

FORMULAS FOR NOETHER'S MAP

Recall the map of precosheaves of dg Lie algebras on $\mathbb{R}_{>0}$

$$\pi_{\mathfrak{g},d} : \Omega_{\mathbb{R}_{>0},c}^* \otimes \mathfrak{g}_d^\bullet \rightarrow r_* \mathcal{G}_d.$$

Specializing this map on an open interval $I \subset \mathbb{R}$ yields a map of dg Lie algebras

$$\pi_{\mathfrak{g},d}(I) : \Omega_c^*(I) \otimes \mathfrak{g}_d^\bullet \rightarrow \mathcal{G}_d(r^{-1}(I)).$$

Since $\Omega_c^*(I) \simeq \mathbb{C}[-1]$ for any open interval, this specialization results in a map of graded Lie algebras

$$H^* \pi_{\mathfrak{g},d}(I) : H^*(\mathfrak{g}_d^\bullet)[-1] = H^*(A_d) \otimes \mathfrak{g}[-1] \rightarrow H^* \mathcal{G}_d(r^{-1}(I))$$

at the level of cohomology.

With this notation set up, we have an explicit description of Noether's map when we evaluate on open sets of the form $r^{-1}(I) \subset \mathbb{C}^d$.

Lemma 0.1. *In cohomology, the composition of Noether's map $r_* J^{\text{cl}}$ with $\pi_{\mathfrak{g},d}(I)$*

$$H^*(A_d) \otimes \mathfrak{g} \xrightarrow{H^* \pi_{\mathfrak{g},d}} H^* r_* \mathbb{U}(\mathcal{G}_d) \xrightarrow{r_* J^{\text{cl}}} H^* r_* \text{Obs}_{\beta\gamma}^q$$

sends an element $\alpha \otimes X \in H^(A_d) \otimes \mathfrak{g}$ to the quadratic observable*

$$(\gamma \otimes v, \beta \otimes v^*) \mapsto \langle v^*, X \cdot v \rangle_V \oint_{S^{2d-1}} \beta \wedge \alpha \wedge \gamma.$$

In the quantum case, we have the following

Lemma 0.2. *In cohomology, the composition of Noether's map $r_* J^q$ with $\mathbb{U} \pi_{\mathfrak{g},d}(I)$*

$$(1) \quad H^* \left(U \widehat{\mathfrak{g}}_{d, \text{ch}_{d+1}^{\mathfrak{g}}(V)} \right) \xrightarrow{H^* \mathbb{U} \pi_{\mathfrak{g},d}} H^* r_* \mathbb{U}_{\text{ch}_{d+1}^{\mathfrak{g}}(V)}(\mathcal{G}_d) \xrightarrow{r_* J^q} H^* \left(r_* \text{Obs}_{\beta\gamma}^q \Big|_{\hbar=(2\pi i)^d} \right)$$

sends a linear element $\alpha \otimes X \in H^(A_d) \otimes \mathfrak{g}$ to the quadratic observable*

$$(\gamma \otimes v, \beta \otimes v^*) \mapsto \langle v^*, X \cdot v \rangle_V \oint_{S^{2d-1}} \beta \wedge \alpha \wedge \gamma.$$

Since each of the maps in (??) is a map of (graded) algebras, the composition is determined by its value on the linear elements $H^*(A_d) \otimes \mathfrak{g}$. We also note that from the formula above, it is clear the composition factors through the cohomology of $U(\mathcal{H}_V)$ to give a map

$$H^* \left(U \widehat{\mathfrak{g}}_{d, \text{ch}_{d+1}^{\mathfrak{g}}(V)} \right) \rightarrow H^* U(\mathcal{H}_V).$$

BW: Maybe we should say why residues live in $U(\mathcal{H}_V)$.