1. THE TWISTED SYMMETRIC ORBIFOLD CFT

Supergravity on $AdS_3 \times S^3 \times X$, where X is either T^4 or a $K3$ surface, is expected to be holographically dual to a particular two-dimensional superconformal field theory (SCFT). Though our primary interest in this note is $K3$, with the T^4 case studied in [**?**], we can be agnostic about X for many aspects of the analysis.

We will briefly review this system of interest, following [**?**] and references therein, with a focus towards applying the holomorphic twist to this system and isolate the 1/4-BPS states. Of course, this SCFT is the IR limit of the field theory that arises from the zero modes of the open strings on the D1-D5 branes. The lowest-lying modes of open strings, which provide an effective field theory description of the D1 and D5-branes, naturally furnish a gauge theory whose IR limit we are primarily interested in. One can study the twist, which is insensitive to RG flow, of both the D1-D5 gauge theory or the symmetric orbifold CFT, and we will sketch salient features of both perspectives.

We recall that the D5-D5 strings give rise to a six-dimensional supersymmetric $U(N_5)$ gauge theory preserving 16 supercharges. When all the D-branes are coincident the gauge theory is in the Higgs phase and when some of the adjoint scalars in the field theory acquire a vev, corresponding to transverse separation of the branes, the theory is in the Coulomb phase. We will focus on the Higgs branch of the gauge theory throughout, which involves turning on a nonvanishing Fayet-Iliopoulos parameter (dually, NS B-field). We reduce four directions of the gauge theory on

$$X = T^4 \quad \text{or} \quad K3$$

which results in an effective two-dimensional $U(N_5)$ gauge theory which preserves 16 supercharges. The D1-D1 strings similarly produce a $U(N_1)$ gauge theory preserving 16 supercharges. More interesting are the D1-D5 and D5-D1 strings, which break the total supersymmetry down to 8 supercharges (though more supersymmetries will be obtained in the near-horizon/low energy limits, so that the dual pair of theories has 16 supersymmetries overall). These strings produce matter multiplets transforming in the bifundamental representations of the gauge groups.

On the Higgs branch, one must solve the vanishing of the bosonic potential (i.e. D-flatness equations) modulo the gauge symmetries $U(N_1) \times U(N_5)$ to obtain the moduli space. If one imagined that both sets of D-branes were supported on a noncompact six-dimensional space, these D-flatness equations can be rewritten to reproduce the ADHM equations for N_1 instantons of a six-dimensional $U(N_5)$ gauge theory a la [**?**].

In fact, it has been argued that the instanton moduli space is the more accurate description of the dual field theory, so that one should study the moduli space of N_1 instantons of a $U(N_5)$ gauge theory on X, for which a useful model is the Hilbert scheme of $N_1 N_5$ points on X [1]. The (conformally invariant limit of the) gauge theory description is expected to only capture the regime of vanishing size instantons (i.e. when the hypermultiplets have small vevs). One can understand that the gauge theory description is approximate by noticing that the Yang-Mills couplings are given in terms of the X volume V and string coupling as $g_1^2 = g_s(2\pi\alpha')$, $g_5^2 = g_s V/(\alpha'(2\pi)^3)$ so for energies much smaller than the inverse string length the gauge theories are strongly coupled [**?**].

To get the SCFT we take an IR limit, which would be dual to a near-horizon limit from the closed string point of view. In this limit, the gauge theory moduli space becomes the target space of the low-energy sigma-model. It has been argued that the correct instanton moduli space is a smooth deformation of the symmetric product theory $Sym^{N_1 N_5}(\tilde{X})/S_{N_1 N_5}$ [2]. Indeed, there is a point in the SCFT moduli space (far from the supergravity point itself) where the theory takes precisely the symmetric orbifold form. The orbifold point is the analogue of free Yang-Mills theory in the perhaps more-familiar $AdS_5 \times S^5/$ 4d $\mathcal{N} = 4$ SYM duality, and is dual to a stringy point in moduli space which has been explored extensively in recent years [**?**].

As usual, one can focus on moduli-independent quantities to provide preliminary matches the supergravity and orbifold points, such as the signed count of 1/4-BPS states at large-N, via the elliptic genus, which matches the corresponding count of (twisted or BPS) supergravity states [**?**]. In principle, this computation can also be done from the twisted CFT perspective using the twist of the symmetric orbifold CFT. This is also known as the half-twist [**?**, **?**]; in particular, the half-twist of a sigma-model can be mathematically formulated as the chiral de Rham complex [**?**, **?**], which is precisely what our holomorphic twist captures, though obtaining a global description of the half-twist on a curved manifold is a nonperturbative computation, and so prohibitively difficult with current technology. We will review some aspects of the twist from the perspective of the UV worldvolume gauge theory, to which we now turn.

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[1] Throughout this note we ignore the center of mass factor of the moduli space that produces a $\tilde{X}$ factor, for some $\tilde{X}$ not necessarily the same as the compactification X. The relationship between the two manifolds in the T^4 case is clarified in [**?**].

[2] Here we are taking both N_1, N_5 large.

1.1. **Branes in twisted supergravity.** We have already recollected the proposal of [**?**] that the twist of type IIB supergravity is equivalent to the topological B-model on a Calabi–Yau fivefold. At the level of branes, this proposal further asserts that D_{2k-1}-branes in type IIB corresponds to topological B-branes. We use that perspective here to deduce the worldvolume CFT of the twist of the $D1/D5$ system in type IIB supergravity.

We consider the system of $D1/D5$ branes in the twist of type IIB on a Calabi–Yau five-fold Z. For simplicity, we assume that we have a collection of $N_1 = N$ $D1$ branes supported along a closed Riemann surface

$$\Sigma \subset Z$$

together with a single $D5$ brane which is parallel to the $D1$ branes.

In topological string theory, one views branes as objects in some category. Morphisms between objects represent open strings stretching between two branes. In particular, a general feature of topological string theory is that the open string fields which start and end on the same brane can be described in terms of the algebra of derived endomorphisms of the object representing the brane. Indeed, following [**?**], one constructs a Chern–Simons theory based off of this derived algebra of endomorphisms where the gauge fields are degree one elements in the algebra of derived endomorphisms. In the B-model, the category is the category of coherent sheaves on the Calabi–Yau manifold. Fields of the corresponding open-string field theory (which start and on on the same brane) are given as holomorphic sections of the sheaf of derived endomorphisms. Following [**?**], we will use a Dolbeault model which resolves a sheaf of holomorphic sections to describe the space of fields as the cohomological shift by one of the Dolbeault resolutions of derived endomorphsism.

We consider $D1$ branes that are a sum of simple branes labeled by the structure sheaf $\mathcal{O}_\Sigma$. In particular N such $D1$ branes are represented by the object $\mathcal{O}_\Sigma^{\oplus N}$ in the category of quasi-coherent sheaves on the Calabi–Yau fivefold Z. A model for the sheaf of derived endomorphisms of $\mathcal{O}_\Sigma$ is the holomorphic sections of the exterior algebra of the normal bundle $\mathcal{N}_\Sigma$ of Σ in Z. A model for the sheaf of derived endomorphisms of a stack of N such branes is therefore

$$\mathrm{Ext}_{\mathcal{O}_Z}\left(\mathcal{O}_\Sigma^{\oplus N}\right) \simeq \mathfrak{gl}(N) \otimes \wedge^\bullet \mathcal{N}_\Sigma. \tag{1.1.1}$$

Thus, the Dolbeault model for the open string fields which stretch between two such $D1$ branes is given by

$$\Omega^{0,\bullet}\left(\Sigma, \mathfrak{gl}(N) \otimes \wedge^\bullet \mathcal{N}_\Sigma\right)[1]. \tag{1.1.2}$$

If we take X to the be the total space of the bundle $\mathcal{N}_\Sigma$ then the Calabi–Yau condition requires $\wedge^4 N_\Sigma = K_\Sigma$. In the case $\Sigma = \mathbf{C}$ and $Z = \mathbf{C}^5$ we can write the open string fields (1.1.2) as

$$\Omega^{0,\bullet}\left(\mathbf{C}, \mathfrak{gl}(N)[\varepsilon_1, \ldots, \varepsilon_4]\right)[1]. \tag{1.1.3}$$

Here the ε_i are odd variables that carry spin $1/4$, meaning they transform as constant sections of the bundle $K_{\mathbf{C}}^{1/4}$. This is precisely the field content of the holomorphic twist of two-dimensional $\mathcal{N} = (8,8)$ pure gauge theory which is the worldvolume theory living on a stack of $D1$ branes in twisted supergravity on flat space.

Next, we consider $D1 - D5$ strings. The open string fields are given by

$$\Omega^{0,\bullet}\left(\Sigma, \underline{\mathrm{Ext}}_{\mathcal{O}_X}\left(\mathcal{O}_Z, \mathcal{O}_\Sigma^{\oplus N}\right)\right). \tag{1.1.4}$$

Again, on flat space with $\Sigma = \mathbf{C}$ this can be written in a more explicit way as

$$\Omega^{0,\bullet}\left(\mathbf{C}, K_{\mathbf{C}}^{1/2}[\varepsilon_3, \varepsilon_4]\right) \otimes \mathrm{Hom}(\mathbf{C}, \mathbf{C}^N) = \Omega^{0,\bullet}\left(\mathbf{C}, K_{\mathbf{C}}^{1/2}[\varepsilon_3, \varepsilon_4]\right) \otimes \mathbf{C}^N. \tag{1.1.5}$$

Together with the $D5 - D1$ strings we get

$$\Omega^{0,\bullet}\left(\mathbf{C}, K_{\mathbf{C}}^{1/2}[\varepsilon_3, \varepsilon_4]\right) \otimes T^*\mathbf{C}^N. \tag{1.1.6}$$

In total, we see that the open-strings of the $D1/D5$ system along $\Sigma = \mathbf{C}$ are given by the Dolbeault complex valued in the following holomorphic vector bundle

$$\left(\mathfrak{gl}(N)[\varepsilon_1, \varepsilon_2][1] \oplus K_{\mathbf{C}}^{1/2} \otimes T^*\mathbf{C}^N\right) \otimes \mathbf{C}[\varepsilon_3, \varepsilon_4]. \tag{1.1.7}$$

If we choose twisting data so that the odd variables carry degree $\deg \varepsilon_1 = \deg \varepsilon_2 = +1$ then the bundle in parentheses can be written as

$$\mathfrak{gl}(N)[1] \oplus K_\Sigma^{1/2} \otimes T^*\left(\mathfrak{gl}(N) \oplus \mathbf{C}^N\right) \oplus \mathfrak{gl}(N)[-1]. \tag{1.1.8}$$

The first summand represents the ghosts of the holomorphic CFT and the last summand the anti-ghosts. The gauge symmetry in the middle term is induced from the standard action of $\mathfrak{gl}(N)$ on $T^*\left(\mathfrak{gl}(N) \oplus \mathbf{C}^N\right)$ by Hamiltonian vector fields (this is induced from the adjoint + fundamental action on the base of the cotangent bundle). Thus, we see that this model describes ($K_\Sigma^{1/2}$-twisted) holomorphic maps from Σ into the well-known GIT description of the symmetric orbifold $\mathrm{Sym}^N \mathbf{C}^2$. That is, the worldvolume theory living on a stack of twisted $D1$ branes is the holomorphic σ-model of maps into the target $\mathrm{Sym}^N \mathbf{C}^2$.

This analysis happened entirely in flat space. The $D1$ branes wrapped

$$\mathbf{C} \times 0 \times 0 \times 0 \times 0 \subset \mathbf{C}^5 \tag{1.1.9}$$

while the $D5$ brane wrapped

$$\mathbf{C} \times \mathbf{C}^2 \times 0 \times 0 \subset \mathbf{C}^5. \tag{1.1.10}$$

If we instead replace this $\mathbf{C}^2$ by a compact Calabi–Yau twofold X then the above computation leads us to the well-established expectation that the worldvolume theory, after twisting, is a holomorphic σ-model with target $\mathrm{Sym}^N X$. BW: elaborate

1.2. **The elliptic genus.** We now briefly recall the elliptic genus computation using the orbifold point in the string moduli space, which reproduces signed counts of 1/4-BPS states in the SCFT. This is formally equal to the partition function of the chiral de Rham complex, or holomorphically twisted theory on the same underlying space. We will take the branes to be supported on $\mathbf{R} \times S^1$ after compactification on X, so that the CFT is defined on the cylinder. On the cylinder, the NS sector corresponds to anti-periodic boundary conditions on the fermions. The sigma model is then the $\mathcal{N} = (4,4)$ theory whose bosonic fields are valued in maps from $S^1 \to Sym^N(X)$.

The physical SCFT has R-symmetries $SO(4) \simeq SU(2)_L \times SU(2)_R$ dual to rotations of the S^3 and symmetries under a global $SO(4)_I \simeq SU(2)_a \times SU(2)_b$ of transverse rotations; this latter symmetry is broken by compactification on X. The latter $SO(4)_I$, although broken by the background, is still often used to organize the field content of the compactified, and acts as an outer automorphism on the $\mathcal{N} = (4,4)$ superconformal algebra. As is well known, the isometries of $AdS_3 \times S^3$ are $SL(2,\mathbf{R}) \times SL(2,\mathbf{R}) \times SO(4)$ form the bosonic part of the supergroup $SU(1,1|2) \times SU(1,1|2)$ which preserve the supergravity vacuum and form the anomaly-free global subalgebra of the $\mathcal{N} = (4,4)$ superconformal algebra.

Part of the underlying chiral algebra of the $\mathcal{N} = (4,4)$ SCFT OPEs is the usual holomorphic $\mathcal{N} = 4$ superconformal algebra with $c = 6N$ (which can be explicitly constructed as a diagonal sum over the N copies of the seed $c = 6$ sigma models):

$$\begin{aligned}
J^a(z)J^b(w) &\sim \frac{c}{12}\frac{\delta^{ab}}{(z-w)^2} + i\epsilon^{ab}_c \frac{J^c(w)}{z-w} \\
J^a(z)G^{\alpha A}(w) &\sim \frac{1}{2}(\sigma^{*a})^\alpha_\beta \frac{G^{\beta A}(w)}{z-w} \\
G^{\alpha A}(z)G^{\beta B}(w) &\sim -\epsilon^{AB}\epsilon^{\alpha\beta}\frac{T(w)}{z-w} - \frac{c}{3}\frac{\epsilon^{AB}\epsilon^{\alpha\beta}}{(z-w)^3} + \epsilon^{AB}\epsilon^{\beta\gamma}(\sigma^{*a})^\alpha_\gamma \left(\frac{2J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{z-w}\right) \\
T(z)J^a(w) &\sim \frac{J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{z-w} \\
T(z)G^{\alpha A}(w) &\sim \frac{\frac{3}{2}G^{\alpha A}(w)}{(z-w)^2} + \frac{\partial G^{\alpha A}(w)}{z-w} \\
T(z)T(w) &\sim \frac{c}{2}\frac{1}{(z-w)^4} + 2\frac{T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}.
\end{aligned}$$

Above, we have written $SU(2)_a \times SU(2)_b$ doublet indices as $A, \dot{B}$ and $SU(2)_L \times SU(2)_R$ doublet indices as $\alpha, \dot{\beta}$.

There is, of course, also a right-moving copy in the full SCFT, though only the chiral half above will be accessible in the holomorphic twist.

It is easy from the above OPEs to get the mode algebra of the $\mathcal{N} = 4$ superconformal algebra. For simplicity, we will just record the mode algebra of the global subalgebra generated by $\left\{J^a_0, G^{\alpha A}_{\pm 1/2}, L_0, L_{\pm 1}\right\}$, which has its Cartan subalgebra generated by J^3_0, L_0:

$$[L_0, L_{\pm 1}] = \mp L_\pm \tag{1.2.1}$$

$$[L_1, L_{-1}] = 2L_0 \tag{1.2.2}$$

$$\left[J^a_0, J^b_0\right] = i\epsilon^{ab}_c J^c_0 \tag{1.2.3}$$

$$\left\{G^{\alpha A}_{1/2}, G^{\beta B}_{-1/2}\right\} = \epsilon^{AB}\epsilon^{\beta\gamma}(\sigma^{*a})^\alpha_\gamma J^a_0 - \epsilon^{AB}\epsilon^{\alpha\beta}L_0 \tag{1.2.4}$$

$$\left\{G^{\alpha A}_{-1/2}, G^{\beta B}_{1/2}\right\} = -\epsilon^{AB}\epsilon^{\beta\gamma}(\sigma^{*a})^\alpha_\gamma J^a_0 - \epsilon^{AB}\epsilon^{\alpha\beta}L_0 \tag{1.2.5}$$

$$\left[L_0, G^{\alpha A}_{\pm 1/2}\right] = \mp G^{\alpha A}_{\pm 1/2} \tag{1.2.6}$$

$$\left[L_1, G^{\alpha A}_{1/2}\right] = \left[L_{-1}, G^{\alpha A}_{-1/2}\right] = 0 \tag{1.2.7}$$

$$\left[L_{\pm 1}, G^{\alpha A}_{\mp 1/2}\right] = \pm G^{\alpha A}_{\pm 1/2} \tag{1.2.8}$$

$$\left[J^a_0, G^{\alpha A}_{\pm n}\right] = \frac{1}{2}(\sigma^{*a})^\alpha_\beta G^{\beta A}_{\pm n} \tag{1.2.9}$$

These commutators generate $\mathfrak{psu}(1,1|2)$. Notice that there is no anomaly $c = 6N$ in the global subalgebra.

Later in the text, we will reproduce the chiral algebra of the symmetric orbifold SCFT using the method of Koszul duality. The global subalgebra can be readily reproduced at tree-level in the planar limit; to obtain the centrally extended chiral algebra, including the holomorphic $\mathcal{N} = 4$ superconformal algebra, we will incorporate a certain class of planar Feynman diagrams sensitive to the backreaction.

It is difficult to perform explicit computations in the holomorphic twist beyond a local (flat space) model, even for a single copy of X. Rather than try to work with the full chiral de Rham complex directly, we will outline the matching of (counts of) states between twisted supergravity and twisted CFT (via the elliptic genus). Then we will turn to the determination of the OPEs in the holomorphically twisted theory in the $N \to \infty$ limit by applying Koszul duality to our twisted supergravity theory.

Consider the chiral half of the $\mathcal{N} = (4,4)$ σ-model on the symmetric orbifold $\mathrm{Sym}^N X$ where X is T^4 or a $K3$ surface. After performing the half-twist, this is all that remains of the supersymmetric σ-model. According to **[?]** we can regard the direct sum of the vacuum modules of the chiral algebras of $\mathrm{Sym}^N X$, for each N, as being itself a Fock space. The generators of this Fock space are given by the single string states. These single string states are the analog of single trace operators in a gauge theory, and will ultimately be matched with single-particle states in the holographic dual.

Let us brefly recall the computation of the elliptic genus. Let $c(n,m)$ be the superdimension of the space of operators in supersymmetric σ-model into X, which are of weight n under L_0 and of weight m under the action of the Cartan of $SU(2)_R$. Let q, y be fugacities for L_0 and the Cartan of $SU(2)_R$, respectively—the elliptic genus $\chi(X;q,y)$ is a series in these variables. Of course, for $X = T^4$ the elliptic genus vanishes [3], so we will now fix $X = K3$.

Introducing another parameter p, which keeps track of the symmetric power, we can consider the generating series

$$\sum_{n\geq 0} p^n \chi(\mathrm{Sym}^n X; q, y) \tag{1.2.10}$$

[3]One could instead consider the modified elliptic genus for T^4, which is enriched with additional insertions of the fermion number operator to absorb the fermionic zero modes.

The main result of [?, ?] is an expression for this generating series

$$\sum_n p^n \chi(\mathrm{Sym}^n X; q, y) = \prod_{l,m\geq 0, n>0} \frac{1}{(1-p^n q^m y^l)^{c(nm,l)}} \tag{1.2.11}$$

where $c(m,l)$ is a function of the quantity $4m-l^2$. In other words, we can interpret the direct sum of the vacuum modules of the $\mathrm{Sym}^n X$ σ-models as being the Fock space generated by a trigraded super-vector space

$$V = \oplus_{n\geq 0, m, l} V_{n,m,l} \tag{1.2.12}$$

where the super-dimension of $V_{n,m,l}$ is $c(nm,l)$.

Setting $V_n = \oplus_{m,l} V_{n,m,l}$, we see that V_n is isomorphic to the vacuum module of the $bc\beta\gamma$ system, or chiral de Rham complex, on the original surface X, except with a different conformal structure: a state of the σ-model into X of spin k is of spin k/n in V_n.

The states in V_N will play the role of the single-trace operators in the large N limit of the $\mathrm{Sym}^N X$ σ-model. These states can be understood geometrically as follows—let us focus just on the S^1-modes of this σ-model. A map $S^1 \to \mathrm{Sym}^N X$ is the same as an N-fold cover $M \to S^1$ together with a map $M \to X$. Therefore, the Hilbert space of the σ-model on $\mathrm{Sym}^N X$ decomposes over sectors corresponding to the topological type of this N-fold cover, which are labelled by partitions of N. The single string sector is the sector that corresponds to M being connected. This means that the monodromy of the cover $M \to S^1$ is conjugate to the length N cycle of type $(1\dots N)$ in the symmetric group S_N.

Since the N-fold cover of S^1 corresponding to the single trace sector is connected, the Hilbert space of the single-trace sector is isomorphic to that of the original σ-model into X. However, the conformal structure is different—a rotation along S^1 in this σ-model rotates the total space $1/N$ times. This tells us that an operator in the single-trace sector carries spin $1/N$ times that of the corresponding state of the original σ-model. The projection onto $\mathbb{Z}_N$-invariant states ultimately restores integrality of the spin. In particular, the generating function of elliptic genera of $\mathrm{Sym}^N X$ decomposes as

$$\sum_{N\geq 0} p^N \chi(\mathrm{Sym}^N X; q, y) = \prod_{n>0} \sum_{N\geq 0} p^{nN} \chi(\mathrm{Sym}^N \mathcal{H}_{(n)}^{\mathbb{Z}_n}; q, y) \tag{1.2.13}$$

with $\sum_{N\geq 0} p^{nN} \chi(\mathrm{Sym}^N \mathcal{H}_{(n)}^{\mathbb{Z}_n}; q, y) = \prod_{l,m\geq 0} \frac{1}{(1-pq^m y^l)^{c(mn,l)}}$. Here, $\mathcal{H}_{(n)}$ is the Hilbert space of a single long string on X of length n with winding number $1/n$.

We can extract the $N \to \infty$ limit of this expression, following the logic employed in [?, ?, ?]. First, in preparation for comparison to supergravity, we perform spectral flow[4] to the NS sector:

$$\sum_{N\geq 0} p^N \chi_{NS}(\mathrm{Sym}^N X; q, y) = \sum_{N\geq 0} p^N \chi(\mathrm{Sym}^N X; q, y\sqrt{q}) y^N q^{N/2}$$
$$= \prod_{\substack{n\geq 0 \\ m\geq 0, m\in\mathbb{Z} \\ l\in\mathbb{Z}}} \frac{1}{(1-p^n q^{m+l/2+n/2} y^{l+n})^{c(nm,l)}}$$
$$= \prod_{\substack{n\geq 0 \\ m'\geq |l'|/2,\ 2m'\in\mathbb{Z}_{\geq 0} \\ l'\in\mathbb{Z},\ m'-l'/2\in\mathbb{Z}_{\geq 0}}} \frac{1}{(1-p^n q^{m'} y^{l'})^{c(nm'-nl'/2, n-l')}}.$$

At any power of q, there will be contributions from terms of the form $\frac{1}{(1-py^{l'})^{c(-l'/2, l'-1)}}$. The only nonvanishing such term in our case when $m' = 0$ is $\frac{1}{(1-p)^2}$. We wish to isolate the coefficients of all terms of the form $q^a y^b p^N$ for $a \ll N$. Taylor expanding $\frac{1}{(1-p)^2}$ and extracting the desired coefficient gives $Nh(a,b) + \mathcal{O}(N^0)$ where $h(a,b)$ is the coefficient of $q^a y^b$ in

$$\prod_{\substack{m'\geq |l'|/2,\ 2m'\in\mathbb{Z}_{\geq 0} \\ l'\in\mathbb{Z},\ m'-l'/2\in\mathbb{Z}_{\geq 0}}} \frac{1}{(1-q^{m'} y^{l'})^{f(m',l')}}$$

with $f(m',l') := \sum_{n>0} c(n(m'-l'/2), l'-n)$. The coefficients $c(M,L)$ vanish for $4M - L^2 < -1$ so for $m' \geq 1$ the sum truncates to $f(m',l') = \sum_{n=1}^{4m'} c(n(m'-l'/2), l'-n)$.

Hence, we can get a finite contribution upon dividing by N.

We can also write out the non-vanishing $f(m',l')$ more explicitly, recalling that the coefficients are constrained to lie in the following range of the Jacobi variable: $-2m' \leq l' \leq 2m', l' \equiv 2m' \mod 2$. Reproducing the elementary manipulations in Appendix A of [?] (in particular, using the fact that $c(N,L)$ depends only on $4N - L^2$ and L mod 2) allows us to rewrite the sum as

$$f(m',l') = \left(\sum_{\tilde{n}\in\mathbb{Z}} c(m'^2 - l'^2/4, \tilde{n})\right) - c(0,l'), \tag{1.2.14}$$

where $n' := n - 2m$ in the first term. The first term is non-vanishing only when $l' = \pm 2m'$ and then it reduces to the Witten index of K3, i.e. $f(m', \pm 2m') = 24$ for general m'. Otherwise, we have $f(m',l') = -c(0,l')$. When $m' \in \mathbb{Z}$ the nonvanishing

[4]We shift the overall power of q by $q^{c/24}$ so that the vacuum occurs at q^0.

such term is $-c(0,0) = -20$, and when $m' \in \mathbb{Z} + 1'/2$ we have $-c(0,1) = -2$ and $-c(0,-1) = -2$.

In sum, we obtain

$$\lim_{N\to\infty} \frac{\chi_{NS}(\mathrm{Sym}^N X; q, y)}{N} = \prod_{k\geq 1} \frac{(1-q^k)^{20}(1-q^{k-1/2}y^{-1})^2(1-q^{k-1/2}y)^2}{(1-q^{k/2}y^k)^{24}(1-q^{k/2}y^{-k})^{24}} \tag{1.2.15}$$

$$= 1 + \left(\frac{22}{y} + 22y\right) q^{1/2} + \left(\frac{277}{y^2} + 464 + 277y^2\right) q + \mathrm{O}(q^{3/2}). \tag{1.2.16}$$

We will denote this large N limit by $\chi_{NS}(\mathrm{Sym}^\infty X; q, y)$. In particular, for there are two bosonic towers corresponding to (anti)chiral primary states and three fermionic towers corresponding to (derivatives of) the states capturing the cohomology of a single copy of K3. At $k = 1$, there is a cancellation to $\frac{(1-q)^{20}}{(1-q^{1/2}y)^{22}(1-q^{1/2}y^{-1})^{22}}$.

We observe that this expression for the large N limit of the elliptic genus agrees exactly with the plethystic exponential of the single particle twisted supergravity index we computed in (**??**). One can easily see this by using the definition of the plethystic exponential

$$\mathrm{PE}[f](q, y) = \exp\left(\sum_{k=1}^{\infty} \frac{f(q^k, y^k)}{k}\right) \tag{1.2.17}$$

and rewriting the infinite-N elliptic genus as $\mathrm{PE}[f_{CFT}](q, y)$ in terms of the function

$$f_{CFT}(q, y) = \sum_{m=1}^{\infty} 24(q^{1/2}y)^m + 24(q^{1/2}y^{-1})^m - 20q^m - 2q^{m-1/2}y - 2q^{m-1/2}y^{-1}, \tag{1.2.18}$$

which can be immediately matched with $\mathrm{PE}[f_{sugra}](q, y)$. This is significant evidence that our twisted supergravity theory is dual to (a point on the moduli space of) the large N limit of the symmetric orbifold CFT.

We can reinstate some of the commented-out text below this line if and only if it is useful when discussing the match to the LQT approach.

1.3. **The large N limit.** BW: Use LQT to give a first-principles description of the large N CFT. Discuss relationship to elliptic genus computed above.