

The holomorphic σ -model and its symmetries

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May 1, 2018

Outline of this talk

1. Rapid overview of the BV formalism.
2. Holomorphic theories, in general. One-loop finiteness and a formula for the general chiral anomaly.
3. The holomorphic σ -model and its factorization algebra.

The BV formalism

The Batalin-Vilkovisky formalism is a technique used to study quantization in field theory. A generalization of the usual problem of *deformation quantization*.

$$\mathrm{SympMfld} \xrightarrow{\mathcal{O}} \mathrm{Alg}_{\mathrm{Poiss}} \xleftarrow{\hbar \rightarrow 0} \mathrm{Alg}_{\mathbb{C}}[[\hbar]]$$

$$(M, \omega) \longmapsto (\mathcal{O}(M), \Pi_{\omega}) \longleftarrow (\mathcal{O}(M)[[\hbar]], \star).$$

In field theory, one works on a smooth manifold X (the spacetime).

$$\mathrm{BV} - \mathrm{Theory}(X) \xrightarrow{\mathrm{Obs}} \mathrm{FactAlg}(X)_{P_0} \xleftarrow{\hbar \rightarrow 0} \mathrm{FactAlg}(X)_{\mathrm{BV}}.$$

Given a classical BV theory we study lifts of the P_0 factorization algebra of classical observables to the BV factorization algebra of quantum observables.

In the one-dimensional case $X = \mathbb{R}$ there exists a classical BV theory associated to a symplectic manifold (M, ω) . In this case, BV quantization recovers ordinary deformation quantization.

The BV formalism (cont.)

In QFT, BV algebras provide a mathematical model for the path integral (see Costello's book on renormalization).

Definition

A BV algebra is a triple (A, Q, Δ) where (A, Q) is a commutative dg algebra, and $\Delta : A \rightarrow A$ is a degree one linear map such that

- (a) $\Delta^2 = [\Delta, Q] = 0$;
- (b) the degree one bilinear map

$$\{a, b\} := \Delta(ab) - \Delta(a)b \pm a\Delta(b)$$

satisfies graded Jacobi, and is a graded biderivation with respect to the commutative product.

Thus $\{-, -\}$ behaves like a Poisson bracket, except with a weird shift. We say an element $I = I_0 + \hbar I_1 + \cdots \in A[[\hbar]]$ satisfies the *quantum master equation* (QME) if

$$(Q + \hbar \Delta)e^{I/\hbar} = 0.$$

We call \hbar the *perturbation* parameter.

The BV formalism (cont.)

When we set $\hbar = 0$, the QME reduces to condition

$$Ql_0 + \frac{1}{2}\{l_0, l_0\} = 0.$$

We call this the classical master equation (CME).

Example

Suppose $A = \mathcal{O}(V) = \text{Sym}(V^*)$ for some graded vector space V . Then a functional l_0 satisfying the CME is equivalent to an data of an L_∞ structure on the graded vector space $V[-1]$.

Most important example of BV algebras in QFT come from (-1) -shifted geometry. Suppose (V, ω) is a (-1) -shifted symplectic vector space.

Then, the symmetric tensor $K_0 := \omega^{-1} \in \text{Sym}^2(V)$ defines an operator (of order two)

$$\Delta_0 = \partial_{K_0} : \mathcal{O}(V) \rightarrow \mathcal{O}(V)$$

by contraction. This operator defines a BV algebra $(\mathcal{O}(V), Q, \Delta_0)$, where Q is the internal differential of V .

The BV formalism (cont.)

Suppose that $P \in \text{Sym}^2(V)$ is a symmetric tensor of degree zero, and define $K_P = K_0 + QP$. One checks that K_P defines another BV algebra based on $\mathcal{O}(V)$.

Given $I \in \mathcal{O}^+(V)$ (at least cubic), define $W(P, I) \in \mathcal{O}(V)[[\hbar]]$ formally by

$$\boxed{e^{W(P, I)/\hbar} = e^{\hbar \partial_P} e^{I/\hbar}}.$$

Lemma

The functional I satisfies the QME relative to K_0 if and only if $W(P, I)$ satisfies the QME relative to K_P .

The functional $W(P, I)$ decomposes as a sum over connected graphs

$$W(P, I) = \sum_{\Gamma} \frac{\hbar^{g(\Gamma)}}{|\text{Aut}(\Gamma)|} W_{\Gamma}(P, I),$$

where W_{Γ} is the *weight* of the graph Γ .

Field theory

A classical field theory on a smooth manifold M is:

- (i) a graded vector bundle E whose sections we denote \mathcal{E} ;
- (ii) a differential operator $Q : \mathcal{E} \rightarrow \mathcal{E}$ of degree one;
- (iii) a graded antisymmetric bundle map $(-, -)_E : E \otimes E \rightarrow \text{Dens}_X$ of degree (-1) that is fiberwise nondegenerate.
- (iv) a *local functional* $I_0 \in \mathcal{O}_{\text{loc}}(\mathcal{E})$ satisfying the CME.

We require that (\mathcal{E}, Q) is an elliptic complex. The pairing $(-, -)_E$ defines a (-1) -shifted symplectic structure via integration

$$\omega = \int_X \circ(-, -)_E.$$

The sheaf of sections \mathcal{E} evaluated on an open set U returns the graded space $\mathcal{E}(U)$ which we refer to as the space of fields supported on U .

Holomorphic field theory

In the world of complex geometry we have the following definition of a *holomorphic* field theory on a complex manifold X :

- (i) a graded holomorphic vector bundle V on X whose sheaf of holomorphic sections we denote \mathcal{V}^{hol} ;
- (ii) a holomorphic differential operator $Q^{hol} : \mathcal{V}^{hol} \rightarrow \mathcal{V}^{hol}$ of degree one;
- (iii) a graded antisymmetric bundle map $(-, -)_V : V \otimes V \rightarrow K_X$ of degree $(d - 1)$ that is fiberwise nondegenerate.
- (iv) a holomorphic Lagrangian \mathcal{J}_0^{hol} satisfying the CME.

Holomorphic theory	BV theory
Holomorphic bundle V	Space of fields $\mathcal{E}_V = \Omega^{0,*}(X, V)$
Holomorphic differential operator Q^{hol}	Linear BRST operator $\bar{\partial} + Q^{hol}$
Non-degenerate pairing $(-, -)_V$	(-1) -symplectic structure ω_V
Holomorphic Lagrangian \mathcal{J}_0^{hol}	Local functional $I_0^{\Omega^{0,*}} \in \mathcal{O}_{loc}(\mathcal{E}_V)$

Table: From holomorphic to BV

Regularization

Let $(\mathcal{E}, Q, \omega, I_0)$ be a classical BV theory. The first thing to do is define the BV operator $\Delta_0 = \omega^{-1}$.

- **Problem:** The tensor ω^{-1} is *distributional*, thus Δ_0 is not well-defined on functionals.

The solution is to find a homotopy replacement for K_0

$$\tilde{K} = K_0 + QP,$$

so that its BV operator is well-defined. (By elliptic regularity, one always exists). Such a regularization is parametrized by a length scale $L > 0$. For each $L < L'$ a regularization scheme prescribes a *propagator* $P_{L < L'}$ such that

$$K_{L'} = K_L + QP_{L < L'}$$

where $K_L, K_{L'}$ are both smooth and $\lim_{L \rightarrow 0} K_L = K_0$.

The definition of a QFT

By definition, a quantization is a family of functionals $\{I[L]\}$ with $I_0 = \lim_{L \rightarrow 0} I[L] \bmod \hbar$ satisfying the following two conditions:

1. the collection of functionals $\{I[L]\}$ are related by *renormalization group flow*

$$I[L'] = W(P_{L < L'}, I[L]).$$

2. for each L , the functional solves the *scale L quantum master equation*

$$(Q + \hbar \Delta_L) e^{I[L]/\hbar} = 0.$$

For abstract reasons, proved by Costello, one can always find a family such that (1) is satisfied. In general, the answer is not constructive and involves choosing counterterms with respect to a renormalization scheme. There may be unavoidable obstructions to solving problem (2).

The naïve definition of $I[L]$ is to apply the operator $P_{0<L}$ to the classical interaction

$$I[L] = W(P_{0<L}, I_0)$$

The problem is that the right-hand side is rarely well-defined (same issue as above). A solution to this, which always exists, is to find counterterms.

Theorem

*There is a regularization scheme for **holomorphic theories** on \mathbb{C}^d such that the limit*

$$I[L] = \lim_{\epsilon \rightarrow 0} W(P_{\epsilon < L}, I_0) \mod \hbar^2$$

exists. In other words, holomorphic theories on \mathbb{C}^d are one-loop finite.

The main ingredient is in the existence of the *gauge fixing operator* $\bar{\partial}^*$.

- ▶ Studying the quantizations of holomorphic theories on \mathbb{C}^d reduces to solving the quantum master equation. This is essentially an algebraic problem.

A corollary of this result is a characterization of the *anomaly*, or obstruction, for a holomorphic theory to solve the QME.

Corollary

The obstruction for a classical holomorphic theory on \mathbb{C}^d to admit a one-loop quantization is given by the following expression:

$$\Theta = \lim_{\epsilon, L \rightarrow 0} \sum_{\Gamma \in \text{Wheel}_{d+1}} W_{\Gamma}(P_{\epsilon < L}, K_{\epsilon}, l_0).$$

Pictorially PICTURE

This gives a holomorphic characterization, and generalization, of the Adler-Bell-Jackiw anomaly for four-dimensional gauge theory.

The holomorphic σ -model

The holomorphic σ -model is a prototypical holomorphic theory. Let X, Y be complex manifolds and consider the mapping space:

$$\mathrm{Map}^{hol}(Y, X) = \{f : Y \rightarrow X \text{ holomorphic}\}.$$

There are a few issues:

1. a classical theory involves a shifted symplectic pairing. The theory we study is of the form

$$T^*[-1] \left(\mathrm{Map}^{hol}(Y, X) \right).$$

In degree zero, the fields consist of a map $\gamma : Y \rightarrow X$ together with a class $\beta \in \Omega^{d, d-1}(Y, \gamma^* T^{*1,0} X)$. The action functional is

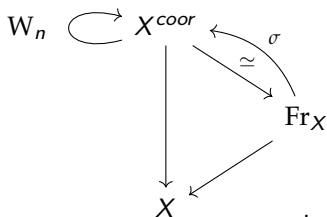
$$S(\beta, \gamma) = \int_Y \beta \wedge \bar{\partial} \gamma.$$

Notice when we vary γ, β we obtain $\bar{\partial} \gamma = 0 = \bar{\partial} \beta$.

2. To make this into a BV theory, we must perturb around a fixed holomorphic map; we look at the formal neighborhood of constant maps $\mathrm{Map}(Y, X)_{const}^\wedge$.

Local-to-global

Our construction of the holomorphic σ -model is local-to-global on the target manifold. We phrase the theory in the style of *formal geometry* due to Gelfand, Kazhdan, Fuks. To every n -dimensional manifold X (smooth, complex, symplectic, etc..) there exists a universal bundle of coordinates:



X^{coord} is a principal Aut_n -bundle together with a transitive action of the Lie algebra of *formal vector fields* in n -dimensions W_n . There is

$$\omega^{coord} \in \Omega^1(X^{coord}, W_n)^{\text{Aut}_n} \xrightarrow{\sigma^*} \Omega^1(Fr_X, W_n)^{\text{GL}_n}$$

satisfying the Maurer-Cartan equation $d\omega^{coord} + \frac{1}{2}[\omega^{coord}, \omega^{coord}] = 0$.

Gelfand-Kazhdan descent

Define a category of “formal vector bundles” on the formal n -disk. In particular, these are (W_n, GL_n) -modules. For each X , there is a functor

$$\begin{array}{ccc}
 \mathcal{V} & \longmapsto & (\mathrm{Fr}_X \times^{GL_n} \mathcal{V}, \nabla^{coord}) \\
 \cap & & \cap \\
 \mathrm{VB}_{\widehat{D}^n} & \xrightarrow{\mathrm{desc}_X} & \mathrm{VB}_X^{flat} \\
 \downarrow & & \downarrow \\
 \mathrm{Mod}_{(W_n, GL_n)} & \longrightarrow & \mathrm{Mod}_{D_X}.
 \end{array}$$

Moreover, there are “formal characteristic classes” that live in the Gelfand-Fuks cohomology. The descent functor determines a transformation of cohomology theories and hence a map of complexes

$$\mathrm{char}_X : C_{\mathrm{Lie}}^*(W_n, GL_n; \mathcal{V}) \rightarrow \Omega^*(X, \mathrm{desc}_X(\mathcal{V})).$$

When $\mathcal{V} = \widehat{\mathcal{O}}_n$ formal power series, $\mathrm{desc}_X(\widehat{\mathcal{O}}_n) = J^\infty \mathcal{O}_X$ equipped with its natural flat connection. Recover all natural bundles in this way.

The formal holomorphic σ -model

Consider the formal disk \widehat{D}^n as a ringed space whose ring of functions is formal power series $\widehat{\mathcal{O}}_n$.

$$Y \longrightarrow \widehat{D}^n \rightrightarrows (W_n, \mathrm{GL}_n).$$

Key idea: study the free theory equivariant for the action of the pair (W_n, GL_n) . Get global target σ -model via descent.

Quantization: holomorphic theory \implies renormalization is simple.

Obstruction is controlled by an element in Gelfand-Fuks cohomology.

Theorem

There is an obstruction to quantizing the formal holomorphic σ -model of maps $\mathbb{C}^d \rightarrow \widehat{D}^n$ given by the class

$$\mathrm{ch}_{d+1}^{\mathrm{GF}}(\widehat{\mathcal{T}}_n) \in C_{\mathrm{Lie}}^{d+1}(W_n, \mathrm{GL}_n; \widehat{\Omega}_{n,cl}^{d+1}).$$

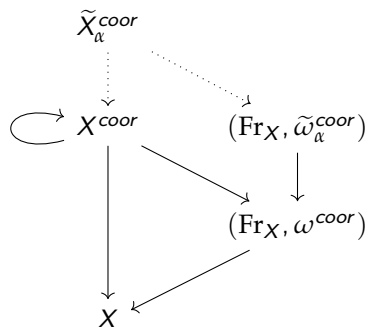
Under characteristic map, this returns the ordinary Chern class.

Determines an L_∞ -extension

$$0 \rightarrow \widehat{\Omega}_{n,cl}^{d+1} \rightarrow \widetilde{W}_{n,d} \rightarrow W_n \rightarrow 0.$$

Extended descent

Given any trivialization α of $\mathrm{ch}_{d+1}(T_X)$ we can lift the structure of the coordinate bundle.



$$\tilde{\omega}_\alpha^{coord} \in \bigoplus_{p+q=1} \Omega^p(\mathrm{Fr}_X, (\tilde{W}_{n,d})^q)$$

$$d\tilde{\omega}_\alpha^{coord} + \sum_{k \geq 2} \ell_k(\tilde{\omega}_\alpha^{coord}) = 0$$

Can find flat connection $\tilde{\omega}_\alpha \in \Omega^1(\mathrm{Fr}_X, \tilde{W}_{n,d})$. Descent functor

$$\widetilde{\mathrm{desc}}_{X,\alpha} : \mathrm{Mod}_{(\tilde{W}_{n,d}, \mathrm{GL}_n)} \rightarrow \mathrm{Mod}_{D_X}.$$

Theorem implies quantization is equivariant for $(\tilde{W}_{n,d}, \mathrm{GL}_n)$. This says that for any trivialization α we obtain a global quantization.

Main result

Explicit GF calculation shows there is a unique $(\tilde{W}_{n,d}, \mathrm{GL}_n)$ -quantization for the formal theory. Extended descent implies the following main result.

Theorem

Suppose $\mathrm{ch}_{d+1}(T_X) = 0$. Then, the space of quantizations respecting certain natural symmetries of the holomorphic σ -model of maps $\mathbb{C}^d \rightarrow \hat{D}^n$ is a torsor for the abelian group $H^d(X, \Omega_X^{d+1, \mathrm{hol}})$.