FORMULAS FOR NOETHER'S MAP

Recall the map of precosheaves of dg Lie algebras on $\mathbb{R}_{>0}$

$$\pi_{\mathfrak{g},d} : \Omega^*_{\mathbb{R}_{>0},c} \otimes \mathfrak{g}^\bullet_d \to r_* \mathcal{G}_d.$$

Specializing this map on an open interval $I \subset \mathbb{R}$ yields a map of dg Lie algebras

$$\pi_{\mathfrak{g},d}(I) : \Omega^*_c(I) \otimes \mathfrak{g}^\bullet_d \to \mathcal{G}_d(r^{-1}(I)).$$

Since $\Omega^*_c(I) \simeq \mathbb{C}[-1]$ for any open interval, this specialization results in a map of graded Lie algebras

$$H^*\pi_{\mathfrak{g},d}(I) : H^*(\mathfrak{g}^\bullet_d)[-1] = H^*(A_d) \otimes \mathfrak{g}[-1] \to H^*\mathcal{G}_d(r^{-1}(I))$$

at the level of cohomology.

With this notation set up, we have an explicit description of Noether's map when we evaluate on open sets of the form $r^{-1}(I) \subset \mathbb{C}^d$.

Lemma 0.1. *In cohomology, the composition of Noether's map $r_* J^{\mathrm{cl}}$ with $\pi_{\mathfrak{g},d}(I)$*

$$H^*(A_d) \otimes \mathfrak{g} \xrightarrow{H^*\pi_{\mathfrak{g},d}} H^* r_* \mathbb{U}(\mathcal{G}_d) \xrightarrow{r_* J^{\mathrm{cl}}} H^* r_* \mathrm{Obs}^{\mathrm{q}}_{\beta\gamma}$$

sends an element $\alpha \otimes X \in H^*(A_d) \otimes \mathfrak{g}$ to the quadratic observable

$$(\gamma \otimes v, \beta \otimes v^*) \mapsto \langle v^*, X \cdot v \rangle_V \oint_{S^{2d-1}} \beta \wedge \alpha \wedge \gamma.$$

In the quantum case, we have the following

Lemma 0.2. *In cohomology, the composition of Noether's map $r_* J^{\mathrm{q}}$ with $\mathbb{U}\pi_{\mathfrak{g},d}(I)$*

$$H^*\left(U\widehat{\mathfrak{g}}_{d,\mathrm{ch}^{\mathfrak{g}}_{d+1}(V)}\right) \xrightarrow{H^*\mathbb{U}\pi_{\mathfrak{g},d}} H^* r_* \mathbb{U}_{\mathrm{ch}^{\mathfrak{g}}_{d+1}(V)}(\mathcal{G}_d) \xrightarrow{r_* J^{\mathrm{q}}} H^*\left(r_* \mathrm{Obs}^{\mathrm{q}}_{\beta\gamma}\Big|_{\hbar = (2\pi i)^d}\right) \tag{1}$$

sends a linear element $\alpha \otimes X \in H^*(A_d) \otimes \mathfrak{g}$ to the quadratic observable

$$(\gamma \otimes v, \beta \otimes v^*) \mapsto \langle v^*, X \cdot v \rangle_V \oint_{S^{2d-1}} \beta \wedge \alpha \wedge \gamma.$$

Since each of the maps in (1) is a map of (graded) algebras, the composition is determined by its value on the linear elements $H^*(A_d) \otimes \mathfrak{g}$. We also note that from the formula above, it is clear the composition factors through the cohomology of $U(\mathcal{H}_V)$ to give a map

$$H^*\left(U\widehat{\mathfrak{g}}_{d,\mathrm{ch}^{\mathfrak{g}}_{d+1}(V)}\right) \to H^* U(\mathcal{H}_V).$$

BW: Maybe we should say why residues live in $U(\mathcal{H}_V)$.