ONE-DIMENSIONAL CHERN-SIMONS AND SHEAVES OF DIFFERENTIAL OPERATORS

CONTENTS

1. INTRODUCTION

2. (TWISTED) DIFFERENTIAL OPERATORS VIA FORMAL GEOMETRY

The purpose of this section is to describe twisted differential operators using the language of Gelfand-Kazhdan formal geometry.

2.1. A quick review of descent.

2.1. A quick review of descent. Our approach to studying topological mechanics relies on a description of the target smooth manifold in the style of formal geometry. As we have discussed, our method is to consider topological mechanics on the *formal disk* that is equivariant for the action of all formal automorphisms. To obtain topological mechanics on a smooth manifold X we apply *Gefland-Kazhdan descent* to this theory on the formal disk.

We will thus rely on a description of twisted differential operators on X in the style of formal geometry. Our formulation is largely motivated by the works of Kontsevich [**?**], Fedosov [**?**], and Nest-Tsygan [**?**] n their approaches to deformation quantization and the algebraic index theorem. It also bears a close relationship to approaches made later on by Cattaneo-Felder-Tomassini [**?**, **?**]. Because of the multitude of references already out there, we make our discussion brief and to the point.

The setting of formal geometry that we consider has its foundations in the work of Gelfand and Kazhdan [**?**]. Categorically, the outcome is a construction of "natural" objects in differential geometry in a functorial way from objects on the formal disk. By natural, we mean that these construction apply uniformly over the category of smooth manifolds (of a fixed dimension) possibly equipped with a local geometry (such as a symplectic or complex structure). Our construction of the one-dimensional theory via Gelfand-Kazhdan descent is very close to the last two author's construction of *chiral differential operators* from BV quantization in [**?**].

The central object, associated to each smooth n-manifold X, is the bundle of coordinates X^{coor}. The fiber over a point $x \in X$ consists of the space of formal coordinates centered at x. This is the space of ∞-jets of local diffeomorphisms of the n-disk D^n to an open neighborhood of x sending $0 \in D^n$ to x. In other words, this is the space of maps from the *formal disk* $\widehat{D}^n$ to the formal completion of X near the point x. The formal disk $\widehat{D}^n$ is defined as the affine formal scheme with ring of functions given by formal power series $\mathbb{R}[[t_1, \ldots, t_n]]$. The bundle $X^{coor} \to X$ is a principal bundle for the group of automorphisms of this formal scheme that fix the maximal ideal, which we denote Aut_n. This principal Aut_n-bundle factors through the ordinary GL_n frame bundle $\mathrm{Fr}_X \to X$, in since there is a natural map of bundles $X^{coor} \to \mathrm{Fr}_X$ that truncates the full ∞-jet to remember only the linear jet.

The crucial point is that this principal Aut_n-bundle carries a *flat connection*. The flat connection is valued in the Lie algebra of *all* formal vector fields W_n. The Lie algebra of Aut_n consists of formal vector fields that vanish at the origin, which is a strict sub Lie algebra of all formal vector fields, so we are leaving the ordinary category of principal bundles with connection.

2.1.1. *Formal to global.* What we call Gelfand-Kazhdan descent is the functorial prescription of taking objects on the formal disk and gluing them over an arbitrary smooth manifold. The descent functor takes as input a geometric object (like a vector bundle) on the formal disk and outputs a universal object on the category of smooth manifolds.

Consider the map that projects onto the linear jet of the formal coordinate

$$\pi_1 : X^{coor} \to \mathrm{Fr}_X.$$

Splittings of this map that are equivariant for the group GL_n of this map always exist. Locally, such a splitting amounts to exponentiating a linear frame to a full formal coordinate.

Remark 2.1. Every BW: torsion free connection ∇ on X defines a global exponentiation and hence a splitting $\sigma_\nabla : \mathrm{Fr}_X \to X^{coor}$ of π_1. BW: reference?

The flat connection on X^{coor} determines an element $\omega \in \Omega^1(X^{coor}; \mathrm{W}_n)$. Using the splitting, we can pull-back this flat connection to obtain a GL_n-invariant element $\omega_\nabla := \sigma_\nabla^* \omega^{coor} \in \Omega^1(\mathrm{Fr}_X; \mathrm{W}_n)$ satisfying

$$\mathrm{d}\omega_\nabla + \frac{1}{2}[\omega_\nabla, \omega_\nabla] = 0$$

where d is the de Rham differential on Fr_X and $[-,-]$ is the Lie bracket of formal vector fields extended to the de Rham complex.

To define descent, we need to fix a category of modules for formal vector fields and GL_n. The class of objects we consider are GL_n-representations $\mathcal{V}$ (possibly infinite dimensional) that have a compatible structure of a topological W_n-module. We denote this category of modules by $\mathrm{Mod}_{(\mathrm{W}_n, \mathrm{GL}_n)}$. Precisely, these are modules for the Harish–Chandra pair $(\mathrm{W}_n, \mathrm{GL}_n)$, whose underlying W_n-module structure is continuous.

We want to consider modules that behave like vector bundles on the formal disk. Let $\widehat{\mathcal{O}}_n = \mathbb{C}[[t_1, \ldots, t_n]]$ denote the ring of formal power series. If $\mathcal{V}$ is a $(\mathrm{W}_n, \mathrm{GL}_n)$-module we require that it admits a GL_n-equivariant splitting

$$\mathcal{V} = \widehat{\mathcal{O}}_n \otimes_{\mathbb{C}} V$$

where GL_n acts on $\widehat{\mathcal{O}}_n$ in the natural way, and V is some finite dimensional GL_n-representation. We refer to such objects as *formal vector bundles*. They form a category that we denote VB_n.

We denote the category of smooth finite dimensional vector bundles on X by VB_X. Let $\mathrm{Pro}(\mathrm{VB}_X)$ denote the category of pro-vector bundles. An object in $\mathrm{Pro}(\mathrm{VB}_X)$ consists of a sequence of vector bundles BW: Did we flip pro and ind objects in the CDO paper?

We are now ready to define the descent functor.

Definition 2.2. Let X be a smooth manifold equipped with a connection. Define the functor

$$\mathrm{desc}_X : \mathrm{VB}_n \to \mathrm{Pro}(\mathrm{VB}_X))_{flat}$$

that sends a formal vector bundle $\mathcal{V} = \widehat{\mathcal{O}}_n \otimes V$ to the pro vector bundle...BW: I'm worried I'm mixing up pro and ind

Given a flat vector bundle we can take its flat sections to obtain a sheaf on X. For the flat bundle $\mathrm{desc}_X(\mathcal{V})$ we denote this sheaf by $\mathcal{D}\mathrm{esc}_X(\mathcal{V})$. This sheaf is automatically locally free and hence corresponds to the sheaf of sections of a finite dimensional vector bundle.

2.1.2. *Characteristic classes.* An exact sequence of formal vector bundles

$$\mathcal{A} \to \mathcal{B} \to \mathcal{C}$$

is classified by an element in the ext-group $\mathrm{Ext}^1_{\mathrm{VB}_n}(\mathcal{C}, \mathcal{A})$. In **[?]** we show that for each k the ext-group $\mathrm{Ext}^k_{\mathrm{VB}_n}(\mathcal{C}, \mathcal{A})$ is canonically identified with the *relative Chevalley–Eilenberg cochains* $\mathrm{C}^k_{\mathrm{Lie}}(W_n, \mathrm{GL}_n; \mathcal{C}^\vee \otimes_{\widehat{\mathcal{O}}_n} \mathcal{A})$, where the $(-)^\vee$ denotes the $\widehat{\mathcal{O}}_n$-linear dual.

In Lemma 2.28 of [**?**] we prove that the functor $\mathcal{D}\mathrm{esc}_X(-)$ is exact. In particular, applied to the exact sequence **??** is an exact sequence of formal vector bundles we obtain a map

$$\mathrm{char}_X : \mathrm{C}^k_{\mathrm{Lie}}(W_n, \mathrm{GL}_n; \mathcal{C}^\vee \otimes_{\widehat{\mathcal{O}}_n} \mathcal{A}) \to H^k(X; \mathcal{D}\mathrm{esc}_X(\mathcal{C})^\vee \otimes_{\mathcal{O}_X} \mathcal{D}\mathrm{esc}_X(\mathcal{A})).$$

that we call the *characteristic map*.

2.2. **Formal Lie algebroids.** In this section we define formal Lie algebroids. These are Lie algebroids on the formal disk, in the same way that the category VB_n provides a model for vector bundles on the formal n-disk.

Definition 2.3. A *formal Lie algebroid* is an object $\mathcal{L} \in \mathrm{VB}_n$ together with

(i) A $\mathbb{C}$-linear bracket $[-,-] : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ and
(ii) an "anchor" map $a : \mathcal{L} \to \widehat{\mathcal{T}}_n$

such that

(1) $[-,-]$ and a are equivariant for the action of $(\mathrm{W}_n, \mathrm{GL}_n)$,
(2) a is $\widehat{\mathcal{O}}$-linear, and
(3) for any $x, y \in \mathcal{L}$ and $f \in \widehat{\mathcal{O}}_n$ one has

$$[x, f \cdot y] - f \cdot [x, y] = a(x) \cdot f.$$

Remark 2.4. Formal Lie algebroids are precisely Lie–Rinehart algebras over the ring $\widehat{\mathcal{O}}_n$ that are equivariant for the pair $(\mathrm{W}_n, \mathrm{GL}_n)$.

We define the category Algd_n to be the full subcategory of VB_n consisting of formal Lie algebroids. The following is an immediate consequence of the functoriality of descent.

Lemma 2.5. *Suppose $\mathcal{L}$ is a formal Lie algebroid. Then, $\mathcal{D}\mathrm{esc}_X(\mathcal{L})$ has the structure of a Lie algebroid on X.*

It is immediate to see that the descent of the formal Lie algebroid $\widehat{\mathcal{T}}_n$ is the standard Lie algebroid.

2.2.1. *Enveloping algebroid.* To any Lie algebra we can assign an associative algebra through its universal enveloping algebra. BW: blah blah

Let $\mathcal{L}$ be a formal Lie algebroid. Consider the left $\widehat{\mathcal{O}}_n$-module $\widehat{\mathcal{O}}_n \oplus \mathcal{L}$. This direct sum has a natural Lie algebra structure given by

$$[(f, x), (g, y)] := (a(x) \cdot f - a(y) \cdot g, [x, y]).$$

Let $U_{\mathrm{Lie}}(\widehat{\mathcal{O}}_n \oplus \mathcal{L})$ be the universal envelope of this Lie algebra. Note that there is a natural augmentation map $\epsilon : U_{\mathrm{Lie}}(\widehat{\mathcal{O}}_n \oplus \mathcal{L}) \to \mathbb{R}$.

Definition 2.6. Let $\mathcal{L}$ be a formal Lie algebroid. Denote by $\overline{U}$ the augmentation ideal with respect to ϵ, and let $I \subset \overline{U}$ be the ideal generated by

$$i(f,0) \otimes i(g,x) - i(fg,fx)$$

for $x \in \mathcal{L}$ and $f,g \in \widehat{\mathcal{O}}_n$. The *enveloping algebra* of the formal Lie algebroid $\mathcal{L}$ is the quotient

$$\mathcal{U}_{\widehat{\mathcal{O}}_n}(\mathcal{L}) := \overline{U}/I.$$

It is immediate to see that $\mathcal{U}_{\widehat{\mathcal{O}}_n}(\mathcal{L})$ is an associative algebra object in the category of formal vector bundles. By monoidality of the descent functor, we $\mathcal{D}\mathrm{esc}_{\mathrm{GK}}(\mathcal{L})$ is a sheaf of associative algebras on the category of n-manifolds.

Example 2.7. Consider the $\widehat{\mathcal{O}}_n$-module $\widehat{\mathcal{T}}_n$, which has a natural formal Lie algebroid structure via the Lie bracket of vector fields. The anchor map is the identity map. It is straightforward to exhibit an isomorphism $\mathcal{U}_{\widehat{\mathcal{O}}_n}(\widehat{\mathcal{T}}_n) \cong \widehat{A}_n$ of algebras in the category VB_n.

We have already mentioned that $\mathcal{D}\mathrm{esc}$ is exact, it in particular preserves finite colimits. An immediate corrollary of this is the following.

Corollary 2.8. *Let $\mathcal{L}$ be a formal Lie algebroid and X a smooth manifold. Then, there is an isomorphism of sheaves of associative algebras on X:*

$$\mathcal{D}\mathrm{esc}_X(\mathcal{U}_{\widehat{\mathcal{O}}_n}\mathcal{L}) \cong \mathcal{U}_{\mathcal{O}_X}(\mathcal{D}\mathrm{esc}(\mathcal{L})).$$

Example 2.9. Let $\widehat{A}_n$ denote the associative Weyl algebra on generators $t_1, \ldots, t_n$ and $\frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_n}$. The relation is $[\frac{\partial}{\partial t_i}, t_j] = \delta_{ij}$. Consider the formal vector bundle $\widehat{\mathcal{T}}_n$, which has a natural formal Lie algebroid structure via the Lie bracket of vector fields. The anchor map is the identity map. It is straightforward to exhibit an isomorphism $\mathcal{U}_{\widehat{\mathcal{O}}_n}(\widehat{\mathcal{T}}_n) \cong \widehat{A}_n$ of algebras in $(\mathrm{W}_n, \mathrm{GL}_n)$-modules.

3. A RAPID RECOLLECTION OF EQUIVARIANT BV

RG: Could be an appendix

3.1. **BV.**

3.2. **Equivariant BV Quantization.** RG: page 72 of CDO. Also, inner versus equivariant QME

3.3. **Observables.**

3.3.1. *Free Theories.*

4. The Formal Theory

OG: This section should also have an introduction that indicates how it is parallel to the purely algebraic construction just described.

We construct one-dimensional Chern-Simons OG: theory with target the formal disk. We will use this theory to study how diffeomorphisms on the target act on the theory, which will be encoded by the structure of W_n-equivariance. OG: This second sentence is befuddling.

We have already seen OG: ? that the formal disk is encoded by the dg Lie algebra $\mathfrak{g}_n = \mathbb{C}^n[-1]$. Consider the dg Lie algebra

$$\mathfrak{g}_n^{\mathbb{R}} := \Omega^*(\mathbb{R}; \mathfrak{g}_n).$$

This dg Lie algebra is abelian and the differential is the de Rham differential d_{dR}. This dg Lie algebra encodes deformations of the constant map to 0 to nearby smooth functions.

The dg Lie algebra that encodes one-dimensional Chern-Simons with target $\widehat{D}^n$ is defined to be the "double"

$$\mathbb{D}\mathfrak{g}_n^{\mathbb{R}} = \Omega^*(\mathbb{R}; \mathfrak{g}_n \oplus \mathfrak{g}_n^\vee[-2]).$$

The Lie bracket is still trivial, and the differential is d_{dR}. This dg Lie algebra encodes the cotangent theory OG: ? that's not widespread terminology and hence should be explained to the elliptic moduli problem OG: also not widespread described by $\mathfrak{g}_n^{\mathbb{R}}$. The pairing is "wedge and integrate". Explicitly, for $\phi, \phi' \in \Omega^*_c(\mathbb{R}; \mathfrak{g}_n)$ and $\psi, \psi' \in \Omega^*_c(\mathbb{R}; \mathfrak{g}_n^\vee[-2])$ we have

$$\langle \phi + \psi, \phi' + \psi' \rangle = \int_{\mathbb{R}} \mathrm{ev}_{\mathfrak{g}_n}(\phi \wedge \psi') + \mathrm{ev}_{\mathfrak{g}_n^\vee}(\psi \wedge \phi').$$

This pairing has cohomological degree -3 and hence determines a classical BV theory OG: no hyphen! BV theory that we call *one-dimensional Chern-Simons with target* $\widehat{D}^n$.

4.1. The W_n action.

4.1.1. We have just introduced the dg Lie algebra $\mathfrak{g}_n$. OG: I find this a weird way to start the subsection. Need to restructure a bit.

Let us denote the generators of $\mathfrak{g}_n$ by $\{\xi_1, \ldots, \xi_n\}$ and the dual generators by $\{t_1, \ldots, t_n\}$. Thus

$$\mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g}_n) = \widehat{\mathrm{Sym}}(\mathfrak{g}_n^\vee[-1]) = \mathbb{C}[[t_1, \ldots, t_n]].$$

There is hence a natural isomorphism

$$\rho^{\mathrm{W}} : \mathrm{W}_n \to \mathrm{Der}(\mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g}_n))$$

sending $f(t_i)\partial_j$ to $f(t_i)\xi_j$. OG: Might help to identify the derivations with something explicit so that these notations make sense.

We can interpret ρ^{W} as expressing an L_∞ action of W_n on $\mathfrak{g}_n$, where

$$\ell_m^W : \mathrm{W}_n \otimes \mathfrak{g}_n^{\otimes m} \to \mathfrak{g}_n$$

has cohomological degree $1-m$ and m ranges over all non-negative integers. These maps are simply the "Taylor components" of ρ_W. For instance, the vector field $\mathfrak{X} = t_1^{m_1} \cdots t_n^{m_n} \partial_j \in \mathrm{W}_n$ acts by zero for any $m \neq m_1 + \cdots + m_n$, and for $m = m_1 + \cdots + m_n$,

$$\ell_m^W \left(\mathfrak{X}, (\xi_1^{\otimes m_1} \otimes \cdots \otimes \xi_n^{\otimes m_n}) \right) = \ell_m^{\mathbf{W}} \left((t_1^{m_1} \cdots t_n^{m_n} \partial_j) \otimes \xi_1^{\otimes m_1} \otimes \cdots \otimes \xi_n^{\otimes m_n} \right) = \xi_j$$

and vanishes on any other basis element $\mathfrak{g}_n^{\otimes m}$.

4.1.2. Let A be a commutative dg algebra. We now show how the dg Lie algebra $A \otimes \mathfrak{g}_n$ inherits a natural L_∞ action of W_n. Here the sequence of maps is

$$\ell_m^{W,A} : \mathrm{W}_n \otimes (A \otimes \mathfrak{g}_n)^{\otimes m} \to A \otimes \mathfrak{g}_n$$

with

$$\ell_m^{W,A}(\mathfrak{X}, (a_1 \otimes x_1) \otimes \cdots \otimes (a_m \otimes x_m)) = \pm (a_1 \cdots a_m) \otimes \ell_m^W(\mathfrak{X}, x_1 \otimes \cdots \otimes x_m),$$

where the sign is determined by Koszul's rule. Equivalently, we can encode the L_∞ action in a Lie algebra map

$$\rho_{W,A} : \mathrm{W}_n \to \mathrm{C}^*_{\mathrm{Lie}}(A \otimes \mathfrak{g}_n, A \otimes \mathfrak{g}_n[-1]),$$

which assembles the ℓ_m^A maps into a "Taylor series." If we set A to be $\Omega^*(\mathbb{R})$, then we obtain an L_∞ action of W_n on $\mathfrak{g}_n^{\mathbb{R}}$. A lift of this action to an L_∞ action of W_n on $\mathbb{D}\mathfrak{g}_n^{\mathbb{R}}$ is uniquely determined by the requirement that the action preserve the pairing of degree -3.

OG: Might be nice to make a small comment about why this construction is relevant (encodes action on a mapping space!

4.2. **The deformation complex.**

Proposition 4.1. *There is a quasi-isomorphism of* W_n*-modules*

$$J : \Omega^1_{n,cl}[1] \xrightarrow{\simeq} (\mathrm{Def}_n)^{\mathbb{R}^\times \times \mathrm{Aff}(\mathbb{R})}.$$

OG: Is it a map of Lie algebras? Thus, taking the invariant part of the W_n*-equivariant deformation complex, we have a quasi-isomorphism*

$$J : \mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n; \widehat{\Omega}^1_{n,cl}[1]) \xrightarrow{\simeq} \left(\mathrm{Def}_n^{\mathrm{W}}\right)^{\mathbb{R}^\times \times \mathrm{Aff}(\mathbb{R})}.$$

4.2.1. *One-forms as local functionals.* We explicitly define the quasi-isomorphism

$$J : \widehat{\Omega}^1_{n,cl}[1] \to \mathrm{Def}_n$$

as follows. First, we note that this quasi-isomorphism will land completely in the $\mathbb{R}^\times$-invariant piece of the deformation complex

$$\mathrm{C}^*_{\mathrm{loc}}(\mathfrak{g}_n^{\mathbb{R}}) \subset \mathrm{Def}_n.$$

Given $\theta \in \widehat{\Omega}^1_n = \mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g}_n; \mathfrak{g}_n^\vee[-1])$ we define

$$J_\theta(\gamma) = \sum_k \int_{\mathbb{R}} \left\langle \theta_k(\gamma^{\otimes k}), \gamma \right\rangle_{\mathfrak{g}}$$

where θ_k is the kth homogenous component of θ, that is $\theta_k \in \mathrm{Sym}^k(\mathfrak{g}_n^\vee[-1]) \otimes \mathfrak{g}_n^\vee[-1]$.

OG: I think we provided a nicer geometric motivation for this construction in the latest version of the CDO paper.

Proposition 4.2. *The map J has the following properties:*

(1) *For each θ, J_θ is a local functional and is invariant with respect to the action of* $\mathrm{Aff}(\mathbb{R})$.
(2) *The assignment $\theta \mapsto J_\theta$ is W_n-equivariant.*
(3) *If θ is closed then $\mathrm{d}_{dR} J_\theta = 0$.*

This proposition implies that we have a W_n-equivariant cochain map

$$J : \widehat{\Omega}^1_{n,cl}[1] \to \mathrm{C}^*_{\mathrm{loc}}(\mathfrak{g}_n^{\mathbb{R}})^{\mathrm{Aff}(\mathbb{R})} = (\mathrm{Def}_n)^{\mathbb{R}^\times \times \mathrm{Aff}(\mathbb{R})}.$$

Note the shift on the left side is due to the fact that J_θ is a local functional of degree -1.

4.2.2. *Proof of Proposition 4.1.* To complete the proof of the proposition it remains to show that J is a quasi-isomorphism. This is a cohomological calculation.

BW: Insert your guys calculation here, adjusted to the formal disk.

Lemma 4.3. *The first cohomology of W_n with coefficients in the module $\widehat{\Omega}^1_{n,cl}$ is one dimensional*

$$\mathrm{H}^1(\mathrm{W}_n; \widehat{\Omega}^1_{n,cl}) \cong \mathbb{C}$$

spanned by the Gelfand-Fuks chern class $c_1^{\mathrm{GF}}(\widehat{\mathcal{T}}_n)$.

4.3. **BV quantization.** BW: We should copy a lot of the analysis from Paper 1 and put in the next 3 sections with modest notational modification.

RG: Equivariant quantization

4.3.1. *Pre-quantization.*

4.3.2. *Vanishing of the obstruction.*

4.4. **Quantum Observables.** RG: We just focus on smooth observables...fix notation below...blah...

Let's start with the classical factorization algebra which assigns to an interval $I \subset \mathbb{R}$ the cochains on the dg Lie algebra $\mathbb{D}\mathfrak{g}_n^I$. If $I \subset J$ is an embedding of open intervals then it follows from the Poincaré Lemma for differential forms on $\mathbb{R}^n$ that the induced map

$$\mathfrak{g}_n^J \xrightarrow{\simeq} \mathfrak{g}_n^I$$

is a homotopy equivalence. It follows that Obs_n^{cl} is a locally constant factorization algebra.

The aim of this section is to prove the following theorem.

Theorem 4.4. *Let $I \subset \mathbb{R}$ be a non-empty interval.*

(a) *There is a quasi-isomorphism $F_I : \widehat{A}_n \xrightarrow{\sim} Obs_n^q(I)$.*
(b) *The quasi-isomorphism extends to a W_n-equivariant quasi-isomorphism.*

Corollary 4.5. *There is an equivalence of factorization algebras $\mathcal{F}_{\widehat{A}_n} \xrightarrow{\sim} Obs_n^q$, from the one-dimensional factorization algebra determined by the associative algebra $\widehat{A}_n$ to the quantum observables of the formal theory on the formal n-disk.*

4.4.1. *The non-equivariant observables.* This follows from Section 3.3.1...

Let f_I be a smooth bump function of integral 1 with supported contained in the interval $I \subset \mathbb{R}$. There is a commutative diagram

$$\begin{tikzcd} \widehat{A}_n \arrow[r, "F_I"] \arrow[d, equal] & Obs_n^q(I) \arrow[d] \\ \mathbb{C}[[t_i, \partial t_i]] \arrow[r, "f_I"] & Obs_n^{cl}(I) \arrow[u, bend right] \end{tikzcd}$$

We abuse notation in the diagram above as the bottom map is determined by the bump function f_I, more explicitly, it is the assignment

$$t_i \mapsto f_I dx \otimes t_i \quad \text{and} \quad \partial t_i \mapsto f_I dx \otimes \partial t_i.$$

4.4.2. W_n*-equivariant observables.* We now prove part (b) of Theorem 4.4. Define a map

$$F_I^{\mathrm{W}_n} : C^*(\mathrm{W}_n; \widehat{A}_n) \to \left(\mathcal{O}(\mathrm{W}_n[1]) \otimes Obs_n^q(I), \widetilde{Q}\right); \quad F_I^{\mathrm{W}_n} = \mathrm{Id} \otimes F_I.$$

Recall from Section 3.3, that the codomain of the map $F_I^{\mathrm{W}_n}$ is our definition of the W_n-equivariant quantum observables on an interval I.

Proposition 4.6. *The map $F_I^{\mathrm{W}_n} : C^*(\mathrm{W}_n; \widehat{A}_n) \to Obs_n^{q,\mathrm{W}_n}(I)$ is a cochain map.*

RG: rewrite the following paragraph

To show that the identification of $\mathcal{A}_n$ with $\widehat{A}_n$ is W_n-equivariant we must show that the BV bracket with t. Let $a \in \mathcal{A}_n$ and pick a lift

$$\widetilde{a} \in \overline{\mathrm{Obs}}^q(I).$$

Running tree level RG-flow we obtain a (non-smeared) observable $\{\widetilde{a}[L]\}$. Since the action of a formal vector field $\mathfrak{X}$ on $\widetilde{a}[L]$ is through the BV bracket it suffices to compute $\{I_{\mathfrak{X}}^{\mathrm{W}}[L], \widetilde{a}[L]\}_L$. Likewise, the action of an element $\theta \in \widehat{\Omega}^1_{n,cl}$ is the BV bracket with J_θ.

Since $I_{\mathfrak{X}}^{\mathrm{W}}$ is a local functional we see that the scale L bracket is equal to $\{I_{\mathfrak{X}}^{\mathrm{W}}, \widetilde{a}\}[L] = W(P_0^L, \{I_{\mathfrak{X}}^{\mathrm{W}}, \widetilde{a}\})$. Likewise for the local functional J and closed one-forms. So, to compute the action of $\mathrm{W}_n \ltimes \widehat{\Omega}^1_{n,cl}$ on the quantum observables it suffices to compute $\{I_{\mathfrak{X}}^{\mathrm{W}} + \hbar J_\theta, -\}$ acting on observables.

Hence, we are left to prove the following lemma.

Lemma 4.7. *Let $\mathfrak{X} \in \mathrm{W}_n$ and $\theta \in \widehat{\Omega}^1_{n,cl}$. At the level of cohomology in $\mathrm{Obs}^q_n(I)$ we have*

$$\left[\{I^{\mathrm{W}}_{\mathfrak{X}}, a\}\right] = \mathfrak{X} \cdot [a]$$

and

$$[\{J_\theta, a\}] = \theta \cdot [a]$$

where on the right hand side we mean the natural action of formal vector fields and closed one-forms on the Weyl algebra $\widehat{A}_n$ by derivations.

Proof. Let us first consider the local functional $I^{\mathrm{W}}_{\mathfrak{X}} \in \mathrm{C}^*_{\mathrm{loc}}(\mathbb{D}\mathfrak{g}^{\mathbb{R}}_n)$ associated to a fixed formal vector field $\mathfrak{X} \in \mathrm{W}_n$. It is of the form $I^{\mathrm{W}}_{\mathfrak{X}} = \int \mathcal{L}_{\mathfrak{X}} \mathrm{d}x$ where $\mathcal{L}_{\mathfrak{X}} : \mathbb{D}\mathfrak{g}_n \to C^\infty(\mathbb{R})$ is the Lagrangian density.

Given any element $\omega \in \Omega^*_c(U)$ supported on an interval U we obtain an observable in $\mathrm{Obs}^{cl}(U)$ by wedging and integrating

$$\int \omega \wedge \mathcal{L}_{\mathfrak{X}} \in \mathrm{Obs}^{cl}_n(U).$$

In particular, suppose f is a bump function for the interval U. That is, $\int_U f \mathrm{d}x = 1$. Then, we can consider the observable

$$\int f \mathrm{d}x \wedge \mathcal{L}_{\mathfrak{X}} \in \mathrm{Obs}^{cl}_n(U).$$

It is easy to see that the cohomology class of this observable coincides with the element $\mathfrak{X} \in \mathcal{A}_n$. BW: should we write down the formulas?

If we view $\mathfrak{X}$ as an element of the formal Weyl algebra then the action $\mathfrak{X} \cdot a$ is equal to the commutator $\frac{1}{\hbar}[\mathfrak{X}, a]$. We wish to show that the image of $\{I^{\mathrm{W}}_{\mathfrak{X}}, \tilde{a}\}$ in cohomology is equal to this commutator, which we proceed to compute.

Suppose U_{-1}, U_0, U_1 are the intervals of length $1/2$ centered at -1, 0, and 1 respectively. Fix a bump function f for the interval U_0 and let

$$(\tau_{-1} f)(x) = f(x+1) \ \ , \ \ (\tau_1 f)(x) = f(x-1).$$

Define the function

$$h(x) = \int_{s=0}^{s=x} \left((\tau_1 f) - (\tau_{-1} f)\right) \mathrm{d}s.$$

Fix a closed element $a \in \mathrm{Obs}^q_n(U_0)$ and consider the observable $\int h \wedge \mathcal{L}_{\mathfrak{X}} \in \mathrm{Obs}^q_n(\mathbb{R})$. The following is a direct calculation

$$\begin{aligned}
(\mathrm{d}_{dR} + \hbar\Delta)\left(\left(\int h \wedge \mathcal{L}_{\mathfrak{X}}\right) \cdot a\right) &= \left(\int (\mathrm{d}_{dR} h) \wedge \mathcal{L}_{\mathfrak{X}}\right) \cdot a + \hbar\Delta\left(\left(\int h \wedge \mathcal{L}_{\mathfrak{X}}\right) \cdot a\right) \\
&= \left(\int (f_1 \mathrm{d}x) \wedge \mathcal{L}_{\mathfrak{X}} - \int (f_{-1} \mathrm{d}x) \wedge \mathcal{L}_{\mathfrak{X}}\right) \cdot a - \{I_{\mathfrak{X}}, a\}.
\end{aligned}$$

We have used the fact that a is a closed quantum observable by assumption so $(\mathrm{d}_{dR} + \hbar\Delta)a = 0$. We have also used the fact that h is identically -1 on the interval U_0 so that $\int h \mathcal{L}_{\mathfrak{X}} = I_{\mathfrak{X}}$ on U_0.

We conclude that in the complex $\mathrm{Obs}_n^q(\mathbb{R})$ the following relation holds

$$\hbar\{I_{\mathfrak{X}}^{\mathrm{W}}, a\} = I_{\mathfrak{X}}^{\mathrm{W}}((\tau_1 f)\mathrm{d}t - (\tau_{-1} f)\mathrm{d}t) \cdot a + \{\text{exact terms}\}.$$

In cohomology, the right hand side becomes $[\mathfrak{X}, [a]]$ as desired, where $[a]$ is the cohomology class of a. □

The proof of Theorem 4.4 is completed by the following lemma.

Lemma 4.8. *The map $F_I^{\mathrm{W}_n}$ is a quasi-isomorphism.*

5. The Global Theory

5.1. **Descending to the Global Classical Theory.** RG: Comparison Theorem to GG theory RG: Fix notation to be consistent

Proposition 5.1. *For any choice of Gelfand-Kazhdan structure σ on a smooth manifold X, there is an isomorphism*

$$\mathbf{desc}(\sigma, \mathfrak{g}_n) \cong \mathfrak{g}_X$$

of sheaves on X of curved L_∞ algebras.

We have already seen how the action of W_n on $\mathfrak{g}_X$ extends to an action of W_n on the dg Lie algebra $\mathbb{D}\mathfrak{g}_n^{\mathbb{R}}$ defining the classical field theory. As a corollary of the above we obtain.

Corollary 5.2. *Let $\mathbb{D}\mathfrak{g}_X^{\mathbb{R}}$ be the curved L_∞ algebra defining one-dimensional Chern-Simons with target X. Then* $\mathrm{desc}(\sigma, \mathbb{D}\mathfrak{g}_n^{\mathbb{R}}) \cong \mathbb{D}\mathfrak{g}_X^{\mathbb{R}}$.

RG: Note that $\Omega^*(S^1) \otimes -$ commutes with descent and the shifted symplectic structure only depends on the source: it's just the Kronecker pairing on the target.

Thus, one-dimensional Chern-Simons OG: theory! with target the formal n-disk descends to one-dimensional Chern-Simons with target any smooth manifold.

5.2. **Descending to the Global Quantum Theory.**

Proposition 5.3. *The obstruction deformation complex descends to the obstruction deformation complex...*

Lemma 5.4. *If the classical descends and the obs def descends than a formal quantization descends (up to homotopy) to an a priori quantization of the global theory.*

Corollary 5.5. *The formal quantization descends (up to homotopy) to the quantization constructed in GG...*

RG: Could alternatively/additionally remark how the propogator and obstruction descend...

6. A Deformation: Twisted Differential Operators

We will now extend our W_n-equivariant theory by the module of deformations (respecting certain symmetries) that we computed above. Thus, we will consider the extension of the Lie algebra W_n by the module $\widehat{\Omega}^1_{n,cl}$ given by $\mathrm{W}_n \ltimes \widehat{\Omega}^1_{n,cl}$. Just as above, we show that there is a quantization of this classical equivariant theory. This theory will admit a useful and interesting class of deformations when we descend over arbitrary target manifolds.

The extended classical theory is simple to describe. The W_n-equivariant theory above was encoded by a map of Lie algebras

$$I^{\mathrm{W}} : \mathrm{W}_n \to \mathrm{C}^*_{\mathrm{loc}}(\mathbb{D}\mathfrak{g}_n^{\mathbb{R}})[-1].$$

Our extended classical theory is encoded by the restriction of this equivariant theory along the map of Lie algebras $p : \mathrm{W}_n \ltimes \widehat{\Omega}^1_{n,cl} \to \mathrm{W}_n$. It is immediate that this defines a $\mathrm{W}_n \ltimes \widehat{\Omega}^1_{n,cl}$-equivariant classical field theory.

Denote by $\widetilde{\mathrm{Def}}_n^{\mathrm{W}}$ the extended equivariant deformation complex. We can realize it as the tensor product

$$\widetilde{\mathrm{Def}}_n^{\mathrm{W}} = \mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n \ltimes \widehat{\Omega}^1_{n,cl}; \mathrm{Def}_n) \cong \mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n \ltimes \widehat{\Omega}^1_n) \otimes_{\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n)} \mathrm{Def}_n^{\mathrm{W}}.$$

Thus, the quasi-isomorphism J extends to a quasi-isomorphism

$$\widetilde{J} : \mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n \ltimes \widehat{\Omega}^1_n; \widehat{\Omega}^1_n) \xrightarrow{\simeq} \left(\widetilde{\mathrm{Def}}_n^{\mathrm{W}}\right)^{\mathbb{R}^\times \times \mathrm{Aff}(\mathbb{R})}. \tag{1}$$

Here, $\widehat{\Omega}^1_{n,cl}$ denotes the restriction of the W_n-module along the map $p : \mathrm{W}_n \ltimes \widehat{\Omega}^1_{n,cl} \to \mathrm{W}_n$.

There is a natural cocycle in $\mathrm{C}^1_{\mathrm{Lie}}(\mathrm{W}_n \ltimes \widehat{\Omega}^1_{n,cl}; \widehat{\Omega}^1_{n,cl})$ given by the identity id_{Ω^1} on one-forms.

Lemma 6.1. *The space* $\mathrm{H}^1(\mathrm{W}_n \ltimes \widehat{\Omega}^1_{n,cl}, \mathrm{GL}_n; \widehat{\Omega}^1_{n,cl})$ *is two dimensional spanned by* $c_1^{\mathrm{GF}}(\widehat{\mathcal{T}}_n)$ *and* id_{Ω^1}.

Thus, the cocycle id_{Ω^1} determines an additional cohomologically non-trivial deformation in the extended case.

6.0.1. *Quantization.* We obtain a quantization for this classical theory in the same way as in the non-extended case. In the W_n-equivariant theory we showed that we have a quantization given by the collection of functionals $\{I^{\mathrm{W}}[L]\} \subset \mathrm{C}^\#_{\mathrm{Lie}}(\mathrm{W}_n; \mathrm{C}^\#_{\mathrm{Lie}}(\mathbb{D}\mathfrak{g}_n^{\mathbb{R}}))[[\hbar]]$. We can restrict these functionals along the morphism $p : \mathrm{W}_n \ltimes \widehat{\Omega}^1_{n,cl}$ to obtain functionals

$$\widetilde{I}^{\mathrm{W}}[L] := p^*(I^{\mathrm{W}}[L]) \in \mathrm{C}^\#_{\mathrm{Lie}}(\mathrm{W}_n \ltimes \widehat{\Omega}^1_{n,cl}; \mathrm{C}^\#_{\mathrm{Lie}}(\mathbb{D}\mathfrak{g}_n^{\mathbb{R}})[[\hbar]].$$

The same argument as above shows that this pre-quantization is actually a quantization; that is, there is no obstruction.

Proposition 6.2. *The functionals* $\{\widetilde{I}^{\mathrm{W}}[L]\}$ *define a quantization for the* $\mathrm{W}_n \ltimes \widehat{\Omega}^1_{n,cl}$*-equivariant theory.*

6.0.2. *A deformation.* We are interested in a deformation of this theory given by the additional cocycle present in the extended case. The map $J : \widehat{\Omega}^1_{n,cl} \to \mathrm{Def}_n$ from OG: add cross ref extends to a map

$$J : \mathrm{W}_n \ltimes \widehat{\Omega}^1_{n,cl} \to \mathrm{Def}_n$$

that sends $(X, \theta) \mapsto J_\theta$. Hence we can realize J as a cocycle in $\mathrm{Def}_n^{\mathrm{W}}$. Tautologically, this cocycle corresponds to the identity cocycle $\mathrm{id}_{\Omega^1} \in \mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n \ltimes \widehat{\Omega}^1_{n,cl}; \widehat{\Omega}^1_{n,cl})$ under the quasi-isomorphism (1).

We then obtain a deformation of the quantum theory as follows. For $L > 0$ define the functional

$$J[L] := \lim_{\epsilon \to 0} \sum_{\substack{\Gamma \in \mathrm{Trees} \\ v \in V(\Gamma)}} W_{\Gamma,v}(P_{\epsilon<L}, I, J)$$

where the sum is over graphs and distinguished vertices. The notation $W_{\Gamma,v}(P_{\epsilon<L}, I^{\mathrm{W}}, J)$ means that we compute the weight of the graph Γ with J placed at the vertex v and I^{W} placed at all other vertices. OG: Add picture!

Proposition 6.3. *BW: ref any of Si's papers here, he always proves this fact For each $L > 0$ the functional $I^{\mathrm{W}}[L] + \hbar J[L]$ satisfies the scale L quantum master equation*

$$(\mathrm{d}_{dR} + \mathrm{d}_{\mathrm{W}_n \ltimes \widehat{\Omega}^1_{n,cl}})(I^{\mathrm{W}}[L] + \hbar J[L]) + \frac{1}{2}\{I^{\mathrm{W}}[L] + \hbar J[L], I^{\mathrm{W}}[L] + \hbar J[L]\}_L + \hbar \Delta_L(I^{\mathrm{W}}[L] + \hbar J[L]) = 0.$$

Thus, the family of functionals $\{\widetilde{I}^{\mathrm{W}}[L] + \hbar J[L]\}_{L>0}$ defines a quantization of the $\mathrm{W}_n \ltimes \widehat{\Omega}^1_{n,cl}$-equivariant theory.

RG: This is from an old draft...

6.1. **Extended descent.** We wish to extend the above construction of Gelfand-Kazhdan descent by an element $\alpha \in \mathrm{H}^1(X; \Omega^1_{cl,X})$. OG: This sentence is a bit cryptic. We don't extend the construction by a 2-form but rather work with an HC extension. This will give us a formal construction of the sheaf of twisted differential operators for any α as above. OG: "Formal" here is needlessly confusing. We mean that we can construct TDOs by a version of formal geometry.

Let $\widehat{\Omega}^1_{n,cl}$ denote the closed one-forms on $\widehat{D}^n$. OG: We should say somewhere (and at least remind here) whether we mean the cochain complex obtained by truncating the de Rham complex or the actual closed one-forms. The $(\mathrm{W}_n, \mathrm{GL}_n)$-structure comes from Lie derivative by vector fields on forms and linear changes of frame. From above, we know that for any choice of a Gelfand-Kazhdan structure σ on X, we have an isomorphism of sheaves $\mathcal{D}\mathrm{esc}(\sigma, \widehat{\Omega}^1_{n,cl}) \cong \Omega^1_{X,cl}$.

Consider the extension

$$0 \to \widehat{\Omega}^1_{n,cl} \to \mathrm{W}_n \ltimes \widehat{\Omega}^1_{n,cl} \to \mathrm{W}_n \to 0$$

where

$$[X, \omega] = L_X \omega$$

for X a vector field OG: We've already used X for the manifold and ω lives in $\widehat{\Omega}^1_{n,cl}$. Since ω is closed this can be written as $L_X\omega = \mathrm{d}(\iota_X\omega)$.

The Lie algebra W_n is a sub-Lie algebra of $\mathrm{W}_n \ltimes \widehat{\Omega}^1_{n,cl}$. The Lie group GL_n acts on W_n by linear changes of frame and this action extends to an action on $\mathrm{W}_n \ltimes \widehat{\Omega}^1_{n,cl}$. We summarize the situation as follows.

Lemma 6.4. $(\mathrm{W}_n \ltimes \widehat{\Omega}^1_{n,cl}, \mathrm{GL}_n)$ *forms a Harish-Chandra pair.*

There is a natural quotient map of HC pairs

$$(\mathrm{W}_n \ltimes \widehat{\Omega}^1_{n,cl}, \mathrm{GL}_n) \to (\mathrm{W}_n, \mathrm{GL}_n).$$

We wish to understand lifts of the $(\mathrm{W}_n, \mathrm{GL}_n)$-bundle $(\mathrm{Fr}_X, \omega^\sigma)$ along this quotient map.

For a fixed GK-structure σ, a lift is the structure of a flat $(\mathrm{W}_n \ltimes \widehat{\Omega}^1_{n,cl}, \mathrm{GL}_n)$-principal bundle on a pair (Fr_X, ω) such that under the map

$$\Omega^1(\mathrm{Fr}_X; \mathrm{W}_n \ltimes \widehat{\Omega}^1_{n,cl}) \to \Omega^1(\mathrm{Fr}_X; \mathrm{W}_n),$$

the one-form ω is sent to ω^σ.

Proposition 6.5. *Fix a GK-structure σ on X. There is a natural bijection between the following two sets:*

- $\mathrm{H}^1(\mathrm{X}, \widehat{\Omega}^1_{\mathrm{X},cl}) \cong \mathrm{H}^2_{\mathrm{dR}}(\mathrm{X})$ *and*
- *the set of lifts of the flat $(\mathrm{W}_n, \mathrm{GL}_n)$-principal bundle $(\mathrm{Fr}_X, \omega^\sigma)$ to a flat $(\mathrm{W}_n \ltimes \widehat{\Omega}^1_{n,cl}, \mathrm{GL}_n)$-principal bundle, up to isomorphism.*

Proof. Consider the sheaf of closed one-forms $\Omega^1_{X,cl}$ and its associated sheaf of ∞-jets $\mathcal{J}(\Omega^1_{X,cl})$. By definition of the coordinate bundle, the pull-back of $\mathcal{J}(\Omega^1_{X,cl})$ along the projection $\pi^{coor} : X^{coor} \to X$ returns the trivial sheaf of formal closed one-forms

$$(\pi^{coor})^* \left(\mathcal{J}(\Omega^1_{X,cl})\right) \cong \underline{\widehat{\Omega}^1_{n,cl}}.$$

Any $\alpha \in \mathrm{H}^1(X; \Omega^1_{X,cl})$ defines an element in $\mathrm{H}^1(X; \mathcal{J}(\Omega^1_{X,cl}))$ and hence determines an element

$$(\pi^{coor})^*\alpha \in \mathrm{H}^1(X^{coor}; \underline{\widehat{\Omega}^1_{n,cl}})$$

via pull-back to the coordinate bundle. In turn, this element classifies a $\widehat{\Omega}^1_{n,cl}$-torsor over X^{coor} that we denote X^{coor}_α. Being a torsor for a constant sheaf of abelian groups, that are contractible as topological spaces, there exists sections $\sigma_\alpha : X^{coor} \to X^{coor}_\alpha$. OG: This last sentence is a little hard to parse. I think "sheaf of vector spaces" would be clearer than "abelian groups." Moreover, don't we have a cochain complex here? Then contractibility sounds weirder. I think we can phrase things so it's a little less distracting and the reader can see that sections obviously exist.

Recall how one constructs the principal $(\mathrm{W}_n, \mathrm{GL}_n)$-bundle structure on Fr_X from a GK-structure σ. We have a sequence of bundle maps

$$X^{coor} \to \mathrm{Fr}_X \to X,$$

and σ determines a splitting of the first map. The flat connection one-form on Fr_X is the pull-back of the Grothendieck connection $\sigma^*\omega^{coor}$. This Grothendieck connection is uniquely determined by the transitive action of W_n on X^{coor}. Now we examine how this process would apply to X^{coor}_α.

While W_n does not act transitively on X^{coor}_α, the extension $\mathrm{W}_n \ltimes \widehat{\Omega}^1_{n,cl}$ does act. Indeed, we have a pull-back diagram of Lie algebras

$$\begin{tikzcd} \mathrm{W}_n \ltimes \widehat{\Omega}^1_{n,cl} \arrow[r, "\theta^{coor}_\alpha"] \arrow[d] & \mathcal{X}(X^{coor}_\alpha) \arrow[d] \\ \mathrm{W}_n \arrow[r, "\theta^{coor}"] & \mathcal{X}(X^{coor}). \end{tikzcd}$$

The map θ^{coor} induces an isomorphism on tangent spaces $\theta^{coor} : \mathrm{W}_n \xrightarrow{\cong} T_\varphi X^{coor}$ and hence so does θ^{coor}_α. The map $(\theta^{coor}_\alpha)^{-1}$ is represented by an element

$$\omega^{coor}_\alpha \in \Omega^1(X^{coor}, \mathrm{W}_n \ltimes \widehat{\Omega}^1_{n,cl})$$

satisfying the Maurer-Cartan equation.

Conversely, if we are given a $(\mathrm{W}_n \ltimes \widehat{\Omega}^1_{n,cl})$-structure (Fr_X, ω), we can consider the characteristic map

$$\mathrm{ch} : \mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n \ltimes \widehat{\Omega}^1_{n,cl}, \mathrm{GL}_n; \widehat{\Omega}^1_{n,cl}) \to \mathbf{desc}((\mathrm{Fr}_X, \omega), \widehat{\Omega}^1_{n,cl}).$$

For any tensor bundle $\mathcal{V}$, we have a quasi-isomorphism

$$\mathbf{desc}((\mathrm{Fr}_X, \omega^\sigma), \mathcal{V}) \simeq \check{\mathrm{C}}^*(X; \mathcal{D}\mathrm{esc}((\mathrm{Fr}_X, \omega^\sigma), \mathcal{V}),$$

since $\mathbf{desc}((\mathrm{Fr}_X, \omega^\sigma), \mathcal{V})$ is an acyclic resolution for $\mathcal{D}\mathrm{esc}((\mathrm{Fr}_X, \omega^\sigma), \mathcal{V})$. Thus, we have

$$\mathbf{desc}((\mathrm{Fr}_X, \omega), \widehat{\Omega}^1_{n,cl}) \simeq \check{\mathrm{C}}^*(X; \mathcal{D}\mathrm{esc}((\mathrm{Fr}_X, \omega), \widehat{\Omega}^1_{n,cl})) = \check{\mathrm{C}}^*(X; \Omega^1_{X,cl}).$$

OG: What does the check mean? Do we mean Čech cohomology? We should figure out what notation we want to use. I thought here we would probably just mean levelwise global sections, since we're talking about a truncated de Rham complex.

Consider the cocycle $\mathrm{id}_{\Omega^1_{n,cl}} \in \mathrm{C}^1_{\mathrm{Lie}}(\mathrm{W}_n \ltimes \widehat{\Omega}^1_{n,cl}; \widehat{\Omega}^1_{n,cl})$. In cohomology, the image of this element under the characteristic map is an element

$$\mathrm{ch}([\mathrm{id}_{\Omega^1_{n,cl}}]) \in \mathrm{H}^1(X; \Omega^1_{X,cl}).$$

These two constructions are inverse to each other, as can be shown by direct computation.

$\square$

6.1.1. *Extended Harish-Chandra modules.* In this section we define the analog of the category of modules $\mathrm{VB}_{(W_n, GL_n)}$ for the pair $(W_n \rtimes \widehat{\Omega}^1_{n,cl}, GL_n)$.

First, consider the category of *all* Harish-Chandra modules for the pair $(W_n \ltimes \widehat{\Omega}^1_{n,cl}, GL_n)$. There is a subcategory of this consisting of those modules that are filtered with respect to the obvious two-step filtration on $W_n \ltimes \widehat{\Omega}^1_{n,cl}$ given by

$$F^1 = W_n \ltimes \widehat{\Omega}^1_{n,cl} \supset F^0 = \widehat{\Omega}^1_{n,cl}.$$

OG: This isn't a subcategory as a filtration is data. Let M be such a module. Taking the associated graded with respect to this filtration we obtain a graded module $\mathrm{gr}\, M$ for the pair (W_n, GL_n).

Example 6.6. We have the formal Atiyah sequence

$$0 \to \widehat{\mathcal{O}}_n \to \widetilde{\mathcal{T}}_n \to \widehat{\mathcal{T}}_n \to 0, \tag{2}$$

where $\widetilde{\mathcal{T}}_n$ is...RG: Finish...this sequence is corresponds to $\mathrm{id}_{\Omega^1_n} \in C^1_{Lie}(W_n \rtimes \widehat{\Omega}^1_{n,cl}, \widehat{\Omega}^1_{n,cl})$. Note that by construction this can be considered as a sequence of $(W_n \ltimes \widehat{\Omega}^1_{n,cl}, GL_n)$-modules.

6.2. **TDO's from descent.**

6.2.1. *Classical theory of TDO's.* We briefly recall a construction of locally trivial TDO's. A standard reference is **[?]** particularly section 2.2 or appendix A of **[?]**.

For any $\alpha \in H^1(X, \Omega^1_{cl})$ we have an associated Atiyah algebra

$$0 \to \mathcal{O}_X \to \mathcal{T}^\alpha_X \to \mathcal{T}_X \to 0.$$

Up to isomorphism, this association is a bijection, i.e., Atiyah algebras are classified by $H^1(X, \Omega^1_{cl})$.

Definition 6.7. The *twisted differential operators* TDO^α_X corresponding to an element $\alpha \in \mathrm{H}^1(X; \Omega^1_{cl})$ is the sheaf of algebras $\mathcal{U}_{\mathcal{O}_X}(\mathcal{T}^\alpha_X)$.

6.2.2. Recall the formal Atiyah sequence

$$0 \to \widehat{\mathcal{O}}_n \to \widetilde{\mathcal{T}}_n \to \widehat{\mathcal{T}}_n \to 0. \tag{3}$$

Proposition 6.8. *Fix a GK-structure σ on X. There is an isomorphism of Lie algebroids on X*

$$\mathcal{D}\mathrm{esc}(\sigma, \alpha, \widetilde{\mathcal{T}}_n) \cong \mathcal{T}^\alpha_X.$$

Proof. Consider the short exact sequence (3) of $(W_n \ltimes \widehat{\Omega}^1_{n,cl}, GL_n)$ modules. It is determined by the cocycle $\mathrm{id}_{\Omega^1_n}$ which maps under the characteristic map to $(\mathrm{Fr}_X, \omega^\sigma_\alpha)$ to the element $\alpha \in \mathrm{H}^1(X; \Omega^1_{X,cl})$. Thus we see that when we apply descent to the short exact sequence of $(W_n \ltimes \widehat{\Omega}, GL_n)$-modules we get a short exact sequence of $\mathcal{O}_X$-modules

$$0 \to \mathcal{O}_X \to \mathcal{D}\mathrm{esc}(\sigma, \alpha, \widetilde{\mathcal{T}}_n) \to \mathcal{T}_X \to 0,$$

which is the Atiyah algebra classified by α. We conclude that as $\mathcal{O}_X$-modules $\mathcal{D}\mathrm{esc}(\sigma, \alpha, \widetilde{\mathcal{T}}_n) \cong \mathcal{T}_X^{\alpha}$. Moreover, by the classification of Atiyah algebras, it is an equivalence as Lie algebroids. $\square$

It then follows from Proposition 2.8 that we obtain twisted differential operators via descent of the formal Weyl algebra.

Corollary 6.9. *Fix a GK-structure σ and let $\alpha \in \mathrm{H}^1(X, \Omega^1_{cl,X})$. Then*

$$\mathcal{D}\mathrm{esc}(\sigma, \alpha, \widehat{A}_n) \cong TDO_X^{\alpha}.$$

That is, the Gelfand-Kazhdan descent of the completed Weyl algebra $\widehat{A}_n$ along the flat $(\mathrm{W}_n \ltimes \widehat{\Omega}^1_{n,cl}, \mathrm{GL}_n)$-bundle $(\mathrm{Fr}_X, \omega_\alpha^\sigma)$ recovers the sheaf of TDO's on X twisted by α.