LOCAL FUNCTIONALS FROM FORMAL VECTOR FIELDS

1. LOCAL FUNCTIONALS FROM GELFAND-FUKS COHOMOLOGY

1.1. **Gelfand-Fuks cohomology.** In this section we recall some facts about the Lie algebra cohomology of formal vector fields W_d on the d-disk with values in certain non-trivial modules. We refer to Section **??** for the requisite notation for objects living on the formal disk.

In Section **??** we have constructed the formal Atiyah class for any formal vector bundle $\mathcal{V}$ on $\widehat{D}^n$. It is an element of the relative Gelfand-Fuks cohomology

$$\mathrm{At}^{\mathrm{GF}}(\mathcal{V}) \in C^1_{\mathrm{Lie}}(W_d, \mathrm{GL}_d; \widehat{\Omega}^1_d \otimes_{\widehat{\mathcal{O}}_d} \mathrm{End}_{\widehat{\mathcal{O}}_d}(\mathcal{V})).$$

From the Atiyah class we have built the formal Chern character using the usual formula

$$\mathrm{ch}^{\mathrm{GF}}(\mathcal{V}) = \mathrm{Tr}\left(\exp\left(\frac{1}{2\pi i}\mathrm{At}^{\mathrm{GF}}(\mathcal{V})\right)\right),$$

and have studied how components of this formal Chern character give rise to L_∞ extensions of W_d that appear as natural universal symmetries of quantizations of higher dimensional holomorphic σ models with target $\widehat{D}^d$.

In this section we arrive at the Lie algebra of formal vector fields, and its cohomology, from a different perspective. Instead of using formal geometry to construct universal objects on the *target* of a σ model, we will see how Gelfand-Fuks classes characterize holomorphic symmetries on the higher *world-sheet*, or source manifold.

The symmetry is that of holomorphic reparametrizations. Infinitesimally, this is described by the Lie algebra of holomorphic vector fields. We have already seen BW: ref that classical theories on a complex manifold X with such a symmetry by holomorphic reparametrizations admit an action by the local Lie algebra $\mathcal{T}_X = \Omega^{0,*}(X, T_X^{1,0})$.

The Gelfand-Fuks classes we will consider in this section appear as anomalies for quantizing an action by the local Lie algebra $\mathcal{T}_X$. In other words, these classes parametrize shifted central extensions of $\mathcal{T}_X$, just as the classes $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$ defined central extensions of the current algebra $\mathfrak{g}^X$. By our usual yoga of studying equivariant quantizations, we know such anomalies live in the local cohomology complex $C^*_{\mathrm{loc}}(\mathcal{T}_X)$.

Definition/Lemma 1. Consider the following two classes of cocycles on W_d.

Chern type: For $1 \leq k \leq n$, let $\tau_k \in C^k_{\mathrm{Lie}}(W_d; \widehat{\Omega}^k_d)$ be the cocycle

$$\tau_k = \sigma_k\left(\mathrm{At}^{\mathrm{GF}}(\widehat{\mathcal{T}}_d)\right) ...finish...$$

GL type: For $1 \leq i \leq d$ let $\xi_i \in C^{2i-1}_{\mathrm{Lie}}(W_d; \widehat{\mathcal{O}}_d)$ be the cocycle

$$\xi_i : (f_{i_1}\partial_{i_1}, \ldots, f_{i_{2i-1}}\partial_{i_{2i-1}}) \mapsto \sum ..$$

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We will use the notation $\widehat{\Omega}_d^{\#} = \oplus_k \widehat{\Omega}_d^k[-k]$ to denote the graded W_d-module with $\widehat{\Omega}_d^k$ sitting in degree k. The wedge product of forms endows this W_d-module with the structure of a graded commutative algebra.

If V is a graded vector space then we use the notation $\mathbb{C}[V]$ to denote the free graded $\mathbb{C}$-algebra on V. If V is spanned by vectors $\{v_i\}$ we will use the shorthand $\mathbb{C}[v_i]$ for this graded algebra.

Theorem 1.1 ([?]). *The bigraded commutative algebra $H^*(W_d; \widehat{\Omega}_d^{\#})$ is isomorphic to the bigraded commutative algebra*

$$(\mathbb{C}[\xi_1, \ldots, \xi_{2d-1}, \tau_1, \ldots, \tau_d]) / \left(c_1^{j_1} \cdots c_d^{j_d} \right),$$

where the quotient is over all indices $\{j_1, \ldots, j_d\}$ that satisfiy $j_1 + 2j_2 + \cdots + dj_d > d$. Here ξ_{2i-1} is in bidegree $(2i-1, 0)$ and τ_j is in bidegree (j, j).

In the above result we have not turned on the de Rham differential $d_{dR} : \widehat{\Omega}_d^k \to \widehat{\Omega}_d^{k+1}$. This endows $\widehat{\Omega}_d^* = (\widehat{\Omega}_d^{\#}, d_{dR})$ with the structure of a dg commutative algebra in W_d-modules. The formal Poincaré lemma asserts that the inclusion of the trivial W_d-module

$$\mathbb{C} \xrightarrow{\simeq} \widehat{\Omega}_d^*$$

is a quasi-isomorphism. In turn, we obtain a quasi-isomorphism of Chevalley-Eilenberg complexes

$$C_{\mathrm{Lie}}^*(W_d) \xrightarrow{\simeq} C_{\mathrm{Lie}}^*(W_d; \widehat{\Omega}_d^*).$$

We may think of the cochain complex $C_{\mathrm{Lie}}^*(W_d; \widehat{\Omega}_d^*)$ as the total complex of the double complex with vertical differential given by the W_d Chevalley-Eilenberg differential for the graded module $\widehat{\Omega}_d^{\#}$, and horizontal differential equal to the de Rham differential.

To any double complex there is a spectral sequence abutting to the cohomology of the total complex. The E_1 page of this spectral sequence is given by the cohomology of the vertical differential. Moreover, if the double complex is a bigraded algebra so are each of the pages. In this case, the E_1 page is precisely the bigraded algebra of Theorem **??** and we have a spectral sequence

(1) $$E_2^{p,q} = \left(H^q(W_d; \widehat{\Omega}_d^p), d_{dR} \right) \implies H^*(W_d; \widehat{\Omega}_d^*) \cong H^*(W_d).$$

Example 1.2. For the case $d = 1$ the spectral sequence collapses at the E_2 page. The only nontrivial cohomology is $\mathbb{C}$ in bidegree $(0,0)$ and $\xi_1 \cdot \tau_1$ in bidegree $(1,2)$. The 1-cocycle valued in formal power series ξ_1 is given by $\xi_1(f_i \partial_i) = \partial_i f_i \in \widehat{\mathcal{O}}_n$. The 1-cocycle valued in formal 1-forms τ_1 is given by $\tau_1(g_j \partial_j) = d_{dR}(\partial_j g_j)$. To obtain the generator of $H^3(W_1)$ we perform the following zig-zag:

$$
\begin{array}{ccc}
C_{\mathrm{Lie}}^3(W_1) & \longrightarrow & C_{\mathrm{Lie}}^3(W_1; \widehat{\mathcal{O}}_1) \\
& & \Big\uparrow{\scriptstyle d_{CE}} \\
& & C_{\mathrm{Lie}}^2(W_1; \widehat{\mathcal{O}}_1) \xrightarrow{d_{dR}} C_{\mathrm{Lie}}^2(W_1; \widehat{\Omega}_1^1).
\end{array}
$$

The de Rham differential kills $\xi_1 \cdot \tau_1$, so there exists an $\alpha \in C_{\mathrm{Lie}}^2(W_1; \widehat{\mathcal{O}}_1)$ such that $d_{dR}\alpha = -\xi_1 \cdot \tau_1$. Now, the class $d_{CE}^{\widehat{\mathcal{O}}}\alpha \in C_{\mathrm{Lie}}^3(W_1; \widehat{\mathcal{O}}_n)$ satisfies

$$d_{dR}(d_{CE}^{\widehat{\mathcal{O}}}\alpha) = -d_{CE}(\xi_1 \tau_1) = 0$$

$$d_{CE} d_{CE}^{\widehat{\mathcal{O}}}\alpha = 0.$$

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Here, $d_{CE}^{\widehat{\mathcal{O}}}$ denote the Chevalley-Eilenberg differential for $C_{\text{Lie}}^*(W_1; \widehat{\mathcal{O}}_1)$ and d_{CE} is the restriction of this Chevalley-Eilenberg differential to $C_{\text{Lie}}^*(W_1)$. The first line says that $d_{CE}\alpha$ lifts to $C_{\text{Lie}}^3(W_1)$, and the second line says that it is a cocycle for the absolute cohomology. Finally, note that $(d_{CE}^{\widehat{\mathcal{O}}} + d_{dR})\alpha = d_{CE}^{\widehat{\mathcal{O}}}\alpha - \xi_1\tau_1$. Thus, in the total complex $d_{CE}^{\widehat{\mathcal{O}}}\alpha$ is homotopic to $\xi_1\tau_1$, and so $[d_{CE}^{\widehat{\mathcal{O}}}\alpha]$ is the generator of $H^3(W_1)$.

For general $d \geq 1$, one can apply this spectral sequence to understand the cohomology $H^*(W_d)$. To describe it, we introduce the following topological space. Let $\text{Gr}(d, n)$ be the complex Grassmannian of d-planes in $\mathbb{C}^n$. Denote by $\text{Gr}(d, \infty)$ the colimit of the natural sequence

$$\text{Gr}(d, d) \to \text{Gr}(d, d+1) \to \cdots .$$

It is a standard fact that $\text{Gr}(d, \infty)$ is a model for the classifying space $BU(d)$ of principal $U(d)$-bundles. Let $EU(d) \to BU(d)$ be the universal principal $U(d)$-bundle. Using the colimit description above, we have a natural skeletal filtration of $BU(d)$ by

$$\text{sk}_k BU(d) = \text{Gr}(d, k).$$

Let X_d denote the restriction of $EU(d)$ over the $2d$-skeleton:

$$
\begin{array}{ccc}
X_d & \longrightarrow & EU(d) \\
\downarrow & & \downarrow \\
\text{sk}_{2d} BU(d) & \longrightarrow & BU(d).
\end{array}
$$

Remark 1.3. Though not the way the Gelfand and Fuks originally proved the result, one can use the computation of the cohomology of W_d with coefficients in $\widehat{\Omega}_d^k$ together with the spectral sequence (??) to prove this description of $H^*(W_d)$. Indeed, the spectral sequence (??) is isomorphic, up to regradings, to the Serre spectral sequence for the principal $U(d)$-bundle $X_d \to \text{sk}_{2d} BU(d)$. In other words, the formal de Rham differential on $\widehat{\Omega}_d^*$ is exactly the E_2 differential for the Serre spectral sequence.

Theorem 1.4 ([?] Theorem 2.2.4). *There is an isomorphism of graded vector spaces*

$$H^*(W_d) \cong H_{dR}^*(X_d).$$

Moreover, the commutative algebra structure on $H^*(W_d)$ is trivial.

As a simple example, note that when $d = 1$ we have $\text{sk}_2 BU(1) = \mathbb{P}^1 \subset \mathbb{P}^\infty = BU(1)$. Moreover, the restriction of the universal bundle is Hopf fibration $U(1) \to S^3 \to \mathbb{P}^1$. In particular, one has $X_1 = S^3$.

1.2. **Local cocycles on holomorphic vector fields.** We now turn to a description of local central extensions of the local Lie algebra of holomorphic vector fields $\mathcal{T}_X = \Omega^{0,*}(X; T_X^{1,0})$ for any complex d-fold X. Recall, such a central extension is determined by a cocycle in complex of local functionals $C_{\text{loc}}^*(\mathcal{T}_X)$. Our main result is to identify such local cocycles with Gelfand-Fuks cocycles we have just discussed.

Our first goal is to construct, from a Gelfand-Fuks class in $C_{\text{Lie}}^*(W_d)$, a local functional on $\mathcal{T}_X$. We have seen that the cochain complex $C_{\text{Lie}}^*(W_d; \widehat{\Omega}_d^*)$, equipped with the total differential

$d_{CE} + d_{dR}$, computes the absolute Gelfand-Fuks cohomology $H^*(W_d)$. We will use this property to represent elements of $H^*(W_d)$ by local cocycles on $\mathcal{T}_X$.

Using the natural framing on the formal disk, we can decompose a class $\alpha \in C_{\text{Lie}}^k(W_d; \widehat{\Omega}_d^*)$ as

$$\alpha = f^I dt_I$$

where the sum is over the multi-index $I = (i_1, \ldots, i_k)$ where $1 \le i_j \le d$, and for each I, f^I is a k multi-linear symmetric functional on W_d valued in $\widehat{\mathcal{O}}_d$

$$f^I : \text{Sym}^k(W_d[1]) \to \widehat{\mathcal{O}}_d.$$

We extend f^I to a functional on the Dolbeault complex $\Omega^{0,*}(\mathbb{C}^d; T^{1,0}\mathbb{C}^d)$ as follows. Using the framing on $\mathbb{C}^d$, every element of the Dolbeualt complex can be written as

$$X^J(z, \bar{z}) d\bar{z}_J$$

where $J = (j_1, \ldots, j_l)$ is a multi-index and X^J is an ordinary holomorphic vector field on $\mathbb{C}^d$. We extend f^I to a Dolbeualt valued functional $\Omega^{0,*}(\mathbb{C}^d; T^{1,0}\mathbb{C}^d)$ via the formula

$$f_{\Omega^{0,*}}^I : \qquad \text{Sym}^k\left(\Omega^{0,*}(\mathbb{C}^d; T^{1,0})\right) \qquad \to \qquad \Omega^{0,*}(\mathbb{C}^d)$$
$$\left(X_1^{J(1)}(z,\bar{z})d\bar{z}_{J(1)}, \ldots, X_k^{J(k)}(z,\bar{z})d\bar{z}_{J(k)}\right) \quad \mapsto \quad f^I(X_1^{J(1)}, \ldots, X_k^{J(k)})d\bar{z}_{J(1)} \wedge \cdots d\bar{z}_{J(k)}$$

The local functional corresponding to the original class $\alpha = f^I dt_I \in C_{\text{Lie}}^*(W_n; \widehat{\Omega}_d^*)$ is defined by the k-multi-linear functional

$$(\xi_1, \ldots, \xi_k) \mapsto \int_{\mathbb{C}^d} f_{\Omega^{0,*}}^I(\xi_1, \ldots, \xi_k) dz_I.$$

Denote this functional by $J^{GF}(\alpha)$. Note that it is only nonzero when the multi-index I is a permutation of $(1, \ldots, d)$. Since it is given by the integral of a some multi-differential operators against a density it is manifestly a local functional.

Proposition 1.5. *Let $C_{\text{loc}}^*(\mathcal{T}_{\mathbb{C}^d})$ be the local functionals of $\mathcal{T}_{\mathbb{C}^d}$ on $\mathbb{C}^d$. The map*

$$J^{GF} : C_{\text{Lie}}^*(W_d; \widehat{\Omega}_n^*)[2d] \to C_{\text{loc}}^*(\mathcal{T}_{\mathbb{C}^d})$$

sending $\alpha \mapsto J^{GF}(\alpha)$ is a map of cochain complexes. Moreover, it is a quasi-isomorphism.

Theorem 1.6. *Let X be a complex d-fold. Then, the map*

$$J^{GF} : \underline{C_{\text{Lie}}^*(W_d; \widehat{\Omega}_n^*)}[2d] \to C_{\text{loc}}^*(\mathcal{T}_X)$$

is a quasi-isomorphism of sheaves. In particular, there is an isomorphism of graded vector spaces

(2) $$H^{*+2d}(W_n) \cong H^*(X, C_{\text{loc}}^*(\mathcal{T}_X)),$$

where the right-hand side denotes the hypercohomology.

Lemma 1.7. *On any complex manifold X of complex dimension d there is a quasi-isomorphism of sheaves of cochain complexes*

$$C_{\text{loc}}^*(\mathcal{T}_X) \simeq \underline{C_{\text{Lie}}^*(W_d)}[2d]$$

Proof. We recall a description of the local cohomology complex using D-modules given in Section 4.5 of [**?**]. We have already seen that

$$C^*_{loc}(\mathcal{L}) = \Omega^{d,d}_X \otimes_{D_X} C^*_{\text{Lie,red}}(JL)$$

where JL is the D_X-module given by taking the ∞-jets of the underlying vector bundle of $\mathcal{L}$. In [**?**] it was shown that $C^*_{\text{Lie,red}}(JL)$ is flat as a D_X-module, thus we can replace the tensor product above by a left-derived tensor product

$$\Omega^{d,d}_X \otimes_{D_X} C^*_{\text{Lie,red}}(JL) \simeq \Omega^{d,d}_X \otimes^{\mathbb{L}}_{D_X} C^*_{\text{Lie,red}}(JL).$$

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To wrap the above discussion up, we obtain a quasi-isormophism

$$C^*_{loc}(\mathcal{L}) \simeq \Omega^* \left(X, C^*_{\text{Lie,red}}(JL) \right) [2d].$$

Any Lie algebra $\mathfrak{g}$ acts on its (reduced) Chevalley-Eilenberg cochains $C^*_{\text{Lie,red}}(\mathfrak{g})$. It is a standard fact that this action is, in fact, homotopically trivial.

This argument transfers over to continuous Lie algebra cohomology. Indeed, applying this to the case W_d, we see that the Lie algebra of formal vector fields acts homotopically trivial on its Gelfand-Fuks cochain complex $C^*_{\text{Lie}}(W_d)$.

Lemma 1.8. *For any $\mathbb{Z}$-graded vector bundle L one has $\mathcal{O}_{red}(JL) \cong J\mathcal{O}(L)$.*

$\square$

Example 1.9. Again, take the case $d = 1$. We can describe the local cocycle corresponding to the generator $H^3(W_1) \cong H^1(\mathcal{T}_{\mathbb{C}})$ explicitly. Recall, the generator of $H^3(W_1)$ came from the element $\xi_1 \tau_1 \in C^2_{\text{Lie}}(W_1; \widehat{\Omega}^1_1)$ on the E_2 page of the spectral sequence (**??**). Using the formulas for ξ_1, τ_1 above, we see that the local functional $J^{GF}(\xi_1 \tau_1)$ is given by

$$\left(f(z,\bar{z}) \frac{\partial}{\partial z}, g(z,\bar{z}) d\bar{z} \frac{\partial}{\partial z} \right) \mapsto \int_{\mathbb{C}} \left(\frac{\partial}{\partial z} f \right) \partial \left(\frac{\partial}{\partial z} g \right) d\bar{z}.$$

For instance, the linear functional $\tau_1 : g(t) \frac{\partial}{\partial t} \mapsto d_{dR}(\partial_t g(t))$ is mapped to the functional on the Dolbeualt complex of holomorphic vector fields given by $g(z,\bar{z}) \frac{\partial}{\partial z} \mapsto \partial(\partial_z g(z,\bar{z}))$.

If we integrate by parts, we can put $J^{GF}(\xi_1 \tau_1)$ in the form $\int f \partial^3_z g dz d\bar{z}$. If one restricts this local functional to the annulus and performs the radial integration, we recover the usual formula for the generator of $H^2(\text{Vect}(S^1))$ BW: citation defining the central extension of the Virasoro Lie algebra. In fact, in [**?**] BW: finish