1. The twisted symmetric orbifold CFT

Supergravity on $AdS_3 \times S^3 \times X$, where X is either T^4 or a $K3$ surface, is expected to be holographically dual to a particular two-dimensional superconformal field theory (SCFT). Here we review this system of interest, following [**?**] and references therein. We also review the notion of the chiral de Rham complex of a manifold, and its generalization to orbifolds, in order to discuss the chiral algebra associated with the half-twist of the $Sym^N(T^4)$ SCFT, putatively dual to Kodaira-Spencer theory on the super-conifold. Of course, this SCFT is the IR limit of the field theory that arises from the zero modes of the open strings on the D1-D5 branes. The lowest-lying modes of open strings, which provide an effective field theory description of the D1 and D5-branes, naturally furnish a gauge theory whose IR limit we are primarily interested in. The D5-D5 strings give rise to a six-dimensional supersymmetric $U(N_5)$ gauge theory preserving 16 supercharges. When all the D-branes are coincident the gauge theory is in the Higgs phase and when some of the adjoint scalars in the field theory acquire a vev, corresponding to transverse separation of the branes, the theory is in the Coulomb phase. We will focus on the Higgs branch of the gauge theory throughout, which involves turning on a nonvanishing Fayet-Iliopoulos parameter (dually, NS B-field). We reduce four directions of the gauge theory on

$$X = T^4 \quad \text{or} \quad K3$$

which results in an effective two-dimensional $U(N_5)$ gauge theory which preserves 16 supercharges. The D1-D1 strings similarly produce a $U(N_1)$ gauge theory preserving 16 supercharges. More interesting are the D1-D5 and D5-D1 strings, which break the total supersymmetry down to 8 supercharges (though more supersymmetries will be obtained in the near-horizon/low energy limits, so that the dual pair of theories has 16 supersymmetries overall). These strings produce matter multiplets transforming in the bifundamental representations of the gauge groups.

On the Higgs branch, one must solve the vanishing of the bosonic potential (i.e. D-flatness equations) modulo the gauge symmetries $U(N_1) \times U(N_5)$ to obtain the moduli space. If one imagined that both sets of D-branes were supported on a noncompact six-dimensional space, these D-flatness equations can be rewritten to reproduce the ADHM equations for N_1 instantons of a six-dimensional $U(N_5)$ gauge theory a la [**?**]. In fact, it has been argued that the instanton moduli space is the more accurate description of the dual field theory, so that one should study the moduli space of N_1 instantons of a $U(N_5)$ gauge theory on T^4, i.e. the Hilbert scheme of N_1N_5 points on

T^4 [1]. The (conformally invariant limit of the) gauge theory description is expected to only capture the regime of vanishing size instantons (i.e. when the hypermultiplets have small vevs). One can understand that the gauge theory description is approximate by noticing that the Yang-Mills couplings are given in terms of the T^4 volume V and string coupling as $g_1^2 = g_s(2\pi\alpha'), g_5^2 = g_s V/(\alpha'(2\pi)^3)$ so for energies much smaller than the inverse string length the gauge theories are strongly coupled [**?**].

To get the SCFT we take an IR limit, which would be dual to a near-horizon limit from the closed string point of view. In this limit, the gauge theory moduli space becomes the target space of the low-energy sigma-model. It has been argued that the correct instanton moduli space is a smooth deformation of the symmetric product theory $Sym^{N_1 N_5}(\tilde{T}^4)/S_{N_1 N_5}$. Indeed, there is a point in the SCFT moduli space (far from the supergravity point itself) where the theory takes precisely the symmetric orbifold form. The orbifold point is the analogue of free Yang-Mills theory in the perhaps more-familiar $AdS_5 \times S^5$/ 4d $\mathcal{N} = 4$ SYM duality, and is dual to a stringy point in moduli space which has been explored extensively in recent years [**?**].

1.1. **Branes in twisted supergravity.** We have already recollected the proposal of [**?**] that the twist of type IIB supergravity is equivalent to the topological B-model on a Calabi–Yau fivefold. At the level of branes, this proposal further asserts that D_{2k-1}-branes in type IIB corresponds to topological B-branes. We use that perspective here to deduce the worldvolume CFT of the twist of the $D1/D5$ system in type IIB supergravity.

We consider the system of $D1/D5$ branes in the twist of type IIB on a Calabi–Yau five-fold Z. For simplicity, we assume that we have a collection of $N_1 = N$ $D1$ branes supported along a closed Riemann surface

$$\Sigma \subset Z$$

together with a single $D5$ brane which is parallel to the $D1$ branes.

We consider $D1$ branes that are a sum of simple branes labeled by the structure sheaf $\mathcal{O}_\Sigma$. The Dolbeault model for the open string fields which stretch between two such $D1$ branes is given by

$$\Omega^{0,\bullet}(\Sigma, \underline{\mathrm{Ext}}_{\mathcal{O}_Z}(\mathcal{O}_\Sigma^{\oplus N}))[1] \simeq \Omega^{0,\bullet}\left(\Sigma, \mathfrak{gl}(N) \otimes \wedge^\bullet \mathcal{N}_\Sigma\right)[1] \tag{1.1.1}$$

[1]Throughout this note we ignore the center of mass factor of the moduli space that produces a $\tilde{T}^4$ factor, for some $\tilde{T}^4$ not necessarily the same as the compactification T^4. The relationship between the two tori is clarified in [**?**].

where $\mathcal{N}_\Sigma$ is the normal bundle to Σ in Z. If we take X to the be the total space of the bundle $\mathcal{N}_\Sigma$ then the Calabi–Yau condition requires $\wedge^4 N_\Sigma = K_\Sigma$. In the case $\Sigma = \mathbf{C}$ and $Z = \mathbf{C}^5$ we can write the open string fields (1.1.1) as

$$\Omega^{0,\bullet}\left(\mathbf{C}, \mathfrak{gl}(N)[\varepsilon_1, \ldots, \varepsilon_4]\right)[1]. \tag{1.1.2}$$

Here the ε_i are odd variables that carry spin $1/4$, meaning they transform as constant sections of the bundle $K_{\mathbf{C}}^{1/4}$. This is precisely the field content of the holomorphic twist of two-dimensional $\mathcal{N} = (8,8)$ pure gauge theory which is the worldvolume theory living on a stack of $D1$ branes in twisted supergravity on flat space.

Next, we consider $D1 - D5$ strings. The open string fields are given by

$$\Omega^{0,\bullet}\left(\Sigma, \underline{\mathrm{Ext}}_{\mathcal{O}_X}\left(\mathcal{O}_Y, \mathcal{O}_\Sigma^{\oplus N}\right)\right). \tag{1.1.3}$$

Again, on flat space $\Sigma = \mathbf{C}$ this can be written in a more explicit way as

$$\Omega^{0,\bullet}\left(\mathbf{C}, K_{\mathbf{C}}^{1/2}[\varepsilon_3, \varepsilon_4]\right) \otimes \mathrm{Hom}(\mathbf{C}, \mathbf{C}^N) = \Omega^{0,\bullet}\left(\mathbf{C}, K_{\mathbf{C}}^{1/2}[\varepsilon_3, \varepsilon_4]\right) \otimes \mathbf{C}^N. \tag{1.1.4}$$

Together with the $D5 - D1$ strings we get

$$\Omega^{0,\bullet}\left(\mathbf{C}, K_{\mathbf{C}}^{1/2}[\varepsilon_3, \varepsilon_4]\right) \otimes T^*\mathbf{C}^N. \tag{1.1.5}$$

In total, we see that the open-strings of the $D1/D5$ system along $\Sigma = \mathbf{C}$ are given by the Dolbeault complex valued in the following holomorphic vector bundle

$$\left(\mathfrak{gl}(N)[\varepsilon_1, \varepsilon_2][1] \oplus K_{\mathbf{C}}^{1/2} \otimes T^*\mathbf{C}^N\right) \otimes \mathbf{C}[\varepsilon_3, \varepsilon_4]. \tag{1.1.6}$$

If we choose twisting data so that the odd variable carry degree $\deg \varepsilon_1 = \deg \varepsilon_2 = +1$ then the bundle in parentheses can be written as

$$\mathfrak{gl}(N)[1] \oplus T^*\left(\mathfrak{gl}(N) \oplus K_\Sigma^{1/2} \otimes \mathbf{C}^N\right) \oplus \mathfrak{gl}(N)[-1]. \tag{1.1.7}$$

Up to the factor of $K_\Sigma^{1/2}$ this is evidently the underlying vector space of the graded Lie algebra controlling Hamiltonian reduction

$$T^*(\mathfrak{gl}(N) \oplus \mathbf{C}^N) /\!\!/ \mathfrak{gl}(N). \tag{1.1.8}$$

This recovers the well-known GIT description of the symmetric orbifold $\mathrm{Sym}^N \mathbf{C}^2$. The total worldvolume theory living on the twisted $D1$ brane is thus a holomorphic σ-model with target this symmetric orbifold.

This analysis happened entirely in flat space. The $D1$ branes wrapped

$$\mathbf{C} \times 0 \times 0 \times 0 \times 0 \subset \mathbf{C}^5 \tag{1.1.9}$$

while the $D5$ brane wrapped

$$\mathbf{C} \times \mathbf{C}^2 \times 0 \times 0 \subset \mathbf{C}^5. \tag{1.1.10}$$

If we instead replace this $\mathbf{C}^2$ by a compact Calabi–Yau twofold X then the above computation leads us to the well-established expectation that the worldvolume theory, after twisting, is a holomorphic σ-model with target $\mathrm{Sym}^N X$.

1.2. **The symmetric orbifold SCFT.** We recall the SCFT description of the orbifold point in the string moduli space. We will be particularly interested in protected, moduli-independent quantities that can still be compared to the supergravity point in moduli space. We will take the branes to be supported on $\mathbf{R} \times S^1$ after T^4 compactification, so that the CFT is defined on the cylinder. On the cylinder, the NS sector corresponds to anti-periodic boundary conditions on the fermions. The sigma model is then the $\mathcal{N} = (4,4)$ theory whose bosonic fields are valued in maps from $S^1 \to Sym^N(T^4)$.

The physical SCFT has R-symmetries $SO(4) \simeq SU(2)_L \times SU(2)_R$ dual to rotations of the S^3 and symmetries under a global $SO(4)_I \simeq SU(2)_a \times SU(2)_b$ of transverse rotations; this latter symmetry is broken by compactification on T^4. The latter $SO(4)_I$, although broken by the background, is still often used to organize the field content of the theory, and acts as an outer automorphism on the $\mathcal{N} = (4,4)$ superconformal algebra. As is well known, the isometries of $AdS_3 \times S^3$ are $SL(2,\mathbf{R}) \times SL(2,\mathbf{R}) \times SO(4)$ form the bosonic part of the supergroup $SU(1,1|2) \times SU(1,1|2)$ which preserve the supergravity vacuum and form the anomaly-free global subalgebra of the $\mathcal{N} = (4,4)$ superconformal algebra.

The orbifold theory can be described in terms of free fields on $N := N_1 N_5$ copies of the T^4 theory. We write $SU(2)_a \times SU(2)_b$ doublet indices as $A, \dot{B}$, $SU(2)_L \times SU(2)_R$ doublet indices as $\alpha, \dot{\beta}$. $SO(4)_I$ vector indices will be denoted by i, j, etc and subscripts $(r), r = 1, \ldots, N$ label the orbifold copy number.

Each T^4 theory has four free bosons $X^i_{(r)}$ and eight free fermions, the left-movers $\psi^{\alpha\dot{A}}_{(r)}(z)$ and right-movers $\bar{\psi}^{\dot{\alpha}\dot{A}}_{(r)}(\bar{z})$, for fixed copy (r) that satisfy the reality conditions

$$\begin{aligned} \psi^{\dagger}_{\alpha\dot{A}} &= -\epsilon_{\alpha\beta}\epsilon_{\dot{A}\dot{B}}\psi^{\beta\dot{B}} \\ \bar{\psi}^{\dagger}_{\dot{\alpha}\dot{A}} &= -\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{\dot{A}\dot{B}}\bar{\psi}^{\dot{\beta}\dot{B}}. \end{aligned}$$

In terms of the free fields, we construct the holomorphic $\mathcal{N} = 4$ superconformal algebra generators (similar expressions hold for the right-movers). In what follows, we have implicitly performed the diagonal sum over the copy index of all fields to obtain $S_{N_1N_5}$-invariant expressions:

$$
\begin{aligned}
J^a(z) &= \frac{1}{4}\epsilon_{\dot{A}\dot{B}}\psi^{\alpha\dot{A}}\epsilon_{\alpha\beta}(\sigma^{*a})^{\beta}_{\gamma}\psi^{\gamma\dot{B}} \\
G^{\alpha A}(z) &= \psi^{\alpha\dot{A}}\,[\partial X]^{\dot{B}A}\,\epsilon_{\dot{A}\dot{B}} \\
T(z) &= \frac{1}{2}\epsilon_{\dot{A}\dot{B}}\epsilon_{AB}\,[\partial X]^{\dot{A}A}\,[\partial X]^{\dot{B}B} + \frac{1}{2}\epsilon_{\alpha\beta}\epsilon_{\dot{A}\dot{B}}\psi^{\alpha\dot{A}}\partial\psi^{\beta\dot{B}}
\end{aligned}
$$

with a an $SU(2)_L$ triplet index and using the notation $[X]^{A\dot{A}} = \frac{1}{\sqrt{2}}X^i(\sigma^i)^{\dot{A}A}$ using the usual Pauli matrices plus $\sigma^4 = i\mathbb{1}_2$.

The free fields are normalized in the usual way,

$$
\begin{aligned}
\langle X^i(z)X^j(w)\rangle &= -2\delta^{ij}\log|z-w| \\
\langle\psi^{\alpha\dot{A}}\psi^{\beta\dot{B}}\rangle &= -\frac{\epsilon^{\alpha\beta}\epsilon^{\dot{A}\dot{B}}}{z-w}
\end{aligned}
$$

using which one can verify that the generators for each copy indeed satisfy the OPEs for the $\mathcal{N}=4$ superconformal algebra with $c=6$ and the diagonal sum for $c=6N$ Kodaira-Spencer theory seems to miss the ∂J terms in the TJ, GG OPEs below. Do they come from the diagrams including a single line sourcing the Beltrami differential?:

$$
\begin{aligned}
J^a(z)J^b(w) &\sim \frac{c}{12}\frac{\delta^{ab}}{(z-w)^2} + i\epsilon^{ab}_{c}\frac{J^c(w)}{z-w} \\
J^a(z)G^{\alpha A}(w) &\sim \frac{1}{2}(\sigma^{*a})^{\alpha}_{\beta}\frac{G^{\beta A}(w)}{z-w} \\
G^{\alpha A}(z)G^{\beta B}(w) &\sim -\epsilon^{AB}\epsilon^{\alpha\beta}\frac{T(w)}{z-w} - \frac{c}{3}\frac{\epsilon^{AB}\epsilon^{\alpha\beta}}{(z-w)^3} + \epsilon^{AB}\epsilon^{\beta\gamma}(\sigma^{*a})^{\alpha}_{\gamma}\left(\frac{2J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{z-w}\right) \\
T(z)J^a(w) &\sim \frac{J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{z-w} \\
T(z)G^{\alpha A}(w) &\sim \frac{\frac{3}{2}G^{\alpha A}(w)}{(z-w)^2} + \frac{\partial G^{\alpha A}(w)}{z-w} \\
T(z)T(w) &\sim \frac{c}{2}\frac{1}{(z-w)^4} + 2\frac{T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}.
\end{aligned}
$$

It is also easy to derive the OPEs of these generators with the basic free primaries:

$$J^a(z)\psi^{\alpha\dot{A}}(w) \sim \frac{1}{2}(\sigma^{*a})^{\alpha}_{\beta}\frac{\psi^{\beta\dot{A}}(w)}{z-w}$$
$$G^{\alpha A}(z)\left[\partial X(w)\right]^{\dot{B}B} \sim \epsilon^{AB}\left(\frac{\psi^{\alpha\dot{B}}(w)}{(z-w)^2}+\frac{\partial\psi^{\alpha\dot{B}}(w)}{z-w}\right)$$
$$G^{\alpha A}(z)\psi^{\beta\dot{A}}(w) \sim \epsilon^{\alpha\beta}\frac{\left[\partial X(w)\right]^{\dot{A}A}}{z-w}$$
$$T(z)\left[\partial X(w)\right]^{\dot{A}A} \sim \frac{\left[\partial X(w)\right]^{\dot{A}A}}{(z-w)^2}+\frac{\left[\partial^2 X(w)\right]^{\dot{A}A}}{z-w}$$
$$T(z)\psi^{\alpha\dot{A}} \sim \frac{\frac{1}{2}\psi^{\alpha\dot{A}}(w)}{(z-w)^2}+\frac{\partial\psi^{\alpha\dot{A}}(w)}{z-w}.$$

As always, we define the modes $\mathcal{O}_m$ of a field $\mathcal{O}(z)$ in terms of its weight Δ:

$$\mathcal{O}_m = \oint \frac{dz}{2\pi i}\mathcal{O}(z)z^{\Delta+m-1}. \tag{1.2.1}$$

It is easy from the above OPEs to get the mode algebra of the $\mathcal{N}=4$ superconformal algebra. For simplicity, we will just record the mode algebra of the global subalgebra generated by $\left\{J^a_0, G^{\alpha A}_{\pm 1/2}, L_0, L_{\pm 1}\right\}$, which has its Cartan subalgebra generated by J^3_0, L_0:

$$[L_0, L_{\pm 1}] = \mp L_{\pm} \tag{1.2.2}$$
$$[L_1, L_{-1}] = 2L_0 \tag{1.2.3}$$
$$\left[J^a_0, J^b_0\right] = i\epsilon^{ab}_{c}J^c_0 \tag{1.2.4}$$
$$\left\{G^{\alpha A}_{1/2}, G^{\beta B}_{-1/2}\right\} = \epsilon^{AB}\epsilon^{\beta\gamma}(\sigma^{*a})^{\alpha}_{\gamma}J^a_0 - \epsilon^{AB}\epsilon^{\alpha\beta}L_0 \tag{1.2.5}$$
$$\left\{G^{\alpha A}_{-1/2}, G^{\beta B}_{1/2}\right\} = -\epsilon^{AB}\epsilon^{\beta\gamma}(\sigma^{*a})^{\alpha}_{\gamma}J^a_0 - \epsilon^{AB}\epsilon^{\alpha\beta}L_0 \tag{1.2.6}$$
$$\left[L_0, G^{\alpha A}_{\pm 1/2}\right] = \mp G^{\alpha A}_{\pm 1/2} \tag{1.2.7}$$
$$\left[L_1, G^{\alpha A}_{1/2}\right] = \left[L_{-1}, G^{\alpha A}_{-1/2}\right] = 0 \tag{1.2.8}$$
$$\left[L_{\pm 1}, G^{\alpha A}_{\mp 1/2}\right] = \pm G^{\alpha A}_{\pm 1/2} \tag{1.2.9}$$
$$\left[J^a_0, G^{\alpha A}_{\pm n}\right] = \frac{1}{2}(\sigma^{*a})^{\alpha}_{\beta}G^{\beta A}_{\pm n} \tag{1.2.10}$$

These commutators generate $\mathfrak{psu}(1,1|2)$. Notice that there is no anomaly $c = 6N$ in the global subalgebra.

1.3. **Chiral primaries & short multiplets in the symmetric orbifold.** Chiral primaries themselves have an OPE $J[m,0]J[n,0] \sim regular$, but still have interesting three point functions from the regular terms $O^i O^j \sim C^{ij}_k O^k$... can we recover the chiral ring coefficients from KS somehow? From studying the algebra above it is easy to derive that a primary ϕ that also satisfies the condition $G^{+A}_{-1/2}|\phi\rangle = 0, A = 1,2$ satisfies $h = j$ (for L_0 eigenvalue h and J^3_0 eigenvalue j) and is called a chiral primary. The quantum numbers and two and three-point functions among chiral primaries are protected quantities as one moves in moduli space and can be matched to the corresponding quantities at the supergravity point. Anti-chiral primaries are defined similarly and satisfy $h = -j$. In the full physical theory, one combines left and right-moving (anti)chiral primaries: $(c,c), (a,a), (a,c), (c,a)$. We will first focus on chiral primaries in the holomorphic half of the SCFT.

Chiral primaries can arise in the twisted and untwisted sectors of the orbifold. In an n-twisted sector (cyclically permuting n copies of the T^4 SCFT), the weights of the chiral primaries are bounded: $\frac{n-1}{2} \leq h \leq \frac{n+1}{2}$. The chiral primaries are explicitly constructed as follows, starting with a twist field. Consider the twist field $\sigma_{l+1}(z)$ which cyclically permutes $l+1$ copies of the holomorphic SCFT as one moves around the point z in the base space; it creates the ground state of the twisted sector by acting on the original NS vacuum. It has weight $h = \frac{6}{24}((l+1) - \frac{1}{l+1})$, but no charge. To make a chiral primary from this state, we must dress it with modes of $J^+ \sim \psi^{+\dot{1}}\psi^{+\dot{2}}$, which carry $SU(2)_L$ charge. In particular, using the fact that operators in the twisted sector are fractionally moded we can build the chiral primaries [**?**]:

$$\sigma^0_{l+1} := J^+_{-\frac{l-1}{l+1}} J^+_{-\frac{l-3}{l+1}} \dots J^+_{-\frac{1}{l+1}} \sigma_{l+1}, \quad l+1 \text{ odd}$$
$$\sigma^0_{l+1} := J^+_{-\frac{l-1}{l+1}} J^+_{-\frac{l-3}{l+1}} \dots J^+_{-\frac{2}{l+1}} S^+_{l+1} \sigma_{l+1}, \quad l+1 \text{ even}$$

which have $h = j = l/2$. The spin fields S^+_{l+1} map the NS sector vacuum to the R sector, in order to restore the overall periodicity of the fermions as it traverses the length of the long closed string. These single long string states map to single particle states in the supergravity theory.

From this basic chiral primary, we can create three additional chiral primaries by acting with the fermions, which also map to single particle supergravity states:

$$
\begin{aligned}
&\sigma^0_{l+1}, && h = j = l/2 \\
&\psi^{+\dot{1}}\sigma^0_{l+1}, && h = j = (l+1)/2 \\
&\psi^{+\dot{2}}\sigma^0_{l+1}, && h = j = (l+1)/2 \\
&\psi^{+\dot{1}}\psi^{+\dot{2}}\sigma^0_{l+1}, && h = j = (l+2)/2.
\end{aligned}
$$

Combining this construction on the left and right enables one to construct 16 (c,c) primaries, which can be mapped to cohomology classes of the target space when viewing the fermions as differential forms. (Again, remember that we are implicitly summing over copy indices so that in the nth twisted sector we have e.g. $\psi^{+\dot{1}} = \sum_{r=1}^{n}\psi^{+\dot{1}}_{(r)}$). Of course, when $l = 0$, the basic chiral primary is just the NS sector vacuum with $h = j = 0$.

The chiral primaries are part of supermultiplets. These $SU(1,1|2)$ multiplets arise from acting on the chiral primaries with modes of the global subalgebra: the chiral primaries are precisely the highest weight states of short $SU(1,1|2)$ representations [2]. Schematically, one can view a short multiplet as associating to each chiral primary c 4 $\mathfrak{sl}(2)$ primary fields that are also $SU(2)_L$ highest weight states: $|c\rangle, G^{-1}_{-1/2}|c\rangle, G^{-2}_{-1/2}|c\rangle$, $G^{-1}_{-1/2}G^{-2}_{-1/2}|c\rangle + \frac{1}{2h}J^-_0 L_{-1}|c\rangle$. To fill out the rest of the short multiplet, one acts on each of these four states with an arbitrary number of L_{-1} generators, as well as with repeated applications of J^-_0 to fill out each $SU(2)_L$ multiplet.

When the chiral primary has weight $h \leq 1/2$, the representation is further truncated and does not contain the $G^{-1}_{-1/2}G^{-2}_{-1/2}|c\rangle + \frac{1}{2h}J^-_0 L_{-1}|c\rangle$ state, so is sometimes called an ultra-short representation.

One can also construct anti-chiral primaries with $h = -j$ in the holomorphic (or anti-holomorphic with $\bar{h} = -\bar{j}$) sector. The construction is almost identical to the chiral primary case (again, see **[?]** for details):

$$
\begin{aligned}
\tilde{\sigma}^0_{l+1} &:= J^-_{-\frac{l-1}{l+1}} J^-_{-\frac{l-3}{l+1}} \dots J^-_{-\frac{1}{l+1}} \sigma_{l+1}, \quad l+1 \text{ odd} \\
\tilde{\sigma}^0_{l+1} &:= J^-_{-\frac{l-1}{l+1}} J^-_{-\frac{l-3}{l+1}} \dots J^-_{-\frac{2}{l+1}} S^-_{l+1}\sigma_{l+1}, \quad l+1 \text{ even.}
\end{aligned}
$$

[2] Long $SU(1,1|2)$ representations can be obtained by acting with the global modes on global primary fields, i.e. fields annihilated by $L_1, G^{\alpha A}_{+1/2}$; there are 16 states per long multiplet.

One can again act on these basic primaries with the fermions. Two point functions of chiral and anti-chiral primaries are non-vanishing (in contrast to c-c and a-a two-point functions) and can always be normalized to unity when the operators are unit-separated.

We also note briefly that the exactly marginal operators, which form a basis for the tangent space of the moduli space, can be found in such multiplets. In particular, marginal operators that preserve $\mathcal{N} = (4,4)$ supersymmetry must be $SU(2)_L$ singlets with $h = \bar{h} = 1$. Therefore, they must be in the multiplets with highest weight states (combining now holomorphic and anti-holomorphic sectors) $\sigma_2^{+\dot{+}}, \psi^{+\dot{A}}\bar{\psi}^{\dot{+}\dot{B}}$. Each of these five operators gives four such states, corresponding to 20 marginal operators.

1.4. **The chiral de Rham complex.** The chiral de Rham complex is a sheaf of super vertex algebras defined on any manifold [**?**, **?**]. To an open coordinate neighborhood in the target, the vertex algebra is a $bc\beta\gamma$ system. Its global sections over the target is a super vertex algebra which is notoriously difficult to describe explicitly [**?**]. On the other hand, properties of this vertex algebra are known. For complex manifolds this vertex algebra is known to have an action by the $\mathcal{N} = 2$ superconformal algebra. The graded character of this super vertex algebra returns the elliptic genus of the complex manifold; in this sense the chiral de Rham complex is a vertex algebra refinement of the elliptic genus. If the complex manifold is Calabi–Yau then this symmetry is further enhanced to the $\mathcal{N} = 4$ superconformal algebra and hence makes manifest the decomposition of the elliptic genus for a Calabi–Yau manifold into characters of irreducible representations of this superconformal algebra.

We review how the chiral de Rham complex arises as the vertex algebra associated to the holomorphic twist, or half twist, which captures capturing 1/4-BPS states of the supersymmetric σ-model on a complex manifold.

One can start with the single-particle states furnished by chiral primaries and their $SU(1,1,|2)$ descendents and construct a Fock space. These are dual to 1/4-BPS states in the full physical SCFT, and their graded dimension gives rise to the elliptic genus of the model. The 1/4-BPS states are captured by the so-called half-twist of the supersymmetric model which we now recall.

The two-dimensional $\mathcal{N} = (2,2)$ σ-model admits a half-twist along the lines of [**?**, **?**] which results in a purely holomorphic theory. For this purpose, it can be convenient to recombine the fermions into vectors, and complexify the bosons X so that we chose local holomorphic and anti-holomorphic coordinates on the target space: $\phi^i, \phi^{\bar{i}}$. Then, explicitly, the fermionic fields are sections of the following bundles:

$$\begin{aligned}
\Psi^i &\in \Gamma(K^{1/2} \otimes \phi^*(T^{(1,0)}M)) \\
\Psi^{\bar{i}} &\in \Gamma(K^{1/2} \otimes \phi^*(T^{(0,1)}M)) \\
\bar{\Psi}^i &\in \Gamma(\bar{K}^{1/2} \otimes \phi^*(T^{(1,0)}M)) \\
\bar{\Psi}^{\bar{i}} &\in \Gamma(\bar{K}^{1/2} \otimes \phi^*(T^{(0,1)}M))
\end{aligned}$$

where as before the left-movers are given by Ψ and the right-movers by $\bar{\Psi}$.

To pass to the half-twisted model, we will restrict to the cohomology of the supercharge $\bar{Q}_+$, after we twist with a certain combination of R-symmetry currents. It is common, as in [**?**], to perform the A-type twist by the current $\frac{1}{2}(J_L - J_R)$ before passing to cohomology. We will instead simply consider a twist by $-J_R$, on the right-movers only, so that the twisted fields live in the following spaces of sections:

$$\begin{aligned}
\Psi^i &\in \Gamma(K^{1/2} \otimes \phi^*(T^{(1,0)}M)) \\
\Psi^{\bar{i}} &\in \Gamma(K^{1/2} \otimes \phi^*(T^{(0,1)}M)) \\
\bar{\Psi}^i &\in \Gamma(\bar{K} \otimes \phi^*(T^{(1,0)}M)) \\
\bar{\Psi}^{\bar{i}} &\in \Gamma(\phi^*(T^{(0,1)}M))
\end{aligned}$$

We then make the standard local identifications of fields in the twisted theory with those of $\dim_{\mathbf{C}} M$ copies of a free $bc\beta\gamma$ system (though again, we stress, the bc fields are just ordinary fermions):

$$\begin{aligned}
\beta_i &\equiv g_{i\bar{j}} \partial_z X^{\bar{j}} \\
\gamma^i &\equiv \phi^i \\
b_i &\equiv g_{i\bar{j}} \Psi^{\bar{j}} \\
c^i &\equiv \Psi^i.
\end{aligned}$$

On a local patch $U \subset M$ we can also take $g_{i\bar{j}} = \delta_{i\bar{j}}$. Notice that in the standard treatment of the half-twisted model, where the twist is performed using the A-model current, the left-moving fermions transform instead as $\Psi^i \in \Gamma(\phi^* T^{(1,0)} M), g_{i\bar{j}} \Psi^{\bar{j}} \in \Gamma(K \otimes \phi^* T^{(0,1)} M)$, rendering the bc fields of spin $(1,0)$, respectively. In our case, the spins remain half-integral.

The nontrivial operators in the $\bar{Q}_+ = g_{i\bar{j}}\tilde{\Psi}^{\bar{j}}\partial_{\bar{z}}X^i$ cohomology are those of left and right-moving conformal dimensions $(n, 0), n \geq 0$ **[?, ?, ?]**. The operators of dimension $(0, 0)$ (i.e. the operators forming the ground ring), in particular, have an interpretation as $(0, k)$-forms on the target space. The operators of dimension $(n > 0, 0)$ are given by $(0, k)$-forms valued in various tensor product bundles arising from insertions of $\partial_z X^i, g_{i\bar{j}}\partial_z X^{\bar{j}}, g_{i\bar{j}}\partial_z \Psi^{\bar{j}}$. In terms of the physical operators (post-twist), the operators in $\bar{Q}_+$-cohomology will composites of 1.) polynomials in the left-moving fermions and in arbitrary numbers of their holomorphic derivatives 2.) some function of the scalar fields and arbitrary numbers of their holomorphic derivatives 3.) The field $\Psi^{\bar{i}}$, though none of its derivatives (since, by its equation of motion, the holomorphic derivatives of $\Psi^{\bar{i}}$ may be expressed in terms of the aforementioned fields and their derivatives only). Call such an operator $\mathcal{F}$. One can further study which such operators can be constructed globally. Using standard techniques from cohomology reveals that that the Dolbeault cohomology describing the local operators $H^{(0,k)}_{\bar{\partial}}(M, \mathcal{F}) = 0$ for $k > 0$, so that we disallow operators $\mathcal{F}$ that contain $\Psi^{\bar{i}}$. Translating this over to the $bc\beta\gamma$ language, we have that the relevant operators are nothing but functions of b, c, β, γ and their holomorphic derivatives.

In the present context, our target space M is given by $\oplus_{a=1}^{\infty} Sym^a(T^4)$ or $\oplus_{a=1}^{\infty} Sym^a(K3)$. These spaces are hyperkahler, so the chiral de Rham complex has (in general, a twisted version of) $\mathcal{N} = 4$ supersymmetry **[?]**.

The basic, non vanishing OPEs for $bc\beta\gamma$ systems are

$$b_i(z)c^j(w) = c^j(z)b_i(w) \sim \frac{\delta_i^j}{z-w}$$
$$\beta_i(z)\gamma^j(w) = -\gamma^j(z)\beta_i(w) \sim \frac{-\delta_i^j}{z-w}$$

As explained in previous sections, odd spin operators in the symmetric orbifold theories will be built up from the Ramond sector vacuum, so we will also need the OPEs between the ghosts and the operator $\Sigma(z)$ (often called the spin field) that maps $|0\rangle_{NS} \to |0\rangle_R$:

$$\beta(z)\Sigma(w) \sim \frac{\tilde{\Sigma}(w)}{(z-w)^{1/2}}$$
$$\gamma(z)\Sigma(w) \sim 0.$$

1.5. **Elliptic genera for K3 surfaces.** Consider the chiral half of the $\mathcal{N} = (4,4)$ σ-model on the symmetric orbifold $\mathrm{Sym}^N X$ where X is T^4 or a $K3$ surface. After performing the half-twist, this is all that remains of the supersymmetric σ-model. According to [**?**] we can regard the direct sum of the vacuum modules of the chiral algebras of $\mathrm{Sym}^N X$ as being itself a Fock space. The generators of this Fock space are given by the single string states. These single string states are the analog of single trace operators in a gauge theory, and will ultimately be matched with single-particle states in the holographic dual.

Let $c(n,m)$ be the super dimension of the space of operators in supersymmetric σ-model into X, which are of weight n under L_0 and of weight m under the action of the Cartan of $SU(2)_R$. Let q, y be fugacities for L_0 and the Cartan of $SU(2)_R$, respectively—the elliptic genus $\chi(X; q, y)$ is a series in these variables. Of course, for $X = T^4$ the elliptic genus vanishes, so it is safe to assume from hereon that X is a K3 surface.

Introducing another parameter p, which keeps track of the symmetric power, we can consider the generating series

$$\sum_{n\geq 0} p^n \chi(\mathrm{Sym}^n X; q, y) \tag{1.5.1}$$

The main result of [**?**, **?**] is an expression for this generating series

$$\sum_n p^n \chi(\mathrm{Sym}^n X; q, y) = \prod_{l,m\geq 0, n>0} \frac{1}{(1 - p^n q^m y^l)^{c(nm,l)}} \tag{1.5.2}$$

where $c(m,l)$ is a function of the quantity $4m - l^2$. In other words, we can interpret the direct sum of the vacuum modules of the $\mathrm{Sym}^n X$ σ-model as being the Fock space generated by a trigraded super vector space

$$V = \oplus_{n\geq 0, m, l} V_{n,m,l} \tag{1.5.3}$$

where the super dimension of $V_{n,m,l}$ is $c(nm, l)$.

Setting $V_n = \oplus_{m,l} V_{n,m,l}$, we see that V_n is isomorphic to the vacuum module of the $bc\beta\gamma$ system on the original surface X, except with a different conformal structure. A state of the σ-model into X of spin k is of spin k/n in V_n.

The states in V_N will play the role of the single-trace operators in the large N limit of the $\mathrm{Sym}^N X$ σ-model. These states can be understood geometrically as follows—let us focus just on the S^1-modes of this σ-model. A map $S^1 \to \mathrm{Sym}^N X$ is the same as an N-fold cover $M \to S^1$ together with a map $M \to X$. Therefore, the Hilbert space of the σ-model on $\mathrm{Sym}^N X$ decomposes over sectors corresponding to the topological type of this N-fold cover, which are labelled by partitions of N. The single string sector is the sector corresponds to M being connected. This means that the monodromy of the cover $M \to S^1$ is conjugate to the length N cycle of type $(1 \dots N)$ in the symmetric group S_N.

Since the N-fold cover of S^1 corresponding to the single trace sector is connected, the Hilbert space of the single-trace sector is isomorphic to that of the original σ-model into X. However, the conformal structure is different—a rotation along S^1 in this σ-model rotates the total space $1/N$ times. This tells us that an operator in the single-trace sector carries spin $1/N$ times that of the corresponding state of the original σ-model. The projection onto $\mathbb{Z}_N$-invariant states ultimately restores integrality of the spin. In particular, the generating function of elliptic genera of $\mathrm{Sym}^N X$ decomposes as

$$\sum_{N\geq 0} p^N \chi(\mathrm{Sym}^N X; q, y) = \prod_{n>0} \sum_{N\geq 0} p^{nN} \chi(\mathrm{Sym}^N \mathcal{H}_{(n)}^{\mathbb{Z}_n}; q, y) \tag{1.5.4}$$

with $\sum_{N\geq 0} p^{nN} \chi(\mathrm{Sym}^N \mathcal{H}_{(n)}^{\mathbb{Z}_n}; q, y) = \prod_{l,m\geq 0} \frac{1}{(1-pq^m y^l)^{c(mn,l)}}$. Here, $\mathcal{H}_{(n)}$ is the Hilbert space of a single long string on X of length n with winding number $1/n$.

We can extract the $N \to \infty$ limit of this expression, following the logic employed in [**?**, **?**, **?**]. First, in preparation for comparison to supergravity, we perform spectral flow[3] to the NS sector:

$$\begin{aligned}
\sum_{N\geq 0} p^N \chi_{NS}(\mathrm{Sym}^N X; q, y) &= \sum_{N\geq 0} p^N \chi(\mathrm{Sym}^N X; q, y\sqrt{q}) y^N q^{N/2} \\
&= \prod_{\substack{n\geq 0 \\ m\geq 0, m\in\mathbb{Z} \\ l\in\mathbb{Z}}} \frac{1}{(1-p^n q^{m+l/2+n/2} y^{l+n})^{c(nm,l)}} \\
&= \prod_{\substack{n\geq 0 \\ m'\geq |l'|/2,\ 2m'\in\mathbb{Z}_{\geq 0} \\ l'\in\mathbb{Z},\ m'-l'/2\in\mathbb{Z}_{\geq 0}}} \frac{1}{(1-p^n q^{m'} y^{l'})^{c(nm'-nl'/2, n-l')}}.
\end{aligned}$$

At any power of q, there will be contributions from terms of the form $\frac{1}{(1-py^{l'})^{c(-l'/2,l'-1)}}$. The only nonvanishing such term in our case when $m' = 0$ is $\frac{1}{(1-p)^2}$. We wish to isolate the coefficients of all terms of the form $q^a y^b p^N$ for $a \ll N$. Taylor expanding $\frac{1}{(1-p)^2}$ and extracting the desired coefficient gives $Nh(a,b) + \mathcal{O}(N^0)$ where $h(a,b)$ is

[3]We shift the overall power of q by $q^{c/24}$ so that the vacuum occurs at q^0.

the coefficient of $q^a y^b$ in

$$\prod_{\substack{m'\geq|l'|/2,\ 2m'\in\mathbb{Z}_{\geq 0}\\ l'\in\mathbb{Z},\ m'-l'/2\in\mathbb{Z}_{\geq 0}}} \frac{1}{(1-q^{m'}y^{l'})^{f(m',l')}}$$

with $f(m',l') := \sum_{n>0} c(n(m'-l'/2), l'-n)$. The coefficients $c(M,L)$ vanish for $4M - L^2 < -1$ so for $m' \geq 1$ the sum truncates to $f(m',l') = \sum_{n=1}^{4m'} c(n(m'-l'/2), l'-n)$.

Hence, we can get a finite contribution upon dividing by N.

We can also write out the non-vanishing $f(m',l')$ more explicitly, recalling that the coefficients are constrained to lie in the following range of the Jacobi variable: $-2m' \leq l' \leq 2m', l' \equiv 2m' \mod 2$. Reproducing the elementary manipulations in Appendix A of [**?**] (in particular, using the fact that $c(N,L)$ depends only on $4N-L^2$ and L mod 2) allows us to rewrite the sum as

$$f(m',l') = \left(\sum_{\tilde{n}\in\mathbb{Z}} c(m'^2 - l'^2/4, \tilde{n})\right) - c(0,l'), \tag{1.5.5}$$

where $n' := n - 2m$ in the first term. The first term is non-vanishing only when $l' = \pm 2m'$ and then it reduces to the Witten index of K3, i.e. $f(m', \pm 2m') = 24$ for general m'. Otherwise, we have $f(m',l') = -c(0,l')$. When $m' \in \mathbb{Z}$ the nonvanishing such term is $-c(0,0) = -20$, and when $m' \in \mathbb{Z} + 1'/2$ we have $-c(0,1) = -2$ and $-c(0,-1) = -2$.

In sum, we obtain

$$\begin{aligned}\lim_{N\to\infty} \frac{\chi_{NS}(\mathrm{Sym}^N X; q, y)}{N} &= \prod_{k\geq 1} \frac{(1-q^k)^{20}(1-q^{k-1/2}y^{-1})^2(1-q^{k-1/2}y)^2}{(1-q^{k/2}y^k)^{24}(1-q^{k/2}y^{-k})^{24}} \\ &= 1 + \left(\frac{22}{y} + 22y\right) q^{1/2} + \left(\frac{277}{y^2} + 464 + 277y^2\right) q + \mathrm{O}(q^{3/2}).\end{aligned}$$

We will denote this large N limit by $\chi_{NS}(\mathrm{Sym}^\infty X; q, y)$. In particular, for there are two bosonic towers corresponding to (anti)chiral primary states and three fermionic towers corresponding to (derivatives of) the states capturing the cohomology of a single copy of K3. At $k=1$, there is a cancellation to $\frac{(1-q)^{20}}{(1-q^{1/2}y)^{22}(1-q^{1/2}y^{-1})^{22}}$.

Throughout this derivation, we have used the coefficients $c(m,l)$ that appear in the expansion of the K3 elliptic genus in the Ramond sector: $\sum_{m\geq 0, l\in\mathbb{Z}} c(m,l) q^m y^l$. We can also rewrite things slightly in terms of the coefficients of the NS sector elliptic genus expansion: $\sum_{2m'\in\mathbb{Z}_{\geq 0}, l'\in\mathbb{Z}} \mathcal{C}(m',l') q^{m'} y^{l'} = 2 + (20/y + 20y)q^{1/2} + (2/y^2 - 128 +$

$2y^2)q + \ldots$ using the half-integral spectral flow relation on the coefficients of the elliptic genera: $\mathcal{C}(m', l') = c(m' - l'/2, l' - 1)$.

Let us unpack those contributions a bit more, starting from the expression

$$
\begin{aligned}
f(m', l') &= \sum_{n=1}^{4m'} c(n(m' - l'/2), l' - n) \\
&= \sum_{n=1}^{4m'} \mathcal{C}(n(m' - l'/2 - 1/2) + l'/2 + 1/2, l' - n + 1).
\end{aligned}
$$

In the second line, we have used spectral flow to rewrite the sum in terms of the NS sector elliptic genus.

Let us henceforth drop the primes on our variables, for ease of notation, and then take for example the states $l = 2m$. The summand is nonvanishing only for $\mathcal{C}(1/2, 1) = 20, \mathcal{C}(0, 0) = 2, \mathcal{C}(1, 2) = 2$ and, of course, have contributions which sum to 24. The solutions to these conditions occur at $n = 2m, n = 1 + 2m, n = 2m - 1 \quad (\forall m > 0, 2m \in \mathbb{Z}_{\geq 0})$ and these values of n do appear in the sum. The corresponding states therefore have quantum numbers $(n, m, l) = 20 \times (j, j/2, j), 2 \times (j + 1, j/2, j), 2 \times (j - 1, j/2, j),\ j \in \mathbb{Z}_{>0}$ and come from chiral primary states in the physical theory. In the next section, we will recall the corresponding states in the physical theory [**?**, **?**]. Similarly, for $l = -2m$ we will obtain contributions from anti-chiral primary states finish.

We can also discuss the origin of the terms of negative multiplicity, which contribute to the numerator of the index. Taking $f(m, l)$ as it is written in **??** and studying $l = 0, k \in \mathbb{Z}_{\geq 1}$, there is a cancellation between the $n = 4k$ term, $\mathcal{C}(4k(k - 1/2) + 1/2, -4k + 1) = 20$, and the rest of the terms, which sum to -40. Similar cancellations apply to $l = \pm 1, m = k - 1/2, k \in \mathbb{Z}_{\geq 1}$, wherein the last terms of the sum are always $\mathcal{C}(4(k-1)^2, 4 - 4k) = 2, \mathcal{C}((1 - 2k)^2, 2 - 4k) = 2$, respectively, and the remainder of the terms produce coefficients summing to -4. Alternatively, one can use manipulations on Jacobi form coefficients to rewrite $f(m, l)$ as in 1.5.5:

$$
f(m', l') = \left(\sum_{\tilde{n}=-\infty}^{\infty} \mathcal{C}(m'^2 - l'^2/4 + (\tilde{n} + 1)/2, \tilde{n} + 1) \right) - \mathcal{C}((l' + 1)/2, l' + 1) \tag{1.5.6}
$$

with $\tilde{n} = n - 2m$ as before. The first term sums to zero when $l = 0, m = k$ and for $l = \pm 1, m = k - 1/2$, and the second gives the desired negative coefficients.

finish discussion of quantum numbers of states contributing to the infinite-N index

1.6. **The large N limit.** BW: Use LQT to give a first-principles description of the large N CFT. Discuss relationship to elliptic genus computed above.