ONE-DIMENSIONAL CHERN-SIMONS AND SHEAVES OF DIFFERENTIAL OPERATORS

CONTENTS

1. INTRODUCTION

2. (TWISTED) DIFFERENTIAL OPERATORS VIA FORMAL GEOMETRY

The purpose of this section is to describe a model for twisted differential operators using the language of Gelfand–Kazhdan formal geometry.

2.1. A quick review of descent.

Our approach to studying topological mechanics relies on a description of the target smooth manifold in the style of formal geometry. As we have discussed, our method is to consider topological mechanics on the *formal disk* that is equivariant for the action of all formal automorphisms. To obtain topological mechanics on a smooth manifold X we apply *formal descent* to this theory on the formal disk.

Our version of descent has appeared in the formal geometry of Gelfand and Kazhdan **[?]** in the construction of "natural" objects in differential geometry. By natural, we mean that these construction apply uniformly over the category of smooth manifolds (of a fixed dimension) possibly equipped with a local geometry (such as a symplectic or complex structure). Our construction of the one-dimensional theory via descent is very close to the last two author's construction of *chiral differential operators* from BV quantization in

[?]. Because of this, our review of formal geometry will be brief, and we refer the reader to more detailed explanations and proofs to our prior work.

The central object, associated to each smooth n-manifold X, is the bundle of coordinates X^{coor}. The fiber over a point $x \in X$ consists of the space of formal coordinates centered at x. This is the space of ∞-jets of local diffeomorphisms of the n-disk D^n to an open neighborhood of x sending $0 \in D^n$ to x. In other words, this is the space of maps from the *formal disk* $\widehat{D}^n$ to the formal completion of X near the point x. The formal disk $\widehat{D}^n$ is defined as the affine formal scheme with ring of functions given by formal power series $\mathbb{R}[[t_1, \ldots, t_n]]$. The bundle $X^{coor} \to X$ is a principal bundle for the group of automorphisms of this formal scheme that fix the maximal ideal, which we denote Aut_n. This principal Aut_n-bundle factors through the ordinary GL_n frame bundle $\mathrm{Fr}_X \to X$, in since there is a natural map of bundles $X^{coor} \to \mathrm{Fr}_X$ that truncates the full ∞-jet to remember only the linear jet.

The crucial point of formal geometry is that this principal Aut_n-bundle carries a *flat connection*. The flat connection is valued in the Lie algebra of *all* formal vector fields W_n. The Lie algebra of Aut_n consists of formal vector fields that vanish at the origin, which is a strict sub Lie algebra of all formal vector fields, so we are leaving the ordinary category of principal bundles with connection.

2.1.1. *The descent functor.* The descent functor takes as input a geometric object (like a vector bundle) on the formal disk and outputs a universal object on the category of smooth manifolds.

It is most convenient for us to use a simplification of the coordinate bundle X^{coor}. Consider the map that projects onto the linear jet of the formal coordinate

$$\pi_1 : X^{coor} \to \mathrm{Fr}_X.$$

Splittings of this map that are equivariant for the group GL_n of this map always exist. Locally, such a splitting amounts to exponentiating a linear frame to a full formal coordinate. In fact, every BW: torsion free connection ∇ on X defines a global exponentiation and hence a splitting $\sigma_\nabla : \mathrm{Fr}_X \to X^{coor}$ of π_1.

The flat connection on X^{coor} determines an element $\omega \in \Omega^1(X^{coor}; \mathrm{W}_n)$. Using the splitting, we can pull-back this flat connection to obtain a GL_n-invariant element $\omega_\nabla := \sigma_\nabla^* \omega^{coor} \in \Omega^1(\mathrm{Fr}_X; \mathrm{W}_n)$ satisfying

$$\mathrm{d}_{dR}\omega_\nabla + \frac{1}{2}[\omega_\nabla, \omega_\nabla] = 0$$

where d_{dR} is the de Rham differential on Fr_X and $[-,-]$ is induced from the Lie bracket of formal vector fields.

To define descent, we need to fix a convenient category of modules for formal vector fields and GL_n. The class of objects we consider are GL_n-representations $\mathcal{V}$ (possibly infinite dimensional) that have a compatible structure of a W_n-module. For instance, if

$X \in \mathrm{W}_n$ ... We denote this category of modules by $\mathrm{Mod}_{(\mathrm{W}_n, \mathrm{GL}_n)}$. In the literature, this is known as the category of modules for the Harish–Chandra pair $(\mathrm{W}_n, \mathrm{GL}_n)$.

We want to consider modules that behave like vector bundles on the formal disk. Let $\widehat{\mathcal{O}}_n = \mathbb{C}[[t_1, \ldots, t_n]]$ denote the ring of formal power series. If $\mathcal{V}$ is a $(\mathrm{W}_n, \mathrm{GL}_n)$-module we require that it admits a GL_n-equivariant splitting

$$\mathcal{V} = \widehat{\mathcal{O}}_n \otimes_{\mathbb{C}} V$$

where GL_n acts on $\widehat{\mathcal{O}}_n$ in the natural way, and V is some finite dimensional GL_n-representation. We refer to such objects as *formal vector bundles*. They form a category that we denote VB_n.

We denote the category of smooth finite dimensional vector bundles on X by VB_X. Let $\mathrm{Pro}(\mathrm{VB}_X)$ denote the category of pro-vector bundles. An object in $\mathrm{Pro}(\mathrm{VB}_X)$ consists of a sequence of vector bundles BW: Did we flip pro and ind objects in the CDO paper?

We are now ready to define the descent functor.

Definition 2.1. Let X be a smooth manifold equipped with a connection. Define the functor

$$\mathrm{desc}_X : \mathrm{VB}_n \to \mathrm{Pro}(\mathrm{VB}_X))_{flat}$$

that sends a formal vector bundle $\mathcal{V} = \widehat{\mathcal{O}}_n \otimes V$ to the pro vector bundle...BW: I'm worried I'm mixing up pro and ind

Given a flat vector bundle we can take its flat sections to obtain a sheaf on X. For the flat bundle $\mathrm{desc}_X(\mathcal{V})$ we denote this sheaf by $\mathcal{D}\mathrm{esc}_X(\mathcal{V})$. This sheaf is automatically locally free and hence corresponds to the sheaf of sections of a finite dimensional vector bundle.

2.1.2. *Characteristic classes.* An exact sequence of formal vector bundles

$$\mathcal{A} \to \mathcal{B} \to \mathcal{C}$$

is classified by an element in the ext-group $\mathrm{Ext}^1_{\mathrm{VB}_n}(\mathcal{C}, \mathcal{A})$. In **[?]** we show that for each k the ext-group $\mathrm{Ext}^k_{\mathrm{VB}_n}(\mathcal{C}, \mathcal{A})$ is canonically identified with the *relative Chevalley–Eilenberg cochains* $\mathrm{C}^k_{\mathrm{Lie}}(W_n, \mathrm{GL}_n; \mathcal{C}^\vee \otimes_{\widehat{\mathcal{O}}_n} \mathcal{A})$, where the $(-)^\vee$ denotes the $\widehat{\mathcal{O}}_n$-linear dual.

In Lemma 2.28 of **[?]** we prove that the functor $\mathcal{D}\mathrm{esc}_X(-)$ is exact. In particular, applied to the exact sequence **??** is an exact sequence of formal vector bundles we obtain a map

$$\mathrm{char}_X : \mathrm{C}^k_{\mathrm{Lie}}(W_n, \mathrm{GL}_n; \mathcal{C}^\vee \otimes_{\widehat{\mathcal{O}}_n} \mathcal{A}) \to H^k(X; \mathcal{D}\mathrm{esc}_X(\mathcal{C})^\vee \otimes_{\mathcal{O}_X} \mathcal{D}\mathrm{esc}_X(\mathcal{A})).$$

that we call the *characteristic map*.

2.2. **Formal Lie algebroids.** In this section we define formal Lie algebroids. These are Lie algebroids on the formal disk, in the same way that the category VB_n provides a model for vector bundles on the formal n-disk.

Definition 2.2. A *formal Lie algebroid* is an object $\mathcal{L} \in \mathrm{VB}_n$ together with

(i) A $\mathbb{C}$-linear bracket $[-,-] : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ and
(ii) an "anchor" map $a : \mathcal{L} \to \widehat{\mathcal{T}}_n$

such that

(1) $[-,-]$ and a are equivariant for the action of $(\mathrm{W}_n, \mathrm{GL}_n)$,
(2) a is $\widehat{\mathcal{O}}$-linear, and
(3) for any $x, y \in \mathcal{L}$ and $f \in \widehat{\mathcal{O}}_n$ one has

$$[x, f \cdot y] - f \cdot [x, y] = a(x) \cdot f.$$

Remark 2.3. Formal Lie algebroids are precisely Lie–Rinehart algebras over the ring $\widehat{\mathcal{O}}_n$ that are equivariant for the pair $(\mathrm{W}_n, \mathrm{GL}_n)$.

We define the category Algd_n to be the full subcategory of VB_n consisting of formal Lie algebroids. The following is an immediate consequence of the functoriality of descent.

Lemma 2.4. *Suppose $\mathcal{L}$ is a formal Lie algebroid. Then, $\mathcal{D}\mathrm{esc}_X(\mathcal{L})$ has the structure of a Lie algebroid on X.*

It is immediate to see that the descent of the formal Lie algebroid $\widehat{\mathcal{T}}_n$ is the standard Lie algebroid.

2.2.1. *Enveloping algebroid.* To any Lie algebra we can assign an associative algebra through its universal enveloping algebra. BW: blah blah

Let $\mathcal{L}$ be a formal Lie algebroid. Consider the left $\widehat{\mathcal{O}}_n$-module $\widehat{\mathcal{O}}_n \oplus \mathcal{L}$. This direct sum has a natural Lie algebra structure given by

$$[(f, x), (g, y)] := (a(x) \cdot f - a(y) \cdot g, [x, y]).$$

Let $U_{\mathrm{Lie}}(\widehat{\mathcal{O}}_n \oplus \mathcal{L})$ be the universal envelope of this Lie algebra. Note that there is a natural augmentation map $\epsilon : U_{\mathrm{Lie}}(\widehat{\mathcal{O}}_n \oplus \mathcal{L}) \to \mathbb{R}$.

Definition 2.5. Let $\mathcal{L}$ be a formal Lie algebroid. Denote by $\overline{U}$ the augmentation ideal with respect to ϵ, and let $I \subset \overline{U}$ be the ideal generated by

$$i(f, 0) \otimes i(g, x) - i(fg, fx)$$

for $x \in \mathcal{L}$ and $f, g \in \widehat{\mathcal{O}}_n$. The *enveloping algebra* of the formal Lie algebroid $\mathcal{L}$ is the quotient

$$\mathcal{U}_{\widehat{\mathcal{O}}_n}(\mathcal{L}) := \overline{U}/I.$$

It is immediate to see that $\mathcal{U}_{\widehat{\mathcal{O}}_n}(\mathcal{L})$ is an associative algebra object in the category of formal vector bundles. By monoidality of the descent functor, we $\mathcal{D}\mathrm{esc}_{\mathrm{GK}}(\mathcal{L})$ is a sheaf of associative algebras on the category of n-manifolds.

Example 2.6. Consider the $\widehat{\mathcal{O}}_n$-module $\widehat{\mathcal{T}}_n$, which has a natural formal Lie algebroid structure via the Lie bracket of vector fields. The anchor map is the identity map. It is straightforward to exhibit an isomorphism $\mathcal{U}_{\widehat{\mathcal{O}}_n}(\widehat{\mathcal{T}}_n) \cong \widehat{A}_n$ of algebras in the category VB_n.

We have already mentioned that $\mathcal{D}\mathrm{esc}$ is exact, it in particular preserves finite colimits. An immediate corrollary of this is the following.

Corollary 2.7. *Let $\mathcal{L}$ be a formal Lie algebroid and X a smooth manifold. Then, there is an isomorphism of sheaves of associative algebras on X:*

$$\mathcal{D}\mathrm{esc}_X(\mathcal{U}_{\widehat{\mathcal{O}}_n}\mathcal{L}) \cong \mathcal{U}_{\mathcal{O}_X}(\mathcal{D}\mathrm{esc}(\mathcal{L})).$$

Example 2.8. Let $\widehat{A}_n$ denote the associative Weyl algebra on generators $t_1, \dots, t_n$ and $\frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_n}$. The relation is $[\frac{\partial}{\partial t_i}, t_j] = \delta_{ij}$. Consider the formal vector bundle $\widehat{\mathcal{T}}_n$, which has a natural formal Lie algebroid structure via the Lie bracket of vector fields. The anchor map is the identity map. It is straightforward to exhibit an isomorphism $\mathcal{U}_{\widehat{\mathcal{O}}_n}(\widehat{\mathcal{T}}_n) \cong \widehat{A}_n$ of algebras in $(\mathrm{W}_n, \mathrm{GL}_n)$-modules.

3. The Formal Theory

OG: This section should also have an introduction that indicates how it is parallel to the purely algebraic construction just described.

We construct one-dimensional Chern-Simons OG: theory with target the formal disk. We will use this theory to study how diffeomorphisms on the target act on the theory, which will be encoded by the structure of W_n-equivariance. OG: This second sentence is befuddling.

We have already seen OG: ? that the formal disk is encoded by the dg Lie algebra $\mathfrak{g}_n = \mathbb{C}^n[-1]$. Consider the dg Lie algebra

$$\mathfrak{g}_n^{\mathbb{R}} := \Omega^*(\mathbb{R}; \mathfrak{g}_n).$$

This dg Lie algebra is abelian and the differential is the de Rham differential d_{dR}. This dg Lie algebra encodes deformations of the constant map to 0 to nearby smooth functions.

The dg Lie algebra that encodes one-dimensional Chern-Simons with target $\widehat{D}^n$ is defined to be the "double"

$$\mathbb{D}\mathfrak{g}_n^{\mathbb{R}} = \Omega^*(\mathbb{R}; \mathfrak{g}_n \oplus \mathfrak{g}_n^\vee[-2]).$$

The Lie bracket is still trivial, and the differential is d_{dR}. This dg Lie algebra encodes the cotangent theory OG: ? that's not widespread terminology and hence should be explained to the elliptic moduli problem OG: also not widespread described by $\mathfrak{g}_n^{\mathbb{R}}$. The pairing is "wedge and integrate". Explicitly, for $\phi, \phi' \in \Omega^*_c(\mathbb{R}; \mathfrak{g}_n)$ and $\psi, \psi' \in \Omega^*_c(\mathbb{R}; \mathfrak{g}_n^\vee[-2])$ we have

$$\langle \phi + \psi, \phi' + \psi' \rangle = \int_{\mathbb{R}} \mathrm{ev}_{\mathfrak{g}_n}(\phi \wedge \psi') + \mathrm{ev}_{\mathfrak{g}_n^\vee}(\psi \wedge \phi').$$

This pairing has cohomological degree -3 and hence determines a classical BV theory OG: no hyphen! BV theory that we call *one-dimensional Chern-Simons with target $\widehat{D}^n$*.

3.1. The W_n action.

3.1.1. We have just introduced the dg Lie algebra $\mathfrak{g}_n$. OG: I find this a weird way to start the subsection. Need to restructure a bit.

Let us denote the generators of $\mathfrak{g}_n$ by $\{\xi_1, \ldots, \xi_n\}$ and the dual generators by $\{t_1, \ldots, t_n\}$. Thus

$$C^*_{\mathrm{Lie}}(\mathfrak{g}_n) = \widehat{\mathrm{Sym}}(\mathfrak{g}_n^\vee[-1]) = \mathbb{C}[[t_1, \ldots, t_n]].$$

There is hence a natural isomorphism

$$\rho^{\mathrm{W}} : \mathrm{W}_n \to \mathrm{Der}(C^*_{\mathrm{Lie}}(\mathfrak{g}_n))$$

sending $f(t_i)\partial_j$ to $f(t_i)\xi_j$. OG: Might help to identify the derivations with something explicit so that these notations make sense.

We can interpret ρ^{W} as expressing an L_∞ action of W_n on $\mathfrak{g}_n$, where

$$\ell_m^W : \mathrm{W}_n \otimes \mathfrak{g}_n^{\otimes m} \to \mathfrak{g}_n$$

has cohomological degree $1 - m$ and m ranges over all non-negative integers. These maps are simply the "Taylor components" of ρ_W. For instance, the vector field $\mathfrak{X} = t_1^{m_1} \cdots t_n^{m_n}\partial_j \in \mathrm{W}_n$ acts by zero for any $m \neq m_1 + \cdots + m_n$, and for $m = m_1 + \cdots + m_n$,

$$\ell_m^W \left(\mathfrak{X}, (\xi_1^{\otimes m_1} \otimes \cdots \otimes \xi_n^{\otimes m_n})\right) = \ell_m^{\mathbf{W}} \left((t_1^{m_1} \cdots t_n^{m_n}\partial_j) \otimes \xi_1^{\otimes m_1} \otimes \cdots \otimes \xi_n^{\otimes m_n}\right) = \xi_j$$

and vanishes on any other basis element $\mathfrak{g}_n^{\otimes m}$.

3.1.2. Let A be a commutative dg algebra. We now show how the dg Lie algebra $A \otimes \mathfrak{g}_n$ inherits a natural L_∞ action of W_n. Here the sequence of maps is

$$\ell_m^{W,A} : \mathrm{W}_n \otimes (A \otimes \mathfrak{g}_n)^{\otimes m} \to A \otimes \mathfrak{g}_n$$

with

$$\ell_m^{W,A}(\mathfrak{X}, (a_1 \otimes x_1) \otimes \cdots \otimes (a_m \otimes x_m)) = \pm(a_1 \cdots a_m) \otimes \ell_m^W(\mathfrak{X}, x_1 \otimes \cdots \otimes x_m),$$

where the sign is determined by Koszul's rule. Equivalently, we can encode the L_∞ action in a Lie algebra map

$$\rho_{W,A} : \mathrm{W}_n \to C^*_{\mathrm{Lie}}(A \otimes \mathfrak{g}_n, A \otimes \mathfrak{g}_n[-1]),$$

which assembles the ℓ_m^A maps into a "Taylor series." If we set A to be $\Omega^*(\mathbb{R})$, then we obtain an L_∞ action of W_n on $\mathfrak{g}_n^{\mathbb{R}}$. A lift of this action to an L_∞ action of W_n on $\mathbb{D}\mathfrak{g}_n^{\mathbb{R}}$ is uniquely determined by the requirement that the action preserve the pairing of degree -3.

OG: Might be nice to make a small comment about why this construction is relevant (encodes action on a mapping space!

3.2. **The deformation complex.**

Proposition 3.1. *There is a quasi-isomorphism of* W_n*-modules*

$$J : \Omega^1_{n,cl}[1] \xrightarrow{\simeq} (\mathrm{Def}_n)^{\mathbb{R}^\times \times \mathrm{Aff}(\mathbb{R})}.$$

OG: Is it a map of Lie algebras? *Thus, taking the invariant part of the* W_n*-equivariant deformation complex, we have a quasi-isomorphism*

$$J : \mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n; \widehat{\Omega}^1_{n,cl}[1]) \xrightarrow{\simeq} \left(\mathrm{Def}_n^{\mathrm{W}}\right)^{\mathbb{R}^\times \times \mathrm{Aff}(\mathbb{R})}.$$

3.2.1. *One-forms as local functionals.* We explicitly define the quasi-isomorphism

$$J : \widehat{\Omega}^1_{n,cl}[1] \to \mathrm{Def}_n$$

as follows. First, we note that this quasi-isomorphism will land completely in the $\mathbb{R}^\times$-invariant piece of the deformation complex

$$\mathrm{C}^*_{\mathrm{loc}}(\mathfrak{g}_n^{\mathbb{R}}) \subset \mathrm{Def}_n.$$

Given $\theta \in \widehat{\Omega}^1_n = \mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g}_n; \mathfrak{g}_n^\vee[-1])$ we define

$$J_\theta(\gamma) = \sum_k \int_{\mathbb{R}} \left\langle \theta_k(\gamma^{\otimes k}), \gamma \right\rangle_{\mathfrak{g}}$$

where θ_k is the kth homogenous component of θ, that is $\theta_k \in \mathrm{Sym}^k(\mathfrak{g}_n^\vee[-1]) \otimes \mathfrak{g}_n^\vee[-1]$.

OG: I think we provided a nicer geometric motivation for this construction in the latest version of the CDO paper.

Proposition 3.2. *The map J has the following properties:*

1. *For each θ, J_θ is a local functional and is invariant with respect to the action of* $\mathrm{Aff}(\mathbb{R})$.
2. *The assignment $\theta \mapsto J_\theta$ is* W_n*-equivariant.*
3. *If θ is closed then $\mathrm{d}_{dR} J_\theta = 0$.*

This proposition implies that we have a W_n-equivariant cochain map

$$J : \widehat{\Omega}^1_{n,cl}[1] \to \mathrm{C}^*_{\mathrm{loc}}(\mathfrak{g}_n^{\mathbb{R}})^{\mathrm{Aff}(\mathbb{R})} = (\mathrm{Def}_n)^{\mathbb{R}^\times \times \mathrm{Aff}(\mathbb{R})}.$$

Note the shift on the left side is due to the fact that J_θ is a local functional of degree -1.

3.2.2. *Proof of Proposition 3.1.* To complete the proof of the proposition it remains to show that J is a quasi-isomorphism. This is a cohomological calculation.

BW: Insert your guys calculation here, adjusted to the formal disk.

Lemma 3.3. *The first cohomology of* W_n *with coefficients in the module* $\widehat{\Omega}^1_{n,cl}$ *is one dimensional*

$$\mathrm{H}^1(\mathrm{W}_n; \widehat{\Omega}^1_{n,cl}) \cong \mathbb{C}$$

spanned by the Gelfand-Fuks chern class $c_1^{\mathrm{GF}}(\widehat{\mathcal{T}}_n)$.

3.3. **BV quantization.** BW: We should copy a lot of the analysis from Paper 1 and put in the next 3 sections with modest notational modification.

RG: Equivariant quantization

3.3.1. *Pre-quantization.*

3.3.2. *Vanishing of the obstruction.*

3.4. **The "extended" theory.** OG: I think it would be good to explain why we want to do this and what it should accomplish, or at least point to some such discussion elsewhere in the paper. Otherwise it's a bit terse.

We will now extend our W_n-equivariant theory by the module of deformations (respecting certain symmetries) that we computed above. Thus, we will consider the extension of the Lie algebra W_n by the module $\widehat{\Omega}^1_{n,cl}$ given by $\mathrm{W}_n \ltimes \widehat{\Omega}^1_{n,cl}$. Just as above, we show that there is a quantization of this classical equivariant theory. This theory will admit a useful and interesting class of deformations when we descend over arbitrary target manifolds.

The extended classical theory is simple to describe. The W_n-equivariant theory above was encoded by a map of Lie algebras

$$I^{\mathrm{W}} : \mathrm{W}_n \to \mathrm{C}^*_{\mathrm{loc}}(\mathbb{D}\mathfrak{g}_n^{\mathbb{R}})[-1].$$

Our extended classical theory is encoded by the restriction of this equivariant theory along the map of Lie algebras $p : \mathrm{W}_n \ltimes \widehat{\Omega}^1_{n,cl} \to \mathrm{W}_n$. It is immediate that this defines a $\mathrm{W}_n \ltimes \widehat{\Omega}^1_{n,cl}$-equivariant classical field theory.

Denote by $\widetilde{\mathrm{Def}}_n^{\mathrm{W}}$ the extended equivariant deformation complex. We can realize it as the tensor product

$$\widetilde{\mathrm{Def}}_n^{\mathrm{W}} = \mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n \ltimes \widehat{\Omega}^1_{n,cl}; \mathrm{Def}_n) \cong \mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n \ltimes \widehat{\Omega}^1_n) \otimes_{\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n)} \mathrm{Def}_n^{\mathrm{W}}.$$

Thus, the quasi-isomorphism J extends to a quasi-isomorphism

$$\widetilde{J} : \mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n \ltimes \widehat{\Omega}^1_n; \widehat{\Omega}^1_n) \xrightarrow{\simeq} \left(\widetilde{\mathrm{Def}}_n^{\mathrm{W}}\right)^{\mathbb{R}^\times \times \mathrm{Aff}(\mathbb{R})}. \tag{1}$$

Here, $\widehat{\Omega}^1_{n,cl}$ denotes the restriction of the W_n-module along the map $p : \mathrm{W}_n \ltimes \widehat{\Omega}^1_{n,cl} \to \mathrm{W}_n$.

There is a natural cocycle in $\mathrm{C}^1_{\mathrm{Lie}}(\mathrm{W}_n \ltimes \widehat{\Omega}^1_{n,cl}; \widehat{\Omega}^1_{n,cl})$ given by the identity id_{Ω^1} on one-forms.

Lemma 3.4. *The space* $\mathrm{H}^1(\mathrm{W}_n \ltimes \widehat{\Omega}^1_{n,cl}, \mathrm{GL}_n; \widehat{\Omega}^1_{n,cl})$ *is two dimensional spanned by* $c_1^{\mathrm{GF}}(\widehat{\mathcal{T}}_n)$ *and* id_{Ω^1}.

Thus, the cocycle id_{Ω^1} determines an additional cohomologically non-trivial deformation in the extended case.

3.4.1. *Quantization.* We obtain a quantization for this classical theory in the same way as in the non-extended case. In the W_n-equivariant theory we showed that we have a quantization given by the collection of functionals $\{I^{\mathrm{W}}[L]\} \subset \mathrm{C}^{\#}_{\mathrm{Lie}}(\mathrm{W}_n; \mathrm{C}^{\#}_{\mathrm{Lie}}(\mathbb{D}\mathfrak{g}_n^{\mathbb{R}}))[[\hbar]]$. We can restrict these functionals along the morphism $p : \mathrm{W}_n \ltimes \widehat{\Omega}^1_{n,cl}$ to obtain functionals

$$\widetilde{I}^{\mathrm{W}}[L] := p^*(I^{\mathrm{W}}[L]) \in \mathrm{C}^{\#}_{\mathrm{Lie}}(\mathrm{W}_n \ltimes \widehat{\Omega}^1_{n,cl}; \mathrm{C}^{\#}_{\mathrm{Lie}}(\mathbb{D}\mathfrak{g}_n^{\mathbb{R}})[[\hbar]].$$

The same argument as above shows that this pre-quantization is actually a quantization; that is, there is no obstruction.

Proposition 3.5. *The functionals $\{\widetilde{I}^{\mathrm{W}}[L]\}$ define a quantization for the $\mathrm{W}_n \ltimes \widehat{\Omega}^1_{n,cl}$-equivariant theory.*

3.4.2. *A deformation.* We are interested in a deformation of this theory given by the additional cocycle present in the extended case. The map $J : \widehat{\Omega}^1_{n,cl} \to \mathrm{Def}_n$ from OG: add cross ref extends to a map

$$J : \mathrm{W}_n \ltimes \widehat{\Omega}^1_{n,cl} \to \mathrm{Def}_n$$

that sends $(X, \theta) \mapsto J_\theta$. Hence we can realize J as a cocycle in $\mathrm{Def}_n^{\mathrm{W}}$. Tautologically, this cocycle corresponds to the identity cocycle $\mathrm{id}_{\Omega^1} \in \mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n \ltimes \widehat{\Omega}^1_{n,cl}; \widehat{\Omega}^1_{n,cl})$ under the quasi-isomorphism (1).

We then obtain a deformation of the quantum theory as follows. For $L > 0$ define the functional

$$J[L] := \lim_{\epsilon \to 0} \sum_{\substack{\Gamma \in \mathrm{Trees} \\ v \in V(\Gamma)}} W_{\Gamma,v}(P_{\epsilon<L}, I, J)$$

where the sum is over graphs and distinguished vertices. The notation $W_{\Gamma,v}(P_{\epsilon<L}, I^{\mathrm{W}}, J)$ means that we compute the weight of the graph Γ with J placed at the vertex v and I^{W} placed at all other vertices. OG: Add picture!

Proposition 3.6. *BW: ref any of Si's papers here, he always proves this fact For each $L > 0$ the functional $I^{\mathrm{W}}[L] + \hbar J[L]$ satisfies the scale L quantum master equation*

$$(\mathrm{d}_{dR} + \mathrm{d}_{\mathrm{W}_n \ltimes \widehat{\Omega}^1_{n,cl}})(I^{\mathrm{W}}[L] + \hbar J[L]) + \frac{1}{2}\{I^{\mathrm{W}}[L] + \hbar J[L], I^{\mathrm{W}}[L] + \hbar J[L]\}_L + \hbar \Delta_L(I^{\mathrm{W}}[L] + \hbar J[L]) = 0.$$

Thus, the family of functionals $\{\widetilde{I}^{\mathrm{W}}[L] + \hbar J[L]\}_{L>0}$ defines a quantization of the $\mathrm{W}_n \ltimes \widehat{\Omega}^1_{n,cl}$-equivariant theory.

4. The Global Theory

4.1. Descending to the Global Classical Theory.

RG: Comparison Theorem to GG theory RG: Fix notation to be consistent

Proposition 4.1. *For any choice of Gelfand-Kazhdan structure σ on a smooth manifold X, there is an isomorphism*

$$\mathbf{desc}(\sigma, \mathfrak{g}_n) \cong \mathfrak{g}_X$$

of sheaves on X of curved L_∞ algebras.

We have already seen how the action of W_n on $\mathfrak{g}_X$ extends to an action of W_n on the dg Lie algebra $\mathbb{D}\mathfrak{g}_n^{\mathbb{R}}$ defining the classical field theory. As a corollary of the above we obtain.

Corollary 4.2. *Let $\mathbb{D}\mathfrak{g}_X^{\mathbb{R}}$ be the curved L_∞ algebra defining one-dimensional Chern-Simons with target X. Then $\mathrm{desc}(\sigma, \mathbb{D}\mathfrak{g}_n^{\mathbb{R}}) \cong \mathbb{D}\mathfrak{g}_X^{\mathbb{R}}$.*

RG: Note that $\Omega^*(S^1) \otimes -$ commutes with descent and the shifted symplectic structure only depends on the source: it's just the Kronecker pairing on the target.

Thus, one-dimensional Chern-Simons OG: theory! with target the formal n-disk descends to one-dimensional Chern-Simons with target any smooth manifold.

4.2. **Descending to the Global Quantum Theory.**

Proposition 4.3. *The obstruction deformation complex descends to the obstruction deformation complex...*

Lemma 4.4. *If the classical descends and the obs def descends than a formal quantization descends (up to homotopy) to an a priori quantization of the global theory.*

Corollary 4.5. *The formal quantization descends (up to homotopy) to the quantization constructed in GG...*

RG: Could alternatively/additionally remark how the propogator and obstruction descend...