The holomorphic σ-model and its symmetries

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Outline of this talk

1. Rapid overview of the Batalin-Vilkovisky (BV) formalism.
2. Holomorphic theories, in general. One-loop finiteness and a formula for the general chiral anomaly.
3. The holomorphic σ-model and its factorization algebra.

The BV formalism

The BV formalism is a technique used to study quantizations of field theories. A generalization of the usual problem of *deformation quantization*.

$$\begin{array}{ccccc} \mathrm{SympMfld} & \xrightarrow{\mathcal{O}} & \mathrm{Alg}_{\mathrm{Poiss}} & \xleftarrow{\hbar \to 0} & \mathrm{Alg}_{\mathbb{C}[[\hbar]]} \\ & & & & \\ (M, \omega) & \longmapsto & (\mathcal{O}(M), \Pi_\omega) & \longleftarrow\!\shortmid & (\mathcal{O}(M)[[\hbar]], \star)\,. \end{array}$$

In field theory, one works on a smooth manifold X (the spacetime).

$$\mathrm{BV}-\mathrm{Theory}(X) \xrightarrow{\mathrm{Obs}} \mathrm{FactAlg}(X)_{P_0} \xleftarrow{\hbar \to 0} \mathrm{FactAlg}(X)_{\mathrm{BV}}.$$

Given a classical BV theory we study lifts of the P_0 factorization algebra of classical observables to the *BV* factorization algebra of quantum observables.
In the one-dimensional case $X = \mathbb{R}$ there exists a classical BV theory associated to a symplectic manifold (M, ω). In this case, BV quantization recovers ordinary deformation quantization.

The BV formalism (cont.)

In QFT, BV algebras provide a mathematical model for the path integral.

Definition

A *BV algebra* is a triple (A, Q, Δ) where (A, Q) is a commutative dg algebra, and $\Delta : A \to A$ is a degee one linear map such that

(a) $\Delta^2 = [\Delta, Q] = 0$;

(b) the degree one bilinear map

$$\{a, b\} := \Delta(ab) - \Delta(a)b \pm a\Delta(b)$$

satisfies graded Jacobi, and is a graded biderivation with respect to the commutative product.

Thus $\{-,-\}$ behaves like a Poisson bracket, except with a weird shift. We say an element $I = I_0 + \hbar I_1 + \cdots \in A[[\hbar]]$ satisfies the *quantum master equation* (QME) if

$$\boxed{(Q + \hbar\Delta)e^{I/\hbar} = 0.}$$

We call $\hbar$ the *perturbation* parameter.

The BV formalism (cont.)

When we set $\hbar = 0$, the QME reduces to condition

$$\boxed{QI_0 + \frac{1}{2}\{I_0, I_0\} = 0}.$$

We call this the classical master equation (CME).

Example

Suppose $A = \mathcal{O}(V) = \mathrm{Sym}(V^*)$ for some graded vector space V. Then a functional I_0 satisfying the CME is equivalent to an data of an L_∞ structure on the graded vector space $V[-1]$.

Most important example of BV algebras in QFT come from (-1)-shifted geometry. Suppose (V, ω) is a (-1)-shifted symplectic vector space. Then, the symmetric tensor $K_0 := \omega^{-1} \in \mathrm{Sym}^2(V)$ defines an operator (of order two)

$$\Delta_0 = \partial_{K_0} : \mathcal{O}(V) \to \mathcal{O}(V)$$

by contraction. This operator defines a BV algebra $(\mathcal{O}(V), Q, \Delta_0)$, where Q is the internal differential of V.

The BV formalism (cont.)

Suppose that $P \in \mathrm{Sym}^2(V)$ is a symmetric tensor of degree zero, and define $K_P = K_0 + QP$. One checks that K_P defines another BV algebra based on $\mathcal{O}(V)$.
Given $I \in \mathcal{O}^+(V)$ (at least cubic), define $W(P,I) \in \mathcal{O}(V)[[\hbar]]$ formally by

$$\boxed{e^{W(P,I)/\hbar} = e^{\hbar\partial_P} e^{I/\hbar}}.$$

Lemma

The functional I satisfies the QME relative to K_0 if and only if $W(P,I)$ satisfies the QME relative to K_P.

The functional $W(P,I)$ decomposes as a sum over connected graphs

$$W(P,I) = \sum_{\Gamma} \frac{\hbar^{g(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} W_\Gamma(P,I),$$

where W_Γ is the *weight* of the graph Γ.

Field theory

A classical field theory on a smooth manifold M is:

(i) a graded vector bundle E whose sections we denote $\mathcal{E}$;

(ii) a differential operator $Q : \mathcal{E} \to \mathcal{E}$ of degree one;

(iii) a graded antisymmetric bundle map $(-,-)_E : E \otimes E \to \mathrm{Dens}_X$ of degree (-1) that is fiberwise nondegenerate.

(iv) a *local functional* $I_0 \in \mathcal{O}_{\mathrm{loc}}(\mathcal{E})$ satisfying the CME.

We require that $(\mathcal{E}, Q)$ is an elliptic complex. The pairing $(-,-)_E$ defines a (-1)-shifted symplectic structure via integration

$$\omega = \int_X \circ (-,-)_E.$$

The sheaf of sections $\mathcal{E}$ evaluated on an open set U returns the graded space $\mathcal{E}(U)$ which we refer to as the space of fields supported on U. The *classical observables* supported on U:

$$\mathrm{Obs}^{\mathrm{cl}}(U) = \left(\mathrm{Sym}(\mathcal{E}(U)^\vee), Q + \{I_0, -\}\right).$$

Holomorphic field theory

In the world of complex geometry we have the following definition of a *holomorphic* field theory on a complex manifold X:

(i) a graded holomorphic vector bundle V on X whose sheaf of holomorphic sections we denote $\mathcal{V}^{hol}$;

(ii) a holomorphic differential operator $Q^{hol} : \mathcal{V}^{hol} \to \mathcal{V}^{hol}$ of degree one;

(iii) a graded antisymmetric bundle map $(-,-)_V : V \otimes V \to K_X$ of degree $(d-1)$ that is fiberwise nondegenerate.

(iv) a holomorphic Lagrangian $\mathcal{I}_0^{hol}$ satisfying the CME.

Holomorphic theory	BV theory
Holomorphic bundle V	Space of fields $\mathcal{E}_V = \Omega^{0,*}(X, V)$
Holomorphic differential operator Q^{hol}	Linear BRST operator $\overline{\partial} + Q^{hol}$
Non-degenerate pairing $(-,-)_V$	(-1)-symplectic structure ω_V
Holomorphic Lagrangian $\mathcal{I}_0^{hol}$	Local functional $I_0^{\Omega^{0,*}} \in \mathcal{O}_{\mathrm{loc}}(\mathcal{E}_V)$

Table: From holomorphic to BV

Regularization

Let $(\mathcal{E}, Q, \omega, I_0)$ be a classical BV theory. The first thing to do is define the BV operator $\Delta_0 = \omega^{-1}$.

- **Problem:** The tensor ω^{-1} is *distributional*, thus Δ_0 is not well-defined on functionals.

The solution is to find a homotopy replacement for K_0

$$\widetilde{K} = K_0 + QP,$$

so that its BV operator is well-defined. (By elliptic regularity, one always exists). Such a regularization is parametrized by a length scale $L > 0$. For each $L < L'$ a regularization scheme prescribes a *propagator* $P_{L<L'}$ such that

$$K_{L'} = K_L + QP_{L<L'}$$

where $K_L, K_{L'}$ are both smooth and $\lim_{L\to 0} K_L = K_0$.

The definition of a QFT

By definition, a quantization is a family of functionals $\{I[L]\}$ with $I_0 = \lim_{L\to 0} I[L] \mod \hbar$ satisfying the following two conditions:

1. the collection of functionals $\{I[L]\}$ are related by *renormalization group flow*
$$I[L'] = W(P_{L<L'}, I[L]);$$
2. for each L, the functional solves the *scale L* quantum master equation
$$(Q + \hbar\Delta_L)e^{I[L]/\hbar} = 0;$$
3. some technical locality conditions.

For abstract reasons, proved by Costello, one can always find a family such that (1) is satisfied. In general, the answer is not constructive and involves choosing counterterms with respect to a renormalization scheme. There may be unavoidable obstructions to solving problem (2).

Holomorphic renormalization

The naïve definition of $I[L]$ is to apply the operator $P_{0<L}$ to the classical interaction

$$I[L] = W(P_{0<L}, I_0)$$

The problem is that the right-hand side is rarely well-defined (same issue as above). A solution to this, which always exists, is to find counterterms.

Theorem

There is a regularization scheme for **holomorphic theories** *on* $\mathbb{C}^d$ *such that the limit*

$$I[L] = \lim_{\epsilon \to 0} W(P_{\epsilon<L}, I_0) \mod \hbar^2$$

exists. In other words, holomorphic theories on $\mathbb{C}^d$ *are one-loop finite.*

The main ingredient is in the existence of the *gauge fixing operator* $\overline{\partial}^*$.

- Studying the quantizations of holomorphic theories on $\mathbb{C}^d$ reduces to solving the quantum master equation. This is essentially an algebraic problem.

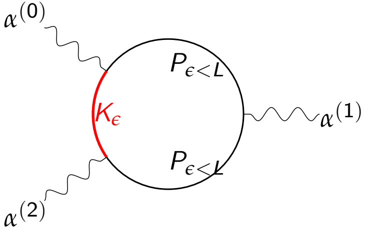
A general formula for the chiral anomaly

A corollary of this result is a characterization of the *anomaly*, or obstruction, for a holomorphic theory to solve the QME.

Corollary

The obstruction for a classical holomorphic theory on $\mathbb{C}^d$ to admit a one-loop quantization is given by the following expression:

$$\Theta = \lim_{\epsilon, L \to 0} \sum_{\Gamma \in \mathrm{Wheel}_{d+1}} W_\Gamma(P_{\epsilon<L}, K_\epsilon, I_0).$$



This gives a holomorphic characterization, and generalization, of the Adler-Bell-Jackiw anomaly for four-dimensional gauge theory.

The holomorphic σ-model

The holomorphic σ-model is a prototypical holomorphic theory. Let X, Y be complex manifolds and consider the mapping space:

$$\mathrm{Map}^{hol}(Y, X) = \{f : Y \to X \ \text{ holomorphic}\}.$$

There are a few issues:

1. a classical theory involves a shifted symplectic pairing. The theory we study is of the form

   $$T^*[-1]\left(\mathrm{Map}^{hol}(Y, X)\right).$$

   In degree zero, the fields consist of a map $\gamma : Y \to X$ together with a class $\beta \in \Omega^{d,d-1}(Y, \gamma^* T^{*1,0}X)$. The action functional is

   $$S(\beta, \gamma) = \int_Y \beta \wedge \bar{\partial}\gamma.$$

   Notice when we vary γ, β we obtain $\bar{\partial}\gamma = 0 = \bar{\partial}\beta$.
2. To make this into a BV theory, we must perturb around a fixed holomorphic map; we look at the formal neighborhood of constant maps $\mathrm{Map}(Y, X)^{\wedge}_{const}$.

Local-to-global

Our construction of the holomorphic σ-model is local-to-global on the target manifold. We phrase the theory in the style of *formal geometry* due to Gelfand, Kazhdan, Fuks. To every n-dimensional manifold X (smooth, complex, symplectic, etc..) there exists a universal bundle of coordinates:

$$\begin{tikzcd}
\mathrm{W}_n \arrow[r, phantom, "\curvearrowright"] & X^{coor} \arrow[dd] \arrow[dr, "\simeq"] & \\
 & & \mathrm{Fr}_X \arrow[dl] \arrow[ul, bend right, "\sigma"'] \\
 & X & .
\end{tikzcd}$$

X^{coor} is a principal Aut_n-bundle together with a transitive action of the Lie algebra of *formal vector fields* in n-dimensions W_n. There is

$$\omega^{coor} \in \Omega^1(X^{coor}, \mathrm{W}_n)^{\mathrm{Aut}_n} \xrightarrow{\sigma^*} \Omega^1(\mathrm{Fr}_X, \mathrm{W}_n)^{\mathrm{GL}_n}$$

satisfying the Maurer-Cartan equation $\mathrm{d}\omega^{coor} + \frac{1}{2}[\omega^{coor}, \omega^{coor}] = 0$.

Gelfand-Kazhdan descent

Define a category of "formal vector bundles" on the formal n-disk. In particular, these are $(\mathrm{W}_n, \mathrm{GL}_n)$-modules. For each X, there is a functor

$$\begin{tikzcd}
\mathcal{V} \arrow[r, mapsto] & \left(\mathrm{Fr}_X \times^{\mathrm{GL}_n} \mathcal{V}, \nabla^{coor}\right) \\
\mathrm{VB}_{\widehat{D}^n} \arrow[u, phantom, "\Cap"] \arrow[r, "\mathrm{desc}_X"] \arrow[d, hook] & \mathrm{VB}_X^{flat} \arrow[u, phantom, "\Cap"] \arrow[d, hook] \\
\mathrm{Mod}_{(\mathrm{W}_n, \mathrm{GL}_n)} \arrow[r] & \mathrm{Mod}_{D_X}.
\end{tikzcd}$$

Moreover, there are "formal characteristic classes" that live in the Gelfand-Fuks cohomology. The descent functor determines a transformation of cohomology theories and hence a map of complexes

$$\mathrm{char}_X : \mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n, \mathrm{GL}_n; \mathcal{V}) \to \Omega^*(X, \mathrm{desc}_X(\mathcal{V})).$$

When $\mathcal{V} = \widehat{\mathcal{O}}_n$ formal power series, $\mathrm{desc}_X(\widehat{\mathcal{O}}_n) = J^\infty \mathcal{O}_X$ equipped with its natural flat connection. Recover all natural bundles in this way.

The formal holomorphic σ-model

Consider the formal disk $\widehat{D}^n$ as a ringed space whose functions are formal power series $\widehat{\mathcal{O}}_n$.

$$Y \longrightarrow \widehat{D}^n \circlearrowleft \quad (\mathrm{W}_n, \mathrm{GL}_n).$$

Key idea: study the free theory *equivariant* for the action of the pair $(\mathrm{W}_n, \mathrm{GL}_n)$. Get global target σ-model via descent.
Quantization: holomorphic theory $\implies$ renormalization is simple.
Obstruction is controlled by an element in Gelfand-Fuks cohomology.

Theorem

There is an obstruction to quantizing the formal holomorphic σ-model of maps $\mathbb{C}^d \to \widehat{D}^n$ given by the class

$$\mathrm{ch}_{d+1}^{\mathrm{GF}}(\widehat{\mathcal{T}}_n) \in \mathrm{C}_{\mathrm{Lie}}^{d+1}(\mathrm{W}_n, \mathrm{GL}_n; \widehat{\Omega}_{n,cl}^{d+1}).$$

Under characteristic map, this returns the ordinary Chern class.
Determines an L_∞*-extension*

$$0 \to \widehat{\Omega}_{n,cl}^{d+1} \to \widetilde{\mathrm{W}}_{n,d} \to \mathrm{W}_n \to 0.$$

Extended descent

Given any trivialization α of $\mathrm{ch}_{d+1}(T_X)$ we can lift the structure of the coordinate bundle.

$$
\begin{tikzcd}
\widetilde{\mathrm{W}}_{n,d} \arrow[d] & \circlearrowright & \widetilde{X}^{coor}_{\alpha} \arrow[d, dotted] \arrow[dr, dotted] & \\
\mathrm{W}_n & \circlearrowright & X^{coor} \arrow[dr] \arrow[dd] & (\mathrm{Fr}_X, \widetilde{\omega}^{coor}_{\alpha}) \arrow[d] \\
 & & & (\mathrm{Fr}_X, \omega^{coor}) \arrow[dl] \\
 & & X &
\end{tikzcd}
$$

$$\widetilde{\omega}^{coor}_{\alpha} \in \oplus_{p \geq 1} \Omega^p(\mathrm{Fr}_X, \widetilde{\mathrm{W}}_{n,d})$$

$$\mathrm{d}\widetilde{\omega}^{coor}_{\alpha} + \textstyle\sum_{k \geq 2} \ell_k(\widetilde{\omega}^{coor}_{\alpha}) = 0$$

Descent functor

$$\widetilde{\mathrm{desc}}_{X,\alpha} : \mathrm{Mod}_{(\widetilde{\mathrm{W}}_{n,d}, \mathrm{GL}_n)} \to \mathrm{Mod}_{D_X}.$$

Theorem implies quantization is equivariant for $(\widetilde{\mathrm{W}}_{n,d}, \mathrm{GL}_n)$. This says that for any trivialization α we obtain a global quantization.

Main result

Explicit GF calculation shows there is a unique $(\widetilde{\mathrm{W}}_{n,d}, \mathrm{GL}_n)$-quantization for the formal theory. Extended descent implies the following main result.

Theorem

Suppose $\mathrm{ch}_{d+1}(T_X) = 0$. Then, the space of quantizations (respecting certain natural symmetries) of the holomorphic σ-model of maps $\mathbb{C}^d \to X$ is a torsor for the abelian group $H^d(X, \Omega_X^{d+1,hol})$.

- Quantizations exist on other source manifolds: affine manifolds, abelian varieties, Hopf manifolds $Y = \mathbb{C}^d \setminus \{0\}/q^{\mathbb{Z}} \cong S^{2d-1} \times S^1$.
- Local calculation of the index produces elliptic Γ-functions. This agrees with the partition function for supersymmetric theories in dimensions $2, 4, 6$. For a general target, this should produced refined invariants generalizing the Witten genus in complex dimension one.

Relation to deformation quantization

Immediate corollary: obtain the following deformation quantization for "sphere algebras". Theory on

$$\begin{tikzcd} \mathbb{C}^d \setminus \{0\} \arrow[r, "\cong"] \arrow[d, "\pi"'] & \mathbb{R}_{>0} \times S^{2d-1} \\ \mathbb{R}_{>0}. \end{tikzcd}$$

Reduction along the sphere:

$$\begin{array}{c} \pi_* \left(\text{Holomorphic } \sigma\text{-model } \mathbb{C}^d \setminus \{0\} \to X\right) \\ \Big\| \\ \text{One dimensional } \sigma\text{-model } \mathbb{R}_{>0} \to T^*\mathrm{Map}^{alg}(S^{2d-1}, X). \end{array}$$

Sphere mapping space is really a derived algebraic version. There is a dg algebra A_d with $A_d^0 \hookrightarrow C^\infty(S^{2d-1})$ densely and

$$A_d \hookrightarrow \Omega^{0,*}(\mathbb{C}^d \setminus \{0\})$$

which is dense in cohomology. When $d = 1$, $A_1 = \mathbb{C}[z, z^{-1}]$ and we get algebraic loop space.

Observables

BV quantization produces a (sheaf of) factorization algebras on $\mathbb{C}^d$. In the one-dimensional reduction, restricts to a factorization algebra on $\mathbb{R}_{>0} \rightsquigarrow$ dg associative algebra. When $\mathrm{ch}_{d+1}(T_X) = 0$ we get a deformation quantization = "differential operators on the sphere mapping space".

$$\begin{tikzcd} \mathcal{O}_\hbar\left(T^*\mathrm{Map}(S^{2d-1}, X)\right) \arrow[r, equal] \arrow[d, "\hbar\to 0"] & : D_\hbar\left(\mathrm{Map}(S^{2d-1}, X)\right) \\ \mathcal{O}\left(T^*\mathrm{Map}(S^{2d-1}, X)\right). & \end{tikzcd}$$

The state space $\mathcal{V}_X$ is equal to the observables supported on the disk in $\mathbb{C}^d$. Factorization product endows $\mathcal{V}_X$ with the structure of a dg module over $D_\hbar(\mathrm{Map}(S^{2d-1}, X))$. It is equal to the "vacuum" module

$$\mathcal{V}_X = D_\hbar \otimes_{D_{\hbar,+}} \mathbb{C}[[\hbar]].$$

Where $D_{\hbar,+} \subset D_\hbar$ is a maximal commutative subalgebra of "positive modes". This plays the role of the Hilbert space in quantum mechanics.

Conclusions and outlook

- Have not discussed much about "source symmetries" of the holomorphic σ-model. Big part of my thesis was to characterize symmetries by holomorphic gauge transformations and by holomorphic diffeomorphisms. Lead to higher dimensional Kac-Moody algebras and Virasoro algebras, respectively.
- In particular, there is a dg Lie algebra central extension of holomorphic vector fields on punctured affine space that embeds inside of $D_{\hbar}$. This central extension is parametrized by a higher dimensional version of "central charge" in CFT.