the parameter space B the restriction $\mathcal{P}|_{X\times\{b\}}$ is a principal G-bundle on $X = X \times \{b\}$. Such families are classified by a map $f_{\mathcal{P}} : B \to \mathrm{Bun}_G(X)$.

Suppose V is a G-representation. Given any principal G-bundle P on X we obtain the vector bundle $P \times^G V$ on X via the Borel construction. Similarly, if $\mathcal{P}$ is a family of G-bundles as above, we obtain a family of vector bundles $\mathcal{V}_{\mathcal{P}}$ over $X \times B$ whose fiber over $X \times \{b\}$ is the vector bundle $\mathcal{V}_b = \mathcal{P}|_{X\times\{b\}} \times^G V$ on X.

Moreover, $\mathcal{V}_{\mathcal{P}}$ is a family of holomorphic vector bundles. In particular, for each b there is a $\bar\partial$-operator

$$\bar\partial_b : \Gamma(X, \mathcal{V}_b) \to \Gamma(X, T^{*0,1}_X \otimes \mathcal{V}_b).$$

We can extend this operator to an elliptic complex

$$\Omega^{0,0}(X, \mathcal{V}_b) \xrightarrow{\bar\partial_b} \Omega^{0,1}(X, \mathcal{V}_b) \xrightarrow{\bar\partial_b} \cdots \xrightarrow{\bar\partial_b} \Omega^{0,d}(X, \mathcal{V}_b)$$

where d is the dimension of X. Denote this elliptic complex by $\Omega^{0,*}(X, \mathcal{V}_b)$. This construction also makes sense in families. We denote the holomorphic family of elliptic complexes by $\Omega^{0,*}(X, \mathcal{V})$ over $X \times B$ whose fiber over $b \in B$ is $\Omega^{0,*}(X, \mathcal{V}_b)$.

4.2. **Quantization in formal families.** BW: this section is probably unnecessary, but I may use something like it it in my thesis. I'll probably remove it.

We will be most concerned with families of QFT's over moduli spaces that are *formal*. There is a Koszul duality between formal moduli spaces and dg Lie algebras. The shifted tangent space of a formal moduli space is a dg Lie algebra, and the Maurer–Cartan elements of this dg Lie algebras completely describe the formal moduli space. This duality allows us to interpret such formal families of theories in terms of symmetries by dg Lie algebras. Before discussing the requisite machinery to talk about symmetries of a QFT in the BV formalism we recount a general algebraic situation. First, we step back and recall symmetries in classical mechanics.

Let $\mathfrak{g}$ be a Lie algebra and (M, ω) a symplectic manifold. A symplectic action of $\mathfrak{g}$ on M is a map of Lie algebras $\rho : \mathfrak{g} \to \mathrm{Vect}^{\mathrm{symp}}(M, \omega)$, where the target is the Lie algebra of symplectic vector fields. Let $C^\infty(M)$ be the commutative algebra of smooth functions on M. An action of the Lie algebra $\mathfrak{g}$ on M induces a map of Lie algebras $\rho : \mathfrak{g} \to \mathrm{Der}(C^\infty(M))$ by derivations. The Poisson bracket $\{-,-\}$ on functions also determines a map of Lie algebras $\mathcal{O}(M) \to \mathrm{Der}(C^\infty(M))$ sending $f \mapsto \{f, -\}$. The symplectic action is *Hamiltonian* if there exists a lifting $\widetilde{\rho} : \mathfrak{g} \to C^\infty(M)$.

The obstruction to lifting a symplectic action to a Hamiltonian one is an element in the cohomology of M. Thus, in the case that M is a symplectic vector space V there is no obstruction to lifting a symplectic action of $\mathfrak{g}$ on V to a Hamiltonian action. If $C^\infty_\hbar(M)$ is a quantization of (M, ω) then it is reasonable to ask for quantizations of the co-moment map. We study the analogous problem in the context of BV quantization.

Recall, a BV quantization of a P_0 algebra A is a BD algebra A^{q} defined over the ring $\mathbb{C}[[\hbar]]$ such that $A^{\mathrm{q}}/(\hbar)$ is isomorphic to A as P_0 algebras. A Hamiltonian action of $\mathfrak{g}$ on A is a map of dg Lie algebras [8]

$$\Phi : \mathfrak{g} \to A[-1].$$

By the universal property of the P_0 envelope this determines a map of P_0 algebras $\Phi : U^{P_0}\mathfrak{g} \to A$.

Definition 4.1. Fix a Hamiltonian action of a dg Lie algebra $\mathfrak{g}$ on a P_0 algebra A with co-moment map $\Phi : \mathfrak{g} \to A[-1]$. A *weak* $\mathfrak{g}$-equivariant quantization is a BV quantization A^{q} together with a map of BD algebras

$$\Phi^{\mathrm{q}} : U^{BD}_\alpha(\mathfrak{g}) \to A^{\mathrm{q}}$$

where $\alpha \in \hbar H^1(\mathfrak{g})[\hbar]$ is a twisting cocycle, that reduces modulo $\hbar$ to the map Φ. A weak $\mathfrak{g}$-equivariant quantization is *strong* if $\alpha = 0$.

The α-twisted BD envelope is defined by

$$U^{BD}_\alpha(\mathfrak{g}) = (\mathrm{Sym}^*(\mathfrak{g}[-1])[[\hbar]], \mathrm{d}_{\mathfrak{g}} + \hbar \mathrm{d}_{CE} + \alpha)\,.$$

Thus, in the case that $\alpha = 0$ we recover the ordinary BD envelope from Section BW: ref.

With the algebraic prerequisites in place, we are ready to discuss lifting this to the level of field theory.

[8] Really, one can imagine an L_∞ morphism.

4.3. **Classical symmetries of a classical BV theory.** In the BV formalism, the data of a classical field theory on X consists of a sheaf of fields $\mathcal{E}$, an action functional $S \in \mathcal{O}_{loc}(\mathcal{E})$ of degree zero, and a (-1)-shifted $\mathbb{C}$-valued pairing on $\mathcal{E}$. The pairing induces a bracket $\{-,-\}$ on the space of local functionals, and this data is required to satisfy the condition $\{S, S\} = 0$. This is known as the *classical master equation*.

Alternatively, we can view the shifted space of local functionals $\mathcal{O}_{loc}(\mathcal{E})$ as a dg Lie algebra. The differential is $\{S, -\}$ and the Lie bracket is $\{-,-\}$. The classical master equation is equivalent to the statement that S is a Maurer–Cartan element of this dg Lie algebra.

Let $\mathcal{L}$ be a local Lie algebra on X. Then, $\mathcal{L}(X)$ is an L_∞ algebra and we can consider its reduced Chevalley–Eilenberg cochain complex $C^*_{\mathrm{Lie,red}}(\mathcal{L}(X))$. This is a commutative dg algebra, so we can tensor with $\mathcal{O}_{loc}(\mathcal{E})[-1]$ to form the new dg Lie algebra $C^*_{\mathrm{Lie,red}}(\mathcal{L}(X)) \otimes \mathcal{O}_{loc}(\mathcal{E})[-1]$. The differential is of the form $\mathrm{d}_{\mathcal{L}} + \{S, -\}$, where $\mathrm{d}_{\mathcal{L}}$ is the CE differential for $\mathcal{L}(X)$, and the bracket is $\mathrm{id}_{\mathcal{L}} \otimes \{-,-\}$.

Definition 4.2. Let $\mathcal{L}$ be a local Lie algebra and $(\mathcal{E}, S)$ a classical theory. Define the dg Lie algebra

$$\mathrm{Act}(\mathcal{L}, \mathcal{E}) := C^*_{loc}(\mathcal{L}) \otimes \mathcal{O}_{loc}(\mathcal{E}) / \left(C^*_{loc}(\mathcal{L}) \oplus \mathcal{O}_{loc}(\mathcal{E})\right)$$

with differential and bracket given by the restriction of $\mathrm{d}_{\mathcal{L}} + \{S, -\}$ and $\{-,-\}$, respectively.

Note that $\mathrm{Act}(\mathcal{L}, \mathcal{E}) \subset C^*_{\mathrm{Lie,red}}(\mathcal{L}(X)) \otimes \mathcal{O}_{loc}(\mathcal{E})[-1]$ is an inclusion of dg Lie algebras. A functional $F \in C^*_{\mathrm{Lie,red}}(\mathcal{L}(X)) \otimes \mathcal{O}_{loc}(\mathcal{E})[-1]$ lives in $\mathrm{Act}(\mathcal{L}, \mathcal{E})$ if and only if:

(1) As a functional of $\mathcal{L}$, F is *local*, and

(2) The functional $S^{\mathcal{L}}$ must depend on both $\mathcal{L}$ and $\mathcal{E}$. We mod out by functionals that are of purely one or the other.

We can now define what it means for a local Lie algebra to be a symmetry.

Definition 4.3. Suppose $\mathcal{L}$ is a local Lie algebra and $(\mathcal{E}, S)$ defines a classical theory. An $\mathcal{L}$-*symmetry* of $\mathcal{E}$ is a functional $S^{\mathcal{L}} \in \mathrm{Act}(\mathcal{L}, \mathcal{E})$ that satisfies the *$\mathcal{L}$-equivariant classical master equation*:

$$\mathrm{d}_{\mathcal{L}} S^{\mathcal{L}} + \{S, S^{\mathcal{L}}\} + \frac{1}{2}\{S^{\mathcal{L}}, S^{\mathcal{L}}\} = 0.$$

Such an element $S^{\mathcal{L}}$ is automatically a Maurer–Cartan element of the dg Lie algebra $C^*_{\mathrm{Lie,red}}(\mathcal{L}(X)) \otimes \mathcal{O}_{loc}(\mathcal{E})[-1]$. By the general yoga of Koszul duality, a Maurer–Cartan element defines a map of L_∞ algebras

$$S^{\mathcal{L}} : \mathcal{L}(X) \to \mathcal{O}_{loc}(\mathcal{E})[-1].$$

This construction has consequences for the classical observables of the theory $\mathcal{E}$.

BW: recall classical obs

Proposition 4.4. *Suppose... BW: ! Then, for each open $U \subset X$, $S^{\mathcal{L}}$ determines a Hamiltonian action of $\mathcal{L}_c(U)$ on the P_0 algebra $\mathrm{Obs}^{\mathrm{cl}}(U)$. Thus, we have a map of dg Lie algebras*

$$\Phi_U : \mathcal{L}_c(U) \to \mathrm{Obs}^{\mathrm{cl}}(U)[-1].$$

Moreover, this map is compatible with inclusions of open sets and so determines a map of precosheaves of dg Lie algebras $\Phi : \mathcal{L}_c \to \mathrm{Obs}^{\mathrm{cl}}[-1]$.

This is the appearance of the co-moment map in the setting of classical BV theory. There is an immediate enhancement of this result to factorization algebras. Indeed, by the universal property of the P_0 envelope the map Φ determines a map of P_0 factorization algebras

$$\Phi : U^{P_0}(\mathcal{L}_c) \to \mathrm{Obs}^{\mathrm{cl}}.$$

4.4. **Quantum symmetries in the BV formalism.** We follow the approach of Costello [**?**] to perturbative QFT based on the Wilsonian renormalization of the path integral. We start with a space of fields $\mathcal{E}$ equipped with a square zero elliptic differential operator Q of cohomological degree zero, and a (-1)-shifted symplectic pairing. This is the data of a *free* theory in the classical BV formalism. A QFT is a family of functionals $\{S[L]\}$ ...

The main result of [**?**] says that associated to any QFT $(\mathcal{E}, S^{\mathrm{q}})$ defined on X there is a factorization algebra $\mathrm{Obs}^{\mathrm{q}}$ on X called the *quantum observables*.

Theorem 4.5 ([**?**] Theorem 12.5.0.1)**.** *Suppose we have an $\mathcal{L}$-symmetry of a QFT $(\mathcal{E}, S^{\mathrm{q}})$. Then, there is a cohomology class $\alpha_{\mathcal{E}} \in H^1_{red,loc}(\mathcal{L})[[\hbar]]$ such that the factorization Lie algebra $\mathcal{L}_c$ acts (up to homotopy) on the factorization algebra of quantum observables $\mathrm{Obs}^{\mathrm{q}}_{\mathcal{E}}[\hbar^{-1}]$ by $\alpha_{\mathcal{E}}$ times the identity.*

We will call $\alpha_{\mathcal{E}}$ the *anomaly cocycle* corresponding to the $\mathcal{L}$-symmetry. This cocycle $\alpha = \alpha_{\mathcal{E}}$ can be viewed as the "local character" for the action of the local Lie algebra $\mathcal{L}$ on the observables. Indeed, this statement implies that for any open set $U \subset X$ we have an action of the L_∞ algebra $\mathcal{L}_c(U)$ on $\mathrm{Obs}^{\mathrm{q}}(U)[\hbar^{-1}]$, and that this action is homotopy equivalent to the trivial action times the character α. Moreover, this homotopy equivalence is compatible with the factorization structure.

There is a convincing way to repackage this action of $\mathcal{L}_c$ on the quantum observables. Let $\mathrm{Obs}^{\mathrm{q}}_\alpha$ denote the $\mathcal{L}_c$-equivariant factorization algebra

$$\mathrm{Obs}^{\mathrm{q}} \otimes_{\mathbb{C}[[\hbar]]} \underline{\mathbb{C}}_\alpha[[\hbar]]$$

where $\mathbb{C}_\alpha[[\hbar]]$ denotes the $\mathbb{C}[[\hbar]]$-linear constant factorization algebra with action of $\mathcal{L}_c$ given by the character α. The theorem implies that there is a quasi-isomorphism of factorization algebras

$$\mathrm{C}^{\mathrm{Lie}}_*(\mathcal{L}_c, \mathrm{Obs}^{\mathrm{q}}_\alpha)[\hbar^{-1}] \simeq \mathrm{C}^{\mathrm{Lie}}_*(\mathcal{L}_c) \otimes \mathrm{Obs}^{\mathrm{q}}[\hbar^{-1}].$$

There is a natural augmentation map of factorization algebras $\epsilon : \mathrm{C}^{\mathrm{Lie}}_*(\mathcal{L}_c) \to \underline{\mathbb{C}}$ that projects onto the Sym^0 component. Furthermore, the unit observable $\mathbb{1} : \underline{\mathbb{C}} \to \mathrm{Obs}^{\mathrm{q}}$ defines a map of factorization algebras

$$\mathbb{1} : \mathrm{C}^{\mathrm{Lie}}_*(\mathcal{L}_c, \underline{\mathbb{C}}_\alpha[[\hbar]]) \to \mathrm{C}^{\mathrm{Lie}}_*(\mathcal{L}_c, \mathrm{Obs}^{\mathrm{q}}_\alpha).$$

Theorem 4.6 ([**?**])**.** *The composition defines a sequence of maps of factorization algebras*

$$\mathrm{C}^{\mathrm{Lie}}_*(\mathcal{L}_c, \mathbb{C}_\alpha[[\hbar]]) \xrightarrow{\mathbb{1}} \mathrm{C}^{\mathrm{Lie}}_*(\mathcal{L}_c, \mathrm{Obs}^{\mathrm{q}}_\alpha)[\hbar^{-1}] \simeq \mathrm{C}^{\mathrm{Lie}}_*(\mathcal{L}_c) \otimes \mathrm{Obs}^{\mathrm{q}}[\hbar^{-1}] \xrightarrow{\epsilon} \mathrm{Obs}^{\mathrm{q}}[\hbar^{-1}].$$

In summary, there is a map of factorization algebras

$$\Phi : \mathrm{C}^{\mathrm{Lie}}_{*,\alpha}(\mathcal{L}_c) \to \mathrm{Obs}^{\mathrm{q}}[\hbar^{-1}]$$

where $\mathrm{C}^{\mathrm{Lie}}_{*,\alpha}(\mathcal{L})$ *is the* $\mathbb{C}[[\hbar]]$*-linear twisted factorization envelope of* $\mathcal{L}$ *by* α*.*

BW: remark about Noether

4.5. **The anomaly for the $\beta\gamma$ system.** In the remainder of this section we examine an instance of the above situation for the current algebra acting on the free $\beta\gamma$ system with coefficients in a vector bundle. Our goal is to arrive at the index theorem over the moduli of principal G-bundles mentioned in the introduction of this section. We will also interpret Theorem 4.6 as providing a higher dimensional version of *free field realization* of the Kac–Moody factorization algebra.

Before discussing the specific example, we recount some facts about BV quantization for *free theories*.

4.5.1. *The quantization of free BV theories.* BW: Owen, please try to point to the correct references and adjust the way I state results so that it fits with your thesis / paper with Rune.

4.5.2. We now introduce the higher dimensional free $\beta\gamma$ system. This is a free BV theory defined on any complex d-fold X. Let V be a finite dimensional vector space. The fields are defined as

$$\mathcal{E}(X,V) = \Omega^{0,*}(X) \otimes V \oplus \Omega^{d,*}(X) \otimes V^\vee[d-1].$$

We denote a general field by (γ, β) according to the above decomposition. The action is

$$S(\gamma,\beta) = \int_X \langle \beta, \bar{\partial}\gamma \rangle$$

where the brackets $\langle -, - \rangle$ denote the obvious pairing between V and its dual.

Now, we are ready to state the main result about the anomaly cocycle for the Kac–Moody symmetry of the higher dimensional $\beta\gamma$ system.

OG: You make a line break before "end thm" so please also put one after "begin thm". It makes it easier to navigate the LaTeX.

Theorem 4.7. *Let* V *be a finite dimensional* $\mathfrak{g}$*-module and* X *any complex* d*-fold. There exists a one-loop exact* $\mathfrak{g}^X$*-symmetry of the quantum* $\beta\gamma$ *system valued in* V *quantizing the natural classical* $\mathfrak{g}^X$*-symmetry. Moreover, the anomaly cocycle* $\alpha_V \in H^1_{\mathrm{loc}}(\mathfrak{g}^X)$ *is identified with the image of*

$$\#\mathrm{ch}_{d+1}(V) \in \mathrm{Sym}^{d+1}(\mathfrak{g}^\vee)^{\mathfrak{g}}$$

under the map $J : \mathrm{Sym}^{d+1}(\mathfrak{g}^\vee)^{\mathfrak{g}}[-1] \to \mathrm{C}^*_{\mathrm{loc}}(\mathfrak{g}^X)$*.*

As a simple corollary, we find the anomaly in a slightly more general situation.

Corollary 4.8. *Let* P *be a principal* G*-bundle on* X*, and* V *a* G*-representation. Then we can consider the* $\mathfrak{g}^X_P = \Omega^{0,*}(X; \mathrm{ad}(P))$*-equivariant theory*

$$\mathcal{E}_{P \to X, V} = T^*[-1](\Omega^{0,*}(X; P \times^G V)).$$

This theory admits a canonical $\mathfrak{g}^X_P$*-equivariant quantization. Moreover, the cohomology class of the obstruction* $[\Theta_V]$ *to an inner action is also identified with* $\#\mathrm{ch}_{d+1}(V)$*.*

We will prove the proposition in the following steps. First, we argue that it suffices to calculate this obstruction on an arbitrary open set in X. Taking this open set to be a disk we see that it is enough to compute the cocycle in the case that $X = \mathbb{C}^d$. In this case, we find a quantization that is actually finite at the one-loop level. This means that there are no counterterms necessary, and we can explicitly calculate the cocycle in terms of the weight of a simple one-loop Feynman diagram.

4.5.3. *The reduction to a disk.* By construction, the data of a classical BV theory on X is sheaf-like on the manifold. That is, we have a sheaf of (-1)-shifted elliptic complexes $\mathcal{E}$ on X together with a local functional $I \in \mathcal{O}_{\mathrm{loc}}(\mathcal{E})(X)$. The space of local functionals $\mathcal{O}_{\mathrm{loc}}(\mathcal{E})$ also forms a sheaf on X, so it makes sense to restrict I to any open set $U \subset X$. In this way, for each open we have a (-1)-shifted elliptic complex $\mathcal{E}(U)$ together with a local functional $I|_U \in \mathcal{O}_{\mathrm{loc}}(\mathcal{E})(U)$ – that is, a classical field theory on $U \subset X$. A fancy way of saying this is that the space of classical field theories on X forms a sheaf.

A very slightly refined version of this takes into account an action of a local Lie algebra. If $\mathcal{L}$ is a local Lie algebra on X then the space of $\mathcal{L}$-equivariant classical BV theories also forms a sheaf on X.

Costello has shown in [**?**] that the space of quantum field theories also form a sheaf on X. In a completely analogous way, one can show that the space of $\mathcal{L}$-equivariant quantum field theories forms a sheaf on X.

We have already seen how the obstruction to lifting a quantum field theory with an action of a local Lie algebra $\mathcal{L}$ to an inner action arises as a failure of satisfying the QME. Since an $\mathcal{L}$-equivariant theory satisfies the QME modulo terms in $C^*_{\mathrm{loc}}(\mathcal{L})(X)$, this obstruction $\Theta(X)$is a degree one cocycle in $C^*_{\mathrm{loc}}(\mathcal{L})(X)$. By the remarks above, we can restrict any $\mathcal{L}$-equivariant field theory to an arbitrary open set $U \subset X$. Hence, for each open $U \subset X$ we have an obstruction element Θ^U. The complex $C^*_{\mathrm{loc}}(\mathcal{L})(X)$ also has a refinement to a sheaf of complexes on X and the obstruction Θ^U is an element in $C^*_{\mathrm{loc}}(\mathcal{L})(U)$. We will need the following elementary fact that the obstruction to having an inner action is natural with respect to the restriction of open sets.

Lemma 4.9. *Let $i_U^V : U \hookrightarrow V$ be any inclusion of open sets in X. Then*

$$(i_U^V)^*([\Theta^V]) = [\Theta^U]$$

where $(i_U^V)^* : C^*_{\mathrm{loc}}(\mathcal{L})(V) \to C^*_{\mathrm{loc}}(\mathcal{L})(U)$ is the restriction map and the brackets $[-]$ denotes the cohomology class of the cocycle. In other words, the map that sends a quantum field theory on X with an $\mathcal{L}$-action to its obstruction to having an inner $\mathcal{L}$-action is a map of sheaves.

For any complex d-fold X we have defined the map $J^X : \mathrm{Sym}^{d+1}(\mathfrak{g}^\vee)^{\mathfrak{g}} \to C^*_{\mathrm{loc}}(\mathfrak{g}^X)$. The complex $C^*_{\mathrm{loc}}(\mathfrak{g}^X)$

Lemma 4.10. *The map*

$$J : \underline{\mathrm{Sym}^{d+1}(\mathfrak{g}^\vee)^{\mathfrak{g}}} \to C^*_{\mathrm{loc}}(\mathfrak{g}^X)$$

defined on each open by $J|_U = J^U$ is a map of sheaves. Here, the underline means the constant sheaf.

Lemma 4.11. *For any open sets $i_U^V : U \subset V$ in X the induced map*

$$(i_U^V)^* : H^1\left(V; \mathrm{C}^*_{\mathrm{loc}}(\mathfrak{g}^X)\right) \to H^1\left(U; \mathrm{C}^*_{\mathrm{loc}}(\mathfrak{g}^X)\right)$$

is injective.

BW: The last key observation is that $(i_U^V)^* J^V = J^U$.

4.5.4. *The theory on a disk.*

4.5.5. *The Heisenberg algebra.* In ordinary classical mechanics, the Heisenberg algebra is a convenient tool to construct the deformation quantization for quadratic Hamiltonians. This construction carries over for symplectic dg vector spaces. We will use it to give a model for the sphere observables of the $\beta\gamma$ system. Furthermore, we provide a map from the sphere Lie algebra $\widehat{\mathfrak{g}}_{d,\theta}$ to a completion of this algebra as a corollary of the Theorem 4.6.

Let A_d be the commutative dg algebra from Section **??** and V a finite dimensional vector space. Consider the dg (0-shfted) symplectic vector space

$$W_d(V) = A_d \otimes V \oplus A_d \otimes V^\vee[d-1]$$

with pairing defined by

$$\omega_W(a \otimes v, b \otimes v^\vee) = \langle v, v^\vee \rangle \oint_{S^{2d-1}} a \wedge b$$

where $\oint_{S^{2d-1}}$ is the higher residue and $\langle v, v^\vee \rangle$ denotes the pairing between V and its dual. Clearly ω_W is non-degenerate and it is immediate to check that $\mathrm{d}\omega = 0$ where d is the differential on A_d, so that ω indeed defines a symplectic structure.

5. Higher Kac–Moody as a boundary theory

In this section we show how the Kac–Moody factorization algebra appears as the boundary of a twist of supersymmetric gauge theory.

This example extrapolates the ubiquitous relationship between Chern–Simons theory on a3-manifold and the Wess-Zumino-Witten conformal field theory. BW: expand on this

The five dimensional gauge theory we consider is obtained as a twist of $\mathcal{N} = 1$ supersymmetric pure gauge theory. This twist is not topological, but it is holomorphic in four real (two complex) directions, and topological in the transverse direction. We write down a boundary condition on manifolds of the form $X \times \mathbb{R}_{\geq 0}$, where X is a Calabi–Yau surface, at $X \times \{0\}$. Recall, the observables of any theory determine a factorization algebra on the manifold in which the theory lives. Likewise, this boundary condition determines a factorization algebra of classical observables supported on the boundary. At the classical level, we find that this factorization algebra is the classical limit Kac–Moody factorization algebra on X. We show that there is a quantization of this theory that returns the Kac–Moody at a specified level.

Remark 5.1. BW: 7d-6d example

5.1. **5d $\mathcal{N} = 1$ supersymmetric gauge theory.** We first provide a description of 5d $\mathcal{N} = 1$ pure gauge theory. The $\mathcal{N} = 1$ supersymmetry algebra in 5d is of the form

$$(\mathfrak{so}(5,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})_R) \ltimes T_{5d}^{\mathcal{N}=1}$$

where $T_{5d}^{\mathcal{N}=1}$ is the super Lie algebra of $\mathcal{N} = 1$ supertranslations. The copy of $\mathfrak{sl}(2,\mathbb{C})_R$ is the R-symmetry Lie algebra. As a super vector space the supertranslations are

$$T_{5d}^{\mathcal{N}=1} = V_{5d} \oplus \Pi(S_{5d} \otimes \mathbb{C}^2_R$$

where $V \cong \mathbb{C}^5$ is the fundamental representation of $\mathfrak{so}(5,\mathbb{C})$ and S is the irreducible spin representation. As a complex vector space S is four-dimensional BW: check that. The Π indicates that S is placed in super degree $+1$. The only non-trivial Lie bracket in $T_{5d}^{\mathcal{N}=1}$ is of the form

$$[-,-] : (S_{5d} \otimes \mathbb{C}^2_R) \otimes (S_{5d} \otimes \mathbb{C}^2_R) \to V_{5d}.$$

To describe it, introduce the exterior wedge product

$$\wedge : S_{5d} \otimes S_{5d} \to V_{5d}.$$

where we have used the spin invariant isomorphism $\wedge^2 S_{5d} \cong V_{5d}$. Also, fix the standard holomorphic symplectic pairing ω on $\mathbb{C}^2_R$. The bracket is defined by $[\psi_1 \otimes v_1, \psi_2 \otimes v_2] = (\psi_1 \wedge \psi_2)\omega(v_1, v_2)$. The vector multiplet of this algebra consists of a vector, a scalar, and a spinor.

Let G be a complex algebraic group and $\mathfrak{g}$ its Lie algebra. The fields of 5d $\mathcal{N} = 1$ pure gauge theory are given by a connection A, a scalar ϕ, and a spinor λ

$$\begin{aligned} &A \in \Omega^1(\mathbb{R}^5) \otimes \mathfrak{g} \\ &\phi \in C^\infty(\mathbb{R}^5) \otimes \mathfrak{g} \\ &\lambda \in C^\infty(\mathbb{R}^5) \otimes (S_{5d} \otimes \mathbb{C}^2_{\mathbb{R}}) \otimes \mathfrak{g}. \end{aligned}$$

The action functional is

$$S_{5d}^{\mathcal{N}=1}(A, \phi, \lambda) = \int_{\mathbb{R}^6} F(A) \wedge \star F(A) + \lambda \not{\partial}_A \lambda + \ldots$$

Proposition 5.2. *There is a twist of 5d $\mathcal{N} = 1$ supersymmetric pure gauge theory that exists on any manifold of the form $X \times S$ where X is a Calabi–Yau surface and S is a real one-dimensional manifold. Choosing local holomorphic coordinates z_i on X and a real coordinate t on S, the fields consist of a $\mathfrak{g}$-valued connection one-form*

$$A = A_1 \mathrm{d}\bar{z}_1 + A_2 \mathrm{d}\bar{z}_2 + A_t \mathrm{d}t \quad , \quad A_i, A_t \in C^\infty(X \times S) \otimes \mathfrak{g},$$

together with a $\mathfrak{g}^*$-valued one-form

$$B = B_1 \mathrm{d}\bar{z}_1 + B_2 \mathrm{d}\bar{z}_2 + B_t \mathrm{d}t \quad , \quad B_i, B_t \in C^\infty(X \times S) \otimes \mathfrak{g}^*.$$

The action functional is

$$S(A, B) = \int_{X \times \mathbb{R}} \Omega \left(B \mathrm{d}A + \frac{1}{3} B[A, A] \right)$$

where Ω is the holomorphic volume form on X.

We obtain this result by a dimensional reduction of a twist of 6d $\mathcal{N} = (1,0)$ pure gauge theory.

5.2. 5d $\mathcal{N}=1$ **from** 6d $\mathcal{N}=(1,0)$**.** It is known in the literature that 5d $\mathcal{N}=1$ gauge theory can be obtained from $\mathcal{N}=(1,0)$ gauge theory in six dimensions via dimensional reduction. BW: pestun lecture notes. there must be more references though At the level of the supersymmetry algebra this is clear to see. BW: do this

In BW: ref Butson, Costello, Gaiotto it is shown that there is a holomorphic twist of 6d $\mathcal{N}=(1,0)$ gauge theory that exists on any Calabi–Yau 3-fold Y. The fields consist of a $(0,1)$-form valued in $\mathfrak{g}$:

$$A \in \Omega^{0,1}(Y) \otimes \mathfrak{g}$$

together with a $(0,1)$-form valued in $\mathfrak{g}^*$:

$$B \in \Omega^{0,1}(Y) \otimes \mathfrak{g}^*.$$

The action functional is

$$S^{twist}_{6\mathrm{d}}(A,B) = \int_Y \Omega_Y \left(\langle B, \bar\partial A\rangle + \langle B, [A,A]\rangle\right)$$

where Ω_Y is the holomorphic volume form on Y.

Remark 5.3. There is a concise geometric description of this twist as an AKSZ type theory. Let Y be a 3-fold equipped with a holomorphic volume form as above. To any holomorphic symplectic manifold Z there is an associated complex three-dimensional AKSZ theory of maps $\mathrm{Map}(Y,Z)$. This is holomorphic version of Rozansky–Witten theory, and is spelled out in [**?**], for instance. Suppose $\mathfrak{g}$ is the Lie algebra of a complex algebraic group G. The theory above is holomorphic Rozansky–Witten theory for the (derived) symplectic reduction $*//G$. [9]

We now see how the reduction of this twisted theory from six dimensions down to five dimensions is equal to the description of our 5d theory in Proposition 5.2. Choose holomorphic coordinates z_1, z_2, z_3 on Y and write $z_3 = t + iy$. We are reducing along the real y-coordinate. Write $A = A_1 \mathrm{d}\bar z_1 + A_2 \mathrm{d}\bar z_2 + A_3 \mathrm{d}\bar z_3$ for the theory on Y. In the reduced theory this becomes $A^{5\mathrm{d}} = A_1^{5\mathrm{d}}\mathrm{d}\bar z_1 + A_2^{5\mathrm{d}}\mathrm{d}\bar z_2 + A_t^{5\mathrm{d}}\mathrm{d}t$ where $A_i^{5\mathrm{d}}$ and $A_t^{5\mathrm{d}}$ are valued in $\mathfrak{g}$. Similarly, the B field reduces to $B^{5\mathrm{d}} = B_1^{5\mathrm{d}}\mathrm{d}\bar z_1 + B_2^{5\mathrm{d}}\mathrm{d}\bar z_2 + B_t^{5\mathrm{d}}\mathrm{d}t$.

Now, consider the quadratic term in the twisted 6d action functional. BW: finish...

We have computed the twist of 5d $\mathcal{N}=1$ at the level of the physical fields. We are interested in a refined version of this, that is, a description of the twist of the classical theory in the BV-BRST formalism including the ghosts, anti-fields, etc..

Proposition 5.4. *The holomorphic/topological twist of* 5d $\mathcal{N}=1$ *in the BV formalism has space of fields*

$$(\alpha,\beta) \in \Omega^{0,*}(X) \otimes \Omega^*(S) \otimes (\mathfrak{g}\oplus\mathfrak{g}^*)[1],$$

where α is a form valued in $\mathfrak{g}$ and β is a form valued in $\mathfrak{g}^*$. The action functional is

$$S(\alpha,\beta) = \frac{1}{2}\int \beta(\mathrm{d}_{dR} + \bar\partial)\alpha \wedge \Omega + \frac{1}{6}\int \beta[\alpha,\alpha]\wedge\Omega$$

[9] Note that this endows the mapping space $\mathrm{Map}(Y,Z)$ with a (-3)-shifted symplectic structure, as opposed to the familiar (-1)-shifted symplectic structure....

We will denote the full complex of fields of the 5d gauge theory by $\mathcal{E}$. As is usual in the BV formalism, there is an associated deformation complex consisting of local functionals $\mathcal{O}_{\mathrm{loc}}(\mathcal{E})$ equipped with the differential $\{S, -\}$. Cocycles in this complex consist of all the possible deformations of the theory.

There is a deformation that is particularly relevant to finding the Kac–Moody factorization algebra on the the boundary of the 5-dimensional theory. Recall, that an invariant polynomial $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$ determines a local cocycle of the current algebra on any complex d-fold. When $d = 2$ we see that such an element θ also determines a deformation of the classical gauge theory.

Lemma 5.5. *Let $\theta \in \mathrm{Sym}^3(\mathfrak{g}^*)^{\mathfrak{g}}$. Define the local functional*

$$F_\theta(\alpha, \beta) = \int_{X \times S} \theta(\alpha \partial \alpha \partial \alpha).$$

Then, F_θ defines a deformation of the classical gauge theory. In other words, the functional $S + F_\theta$ satisfies the classical master equation

$$\{S + F_\theta, S + F_\theta\} = 0.$$

Remark 5.6. It is immediate to check that the degree of F_θ in $\mathcal{O}_{\mathrm{loc}}(\mathcal{E})$ is zero. If we were only writing the part of F_θ involving the physical fields it would be of the form $\int \theta(A \partial A \partial A)$. Also, our convention for evaluating $\theta(\alpha \partial \alpha \partial \alpha)$ is the same as above. We take the wedge product of the form component and evaluate θ on the Lie algebra component.

5.3. **The classical boundary observables.** We now turn to studying the boundary observables of the 5-dimensional gauge theory introduced in the previous sections. We place the theory on a manifold of the form $X \times \mathbb{R}_{\geq 0}$ where X is a Calabi–Yau surface.

To specify this classical theory we need to choose a boundary condition at $X \times \mathbb{R}_{\geq 0}$. The space of fields restricted to the boundary is

$$\mathcal{E}^\partial = \Omega^{0,*}(X) \otimes (\mathfrak{g} \oplus \mathfrak{g}^*)[1]$$

Denote by $\alpha^\partial, \beta^\partial$ the restriction of the fields α, β to the boundary. Note that space of fields restricted to the boundary is a sheaf of sections of a graded vector bundle on X. Moreover, $\mathcal{E}^\partial$ is equipped with a (0-shifted) symplectic structure given by

$$\omega^\partial(\alpha^\partial, \beta^\partial) = \int_X \alpha^\partial \beta^\partial \Omega.$$

The boundary condition is given by setting $\alpha|_{X \times \{0\}} = \alpha^\partial = 0$. Equivalently, we represent the boundary condition by the Lagrangian subspace

$$\mathcal{L} = \Omega^{0,*}(X) \otimes \mathfrak{g}^*[1] \hookrightarrow \mathcal{E}^\partial.$$

Proposition 5.7. *Consider the 5-dimensional theory $(\mathcal{E}, S)$ placed on the manifold $X \times \mathbb{R}_{\geq 0}$ with X Calabi–Yau. The factorization algebra of classical boundary observables with respect to the Lagrangian $\mathcal{L}$ is equivalent to the classical limit of the Kac–Moody factorization algebra on X from BW: ref.*

Recall that one can endow the structure of a P_0 factorization algebra on the classical limit of the Kac–Moody for every degree one local cocycle of the current algebra.

Proposition 5.8. *Fix an element $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$. If we turn on the deformation F_θ, the factorization algebra of boundary observables is equivalent as a P_0-factorization algebra to the classical limit of the Kac–Moody factorization algebra with P_0 structure determined by the local cocycle $J(\theta)$.*

BW: Enhancement to arbitrary principal bundle. Gauge theory will be valued in the adjoint bundle.

5.4. **The quantum boundary observables.** We now turn to the quantum observables on the boundary of the 5-dimensional gauge theory. Classically, we have just seen that on the boundary we find the P_0-envelope of the current algebra. There is a naïve quantization of this P_0-envelope given by the factorization enveloping algebra.

Theorem 5.9. *There exists an exact one-loop quantization of the holomorphic/topological twist of 5-dimensional $\mathcal{N} = 1$ gauge theory deformed by the term F_θ on $\mathbb{C}^2 \times \mathbb{R}_{\geq 0}$. The factorization algebra of quantum boundary observables on $\mathbb{C}^2$ is equivalent to the Kac–Moody factorization algebra $\mathbb{U}_{\theta_\hbar}(\Omega^{0,*}(\mathbb{C}^2) \otimes \mathfrak{g})$ where the $\hbar$-dependent level is*

$$\theta_\hbar = \theta + \#\hbar \mathrm{ch}_3^{\mathfrak{g}}(\mathfrak{g}) \in \mathrm{Sym}^3(\mathfrak{g}^*)^{\mathfrak{g}}[\hbar].$$

BW: this is analogous to the usual *shift* by the critical level in the quantization of CS/WZW

Appendix A. L_∞ algebras and their modules

BW: this may be an unnecessary section. Want to stress that KHF do not write down an explicit L_∞-model but it will often be convenient for us to use one.

OG: I think we should skip this.

Suppose V is a dg vector space. Then, the symmetric algebra

$$\mathrm{Sym}(V) := \prod_k \mathrm{Sym}^k(V)$$

has the natural structure of a dg cocommutative coalgebra.

Definition A.1. An L_∞ *algebra* is a dg vector space V together with a coderivation

$$D : \mathrm{Sym}(V) \to \mathrm{Sym}(V).$$

A *morphism* of L_∞ algebras $f : (V, D) \to (V', D')$ is a morphism of dg cocommutative coalgebras

$$f : (\mathrm{Sym}(V), D) \to (\mathrm{Sym}(V'), D') .$$

Denote the category of L_∞ algebras by $L_\infty\mathrm{Alg}$.

The complex $(\mathrm{Sym}(V), D)$ is the complex of Chevalley-Eilenberg chains of the L_∞ algebra $\mathfrak{g} = (V, D)$. In the case of a dg Lie algebra this is the usual complex of Chevalley-Eilenberg chains. Without loss of generality we denote this complex by $\mathrm{C}^{\mathrm{Lie}}(\mathfrak{g})$ just as in the classical case.

We may a remark about dg Lie algebras and their close relatives, L_∞ algebras.

Theorem A.2. BW: Kriz and May? *Every L_∞ algebra (V, D) is quasi-isomorphic (in the category $L_\infty\mathrm{Alg}$) to a dg Lie algebra.*

By an L_∞ algebra model for a dg Lie algebra $\mathfrak{g}$, we mean an L_∞ algebra (L, D) together with a quasi-isomorphism $(L, D) \simeq \mathfrak{g}$.

A.1. **Extensions from cocycles.** Suppose $\mathfrak{g}$ is a dg Lie algebra. Let $\theta \in \mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g})$ be a cocycle of degree 2, so its cohomology class is an element $[\theta] \in H^2_{\mathrm{Lie}}(\mathfrak{g})$. By BW: ref, we know that θ determines a central extension in the category of dg Lie algebras:

$$0 \to \mathbb{C} \cdot K \to \widehat{\mathfrak{g}} \to \mathfrak{g} \to 0$$

that only depends, up to isomorphism, on the cohomology class of θ.

The explicit dg Lie algebra structure on $\widehat{\mathfrak{g}}$ may be tricky to describe. However, if we are willing to work in the category of L_∞ algebras, there is an explicit model for $\mathfrak{g}$ as an L_∞ algebra. The underlying dg vector space for the L_∞ algebra is the same as that of the dg Lie algebra, $\widehat{\mathfrak{g}} \oplus \mathbb{C} \cdot K$. To equip this with an L_∞ structure we need to provide a coderivation $D = D_1 + D_2 + \cdots$ for the cocommutative coalgebra $\mathrm{Sym}(\mathfrak{g} \oplus \mathbb{C} \cdot K) = \prod_k \mathrm{Sym}^k(\mathfrak{g} \oplus \mathbb{C} \cdot K)$. Indeed, we define

$$\begin{aligned} D_1(X_1) &= \mathrm{d}_{\mathfrak{g}}(X_1) + \theta(X_1) \\ D_2(X_1, X_2) &= [X_1, X_2]_{\mathfrak{g}} + \theta(X_1, X_2) \\ D_k(X_1, \ldots, X_k) &= \theta(X_1, \ldots, X_k) \ , \ \text{for } k \geq 3. \end{aligned}$$

One immediately checks that $(\mathfrak{g} \oplus \mathbb{C}, D)$ is an L_∞ model for $\widehat{\mathfrak{g}}$.

Example A.3. As an example, consider the following L_∞ model for the dg Lie algebra $\widehat{\mathfrak{g}}_{d,\theta}$. As a dg vector space $\widehat{\mathfrak{g}}_{d,\theta}$ is of the form $A_d \otimes \mathfrak{g} \oplus \mathbb{C} \cdot K$. The only nonzero components of the coderivation determining the L_∞ structure are D_1,D_2, and D_{d+1} and they are determined by $D_1(aX) = (\bar{\partial} a)X$, $D_2(aX, bY) = (a \wedge b)[X, Y]_{\mathfrak{g}}$, and

$$D_{d+1}(a_0 X_0, \ldots, a_d X_d) = \operatorname*{Res}_{z=0} (a_0 \wedge \partial a_1 \wedge \cdots \wedge \partial a_d)\, \theta(X_0, \ldots, X_d) \cdot K.$$

Lemma A.4. *Suppose $\mathfrak{g}$ is an L_∞ algebra and we are given two central extensions*

$$0 \to \mathbb{C} \cdot K[k] \to \widetilde{\mathfrak{g}}, \widetilde{\mathfrak{g}}' \to \mathfrak{g} \to 0$$

of L_∞ algebras by the trivial module placed in degree $-k$. Suppose that the cocycles determining the central extensions differ by an exact cocycle of the form $\mathrm{d}\eta \in \mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g})$ where η is a cochain of degree $k+1$. Then, the map

$$\mathrm{id} + \eta \cdot K : \mathrm{C}_*^{\mathrm{Lie}}(\widetilde{\mathfrak{g}}) \to \mathrm{C}_*^{\mathrm{Lie}}(\widetilde{\mathfrak{g}}')$$

determines an L_∞-isomorphism $\widetilde{\mathfrak{g}} \cong \widetilde{\mathfrak{g}}'$.

In the lemma above the map $\mathrm{id} + \eta$ sends the element $X_1 \cdots X_n \in \mathrm{Sym}^n(\mathfrak{g})$ to $X_1 \cdots X_n + \eta(X_1, \ldots, X_n) \cdot K$ and is the identity on the subspace generated by the central element K.

Appendix B. Homotopy Poisson structures