ONE-LOOP RENORMALIZATION

In this section we study the renormalization of holomorphically translation invariant field theories on $\mathbb{C}^d$ for any $d \geq 1$. We start with a classical interacting holomorphic theory on $\mathbb{C}^d$ and consider one-loop homotopy RG flow from some finite scale ϵ to scale L. That is, we consider the sum over graphs of genus zero and one where at each vertex we place the holomorphic interaction. To obtain a prequantization of a classical theory one must make sense of the $\epsilon \to 0$ limit of this construction. In general, this involves introducing a family of counterterms. Our main result is that for a holomorphic theory no such counterterms are required, and one obtains a well-defined $\epsilon \to 0$ limit.

We can write the fields of a holomorphic theory on $\mathbb{C}^d$ as

$$\mathcal{E} = \left(\Omega^{0,*}(\mathbb{C}^d, V), \bar{\partial} + Q^{hol}\right)$$

where V is a graded holomorphic vector bundle and Q^{hol} is a holomorphic differential operator.

Since the theory is holomorphically translation invariant we have an identification $\Omega^{0,*}(\mathbb{C}^d, V) \cong \Omega^{0,*}(\mathbb{C}^d) \otimes_{\mathbb{C}} V_0$ where V_0 is the fiber of V over $0 \in \mathbb{C}^d$. Further, we can write the (-1)-shifted symplectic structure defining the classical BV theory in the form

$$\omega(\alpha \otimes v, \beta \otimes w) = (v, w)_{V_0} \int \mathrm{d}^d z (\alpha \wedge \beta)$$

where $(-,-)_{V_0}$ is a degree $(d-1)$-shifted BW: check pairing on the finite dimensional vector space V_0.

0.1. **Holomorphic gauge fixing.** To begin the process of renormalization we must fix the data of a gauge fixing operator. A gauge fixing operator is an operator on fields

$$Q^{GF} : \mathcal{E} \to \mathcal{E}[1]$$

of cohomological degree -1 such that $[Q, Q^{GF}]$ is a generalized Laplacian on $\mathcal{E}$ where Q is the linearized BRST operator. For a full definition of this see Definition ?? **??**.

For holomorphic theories there is a convenient choice for a gauge fixing operator. To construct it we fix the standard flat metric on $\mathbb{C}^d$. Doing this, we let $\bar{\partial}^*$ be the adjoint of the operator $\bar{\partial}$. Using the coordinates on $(z_1, \ldots, z_d) \in \mathbb{C}^d$ we can write this operator as

$$\bar{\partial}^* = \sum_{i=1}^{d} \frac{\partial}{\partial(\mathrm{d}\bar{z}_i)} \frac{\partial}{\partial z_i}.$$

Equivalently $\frac{\partial}{\partial(\mathrm{d}\bar{z}_i)}$ is equal to contraction with the anti-holomorphic vector field $\frac{\partial}{\partial \bar{z}_i}$. The operator $\bar{\partial}^*$ extends to the complex of fields via the formula

$$Q^{GF} = \bar{\partial}^* \otimes \mathrm{id}_V : \Omega^{0,*}(X, V) \to \Omega^{0,*-1}(X, V),$$

We claim that this is a gauge fixing operator for our holomorphic theory. Indeed, since Q^{hol} is a translation invariant holomorphic differential operator we have

$$[\bar{\partial} + Q^{hol}, Q^{GF}] = [\bar{\partial}, \bar{\partial}^*] \otimes \mathrm{id}_V.$$

The operator $[\bar{\partial}, \bar{\partial}^*]$ is simply the Dolbeualt Laplacian on $\mathbb{C}^d$, which is certainly a generalized Laplacian. In coordinates it is

$$[\bar{\partial}, \bar{\partial}^*] = -\sum_{i=1}^{d} \frac{\partial}{\partial \bar{z}_i} \frac{\partial}{\partial z_i}$$

By definition, the heat kernel is the dual BW: check factor

Pick a basis $\{e_i\}$ of V_0 and let

$$\mathbf{C}_{V_0} = \sum_{i,j} \omega_{ij} (e_i \otimes e_j) \in V_0 \otimes V_0$$

be the quadratic Casimir. Here, (ω_{ij}) is the inverse matrix to the pairing $(-,-)_{V_0}$. The regularized heat kernel then takes the form

$$K_\epsilon(z, w) = K^{an}(z, w) \cdot \mathbf{C}_{V_0}$$

Lemma 0.1. *If Γ is a tree then $\lim_{\epsilon \to 0} W_\Gamma(P_{\epsilon < L}, I)$ exists.*

0.2. **One-loop weights.**

Definition 0.2. Let $\epsilon, L > 0$. In addition, fix the following data.

1. An integer $k \geq 1$ that will be the number of vertices of the graph.
2. For each $\alpha = 1, \ldots, k$ a sequence of integers
$$\vec{n}^\alpha = (n_1^\alpha, \ldots, n_d^\alpha).$$
We denote by $(\vec{n}) = (n_i^j)$ the corresponding $d \times k$ matrix of integers.
3. A smooth compactly supported function $\Phi \in C_c^\infty((\mathbb{C}^d)^k) = C_c^\infty(\mathbb{C}^{dk})$.

The analytic weight associated to the triple $(k, (\vec{n}), \Phi)$ is

$$W_{k,(\vec{n})}^{\Phi}(\epsilon, L) = \int_{(z^1, \ldots, z^k) \in (\mathbb{C}^d)^k} \prod_{\alpha=1}^{k} \mathrm{d}^d z^\alpha \Phi(z^1, \ldots, z^\alpha) \prod_{\alpha=1}^{k} \left(\frac{\partial}{\partial z^i} \right)^{\vec{n}^\alpha} P_{\epsilon < L}^{an}(z^i, z^{i+1}). \tag{1}$$

In the above expression, we use the convention that $z^{k+1} = z^1$.

We will refer to the collection of data $(k, (\vec{n}), \Phi)$ in the definition as *wheel data*. The motivation for this is that the weight $W_{k,(\vec{n})}^{\Phi}(\epsilon, L)$ is the analytic part of the full weight $W_\Gamma(P_{\epsilon<L}, I)$ where Γ is a wheel with k vertices.

We have reduced the proof of Theorem **??** to showing that the $\epsilon \to 0$ limit of the analytic weight $W_{k,(\vec{n})}^{\Phi}(\epsilon, L)$ exists for any tripe of wheel data $(k, (\vec{n}), \Phi)$. To do this, there are two steps. First, we show a vanishing result that says when $k \geq d$ the weights vanish for purely algebraic reasons. The second part is the most technical aspect of the chapter where we show that for $k > d$ the weights have nice asymptotic behavior as a function of ϵ.

Lemma 0.3. *Let $(k, (\vec{n}), \Phi)$ be a triple of wheel data. If the number of vertices k satisfies $k \leq d$ then*

$$W_{k,(\vec{n})}^{\Phi}(\epsilon, L) = 0$$

for any $\epsilon, L > 0$.

Proof. In the integral expression for the weight (1) there is the following factor involving the product over the edges of the propagators:

$$\prod_{\alpha=1}^{k}\left(\frac{\partial}{\partial z^i}\right)^{\vec{n}^\alpha} P^{an}_{\epsilon<L}(z^i, z^{i+1}). \tag{2}$$

We will show that this expression is identically zero. To simplify the expression we first make the following change of coordinates on $\mathbb{C}^{dk}$:

$$w^i = z^{\alpha+1} - z^\alpha \quad , \quad 1 \le \alpha < k \tag{3}$$

$$w^k = z^k. \tag{4}$$

Introduce the following operators

$$\eta^\alpha = \sum_{i=1}^{d} \bar{w}_i^\alpha \frac{\partial}{\partial(\mathrm{d}\bar{w}_i^\alpha)}$$

acting on differential forms on $\mathbb{C}^{dk}$. The operator η^α lowers the anti-holomorphic Dolbuealt type by one : $\eta : (p,q) \to (p, q-1)$. Equivalently, η^α is contraction with the anti-holomorphic Euler vector field $\bar{w}_i^\alpha \partial/\partial\bar{w}_i^\alpha$.

Once we do this, we see that the expression (2) can be written as

$$\left(\left(\sum_{\alpha=1}^{k-1}\eta^\alpha\right)\prod_{i=1}^{d}\left(\sum_{\alpha=1}^{k-1}\mathrm{d}\bar{w}_i^\alpha\right)\right)\prod_{\alpha=1}^{k-1}\left(\eta^\alpha\prod_{i=1}^{d}\mathrm{d}\bar{w}_i^\alpha\right).$$

Note that only the variables $\bar{w}_i^\alpha$ for $i = 1, \ldots, d$ and $\alpha = 1, \ldots, k-1$ appear. Thus we can consider it as a form on $\mathbb{C}^{d(k-1)}$. As such a form it is of Dolbeualt type $(0, (d-1) + (k-1)(d-1)) = (0, (d-1)k)$. If $k < d$ then clearly $(d-1)k > d(k-1)$ so the form has greater degree than the dimension of the manifold and hence it vanishes.

The case left to consider is when $k = d$. In this case, the expression in (2) can be written as

$$\left(\left(\sum_{\alpha=1}^{d-1}\eta^\alpha\right)\prod_{i=1}^{d}\left(\sum_{\alpha=1}^{d-1}\mathrm{d}\bar{w}_i^\alpha\right)\right)\prod_{\alpha=1}^{d-1}\left(\eta^\alpha\prod_{i=1}^{d}\mathrm{d}\bar{w}_i^\alpha\right). \tag{5}$$

Again, since only the variables $\bar{w}_i^\alpha$ for $i = 1, \ldots, d$ and $\alpha = 1, \ldots, d-1$ appear, we can view this as a differential form on $\mathbb{C}^{d(d-1)}$. Furthermore, it is a form of type $(0, d(d-1))$. For any vector field X on $\mathbb{C}^{d(d-1)}$ the interior derivative i_X is a graded derivation. Suppose ω_1, ω_2 are two $(0,*)$ forms on $\mathbb{C}^{d(d-1)}$ such that the sum of their degrees is equal to d^2. Then, $\omega_1 \imath_X \omega_2$ is a top form for any vector field on $\mathbb{C}^{d(d-1)}$. Since $\omega_1\omega_2 = 0$ for form type reasons, we conclude that $\omega_1 \imath_X \omega_2 = \pm(i_X\omega_1)\omega_2$ with sign depending on the dimension d. Applied to the vector field $\bar{z}_i^1\partial/\partial\bar{w}_i^1$ in ([**?**]) we see that the expression can be written (up to a sign) as

$$\eta^1\left(\sum_{\alpha=1}^{d-1}\eta^\alpha\prod_{i=1}^{d}\left(\sum_{\alpha=1}^{d-1}\mathrm{d}\bar{w}_i^\alpha\right)\right)\left(\prod_{i=1}^{d}\mathrm{d}\bar{w}_i^1\right)\prod_{\alpha=2}^{d-1}\left(\eta^\alpha\prod_{i=1}^{d}\mathrm{d}\bar{w}_i^\alpha\right).$$

Repeating this, for $\alpha = 2, \ldots, k-1$ we can write this expression (up to a sign) as

$$\left(\eta_{k-1}\cdots\eta_2\eta_1\sum_{\alpha=1}^{k-1}\eta^\alpha\prod_{i=1}^{d}\left(\sum_{\alpha=1}^{k-1}\mathrm{d}\bar{w}_i^\alpha\right)\right)\prod_{\alpha=1}^{k-1}\prod_{i=1}^{d}\mathrm{d}\bar{w}_i^\alpha$$

The expression inside the parentheses is zero since each term in the sum over α involves a term like $\eta^\beta\eta^\beta = 0$. This completes the proof for $k = d$. $\square$

Lemma 0.4. *Let $(k, (\vec{n}), \Phi)$ be a triple of wheel data such that $k > d$. Then the $\epsilon \to 0$ limit of the analytic weight*

$$\lim_{\epsilon \to 0} W^{\Phi}_{k,(\vec{n})}(\epsilon, L)$$

exists.

Proof. We will bound the absolute value of the weight in Equation (1) and show that it has a well-defined $\epsilon \to 0$ limit. First, consider the change of coordinates as in Equations (3),(4). The weight can be written as

(6)

$$\int_{w^k \in \mathbb{C}^d} \mathrm{d}^d w^k \int_{(w_1,\ldots,w_{k-1}) \in (\mathbb{C}^d)^{k-1}} \left(\prod_{\alpha=1}^{k-1} \mathrm{d}^d w^\alpha \right) \Phi(w^1, \ldots, w^k) \left(\prod_{\alpha=1}^{k-1} \left(\frac{\partial}{\partial w^\alpha} \right)^{\vec{n}^\alpha} P^{an}_{\epsilon<L}(w^\alpha) \right) \sum_{\alpha=1}^{k-1} \left(\frac{\partial}{\partial w^\alpha} \right)^{\vec{n}^k} P^{an} \left(\sum_{\alpha=1}^{k-1} w^\alpha \right).$$

For $\alpha = 1, \ldots, k-1$ the notation

$$P^{an}_{\epsilon<L}(w^\alpha) = \int_{t_\alpha=\epsilon}^{L} \frac{\mathrm{d}t_\alpha}{4\pi t_\alpha} \bar{\partial}^* \textit{BW :FINISH}$$

makes sense since $P^{an}_{\epsilon<L}(z^\alpha, z^{\alpha+1})$ is only a function of $w^\alpha = z^{\alpha+1} - z^\alpha$. Similarly $P^{an}_{\epsilon<L}(z^{k+1}, z^1)$ is a function of

$$z^k - z^1 = \sum_{\alpha=1}^{k-1} w^\alpha.$$

Expanding out the propagators the weight takes the form

$$\begin{aligned}
&\int_{w^k \in \mathbb{C}^d} \mathrm{d}^{2d} w^k \int_{(w_1,\ldots,w_{k-1}) \in (\mathbb{C}^d)^{k-1}} \left(\prod_{\alpha=1}^{k-1} \mathrm{d}^{2d} w^\alpha \right) \Phi(w^1, \ldots, w^k) \int_{(t_1,\ldots,t_k) \in [\epsilon, L]^k} \prod_{\alpha=1}^{k} \frac{\mathrm{d}t_\alpha}{4\pi t_\alpha} \\
&\times \sum_{i_1,\ldots,i_{k-1}=1}^{d} \left(\frac{\bar{w}^1_{i_1}}{t_1} \frac{(\bar{w}^1)^{n^1}}{t^{|n^1|}} \right) \cdots \left(\frac{\bar{w}^{k-1}_{i_{k-1}}}{t_{k-1}} \frac{(\bar{w}^{k-1})^{n^{k-1}}}{t^{|n^{k-1}|}} \right) \left(\sum_{\alpha=1}^{k-1} \frac{\bar{w}^\alpha_{i_k}}{t_k} \cdot \frac{1}{t^{|n^k|}} \left(\sum_{\alpha=1}^{k-1} \bar{w}^\alpha \right)^{n^k} \right) \\
&\times \exp \left(- \sum_{\alpha=1}^{k-1} \frac{|w^\alpha|^2}{t_\alpha} - \frac{1}{t_k} \left| \sum_{\alpha=1}^{k-1} w^\alpha \right|^2 \right)
\end{aligned}$$

The notation used above warrants some explanation. Recall, for each α the vector of integers is defined as $n^\alpha = (n^\alpha_1, \ldots, n^\alpha_d)$. We use the notation

$$(\bar{w}^\alpha)^{n^\alpha} = \bar{w}_1^{n_1^\alpha} \cdots \bar{w}_d^{n_d^\alpha}.$$

Furthermore, $|n^\alpha| = n^\alpha_1 + \cdots + n^\alpha_d$. Each factor of the form $\frac{\bar{w}^\alpha_{i_\alpha}}{t_\alpha}$ comes from the application of the operator $\frac{\partial}{\partial z_i}$ in $\bar{\partial}^*$ applied to the propagator. The factor $\frac{(\bar{w}^\alpha)^{n^\alpha}}{t^{|n^\alpha|}}$ comes from applying the operator $\left(\frac{\partial}{\partial w}\right)^{n^\alpha}$ to the propagator. Note that $\bar{\partial}^*$ commutes with any translation invariant holomorphic differential operator, so it doesn't matter which order we do this.

To bound this integral we will recognize each of the factors

$$\frac{\bar{w}^\alpha_{i_\alpha}}{t_\alpha} \frac{(\bar{w}^\alpha)^{n^\alpha}}{t^{|n^\alpha|}}$$

as coming from the application of a certain holomorphic differential operator to the exponential in the last line. We will then integrate by parts to obtain a simple Gaussian integral which will give us the necessary bounds in the t-variables. Let us denote this Gaussian factor by

$$E(w,t) := \exp\left(-\sum_{\alpha=1}^{k-1}\frac{|w^\alpha|^2}{t_\alpha} - \frac{1}{t_k}\left|\sum_{\alpha=1}^{k-1} w^\alpha\right|^2\right)$$

For each α, i_α introduce the $t = (t_1, \ldots, t_k)$-dependent holomorphic differential operator

$$D_{\alpha,i_\alpha}(t) := \left(\frac{\partial}{\partial w^\alpha_{i_\alpha}} - \sum_{\beta=1}^{k-1}\frac{t_\beta}{t_1+\cdots+t_k}\frac{\partial}{\partial w^\beta_{i_\alpha}}\right)\prod_{j=1}^{d}\left(\frac{\partial}{\partial w^\alpha_j} - \sum_{\beta=1}^{k-1}\frac{t_\beta}{t_1+\cdots+t_k}\frac{\partial}{\partial w^\beta_j}\right)^{n^\alpha_j}.$$

The following lemma is an immediate calculation

Lemma 0.5. *One has*

$$D_{\alpha,i_\alpha}E(w,t) = \frac{\bar{w}^\alpha_{i_\alpha}}{t_\alpha}\frac{(\bar{w}^\alpha)^{n^\alpha}}{t^{|n^\alpha|}}E(w,t).$$

Note that all of the D_{α,i_α} operators mutually commute. Thus, we can integrate by parts iteratively to obtain the following expression for the weight:

$$\begin{aligned}&\pm\int_{w^k\in\mathbb{C}^d}\mathrm{d}^{2d}w^k\int_{(w_1,\ldots,w_{k-1})\in(\mathbb{C}^d)^{k-1}}\left(\prod_{\alpha=1}^{k-1}\mathrm{d}^{2d}w^\alpha\right)\int_{(t_1,\ldots,t_k)\in[\epsilon,L]^k}\prod_{\alpha=1}^{k}\frac{\mathrm{d}t_\alpha}{4\pi t_\alpha}\\ &\times\left(\sum_{i_1,\ldots,i_d} D_{1,i_1}\cdots D_{k-1,i_{k-1}}\sum_{\alpha=1}^{k-1}D_{\alpha,i_k}\Phi(w^1,\ldots,w^k)\right)\times\exp\left(-\sum_{\alpha=1}^{k-1}\frac{|w^\alpha|^2}{t_\alpha}-\frac{1}{t_k}\left|\sum_{\alpha=1}^{k-1}w^\alpha\right|^2\right).\end{aligned}$$

Now, since $t_\alpha / \sum_\beta t_\beta < 1$ for each α we have the following bound for the operators D_{α,i_α}: □