

0.1. Holomorphically translation invariant theories. We restrict ourselves to considering theories on the manifold $\mathbb{C}^d = \mathbb{R}^{2n}$ equipped with its standard complex structure. Fix a holomorphic vector bundle V on \mathbb{C}^d and an identification of bundles

$$V \cong \mathbb{C}^d \times V_0$$

where V_0 is the fiber of V at $0 \in \mathbb{C}^d$. We want to consider a classical theory with space of fields given by $\Omega^{0,*}(\mathbb{C}^d, V)$. Moreover, we want this theory to be invariant with respect to the group of translations on \mathbb{C}^d . Per usual, it is easiest to work with the corresponding Lie algebra of translations. Moreover, using the complex structure, we choose a presentation for this Lie algebra as

$$\mathbb{C}^d \cong \text{span} \left\{ \frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i} \right\}_{1 \leq i \leq d}.$$

The first thing we need to do is fix a (-1) -shifted symplectic pairing. To do this, we suppose that we have a translation invariant pairing on V valued in the canonical bundle $K_{\mathbb{C}^d}$. That is, suppose

$$(1) \quad \langle \cdot, \cdot \rangle_V : V \otimes V \rightarrow K_{\mathbb{C}^d}[d-1]$$

is a skew-symmetric bundle map that is equivariant for the Lie algebra of translations. The shift is so that the resulting pairing on the Dolbeault complex is of the appropriate degree. Here, equivariance means that for sections v, v' we have

$$\left\langle \frac{\partial}{\partial z_i} v, v' \right\rangle_V = L_{\partial z_i} \langle v, v' \rangle_V$$

where the right-hand side denotes the Lie derivative applied to $\langle v, v' \rangle_V \in K_{\mathbb{C}^d}$. There is a similar relation for the anti-holomorphic derivatives. We obtain a \mathbb{C} -valued pairing on $\Omega_c^{0,*}(\mathbb{C}^d, V)$ via integration:

$$\int_{\mathbb{C}^d} \circ \langle \cdot, \cdot \rangle_V : \Omega_c^{0,*}(\mathbb{C}^d, V) \otimes \Omega_c^{0,*}(\mathbb{C}^d, V) \xrightarrow{\wedge \cdot \langle \cdot, \cdot \rangle_V} \Omega^{d,*}(\mathbb{C}^d) \xrightarrow{\int} \mathbb{C}.$$

The first arrow is the wedge product of forms combined with the pairing on V . The second arrow is only nonzero on forms of type $\Omega^{d,d}$. Clearly, integration is translation invariant, so that the composition is as well.

This pairing $\Omega^{0,*}(\mathbb{C}^d, V)$ together with the differential $\bar{\partial}$ are enough to define a free theory. However, it is convenient to consider a slightly generalized version of this situation. We want to allow deformations of the differential $\bar{\partial}$ on Dolbeault forms of the form

$$Q = \bar{\partial} + Q^{hol}$$

where Q^{hol} is a holomorphic differential operator of the form

$$(2) \quad Q^{hol} = \sum_I \frac{\partial}{\partial \bar{z}^I} \mu_I$$

where I is some multi-index and $\mu_I : V \rightarrow V$ is a linear map of cohomological degree $+1$. Note that we have automatically written Q^{hol} in a way that it is translation invariant. Of course, for this differential to define a free theory there needs to be some compatibility with the pairing on V . We can summarize this in the following definition, which should be viewed as a slight modification of a free theory to this translation invariant holomorphic setting.

Definition 0.1. A *holomorphically translation invariant free BV theory* is the data of a holomorphic vector bundle V together with

- (1) an identification $V \cong \mathbb{C}^d \times V_0$;
- (2) a translation invariant skew-symmetric pairing $\langle -, - \rangle_V$ as in (1);
- (3) a holomorphic differential operator Q^{hol} as in (2);

such that the following conditions hold

- (1) the induced \mathbb{C} -valued pairing $\int \circ \langle -, - \rangle_V$ is non-degenerate;
- (2) the operator Q^{hol} satisfies $(\bar{\partial} + Q^{hol})^2 = 0$ and

$$\int \langle Q^{hol} v, v' \rangle_V = \pm \int \langle v, Q^{hol} v' \rangle.$$

The first condition is required so that we obtain an actual (-1) -shifted symplectic structure on $\Omega^{0,*}(\mathbb{C}^d, V)$. The second condition implies that the derivation $Q = \bar{\partial} + Q^{hol}$ defines a cochain complex

$$\mathcal{E}_V = (\Omega^{0,*}(\mathbb{C}^d, V), \bar{\partial} + Q^{hol}),$$

and that Q is skew self-adjoint for the symplectic structure. Thus, in particular, \mathcal{E}_V together with the pairing define a free BV theory in the ordinary sense. In the usual way, we obtain the action functional via

$$S(\varphi) = \int \langle \varphi, (\bar{\partial} + Q^{hol}) \varphi \rangle_V.$$

Before going further, we will list some familiar examples.

Example 0.2. The free $\beta\gamma$ system on \mathbb{C}^d . Suppose that

$$V = \underline{\mathbb{C}} \oplus K_{\mathbb{C}^d}[d-1].$$

Let \langle , \rangle_V be the pairing

$$(\underline{\mathbb{C}} \oplus K_{\mathbb{C}^d}) \otimes (\underline{\mathbb{C}} \oplus K_{\mathbb{C}^d}) \rightarrow K_{\mathbb{C}^d} \oplus K_{\mathbb{C}^d} \rightarrow K_{\mathbb{C}^d}$$

sending $(\lambda, \mu) \otimes (\lambda', \mu') \mapsto (\lambda\mu', \lambda'\mu) \mapsto \lambda\mu' + \lambda'\mu$. Finally, let $Q^{hol} = 0$. One immediately checks that this is a holomorphically translation invariant free theory as above. The space of fields can be written as

$$\Omega^{0,*}(\mathbb{C}^d) \oplus \Omega^{d,*}(\mathbb{C}^d)[d-1].$$

We write γ for a field in the first component, and β for a field in the second component. The action functional reads

$$S(\gamma + \beta, \gamma' + \beta') = \int_{\mathbb{C}^d} \beta \wedge \bar{\partial} \gamma' + \beta' \wedge \bar{\partial} \gamma.$$

When $d = 1$ this reduces to the ordinary chiral $\beta\gamma$ system from conformal field theory [BW: ref.](#) We will discuss this higher dimensional version further in Section [BW:](#). For instance, we will see how this theory is the starting block for constructing general holomorphic σ -models.

Of course, there are many variants of the $\beta\gamma$ system that we can consider. For instance, if E is any holomorphic vector bundle we can take

$$V = E \oplus K_{\mathbb{C}^d} \otimes E^\vee$$

where E^\vee is the linear dual bundle. The pairing is constructed as in the case above where we also use the evaluation pairing between E and E^\vee , $\text{ev}_E : E \otimes E^\vee \rightarrow \mathbb{C}$. In this case, the fields are $\gamma \in \Omega^{0,*}(\mathbb{C}^d, E)$ and $\beta \in \Omega^{d,*}(\mathbb{C}^d, E^\vee)[d-1]$. The action functional is simply

$$S(\gamma, \beta) = \int_{\mathbb{C}^d} \text{ev}_E(\beta \wedge \bar{\partial}\gamma).$$

Example 0.3. Topological ... Consider the above example with $Q^{\text{hol}} = \partial$. **BW: finish.**

0.1.1. *Interacting holomorphic theories.* It is convenient to introduce the following set of degree -1 derivations of $\Omega^{0,*}(\mathbb{C}^d)$ given by

$$\bar{\eta}_i := \frac{\partial}{\partial(\text{d}\bar{z}_i)}.$$

The right-hand side is sometimes written using the interior derivative notation $\iota_{\partial/\partial\bar{z}_i}$. By a holomorphic version of “Cartan’s magic formula” these derivations satisfy the relation

$$L_{\frac{\partial}{\partial\bar{z}_i}} = \bar{\partial}\bar{\eta}_i + \bar{\eta}_i\bar{\partial}.$$

In addition, they serve to define homotopies for the following holomorphic version of Poincaré’s lemma. First, consider the algebraic case. Let \mathbb{A}^d be the complex d -dimensional affine space with space of smooth algebraic functions $\mathcal{O}^{\text{alg}, \text{sm}}(\mathbb{A}^d) = \mathbb{C}[z_1, \dots, z_d, \bar{z}_1, \dots, \bar{z}_d]$. Then, we can build the algebraic Dolbeault complex

$$\Omega_{\text{alg}}^{0,*}(\mathbb{A}^d) = \mathbb{C}[z_1, \dots, z_d, \bar{z}_1, \dots, \bar{z}_d][\text{d}\bar{z}_1, \dots, \text{d}\bar{z}_d]$$

where the $\text{d}\bar{z}_i$ ’s are in cohomological degree 1. The Dolbeault differential $\bar{\partial}$ is defined in the same way. Note that the operators $\bar{\eta}_i$ also make sense on $\Omega_{\text{alg}}^{0,*}(\mathbb{A}^d)$.

Lemma 0.4. *The map*

$$\mathbb{C}[z_1, \dots, z_d] \hookrightarrow \Omega_{\text{alg}}^{0,*}(\mathbb{A}^d)$$

is a quasi-isomorphism.

Proof. Note that

$$\Omega_{\text{alg}}^{0,*}(\mathbb{A}^d) \cong \mathbb{C}[z_1, \dots, z_d] \otimes_{\mathbb{C}} \mathbb{C}[\bar{z}_1, \dots, \bar{z}_d][\text{d}\bar{z}_1, \dots, \text{d}\bar{z}_d].$$

The right-hand term is one-dimensional, concentrated in degree zero by the ordinary Poincaré lemma. An explicit homotopy for a **BW: ...** □

The analogous result holds with \mathbb{A}^d replaced by the formal d -disk \widehat{D}^d .

For the general case...

Lemma 0.5. ***BW: find good citation. should I just state the general Stein result?** The map*

$$\mathcal{O}^{\text{hol}}(\mathbb{C}^d) \hookrightarrow \Omega^{0,*}(\mathbb{C}^d)$$

is a quasi-isomorphism.

BW: Include equivalence with certain structured local Lie algebras. Namely holomorphically translation invariant local Lie algebras.

Local Lie algebras¹ provide a convenient language to cast the data of an interacting classical field theory. In [BW: ref](#) it is shown that the data of a local Lie algebra together with a non-degenerate pairing of degree -3 is equivalent to the data of a classical interacting BV theory. It will be convenient for us to formulate, under this equivalence, a Lie theoretic interpretation of holomorphically translation invariant interacting BV theories. First, we introduce the following definition. Recall, data of a local Lie algebra is a \mathbb{Z} -graded vector bundle L together with poly-differential operators

$$\ell_n : L \otimes \cdots \otimes L \rightarrow L$$

for $n \geq 1$, satisfying some conditions. The sheaf of smooth sections of this bundle will be denoted by \mathcal{L} , which inherits from the operators $\{\ell_n\}$ the structure of a sheaf of L_∞ algebras. We will often refer to the local Lie algebra simply by its sheaf of sections.

Definition 0.6. A translation invariant local Lie algebra on \mathbb{R}^n is a local Lie algebra \mathcal{L} , together with an identification $L = \mathbb{R}^n \times L_0$ such that for each n the structure map

$$\ell_n : \mathbb{R}^n \times (L_0 \otimes \cdots \otimes L_0) \rightarrow \mathbb{R}^n \times L_0$$

is compatible with translations.

0.2. Holomorphic theories on general complex manifolds. There is a weaker notion of a holomorphic theory that makes sense on a general complex manifold. It is most convenient to formulate this in terms of L_∞ algebras. We start with the definition of a *holomorphic local Lie algebra* on a complex manifold. Enhancing this to a definition of a holomorphic classical BV theory will be immediate.

Fix a complex manifold X , of complex dimension d . The starting data of a local Lie algebra on X is a \mathbb{Z} -graded vector bundle L on X . By a *holomorphic \mathbb{Z} -graded vector bundle* we mean a \mathbb{Z} -graded vector bundle $L = \bigoplus_i L^i$ such that each graded piece L^i is a holomorphic vector bundle.

If E and F are two holomorphic vector bundles on X , we can speak of holomorphic differential operators between E and F . First, note that the Hom-bundle $\text{Hom}(E, F)$ inherits a natural holomorphic structure. By definition, a holomorphic differential operator of order m is a linear map

$$D : \Gamma^{hol}(X; E) \rightarrow \Gamma^{hol}(X; F)$$

such that, with respect to a holomorphic coordinate chart $\{z_i\}$ on X , D can be written as

$$(3) \quad D|_{\{z_i\}} = \sum_{|I| \leq m} a_I(z) \frac{\partial^{|I|}}{\partial z_I}$$

where $a_I(z)$ is a local holomorphic section of $\text{Hom}(E, F)$. Here, the sum is over all multi-indices $I = (i_1, \dots, i_d)$ and

$$\frac{\partial^{|I|}}{\partial z_I} := \frac{\partial^{i_k}}{\partial z_k^{i_k}}.$$

The length is defined by $|I| := i_1 + \cdots + i_d$.

¹Local Lie algebras will mean L_∞ ...

The most basic example of a holomorphic differential operator is the ∂ operator which, for each $1 \leq \ell \leq d$, is a holomorphic differential operator from $E = \wedge^\ell T^{1,0*}X$ to $F = \wedge^{\ell+1} T^{1,0*}X$. Locally, of course, it has the form

$$\partial = \sum_{i=1}^d (dz_i \wedge (-)) \frac{\partial}{\partial z_i},$$

where $dz_i \wedge (-)$ is the vector bundle homomorphism $\wedge^\ell T^{1,0*}X \rightarrow \wedge^{\ell+1} T^{1,0*}X$ sending $\alpha \mapsto dz_i \wedge \alpha$.

Definition 0.7. A *holomorphic local Lie algebra* on a complex manifold X is the data

- (1) A \mathbb{Z} -graded holomorphic vector bundle L on X ;
- (2) For each $n \geq 1$ a holomorphic differential operator

$$\ell_n : L^{\boxtimes n} \rightarrow L;$$

such that $\{\ell_n\}$ endow the sheaf of holomorphic sections $\Gamma_{hol}(L)$ with the structure of a sheaf of L_∞ algebras.

A holomorphic local Lie algebra is *not* a local Lie algebra in the usual sense. The problem is that, in the definition, we have utilized the space of *holomorphic sections*, instead of the space of all smooth sections. There is a natural resolution that will allow us to turn every holomorphic local Lie algebra into an ordinary local Lie algebra. Of course, given a holomorphic vector bundle L we can consider its Dolbeault complex $\text{Dol}(L) = \Omega^{0,*}(X, L)$. For a holomorphic local Lie algebra L this is really the object we want to consider.

Applying the Dolbeault complex functor to the L_∞ maps $\ell_n : L^{\boxtimes n} \rightarrow L$ we obtain maps

$$\text{Dol}(\ell_n) : \Omega^{0,*}(X, L)^{\otimes n} \rightarrow \Omega^{0,*}(X, L).$$

Lemma 0.8. Suppose L is a holomorphic local Lie algebra. Then, $\text{Dol}(L) = \Omega^{0,*}(X, L)$ is endowed with the structure of a local Lie algebra with structure maps given by $\ell_1 = \bar{\partial} + \text{Dol}(\ell_1^L)$, and $\ell_n = \text{Dol}(\ell_n^L)$ for $n \geq 2$.

Proof. We need to check that for each $n \geq 1$ the map

$$\text{Dol}(\ell_n^L) : \Omega^{0,*}(X, L)^{\otimes n} = \Omega^{0,*}(X^{\times n}, L^{\boxtimes n}) \rightarrow \Omega^{0,*}(X, L)$$

is compatible with the Dolbeault differential. This is a local calculation, so it suffices to assume all operators ℓ_n^L are of the form (3). Indeed, since each of the coefficients $a_I(z)$ are holomorphic we see that $\text{Dol}(\ell_n^L)$ is compatible with $\bar{\partial}$. \square

0.3. Holomorphic deformations.

0.3.1. *Translation invariant deformations on \mathbb{C}^d .* Any local Lie algebra on a manifold endows the structure of an L_∞ algebra on its fibers. Suppose \mathcal{L} is a holomorphically translation invariant local Lie algebra on \mathbb{C}^d of the form $\Omega^{0,*}(\mathbb{C}^d, L)$ where L is a graded holomorphic vector bundle....

Proposition 0.9. Suppose \mathcal{L} is a holomorphically translation invariant local Lie algebra on \mathbb{C}^d such that $\ell_1 = \bar{\partial}$. Then, one has

$$\mathcal{C}_{\text{loc}}^*(\mathcal{L})^{\mathbb{C}^d} \simeq K_{\mathbb{C}^d} \otimes_{\mathbb{C}[\frac{\partial}{\partial z_i}]}^{\mathbb{L}} \mathcal{C}_{\text{Lie,red}}^*(L_0[[z_1, \dots, z_d]]).$$

For instance, if $L = \underline{\mathfrak{g}}$ is the constant bundle on \mathbb{C}^d where \mathfrak{g} is an ordinary Lie (or L_∞) algebra one has $L_0 = \mathfrak{g}$ so that

$$C_{\text{loc}}^*(\Omega^{0,*}(\mathbb{C}^d, \mathfrak{g}))^{\mathbb{C}^d} \simeq \mathbb{C} \cdot d^d z \otimes_{\mathbb{C}[\frac{\partial}{\partial z_i}]}^{\mathbb{L}} C_{\text{Lie,red}}^*(\mathfrak{g}[[z_1, \dots, z_d]]).$$

0.3.2. *Holomorphic deformations on an arbitrary complex manifold.* There is a more general result that holds on an arbitrary complex manifold. Recall, that on a complex manifold X we have introduced the notion of a holomorphic local Lie algebra L . Let $\mathcal{L} = \text{Dol}(L) = \Omega^{0,*}(X, L)$ be its associated local Lie algebra. Ordinarily, the local Lie algebra cohomology of a local Lie algebra is computed in terms of the Lie algebra cohomology of the associated jet bundle. With holomorphicity and some mild assumption, we can, up to quasi-isomorphism, exhibit a smaller complex computing this local cohomology.

Proposition 0.10. *Suppose L is a holomorphic local Lie algebra with $\ell_1 = 0$, and let $\mathcal{L} = \Omega^{0,*}(X, L)$ be its associated local Lie algebra. There is a quasi-isomorphism of sheaves of cochain complexes*

$$C_{\text{loc}}^*(\mathcal{L}) \simeq \Omega_X^{d,hol} \otimes_{D_X^{hol}}^{\mathbb{L}} C_{\text{Lie}}^*(J\mathcal{L}^{hol}).$$

Proof. Recall, the definition of $C_{\text{loc}}^*(\mathcal{L})$ is given in terms of D -module data by

$$C_{\text{loc}}^*(\mathcal{L}) = \Omega_X^{d,d} \otimes_{D_X}^{\mathbb{L}} C_{\text{Lie}}^*(J\mathcal{L})$$

where $J\mathcal{L}$ denotes the ∞ -jet bundle of $\Omega^{0,*}(X, L)$. Of course, $J\mathcal{L}$ is a bundle equipped with a natural flat connection, and hence the structure of a D_X -module. The Chevalley-Eilenberg complex $C_{\text{Lie}}^*(J\mathcal{L})$ inherits this D_X -module structure.

On the other hand, if we view L as a holomorphic vector bundle, it makes sense to look at the holomorphic jet bundle $J^{hol}L$. This holomorphic vector bundle is equipped with a holomorphic flat connection, and hence is a module for the sheaf of holomorphic differential operators D_X^{hol} .

Lemma 0.11. *Let $(J^{hol}L)^{C^\infty}$ be the D_X^{hol} -module $J^{hol}L$ viewed, via the forgetful functor, as a D_X -module. Then, there is a quasi-isomorphism of dg D_X -modules $(J^{hol}L)^{C^\infty} \simeq J\Omega^{0,*}(X, L)$. Furthermore, this quasi-isomorphism is compatible with the L_∞ structures, so that we obtain a quasi-isomorphism of dg D_X -modules $C_{\text{Lie}}^*(J^{hol}L)^{C^\infty} \simeq C_{\text{Lie}}^*(J\mathcal{L})$.*

Proof. For any holomorphic vector bundle E , the Dolbeault complex of E is a resolution of the sheaf of holomorphic sections of E . Thus, there is an equivalence of sheaves on X

$$\Omega_X^{0,*}(E) \simeq \Gamma_X^{hol}(E).$$

Thus, there is an equivalence of sheaves on X :

$$J^{hol}L \simeq J\Omega^{0,*}(X, L).$$

We need to see that this equivalence respects the D_X -module structure present on both sides.... □

To finish the proof, we verify the following general lemma.

Lemma 0.12. *Suppose V is a D_X^{hol} -module, and let V^{C^∞} denote its underlying D_X -module. Then, there is a quasi-isomorphism of sheaves of cochain complexes*

$$\Omega_X^{d,hol} \otimes_{D_X^{hol}}^{\mathbb{L}} V[d] \simeq \Omega_X^{d,d} \otimes_{D_X}^{\mathbb{L}} V^{C^\infty}.$$

