COHOMOLOGIES OF LIE ALGEBRA OF VECTOR FIELDS WITH NONTRIVIAL COEFFICIENTS

I. M. Gel'fand and D. B. Fuks

In this paper we continue the investigation which we began in [1-3] of the cohomologies of the Lie algebra of formal as well as smooth vector fields (on a smooth manifold). Specifically we study the cohomologies of the algebras with coefficients in various representations, mainly with coefficients in the spaces of exterior differential forms, formal or smooth, respectively. The results obtained in these papers, involving cohomologies with coefficients in spaces of forms of degree 0 (i.e., in the spaces of smooth functions or of formal power series), were also contained in the work of M. V. Losik [4]. It is true that theorems about the algebra of formal vector fields were not singled out, but were essentially contained in them (see [4], §2). We shall not rely here on the results of M. V. Losik since we will prove them again. Our proof in the corresponding part is not, in principle, different from the proof of M. V. Losik, although it is considerably shorter. The investigation of cohomologies with coefficients in forms of degree greater than zero encounters a series of new difficulties which are overcome with the application of the results of [3].

The first part of the present work is devoted to the algebra of formal vector fields. Using the concept of induced representation we reduce the investigation of the cohomologies of this algebra with coefficients in the spaces of formal exterior differential forms to the study of the cohomologies of the subalgebra of an algebra originating from the fields having a singularity at the origin of coordinates with coefficients now in a finite dimensional module. The latter problem is connected with this problem, now solved in [3]: the desired cohomologies are found through isomorphisms with the displacement of the dimensions over the columns of the initial term of the spectral sequence to be investigated in this paper. In particular, the cohomologies of the Lie algebra of formal vector fields with coefficients in formal exterior differential forms turned out to be finite-dimensional.

In the second part the Lie algebras of smooth vector fields on smooth manifolds are investigated. In the standard complex of cochains with coefficients in the space of smooth exterior differential forms (in general in the space of sections of a finite-dimensional vector fiber induced by a tangent fiber) we select a subcomplex which, in analogy with [1], we shall call diagonal. We emphasize that the diagonal complex to be studied here and the diagonal complex of [1], although they are defined similarly, play quite different roles. If, in the case of constant coefficients, a diagonal complex, in some sense, multiplicatively generates the entire complex of cochains then, for example, in the case of coefficients in the space of exterior differential forms, the diagonal subcomplex is the subring of the whole standard complex. Thus the ring of cohomologies of the algebra of smooth vector fields with coefficients in the full ring of smooth exterior differential forms is a module over the ring of cohomologies of the corresponding diagonal complex.*

The main result of the second part is the calculation of the ring of cohomologies of the diagonal complex with coefficients in the ring of smooth exterior differential forms. These cohomologies are found to be finite-dimensional: they depend only on the rational cohomologies of the manifold and on the rational classes of Pontryagin. In the general case (i.e., in the case when the coefficients are taken in he space of smooth sections of a fiber induced by tangent) the constructed spectral sequence converges to the homologies of the diagonal complex. The second term of this spectral sequence is a tensor product of real

*This ring was designated by M. V. Losik as the ring of differential cohomologies of the algebra of vector fields.

Moscow State University. Translated from Funktional'nyi Analiz i Ego Prilozheniya, Vol. 4, No. 3, pp. 10-25, July-September, 1970. Original article submitted March 31, 1970.

cohomologies of the manifold and the cohomologies of the algebra of formal vector fields with appropriate coefficients. In particular if the latter cohomologies are finite-dimensional, then the homologies of the diagonal complex are also finite-dimensional.

The question over the arrangement of homologies of a completely standard complex of the algebra of smooth vector fields with nontrivial coefficients remains completely open.

I. ALGEBRAS OF FORMAL FIELDS

§1. Cohomologies of Algebra L_0 with Coefficients in Skew-Symmetric Representations

1.1. We shall denote by W the Lie algebra of formal vector fields at the point 0 of the point $\mathbf{R}^n$. The subalgebra of the algebra W is denoted by L_k $k = 0, 1, \ldots$), consisting of those fields $\sum_{i=1}^{n} a_i(x_1, \ldots, x_n)e_i$ (where $e_1, \ldots, e_n$ are vectors of a standard basis of space $\mathbf{R}^n$) for which in the power series $a_i(x_1, \ldots, x_n)$ terms of degree less than k are totally absent. It is evident that the space W/L_k is finite-dimensional and that $L_1, L_2, \ldots$ is the normal subalgebra of the algebra L_0. The factor algebra L_0/L_1 is isomorphic to $\mathfrak{gl}(n, \mathbf{R})$, moreover, there exists an imbedding of the algebra $\mathfrak{gl}(n, \mathbf{R})$ into L_0 whose superposition is identical with the projection $L_0 \to L_0/L_1 = \mathfrak{gl}(n, \mathbf{R})$: this imbedding relates the element $A_{ij} \in \mathfrak{gl}(n, \mathbf{R})$ to the element $x_i e_j \in L_0$ (here A_{ij} is a matrix in which the only nonvanishing element is at the intersection of the i-th row with the j-th column and equals unity).

Any representation of the algebra $\mathfrak{gl}(n, \mathbf{R})$ induces, owing to the existence of the projection $L_0 \to \mathfrak{gl}(n, \mathbf{R})$, a representation of the algebra L_0. In particular we will consider as the L_0-modules the spaces $\Lambda^r(\mathbf{R}^n)'$ of exterior forms of degree r of the space $\mathbf{R}^n$ (in which the algebra $\mathfrak{gl}(n, \mathbf{R})$ acts in a natural manner). It is evident that the action of the algebra L_0 in $\Lambda^*(\mathbf{R}^n)' = \sum_{r=0}^{n} \Lambda^r(\mathbf{R}^n)'$ is congruent with the ring structure in $\Lambda^*(\mathbf{R}^n)'$ since the cohomologies of the algebra L_0 with coefficients in $\Lambda^*(\mathbf{R}^n)'$ represent the same algebra over $\mathbf{R}$. In what follows we shall write $\Lambda^r(\mathbf{R}^n)'$, $\Lambda^*(\mathbf{R}^n)'$ as Λ^r, Λ^*.

1.2. We formulate the main results of this paragraph.

<u>THEOREM.</u> A bigraded ring $H^*(L_0; \Lambda^*)$ is multiplicatively generated by 2n generators

$$\rho_i \in H^{2i-1}(L_0; \Lambda^0) \quad (i = 1, \ldots, n), \qquad \tau_j \in H^j(L_0; \Lambda^j) \quad (j = 1, \ldots, n).$$

These generators are connected only through such relations:

$$\rho_k\rho_l = -\rho_l\rho_k; \quad \rho_k\tau_l = \tau_l\rho_k; \quad \tau_k\tau_l = \tau_l\tau_k; \quad \tau_1^{j_1}\tau_2^{j_2} \ldots \tau_n^{j_n} = 0 \text{ for } j_1 + 2j_2 + \ldots + nj_n > n.$$

In particular the ring $H^*(L_0; \mathbf{R}) = H^*(L_0; \Lambda^0)$ is an exterior algebra from the generators with dimensions of $1, 3, 5, \ldots, 2n-1$; i.e., $H^*(L_0; \mathbf{R}) = H^*(\mathfrak{gl}(n, \mathbf{R}); \mathbf{R})$. Moreover,

$$H^q(L_0; \Lambda^r) = \begin{cases} 0 & \text{for } q < r, \\ H^r(L_0; \Lambda^r) \otimes H^{q-r}(\mathfrak{gl}(n, \mathbf{R}); \mathbf{R}) & \text{for } q \geqslant r, \end{cases}$$

and the dimension of the space $H^r(L_0; \Lambda^r)$ is equal to the number of different representations of the number r in the form of natural terms.

Proof of this theorem is contained in §1.3-1.7.

The plan of the proof is as follows. We compare two spectral sequences which converge to the cohomologies of the algebra W with coefficients in $\mathbf{R}$. Essentially, this is the spectral sequence of Serre-Hochshild [6] corresponding to subalgebras $\mathfrak{gl}(n, \mathbf{R})$ and L_0 of the algebra W. Next we shall prove that E_1 terms of these spectral sequences are isomorphic (up to a permutation of the indices) and at the same time one of these is $H^*(L_0; \Lambda^*)$, and the other was calculated by us in [3].

1.3. We shall begin by recapitulating some facts from [3].

The elements of the space $C^q(W; \mathbf{R})$, i.e., the continuous skew-symmetric q-linear real functionals on W, can be identified with the functions depending on q vectors $\beta_1, \ldots, \beta_q \in \mathbf{R}^n$ and on q covectors $\alpha_1, \ldots, \alpha_q \in (\mathbf{R}^n)'$, multilinear in $\beta_1, \ldots, \beta_q$, polynomials in the coordinates of the covectors $\alpha_1, \ldots, \alpha_q$, and change sign under the simultaneous interchange of α_i with α_j, β_i with β_j. Namely, to the function

$$P(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_q) = \sum A_{m_{11}, \ldots, m_{qn}; i_1, \ldots, i_q} \alpha_{11}^{m_{11}} \ldots \alpha_{qn}^{m_{qn}} \beta_{1 i_1} \ldots \beta_{q i_q},$$

where $\alpha_{i1}, \ldots, \alpha_{in}$ are the coordinates of the covector α_i, and $\beta_{i1}, \ldots, \beta_{in}$ are the coordinates of the vector β_i, there corresponds a cochain $p \in C^q$ (W; R) definable by the formula

$$p\left(\sum a_{1i} e_i, \ldots, \sum a_{qi} e_i\right) = \sum A_{m_{11}, \ldots, m_{qn}; i_1, \ldots, i_q} \left[\prod_{s=1}^{n} \frac{\partial^{m_{s1} + \ldots + m_{sn}} a_{s i_s}}{\partial^{m_{s1}} x_1 \ldots \partial^{m_{sn}} x_n} \right]_{\substack{x_1 = 0 \\ \cdots \\ x_n = 0}} \tag{1}$$

The number $m_i = m_{i1} + \ldots + m_{in}$ is called the degree of the monomial $\alpha_{11}^{m_{11}} \ldots \alpha_{qn}^{m_{qn}} \beta_{1 i_1} \ldots \beta_{q i_q}$ in α_i. We denote through C_0^*(W; R) a subcomplex of the complex C^*(W; R) consisting of monomials in which $m_1 + \ldots + m_q = q$. It was shown (see [3], §1.4; [1] §6.2) that the imbedding induces an isomorphism of the homologies.

The subspace F_s of the space C_0^*(W; R) is defined as the set of those polynomials, the mean degree of each of their monomials in $\alpha_1, \ldots, \alpha_q$ is not less than s, different from 1. The spaces F_s make up a filtration

$$C_0^*(W; \mathbf{R}) = F_0 \supset F_1 \supset F_2 \supset \ldots$$

This filtration is identical with the differential [see [3], §2.2] and generates therefore a spectral sequence converging to the homologies of the complex C_0^*(W; R), i.e., to H^*(W; R). We denote this spectral sequence by $\mathscr{E} = \{E_r^{s,t}, d_r^{s,t}\}$.

The ring $E_1 = \sum E_1^{s,t}$ is generated by the generators $\varphi_i \in E_1^{0,2i-1}$ (i = 1, ..., n), $\psi_j \in E_1^{2j,0}$ (j = 1, ..., n), which are connected by the relations

$$\varphi_k \varphi_l = -\varphi_l \varphi_k; \quad \varphi_k \psi_i = \psi_i \varphi_k; \quad \psi_i \psi_k = \psi_k \psi_i; \quad \psi_1^{j_1} \ldots \psi_n^{j_n} = 0 \text{ for } j_1 + 2j_2 + \ldots + nj_n > n.$$

Proof is carried out in §5 of [3].

1.4. Alongside the filtration $\{F_s\}$ we shall investigate in the complex C_0^*(W; R) the filtration $\{F'_s\}$ defined as follows. We denote by F'_{2r} the subspace of the space C_0^*(W; R) consisting of those polynomials, among each of whose monomials at least r vanish, and we put $F'_{2r-1} = F'_{2r}$. The filtration

$$C_0^*(W; \mathbf{R}) = F'_0 \supset F'_1 \overset{(=)}{\supset} F'_2 \supset F'_3 \overset{(=)}{\supset} F'_4 \supset \ldots$$

also agrees with the differential in the complex C_0^*(W; R) [this follows directly from the formula (1)] and therefore generates a spectral sequence which converges to H^*(W; R). We denote this spectral sequence by $\mathscr{E}' = \{'E_r^{s,t}, 'd_r^{s,t}\}$. It is clear that $'E_r^{s,t} = 0$ if s is odd.

There exists an imbedding $F_s \supset F'_s$; indeed if among nonnegative integers $m_1, \ldots, m_q$ with $m_1 + \ldots + m_q = q$ at least s are different from unity then among these at least $[(s+1)/2]$ are zero. This imbedding induces a homomorphism of the spectral sequence $\mathscr{E}$ in the spectral sequence $\mathscr{E}'$; we denote it by $\varkappa = \{\varkappa_r : E_r \to {}'E_r\}$.

1.5. The homomorphism $\varkappa_r : E_r \to {}'E_r$ is an isomorphism for $r \geq 1$.

Proof. In each of the spaces C_0^*(W; R), E_0, $'E_0$ it is possible to take as basis the set of skew-symmetric monomials of the form

$$\alpha_{11}^{m_{11}} \ldots \alpha_{qn}^{m_{qn}} \beta_{1 i_1} \ldots \beta_{q i_q}. \tag{2}$$

In the differential dP of such a skew-symmetric monomial P of degrees $m_1, \ldots, m_q$ in $\alpha_1, \ldots, \alpha_q$ there enter monomials whose degrees in $\alpha_1, \ldots, \alpha_{q+1}$ (taken in some order) are equal to $m_1, \ldots, \hat{m}_i, \ldots, m_q, m'_i, m''_i$, where $m'_i + m''_i = m_i + 1$, $m'_i \geq m''_i$ (this follows from p. 1.3 of [3]). The differential $'d_0P$ of this same monomial in $'E_0$ is obtained if we choose from dP those monomials in which $m''_i = 0$, $m_i \neq 0$; the differential d_0P of the element P in E_0 is obtained if we leave in dP only those terms in which one of the numbers m'_i, m''_i is equal to m_i, and the remaining are unity.

The complex E_0 can be decomposed into a direct sum of two subcomplexes: $E_0^{(1)}$ and $E_0^{(2)}$; the first is generated by skew-symmetric monomials, each of whose degree is less than or equal to 2, the second by the remaining. From §4.6 of [3] it can be seen that all the elements from E_1, i.e., all classes of homologies of the complex E_0 can be represented by polynomials consisting of monomials whose degree is less than or equal to 2. Hence it follows that the imbedding $E_0^{(1)} \to E_0$ induces a cohomological isomorphism and the complex $E_0^{(2)}$ is acyclic.

The homomorphism $\varkappa_0$ isomorphically maps $E_0^{(1)}$ onto some subcomplex $'E_0^{(1)}$ of the complex E_0 and identically vanishes on $E_0^{(1)}$. Since $\varkappa_1$ is a homological homomorphism induced by the homomorphism $\varkappa_0$, our statement is reduced to the fact that the imbedding $'E_0^{(1)} \to 'E_0$ induces an isomorphism of the homologies. The latter, in view of the exactness of the pair sequence $('E_0, 'E_0^{(1)})$ is equivalent to the acyclicity of the complex $'E_0^{(2)} = 'E_0/'E_0^{(1)}$.

Let us compare the complexes $E_0^{(2)}$ and $'E_0^{(2)}$. They have the same basis – the set of skew-symmetric monomials of the form (2), among which there is none with degree less than 3. We denote a filtration of a skew-symmetric monomial P with degrees $m_1, \ldots, m_q$ by the quantity $\varphi(P) = m_1^2 + \ldots + m_q^2 - q$. It is clear that $'d_0P$ is a linear combination of monomials of a filtration less than equal to $\varphi(P)$ and d_0P is obtained from $'d_0P$ by the crossing out of monomials of a filtration smaller than $\varphi(P)$. (Both the statements follow from the fact that for any natural m', m", and also for m' = 0, m" = 1, there exists the inequality $(m')^2 + (m'')^2 - 1 \geq (m' + m'' - 1)^2$, where the equality is attained only when one of the numbers m', m" equals unity.) Hence it can be seen that $E_0^{(2)}$ is none other than the initial term of the spectral sequence corresponding to this filtration in the complex $'E_0^{(2)}$. Hence, from the acyclicity of the complex $E_0^{(2)}$ follows the acyclicity of the complex $'E_0^{(2)}$. The proposition is proved.

1.6. From §1.5, by having taken $F'_{2r-1} = F'_{2r}$, we have naturally stretched the enumeration in the definition of the filtration $\{F'_s\}$. We will however consider the spectral sequence $\mathscr{E}'' = \{''E_r^{s,t}, ''d_r^{s,t}\}$, corresponding to the "nonexpanding" of the filtration

$$C_0^*(W; \mathbf{R}) = F_0'' \supset F_1'' \supset F_2'' \supset \ldots,$$

where $F_s'' = F_{2s}'$. It is evident that the spectral sequences $\mathscr{E}'$ and $\mathscr{E}''$ differ only in enumerations (see Fig. 1):

$$'E_r^{s,t} = \begin{cases} 0, & \text{if } s \text{ is odd,} \\ ''E_{[(r+1)/2]}^{s/2,\, t+(s/2)}, & \text{if } s \text{ is even} \end{cases}$$

$$'d_r^{s,t} = \begin{cases} 0, & \text{if } s \text{ is odd,} \\ ''d_{[(r+1)/2]}^{s/2,\, t+(s/2)}, & \text{if } s \text{ is even.} \end{cases}$$

Thus the isomorphism $''E_{[(r+1)/2]} \to 'E_r$, translating $''E_{[(r+1)/2]}^{u,v}$ into $'E_r^{2u,v-u}$, is multiplicative.

1.7. There exists an isomorphism of bigraded rings

$$H^*(L_0; \Lambda^*) = \sum_{q,r} H^q(L_0; \Lambda^r) \quad u \quad ''E_1 = \sum_{q,r} {''E_1^{q,r}}.$$

To prove this proposition we shall complete, in view of §1.3, 1.5, 1.6, the proof of Theorem 1.2. We denote through $\tilde{F}_r''$ a subcomplex of the complex $C^*(W; \mathbf{R})$ consisting of such polynomials that among the powers of each of these monomials at least r vanish. Evidently $F_r'' = \tilde{F}_r'' \cap C_0^*(W; \mathbf{R})$. The imbedding $F_r'' \to \tilde{F}_r''$ induces an isomorphism of the homologies (this follows from § 6.2 of [1] and §1.6 of [3]). From the lemma over the five homologies it follows that an isomorphism of homologies also induces the imbedding $F_r''/F_{r+1}'' \to \tilde{F}_r''/\tilde{F}_{r+1}''$.

The subcomplex $\tilde{F}_r''$ of the complex $C^*(W; \mathbf{R})$ can be defined also as consisting of such rings $p \in C^*(W; \mathbf{R})$ that $p(\xi_1, \ldots, \xi_q) = 0$ for $\xi_r, \ldots, \xi_q \in L_0$ (here $q = \dim p$). The equivalence of the two definitions of the subcomplex $\tilde{F}_r''$ immediately follows from the formula (1).

For all the formal vector fields $\xi = \sum a_i(x_1, \ldots, x_n)e_i \in W$ we denote by ξ^0 its free term (i.e., the vector $\sum a_i(0, \ldots, 0)e_i$ and by $\tilde{\xi}$ the difference $\xi - \xi^0 \in L_0$. We define the mapping $g : C^q(L_0; \Lambda^r) \to C^{q+r}(W; \mathbf{R})$, by putting

$$[g(P)](\xi_1, \ldots, \xi_{q+r}) = \sum_{1 \leqslant i_1 < \ldots < i_q \leqslant q+r} (-1)^{i_1 + \ldots + i_q - q(q+1)/2} [P(\tilde{\xi}_{i_1}, \ldots, \tilde{\xi}_{i_q})](\xi^0_{j_1}, \ldots, \xi^0_{j_r}),$$

where $P \in C^q(L_0; \Lambda^r)$, $\xi_1, \ldots, \xi_{q+r} \in W$, and $\{j_1, \ldots, j_r\}$ is the complement of the set $\{i_1, \ldots, i_q\}$ in the set $\{1, \ldots, q+r\}$, where $j_1 < \ldots < j_r$.

It is clear that $g(C^*(L_0;\ \Lambda^r)) \subset \tilde{F}_r^n$. The composition g_0 of the mapping g with the projection $\tilde{F}_r^n \to \tilde{F}_r^n/\tilde{F}_{r+1}^n$ is the isomorphism $C^*(L_0;\ \Lambda^r) \to \tilde{F}_r^n/\tilde{F}_{r+1}^n$. Indeed the mapping $h : \tilde{F}_r^n \to C^*(L_0;\ \Lambda^r)$, defined by the formula

$$[h(P')(\xi_1, \ldots, \xi_q)](\varepsilon_1, \ldots, \varepsilon_r) = P'(\xi_1, \ldots, \xi_q, \varepsilon_1, \ldots, \varepsilon_r),$$

where $\xi_1, \ldots, \xi_q \in L_0 \subset W$, $\varepsilon_1, \ldots, \varepsilon_r \in R^n \subset W$, is trivial on $\tilde{F}_{r+1}^n$ and thus $h \circ g = 1$, $\mathrm{Im}[(g \circ h) - 1] \subset \tilde{F}_{r+1}^n$. It is easy to verify that the mapping h is permutable with bounded operators so that g_0 is an isomorphism of the complexes. Consequently the mapping g_0 induces an isomorphism of q-dimensional cohomologies of the algebra L_0 with coefficients in Λ^r and (q + r)-dimensional homologies of the complex $\tilde{F}_r^n/\tilde{F}_{r+1}^n$, i.e., an isomorphism between $H^q(L_0;\ \Lambda^r)$ and $E_1^{q,r}$. The multiplicativity resulting from the isomorphism $H^*(L_0;\ \Lambda^*) \to E_1$ is seen, for example, from the definition of the mapping h.

§1.7 is proved and with it Theorem 1.2 is also proved.

Remark. The coincidence of the homologies of the complex $\tilde{F}_r^n/\tilde{F}_{r+1}^n$ with $H^*(L_0;\ \Lambda^r)$ can be derived also from the well-known Serre-Hochshild theorem [6].

1.8. The imbedding $\mathfrak{gl}(n, R) \to L_0$ induces an isomorphism $H^*(\mathfrak{gl}(n, R)) \to H^*(L_0;\ R)$.

The correctness of this version of Theorem 1.2 can be discerned from the proof of the latter.

2. Relations Between the Cohomologies of the Lie Algebras L_0 and W

In this paragraph the results of §1 are processed into information over the cohomologies of the algebra W. All the propositions proved here are corollaries of the corresponding assertions of §1 and the classical theorem over the cohomologies of the Lie algebra. However, since the general theory of the cohomologies of the topological Lie algebras has not been set forth anywhere, we shall briefly give a straightforward proof.

2.1. Let M be a finite-dimensional module over the algebra L_0. We denote by $\tilde{M}$ the space of formal power series of the variables $x_1, \ldots, x_n$ (identifiable with the coordinates in the space R^n) with coefficients from M. The action of the algebra W on $\tilde{M}$ is defined as follows. If $a(x_1, \ldots, x_n)$ is a power series (with real coefficients) and m is an element of M then the element $ma \in \tilde{M}$ under the action of the vector field $\xi \in W$ goes over into the element

$$m\xi(a) + \sum_{q_1 \geq 0, \ldots, q_n \geq 0} \frac{1}{q_1! \ldots q_n!} \left[\left(\widetilde{\frac{\partial^{q_1+\ldots+q_n}\xi}{\partial x_1^{q_1} \ldots \partial x_n^{q_n}}} \right) m \right] x_1^{q_1} \ldots x_n^{q_n}$$

of the module $\tilde{M}$. Here $\xi(a)$ is the result of the application of the differential operator ξ to the power series a, the differentiation of the vector field is with respect to the coordinates, the tilde means the same as in §1.7.

It is immediately verified that the mentioned action of the algebra W on $\tilde{M}$ agrees with the operation of commutation in W. Thus $\tilde{M}$ is equipped with the structure of the W-module. One says that the W-module of $\tilde{M}$ is induced by an L_0-module of M.

Evidently if M is a ring and mappings of the algebra L_0 are through its differentiations then $\tilde{M}$ is a ring and the mappings of the algebra W are also through its differentiations.

Example. The module Λ^r over the algebra L_0 induces a W-module Ω^r of the formal exterior differential forms of degree r with the ordinary action of the algebra W; in particular the trivial L_0-module of W induces a W-module of the power series.

Remark. We leave it to the reader to prove that our definition of the W-module agrees with the ordinary definition of the induced module which is as follows. If A is a Lie algebra, and B its subalgebra, then the A-module, induced by the B-module of M is defined as $\mathrm{Hom}_{[B]}([A], M)$, where [A], [B] are the enveloping algebras for A, B.

2.2. For every finite dimensional L_0-module of M there exists an isomorphism between the cohomologies of the algebra L_0 with coefficients in M and the cohomologies of the algebra W with coefficients in $\tilde{M}$. If M is a ring and the action of the algebra L_0 is multiplicative then the mentioned isomorphism is annular.

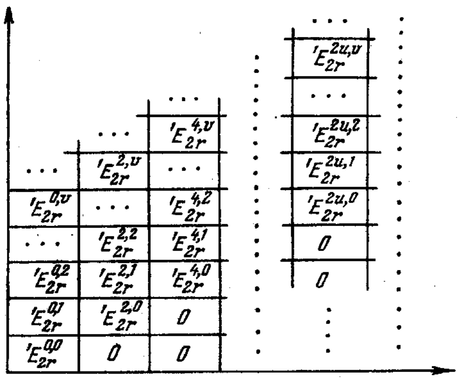

Fig. 1

In particular the bigraded rings $H^*(W;\ \Omega^r)$ and $H^*(L_0;\ \Lambda^r)$ are isomorphic. Thus Theorem 1.2 gives a description of the ring $H^*(W;\ \Omega^r)$.

The reciprocally inverse homomorphisms $H^q(W;\ \Omega^r) \to H^q(L_0;\ \Lambda^r)$, $H^q(L_0;\ \Lambda^r) \to H^r(W;\ \Omega_r)$ are induced by the mappings

$$f^*: C^q(W;\ \Omega^r) \to C^q(L_0;\ \Lambda^r), \qquad g: C^q(L_0;\ \Lambda^r) \to C^q(W;\ \Omega^r),$$

acting according to the formula

$$[f(P)](\xi_1, \ldots, \xi_q) = \nu_0(P(\xi_1; \ldots, \xi_q))$$

(here $P \in C^q(W;\ \widetilde{M})$; $\xi_1, \ldots, \xi_q \in L_0 \subset W$; ν_0 is a function relating a power series to its free term);

$$[g(Q)](\xi_1, \ldots, \xi_q) = \sum \frac{1}{r_{11}! \ldots r_{qn}!} Q\left(\widetilde{\frac{\partial^{r_{11}+\ldots+r_{1n}}\xi_1}{\partial^{r_{11}}x_1 \ldots \partial^{r_{1n}}x_n}}, \ldots, \frac{\partial^{r_{q1}+\ldots+r_{qn}}\xi_q}{\partial^{r_{q1}}x_1 \ldots \partial^{r_{qn}}x_n}\right) x_1^{q_1} \ldots x_n^{q_n}$$

(here $Q \in C^q(L_0;\ M)$; $\xi_1, \ldots, \xi_p \in W$; the summation is carried out over all nonnegative integers $r_{11}, \ldots, r_{1n}, \ldots, r_{q1}, \ldots, r_{qn}, q_1, \ldots, q_n$ such that $r_{11} + \ldots + r_{q1} = q_1, \ldots, r_{1n} + \ldots + r_{qn} = q_n$).

Remark. The equality $H^*(B;\ M) = H^*(A;\ \mathrm{Hom}_{[B]}([A], M))$ (the notations are borrowed from the remark to §2.1) in the classical theory of the cohomologies of the Lie algebra is well known (see for example [5], § XIII, 4.2).

2.3. The spectral sequence $\mathscr{E}''$ (see, §1.6) can be interpreted as a spectral sequence, corresponding to no acyclic resolvent

$$0 \to \mathbf{R} \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \to \ldots \to \Omega^n \to 0$$

of the W-module of $\mathbf{R}$ (grossly speaking as the second spectral sequence of the double complex $C^*(W;\ \Omega^*)$; see [5], XV, § 6). Proof is left to the reader.

II. ALGEBRAS OF SMOOTH FIELDS

§3. The Diagonal Complex

3.1. Let A be a finite-dimensional GL(n, R)-module (i.e., a finite-dimensional real linear space in which a representation of the group GL(n, R) is given) and let M be a smooth (class C^∞) connected manifold. We emphasize that we assume M to be neither orientable nor compact, nor having an edge. By α we denote a vector fiber over M with a layer isomorphic to A, induced through the tangent fiber by means of a representation of the group GL(n, R) in A. By $\mathscr{A}$ we denote the space of smooth sections of the fiber α. Every diffeomorphism of the manifold M induces a layerwise diffeomorphism of the space of tangent fiber into itself, and hence a diffeomorphism of the space of fiber α into itself. As a result the space $\mathscr{A}$ is equipped with the structure of a module over the group of diffeomorphisms of the manifold M and hence over the Lie algebra $\mathfrak{A}(M)$ of smooth vector fields of the manifold M.

Our purpose is to study the cohomologies of the algebra $\mathfrak{A}(M)$ with coefficients in the $\mathfrak{A}(M)$-module of $\mathscr{A}$.

We note that if A is an algebra over **R** and operators of GL(n, R) are automorphisms of this algebra (we shall call this case in the subsequent as simply multiplicative) then $\mathscr{A}$ is also an algebra over **R** and operators from $\mathfrak{A}(M)$ are diffeomorphisms of this algebra. In this case the spaces $C^*(\mathfrak{A}(M);\ \mathscr{A})$ and $H^*(\mathfrak{A}(M);\ \mathscr{A})$ are equipped with natural structures of graded real algebras.

3.2. We mention an important reciprocal connection existing between the $\mathfrak{A}(M)$-module $\mathscr{A}$ and the GL(n, R)-module of A. We choose in the neighborhood U of a point $x \in M$ a local coordinate system. We obtain a canonical layerwise diffeomorphism of the total image of the neighborhood U in the space of the fiber α onto the product $U \times A$. We identify, with the help of this diffeomorphism, the layer of the fiber α at the point x with A. Let ξ be a vector field on M vanishing at the point x and let f be a smooth section of the fiber α. Since $\mathscr{A}$ is a $\mathfrak{A}(M)$-module, the element $\xi f \in \mathscr{A}$ is defined. The points $f(x)$ and $[\xi f](x)$ are elements of the space A. As can be verified $[\xi f](x) = \Xi f(x)$, where $\Xi \in \mathfrak{gl}(n, \mathbf{R})$ is a matrix consisting of products of coordinate functions of the field ξ in terms of local coordinates at the point x. Hence it can be seen that the structure of the $\mathfrak{gl}(n, R)$-module (and hence the structure of the GL(n, R)-module) in A can be reconstructed through the structure of the $\mathfrak{A}(M)$-module in $\mathcal{A}$.

3.3. We single out from what has been said the following important assertion. If a vector field $\xi \in \mathfrak{A}(M)$ and the section $f \in \mathcal{A}$ of the fiber α vanish at the point $x \in M$ then the section ξf also vanishes at the point x.

3.4. We denote through A(x) the space of formal sections of the fiber α at the point x by W(x) the space of formal vector fields at this point. Evidently A(x) is within the W(x)-module. Fixing in the neighborhood of the point x a local coordinate system we obtain a canonical isomorphism W(x) = W. The W-module A(x) is none other than W-module induced by the L_0-module of A (the action of the algebra L_0 on A is defined by the projection $L_0 \to \mathfrak{gl}(n, R)$). This assertion is not employed in the following, and its proof is left to the reader.

3.5. All of what follows can, without difficulty, be carried over to the case when $\mathcal{A}$ is the space of smooth sections of a vector fiber over M in which the connectedness is specified. It is clear that such a fiber can not be induced by tangents, so that this case is more general than that described in §3.1 although it requires the introduction of additional structure, namely, connectedness.

3.6. We proceed to the definition in $C^*(\mathfrak{A}(M); \mathcal{A})$ of a filtration, analogous to that considered in [1] (see [1], §1.2). We recall that $C^q(\mathfrak{A}(M); \mathcal{A})$ is the space of q-linear continuous functionals defined on $\mathfrak{A}(M)$ and with starting values in $\mathcal{A}$.

We shall say that the ring $P \in C^q(\mathfrak{A}(M); \mathcal{A})$ has a filtration no larger than k, if, for arbitrary vector fields $\xi_1, \ldots, \xi_q \in \mathfrak{A}(M)$ such that the section $P(\xi_1, \ldots, \xi_q)$ of the fiber α is different from zero at some point $x \in M$, it is required to find points $x_1, \ldots, x_k \in M$ such that each of the fields $\xi_1, \ldots, \xi_q$ does not identically vanish in an arbitrarily small neighborhood of one of the points $x_1, \ldots, x_k$, x.

For example a filtration, not greater than −1, has only a zero ring; a filtration not greater than 0 has a ring P if and only if the section $P(\xi_1, \ldots, \xi_q)$ vanishes at every point of the manifold M in a neighborhood of which one of the fields $\xi_1, \ldots, \xi_q$ vanishes.

The set of q-dimensional rings of a filtration, not exceeding k, will be denoted by $C^q_k(\mathfrak{A}(M); \mathcal{A}$. It is clear that $C^q_k(\mathfrak{A}(M); \mathcal{A})$ is the subspace of the space $C^q(\mathfrak{A}(M); \mathcal{A})$ and that $d(C^q_k(\mathfrak{A}(M); \mathcal{A})) \subset C^{q+1}_k(\mathfrak{A}(M); \mathcal{A})$.

The subcomplex $C^*_0 = \bigoplus_q C^q_0(\mathfrak{A}(M); A)$ of the complex $C^*(\mathfrak{A}(M); \mathcal{A})$ is called <u>diagonal</u> and is denoted by C^q_Δ. Evidently the ring P belongs to the diagonal complex if and only if the value of the section $P(\xi_1, \ldots, \xi_q)$ of the fiber α at any point $x \in M$ depends only on the germs of the fields $\xi_1, \ldots, \xi_q$ at the point x.

In the multiplicative case the introduced filtration is multiplicative: if $\alpha \in C^q_k(\mathfrak{A}(M); \mathcal{A})$, $\beta \in C^{q'}_{k'}(\mathfrak{A}(M); \mathcal{A})$, then $\alpha\beta \in C^{q+q'}_{k+k'}(\mathfrak{A}(M); \mathcal{A})$. In particular, the diagonal complex is a multiplicative subcomplex of the whole complex $C^*(\mathfrak{A}(M); \mathcal{A})$. The main difference of the considered case from the case of constant coefficients was investigated in [1]: here the filtration was also multiplicative but for the diagonal complex the complex $C^*_1(\mathfrak{A}(M); R)$, which was not multiplicative, was taken.

3.7. In this paper we limit ourselves to the investigation of the ring $H^*_\Delta(\mathfrak{A}(M); \mathcal{A})$ of cohomologies of the diagonal complex $C^*_\Delta(\mathfrak{A}(M); \mathcal{A})$. We shall leave open the question of the relation of the ring $H^*_\Delta(\mathfrak{A}(M); \mathcal{A})$ with the ring $H^*(\mathfrak{A}(M); \mathcal{A})$ of homologies of the whole complex $C^*(\mathfrak{A}(M); \mathcal{A})$. We shall only remark that $H^*(\mathfrak{A}(M); \mathcal{A})$ is a $H^*_\Delta(\mathfrak{A}(M); \mathcal{A})$-module.

4. The Spectral Sequence

We shall begin the study of the homologies of the complex $C^*_\Delta(\mathfrak{A}(M); \mathcal{A})$ by defining in this complex a filtration and investigate the spectral sequence arising therefrom.

4.1. We shall denote by Φ^q_r the subsequence of the space $C^q_\Delta(\mathfrak{A}(M); \mathcal{A})$, consisting of rings P such that if q + 1 − r of vector fields $\xi_1, \ldots, \xi_q$ vanish at some point $x \in M$, then the section $P(\xi_1, \ldots, \xi_q)$ of the fiber α also vanishes at the point x. We also put $\Phi_r = \bigoplus_q \Phi^q_r$. It is evident that

$$C^*_\Delta(\mathfrak{A}(M); \mathcal{A}) = \Phi_0 \supset \Phi_1 \supset \Phi_2 \supset \ldots$$

and that $\Phi_r^q =$ for $q < r$.

4.2. The filtration $\{\Phi_r\}$ agrees with a diffeomorphism.

Proof. Let $P \in \Phi_r^q$ and let $\xi_1, \ldots, \xi_{q+1} \in \mathfrak{A}(M)$ be vector fields where $\xi_1, \ldots, \xi_{q+2-r}$ vanish at the point $x \in M$. Then

$$(dP)(\xi_1, \ldots, \xi_{q+1}) = \sum_{1 \leqslant s < t \leqslant q+1} (-1)^{s+t-1} P([\xi_s, \xi_t], \xi_1, \ldots \hat{\xi}_s \ldots \hat{\xi}_t \ldots, \xi_{q+1})$$

$$+ \sum_{1 \leqslant s \leqslant q+1} (-1)^{s-1} \xi_s P(\xi_1, \ldots \hat{\xi}_s \ldots, \xi_{q+1}).$$

The first sum on the right-hand order vanishes at the point x by virtue of the fact that if two vector fields vanish at some point, then their commutator also vanishes at the same point. From Proposition 3.3 it follows the vanishing at the point x of terms of the second sum corresponding to the values of s less than or equal to $q + r - 2$. Finally the vanishing of the remaining terms at the point x follows from the following lemma.

LEMMA. If $P \in \Phi_r^m$ and $\xi \in \mathfrak{A}(M)$, then $\xi P \in \Phi_{r-1}^m$ (here ξP is defined by the formula $(\xi P)(\xi_1, \ldots, \xi_m) = \xi[P(\xi_1, \ldots, \xi_m)]$).

Proof of the Lemma. Evidently it is possible to limit ourselves to the case when ξ has a compact carrier.

If φ is a diffeomorphism of the manifold into itself and $P \in C_\Delta^m(\mathfrak{A}(M); \mathcal{A})$, then we denote by φP the ring given by the formula

$$(\varphi P)(\xi_1, \ldots, \xi_m) = \varphi[P(\varphi^{-1}\xi_1, \ldots, \varphi^{-1}\xi_m)].$$

It is clear that if $P \in \Phi_r$, then also $\varphi P \in \Phi_r$.

Let now $\xi, \xi_1, \ldots, \xi_q \in \mathfrak{A}(M)$, where the fields $\xi_1, \ldots, \xi_{q+2-r}$ vanish at the point x. The vector field ξ defines a set of diffeomorphisms φ_t: $M \to M$. We have

$$[\xi P(\xi_1, \ldots, \xi_q)](x) = \frac{d}{dt}[\varphi_t(P(\xi_1, \ldots, \xi_q))](x),$$

$$\varphi_t[P(\xi_1, \ldots, \xi_q)](x) = [(\varphi_t P)(\varphi_t \xi_1, \ldots, \varphi_t \xi_q)](x) = (\varphi_t P)(\xi_1, \ldots, \xi_q)(x)$$

$$+ t \sum_s (\varphi_t P)\left(\xi_1, \ldots, \xi_{s-1}, \frac{\partial \varphi_t \xi_s}{\partial t}, \xi_{s+1}, \ldots, \xi_q\right)(x) + o(t).$$

Since $\varphi_t P \in \Phi_r$, the first two terms of the last sum vanish. Thus $\xi P \in \Phi_{r-1}$.

4.3. If A is the algebra and the operators from GL(n, R) are automorphisms then the filtration $\{\Phi_r\}$ is multiplicative. This is evident.

4.4. In view of §4.2, 4.3 the filtration $\{\Phi_r\}$ generates a spectral sequence which is multiplicative in the multiplicative case converging to $H_\Delta^*(\mathfrak{A}(M); \mathcal{A})$. We denote this spectral sequence by $\{E_r^{s,t}, d_r^{s,t}\}$.

We begin with the zeroth term of this spectral sequence. We denote through A(x) a layer of the fiber α over the point x, through $L_0(x)$ the algebra of formal vector fields at the point x vanishing at this point. In agreement with 3.2 A(x) is an $L_0(x)$-module where, if we fix a local coordinate system in the neighborhood of the point x there arises the isomorphisms $A(x) = A$, $L_0(x) = L_0$ wherefrom is obtained the structure of the L_0-module on A induced by the structure of $\mathfrak{gl}(n, R)$-module on A by means of the projection $L_0 \to \mathfrak{gl}(n, R)$. We denote through $G^{s,t}(x)$ the space of linear functionals acting from the s-th exterior series $\Lambda^s \tau(x)$ of the tangent space to M at the point x in the space $C^t(L_0(x);\ A(x))$ of rings of dimensions t of the algebra $L_0(x)$ with coefficients in A(x). The union of the spaces $G^{s,t}(x)$, for all $x \in M$ in an obvious fashion, is equipped with the structure of a smooth vector fiber with the basis M. We denote it through $\mathcal{G}^{s,t}$.

We shall construct a canonical isomorphism of the space $E_0^{s,t}$ on the space $\Gamma(\mathcal{G}^{s,t})$ of smooth sections of the fiber $\mathcal{G}^{s,t}$.

We fix on M a Riemannian metric. Let h: $M \to R$ be such a positive continuous function on M that in the h(x)-neighborhood of the point x any two points are connected by a single geodesic; further, let g: $M \times M \to R$ be a smooth function such that $0 \leq g(x, y) \leq 1$ for any x, y; $g(x, y) = 1$ for $\rho(x, y) < (h(x)/2)$;

$g(x, y) = 0$ for $\rho(x, y) > h(x)$. If ε is a tangent vector to the manifold M at the point x then we define a vector field $\widetilde{\varepsilon}$ in the following manner. The exponential mapping exp: $\tau(x) \to M$ is a diffeomorphism on the original $h(x)$-neighborhood U of the point x. We choose in $\exp^{-1}(U) \subset \tau(x)$ a vector field obtained by a parallel displacement of the vector $\exp^{-1}\varepsilon$. We translate this vector field in U by means of an exp mapping, we multiply a vector of this field at every point $y \in U$ by $g(x, y)$ and in the obtained vector field in U we define $M\setminus U$ through the zero.

If ξ is a smooth vector field on M and $x \in M$ is a point then we define the element $\xi^*(x) \in L_0(x)$ as a formal vector field at the point x corresponding to a smooth vector field $\xi - \widetilde{\xi(x)}$.

We now define a homomorphism F: $\Gamma(\mathcal{G}^{s,t}) \to E_0^{s,t}$. Let $\varphi \in \Gamma(\mathcal{G}^{s,t})$. We denote through $\widetilde{\varphi}$ the ring of $C^{s+t}(\mathfrak{A}(M); \mathcal{A})$, corresponding to vector fields $\xi_1, \ldots, \xi_q \in \mathfrak{A}(M)$ of the fiber section α assuming at the point $x \in M$ the value

$$\sum (-1)^{i_1+\ldots+i_s - s(s+1)/2} \{[(\varphi(x)](\xi_{i_1}(x), \ldots, \xi_{i_s}(x))\} (\xi^*_{j_1}(x), \ldots, \xi^*_{j_t}(x)) \in A(x).$$

Here $\varphi(x) \in \mathrm{Hom}(\Lambda^s\tau(x), C^t(L_0(x), A(x)))$ is the value of the section φ at the point x; summation is carried out over all the partitions of the set $\{1, \ldots, s + t\}$ on the subsets $\{i_1, \ldots, i_s\}$, $\{j_1, \ldots, j_t\}$, where $i_1 < \ldots < i_s$, $j_1 < \ldots < j_t$. The ring φ belongs to Φ_s^{s+t}: if $q + 1 - s$ of the fields $\xi_1, \ldots, \xi_q$ vanish at the point x then one of the vectors $\xi_{i_1}(x), \ldots, \xi_{i_s}(x)$ must necessarily vanish.

The residue class of the ring $\widetilde{\varphi}$ with respect to Φ_{s+1}^{s+t} is an element of $\Phi_s^{s+t}/\Phi_{s+1}^{s+t} = E_0^{s,t}$. We denote it through $F(\varphi)$.

The monomorphicity of the mapping F is almost evident: if the section φ differs from zero, i.e., if there is a point $x \in X$, vectors $\varepsilon_1, \ldots, \varepsilon_s \in \tau(x)$ and elements $\eta_1, \ldots, \eta_t \in L_0(x)$ such that

$$\{[\varphi(x)](\varepsilon_1, \ldots, \varepsilon_s)\}(\eta_1, \ldots, \eta_t) \neq 0,$$

then the ring $\widetilde{\varphi}$ takes a value different from zero at the point x on the vector fields $\widetilde{\varepsilon}_1, \ldots, \widetilde{\varepsilon}_s, \zeta_1, \ldots, \zeta_t$, where $\zeta_1, \ldots, \zeta_t$ are smooth vector fields defining at the point x formal vector fields $\eta_1, \ldots, \eta_t$. Hence it follows the ring $\widetilde{\varphi}$ not only is different from zero but does not belong to Φ_{s+1}, i.e., $F(\varphi) \neq 0$.

Finally, we prove that the mapping F is epimorphic. Let $\psi \in \Phi_s^{s+t}$. We denote through φ the section of the fiber $\mathcal{G}^{s,t}$, defined by the formula

$$\{[\varphi(x)](\varepsilon_1, \ldots, \varepsilon_s)\} (\eta_1, \ldots, \eta_t) = \psi(\widetilde{\varepsilon}_1, \ldots, \widetilde{\varepsilon}_s, \zeta_1, \ldots, \zeta_t)(x),$$

where $\zeta_1, \ldots, \zeta_t$ are vector fields which define at the point x formal vector fields $\eta_1, \ldots, \eta_t$ (in view of the continuity of the functional ψ, $\psi(\varepsilon_1, \ldots, \varepsilon_s, \zeta_1, \ldots, \zeta_t)$ (x) does not depend on the choice of these fields $\zeta_1, \ldots, \zeta_t$). The difference $\psi - \widetilde{\varphi}$ belongs to Φ_{r+1}: if $q - s$ of these fields $\xi_1, \ldots, \xi_q$, say, $\xi_{s+1}, \ldots, \xi_q$ vanish at the point x, then

$$]\psi - \widetilde{\varphi}](\xi_1, \ldots, \xi_q)(x) = [\psi(\xi_1, \ldots, \xi_q)]x - \sum \{[\varphi(x)](\xi_{i_1}(x), \ldots, \xi_{i_s}(x))\}(\xi^*_{j_1}(x), \ldots, \xi^*_{j_{q-s}}(x))$$

$$= [\psi((\xi_1 - \widetilde{\xi_1(x)}) + \widetilde{\xi_1(x)}, \ldots, (\xi_s - \widetilde{\xi_s(x)}) + \widetilde{\xi_s(x)}, \xi_{s+1}, \ldots, \xi_q) - \psi(\widetilde{\xi_1(x)}, \ldots, \widetilde{\xi_s(x)}, \xi_{s+1}, \ldots, \xi_q)](x).$$

But since $\psi \in \Phi_r$ and the vector fields $\xi_1 - \widetilde{\xi_1(x)}, \ldots, \xi_s - \widetilde{\xi_s(x)}, \xi_{s+1}, \ldots, \xi_q$ vanish at the point x, then

$$[\psi(\xi_1, \ldots, \xi_q)](x) = [\psi(\widetilde{\xi_1(x)}, \ldots, \widetilde{\xi_s(x)}, \xi_{s+1}, \ldots, \xi_q)](x).$$

Thus the class of the element ψ in $E_r^{s,t}$ is equal to $F(\varphi)$. The isomorphicity of the mapping F is proved.

4.5. The differential $d_0^{s,t}: E_0^{s,t} \to E_0^{s,t+1}$ coincides up to a sign with the homomorphism $\Gamma(\mathcal{G}^{s,t}) \to \Gamma(\mathcal{G}^{s,t+1})$, induced by the differentials d: $C^t(L_0(x); A(x)) \to C^{t+1}(L_0(x); A(x))$.

<u>Proof.</u> We choose in the section $\varphi \in \Gamma(\mathcal{G}^{s,t})$, the point $x \in X$, vectors $\varepsilon_1, \ldots, \varepsilon_s \in \tau(x)$ and the elements $\eta_1, \ldots, \eta_{t+1} \in L_0(x)$. We have

$$\begin{aligned}\{[d_0^{s,t}\varphi](\varepsilon_1, \ldots, \varepsilon_s)\}(\eta_1, \ldots, \eta_{t+1})(x) &= [d\widetilde{\varphi}](\widetilde{\varepsilon}_1, \ldots, \widetilde{\varepsilon}_s, \zeta_1, \ldots, \zeta_{t+1})(x)\\ &= \sum_{1 \leq i < j \leq t+1} (-1)^{i+j+s-1} \widetilde{\varphi}(\widetilde{\varepsilon}_1, \ldots, \widetilde{\varepsilon}_s, [\zeta_i, \zeta_j], \zeta_1, \ldots \hat{\zeta}_i \ldots \hat{\xi}_j \ldots, \zeta_{t+1})(x)\\ &+ \sum_{1 \leq i \leq s,\ 1 \leq j \leq t+1} (-1)^{i+j+s-1} \widetilde{\varphi}([\widetilde{\varepsilon}_i, \zeta_j], \widetilde{\varepsilon}_1, \ldots \hat{\widetilde{\varepsilon}}_i \ldots, \widetilde{\varepsilon}_s, \zeta_1, \ldots \hat{\zeta}_j \ldots, \zeta_{t+1})(x)\end{aligned}$$

$$+\sum_{1\leqslant i\leqslant s}(-1)^{i-1}\tilde{\varepsilon}_i[\tilde{\varphi}(\tilde{\varepsilon}_1,\ldots\hat{\tilde{\varepsilon}}_i\ldots,\tilde{\varepsilon}_s,\zeta_1,\ldots,\zeta_{t+1})](x)+$$

$$+\sum_{1\leqslant j\leqslant t+1}(-1)^{j+s-1}\zeta_j[\tilde{\varphi}(\tilde{\varepsilon}_1,\ldots,\tilde{\varepsilon}_s,\zeta_1,\ldots\hat{\zeta}_j\ldots,\zeta_{t+1})](x)$$

(here $\zeta_1, \ldots, \zeta_{t+1}$ are smooth vector fields which define at the point x formal vector fields $\eta_1, \ldots, \eta_{t+1}$).

The second and the third terms in the sum vanish at the point x (proof is analogous to the proof of lemma of § 4.2); the value of the remaining part of the sum at the point x is equal to

$$(-1)^s\Big[\sum_{1\leqslant i<j\leqslant t+1}(-1)^{i+j-1}\{[\varphi(x)](\varepsilon_1,\ldots,\varepsilon_s)\}([\eta_i,\eta_j],\eta_1,\ldots\hat{\eta}_i\ldots\hat{\eta}_j\ldots,\eta_{t+1})$$

$$+\sum_{1\leqslant i\leqslant t+1}(-1)^{i-1}\eta_i\{[\varphi(x)](\varepsilon_1,\ldots,\varepsilon_s)\}(\eta_1,\ldots\hat{\eta}_i\ldots,\eta_{t+1}),$$

i.e., it is equal to the value of the differential of the ring $[\varphi(x)](\varepsilon_1, \ldots, \varepsilon_s) \in C^t(L_0(x);\ A(x))$ on the fields $\eta_1, \ldots, \eta_{t+1}$. The proposition is proved.

4.6. From Proposition 4.5 it follows that $E_1^{s,t}$ in the space of smooth sections of the fiber is composed of spaces $\mathrm{Hom}(\Lambda^s\tau(x), H^t(L_0(x);\ A(x)))$ for all $x \in M$. Since the spaces $H^t(L_0(x);\ A(x))$ are canonically isomorphic to the space $H^t(L_0;\ A)$ (this follows from the fact that every Lie algebra acts trivially on its cohomologies with arbitrary coefficients) there exists the canonical isomorphism,

$$E_1^{s,t}=\Omega^s(M)\otimes H^t(L_0;A),$$

where $\Omega^s(M)$ is the space of smooth exterior differential forms of degree s on M.

4.7. The differential $d_1^{s,t} : E_1^{s,t} \to E_1^{s+1,t}$ is induced by the exterior differential form $d : \Omega^s(M) \to \Omega^{s+1}(M)$.

Proof. We take the elements $\beta \in \Omega^s(M)$, $\gamma \in H^t(L_0;\ A)$ and choose a cocycle $P \in C^t(L_0;\ A)$ representing γ. We fix a point $x \in M$ and choose in its neighborhood U the same coordinate system as in 4.4. There arises a layerwise diffeomorphism of the original neighborhood U in the fiber α on $U \times A$. The element $\beta \otimes \gamma \in \Omega^s(M) \otimes H^t(L_0;\ A) = E_1^{s,t}$ represents in $E_0^{s,t}$ the section φ of the fiber $\mathscr{G}^{s,t}$, which, in U, is specified by the formula

$$\{[\varphi(y)](\varepsilon_1,\ldots,\varepsilon_s)\}(\eta_1,\ldots,\eta_t)=\beta(\varepsilon_1,\ldots,\varepsilon_s)P(\eta_1,\ldots,\eta_t),$$

where $y \in U$; $\varepsilon_1, \ldots, \varepsilon_s \in \tau(y)$. We must find out which section defines the ring $d\tilde{\varphi}$ which is no longer in $\mathscr{G}^{s,t+1}$, but in $\mathscr{G}^{s+1,t}$, i.e., must define

$$d\tilde{\varphi}(\tilde{\nu}_1,\ldots,\tilde{\nu}_{s+1},\xi_1,\ldots,\xi_t)(x),$$

where $\tilde{\nu}_1, \ldots, \tilde{\nu}_{s+1} \in \tau(x)$; $\xi_1, \ldots, \xi_t \in \mathfrak{A}(M)$ are fields vanishing at the point x. We have

$$d\tilde{\varphi}(\tilde{\nu}_1,\ldots,\tilde{\nu}_{s+1},\xi_1,\ldots,\xi_t)(x)=\sum_{\substack{1\leqslant i\leqslant s+1\\1\leqslant j\leqslant t}}(-1)^{i+j-1}\tilde{\varphi}(\tilde{\nu}_1,\ldots\hat{\tilde{\nu}}_i\ldots,\tilde{\nu}_{s+1},[\tilde{\nu}_i,\xi_j],\xi_1,\ldots\hat{\xi}_j\ldots,\xi_t)(x)$$

$$+\sum_{1\leqslant i\leqslant s+1}(-1)^{i-1}\tilde{\nu}_i[\varphi(\tilde{\nu}_1,\ldots\hat{\tilde{\nu}}_i\ldots,\tilde{\nu}_{s+1},\xi_1,\ldots,\xi_t)](x)$$

(the remaining terms vanish at the point x for obvious reasons)

$$=\sum_{1\leqslant i\leqslant s+1}(-1)^{i-1}\{\tilde{\nu}_i[\beta(\tilde{\nu}_1,\ldots\hat{\tilde{\nu}}_i\ldots,\tilde{\nu}_{s+1})P(\xi_1,\ldots,\xi_t)]$$

$$-\beta(\tilde{\nu}_1,\ldots\hat{\tilde{\nu}}_i\ldots,\tilde{\nu}_{s-1})\tilde{\nu}_iP(\xi_1,\ldots,\xi_t)\}(x)=$$

(the application of the form β and the cocycle P to the smooth vector fields gives smooth functions in U; more explicitly the function $P(\xi_1, \ldots, \xi_t)$ assumes at the point $y \in U$ a value equal to

$$=\sum_{1\leqslant i\leqslant s+1}(-1)^{i-1}\{[\tilde{\nu}_i\beta(\tilde{\nu}_1,\ldots\hat{\tilde{\nu}}_i\ldots,\tilde{\nu}_{s+1})]P(\xi_1^*,\ldots,\xi_t^*)\}(x)$$

$$=d\beta(\nu_1,\ldots,\nu_{s+1})P(\xi_1^*,\ldots,\xi_t^*).$$

The proposition is proved.

4.8. There exists the canonical isomorphism $E_2^{s,t} = H^s(M;\ \mathbf{R}) \otimes H^t(L_0;\ A)$.

This isomorphism is obtained from §4.6 and 4.7. We emphasize that in the multiplicative case this isomorphism is annular.

With this we complete investigation of the general case. From Theorem 4.8 it follows that if the spaces $H^*(L_0;\ A)$, $H^*(M;\ \mathbf{R})$ are finite-dimensional in the large or in every dimension, then the analogous assertion is valid also for the spaces $H_\Delta^*(\mathfrak{A}(M);\ \mathcal{A})$. Applying then 1.2 we can reach a conclusion about the finite dimensionality of the cohomologies of the diagonal complex with coefficients in the space of smooth exterior differential forms. This result, however, will be covered in §5 where the mentioned cohomologies will be calculated.

§5. Cohomologies with Coefficients in Exterior Forms

5.1. Theorem. Let M be a smooth connected n-dimensional manifold, $\Omega^*(M) = \sum \Omega^q(M)$ is the algebra of smooth exterior differential forms on M. A bigraded ring $H_\Delta^*(\mathfrak{A}(M);\ \Omega^*(M))$ is isomorphic to the tensor product of the two of its subrings: the ring $H_\Delta^*(\mathfrak{A}(M);\ \Omega^0)$ and the ring $\sum_q H_\Delta^q(\mathfrak{A}(M);\ \Omega^q)$.

A graded ring $H_\Delta^*(\mathfrak{A}(M);\ \Omega^0)$ is isomorphic to the ring of real cohomologies of the space T of a smooth U(n)-fiber induced by the complexification of the tangent fiber.*

A graded ring $\sum_q H_\Delta^q(\mathfrak{A}(M); \Omega^q(M))$ is isomorphic to the ring $\sum_q H^q(L_0; \Lambda^q)$, i.e., is generated by the commutations of the generators $\lambda_q \in H^q(L_0;\ \Lambda^q)$ $(q = 1, \ldots, n)$, connected by the relations $\lambda_1^{s_1} \ldots \lambda_n^{s_n} = 0$ for $s_1 + 2s_2 + \ldots + ns_n > n$. Thus

$$H_\Delta^r(\mathfrak{A}(M); \Omega^q(M)) = \begin{cases} 0 & \text{for } r < q, \\ H_\Delta^q(\mathfrak{A}(M);\ \Omega^q(M)) \otimes H^{r-q}(T;\ \mathbf{R}) & \text{for } r \geqslant q, \end{cases}$$

where the dimension of the space $H_\Delta^q(\mathfrak{A}(M);\ \Omega^q(M))$ is equal to the number of partitions of the number q into a sum of natural terms.

Proof. According to §4 there exists a spectral sequence $\{E_r^{s,t}, d_r^{s,t}\}$, converging to the ring $H_0^*(\mathfrak{A}(M);\ \Omega^*(M))$ with $E_2^{s,t} = H^s(M;\ \mathbf{R}) \otimes H^t(L_0;\ \Lambda^*)$. Under imbedding the spectral sequence corresponding to the subring $C^\infty(M) = \Omega^0(M) \subset \Omega^*(M)$. To begin with we study this subsequence.

The space T is a manifold on which the group U(n) acts. Since the group U(n) is compact the cohomologies of the space T are isomorphic to the homologies of the subcomplex $\widetilde{\Omega}^*(T)$ of the de Rahm complex of this manifold composed of U(n)-invariant forms. We denote through τ^* the vector fiber over M with the layer $\mathfrak{gl}(n, \mathbf{R})$, induced by the tangent fiber. Since the space of left-invariant exterior forms on U(n) is canonically isomorphic to the space of linear skew-symmetric forms on $\mathfrak{gl}(n, \mathbf{R})$, the elements of the space $\widetilde{\Omega}^*(T)$ can be regarded as smooth functions corresponding to each point x of the manifold M with a linear skew-symmetric form on the space $\tau(x) \oplus \tau^*(x)$. We remark now that a smooth vector field on M (more accurately its 1-jet) defines a smooth section of the fiber $\tau \oplus \tau^*$. Therefore, if q vector fields $\xi_1, \ldots, \xi_q$ are specified on M, and the element $\alpha \in \widetilde{\Omega}^q(t)$, then by calculating the value of the form α on the 1-jets of fields $\xi_1, \ldots, \xi_q$ at every point of the manifold we obtain a smooth function on M. Thus we construct the homomorphism

$$\chi : \widetilde{\Omega}^q(T) \to C_\Delta^q(\mathfrak{A}(M); C^\infty(M)).$$

It is easy to verify that it permutes with the differentials.

In the complex $\widetilde{\Omega}^*(T)$ there is a filtration due to the presence in T of the structure of a fiber space (see for example [7], p. 178-180). The homomorphism χ coincides with filtrations existing in both the complexes (this follows from the definition of filtrations). The first terms induced by these filtrations (of the spectral sequences) are isomorphic to $\Omega^*(M) \otimes H^*(\mathfrak{gl}(n, \mathbf{R});\ \mathbf{R})$, $\Omega^*(M) \otimes H^*(L_0;\ \mathbf{R})$, respectively. From the construction of these isomorphisms it can be seen that the homomorphism, induced by the homomorphism χ on $\Omega^*(M)$, is an identity but on $H^*(\mathfrak{gl}(n, \mathbf{R});\ \mathbf{R})$ defines a mapping into $H^*(L_0;\ \mathbf{R})$ coinciding with the homomorphism induced by the imbedding $\mathfrak{gl}(n, \mathbf{R}) \to L_0$. Thus from §1.8 it follows that the homomorphism χ

*This result is due to M. V. Losik [4].

induces an isomorphism of the first terms of the spectral sequences and hence it induces an isomorphism of the rings of cohomologies. The part of the theorem relating to $H^*(\mathfrak{A}(M);\ \Omega^0(M))$ is proved.

The spectral sequence $\{E_r^{s,t}, d_r^{s,t}\}$ is decomposed into a sum of spectral sequences $\{{}^{(q)}E_r^{s,t}, {}^{(q)}d_r^{s,t}\}$, corresponding to the submodules $\Omega^q(M) \subset \Omega^*(M)$. In accordance with § 4.8 ${}^{(q)}E_2^{s,t} = H^s(M;\ R) \otimes H^t(L_0;\ \Lambda^q)$. As is known (Theorem 1.2), $H^*(L_0;\ \Lambda^q)$ is a free $H^*(L_0;\ R)$-module generated by the additive basis of the space $H^q(L_0;\ \Lambda^q)$.† Using the multiplicative structure in the spectral sequence $\{E_r^{s,t}, d_r^{s,t}\}$, we may reconstruct through the differentials within the spectral sequence $\{{}^{(0)}E_r^{s,t}, {}^{(0)}d_r^{s,t}\}$ (but this spectral sequence coincides with the spectral sequence of the fiber $T \to M$) the differentials in the spectral sequence $\{{}^{(q)}E_r^{s,t}, {}^{(q)}d_r^{s,t}\}$. We obtain that ${}^{(q)}d_r^{0,q} = 0$ for $r \geq 2$, that $H_\Delta^q(\mathfrak{A}(M);\ \Omega^q) = {}^{(q)}E_\infty^{0,q} = {}^{(q)}E_2^{0,q}) = H^q(L_0;\ \Lambda^q)$, and that $H_\Delta^*(\mathfrak{A}(M);\ \Omega^q(M))$ is a free $H_\Delta^*(\mathfrak{A}(M), \Omega^0(M))$-module, generated by an additive basis of the space $H_\Delta^q(\mathfrak{A}(M);\ \Omega^q(M))$. Thus $H_\Delta^*(\mathfrak{A}(M);\ \Omega^*(M))$ is a tensor product of the subrings $H_\Delta^*(\mathfrak{A}(M);\ \Omega^0(M))$, $\sum_q H_\Delta^q(\mathfrak{A}(M);\Omega^q(M))$.

It remains to remark that the isomorphism $\sum_q H_\Delta^q(\mathfrak{A}(M);\Omega^q(M)) = \sum_q H^q(L_0;\Lambda^q)$ is evidently multiplicative.

LITERATURE CITED

1. I. M. Gel'fand and D. B. Fuks, "Cohomologies of Lie algebra of tangential vector fields of a smooth manifold," Funktsional'. Analiz i Ego Prilozhen., 3, No. 3, 32-52 (1969).
2. I. M. Gel'fand and D. B. Fuks, "Cohomologies of Lie algebra of tangential vector fields of a smooth manifold. II," Funktsional'. Analiz i Ego Prilozhen., 4, No. 2, 23-31 (1970).
3. I. M. Gel'fand and D. B. Fuks, "Cohomologies of Lie algebra of formal vector fields," Izv. AN SSSR, seriya matem., 34, No. 2, 322-337 (1970).
4. M. V. Losik, "On the cohomologies of infinite dimensional Lie algebras of formal vector fields," Funktsional'. Analiz i Ego Prilozhen., 4, No. 2, 43-53 (1970).
5. H. Cartan and S. Eilenberg, Cohomological Algebra [Russian translation], IL, Moscow (1960).
6. J. P. Serre and C. Hochshild, Cohomology of group extensions, Trans. Amer. Math. Soc., 74 (1953), 110-134.
7. D. B. Fuks, "Spectral Sequences of Fibers," Uspekhi Matem. Nauk, 21, No. 5, 149-181 (1966).

†Hence it follows that ${}^{(q)}E_2$ is the free ${}^{(0)}E_2$-module generated by the additive basis of the space ${}^{(q)}E^{0q}$.