1. ENUMERATING TWISTED SUPERGRAVITY STATES

1.1. **Enumerating states in Kodaira–Spencer theory.** BW: just review work of Costello and Gaiotto

To describe the boundary conditions we will use the partial compactification of the extended deformed conifold as described in §**??**.

Recall that there are three fundamental fields for Kodaira–Spencer theory. Two fundamental fields α, γ are Dolbeault forms of type $(0, \bullet)$. The last fundamental field μ is a $(0, \bullet)$ form valued in in the holomorphic tangent bundle. We can use the Calabi–Yau form to view μ as a Dolbeault form of type $(2, \bullet)$.

- The vacuum boundary condition for the fields α, γ is that each are divisible by the coordinate n.
- The vacuum boundary condition for the field μ is that when viewing it as a Dolbeault form of type $(2, \bullet)$ it can be expressed as a sum of terms which are each wedge products of $\mathrm{d}\log n, \mathrm{d}w, \mathrm{d}z, \mathrm{d}\overline{n}, \mathrm{d}\overline{w}, \mathrm{d}\overline{z}$ with coefficients that are regular at $n = 0$.

Denote by $\left(\frac{\mathbf{m}}{\mathbf{2}}\right)_S$ the short representation of $\mathfrak{psu}(1,1|2)$ whose highest weight vector has (J_0^3, L_0) eigenvalues $(\frac{m}{2}, \frac{m}{2})$. Denote by y the fugacity for the $U(1)$ symmetry $2J_0^3$ and q the fugacity for the $U(1)$ symmetry L_0. Let

$$D = (1-q)(1-q^{1/2}y)(1-q^{-1/2}y^{-1}). \tag{1.1.1}$$

Let's first consider the case where the internal manifold is just a point. This is just the topological string with B-branes wrapping

$$\mathbf{C} \subset \mathbf{C}^3. \tag{1.1.2}$$

1.1.1 Proposition. *The single particle states for Kodaira–Spencer theory on $\mathbf{C}^3$ with branes wrapping $\mathbf{C} \subset \mathbf{C}^3$ decompose as*

$$\oplus_{m \geq 1} \left(\frac{\mathbf{m}}{\mathbf{2}}\right)_S \tag{1.1.3}$$

From this, we deduce that the single particle index for Kodaira–Spencer theory on $\mathbf{C}^3$ is

$$\frac{q^2 - 3q + q^{1/2}(y + y^{-1})}{D}. \tag{1.1.4}$$

- State $\mu \sim n^{-k} \mathrm{d} \log n \mathrm{d} w_1 \delta_{z=0}$. For $k \geq 1$ these even states and their descendants contribute

$$\frac{yq^{1/2}}{D} \tag{1.1.5}$$

to the single particle index.

- Lowest lying state $\mu \sim n^{-k} \mathrm{d} \log n \mathrm{d} w_2 \delta_{z=0}$. For $k \geq 1$ these even states and their descendants contribute

$$\frac{y^{-1}q^{1/2}}{D} \tag{1.1.6}$$

to the single particle index.

- Lowest lying state $\mu \sim n^{-k} \mathrm{d} \log n \mathrm{d} z \delta_{z=0}$. For $k \geq 2$ these even states and their descendants contribute

$$\frac{q^2}{D} \tag{1.1.7}$$

to the single particle index.

- State $\alpha \sim n^{1-k} \delta_{z=0}$. For $k \geq 1$ these odd states and their descendants contribute

$$-\frac{q}{D}. \tag{1.1.8}$$

to the single particle index.

- State $\gamma \sim n^{1-k} \delta_{z=0}$. For $k \geq 1$ these odd states and their descendants contribute

$$-\frac{q}{D}. \tag{1.1.9}$$

to the single particle index.

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- State $\nu \sim n^{1-k} \delta_{z=0}$. For $k \geq 1$ these odd states contribute

$$-\frac{q}{D}. \tag{1.1.10}$$

to the single particle index.

In total we find that the multi particle gravitational index is

$$\frac{q^2 - 3q + q^{1/2}(y + y^{-1})}{(1-q)(1-q^{1/2}y)(1-q^{-1/2}y^{-1})} = \frac{yq^{1/2}}{1-yq^{1/2}} + \frac{y^{-1}q^{1/2}}{1-y^{-1}q^{1/2}} - \frac{q}{1-q}. \tag{1.1.11}$$

1.2. **The twisted supergravity elliptic genus.** The supergravity states were enumerated in [?]. We briefly recall the results here.

The twisted supergravity states organize into a representation for the super Lie algebra $\mathfrak{psu}(1,1|2)$. The bosonic factor of this super Lie algebra is $\mathfrak{su}(2)_L \times \mathfrak{su}(2)_R$. The first copy is the global conformal transformations in the z-plane and the second copy is the R-symmetry algebra which rotates the w-coordinate. We take the Cartan of this Lie algebra to be generated by (L_0, J_0^3).

Denote by $(\frac{\mathbf{m}}{\mathbf{2}})_S$ the short representation of $\mathfrak{psu}(1,1|2)$ whose highest weight vector has (L_0, J_0^3) eigenvalue $(m/2, m/2)$ [?]. As an example, the short representation $(\mathbf{1})_S$ consists of a boson with weight $(L_0 = 1, J_0^3 = 1)$, which in our notation corresponds to

$$\mu \sim n^{-2} \mathrm{d} \log n \mathrm{d} z \delta_{z=0}. \tag{1.2.1}$$

There are also two fermions in $(\mathbf{1})_S$ with weights $(3/2, 1/2)$ corresponding to the states

$$\alpha \sim n^{-1} \delta_{z=0} + \cdots, \quad \gamma \sim n^{-1} \delta_{z=0} + \cdots \tag{1.2.2}$$

and another boson of weight $(2, 0)$ corresponding to

$$\mu \sim n^{-2} \mathrm{d} \log n \mathrm{d} w \delta_{z=0} + \cdots . \tag{1.2.3}$$

We consider twisted type IIB supergravity on a Calabi–Yau surface X, where X could be T^4 or a $K3$ surface.

1.2.1 Proposition ([?]). *The supergravity states for the $D1 - D5$ brane system in twisted type IIB supergravity on a compact Calabi–Yau surface X decompose as*

$$\bigoplus_{m \geq 1} (\frac{\mathbf{m}}{\mathbf{2}})_S \otimes H^\bullet(X) = \bigoplus_{m \geq 1} \bigoplus_{i,j} (\frac{\mathbf{m}}{\mathbf{2}})_S \otimes H^{i,j}(X). \tag{1.2.4}$$

In particular, when X is a $K3$ surface the single particle twisted supergravity index is

$$f_{KS}(q, y) = 24 \frac{q^2 - 3q + q^{1/2}(y + y^{-1})}{D}. \tag{1.2.5}$$

This result should be compared to [?], where the space of supergravity states upon supersymmetric localization (that is, the chiral half of the supergravity states) is found to be

$$\bigoplus_{m \geq 0} \bigoplus_{i,j} (\frac{\mathbf{m} + \mathbf{i}}{\mathbf{2}})_S \otimes H^{i,j}(X). \tag{1.2.6}$$

The answers agree in the range where the highest weight of the short representation is at least two. The low weight discrepancies break up into two types:

- In [?] there is an extra factor of $(\mathbf{0})_S \otimes H^{0,i}(X)$. So, in the case that X is a $K3$ surface there are two extra bosonic operators in the analysis of [?]. In [?] is was pointed out that these are topological operators, annihilated by L_{-1}, and have nonsingular OPE with all remaining operators.
- In our analysis there is an extra factor of $(\frac{\mathbf{1}}{\mathbf{2}})_S \otimes H^{2,j}(X)$. In the case that X is a $K3$ surface one can remove these two bosonic states while maintaining an $SO(21)$ symmetry.

Denote the single particle index of the supergravity states, described in equation (**??**), by $f_{sugra}(q, y)$. One of the main results of [?] is that the corresponding multiparticle index agrees with the large N elliptic genus of the orbifold CFT of a $K3$ surface

$$\chi_{NS}(\mathrm{Sym}^\infty X; q, y) = \mathrm{PExp}\left[f_{sugra}(q, y)\right] \tag{1.2.7}$$

where PExp is the plethystic exponential. For X a $K3$ surface, the states $(\frac{\mathbf{1}}{\mathbf{2}})_S \otimes H^{2,\bullet}(X)$ contribute the single particle index

$$2f_1(q, y) = \frac{2}{1-q}\left(-2q + q^{1/2}(y + y^{-1})\right). \tag{1.2.8}$$

If we subtract this from the supergravity index we find an exact match with the supergravity index computed by [?]

$$f_{sugra}(q, y) = f_{KS}(q, y) - 2f_1(q, y). \tag{1.2.9}$$

1.3. **Global symmetry algebra.** For $F \in H^2(Y_{K3})$, let X_F^0 be the extended deformed conifold as defined in §**??**. Recall that $Z = \mathrm{Spec}(A)$ is the affine variety described by the commutative algebra $A = H^\bullet(Y_{K3})$. Previously, we saw that the space X_F^0 admits a fibration $X_F^0 \to Z$ fibered in Calabi–Yau three-folds. Let $\mathcal{O}(X_F^0)$ be the algebra of functions on X_F^0. By Hartog's theorem this is the algebra generated by the bosonic linear functions $u_i, w_j, \eta, \overline{\eta}, \eta_a$ where $i, j = 1, 2$, $a = 1, \ldots, 20$ subject to the relations

$$\eta^2 = \overline{\eta}^2 = \eta_a\eta_b - h_{ab}\eta\overline{\eta} = 0, \qquad \epsilon^{ij}u_i w_j = F.$$

Let