

# KÄHLER-CHERN-SIMONS THEORY

V.P.NAIR

*Physics Department, Columbia University  
New York, NY 10027*

## ABSTRACT

Kähler-Chern-Simons theory describes antiself-dual gauge fields on a four-dimensional Kähler manifold. The phase space is the space of gauge potentials, the symplectic reduction of which by the constraints of antiself-duality leads to the moduli space of antiself-dual instantons. We outline the theory highlighting symmetries, their canonical realization and some properties of the quantum wave functions. The relationship to integrable systems *via* dimensional reduction is briefly discussed.

In this talk, I shall describe some recent work done in collaboration with Jeremy Schiff on what we refer to as Kähler-Chern-Simons (KCS) theory.<sup>1</sup> The theory basically provides an action description of antiself-dual gauge fields, i.e. instantons on four-dimensional Kähler manifolds. The motivation for seeking such a theory is essentially twofold. There is considerable evidence that antiself-dual gauge fields may be considered as a ‘master’ integrable system.<sup>2</sup> For example, we can consider  $\mathbf{R}^4$  as a Kähler manifold, pairing up the standard coordinates into complex ones as  $z = x_2 + ix_1$ ,  $w = x_4 + ix_3$ . The conditions of antiself-duality are then given by

$$F_{zw} = F_{\bar{z}\bar{w}} = F_{z\bar{z}} + F_{w\bar{w}} = 0 \quad (1)$$

where  $F_{ab}$  denotes the  $(ab)$  component of the field strength, which is as usual valued in the algebra of a Lie group  $G$ . Dimensional reduction of these equations, such as the requirement of the fields being independent of, say  $\bar{w}$ , gives a large class of known two-dimensional integrable theories such as the Korteweg-de Vries (KdV), nonlinear Schrödinger, Boussinesq and other equations.<sup>3,4</sup> It is likely that all integrable theories can be considered as special cases of the antiself-dual gauge theory. Now, integrable theories themselves are of interest because they may describe certain types of perturbations of conformal field theories and also because the Poisson bracket structures associated with integrable theories are related to the Virasoro and  $W_N$  algebras.<sup>5</sup> These algebras in turn can be the chiral algebras of conformal field theories. Independently, the study of gravity theories associated to these algebras also seem to lead to four-dimensional Kähler manifolds.<sup>6</sup> A unified description of integrable theories in terms of antiself-dual gauge fields, especially in a Lagrangian framework with associated Hamiltonian and Poisson bracket structures, can thus be useful in understanding these theories and algebras.

Another way to introduce KCS theory would be as a generalization of Chern-Simons theory in  $2+1$  dimensions. Let me explain by recalling that Chern-Simons theory is described by the action

$$S = -\frac{k}{4\pi} \int_{\Sigma \times \mathbf{R}} \text{Tr}(AdA + \frac{2}{3}A^3) \quad (2)$$

In the Hamiltonian quantization of this theory that we want to focus on,  $\Sigma$  is in general a Riemann surface and the coordinate representing  $\mathbf{R}$  will be taken as time. The wave functionals obtained upon quantization of this theory are the chiral blocks of a conformal field theory (specified by choice of  $k$  and gauge group  $G$ ) on  $\Sigma$ . Thus they carry a representation of the holomorphic current algebra.<sup>7</sup> A natural higher dimensional generalization would be a four-dimensional gauge theory with a holomorphic symmetry algebra.

At first glance, these two motivations seem somewhat disjoint, but it is easy to see that the same theory would be the result. The phase space of the Chern-Simons theory is given by the Wilson lines or holonomies associated with the noncontractible paths in  $\Sigma$ , i.e. by the homomorphisms  $\pi_1(\Sigma) \rightarrow G$ . This space can also be identified as the moduli space of (stable) holomorphic vector bundles of rank  $N$  on  $\Sigma$  (for  $G = SU(N)$ ).<sup>8</sup> The four-dimensional generalization of this result is that the moduli space of (stable) holomorphic vector bundles of rank  $N$  on a compact Kähler manifold  $M$  is essentially the moduli space of (irreducible) anti-self-dual gauge fields on  $M$  (for  $G = SU(N)$ ).<sup>9</sup> Thus a generalization of Chern-Simons theory in terms of the nature of its phase space would naturally lead us to anti-self-dual gauge fields.

The notion of self-duality or anti-self-duality is also crucial in the dynamics of  $N = 2$  strings.<sup>10</sup> For gauge fields, the most appropriate theory would be  $N = 2$  heterotic strings. The KCS theory is an effective Lagrangian description, in terms of fields on the target space, of this theory. (Of course, this is for the case when there are only gauge fields; our description of the target space dynamics has to be augmented when gravitational excitations are also present.)

The action for the KCS theory can be written as

$$S = \int_{M \times \mathbf{R}} -\frac{k}{4\pi} \text{Tr}(AdA + \frac{2}{3}A^3)\omega + \text{Tr}(\Phi\mathcal{F} + \bar{\Phi}\mathcal{F}) \quad (3)$$

where  $\mathcal{F}$  is the field strength on  $M \times \mathbf{R}$ ,  $F$  will denote the field strength on  $M$  when needed.  $M$  is a Kähler manifold of real dimension four,  $\omega$  is the Kähler form, related as usual to the metric by  $ds^2 = g_{a\bar{a}}dz^a d\bar{z}^{\bar{a}}$ ,  $\omega = \frac{i}{2}g_{a\bar{a}}dz^a \wedge d\bar{z}^{\bar{a}} = i\partial\bar{\partial}K$ , where  $z^a, a = 1, 2$  are local complex coordinates on  $M$  and  $K$  is the Kähler potential. Exterior products of the differential forms in (3) are understood.  $\Phi$  is a Lie algebra valued  $(2,0)$  form on  $M$  and a one-form on  $\mathbf{R}$ ; i.e. in local coordinates

$$\Phi = \phi dt = \frac{1}{2}\phi_{ab}dz^a dz^b dt \quad (4)$$

The fields have the standard gauge transformation properties.

$$A^u = uAu^{-1} - du u^{-1}, \quad \Phi^u = u\Phi u^{-1} \quad (5)$$

where  $u$  is a locally defined  $G$ -valued function on  $M \times \mathbf{R}$ . Since  $\omega$  is a closed two-form, the action (3) is indeed invariant under gauge transformations which are homotopic to the identity. Invariance under gauge transformations which are homotopically nontrivial, if they exist, will require  $k$  to be an integer.

The equations of motion for the action (3) are

$$F^{(0,2)} = F^{(2,0)} = 0 \quad (6a)$$

$$F \wedge \omega = 0 \quad (6b)$$

$$\frac{k}{4\pi} \dot{A}_a = -ig^{\bar{a}b} \bar{\nabla}_{\bar{a}} \phi_{ba} \quad (6c)$$

$$\frac{k}{4\pi} \dot{A}_{\bar{a}} = ig^{\bar{b}a} \nabla_a \bar{\phi}_{\bar{b}\bar{a}} \quad (6d)$$

Here  $\nabla$ ,  $\bar{\nabla}$  denote the gauge and Levi-Civita covariant derivatives and  $\dot{A}$  denotes the time derivative of  $A$ ; we have used the  $A_t = 0$  gauge in (6). Equations (6a,b) follow from variation with respect to  $\Phi$ ,  $\bar{\Phi}$  and the time component of the gauge potential  $A_t$ . They are constraints on the initial data for gauge fields, i.e. the fields on  $M$ , requiring them to be antiself-dual. The time evolution equations (6c,d) follow from variation with respect to the space ( $M$ -)components of the potential. Since the constraints (6a,b) hold for all time, we have  $\dot{F}^{(0,2)} = 0$ ; this leads to the gauge covariant Laplace equation for  $\phi$

$$\nabla * \bar{\nabla} * \phi = 0 \quad (7)$$

Here  $*$  denotes the Hodge star operation. On manifolds with  $\partial M = \emptyset$ , using a Bochner-type argument, one can show that there is no solution for  $\phi$ , at least for manifolds of positive scalar curvature.<sup>11</sup> Thus  $\dot{A} = 0$ . Time evolution is trivial and the set of classical solutions is given by antiself-dual gauge fields on  $M$ . For a general  $M$ , there can be nontrivial solutions for  $\phi$ ; it will turn out that these can be related to Bäcklund transformations. In any case, the Hamiltonian is a sum of the constraints  $F^{(0,2)}$ ,  $F^{(2,0)}$  and  $F \wedge \omega$  and therefore any time evolution is a change of ‘gauge’ in the sense of Dirac’s theory of constraints; it is thus effectively trivial.

The KCS theory (3) thus provides an action description of antiself-dual gauge fields. The Hamiltonian structure of the theory is as follows. The phase space is the space  $\mathcal{A}$  of gauge potentials on  $M$ . The symplectic two-form on this space can be easily read off from the action since the latter is first order in time derivatives and is given by

$$\Omega = \frac{k}{4\pi} \int_M \text{Tr}(\delta A \delta A) \omega = \frac{ik}{2\pi} \int_M dV g^{\bar{a}a} \delta A_{\bar{a}}^i \delta A_a^i \quad (8)$$

where  $\delta$  denotes exterior differentiation on  $\mathcal{A}$  and  $dV$  is the volume element on  $M$ . The superscript on the potentials refers to the component with respect to a chosen basis for the Lie algebra of  $G$ . The cohomology class of  $\Omega$  is unchanged by gauge transformations of  $A$  or by the addition of exact terms to  $\omega$ . It is an example of the Donaldson map  $H^2(M) \rightarrow H^2(\mathcal{A}/\mathcal{G})$  on the cohomology groups,<sup>12</sup> where  $\mathcal{G}$  denotes the group of gauge transformations. The Poisson brackets corresponding to (8) are given by

$$[A_a^i(x), A_{\bar{a}}^j(y)] = \frac{2\pi}{ik} g_{a\bar{a}} \delta^{ij} \frac{\delta^{(4)}(x-y)}{\det(g)} \quad (9)$$

One has to further take care of the constraints (6a,b) on  $\mathcal{A}$ . This of course involves the symplectic or Hamiltonian reduction of  $\mathcal{A}$  essentially by the constraints  $F^{(0,2)}$  and  $F \wedge \omega$ . The reduced phase space is the moduli space of antiself-dual gauge fields on  $M$ .

For the remainder of this talk I shall discuss the constraints and the reduction of phase space and some aspects of the quantum theory and dimensional reduction on  $\mathbf{R}^4$ . The algebra of the constraints is a necessary prerequisite for proper reduction of the phase space. It turns out that  $F \wedge \omega$  and either of the other two, say  $F^{(0,2)}$ , are the relevant first class constraints.  $F \wedge \omega$  is the generator of gauge transformations;  $F^{(0,2)}$  generates infinitesimal Bäcklund transformations in those cases where the Laplace equation (7) has nontrivial solutions. The wave functionals in the quantum theory have a factor  $e^{i\mathcal{S}[U]}$  where  $U$  is a locally defined function taking values in  $G^{\mathbf{C}}$ , the complexification of  $G$ .  $\mathcal{S}[U]$  is a generalization of the Wess-Zumino-Witten (WZW) action in two dimensions<sup>13</sup> and obeys an analogous Polyakov-Wiegmann factorization property. This property leads to holomorphic and antiholomorphic symmetries for  $\mathcal{S}[U]$  and associated current algebras. Finally we shall consider dimensional reduction of the conditions of antiself-duality for the case of  $M$  being  $\mathbf{R}^4$ . In the framework of reduction of  $\mathcal{A}$  by  $F^{(0,2)}$  followed by  $F \wedge \omega$ , we shall see that many of the different ansätze which have been used by various authors are indeed obtained by suitable gauge choices and are related to each other.

A remark before we turn to the details: notice that there is considerable similarity between the present discussion and 2 + 1 dimensional Chern-Simons theory. In the latter case, the holomorphic and antiholomorphic components of the gauge potential, *viz.*  $A_z$  and  $A_{\bar{z}}$  are canonically conjugate to each other, analogous to (9). The wave functionals likewise have a factor  $e^{i\mathcal{S}[U]}$  where  $\mathcal{S}[U]$  is the WZW action. The holomorphic and antiholomorphic current algebras are, of course, the Kac-Moody algebras.<sup>7</sup>

We now turn to the reduction of  $\mathcal{A}$ . The first step is the algebra of constraints. We introduce test functions  $\varphi$ ,  $\bar{\varphi}$  and  $\theta$  which take values in the Lie algebra of  $G$  and serve as parameters for the transformations generated by  $F^{(0,2)}$ ,  $F^{(2,0)}$  and  $F \wedge \omega$  respectively.  $\bar{\varphi}$  is a (0,2) form and  $\theta$  is a scalar. The generators are collected

as

$$E(\bar{\varphi}) = -\frac{k}{2\pi} \int_M \text{Tr}(\bar{\varphi}F) \quad \bar{E}(\varphi) = -\frac{k}{2\pi} \int_M \text{Tr}(\varphi F) \quad (10)$$

$$G(\theta) = -\frac{k}{2\pi} \int_M \text{Tr}(\theta F) \quad (11)$$

The Poisson brackets of these generators and the potential  $A$  are given by

$$\begin{aligned} [G(\theta), A_a^i(x)] &= -(\nabla\theta)_a^i(x) & [G(\theta), A_{\bar{a}}^i(x)] &= -(\bar{\nabla}\theta)_{\bar{a}}^i(x) \\ [E(\bar{\varphi}), A_a^i(x)] &= 0 & [E(\bar{\varphi}), A_{\bar{a}}^i(x)] &= i(*\nabla * \bar{\varphi})_{\bar{a}}^i(x) \\ [\bar{E}(\varphi), A_a^i(x)] &= 0 & [\bar{E}(\varphi), A_{\bar{a}}^i(x)] &= -i(*\bar{\nabla} * \varphi)_{\bar{a}}^i(x) \end{aligned} \quad (12)$$

$$\begin{aligned} [G(\theta), G(\theta')] &= G(\theta \times \theta') \\ [G(\theta), E(\bar{\varphi})] &= E(\theta \times \bar{\varphi}) \\ [G(\theta), \bar{E}(\varphi)] &= \bar{E}(\theta \times \varphi) \\ [\bar{E}(\varphi), E(\bar{\varphi}')] &= \frac{ik}{2\pi} \int_M \text{Tr}(\bar{\varphi}' \nabla * \bar{\nabla} * \varphi) \end{aligned} \quad (13)$$

Here the cross product is in the Lie algebra, i.e.  $\theta \times \theta'^i = f^{ijk} \theta^j \theta'^k$ ,  $f^{ijk}$  being the structure constants of the Lie algebra. The first set of equations show that  $G(\theta)$  is the generator of gauge transformations and that  $\bar{E}$  and  $E$  generate changes in the potentials, respectively, of the form

$$\begin{aligned} A_a &\rightarrow A_a - i(*\bar{\nabla} * \varphi)_a \\ A_{\bar{a}} &\rightarrow A_{\bar{a}} + i(*\nabla * \bar{\varphi})_{\bar{a}} \end{aligned} \quad (14)$$

Consider now the reduction of  $\mathcal{A}$ . We start by setting  $\bar{E}$  or  $F^{(0,2)}$  to zero. The solution set for this condition is given by

$$A = (A_a, -\partial_{\bar{a}} U U^{-1}) \quad (15)$$

where  $U$  is a locally defined  $G^{\mathbb{C}}$ -valued function. If the equation

$$\nabla * \bar{\nabla} * \varphi = 0 \quad (16)$$

has no solution, then  $\bar{E}$  and  $E$  have nonvanishing Poisson brackets, as seen from (13), and  $E = 0$  can be used as gauge fixing condition for the flow generated by

$\overline{E}$ . The reduced phase space is characterized by the vanishing of  $\overline{E}$  and  $E$ . Further reduction is achieved by setting  $G(\theta)$  to zero; this is equivalent to

$$g^{a\bar{a}} \partial_{\bar{a}} (J^{-1} \partial_a J) = 0 \quad (17)$$

where  $J = U^\dagger U$ . The reduced fields are antiself-dual gauge fields and the reduced phase space is the moduli space of antiself-dual fields.

Consider now the case when (16) has nontrivial solutions. The flow generated by  $\overline{E}$  on the  $\overline{E} = 0$  subspace is given by

$$U \rightarrow U$$

$$A_a \rightarrow A'_a = A_a - i(*\overline{\nabla} * \varphi)_a \quad (18)$$

For those  $\varphi$  which are solutions to (16), we see that both  $A_a$  and  $A'_a$  satisfy the condition  $E = 0$ . Writing  $*\overline{\nabla} * \varphi = \nabla \sigma$ , we get  $A_a = (U^{\dagger-1} \partial_a U^\dagger)$ ,  $A'_a = (U'^{\dagger-1} \partial_a U'^\dagger)$ , where  $U'^\dagger = U^\dagger e^{-i\sigma}$ . We can rewrite these as

$$(J^{-1} \partial_a J) - (J'^{-1} \partial_a J') = i(U^{-1} \nabla \sigma U) \quad (19)$$

We can check easily that if  $J$  satisfies (17), then  $J'$  defined by the solution of the first order equation (19) also satisfies (17). Thus the transformations generated by  $\overline{E}$ , with parameters which are solutions to (16), generate solutions to the antiself-dual conditions from solutions; i.e. it is an infinitesimal Bäcklund transformation. It is important that these are now canonically realized.

In the case of  $\mathbf{R}^4$ , it is possible to choose  $\varphi_{ab} = \frac{1}{2} \lambda \epsilon_{ab} \sigma$ ; equation (19) then becomes

$$(J^{-1} \partial_a J) - (J'^{-1} \partial_a J') = -\lambda g^{\bar{a}b} \epsilon_{ba} \partial_{\bar{a}} (J^{-1} J') \quad (20)$$

This is the more conventional way of writing Bäcklund transformations.<sup>14</sup>

As regards the quantum theory, we shall concentrate on the question of holomorphic current algebra. Since  $A_a$  and  $A_{\bar{a}}$  are canonically conjugate, we can take the wave functions to be functionals of only one of these sets, say  $A_{\bar{a}}$ . Once we require  $\overline{E} = 0$ , i.e. with  $A_{\bar{a}} = -\partial_{\bar{a}} U U^{-1}$ , we can take the wave functions to be functionals of  $U$ . The inner product is then given by

$$\langle \Psi_1 | \Psi_2 \rangle = \int [dU] e^{-\tilde{K}} \Psi_1^* \Psi_2 \quad (21)$$

where  $\tilde{K}$  is the Kähler potential associated with  $\Omega$ ,  $\tilde{K} = \frac{k}{2\pi} \int dV A_a^i A_{\bar{a}}^i g^{a\bar{a}}$ . For the purposes of discussing the current algebra, it suffices to consider the case of zero instanton number, that is, the case for which the second Chern class of the gauge fields on  $M$  is trivial; in this case  $U$  can be globally defined. The second stage of reduction is performed by requiring

$$G(\theta) \Psi[U] = 0 \quad (22)$$

The solution to this equation is given by  $\Psi[U] = e^{i\mathcal{S}[U]}$ , where

$$\mathcal{S}[U] = \frac{k}{2\pi} \int_M dV g^{\bar{a}a} \text{Tr}(\partial_a U \partial_{\bar{a}} U^{-1}) + \frac{ik}{12\pi} \int_{M^5} \text{Tr}(U^{-1} dU)^3 \omega \quad (23)$$

$M^5$  is taken to be  $M \times [0, 1]$ ; one component of the boundary of  $M^5$  is identified as our space  $M$ . The field  $U$  which is defined on  $M$  is extended to  $M^5$  in such a way that it goes to a fixed function  $U_0$  on the other component, same for all  $U$  within the same homotopy class.  $\mathcal{S}[U]$  is the analogue of the WZW action in two dimensions.<sup>13</sup> It obeys a Polyakov-Wiegmann type factorization formula

$$\mathcal{S}[U_1 U_2] = \mathcal{S}[U_1] + \mathcal{S}[U_2] - \frac{k}{\pi} \int_M dV g^{\bar{a}a} \text{Tr}(U_1^{-1} (\partial_a U_1) (\partial_{\bar{a}} U_2) U_2^{-1}) \quad (24)$$

This formula shows that transformations on  $U$  of the form  $U \rightarrow h U \tilde{h}$ , where  $h$  is antiholomorphic and  $\tilde{h}$  is holomorphic, is a symmetry of  $\mathcal{S}$ . As in the case of WZW action in two dimensions, one can think of  $\mathcal{S}$  as the starting point, choosing, say  $z_1$  as the time coordinate and obtain the generators and algebra of these symmetries. The algebra one obtains, for the generators  $Q^i(\bar{z})$  of the antiholomorphic transformations, is

$$[Q^i(\bar{z}), Q^j(\bar{z}')] = f^{ijk} Q^k(\bar{z}) \delta^2(\bar{z} - \bar{z}') - \frac{k}{4\pi} \delta^{ij} C(\bar{z}, \bar{z}') \quad (25)$$

$$C(\bar{z}, \bar{z}') = \int dz_2 \det(g) g^{\bar{a}1} \partial_{\bar{a}} \delta^{(2)}(\bar{z} - \bar{z}')$$

This algebra, despite its similarity to the Kac-Moody algebra, is very limited in its utility in solving the theory defined by (23), since, unlike its two-dimensional counterpart, the classical solutions do not factorize into holomorphic and antiholomorphic matrices.

We now turn to dimensional reduction on  $\mathbf{R}^4$ . Our remarks earlier concerning gauge choices can be illustrated by considering  $G = SL(2, \mathbf{C})$ . Starting with the four coordinates  $z, \bar{z}, w$  and  $\bar{w}$ , we do a dimensional reduction by considering gauge potentials which are independent of one of them, say  $\bar{w}$ , in some gauge. This still leaves us the freedom of gauge transformations which are independent of  $\bar{w}$ ; under these  $A_{\bar{w}} \rightarrow u A_{\bar{w}} u^{-1}$ , where  $u$  is the gauge transformation matrix which now depends only on  $z, \bar{z}, w$ . Potentials  $A_{\bar{w}}$  fall into two classes, characterized by the following canonical forms, to which they can be brought by use of the above gauge freedom.

$$A_{\bar{w}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{or} \quad \kappa \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (26)$$

The first of these leads to the KdV and the modified KdV equations, the second gives the nonlinear Schrödinger equation. We now impose

$$F_{\bar{z}\bar{w}} = 0 \quad F_{z\bar{z}} + F_{w\bar{w}} = 0 \quad (27)$$

These equations can be solved for the other components of the potential; for example, for the KdV choice

$$A_w = \begin{pmatrix} (j_z - f_{\bar{z}} - 2dj)/2 & d_{\bar{z}} - j \\ c & -(j_z - f_{\bar{z}} - 2dj)/2 \end{pmatrix}$$

$$A_z = \begin{pmatrix} d & 1 \\ f & -d \end{pmatrix} \quad A_{\bar{z}} = \begin{pmatrix} 0 & 0 \\ j & 0 \end{pmatrix} \quad (28)$$

$c, d, f, j$  are arbitrary functions of  $z, \bar{z}, w$  and the subscripts denote differentiation with respect to the coordinates indicated. There is some freedom in choosing a solution to (27) due to integration constants, etc; we have made a specific choice. There is also some gauge freedom left; thus for the KdV choice of  $A_{\bar{w}}$ , we can still do gauge transformations by matrices of the form

$$u = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \quad (29)$$

By a suitable choice of  $\gamma$  we can set  $j$  to zero whereupon the last of the antiself-duality conditions, *viz.*  $F_{zw} = 0$  becomes the KdV equation. This is the choice made by Mason and Sparling.<sup>3</sup> Another choice, made by Bakas and Depireux,<sup>4</sup> is to set  $d$  to zero; this also leads to the KdV equation. A third possibility, *viz.* choosing  $f$  to be zero gives the modified KdV equation. Notice that the various ansätze for the potentials are not *ad hoc* choices for us; the gauge freedom naturally leads us to these. The relationships among the various choices is also clear.

## References

1. V.P.Nair and J.Schiff, *Phys.Lett.* **B246** (1990) 423; *Kähler-Chern-Simons Theory and Symmetries of Antiself-Dual Gauge Fields*, Columbia Preprint CU-TP-521 (1991).
2. R.S.Ward, *Phil.Trans.Roy.Soc.Lond.* **A315** (1985) 451.
3. L.J.Mason and G.A.J.Sparling, *Phys.Lett.* **A137** (1989) 29.
4. I.Bakas and D.Depireux, *Mod.Phys.Lett.* **A6** (1991) 399; Maryland Preprints UMD-PP91-111,168.
5. J.L.Gervais, *Phys.Lett.* **B160** (1985) 277; for further reference, see P.di Francesco, C.Itzykson and J.B.Zuber, Santa Barbara, Princeton and Saclay Preprint NSF-ITP-90-193, PUPT-1211, SPhT/90-149.
6. K.Schoutens. A.Sevrin and P.van Nieuwenhuizen, Stony Brook Preprints ITP-SB-90-19,39,62.



7. E.Witten, *Comm.Math.Phys.* **121** (1989) 351; M.Bos and V.P.Nair, *Phys.Lett.* **B223** (1989) 61; *Int.J.Mod.Phys.* **A5** (1990) 959; S.Elitzur *et al*, *Nucl.Phys.* **B 326** (1989) 108; J.M.F.Labastida and A.V.Ramallo, *Phys.Lett.* **B227** (1989) 92; H.Murayama, *Z.Phys.* **C48** (1990) 79; T.Killingback, *Phys.Lett.* **B219** (1989) 448; A.Polychronakos, *Ann.Phys.* **203** (1990) 231; Florida Preprint UFIFT-HEP-89-9; T.R.Ramadas, I.M.Singer and J.Weitsman, MIT Mathematics Preprint (1989).
8. M.F.Atiyah and R.Bott, *Phil.Trans.Roy.Soc.Lond.* **A308** (1982) 523.
9. S.K.Donaldson, *Proc.Lond.Math.Soc.*(3) **50** (1985) 1.
10. H.Ooguri and C.Vafa, Harvard Preprints HUTP-91/A003 and A004.
11. M.Itoh, *Publ. RIMS, Kyoto Univ.* **19** (1983) 15; *J.Math.Soc.Japan* **40** (1988) 9.
12. P.Braam, *The Fascinating Relations between 3- and 4-Manifolds and Gauge Theory*, Lectures at the Fifth Annual University of California Summer School on Nonlinear Science, *Physics and Geometry*, Lake Tahoe, 1989.
13. E.Witten, *Comm.Math.Phys.* **92** (1984) 455.
14. L.Dolan, *Phys.Rep.* **109** (1983) 1; L.L.Chau in *Integrable Systems*, X.C.Song (ed.) (World Scientific, 1990); L.Crane, *Comm.Math.Phys.* **110** (1987) 391.