1. TREE-LEVEL OPEs

In this section we initiate our computation of OPEs of the gravitational side chiral algebra, using the same techniques as in [**?**].

1.1. **Explicit expressions for supergravity states.** Recall that the full classical coupling of Kodaira–Spencer theory is

$$\int_{\mathbf{C}^{3|4}} \mu_1\mu_2\mu_z \, \mathrm{d}z\mathrm{d}^2w\mathrm{d}^4\eta + \int_{\mathbf{C}^{3|4}} \alpha\mu_i\partial_{w_i}\gamma \, \mathrm{d}z\mathrm{d}^2w\mathrm{d}^4\eta + \int_{\mathbf{C}^{3|4}} \alpha\mu_z\partial_z\gamma \, \mathrm{d}z\mathrm{d}^2w\mathrm{d}^4\eta.$$

We will use the notation $D_{r,s}$ to denote the holomorphic differential operator

$$D_{r,s} = \frac{1}{r!}\frac{1}{s!}\partial_{w_1}^r\partial_{w_2}^s.$$

1.2. $\widetilde{J}\widetilde{J}$ **OPE.** We first compute the OPE of the off-shell operators $\widetilde{J}^i[r,s]$ and then impose constraints to determine the OPE of the on-shell operators $J[r,s]$.

The coefficient of $\widetilde{J}^1[k,l]$ in the OPE will be determined by the terms in the BRST variation of μ_1 which involve $\mathfrak{c}_1$ and μ_1, $\mathfrak{c}_1$ and μ_2, or $\mathfrak{c}_2$ and μ_1.

Consider the gauge variation of

$$\int_{(z,\eta_a)\in\mathbf{C}^{1|4}} \widetilde{J}^1[r,s](z,\eta_a)D_{r,s}\mu_1(z,w_i=0,\eta_a). \tag{1.2.1}$$

The gauge variation of μ_1 is

$$\begin{aligned} Q\mu_1 = {} & \overline{\partial}\mathfrak{c}_1 + \mu_i\partial_{w_i}\mathfrak{c}_1 + \mu_z\partial_z\mathfrak{c}_1 - \mathfrak{c}_i\partial_{w_i}\mu_1 - \mathfrak{c}_z\partial_z\mu_1 \\ & + \partial_{w_2}\mathfrak{c}_\gamma\partial_z\alpha - \partial_z\mathfrak{c}_\gamma\partial_{w_2}\alpha + \partial_{w_2}\mathfrak{c}_\alpha\partial_z\gamma - \partial_z\mathfrak{c}_\alpha\partial_{w_2}\gamma. \end{aligned}$$

For now, we can disregard the terms involving $\mathfrak{c}_\gamma$ and α or $\mathfrak{c}_\alpha$ and γ. These will play a role later on when we constrain the OPE's involving the operators G_α, G_γ.

Inserting this gauge variation into the coupling to $\widetilde{J}^i[r,s]$, we see that the first term, $\overline{\partial}\mathfrak{c}_1$, vanishes by integration by parts. Cancellation of the remaining terms will give us constraints on the OPE coefficients. The remaining terms are

$$\int_{z,\eta_a} \widetilde{J}^1[r,s](z,\eta_a)D_{r,s}\left(\mu_i\partial_{w_i}\mathfrak{c}_1 + \mu_z\partial_z\mathfrak{c}_1 - \mathfrak{c}_i\partial_{w_i}\mu_1 - \mathfrak{c}_z\partial_z\mu_1\right)(z,w_i=0,\eta_a).$$

Let us focus on the term in this expression which involves the fields μ_1 and $\mathfrak{c}_1$. This is

$$\int_{z,\eta_a} \widetilde{J}^1[r,s](z,\eta_a) D_{r,s} \left(\mu_1 \partial_{w_1} \mathfrak{c}_1 - \mathfrak{c}_1 \partial_{w_1} \mu_1\right)(z, w_i = 0, \eta_a).$$

Because this expression involves both $\mathfrak{c}_1$ and μ_1, which are fields (and a corresponding ghost) that couple to $\widetilde{J}^1$, we find that it can only be cancelled by a gauge variation of an integral involving two copies of the operators $\widetilde{J}^1$, at separate points z, z':

$$\tfrac{1}{2} \int_{z,z',\eta_a,\eta_a'} \widetilde{J}^1[k,l](z,\eta_a) D_{k,l} \mu_1(z, w_i = 0, \eta_a) \widetilde{J}^1[r,s](z',\eta_a') D_{r,s} \mu_1(z', w_i' = 0, \eta_a').$$

Applying the gauge variation of μ_1 to this expression, and retaining only the terms involving $\overline{\partial}\mathfrak{c}_1$, gives us

$$\int_{z,z',\eta_a,\eta_a'} \widetilde{J}^1[k,l](z,\eta_a) D_{k,l} \mu_1(z, w_i = 0, \eta_a) \widetilde{J}^1[r,s](z',\eta_a') D_{r,s} \overline{\partial}\mathfrak{c}_1(z', w_i' = 0, \eta_a').$$

Here the $\overline{\partial}$ operator only involves the z-component because restricting to $w_i = 0$ sets any $\mathrm{d}\overline{w}_i$ to zero. We can integrate by parts to move the location of the $\overline{\partial}$ operator. Every field μ_i contains a $\mathrm{d}\overline{z}$, as otherwise it would restrict to zero at $w_i = 0$, so that $\partial_{\overline{z}} \mu_i = 0$.

This discussion shows that in order for the anomaly to cancel we need

(1.2.2)
$$\begin{aligned}\int_{z,z',\eta_a,\eta_a'} & \overline{\partial}_{\overline{z}} \left(\widetilde{J}^1[k,l](z,\eta_a) \widetilde{J}^1[r,s](z',\eta_a') \right) D_{m,n} \mu_1(z, w_i = 0, \eta_a) D_{r,s} \mathfrak{c}_1(z', w_i' = 0, \eta_a') \\ &= \int_{z'',\eta_a''} \widetilde{J}^1[m,n](z'',\eta_a'') D_{m,n} \left(\mu_1 \partial_{w_1} \mathfrak{c}_1 - \mathfrak{c}_1 \partial_{w_1} \mu_1\right)(z'', w_i = 0, \eta_a'').\end{aligned}$$

In these expressions, we sum over the indices r, s, k, l, m, n. This equation must hold for all values of the field μ_1, $\mathfrak{c}_1$. To constrain the OPEs, we can test the equation by setting

$$\begin{aligned}\mu_1 &= G(z, \overline{z}, \eta_a) \mathrm{d}\overline{z} w_1^k w_2^l \\ \mathfrak{c}_1 &= H(z, \overline{z}, \eta_a) w_1^r w_2^s\end{aligned}$$

for G, H arbitrary smooth functions of the variables $z, \overline{z}, \eta_a$.

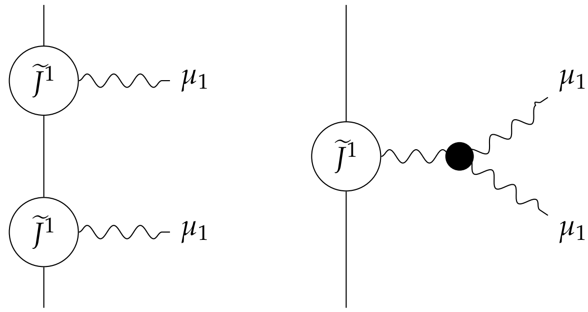

FIGURE 1. Cancellation of the gauge anomaly of these two diagrams leads to the equation for the self OPE of the currents $\widetilde{J}^1[k,l]$.

Inserting these values for the fields into the anomaly-cancellation condition gives

$$\int_{z,z',\eta_a,\eta'_a} \overline{\partial}_{\overline{z}} \left(\widetilde{J}^1[k,l](z,\eta_a)\widetilde{J}^1[r,s](z',\eta'_a)\right) G(z,\overline{z},\eta_a)H(z',\overline{z}',\eta'_a)$$
$$= \int_{z'',\eta''_a} (r-k)\widetilde{J}^1[k+r-1,l+s](z'',\eta''_a)G(z'',\overline{z}'',\eta''_a)H(z'',\overline{z}'',\eta''_a). \tag{1.2.3}$$

Since this must hold for all values of the functions G, H we get an identity of the integrands:

$$\overline{\partial}_{\overline{z}} \left(\widetilde{J}^1[k,l](z,\eta_a)\widetilde{J}^1[r,s](z',\eta'_a)\right) = \delta_{z=z',\overline{z}=\overline{z}'}\delta_{\eta_a=\eta'_a}(r-m)\widetilde{J}^1[k+r-1,l+s].$$

(Recall that the fermionic δ-function $\delta_{\eta_a=\eta'_a}$ has the simple expression $\prod_a(\eta_a-\eta'_a)$).

This in turn leads to the OPE:

$$\widetilde{J}^1[k,l](0,\eta_a)\widetilde{J}^1[r,s](z,\eta'_a) \simeq \frac{1}{z}(r-k)\widetilde{J}^1[k+r-1,l+s](0,\eta_a)\delta_{\eta_a=\eta'_a}.$$

We apply the fermionic Fourier transform to write this expression in terms of the operators $\widetilde{J}^1[k,l](0,\hat{\eta}^a)$. We find

$$\widetilde{J}^1[k,l](0,\hat{\eta}^a)\widetilde{J}^1[r,s](z,\hat{\eta}'^a) \simeq \frac{1}{z}(r-k)\widetilde{J}^1[k+r-1,l+s](0,\hat{\eta}^a+\hat{\eta}'^a).$$

Diagrammatically, the OPE we have just deduced follows from the cancellation of the gauge anomaly in Figure 1.

1.2.1. Similarly, we have the $\widetilde{J}^2\widetilde{J}^2$ OPE

$$\widetilde{J}^2[r,s](0,\hat{\eta}^a)\widetilde{J}^2[k,l](z,\hat{\eta}'^a) \simeq \frac{1}{z}(l-s)\widetilde{J}^2[r+k,s+l-1](0,\hat{\eta}^a+\hat{\eta}'^a).$$

the $\widetilde{J}^1\widetilde{J}^2$ OPE

$$\widetilde{J}^1[r,s](0,\hat{\eta}^a)\widetilde{J}^2[k,l](z,\hat{\eta}'^a) \simeq -\frac{1}{z}s\widetilde{J}^1[r+k,l+s-1](0,\hat{\eta}^a+\hat{\eta}'^a)+\frac{1}{z}k\widetilde{J}^2[k+r-1,l+s](0,\hat{\eta}^a+\hat{\eta}'^a)$$

and the $\widetilde{J}^2\widetilde{J}^1$ OPE

$$\widetilde{J}^2[r,s](0,\hat{\eta}^a)\widetilde{J}^1[k,l](z,\hat{\eta}'^a) \simeq -\frac{1}{z}r\widetilde{J}^2[r+k-1,l+s](0,\hat{\eta}^a+\hat{\eta}'^a)+\frac{1}{z}l\widetilde{J}^1[k+r,l+s-1](0,\hat{\eta}^a+\hat{\eta}'^a).$$

1.2.2. Let us use the these calculations to calculate the OPEs of the on-shell operators

$$J[r,s] = r\widetilde{J}^2[r-1,s] - s\widetilde{J}^1[r,s-1].$$

We find

$$\begin{aligned}(1.2.4)\quad J[r,s](0,\hat{\eta}^a)J[k,l](z,\hat{\eta}'^a) &= \frac{1}{z}(l-s)kr\widetilde{J}^2[k+r-2,l+s-1]\\ &+\frac{1}{z}ls(k-r)\widetilde{J}^1[k+r-1,l+s-2]\\ +\frac{1}{z}r(r-1)l\widetilde{J}^2[r+k-2,l+s-1] &- \frac{1}{z}l(l-1)r\widetilde{J}^1[k+r-1,l+s-2]\\ +\frac{1}{z}ks(s-1)\widetilde{J}^1[r+k-1,l+s-2] &- \frac{1}{z}ks(k-1)\widetilde{J}^2[k+r-2,l+s-1]\end{aligned}$$

(On the right hand side, all operators are evaluated at $z=0$ and with the fermionic variables $\hat{\eta}^a+\hat{\eta}'^a$. We have dropped this dependence for clarity.)

Collecting the terms, we find the OPE is

$$\begin{aligned}&\frac{1}{z}\left((l-s)kr + r(r-1)l - ks(k-1)\right)\widetilde{J}^2[k+r-2,l+s-1]\\ &+\frac{1}{z}\left(ls(k-r) - l(l-1)r + ks(s-1)\right)\widetilde{J}^1[k+r-1,l+s-2].\end{aligned}$$

Since

$$J[k+r-1,l+s-1] = (k+r-1)\widetilde{J}^2[k+r-2,l+s-1]-(l+s-1)\widetilde{J}^1[k+r-1,l+s-2]$$

we find that the OPE is

$$J[r,s](0,\hat{\eta}^a)J[k,l](z,\hat{\eta}'^a) = \frac{1}{z}(rl-ks)J[r+k-1,l+s-1](z,\hat{\eta}^a+\hat{\eta}'^a).$$

Note that the operators with $r+s=2$ which are independent of $\hat{\eta}^a$ satisfy the OPE of the $\mathfrak{su}(2)$ Kac-Moody algebra at level zero. We will get a nontrivial level once we include the contribution from the back reaction, which we do in §**??**.

As was pointed out in **[?]**, the mode algebra corresponding to this full collection of OPE's can be expressed as the super loop space of the Lie algebra $\mathfrak{w}_\infty$ of Hamiltonoian vector fields on $\mathbf{C}^2$.[1] This is the Lie algebra $\mathcal{L}^{1|4}\mathfrak{w}_\infty$. Explicitly, elements of this super Lie algebra can be expressed as

$$z^n f(w_1, w_2; \eta_a) \in \mathcal{L}^{1|4}\mathfrak{w}_\infty \tag{1.2.5}$$

for $n \in \mathbb{Z}$ and where $f(w_1, w_2; \eta_a) \in \mathbf{C}[w_1, w_2, \eta_a]/\mathbf{C}$. The bracket is

$$[z^n f, z^m g] = z^{n+m}\epsilon^{ij}\partial_{w_i} f \partial_{w_j} g. \tag{1.2.6}$$

1.3. TJ **OPE.** We turn to the tree-level OPE between the on-shell operators T and J. First, we compute the tree-level OPE between the off-shell operators $\widetilde{J}$ and $\widetilde{T}$.

The coefficient of $\widetilde{J}^1$, for instance, in this OPE will be determined by the terms in the BRST variation of μ_1 which involve $\mathfrak{c}_1$ and μ_z or $\mathfrak{c}_z$ and μ_1. We collect such terms in the gauge variation of (1.2.1) and

$$\int_{(z,\eta_a)\in\mathbf{C}^{1|4}} \widetilde{T}[m,n](z,\eta_a) D_{m,n}\mu_z(z, w_i = 0, \eta_a). \tag{1.3.1}$$

Recall that the gauge variation of μ_z is

$$\begin{aligned} Q\mu_z = \overline{\partial}\mathfrak{c}_z + \mu_i\partial_{w_i}\mathfrak{c}_z + \mu_z\partial\mathfrak{c}_z - \mathfrak{c}_i\partial_{w_i}\mu_z - \mathfrak{c}_z\partial_z\mu_z \\ - \epsilon_{ij}\partial_i\mathfrak{c}_\gamma\partial_j\alpha - \epsilon_{ij}\partial_i\mathfrak{c}_\alpha\partial_j\gamma. \end{aligned}$$

For now, we can disregard the terms involving α and $\mathfrak{c}_\gamma$ or $\mathfrak{c}_\alpha$ and γ.

The terms in the variations of (1.2.1) and (1.3.1) involving $\mathfrak{c}_1$ and μ_z or $\mathfrak{c}_z$ and μ_1 is

$$\begin{aligned} &\int_{z,\eta} \widetilde{J}^1[m,n](z,\eta_a) D_{m,n}(\mu_z\partial_z\mathfrak{c}_1 - \mathfrak{c}_z\partial_z\mu_1)(z, w_i = 0, \eta_a) \\ &+ \int_{z,\eta} \widetilde{T}[m,n](z,\eta_a) D_{m,n}(\mu_1\partial_{w_1}\mathfrak{c}_z - \mathfrak{c}_1\partial_{w_1}\mu_z)(z, w_i = 0, \eta_a). \end{aligned}$$

The coefficient of $\mathfrak{c}_z$ can only be cancelled by a gauge variation of

$$\int_{z,z',\eta_a,\eta'_a} \widetilde{J}^1[r,s](z,\eta_a) D_{r,s}\mu_1(z, w_i = 0, \eta_a)\widetilde{T}[k,l](z',\eta'_a) D_{k,l}\mu_z(z', w'_i = 0, \eta'_a).$$

[1] This is the quotient of the Lie algebra of functions on $\mathbf{C}^2$, which equipped with the standard Poisson bracket, by its center consisting of the constant functions.

By similar manipulation as above, we find that the gauge variation of this expression is

$$\int\limits_{z,z',\eta_a,\eta'_a} \overline{\partial}_z \left(\widetilde{J}^1[r,s](z,\eta_a)\widetilde{T}[k,l](z',\eta'_a)\right) D_{r,s}\mathfrak{c}_1(z,w_i=0,\eta_a)D_{k,l}\mu_z(z',w'_i=0,\eta'_a)$$
$$+\int\limits_{z,z',\eta_a,\eta'_a} \overline{\partial}_{z'} \left(\widetilde{J}^1[r,s](z,\eta_a)\widetilde{T}[k,l](z',\eta'_a)\right) D_{r,s}\mu_1(z,w_i=0,\eta_a)D_{k,l}\mathfrak{c}_z(z',w'_i=0,\eta'_a).$$

To constrain the OPEs, we use the test functions $\mu_z=0$, $\mathfrak{c}_1=0$, $\mu_1=G(z,\overline{z},\eta_a)\mathrm{d}\overline{z}w_1^k w_2^l$, $\mathfrak{c}_z=H(z,\overline{z},\eta_a)w_1^r w_2^s$ for G,H arbitrary smooth functions of the variables $z,\overline{z},\eta_a$. This yields the anomaly cancellation condition

$$\int\limits_{z,z',\eta_a,\eta'_a} \overline{\partial}_{z'} \left(\widetilde{J}^1[r,s](z,\eta_a)\widetilde{T}[k,l](z',\eta'_a)\right) G(z,\overline{z},\eta_a)H(z',\overline{z}',\eta'_a) = -\int\limits_{z'',\eta''_a} \widetilde{J}^1[r+k,s+l](z'',\eta''_a)H(z'',\overline{z}'',\eta''_a)\partial_{z''}G(z'',\overline{z}'',\eta''_a) + r\int\limits_{z'',\eta''_a} \widetilde{T}[r+k-1,s+l](z'',\eta''_a)G(z'',\overline{z}'',\eta''_a)H(z'',\overline{z}'',\eta''_a). \tag{1.3.2}$$

Integrating the right hand side by parts gives us

$$\int\limits_{z'',\eta''_a} \partial_{z''}\widetilde{J}^1[r+k,s+l](z'',\eta''_a)H(z'',\overline{z}'',\eta''_a)G(z'',\overline{z}'',\eta''_a) + \int\limits_{z'',\eta''_a} \widetilde{J}^1[r+k,s+l](z'',\eta''_a)\partial_{z''}H(z'',\overline{z}'',\eta''_a)G(z'',\overline{z}'',\eta''_a) + r\int\limits_{z'',\eta''_a} \widetilde{T}[r+k-1,s+l](z'',\eta''_a)G(z'',\overline{z}'',\eta''_a)H(z'',\overline{z}'',\eta''_a) \tag{1.3.3}$$

Because G,H are arbitrary functions, we arrive at the OPE

$$\widetilde{T}[r,s](0,\eta_a)\widetilde{J}^1[k,l](z,\eta'_a) \simeq \delta_{\eta_a=\eta'_a}\frac{1}{z}\partial_z\widetilde{J}^1[r+k,s+l](0,\eta_a)+\delta_{\eta_a=\eta'_a}\frac{1}{z^2}\widetilde{J}^1[r+k,s+l](0,\eta_a) + r\delta_{\eta_a=\eta'_a}\widetilde{T}[r+k-1,s+l](0,\eta_a). \tag{1.3.4}$$

Switching the η_a variables to $\hat{\eta}^a$ variables by applying the odd Fourier transform we can write this OPE as

(1.3.5)
$$\widetilde{T}[r,s](0,\hat{\eta}^a)\widetilde{J}^1[k,l](z,\hat{\eta}'^a) \simeq \frac{1}{z}\partial_z\widetilde{J}^1[r+k,s+l](0,\hat{\eta}^a+\hat{\eta}'^a) + \frac{1}{z^2}\widetilde{J}^1[r+k,s+l](0,\hat{\eta}^a+\hat{\eta}'^a) + r\widetilde{T}[r+k-1,s+l](0,\hat{\eta}^a+\hat{\eta}'^a).$$

1.3.1. In a completely similar way one can deduce the $\widetilde{T}\widetilde{J}^2$ OPE

(1.3.6)
$$\widetilde{T}[r,s](0,\hat{\eta}^a)\widetilde{J}^2[k,l](z,\hat{\eta}'^a) \simeq \frac{1}{z}\partial_z\widetilde{J}^2[r+k,s+l](0,\hat{\eta}^a+\hat{\eta}'^a) + \frac{1}{z^2}\widetilde{J}^2[r+k,s+l](0,\hat{\eta}_a+\hat{\eta}_a{}') + s\widetilde{T}[r+k,s+l-1](0,\hat{\eta}^a+\hat{\eta}'^a).$$

1.3.2. Using the $\widetilde{T}\widetilde{J}^i$ and $\widetilde{J}^i\widetilde{J}^2$ OPE's that we have computed, we deduce the OPE's between the on-shell operators T and J^i. Recall that

$$\begin{aligned} T[r,s] &:= \widetilde{T}[r,s] - \frac{1}{2(r+1)}\partial_z\widetilde{J}^1[r+1,s] - \frac{1}{2(s+1)}\partial_z\widetilde{J}^2[r,s+1] \\ J[k,l] &:= k\widetilde{J}^2[k-1,l] - l\widetilde{J}^1[k,l-1] \end{aligned} \tag{1.3.7}$$

Thus

(1.3.8) $T[r,s](0,\eta_a)J[k,l](z,\hat{\eta}'^a) \simeq$
$$\begin{aligned} & k\widetilde{T}[r,s](0,\eta_a)\widetilde{J}^2[k-1,l] - l\widetilde{T}[k,l](0,\eta_a)\widetilde{J}^1[k,l-1] \\ & - \frac{k}{2(r+1)}\partial_z\widetilde{J}^1[r+1,s](0,\eta_a)\widetilde{J}^2[k-1,l] + \frac{l}{2(r+1)}\partial_z\widetilde{J}^1[r+1,s]\widetilde{J}^1[k,l-1] \\ & - \frac{k}{2(s+1)}\partial_z\widetilde{J}^2[r+1,s](0,\eta_a)\widetilde{J}^2[k-1,l] + \frac{l}{2(s+1)}\partial_z\widetilde{J}^2[r+1,s]\widetilde{J}^1[k,l-1]. \end{aligned}$$

(On the right hand side, all operators are evaluated at $z = 0$ and with the fermionic variables $\hat{\eta}^a + \hat{\eta}'^a$. We have dropped this dependence for clarity.)

1.4. TT **OPE.** Following the same logic we constrain the $\widetilde{T}\widetilde{T}$ OPE. These OPE's are determined by terms in the BRST variation of μ_z which involve c_z and μ_z.

Proceeding as above we set

$$\begin{aligned} \mu_z &= G(z,\bar{z},\eta_a)\mathrm{d}\bar{z}w_1^k w_2^l \\ \mathfrak{c}_1 &= H(z,\bar{z},\eta_a)w_1^r w_2^s \end{aligned}$$

to arrive at the anomaly constraint

(1.4.1)
$$\int_{z,z',\eta_a,\eta'_a} \bar{\partial}_{z'} \left(\widetilde{T}[r,s](z,\eta_a)\widetilde{T}[k,l](z',\eta'_a) \right) G(z,\bar{z},\eta_a)H(z',\bar{z}',\eta'_a)$$
$$= \int_{z'',\eta''_a} \widetilde{T}[r+k,s+l](z'',\eta''_a) \left(G(z'',\bar{z}'',\eta''_a)\partial_{z''}H(z'',\bar{z}'',\eta''_a) - H(z'',\bar{z}'',\eta''_a)\partial_{z''}G(z'',\bar{z}'',\eta''_a) \right)$$

Integrating by parts and switching to the Fourier dual odd coordinates, we find the OPE

(1.4.2)
$$\widetilde{T}[r,s](0,\hat{\eta}^a)\widetilde{T}[k,l](z,\hat{\eta}'^a) \simeq \frac{1}{z}\partial_z\widetilde{T}[r+k,s+l](0,\hat{\eta}^a+\hat{\eta}'^a) + 2\frac{1}{z^2}\widetilde{T}[r+k,s+l](0,\hat{\eta}^a+\hat{\eta}'^a).$$

1.5. *GG* **OPE.** To constrain the G_α, G_γ OPE we consider terms in the gauge variations of the classical couplings involving α and $\mathfrak{c}_\gamma$ or γ and $\mathfrak{c}_\alpha$ (we have disregarded those terms in the analysis above as they played no role in the previous OPE calculations).

The term in the gauge variation of μ_i involving the fields α and $\mathfrak{c}_\gamma$ is $\epsilon_{ij}\partial_j\mathfrak{c}_\gamma\partial_z\alpha - \epsilon_{ij}\partial_z\mathfrak{c}_\gamma\partial_j\alpha$. Therefore, the gauge variation of $\int \widetilde{J}^i[m,n]D_{m,n}\mu_i$ involving such terms is

$$\int \widetilde{J}^i[m,n]D_{m,n}\left(\epsilon_{ij}\partial_{w_j}\mathfrak{c}_\gamma\partial_z\alpha - \epsilon_{ij}\partial_z\mathfrak{c}_\gamma\partial_{w_j}\alpha\right).$$

The term in the gauge variation of μ_z involving α and $\mathfrak{c}_\gamma$ is $-\epsilon_{ij}\partial_{w_i}\mathfrak{c}_\gamma\partial_{w_j}\alpha$. Therefore, the gauge variation of $\int \widetilde{T}[m,n]D_{m,n}\mu_z$ involving such terms is

$$\int \widetilde{T}[m,n]D_{m,n}(-\epsilon_{ij}\partial_{w_i}\mathfrak{c}_\gamma\partial_{w_j}\alpha).$$

The sum of these anomalies can only be cancelled by a gauge variation of a term of the form

$$\int_{z,z',\eta_a,\eta'_a} G_\alpha[r,s](z,\eta_a)D_{r,s}\alpha(z,w_i=0,\eta_a)G_\gamma[k,l](z',\eta'_a)D_{k,l}\gamma(z',w'_i=0,\eta'_a).$$

The gauge variation of this expression involving the terms $\mathfrak{c}_\gamma$ and α is

$$\int_{z,z',\eta_a,\eta'_a} \bar{\partial}_{z'}\left(G_\alpha[r,s](z,\eta_a)G_\gamma[k,l](z',\eta'_a)\right) D_{r,s}\alpha(z,w_i=0,\eta_a)D_{k,l}\mathfrak{c}_\gamma(z',w'_i=0,\eta_a).$$

Let us plug in test fields $\alpha = d\bar{z} w_1^r w_2^s G(z, \bar{z}, \eta_a)$ and $\mathfrak{c}_\gamma = w_1^k w_2^l H(z, \bar{z}, \eta_a)$ where G, H are arbitrary functions. Cancellation of these gauge anomalies requires

$$\int_{z,z',\eta_a,\eta'_a} \bar{\partial}_{z'} \left(G_\alpha[r,s](z,\eta_a) G_\gamma[k,l](z',\eta'_a) \right) G(z,\bar{z},\eta_a) H(z',\bar{z}',\eta'_a) = \tag{1.5.1}$$
$$l \int_{z'',\eta''_a} \widetilde{J}^1[r+k, s+l-1](z'',\eta''_a) H(z'',\bar{z}'',\eta''_a) \partial_{z''} G(z'',\bar{z}'',\eta''_a)$$
$$- k \int_{z'',\eta''_a} \widetilde{J}^2[r+k-1, s+l](z'',\eta''_a) H(z'',\bar{z}'',\eta''_a) \partial_{z''} G(z'',\bar{z}'',\eta''_a)$$
$$- s \int_{z'',\eta''_a} \widetilde{J}^1[r+k, s+l-1](z'',\eta''_a) \partial_{z''} H(z'',\bar{z}'',\eta''_a) G(z'',\bar{z}'',\eta''_a)$$
$$+ r \int_{z'',\eta''_a} \widetilde{J}^2[r+k-1, s+l](z'',\eta''_a) \partial_{z''} H(z'',\bar{z}'',\eta''_a) G(z'',\bar{z}'',\eta''_a)$$
$$- \int_{z'',\eta''_a} \widetilde{T}[r+k-1, s+l-1](z'',\eta''_a) H(z'',\bar{z}'',\eta''_a) G(z'',\bar{z}'',\eta''_a).$$

We integrate by parts to rewrite the right hand side as

$$\int_{z'',\eta''_a} \Big(- l \partial_{z''} \widetilde{J}^1[r+k, s+l-1] + k \partial_{z''} J[r+k-1, s+l] \tag{1.5.2}$$
$$- \widetilde{T}[r+k-1, s+l-1] \Big) (z'',\eta''_a) H(z'',\bar{z}'',\eta''_a) G(z'',\bar{z}'',\eta''_a)$$
$$- (s+l) \int_{z'',\eta''_a} \widetilde{J}^1[r+k, s+l-1](z'',\eta''_a) \partial_{z''} H(z'',\bar{z}'',\eta''_a) G(z'',\bar{z}'',\eta''_a)$$
$$+ (r+k) \int_{z'',\eta''_a} \widetilde{J}^2[r+k-1, s+l](z'',\eta''_a) \partial_{z''} H(z'',\bar{z}'',\eta''_a) G(z'',\bar{z}'',\eta''_a).$$

From these expressions we can read off the OPE's just as above. We obtain

(1.5.3)

$$G_\alpha[r,s](0,\hat{\eta}_a) G_\gamma[k,l](z,\hat{\eta}'_a) \simeq -(s+l)\frac{1}{z^2}\widetilde{J}^1[r+k, s+k-1] + (r+k)\frac{1}{z^2}\widetilde{J}^2[r+k-1, s+l]$$
$$- l\frac{1}{z}\partial_z \widetilde{J}^1[r+k, s+l-1] + k\frac{1}{z}\partial_z \widetilde{J}^2[r+k-1, s+l] + (rl - sk)\frac{1}{z}\widetilde{T}[r+k-1, s+l-1].$$

(On the right hand side, all operators are evaluated at $z = 0$ and with the fermionic variables $\hat{\eta}^a + \hat{\eta}'^a$. We have dropped this dependence for clarity.)

1.6. TG **OPE.**