0.1. **The disk observables.** In this section we give a description of the observables of the $\beta\gamma$ system supported on a d-disk inside $\mathbb{C}^d$. For now, we will only consider the free $\beta\gamma$ system with target a complex vector space V.

0.1.1. *Primaries of a CFT.* In ordinary chiral conformal field theory, there is a collection of operators that, in some sense, generate all other operators. These are called "primary operators" (or primary fields), and are defined by those operators that are killed by the positive part of the Virasoro algebra [**?**], that is, the "lowering operators". To obtain all of the operators one considers the descendants of the primary operators which are obtained by applying the negative part of the Virasoro algebra, or the "raising operators", to the primaries. For example, in the $d = 1$ $\beta\gamma$ system, there are two primary operators:

$$\mathcal{O}_{\gamma,0}(w) : \gamma \mapsto \gamma(w) = \int_{z \in C_w} \frac{\gamma(z)}{z-w} \mathrm{d}z$$
$$\mathcal{O}_{\beta,-1}(w) : \beta \mathrm{d}z \mapsto \beta(w) = \int_{z \in C_w} \frac{\beta(z)}{z-w} \mathrm{d}z,$$

where C_w is any closed contour surrounding w. (The indices $0, -1$ are to indicate the conformal weight.) Consider the operators placed at $w = 0$. We notice that each of these operators are annihilated by the positive half of the Virasoro $L_n = z^{n+1}\partial_z$, $n \geq 0$. The descendants are obtained by iteratively applying the raising operator $L_{-1} = \partial_z$, which in this case is just the infinitesimal translations. Indeed, for each $n \geq 0$ we obtain

$$\mathcal{O}_{\gamma,-n}(w) = \frac{1}{n!}\partial^n \mathcal{O}_{\gamma,0}(w) : \gamma \mapsto \partial_z^n \gamma(z = w)$$
$$\mathcal{O}_{\beta,-n-1}(w) = \frac{1}{n!}\partial^n \mathcal{O}_{\beta,1}(w) : \beta \mathrm{d}z \mapsto \partial_z^n \beta(z = w).$$

There is an S^1 action on $\mathbb{C}$ given by rotations, and this extends to an S^1 action on the $\beta\gamma$ system. In terms of the Virasoro algebra, the infinitesimal action of S^1 is given by the Euler vector field $L_0 = z\partial_z$. There is an induced grading on the factorization algebra of the one-dimensional free $\beta\gamma$ system by the eigenvalues of this S^1 action. Applied to the disk, or local, observables this is precisely the $\mathbb{Z}_{\geq 0}$ conformal weight grading of the chiral CFT. For instance, the operators $\mathcal{O}_{\gamma,-n}(w), \mathcal{O}_{\beta,-n}$ lie in the weight n subspace of the factorization algebra applied to $D(w,r)$ (for any $r > 0$). We will see a similar grading in the higher dimensional holomorphic case.

0.1.2. The $\beta\gamma$ system on $\mathbb{C}^d$ has a symmetry by the group unitary group $U(d)$. Indeed, the fields of the $\beta\gamma$ system are built from sections of certain natural holomorphic vector bundles on $\mathbb{C}^d$. The group $U(d)$ acts by automorphisms on every holomorphic vector bundle, hence it acts on sections via the pull-back.

There is another symmetry that we wish to contemplate. Introduce an action of $U(1)$ on the fields of the theory such that V has weight $q_f \in \mathbb{Z}$ and V^* has weight $-q_f$. The value of the fields γ lie in the vector space V, so these fields are of weight q_f. Conversely, the fields β lie in V^*, so have weight $-q_f$. Since the pairing defining the free theory is only non-zero between a single γ and single β field, the theory is invariant under this symmetry. In the physics literature, this is a so-called "flavor symmetry" of the theory, and so to distinguish it from the other symmetry we will denote this group by $U(1)_f$. This symmetry will be especially relevant when we compute the character of the $\beta\gamma$ system.

Lemma 0.1. *The symmetry by $U(d) \times U(1)_f$ on the classical $\beta\gamma$ system with values in the complex vector space V extends to a symmetry of the factorization algebra of quantum observables* Obs^q.

Proof. The differential on the factorization algebra is of the form $\bar{\partial} + \hbar\Delta$. The operator $\bar{\partial}$ is manifestly equivariant for the action of $U(d)$. Since $U(1)_f$ does not act on spacetime, $\bar{\partial}$ trivially commutes with its action. Further, the action of $U(d)$ is through linear automorphisms, and since the BV Laplacian Δ is a second order differential operator, it certainly commutes with the action of $U(d)$. Likewise, since $U(1)_f$ is compatible with the (-1)-symplectic pairing, it automatically is compatible with Δ. □

We are now ready for the following definition, which we single out the class of local operators we will be most interested in. The eigenvectors of $U(d)$ are labeled by the eigenvectors of a maximal torus, which we will take to be given by the subgroup

$$T^d = \{\mathrm{diag}(q_1, \ldots, q_d) \mid |q_i| = 1\} \subset U(d).$$

We say that an element v of the factorization algebra has weight $(n_1, \ldots, n_k)$ if $(q_1, \ldots, q_d) \cdot v = q_1^{n_1} \cdots q_d^{n_d} v$. We will use the shorthand $\vec{n} = (n_1, \ldots, n_d)$.

Definition 0.2. (1) Let $w \in \mathbb{C}^d$ and $r > 0$. For any vector of non-negative integers $\vec{n} = (n_1, \ldots, n_d)$ denote by

$$\mathrm{Obs}_V^{\mathrm{q}}(r)^{(\vec{n})} \subset \mathrm{Obs}_V^{\mathrm{q}}(D(w, r))$$

the subcomplex of weight $\vec{n}$ elements.

(2) Let

$$\mathrm{Obs}_V^{\mathrm{q}}(r) := \bigoplus_{\vec{n}} \mathrm{Obs}_V^{\mathrm{q}}(r)^{(\vec{n})}$$

where the direct sum is over all vectors of non-negative integers.

Remark 0.3. Note that we have excluded $w \in \mathbb{C}^d$ from the notation above. This is because the $\beta\gamma$ system is translation invariant. BW: say more?

We now introduce the following operators that will be of most relevance for our study of the operator product expansion.

Definition 0.4. BW: define $\mathcal{O}_{\gamma,-\vec{n}}(w; v^*)$

Remark 0.5. The minus sign in $\mathcal{O}_{\gamma,-\vec{n}}$ is purely conventional, and meant to match up with the physics and vertex algebra literature BW: ref.

Lemma 0.6. *Let $r < s$. Then, the factorization structure map for including disks $D(0, r) \subset D(0, s)$ induces a diagram*

$$\begin{tikzcd} \mathrm{Obs}_V^{\mathrm{q}}(D(0,r)) \arrow[r] & \mathrm{Obs}_V^{\mathrm{q}}(D(0,s)) \\ \mathrm{Obs}_V^{\mathrm{q}}(r) \arrow[u] \arrow[r, "\simeq"] & \mathrm{Obs}_V^{\mathrm{q}}(s) \arrow[u] \end{tikzcd}$$

Further, the bottom horizontal map is a quasi-isomorphism.

Proof. The two vertical maps are the inclusions of the $U(d)$-eigenspaces of the observables supported on disks of radius r and s respectively. It follows from Lemma 0.1 that the factorization algebra is $U(d)$-equivariant, so in particular the factorization algebra structure map for the inclusion of disks $D(0,r) \hookrightarrow D(0,s)$ is a map of $U(d)$-representations. Hence, the map restricts to each of the eigenspaces, yielding the diagram.

In **[?]** it is shown in Corollary 5.3.6.4 that for the one-dimensional $\beta\gamma$ system, the lower map above is a quasi-isomorphism. In fact, a similar argument applies to the $\beta\gamma$ system on $\mathbb{C}^d$. Indeed, consider the collection

$$\{\mathcal{O}_{\gamma,-\vec{n}_1}(0;v_1^*)\cdot\mathcal{O}_{\gamma,-\vec{n}_k}(0;v_k^*)\cdot\mathcal{O}_{\beta,-\vec{m}_1}(0;v_1)\cdots\mathcal{O}_{\beta,-\vec{m}_l}(0;v_l)\}.$$

The collection runs over non-negative integers k,l and sequences $\vec{n}_i = (n_{i,1},\ldots,n_{i,d})$, $n_{i,j} \geq 0$ and $\vec{m}_i = (m_{i,1},\ldots,m_{i,d})$, $m_{i,1} \geq 1$. It also runs over vectors v_i, v_j^* in V and V^*, respectively. Now, it follows from Lemma 5.3.6.2 of **[?]** that the above collection form a basis for the cohomology

$$H^*\mathrm{Obs}^{\mathrm{q}}_V(r)^{(\vec{N})} \subset H^*\mathrm{Obs}^{\mathrm{q}}(D(0,r))$$

for any r, where $\vec{N} = (N_1,\ldots,N_d)$

$$N_j = \left(n_{1,j}+\cdots+n_{k,j}\right) + \left(m_{1,j}+\cdots+m_{l,j}\right).$$

The result follows. ∎

0.2. **The character.** In this section we compute the character of the action of $U(d)\times U(1)_f$ on the local observables of the free $\beta\gamma$ system with values in V. By definition, the character is conjugation invariant, so it is completely determined by its value on the subgroup $T^d \times U(1)_f \subset U(d)\times U(1)_f$. Choose the following basis for the maximal torus of $U(d)$:

$$T^d = \{\mathrm{diag}(q_1,\ldots,q_d) \mid |q_i| = 1\} \subset U(d).$$

We label the coordinate on $U(1)_f$ by u. BW: something about filtrations. I.e., why does the "formal character" make sense? We conclude that the character is valued in the power series ring $\mathbb{C}[[q_i^{\pm}, u^{\pm q_f}]]$.

We now turn to the case that the complex dimension $d = 2$, with an aim to compare to the formula for the character of the $\mathcal{N}=1$ supersymmetric chiral multiplet on $\mathbb{R}^4$.

The local operators of the theory are equal to the observables on a complex 2-disk $D^2 \subset \mathbb{C}^2$. By translation invariance it suffices to consider a disk centered at the origin $0 \in \mathbb{C}^2$. When $d = 2$ we use Proposition **??** to read off the cohomology of the disk observables $H^*\mathrm{Obs}^q(D^2)$:

$$\mathrm{Sym}\left((\mathcal{O}^{hol}(D^2)\otimes V)^\vee\right)\otimes\mathrm{Sym}\left((\Omega^{2,hol}(D^2)\otimes V^*)^\vee[-1]\right).$$

Proposition 0.7. *The $U(2)\times U(1)_f$ character of the local operators of the $\beta\gamma$ system on $\mathbb{C}^2$ is equal to*

$$\prod_{n_1,n_2\geq 0}\frac{1-u^{q_f}q_1^{n_1-1}q_2^{n_2-1}}{1-u^{-q_f}q_1^{n_1}q_2^{n_2}} \in \mathbb{C}[[q_1^{\pm},q_2^{\pm},u^{\pm q_f}]]$$

Proof. We will write down a basis for a dense subspace of the observables on a 2-disk. For integers $n_1, n_2 \geq 0$ and elements $v \in V$, $v^* \in V^*$ consider the following linear observables on the 2-disk:

$$\begin{array}{lllll} O_\gamma(n_1, n_2; v^*) & : & \gamma \otimes w & \in \mathcal{O}^{hol}(D^2) \otimes V & \mapsto & \mathrm{ev}(v^*, w) \frac{\partial^{n_1}}{\partial z_1^{n_1}} \frac{\partial^{n_2}}{\partial z_2^{n_2}} \gamma(0) \\ O_\beta(n_1+1, n_2+1; v) & : & \beta \mathrm{d}z_1 \mathrm{d}z_2 \otimes w^* & \in \Omega^{2,hol}(D^2) \otimes V^* & \mapsto & \mathrm{ev}(w^*, v) \frac{\partial^{n_1}}{\partial z_1^{n_1}} \frac{\partial^{n_2}}{\partial z_2^{n_2}} \beta(0). \end{array}$$

Since the field $\gamma \otimes w \in \mathcal{O}^{hol} \otimes V$ has $U(2)$ weight zero, we see that the

For fixed $n_1, n_2 \geq 0$, let $V^*_{n_1,n_2}$ denote the linear span of operators $O_\gamma(n_1, n_2; v^*)$. As a vector space $V^*_{n_1,n_2} \cong V^*$, but we want to remember the weights under $U(2)$. Likewise, for $n_1, n_2 > 0$, let $V_{n_1,n_2} \cong V$ be the linear span of the operators $O_\beta(n_1, n_2; v)$.

There is an injective map of graded vector spaces

$$\mathrm{Sym}\left(\left(\bigoplus_{n_1,n_2 \geq 0} V^*_{n_1,n_2}\right) \oplus \left(\bigoplus_{n_1,n_2>0} V_{n_1,n_2}[-1]\right)\right) \to \mathrm{Sym}\left(\left(\mathcal{O}^{hol}(D^2) \otimes V\right)^\vee \oplus \left(\Omega^{2,hol}(D^2) \otimes V^*\right)^\vee [-1]\right),$$

where the right-hand side is the cohomology of the observables on D^2. Moreover, this map is *dense*. BW: explain

Thus, to compute the character of the local operators it suffices to compute it on the vector space

$$\mathrm{Sym}\left(\left(\bigoplus_{n_1,n_2 \geq 0} V^*_{n_1,n_2}\right) \oplus \left(\bigoplus_{n_1,n_2>0} \oplus V_{n_1,n_2}[-1]\right)\right) \cong \mathrm{Sym}\left(\bigoplus_{n_1,n_2 \geq 0} V^*_{n_1,n_2}\right) \otimes \bigwedge\left(\bigoplus_{n_1,n_2>0} V_{n_1,n_2}\right).$$

We have used the convention that as (ungraded) vector spaces the symmetric algebra of a vector space in odd degree is the exterior algebra. For instance, $\mathrm{Sym}(W[-1]) = \bigwedge(W)$ as ungraded vector spaces. We can further simplify the right-hand side as

$$\bigotimes_{n_1,n_2 \geq 0} (\mathrm{Sym}(V^*_{n_1,n_2})) \bigotimes \bigotimes_{n_1,n_2>0} \left(\bigwedge(V_{n_1,n_2})\right).$$

The character of the symmetric algebra $\mathrm{Sym}(V^*_{n_1,n_2})$ is equal to $(1 - u^{-q_f} q_1^{n_1} q_2^{n_2})^{-1}$ and the character of $\bigwedge(V_{n_1,n_2})$ is equal to $(1 - u^{q_f} q_1^{n_1} q_2^{n_2})$. The formula for character in the statement of the proposition follows from the fact that the character of a tensor product is the product of the characters. $\square$

We have seen in Proposition BW: ref that when the complex dimension $d = 2$, the free $\beta\gamma$ system is equivalent to the holomorphic twist of the free $\mathcal{N} = 1$ chiral multiplet in four dimensions. In [**?**] Equation 5.58 the index for the $\mathcal{N} = 1$ chiral multiplet is computed, and our answer is easily seen to agree with theirs. We conclude that in this instance that under the holomorphic twist the superconformal index was sent to the character of the local observables of the holomorphic theory. We will see BW: ref that this is a general fact about superconformal indices.

BW: Do general case. Relate to elliptic gamma functions. Relate to Witten index, which is the parition function on $S^3 \times S^1$.