Versal deformation of Hopf surfaces

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In 1948 H. Hopf constructed a class of compact complex manifolds which admit no embedding into complex projective space and are not even homeomorphic to an algebraic variety. For example Hopf succeeded to provide the topological manifold $S^1 \times S^3$ with a complex structure; a slight generalization of his original idea even yielded a series of different complex structures. Later on his examples were called Hopf surfaces and their study was continued by K. Kodaira [3] within the frame of his fundamental work on compact complex surfaces. In particular Kodaira proved in [4] that every complex structure on $S^1 \times S^3$ is given by a Hopf surface. In collaboration with D. Spencer he developed a theory of deformation of complex structures. Section 15 of their paper [5] deals with the versal deformation of Hopf surfaces. The family they construct on page 427, however, is not everywhere complete and therefore Theorem 15. 4. is not correct. This incorrect statement is repeated by A. Douady [2], Proposition 3 and by V. Palamodov [8], § 9.

In the exceptional cases the Hopf surface X can be defined by a linear transformation having two eigenvalues α and δ which satisfy a relation

$$\alpha = \delta^p, \; p \in \mathbb{N} \setminus \{0, 1\}.$$

Moreover, M. Namba [7] has shown that there is a one dimensional subspace of vector fields on X having coefficients which are not linear functions.

The study of the exceptional case gave rise to the present paper. Theorem 2 will give a complete survey on the versal deformation of Hopf surfaces.

In the following the expression versal deformation will always mean a family which is complete with injective Kodaira-Spencer map at the distinguished point. The notation is not generally accepted, other authors use the terms "effective versal" or "semi-universal" instead of versal. By a theorem of Kodaira-Spencer [6] a family with smooth base is a versal deformation if the Kodaira-Spencer map is bijective.

I. Vector fields and cohomology groups of Hopf surfaces

Definition. A compact complex surface X is called (primary) Hopf surface, if the universal covering is biholomorphic equivalent to the domain $W := \mathbb{C}^2 \setminus \{0\}$ and the fundamental group equals the infinite cyclic group $\mathbb{Z}$.

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Remarks. 1. The group of covering transformations of a Hopf surface is generated by a contraction

$$f\colon \mathbb{C}^2 \to \mathbb{C}^2,\ f(0)=0.$$

After a suitable choice of coordinates in $\mathbb{C}^2$ the contraction has the normal form

$$f(z, w) = (\alpha z + \lambda w^p, \delta w),$$

where $p \in \mathbb{N} \setminus \{0\}$ and $\alpha, \delta, \lambda \in \mathbb{C}$ are constants subject to the restrictions

$$0 < |\alpha| \leqq |\delta| < 1 \quad \text{and} \quad (\alpha - \delta^p)\lambda = 0.$$

2. Every Hopf surface X is homeomorphic to $S^1 \times S^3$. From this one gets the topological result

$$H^j(X, \mathbb{Z}) \cong \mathbb{Z} \quad \text{for } j = 0, 1, 3, 4 \quad \text{and} \quad H^2(X, \mathbb{Z}) = 0,$$

which implies the vanishing of the first Chern class

$$c_1 = 0 \in H^2(X, \mathbb{Z}).$$

On any compact complex manifold M of dimension n the n-th Chern class c_n equals the Euler class

$$e[M] = \sum_{j \geqq 0} (-1)^j \dim H^j(M, \mathbb{C}).$$

Hence for a Hopf surface we get $c_2 = 0$.

3. On any compact complex surface X the Hodge spectral sequence

$$E_1^{p,q} = H^q(X, \Omega^p) \Rightarrow H^{p+q}(X, \mathbb{C})$$

degenerates, i.e.

$$\sum_{p+q=r} h^{p,q} = b_r \quad \text{for all } r \in \mathbb{N}$$

with $h^{p,q} = \dim H^q(X, \Omega^p)$ and $b_r = \dim H^r(X, \mathbb{C})$.

This follows, even if the surface is non Kählerian, from the relations

$$b_1 \text{ even}\colon\ h^{0,1} = h^{1,0} \quad \text{and} \quad b_1 = 2h^{0,1},$$

$$b_1 \text{ odd}\colon\ h^{0,1} = h^{1,0} + 1 \quad \text{and} \quad b_1 = h^{0,1} + h^{1,0}$$

as shown by Kodaira [3]. Hence we get for a Hopf surface

$$h^{0,0} = h^{0,1} = h^{2,1} = h^{2,2} = 1 \quad \text{and} \quad h^{p,q} = 0 \quad \text{for all other } (p, q),$$

and this result depends only on the topological structure.

Theorem 1. *The following table shows the classification of Hopf surfaces $X = W/\langle f \rangle$ in the complex analytic case. Hopf surfaces of a fixed type are classified by the different values, which the parameters of the contraction*

$$f\colon \mathbb{C}^2 \to \mathbb{C}^2,\ f(0)=0,$$

can take.

type	$\dim H^0(X, \Theta)$	$f(z, w)$	parameters
IV	4	$(\alpha z, \alpha w)$	$0<\lvert\alpha\rvert<1$
III	3	$(\delta^p z, \delta w)$	$p \in \mathbb{N} \setminus \{0, 1\}, 0<\lvert\delta\rvert<1$
II_a	2	$(\delta^p z + w^p, \delta w)$	$p \in \mathbb{N} \setminus \{0, 1\}, 0<\lvert\delta\rvert<1$
II_b	2	$(\alpha z + w, \alpha w)$	$0<\lvert\alpha\rvert<1$
II_c	2	$(\alpha z, \delta w)$	$0<\lvert\alpha\rvert<\lvert\delta\rvert<1,\ \alpha \neq \delta^p$ for all $p \in \mathbb{N}$

Remarks. 1. At this point we want to call the attention to the following relations, which are valid for any Hopf surface X and will be proven subsequently in Lemma 1

$$\dim H^0(X, \Theta) = \dim H^1(X, \Theta) \quad \text{and} \quad H^2(X, \Theta) = 0.$$

These vector spaces are of great importance in deformation theory: The vanishing of $H^2(X, \Theta)$ implies that the base of the versal deformation is isomorphic to an open neighborhood of zero in $H^1(X, \Theta)$.

2. For the convenience of the reader the following table specifies a basis of the vector space $H^0(X, \Theta)$ and the group $\mathrm{Aut}(X)$ of holomorphic automorphisms for the Hopf surfaces $X = W/\langle f \rangle$ of each type respectively.

type	basis of $H^0(X, \Theta)$	$\mathrm{Aut}(X)$
IV	$z\dfrac{\partial}{\partial z},\ w\dfrac{\partial}{\partial z},\ z\dfrac{\partial}{\partial w},\ w\dfrac{\partial}{\partial w}$	$GL(2, \mathbb{C})/\langle f \rangle$
III	$z\dfrac{\partial}{\partial z},\ w\dfrac{\partial}{\partial w},\ w^p\dfrac{\partial}{\partial z}$	$\{(z, w) \mapsto (az + bw^p, dw) : ad \neq 0, b \in \mathbb{C}\}/\langle f \rangle$
II_a	$pz\dfrac{\partial}{\partial z} + w\dfrac{\partial}{\partial w},\ w^p\dfrac{\partial}{\partial z}$	$\{(z, w) \mapsto (a^p z + bw^p, aw) : a \in \mathbb{C}^*, b \in \mathbb{C}\}/\langle f \rangle$
II_b	$z\dfrac{\partial}{\partial z} + w\dfrac{\partial}{\partial w},\ w\dfrac{\partial}{\partial z}$	$\{(z, w) \mapsto (az + bw, aw) : a \in \mathbb{C}^*, b \in \mathbb{C}\}/\langle f \rangle$
II_c	$z\dfrac{\partial}{\partial z},\ w\dfrac{\partial}{\partial w}$	$\{(z, w) \mapsto (az, dw) : ad \neq 0\}/\langle f \rangle$

Proof. The statement of Theorem 1 and Remark 2 is essentially contained in the paper of Namba [7]. The proof uses standard techniques in the theory of covering spaces. Every biholomorphic map from a Hopf surface $W/\langle f_1 \rangle$ onto another one $W/\langle f_2 \rangle$ is induced by an automorphism of the universal covering $W = \mathbb{C}^2 \setminus \{0\}$ which extends to a biholomorphic map

$$\varphi : \mathbb{C}^2 \to \mathbb{C}^2,\ \varphi(0) = 0.$$

Because f_1 as well as f_2 are contractions the map φ satisfies the compatibility relation

$$\varphi \circ f_1 = f_2 \circ \varphi.$$

After expanding the functions on both sides of the equation into their Taylor series and comparing coefficients of the same degree one gets necessary and sufficient conditions of f_1, f_2 and φ. Derivating the one-parameter subgroups of the complex Lie group $\mathrm{Aut}(X)$ one gets the infinitesimal generators, i.e. the elements of $H^0(X, \Theta)$.

Lemma 1. *Let X be a Hopf surface and denote by Θ the sheaf of holomorphic vector fields on X. Then*

$$\dim H^0(X, \Theta) = \dim H^1(X, \Theta) \quad \textit{and} \quad H^2(X, \Theta) = 0.$$

Proof. From Kodaira-Serre duality we get

$$H^2(X, \Theta)^* \cong H^0(X, \Theta^* \otimes \Omega^2)$$

denoting by Ω^2 the sheaf of holomorphic 2-forms on X. In order to prove the vanishing of $H^0(X, \Theta^* \otimes \Omega^2)$ we consider the evaluation map

$$\langle -, - \rangle : H^0(X, \Theta) \times H^0(X, \Theta^* \otimes \Omega^2) \to H^0(X, \Omega^2).$$

The table in Remark 2 shows the existence of two vector fields

$$\theta_1, \theta_2 \in H^0(X, \Theta)$$

defining a base of the tangent space at the points of an dense open subset of X (this means that every Hopf surface is almost homogeneous; cf. J. Potters [9]). Assume there is a section

$$\sigma \in H^0(X, \Theta^* \otimes \Omega^2)$$

different from zero. Then there are complex numbers λ_1, λ_2 with

$$0 \neq \langle \alpha_1 \theta_1 + \alpha_2 \theta_2, \sigma \rangle \in H^0(X, \Omega^2).$$

On the other hand we have

$$H^0(X, \Omega^2) = 0$$

since there exists a meromorphic two-form having only poles but no zeros, cf. Kodaira [3]. This contradiction implies the vanishing of $H^2(X, \Theta)$. (The vanishing of $H^0(X, \Omega^2)$ has already been demonstrated in Remark 3. It depends only on the topology of $X \cong S^1 \times S^3$, but the proof uses deep results from the theory of compact complex surfaces.)

In order to prove the first part of the assertion we use the theorem of Riemann-Roch

$$\chi(\Theta) := \sum_{j=0}^{2} (-1)^j \dim H^j(X, \Theta) = (\mathrm{ch}\,\Theta \cdot \mathrm{td}\,X)\,[X].$$

From the explicit formulas

$$\mathrm{ch}\,\Theta = 2 + c_1 + \tfrac{1}{2}(c_1^2 - 2c_2),$$
$$\mathrm{td}\,X = 1 + \tfrac{1}{2}c_1 + \tfrac{1}{12}(c_1^2 + c_2)$$

and the vanishing of all Chern numbers we get $\chi(\Theta) = 0$. The equation $H^2(X, \Theta) = 0$ demonstrated in part one implies

$$\dim H^0(X, \Theta) = \dim H^1(X, \Theta).$$

II. Calculations on the universal covering

In the following we want to compare the cohomology groups of the Hopf surface $X = W/\langle f \rangle$ with the cohomology of the universal covering $W = \mathbb{C}^2 \setminus \{0\}$. Therefore we construct a suitable sequence of complexes; cf. Douady [2]. Take a finite covering $\mathfrak{U} = (U_i)_{i \in I}$ of X where every U_i is an open Stein subset of X and has the additional property: the inverse image $\tilde{U}_i := \nu^{-1}(U_i)$ relative to the canonical projection

$$\nu\colon W \to X$$

splits up into a disjoint union

$$\tilde{U}_i = \dot{\bigcup_{m \in \mathbb{Z}}} f^m(U_i'),$$

and the canonical projection induces a homeomorphism

$$\nu | U_i' \to U_i.$$

The family $\tilde{\mathfrak{U}} := (\tilde{U}_i)_{i \in I}$ constitutes an open covering of W and the mapping $f\colon W \to W$ induces a morphism

$$f_*\colon \Gamma(\tilde{U}_i, \Theta) \to \Gamma(\tilde{U}_i, \Theta)$$

for every $i \in I$. There results a short exact sequence

$$0 \longrightarrow C^\bullet(\mathfrak{U}, \Theta) \longrightarrow C^\bullet(\tilde{\mathfrak{U}}, \Theta) \xrightarrow{\mathrm{id} - f_*} C^\bullet(\tilde{\mathfrak{U}}, \Theta) \longrightarrow 0$$

which determines the long exact sequence

$$(1)\quad 0 \longrightarrow H^0(X, \Theta) \longrightarrow H^0(W, \Theta) \xrightarrow{\mathrm{id} - f_*} H^0(W, \Theta) \longrightarrow H^1(X, \Theta) \longrightarrow \cdots.$$

By Hartogs theorem every element of $H^0(W, \Theta)$ extends to a holomorphic vector field on $\mathbb{C}^2$, which has a convergent power series expansion. Taking

$$\theta = g \frac{\partial}{\partial z} + h \frac{\partial}{\partial w} \in H^0(W, \Theta)$$

as given we want to calculate

$$(\mathrm{id} - f_*)\theta = \tilde{g} \frac{\partial}{\partial z} + \tilde{h} \frac{\partial}{\partial w} \in H^0(W, \Theta)$$

hereby denoting

$$g(z, w) = \sum_{\mu, \nu} a_{\mu\nu} z^\mu w^\nu, \quad h(z, w) = \sum_{\mu, \nu} b_{\mu\nu} z^\mu w^\nu,$$

$$\tilde{g}(z, w) = \sum_{\mu, \nu} \tilde{a}_{\mu\nu} z^\mu w^\nu, \quad \tilde{h}(z, w) = \sum_{\mu, \nu} \tilde{b}_{\mu\nu} z^\mu w^\nu.$$

For the two cases relative to the classification of Theorem 1, we are interested in, a short calculation shows:

Type III, i.e. $f(z, w) = (\delta^p z, \delta w)$ with $p \in \mathbb{N} \setminus \{0, 1\}$

$$\tilde{a}_{\mu\nu} = a_{\mu\nu}[1 - \delta^{p(1-\mu)-\nu}], \quad \tilde{b}_{\mu\nu} = b_{\mu\nu}[1 - \delta^{-p\mu+1-\nu}].$$

Type IIa, i.e. $f(z, w) = (\delta^p z + w^p, \delta w)$ with $p \in \mathbb{N} \setminus \{0, 1\}$

$$\tilde{a}_{\mu\nu} = a_{\mu\nu} - \sum \binom{m}{\mu} (-1)^{m-\mu} \delta^{p(1-m)-\nu} a_{m, \nu + p(\mu - m)}$$

$$- p \sum \binom{m}{\mu} (-1)^{m-\mu} \delta^{p(1-m)-\nu-1} b_{m, \nu+1+p(\mu-m-1)}$$

$$\tilde{b}_{\mu\nu} = b_{\mu\nu} - \sum \binom{m}{\mu} (-1)^{m-\mu} \delta^{-pm+1-\nu} b_{m, \nu+p(\mu-m)},$$

the summation is restricted to the conditions

$$m \geqq \mu \quad \text{and} \quad a_{k\lambda} = b_{k\lambda} = 0 \quad \text{if} \quad k < 0 \quad \text{or} \quad \lambda < 0.$$

Lemma 2. *The connecting homomorphism*

$$\sigma \colon H^0(W, \Theta) \to H^1(X, \Theta)$$

in the above-mentioned sequence (1) *is surjective for any Hopf surface X. In particular for*

$$M := \operatorname{coker}(\mathrm{id} - f_* \colon H^0(W, \Theta) \to H^0(W, \Theta))$$

the following representation is valid:

$$\text{Type III} \quad M \cong \operatorname{span}_{\mathbb{C}} \left\langle z \frac{\partial}{\partial z}, \; w^p \frac{\partial}{\partial z}, \; w \frac{\partial}{\partial w} \right\rangle,$$

$$\text{Type II}_a \quad M \cong \operatorname{span}_{\mathbb{C}} \left\langle (\delta^p z - w^p) \frac{\partial}{\partial z}, \; w \frac{\partial}{\partial w} \right\rangle.$$

Proof. Using the previously stated formulas to calculate for type III and II_a the image

$$(\mathrm{id} - f_*)\theta \quad \text{where } \theta \in H^0(W, \Theta)$$

one shows that the vector fields given in the Lemma are linearly independent modulo $\operatorname{im}(\mathrm{id} - f_*)$. Hence Theorem 1 implies the inequality

$$\dim M \geqq \dim H^1(X, \Theta).$$

On the other hand one obviously has

$$\dim M = \dim(\operatorname{im} \sigma) \leqq \dim H^1(X, \Theta),$$

therefore, the assertion

$$M \cong H^1(X, \Theta)$$

follows for types III and II_a.

In any other case of the classification the space $H^0(X, \Theta)$ is already contained in the subspace L of those vector fields on W having linear coefficients,

$$L = \left\{ g \frac{\partial}{\partial z} + h \frac{\partial}{\partial w} \in H^0(W, \Theta) \colon g, h \text{ polynomial of total degree} \leqq 1 \right\}.$$

The set L is a finite dimensional subspace of $H^0(W, \Theta)$; a complement is given by the vector fields having a zero of order at least two at the point $0 \in W$:

$$N = \left\{ g \frac{\partial}{\partial z} + h \frac{\partial}{\partial w} \in H^0(W, \Theta) \colon g(0) = h(0) = 0 \quad \text{and} \quad Dg(0) = Dh(0) = 0 \right\}.$$

Because the Hopf surface is now defined by a linear automorphism f of W, the mapping $\mathrm{id} - f_*$ leaves invariant both vector spaces L and N, hence

$$\ker(\sigma|L) = L \cap \ker\sigma .$$

The surjectivity of the map σ results from the exact sequence

$$0 \longrightarrow H^0(X, \Theta) \longrightarrow L \xrightarrow{\mathrm{id}-f_*} L \xrightarrow{\sigma|L} H^1(X, \Theta)$$

and the equality

$$\dim H^0(X, \Theta) = \dim H^1(X, \Theta)$$

derived in Lemma 1.

III. The versal deformation of Hopf surfaces

We are now ready to describe the versal deformation of all Hopf surfaces.

Theorem 2. *The versal deformation*

$$\pi\colon Y \longrightarrow (S, s_0)$$

of a Hopf surface $X = W/\langle f\rangle$ can be constructed as follows: The base S is a smooth manifold. There exists a biholomorphic map

$$W \times S \longrightarrow W \times S, \quad (x, s) \mapsto \big(F(x, s), s\big) \quad \textit{with} \quad f(x) = F(x, s_0)$$

generating an infinite cyclic group G. This group acts properly discontinuous and without fixed points on $W \times S$ and induces

$$\pi\colon Y \longrightarrow (S, s_0)$$

as canonical projection from the factor space $Y = (W \times S)/G$. According to the type of X as defined by Theorem 1 the base S and the map

$$F\colon W \times S \longrightarrow W$$

are given by the explicit formulas:

type	
IV	$S = \{s \in GL(2, \mathbb{C})\colon$ all eigenvalues of modulus $< 1\}$ $F(x, s) = sx$
III	$S = \{(\alpha, \delta, \lambda) \in \mathbb{C}^3\colon 0 < \lvert\alpha\rvert < \lvert\delta\rvert < 1\}$ $F\big((z, w), (\alpha, \delta, \lambda)\big) = (\alpha z + \lambda w^p, \delta w)$
$\mathrm{II_a}$	$S = \{(\alpha, \delta) \in \mathbb{C}^2\colon 0 < \lvert\alpha\rvert < \lvert\delta\rvert < 1\}$ $F\big((z, w), (\alpha, \delta)\big) = (\alpha z + w^p, \delta w)$
$\mathrm{II_b}$	$S = \left\{s = \begin{pmatrix} \alpha & 1 \\ \beta & \alpha \end{pmatrix} \in GL(2, \mathbb{C})\colon \text{all eigenvalues of modulus} < 1\right\}$ $F(x, s) = sx$
$\mathrm{II_c}$	$S = \left\{s = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \in GL(2, \mathbb{C})\colon 0 < \lvert\alpha\rvert < \lvert\delta\rvert < 1\right\}$ $F(x, s) = sx$

In order to prove the theorem we have to use the following Lemma.

Lemma 3 (Kodaira-Spencer map of the versal deformation). *Under the hypotheses of Theorem 2 assume $s \in S$, let*

$$f\colon \mathbb{C}^2 \to \mathbb{C}^2$$

be given by $f(x) := F(x, s)$ and set $X := W/\langle f\rangle$. Denote by

$$\rho\colon T_sS \to H^1(X, \Theta)$$

the Kodaira-Spencer map of the family

$$\pi\colon Y \to S$$

at the basepoint $s \in S$. Now a linear map

$$\tau\colon T_sS \to H^0(W, \Theta)$$

can be defined by

$$\tau(v)\,(x) := a \cdot {}^t\!A\bigl(f^{-1}(x)\bigr) \cdot \frac{\partial}{\partial x},$$

where $v = a\dfrac{\partial}{\partial s} \in T_sS$ and $A(y) := \dfrac{\partial F}{\partial s}(y, s)$, which renders the following diagram commutative

$$\begin{tikzcd} H^0(W, \Theta) \arrow[r, "\sigma"] & H^1(X, \Theta) \\ T_sS \arrow[u, "\tau"] \arrow[ur, "\rho"'] & \end{tikzcd}$$

Proof. Every tangent vector $v \in T_sS$ can be lifted to a cochain of projectable vector fields on a suitable covering of the fibre X. On the intersections the coboundary defines a vector field which is tangent to the fibre and the resulting cocycle represents the image $\rho(v) \in H^1(X, \Theta)$.

We use the above mentioned coverings

$$\mathfrak{U} = (U_i)_{i \in I} \text{ of } X \text{ and } \tilde{\mathfrak{U}} = (\tilde{U}_i)_{i \in I} \text{ of } W,$$

where $\tilde{U}_i = \dot{\bigcup\limits_m} f^m U_i'$.

If

$$v = a\frac{\partial}{\partial s} \in T_sS$$

is given, we construct a family of projectable vector fields

$$\tilde{\eta}_i \in \Gamma(\tilde{U}_i, \Pi), \quad i \in I,$$

setting

$$\tilde{\eta}_i := a\frac{\partial}{\partial s} \quad \text{on} \quad U_i', \text{ resp.}$$

$$\tilde{\eta}_i := G_*^m \tilde{\eta}_i \quad \text{on} \quad f^m U_i',\ m \in \mathbb{Z}, \text{ where } G(x, s) := \bigl(F(x, s), s\bigr).$$

The explicit notation

$$\tilde{\eta}_i = (\beta_i \quad a)\begin{pmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial s} \end{pmatrix}$$

shows that we have defined a cochain

$$\beta = \left(\beta_i \frac{\partial}{\partial x}\right) \in C^0(\tilde{\mathfrak{U}}, \Theta).$$

From

$$G_* \tilde{\eta}_i = \tilde{\eta}_i \quad \text{and} \quad \partial G = \begin{pmatrix} \dfrac{\partial f}{\partial x} & A \\ 0 & 1 \end{pmatrix}, \quad A := \frac{\partial F}{\partial s},$$

follows the transformation law

$$(\mathrm{id} - f_*)\beta = a \cdot {}^t A(f^{-1}) \cdot \frac{\partial}{\partial x}.$$

In particular the cocycle

$$(\beta_j - \beta_i) \frac{\partial}{\partial x} \in Z^1(\tilde{\mathfrak{U}}, \Theta)$$

is invariant with respect to the map $\mathrm{id} - f_*$. By definition of the connecting morphism σ the cocycle represents the image

$$\sigma\left(a \cdot {}^t A(f^{-1}) \cdot \frac{\partial}{\partial x}\right) \in H^1(\mathfrak{U}, \Theta) \cong H^1(X, \Theta).$$

On the other hand the invariant cocycle

$$\tilde{\theta} = (\tilde{\theta}_{ij}) \in Z^1(\tilde{\mathfrak{U}}, \Theta) \quad \text{given by} \quad \tilde{\theta}_{ij} := (\tilde{\eta}_j - \tilde{\eta}_i) \frac{\partial}{\partial x} = (\beta_j - \beta_i) \frac{\partial}{\partial x}$$

defines the cocycle $\theta \in Z^1(\mathfrak{U}, \Theta)$ on X which represents the image

$$[\theta] = \rho(v) \in H^1(X; \Theta)$$

of the Kodaira-Spencer map.

Proof of Theorem 2. If the Hopf surface X belongs to type IV, II_b or II_c the theorem is due to Kodaira-Spencer [5]. Douady [2] gives a short proof of this fact. In the following we will generalize his ideas to include types III and II_a.

Denote by $X = W/\langle f \rangle$ a Hopf surface of type III. The defining automorphism is given by

$$f\colon W \to W, \; f(z, w) = (\delta^p z, \delta w)$$

with constants

$$0 < |\delta| < 1 \quad \text{and} \quad p \in \mathbb{N} \setminus \{0, 1\}.$$

We set

$$s_0 = (\delta^p, \delta, 0) \in S$$

and consider the commutative diagram from Lemma 3

$$\begin{tikzcd}
H^0(W, \Theta) \arrow[r, "\sigma"] & H^1(X, \Theta) \\
T_{s_0}S \arrow[u, "\tau"] \arrow[ur, "\rho"'] &
\end{tikzcd}$$

Explicit calculation shows that τ induces a biholomorphic map

$$\tau\colon T_{s_0}S \to M := \operatorname{span}_{\mathbb{C}} \left\langle z \frac{\partial}{\partial z}, w^p \frac{\partial}{\partial z}, w \frac{\partial}{\partial w} \right\rangle.$$

By Lemma 2 the restriction

$$\sigma | M \to H^1(X, \Theta)$$

and, therefore, the Kodaira-Spencer map $\rho = \sigma \circ \tau$ is an isomorphism as well. Hence the family

$$\pi\colon Y \to (S, s_0)$$

is versal at the point $s_0 \in S$.

If $X = W/\langle f \rangle$ is a Hopf surface of type II_a we have

$$f(z, w) = (\delta^p z + w^p, \delta w)$$

with constants

$$0 < |\delta| < 1 \quad \text{and} \quad p \in \mathbb{N} \setminus \{0, 1\}.$$

In order to show the versality of the family

$$\pi\colon Y \to (S, s_0)$$

at the point $s_0 := (\delta^p, \delta) \in S$ one has to verify the bijectivity of the map

$$\tau\colon T_{s_0}S \to M := \operatorname{span}_{\mathbb{C}} \left\langle (\delta^p z - w^p) \frac{\partial}{\partial z}, w \frac{\partial}{\partial w} \right\rangle.$$

The rest of the proof goes along the same lines as in the previous case.

Concluding remark. There is a rough classification of Hopf surfaces according to the dimension of the space of holomorphic vector fields. The possibility to transform one type into another by small deformation is confined to the following pattern:

$$\begin{tikzcd}
\mathrm{IV} \arrow[r] \arrow[drr] & \mathrm{II}_b \arrow[dr] & & \mathrm{II}_a \arrow[dl] & \mathrm{III} \arrow[l] \arrow[dll] \\
& & \mathrm{II}_c & &
\end{tikzcd}$$

To each Hopf surface we have constructed in Theorem 2 a deformation which is versal at the distinguished point. By a general theorem the family is complete even in an open neighborhood, cf. J. Bingener [1]. The versality of the families constructed to Hopf surfaces of type IV and III is, however, restricted to the distinguished points because the number of infinitesimal deformations of the fibre takes different values on the complement of a proper analytic subset.

Added in Proof. After completion of the present paper I came to know the manuscript of

Borcea, C.: Some remarks on Deformations of Hopf Manifolds; Bucharest.

Herein the author constructs inter alia the versal deformation of Hopf manifolds defined by diagonal matrices.

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