LOCAL SYMMETRIES OF HOLOMORPHIC THEORIES

CONTENTS

In this chapter we investigate the symmetries that generic holomorphic quantum field theories possess. Our overarching goal is to develop tools for understanding such symmetries that provide a systematic generalization of methods used in chiral conformal field theory on Riemann surfaces, especially for the Kac-Moody and Virasoro vertex algebras. We will utilize the tools of BV quantization and factorization algebras that has already heavily percolated this thesis.

We will focus on two main types of symmetries: holomorphic gauge symmetries and symmetries by holomorphic diffeomorphisms. An ordinary gauge symmetry is characterized as being local on the spacetime manifold. Each of the types of symmetries we consider share this characteristic, but they also enjoy an additional structure: they are holomorphic (up to homotopy) on the spacetime manifold. This means that they are specific to the type of theories we consider. Moreover, they store more interesting information about the geometry of the underlying manifold as compared to the smooth version of such symmetries.

Infinitesimally speaking, a symmetry is encoded by the action of a Lie algebra. For the holomorphic gauge symmetry this will become a sort of current algebra which is equivalent to holomorphic functions on the complex manifold with values in a Lie algebra. For the holomorphic diffeomorphisms this Lie algebra is that of holomorphic vector fields. Locality implies that this actually extends to a symmetry by a sheafy version of a Lie algebra. The precise sheafy version we mean is called a *local Lie algebra*, which we will recall in the main body of the text. To every local Lie algebra we can assign a factorization algebra through the so-called factorization enveloping algebra:

$$\mathbb{U} : \mathrm{Lie}_X \to \mathrm{Fact}_X.$$

Here, Lie_X is the category of local Lie algebras which we will recall in the main body of the text. By this construction, we see that the symmetries themselves of field theories give rise to factorization algebras.

One compelling reason for constructing a factorization algebra model for Lie algebras encoding the symmetries of a theory is that it allows one to consider universal versions of such objects. In the case of the symmetry by a current algebra of a Lie algebra in chiral conformal field theory this has been spelled out in the book [?]. For the case of conformal symmetry our work in [?] provides a factorization algebra lift of the ordinary Virasoro vertex algebra that exists uniformly on the site of Riemann surfaces. In this chapter, we extend each of these objects to arbitrary complex dimensions. Our formulation lends itself to an explicit computation of the factorization homology along certain complex manifolds, for which we will focus on several examples.

Studying such local symmetries involves rich geometric input even at the classical level, but the skeptical mathematician may view this as a repackaging of already familiar objects in complex geometry. The main advantage of working with factorization algebra analogs of such symmetries is in their relationship to studying quantizations of field theories. A similar obstruction deformation theory for studying quantizations of classical field theories also allows us to study the problem of *quantizing* local symmetries of a field theory. Moreover, we already know that factorization algebras describe the operator product expansion of the observables of a QFT. A formulation of Noether's theorem in [?] makes the relationship between the associated factorization algebra of a symmetry and the factorization algebra of observables of a theory.

Of course, quantizing a symmetry of a field theory may not always exist. In fact, this failure sheds light into subtle field theoretic phenomena of the underlying system. For example, in the case of conformal symmetries of a conformal field theory, the failure is exactly measured by the *central charge* of the theory. It is well established that the central charge is a very important characterization of a conformal field theory. At the Lie theoretic level, this failure is measured by a cocycle which in turn defines a central extension of the Lie algebra. It is this central extension that acts on the theory.

For this reason, an essential aspect of studying the local symmetries of holomorphic field theories we mentioned above is to characterize the possible cocycles that give rise to central extensions. As we have already mentioned, for vector fields in complex dimension one this is related to the central charge and the central extension of the Witt algebra (vector fields on the circle) known as the Virasoro Lie algebra. In the case of a current algebra associated to a Lie algebra, central

extensions are related to the *level* and the corresponding central extensions are called affine algebras.

Theorem 0.1. *The following is true about the local Lie algebras associated to holomorphic diffeomorphisms and holomorphic gauge symmetries.*

(1) Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{g}^X$ is associated current algebra defined on any complex manifold X. There is an embedding of the cohomology $H^*_{Lie}(\mathfrak{g}, \mathrm{Sym}^{d+1}g^\vee[-d-1])$ inside of the local cohomology of $\mathfrak{g}^X$.

(2) There is an isomorphism between the local cohomology of holomorphic vector fields on any complex manifold X of dimension d and $H^*_{dR}(X) \otimes H^*_{GF}(W_d)[2d]$, where $H^*_{GF}(W_d)$ is the Gelfand-Fuks cohomology of vector fields on the formal disk.

The central extensions we are interested in come from classes of degree $+1$ of the above local Lie algebras. In the case of holomorphic vector fields the result above implies that all such extensions are parametrized by $H^{2d+1}(W_d)$. It is a classical result of Fuks [?] that this cohomology is isomorphic to $H^{2d+2}(BU(d))$. In complex dimension one this cohomology is one dimensions corresponding to the class c_1^2. In general we obtain new classes, which are shown to agree with calculations in the physics literature in dimensions four and six.

In general, any of these cohomology classes define factorization algebras by twisting the factorization enveloping algebra. We especially focus on this construction in the case that the complex d-fold is equal to affine space $\mathbb{C}^d$, or some of its natural submanifolds. In the case of the current algebra, our result is compatible with recent work of Kapranov et. al. in [?] where they study higher dimensional versions of affine algebras, and their relationship to the (derived) moduli space of G-bundles in an analogous way that affine algebras are related to the moduli of bundles on curves via Kac-Moody uniformization. Our second main result shows how to recover these higher affine algebras from our factorization algebra on punctured affine space $\mathbb{C}^d \setminus \{0\}$, see Theorem **??**.

The extensions of part (1) of Theorem 0.1 are related to cohomology classes in the moduli of G-bundles on complex d-folds. We will show how techniques in equivariant BV quantization lead to natural families of QFTs defined over formal neighborhoods in the moduli space of G-bundles. Our techniques allow us to study quantizations of such families, in particular there are anomalies to quantization. An explicit analysis of Feynman diagrams leads to a computation of certain classes in the local cohomology which we relate to Chern classes of natural line bundles on $\mathrm{Bun}_G(X)$. This leads us to our next main result which is to prove a version of the Grothendieck-Riemann-Roch (GRR) theorem using the aforementioned methods of BV quantization, see Theorem BW: ref.

1. The current algebra

1.1. **Definitions.** We recall the definition of a local L_∞ algebra on a manifold X.

Definition 1.1. A *local L_∞ algebra* on X is the following data:

(i) a $\mathbb{Z}$-graded vector bundle L on X, whose sheaf of smooth sections we denote $\mathcal{L}^{sh}$, and

(ii) for each positive integer n, a polydifferential operator in n inputs

$$\ell_n : \underbrace{\mathcal{L}^{sh} \times \cdots \times \mathcal{L}^{sh}}_{n \text{ times}} \to \mathcal{L}[2-n]$$

such that the collection $\{\ell_n\}_{n \in \mathbb{N}}$ satisfy the conditions of an L_∞ algebra. Thus, $\mathcal{L}^{sh}$ is a sheaf of L_∞ algebras.

In practice, we prefer to work with the compactly supported sections of L, as explained in BW: cross ref, for which we reserve the more succinct notation $\mathcal{L}$.

Definition 1.2. Given a local L_∞ algebra $(L, \{\ell_n\})$ on X, let $\mathcal{L}$ denote the precosheaf of L_∞ algebras that assigns compactly supported sections of L to each open of X.

We typically refer to the local L_∞ algebra $(L, \{\ell_n\})$ by $\mathcal{L}$. We will often use local *Lie* algebra, especially if $\mathcal{L}$ is a precosheaf of dg Lie algebras and hence has trivial $\ell_{n \geq 3}$.

Example 1.3. Let $P \to X$ be a principal G-bundle. The adjoint bundle is a bundle of Lie algebras that we denote $\mathrm{ad}(P) \to X$. We will hereafter use $\mathcal{A}\mathrm{d}(P)$ to denote the *cosheaf* of compactly supported sections of Dolbeault complex of $\mathrm{ad}(P)$

$$\mathcal{A}\mathrm{d}(P)(U) = \Omega_c^{0,*}(U; \mathrm{ad}(P)).$$

In keeping with our conventions, $\mathcal{A}\mathrm{d}(P)^{sh}$ will denote the corresponding *sheaf* of sections of the Dolbeault complex

$$\mathcal{A}\mathrm{d}(P)^{sh}(U) = \Omega^{0,*}(U; \mathrm{ad}(P)).$$

The Dolbeualt differential $\bar{\partial}$ and the fiberwise Lie bracket on $\mathrm{ad}(P)$ endow $\mathcal{A}\mathrm{d}(P)^{sh}$ with the structure of a sheaf of dg Lie algebras on X.

The following lemma follows from tracing through definitions.

Lemma 1.4. *For any holomorphic principal bundle $P \to X$, the Dolbeualt complex of forms with values in $\mathrm{ad}(P)$ is a local Lie algebra.*

Example 1.5. Another key local Lie algebra makes sense on an arbitrary complex d-fold. Let $\mathfrak{g}$ be an ordinary Lie algebra, such as $\mathfrak{sl}_n$. There is a natural assignment

$$\mathcal{G}^{sh} : X \mapsto \Omega^{0,*}(X) \otimes \mathfrak{g},$$

where X is a complex d-fold. In fact, this assignment defines a sheaf of dg Lie algebras on the category of complex d-folds and local biholomorphisms,[1] and $\mathcal{G}$ to denote the cosheaf of compactly supported sections $\Omega_c^{0,*} \otimes \mathfrak{g}$. For any $\mathfrak{g}$, $\mathcal{G}$ defines a local Lie algebra on the category of d-folds, though we don't elaborate on the requisite categorical machinery to make this precise. We use $\mathcal{G}_X$ to denote the restriction of $\mathcal{G}$ to a fixed complex d-fold X. This defines a local Lie algebra whose associated cosheaf of sections is $U \subset X \mapsto \Omega_c^{0,*}(U) \otimes \mathfrak{g}$. Note that in the case of the trivial holomorphic principal G-bundle on X one as $\mathcal{G}_X = \mathcal{A}\mathrm{d}(\mathrm{triv})$.

1.2. **Factorization Lie algebras.**

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[1]A biholomorphism is a map $\phi : X \to Y$ that is biijective and both ϕ and ϕ^{-1} are holomorphic. A *local* biholomorphism means a map $\phi : X \to Y$ such that for every point $x \in X$ has a neighborhood on which ϕ is a biholomorphism.

1.3. **Local cohomology.** In this section we study the cohomology of the local Lie algebra $\mathcal{A}d(P)$. As we have already encountered many times in this thesis, the cohomology we are interested in consists of those functionals on the local Lie algebra that are *local*. From the perspective of local Lie algebras, one appealing aspect of this class of functionals is that they give rise to local Lie algebra extensions of the current algebra. These extensions will be appear when we quantize holomorphic gauge symmetries.

In Section **??** we have discussed local cohomology of a local Lie algebra, but we briefly recall it here. The basic idea is that a local cochain is a functional on the local Lie algebra obtained by integrating a polydifferential operator applied to an element in the local Lie algebra. If L is a graded vector bundle, let JL denote the corresponding ∞-jet bundle. If L is the underlying vector bundle of a local Lie algebra then JL has the structure of a bundle of Lie algebras. Thus, we may consider its reduced Chevalley-Eilenberg cochain complex $C^*_{\text{Lie,red}}(JL)$[2]. For any vector bundle JL has the structure of a D_X-module. In the case of a local Lie algebra, JL is a Lie algebra object in D_X-modules. Thus, $C^*_{\text{Lie}}(JL)$ is a commutative dg algebra in D_X-modules. The local cochain complex is obtained by tensoring the right D_X-modules of densities on X over D_X with this D_X-module.

Definition 1.6. Let $\mathcal{L}$ be a local Lie algebra on X. The local cohomology of $\mathcal{L}$ is defined as

$$C^*_{\text{loc}}(\mathcal{L}) = \Omega_X^{d,d} \otimes_{D_X} C^*_{\text{Lie,red}}(JL).$$

This is a sheaf of cochain complexes on X whose global sections we will denote by $C^*_{\text{loc}}(\mathcal{L}(X))$.

We note that the cochain complex of local functionals is a subcomplex of $C^*_{\text{Lie,red}}(\mathcal{L}(X))$, the reduced Lie algebra cochains of the global sections $\mathcal{L}(X)$. The differential on local functionals is, in essence, just precomposition with the polydifferentials defining the brackets of $\mathcal{L}$. Altogether $C^*_{\text{loc}}(\mathcal{L})$ is just a version of diagonal Gelfand-Fuks cohomology [**?**] for this kind of Lie algebra. We will discuss this further when we approach the local Lie algebra of holomorphic vector fields.

In ordinary Lie theory, central extensions are parametrized by cocycles on the Lie algebra valued in the trivial module. Similarly, local cocycles define central extensions of local Lie algebras.

Definition 1.7. A cocycle θ of degree $2 + k$ in $C^*_{\text{loc}}(\mathcal{L})$ determines a *k-shifted central extension*

(1) $$0 \to \mathbb{C}[k] \to \widehat{\mathcal{L}}_\theta \to \mathcal{L} \to 0$$

of precosheaves of L_∞ algebras, where the L_∞ structure maps are defined by

$$\widehat{\ell}_n(x_1, \ldots, x_n) = (\ell_n(x_1, \ldots, x_n), \theta(x_1, \ldots, x_n)).$$

Cohomologous cocycles determine quasi-isomorphic extensions.

Example 1.8. Let X be a Riemann surface, i.e., a complex 1-fold, and let $\mathfrak{g}$ be a simple Lie algebra with Killing form κ. Consider the local Lie algebra $\mathcal{G}_X$ on X. There is a natural cocycle depending precisely on two inputs:

$$\theta(\alpha \otimes x, \beta \otimes y) = \kappa(x,y) \int_X \alpha \wedge \partial\beta,$$

where $\alpha, \beta \in \Omega_c^{0,*}(X)$ and $x, y \in \mathfrak{g}$. As explained in BW: cross ref and Section ??? of [**?**], this cocycle determines an affine Kac-Moody algebra extending the loop algebra $L\mathfrak{g} = \mathfrak{g}[z, z^{-1}]$.

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[2] A local functional will always be defined modulo constants, hence we look at reduced cochains.

Much of the rest of the section is devoted to constructing and analyzing various cocycles and the resulting extensions.

1.3.1.   There is a particular family of local cocycles that has special importance for us and that generalizes the preceding example.

Let θ be an invariant polynomial on $\mathfrak{g}$ of homogenous degree $d + 1$. That is, θ is an element of $\mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$. We can extend θ to a functional $J_X(\theta)$ on $\mathcal{G}(X) = \Omega_c^{0,*}(X) \otimes \mathfrak{g}$ by the formula

$$(2) \qquad J(\theta)(\omega_1 \otimes Y_1, \ldots, \omega_{d+1} \otimes Y_{d+1}) = \theta(Y_1, \ldots, Y_{d+1}) \int_X \omega_1 \wedge \partial \omega_2 \cdots \wedge \partial \omega_{d+1}.$$

Note that we use d copies of the holomorphic derivative $\partial : \Omega^{0,*} \to \Omega^{1,*}$ to obtain an element of $\Omega_c^{d,*}$ in the integrand (and hence something that has a chance of being integrated).

This formula clearly makes sense for any complex d-fold X, and since integration is local on X, it intertwines nicely with the structure maps of $\mathcal{G}$.

Proposition 1.9. *For any complex d-fold X, the assignment*

$$J_X : \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}[-1] \to C_{\mathrm{loc}}^*(\mathcal{G}(X))$$

is an cochain map.

Proof. The functional $J_X(\theta)$ is local as it is expressed as the integral of a multilinear map composed with a product of differential operators. We need to show that $J_X(\theta)$ is closed for the differential on $C_{\mathrm{loc}}^*(\mathcal{G}|_X)$. The total differential splits as a sum $\bar{\partial} + d_{\mathfrak{g}}$ where $\bar{\partial}$ denotes the induced $\bar{\partial}$ differential on functionals and $d_{\mathfrak{g}}$ is constructed from the Lie bracket on $\mathfrak{g}$. We observe that

$$\bar{\partial} J_X(\theta) = 0$$
$$d_{\mathfrak{g}} J_X(\theta) = 0.$$

The first line follows from the fact that $\bar{\partial}$ and ∂ are graded commutative. The differential $d_{\mathfrak{g}}$ is obtained from the differential in the Chevalley–Eilenberg complex of $\mathfrak{g}$ in a natural way. The second line follows from the fact that the homogenous polynomial $\theta : \mathfrak{g} \times \cdots \times \mathfrak{g} \to \mathbb{C}$ is closed in the Chevalley–Eilenberg complex for $\mathfrak{g}$. $\qquad \square$

Having the fundamental construction of the cocycle down, we discuss two modest extensions of the construction. First is to consider an arbitrary G-bundle P on X. Suppose $\mathrm{ad}(P)$ is trivialized over an open set $U \subset X$. On this open set, we can write an element $\alpha \in \mathcal{A}d(P)(U) = \Omega_c^{0,*}(U, \mathrm{ad}(P))$ as $\alpha = \omega \otimes X$ where $\omega \in \Omega^{0,*}(X)$ and $X \in \mathfrak{g}$. Thus, the formula above for $J_X(\theta)$ still makes sense on $\mathcal{A}d(P)(U)$. Since the expression for the cocycle is clearly independent of the choice of a coordinate it glues to define a global section. Thus, for any principal bundle we have a cochain map

$$J_X^P : \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}[-1] \to C_{\mathrm{loc}}^*(\mathcal{A}d(P)(X))$$

given by the same formula as in (2).

If $\mathfrak{g}$ is the Lie algebra of a group G, there is an interpretation of the space of extensions $\mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$ in terms of G.

Proposition 1.10. **[?]** *Let G be an affine algebraic group scheme (such as* $\mathrm{GL}_n(\mathbb{C})$*). Then, there is an isomorphism*

$$H^i(BG, \Omega^j) \cong H^{i-j}(G, \mathrm{Sym}^j(\mathfrak{g}^*)).$$

In the case that $i = j = d + 1$ we find that $H^{d+1}(BG, \Omega^{d+1}) = \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$. Thus the central extensions have a nice interpretation in terms of the Dolbeualt cohomology of BG.

Now, suppose that $\mathfrak{g}$ is a an arbitrary dg Lie algebra, or more generally an L_∞ algebra. There is an analog of the construction above that still makes sense. Recall, that functions on the formal moduli space $B\mathfrak{g}$ associated to $\mathfrak{g}$ is given by $\mathcal{O}(B\mathfrak{g}) = C^*_{\mathrm{Lie}}(\mathfrak{g})$. Similarly, the k-forms on $B\mathfrak{g}$ have a Lie algebraic expression:

$$\Omega^k(B\mathfrak{g}) = C^*_{\mathrm{Lie}}\left(\mathfrak{g}; \mathrm{Sym}^k(\mathfrak{g}^*)[-k]\right)$$

The space of *closed k-forms* is defined by the complex

$$\Omega^k_{cl}(B\mathfrak{g}) = \dots$$

BW: finish

Lemma 1.11. *This defines a cochain map* $J_X : \widehat{\Omega}^{d+1}_{cl}[d] \to C^*_{\mathrm{loc}}(\mathcal{G}(X))$.

On $X = \mathbb{C}^d$ the functional $J_{\mathbb{C}^d}$ gives us the following complete description of a natural subcomplex of local cochains. On $\mathbb{C}^d$ exists a natural action by the group $U(d) \ltimes \mathbb{C}^d$, where $\mathbb{C}^d$ acts by translations.

Proposition 1.12. *The map* $J_{\mathbb{C}^d} : \widehat{\Omega}^{d+1}_{cl}[d] \to C^*_{\mathrm{loc}}(\mathcal{G}(X))$ *factors through the subcomplex of local functionals that are invariant for the group* $U(d) \ltimes \mathbb{C}^d$ *to define a quasi-isomorphism*

$$J_{\mathbb{C}^d} : \widehat{\Omega}^{d+1}_{cl}[d] \xrightarrow{\simeq} C^*_{\mathrm{loc}}(\mathcal{G}(X))^{U(d) \ltimes \mathbb{C}^d}$$

The proof of this fact has already been provided in Section **??**. We refer the reader to that section for details.

1.4. **The chiral enveloping algebra.** So far we have discussed a natural Lie algebra motivated by considering holomorphic gauge symmetries of holomorphic principal bundles. The sheaf-like nature of this object was formalized to the structure of a local Lie algebra, which in turn defines a factorization Lie algebra. For us, it is more natural to consider factorization algebras

To every factorization Lie algebra we can associate a factorization algebra

The chiral enveloping algebra of a Lie algebra is a special case of the

2. LOCAL STRUCTURES AND THE KAC-MOODY FACTORIZATION ALGEBRA

BW: recall vertex algebras and annular algebra

In this section we specialize the Kac-Moody factorization algebra on two particular complex manifolds of dimension d, $X = \mathbb{C}^d \setminus \{0\}$ and $X = (\mathbb{C} \setminus \{0\})^d$. In each case we show how the factorization product encodes the structure of E_n-algebra for some integer n.

In the case of $X = \mathbb{C}^d \setminus \{0\}$ we will restrict the factorization product to open sets equal to neighborhoods of $(2d - 1)$-spheres and find an $E_1 = A_\infty$-algebra. This means that we are reducing our factorization algebra along $S^{2d-1} \subset \mathbb{C}^d \setminus \{0\}$ to obtain a one-dimension factorization algebra via pushing forward along the radial projection map $\mathbb{C}^d \setminus \{0\} \to \mathbb{R}_{>0}$. Embedded inside

of this factorization algebra is a locally constant factorization algebra, which will define for us our A_∞-algebra.

We will recognize this A_∞-algebra as the universal enveloping algebra of an L_∞ algebra which is obtained as a central extension of an algebraic version of the sphere algebra

$$\mathrm{Map}(S^{2d-1}, \mathfrak{g}).$$

BW: explain algebraic version

BW: kapranov Lie algebra $\mathfrak{g}_{d,\theta}$

Theorem 2.1. *Consider the reduction $\mathcal{F}_{1d}$ of the Kac-Moody factorization algebra $\mathbb{U}_\alpha \left(\mathcal{G}_{\mathbb{C}^d \setminus \{0\}} \right)$ along the $(2d-1)$-sphere in $\mathbb{C}^d \setminus \{0\}$. There is a dense subfactorization algebra $\mathcal{F}_{1d}^{lc} \subset \mathcal{F}_{1d}$ that is locally constant. As a one-dimensional locally constant factorization algebra $\mathcal{F}_{1d}^{lc}$ is equivalent to the A_∞ algebra $U(\mathfrak{g}_{d,\theta})$ of [?].*

In a similar way in Section **??** we will see how the Kac-Moody factorization algebra on $(\mathbb{C} \setminus \{0\})^d$ are related to extensions of higher loop Lie algebras

$$L^d \mathfrak{g} = L(\cdots (L\mathfrak{g}) \cdots) = \mathrm{Map}(S^1 \times S^1, \mathfrak{g}).$$

2.1. **The higher sphere algebras.** The affine algebra associated to a Lie algebra $\mathfrak{g}$ together with an invariant pairing is defined as a central extension of the loop algebra of $\mathfrak{g}$

$$\mathbb{C} \to \widehat{\mathfrak{g}} \to Lg$$

where we use the algebraic loop algebra $L\mathfrak{g} = g[z, z^{-1}]$. The central extension is determined by the cocycle

$$(f \otimes X, g \otimes Y) \mapsto \oint f \mathrm{d}g \langle X, Y \rangle.$$

We use the punctured algebraic disk $D^{1\times} = \mathrm{Spec}\, \mathbb{C}[z, z^{-1}]$, but the definition also makes sense for the puncture formal disk (formal loops).

Let $D^d = \mathrm{Spec}\, \mathbb{C}[z_1, \ldots, z_d]$ be the d-dimensional algebraic disk. The key difference with the 1-dimensional case is that the punctured d-disk is no longer affine for $d > 1$. In fact, its cohomology is given by

$$H^*(D^{d\times}, \mathcal{O}) = BW\text{:}copypasta$$

Instead of working with the naive commutative algebra $\Gamma(D^{d\times}, \mathcal{O})$ we will use the dg commutative algebra of *derived* sections $\mathbb{R}\Gamma(D^{d\times}, \mathcal{O})$. An explicit model for this has been written down in [?] based on the Jouanolou method for resolving singularities. We recall its definition BW: I should put this in the appendix to the bg chapter

Definition 2.2. Let A_d be the commutative dg algebra generated by elements

$$z_1, \ldots, z_d, z_1^*, \ldots, z_d^*, (z_1 z_1^*)^{-1}, \ldots (z_1 z_d^*)^{-1}$$

in degree zero and

$$\mathrm{d}z_1, \ldots, \mathrm{d}z_d, \mathrm{d}z_1^*, \ldots, \mathrm{d}z_d^*$$

in degree one. Introduce a $*$-weight, so that $z_i^*, \mathrm{d}z_i^*$ have $*$-weight $+1$ and $(z_i^*)^{-1}$ has $*$-weight -1. We require that:

(i) every element is of total $*$-weight zero and

(ii) the contraction of every element with the Euler vector field $\sum_i z_i^* \partial_{z_i^*}$ vanishes.

The key properties of the dg algebra A_d we will utlilize are summarized in the following result of [?].

Proposition 2.3 ([?] Proposition 1.3.1). *The commutative dg algebra A_d is a model for $\mathbb{R}\Gamma(D^{d\times}, \mathcal{O})$. Moreover, there is a dense map of commutative dg algebras*

$$j : A_d \to \Omega^{0,*}(\mathbb{C}^d \setminus 0)$$

sending $z_i \mapsto z_i$, $z_i^* \mapsto \bar{z}_i$, and $\mathrm{d}z_i^* \mapsto \mathrm{d}\bar{z}_i$.

We are interested in the dg Lie algebra $A_d \otimes \mathfrak{g}$. In [?] they show, via knowledge of the Lie algebra cohomology, that there is a central extension of this BW: not sure what to say

Definition 2.4. Fix an element $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g})^{\mathfrak{g}}$. Let $\widehat{\mathfrak{g}}_{d,\theta}$ be the L_∞ central extension

$$\mathbb{C} \to \widehat{\mathfrak{g}}_{d,\theta} \to A_d \otimes \mathfrak{g}$$

determined by the degree two cocycle $\theta_{\mathrm{FHK}} \in \mathrm{C}^*_{\mathrm{Lie}}(A_d \otimes \mathfrak{g})$ defined by

$$\theta_{\mathrm{FHK}}(a_0 \otimes X_0, \dots, a_d \otimes X_d) = \underset{z=0}{\mathrm{Res}}\, (a_0 \wedge \mathrm{d}a_1 \wedge \cdots \wedge \mathrm{d}a_d)\, \theta(X_0, \dots, X_d)$$

where $a_i \otimes X_i \in A_d \otimes \mathfrak{g}$.

2.2. **The strategy.** We consider the restriction of the factorization algebra $\mathbb{U}_\theta(\mathcal{G})$ on $\mathbb{C}^d \setminus \{0\}$ to the collection of open sets diffeomorphic to spherical shells. This restriction has the structure of a one-dimensional factorization algebra corresponding to the iterated nesting of spherical shells. We show that there is a dense subfactorization algebra that is locally constant, hence corresponds to an E_1 algebra. We conclude by identifying an A_∞ model for this algebra as the universal enveloping algebra of a certain L_∞ algebra, that agree with the higher dimensional affine algebras of [?]

Introduce the radial projection map

$$\rho : \mathbb{C}^d \setminus 0 \to \mathbb{R}_{>0}$$

sending $z = (z_1, \dots, z_d)$ to $|z| = \sqrt{|z_1|^2 + \cdots + |z_d|^2}$. We will restrict our factorization algebra to spherical shells by pushing forward the factorization algebra along this map. Indeed, the preimage of an open interval is such a spherical shell, and the factorization product on the line is equivalent to the nesting of shells.

2.2.1. *The case of zero level.* First we will consider the higher Kac-Moody factorization algebra on $\mathbb{C}^d \setminus \{0\}$ "at level zero". That is, the factorization algebra $\mathbb{U}(\mathcal{G}_{\mathbb{C}^d \setminus \{0\}}$. In this section we will omit $\mathbb{C}^d \setminus \{0\}$ from the notation, and simply refer to the factorization algebra by $\mathbb{U}(\mathcal{G})$.

Let $\rho_*(\mathbb{U}\mathcal{G})$ be the factorization algebra on $\mathbb{R}_{>0}$ obtained by pushing forward along the radial projection map. Explicitly, to an open set $I \subset \mathbb{R}_{>0}$ this factorization algebra assigns the dg vector space

$$\mathrm{C}^{\mathrm{Lie}}_* \left(\Omega^{0,*}_c(\rho^{-1}(I)) \otimes \mathfrak{g} \right).$$

Let $I \subset \mathbb{R}_{>0}$ be an open subset. There is the natural map $\rho^* : \Omega^*_c(I) \to \Omega^*_c(\rho^{-1}(I))$ given by the pull back of differential forms. We can post compose this with the natural projection $\mathrm{pr}_{\Omega^{0,*}}$:

$\Omega_c^* \to \Omega_c^{0,*}$ to obtain a map of commutative algebras $\mathrm{pr}_{\Omega^{0,*}} \circ \rho^* : \Omega_c^*(I) \to \Omega_c^{0,*}(\rho^{-1}(I))$. The map j from Proposition **??** determines a map of dg commutative algebras $j : A_d \to \Omega^{0,*}(\rho^{-1}(I))$. Thus, we obtain a map

$$\Phi(I) = (\mathrm{pr}_{\Omega^{0,*}} \circ \rho^*) \otimes j : \quad \Omega_c^*(I) \otimes A_d \quad \to \quad \Omega_c^{0,*}\left((\rho^{-1}(I))\right)$$
$$\varphi \otimes a \quad \mapsto \quad ((\mathrm{pr}_{\Omega^{0,*}} \circ \rho^*)\varphi) \wedge j(a)$$

Since this is a map of commutative dg algebras it defines a map of dg Lie algebras

$$\Phi(I) \otimes \mathrm{id}_{\mathfrak{g}} : (\Omega_c^*(I) \otimes A_d) \otimes \mathfrak{g} = \Omega_c^*(I) \otimes (A_d \otimes \mathfrak{g}) \to \Omega^{0,*}(\rho^{-1}(I)) \otimes \mathfrak{g}$$

which maps $(\varphi \otimes a) \otimes X \mapsto \Phi(\varphi \otimes a) \otimes X$. We will drop the $\mathrm{id}_{\mathfrak{g}}$ from the notation and will denote this map simply by $\Phi(I)$. Note that $\Phi(I)$ is compatible with inclusions of open sets, hence extends to a map of cosheaves of dg Lie algebras that we will call Φ.

We can summarize the results as follows.

Proposition 2.5. *The map Φ extends to a map of factorization Lie algebras*

$$\Phi : \Omega_{\mathbb{R}_{>0},c}^* \otimes (A_d \otimes \mathfrak{g}) \to \rho_* \left(\Omega_{\mathbb{C}^d \setminus 0,c}^{0,*} \otimes \mathfrak{g} \right).$$
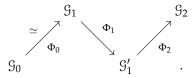
Hence, it defines a map of factorization algebras

$$C_*(\Phi) : U^{fact} \left(\Omega_{\mathbb{R}_{>0}}^* \otimes (A_d \otimes \mathfrak{g}) \right) \to \rho_* \left(\mathbb{U}\mathcal{G} \right).$$

The fact that we obtain a map of factorization algebras follows from the universal property of the universal enveloping factorization algebra we discussed in Section **??**.

2.2.2. *The case of non-zero level.* We now proceed to the proof of Theorem . The dg Lie algebra $\mathfrak{g}_{d,\theta}$ determines a dg associative algebra via its universal enveloping algebra $U(\mathfrak{g}_{d,\theta})$. BW: define it? By BW: ref this dg algebra determines a factorization algebra on the one-manifold $\mathbb{R}_{>0}$ that assigns to every open interval $I \subset \mathbb{R}_{>0}$ the dg vector space $U(A_d \otimes \mathfrak{g})$. The factorization product is uniquely determined by the algebra structure. Henceforth, we denote this factorization algebra by $U(\mathfrak{g}_{d,\theta})^{fact}$.

To prove the theorem we will construct a sequence of maps of factorization Lie algebras on $\mathbb{R}_{>0}$:

$$\begin{array}{ccccc} & & \mathcal{G}_1 & & \mathcal{G}_2 \\ & \overset{\simeq}{\nearrow} \quad \overset{\Phi_0}{} & & \overset{\Phi_1}{\searrow} \quad \overset{\Phi_2}{\nearrow} & \\ \mathcal{G}_0 & & & \mathcal{G}_1' & . \end{array}$$

The factorization envelope of $\mathcal{G}_0$ is equivalent to the factorization algebra $U(\widehat{\mathfrak{g}}_{d,\theta})^{fact}$. Moreover, the factorization envelope of $\mathcal{G}_2$ is the push-forward of of the higher Kac–Moody factorization algebra $\rho_* \mathbb{U}\mathcal{G}$. Hence, the desired map of factorization algebras is produced by applying the factorization envelope functor to the above composition of factorization Lie algebras.

First, we introduce the factorization Lie algebra $\mathcal{G}_0$. To an open set $I \subset \mathbb{R}$, it assigns the dg Lie algebra $\mathcal{G}_0(I) = \Omega_c^*(I) \otimes \widehat{\mathfrak{g}}_{d,\theta}$, where $\widehat{\mathfrak{g}}_{d,\theta}$ is the central extension from BW: ref. The differential and Lie bracket are determined by the fact that we are tensoring a commutative dg algebra

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with a dg Lie algebra. A slight variant of Proposition 3.4.0.1 in [?], which shows that the one-dimensional factorization envelope of an ordinary Lie algebra produces its ordinary universal enveloping algebra, shows that there is a quasi-isomorphism of factorization algebras on $\mathbb{R}$,

$$(U\widehat{\mathfrak{g}}_{d,\theta})^{fact} \xrightarrow{\simeq} C_*^{\mathrm{Lie}}(\mathcal{G}_0).$$

The factorization Lie algebra $\mathcal{G}_0$ is a central extension of the factorization Lie algebra $\Omega_{\mathbb{R},c}^* \otimes (A_d \otimes \mathfrak{g})$ by the trivial module $\Omega_c^* \oplus \mathbb{C} \cdot K$. Indeed, the cocycle determining the central extension is given by

$$\theta_0(\varphi_0\alpha_0, \ldots, \varphi_d\alpha_d) = (\varphi_0 \wedge \cdots \wedge \varphi_d)\theta_{A_d}(\alpha_1, \ldots, \alpha_d).$$

The factorization Lie algebra $\Omega_{\mathbb{R},c}^* \otimes (A_d \otimes \mathfrak{g})$ is the compactly supported sections of the local Lie algebra $\Omega_{\mathbb{R}}^* \otimes (A_d \otimes \mathfrak{g})$ and this cocycle determining the extension is a local cocycle.

Next, we define the factorization dg Lie algebra $\mathcal{G}_1$ on $\mathbb{R}$. This is also obtained as a central extension of the factorization Lie algebra $\Omega_{\mathbb{R},c}^* \otimes (A_d \otimes \mathfrak{g})$:

$$0 \to \mathbb{C} \cdot K[-1] \to \mathcal{G}_1 \to \Omega_{\mathbb{R},c}^* \otimes (A_d \otimes \mathfrak{g}) \to 0$$

determined by the following cocycle. For an open interval I write $\varphi_i \in \Omega_c^*(I)$, $\alpha_i \in A_d \otimes \mathfrak{g}$. The cocycle is defined by

$$(3) \qquad \theta_1(\varphi_0\alpha_0, \ldots, \varphi_d\alpha_d) = \left(\int_I \varphi_0 \wedge \cdots \varphi_d\right) \theta_{\mathrm{FHK}}(\alpha_0, \ldots, \alpha_d)$$

where θ_{FHK} was defined in Definition **??**.

The functional θ_1 determines a local cocycle in $C_{\mathrm{loc}}^*\left(\Omega_{\mathbb{R}}^* \otimes (A_d \otimes \mathfrak{g})\right)$ of degree one.

We now define a map of factorization Lie algebras $\Phi_0 : \mathcal{G}_0 \to \mathcal{G}_1$. On and open set $I \subset \mathbb{R}$, we define the map $\Phi_0(I) : \mathcal{G}_0(I) \to \mathcal{G}_1(I)$ by

$$\Phi_0(I)(\varphi\alpha, \psi K) = \left(\varphi\alpha, \int \psi \cdot K\right).$$

For a fixed open set $I \subset \mathbb{R}$, the map Φ_0 fits into the commutative diagram of short exact sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega_c^*(I) \otimes \mathbb{C} \cdot K & \longrightarrow & \mathcal{G}_0(I) & \longrightarrow & \Omega_c^*(I) \otimes (A_d \otimes \mathfrak{g}) & \longrightarrow & 0 \\
& & \simeq \downarrow \int & & \downarrow \Phi_0(I) & & \| & & \\
0 & \longrightarrow & \mathbb{C} \cdot K[-1] & \longrightarrow & \mathcal{G}_1(I) & \longrightarrow & \Omega_c^*(I) \otimes (A_d \otimes \mathfrak{g}) & \longrightarrow & 0.
\end{array}
$$

To see that $\Phi_0(I)$ is a map of dg Lie algebras we simply observe that the cocycles determining the central extensions are related by $\theta_1 = \int \circ \theta_0$, where $\int : \Omega_c^*(I) \to \mathbb{C}$ as in the diagram above. Since $\int$ is a quasi-isomorphism, the map $\Phi_0(I)$ is as well. It is clear that as we vary the interval I we obtain a quasi-isomorphism of factorization Lie algebras $\Phi_0 : \mathcal{G}_0 \xrightarrow{\simeq} \mathcal{G}_1$.

We now define the factorization dg Lie algebra $\mathcal{G}_1'$. Like $\mathcal{G}_0$ and $\mathcal{G}_0$, it is a central extension of $\Omega_{\mathbb{R},c}^* \otimes (A_d \otimes \mathfrak{g})$. The cocycle determining the central extension is defined by

$$\theta_1'(\varphi_0 a_0 X_0, \ldots, \ldots, \varphi_d a_d X_d) = \theta_1(\varphi_0 a_0 X_0, \ldots, \ldots, \varphi_d a_d X_d) + \widetilde{\theta}_1(\varphi_0 a_0 X_0, \ldots, \ldots, \varphi_d a_d X_d)$$

where θ_1 was defined in Equation (3). Before writing down the explicit formula for $\widetilde{\theta}_1$ we introduce some notation. Set

$$E = r\frac{\partial}{\partial r},$$

$$\mathrm{d}\vartheta = \sum_i \frac{\mathrm{d}z_i}{z_i}.$$

We view E as a vector field on $\mathbb{R}_{>0}$ and $\mathrm{d}\vartheta$ as a $(1,0)$-form on $\mathbb{C}^d \setminus 0$. Define the functional

$$\widetilde{\theta}_1(\varphi_0 a_0 X_0, \ldots, \varphi_d a_d X_d) = \frac{1}{2}\sum_{i=1}^d \left(\int_I \varphi_0(E\cdot\varphi_i)\varphi_1\cdots\widehat{\varphi_i}\cdots\varphi_d\right)\left(\oint (a_0 a_i \mathrm{d}\vartheta)\,\partial a_1\cdots\widehat{\partial a_i}\cdots\partial a_d\right)\theta(X_0,\ldots,X_d).$$

The functional $\widetilde{\theta}$ defines a local functional in $\mathrm{C}^*_{\mathrm{loc}}\left(\Omega^*_{\mathbb{R}_{>0}}\otimes(A_d\otimes\mathfrak{g})\right)$ of cohomological degree one. One immediately checks that it is a cocycle. This completes the definition of the factorization Lie algebra $\mathcal{G}'_1$.

The factorization Lie algebras $\mathcal{G}_1$ and $\mathcal{G}'_1$ are identical as precosheaves of vector spaces. In fact, if we put a filtration on $\mathcal{G}_1$ and $\mathcal{G}'_1$ where the central element K has filtration degree one, then the associated graded factorization Lie algebras $\mathrm{Gr}\,\mathcal{G}_1$ and $\mathrm{Gr}\,\mathcal{G}'_1$ are also identified. The only difference in the Lie algebra structures comes from the deformation of the cocycle determining the extension of $\mathcal{G}'_1$ given by $\widetilde{\theta}_1$.

In fact, we will show that $\widetilde{\theta}_1$ is actually an exact cocycle via the cobounding element $\eta \in \mathrm{C}^*_{\mathrm{loc}}\left(\Omega^*_{\mathbb{R}_{>0}}\otimes(A_d\otimes\mathfrak{g})\right)$ defined by

$$\eta(\varphi_0 a_0 X_0, \ldots, \varphi_d a_d X_d) = \sum_{i=1}^d \left(\int_I \varphi_0\left(\iota_E\varphi_i\right)\varphi_1\cdots\widehat{\varphi_i}\cdots\varphi_d\right)\left(\oint (a_0 a_i \mathrm{d}\vartheta)\,\partial a_1\cdots\widehat{\partial a_i}\cdots\partial a_d\right)\theta(X_0,\ldots,X_d).$$

Lemma 2.6. *One has $\mathrm{d}\eta = \widetilde{\theta}_1$, where d is the differential for the cochain complex $\mathrm{C}^*_{\mathrm{loc}}(\Omega^*_{\mathbb{R}_{>0}}\otimes(A_d\otimes\mathfrak{g}))$. In particular, the factorization Lie algebras $\mathcal{G}_1$ and $\mathcal{G}'_1$ are quasi-isomorphic (as L_∞ algebras). An explicit quasi-isomorphism is given by the L_∞ map $\Phi_1 : \mathcal{G}_1 \to \mathcal{G}'_1$ that sends the central element K to itself and an element $(\varphi_0 a_0 X_0, \ldots, \varphi_d a_d X_d) \in \mathrm{Sym}^{d+1}(\Omega^*_c\otimes(A_d\otimes\mathfrak{g}))$ to*

$$(\varphi_0 a_0 X_0, \ldots, \varphi_d a_d X_d) + \eta(\varphi_0 a_0 X_0, \ldots, \varphi_d a_d X_d)\cdot K \in \mathrm{Sym}^{d+1}(\Omega^*_c\otimes(A_d\otimes\mathfrak{g}))\oplus\mathbb{C}\cdot K.$$

Finally, we define the factorization Lie algebra $\mathcal{G}_2$. We have already seen that the local cocycle $J(\theta) \in \mathrm{C}^*_{\mathrm{loc}}(\mathfrak{g}^{\mathbb{C}^d})$ determines a central extension of factorization Lie algebras

$$0 \to \mathbb{C}\cdot K[-1] \to \mathcal{G}_{J(\theta)} \to \Omega^{0,*}_{\mathbb{C}^d,c}\otimes\mathfrak{g} \to 0.$$

Of course, we can restrict $\mathcal{G}_{J(\theta)}$ to a factorization algebra on $\mathbb{C}^d \setminus 0$. The factorization algebra $\mathcal{G}_2$ is defined as the pushforward of this restriction along the radial projection: $\mathcal{G}_2 := \rho_*\left(\mathcal{G}_{J(\theta)}|_{\mathbb{C}^d\setminus 0}\right)$.

Recall the map $\Phi : \Omega^*_{\mathbb{R}_{>0},c}\otimes(A_d\otimes\mathfrak{g}) \to \rho_*(\Omega^{0,*}_{\mathbb{C}^d\setminus 0,c}\otimes\mathfrak{g})$ defined in BW: ref. On each open set $I \subset \mathbb{R}_{>0}$ we can extend Φ by the identity on the central element to a linear map $\Phi_2 : \mathcal{G}'_1(I) \to \mathcal{G}_2(I)$.

Lemma 2.7. *The map $\Phi_2 : \mathcal{G}'_1(I) \to \mathcal{G}_2(I)$ is a map of dg Lie algebras. Moreover, it extends to a map of factorization Lie algebras $\Phi_2 : \mathcal{G}'_1 \to \mathcal{G}_2$.*

Proof. Modulo the central element Φ_2 reduces to the map Φ, which we have already seen is a map of factorization Lie algebras in Proposition BW: ref. Thus, to show that Φ_2 is a map of factorization Lie algebras we need to show that it is compatible with the cocycles determing the respective central extensions. That is, we need to show that

(4) $$\theta_1'(\varphi_0 a_0 X_0, \ldots, \varphi_d a_d X_d) = \theta_2(\Phi(\varphi_0 a_0 X_0), \ldots, \Phi(\varphi_d a_d X_d))$$

for all $\varphi_i a_i X_i \in \Omega_c^*(I) \otimes (A_d \otimes \mathfrak{g})$. The cocycle θ_1' is only nonzero if one of the φ_i inputs is a 1-form. We evaluate the left-hand side on the $(d+1)$-tuple $(\varphi_0 dr a_0 X_0, \varphi_1 a_1 X_1, \ldots, \varphi_d a_d X_d)$ where $\varphi_i \in C_c^\infty(I)$, $a_i \in A_d$, $X_i \in \mathfrak{g}$ for $i = 0, \ldots, d$. The result is

(5) $$\left(\int_I \varphi_0 \cdots \varphi_d dr \right) \left(\oint a_0 \partial a_1 \cdots \partial a_d \right) \theta(X_0, \ldots, X_d)$$

(6) $$+ \quad \frac{1}{2} \sum_{i=1}^d \left(\int_I \varphi_0 (E \cdot \varphi_i) \varphi_1 \cdots \widehat{\varphi_i} \cdots \varphi_d dr \right) \left(\oint (a_0 a_i d\vartheta) \partial a_1 \cdots \widehat{\partial a_i} \cdots \partial a_d \right) \theta(X_0, \ldots, X_d)$$

We wish to compare this to the right-hand side of Equation (4). Recall that $\Phi(\varphi_0 dr a_0 X_0) = \varphi(r) dr a_0(z) X_0$ and $\Phi(\varphi_i a_i X_i) = \varphi(r) a_i(z) X_i$. Plugging this into the explicit formula for the cocycle θ_2 we see the right-hand side of (4) is

(7) $$\left(\int_{\rho^{-1}(I)} \varphi_0(r) dr a_0(z) \partial(\varphi_1(r) a_1(z)) \cdots \partial(\varphi_d(r) a_d(z)) \right) \theta(X_0, \ldots, X_d).$$

We pick out the term in (7) in which the ∂ operators only act on the elements $a_i(z)$, $i = 1, \ldots, d$. This term is of the form

$$\int_{\rho^{-1}(I)} \varphi_0(r) \cdots \varphi_d(r) dr a_0(z) \partial(a_1(z)) \cdots \partial(a_d(z)) \theta(X_0, \ldots, X_d).$$

Separating variables we find that this is precisely the first term (5) in the expansion of the left-hand side of (4).

Now, note that we can rewrite the ∂-operator in terms of the radius r as

$$\partial = \sum_{i=1}^d dz_i \frac{\partial}{\partial z_i} = \sum_{i=1}^d dz_i \bar{z}_i \frac{\partial}{\partial(r^2)} = \sum_{i=1}^d dz_i \frac{r^2}{2z_i} \frac{\partial}{\partial r}.$$

The remaining terms in (7) correspond to the expansion of

$$\partial(\varphi_1(r) a_1(z)) \cdots \partial(\varphi_d(r) a_d(z)),$$

using the Leibniz rule, for which the ∂ operators act on at least one of the functions $\varphi_1, \ldots, \varphi_d$. In fact, only terms in which ∂ acts on precisely one of the functions $\varphi_1, \ldots, \varphi_d$ will be nonzero. For instance, consider the term

(8) $$(\partial \varphi_1) a_1(z) (\partial \varphi_2) a_2(z) \partial(\varphi_3(z) a_3(z)) \cdots \partial(\varphi_d(z) a_d(z)).$$

Now, $\partial \varphi_i(r) = \omega \frac{\partial \varphi}{\partial r}$ where ω is the one-form $\sum_i (r^2/2z_i) dz_i$. Thus, (8) is equal to

$$\left(\omega \frac{\partial \varphi_1}{\partial r} \right) a_1(z) \left(\omega \frac{\partial \varphi_2}{\partial r} \right) a_2(z) \partial(\varphi_3(z) a_3(z)) \cdots \partial(\varphi_d(z) a_d(z),$$

which is clearly zero as ω appears twice.

We observe that terms in the expansion of (7) for which ∂ acts on precisely one of the functions $\varphi_1, \ldots, \varphi_d$ can be written as

$$\sum_{i=1}^d \int_{\rho^{-1}(I)} \varphi_0(r) \left(r \frac{\partial}{\partial r} \varphi_i(r) \right) \varphi_1(r) \cdots \widehat{\varphi_i(r)} \cdots \varphi_d(r) \mathrm{d}r \frac{r}{2z_i} \mathrm{d}z_i a_0(z) a_i(z) \partial a_1(z) \cdots \widehat{\partial a_i(z)} \cdots \partial a_d(z).$$

Finally, notice that the function $z_i/2r$ is independent of the radius r. Thus, separating variables we find the integral can be written as

$$\frac{1}{2} \sum_{i=1}^d \left(\int_I \varphi_0 \left(r \frac{\partial}{\partial r} \varphi_i \right) \varphi_1 \cdots \widehat{\varphi_i} \cdots \varphi_d \mathrm{d}r \right) \left(\oint \frac{\mathrm{d}z_i}{z_i} a_0 a_i \partial a_2 \cdots \widehat{\partial a_i} \cdots \partial a_d \right).$$

This is precisely equal to the second term (6) above. Hence, the cocycles are compatible and the proof is complete.

$\square$

2.3. The E_n algebra from tori.

We now place the Kac-Moody factorization algebra on the complex d-fold $(\mathbb{C}^\times)^d$. We show, in a similar way as above, how to extract the structure of an E_d algebra from considering the nesting of "polyannuli" in $(\mathbb{C}^\times)^d$. These E_d-algebras are related to interesting extensions of the Lie algebra $L^d \mathfrak{g}$ introduce in the beginning of this section.

When $d = 1$, we have seen that the nesting of ordinary annuli give rise to the structure of an associative algebra. For $d > 1$, a polyannulus is a complex submanifold of the form $\mathbb{A}_1 \times \cdots \times \mathbb{A}_d \subset (\mathbb{C}^\times)^d$ where each $\mathbb{A}_i \subset \mathbb{C}^\times$ is an ordinary annulus. Equivalently, a polyannulus is the complement of a closed polydisk inside of a larger open polydisk. We will see how the nesting of annuli in each component gives rise to the structure of a locally constant factorization algebra in d *real* dimensions, and hence defines an E_d algebra.

A result of Knudsen [**?**], which we recall in Section **??**, states that every dg Lie algebra determines an E_d-algebra, for any $d > 1$, called the universal E_d enveloping algebra. This agrees with the ordinary universal enveloping algebra in the case $d = 1$. For the dg Lie algebra $\widehat{L^d \mathfrak{g}_\theta}$, we denote this E_d algebra by $U^{E_d}(\widehat{L^d \mathfrak{g}_\theta})$. Its associated locally constant factorization algebra on $\mathbb{R}^d$ is denoted $U^{E_d}(\widehat{L^d \mathfrak{g}_\theta})^{fact}$.

The Kac–Moody factorization algebra on the d-fold $(\mathbb{C}^\times)^d$ determines a real d-dimensional factorization algebra by considering the radius in each complex direction. We denote this factorization algebra on $\mathbb{R}^d$ by $\vec{\rho}_* (\mathcal{G}_{\mathbb{C}^{\times d}})$.

Theorem 2.8. *There is a dense injective map of factorization algebras on $\mathbb{R}^d$:*

$$\Phi^{L^d} : \left(U_{E_d} \left(\widehat{L^d \mathfrak{g}_\theta} \right) \right)^{fact} \to \vec{\rho}_* (\mathcal{G}_{\mathbb{C}^{\times d}}).$$

3. THE KAC-MOODY FACTORIZATION ALGEBRA ON GENERAL MANIFOLDS

3.1. The P_0-structure.

Every associative algebra determines a Lie algebra via the commutator. There is a left adjoint to this forgetful functor given by the enveloping algebra of a Lie algebra. Given a Lie algebra $\mathfrak{g}$, this enveloping algebra $U\mathfrak{g}$ can also be thought of as a quantization of a certain Poisson algebra. The Poincaré-Birkhoff-Witt theorem says that the associated graded $\mathrm{Gr}\, U\mathfrak{g}$ by the filtration given by symmetric degree is precisely $\mathbb{C}[\mathfrak{g}^*]$. It is a classical fact that the linear dual $\mathfrak{g}^*$ of a Lie algebra has the structure of a Poisson manifold. The Poisson bracket on

$\mathbb{C}[\mathfrak{g}^*] = \mathrm{Sym}(\mathfrak{g})$ is defined by extending the Lie bracket on the quadratic functions by the Leibniz rule.

In a completely analogous way, the factorization enveloping algebra of a local Lie algebra has a "classical limit" given by a P_0 factorization algebra. Recall, the factorization enveloping algebra of a local Lie algebra $\mathcal{L}$ evaluated on an open set U is given by the Chevalley-Eilenberg complex of the compactly supported sections on U

$$\mathbb{U}(\mathcal{L})(U) = \mathrm{C}^{\mathrm{Lie}}_*(\mathcal{L}(U)) = (\mathrm{Sym}^*(\mathcal{L}(U)[1]), \mathrm{d}_{\mathcal{L}} + \mathrm{d}_{CE}).$$

There is a filtration of this complex defined by $F^k = \mathrm{Sym}^{\geq k}(\mathcal{L}(U)[1])$. Moreover, this defines a filtration of the factorization algebra $\mathbb{U}(\mathcal{L})$.

Lemma 3.1. *Let $\mathcal{L}$ be a local Lie algebra. Then, the associated graded factorization algebra $\mathrm{Gr}\,\mathbb{U}(\mathcal{L})$ has the structure of a P_0 factorization algebra. Similarly, if $\alpha \in \mathrm{C}^*_{\mathrm{loc}}(\mathcal{L})$ is a cocycle of cohomological degree one then $\mathrm{Gr}\,\mathbb{U}_\alpha(\mathcal{L})$ has the structure of a P_0 factorization algebra.*

Up to issues of functional analysis, one should think of the P_0 algebra $\mathrm{Gr}\,\mathbb{U}(\mathcal{L})$ as the algebra of functions on the sheaf of dg vector spaces $\mathcal{L}^\vee[-1]$ with differential induced from that on $\mathcal{L}$. The P_0 algebra $\mathrm{Gr}\,\mathbb{U}_\alpha(\mathcal{L})$ is equal to functions on the same sheaf of dg vector spaces but with bracket modified by α.

Corollary 3.2. *For any principal G-bundle $P \to X$ consider the associated graded factorization algebra*

$$\mathrm{Gr}\,\mathbb{U}(\mathcal{A}\mathrm{d}(P)) : U \mapsto \left(\mathrm{Sym}^*(\Omega^{0,*}_c(U)[1]), \bar{\partial}\right).$$

Then, any element $\alpha \in H^1_{\mathrm{loc}}(\mathcal{A}\mathrm{d}(P))$ determines the structure of a P_0 factorization algebra on $\mathrm{Gr}\,\mathbb{U}(\mathcal{A}\mathrm{d}(P))$.

In the case that $\alpha = J_X(\theta)$ is the local cocycle corresponding to a symmetric polynomial $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$ the Poisson structure can be described explicitly as follows. The Poisson tensor is of the form $\Pi = \Pi_{[-,-]} + \Pi_\theta$ where

$$\Pi_{[-,-]} = \wedge \otimes [-,-] : \left(\Omega^{d,*}_X \otimes \mathfrak{g}\right) \otimes \left(\Omega^{0,*}_X \otimes \mathfrak{g}\right) \to \Omega^{d,*}_X \otimes \mathfrak{g}$$

and

$$\Pi_\theta : \left(\Omega^{0,*}_X \otimes \mathfrak{g}\right)^{\otimes d} \to \Omega^{d,*}_X \otimes \mathfrak{g}$$

sends $\alpha_1 \otimes \cdots \otimes \alpha_d \mapsto \partial\alpha_1 \wedge \cdots \wedge \partial\alpha_d$.

3.2. **Factorization homology along Hopf surfaces.**

Proposition 3.3. *Let X be a Hopf manifold and suppose $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$ is any $\mathfrak{g}$-invariant polynomial of degree $(d+1)$. Then, there is a quasi-isomorphism*

$$\int_X \mathbb{U}_\theta(\mathcal{G}_X) \simeq \mathrm{Hoch}_*(U\mathfrak{g})[K]$$

where K is the central parameter of cohomological degree zero.

Proof. Let's first consider the untwisted case. In this case, we must show $\int_X \mathbb{U}(\mathcal{G}_X) \simeq \mathrm{Hoch}_*(U\mathfrak{g})$. The factorization homology on the left hand side is computed by

$$\int_X \mathbb{U}(\mathcal{G}_X) = \mathrm{C}^{\mathrm{Lie}}_*(\Omega^{0,*}(X) \otimes \mathfrak{g}).$$

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We have already seen in Section **??** that every Hopf manifold is Dolbeualt formal. Thus, there is a quasi-isomorphism

$$\left(H^{0,*}(X), 0\right) \simeq \left(\Omega^{0,*}(X), \bar{\partial}\right).$$

In fact, we have written down a preferred presentation for the cohomology ring of X given by $H^{0,*}(X) = \mathbb{C}[\delta]$ where $|\delta| = 1$. A particular Dolbeault representative for δ given by

$$\bar{\partial}(\log |z|^2) = \sum_i \frac{z_i \mathrm{d}\bar{z}_i}{|z|^2}$$

where $z = (z_1, \ldots, z_d)$ is the coordinate on $\mathbb{C}^d \setminus \{0\}$.

Applied to the global sections of the Kac-Moody we see that there is a quasi-isomorphism

$$\int_X \mathbb{U}(\mathcal{G}_X) \simeq C_*^{\mathrm{Lie}}(\mathbb{C}[\delta] \otimes \mathfrak{g}).$$

Now, note that $C_*^{\mathrm{Lie}}(\mathbb{C}[\delta] \otimes \mathfrak{g}) = C_*^{\mathrm{Lie}}(\mathfrak{g} \oplus \mathfrak{g}[-1]) = C_*^{\mathrm{Lie}}(\mathfrak{g}, \mathrm{Sym}(\mathfrak{g}))$, where $\mathrm{Sym}(\mathfrak{g})$ is the symmetric product of the adjoint action of $\mathfrak{g}$ on itself. By Poincaré-Birkoff-Witt there is an isomorphism of vector spaces $\mathrm{Sym}(\mathfrak{g}) = U\mathfrak{g}$, so we can write this as $C_*^{\mathrm{Lie}}(\mathfrak{g}, \mathrm{Sym}(\mathfrak{g}))$.

Now, any $U(\mathfrak{g})$-bimodule M is automatically a module for the Lie algebra $\mathfrak{g}$ by the formula $x \cdot m = xm - mx$ where $x \in \mathfrak{g}$ and $m \in M$. Moreover, for any such bimodule there is a quasi-isomorphism of cochain complexes

$$C_*^{\mathrm{Lie}}(\mathfrak{g}, M) \simeq \mathrm{Hoch}_*(U\mathfrak{g}, M).$$

This is proved, for instance, in Section 2.3 of [**?**]. Applied to the bimodule $M = U\mathfrak{g}$ itself we obtain $C^{\mathrm{Lie}}(\mathfrak{g}, U\mathfrak{g}) \simeq \mathrm{Hoch}(U\mathfrak{g})$.

The twisted case is similar. Let θ be as in the statement. Then, the factorization homology is equal to

$$\int_X \mathbb{U}_\theta(\mathcal{G}_X) = \left(\mathrm{Sym}(\Omega^{0,*}(X) \otimes \mathfrak{g})[K], \bar{\partial} + \mathrm{d}_{CE} + \mathrm{d}_\theta\right).$$

Applying Dolbeualt formality again we see that this is quasi-isomorphic to the cochain complex

(9) $$(\mathrm{Sym}(\mathfrak{g}[\delta])[K], \mathrm{d}_{CE} + \mathrm{d}_\theta).$$

We note that d_θ is identically zero on $\mathrm{Sym}(\mathfrak{g}[\delta])$. Indeed, for degree reasons, at least one of the inputs must be from $\mathfrak{g} \hookrightarrow \mathfrak{g}[\delta] = \mathfrak{g} \oplus \mathfrak{g}[-1]$, which consists of constant functions on X with values in the Lie algebra $\mathfrak{g}$. In the formula for the local cocycle (**??**) associated to θ it is clear that if any one of the inputs is constant the cocycle vanishes. Indeed, one can integrate by parts to put it in the form $\int \partial \alpha \cdots \partial \alpha$, which is the integral of a total derivative, hence zero since X has no boundary. Thus (9) just becomes the Chevalley-Eilenberg complex with values in the trivial module $\mathbb{C}[K]$. By the same argument as in the untwisted case, we conclude that in this case the factorization homology is quasi-isomorphic to $\mathrm{Hoch}_*(U\mathfrak{g})[K]$ as desired. $\square$

There is an interesting consequence of this calculation to the Hochschild homology for the A_∞ algebra $U(\widehat{\mathfrak{g}}_{d,\theta})$. It is easiest to state this when X is a Hopf manifold of the form $(\mathbb{C}^d \setminus \{0\})/q^{\mathbb{Z}}$ for a single $q \in D(0,1)^\times$ where the quotient is by the relation $(z_1, \ldots, z_d) \simeq (q^{\mathbb{Z}} z_1, \ldots, q^{\mathbb{Z}})$. Let

$p_q : \mathbb{C}^d \setminus \{0\} \to X$ be the quotient map. Consider the following diagram

$$
\begin{array}{ccc}
\mathbb{C}^d \setminus \{0\} & \xrightarrow{\ p_q\ } & X \\
\downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \bar{\rho}} \\
\mathbb{R}_{>0} & \xrightarrow{\ \bar{p}_q\ } & S^1
\end{array}
$$

Here, ρ is the radial projection map and $\bar{\rho}$ is the induced map defined by the quotient. The action of $\mathbb{Z}$ on $\mathbb{C}^d \setminus \{0\}$ gives $\mathcal{G}_{\mathbb{C}^d \setminus \{0\}}$ the structure of a $\mathbb{Z}$-equivariant factorization algebra. In turn, this determines an action of $\mathbb{Z}$ on pushforward factorization algebra $\rho_* \mathcal{G}_{\mathbb{C}^d \setminus \{0\}}$. We have seen that there is a dense locally constant subfactorization algebra on $\mathbb{R}_{>0}$ of the pushforward that is equivalent as an E_1 algebra to $U(\widehat{\mathfrak{g}}_{d,\theta})$. ...BW: finish

Thus, we find that there is a quasi-isomorphism

$$
\mathrm{Hoch}_*(U(\mathfrak{g}_{d,\theta}), q) \simeq \int_{S^1} \bar{\rho}_* \mathbb{U}_\alpha(\mathcal{G}_X) \xrightarrow{\simeq} \int_X \mathbb{U}_\alpha(\mathcal{G}_X).
$$

We have just shown that the factorization homology of $\mathcal{G}_X$ is equal to the Hochschild homology of $U\mathfrak{g}$ so that

$$
\mathrm{Hoch}_*(U(\mathfrak{g}_{d,\theta}), q) \simeq \mathrm{Hoch}_*(U\mathfrak{g})[K].
$$

This statement is purely algebraic and independent of the manifold X. It may be easiest to understand in the case $d = 1$ and $\theta = 0$. Then $\mathfrak{g}_{d,\theta}$ is simply the loop algebra $L\mathfrak{g} = g[z, z^{-1}]$. The action of $\mathbb{Z}$ on $L\mathfrak{g}$ simply rotates the loop parameter: for $z^n \otimes \mathfrak{g} \in L\mathfrak{g} = \mathbb{C}[z, z^{-1}] \otimes \mathfrak{g}$ the action of $1 \in \mathbb{Z}$ is $1 \cdot (z^n \otimes \mathfrak{g}) = q^n z^n \otimes \mathfrak{g}$. In turn, the bimodule structure of $U(\mathfrak{g}[z, z^{-1}])$ on itself, which we denote $U(\mathfrak{g}[z, z^{-1}])_q$ is the ordinary one on the left and on the right is given by twisting by the automorphism corresponding to $1 \in \mathbb{Z}$. The complex $\mathrm{Hoch}_*(U(g[z, z^{-1}]), q)$ is the Hochschild homology of $U(\mathfrak{g}[z, z^{-1}])$ with values in this bimodule so the statement implies

$$
\mathrm{Hoch}_* \left(U(\mathfrak{g}[z, z^{-1}]), U(\mathfrak{g}[z, z^{-1}])_q \right) \simeq \mathrm{Hoch}(U\mathfrak{g}).
$$

4. Universal Grothendieck-Riemann-Roch from BV quantization

The main goal of the BV formalism developed in [**?**] is to rigorously construct quantum field theories using a combination of homological methods and a rigorous model for renormalization. A particular nicety of this approach is the ability to study *families* of field theories, which we will turn into an equivariant version of BV quantization, see Section **??**. In this section we will consider a family of QFT's parametrized by the moduli space of principal G-bundles. Our main result is to interpret a certain anomaly coming from BV quantization as a families index over $\mathrm{Bun}_G(X)$. This anomaly is computed via an explicit Feynman diagrammatic calculation and is related to a local cocycle of the current algebra discussed in Section BW: ref. We will interpret this as a formal universal version of the Grothendieck-Riemann-Roch theorem over the moduli space of bundles.

We will arrive at the result in a way that is local-to-global on spacetime which we formulate in terms of factorization algebras. The main them of Costello-Gwilliam's approach to QFT is that the observables of a QFT determine a factorization algebra. We study the associated family of factorization algebras associated to the family of QFT's over the moduli space of G-bundles. We will recollect a formulation of Noether's theorem for symmetries of a theory in terms of factorization algebras developed in Chapter **??** of [**?**]. The central object in this discussion is a "local index"

which describes how the Kac–Moody factorization algebra acts on the observables of the QFT. Locally on spacetime we see how Noether's theorem provides a *free field realization* of the Kac-Moody factorization algebra generalizing that of the Kac–Moody vertex algebra in chiral conformal field theory [**?**].

We now give a brief summary of the results, with a background for the situation we consider. Suppose that P is a fixed holomorphic G-bundle on a complex manifold X. We have already seen how to express the formal deformation space of P inside of the moduli of G-bundles using the dg Lie algebra $\mathcal{A}d(P)(X) = \Omega^{0,*}(X, \mathrm{ad}(P))$. In particular, any Maurer-Cartan element of $\mathcal{A}d(P)(X)$ defines a deformation of P. We have seen that there is a refinement of this dg Lie algebra to a local Lie algebra $\mathcal{A}d(P)$ whose factorization envelope defines the higher Kac-Moody factorization algebra above. To any G-representation V we will define a holomorphic theory with fields $\mathcal{E}_V$ that is equivariant for this local Lie algebra. Equivalently, we can think of $\mathcal{E}_V$ as defining a family of theories over the formal completion of P in the moduli of G-bundles

$$
\begin{array}{ccc}
\mathcal{E}_V|_{P'} & \longrightarrow & \mathcal{E}_V \\
\downarrow & & \downarrow \\
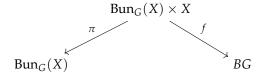\{P'\} & \longrightarrow & \mathrm{Bun}_G(X)^{\wedge}_P.
\end{array}
$$

Over each fiber P' the theory $\mathcal{E}_V|_{P'}$ is a *free* theory, so admits a canonical BV quantization. Our formulation of equivariant BV quantization is codification of the problem of gluing together these quantizations in a compatible way. We will show how this presents itself in the failure of the BV quantization to be a *flat* family. Our main result is the following.

Theorem 4.1. *Let P be any principal G-bundle over a compact affine complex manifold X of dimension d. Suppose V is a G-representation. Then, the factorization homology $\int_X \mathrm{Obs}^{\mathrm{q}}_V$ defines a line bundle over the formal neighborhood of P inside of the moduli of G-bundles. Moreover, its first Chern class is*

$$
c_1 \left(\int_X \mathrm{Obs}^{\mathrm{q}}_V \right) = C\mathrm{ch}^{\mathfrak{g}}_{d+1}(V)
$$

under the identification of $\mathrm{ch}^{\mathfrak{g}}_{d+1}(V)$ as a cohomology class on the formal neighborhood of P inside of the moduli of G-bundles in Equation ([10]) explained below. Here, C is some nonzero complex number.

There is an elucidating geometric description of how the classes $\mathrm{ch}_{d+1}(V)$ appear: they describe curvatures of line bundles over the moduli of G-bundles. Let $\mathrm{Bun}_G(X)$ denote the moduli space of G-bundles on the complex d-fold X. [3] Over the space $\mathrm{Bun}_G(X) \times X$ there is the *universal G-bundle*. If $P \to X$ is a G-bundle, the fiber over the point $\{[P]\} \times X$ is precisely the G-bundle $P \to X$. This universal G-bundle is classified by a map $f : \mathrm{Bun}_G(X) \times X \to BG$. Consider the following diagram

$$
\begin{array}{ccc}
& \mathrm{Bun}_G(X) \times X & \\
{}^{\pi}\swarrow & & \searrow^{f} \\
\mathrm{Bun}_G(X) & & BG
\end{array}
$$

---

[3] For $d > 1$ [**?**] have constructed a global smooth derived realization of this space, but its full structure will not be used in this discussion.

where $\pi : \mathrm{Bun}_G(X) \times X \to \mathrm{Bun}_G(X)$ denotes the projection. If $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}} \cong H^{d+1}(G, \Omega^{d+1}) \subset H^{2d+2}(BG)$ then we obtain via push-pull in the diagram above

$$\int_\pi \circ f^* \theta \in H^2(\mathrm{Bun}_G(X)).$$

The formal moduli space $B\mathfrak{g}^X$ describes a formal neighborhood of the trivial bundle inside of the moduli of G-bundles. So the theorem above can be viewed as a formal version of the universal GRR theorem over the moduli of G-bundles.

4.1. **The classical family.** In this section, we consider a BV theory that is equivariant for the local Lie algebra $\mathcal{A}\mathrm{d}(P)$ in the language of Section **??**. Let V be any G-representation, and define the associated vector bundle $\mathcal{V}_P = P \times^G V$ on X. The holomorphic theory we consider is based on the graded holomorphic vector bundle $\mathcal{V}_P \oplus K_X \otimes \mathcal{V}_P^*[d-1]$, where $\mathcal{V}_P^*$ is the linear dual bundle. The fields of the associated free BV theory are

$$\mathcal{E}_V = \Omega^{0,*}(X, \mathcal{V}_P) \oplus \Omega^{d,*}(X, \mathcal{V}_P^*)[d-1].$$

This is simply the $\beta\gamma$ system on X twisted by the vector bundle $\mathcal{V}_P$. The action functional is $\int \langle \beta, \bar{\partial}\gamma \rangle_V$ where the pairing is between V and its dual. In particular, the theory $\mathcal{E}_V$ is free.

The action of $\mathfrak{g}$ on V extends to an action of the local Lie algebra $\mathcal{A}\mathrm{d}(P)$ on this classical BV theory. To define this equivariance we need to presribe a Noether current.

Lemma 4.2. *The local Noether current* $I^{\mathfrak{g}} \in \mathrm{C}^*_{\mathrm{loc}}(\mathcal{A}\mathrm{d}(P)) \otimes \mathcal{O}_{\mathrm{loc}}(\mathcal{E}_V)$ *defined by*

$$I^{\mathfrak{g}}(\alpha, \gamma, \beta) = \int_X \langle \beta, \alpha \cdot \gamma \rangle_V$$

satisfies the equivariant classical master equation

$$(\mathrm{d}_{\mathfrak{g}} + \bar{\partial})I^{\mathfrak{g}} + \frac{1}{2}\{I^{\mathfrak{g}}, I^{\mathfrak{g}}\} = 0,$$

where $\mathrm{d}_{\mathfrak{g}}$ *encodes the Lie algebra structure on* $\mathcal{A}\mathrm{d}(P)$. *Hence,* $I^{\mathfrak{g}}$ *gives* $\mathcal{E}_V$ *the structure of a classical* $\mathcal{A}\mathrm{d}(P)$-*equivariant theory.*

Proof. If α is an element in $\mathcal{A}\mathrm{d}(P)$ and $\gamma \in \Omega^{0,*}(X, \mathcal{V}_P)$ we define $\alpha \cdot \gamma$ through the $\mathfrak{g}$-module structure of $\mathfrak{g}$ on V combined with the wedge product of Dolbeault forms. Note that $I^{\mathfrak{g}}$ is arises from holomorphic differential operators so that $\bar{\partial}I^{\mathfrak{g}} = 0$. From the definition of the bracket we see that for α_1, α_2 one has $\{\int \langle \beta, \alpha_1 \cdot \gamma \rangle, \int \langle \beta, \alpha_2 \cdot \gamma \rangle\} = \int \langle \beta, [\alpha_1, \alpha_2] \cdot \gamma \rangle$ which cancels the term coming from $\mathrm{d}_{\mathfrak{g}}$. $\square$

4.2. **BV quantization in families.** The main technique we employ is equivariant BV quantization, which we have reviewed in Section **??**. Our main result holds for a compact affine manifold, which we will view as coming from a quotient of an open set in affine space $\mathbb{C}^d$. Thus, we will mostly work with the theory defined on $\mathbb{C}^d$ and afterwards deduce our main result on the quotient via descent. Thus, we will work with the $\beta\gamma$ system

$$\mathcal{E}_V = \Omega^{0,*}(\mathbb{C}^d, V) \oplus \Omega^{d,*}(\mathbb{C}^d, V^*)[d-1]$$

where V is some $\mathfrak{g}$-module. The local Lie algebra which acts on this theory is $\mathcal{G} = \Omega^{0,*}(\mathbb{C}^d, \mathfrak{g})$.

Our first step is to construct an equivariant effective prequantization. for the $\mathcal{G}$-equivariant theory. As has been the case over and over again in this thesis, our situation for constructing the

prequantization is vastly simplified since our theory comes from holomorphic data. Indeed, the equivariant $\beta\gamma$ system is a holomorphic theory on $\mathbb{C}^d$ so that we can apply Lemma **??**. As an immediate corollary, the following definition is well-defined.

Definition 4.3. For $L > 0$, let

$$I^{\mathfrak{g}}[L] := \lim_{\epsilon \to 0} W(P_{\epsilon < L}, I^{\mathfrak{g}}) = \lim_{\epsilon \to 0} \sum_{\Gamma} \frac{\hbar^{g(\Gamma)}}{|\text{Aut}(\Gamma)|} W_{\Gamma}(P_{\epsilon < L}, I^{\mathfrak{g}}).$$

Here the sum is over all isomorphism classes of stabled connected graphs, but only graphs of genus ≤ 1 contribute nontrivially. By construction, the collection satisfies the RG flow equation and its tree-level $L \to 0$ limit is manifestly $I^{\mathfrak{g}}$. Hence $\{I^{\mathfrak{g}}[L]\}_{L \in (0,\infty)}$ is a $\mathcal{G}$-*equivariant prequantization*.

Our next step is to compute the obstruction to quantization of the $\mathcal{G}$-equivariant theory. By definition, the scale L *obstruction cocycle* $\Theta_V[L]$ is the failure for the interaction $I^{\mathfrak{g}}[L]$ to satisfy the scale L equivariant quantum master equation. Explicitly, one has

$$\hbar\Theta_V[L] = (d_{\mathfrak{g}} + Q)I^{\mathfrak{g}}[L] + \hbar\Delta_L I^{\mathfrak{g}}[L] + \{I^{\mathfrak{g}}[L], I^{\mathfrak{g}}[L]\}_L.$$

A completely analogous argument as in Corollary 16.0.5 of **[?]** we see that the scale L obstruction is given by a sum over wheels.

Lemma 4.4. *Only wheels contribute to the anomaly cocycle* $\Theta_V[L]$. *Moreover, one has*

$$\Theta_V[L] = \sum_{\substack{\Gamma \in \text{Wheels} \\ e \in \text{Edge}(\Gamma)}} W_{\Gamma,e}(P_{\epsilon < 1}, K_{\epsilon}, I^{\mathfrak{g}}[\epsilon]),$$

where the sum is over wheels and distinguished edges. The notation $W_{\Gamma,e}(P_{\epsilon < 1}, K_{\epsilon}, I^{\mathfrak{g}}[\epsilon])$ *means we place the propagator at all edges besides the distinguished one, where we place* K_{ϵ}.

The only fields that propagate are the $\beta\gamma$ fields with values in V. Thus, for purely combinatorial reasons we see that the anomaly cocycle is only a functional of the background fields $\mathcal{G}$, see Figure BW: ref. In other words, the external edges of any closed wheel must be labeled by $\mathcal{G}$. As an immediate consequence we have the following.

Lemma 4.5. *The effective family* $\{I^{\mathfrak{g}}[L]\}$ *defines a one-loop exact $\mathcal{G}$-equivariant quantum field theory. In other words, it satisfies the $\mathcal{G}$-equivariant quantum master equation modulo functionals purely of the background fields $\mathcal{G}$.*

It follows that the anomaly $\{\Theta[L]\}$ measures the obstruction to $\{I^{\mathfrak{g}}[L]\}$ to defining an *inner* action.

4.2.1. *The anomaly calculation.* We now perform the main technical calculation of the anomaly cocycle.

Proposition 4.6. *The $L \to 0$ limit of the anomaly cocycle* $\Theta = \lim_{L \to 0} \Theta_V[L] \in C^*_{\text{loc}}(\mathcal{G})$ *is of the form*

$$\Theta_V = C \cdot J_{\mathbb{C}^d}(\text{ch}^{\mathfrak{g}}_{d+1}(V)),$$

where $\text{ch}^{\mathfrak{g}}_{d+1}(V) \in \text{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$ *and where* $J_{\mathbb{C}^d} : \text{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}} \to C^*_{\text{loc}}(\mathcal{G})$ *is the map of Lemma 1.9 and where C some constant only depending on the dimension d.*

To compute the anomaly we refer to the following result about the expression for the anomaly cocycle in terms of the Feynman diagram expansion. This is proved in direct analogy to Lemma **??**. We have already seen in Lemma **??** that only wheels contribute to the anomaly cocycle. Then an explicit analysis of the analytic behavior shows that in the $\epsilon \to 0$ limit only the wheel with $(d+1)$-vertices contributes.

Lemma 4.7. *The limit $\Theta_V := \lim_{L\to 0} \Theta_V[L]$ exists and is an element of degree one in $C^*_{\mathrm{Lie}}(W_n, C^*_{\mathrm{loc}}(\mathfrak{g}_n^{\mathbb{C}}))$. Moreover, it is given by*

$$\Theta_V = \lim_{\epsilon \to 0} \sum_{\substack{\Gamma \in (d+1)\text{-vertex wheels} \\ e \in \mathrm{Edge}(\Gamma)}} W_{\Gamma,e}(P_{\epsilon<1}, K_\epsilon, I^{\mathfrak{g}}[\epsilon]),$$

where the sum is over wheels Γ with two vertices and a distinguished inner edge e.

The lemma implies that we only need to consider the wheel with $d+1$ vertices. Each trivalent vertex is labeled by both an analytic factor and Lie algebraic factor. The Lie algebraic part of each vertex can be thought of as the defining map of the representation $\rho : \mathfrak{g} \to \mathrm{End}(V)$. The diagrammitcs of the wheel amounts to taking the trace of the symmetric $(d+1)$st power of this Lie algebra factor. Thus, the Lie algebraic factor of the weight of the wheel is the $(d+1)$st component of the character of the representation

$$\mathrm{ch}^{\mathfrak{g}}_{d+1}(V) = \frac{1}{(d+1)!} \mathrm{Tr}\left(\rho(X)^{d+1}\right) \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*).$$

To finish the calculation we must compute the analytic weight of the wheel with $d+1$ vertices. Recall, our goal is to identify the anomaly Θ with the image of $\mathrm{ch}^{\mathfrak{g}}_{d+1}(V)$ under the map

$$J : \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}} \to C^*_{\mathrm{loc}}(\Omega^{0,*}(\mathbb{C}^d) \otimes \mathfrak{g})$$

that sends an element θ to the local functional $\int \theta(\alpha \partial \alpha \cdots \partial \alpha)$. We have just seen that the Lie algebra factor in local functional representing the anomaly agrees with the $(d+1)$st Chern character. Thus, to finish we must show the following.

Lemma 4.8. *As a functional on the abelian dg Lie algebra $\Omega^{0,*}(\mathbb{C}^d)$, the analytic factor of the weight $\lim_{L\to 0} \lim_{\epsilon\to 0} W_{\Gamma,e}(P_{\epsilon<L}, K_\epsilon, I^{\mathfrak{g}})$ is equal to a multiple of the local functional*

$$\int \alpha \partial \alpha \cdots \partial \alpha \in C^*_{\mathrm{loc}}(\Omega^{0,*}(\mathbb{C}^d)).$$

Proof. Let's fix some notation. We enumerate the vertices by integers $a = 0, \ldots, d$. Label the coordinate at the ith vertex by $z^{(a)} = (z_1^{(a)}, \ldots, z_d^{(a)})$. The incoming edges of the wheel will be denoted by homogeneous Dolbeault forms

$$\alpha^{(a)} = \sum_J A_J^{(a)} d\bar{z}_J^{(a)} \in \Omega_c^{0,*}(\mathbb{C}^d).$$

where the sum is over the multiindex $J = (j_1, \ldots, j_k)$ where $j_a = 1, \ldots, d$ and $(0, k)$ is the homogenous Dolbeault form type. For instance, if α is a $(0, 2)$ form we would write

$$\alpha = \sum_{j_1 < j_2} A_{(j_1, j_2)} d\bar{z}_{j_1} d\bar{z}_{j_2}.$$

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Denote the functional obtained as the $\epsilon \to 0$ weight of the wheel with $(d+1)$ vertices from Lemma **??** by W_L. The $L \to 0$ limit of W_L is the local functional representing the one-loop anomaly Θ.

The weight has the form

$$W_L(\alpha^{(0)}, \ldots, \alpha^{(d)}) = \pm \lim_{\epsilon \to 0} \int_{\mathbb{C}^{d(d+1)}} \left(\alpha^{(0)}(z^{(0)}) \cdots \alpha^{(d)}(z^{(d)}) \right) K_\epsilon(z^{(0)}, z^{(d)}) \prod_{a=1}^{d} P_{\epsilon,L}(z^{(a-1)}, z^{(a)}).$$

We introduce coordinates

$$w^{(0)} = z^{(0)}$$
$$w^{(a)} = z^{(a)} - z^{(a-1)} \quad 1 \le a \le d.$$

The heat kernel and propagator part of the integral is of the form

$$\begin{aligned}
K_\epsilon(w^{(0)}, w^{(d)}) \prod_{a=1}^{d} P_{\epsilon,L}(w^{(a-1)}, w^{(a)}) &= \frac{1}{(4\pi\epsilon)^d} \int_{t_1,\ldots,t_d=\epsilon}^{L} \frac{dt_1 \cdots dt_d}{(4\pi t_1)^d \cdots (4\pi t_d)^d} \frac{1}{t_1 \cdots t_d} \\
&\times d^d w^{(0)} \prod_{i=1}^{d} (d\bar{w}_i^{(1)} + \cdots + d\bar{w}_i^{(d)}) \prod_{a=1}^{d} d^d w^{(a)} \left(\sum_{i=1}^{d} \bar{w}_i^{(a)} \prod_{j\ne i} d\bar{w}_j^{(a)} \right) \\
&\times e^{-\sum_{a,b=1}^{d} M_{ab} w^{(a)} \cdot \bar{w}^{(b)}}.
\end{aligned}$$

Here, M_{ab} is the $d \times d$ square matrix satisfying

$$\sum_{a,b=1}^{d} M_{ab} w^{(a)} \cdot \bar{w}^{(b)} = |\sum_{a=1}^{d} w^{(a)}|^2/\epsilon + \sum_{a=1}^{d} |w^{(a)}|^2/t_a.$$

Note that

$$\prod_{i=1}^{d} (d\bar{w}_i^{(1)} + \cdots + d\bar{w}_i^{(d)}) \prod_{a=1}^{d} \left(\sum_{i=1}^{d} \bar{w}_i^{(a)} \prod_{j\ne i} d\bar{w}_j^{(a)} \right) = \left(\sum_{i_1,\ldots i_d} \epsilon_{i_1 \cdots i_d} \prod_{a=1}^{d} \bar{w}_{i_a}^{(a)} \right) \prod_{a=1}^{d} d^d \bar{w}^{(a)}.$$

In particular, only the $dw_i^{(0)}$ components of $\alpha^{(0)} \cdots \alpha^{(d)}$ can contribute to the weight.

Let $\Phi = $ BW: some contraction of $\alpha^{(0)} \cdots \alpha^{(d)}$ by a antiholomorphic vector field. Then, the weight can be written as

$$\begin{aligned}
W(\alpha^{(0)}, \ldots, \alpha^{(d)}) &= \lim_{\epsilon \to 0} \int_{\mathbb{C}^{d(d+1)}} \left(\prod_{a=0}^{d} d^d w^{(a)} d^d \bar{w}^{(a)} \right) \Phi \\
&\times \frac{1}{(4\pi\epsilon)^d} \int_{t_1,\ldots,t_d=\epsilon}^{L} \frac{dt_1 \cdots dt_d}{(4\pi t_1)^d \cdots (4\pi t_d)^d} \frac{1}{t_1 \cdots t_d} \sum_{i_1,\ldots,i_d} \epsilon_{i_1 \cdots i_d} \bar{w}_{i_1}^{(1)} \cdots \bar{w}_{i_d}^{(d)} e^{-\sum_{a,b=1}^{d} M_{ab} w^{(a)} \cdot \bar{w}^{(b)}}
\end{aligned}$$

Applying Wick's lemma in the variables $w^{(1)}, \ldots, w^{(d)}$, together with some elementary analytic bounds, we find that the weight above becomes to the following integral over $\mathbb{C}^d$

$$f(L) \int_{w^{(0)} \in \mathbb{C}^d} d^d w^{(0)} d^d \bar{w}^{(0)} \sum_{i_1,\ldots,i_d} \epsilon_{i_1 \cdots i_d} \left(\frac{\partial}{\partial w_{i_1}^{(1)}} \cdots \frac{\partial}{\partial w_{i_d}^{(d)}} \Phi \right) \Big|_{w^{(1)}=\cdots=w^{(d)}=0}$$

where

$$f(L) = \lim_{\epsilon \to 0} \int_{t_1,\ldots,t_d=\epsilon}^{L} \frac{\epsilon}{(\epsilon + t_1 + \cdots + t_d)^{d+1}} d^d t.$$

In fact, $f(L)$ is independent of L and is equal to some nonzero constant $C \neq 0$. Finally, plugging in the forms $\alpha^{(0)}, \dots, \alpha^{(d)}$, we observe that the integral over $w^{(0)} \in \mathbb{C}^d$ simplifies to

$$C \int_{\mathbb{C}^d} \alpha^{(0)} \partial \alpha^{(1)} \cdots \partial \alpha^{(d)}$$

as desired. $\qquad\square$

This completes the proof of Proposition **??**.

4.3. **Local to global.** In this section we finish the proof of our main result Theorem 4.1 by showing how our local calculation above implies the formula for the anomaly on a general compact affine manifold X. By an complex affine manifold, we mean a quotient

$$q : U \subset \mathbb{C}^d \to X$$

of an open subset $U \subset \mathbb{C}^d$ by a free and proper action of a discrete subgroup of the affine group $U(d) \ltimes \mathbb{C}^d$. We consider affine manifolds that are also compact. To deduce our main theorem we will show that the theory and the anomaly above also exhibit equivariance for the affine group on $\mathbb{C}^d$, thus it will descend to any affine manifold.

We have stated the main result for an arbitrary principal G-bundle P on the affine manifold X. Suppose the discrete subgroup $\Gamma \leq U(d) \ltimes \mathbb{C}^d$ defines the affine manifold $q : U \to X = U/\Gamma$ as above. Then, principal G-bundles on X are equivalent to Γ-equivariant principal G-bundles on U.

First, notice that the classical free theory $\mathcal{E}_V$ is equivariant for the affine group $U(d) \ltimes \mathbb{C}^d$. Thus, it defines a classical theory on any affine manifold X. Similarly, the functional $I^{\mathfrak{g}}$ is equivariant for the affine group, so that it also descents to a functional on X. Thus, the Noether current $I^{\mathfrak{g}}$ defines a BW: finish

The local Lie algebra cohomology of any local Lie algebra embeds inside its ordinary (reduced) Lie algebra cohomology of global sections. Thus we have an embedding of cochain complexes

$$C^*_{\mathrm{loc}}(\mathfrak{g}^X) \hookrightarrow C^*_{\mathrm{Lie,red}}(\mathfrak{g}^X(X)) = C^*_{\mathrm{Lie,red}}(\Omega^{0,*}(X) \otimes \mathfrak{g})$$

By Kodaira-Spencer theory have already seen that the global sections of the local Lie algebra $\mathfrak{g}^X$ is a model for the formal neighborhood of the trivial G-bundle inside of G-bundles. Thus we have an identification

$$C^*_{\mathrm{Lie,red}}(\Omega^{0,*}(X) \otimes \mathfrak{g}) = \mathcal{O}_{red}\left(\mathrm{Bun}_G(X)^{\wedge}_{triv}\right) \cong \Omega^1_{cl}\left(\mathrm{Bun}_G(X)^{\wedge}_{triv}\right)$$

where we have used the equivalence of reduced functions and closed one-forms which makes sense on any formal moduli space. The same holds when we work around any holomorphic principal bundle P on X, so that we have an embedding of cochain complexes

$$C^*_{\mathrm{loc}}(\mathfrak{g}^X_P) \hookrightarrow \Omega^1_{cl}\left(\mathrm{Bun}_G(X)^{\wedge}_P\right).$$

Further, we have the cochain map $J^X_P : \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}[-1] \to C^*_{\mathrm{loc}}(\mathfrak{g}^X_P)$ constructed in Section **??** In this section we will compute an anomaly to quantizing a certain family of theories defined over the moduli of G-bundles. The anomaly is a degree $+1$ cocycle in the local cohomology of $\mathfrak{g}^X_P$. At the level of H^1 the above maps determine a composition we obtain by composition of the above cochain complexes a map

$$(10) \qquad \operatorname{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}} \xrightarrow{J_P^X} H_{\mathrm{loc}}^1(\mathfrak{g}_P^X) \to H^1(\Omega_{cl}^1\left(\operatorname{Bun}_G(X)_P^{\wedge}\right)).$$

4.4. **A module for the higher Kac-Moody.** The last part of this section we diverge to deduce a consequence of the quantum Noether theorem using our analysis above by exhibiting a module for the higher affine algebras from the previous section. For convenience, we fix the trivial $\mathfrak{g}$-bundle $P = \operatorname{triv}$ so that $\mathcal{A}d(P) = \mathcal{G}_X$.

On any manifold X, the quantum Noether theorem provides a map of factorization algebras

$$\Phi : \mathbb{U}_\alpha(\mathcal{G}_X) \to \operatorname{Obs}_V^{\mathsf{q}}.$$

We consider, as we did above, the case $X = \mathbb{C}^d \setminus \{0\}$. By functoriality of pushforwards, the quantum Noether theorem produces a map of one-dimensional factorization algebras

$$\rho_*\Phi : \rho_*\mathbb{U}_{\operatorname{ch}_{d+1}(V)}(\mathcal{G}_{\mathbb{C}^d\setminus\{0\}}) \to \rho_*\operatorname{Obs}_V^{\mathsf{q}}.$$

We have exhibited a locally constant dense subfactorization algebra $\mathcal{F}_{1d}^{lc}$ of $\rho_*\mathbb{U}_\alpha(\mathcal{G}_X)$ which is equivalent, as an E_1-algebra, to $U\widehat{\mathfrak{g}}_{d,\operatorname{ch}_{d+1}(V)}$. Similarly, in Section **??** we have shown that there is a locally constant dense subfactorization algebra that is equivalent to the dg algebra $\mathcal{A}_V$.

The map $\rho_*\Phi$ restricts to these dense subfactorization algebras and so defines a map of E_1 algebras

$$\rho_*\Phi : U\widehat{\mathfrak{g}}_{d,\operatorname{ch}_{d+1}(V)} \to \mathcal{A}_V.$$

Also, in Section **??** we have shown how the disk operators $\mathcal{V}_V$ form a module, through the factorization product, for the dg algebra $\mathcal{A}_V$. This is essentially the Fock module of the algebra $\mathcal{A}_V$, thus we should view the above map $\rho_*\Phi$ as being a higher dimensional analog of the "free field realization" for the higher dimensional affine algebras.

Further, by induction along the map $\rho_*\Phi$, we obtain the following.

Proposition 4.9. *The map $\rho_*\Phi$ endows the space $\mathcal{V}_V$ with the structure of a module over the E_1-algebra $U\widehat{\mathfrak{g}}_{d,\operatorname{ch}_{d+1}(V)}$. Equivalently, $\mathcal{V}_V$ is an A_∞-module for $U\widehat{\mathfrak{g}}_{d,\operatorname{ch}_{d+1}(V)}$.*

This is the vacuum Verma module...BW: finish.

5. Holomorphic diffeomorphisms

5.1. **Gelfand-Fuks cohomology.** In this section we recall some facts about the Lie algebra cohomology of formal vector fields W_d on the d-disk with values in certain non-trivial modules. We refer to Section **??** for the requisite notation for objects living on the formal disk.

In Section **??** we have constructed the formal Atiyah class for any formal vector bundle $\mathcal{V}$ on $\widehat{D}^n$. It is an element of the relative Gelfand-Fuks cohomology

$$\operatorname{At}^{\mathrm{GF}}(\mathcal{V}) \in \mathrm{C}_{\mathrm{Lie}}^1(W_d, \mathrm{GL}_d; \widehat{\Omega}_d^1 \otimes_{\widehat{\mathcal{O}}_d} \operatorname{End}_{\widehat{\mathcal{O}}_d}(\mathcal{V})).$$

From the Atiyah class we have built the formal Chern character using the usual formula

$$\operatorname{ch}^{\mathrm{GF}}(\mathcal{V}) = \operatorname{Tr}\left(\exp\left(\frac{1}{2\pi i}\operatorname{At}^{\mathrm{GF}}(\mathcal{V})\right)\right),$$

and have studied how components of this formal Chern character give rise to L_∞ extensions of W_d that appear as natural universal symmetries of quantizations of higher dimensional holomorphic σ models with target $\widehat{D}^d$.

In this section we arrive at the Lie algebra of formal vector fields, and its cohomology, from a different perspective. Instead of using formal geometry to construct universal objects on the *target* of a σ model, we will see how Gelfand-Fuks classes characterize holomorphic symmetries on the higher *world-sheet*, or source manifold.

The symmetry is that of holomorphic reparametrizations. Infinitesimally, this is described by the Lie algebra of holomorphic vector fields. We have already seen BW: ref that classical theories on a complex manifold X with such a symmetry by holomorphic reparametrizations admit an action by the local Lie algebra $\mathcal{T}_X = \Omega^{0,*}(X, T_X^{1,0})$.

The Gelfand-Fuks classes we will consider in this section appear as anomalies for quantizing an action by the local Lie algebra $\mathcal{T}_X$. In other words, these classes parametrize shifted central extensions of $\mathcal{T}_X$, just as the classes $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$ defined central extensions of the current algebra $\mathfrak{g}^X$. By our usual yoga of studying equivariant quantizations, we know such anomalies live in the local cohomology complex $C^*_{\mathrm{loc}}(\mathcal{T}_X)$.

Definition/Lemma 1. Consider the following two classes of cocycles on W_d.

Chern type: For $1 \leq k \leq n$, let $\tau_k \in C^k_{\mathrm{Lie}}(W_d; \widehat{\Omega}^k_d)$ be the cocycle
$$\tau_k = \sigma_k \left(\mathrm{At}^{\mathrm{GF}}(\widehat{\mathcal{T}}_d) \right) \,...finish...$$

GL type: For $1 \leq i \leq d$ let $\xi_i \in C^{2i-1}_{\mathrm{Lie}}(W_d; \widehat{\mathcal{O}}_d)$ be the cocycle
$$\xi_i : (f_{i_1} \partial_{i_1}, \ldots, f_{i_{2i-1}} \partial_{i_{2i-1}}) \mapsto \sum ..$$

We will use the notation $\widehat{\Omega}^\#_d = \oplus_k \widehat{\Omega}^k_d[-k]$ to denote the graded W_d-module with $\widehat{\Omega}^k_d$ sitting in degree k. The wedge product of forms endows this W_d-module with the structure of a graded commutative algebra.

If V is a graded vector space then we use the notation $\mathbb{C}[V]$ to denote the free graded $\mathbb{C}$-algebra on V. If V is spanned by vectors $\{v_i\}$ we will use the shorthand $\mathbb{C}[v_i]$ for this graded algebra.

Theorem 5.1 ([?]). *The bigraded commutative algebra $H^*(W_d; \widehat{\Omega}^\#_d)$ is isomorphic to the bigraded commutative algebra*
$$(\mathbb{C}[\xi_1, \ldots, \xi_{2d-1}, \tau_1, \ldots, \tau_d]) / \left(c_1^{j_1} \cdots c_d^{j_d} \right),$$
where the quotient is over all indices $\{j_1, \ldots, j_d\}$ that satisfiy $j_1 + 2j_2 + \cdots + dj_d > d$. Here ξ_{2i-1} is in bidegree $(2i-1, 0)$ and τ_j is in bidegree (j, j).

In the above result we have not turned on the de Rham differential $d_{dR} : \widehat{\Omega}^k_d \to \widehat{\Omega}^{k+1}_d$. This endows $\widehat{\Omega}^*_d = (\widehat{\Omega}^\#_d, d_{dR})$ with the structure of a dg commutative algebra in W_d-modules. The formal Poincaré lemma asserts that the inclusion of the trivial W_d-module
$$\mathbb{C} \xrightarrow{\simeq} \widehat{\Omega}^*_d$$
is a quasi-isomorphism. In turn, we obtain a quasi-isomorphism of Chevalley-Eilenberg complexes
$$C^*_{\mathrm{Lie}}(W_d) \xrightarrow{\simeq} C^*_{\mathrm{Lie}}(W_d; \widehat{\Omega}^*_d).$$

We may think of the cochain complex $C^*_{\text{Lie}}(W_d; \widehat{\Omega}^*_d)$ as the total complex of the double complex with vertical differential given by the W_d Chevalley-Eilenberg differential for the graded module $\widehat{\Omega}^\#_d$, and horizontal differential equal to the de Rham differential.

To any double complex there is a spectral sequence abutting to the cohomology of the total complex. The E_1 page of this spectral sequence is given by the cohomology of the vertical differential. Moreover, if the double complex is a bigraded algebra so are each of the pages. In this case, the E_1 page is precisely the bigraded algebra of Theorem 5.1 and we have a spectral sequence

$$(11) \qquad E^{p,q}_2 = \left(H^q(W_d; \widehat{\Omega}^p_d), d_{dR} \right) \implies H^*(W_d; \widehat{\Omega}^*_d) \cong H^*(W_d).$$

Example 5.2. For the case $d = 1$ the spectral sequence collapses at the E_2 page. The only nontrivial cohomology is $\mathbb{C}$ in bidegree $(0,0)$ and $\xi_1 \cdot \tau_1$ in bidegree $(1,2)$. The 1-cocycle valued in formal power series ξ_1 is given by $\xi_1(f_i \partial_i) = \partial_i f_i \in \widehat{\mathcal{O}}_n$. The 1-cocycle valued in formal 1-forms τ_1 is given by $\tau_1(g_j \partial_j) = d_{dR}(\partial_j g_j)$. To obtain the generator of $H^3(W_1)$ we perform the following zig-zag:

$$
\begin{array}{ccc}
C^3_{\text{Lie}}(W_1) & \longrightarrow & C^3_{\text{Lie}}(W_1; \widehat{\mathcal{O}}_1) \\
& & \uparrow{\scriptstyle d_{CE}} \\
& C^2_{\text{Lie}}(W_1; \widehat{\mathcal{O}}_1) & \xrightarrow{d_{dR}} C^2_{\text{Lie}}(W_1; \widehat{\Omega}^1_1).
\end{array}
$$

The de Rham differential kills $\xi_1 \cdot \tau_1$, so there exists an $\alpha \in C^2_{\text{Lie}}(W_1; \widehat{\mathcal{O}}_1)$ such that $d_{dR}\alpha = -\xi_1 \cdot \tau_1$. Now, the class $d^{\widehat{\mathcal{O}}}_{CE}\alpha \in C^3_{\text{Lie}}(W_1; \widehat{\mathcal{O}}_n)$ satisfies

$$d_{dR}(d^{\widehat{\mathcal{O}}}_{CE}\alpha) = -d_{CE}(\xi_1 \tau_1) = 0$$

$$d_{CE} d^{\widehat{\mathcal{O}}}_{CE}\alpha = 0.$$

Here, $d^{\widehat{\mathcal{O}}}_{CE}$ denote the Chevalley-Eilenberg differential for $C^*_{\text{Lie}}(W_1; \widehat{\mathcal{O}}_1)$ and d_{CE} is the restriction of this Chevalley-Eilenberg differential to $C^*_{\text{Lie}}(W_1)$. The first line says that $d_{CE}\alpha$ lifts to $C^3_{\text{Lie}}(W_1)$, and the second line says that it is a cocycle for the absolute cohomology. Finally, note that $(d^{\widehat{\mathcal{O}}}_{CE} + d_{dR})\alpha = d^{\widehat{\mathcal{O}}}_{CE}\alpha - \xi_1 \tau_1$. Thus, in the total complex $d^{\widehat{\mathcal{O}}}_{CE}\alpha$ is homotopic to $\xi_1 \tau_1$, and so $[d^{\widehat{\mathcal{O}}}_{CE}\alpha]$ is the generator of $H^3(W_1)$.

For general $d \geq 1$, one can apply this spectral sequence to understand the cohomology $H^*(W_d)$. To describe it, we introduce the following topological space. Let $\text{Gr}(d, n)$ be the complex Grassmannian of d-planes in $\mathbb{C}^n$. Denote by $\text{Gr}(d, \infty)$ the colimit of the natural sequence

$$\text{Gr}(d, d) \to \text{Gr}(d, d+1) \to \cdots.$$

It is a standard fact that $\text{Gr}(d, \infty)$ is a model for the classifying space $BU(d)$ of principal $U(d)$-bundles. Let $EU(d) \to BU(d)$ be the universal principal $U(d)$-bundle. Using the colimit description above, we have a natural skeletal filtration of $BU(d)$ by

$$\text{sk}_k BU(d) = \text{Gr}(d, k).$$

Let X_d denote the restriction of $EU(d)$ over the $2d$-skeleton:

$$\begin{array}{ccc} X_d & \longrightarrow & EU(d) \\ \downarrow & & \downarrow \\ \mathrm{sk}_{2d} BU(d) & \longrightarrow & BU(d). \end{array}$$

Remark 5.3. Though not the way the Gelfand and Fuks originally proved the result, one can use the computation of the cohomology of W_d with coefficients in $\widehat{\Omega}_d^k$ together with the spectral sequence (11) to prove this description of $H^*(\mathsf{W}_d)$. Indeed, the spectral sequence (11) is isomorphic, up to regradings, to the Serre spectral sequence for the principal $U(d)$-bundle $X_d \to \mathrm{sk}_{2d} BU(d)$. In other words, the formal de Rham differential on $\widehat{\Omega}_d^*$ is exactly the E_2 differential for the Serre spectral sequence.

Theorem 5.4 ([**?**] Theorem 2.2.4). *There is an isomorphism of graded vector spaces*

$$H^*(\mathsf{W}_d) \cong H_{dR}^*(X_d).$$

Moreover, the commutative algebra structure on $H^*(\mathsf{W}_d)$ is trivial.

As a simple example, note that when $d = 1$ we have $\mathrm{sk}_2 BU(1) = \mathbb{P}^1 \subset \mathbb{P}^\infty = BU(1)$. Moreover, the restriction of the universal bundle is Hopf fibration $U(1) \to S^3 \to \mathbb{P}^1$. In particular, one has $X_1 = S^3$.

5.2. **The local cohomology of holomorphic vector fields.** Our main result in this section is a complete classification of the local cohomology of the sheaf of Dolbeualt complex of holomorphic vector fields $\mathcal{T}_X = \Omega^{0,*}(X; T_X^{1,0})$ on any complex manifold. This description involves the Gelfand-Fuks cohomology of formal vector fields that we have just discussed.

Theorem 5.5. *Let X be a complex d-fold. There is a quasi-isomorphism of sheaves of cochain complexes*

$$\mathrm{C}_{\mathrm{loc}}^*(\mathcal{T}_X) \; \simeq \; \Omega_X^* \otimes \mathrm{C}_{\mathrm{Lie,red}}^*(\mathsf{W}_d)[2d]$$

where Ω_X^* is the sheaf of de Rham forms on X.

The core of the argument is in interpretting the local Lie algebra cohomology as the cohomology of vector fields on the formal disk through the process of *Gelfand-Kazhdan descent* that we introduced in Chapter **??** Before moving on to the proof, we have the immediate cohomological interpretation of the calculation. Recall that when we study classical BV theories equivariant for a local Lie algebra $\mathcal{L}$, the space the failure for quantizing the BV theory in a way that is equivariant for the Lie algebra is measured by an anomaly class in the local cohomology. For holomorphic diffeomorphisms, we obtain the following.

Corollary 5.6. *For X any complex manifold of complex dimension d one has at the level of cohomology*

$$H_{\mathrm{loc}}^k(\mathcal{T}_X) \; \cong \; \bigoplus_{i=0}^{2d} H_{dR}^i(X) \otimes H_{\mathrm{Lie,red}}^{2d+k-i}(\mathsf{W}_d).$$

In particular, if the manifold is connected the space of anomalies for holomorphic diffeomorphisms for a theory defined on X is:

$$H_{\mathrm{loc}}^1(\mathcal{T}_X) = H_{\mathrm{Lie}}^{2d+1}(\mathsf{W}_d),$$

which is independent of the complex manifold.

The corollary implies that the cohomology $H^{2d+1}_{\text{Lie}}(W_d)$ deserves to be thought of as the space of "higher dimensional central charges" of a classically holomorphic diffeomorphism invariant theory. After the proof of our main result we will how this relates to the central extensions of holomorphic vector fields and the role of these extensions in quantum field theory.

Proof. We recall a description of the local cohomology complex using D-modules given in Section 4.5 of [?]. Let $\mathcal{L}$ be any local Lie algebra on X with associated graded vector bundle L. The local Lie algebra cohomology is defined as the sheaf

$$C^*_{\text{loc}}(\mathcal{L}) = \Omega^{d,d}_X \otimes_{D_X} C^*_{\text{Lie,red}}(JL)$$

where JL is the D_X-module given by taking the ∞-jets of the underlying vector bundle of $\mathcal{L}$. In [?] it was shown that $C^*_{\text{Lie,red}}(JL)$ is flat as a D_X-module, thus we can replace the tensor product above by a left-derived tensor product

$$(12) \qquad \Omega^{d,d}_X \otimes_{D_X} C^*_{\text{Lie,red}}(JL) \simeq \Omega^{d,d}_X \otimes^{\mathbb{L}}_{D_X} C^*_{\text{Lie,red}}(JL).$$

The Spenser resolution is a free resolution of $\Omega^{d,d}_X$ as a right D_X-module (by D_X we mean smooth differential operators) given by

$$M^* = \left(\cdots \to \Omega^{2d-1} \otimes_{C^\infty_X} D_X \xrightarrow{\nabla_D} \Omega^{d,d} \otimes_{C^\infty_X} D_X \right)$$

The differential ∇_D is determined by the natural flat connection on D_X. This complex M^* is concentrated in degree $-2d, \ldots, 0$. Via this resolution, we see that (12) is quasi-isomorphic to

$$M^* \otimes_{D_X} C^*_{\text{Lie,red}}(JL) \simeq \left(\cdots \to \Omega^{2d-1} \otimes_{C^\infty_X} C^*_{\text{Lie,red}}(JL) \xrightarrow{\nabla_D} \Omega^{d,d} \otimes_{C^\infty_X} C^*_{\text{Lie,red}}(JL) \right).$$

The right-hand side is, by definition, the shifted de Rham complex of the D_X-module $C^*_{\text{Lie,red}}(JL)$ so we obtain

$$(13) \qquad C^*_{\text{loc}}(\mathcal{L}) \simeq \Omega^*(X, C^*_{\text{Lie,red}}(JL))[2d].$$

Now, suppose that $\mathcal{L}$ is a holomorphic local Lie algebra of the form $\Omega^{0,*}(X, L^{hol})$ where L^{hol} is a holomorphic vector bundle. In the above notation, the underlying smooth vector bundle is $L = \bigwedge^* T^{0,1^*}X \otimes L^{hol}$.

We have used the notation JE to denote the smooth sections of the infinite rank vector bundle $\text{Jet}(E)$. If E is a holomorphic vector bundle let $\text{Jet}^{hol}(E)$ denote the infinite rank holomorphic vector bundle of holomorphic jets. Similarly, let $J^{hol}E$ be the holomorphic sections of this bundle. This is a D_X^{hol}-module where D_X^{hol} is the sheaf of holomorphic differential operators. Equivalently, a D_X^{hol}-module is a holomorphic vector bundle with a holomorphic flat connection. Of course, any D_X^{hol}-module E forgets to a smooth D_X-module that we denote E^{C^∞}.

Lemma 5.7. *Let L, L^{hol} be as above. There is a quasi-isomorphism of D_X-modules $JL \simeq (J^{hol}L^{hol})^{C^\infty}$.*

Proof. Let $\mathcal{L}$ be the sheaf of sections of L. The Dolbeualt complex is a resolution of the sheaf of holomorphic sections; thus there is a quasi-isomorphism $\mathcal{L} \simeq \mathcal{L}^{hol}$ of $\mathcal{O}_X^{hol}$-modules. $\qquad \square$

This means that we can further reduce the expression for the local cohomology in (13) to

$$
(14) \qquad C^*_{\text{loc}}(\mathcal{L}) \simeq \Omega^*(X, C^*_{\text{Lie,red}}(J^{hol} L^{hol}))[2d].
$$

We have dropped the notation $(-)^{C^\infty}$ for convenience.

We now turn to the local Lie algebra in question, namely $\mathcal{T}_X$. This is, of course, a holomorphic local Lie algebra as it is given by $\mathcal{T}_X = \Omega^{0,*}(X, TX)$. The underlying holomorphic vector bundle is the holomorphic tangent bundle $T^{1,0}X$.

Suppose that $\mathcal{V}$ is any (W_d, GL_d)-module. Then, Gelfand-Kazhdan descent along the complex manifold X yields the D_X-module $\text{desc}_X(\mathcal{V})$. In the case that $\mathcal{V} = \widehat{\mathcal{T}}_d$ we have seen that the D_X-module $\text{desc}_X(\widehat{\mathcal{T}}_d)$ is equivalent to the D_X-module $J^{hol} T^{1,0}X$.

Lemma 5.8. *Gelfand-Kazhdan descent is symmetric monoidal. That is, if $\mathcal{V}, \mathcal{V}'$ are two (W_d, GL_d)-modules, then*

$$
\mathcal{V} \otimes_{\widehat{\mathcal{O}}_n} \mathcal{V}' \simeq \text{desc}_X(\mathcal{V}) \otimes_{J^{hol} \mathcal{O}^{hol}_X} \text{desc}_X(\mathcal{V}').
$$

This implies that there is a string of isomorphisms of D_X-modules

$$
J^{hol} C^*_{\text{Lie,red}}(T^{1,0}X) = \text{desc}(C^*_{\text{Lie,red}}(W_d)) \cong C^*_{\text{Lie,red}}(\text{desc}(W_d)) = C^*_{\text{Lie,red}}(J^{hol} T^{1,0}X).
$$

Equivalently, by **??** we know that the functor of jets is symmetric monoidal, so the same result follows.

To summarize we see that the Gelfand-Kazhdan descent of the (W_n, GL_d)-module $C_{\text{Lie,red}}(W_d)$ is equal to the D_X-module $C^*_{\text{Lie,red}}(J^{hol} T^{1,0}_X)$. This is precisely the D_X-module present in the definition of the local cohomology of $\mathcal{T}_X$. Indeed, by Lemma 5.7 we have

$$
C^*_{\text{loc}}(\mathcal{T}_X) \simeq \Omega^* \left(X, C^*_{\text{Lie,red}}(J^{hol} T^{1,0}_X) \right).
$$

Thus, the de Rham complex of the D_X-module given by descent is precisely the local cohomology

$$
C^*_{\text{loc}}(\mathcal{T}_X) \simeq \Omega^* \left(X, \text{desc}_X(C^*_{\text{Lie,red}}(W_d)) \right)
$$

The interpretation via descent will allow us to describe this de Rham complex explicitly. Suppose that $\mathfrak{g}$ is any Lie algebra. Then $\mathfrak{g}$ acts on itself (and its dual) via the adjoint action. This extends to an action of $\mathfrak{g}$ on its Chevalley-Eilenberg complex $C^*_{\text{Lie}}(\mathfrak{g}; M)$, where M is any $\mathfrak{g}$-module via the formula

$$
(x \cdot \varphi)(x_1, \ldots, x_k) = \sum_i \varphi(x_1, \ldots, [x, x_i], \ldots, x_k) - x \cdot \varphi(x_1, \ldots, x_k)
$$

Here, $x, x_i \in \mathfrak{g}$ and φ is a k-cochain with values in M. The $[-, -]$ denotes adjoint action, and the $\cdot$ is the $\mathfrak{g}$-module structure on M. The following lemma is well-known. The same formula holds for the reduced cochains.

Lemma 5.9. *The $\mathfrak{g}$-module structure on the cochain complexes $C^*_{\text{Lie}}(\mathfrak{g})$ and $C^*_{\text{Lie,red}}(\mathfrak{g})$ is homotopically trivial.*

For the case of an infinite dimensional Lie algebra, such as W_d, the same result holds when we use the continuous, or Gelfand-Fuks, Lie algebra cohomology. Thus, W_d acts homotopically trivial on $C^*_{\text{Lie,red}}(W_d)$.

This implies that the descent $\mathrm{desc}_X(W_d)$ has a homotopically trivial D_X-module structure. Equivalently, this means that the flat connection on $C^*_{\mathrm{Lie,red}}(J^{hol}T^{1,0}X)$ is gauge equivalent to the trivial connection. Thus, there is a quasi-isomorphism of de Rham complexes

$$\Omega^* \left(X, C^*_{\mathrm{Lie,red}}(J^{hol}T^{1,0}_X) \right) \simeq \Omega^*(X) \otimes_{C^\infty_X} \underline{C^*_{\mathrm{Lie,red}}(W_d)}_X$$

where the underline denotes the sections of the trivial bundle with fiber $C^*_{\mathrm{Lie,red}}(W_d)$. We have identified the left hand side with the local cohomology complex, so we are done. □

5.2.1. *An explicit description of the local cocycles.* We'd like to leverage our knowledge of the the Gelfand-Fuks cohomology of formal vector fields to provide an explicit description of the local cocycles. The theorem in the previous section gives a very general equivalence of the local cohomology on any complex manifold with the Gelfand-Fuks cohomology, but writing down the form of the local cocycle from the description on a formal disk is not so obvious.

For instance, consider the case $d = 1$ and we work on $X = \mathbb{C}$. The cohomology $H^*_{\mathrm{Lie,red}}(W_1)$ is one-dimensional concentrated in degree 3. We'd like to describe the local cocycle corresponding to the generator of $H^3(W_1) \cong H^1_{\mathrm{loc}}(\mathcal{T}_\mathbb{C})$ explicitly. Recall, using the formal Hodge-to-de Rham spectral sequence we saw that the generator of $H^3(W_1)$ came from the element $a_1\tau_1 \in H^2_{\mathrm{Lie}}(W_1; \widehat{\Omega}^1_1)$ on the E_2 page of the spectral sequence (11).

Now, the 1-cocycles a_1, τ_1 can both be interpreted as functionals on the Dolbeault complex $\Omega^{0,*}(\mathbb{C}, T^{1,0}\mathbb{C})$. Indeed, if $\xi = \alpha(z,\bar{z})\partial_z$ is an element of the Dolbeault complex we can define

$$\widetilde{a}_1(\xi) = \partial_z\alpha(z,\bar{z}) \in \Omega^{0,*}(\mathbb{C})$$
$$\widetilde{\tau}_1(\xi) = \partial(\partial_z\alpha(z,\bar{z})) \in \Omega^{1,*}(\mathbb{C}).$$

Each of these cocycles clearly only depends on the jet of the vector field $\alpha\partial_z$. Similarly, the product is $\widetilde{a}_1\widetilde{\tau}_1$ is the bilinear functional on jets of $\mathcal{T}_\mathbb{C}$:

$$\widetilde{a}_1\widetilde{\tau}_1(\xi_1, \xi_2) = \partial_z\alpha_1(z,\bar{z})\partial(\partial_z\alpha_2(z,\bar{z})) \in \Omega^{1,*}(\mathbb{C})$$

This is a density precisely when $|\alpha_1| + |\alpha_2| = 1$. Thus, $\widetilde{a}_1\widetilde{\tau}_1$ determines a degree $+1$ density valued cochain on $J\mathcal{T}_\mathbb{C}$; in other other words an element of $C^*_{\mathrm{loc}}(\mathcal{T}_\mathbb{C})$ that we write as

$$\int_\mathbb{C} \partial_z\alpha_1(z,\bar{z})\partial(\partial_z\alpha_2(z,\bar{z})).$$

If we integrate by parts, we can put this local functional in the form $\int f\partial_z^3 g\mathrm{d}z\mathrm{d}\bar{z}$. If one restricts this local functional to the annulus and performs the radial integration, we recover the usual formula for the generator of $H^2(\mathrm{Vect}(S^1))$ BW: citation defining the central extension of the Virasoro Lie algebra. In fact, in [**?**] BW: finish

This can be generalized to arbitrary dimensions in a natural way.

Our first goal is to construct, from a Gelfand-Fuks classes in $C^*_{\mathrm{Lie}}(W_d)$ and $C^*_{\mathrm{Lie}}(W_n; \widehat{\Omega}^*_d)$, a local functional on $\mathcal{T}_X$. We have seen that the cochain complex $C^*_{\mathrm{Lie}}(W_d; \widehat{\Omega}^*_d)$, equipped with the total differential $\mathrm{d}_{CE} + \mathrm{d}_{dR}$, computes the absolute Gelfand-Fuks cohomology $H^*(W_d)$ through the formal Hodge to de Rham spectral sequence. We will use this property to represent elements of $H^*(W_d)$ by local functionals on $\mathcal{T}_X$ by first representing elements in $C^*_{\mathrm{Lie}}(W_n; \widehat{\Omega}^*_d)$ by local functionals.

We can decompose an element $\alpha \in C_{\text{Lie}}^k(W_d; \widehat{\Omega}_d^*)$ as

$$\alpha = f^I \mathrm{d}t_I.$$

The sum is over the multi-index $I = (i_1, \ldots, i_k)$, where $1 \leq i_j \leq d$. For each I, f^I is a k multi-linear symmetric functional on W_d valued in $\widehat{\mathcal{O}}_d$

$$f^I : \text{Sym}^k(W_d[1]) \to \widehat{\mathcal{O}}_d.$$

We extend f^I to a functional on the Dolbeault complex $\Omega^{0,*}(\mathbb{C}^d; T^{1,0}\mathbb{C}^d)$ as follows. Using the framing on $\mathbb{C}^d$, every element of the Dolbeualt complex can be written as

$$X^J(z, \bar{z})\mathrm{d}\bar{z}_J$$

where $J = (j_1, \ldots, j_l)$ is a multi-index and X^J is an ordinary holomorphic vector field on $\mathbb{C}^d$. We extend f^I to a Dolbeualt valued functional $\Omega^{0,*}(\mathbb{C}^d; T^{1,0}\mathbb{C}^d)$ via the formula

$$f_{\Omega^{0,*}}^I : \qquad \text{Sym}^k\left(\Omega^{0,*}(\mathbb{C}^d; T^{1,0})\right) \qquad \to \qquad \Omega^{0,*}(\mathbb{C}^d)$$
$$\left(X_1^{J(1)}(z,\bar{z})\mathrm{d}\bar{z}_{J(1)}, \ldots, X_k^{J(k)}(z,\bar{z})\mathrm{d}\bar{z}_{J(k)}\right) \quad \mapsto \quad f^I(X_1^{J(1)}, \ldots, X_k^{J(k)})\mathrm{d}\bar{z}_{J(1)} \wedge \cdots \mathrm{d}\bar{z}_{J(k)}$$

The local functional corresponding to the original element $\alpha = f^I \mathrm{d}t_I \in C_{\text{Lie}}^*(W_n; \widehat{\Omega}_d^*)$ is defined by the k-multi-linear functional

$$(\xi_1, \ldots, \xi_k) \mapsto \int_{\mathbb{C}^d} f_{\Omega^{0,*}}^I(\xi_1, \ldots, \xi_k)\mathrm{d}z_I.$$

Denote this functional by $J^{GF}(\alpha)$. Note that it is only nonzero when the multi-index I is a permutation of $(1, \ldots, d)$. Since it is given by the integral of a some multi-differential operators against a density it is manifestly a local functional.

Lemma 5.10. *Let $C_{\text{loc}}^*(\mathcal{T}_{\mathbb{C}^d})^{\mathbb{C}^d}$ denote the translation invariant local cochain complex of the local Lie algebra $\mathcal{T}_{\mathbb{C}^d}$. Then, there is a quasi-isomorphism*

$$C_{\text{Lie}}^*(W_d)[2d] \xrightarrow{\cong} C_{\text{loc}}^*(\mathcal{T}_{\mathbb{C}^d})^{\mathbb{C}^d}.$$

Proof. By Lemma **??** we know that if $\mathcal{L} = \Gamma(L)$ is any holomorphically translation invariant local Lie algebra on $\mathbb{C}^d$ then there is a quasi-isomorphism

$$C_{\text{loc}}^*(\mathcal{L})^{\mathbb{C}^d} \simeq \mathbb{C} \cdot \mathrm{d}^d z \otimes_{\mathbb{C}[\partial_1, \ldots, \partial_d]}^{\mathbb{L}} C_{\text{Lie}}^*(J_0 L)[d]$$

where $J_0 L$ denotes the ∞-jets of L at $0 \in \mathbb{C}^d$. There is a Koszul resolution of the trivial right $\mathbb{C}[\partial_1, \ldots, \partial_d] = \mathbb{C}\left[\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_d}\right]$ given by

$$\left(\mathbb{C}\left[\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_d}\right][\epsilon_1, \ldots, \epsilon_d], \mathrm{d}\epsilon_i = \frac{\partial}{\partial z_i}\right)$$

where ϵ_i put in cohomological degree -1. Thus, we can write

$$C_{\text{loc}}^*(\mathcal{L})^{\mathbb{C}^d} \simeq C_{\text{Lie}}^*(J_0 L)[d]$$

$\square$

5.3. **The holomorphic anomaly for σ-models.**