**HIGHER KAC-MOODY**

BW: Add intro comparing to Kapranov-Hennion-Faonte.

## 1. The local Lie algebra

Let $X$ be a fixed complex $d$-fold and let $\mathfrak{g}$ be a Lie algebra. (We assume it is an ordinary Lie algebra, but slight modifications will allow one to handle dg Lie or $L_\infty$ algebras.) For each open set $U \subset X$ define

$$\mathfrak{g}^X(U) = \Omega^{0,*}(U) \otimes \mathfrak{g}.$$

The $\bar{\partial}$ differential for $U$ extended naturally to $\mathfrak{g}^X(U)$ by $\bar{\partial} \otimes 1$. Moreover, $\mathfrak{g}^X(U)$ has a natural Lie bracket defined by the rule

$$[\omega \otimes X, \omega' \otimes X'] = \omega \wedge \omega' \otimes [X, X']_{\mathfrak{g}}$$

where $[-,-]_{\mathfrak{g}}$ is the Lie bracket for $\mathfrak{g}$. Thus, $\mathfrak{g}^X(U)$ has the structure of a dg-Lie algebra.

**Lemma 1.1.** *The assignment $\mathfrak{g}^X : U \mapsto \mathfrak{g}^X(U)$ defines a local Lie algebra.*

1.1. **dg vs $L_\infty$.** BW: this may be an unnecessary section. Want to stress that KHF do not write down an explicit $L_\infty$-model but it will often be convenient for us to use one.

Suppose $V$ is a dg vector space. Then, the symmetric algebra

$$\mathrm{Sym}(V) := \prod_k \mathrm{Sym}^k(V)$$

has the natural structure of a dg cocommutative coalgebra.

**Definition 1.2.** An $L_\infty$ *algebra* is a dg vector space $V$ together with a coderivation

$$D : \mathrm{Sym}(V) \to \mathrm{Sym}(V).$$

A *morphism* of $L_\infty$ algebras $f : (V, D) \to (V', D')$ is a morphism of dg cocommutative coalgebras

$$f : (\mathrm{Sym}(V), D) \to \left(\mathrm{Sym}(V'), D'\right).$$

Denote the category of $L_\infty$ algebras by $L_\infty\mathrm{Alg}$.

We may a remark about dg Lie algebras and their close relatives, $L_\infty$ algebras.

**Theorem 1.3.** BW: Kriz and May? *Every $L_\infty$ algebra $(V, D)$ is quasi-isomorphic (in the category $L_\infty\mathrm{Alg}$) to a dg Lie algebra.*

By an $L_\infty$ algebra model for a dg Lie algebra $\mathfrak{g}$, we mean an $L_\infty$ algebra $(L, D)$ together with a quasi-isomorphism $(L, D) \simeq \mathfrak{g}$.

Suppose $\mathfrak{g}$ is a dg Lie algebra. Let $\theta \in C^*_{\mathrm{Lie}}(\mathfrak{g})$ be a cocycle of degree 2, so its cohomology class is an element $[\theta] \in H^2_{\mathrm{Lie}}(\mathfrak{g})$. By BW: ref, we know that $\theta$ determines a central extension in the category of dg Lie algebras:

$$0 \to \mathbb{C} \cdot K \to \widehat{\mathfrak{g}} \to \mathfrak{g} \to 0$$

that only depends, up to isomorphism, on the cohomology class of $\theta$.

The explicit dg Lie algebra structure on $\widehat{\mathfrak{g}}$ may be tricky to describe. However, if we are willing to work in the category of $L_\infty$ algebras, there is an explicit model for $\mathfrak{g}$ as an $L_\infty$ algebra. The underlying dg vector space for the $L_\infty$ algebra is the same as that of the dg Lie algebra, $\widehat{\mathfrak{g}} \oplus \mathbb{C} \cdot K$. To equip this with an $L_\infty$ structure we need to provide a coderivation $D = D_1 + D_2 + \cdots$ for the cocommutative coalgebra $\mathrm{Sym}(\mathfrak{g} \oplus \mathbb{C} \cdot K) = \prod_k \mathrm{Sym}^k(\mathfrak{g} \oplus \mathbb{C} \cdot K)$. Indeed, we define

$$\begin{array}{lcl} D_1(X_1) & = & \mathrm{d}_{\mathfrak{g}}(X_1) + \theta(X_1) \\ D_2(X_1, X_2) & = & [X_1, X_2]_{\mathfrak{g}} + \theta(X_1, X_2) \\ D_k(X_1, \ldots, X_k) & = & \theta(X_1, \ldots, X_k) \ , \ \text{for } k \geq 3. \end{array}$$

One immediately checks that $(\mathfrak{g} \oplus \mathbb{C}, D)$ is an $L_\infty$ model for $\widehat{\mathfrak{g}}$.

*Example* 1.4. As an example, consider the following $L_\infty$ model for the dg Lie algebra $\widehat{\mathfrak{g}}_{d,\theta}$. As a dg vector space $\widehat{\mathfrak{g}}_{d,\theta}$ is of the form $A_d \otimes \mathfrak{g} \oplus \mathbb{C} \cdot K$. The only nonzero components of the coderivation determining the $L_\infty$ structure are $D_1$,$D_2$, and $D_{d+1}$ and they are determined by $D_1(aX) = (\bar\partial a)X$, $D_2(aX, bY) = (a \wedge b)[X, Y]_{\mathfrak{g}}$, and

$$D_{d+1}(a_0X_0, \dots, a_dX_d) = \operatorname*{Res}_{z=0} (a_0 \wedge \partial a_1 \wedge \cdots \wedge \partial a_d)\, \theta(X_0, \dots, X_d) \cdot K.$$

## 2. Local cocycles from polynomials

Being a local Lie algebra we can consider its local Chevalley-Eilenberg complex. It defined as

$$\mathrm{C}^*_{\mathrm{loc}}(\mathfrak{g}^X) =$$

Recall, a local $k$-cocycle of a local Lie algebra determines a $(k-2)$-shifted central extension, by the constant sheaf $\underline{\mathbb{C}}$. We are interested in $(-1)$-shifted central extensions, and hence, local 1-cocycles. For $\mathfrak{g}^X$ we can describe such a family of 1-cocycles.

Let $P$ be an invariant polynomial of $\mathfrak{g}$ of homogenous degree $d+1$. That is, $P \in \mathrm{Sym}^{d+1}(\mathfrak{g}^\vee)^{\mathfrak{g}}$. We can extend $P$ to a functional on $\Omega^{0,*}(X) \otimes \mathfrak{g}$ by the rule

$$\begin{array}{cccc} P^X : & \mathrm{Sym}^{d+1}(\Omega^{0,*}(X) \otimes \mathfrak{g}) & \to & \mathbb{C} \\ & (\omega_1 \otimes X_1, \dots, \omega_{d+1} \otimes X_{d+1}) & \mapsto & (\omega_1 \wedge \cdots \wedge \omega_{d+1}) P(X_1, \dots, X_{d+1}) \end{array}$$

**Proposition 2.1.** *The assignment*

$$J : \mathrm{Sym}^{d+1}(\mathfrak{g}^\vee)^{\mathfrak{g}}[-1] \to \mathrm{C}^*_{\mathrm{loc}}(\mathfrak{g}^X)$$

*sending and invariant polynomial $P$, of homogeneous degree $d+1$, to the local functional*

$$(\alpha_1, \dots, \alpha_{d+1}) \mapsto \int P^X (\alpha_1, \partial\alpha_2, \dots, \partial\alpha_{d+1})$$

*is a cochain map. Moreover, it is injective at the level of cohomology.*

*Remark* 2.2. We extend the operator $\partial : \Omega^{k,l} \to \Omega^{k+1,l}$ to $\Omega^{0,*}(X) \otimes \mathfrak{g} \to \Omega^{1,*}(X) \otimes \mathfrak{g}$ by the operator $\partial \otimes 1$.

## 3. The factorization algebra

Given any cocycle $\theta \in \mathrm{C}^*_{\mathrm{loc}}(\mathfrak{g}^X)$ of degree one we define a factorization algebra on $X$.

**Definition 3.1.** Let $\theta$ be a local cocycle of $\mathfrak{g}^X$ of cohomological degree one. Define $\mathrm{KM}^X_{\mathfrak{g},\theta}$ to be the factorization algebra on $X$ that assigns to an open set $U \subset X$ the cochain complex $\mathrm{C}^{\mathrm{Lie},\theta}_* \left(\mathfrak{g}^X(U)\right)$. In other words, $\mathrm{KM}^X_{\mathfrak{g},\theta}$ is the twisted factorization envelope $\mathrm{U}^{\mathrm{fact}}_\theta(\mathfrak{g}^X)$.

Explicitly, on an open set $U \subset X$, the cochain complex $\mathrm{KM}^X_{\mathfrak{g},\theta}(U)$ has as its underlying graded vector space

$$\mathrm{Sym}\left(\mathfrak{g}^X_c(U)[1] \oplus \mathbb{C} \cdot K\right)$$

and the differential is given by $\bar\partial + \mathrm{d}_{\mathfrak{g}} + \theta$ where $\mathrm{d}_{\mathfrak{g}}$ is the extension of the Chevalley-Eilenberg differential for $\mathfrak{g}$ to the Dolbeault complex, and where $\theta$ is extended to the full symmetric algebra by the rule that it is a (graded) derivation.

*Example* 3.2. As an example, using the map $J$ of Proposition 2.1, we can construct a factorization algebra on $X$ for any invariant polynomial $P \in \mathrm{Sym}^{d+1}(\mathfrak{g}^\vee)^{\mathfrak{g}}$. Since $j$ is injective, we obtain a unique factorization algebra for every such polynomial, hence it makes sense to denote $\mathrm{KM}^X_{\mathfrak{g},P} := \mathrm{KM}^X_{\mathfrak{g},j(P)}$.

## 4. Higher loop algebras

In this section we restrict to the complex manifold $X = \mathbb{C}^d$. We will extract from the Kac-Moody factorization algebra on $\mathbb{C}^d$ an associative algebra...

4.1. **A model for the annulus.** BW: Facts about the Dolbeault cohomology of the higher annulus. It is not Stein! Recall the Jouanolou model, denoted $A_d$.

Consider the radial projection map

$$\mathrm{rad} : \mathbb{C}^d \setminus 0 \to \mathbb{R}_{>0}$$

sending $z = (z_1, \ldots, z_d)$ to $|z| = \sqrt{|z_1|^2 + \cdots + |z_d|^2}$.

BW: This is essentially in KHF, should we recall it?

**Lemma 4.1.** *There is a map of commutative dg algebras*

$$j : A_d \to \Omega^{0,*}(\mathbb{C}^d \setminus 0)$$

*that induces a quasi-isomorphism* $A_d \simeq \oplus_{k \in \mathbb{Z}} \Omega^{0,*}(\mathbb{C}^d \setminus 0)^{(k)}$.

Note that $j$ induces a map of commutative dg algebras $j : A_d \to \Omega^{0,*}(\mathrm{rad}^{-1}(I))$ where $I \subset \mathbb{R}_{>0}$ is any interval. If $a \in A_d$ we will denote the resulting element in the Dolbeault complex by $a(z) := j(a)$.

4.2. **The case of zero level.** BW: only look at annular part

First we will consider the higher Kac-Moody factorization algebra on $\mathbb{C}^d$ "at level zero". That is, the factorization algebra $\mathrm{KM}^{\mathbb{C}^d}_{\mathfrak{g},0}$.

We obtain a factorization algebra on $\mathbb{R}_{>0}$ via pushing forward the higher Kac-Moody factorization algebra along the radial projection map $\mathrm{rad}_* \left(\mathrm{KM}^{\mathbb{C}^d \setminus 0}_{\mathfrak{g},0}\right)$. Explicitly, to an open set $I \subset \mathbb{R}_{>0}$ this factorization algebra assigns the dg vector space

$$C_*^{\mathrm{Lie}}\left(\Omega^{0,*}_c(\mathrm{rad}^{-1}(I)) \otimes \mathfrak{g}\right).$$

When $I$ is an interval, the subset $\mathrm{rad}^{-1}(I) \subset \mathbb{C}^d$ is a higher dimensional annulus as mentioned above. It is homeomorphic to $S^{2d-1} \times I$.

We wish to compare this one-dimensional factorization algebra to the higher current Lie algebra $A_d \otimes \mathfrak{g}$, or more accurately, its universal enveloping algebra $U(A_d \otimes \mathfrak{g})$. The universal enveloping algebra has the structure of a dg associative algebra and so defines a factorization algebra on any one-manifold. Let $U(A_d \otimes \mathfrak{g})^{\mathrm{fact}}$ be the corresponding factorization algebra on the manifold $\mathbb{R}_{>0}$.

Let $I \subset \mathbb{R}_{>0}$ be an open subset. There is the natural map $\mathrm{rad}^* : \Omega^*_c(I) \to \Omega^*_c(\mathrm{rad}^{-1}(I))$ given by pulling back differential forms. We can post-compose this with the natural projection $\mathrm{pr}_{\Omega^{0,*}} : \Omega^*_c \to \Omega^{0,*}_c$ to obtain a map of commutative algebras $\mathrm{pr}_{\Omega^{0,*}} \circ \mathrm{rad}^* : \Omega^*_c(I) \to \Omega^{0,*}_c(\mathrm{rad}^{-1}(I))$. Using the map $j$ defined in Section BW: ref we obtain a map of commutative dg algebras

$$\begin{array}{rccc} \Phi(I) = (\mathrm{pr}_{\Omega^{0,*}} \circ \mathrm{rad}^*) \otimes j : & \Omega^*_c(I) \otimes A_d & \to & \Omega^{0,*}_c\left((\rho^{-1}(I)\right) \\ & \varphi \otimes a & \mapsto & ((\mathrm{pr}_{\Omega^{0,*}} \circ \mathrm{rad}^*)\varphi) \wedge j(a) \end{array}$$

Since this is a map of commutative dg algebras it defines a map of dg Lie algebras

$$\Phi(I) \otimes \mathrm{id}_{\mathfrak{g}} : (\Omega^*_c(I) \otimes A_d) \otimes \mathfrak{g} = \Omega^*_c(I) \otimes (A_d \otimes \mathfrak{g}) \to \Omega^{0,*}(\mathrm{rad}^{-1}(I)) \otimes \mathfrak{g}$$

which maps $\varphi \otimes a \otimes X \mapsto \Phi(\varphi \otimes a) \otimes X$. BW: Explicitly... . We will drop the $\mathrm{id}_{\mathfrak{g}}$ from the notation and will denote this map simply by $\Phi(I)$. Note that $\Phi(I)$ is compatible with inclusions of open sets, hence extends to a map of cosheaves of dg Lie algebras that we will call $\Phi$.

**Proposition 4.2.** *The map* $\Phi$ *extends to a map of factorization Lie algebras*

$$\Phi : \Omega^*_{\mathbb{R}_{>0},c} \otimes (A_d \otimes \mathfrak{g}) \to \mathrm{rad}_*\left(\Omega^{0,*}_{\mathbb{C}^d \setminus 0, c} \otimes \mathfrak{g}\right).$$

*Hence, it defines a map of factorization algebras*

$$C_*(\Phi) : (U(A_d \otimes \mathfrak{g}))^{fact} \to \mathrm{rad}_*\left(\mathrm{KM}^{\mathbb{C}^d \setminus 0}_{\mathfrak{g},0}\right).$$

4.3. **The case of non-zero level.**

**Theorem 4.3.** *There is a map of factorization algebras on* $\mathbb{R}_{>0}$

$$(U\widehat{\mathfrak{g}}_{d,\theta})^{fact} \to \mathrm{rad}_*\left(\mathrm{KM}^{\mathbb{C}^d}_{\mathfrak{g},\theta}|_{\mathbb{C}^d\setminus 0}\right).$$

*Moreover, its image is quasi-isomorphic to the subfactorization algebra consisting of the* $S^1$*-eigenspaces*

$$\mathcal{A}_{d,\mathfrak{g},\theta} := \bigoplus_{k\in\mathbb{Z}} \mathrm{rad}_*\left(\mathrm{KM}^{\mathbb{C}^d}_{\mathfrak{g},\theta}|_{\mathbb{C}^d\setminus 0}\right)^{(k)} \subset \mathrm{rad}_*\left(\mathrm{KM}^{\mathbb{C}^d}_{\mathfrak{g},\theta}|_{\mathbb{C}^d\setminus 0}\right).$$

We will write down a sequence of maps of factorization Lie algebras

$$\begin{tikzcd} & \mathcal{G}_1 & & \mathcal{G}_2 \\ \mathcal{G}_0 \arrow[ur, "\simeq", "\Phi_1"'] & & \mathcal{G}_1' \arrow[ul] \arrow[ur, "\simeq", "\Phi_2"'] & \end{tikzcd}.$$

First, we introduce the factorization Lie algebra $\mathcal{G}_0 := \Omega^*_{\mathbb{R},c} \otimes \widehat{\mathfrak{g}}_{d,\theta}$. To an open set $I \subset \mathbb{R}$, it assigns the dg Lie algebra $\mathcal{G}_0(I) = \Omega^*_c(I) \otimes \widehat{\mathfrak{g}}_{d,\theta}$. The differential and Lie bracket are determined by the fact that we are tensoring a commutative dg algebra with a dg Lie algebra. This factorization Lie algebra is a central extension of the factorization Lie algebra $\Omega^*_{\mathbb{R},c} \otimes (A_d \otimes \mathfrak{g})$ by the trivial module $\Omega^*_c \oplus \mathbb{C} \cdot K$. The cocycle determining the central extension is given by

$$\theta_0(\varphi_0\alpha_0, \ldots, \varphi_d\alpha_d) = (\varphi_0 \wedge \cdots \wedge \varphi_d)\theta_{A_d}(\alpha_1, \ldots, \alpha_d).$$

A slight variant of Proposition 3.4.0.1 in [**?**] shows that there is a quasi-isomorphism of factorization algebras on $\mathbb{R}$

$$(U\widehat{\mathfrak{g}}_{d,\theta})^{fact} \xrightarrow{\simeq} \mathrm{C}^{\mathrm{Lie}}_*(\mathcal{G}_0).$$

We define the factorization dg Lie algebra $\mathcal{G}_1$ on $\mathbb{R}$. First, consider the factorization dg Lie algebra on $\mathbb{R}$ given by $\Omega^*_{\mathbb{R},c} \otimes (A_d \otimes \mathfrak{g})$. This assigns to an open set $I \subset \mathbb{R}$ the dg Lie algebra $\Omega^*_c(I) \otimes (A_d \otimes \mathfrak{g})$. Equivalently, this is the compactly supported sections of the local Lie algebra $\Omega^*_{\mathbb{R}} \otimes (A_d \otimes \mathfrak{g})$.

The factorization dg Lie algebra $\mathcal{G}_1$ is a central extension of $\Omega^*_{\mathbb{R},c} \otimes (A_d \otimes \mathfrak{g})$

$$0 \to \mathbb{C} \cdot K[-1] \to \mathcal{G}_1 \to \Omega^*_{\mathbb{R},c} \otimes (A_d \otimes \mathfrak{g}) \to 0$$

determined by the following cocycle. For an open interval $I$ write $\varphi_i \in \Omega^*_c(I)$, $\alpha_i \in A_d \otimes \mathfrak{g}$. The cocycle is defined by

$$\theta_1(\varphi_0\alpha_0, \ldots, \varphi_d\alpha_d) = \left(\int_I \varphi_0 \wedge \cdots \varphi_d\right) \theta_{A_d}(\alpha_0, \ldots, \alpha_d) \tag{1}$$

Recall, if we write $\alpha_i = a_i X_i$ for $a_i \in A_d$, $X_i \in \mathfrak{g}$ the cocycle $\theta_{A_d}$ is given by

$$\theta_{A_d}(a_0X_0, \ldots, \ldots, a_dX_d) = \operatorname*{Res}_{z=0}\,(a_0 \wedge \partial a_1 \wedge \cdots \wedge \partial a_d)\,\theta(X_0, \ldots, X_d).$$

The functional $\theta_1$ actually determines a local cocycle in $\mathrm{C}^*_{\mathrm{loc}}\left(\Omega^*_{\mathbb{R}} \otimes (A_d \otimes \mathfrak{g})\right)$ of degree one. As above, if

We define a map of factorization Lie algebras $\Phi_1 : \mathcal{G}_0 \to \mathcal{G}_1$. On and open set $I \subset \mathbb{R}$, we define

$$\Phi_1(\varphi\alpha, \psi K) = \left(\varphi\alpha, \int \psi \cdot K\right)$$

For a fixed open set $I \subset \mathbb{R}$, the map $\Phi_1$ fits into the commutative diagram of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \Omega^*_c(I) \otimes \mathbb{C} \cdot K \arrow[r] \arrow[d, "\int"] & \mathcal{G}_0(I) \arrow[r] \arrow[d, "\Phi_1"] & \Omega^*_c(I) \otimes (A_d \otimes \mathfrak{g}) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathbb{C} \cdot K[-1] \arrow[r] & \mathcal{G}_1(I) \arrow[r] & \Omega^*_c(I) \otimes (A_d \otimes \mathfrak{g}) \arrow[r] & 0. \end{tikzcd}$$

To see that $\Phi_1$ is a map of dg Lie algebras we simply observe that the cocycles determining the central extensions are related by $\theta_1 = \int \circ\, \theta_0$, where $\int : \Omega^*_c(I) \to \mathbb{C}$ as in the diagram above.

We now define the factorization dg Lie algebra $\mathcal{G}_1'$. Like $\mathcal{G}_1$, it is a central extension of $\Omega^*_{\mathbb{R},c} \otimes (A_d \otimes \mathfrak{g})$. The cocycle determining the central extension is defined by

$$\theta_1'(\varphi_0 a_0 X_0, \ldots, \ldots, \varphi_d a_d X_d) = \theta_1(\varphi_0 a_0 X_0, \ldots, \ldots, \varphi_d a_d X_d) + \widetilde{\theta}_1(\varphi_0 a_0 X_0, \ldots, \ldots, \varphi_d a_d X_d)$$

where $\theta_1$ was defined in Equation (1). Before writing down the explicit formula for $\widetilde{\theta}_1$ we introduce some notation. Set

$$E = r\frac{\partial}{\partial r},$$
$$\mathrm{d}\vartheta = \sum_i \frac{\mathrm{d}z_i}{z_i}.$$

We view $E$ as a vector field on $\mathbb{R}_{>0}$ and $\mathrm{d}\vartheta$ as a $(1,0)$-form on $\mathbb{C}^d \setminus 0$. The functional

$$\widetilde{\theta}_1(\varphi_0 a_0 X_0, \ldots, \varphi_d a_d X_d) = \frac{1}{2}\sum_{i=1}^d \left(\int_I \varphi_0 (E \cdot \varphi_i)\varphi_1 \cdots \widehat{\varphi_i} \cdots \varphi_d\right)\left(\oint (a_0 a_i \mathrm{d}\vartheta)\, \partial a_1 \cdots \widehat{\partial a_i} \cdots \partial a_d\right)\theta(X_0, \ldots, X_d).$$

The functional $\widetilde{\theta}$ defines a local functional in $\mathrm{C}^*_{\mathrm{loc}}\left(\Omega^*_{\mathbb{R}_{>0}} \otimes (A_d \otimes \mathfrak{g})\right)$ of cohomological degree one. One immediately checks that it is a cocycle.

In fact, we will show that $\widetilde{\theta}_1$ is actually an exact cocycle. We will see this by displaying an explicit cobounding functional. Define the local functional

$$\eta(\varphi_0 a_0 X_0, \ldots, \varphi_d a_d X_d) = \sum_{i=1}^d \left(\int_I \varphi_0 \left(\iota_E \varphi_i\right) \varphi_1 \cdots \widehat{\varphi_i} \cdots \varphi_d\right)\left(\oint (a_0 a_i \mathrm{d}\vartheta)\, \partial a_1 \cdots \widehat{\partial a_i} \cdots \partial a_d\right)\theta(X_0, \ldots, X_d).$$

**Lemma 4.4.** *One has $\mathrm{d}\eta = \widetilde{\theta}_1$, where $\mathrm{d}$ is the differential for the cochain complex $\mathrm{C}^*_{\mathrm{loc}}(\Omega^*_{\mathbb{R}_{>0}} \otimes (A_d \otimes \mathfrak{g}))$. In particular, the factorization Lie algebras $\mathcal{G}_1$ and $\mathcal{G}_1'$ are quasi-isomorphic. An explicit quasi-isomorphism is given by BW: ....*

Finally, we define the factorization Lie algebra $\mathcal{G}_2$. We have already seen that the local cocycle $J(\theta) \in \mathrm{C}^*_{\mathrm{loc}}(\mathfrak{g}^{\mathbb{C}^d})$ determines a central extension of factorization Lie algebras

$$0 \to \mathbb{C} \cdot K[-1] \to \mathcal{G}_{J(\theta)} \to \Omega^{0,*}_{\mathbb{C}^d,c} \otimes \mathfrak{g} \to 0.$$

Of course, we can restrict $\mathcal{G}_{J(\theta)}$ to a factorization algebra on $\mathbb{C}^d \setminus 0$. The factorization algebra $\mathcal{G}_2$ is defined as the pushforward of this restriction along the radial projection: $\mathcal{G}_2 := \mathrm{rad}_*\left(\mathcal{G}_{J(\theta)}|_{\mathbb{C}^d \setminus 0}\right)$.

Recall the map $\Phi : \Omega^*_{\mathbb{R}_{>0},c} \otimes (A_d \otimes \mathfrak{g}) \to \mathrm{rad}_*(\Omega^{0,*}_{\mathbb{C}^d \setminus 0,c} \otimes \mathfrak{g})$ defined in BW: ref. On each open set $I \subset \mathbb{R}_{>0}$ we can extend $\Phi$ by the identity on the central element to a linear map $\Phi_2 : \mathcal{G}_1'(I) \to \mathcal{G}_2(I)$.

**Lemma 4.5.** *The map $\Phi_2 : \mathcal{G}_1'(I) \to \mathcal{G}_2(I)$ is a map of dg Lie algebras. Moreover, it extends to a map of factorization Lie algebras $\Phi_2 : \mathcal{G}_1' \to \mathcal{G}_2$.*

*Proof.* Modulo the central element $\Phi_2$ reduces to the map $\Phi$, which we have already seen is a map of factorization Lie algebras in Proposition BW: ref. Thus, to show that $\Phi_2$ is a map of factorization Lie algebras we need to show that it is compatible with the cocycles determing the respective central extensions. That is, we need to show that

$$\theta_1'(\varphi_0 a_0 X_0, \ldots, \varphi_d a_d X_d) = \theta_2(\Phi(\varphi_0 a_0 X_0), \ldots, \Phi(\varphi_d a_d X_d)) \tag{2}$$

for all $\varphi_i a_i X_i \in \Omega^*_c(I) \otimes (A_d \otimes \mathfrak{g})$. The cocycle $\theta_1'$ is only nonzero if one of the $\varphi_i$ inputs is a 1-form. We evaluate the left-hand side on the $(d+1)$-tuple $(\varphi_0 \mathrm{d}r a_0 X_0, \varphi_1 a_1 X_1, \ldots, \varphi_d a_d X_d)$ where $\varphi_i \in C^\infty_c(I)$, $a_i \in A_d$, $X_i \in \mathfrak{g}$ for $i = 0, \ldots, d$. The result is

$$\left(\int_I \varphi_0 \cdots \varphi_d \mathrm{d}r\right)\left(\oint a_0 \partial a_1 \cdots \partial a_d\right)\theta(X_0, \ldots, X_d) \tag{3}$$

$$+ \quad \frac{1}{2}\sum_{i=1}^d \left(\int_I \varphi_0 (E \cdot \varphi_i)\varphi_1 \cdots \widehat{\varphi_i} \cdots \varphi_d \mathrm{d}r\right)\left(\oint (a_0 a_i \mathrm{d}\vartheta)\, \partial a_1 \cdots \widehat{\partial a_i} \cdots \partial a_d\right)\theta(X_0, \ldots, X_d) \tag{4}$$

We wish to compare this to the right-hand side of Equation (2). Recall that $\Phi(\varphi_0 \mathrm{d}r a_0 X_0) = \varphi(r)\mathrm{d}r a_0(z) X_0$ and $\Phi(\varphi_i a_i X_i) = \varphi(r) a_i(z) X_i$. Plugging this into the explicit formula for the cocycle $\theta_2$ we see the right-hand side of (2) is

$$\left(\int_{\mathrm{rad}^{-1}(I)} \varphi_0(r)\mathrm{d}r a_0(z)\partial(\varphi_1(r)a_1(z))\cdots\partial(\varphi_d(r)a_d(z))\right)\theta(X_0,\ldots,X_d). \tag{5}$$

We pick out the term in (5) in which the $\partial$ operators only act on the elements $a_i(z)$, $i = 1,\ldots,d$. This term is of the form

$$\int_{\mathrm{rad}^{-1}(I)} \varphi_0(r)\cdots\varphi_d(r)\mathrm{d}r a_0(z)\partial(a_1(z))\cdots\partial(a_d(z))\theta(X_0,\ldots,X_d).$$

Separating variables we find that this is precisely the first term (3) in the expansion of the left-hand side of (2).

Now, note that we can rewrite the $\partial$-operator in terms of the radius $r$ as

$$\partial = \sum_{i=1}^d \mathrm{d}z_i \frac{\partial}{\partial z_i} = \sum_{i=1}^d \mathrm{d}z_i \bar{z}_i \frac{\partial}{\partial(r^2)} = \sum_{i=1}^d \mathrm{d}z_i \frac{r^2}{2z_i}\frac{\partial}{\partial r}.$$

The remaining terms in (5) correspond to the expansion of

$$\partial(\varphi_1(r)a_1(z))\cdots\partial(\varphi_d(r)a_d(z)),$$

using the Leibniz rule, for which the $\partial$ operators act on at least one of the functions $\varphi_1,\ldots,\varphi_d$. In fact, only terms in which $\partial$ acts on precisely one of the functions $\varphi_1,\ldots,\varphi_d$ will be nonzero. For instance, consider the term

$$(\partial\varphi_1)a_1(z)(\partial\varphi_2)a_2(z)\partial(\varphi_3(z)a_3(z))\cdots\partial(\varphi_d(z)a_d(z)). \tag{6}$$

Now, $\partial\varphi_i(r) = \omega\frac{\partial\varphi}{\partial r}$ where $\omega$ is the one-form $\sum_i (r^2/2z_i)\mathrm{d}z_i$. Thus, (6) is equal to

$$\left(\omega\frac{\partial\varphi_1}{\partial r}\right)a_1(z)\left(\omega\frac{\partial\varphi_2}{\partial r}\right)a_2(z)\partial(\varphi_3(z)a_3(z))\cdots\partial(\varphi_d(z)a_d(z),$$

which is clearly zero as $\omega$ appears twice.

We observe that terms in the expansion of (5) for which $\partial$ acts on precisely one of the functions $\varphi_1,\ldots,\varphi_d$ can be written as

$$\sum_{i=1}^d \int_{\mathrm{rad}^{-1}(I)} \varphi_0(r)\left(r\frac{\partial}{\partial r}\varphi_i(r)\right)\varphi_1(r)\cdots\widehat{\varphi_i(r)}\cdots\varphi_d(r)\mathrm{d}r\frac{r}{2z_i}\mathrm{d}z_i a_0(z)a_i(z)\partial a_1(z)\cdots\widehat{\partial a_i(z)}\cdots\partial a_d(z).$$

Finally, notice that the function $z_i/2r$ is independent of the radius $r$. Thus, separating variables we find the integral can be written as

$$\frac{1}{2}\sum_{i=1}^d\left(\int_I \varphi_0\left(r\frac{\partial}{\partial r}\varphi_i\right)\varphi_1\cdots\widehat{\varphi}_i\cdots\varphi_d\mathrm{d}r\right)\left(\oint\frac{\mathrm{d}z_i}{z_i}a_0 a_i\partial a_2\cdots\widehat{\partial a_i}\cdots\partial a_d\right).$$

This is precisely equal to the second term (4) above. Hence, the cocycles are compatible and the proof is complete.

□