THE HOLOMORPHIC TWIST OF $\mathcal{N} = 1$ SUPERMULTIPLET IN FOUR DIMENSIONS

In this section we show that there exists a twist of the $\mathcal{N} = 1$ chiral supermultiplet in four dimensions that is equivalent to our prototypical example of a holomorphic theory: the two dimensional $\beta\gamma$ system.

0.1. $\mathcal{N} = 1$ supersymmetry.

0.1.1. *The $\mathcal{N} = 1$ SUSY algebra.* First, we set up some representation theoretic notations. Let S_+, S_- be the irreducible spin representations. With respect to the decomposition, $\mathrm{Spin}(4) = SU(2) \times SU(2)$ they are the two defining representations of $SU(2)$, hence $S_\pm$ are each complex two dimensional. Let $V_\mathbb{R}$ be the real vector representation of $\mathrm{Spin}(4)$. This is a real four dimensional representation. We denote by $V_\mathbb{C} = V_\mathbb{R} \otimes_\mathbb{R} \mathbb{C} \cong \mathbb{C}^4$ its complexification.

There are $SU(2)$-invariant pairings

$$\langle -, - \rangle : S_\pm \otimes S_\pm \to \mathbb{C},$$

and a $\mathrm{Spin}(4)$-invariant vector valued pairing

$$\Gamma : S_+ \otimes S_- \xrightarrow{\cong} V_\mathbb{C}.$$

If we choose a basis for V one can think of the components of Γ as the usual gamma matrices $\Gamma^i_{\alpha\dot\beta}$ where $\alpha, \dot\beta = 1, 2$ run over a basis for S_+, S_- respectively, and $i = 1, \ldots, 4$ runs over a basis for V.

Define the super Lie algebra

$$T^{\mathcal{N}=1} = V_\mathbb{C} \oplus \Pi(S) = V_\mathbb{C} \oplus \Pi(S_+ \oplus S_-)$$

with only non-trivial bracket

$$[(v, Q), (v', Q')] = \Gamma(Q \otimes Q').$$

Note that $V_\mathbb{C}$ is central. This is the $d = 4, \mathcal{N} = 1$ supertranslation Lie algebra.

A supersymmetric theory on $\mathbb{R}^4$ is one that is invariant under the spin extension of the Poincaré group $\mathrm{Spin}(4) \ltimes \mathbb{R}^n$ such that the complexified infinitesimal action of the complex Lie algebra $\mathfrak{so}(4, \mathbb{C}) \ltimes \mathbb{C}^4$ lifts to an action of the super Lie algebra

$$\mathfrak{so}(4, \mathbb{C}) \ltimes T^{\mathcal{N}=1}.$$

0.1.2. *The chiral multiplet.* We work on the four manifold $\mathbb{R}^4$ equipped with the flat metric and coordinates $\{x_1, \ldots, x_4\}$. We will write Ω^i for the space of complexified smooth i-forms on $\mathbb{R}^4$. The complexified space of sections of the spinor bundles $S_\pm$ are denoted by $\mathcal{S}_\pm$. Write $\mathrm{d}^4x = \mathrm{d}x_1 \cdots \mathrm{d}x_4$ and $\star$ for the Hodge star operator.

We start by recalling the physical description of the $\mathcal{N} = 1$ chiral multiplet, then show how to embed the theory into the classical BV formalism. In the next section we will discuss twisting the BV theory.

The chiral matter we consider will take values in a complex vector space V (not to be confused with $V_\mathbb{C}$). The physical fields consist of bosonic fields $\varphi_\pm, F_\pm$ fermionic fields $\psi_\pm \in \mathcal{S}_\pm$.

The chiral scalar φ_+ takes is a function with values in V, $\varphi \in C^\infty(\mathbb{R}^4) \otimes V$, and the chiral spinor is an element $\psi_+ \in \mathcal{S}_+ \otimes V$. The auxillary scalar field $F \in C^\infty(\mathbb{R}^4) \otimes V$ does not propagate, but exists to ensure the closure of the SUSY algebra.

Similarly, the antichiral fields are elements $\varphi_- \in C^\infty(\mathbb{R}^4) \otimes V^*$, $\psi_- \in \mathcal{S}_- \otimes V^*$, $F \in C^\infty(\mathbb{R}^4) \otimes V^*$.

The action functional is given by

$$S^{\mathcal{N}=1}(\varphi_\pm, \psi_\pm, F_\pm) = \int \varphi_- \Delta \varphi_+ \mathrm{d}^4 x + \int \langle \psi_+, \partial\!\!\!/ \psi_- \rangle \mathrm{d}^4 x + \int F_+ F_- \mathrm{d}^4 x.$$

Here, $\langle -, - \rangle$ denotes the standard symplectic pairing $S_+ \otimes S_+ \to \mathbb{C}$. We are also implicitly using the pairing between V and V^*.

It is convenient to introduce the $\mathcal{N} = 1$ superfield formalism to organize the theory. Choose a basis $\{\theta_\alpha\} = \{\theta_1, \theta_2\}$ for the positive spin representation S_+ and a basis $\{\theta_{\dot\alpha}\}$ for the negative spin representation S_-. The chiral multiplet consists of a pair of superfields $\Phi_+ \in C^\infty(\mathbb{R}^4)[\theta_\alpha], \Phi_- \in C^\infty(\mathbb{R}^4)[\theta_{\dot\alpha}]$. The variables $\theta_\alpha, \theta\dot\alpha$ are odd and hence $C^\infty(\mathbb{R}^4)[\theta_\alpha], C^\infty(\mathbb{R}^4)[\theta_{\dot\alpha}]$ are considered as superalgebras.

In the notation above, the superfields can be written as

$$\begin{aligned} \Phi_+ &= \varphi_+ + \theta_\alpha \psi_{+,\alpha} + \theta_1\theta_2 F_+ \\ \Phi_- &= \varphi_- + \theta_{\dot\alpha} \psi_{-,\dot\alpha} + \theta_{\dot 1}\theta_{\dot 2} F_-. \end{aligned}$$

The action functional can be written as

$$S^{\mathcal{N}=1}(\Phi_\pm) = \int \mathrm{d}^4\theta \mathrm{d}^4 x \Phi_+ D^2 \overline{D}^2 \Phi_-$$

Here, $\int \mathrm{d}^4\theta$ is the Berezin integral.

We will write the chiral supermultiplet in the BV-BRST formalism. To do this, we must introduce anti-fields that pair with the physical fields above. There are no gauge symmetries, so this is all we have to do. In superfield notation, let

$$\begin{aligned} \Phi'_+ &= (\varphi'_+, \psi'_+, F'_+) \in \Omega^4 \otimes V \oplus \mathcal{S}_- \otimes V \oplus \Omega^4 \otimes V \\ \Phi'_- &= (\varphi'_-, \psi'_-, F'_-) \in \Omega^4 \otimes V^* \oplus \mathcal{S}_- \otimes V^* \oplus \Omega^4 \otimes V^*. \end{aligned}$$

We note that the anti-field to a positive spinor $\psi_+ \in \mathcal{S}_\pm$ is a spinor $\psi_\pm^\vee \in \mathcal{S}_+$ of the same chirality. The prime simply indicates the same underlying spinor bundle except we are viewing it as an anti-field. Using this notation, we can define the classical theory in the BV formalism in a succinct way.

The fields together with the anti-fields fit into the complex. We describe the theory S^{FO} as a classical theory in the BV formalism. The space of fields

$$
\begin{tikzcd}[row sep=small]
 & \underline{0} & \underline{1} \\
\text{Fermion degree } \underline{0} & C^\infty(\mathbb{R}^4)\otimes V \arrow[r, "\Delta"] \arrow[d, dotted, "Q"] & \left(\Omega^4\otimes V\right)' \\
\text{Fermion degree } \underline{1} & \mathcal{S}_+\otimes V \arrow[r, "\not{\partial}"] \arrow[d, dotted, "Q"] & (\mathcal{S}_-\otimes V)' \\
\text{Fermion degree } \underline{0} & (C^\infty(\mathbb{R}^4)\otimes V) & \left(\Omega^4\otimes V\right)' \\
\text{Fermion degree } \underline{0} & C^\infty(\mathbb{R}^4)\otimes V* \arrow[r, "\Delta"] \arrow[d, dotted, "Q"] & \left(\Omega^4\otimes V^*\right)' \\
\text{Fermion degree } \underline{1} & \mathcal{S}_-\otimes V^* \arrow[r, "\not{\partial}"] \arrow[d, dotted, "Q"] & (\mathcal{S}_+\otimes V^*)' \\
\text{Fermion degree } \underline{0} & (C^\infty(\mathbb{R}^4)\otimes V^*) & \left(\Omega^4\otimes V^*\right)'
\end{tikzcd}
\tag{1}
$$

The first three rows comprise the chiral superfield $\Phi_+ = (\varphi_+, \psi_+, F_+)$ in degree zero, and its antifield $\Phi'_+ = (\varphi'_+, \psi', F'_+)$. We use the prime label in the diagram to stress that the space consists of antifields, this will be a useful bookkeeping device once we twist. The last three rows comprise the anti-chiral superfield and its anitfield Φ_-, Φ'_-. The dotted arrows denote the SUSY transformations which we describe momentarily.

Definition 0.1. The free $\mathcal{N} = 1$ chiral supermultiplet on $\mathbb{R}^4$ in the BV formalism is the $\mathbb{Z}\times\mathbb{Z}/2$ graded theory with complex of fields in (1) (forgetting the dotted arrows). The (-1)-symplectic pairing is given by BW: define this.

We now turn to showing that this theory is actually supersymmetric. That is, we construct an invariant action by the super Lie algebra $\mathfrak{so}(4;\mathbb{C}) \ltimes T^{\mathcal{N}=1}$. The action of $\mathfrak{so}(4;\mathbb{C})$ is the natural one by the defining representation on spacetime. The action of the bosonic part $V_{\mathbb{C}} = \mathbb{C}^4$ of the supertranslations is by the constant vector fields $\frac{\partial}{\partial x_i}$. It is easiest to define the action of the odd part of the supertranslations using the superfield notation. Define the differential operators on superspace $D_\alpha, D_{\dot\alpha}$ by

$$
\begin{aligned}
D_\alpha &= \tfrac{\partial}{\partial\theta_\alpha} + \Gamma_i^{\alpha\dot\beta}\theta_{\dot\beta}\tfrac{\partial}{\partial x_i} \\
D_{\dot\alpha} &= \tfrac{\partial}{\partial\theta_{\dot\alpha}} + \Gamma_i^{\beta\dot\alpha}\theta_{\beta}\tfrac{\partial}{\partial_i}.
\end{aligned}
$$

Note that $\Gamma_i^{\alpha\dot\beta}\partial_{x_i}$ are the components of the Dirac operator $\not{\partial} : \mathcal{S}_+ \to \mathcal{S}_-$. Choose a basis $\{Q_1, Q_2\}$ and $\{Q_{\dot 1}, Q_{\dot 2}\}$ for S_+ and S_-, respectively. The action of $Q_\alpha, Q_{\dot\alpha}$ on the theory is by the operators by $D_\alpha, D_{\dot\alpha}$, respectively.

We are most interested for the explicit form in which an element $Q_- \in S_-$ acts on the theory. For any $Q_- \in \mathcal{S}_-$ we have

$$Q_+ \cdot (\varphi_+, \psi_+, F) = (0, \Gamma_+(Q \otimes \mathrm{d}\varphi_+), \langle Q, \partial\!\!\!/ \psi_+ \rangle).$$

The right-hand side requires a bit of explanation. The gamma matrices identify $\Gamma : S_+ \otimes S_- \cong V$. Together with the spin pairing, this defines a map

$$\Gamma_+ : S_- \otimes \Omega^1(\mathbb{R}^4) \to \mathcal{S}_+.$$

Thus $\Gamma_+(Q \otimes \mathrm{d}\varphi_+) \in \mathcal{S}_+$. Also, the spin pairing defines a map $\langle -, - \rangle : S_- \otimes \mathcal{S}_- \to C^\infty(\mathbb{R}^4)$ so $\langle Q, \partial\!\!\!/ \psi_+ \rangle \in C^\infty(\mathbb{R}^4)$.

The R-charge of an anti-field is opposite to that of the corresponding field. Thus, $\psi_\pm^\vee$ has R-charge ∓ 1 and $F_\pm^\vee$ has R-charge ∓ 2.

We will perform a twist of the free chiral supermultiplet by a fixed constant spinor $Q \in \mathcal{S}_-$. This element acts on the fields $\Phi = (\varphi_+, \psi_+, F)$ as above:

$$Q \cdot (\varphi_+, \psi_+, F) = (0, Q \cdot (\mathrm{d}\varphi_+), \langle Q, \partial\!\!\!/ \psi_+ \rangle).$$

The action of Q on the anti-fields reads is determined by compatibility with the (-1)-shifted symplectic pairing. Explicitly it is

$$Q \cdot (\varphi_+^\vee, \psi_+^\vee, F) = (\langle Q, \partial\!\!\!/ \psi_+^\vee \rangle, Q \cdot (\star F_+^\vee), 0).$$

We have arrived at the following.

Proposition 0.2. *The twist of the $N = 1$ free chiral supermultiplet on $\mathbb{R}^4$ with values in the hermitian vector space V by an element $Q \in \mathcal{S}_-$ is equivalent to the free $\beta\gamma$ system on $\mathbb{C}^2$ with values in V:*

$$(\gamma, \beta) \in \Omega^{0,*}(\mathbb{C}^2; V) \oplus \Omega^{1,*}(\mathbb{C}^2; V^\vee)[1].$$

The action functional is $S(\gamma, \beta) = \int \langle \beta, \bar{\partial}\gamma \rangle$ where $\langle -, - \rangle$ is the evaluation pairing on V.

0.1.3. We will introduce the following "first-order" reformulation of the chiral supermultiplet that will be convenient for our description of it as a BV theory. Introduce additional scalar fields of the form

$$B \in \Omega^3 \otimes V$$

and define the action

$$\begin{aligned} S^{\mathrm{FO}}(\varphi, B, \psi, F) &= \int h(B \wedge \mathrm{d}\varphi) - \frac{1}{2} \int h(B \wedge \star B) \\ &+ \int h(\langle \psi_+, \partial\!\!\!/ \psi_- \rangle) \mathrm{dvol} \end{aligned}$$

Lemma 0.3. *The theories S^{SO} and S^{FO} are classically equivalent.*

Proof. We show that the spaces of solutions to the classical equations of motion are equivalent. The equations of motion for the chiral supermultiplet read

$$\begin{aligned} \Delta\varphi &= 0 \\ \partial\!\!\!/ \psi &= 0 \\ F &= 0. \end{aligned}$$

Next, consider S^{FO}. The pieces of the action functional involving ψ, F are identical. We use the variation of the scalar field $\varphi \mapsto \varphi + \delta\varphi$ to obtain the equation of motion $\mathrm{d}B = 0$. The variation $B \mapsto B + \delta B$ yields the equation $\mathrm{d}\varphi - h^\vee(\star B)$. This equation is equivalent to $\star\mathrm{d}\varphi = h^\vee(B)$. Applying d to this equation and using the equation $\mathrm{d}B = 0$ we obtain $\mathrm{d}\star\mathrm{d}\varphi = \Delta\varphi = 0$, as desired.

An explicit equivalence at the level of fields can be written as follows. BW: Want $\varphi \mapsto \varphi, B \mapsto \star h^\vee(\mathrm{d}\varphi) + h^\vee(F)$. □

BW: physics description of susy action

0.1.4.