THE HOLOMORPHIC TWIST OF $\mathcal{N}=1$ IN FOUR DIMENSIONS

In this section we describe how the $\mathcal{N}=1$ chiral supermultiplet in four dimensions is related to the two dimensional $\beta\gamma$ system.

0.1.

0.1.1. We work on the four manifold $\mathbb{R}^4$ equipped with the flat metric. We will write Ω^i for the space of smooth i-forms on $\mathbb{R}^4$. The space of sections of the spinor bundles are denoted by $\mathcal{S}_\pm = C^\infty(\mathbb{R}^4; S_\pm)$ where $S_\pm$ are the defining representations of the two copies of $\mathrm{SU}(2)$ in $\mathrm{Spin}(4)$. Write $\mathrm{dvol} = \mathrm{d}x_1 \cdots \mathrm{d}x_4$ and $\star$ for the Hodge star operator.

We fix an even dimensional real vector space V equipped with a complex structure, denoted by J, and a Hermitian inner product, denoted by h.

0.1.2.

Definition 0.1. The $N=1$ chiral supermultiplet on $\mathbb{R}^4$ with values in the complex vector space V equipped with a Hermitian pairing h has space of fields

$$\begin{aligned}
\Phi_+ &= (\varphi_+, \psi_+, F_+) \in \Omega^0 \otimes V \oplus \mathcal{S}_+ \otimes V \oplus \Omega^2 \otimes V \\
\Phi_- &= (\varphi_-, \psi_-, F_-) \in \Omega^0 \otimes V \oplus \mathcal{S}_- \otimes V \oplus \Omega^2 \otimes V
\end{aligned}$$

and action functional given by

$$S^{\mathrm{SO}}(\varphi, \psi, F) = \int h(\varphi \otimes \Delta\varphi)\mathrm{dvol} + \int h(\langle \psi_+, \partial\!\!\!/ \psi_- \rangle)\mathrm{dvol} + \int F^2$$

where $\langle -, - \rangle$ denotes the standard symplectic pairing $S_+ \otimes S_+ \to \mathbb{C}$.

We have written the fields as a pair of superfields, a chiral one Φ_+ and an anti-chiral one Φ_-.

It will be useful for our purposes to write the chiral supermultiplet in the BV-BRST formalism. There are no gauge symmetries, so all that this amounts to is the the introduction of the anti-fields to the fields we have already written.

$$\begin{aligned}
\Phi_+^\vee &= (\varphi_+^\vee, \psi_+^\vee, F_+^\vee) \in \Omega^4 \otimes V \oplus \mathcal{S}_-' \otimes V \oplus \Omega^2 \otimes V \\
\Phi_-^\vee &= (\varphi_-^\vee, \psi_-^\vee, F_-^\vee) \in \Omega^4 \otimes V \oplus \mathcal{S}_-' \otimes V \oplus \Omega^2 \otimes V.
\end{aligned}$$

We note that the anti-field to a positive spinor $\psi_+ \in \mathcal{S}_\pm$ is a spinor $\psi_\pm^\vee \in \mathcal{S}_+$ of the same chirality. The prime simply indicates the same underlying spinor bundle except we are viewing it as an anti-field. Using this notation, we can define the classical theory in the BV formalism in a succinct way.

Definition 0.2. The $N=1$ chiral supermultiplet on $\mathbb{R}^4$ in the BV formalism is the $\mathbb{Z} \times \mathbb{Z}/2$ graded theory with complex of fields given by

The R-charge of an anti-field is opposite to that of the corresponding field. Thus, $\psi_\pm^\vee$ has R-charge ∓ 1 and $F_\pm^\vee$ has R-charge ∓ 2.

We will perform a twist of the free chiral supermultiplet by a fixed constant spinor $Q \in \mathcal{S}_-$. This element acts on the fields $\Phi = (\varphi_+, \psi_+, F)$ as above:

$$Q \cdot (\varphi_+, \psi_+, F) = (0, Q \cdot (\mathrm{d}\varphi_+), \langle Q, \partial\!\!\!/ \psi_+ \rangle).$$

The action of Q on the anti-fields reads is determined by compatibility with the (-1)-shifted symplectic pairing. Explicitly it is

$$Q \cdot (\varphi_+^\vee, \psi_+^\vee, F) = (\langle Q, \partial\!\!\!/ \psi_+^\vee \rangle, Q \cdot (\star F_+^\vee), 0).$$

We have arrived at the following.

Proposition 0.3. *The twist of the $N = 1$ free chiral supermultiplet on $\mathbb{R}^4$ with values in the hermitian vector space V by an element $Q \in \mathcal{S}_-$ is equivalent to the free $\beta\gamma$ system on $\mathbb{C}^2$ with values in V:*

$$(\gamma, \beta) \in \Omega^{0,*}(\mathbb{C}^2; V) \oplus \Omega^{1,*}(\mathbb{C}^2; V^\vee)[1].$$

The action functional is $S(\gamma, \beta) = \int \langle \beta, \bar\partial \gamma \rangle$ where $\langle -, - \rangle$ is the evaluation pairing on V.

0.1.3. We will introduce the following "first-order" reformulation of the chiral supermultiplet that will be convenient for our description of it as a BV theory. Introduce additional scalar fields of the form

$$B \in \Omega^3 \otimes V$$

and define the action

$$\begin{aligned} S^{\mathrm{FO}}(\varphi, B, \psi, F) &= \int h(B \wedge \mathrm{d}\varphi) - \frac{1}{2} \int h(B \wedge \star B) \\ &+ \int h(\langle \psi_+, \partial\!\!\!/ \psi_- \rangle) \mathrm{dvol} \end{aligned}$$

Lemma 0.4. *The theories S^{SO} and S^{FO} are classically equivalent.*

Proof. We show that the spaces of solutions to the classical equations of motion are equivalent. The equations of motion for the chiral supermultiplet read

$$\begin{aligned} \Delta\varphi &= 0 \\ \partial\!\!\!/ \psi &= 0 \\ F &= 0. \end{aligned}$$

Next, consider S^{FO}. The pieces of the action functional involving ψ, F are identical. We use the variation of the scalar field $\varphi \mapsto \varphi + \delta\varphi$ to obtain the equation of motion $\mathrm{d}B = 0$. The variation $B \mapsto B + \delta B$ yields the equation $\mathrm{d}\varphi - h^\vee(\star B)$. This equation is equivalent to $\star \mathrm{d}\varphi = h^\vee(B)$. Applying d to this equation and using the equation $\mathrm{d}B = 0$ we obtain $\mathrm{d} \star \mathrm{d}\varphi = \Delta\varphi = 0$, as desired.

An explicit equivalence at the level of fields can be written as follows. BW: Want $\varphi \mapsto \varphi, B \mapsto \star h^\vee(\mathrm{d}\varphi) + h^\vee(F)$. □

BW: physics description of susy action

0.1.4. We describe the theory S^{FO} as a classical theory in the BV formalism. The space of fields

$$\begin{array}{lccc}
 & \underline{0} & & \underline{1} \\
\text{Fermion degree } \underline{0} & \Omega^0 \otimes V & \xrightarrow{\mathrm{d}_+} & (\Omega^1 \otimes V)_+ \\
\text{Fermion degree } \underline{0} & (\Omega^3 \otimes V)_- & \xrightarrow{\mathrm{d}} & \Omega^4 \otimes V \\
\text{Fermion degree } \underline{1} & (\mathcal{S}_+ \oplus \mathcal{S}_-) \otimes V & \xrightarrow{\not{\partial}} & (\mathcal{S}'_- \oplus \mathcal{S}'_+) \otimes V
\end{array}$$