HIGHER DIMENSIONAL KAC-MOODY SYMMETRIES

Contents

Kevin: This is very incomplete, as I've been distracted by writing up the other projects and focusing on applications. The main goals I want to accomplish in this are: 1) construct the universal Kac-Moody in any dimension, 2) show how to recover the sphere and iterated loop algebras and compare the sphere algebra to Faonte-Hennion-Kapranov, 3) prove a version of GRR over the formal moduli of G-bundles by an explicit calculation of the anomaly of higher Kac-Moody acting on beta-gamma with coefficients in a module, 4) present the realization of these higher factorization algebras as the boundary of both 5d and 7d supersymmetric gauge theories.

I believed I've worked out all of these, and my goal is to have this on the arXiv by the end of October, in time for application decisions. (I wanted to include a formula for the OPE in general dimensions, but I think I'll just include that in my thesis and wait until I have a better idea of the full higher vertex algebra structure.)

BW: Add intro comparing to Kapranov-Hennion-Faonte.

1. Universal holomorphic factorization algebras

1.1. Local Lie algebras and factorization.

1.1.1. *A recollection of local Lie algebras.*

Definition 1.1. A *local Lie algebra* (or *local L_∞ algebra*) on X is the following data:

(i) a $\mathbb{Z}$-graded vector bundle L on X, with sheaf of sections that we denote $\mathcal{L}$;

(ii) for each $n \in \mathbb{Z}$ a polydifferential operator

$$\ell_n : \mathcal{L}^{\otimes n} \to \mathcal{L}[2-n];$$

such that the collection $\{\ell_n\}$ endow $\mathcal{L}$ with the structure of a sheaf of L_∞ algebras.

We often refer to a local Lie algebra $(L, \{\ell_n\})$ simply by its sheaf of sections $\mathcal{L}$. A local Lie algebra defines the sheaf of complexes $C_*^{\mathrm{Lie}}(\mathcal{L})$ that sends an open set $U \subset X$ to the complex $C_*^{\mathrm{Lie}}(\mathcal{L}(U))$.

Note that $C_*^{\mathrm{Lie}}(\mathcal{L})$ is itself the sheaf of sections of a graded vector bundle and that it has the structure of a sheaf of cocommutative coalgebras.

Definition 1.2. A map $f : \mathcal{L} \to \mathcal{L}'$ of local Lie algebras on X is a polydifferential operator

$$f : C_*^{\mathrm{Lie}}(\mathcal{L}) \to C_*^{\mathrm{Lie}}(\mathcal{L}')$$

that is, in addition, a map of sheaves of cocommutative coalgebras.

1.1.2. *Universal objects.* Let CplxMan be the category of complex manifolds with holomorphic maps. There is a fibered category VB of holomorphic vector bundles over CplxMan. Likewise, there is a category of local Lie algebras fibered over CplxMan. Its objects are pairs (X, L) consisting of a complex manifold X together with a local Lie algebra L on X. Maps between $(f, F) : (X, L) \to (X', L')$ is a holomorphic map $f : X \to X'$ together with a map of local Lie algebras on X, $F : L \to f^* L'$. ...

Given a local Lie algebra with underlying $\mathbb{Z}$-graded vector bundle L we can consider both its sheaf of sections $\mathcal{L}$. This has the structure of a sheaf of L_∞ algebras. We can also consider its cosheaf of compactly supported sections, that we denote $\mathcal{L}_c$. The cosheaf of compactly supported sections is not, however, a cosheaf of Lie algebras. It does, however, have a certain "factorization" property that we will exploit to define factorization algebras on the underlying manifold.

Definition 1.3. A *prefactorization Lie algebra* $\mathcal{G}$ on a manifold X is the data:

(i) for each open set $U \subset X$ an L_∞ algebra $\mathcal{G}(U)$;

(ii) for each pairwise disjoint collection of open sets $U_1, \ldots, U_n$ contained inside some open set $V \subset X$ a map of L_∞ algebras

$$\mathcal{G}(U_1) \oplus \cdots \oplus \mathcal{G}(U_n) \to \mathcal{G}(V).$$

There is a symmetric monoidal structure on the category of L_∞ algebras $L_\infty\mathrm{Alg}$ given by the direct sum $\oplus$ of underlying chain complexes. Thus, a prefactorization Lie algebra is simply a symmetric monoidal functor

$$\mathcal{G} : \mathrm{Op}(X)^{\sqcup} \to L_\infty\mathrm{Alg}^{\oplus}.$$

In particular, $\mathcal{G}$ is a precosheaf of L_∞ algebras.

In the holomorphic setting the above definition makes sense in a wider context, where we consider all complex manifolds of a fixed dimension uniformly.

Definition 1.4. A *universal holomorphic prefactorization Lie algebra* of dimension d is a symmetric monoidal functor

$$\mathcal{G} : \mathrm{Hol}_d^{\sqcup} \to L_\infty\mathrm{Alg}^{\oplus}$$

from the symmetric monoidal category of holomorphic manifolds with embeddings equipped with disjoint union to the category of L_∞ algebras equipped with direct sum.

Just like in the case of factorization algebras, we have the following definition.

Definition 1.5. A *factorization Lie algebra* on X is a prefactorization Lie algebra satisfying descent for Weiss covers on X. Likewise, a *universal holomorphic factorization Lie algebra* is a universal holomorphic prefactorization Lie algebra satisfying descent for Weiss covers in Hol_d.

Local Lie algebras provide a nice class of factorization Lie algebras.

Lemma 1.6. *Suppose L is a local Lie algebra on X. Then the precosheaf of compactly supported sections $\mathcal{L}_c$ is a factorization Lie algebra on X. Similarly, if L is a universal holomorphic local Lie algebra then its functor of compactly supported sections $\mathcal{L}_c$ is a universal holomorphic factorization Lie algebra.*

We briefly elaborate by what we mean by the compactly supported sections of a universal local Lie algebra L. Such an object determines a functor

$$\mathcal{L}_c : \mathrm{Hol}_d \to L_\infty \mathrm{Alg}$$

defined by sending a complex d-fold X to the space of compactly supported sections of the bundle $L(X)$. This has the structure of an L_∞ algebra by definition. Given a holomorphic embedding $f : X \to Y$ one defines the map

$$f_c : \mathcal{L}_c(X) \to \mathcal{L}_c(Y)$$

by BW: finish...

Given a Lie algebra $\mathfrak{g}$ one can define the cocoummutative coalgebra $\mathrm{C}_*^{\mathrm{Lie}}(\mathfrak{g})$ of Chevalley–Eilenberg chains. This is the cochain complex computing Lie algebra homology.

From a factorization Lie algebra, we construct a factorization algebra in a similar way. We show that the construction also works to define, from universal Lie algebras, universal factorization algebras. Much of this section is a recollection of the material in Section 3.6 of [**?**].

Lemma 1.7. *Suppose $\mathcal{G}$ is a factorization Lie algebra on X. Then, the assignment*

$$\mathrm{C}_*^{\mathrm{Lie}}(\mathcal{G}) : U \mapsto \mathrm{C}_*^{\mathrm{Lie}}(\mathcal{G}(U))$$

defines a factorization algebra on X. If $\mathcal{G}$ is a universal holomorphic factorization Lie algebra then $\mathrm{C}_*^{\mathrm{Lie}}(\mathcal{G})$ defines a universal holomorphic factorization algebra.

1.2. **The Kac–Moody factorization algebra.** In this section we introduce the local Lie algebra that will be the main focus of the paper. The local Lie algebra will be defined on any complex manifold and is constructed using the data of a Lie algebra $\mathfrak{g}$. For most of this paper we will assume that we have an ordinary Lie algebra, but a very slight generalization can be used to handle dg Lie or L_∞ algebras.

Fix a complex manifold X of complex dimension d. The complex structure determines a splitting of the tangent bundle $TX = TX^{1,0} \oplus TX^{0,1}$ into its holomorphic and anti-holomorphic subbundles. Likewise, the cotangent bundle splits as $T^*X = TX^{1,0} \oplus TX^{0,1}$. Define the following $\mathbb{Z}$-graded vector bundle on X

$$\mathfrak{g}(X) := \wedge^* T^* X^{0,1} \otimes \underline{\mathfrak{g}} = \oplus_{i=0}^{d} \wedge^i T^* X^{0,1}[-i]$$

where $\underline{\mathfrak{g}}$ denotes the trivial vector bundle on X with fiber $\mathfrak{g}$. The differential operator $\bar{\partial}$ on X extends to a degree one operator on $\mathfrak{g}(X)$. On the ith graded piece it is defined by

$$\bar{\partial} \otimes \mathrm{id}_{\mathfrak{g}} : \wedge^i T^* X^{0,1} \otimes \underline{\mathfrak{g}} \to \wedge^{i+1} T^* X^{0,1} \otimes \underline{\mathfrak{g}}.$$

The Lie bracket on $[-,-]_{\mathfrak{g}}$ on $\mathfrak{g}$ extends to a polydifferential operator on $\mathfrak{g}(X)$ of degree zero

$$[-,-] := \wedge \otimes [-,-]_{\mathfrak{g}} : \left(\wedge^i T^* X^{0,1} \otimes \underline{\mathfrak{g}}\right) \otimes \left(\wedge^j T^* X^{0,1} \otimes \underline{\mathfrak{g}}\right) = \left(\wedge^i T^* X^{0,1} \otimes \wedge^j T^* X^{0,1}\right) \otimes (\underline{\mathfrak{g}} \otimes \underline{\mathfrak{g}}) \to \wedge^{i+j} T^* X^{0,1} \otimes \underline{\mathfrak{g}}.$$

Here $\wedge$ denotes the wedge product of differential forms. The sheaf of sections of $\wedge^i T^* X^{0,1}$ is denoted $\Omega_X^{0,*}$ and we write the sheaf of sections of $\mathfrak{g}(X)$ as $\mathfrak{g}^X = \Omega_X^{0,*} \otimes \mathfrak{g}$.

Definition/Lemma 1. The $\mathbb{Z}$-graded bundle $\mathfrak{g}(X)$ together with the polydifferential operators $\bar{\partial}, [-,-]$ determine the structure of local Lie algebra on X. We call $\mathfrak{g}(X)$, or its sheaf of sections $\mathfrak{g}^X$, the *holomorphic $\mathfrak{g}$-current algebra* on X.

Proof. It suffices to show that $\mathfrak{g}^X$ is a presheaf of dg Lie algebras. For each open $U \subset X$ the restriction of the polydifferential operators $\bar{\partial}$ and $[-,-]$ to the vector space $\mathfrak{g}^X(U)$ coincides with structure of a dg Lie algebra obtained by tensoring the dg commutative algebra $\Omega^{0,*}(U)$ with the Lie algebra $\mathfrak{g}$. Now, if $U \hookrightarrow V$ is an inclusion of open sets we need to show that the induced map $\Omega^{0,*}(V) \otimes \mathfrak{g} \to \Omega^{0,*}(U) \otimes \mathfrak{g}$ is a map of dg Lie algebras. This follows from the general fact that if $f : A \to B$ to is a map of commutative dg algebras then the induced map $f \otimes \mathrm{id}_{\mathfrak{g}} : A \otimes \mathfrak{g} \to B \otimes \mathfrak{g}$ is a map of dg Lie algebras (where the dg Lie structure on $A \otimes \mathfrak{g}$ and $B \otimes \mathfrak{g}$ is the one mentioned above). $\square$

Remark 1.8. The sheaf of dg Lie algebras $\mathfrak{g}^X$ has the following geometric description. Any dg Lie algebra $\mathfrak{h}$ can be interpreted as a formal moduli problem $B\mathfrak{h}$. If $U \subset X$ is an open set, the dg Lie algebra $\mathfrak{g}^X(U)$ describes the formal neighborhood of the trivial bundle inside the moduli space of holomorphic G-bundles on X that are trivialized away from U. In particular, $\mathfrak{g}^X(X)$ describes the moduli space of holomorphic G-bundles on X. Suppose $\alpha \in \mathfrak{g}^X(X)$ is a Maurer–Cartan element. That is, α is a $\mathfrak{g}$-valued $(0,1)$-form satisfying the Maurer–Cartan equation $\bar{\partial}\alpha + \frac{1}{2}[\alpha, \alpha] = 0$. We obtain a connection on the trivial G-bundle of the form $\bar{\partial} + \alpha$. In fact, all first order deformations of the trivial G-bundle are of this form. The gauge transformations are of the form $\alpha \mapsto \alpha + \bar{\partial}\lambda + [\lambda, \alpha]$ where $\lambda : X \to \mathfrak{g}$ is a smooth map. In general $H^2_{\mathrm{Lie}}(\mathfrak{g}^X(X))$ is non-trivial, except when $d = 1$ in which it vanishes for degree reasons. This reflects the possibility for obstructions to first-order deformations of the trivial bundle.

The work in **[?]** has made this perspective precise in general complex dimension by giving a derived model for the moduli space of G-bundles. They show that the cohomology of the shifted tangent space at the trivial bundle is the $\bar{\partial}$-cohomology of $\mathfrak{g}^X(X)$:

$$H^* \left(T_{triv} \mathrm{Bun}_G(X)\right)[-1] \cong H^*_{\bar{\partial}}(\mathfrak{g}^X(X)) = H^*(X, \mathcal{O}^{hol}) \otimes \mathfrak{g}$$

as graded Lie algebras.

Given the local Lie algebra $\mathfrak{g}(X)$ we obtain a factorization Lie algebra on X by considering its compactly supported sections $\mathfrak{g}_c^X : U \subset X \mapsto \Omega_c^{0,*}(U) \otimes \mathfrak{g}$.

The local Lie algebra $\mathfrak{g}(X)$ makes sense on any complex manifold and is functorial in the universal sense discussed above. That is, we have a bundle $\mathfrak{g}(-)$ on the category of all complex dimensional d-folds. Thus, its compactly supported sections restricted to the subcategory Hol_d defines a universal holomorphic factorization Lie algebra. Explicitly, this is the functor

$$\mathfrak{g}_c^d : \mathrm{Hol}_d \to L_\infty \mathrm{Alg}$$

sending $X \to \mathfrak{g}_c^X(X)$.

In fact, there is a certain functoriality in the complex manifold that we now describe

1.3. **Central extensions from local cocycles.** In this section we describe the extensions of the local Lie algebra $\mathfrak{g}^X$. Let $\underline{\mathbb{C}}[k]$ be the local Lie algebra defined on any complex manifold X given by the constant bundle concentrated in cohomological degree $-k$. We wish to describe extensions of a local Lie algebra $\mathcal{L}$ on X by the constant Lie algebra $\mathbb{C}[k]$. This is a local Lie algebra $\widehat{\mathcal{L}}$ that fits into an exact sequence of local Lie algebras

$$0 \to \underline{\mathbb{C}}[k] \to \widehat{\mathcal{L}} \to \mathcal{L} \to 0. \tag{1}$$

Every cocycle $\alpha \in \mathrm{C}^*_{\mathrm{loc}}(\mathcal{L})(X)$ of total degree $2+k$ determines a central extension as in (1) as follows. The underlying vector bundle for the extended local Lie algebra is given by $L \oplus \underline{\mathbb{C}}[k]$.

Moreover, any two cohomologous cocycles determine quasi-isomorphic extensions.

Lemma 1.9. *The space of k-shifted central extensions as in Equation (1) is a torsor for the abelian group $H^{2+k}(\mathcal{L})(X)$.*

$$\mathrm{C}^*_{\mathrm{loc}}(\mathfrak{g}^X) =$$

Recall, a local k-cocycle of a local Lie algebra determines a $(k-2)$-shifted central extension, by the constant sheaf $\underline{\mathbb{C}}$. We are interested in (-1)-shifted central extensions, and hence, local 1-cocycles. If θ is such a local cocycle, denote by $\mathfrak{g}^X_\theta$ the corresponding centrally extended local Lie algebra.

There is a particular family of local cocycles that we will be especially interested in. Let P be an invariant polynomial of $\mathfrak{g}$ of homogenous degree $d+1$. That is, $P \in \mathrm{Sym}^{d+1}(\mathfrak{g}^\vee)^{\mathfrak{g}}$. We can extend P to a functional on $\Omega^{0,*}(X) \otimes \mathfrak{g}$ by the rule

$$\begin{array}{cccc} P^X: & \mathrm{Sym}^{d+1}(\Omega^{0,*}(X) \otimes \mathfrak{g}) & \to & \mathbb{C} \\ & (\omega_1 \otimes X_1, \ldots, \omega_{d+1} \otimes X_{d+1}) & \mapsto & (\omega_1 \wedge \cdots \wedge \omega_{d+1}) P(X_1, \ldots, X_{d+1}) \end{array}$$

Proposition 1.10. *The assignment*

$$J : \mathrm{Sym}^{d+1}(\mathfrak{g}^\vee)^{\mathfrak{g}}[-1] \to \mathrm{C}^*_{\mathrm{loc}}(\mathfrak{g}^X)$$

sending and invariant polynomial P, of homogeneous degree $d+1$, to the local functional

$$(\alpha_1, \ldots, \alpha_{d+1}) \mapsto \int P^X(\alpha_1, \partial\alpha_2, \ldots, \partial\alpha_{d+1})$$

is a cochain map. Moreover, it is injective at the level of cohomology.

Remark 1.11. We extend the operator $\partial : \Omega^{k,l} \to \Omega^{k+1,l}$ to $\Omega^{0,*}(X) \otimes \mathfrak{g} \to \Omega^{1,*}(X) \otimes \mathfrak{g}$ by the operator $\partial \otimes 1$.

1.4. **The factorization algebra.** Given any cocycle $\theta \in C^*_{\mathrm{loc}}(\mathfrak{g}^X)$ of degree one we define a factorization algebra on X.

Definition 1.12. Let θ be a local cocycle of $\mathfrak{g}^X$ of cohomological degree one. Define $\mathcal{F}^X_{\mathfrak{g},\theta}$ to be the factorization algebra on X to be the twisted factorization envelope $U^{\mathrm{fact}}_{\theta}(\mathfrak{g}^X)$. Equivalently, this is the factorization envelope of the extended Lie algebra $\widehat{\mathfrak{g}}^X_{\theta}$ determined by θ.

Explicitly, on an open set $U \subset X$, the cochain complex $\mathcal{F}^X_{\mathfrak{g},\theta}(U)$ has as its underlying graded vector space

$$\mathrm{Sym}\left(\mathfrak{g}^X_c(U)[1] \oplus \mathbb{C} \cdot K\right)$$

and the differential is given by $\bar{\partial} + \mathrm{d}_{\mathfrak{g}} + \theta$ where $\mathrm{d}_{\mathfrak{g}}$ is the extension of the Chevalley-Eilenberg differential for $\mathfrak{g}$ to the Dolbeault complex, and where θ is extended to the full symmetric algebra by the rule that it is a (graded) derivation.

Example 1.13. As an example, using the map J of Proposition 1.10, we can construct a factorization algebra on X for any invariant polynomial $P \in \mathrm{Sym}^{d+1}(\mathfrak{g}^\vee)^{\mathfrak{g}}$. Since j is injective, we obtain a unique factorization algebra for every such polynomial, hence it makes sense to denote $\mathcal{F}^X_{\mathfrak{g},P} := \mathcal{F}^X_{\mathfrak{g},j(P)}$.

1.4.1. *Arbitrary principal bundle.* There is a local Lie algebra related to $\mathfrak{g}^X$ associated to any principal G bundle. Formally speaking, one can understand $\mathfrak{g}^X$, or rather its global sections $\mathfrak{g}^X(X)$, as being the dg Lie algebra describing the formal neighborhood of the *trivial* G-bundle inside the derived moduli stack of G-bundle on X. Indeed, if triv denotes the trivial bundle then one has

$$\widehat{\mathrm{triv}} = B\mathfrak{g}^X(X)$$

where the hat denotes formal completion. In other words, the (-1)-shifted tangent space of the moduli stack of G-bundles is identified with the dg Lie algebra $\mathfrak{g}^X(X)$. At an arbitrary principal G bundle P, the dg Lie algebra describing the formal completion $\widehat{P}$ is also the global sections of a local Lie algebra that we now.

Let $\mathrm{ad}(P)$ denote the bundle of Lie algebras on X associated to P. We define the local Lie algebra by

$$\mathfrak{g}^{P\to X} := \Omega^{0,*}(X; \mathrm{ad}(P)),$$

i.e. the $(0,*)$-forms on X with coefficients in the bundle $\mathrm{ad}(P)$. The Lie bracket on $\mathrm{ad}(P)$ together with the Dolbeault operator $\bar{\partial}$ define the structure of the local Lie algebra. The global sections of this local Lie algebra describe the formal completion of P in the moduli of G bundles: $\widehat{P} = B\mathfrak{g}^{P\to X}(X)$.

1.4.2. *A variant on the construction.* The definition of the following flavor of factorization algebras have appeared in Section 3.6 of [**?**], but we wish to further analyze them here. As in the cases above, we work on a complex d-fold X and consider the local Lie algebra $\mathfrak{g}^X = \Omega^{0,*}(X; \mathfrak{g})$. The variant we discuss in this section involves a different (-1)-shifted central extension of this local Lie algebra. In this section, we fix an invariant pairing $\langle -, - \rangle$ on the Lie algebra $\mathfrak{g}$.

Fix a closed $(d-1,d-1)$-form $\omega \in \Omega^{d-1,d-1}(X)$. Define the quadratic functional on $\mathfrak{g}^X$ by

$$\phi_\omega(\alpha, \beta) = \int_X \omega \wedge \langle \alpha, \partial \beta \rangle.$$

Lemma 1.14. *The functional ϕ_ω is a local cocycle of degree one in $\mathrm{C}^*_{\mathrm{loc}}(\mathfrak{g}^X)$.*

Proof. Clearly ϕ_ω is local and degree one. The differential on $\mathrm{C}^*_{\mathrm{loc}}(\mathfrak{g}^X)$ is of the form $\bar{\partial} + \mathrm{d}_{\mathfrak{g}}$ where $\mathrm{d}_{\mathfrak{g}}$ is the Chevalley-Eilenberg differential on $\mathfrak{g}$ extended to $(0,*)$-forms. Since the pairing is invariant one has $\mathrm{d}_{\mathfrak{g}}(\phi_\omega) = 0$. Finally, to see that it is a cocycle we note that

$$\int_X \mathrm{d}_{dR}(\omega \wedge \langle \alpha, \partial \beta \rangle) = \int_X \omega \wedge \langle \bar{\partial} \alpha, \partial \beta \rangle \pm \int_X \omega \wedge \langle \alpha, \bar{\partial} \partial \beta \rangle$$

using the fact that ω is closed and $\omega \wedge \langle \alpha, \partial \beta \rangle$ is ∂-closed. □

Definition 1.15. Let X be a complex d-fold and $\omega \in \Omega^{d-1,d-1}(X)$ a closed form. Define the factorization algebra $\mathcal{U}^{fact}_\omega(\mathfrak{g}^X)$ on X as the twisted factorization envelope of $\mathfrak{g}^X$ twisted by the cocycle ϕ_ω.

Example 1.16. Suppose that X is a Kähler d-fold and let $\omega \in \Omega^{1,1}(X)$ be the Kähler form. We can then take the $(d-1,d-1)$-form above to be the $(d-1)$st power of the Kähler form ω^{d-1}. We will refer to the factorization algebra

$$\mathcal{F}^{(X,\omega)} := \mathcal{U}^{fact}_{\omega^{d-1}}(\mathfrak{g}^X)$$

as the *Kähler-Kac-Moody* factorization algebra on X. In the case that $d = 2$, the factorization algebra is related to the four-dimensional generalization of the Wess-Zumino-Witten model studied by Nair and Schiff in [**?**] and later by Nekrasov et. al. in [**?**, **?**]. We will return to this example later to describe its local operators as a consequence of its factorization algebra structure and to give an interpretation of it as a boundary of a certain Chern-Simons–like gauge theory.

1.5. **Relation to the ordinary Kac-Moody on Riemann surfaces.** In this section we pause to discuss a direct relationship of the higher dimensional Kac-Moody factorization algebras discussed above to the familiar Kac-Moody vertex algebras which are defined on one-dimensional complex manifolds.

Throughout this section we fix a Riemann surface Σ and consider a holomorphic family of complex $(d-1)$-folds over it. That is, we have a holomorphic fibration $\pi : X \to \Sigma$ whose fibers $\pi^{-1}(x)$, $x \in \Sigma$ are $(d-1)$-dimensional. For a fixed Lie algebra $\mathfrak{g}$ we put the higher dimensional Kac-Moody on X and consider its pushforward along π to get some factorization algebra on Σ. We will see how this pushforward is related to the one-dimensional Kac-Moody factorization (and vertex) algebra on Σ.

1.5.1. *A reminder of the ordinary current algebra.* The affine algebra $\widehat{\mathfrak{g}}$ of a Lie algebra $\mathfrak{g}$ together with a invariant pairing $\langle -, - \rangle_{\mathfrak{g}}$ is defined as a Lie algebra central extension of the loop algebra $L\mathfrak{g} = \mathfrak{g}[t, t^{-1}]$ defined by the cocycle $(f, g) \mapsto \mathrm{Res}_0(f \partial g)$. There is a slight generalization of this construction defined for any dg Lie algebra $(\mathfrak{g}, \mathrm{d}_{\mathfrak{g}})$. We take as the input data a $\mathfrak{g}$-invariant pairing $\langle -, - \rangle_{\mathfrak{g}}$ that is closed for the differential $\mathrm{d}_{\mathfrak{g}}$. This means that for any $X, Y \in \mathfrak{g}$ we have

$\langle d_{\mathfrak{g}} X, Y\rangle + (-1)^{|X|}\langle X, d_{\mathfrak{g}} Y\rangle = 0$ where $|X|$ is the cohomological degree of X in $\mathfrak{g}$. Equivalently, $\langle -, -\rangle$...

The loop algebra of a dg Lie algebra $L\mathfrak{g} = \mathfrak{g}[t, t^{-1}]$ is still defined and from the $d_{\mathfrak{g}}$-closed invariant pairing we get a 2-cochain on $L\mathfrak{g}$ defined by the same formula as in the ordinary case. The fact that it is a cocycle comes from being closed for both the differential $d_{\mathfrak{g}}$ and the Chevalley-Eilenberg differential for $L\mathfrak{g}$ (by invariance). Thus, we obtain a dg Lie algebra central extension $\widehat{\mathfrak{g}}$ of $L\mathfrak{g}$.

From the affine algebra associated to $\mathfrak{g}$ one builds the Kac-Moody vertex algebra by inducing the trivial module for $\widehat{\mathfrak{g}}$ up via the subalgebra of positive loops $L_+\mathfrak{g} \subset L\mathfrak{g}$. It is immediate that the same construction carries over for the case of a dg Lie algebra. One obtains, in this way, a *dg vertex algebra*. That is, a vertex algebra in the category of cochain complexes. We denote the level κ vacuum Kac-Moody dg vertex algebra obtained in this way by $\widehat{\mathfrak{g}}_\kappa$.

1.5.2. *Level zero.*

1.5.3.

Corollary 1.17. *Fix a Lie algebra with invariant pairing $\langle -, -\rangle$. Let Σ be an arbitrary Riemann surface and $d > 1$. Consider the volume form $\omega \in \Omega^{d-1,d-1}(\mathbb{P}^{d-1})$. Then, the pushforward of the factorization algebra $\mathcal{F}_\omega^{\Sigma\times\mathbb{P}^{d-1}}$ along the projection $\pi : \Sigma \times \mathbb{P}^{d-1} \to \Sigma$ is quasi-isomorphic to the ordinary Kac-Moody factorization algebra of central charge* $\mathrm{vol}(\omega)$

$$\pi_* \mathcal{F}_\omega^{\Sigma\times\mathbb{P}^{d-1}} \simeq \mathcal{F}^{\Sigma}_{\mathrm{vol}(\omega)}.$$

2. Sphere and loop algebras

We have defined the Kac–Moody factorization algebra as a universal holomorphic factorization algebra in any dimension. In this section we focus on the restriction of the factorization algebra to two complex manifolds of dimension d, $X = \mathbb{C}^d \setminus \{0\}$ and $X = (\mathbb{C} \setminus \{0\})^d$. In each case we show how the factorization product encodes the structure of a dg Lie algebra. Our main results in this section identify these dg Lie algebras with higher dimensional generalizations of loop and affine algebras.

We first consider a dg Lie algebra $\widehat{\mathfrak{g}}_{d,\theta}$, labeled by the dimension and a parameter $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$, whose zeroeth cohomology is a Lie algebra extension of the $(2d-1)$-sphere algebra

$$\mathrm{Map}(S^{2d-1}, \mathfrak{g}).$$

At the level of cohomology this extension is trivial, but at the level of cochain complexes it is non-trivial.

The dg Lie algebra determines a dg associative algebra via the universal enveloping algebra $U(\widehat{\mathfrak{g}}_{d,\theta})$. Our first main result in this section relates this associative algebra to the Kac–Moody factorization algebra.

Theorem 2.1. *The associative algebra $U(\widehat{\mathfrak{g}}_{d,\theta})$ determines a locally constant factorization algebra on the real one-manifold $\mathbb{R}$ that we denote $U(\widehat{\mathfrak{g}}_{d,\theta})^{fact}$. Moreover, there is an injective dense map of factorization algebras on $\mathbb{R}$:*

$$\Phi^{S^{2d-1}} : (U\widehat{\mathfrak{g}}_{d,\theta})^{fact} \to \rho_* \left(\mathcal{F}^{\mathbb{C}^d \setminus \{0\}}_{\mathfrak{g},\theta} \right).$$

where the right-hand side is the push-forward of the Kac–Moody factorization algebra on $\mathbb{C}^d \setminus \{0\}$ along the radial projection map.

Next, we consider the higher loop Lie algebra

$$L^d\mathfrak{g} = L(\cdots (L\mathfrak{g})\cdots) = \mathrm{Map}(S^1 \times S^1, \mathfrak{g}).$$

We study a class of *shifted* central extension of this Lie algebra, also parametrized by $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$, that we denote by $\widehat{L^d\mathfrak{g}}_\theta$.

A result of Knudsen [**?**], which we recall in Section **??**, states that every dg Lie algebra determines an E_d-algebra, for any $d > 1$, called the universal E_d enveloping algebra. This agrees with the ordinary universal enveloping algebra in the case $d = 1$. For the dg Lie algebra $\widehat{L^d\mathfrak{g}}_\theta$, we denote this E_d algebra by $U^{E_d}(\widehat{L^d\mathfrak{g}}_\theta)$. Its associated locally constant factorization algebra on $\mathbb{R}^d$ is denoted $U^{E_d}(\widehat{L^d\mathfrak{g}}_\theta)^{fact}$.

The Kac–Moody factorization algebra on the d-fold $(\mathbb{C}^\times)^d$ determines a real d-dimensional factorization algebra by considering the radius in each complex direction. We denote this factorization algebra on $\mathbb{R}^d$ by $\vec{\rho}_* \left(\mathcal{F}^{\mathbb{C}^{\times d}}_{\mathfrak{g},\theta} \right)$.

Theorem 2.2. *There is a dense injective map of factorization algebras on $\mathbb{R}^d$:*

$$\Phi^{L^d} : \left(U_{E_d} \left(\widehat{L^d g}_\theta \right) \right)^{fact} \to \vec{\rho}_* \left(\mathcal{F}^{\mathbb{C}^{\times d}}_{\mathfrak{g},\theta} \right).$$

2.1. **The higher sphere algebra.** The affine algebra associated to a Lie algebra $\mathfrak{g}$ together with an invariant pairing is defined as a central extension of the loop algebra of $\mathfrak{g}$

$$\mathbb{C} \to \widehat{\mathfrak{g}} \to Lg$$

where loop algebra is equal to $\mathcal{O}(D^{1\times}) \otimes \mathfrak{g} = \mathfrak{g}[z, z^{-1}]$. The central extension is determined by the cocycle

$$f \otimes X, g \otimes Y \mapsto \oint f \mathrm{d}g \langle X, Y \rangle.$$

We use the punctured algebraic disk $D^{1\times} = \mathrm{Spec}\ \mathbb{C}[z, z^{-1}]$, but the definition also makes sense for the puncture formal disk (formal loops).

Let $D^d = \mathrm{Spec}\ \mathbb{C}[z_1, \ldots, z_d]$ be the d-dimensional algebraic disk. The punctured d-disk is no longer affine, in fact its cohomology is given by

$$H^*(D^{d\times}, \mathcal{O}) =$$

Instead of working with the naive commutative algebra $\Gamma(D^{d\times}, \mathcal{O})$ we will use the dg commutative algebra of *derived* sections $\mathbb{R}\Gamma(D^{d\times}, \mathcal{O})$. An explicit model for this has been written down in [**?**] based on the Jouanolou method for resolving singularities. We recall its definition.

Definition 2.3. Let A_d be the commutative dg algebra generated by elements

$$z_1, \ldots, z_d, z_1^*, \ldots, z_d^*, (z_1 z_1^*)^{-1}, \ldots (z_1 z_d^*)^{-1}$$

in degree zero and

$$\mathrm{d}z_1, \ldots, \mathrm{d}z_d, \mathrm{d}z_1^*, \ldots, \mathrm{d}z_d^*$$

in degree one. Introduce a $*$-weight, so that $z_i^*, \mathrm{d}z_i^*$ have $*$-weight $+1$ and $(z_i^*)^{-1}$ has $*$-weight -1. We require that:

(i) every element is of total $*$-weight zero and

(ii) the contraction of every element with the Euler vector field $\sum_i z_i^* \partial_{z_i^*}$ vanishes.

The key properties of the dg algebra A_d we will utlilize are summarized in the following result of **[?]**.

Proposition 2.4 (**[?]** Proposition 1.3.1)**.** *The commutative dg algebra A_d is a model for $\mathbb{R}\Gamma(D^{d\times}, \mathcal{O})$. Moreover, there is a dense map of commutative dg algebras*

$$j : A_d \to \Omega^{0,*}(\mathbb{C}^d \setminus 0)$$

sending $z_i \mapsto z_i$, $z_i^* \mapsto \bar{z}_i$, and $\mathrm{d}z_i^* \mapsto \mathrm{d}\bar{z}_i$.

We are interested in the dg Lie algebra $A_d \otimes \mathfrak{g}$. In **[?]** they show, via knowledge of the Lie algebra cohomology, that there is a central extension of this BW: not sure what to say

Definition 2.5. Fix an element $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g})^{\mathfrak{g}}$. Let $\widehat{\mathfrak{g}}_{d,\theta}$ be the dg Lie algebra central extension of $A_d \otimes \mathfrak{g}$ determined by the degree two cocycle $\theta_{\mathrm{FHK}} \in \mathrm{C}^*_{\mathrm{Lie}}(A_d \otimes \mathfrak{g})$ defined by

$$\theta_{\mathrm{FHK}}(a_0 \otimes X_0, \ldots, a_d \otimes X_d) = \operatorname*{Res}_{z=0} (a_0 \wedge \mathrm{d}a_1 \wedge \cdots \wedge \mathrm{d}a_d)\, \theta(X_0, \ldots, X_d)$$

where $a_i \otimes X_i \in A_d \otimes \mathfrak{g}$.

2.2. **The strategy.** We consider the restriction of the factorization algebra $\mathcal{F}_{\mathfrak{g},\theta}$ on $\mathbb{C}^d \setminus \{0\}$ to the collection of open sets diffeomorphic to spherical shells. This restriction has the structure of a one-dimensional factorization algebra corresponding to the iterated nesting of spherical shells. We show that there is a dense subfactorization algebra that is locally constant, hence corresponds to an A_∞ algebra. We conclude by identifying this A_∞ algebra as a the universal enveloping algebra of a certain L_∞ algebra, that agree with the higher dimensional affine algebras of **[?]**

Introduce the radial projection map

$$\rho : \mathbb{C}^d \setminus 0 \to \mathbb{R}_{>0}$$

sending $z = (z_1, \ldots, z_d)$ to $|z| = \sqrt{|z_1|^2 + \cdots + |z_d|^2}$. We will restrict our factorization algebra to spherical shells by pushing forward the factorization algebra along this map. Indeed, the preimage of an open interval is such a spherical shell, and the factorization product on the line is equivalent to the nesting of shells.

2.2.1. *The case of zero level.* First we will consider the higher Kac-Moody factorization algebra on $\mathbb{C}^d \setminus \{0\}$ "at level zero". That is, the factorization algebra $\mathcal{F}_{\mathfrak{g},0}^{\mathbb{C}^d\setminus\{0\}}$. In this section we will omit $\mathbb{C}^d \setminus \{0\}$ from the notation, and simply refer to the factorization algebra by $\mathcal{F}_{\mathfrak{g},0}$.

Let $\rho_*(\mathcal{F}_{\mathfrak{g},0})$ be the factorization algebra on $\mathbb{R}_{>0}$ obtained by pushing forward along the radial projection map. Explicitly, to an open set $I \subset \mathbb{R}_{>0}$ this factorization algebra assigns the dg vector space

$$C_*^{\mathrm{Lie}}\left(\Omega_c^{0,*}(\rho^{-1}(I)) \otimes \mathfrak{g}\right).$$

Let $I \subset \mathbb{R}_{>0}$ be an open subset. There is the natural map $\rho^* : \Omega_c^*(I) \to \Omega_c^*(\rho^{-1}(I))$ given by the pull back of differential forms. We can post compose this with the natural projection $\mathrm{pr}_{\Omega^{0,*}} : \Omega_c^* \to \Omega_c^{0,*}$ to obtain a map of commutative algebras $\mathrm{pr}_{\Omega^{0,*}} \circ \rho^* : \Omega_c^*(I) \to \Omega_c^{0,*}(\rho^{-1}(I))$. The map j from Proposition **??** determines a map of dg commutative algebras $j : A_d \to \Omega^{0,*}(\rho^{-1}(I))$. Thus, we obtain a map

$$\begin{array}{rccc} \Phi(I) = (\mathrm{pr}_{\Omega^{0,*}} \circ \rho^*) \otimes j : & \Omega_c^*(I) \otimes A_d & \to & \Omega_c^{0,*}\left((\rho^{-1}(I)\right) \\ & \varphi \otimes a & \mapsto & \left((\mathrm{pr}_{\Omega^{0,*}} \circ \rho^*)\varphi\right) \wedge j(a) \end{array}$$

Since this is a map of commutative dg algebras it defines a map of dg Lie algebras

$$\Phi(I) \otimes \mathrm{id}_{\mathfrak{g}} : (\Omega_c^*(I) \otimes A_d) \otimes \mathfrak{g} = \Omega_c^*(I) \otimes (A_d \otimes \mathfrak{g}) \to \Omega^{0,*}(\rho^{-1}(I)) \otimes \mathfrak{g}$$

which maps $(\varphi \otimes a) \otimes X \mapsto \Phi(\varphi \otimes a) \otimes X$. We will drop the $\mathrm{id}_{\mathfrak{g}}$ from the notation and will denote this map simply by $\Phi(I)$. Note that $\Phi(I)$ is compatible with inclusions of open sets, hence extends to a map of cosheaves of dg Lie algebras that we will call Φ.

We can summarize the results as follows.

Proposition 2.6. *The map Φ extends to a map of factorization Lie algebras*

$$\Phi : \Omega^*_{\mathbb{R}_{>0},c} \otimes (A_d \otimes \mathfrak{g}) \to \rho_*\left(\Omega^{0,*}_{\mathbb{C}^d\setminus 0,c} \otimes \mathfrak{g}\right).$$

Hence, it defines a map of factorization algebras

$$C_*(\Phi) : U^{fact}\left(\Omega^*_{\mathbb{R}_{>0}} \otimes (A_d \otimes \mathfrak{g})\right) \to \rho_*\left(\mathcal{F}_{\mathfrak{g},0}^{\mathbb{C}^d\setminus 0}\right).$$

The fact that we obtain a map of factorization algebras follows from the universal property of the universal enveloping factorization algebra we discussed in Section **??**.

2.2.2. *The case of non-zero level.* We now proceed to the proof of Theorem . The dg Lie algebra $\mathfrak{g}_{d,\theta}$ determines a dg associative algebra via its universal enveloping algebra $U(\mathfrak{g}_{d,\theta})$. BW: define it? By BW: ref this dg algebra determines a factorization algebra on the one-manifold $\mathbb{R}_{>0}$ that assigns to every open interval $I \subset \mathbb{R}_{>0}$ the dg vector space $U(A_d \otimes \mathfrak{g})$. The factorization product is uniquely determined by the algebra structure. Henceforth, we denote this factorization algebra by $U(\mathfrak{g}_{d,\theta})^{fact}$.

To prove the theorem we will construct a sequence of maps of factorization Lie algebras on $\mathbb{R}_{>0}$:

$$\begin{tikzcd} & \mathcal{G}_1 \arrow[dr, "\Phi_1"] & & \mathcal{G}_2 \\ \mathcal{G}_0 \arrow[ur, "\simeq", "\Phi_0"'] & & \mathcal{G}_1' \arrow[ur, "\Phi_2"] & . \end{tikzcd}$$

The factorization envelope of $\mathcal{G}_0$ is equivalent to the factorization algebra $U(\widehat{\mathfrak{g}}_{d,\theta})^{fact}$. Moreover, the factorization envelope of $\mathcal{G}_2$ is the push-forward of of the higher Kac–Moody factorization algebra $\rho_*\mathcal{F}_{\mathfrak{g},\theta}$. Hence, the desired map of factorization algebras is produced by applying the factorization envelope functor to the above composition of factorization Lie algebras.

First, we introduce the factorization Lie algebra $\mathcal{G}_0$. To an open set $I \subset \mathbb{R}$, it assigns the dg Lie algebra $\mathcal{G}_0(I) = \Omega^*_c(I) \otimes \widehat{\mathfrak{g}}_{d,\theta}$, where $\widehat{\mathfrak{g}}_{d,\theta}$ is the central extension from BW: ref. The differential and Lie bracket are determined by the fact that we are tensoring a commutative dg algebra with a dg Lie algebra. A slight variant of Proposition 3.4.0.1 in **[?]**, which shows that the one-dimensional factorization envelope of an ordinary Lie algebra produces its ordinary universal enveloping algebra, shows that there is a quasi-isomorphism of factorization algebras on $\mathbb{R}$,

$$(U\widehat{\mathfrak{g}}_{d,\theta})^{fact} \xrightarrow{\simeq} C^{\mathrm{Lie}}_*(\mathcal{G}_0).$$

The factorization Lie algebra $\mathcal{G}_0$ is a central extension of the factorization Lie algebra $\Omega^*_{\mathbb{R},c} \otimes (A_d \otimes \mathfrak{g})$ by the trivial module $\Omega^*_c \oplus \mathbb{C} \cdot K$. Indeed, the cocycle determining the central extension is given by

$$\theta_0(\varphi_0\alpha_0, \ldots, \varphi_d\alpha_d) = (\varphi_0 \wedge \cdots \wedge \varphi_d)\theta_{A_d}(\alpha_1, \ldots, \alpha_d).$$

The factorization Lie algebra $\Omega^*_{\mathbb{R},c} \otimes (A_d \otimes \mathfrak{g})$ is the compactly supported sections of the local Lie algebra $\Omega^*_{\mathbb{R}} \otimes (A_d \otimes \mathfrak{g})$ and this cocycle determining the extension is a local cocycle.

Next, we define the factorization dg Lie algebra $\mathcal{G}_1$ on $\mathbb{R}$. This is also obtained as a central extension of the factorization Lie algebra $\Omega^*_{\mathbb{R},c} \otimes (A_d \otimes \mathfrak{g})$:

$$0 \to \mathbb{C} \cdot K[-1] \to \mathcal{G}_1 \to \Omega^*_{\mathbb{R},c} \otimes (A_d \otimes \mathfrak{g}) \to 0$$

determined by the following cocycle. For an open interval I write $\varphi_i \in \Omega^*_c(I)$, $\alpha_i \in A_d \otimes \mathfrak{g}$. The cocycle is defined by

$$\theta_1(\varphi_0\alpha_0, \ldots, \varphi_d\alpha_d) = \left(\int_I \varphi_0 \wedge \cdots \varphi_d\right) \theta_{\mathrm{FHK}}(\alpha_0, \ldots, \alpha_d) \tag{2}$$

where θ_{FHK} was defined in Definition **??**.

The functional θ_1 determines a local cocycle in $C^*_{\mathrm{loc}}\left(\Omega^*_{\mathbb{R}} \otimes (A_d \otimes \mathfrak{g})\right)$ of degree one.

We now define a map of factorization Lie algebras $\Phi_0 : \mathcal{G}_0 \to \mathcal{G}_1$. On and open set $I \subset \mathbb{R}$, we define the map $\Phi_0(I) : \mathcal{G}_0(I) \to \mathcal{G}_1(I)$ by

$$\Phi_0(I)(\varphi\alpha, \psi K) = \left(\varphi\alpha, \int \psi \cdot K\right).$$

For a fixed open set $I \subset \mathbb{R}$, the map Φ_0 fits into the commutative diagram of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \Omega^*_c(I) \otimes \mathbb{C} \cdot K \arrow[r] \arrow[d, "\int", "\simeq"'] & \mathcal{G}_0(I) \arrow[r] \arrow[d, "\Phi_0(I)"] & \Omega^*_c(I) \otimes (A_d \otimes \mathfrak{g}) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathbb{C} \cdot K[-1] \arrow[r] & \mathcal{G}_1(I) \arrow[r] & \Omega^*_c(I) \otimes (A_d \otimes \mathfrak{g}) \arrow[r] & 0. \end{tikzcd}$$

To see that $\Phi_0(I)$ is a map of dg Lie algebras we simply observe that the cocycles determining the central extensions are related by $\theta_1 = \int \circ\, \theta_0$, where $\int : \Omega^*_c(I) \to \mathbb{C}$ as in the diagram above. Since $\int$ is a quasi-isomorphism, the map $\Phi_0(I)$ is as well. It is clear that as we vary the interval I we obtain a quasi-isomorphism of factorization Lie algebras $\Phi_0 : \mathcal{G}_0 \xrightarrow{\simeq} \mathcal{G}_1$.

We now define the factorization dg Lie algebra $\mathcal{G}_1'$. Like $\mathcal{G}_0$ and $\mathcal{G}_0$, it is a central extension of $\Omega^*_{\mathbb{R},c} \otimes (A_d \otimes \mathfrak{g})$. The cocycle determining the central extension is defined by

$$\theta_1'(\varphi_0 a_0 X_0, \ldots, \ldots, \varphi_d a_d X_d) = \theta_1(\varphi_0 a_0 X_0, \ldots, \ldots, \varphi_d a_d X_d) + \widetilde{\theta}_1(\varphi_0 a_0 X_0, \ldots, \ldots, \varphi_d a_d X_d)$$

where θ_1 was defined in Equation (2). Before writing down the explicit formula for $\widetilde{\theta}_1$ we introduce some notation. Set

$$E = r\frac{\partial}{\partial r},$$
$$\mathrm{d}\vartheta = \sum_i \frac{\mathrm{d}z_i}{z_i}.$$

We view E as a vector field on $\mathbb{R}_{>0}$ and $\mathrm{d}\vartheta$ as a $(1,0)$-form on $\mathbb{C}^d \setminus 0$. Define the functional

$$\widetilde{\theta}_1(\varphi_0 a_0 X_0, \ldots, \varphi_d a_d X_d) = \frac{1}{2}\sum_{i=1}^d \left(\int_I \varphi_0 (E \cdot \varphi_i) \varphi_1 \cdots \widehat{\varphi}_i \cdots \varphi_d\right)\left(\oint (a_0 a_i \mathrm{d}\vartheta)\, \partial a_1 \cdots \widehat{\partial a_i} \cdots \partial a_d\right) \theta(X_0, \ldots, X_d).$$

The functional $\widetilde{\theta}$ defines a local functional in $C^*_{\mathrm{loc}}\left(\Omega^*_{\mathbb{R}_{>0}} \otimes (A_d \otimes \mathfrak{g})\right)$ of cohomological degree one. One immediately checks that it is a cocycle. This completes the definition of the factorization Lie algebra $\mathcal{G}_1'$.

The factorization Lie algebras $\mathcal{G}_1$ and $\mathcal{G}_1'$ are identical as precosheaves of vector spaces. In fact, if we put a filtration on $\mathcal{G}_1$ and $\mathcal{G}_1'$ where the central element K has filtration degree one, then the associated graded factorization Lie algebras $\mathrm{Gr}\ \mathcal{G}_1$ and $\mathrm{Gr}\ \mathcal{G}_1'$ are also identified. The only difference in the Lie algebra structures comes from the deformation of the cocycle determining the extension of $\mathcal{G}_1'$ given by $\widetilde{\theta}_1$.

In fact, we will show that $\widetilde{\theta}_1$ is actually an exact cocycle via the cobounding element $\eta \in C^*_{\mathrm{loc}}\left(\Omega^*_{\mathbb{R}_{>0}} \otimes (A_d \otimes \mathfrak{g})\right)$ defined by

$$\eta(\varphi_0 a_0 X_0, \ldots, \varphi_d a_d X_d) = \sum_{i=1}^d \left(\int_I \varphi_0\, (\iota_E \varphi_i)\, \varphi_1 \cdots \widehat{\varphi}_i \cdots \varphi_d\right)\left(\oint (a_0 a_i \mathrm{d}\vartheta)\, \partial a_1 \cdots \widehat{\partial a_i} \cdots \partial a_d\right) \theta(X_0, \ldots, X_d).$$

Lemma 2.7. *One has $\mathrm{d}\eta = \widetilde{\theta}_1$, where d is the differential for the cochain complex $C^*_{\mathrm{loc}}(\Omega^*_{\mathbb{R}_{>0}} \otimes (A_d \otimes \mathfrak{g}))$. In particular, the factorization Lie algebras $\mathcal{G}_1$ and $\mathcal{G}_1'$ are quasi-isomorphic (as L_∞ algebras). An explicit quasi-isomorphism is given by the L_∞ map $\Phi_1 : \mathcal{G}_1 \to \mathcal{G}_1'$ that sends the central element K to itself and an element $(\varphi_0 a_0 X_0, \ldots, \varphi_d a_d X_d) \in \mathrm{Sym}^{d+1}(\Omega^*_c \otimes (A_d \otimes \mathfrak{g})$ to*

$$(\varphi_0 a_0 X_0, \ldots, \varphi_d a_d X_d) + \eta(\varphi_0 a_0 X_0, \ldots, \varphi_d a_d X_d) \cdot K \in \mathrm{Sym}^{d+1}(\Omega^*_c \otimes (A_d \otimes \mathfrak{g})) \oplus \mathbb{C} \cdot K.$$

Finally, we define the factorization Lie algebra $\mathcal{G}_2$. We have already seen that the local cocycle $J(\theta) \in C^*_{\mathrm{loc}}(\mathfrak{g}^{\mathbb{C}^d})$ determines a central extension of factorization Lie algebras

$$0 \to \mathbb{C} \cdot K[-1] \to \mathcal{G}_{J(\theta)} \to \Omega^{0,*}_{\mathbb{C}^d,c} \otimes \mathfrak{g} \to 0.$$

Of course, we can restrict $\mathcal{G}_{J(\theta)}$ to a factorization algebra on $\mathbb{C}^d \setminus 0$. The factorization algebra $\mathcal{G}_2$ is defined as the pushforward of this restriction along the radial projection: $\mathcal{G}_2 := \rho_* \left(\mathcal{G}_{J(\theta)}|_{\mathbb{C}^d \setminus 0} \right)$.

Recall the map $\Phi : \Omega^*_{\mathbb{R}_{>0},c} \otimes (A_d \otimes \mathfrak{g}) \to \rho_*(\Omega^{0,*}_{\mathbb{C}^d\setminus 0,c} \otimes \mathfrak{g})$ defined in BW: ref. On each open set $I \subset \mathbb{R}_{>0}$ we can extend Φ by the identity on the central element to a linear map $\Phi_2 : \mathcal{G}'_1(I) \to \mathcal{G}_2(I)$.

Lemma 2.8. *The map $\Phi_2 : \mathcal{G}'_1(I) \to \mathcal{G}_2(I)$ is a map of dg Lie algebras. Moreover, it extends to a map of factorization Lie algebras $\Phi_2 : \mathcal{G}'_1 \to \mathcal{G}_2$.*

Proof. Modulo the central element Φ_2 reduces to the map Φ, which we have already seen is a map of factorization Lie algebras in Proposition BW: ref. Thus, to show that Φ_2 is a map of factorization Lie algebras we need to show that it is compatible with the cocycles determing the respective central extensions. That is, we need to show that

$$\theta'_1(\varphi_0 a_0 X_0, \ldots, \varphi_d a_d X_d) = \theta_2(\Phi(\varphi_0 a_0 X_0), \ldots, \Phi(\varphi_d a_d X_d)) \tag{3}$$

for all $\varphi_i a_i X_i \in \Omega^*_c(I) \otimes (A_d \otimes \mathfrak{g})$. The cocycle θ'_1 is only nonzero if one of the φ_i inputs is a 1-form. We evaluate the left-hand side on the $(d+1)$-tuple $(\varphi_0 \mathrm{d}r a_0 X_0, \varphi_1 a_1 X_1, \ldots, \varphi_d a_d X_d)$ where $\varphi_i \in C^\infty_c(I)$, $a_i \in A_d$, $X_i \in \mathfrak{g}$ for $i = 0, \ldots, d$. The result is

$$\left(\int_I \varphi_0 \cdots \varphi_d \mathrm{d}r \right) \left(\oint a_0 \partial a_1 \cdots \partial a_d \right) \theta(X_0, \ldots, X_d) \tag{4}$$

$$+ \quad \frac{1}{2} \sum_{i=1}^{d} \left(\int_I \varphi_0 (E \cdot \varphi_i) \varphi_1 \cdots \widehat{\varphi_i} \cdots \varphi_d \mathrm{d}r \right) \left(\oint (a_0 a_i \mathrm{d}\vartheta)\, \partial a_1 \cdots \widehat{\partial a_i} \cdots \partial a_d \right) \theta(X_0, \ldots, X_d) \tag{5}$$

We wish to compare this to the right-hand side of Equation (3). Recall that $\Phi(\varphi_0 \mathrm{d}r a_0 X_0) = \varphi(r) \mathrm{d}r a_0(z) X_0$ and $\Phi(\varphi_i a_i X_i) = \varphi(r) a_i(z) X_i$. Plugging this into the explicit formula for the cocycle θ_2 we see the right-hand side of (3) is

$$\left(\int_{\rho^{-1}(I)} \varphi_0(r) \mathrm{d}r a_0(z) \partial(\varphi_1(r) a_1(z)) \cdots \partial(\varphi_d(r) a_d(z)) \right) \theta(X_0, \ldots, X_d). \tag{6}$$

We pick out the term in (6) in which the ∂ operators only act on the elements $a_i(z)$, $i = 1, \ldots, d$. This term is of the form

$$\int_{\rho^{-1}(I)} \varphi_0(r) \cdots \varphi_d(r) \mathrm{d}r a_0(z) \partial(a_1(z)) \cdots \partial(a_d(z)) \theta(X_0, \ldots, X_d).$$

Separating variables we find that this is precisely the first term (4) in the expansion of the left-hand side of (3).

Now, note that we can rewrite the ∂-operator in terms of the radius r as

$$\partial = \sum_{i=1}^{d} \mathrm{d}z_i \frac{\partial}{\partial z_i} = \sum_{i=1}^{d} \mathrm{d}z_i \bar{z}_i \frac{\partial}{\partial (r^2)} = \sum_{i=1}^{d} \mathrm{d}z_i \frac{r^2}{2 z_i} \frac{\partial}{\partial r}.$$

The remaining terms in (6) correspond to the expansion of

$$\partial(\varphi_1(r)a_1(z))\cdots\partial(\varphi_d(r)a_d(z)),$$

using the Leibniz rule, for which the ∂ operators act on at least one of the functions $\varphi_1,\ldots,\varphi_d$. In fact, only terms in which ∂ acts on precisely one of the functions $\varphi_1,\ldots,\varphi_d$ will be nonzero. For instance, consider the term

$$(\partial\varphi_1)a_1(z)(\partial\varphi_2)a_2(z)\partial(\varphi_3(z)a_3(z))\cdots\partial(\varphi_d(z)a_d(z)). \tag{7}$$

Now, $\partial\varphi_i(r) = \omega\frac{\partial\varphi}{\partial r}$ where ω is the one-form $\sum_i(r^2/2z_i)\mathrm{d}z_i$. Thus, (7) is equal to

$$\left(\omega\frac{\partial\varphi_1}{\partial r}\right)a_1(z)\left(\omega\frac{\partial\varphi_2}{\partial r}\right)a_2(z)\partial(\varphi_3(z)a_3(z))\cdots\partial(\varphi_d(z)a_d(z),$$

which is clearly zero as ω appears twice.

We observe that terms in the expansion of (6) for which ∂ acts on precisely one of the functions $\varphi_1,\ldots,\varphi_d$ can be written as

$$\sum_{i=1}^d\int_{\rho^{-1}(I)}\varphi_0(r)\left(r\frac{\partial}{\partial r}\varphi_i(r)\right)\varphi_1(r)\cdots\widehat{\varphi_i(r)}\cdots\varphi_d(r)\mathrm{d}r\frac{r}{2z_i}\mathrm{d}z_i a_0(z)a_i(z)\partial a_1(z)\cdots\widehat{\partial a_i(z)}\cdots\partial a_d(z).$$

Finally, notice that the function $z_i/2r$ is independent of the radius r. Thus, separating variables we find the integral can be written as

$$\frac{1}{2}\sum_{i=1}^d\left(\int_I\varphi_0\left(r\frac{\partial}{\partial r}\varphi_i\right)\varphi_1\cdots\widehat{\varphi}_i\cdots\varphi_d\mathrm{d}r\right)\left(\oint\frac{\mathrm{d}z_i}{z_i}a_0a_i\partial a_2\cdots\widehat{\partial a_i}\cdots\partial a_d\right).$$

This is precisely equal to the second term (5) above. Hence, the cocycles are compatible and the proof is complete.

□

2.3. **Higher loop algebras.** We now put the Kac-Moody factorization algebra on the d-fold $(\mathbb{C}^\times)^d$. Our main result in this section involves extracting the structure of an E_d algebra from considering the nesting of "polyannuli" in $(\mathbb{C}^\times)^d$. When $d=1$, we have seen that the nesting of ordinary annuli give rise to the structure of an associative algebra. For $d>1$, a polyannulus is a complex submanifold of the form $\mathbb{A}_1\times\cdots\times\mathbb{A}_d\subset(\mathbb{C}^\times)^d$ where each $\mathbb{A}_i\subset\mathbb{C}^\times$ is an ordinary annulus. Equivalently, a polyannulus is the complement of a closed polydisk inside of a larger open polydisk. We will see how the nesting of annuli in each component gives rise to the structure of a locally constant factorization algebra in d *real* dimensions, and hence defines an E_d algebra.

2.3.1. Define the commutative algebra

$$B_d = \mathbb{C}[z_1,z_1^{-1}]\otimes\cdots\otimes\mathbb{C}[z_d,z_d^{-1}].$$

If $\mathfrak{g}$ is any Lie algebra we define the Lie algebra $L^d\mathfrak{g} := B_d\otimes\mathfrak{g}$. This is the algebraic version of the d-fold loop space of the Lie algebra $\mathfrak{g}$:

$$L(L(\cdots L(\mathfrak{g})\ldots)) = \mathrm{Map}((S^1)^{\times d},\mathfrak{g}).$$

We will write elements as $f\otimes X\in B_d\otimes\mathfrak{g}$ for $f=f(z_1,\ldots,f_d)\in B_d$ and $X\in\mathfrak{g}$.

In the commutative algebra B_d there are derivations $\partial/\partial z_1, \ldots, \partial/\partial z_d$. Let $\Omega^1_{B_d} = B_d[\mathrm{d}z_1, \ldots, \mathrm{d}z_d]$ be the vector space of algebraic differentials. Similarly, define $\Omega^k_{B_d}$ by $B_d \otimes \wedge^k \mathbb{C}\{\mathrm{d}z_1, \ldots, \mathrm{d}z_d\}$. There is a universal algebraic differential $\partial : B_d \to \Omega^1_{B_d}$ given in coordinates by $\partial = \sum_i \frac{\partial}{\partial z_i} \mathrm{d}z_i$.

We note that the space of d-forms $\Omega^d_{B_d}$ admits a residue map defined by taking d-fold iterated one-dimensional residues:

$$\oint_{|z_1|=1} \cdots \oint_{|z_d|=1} : \Omega^d_{B_d} \to \mathbb{C}.$$

Explicitly, if $f \mathrm{d}z_1 \cdots \mathrm{d}z_d$ is a top form then

$$\oint_{|z_1|=1} \cdots \oint_{|z_d|=1} f \mathrm{d}z_1 \cdots \mathrm{d}z_d = (2\pi i)^n \times \{\text{coefficient of } z_1^{-1} \cdots z_d^{-1}\}.$$

Given a homogenous degree d invariant polynomial on $\mathfrak{g}$ there is a shifted extension of $L^d\mathfrak{g}$ that is closely related to the extension we discussed in the previous section.

Proposition 2.9. *Given any $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g}^\vee)^{\mathfrak{g}}$ there is $(d-1)$-shifted L_∞-central extension of $L^d\mathfrak{g}$*

$$0 \to \mathbb{C}[d-1] \to \widehat{L^d\mathfrak{g}}_\theta \to L^d\mathfrak{g} \to 0$$

with brackets given by $\ell_2 = [-,-]_{L^d\mathfrak{g}}$ and

$$\ell_{d+1}(f_0 \otimes X_0, \cdots, f_d \otimes X_d) = \theta(X_1, \ldots, X_d) \oint_{|z_1|=1} \cdots \oint_{|z_d|=1} f_0 \partial f_1 \cdots \partial f_d \cdot K$$

and all other brackets zero. Here, K is the generator of the central part of the Lie algebra of degree $-d+1$.

2.3.2. Given any Lie algebra $\mathfrak{h}$ we can define the universal enveloping algebra $U\mathfrak{h}$ which is an associative. In fact, the functor $\mathfrak{h} \mapsto U\mathfrak{h}$ from Lie algebras to associative algebras is left adjoint to the forgetful functor obtained by forming the commutator in the associative algebra. The homotopical generalization of associative algebras are E_1-algebras which are algebras over the operad of little 1-disks.

Theorem 2.10 ([?]). *There is a forgetful functor $F : \mathrm{Alg}_{E_d} \to \mathrm{dgLie}_{\mathbb{C}}$ and it admits a left adjoint*

$$U_{E_d} : \mathrm{dgLie}_{\mathbb{C}} \to \mathrm{Alg}_{E_d}$$

called the E_d-universal enveloping algebra. If $\mathfrak{h}$ is an ordinary Lie algebra the E_d-algebra has underlying graded vector space

$$U_{E_d}(\mathfrak{h}) = \mathrm{Sym}\left(\mathfrak{h}[1-d]\right).$$

There is an equivalence of categories between E_d algebras and locally constant factorization algebras on $\mathbb{R}^d$. If A is an E_d algebra we denote by A^{fact} its associated locally constant factorization algebra on $\mathbb{R}^d$.

Proposition 2.11. *Suppose $\mathfrak{h}$ is a dg Lie algebra. Then, there is a quasi-isomorphism of factorization algebras on $\mathbb{R}^d$:*

$$\left(U_{E_d}\mathfrak{h}\right)^{fact} \simeq \mathrm{C}^{\mathrm{Lie}}_*(\Omega^*_{c,\mathbb{R}^d} \otimes \mathfrak{h})$$

We now explain how the higher dimensional Kac-Moody factorization algebra is related to the universal E_d enveloping algebra of the Lie algebra $B_d \otimes \mathfrak{g}$ (and its central extension). We will consider the factorization algebra restricted to the complex manifold $(\mathbb{C}^\times)^d \subset \mathbb{C}^d$. Throughout this section we will denote the factorization algebra $\mathcal{F}_{\mathfrak{g},\theta}^{(\mathbb{C}^\times)^d}$ on $(\mathbb{C}^\times)^d$ simply by $\mathcal{F}_{\mathfrak{g},\theta}$.

Let $\vec{\rho} : (\mathbb{C}^\times)^d \to (\mathbb{R}_{>0})^d$ be the map sending $(z_1, \ldots, z_d) \mapsto (|z_1|, \ldots, |z_d|)$. If $I_1, \ldots, I_d \subset \mathbb{R}_{>0}$ is any collection of intervals we see that $\vec{\rho}^{-1}(I_1 \times \cdots \times I_d) \subset (\mathbb{C}^\times)^d$ is a polyannulus. Thus, to understand the behavior of a factorization algebra $\mathcal{F}$ on $(\mathbb{C}^\times)^d$ with respect to the nesting of polyannuli, as discussed in the beginning of this section, it suffices to understand the factorization product of cubes of the pushforward of the factorization algebra $\vec{\rho}_* \mathcal{F}$ on $(\mathbb{R}_{>0})^d$.

A general factorization algebra $\mathcal{F}$ on $(\mathbb{C}^\times)^d$ does not define a E_d algebra in the way we have just described. Indeed, even in the case of a holomorphic factorization algebra, it is reasonable to expect that the pushforward factorization algebra will be sensitive to the length of the sides of the cubes in $\mathbb{R}_{>0}$. Just as in the case of the previous section, where we considered compactification along the $2d-1$ sphere in $\mathbb{C}^d \setminus 0$, we will show that there is a well-behaved sub-factorization algebra that *is* locally constant and hence does define the structure of an E_d algebra.

There is a holomorphic action of the d-torus $T^d = S^1 \times \cdot \times S^1$ on the complex manifold $(\mathbb{C}^\times)^d$ by rotating component-wise. Hence, there is an induced action of T^d on the Dolbeault complex $\Omega^{0,*}((\mathbb{C}^\times)^d) \cong \Omega^{0,*}(\mathbb{C}^\times)^{\otimes d}$. The action of the torus is induced from a tensor product of S^1 representations with respect to this decomposition. For an integer $n \in \mathbb{Z}$ let $\Omega^{0,*}(\mathbb{C}^\times)^{(n)} \subset \Omega^{0,*}(\mathbb{C}^\times)$ be the dg subspace consisting of all forms with eigenvalue n. Similarly, for each sequence of integers $(n_1, \ldots, n_d)$ we let

$$\Omega^{0,*}\left((\mathbb{C}^\times)^d\right)^{(n_1,\ldots,n_d)} \subset \Omega^{0,*}\left((\mathbb{C}^\times)^d\right)$$

be the tensor product $\Omega^{0,*}(\mathbb{C}^\times)^{(n_1)} \otimes \cdots \otimes \Omega^{0,*}(\mathbb{C}^\times)^{(n_d)}$.

For each open set $U \subset (\mathbb{C}^\times)^d$ we can define, in a completely analogous way, the subspace

$$\mathcal{F}_{\mathfrak{g},\theta}^{(\mathbb{C}^\times)^d}(U)^{(n_1,\ldots,n_d)} \subset \mathcal{F}_{\mathfrak{g},\theta}^{(\mathbb{C}^\times)^d}(U).$$

3. Twisted D-modules on the moduli of G-bundles

Suppose X is a complex curve and G is a simple Lie group. If $x \in X$, denote by $\widehat{\mathcal{O}}_x$ the completed local ring at x which is non-canonically isomorphic to the ring of power series $\mathbb{C}[[t]]$. Let $\widehat{\mathcal{K}}_x$ denote its field of fractions, which can be identified with Laurent series $\mathbb{C}((t))$. The corresponding formal disk and formal punctured disk are denoted by $\widehat{D}_x = \mathrm{Spec}(\widehat{\mathcal{O}}_x)$, $\widehat{D}_x^\times = \mathrm{Spec}(\widehat{\mathcal{K}}_x)$. Let $G(\widehat{\mathcal{O}}_x)$ be the group of maps $\widehat{D}_x \to G$ and $G(\widehat{\mathcal{K}}_x)$ be the group of maps $\widehat{D}_x^\times \to G$. The latter is sometimes called the formal loop group of G.

There is a subgroup G_{out} of $G(\widehat{\mathcal{K}}_x)$ consisting of the maps $X \setminus x \to G$. A result of BW: ref identifies the moduli space of G-bundles on X with the double quotient

$$\mathrm{Bun}_G(X) \cong G_{\mathrm{out}}??G(\widehat{\mathcal{K}}_x)/G(\widehat{\mathcal{O}}_x).$$

Let $\mathrm{Bun}_G(X)$ denote the moduli space of G-bundles on the complex d-fold X. For $d > 1$ **[?]** have constructed a global smooth derived realization of this space, but its full structure will not be used in this discussion. We recall the definition of the determinant line bundle associated to a representation as a functor

$$\kappa : \mathrm{Rep}(G) \to \mathrm{Pic}(\mathrm{Bun}_G(X)).$$

Consider the universal G-bundle $\mathcal{B}\mathrm{un}_G(X)$ over the space $\mathrm{Bun}_G(X) \times X$ whose fiber over $\{P \to X\} \times X$ is equal to the bundle P itself:

$$\begin{tikzcd} P \arrow[r] \arrow[d] & \mathcal{B}\mathrm{un}_G(X) \arrow[d, "G"] \\ \{P\} \times X \arrow[r] & \mathrm{Bun}_G(X) \times X. \end{tikzcd}$$

Given a representation V consider the associated vector bundle $\mathcal{V} = \mathcal{B}\mathrm{un}_G(X) \times^G V$ over $\mathrm{Bun}_G(X) \times X$. If $p_1 : \mathrm{Bun}_G(X) \times X \to \mathrm{Bun}_G(X)$ is the projection, the determinant line bundle associated to V is defined by

$$\kappa_V := \det(\mathbb{R}p_{1*}\mathcal{V})$$

where $\mathbb{R}p_{1*}$ is the derived pushforward, and the determinant is interpreted in the graded sense. For instance, if $W = W_0 + W_1[-1]$ is a graded vector space concentrated in degree zero and one then $\det(W) = \det(W_0) \otimes \det(W_1)^{-1}$.

3.1. **Symmetries of BV theories.** First, we review what symmetries look like in classical mechanics. Let $\mathfrak{g}$ be a Lie algebra and (M, ω) a symplectic manifold. A symplectic action of $\mathfrak{g}$ on M is a map of Lie algebras $\rho : \mathfrak{g} \to \mathrm{Vect}^{\mathrm{symp}}(M, \omega)$, where the target is the Lie algebra of symplectic vector fields. Let $\mathcal{O}(M)$ be the commutative algebra of smooth functions on M. An action of the Lie algebra $\mathfrak{g}$ on M induces a map of Lie algebras $\rho : \mathfrak{g} \to \mathrm{Der}(\mathcal{O}(M))$ by derivations. The Poisson bracket $\{-,-\}$ on functions also determines a map of Lie algebras $\mathcal{O}(M) \to \mathrm{Der}(\mathcal{O}(M))$ sending $f \mapsto \{f, -\}$. The symplectic action is *Hamiltonian* if there exists a lifting $\widetilde{\rho} : \mathfrak{g} \to C^\infty(M)$.

A classical field theory is determined by an (infinite-dimensional) space of fields with a shifted symplectic structure. We recall how ...

3.1.1. *Classical symmetries.* In the BV formalism, the data of a classical field theory on X consists of a sheaf of fields $\mathcal{E}$, an action functional $S \in \mathcal{O}_{\mathrm{loc}}(\mathcal{E})$ of degree zero, and a (-1)-shifted $\mathbb{C}$-valued pairing on $\mathcal{E}$. The pairing induces a bracket $\{-,-\}$ on the space of local functionals, and this data is required to satisfy the condition $\{S, S\} = 0$. This is known as the *classical master equation*.

Alternatively, we can view the shifted space of local functionals $\mathcal{O}_{\mathrm{loc}}(\mathcal{E})$ as a dg Lie algebra. The differential is $\{S, -\}$ and the Lie bracket is $\{-,-\}$. The classical master equation is equivalent to the statement that S is a Maurer–Cartan element of this dg Lie algebra.

Let $\mathcal{L}$ be a local Lie algebra on X. Then, $\mathcal{L}(X)$ is an L_∞ algebra and we can consider its reduced Chevalley–Eilenberg cochain complex $\mathrm{C}^*_{\mathrm{Lie,red}}(\mathcal{L}(X))$. This is a commutative dg algebra, so we can tensor with $\mathcal{O}_{\mathrm{loc}}(\mathcal{E})[-1]$ to form the new dg Lie algebra $\mathrm{C}^*_{\mathrm{Lie,red}}(\mathcal{L}(X)) \otimes \mathcal{O}_{\mathrm{loc}}(\mathcal{E})[-1]$. The differential is of the form $\mathrm{d}_{\mathcal{L}} + \{S, -\}$, where $\mathrm{d}_{\mathcal{L}}$ is the CE differential for $\mathcal{L}(X)$, and the bracket is $\mathrm{id}_{\mathcal{L}} \otimes \{-,-\}$.

Definition 3.1. Let $\mathcal{L}$ be a local Lie algebra and $(\mathcal{E}, S)$ a classical theory. Define the dg Lie algebra

$$\mathrm{Act}(\mathcal{L}, \mathcal{E}) := \mathrm{C}^*_{\mathrm{loc}}(\mathcal{L}) \otimes \mathcal{O}_{\mathrm{loc}}(\mathcal{E}) / \left(\mathrm{C}^*_{\mathrm{loc}}(\mathcal{L}) \oplus \mathcal{O}_{\mathrm{loc}}(\mathcal{E})\right)$$

with differential and bracket given by the restriction of $\mathrm{d}_{\mathcal{L}} + \{S, -\}$ and $\{-,-\}$, respectively.

Note that $\mathrm{Act}(\mathcal{L}, \mathcal{E}) \subset \mathrm{C}^*_{\mathrm{Lie,red}}(\mathcal{L}(X)) \otimes \mathcal{O}_{\mathrm{loc}}(\mathcal{E})[-1]$ is an inclusion of dg Lie algebras. A functional $F \in \mathrm{C}^*_{\mathrm{Lie,red}}(\mathcal{L}(X)) \otimes \mathcal{O}_{\mathrm{loc}}(\mathcal{E})[-1]$ lives in $\mathrm{Act}(\mathcal{L}, \mathcal{E})$ if and only if:

(1) As a functional of $\mathcal{L}$, F is *local*, and

(2) The functional $S^{\mathcal{L}}$ must depend on both $\mathcal{L}$ and $\mathcal{E}$. We mod out by functionals that are of purely one or the other.

We can now define what it means for a local Lie algebra to be a symmetry.

Definition 3.2. Suppose $\mathcal{L}$ is a local Lie algebra and $(\mathcal{E}, S)$ defines a classical theory. An $\mathcal{L}$-*symmetry* of $\mathcal{E}$ is a functional $S^{\mathcal{L}} \in \mathrm{Act}(\mathcal{L}, \mathcal{E})$ that satisfies the *$\mathcal{L}$-equivariant classical master equation*:

$$\mathrm{d}_{\mathcal{L}} S^{\mathcal{L}} + \{S, S^{\mathcal{L}}\} + \frac{1}{2}\{S^{\mathcal{L}}, S^{\mathcal{L}}\} = 0.$$

Such an element $S^{\mathcal{L}}$ is automatically a Maurer–Cartan element of the dg Lie algebra $\mathrm{C}^*_{\mathrm{Lie,red}}(\mathcal{L}(X)) \otimes \mathcal{O}_{\mathrm{loc}}(\mathcal{E})[-1]$. By the general yoga of Koszul duality, a Maurer–Cartan element defines a map of L_∞ algebras

$$S^{\mathcal{L}} : \mathcal{L}(X) \to \mathcal{O}_{\mathrm{loc}}(\mathcal{E})[-1].$$

3.1.2. *Quantum symmetries.* We now discuss how to lift symmetries to the quantum level. We follow the approach of Costello [**?**] to perturbative QFT based on the Wilsonian renormalization of the path integral. Suppose $\mathcal{E}$ is the space of fields. A QFT is a family of functionals $\{S[L]\}$ ...

The main result of [**?**] says that associated to any QFT $(\mathcal{E}, S^{\mathrm{q}})$ defined on X there is a factorization algebra $\mathrm{Obs}^{\mathrm{q}}$ on X called the *quantum observables*.

Theorem 3.3 ([**?**] Theorem 12.5.0.1)**.** *Suppose we have an $\mathcal{L}$-symmetry of a QFT $(\mathcal{E}, S^{\mathrm{q}})$. Then, there is a cohomology class $\alpha_{\mathcal{E}} \in H^1_{red,loc}(\mathcal{L})[[\hbar]]$ such that the factorization Lie algebra $\mathcal{L}_c$ acts (up to homotopy) on the factorization algebra of quantum observables $\mathrm{Obs}^{\mathrm{q}}_{\mathcal{E}}[\hbar^{-1}]$ by $\alpha_{\mathcal{E}}$ times the identity.*

We can view $\alpha_{\mathcal{E}}$ as the "local character" for the action of the local Lie algebra $\mathcal{L}$ on the observables. Indeed, this statement implies that for any open set $U \subset X$ we have an action of the L_∞ algebra $\mathcal{L}_c(U)$ on $\mathrm{Obs}^{\mathrm{q}}(U)[\hbar^{-1}]$, and that this action is homotopy equivalent to the trivial action times the character $\alpha_{\mathcal{E}}$. Moreover, this homotopy equivalence is compatible with the factorization structure.

In the remainder of this section we interpret the implication of this for the Kac–Moody

3.2. **Free theories and the $\beta\gamma$ system.** BW: statement about determinants a la Gwilliam-Haugseng

3.3. **Computing the anomaly.**

Proposition 3.4. *Let V be a $\mathfrak{g}$ module and X a complex d-fold. The classical $\mathfrak{g}^X$-equivariant theory*

$$\mathcal{E}_{X,V} = T^*[-1](\Omega^{0,*}(X;V))$$

admits a canonical $\mathfrak{g}^X$-equivariant quantization. The cohomology class of the obstruction $[\Theta_V] \in H^1(\mathrm{C}^*_{\mathrm{loc}}(\mathfrak{g}^X))$ to lifting this to an inner action by the local Lie algebra $\mathfrak{g}^X$ is identified with the image of

$$\#\mathrm{ch}_{d+1}(V) \in \mathrm{Sym}^{d+1}(\mathfrak{g}^\vee)^{\mathfrak{g}}$$

under the map $J : \mathrm{Sym}^{d+1}(\mathfrak{g}^\vee)^{\mathfrak{g}}[-1] \to \mathrm{C}^*_{\mathrm{loc}}(\mathfrak{g}^X)$.

As a simple corollary we find the anomaly in a slightly more general situation.

Corollary 3.5. *Let P be a principal G-bundle on X, and V a G-representation. Then we can consider the $\mathfrak{g}_P^X = \Omega^{0,*}(X;\mathrm{ad}(P))$-equivariant theory*

$$\mathcal{E}_{P\to X,V} = T^*[-1](\Omega^{0,*}(X; P \times^G V)).$$

This theory admits a canonical $\mathfrak{g}_P^X$-equivariant quantization. Moreover, the cohomology class of the obstruction $[\Theta_V]$ to an inner action is also identified with $\#\mathrm{ch}_{d+1}(V)$.

We will prove the proposition in the following steps. First, we argue that it suffices to calculate this obstruction on an arbitrary open set in X. Taking this open set to be a disk we see that it suffices to compute the cocycle in the case that $X = \mathbb{C}^d$. This calculation is done explicitly in terms of one-loop Feynman diagrams.

3.3.1. By construction, the data of a classical BV theory on X is sheaf-like on the manifold. That is, we have a sheaf of (-1)-shifted elliptic complexes $\mathcal{E}$ on X together with a local functional $I \in \mathcal{O}_{\mathrm{loc}}(\mathcal{E})(X)$. The space of local functionals $\mathcal{O}_{\mathrm{loc}}(\mathcal{E})$ also forms a sheaf on X, so it makes sense to restrict I to any open set $U \subset X$. In this way, for each open we have a (-1)-shifted elliptic complex $\mathcal{E}(U)$ together with a local functional $I|_U \in \mathcal{O}_{\mathrm{loc}}(\mathcal{E})(U)$ – that is, a classical field theory on $U \subset X$. A fancy way of saying this is that the space of classical field theories on X forms a sheaf.

A very slightly refined version of this takes into account an action of a local Lie algebra. If $\mathcal{L}$ is a local Lie algebra on X then the space of $\mathcal{L}$-equivariant classical BV theories also forms a sheaf on X.

Costello has shown in [**?**] that the space of quantum field theories also form a sheaf on X. In a completely analogous way, one can show that the space of $\mathcal{L}$-equivariant quantum field theories forms a sheaf on X.

We have already seen how the obstruction to lifting a quantum field theory with an action of a local Lie algebra $\mathcal{L}$ to an inner action arises as a failure of satisfying the QME. Since an $\mathcal{L}$-equivariant theory satisfies the QME modulo terms in $\mathrm{C}^*_{\mathrm{loc}}(\mathcal{L})(X)$, this obstruction $\Theta(X)$is a degree one cocycle in $\mathrm{C}^*_{\mathrm{loc}}(\mathcal{L})(X)$. By the remarks above, we can restrict any $\mathcal{L}$-equivariant field theory to an arbitrary open set $U \subset X$. Hence, for each open $U \subset X$ we have an obstruction element Θ^U. The complex $C^*_{\mathrm{loc}}(\mathcal{L})(X)$ also has a refinement to a sheaf of complexes on X and the obstruction Θ^U is an element in $C^*_{\mathrm{loc}}(\mathcal{L})(U)$. We will need the following elementary fact that the obstruction to having an inner action is natural with respect to the restriction of open sets.

Lemma 3.6. *Let $i_U^V : U \hookrightarrow V$ be any inclusion of open sets in X. Then*

$$(i_U^V)^*([\Theta^V]) = [\Theta^U]$$

where $(i_U^V)^* : C^*_{\mathrm{loc}}(\mathcal{L})(V) \to C^*_{\mathrm{loc}}(\mathcal{L})(U)$ is the restriction map and the brackets $[-]$ denotes the cohomology class of the cocycle. In other words, the map that sends a quantum field theory on X with an $\mathcal{L}$-action to its obstruction to having an inner $\mathcal{L}$-action is a map of sheaves.

For any complex d-fold X we have defined the map $J^X : \mathrm{Sym}^{d+1}(\mathfrak{g}^\vee)^{\mathfrak{g}} \to C^*_{\mathrm{loc}}(\mathfrak{g}^X)$. The complex $C^*_{\mathrm{loc}}(\mathfrak{g}^X)$

Lemma 3.7. *The map*

$$J : \underline{\mathrm{Sym}^{d+1}(\mathfrak{g}^\vee)^{\mathfrak{g}}} \to C^*_{\mathrm{loc}}(\mathfrak{g}^X)$$

defined on each open by $J|_U = J^U$ is a map of sheaves. Here, the underline means the constant sheaf.

Lemma 3.8. *For any open sets $i_U^V : U \subset V$ in X the induced map*

$$(i_U^V)^* : H^1\left(V; C^*_{\mathrm{loc}}(\mathfrak{g}^X)\right) \to H^1\left(U; C^*_{\mathrm{loc}}(\mathfrak{g}^X)\right)$$

is injective.

BW: The last key observation is that $(i_U^V)^* J^V = J^U$.

4. Higher Kac–Moody as a boundary theory

In this section we show how the Kac–Moody factorization algebra appears as the boundary of a class of supersymmetric gauge theories. We choose to focus on two examples, the four dimensional boundary of a five dimensional gauge theory, and the six dimensional boundary of a seven dimensional gauge theory.

These examples extrapolate a ubiquitous relationship between Chern–Simons theory and the Wess-Zumino-Witten conformal field theory. BW: expand on this

The five dimensional gauge theory we consider is obtained as a twist of $\mathcal{N} = 1$ supersymmetric pure gauge theory. This twist is *not* topological, but it is holomorphic in four real (two complex) directions, and topological in the transverse direction. We will show how a deformation of this theory yields a boundary condition on manifolds of the form $X \times [0, 1]$, where X is a Calabi–Yau surface, at $X \times \{0\}$. Moreover, this boundary condition determines a factorization algebra of classical observables supported on the boundary that is equal to a certain degenerate classical limit of the Kac–Moody factorization algebra. We show that there is a quantization of this theory that returns the Kac–Moody at a certain level.

The seven dimensional theory similarly appears as a twist, this time of maximally supersymmetric gauge theory. We perform a similar analysis to show how to find the higher Kac–Moody on a Calabi–Yau three-fold in the manner sketched above.

4.1. **The P_0 structure.** In ordinary classical mechanics, the symplectic structure on the phase space induces the structure of a Poisson algebra on the operators of the theory. Classically, the data of a field theory in the BV–formalism involves a (-1)-shifted symplectic form on the space of fields. It is shown in [**?**] that this induces the factorization algebra of classical observables with the structure of a strict P_0-algebra. A P_0-algebra is a shifted version of a Poisson algebra in this graded setting. Indeed, the data of such an algebra includes a commutative dg product together with a bracket of cohomological degree $+1$. These

In this section we will describe the P_0 structure on the higher dimensional Kac–Moody factorization algebra at level zero. We will give an interpretation of this P_0 structure as coming from a Poisson structure on a particular formal moduli space.

4.1.1. Suppose $\mathfrak{h}$ is any L_∞ algebra. Then, we can define the commutative dg algebra of Chevalley–Eilenberg cochains on $C^*_{\rm Lie}(\mathfrak{h})$. We formulate a convenient way to define homotopy Poisson structures on this commutative dg algebra. The L_∞ algebra $\mathfrak{h}$ acts on $\mathfrak{h}[1]$ via the adjoint representation, and this extends to an action on the completed symmetric algebra $\widehat{\rm Sym}(\mathfrak{h}[1])$. Consider an element $\Pi \in C^*_{\rm Lie}(\mathfrak{h}; \widehat{\rm Sym}(\mathfrak{h}[1]))$ of total degree $1-n$ BW: or $n-1$.

This P_0 algebra is induced from a *local* Poisson structure on a certain moduli space that we now discuss.

First, we introduce the following local L_∞ algebra on X,

$$\mathcal{L} = \Omega^{d,*}_X \otimes \mathfrak{g}[d-2]\ , \qquad \ell_1 = \bar\partial \otimes {\rm id}_{\mathfrak{g}}, \quad \ell_n = 0 \ \text{ for } n > 1.$$

Thus, this an abelian L_∞ algebra concentrated in degrees $-d+2$ to 2.

We have already discussed how local Lie algebras define factorization algebras via the enveloping construction. There is another construction of a factorization algebra that is "Fourier dual" to this. On an open set $U \subset X$ we assign the complex of Chevalley–Eilenberg cochains on $\mathcal{L}(U)$, $C^*_{\rm Lie}(\mathcal{L}(U))$. The product maps are defined in a natural way. For more details see BW: ref in [**?**].

For each open $U \subset X$ we have a formal moduli problem $B\mathcal{L}(U)$ whose functions is commutative dg ring $C^*_{\rm Lie}(\mathcal{L}(U))$. These formal moduli problems glue together to define a *local* moduli problem $B\mathcal{L}$ on X [**?**]. The induced factorization algebra of functions on the local moduli problem will be denoted by $\mathcal{O}(B\mathcal{L})$.

Proposition 4.1. *The local moduli problem $B\mathcal{L}$ satisfies $\mathcal{O}(B\mathcal{L}) = \mathcal{F}_{\mathfrak{g},0}$. Moreover, there is a local (-1)-shifted Poisson structure on $B\mathcal{L}$ defined by the Poisson tensor $\Pi = \Pi_{1,2} + \Pi_{0,d+1}$ where*

$$\Pi_{1,2} = [-,-] : \left(\Omega^{d,*}_X \otimes \mathfrak{g}\right) \otimes \left(\Omega^{0,*}_X \otimes \mathfrak{g}\right) \to \Omega^{d,*}_X \otimes \mathfrak{g}$$

and

$$\Pi_{0,d+1} : \left(\Omega^{0,*}_X \otimes \mathfrak{g}\right)^{\otimes d} \to \Omega^{d,*}_X \otimes \mathfrak{g}$$

sends $\alpha_1 \otimes \cdots \otimes \alpha_d \mapsto \partial\alpha_1 \wedge \cdots \wedge \partial\alpha_d$. In particular, $\mathcal{F}_{\mathfrak{g},0}$ has the structure of a P_0 factorization algebra.

4.2. **5d $N = 1$ supersymmetric gauge theory.** BW: discuss twist

Proposition 4.2. *The twist of 5d $N = 1$ supersymmetric pure gauge theory exists on any manifold of the form*

$$\mathbb{R} \times X$$

where X is a Calabi-Yau surface. The fields of the theory are

$$\mathcal{E}_{5d} = \Omega^*(\mathbb{R}) \otimes \Omega^{0,*}(X; \mathfrak{g} \oplus \mathfrak{g}^*)[1]$$

and the action is

$$S(\alpha, \beta) = \frac{1}{2} \int \beta(\mathrm{d}_{dR} + \bar{\partial})\alpha \wedge \Omega + \frac{1}{6} \int \beta[\alpha, \alpha] \wedge \Omega$$

where α is valued in $\mathfrak{g}$ and β is valued in $\mathfrak{g}^*$. Here, Ω is the holomorphic volume form on X, and we have used the evaluation pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$.

Theorem 4.3. *Consider the twisted theory $\mathcal{E}_{5d}$ on the manifold $\mathbb{R}_{\geq 0} \times X$, where X is a Calabi-Yau surface. Then:*

1. *there is a boundary condition at $\{0\} \times X$ whose associated degenerate field theory is equivalent to the Kac-Moody on X at level zero with its P_0 structure from Section **??**, and*
2. *there exists a one-loop quantization of the 5d theory with boundary factorization algebra given by the by the Kac-Moody factorization algebra with level given by the local cocycle corresponding to $\#\mathrm{ch}_3 \in \mathrm{Sym}^3(\mathfrak{g}^*)^{\mathfrak{g}}$ under the map J above.*

4.3. **Maximally supersymmetric 7d gauge theory.** In this section we will see how the six-dimensional Kac-Moody degenerate field theory arises as the boundary of a supersymmetric gauge theory in seven dimensions.

Proposition 4.4. *The twist of maximally supersymmetric 7d pure gauge theory exists on any manifold of the form*

$$\mathbb{R} \times X$$

where X is a Calabi-Yau 3-fold. The fields of the theory are

$$\mathcal{E}_{7d} = \Omega^*(\mathbb{R}) \otimes \Omega^{0,*}(X) \otimes \mathfrak{g}[\epsilon][1]$$

where ϵ is a formal parameter of cohomological degree -1. If we write the fields as $\alpha + \epsilon\beta$ the action has the form

$$S(\alpha + \epsilon\beta) = \frac{1}{2} \int \beta(\mathrm{d}_{dR} + \bar{\partial})\alpha \wedge \Omega + \frac{1}{3} \int (\beta[\alpha, \alpha] + \alpha[\alpha, \beta]) \wedge \Omega.$$

Here, Ω is the holomorphic volume form on X.

Theorem 4.5. *Consider the twisted theory $\mathcal{E}_{7d}$ on the manifold $\mathbb{R}_{\geq 0} \times X$, where X is a Calabi-Yau 3-fold. Then:*

1. *there is a boundary condition at $\{0\} \times X$ whose associated degenerate field theory is equivalent to the Kac-Moody on X at level zero with its P_0 structure from Section **??**, and*
2. *there exists a one-loop quantization of the 7d theory with boundary factorization algebra given by the by the Kac-Moody factorization algebra with level given by the local cocycle corresponding to $\#\mathrm{ch}_4 \in \mathrm{Sym}^4(\mathfrak{g}^*)^{\mathfrak{g}}$ under the map J above.*

4.3.1. The gauge theory we consider arises as a deformation of a partial twist of maximally supersymmetric Yang-Mills gauge theory in seven dimensions.

4.3.2.

Theorem 4.6. *Suppose we put $\widetilde{\mathcal{Y}}_\theta$, the deformation of the twisted $N = 2$ gauge theory we considered above, on a seven manifold of the form $X \times \mathbb{R}_{\geq 0}$ where X is a Calabi-Yau six-fold. Then, there is a boundary condition on $X \times \{0\} \subset X \times \mathbb{R}_{\geq 0}$ whose associated boundary theory is equivalent to the degenerate field theory $\mathcal{K}_\theta$ on X.*

Appendix A. L_∞ algebras and their modules

BW: this may be an unnecessary section. Want to stress that KHF do not write down an explicit L_∞-model but it will often be convenient for us to use one.

Suppose V is a dg vector space. Then, the symmetric algebra

$$\mathrm{Sym}(V) := \prod_k \mathrm{Sym}^k(V)$$

has the natural structure of a dg cocommutative coalgebra.

Definition A.1. An L_∞ *algebra* is a dg vector space V together with a coderivation

$$D : \mathrm{Sym}(V) \to \mathrm{Sym}(V).$$

A *morphism* of L_∞ algebras $f : (V, D) \to (V', D')$ is a morphism of dg cocommutative coalgebras

$$f : (\mathrm{Sym}(V), D) \to \left(\mathrm{Sym}(V'), D'\right).$$

Denote the category of L_∞ algebras by $L_\infty\mathrm{Alg}$.

The complex $(\mathrm{Sym}(V), D)$ is the complex of Chevalley-Eilenberg chains of the L_∞ algebra $\mathfrak{g} = (V, D)$. In the case of a dg Lie algebra this is the usual complex of Chevalley-Eilenberg chains. Without loss of generality we denote this complex by $\mathrm{C}^{\mathrm{Lie}}(\mathfrak{g})$ just as in the classical case.

We may a remark about dg Lie algebras and their close relatives, L_∞ algebras.

Theorem A.2. BW: Kriz and May? *Every L_∞ algebra (V, D) is quasi-isomorphic (in the category $L_\infty\mathrm{Alg}$) to a dg Lie algebra.*

By an L_∞ algebra model for a dg Lie algebra $\mathfrak{g}$, we mean an L_∞ algebra (L, D) together with a quasi-isomorphism $(L, D) \simeq \mathfrak{g}$.

A.1. **Extensions from cocycles.** Suppose $\mathfrak{g}$ is a dg Lie algebra. Let $\theta \in C^*_{\rm Lie}(\mathfrak{g})$ be a cocycle of degree 2, so its cohomology class is an element $[\theta] \in H^2_{\rm Lie}(\mathfrak{g})$. By BW: ref, we know that θ determines a central extension in the category of dg Lie algebras:

$$0 \to \mathbb{C} \cdot K \to \widehat{\mathfrak{g}} \to \mathfrak{g} \to 0$$

that only depends, up to isomorphism, on the cohomology class of θ.

The explicit dg Lie algebra structure on $\widehat{\mathfrak{g}}$ may be tricky to describe. However, if we are willing to work in the category of L_∞ algebras, there is an explicit model for $\mathfrak{g}$ as an L_∞ algebra. The underlying dg vector space for the L_∞ algebra is the same as that of the dg Lie algebra, $\widehat{\mathfrak{g}} \oplus \mathbb{C} \cdot K$. To equip this with an L_∞ structure we need to provide a coderivation $D = D_1 + D_2 + \cdots$ for the cocommutative coalgebra $\mathrm{Sym}(\mathfrak{g} \oplus \mathbb{C} \cdot K) = \prod_k \mathrm{Sym}^k(\mathfrak{g} \oplus \mathbb{C} \cdot K)$. Indeed, we define

$$\begin{aligned} D_1(X_1) &= \mathrm{d}_{\mathfrak{g}}(X_1) + \theta(X_1) \\ D_2(X_1, X_2) &= [X_1, X_2]_{\mathfrak{g}} + \theta(X_1, X_2) \\ D_k(X_1, \ldots, X_k) &= \theta(X_1, \ldots, X_k) \;\; , \;\; \text{for } \; k \geq 3. \end{aligned}$$

One immediately checks that $(\mathfrak{g} \oplus \mathbb{C}, D)$ is an L_∞ model for $\widehat{\mathfrak{g}}$.

Example A.3. As an example, consider the following L_∞ model for the dg Lie algebra $\widehat{\mathfrak{g}}_{d,\theta}$. As a dg vector space $\widehat{\mathfrak{g}}_{d,\theta}$ is of the form $A_d \otimes \mathfrak{g} \oplus \mathbb{C} \cdot K$. The only nonzero components of the coderivation determining the L_∞ structure are D_1,D_2, and D_{d+1} and they are determined by $D_1(aX) = (\bar{\partial} a)X$, $D_2(aX, bY) = (a \wedge b)[X, Y]_{\mathfrak{g}}$, and

$$D_{d+1}(a_0 X_0, \ldots, a_d X_d) = \operatorname*{Res}_{z=0} (a_0 \wedge \partial a_1 \wedge \cdots \wedge \partial a_d)\, \theta(X_0, \ldots, X_d) \cdot K.$$

Lemma A.4. *Suppose $\mathfrak{g}$ is an L_∞ algebra and we are given two central extensions*

$$0 \to \mathbb{C} \cdot K[k] \to \widetilde{\mathfrak{g}}, \widetilde{\mathfrak{g}}' \to \mathfrak{g} \to 0$$

of L_∞ algebras by the trivial module placed in degree $-k$. Suppose that the cocycles determining the central extensions differ by an exact cocycle of the form $\mathrm{d}\eta \in C^*_{\rm Lie}(\mathfrak{g})$ where η is a cochain of degree $k+1$. Then, the map

$$\mathrm{id} + \eta \cdot K : C_*^{\rm Lie}(\widetilde{\mathfrak{g}}) \to C_*^{\rm Lie}(\widetilde{\mathfrak{g}}')$$

determines an L_∞-isomorphism $\widetilde{\mathfrak{g}} \cong \widetilde{\mathfrak{g}}'$.

In the lemma above the map $\mathrm{id} + \eta$ sends the element $X_1 \cdots X_n \in \mathrm{Sym}^n(\mathfrak{g})$ to $X_1 \cdots X_n + \eta(X_1, \ldots, X_n) \cdot K$ and is the identity on the subspace generated by the central element K.

Appendix B. Homotopy Poisson structures