1. Twisted supergravity in six dimensions

1.1. Generalities on twisting.

1.2. Twisted supergravity.

1.3. Kodaira–Spencer theory and IIB supergravity. We begin with a description of the holomorphic twist of type IIB supergravity in ten dimensions. The holomorphic supercharge is invariant under $SU(5) \subset Spin(10)$, and so can be defined on any Calabi–Yau fivefold X. In [**?**], it was conjectured that the holomorphic twist of IIB supergravity is equivalent to a certain truncation of the topological B-model on X.[1] In [**?**] this conjecture was verified at the level of the free limit of type IIB supergravity. We will assume this conjecture throughout the paper, and we will provide further justification in section **??**.

The fields of Kodaira–Spencer theory on the Calabi–Yau fivefold X are given in terms of the Dolbeault complex of polyvector fields on X; that is, sections of exterior powers of the holomorphic tangent bundle with values in $(0,\bullet)$ Dolbeault forms:

$$\mathrm{PV}^{i,j}(X) = \Omega^{0,j}(X, \wedge^i TX). \tag{1.3.1}$$

In local holomorphic coordinated $z_1, \ldots, z_5$ such a polyvector field can be expressed as

$$\mu = \mu^{\bar{i}_1 \cdots \bar{i}_5}_{j_1 \cdots j_5} \mathrm{d}\bar{z}_{\bar{i}_1} \cdots \mathrm{d}\bar{z}_{\bar{i}_5} \partial_{z_{j_1}} \cdots \partial_{z_{j_5}}. \tag{1.3.2}$$ [2]

It is convenient to express polyvector fields in terms of a single super-field. To do this, we rename $\mathrm{d}\bar{z}_{\bar{i}}$ as $\bar{\theta}_{\bar{i}}$ and ∂_{z_j} as θ^j. Bear in mind that θ transforms as a holomorphic vector while $\bar{\theta}$ transforms as an anti-holomorphic covector. With this notation, a general polyvector field

$$\mu \in \mathrm{PV}(X) = \oplus_{i,j} \mathrm{PV}^{i,j}(X) \tag{1.3.3}$$

can be thought of as a smooth function

$$\mu = \mu(z_i, \bar{z}_{\bar{i}}, \theta^i, \bar{\theta}_{\bar{i}}) \tag{1.3.4}$$

on the superspace $\mathbf{C}^{5|5+5}$ where the odd cordinates are $\theta^i, \bar{\theta}_{\bar{i}}$ for $i, \bar{i} = 1, \ldots, 5$.

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[1] This truncation was referred to as 'minimal' Kodaira–Spencer theory in *loc. cit.*. It effectively throws out on the non-propagating fields.

[2] Notationally, we will always omit the wedge product symbol $\wedge$ for simplicity.

The fields of Kodaira–Spencer theory are not all polyvector fields: they are polyvector fields which satisfy the constraint that they are divergence-free with respect to the holomorphic volume form Ω. Geometrically, this means that $L_\mu \Omega = 0$ where L_μ is the Lie derivative; equivalently this is the condition $\partial_\Omega \mu = 0$ where ∂_Ω is the divergence operator. In coordinates this reads

$$\partial_\Omega = \sum_i \partial_{\theta^i} \partial_{z_i}. \tag{1.3.5}$$

In addition to $\partial_\Omega \mu = 0$ we also require that

$$\partial_{\theta^1} \cdots \partial_{\theta^5} \mu = 0, \tag{1.3.6}$$

which effectively throws away the top power of T_X.

To define the action functional we utilize an integration map

$$\int_X^\Omega : \mathrm{PV}^{5,5}(X) \simeq C^\infty(X) \theta^1 \cdots \theta^5 \overline{\theta}_1 \cdots \overline{\theta}_5 \to \mathbf{C} \tag{1.3.7}$$

which is simply $\int (\mu \vee \Omega) \wedge \Omega$, with Ω the Calabi–Yau form. In terms of the superspace description this is simply the usual integration along X together with the Berezinian integral along the odd directions.

Appearing the Lagrangian is a non-local term involving a term proportional to ${\partial_\Omega}^{-1} \mu$. While this is not globally well-defined, by the condition that μ be in the kernel of ∂_Ω, there exists locally such a polyvector field.

In summary, the fields of Kodaira–Spencer theory are

$$\mathrm{PV}(X) \cap \ker \partial_\Omega. \tag{1.3.8}$$

The Lagrangian is

$$\frac{1}{2} \int_X^\Omega \mu \overline{\partial} \partial^{-1} \mu + \frac{1}{6} \int_X^\Omega \mu^3 \tag{1.3.9}$$

The conjecture originally put forth in [**?**] is that this Lagrangian captures the supersymmetric sector of IIB supergravity as described above. The super-field μ captures all the original fields, anti-fields, ghosts, etc. of type IIB supergravity after integrating out those fields which become massive in the holomorphic twist. Since the field μ includes anti-fields and anti-ghosts, we can describe the BV anti-bracket in this notation.

The BV anti-bracket of two super-fields is

$$\{\mu(z,\bar{z},\theta,\bar{\theta}),\mu(w,\overline{w},\eta,\bar{\eta}) = \partial_{z_i}\partial_{\theta^i}\delta(z-w)\delta(\bar{z}-\overline{w})(\bar{\theta}-\bar{\eta})(\theta-\eta)\mathrm{Id}. \tag{1.3.10}$$

The appearance of the holomorphic derivative ∂_{z_i} in the expression above is one way to understand the appearance of the non-local kinetic term in the Lagrangian.

From this BV anti-bracket it is clear that the component of the super-field μ proportional to the top polyvector $\partial_{\theta^1}\cdots\partial_{\theta^5}$ does not propagate. It is therefore convenient to impose the additional constraint

$$\partial_{\theta^1}\cdots\partial_{\theta^5}\mu = 0 \tag{1.3.11}$$

on the fields of Kodaira–Spencer theory.

We can avoid part of the non-locality appearing in the action by introducing a field $\hat{\mu}_{i_1\cdots i_4} \in \mathrm{PV}^{4,\bullet}$ which satisfies

$$(\partial_\Omega \hat{\mu})^\bullet_{i_1 i_2 i_3} = \mu^\bullet_{i_1 i_2 i_3}, \tag{1.3.12}$$

where the bullet denotes arbitrary anti-holomorphic form type. We can do this because we have the constraint $\partial_\Omega \mu = 0$. Then, the kinetic term in the Lagrangian above can be written as

$$\int \epsilon^{i_1\cdots i_5}\epsilon_{\bar{j}_1\cdots\bar{j}_5}\mu_{i_1}\bar{\partial}\hat{\mu}_{i_2\cdots i_5} + \frac{1}{2}\int \epsilon^{i_1\cdots i_5}\mu_{i_1 i_2}(\bar{\partial}\partial_\Omega^{-1}\mu)_{i_3 i_4 i_5}. \tag{1.3.13}$$

This Lagrangian is still non-local, but the only non-locality involves the field $\mathrm{PV}^{2,\bullet}(X)$. We will see the significance of this field from the perspective of supergravity in the next subsection.

1.4. **Matching supergravity with Kodaira–Spencer theory.** At the level of free fields, the match between the holomorphic twist of type IIB supergravity on $\mathbb{R}^{10}$ and Kodaira–Spencer theory has been done in **[?]**. Here, we spell out a precise relationship between the fields of Kodaira–Spencer theory and those of supergravity. For simplicity we will work on flat space near the flat Kähler metric $g_0^{i\bar{j}} = \delta^{i\bar{j}}$.

In type IIB supergravity there is a subsector known as type I supergravity. This theory is equipped with $N=(1,0)$ supersymmetry rather than $N=(2,0)$ supergravity for IIB supergravity. This theory is anomalous at one-loop, but we will only be concerned with the classical limit. In terms of Kodaira–Spencer theory, the fields of type I supergravity correspond to $\mathrm{PV}^{1,\bullet}(\mathbf{C}^5)$ and $\mathrm{PV}^{3,\bullet}(\mathbf{C}^5)$ whose fields we denote by $\mu_i^\bullet$, $\mu_{ijk}^\bullet$ respectively.

The bosonic fields of type I supergravity include a metric tensor. As representations of $SU(5)$, the metric tensor breaks into three components $g^{ij}, g^{i\bar{j}}, g^{\bar{i}\bar{j}}$. To leading order, the components $g^{ij}, g^{i\bar{j}}$ are rendered massive in the twist and can hence be removed. The remaining component of the metric corresponds to the field $\mu_k^{\bar{j}}$ in Kodaira–Spencer theory via the Kähler form

$$g^{\bar{i}\bar{j}} \mapsto \delta^{k\bar{i}} \mu_k^{\bar{j}}. \tag{1.4.1}$$

The fermionic fields of type I supergravity include a gravitino. In the untwisted theory the gravitino has a spinor index and a vector index. As a $SU(5)$ representation, the 16-dimensional spinor representation S_+ of $SO(10)$ decomposes as a sum of three irreducible representations: the trivial representation, the exterior square of the anti-fundamental representation, and the fourth exterior power of the anti-fundamental representation

$$S_+ \simeq_{SU(5)} \mathbf{C} \oplus \wedge^2 \overline{\mathbf{C}}^5 \oplus \wedge^4 \overline{\mathbf{C}}^5. \tag{1.4.2}$$

The component which survives the twist is the holomorphic vector valued in the exterior square in the above equation, and we denote this field by

$$\lambda_i^{\bar{j}_1\bar{j}_2}, \tag{1.4.3}$$

which we can view as an element $\mathrm{PV}^{1,2}(\mathbf{C}^5)$.

The antifield to the component of the gravitino $\lambda_i^{\bar{j}_1\bar{j}_2}$ is a tensor of the form $\lambda^{*k}_{\bar{l}_1\bar{l}_2}$, where the $*$ just indicates that this is an anti-field in the physical theory. Since the gravitino is an odd field, its anti-field has overall even parity. It turns out that it is the derivative of this anti-field that corresponds to a field of Kodaira–Spencer theory

$$\partial_{z_{k_1}} \lambda^{*k_2}_{\bar{l}_1\bar{l}_2} \mapsto \epsilon^{k_1k_2i_1i_2i_3} \epsilon_{\bar{l}_1\bar{l}_2\bar{j}_1\bar{j}_2\bar{j}_3} \mu_{i_1i_2i_3}^{\bar{j}_1\bar{j}_2\bar{j}_3}. \tag{1.4.4}$$

That is, we view the derivative of the anti-field as an element of $\mathrm{PV}^{3,3}$. Following the discussion above, we can use the equation $\partial_\Omega \mu = 0$ to replace the field $\mu_{i_1i_2i_3}^{\bar{j}_1\bar{j}_2\bar{j}_3}$ by a field $\hat{\mu}$ satisfying

$$\mu_{i_1i_2i_3}^{\bar{j}_1\bar{j}_2\bar{j}_3} = \partial_{z_j} \hat{\mu}_{ji_1i_2i_3}^{\bar{j}_1\bar{j}_2\bar{j}_3}. \tag{1.4.5}$$

Note that $\hat{\mu}_{ji_1i_2i_3}^{\bar{j}_1\bar{j}_2\bar{j}_3}$ is a field of type $\mathrm{PV}^{4,3}$. Using this modified field in Kodaira–Spencer theory, we can more easily match with the anti-gravitino via

$$\lambda^{*k}_{\bar{l}_1\bar{l}_2} \mapsto \epsilon^{ki_1i_2i_3i_4} \epsilon_{\bar{l}_1\bar{l}_2\bar{j}_1\bar{j}_2\bar{j}_3} \mu_{ki_1i_2i_4}^{\bar{j}_1\bar{j}_2\bar{j}_3}. \tag{1.4.6}$$

So far we have just discussed matching Kodaira–Spencer theory with type I supergravity. In type IIB supergravity there is an important bosonic field called the chiral four-form $C \in \Omega^4(\mathbb{R}^{10})$. Chiral means that the field strength five-form $F = \mathrm{d}C$ is required to be self-dual. It is components of the field strength that appears as a fundamental field in Kodaira–Spencer theory. Specifically, the component

$$F^{\bar{i}_1\bar{i}_2 j_1 j_2 j_3} \in \Omega^{3,2}(\mathbf{C}^5) \tag{1.4.7}$$

survives the twist. Using the holomorphic volume form these components are identified with the fields

$$F^{\bar{i}_1\bar{i}_2 j_1 j_2 j_3} \mapsto \epsilon^{j_1 j_2 j_3 j_4 j_5} \mu^{\bar{i}_1\bar{i}_2}_{j_4 j_5} \tag{1.4.8}$$

which we view as a polyvector field of type $\mathrm{PV}^{2,2}(\mathbf{C}^5)$. Notice that the Bianchi identity $\mathrm{d}F = 0$ becomes the constraint $\partial_j \mu^{\bar{i}_1\bar{i}_2}_{jk} = 0$ that this polyvector field be divergence-free.

We continue to match with Kodaira–Spencer theory at the level of the kinetic term in the Lagrangian.

1.5. **Compactification of Kodaira–Spencer theory.** Let Y be a complex surface (which we will soon take to be compact) with a fixed holomorphic symplectic structure. A general field of Kodaira–Spencer theory on $\mathbf{C}^3 \times Y$ is a Dolbeault-valued polyvector field which is annihilated by the divergence operator with respect to the holomorphic volume form. A Dolbeault valued polyvector field $\alpha^{k,\bullet}$ on $\mathbf{C}^3 \times Y$ of type $(k, \bullet)$ can be written as a tensor product of one on $\mathbf{C}^3$ with one on X

$$\alpha^{k,\bullet} = \sum_{i+j=k} \beta^{i,\bullet} \otimes \gamma^{j,\bullet} \tag{1.5.1}$$

where $\beta^{i,\bullet}, \gamma^{j,\bullet}$ are polyvector fields of type $(i, \bullet), (j, \bullet)$ on $\mathbf{C}^3$, Y respectively. Polyvector fields on Y are the same as differential forms, because the holomorphic symplectic form on Y identifies the tangent and cotangent bundles. In particular, the harmonic polyvector fields are given simply by the de Rham cohomology of Y. Furthermore, polyvector fields on Y which are harmonic are automatically in the kernel of the divergence operator ∂_Ω, by standard Hodge theory arguments. Summarizing there is an equivalence of graded algebras

$$\mathrm{PV}(\mathbf{C}^3) \otimes \left(\ker \partial_\Omega|_{\mathrm{PV}(Y)} \right) \simeq \mathrm{PV}(\mathbf{C}^3) \otimes H^\bullet(Y).$$

We will use this equivalence to describe the fields of the theory on $\mathbf{C}^3$ upon compactification along Y.

Let $A = H^\bullet(Y)$ denote the cohomology ring of Y. We are mostly interested in the case that Y is a K3 surface in which case this algebra is generated by even elements $\eta, \overline{\eta}, \eta_a$ for $a = 1, \ldots 20$ subject to the relations

$$\begin{aligned} \eta^2 = \bar{\eta}^2 = 0 \\ \eta_a \eta_b = h_{ab} \eta \bar{\eta} \end{aligned} \tag{1.5.2}$$

where h_{ab} is a non-degenerate symmetric pairing on $\mathbf{C}^{20}$. Let I denote the ideal generated by these equations so that $A = \mathbf{C}[\eta, \bar{\eta}, \eta_a]/I$.

As before, we write the polyvector fields on $\mathbf{C}^3$ in terms of a superspace by introducing odd variables θ^i, $\overline{\theta}_{\bar{j}}$. Here, θ^i represents the coordinate vector field ∂_{z_i} and $\overline{\theta}_i$ represents the coordinate Dolbeault form $\mathrm{d}\overline{z}_i$. Then we can write the field content as a collection of superfields

$$\mu(z, \overline{z}, \theta^i, \overline{\theta}_{\bar{i}}, \eta) \in \oplus_{i,j} \mathrm{PV}^{i,j}(\mathbf{C}^3) \otimes A. \tag{1.5.3}$$

Here, we are using the shorthand η to inform that there is a dependence on $\eta, \overline{\eta}$, and η_a, $a = 1, \ldots, 20$. As such, such a superfield decomposes in its dependencies on the generators of the cohomology of X as

$$\begin{aligned} \mu(z, \overline{z}, \theta^i, \overline{\theta}_{\bar{i}}) \\ + \mu_\eta(z, \overline{z}, \theta^i, \overline{\theta}_{\bar{i}})\eta + \mu_{\overline{\eta}}(z, \overline{z}, \theta^i, \overline{\theta}_{\bar{i}})\overline{\eta} + \mu^a(z, \overline{z}, \theta^i, \overline{\theta}_{\bar{i}})\eta_a \\ + \mu_{\eta\bar{\eta}}(z, \overline{z}, \theta^i, \overline{\theta}_{\bar{i}})\eta\bar{\eta}. \end{aligned} \tag{1.5.4}$$

We emphasize that the η-variables represent harmonic polyvector fields on X and so are not acted on by any differential operators along $\mathbf{C}^3$.

The superfield satisfies the equation $\partial\mu = 0$ where, in the superspace formulation,

$$\overline{\partial} = \overline{\theta}_{\bar{j}} \partial_{\overline{z}_{\bar{j}}} \tag{1.5.5}$$

$$\partial = \partial_{\theta^i} \partial_{z_i}. \tag{1.5.6}$$

We denote by

$$\int_{\mathbf{C}^3}^{\Omega} (-)|_{\eta\overline{\eta}} \colon \mathrm{PV}^{3,3} \otimes A \to \eta\overline{\eta}\mathrm{PV}^{3,3} \to \mathbf{C} \tag{1.5.7}$$

the projection onto the summand $\mathbf{C}\eta\overline{\eta} \subset A$ followed by integration as in (1.3.7).

The Lagrangian is

$$\frac{1}{2}\int_{\mathbf{C}^3}^{\Omega} \mu\bar{\partial}\partial^{-1}\mu|_{\eta\overline{\eta}} + \frac{1}{6}\int_{\mathbf{C}^3}^{\Omega} \mu^3|_{\eta\overline{\eta}} \tag{1.5.8}$$

where the $(-)|_{\eta\overline{\eta}}$ means we pick up only the $\eta\overline{\eta}$ component.

We can simplify the field content somewhat, following **[?]**. We note that the coefficient of $\theta^1\theta^2\theta^3$ does not appear in the kinetic term in the action. This field does not propagate, so we can (and will) impose the additional constraint

$$\partial_{\theta^1}\partial_{\theta^2}\partial_{\theta^3}\mu(z,\overline{z},\theta,\overline{\theta},\eta) = 0. \tag{1.5.9}$$

Next, let us expand the superfield μ only in the θ^i variables:

$$\mu = \mu(z,\overline{z},\overline{\theta},\eta) + \mu_i(z,\overline{z},\overline{\theta},\eta)\theta^i + \dots \tag{1.5.10}$$

We note that the constraint $\partial\mu_{ij} = 0$ implies that there is some super-field

$$\hat{\mu}_{ijk}(z,\overline{z},\overline{\theta},\eta) = \alpha(z,\overline{z},\overline{\theta},\eta)\epsilon_{ijk} \tag{1.5.11}$$

so that $\partial_{z_i}\hat{\mu}_{ijk} = \mu_{jk}$.

It is convenient to rephrase the theory in terms of the field $\alpha(z,\overline{z},\overline{\theta},\eta)$, which has no holomorphic index. We will also change notation and let $\gamma(z,\overline{z},\overline{\theta},\eta)$ be the term with no θ^i dependence in the superfield $\mu(z,\overline{z},\theta,\overline{\theta},\eta)$.

In summary, we have the following fundamental superfields in the compactified theory on $\mathbf{C}^3$:

- $\mu_i(z,\overline{z},\overline{\theta},\eta)\theta^i$ which we identify with an element in the graded space

$$\mu \in \mathrm{PV}^{1,\bullet}(\mathbf{C}^3) \otimes A. \tag{1.5.12}$$

- $\alpha(z,\overline{z},\overline{\theta},\eta)$ which we identify with an element of the graded space

$$\alpha \in \Omega^{0,\bullet}(\mathbf{C}^3) \otimes A. \tag{1.5.13}$$

- $\gamma(z,\overline{z},\overline{\theta},\eta)$ which we also identify with an element of the graded space

$$\gamma \in \Omega^{0,\bullet}(\mathbf{C}^3) \otimes A. \tag{1.5.14}$$

In terms of these fields, the Lagrangian is

$$\frac{1}{2}\int_{\mathbf{C}^3} \epsilon^{ijk}\overline{\partial}\mu_i(\partial^{-1}\mu)_{jk}\,\mathrm{d}^3z|_{\eta\overline{\eta}} + \int_{\mathbf{C}^3} \alpha\overline{\partial}\gamma\mathrm{d}^3z|_{\eta\overline{\eta}} + \frac{1}{6}\int_{\mathbf{C}^3} \epsilon_{ijk}\mu_i\mu_j\mu_k\,\mathrm{d}^3z|_{\eta\overline{\eta}} +_{\mathbf{C}^3} \alpha\mu_i\partial_{z_i}\gamma\,\mathrm{d}^3z|_{\eta\overline{\eta}}. \tag{1.5.15}$$

In this expression we project onto the component $\eta\overline{\eta}$ as before.

Just as when we twist a field theory, when we twist a supergravity theory the ghost number of the twisted theory is a mixture of the ghost number and a $U(1)_R$-charge of the original physical theory. To define a consistent ghost number, one can choose any $U(1)_R$ in the physical theory under which the supercharge has weight 1. In general, there are many ways to do this. It is convenient for us to make the following assignments of ghost number.

(1) The variables η_a BW: finish
(2) The anti-commuting variables $\overline{\theta}_i$ have ghost number 1.
(3) The fields α, γ have ghost number -1, and are fermionic.

1.6. **Compactification and twisted multiplets.** In six-dimensional $\mathcal{N} = (1,0)$ supersymmetry there are four multiplets which appear in the compactifications we will discuss: (i) the graviton multiplet, (ii) the vector multiplet, (iii) the tensor (or chiral two-form [**?**]) multiplet, and (iv) the hypermultiplet. Through the work in [**?**] the holomorphic twists of each of the theories associated to each of these multiplets has been characterized. By the nature of being holomorphic, each theory shares a linear gauge symmetry by the $\overline{\partial}$ operator; schematically of the form $\delta\Phi = \overline{\partial}\Phi$.

Here, we recall the field content of each of the twisted six-dimensional multiplets.

(i) The holomorphic twist of the the graviton multiplet has two fundamental fields

$$(\mu, \rho, \widetilde{\alpha}) \in \left(\mathrm{PV}^{1,\bullet}(\mathbf{C}^3) \cap \ker\partial_\Omega\right)^{\oplus 3}. \tag{1.6.1}$$

(ii) The holomorphic twist of the vector multiplet is three-dimensional holomorphic BF theory with two fundamental fields

$$(A, B) \in \Omega^{0,\bullet}(\mathbf{C}^3)[1] \tag{1.6.2}$$

(iii) The holomorphic twist of the tensor multiplet has as its underlying complex of fields

$$\alpha \in \left(\Omega^{2,\bullet}(\mathbf{C}^3)[1]\right) \cap \ker \partial. \tag{1.6.3}$$

(iv) The holomorphic twist of the hypermultiplet is the higher dimensional $\beta - \gamma$ system with two fundamental fields

$$(\gamma, \beta) \in \Omega^{0,\bullet}(\mathbf{C}^3). \tag{1.6.4}$$

In this paper we are concerned with the compactification of the holomorphic twist of type IIB supergravity on a K3 surface. Before turning towards IIB supergravity, we can instead look at the simpler type I supergravity. Its compactification on a K3 surface is a model with 6d $\mathcal{N} = (1,0)$ supersymmetry, and hence its twist should be expressed in terms of the multiplets above.

In [**?**, **?**] it is argued that the holomorphic twist of type I supergravity on a Calabi–Yau fivefold X has two fundamental fields

$$\mu_{type\,I} \in \Pi \mathrm{PV}^{1,\bullet}(X) \cap \ker \partial_\Omega$$
$$\rho_{type\,I} \in \Pi \mathrm{PV}^{3,\bullet}(X) \cap \ker \partial_\Omega.$$

On a fivefold of the form $X = \mathbf{C}^3 \times Y$ where Y is a K3 surface, the field $\mu_{type\,I}$ decomposes as

$$\mu_{type\,I} = (\mu, \alpha; \gamma) \in \left(\mathrm{PV}^{1,\bullet}(\mathbf{C}^3)[1] \oplus \mathrm{PV}^{1,\bullet}(\mathbf{C}^3)[1] \otimes H^{0,2}(Y)\right) \cap \ker \partial_\Omega \oplus \Omega^{0,\bullet}(\mathbf{C}^3) \otimes H^{1,1}(Y), \tag{1.6.5}$$

where the divergence is with respect to the CY form on $\mathbf{C}^3$. Similarly, if we neglect topological degrees of freedom, the field $\rho_{type\,I}$ decomposes as

$$\rho_{type\,I} = (\rho, \widetilde{\alpha}, \beta) \in \left(\mathrm{PV}^{1,\bullet}(\mathbf{C}^3)[1] \otimes H^{2,2}(Y) \oplus PV^{1,\bullet}(\mathbf{C}^3)[1] \otimes H^{2,2}(Y)\right) \cap \ker \partial_\Omega \oplus \Omega^{0,\bullet}(\mathbf{C}^3) \otimes H^{1,1}(Y). \tag{1.6.6}$$

The fields $(\mu, \rho, \widetilde{\alpha})$ comprise the holomorphic twist of the six-dimensional graviton multiplet, denoted (i) above. The field

$$\alpha \in \mathrm{PV}^{1,\bullet}(\mathbf{C}^3)[1] \cap \ker \partial_\Omega \simeq \Omega^{2,\bullet}(\mathbf{C}^3)[1] \cap \ker \partial \tag{1.6.7}$$

comprises the holomorphic twist of a single tensor multiplet, denoted (iii) above. Finally the fields $(\gamma, \beta) = (\gamma_a, \beta_a)$, $a = 1, \ldots, 20 = \dim H^{1,1}(Y)$ comprise the holomorphic twist of 20 hyper multiplets, denoted (iv) above. In particular, we see that in terms of multiplets the compactification of type I supergravity on a K3 surface decomposes as

$$\text{type I supergravity} \rightsquigarrow (i) + (iii) + 20\,(iv), \tag{1.6.8}$$

which is compatible with the description of the K3 compactification of the physical type I supergravity [**?**].

1.7. **Backreaction as an infinitesimal deformation.** In the type IIB theory on $\mathbf{C}^3 \times K3$ we consider a D1–D5 system where the D1 branes wrap

$$\mathbf{C} \times 0 \times \{x\} \subset \mathbf{C} \times \mathbf{C}^2 \times K3$$

for some $y \in K3$ and the D5 branes wrap

$$\mathbf{C} \times 0 \times K3 \subset \mathbf{C} \times \mathbf{C}^2 \times K3.$$

We can apply a duality to turn this into a D3 brane system which wraps

$$\mathbf{C} \times 0 \times \Sigma \subset \mathbf{C} \times \mathbf{C}^2 \times K3$$

for some two-cycle $\Sigma \subset K3$. BW: say this precisely

In the last section, we argued that the dimensional reduction along a $K3$ surface becomes an extended version of Kodaira–Spencer theory where the extra fields are labeled by the cohomology of the surface. Upon compactification, the D3 system becomes a system of B-type branes in this extended version of Kodaira–Spencer theory.

The charge of these branes is labeled by a cohomology class

$$F \in H^2(K3) \subset A. \tag{1.7.1}$$

We denote

$$N \overset{\text{def}}{=} \langle F, F \rangle \tag{1.7.2}$$

using the inner product on $H^2(Y)$. Explicitly, if $F = f\eta + \overline{f}\overline{\eta} + f^a \eta_a$ for $f, \overline{f}, f_a$ complex numbers, then $N = f\overline{f} + f^a f^b h_{ab}$ where h_{ab} is the fixed non-degenerate symmetric pairing. BW: Can we say how N is related to N_1 and N_5? This would require tracing through some dualities carefully...

Let's choose coordinates z, w_1, w_2 on $\mathbf{C}^3$, where the branes wrap the z-plane along $w_1 = w_2 = 0$. Including the backreaction will deform the geometry away from the locus of the brane. Before backreacting, we should say what geometry is actually being deformed. Recall that in the case of ordinary Kodaira–Spencer theory on $\mathbf{C}^3$, it was shown in [**?**] that the backreaction of B-branes along $\mathbf{C} \subset \mathbf{C}^3$ deformed the complex structure on $\mathbf{C}^3 \setminus \mathbf{C}$ to the *deformed conifold* $SL_2(\mathbf{C})$.

Our case is similar in that the branes are supported along the same locus as in **[?]**. The difference is that we are working with a bigger space of fields, roughly extended by the cohomology of the $K3$ surface. We will now show that in the case of type IIB compactified on a K3 surface, the backreaction determines an *infinitesimal* deformation of the complex manifold $\mathbf{C}^3 \setminus \mathbf{C}$ over the fat point Spec A where $A = H^\bullet(K3)$.

If A is any local ring, an infinitesimal deformation of a complex manifold M_0 over Spec A is an element

$$\mu_{def} \in \mathrm{PV}^{1,1}(M_0) \otimes \mathfrak{m}_A \tag{1.7.3}$$

satisfying the Maurer–Cartan equation. In our case, $M_0 = \mathbf{C}^3 \setminus \mathbf{C}$ and μ_{def} is a supergravity field sourced by the branes. The Maurer–Cartan equation is the equation of motion for μ_{def}. The cohomology ring A of a $K3$ surface is a local ring. Following **[?]**, the backreaction of this system of branes introduces a twisted supergravity field

$$\mu_{BR} \in \overline{\mathrm{PV}}^{1,1}(\mathbf{C}^3) \otimes A$$

which we can identify with an element of $\overline{\Omega}^{2,1}(\mathbf{C}^3) \otimes A$ using the Calabi–Yau form on $\mathbf{C}^3$. This field satisfies the following equations of motion

$$\begin{aligned} \bar{\partial}\mu_{BR} &= F\,\delta_{\mathbf{C}\subset\mathbf{C}^3} \\ \partial\mu_{BR} &= 0. \end{aligned} \tag{1.7.4}$$

For quantization we will also impose the gauge fixing condition that $\bar{\partial}^*\mu_{BR}(\eta_a) = 0$. There is a unique solution to the above equations given by

$$\mu_{BR} = \frac{\epsilon^{ij}\overline{w}_i \mathrm{d}\overline{w}_j}{|w|^4}\partial_z \otimes F. \tag{1.7.5}$$

Note that this field is of the form $\mu_{BR,0} \otimes F$ where $\mu_{BR,0}$ is the Beltrami differential which gives rise to the deformed conifold **[?]**—all of the dependence on the parameters $\eta, \overline{\eta}, \eta_a$ is in the cohomology class F.

Equations (1.7.4) imply that μ_{BR} determines an infinitesimal deformation of $\mathbf{C}^3 \setminus \mathbf{C}$ over Spec A. The Kodaira–Spencer map associated to this infinitesimal deformation is of the form

$$KS\colon T_{\mathrm{Spec}\,A} \to H^1(\mathbf{C}^3 \setminus \mathbf{C}, T)$$

and simply maps a derivation δ of A to the class

$$\delta(F)\left[\frac{\epsilon^{ij}\overline{w}_i \mathrm{d}\overline{w}_j}{|w|^4}\partial_z\right] \in H^1(\mathbf{C}^3 \setminus \mathbf{C}, T).$$

BW: what more to say?