

A CHIRAL ALGEBRAIC INDEX THEOREM

1. A CHIRAL ALGEBRAIC INDEX THEOREM

In this section we would like to leverage our analysis of the holomorphic σ -model to formulate an index theorem for operators on the higher sphere spaces, $\text{Map}(S^{2d-1}, X)$, of the target manifold. We are inspired, in part, by Witten's original study of elliptic genera appearing as the index of Dirac operators on the loop space $LX = \text{Map}(S^1, X)$ of X [?, ?]. First, we will focus on this $d = 1$ case by giving a mathematical formulation of Witten's result that has the flavor of an "algebraic index theorem" in the sense of Fedosov, Nest-Tsygan [?, ?].

The method we employ to formulate the chiral index theorem is reminiscent of Nest-Tsygan's [?] approach based on formal geometry to prove the original algebraic index theorem, which we recall now. This approach is further clarified in [?] for the case of a general deformation quantization. There are two main steps. The first is to prove a "local" algebraic index theorem which characterizes the unique trace present on the Weyl algebra in any number of variables. Precisely, the statement is that the Hochschild homology of the n -dimensional Weyl algebra is one-dimensional and concentrated in degree $-2n$.

The next step uses a version of Gelfand-Kazhdan formal geometry to "descend" this result over any manifold. The crucial fact that makes this procedure possible is that the cocycle representing the degree $-2n$ Hochschild homology of the Weyl algebra is equivariant for formal vector fields. The cocycle then yields a differential form on the manifold, and in the case the manifold is closed, one can integrate it. The global index theorem is obtained by explicitly evaluating the integral of this descended cocycle, which is where the \hat{A} -class appears.

BW: make this better no surprise that our approach to constructing σ -models in the BV formalism using formal geometry is applicable to index theorems involving Our technique is to construct a local version of the index theorem on the formal n -disk, and then use the technique of descent to deduce the index theorem on a general manifold (satisfying the usual anomaly conditions for the chiral case).

The above process should appear familiar in our construction of the quantization of σ -models. We first study the BV quantization with target the formal disk, and then analyze how the quantization is equivariant for formal vector fields by placing them in the background. Before embarking on the chiral index theorem, we recall how the ordinary algebraic index theorem is related to the study of one-dimensional quantum field theory, that is, quantum mechanics.

1.1. The algebraic index theorem and QM. Recently, in [?] it is shown how to deduce the algebraic index theorem using the Batalin-Vilkovisky approach to perturbative quantum field theory. Our summary here highlights the application of formal geometry, **BW: finish**

Topological quantum mechanics is a one-dimensional σ -model which we will write in the BV formalism. We start with the case in which the target is an n -disk D^n . The complex of fields is

$$(f, g) \in \Omega^*(\mathbb{R}, \mathbb{R}^n \oplus (\mathbb{R}^n)^*).$$

The dual $(-)^*$ is to remind ourselves that f and g are conjugate to each other. The BRST differential is simply the de Rham operator. The action functional reads

$$S_{1d}(f, g) = \int_{\mathbb{R}} g df,$$

where we are implicitly using the evaluation pairing on \mathbb{R}^n . The equations of motion on \mathbb{R} reduce to looking just at constant maps. Being a free theory, it has a unique quantization. The factorization algebra of quantum observables Obs_{1d} is a locally constant factorization algebra on \mathbb{R} , which is equivalent as an associative algebra to the Weyl algebra Weyl_n^{\hbar} on generators $\{t_1, \dots, t_n, \partial_{t_1}, \dots, \partial_{t_n}\}$. The commutation relation is $[\partial_{t_i}, t_j] = \hbar \delta_{ij}$.

As a factorization algebra on \mathbb{R} , Obs_{1d} is translation invariant. Thus, we can place this factorization algebra on the circle S^1 , viewing $S^1 = \mathbb{R}/\mathbb{Z}$. Equivalently, this is the quantization of the σ -model of maps $S^1 \rightarrow D^n$. Now, the factorization homology, or global sections, of any topological factorization algebra on the circle S^1 is equivalent to the Hochschild homology of the underlying algebra. In fact, in [?], an explicit cochain homotopy equivalence is written down

$$\Phi : \text{Hoch}(\text{Weyl}_n^{\hbar}) \xrightarrow{\sim} \int_{S^1} \text{Obs}_{1d}.$$

We will explain this equivalence further in Section [BW: ref](#).

Using the standard flat metric on S^1 we can consider the finite dimensional subspace of harmonic forms on the circle $\mathbb{H}(S^1) \subset \Omega^*(S^1)$. It follows that there is quasi-isomorphism of cochain complexes

$$\int_{S^1} \text{Obs}_{1d} \xrightarrow{\sim} \left(\mathcal{O}(\mathbb{H}(S^1) \otimes (\mathbb{R}^n \oplus (\mathbb{R}^n)^*))[[\hbar]], \hbar \Delta_{\infty} \right).$$

Here Δ_{∞} [BW: how to say this best](#). We will see how we can interpret this map as taking homotopy RG flow to “scale ∞ ”.

In [?] this theory is called one-dimensional abelian Chern-Simons theory. It makes sense when we replace the target D^n with any L_{∞} space. In particular, if X is a smooth manifold, we can substitute the L_{∞} space \mathfrak{g}_X encoding the smooth structure to obtain a BV formulation of the perturbative σ -model of maps $\mathbb{R} \rightarrow X$ that are infinitesimally close to the constant maps

$$T^*[-1]\widehat{\text{Map}}_0(\mathbb{R}, X).$$

Here Map_0 denotes the formal neighborhood of constant maps inside the space of all smooth maps $\mathbb{R} \rightarrow X$.

Alternatively, which is the approach we will take in the chiral case, we can replace the target disk D^n by the formal disk \widehat{D}^n and formulate this theory using Gelfand-Kazhdan descent. This means that we need to study how the free theory is equivariant for the Lie algebra of formal vector fields W_n on the n -disk. The following theorem will appear in joint work with Grady and Gwilliam.

Theorem 1.1 ([?]). *There is a W_n -equivariant quantization of topological quantum mechanics on \mathbb{R} with target the formal disk \widehat{D}^n . In turn, the factorization algebra Obs_{1d} is W_n -equivariant and hence determines a sheaf Obs_X of locally constant factorization algebras on any smooth n -manifold X . This sheaf is equivalent to the sheaf of differential operators D_X as associative algebras.*

BW: finish

1.2. Stating the chiral algebraic index theorem. We now proceed to formulate the chiral algebraic index theorem. Our starting point is the holomorphic σ -model of maps from a Riemann surface to a complex manifold X that we have just considered earlier in this thesis. We will propose a higher dimensional version of this story in the end of this section.

So far, in our analysis of the holomorphic σ -model we have been considered with the local theory so we have taken the source to be of the form \mathbb{C}^d . For the $d = 1$ chiral algebraic index theorem we are most interested in the case that the source of this σ -model is an elliptic curve

$$E_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$$

where τ lies in the upper-half plane \mathbb{H} . It will sometimes be convenient to use the “ q -presentation” where $q = e^{2\pi i\tau}$ and write the elliptic curve as a quotient of \mathbb{C}^\times by the integers $E_q = \mathbb{C}^\times / q^\mathbb{Z}$. Of course, the exponential induces an isomorphism $E_\tau \cong E_q$.

As a smooth manifold, the elliptic curve is just a torus $S^1 \times S^1$. We may therefore write the fields of the σ -model on E_τ with target X as $T^*[-1]\text{Map}(E_\tau, X) = T^*[-1]\text{Map}(S^1, LX)$ where LX is the free loop space of the manifold X . Thus, via compactification along one of the circles, we obtain a one-dimensional σ -model akin to the one we have just discussed in the previous section. Heuristically, one can think of this as quantum mechanics on the free loop space. We will state the precise relationship between the two-dimensional chiral theory and this one-dimensional quantum mechanics in Section ??.

Of course, the space LX is not a smooth manifold in the ordinary sense. Consequently, there are many analytic difficulties when doing quantum mechanics. In [?], Witten provides a physical description of the path integral for quantum mechanics on the free loop space. One of the main outcomes of his work is the partition function of this system, from which he extracts the so-called *Witten genus* of the manifold X . We will not work with the space of topological loops LX , but rather work with an algebraic replacement that we will call the algebraic loop space LX^{alg} . There is an associated sheaf of associative algebras on X that deserves to be called differential operators on LX^{alg} , denoted $D_{LX^{alg}}^\hbar$ that we will define in Section BW: ref. The fundamental object in the chiral algebraic index theorem will be a q -twisted version of a trace on this sheaf of algebras.

In [?, ?] it is shown how the partition function of the holomorphic σ -model along an elliptic curve E_τ recovers the Witten genus of the target manifold. We will see how to arrive at this result through the lens of formal geometry, which will be the key step in deducing the global chiral algebraic index theorem. The one main advantage of working directly with the elliptic curve, as opposed to doing quantum mechanics on the loop space, is that the modular properties of the Witten genus are manifest.

The advantage of the perspective of quantum mechanics is that we have at our disposal many algebraic tools that will allow us to view the calculation of the Witten genus on an elliptic curve as a “chiral index theorem”.

Theorem 1.2. (*Chiral algebraic index theorem*) Let $q \in D(0,1)^\times$, X be a compact complex manifold, and α a trivialization of $\text{ch}_2(TX) \in H^2(X, \Omega_{cl}^2)$, so that the factorization algebra $\text{Obs}_{X,\alpha}^q$ of quantum observables of the two-dimensional holomorphic σ -model from E_q to X is defined (Theorem ??). Then:

(1) there is quasi-isomorphism of sheaves of cochain complexes on X

$$\Phi^q : \text{Hoch}(D_{LX^{alg}}^\hbar; q) \xrightarrow{\sim} \int_{E_q} \text{Obs}_{X,\alpha}^q;$$

(2) there is a q -twisted trace map

$$\text{Tr}_X^q : \text{Hoch}(D_{LX^{alg}}^\hbar; q) \rightarrow \mathbb{C}[[\hbar, \hbar^{-1}];$$

BW: some uniqueness?

(3) if

$$1 \in H^* \int_{E_q} \text{Obs}_{X,\alpha}^q \cong HH^*(D_{LX^{alg}}^\hbar; q)$$

denotes the unit observable, then in cohomology the trace map satisfies

$$\text{Tr}_X^q(1) = \int_X \text{Wit}(X, q).$$

Item (1) will be a formal consequence of the quantization of the two-dimensional holomorphic σ -model and will be proved in [BW: ref](#). Item (2) is a q -twisted version of the trace present in ordinary deformation quantization, which we will be a consequence of Proposition [BW: ref](#). Finally, item (3) will follow from an explicit Feynman diagram calculation which is akin to Costello's original calculation of the Witten genus as the partition function of the holomorphic σ -model.

1.3. The local chiral algebraic index theorem. In this section we prove a local version of the chiral algebraic index theorem in the case the target manifold is the complex n -disk. For now, we take the source of our theory to be the punctured complex line \mathbb{C}^\times . Eventually, we will descend this to the elliptic curve E_q . The two-dimensional chiral theory describing this holomorphic σ -model is the usual $\beta\gamma$ system on \mathbb{C}^\times with values in the n -dimensional affine space \mathbb{C}^n . In this section it will be convenient to replace \mathbb{C}^n with an arbitrary finite dimensional complex vector space V .

The fields of the theory are of the form

$$(\gamma, \beta) \in \Omega^{0,*}(\mathbb{C}^\times) \otimes V \oplus \Omega^{1,*}(\mathbb{C}^\times) \otimes V^*$$

with the usual action $\int \beta \bar{\partial} \gamma$. As in Section [BW: ref](#) it is convenient to use a smoothed version of the factorization algebra of observables that we denote by $\text{Obs}_V^{q,sm}$. To an open set $U \subset \mathbb{C}^\times$ the smoothed observables assign the cochain complex

$$\text{Obs}_V^{q,sm}(U) = (\text{Sym}(\Omega_c^{1,*}(U) \otimes V[1] \oplus \Omega_c^{0,*}(U) \otimes V[1])[[\hbar]], \bar{\partial} + \hbar \Delta).$$

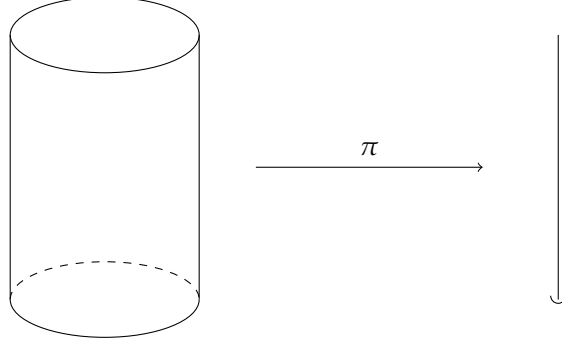
Note that the naive BV Laplacian is well-defined on the smoothed observables.

Consider the vector field $L_0 = z \frac{\partial}{\partial z}$. This is simply the Euler vector field, which represents infinitesimal scaling automorphism on \mathbb{C}^\times . There is a natural action of L_0 on the fields of the $\beta\gamma$ system by Lie derivative. This further induces an action on the factorization algebra of quantum observables.

1.3.1. *Quantum mechanics on the loop space.* We consider a reduction of the two dimensional theory on \mathbb{C}^\times to a one-dimensional theory. It is convenient to view \mathbb{C}^\times as a cylinder

$$\mathbb{C}^\times \cong \mathbb{R}_{>0} \times S^1.$$

We can view the radial direction $\mathbb{R}_{>0}$ as a “time” parameter and the S^1 as the “loop” parameter. We consider the reduction of our theory “along the circle S^1 ”. At the level of factorization algebras, this is equivalent to pushing forward along the radial projection map $\pi : \mathbb{C}^\times \rightarrow \mathbb{R}_{>0}$ which we can view pictorially as



Inside of the pushforward $\text{Obs}_V^{\text{qsm}}$ along π is a locally constant factorization algebra on $\mathbb{R}_{>0}$ that we will see can be viewed as the observables of topological quantum mechanics with target the algebraic loop space of V . To see this, we will express our two-dimensional theory in terms of the natural coordinates on the cylinder.

We will use the coordinates on \mathbb{C}^\times coming from the flat coordinates on \mathbb{C} via the exponential map

$$\mathbb{C} \rightarrow \mathbb{C}^\times, \quad w \mapsto e^{2\pi i w}.$$

We will use the coordinate $z = e^{2\pi i w} \in \mathbb{C}^\times$. Thus the constant holomorphic and anti-holomorphic one-forms read

$$\begin{aligned} dw &= \frac{1}{2\pi i} \frac{dz}{z} \\ d\bar{w} &= -\frac{1}{2\pi i} \frac{d\bar{z}}{\bar{z}}. \end{aligned}$$

Using this coordinate we fix trivializations

$$\begin{aligned} \Omega^{0,*}(\mathbb{C}^\times) &\cong C^\infty(\mathbb{C}^\times) \left[\frac{d\bar{z}}{2\pi i z} \right] \\ \Omega^{1,*}(\mathbb{C}^\times) &\cong \frac{dz}{2\pi i} C^\infty(\mathbb{C}^\times) \left[\frac{d\bar{z}}{2\pi i z} \right]. \end{aligned}$$

The zero degree piece of the γ field is a map from \mathbb{C}^\times to a vector space V . In terms of the coordinates $(r, e^{i\theta}) \in \mathbb{R}_{>0} \times S^1$ we can expand γ as:

$$\gamma(z, \bar{z}) = \sum_{n \in \mathbb{Z}} f_n(r) e^{in\theta}$$

where $f_n : \mathbb{R}_{>0} \rightarrow V$ is defined by $f_n(r) = \frac{1}{2\pi} \int_{S^1} d\theta \gamma e^{-in\theta}$. There is a similar mode expansion of $\beta \frac{dz}{z} \in \Omega^{1,0}(\mathbb{C}^\times) \otimes V^*$.

$$\beta(z, \bar{z}) \frac{dz}{2\pi i z} = \sum_{n \in \mathbb{Z}} g_n(r) e^{in\theta} \frac{1}{2\pi i} \left(\frac{dr}{r} + i d\theta \right),$$

where we used the fact $dz/z = dr/r + i d\theta$.

With respect to the mode expansion, the action functional becomes

$$\begin{aligned} (1) \quad S(\gamma, \beta) &= \int_{\mathbb{C}^\times} \left(\beta \frac{dz}{2\pi i z} \right) \bar{\partial} \gamma \\ (2) \quad &= \frac{1}{2} \sum_{n \in \mathbb{Z}} \int_{r \in \mathbb{R}_{>0}} dr g_{-n}(r) \left(\partial_r - \frac{n}{r} \right) f_n(r), \end{aligned}$$

where we have performed an integration over the loop parameter θ .

We can rewrite this action functional as $S = \sum_{n \in \mathbb{Z}} S^{(n)}$ where $S^{(n)}(g_{-n}, f_n) = \int_{\mathbb{R}_{>0}} g_{-n} \nabla^{(n)} f_n$, using the connection $\nabla^{(n)} = d - \frac{n}{r} dr$ on the trivial bundle on $\mathbb{R}_{>0}$. We think of the collection $\{S^{(n)}\}_{n \in \mathbb{Z}}$ defining a family of quantum mechanics theories, one for each mode number $n \in \mathbb{Z}$.

The algebraic loop space of the vector space V is defined by $LV = V[u, u^{-1}]$ where u is a formal loop parameter. What we have written down is a version of quantum mechanics on $\mathbb{R}_{>0}$ with target this algebraic loop space LV . The piece of the action functional $S^{(n)}$ comes restricting the target to $z^n V \subset LV$. Note that S does not define a one-dimensional BV theory in the usual sense, since the fields are not the sections of a finite dimensional vector bundle. Nevertheless, we can still make sense of the observables of this theory.

For each n the action functional $S^{(n)}$ describes a free theory. This theory is described by the (-1) -symplectic elliptic complex

$$\left(\Omega^*(\mathbb{R}_{>0}) \otimes (u^n V \oplus u^{-n} V^*), \nabla^{(n)} \right),$$

where $\nabla^{(n)}$ is as above. Of course, as a vector space $u^n V = V$, but the notation reminds us of the mode number. Correspondingly, there is a factorization algebra of smoothed quantum observables $\text{Obs}_{z^n V}^{\text{q,sm}}$, which assigns to an interval $I \subset \mathbb{R}_{>0}$ the cochain complex

$$(3) \quad \left(\text{Sym}(\Omega_c^*(I) \otimes (u^n V \oplus u^{-n} V^*)^*), \nabla^{(n)} + \hbar \Delta \right).$$

Definition 1.3. Define the factorization algebra (valued in pro-cochain complexes) of *smoothed quantum observables for topological quantum mechanics on LV* to be the colimit

$$(4) \quad \text{Obs}_{LV}^{\text{q,sm}} := \text{colim}_{N \rightarrow \infty} \bigotimes_{|n| \leq N} \text{Obs}_{z^n V}^{\text{q,sm}}.$$

Heuristically, we think of the factorization algebra $\text{Obs}_{LV}^{\text{q,sm}}$ as being associated to the BV theory of quantum mechanics on the loop space. To make this precise, we'd have to modify our definition of a BV theory to include fields that are sections of infinite dimensional vector bundles, but we omit that discussion here. For now, we will work with the above well-posed definition.

We can associate to any vector space V the following associative algebra. Consider first the one-dimensional central extension of the abelian Lie algebra $V \oplus V^*$

$$0 \rightarrow \mathbb{C} \cdot \hbar \rightarrow \mathcal{H}(V) \rightarrow V^* \oplus V \rightarrow 0$$

with bracket given by $[v^*, v] = \hbar \text{ev}(v, v^*)$. This is the Heisenberg algebra on V . The Weyl algebra on V is, by definition, the universal enveloping algebra of $\mathcal{H}(V)$, which we denote by D_V^\hbar . Of course, D_V^\hbar is simply the associative algebra of polynomial differential operators on V .

Definition 1.4. Let $D_{z^n V}^\hbar$ denote the Weyl algebra on the symplectic vector space $z^n V \oplus z^{-n} V^*$. Define the *Weyl algebra of LV* by the colimit:

$$(5) \quad D_{LV}^\hbar := \text{colim}_{N \rightarrow \infty} \bigotimes_{|n| \leq N} D_{z^n V}^\hbar.$$

The algebra D_{LV}^\hbar has a natural weight filtration induced by the operator $L_0 = z\partial_z$ on LV . For any n , $D_{z^n V}^\hbar$ is isomorphic as an algebra to the ordinary Weyl algebra on V . The notation is to keep track of the L_0 weight n .

We can describe this Weyl algebra in the following equivalent way. Consider the following bilinear map

$$\omega : LV \times LV^* \rightarrow \mathbb{C} \ , \ (v \otimes f, v^* \otimes g) \mapsto \text{ev}(v, v^*) \text{Res}_{z=0}(fdg) \ .$$

Here, $\text{ev}(v, v^*)$ denotes the evaluation pairing between V and V^* . Note that $\omega(v \otimes z^n, v^* \otimes z^m)$ is only nonzero when $n + m = 0$. This equips $LV \oplus LV^*$ with the structure of a symplectic vector space.

For each $N \geq 0$ denote by $LV^{(N)}$ be the finite sum

$$LV^{(N)} = \bigoplus_{n \leq N} z^n \cdot V \subset LV.$$

Then, ω restricts to a symplectic form on the finite dimensional vector space $LV^{(N)} \oplus LV^{*(N)}$. Denote by $\text{Weyl}(LV^{(N)})$ the corresponding Weyl algebra on this finite dimensional vector space. There is a L_0 -equivariant equivalence of algebras

$$D_{LV}^\hbar \cong \text{colim}_{N \rightarrow \infty} \text{Weyl}(LV^{(N)}).$$

Lemma 1.5. *The following hold about the associative algebra D_{LV}^\hbar and the factorization algebras $\text{Obs}_{LV}^{\text{q},sm}$, $\text{Obs}_V^{\text{q},sm}$.*

- (1) *The factorization algebra $\text{Obs}_{LV}^{\text{q},sm}$ is locally constant and is equivalent, as an associative algebra, to D_{LV}^\hbar . Moreover, this equivalence is L_0 -equivariant.*
- (2) *There is a map of factorization algebras on $\mathbb{R}_{>0}$*

$$(6) \quad \iota : \text{Obs}_{LV}^{\text{q},sm} \rightarrow \pi_* \text{Obs}_V^{\text{q},sm} \ ,$$

which is dense inclusion when applied to each interval.

Proof. Consider $\text{Obs}_{z^0 V}^{\text{q},sm}$, the factorization algebra associated to topological quantum mechanics from the first part of this section. Equivalently, this is the zero mode part of the theory on the loop space described by the action $S^{(0)}$. It is clear that this factorization algebra is locally constant. Moreover, in Section 4.4 of [?] it is shown that the corresponding associative algebra is equivalent to the Weyl algebra. To see (1) we observe that since $d_{dR}(n \log r) = \frac{n}{r} dr$, for each n the connection $\nabla^{(n)} = d_{dR} - \frac{n}{r} dr$ is gauge equivalent to the de Rham differential d_{dR} . This implies that the classical observables of the theory $S^{(n)}$ are quasi-isomorphic to the classical observables of $S^{(0)}$. Since the BV Laplacian Δ commutes with this gauge transformation, we see that this implies that

the quantum observables are also quasi-isomorphic. Thus, for each n , $\text{Obs}_{z^n V}^{\text{q},sm} \simeq \text{Obs}_{z^0 V}^{\text{q},sm}$. At each N the factorization algebra

$$\bigotimes_{|n| \leq N} \text{Obs}_{z^n V}^{\text{q},sm}$$

is thus equivalent to the associative algebra $\bigotimes_{|n| \leq N} D_{z^n V}^{\hbar}$. Moreover, this equivalence preserves the L_0 -equivariance. Both the observables $\text{Obs}_{LV}^{\text{q},sm}$ and the Weyl algebra D_{LV}^{\hbar} we are considering are defined as colimits (4), (5), so the result follows.

The claim (2) is a general fact about pushing forward holomorphic theories along the radial projection. First, consider the cosheaf of compactly supported Dolbeault forms on \mathbb{C}^\times , $\Omega_c^{0,*}$. We can pushforward this cosheaf along $\pi : \mathbb{C}^\times \rightarrow \mathbb{R}_{>0}$ whose value on an interval $I \subset \mathbb{R}_{>0}$ is $(\pi_* \Omega_c^{0,*})(I) = \Omega_c^{0,*}(\pi^{-1}(I))$. Let Ω_c^* be the cosheaf of compactly supported forms on $\mathbb{R}_{>0}$. Pull-back induces a map of cosheaves $\pi^* : \Omega_c^* \rightarrow \pi_* \Omega_c^{0,*}$. Moreover, for every n we can define another map of cosheaves

$$\pi_n^* : \Omega_c^* \rightarrow \pi_* \Omega_c^{0,*}$$

that sends a function $f(r)$ to $z^n f(|z|)$ and a one-form $f(r)dr$ to the one-form $(z^n f(|z|))d(|z|)$. We can then consider the direct sum $\bigoplus_{n \in \mathbb{Z}} \pi_n^* : \bigoplus_n \Omega_c^* \rightarrow \pi_* \Omega_c^{0,*}$. Now, every open set in \mathbb{C} is Stein, so that the cohomology of $H^*(\Omega_c^{0,*}(\pi^{-1}(I)))$ is concentrated in degree one and given by

$$H^1(\Omega_c^{0,*}(\pi^{-1}(I))) \cong \left(\Omega_{hol}^1(\pi^{-1}(I)) \right)^\vee.$$

In cohomology, $\bigoplus_n \pi_n^*$ exhibits a dense map $\mathbb{C}[z, z^{-1}] \rightarrow (\Omega_{hol}^1(\pi^{-1}(I)))^\vee$. The same argument goes through to show that the smoothed observables of quantum mechanics on the loop space, which is built from the compactly supported forms on $\mathbb{R}_{>0}$, maps densely in cohomology to the push forward of the observables of the $\beta\gamma$ system as desired. For a more detailed argument that is similar to this, see Section [BW: ref Kac moody section](#). \square

1.3.2. Global sections. We now turn to the statement of the local chiral algebraic index theorem using factorization homology. We will use the results above about the factorization algebras on \mathbb{C}^\times and $\mathbb{R}_{>0}$ to deduce statements about the factorization homologies along E_q and S^1 , respectively.

First, we recall the general fact about discretely equivariant factorization algebras. Let \mathcal{F} be any factorization algebra on X , and that G is a discrete group acting on X . We say that \mathcal{F} is G -equivariant if, for every $g \in G$, there is a quasi-isomorphism of factorization algebras

$$\rho_g : g^*(\mathcal{F}) \xrightarrow{\sim} \mathcal{F}$$

on X such that for any g, h $\rho_{gh} = \rho_h \circ h^* \rho_g$. We note that the pull-back of factorization algebras is defined since the action by $g : X \rightarrow X$ is a homeomorphism. In fact, as long as $f : X \rightarrow Y$ is an open immersion, the pullback $f^* \mathcal{G}$ of a factorization algebra \mathcal{G} on Y is defined [\[?\] BW: place?](#). We denote by Fact_X^G the category of G -equivariant factorization algebras. Note that this definition is sufficiently simpler than the case where G is a continuous, or Lie, group since we do not need to impose any smoothness conditions.

Proposition 1.6 ([\[?\] Proposition BW: ??](#)). *Suppose G acts properly discontinuously on X and let $p : X \rightarrow X/G$ be the resulting open immersion. Then, the induced functor $p^* : \text{Fact}_{X/G} \rightarrow \text{Fact}_X^G$ is an equivalence of categories.*

Remark 1.7. This equivalence of categories factors through an equivalence of locally constant factorization algebras.

In our situation, for each $q \in D(0,1)^\times$, we have an action of the discrete group \mathbb{Z} on two different spaces: \mathbb{C}^\times and $\mathbb{R}_{>0}$. On \mathbb{C}^\times the action is by $n \in \mathbb{Z} : z \mapsto q^n z$. Equivalently, this is induced by the exponentiation of the Euler vector field $L_0 = z \frac{\partial}{\partial z}$. We have already seen that the factorization algebra $\text{Obs}_V^{q,sm}$ on \mathbb{C}^\times is equivariant for L_0 . This induces, by exponentiation an action of \mathbb{Z} on the factorization algebra by the automorphism q^{L_0} . By Proposition 1.6 this implies that $\text{Obs}_V^{q,sm}$ descends along $p_q : \mathbb{C}^\times \rightarrow E_q$ to a factorization algebra on $E_q = \mathbb{C}^\times / q^\mathbb{Z}$ for any q . The factorization homology along E_q is quasi-isomorphic to the global sections

$$\int_{E_q} \text{Obs}_V^{q,sm} \simeq \left(\text{Sym}(\Omega^{1,*}(E_q) \otimes V[1] \oplus \Omega^{0,*}(E_q) \otimes V[1])[[\hbar]], \bar{\partial} + \hbar \Delta \right).$$

Next, we consider our locally constant factorization algebra on $\mathbb{R}_{>0}$. Consider the quotient map $p : \mathbb{R}_{>0} \rightarrow S_q^1 = \mathbb{R}_{>0} / |q|^\mathbb{Z}$. Here, we identify $r \in \mathbb{R}_{>0}$ with $|q|r$. For each n , the connection $\nabla^{(n)} = d - \frac{n}{r} dr$ is left invariant by this action of \mathbb{Z} , as is the BV Laplacian Δ . Thus, the factorization algebra $\text{Obs}_{LV}^{q,sm}$ is \mathbb{Z} -equivariant and hence determines a factorization algebra on S_q^1 .

Furthermore, the map of factorization algebras (6) induces a map of factorization homologies

$$\iota : \int_{S_q^1} \text{Obs}_{LV}^{q,sm} \rightarrow \int_{E_q} \text{Obs}_V^{q,sm}.$$

[BW: should this go here?](#)

Lemma 1.8. *This map is a quasi-isomorphism.*

1.3.3. Twisted Hochschild homology. The next piece of input we need to formulate the local chiral algebraic index theorem is a map from the Hochschild homology of the Weyl algebra on the the loop space to quantum mechanics. If \mathcal{F} is any locally constant factorization algebra on S^1 , then $p^*\mathcal{F}$ is a \mathbb{Z} -equivariant factorization algebra on $\mathbb{R}_{>0}$ where $p : \mathbb{R}_{>0} \rightarrow S^1$ is the quotient map. In particular, the element $1 \in \mathbb{Z}$ defines an equivalence $\rho_1 : 1^*p^*\mathcal{F} \xrightarrow{\sim} p^*\mathcal{F}$. In addition, since \mathcal{F} is locally constant this implies that $1 \in \mathbb{Z}$ defines an automorphism $\rho_1 : p^*\mathcal{F} \rightarrow p^*\mathcal{F}$. This, in turn induces an automorphism of the underlying associative algebra defined by $p^*\mathcal{F}$. Thus, to prescribe a locally constant factorization algebra \mathcal{F} on S^1 , it is equivalent to give the data of an associative algebra $A_{\mathcal{F}}$ (really an E_1 -algebra) together with an algebra automorphism σ (really an E_1 -automorphism). It is a standard result of [?, ?] that the factorization homology along S^1 of an associative algebra is equivalent to its Hochschild complex. The following is true about this twisted case.

Proposition 1.9 ([?] 5.3.3). *Let \mathcal{F} be a locally constant factorization algebra on S^1 and $(A_{\mathcal{F}}, \sigma)$ the corresponding pair of an E_1 algebra and an E_1 algebra automorphism. Then there is a quasi-isomorphism*

$$\int_{S^1} \mathcal{F} \simeq \text{Hoch}_*(A_{\mathcal{F}}, \sigma)$$

where $\text{Hoch}_*(A_{\mathcal{F}}, \sigma)$ is the Hochschild cochain complex twisted by σ .

Remark 1.10. [BW: define twisted Hochschild](#)

Example 1.11. Suppose \mathfrak{g} is a Lie algebra and consider its universal enveloping algebra $U\mathfrak{g}$. There is a convenient model for $U\mathfrak{g}$ as a locally constant factorization algebra on \mathbb{R} . One starts with the cosheaf $\Omega_c^* \otimes \mathfrak{g}$, where Ω_c^* is the cosheaf of compactly supported sections on \mathbb{R} . This is a cosheaf of Lie algebras, hence we may consider its Chevalley-Eilenberg chains $C_{\text{Lie}}^*(\Omega_c^* \otimes \mathfrak{g})$. In [?] it is shown that this is a locally constant factorization algebra on \mathbb{R} which is equivalent, as an associative algebra, to $U\mathfrak{g}$.

Now, suppose we fix a derivation D of \mathfrak{g} . Then, D induces a factorization algebra derivation of $C_{\text{Lie}}^*(\Omega_c^* \otimes \mathfrak{g})$. We can deform the factorization algebra by the derivation D in the following way: for every open set U the deformed factorization algebra assigns the cochain complex

$$(7) \quad (\text{Sym}(\Omega_c^*(U) \otimes \mathfrak{g}[1]), d_{dR} + d_{CE} + dtD).$$

The operator $d_{dR} + d_{CE}$ is the ordinary differential for the cochain complex $C_{\text{Lie}}^*(\Omega_c^*(U) \otimes \mathfrak{g})$. Since $(dt)^2 = 0$ and D commutes with d_{dR}, d_{CE} , the operator $d_{dR} + d_{CE} + dtD$ still defines a differential. Moreover, this deformed factorization algebra is still translation invariant so we may consider its global sections along S^1 .

Note that D determines a derivation of the associative algebra $U\mathfrak{g}$. If D is nilpotent, it defines an automorphism $\sigma_D = e^D = 1 + D + \frac{1}{2}D^2 + \dots$. In this case, the global sections of the deformed factorization algebra provide a nice model for the twisted Hochschild homology of the universal enveloping algebra:

$$\text{Hoch}_*(U\mathfrak{g}, \sigma_D) \simeq (\text{Sym}(\Omega^*(S^1) \otimes \mathfrak{g}[1]), d_{dR} + d_{CE} + dtD).$$

Both of these are equivalent to the factorization homology of the factorization algebra on S^1 determined by $U\mathfrak{g}$ together with the automorphism φ_D .

In our situation we can make this equivalence of the factorization homology with the twisted Hochschild complex more precise by exhibiting an explicit quasi-isomorphism.

We first recount a construction that's implicit in [?] relating ordinary Hochschild homology to quantum mechanics on S^1 .

Proposition 1.12. [?] *The map $\Phi : \text{Hoch}_*(D_{LV}^{\hbar}) \xrightarrow{\sim} \int_{S^1} \text{Obs}_{LV}^{qsm}$ is a quasi-isomorphism.*

1.3.4. BV integration. To the vector space V we can associate the (-1) -shifted symplectic vector space $T^*[-1]V = V \oplus V^*[-1]$. Then, the space of functions $\mathcal{O}(T^*[-1]V)$ has the structure of a P_0 -algebra. In this case, there is a natural quantization of this P_0 -algebra to a BV algebra, defined over $\mathbb{C}[[\hbar]]$. Choose a basis $\{x_i\}$ for V and a dual basis $\{\zeta_i\}$ for V^* . Note that in $T^*[-1]V$, the variables ζ_i are odd. The BV operator is defined by

$$\Delta = \sum_i \frac{\partial}{\partial x_i} \frac{\partial}{\partial \zeta_i}.$$

The BV algebra is $(\mathcal{O}(T^*[-1]V)[[\hbar]], \hbar\Delta, \cdot)$ where \cdot is the ordinary commutative product of functions.

One motivation for considering BV structures is that it gives us a setting to do integration. **BW:** ref schwarz, kevin, etc. There is a pairing between homology classes of Lagrangian super-submanifolds (in this setting, these are submanifolds of the (-1) -shifted symplectic vector space $T^*[-1]V$) and the corresponding BV cohomology.

In this case, there is essentially a unique super-submanifold of $T^*[-1]V$:

$$V^*[-1] \subset T^*[-1]V = V \oplus V^*[-1].$$

The corresponding “integration map” $\int_{BV} : \mathcal{O}(T^*[-1]V)[[\hbar]] \rightarrow \mathbb{C}[[\hbar]]$ is the Berezin integral over $V^*[-1]$ as a supermanifold. This is the map that simply picks out the “top fermion” and is normalized by the condition

$$\int_{BV} \xi_1 \cdots \xi_{\dim V} = 1.$$

Ordinarily, this toy model does not work for BV algebras coming from quantum field theory, as the BV algebra we obtain is not finite dimensional. One solution is to perform regularization, but there another approach that allows us to directly apply the discussion above.

The BV algebra we are interested in is the global observables of the chiral theory on the elliptic curve E_q . Let us fix the flat metric on E_q , which is descended from the one we used on \mathbb{C} to find a gauge fixing condition in order to regularize the theory. Inside of the Dolbeault complex $\Omega^{0,*}(E_q)$ we have the subspace of harmonic forms

$$\mathcal{H}(E_q) \subset \Omega^{0,*}(E_q).$$

This induces an inclusion of cochain complexes

$$(8) \quad \mathcal{H}(E_q) \otimes (V \oplus \frac{dz}{z} V^*) \hookrightarrow \Omega^{0,*}(E_q) \otimes (V \oplus \frac{dz}{z} V^*),$$

where the right-hand side is an expression for the fields $\mathcal{E}_V(E_q)$ of the chiral theory on E_q after using the frame $\frac{dz}{z}$ for the canonical bundle. At the level of functions we then obtain a projection

$$\mathcal{O} \left(\mathcal{H}(E_q) \otimes (V \oplus \frac{dz}{z} V^*) \right) \xleftarrow{\pi} \mathcal{O}(\mathcal{E}_V(E_q)).$$

On the right-hand side, we mean the continuous functions, so this is inherently built from distributional differential forms on E_q . Using this projection, we obtain the following relationship to the smoothed global observables.

Lemma 1.13. *Homotopy RG flow induces the quasi-isomorphism π_∞ from the smoothed global observables on E_q to the harmonic observables equipped with the scale ∞ BV Laplacian Δ_∞ :*

$$\begin{array}{ccc} \text{Obs}_V^{\text{q},sm}(E_q) & \xrightarrow[\cong]{W_0^L} & \text{Obs}_V^{\text{q},sm}(E_q)[L] \\ \vdots & & \simeq \downarrow i[L] \\ \pi_\infty \downarrow \simeq & & \text{Obs}^{\text{q}}(E_q)[L] \\ & & \cong \downarrow W_L^\infty \\ \left(\mathcal{O} \left(\mathcal{H}(E_q) \otimes (V \oplus \frac{dz}{z} V^*) \right) [[\hbar]], \hbar \Delta_\infty \right) & \xleftarrow[\pi]{} & \text{Obs}_V^{\text{q}}[\infty]. \end{array}$$

Proof. Observe that homotopy RG flow from scale zero to an arbitrary scale L is well defined on the smoothed observables, which defines the top horizontal isomorphism W_0^L . The map $i[L]$ is inclusion of the smoothed observables at scale L inside of all observables at scale L . The space of all observables at scale L is isomorphic through W_L^∞ to the observables at scale ∞ . The scale ∞ observables $\text{Obs}_V^{\text{q}}[\infty]$ is equipped with the differential Δ_∞ , which we observe restricts to functions

on the harmonic fields. This is because the kernel K_∞ defining the scale ∞ BV Laplacian actually lies in the subspace $\mathcal{H}(E_q) \otimes \mathcal{H}(E_q)$. \square

We observe that $\mathcal{H}(E_q) = \mathbb{C}[\delta]$, where δ is a formal parameter of degree $+1$ representing the $(0,1)$ form $\frac{dz}{z}$ generating $H^1(E_q, \mathcal{O}) \cong \mathbb{C}$. We will use the natural (-1) -shifted symplectic structure on this polynomial algebra given by

$$(f(\delta), g(\delta)) \mapsto \frac{\partial}{\partial \delta} (f(\delta)g(\delta)).$$

Using the usual symplectic pairing between V and V^* we see that $\mathcal{H}(E_q) \otimes (V \oplus \frac{dz}{z} V^*)$ is a finite dimensional (-1) -shifted symplectic vector space. This is clearly compatible with the map (8) using the (-1) -shifted symplectic structure defining the chiral theory.

Choosing the odd super-submanifold $\delta(V \oplus \frac{dz}{z} V^*) \subset (V \oplus \frac{dz}{z} V^*)[\delta] \cong \mathcal{H}(E_q) \otimes (V \oplus \frac{dz}{z} V^*)$ we obtain the BV integration map

$$\int_{BV} : \left(\mathcal{O} \left((V \oplus \frac{dz}{z} V^*)[\delta] \right) [[\hbar]], \hbar \Delta_\infty \right) \rightarrow \mathbb{C}[[\hbar]].$$

For any vector space W we can make the identification $\mathcal{O}(W[\delta]) = \Omega^{-*}(W)$, the negatively graded de Rham forms on W . For instance, the constant coefficient one-forms are identified with $\delta^* W^*$, which sits in degree -1 . It is convenient to identify $V \oplus \frac{dz}{z} V^* \cong T^*V$ so that

$$\mathcal{O} \left((V \oplus \frac{dz}{z} V^*)[\delta] \right) \cong \Omega^{-*}(T^*V)$$

The BV Laplacian Δ_∞ is identified with the Lie derivative L_π with respect to the natural Poisson bivector on T^*V . Making these identifications, the BV integration map

$$\int_{BV} : (\Omega^{-*}(T^*V)[[\hbar]], \hbar L_\pi) \rightarrow \mathbb{C}[[\hbar]]$$

simply picks out the top de Rham form sitting in degree $-2n$, where $n = \dim_{\mathbb{C}} V$.

Remark 1.14. The $(\Omega^{-*}(T^*V)[[\hbar]], \hbar L_\pi) \dots$ Poisson homology [BW: finish](#).

Theorem 1.15. (*Local chiral algebraic index theorem*) *For any $q \in D(0,1)^\times$ there is a commuting diagram of quasi-isomorphisms*

$$\begin{array}{ccc} \int_{E_q} \text{Obs}_V^{q,sm} & \xrightarrow[\simeq]{\pi_\infty} & (\Omega^{-*}(T^*V)[[\hbar]], \hbar L_\pi) \\ \Phi \uparrow \simeq & & \simeq \downarrow \int_{BV} \\ \text{Hoch} \left(\text{Weyl}_{LV}^\hbar; q \right) & \xrightarrow[\simeq]{\tau^q} & \mathbb{C}[[\hbar]][2n]. \end{array}$$

1.4. The global chiral algebraic index theorem.