The holomorphic σ-model and its symmetries

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Outline of this talk

1. Rapid overview of the Batalin-Vilkovisky (BV) formalism.
2. Holomorphic theories, in general. One-loop finiteness and a formula for the general chiral anomaly.
3. The holomorphic σ-model and its factorization algebra.

The BV formalism

The BV formalism is a technique used to study quantizations of field theories. A generalization of the usual problem of *deformation quantization*.

$$\begin{array}{ccccc} \mathrm{SympMfld} & \xrightarrow{\mathcal{O}} & \mathrm{Alg}_{\mathrm{Poiss}} & \xleftarrow{\hbar \to 0} & \mathrm{Alg}_{\mathbb{C}[[\hbar]]} \\ & & & & \\ (M, \omega) & \longmapsto & (\mathcal{O}(M), \Pi_\omega) & \longleftarrow\!\shortmid & (\mathcal{O}(M)[[\hbar]], \star). \end{array}$$

In field theory, one works on a smooth manifold X (the spacetime).

$$\mathrm{BV}-\mathrm{Theory}(X) \xrightarrow{\mathrm{Obs}} \mathrm{FactAlg}(X)_{P_0} \xleftarrow{\hbar \to 0} \mathrm{FactAlg}(X)_{\mathrm{BV}}.$$

Given a classical BV theory we study lifts of the P_0 factorization algebra of classical observables to the *BV* factorization algebra of quantum observables.

In the one-dimensional case $X = \mathbb{R}$ there is a classical BV theory associated to a symplectic manifold (M, ω). In this case, BV quantization recovers ordinary deformation quantization.

Classical field theory

Start with a manifold X (the spacetime). A *classical theory* on X is the data of a sheaf of dg Lie (really L_∞) algebras $\mathcal{E}$ equipped with a pairing $\omega \in \bigwedge^2 \mathcal{E}^*$ satisfying:

1. Locality. The structure maps are given by polydifferential operators (in particular $\mathcal{E}$ comes as the smooth sections of a graded vector bundle on X);
2. Ellipticity. $(\mathcal{E}, \ell_1 = Q)$ is an elliptic complex $(Q = \mathrm{d}_{dR}, \Delta, \bar{\partial}, \ldots)$;
3. ω equips $B\mathcal{E} = \mathcal{E}[1]$ with a (-1)-shifted symplectic structure.

(1), (3) are equivalent to prescribing $I_0 \in \mathcal{O}^{\geq 3}(B\mathcal{E}) = \mathrm{Sym}^{\geq 3}(\mathcal{E}^*[-1])$ satisfying the MC equation

$$\boxed{QI_0 + \frac{1}{2}\{I_0, I_0\} = 0}.$$

The *classical observables* supported on $U \subset X$:

$$\mathrm{Obs}^{\mathrm{cl}}(U) = \left(\mathrm{Sym}(\mathcal{E}(U)^\vee), Q + \{I_0, -\}\right).$$

The quantum BV formalism

Want to find $I \in \mathcal{O}(\mathcal{E})[[\hbar]]$, $I_0 = I \mod \hbar$ satisfying the *quantum master equation*

$$(Q + \hbar\Delta)e^{I/\hbar} = 0,$$

where $\Delta = \partial_{K_0} \in \mathrm{End}(\mathcal{O}(\mathcal{E}))$, $K_0 = \omega^{-1}$.

- **Problem:** The tensor ω^{-1} is *distributional*, thus Δ_0 is not well-defined on functionals. **Solution:** Find a homotopy replacement $K_L = K_0 + QP_{0<L}$ that so that $\Delta_L = \partial_{K_L}$ *is* well-defined on functionals.

A *quantization* is a family of functionals $\{I[L]\}_{L>0} \subset \mathcal{O}(\mathcal{E})[[\hbar]]$ such that:

1. the collection of functionals are related by *renormalization group flow*

$$\boxed{I[L'] = W(P_{L<L'}, I[L])};$$

2. for each L, the functional solves the *scale L* QME

$$\boxed{(Q + \hbar\Delta_L)e^{I[L]/\hbar} = 0};$$

3. some technical locality conditions.

For abstract reasons, proved by Costello, one can always solve (1), (3). Subtle analysis in (1), involves counterterms.

Holomorphic field theory

In the world of complex geometry we have the following definition of a *holomorphic* field theory on a complex manifold X:

(i) a graded holomorphic vector bundle V on X whose sheaf of holomorphic sections we denote $\mathcal{V}^{hol}$;

(ii) a holomorphic differential operator $Q^{hol} : \mathcal{V}^{hol} \to \mathcal{V}^{hol}$ of degree one;

(iii) a graded antisymmetric bundle map $(-,-)_V : V \otimes V \to K_X$ of degree $(d-1)$ that is fiberwise nondegenerate.

(iv) a holomorphic Lagrangian $\mathcal{I}_0^{hol}$ satisfying the CME.

Holomorphic theory	BV theory
Holomorphic bundle V	Space of fields $\mathcal{E}_V = \Omega^{0,*}(X, V)$
Holomorphic differential operator Q^{hol}	Linear BRST operator $\overline{\partial} + Q^{hol}$
Non-degenerate pairing $(-,-)_V$	(-1)-symplectic structure ω_V
Holomorphic Lagrangian $\mathcal{I}_0^{hol}$	Local functional $I_0^{\Omega^{0,*}} \in \mathcal{O}_{\mathrm{loc}}(\mathcal{E}_V)$

Table: From holomorphic to BV

Holomorphic renormalization

The naïve definition of $I[L]$ is to apply the operator $P_{0<L}$ to the classical interaction

$$I[L] = W(P_{0<L}, I_0).$$

The problem is that the right-hand side is rarely well-defined (same issue as above). A solution to this, which always exists, is to find counterterms.

Theorem (W.)

There is a regularization for **holomorphic theories** *on $\mathbb{C}^d$ such that the limit*

$$I[L] = \lim_{\epsilon \to 0} W(P_{\epsilon<L}, I_0) \mod \hbar^2$$

exists. In other words, holomorphic theories on $\mathbb{C}^d$ are one-loop finite.

The main ingredient is in the existence of the *gauge fixing operator* $\bar{\partial}^*$.

- Studying the quantizations of holomorphic theories on $\mathbb{C}^d$ reduces to solving the quantum master equation. This is essentially an algebraic problem.

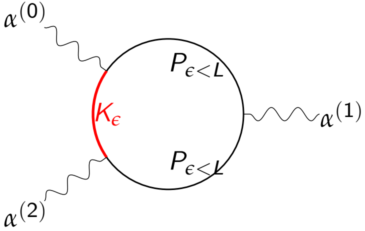
A general formula for the chiral anomaly

A corollary of this result is a characterization of the *anomaly*, or obstruction, for a holomorphic theory to solve the QME.

Corollary (W.)

The obstruction for a classical holomorphic theory on $\mathbb{C}^d$ to admit a one-loop quantization is given by the following expression:

$$\Theta = \lim_{\epsilon, L \to 0} \sum_{\Gamma \in \mathrm{Wheel}_{d+1}} W_\Gamma(P_{\epsilon<L}, K_\epsilon, I_0).$$



This gives a holomorphic characterization, and generalization, of the Adler-Bell-Jackiw anomaly for four-dimensional gauge theory.

The holomorphic σ-model

The holomorphic σ-model is a prototypical holomorphic theory. Let X, Y be complex manifolds and consider the mapping space:

$$\mathrm{Map}^{hol}(Y, X) = \{f : Y \to X \ \text{ holomorphic}\}.$$

There are a few issues:

1. a classical theory involves a shifted symplectic pairing. The theory we study is of the form

   $$T^*[-1]\left(\mathrm{Map}^{hol}(Y, X)\right).$$

   In degree zero, the fields consist of a map $\gamma : Y \to X$ together with a class $\beta \in \Omega^{d,d-1}(Y, \gamma^* T^{*1,0} X)$. The action functional is

   $$S(\beta, \gamma) = \int_Y \beta \wedge \bar{\partial}\gamma.$$

   Notice when we vary γ, β we obtain $\bar{\partial}\gamma = 0 = \bar{\partial}\beta$.
2. To make this into a BV theory, we must perturb around a fixed holomorphic map; we look at the formal neighborhood of constant maps $\mathrm{Map}(Y, X)^{\wedge}_{const}$.

Local-to-global

Our construction of the holomorphic σ-model is local-to-global on the target manifold. We phrase the theory in the style of *formal geometry* due to Gelfand, Kazhdan, Fuks. To every n-dimensional manifold X (smooth, complex, symplectic, etc..) there exists a universal bundle of coordinates:

$$\begin{tikzcd}
\mathrm{W}_n \arrow[r, phantom, "\circlearrowright"] & X^{coor} \arrow[dr, "\simeq"] \arrow[dd] & \\
& & \mathrm{Fr}_X \arrow[ul, bend right, "\sigma"'] \arrow[dl] \\
& X &
\end{tikzcd}.$$

X^{coor} is a principal Aut_n-bundle together with a transitive action of the Lie algebra of *formal vector fields* in n-dimensions W_n. There is

$$\omega^{coor} \in \Omega^1(X^{coor}, \mathrm{W}_n)^{\mathrm{Aut}_n} \xrightarrow{\sigma^*} \Omega^1(\mathrm{Fr}_X, \mathrm{W}_n)^{\mathrm{GL}_n}$$

satisfying the Maurer-Cartan equation $\mathrm{d}\omega^{coor} + \frac{1}{2}[\omega^{coor}, \omega^{coor}] = 0$.

Gelfand-Kazhdan descent

Define a category of "formal vector bundles" on the formal n-disk. In particular, these are $(\mathrm{W}_n, \mathrm{GL}_n)$-modules. For each X, there is a functor

$$\begin{tikzcd}
\mathcal{V} \arrow[rr, mapsto] & & \left(\mathrm{Fr}_X \times^{\mathrm{GL}_n} \mathcal{V}, \nabla^{coor}\right) \\
\pitchfork & & \pitchfork \\
\mathrm{VB}_{\widehat{D}^n} \arrow[rr, "\mathrm{desc}_X"] \arrow[d, hook] & & \mathrm{VB}_X^{flat} \arrow[d, hook] \\
\mathrm{Mod}_{(\mathrm{W}_n, \mathrm{GL}_n)} \arrow[rr] & & \mathrm{Mod}_{D_X}.
\end{tikzcd}$$

Moreover, there are "formal characteristic classes" that live in the Gelfand-Fuks cohomology. The descent functor determines a transformation of cohomology theories and hence a map of complexes

$$\mathrm{char}_X : \mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n, \mathrm{GL}_n; \mathcal{V}) \to \Omega^*(X, \mathrm{desc}_X(\mathcal{V})).$$

When $\mathcal{V} = \widehat{\mathcal{O}}_n$ formal power series, $\mathrm{desc}_X(\widehat{\mathcal{O}}_n) = J^\infty \mathcal{O}_X$ equipped with its natural flat connection. Recover all natural bundles in this way.

The formal holomorphic σ-model

Consider the formal disk $\widehat{D}^n$ as a ringed space whose functions are formal power series $\widehat{\mathcal{O}}_n$.

$$Y \longrightarrow \widehat{D}^n \circlearrowleft \quad (\mathrm{W}_n, \mathrm{GL}_n).$$

Key idea: study the free theory *equivariant* for the action of the pair $(\mathrm{W}_n, \mathrm{GL}_n)$. Get global target σ-model via descent.
Quantization: holomorphic theory $\implies$ renormalization is simple.
Obstruction is controlled by an element in Gelfand-Fuks cohomology.

Theorem (W.)

There is an obstruction to quantizing the formal holomorphic σ-model of maps $\mathbb{C}^d \to \widehat{D}^n$ given by the class

$$\mathrm{ch}^{\mathrm{GF}}_{d+1}(\widehat{\mathcal{T}}_n) \in \mathrm{C}^{d+1}_{\mathrm{Lie}}(\mathrm{W}_n, \mathrm{GL}_n; \widehat{\Omega}^{d+1}_{n,cl}).$$

Under characteristic map, this returns the ordinary Chern class.
Determines an L_∞*-extension*

$$0 \to \widehat{\Omega}^{d+1}_{n,cl}[d-1] \to \widetilde{\mathrm{W}}_{n,d} \to \mathrm{W}_n \to 0.$$

Extended descent

Given any trivialization α of $\mathrm{ch}_{d+1}(T_X)$ we can lift the structure of the coordinate bundle.

$$
\begin{tikzcd}
\widetilde{\mathrm{W}}_{n,d} \arrow[d] \arrow[r, phantom, "\curvearrowright"] & \widetilde{X}^{coor}_{\alpha} \arrow[d, dotted] \arrow[dr, dotted] & \\
\mathrm{W}_n \arrow[r, phantom, "\curvearrowright"] & X^{coor} \arrow[dd] \arrow[dr] & (\mathrm{Fr}_X, \widetilde{\omega}^{coor}_{\alpha}) \arrow[d] \\
& & (\mathrm{Fr}_X, \omega^{coor}) \arrow[dl] \\
& X &
\end{tikzcd}
$$

$$\widetilde{\omega}^{coor}_{\alpha} \in \oplus_{p \geq 1} \Omega^p(\mathrm{Fr}_X, \widetilde{\mathrm{W}}_{n,d})$$

$$\mathrm{d}\widetilde{\omega}^{coor}_{\alpha} + \textstyle\sum_{k \geq 2} \ell_k(\widetilde{\omega}^{coor}_{\alpha}) = 0$$

Descent functor

$$\widetilde{\mathrm{desc}}_{X,\alpha} : \mathrm{Mod}_{(\widetilde{\mathrm{W}}_{n,d}, \mathrm{GL}_n)} \to \mathrm{Mod}_{D_X}.$$

Theorem implies quantization is equivariant for $(\widetilde{\mathrm{W}}_{n,d}, \mathrm{GL}_n)$. This says that for any trivialization α we obtain a global quantization.

Main result

Explicit GF calculation shows there is a unique $(\widetilde{\mathrm{W}}_{n,d}, \mathrm{GL}_n)$-quantization for the formal theory. Extended descent implies the following main result.

Theorem (W.)

Suppose $\mathrm{ch}_{d+1}(T_X) = 0$*. Then, the space of quantizations (respecting certain natural symmetries) of the holomorphic* σ*-model of maps* $\mathbb{C}^d \to X$ *is a torsor for the abelian group* $H^d(X, \Omega_X^{d+1,hol})$*.*

- Get sheaf on X of factorization algebras on $\mathbb{C}^d$. "Higher dimensional chiral differential operators". Gwilliam-Gorbounov-W. in $\dim_{\mathbb{C}} = 1$.
- Quantizations exist on other source manifolds: affine manifolds, abelian varieties, Hopf manifolds $Y = \mathbb{C}^d \setminus \{0\}/q^{\mathbb{Z}} \cong S^{2d-1} \times S^1$.
- Local calculation of the index produces elliptic Γ-functions. This agrees with the partition function for supersymmetric theories in dimensions $2, 4, 6$. For a general target, this produces refined invariants generalizing the Witten genus in complex dimension one.

Relation to deformation quantization

Immediate corollary: obtain the following deformation quantization for "sphere algebras". Theory on

$$\begin{array}{ccc} \mathbb{C}^d \setminus \{0\} & \xrightarrow{\cong} & \mathbb{R}_{>0} \times S^{2d-1} \\ \pi\big\downarrow & & \\ \mathbb{R}_{>0}. & & \end{array}$$

Reduction along the sphere:

$$\begin{array}{c} \pi_* \left(\text{Holomorphic } \sigma\text{−model } \mathbb{C}^d \setminus \{0\} \to X\right) \\ \Big\| \\ \text{One dimensional } \sigma\text{−model } \mathbb{R}_{>0} \to T^*\mathrm{Map}^{alg}(S^{2d-1}, X). \end{array}$$

Sphere mapping space is really a derived algebraic version. There is a dg algebra A_d with $A_d^0 \hookrightarrow C^\infty(S^{2d-1})$ densely and

$$A_d \hookrightarrow \Omega^{0,*}(\mathbb{C}^d \setminus \{0\})$$

which is dense in cohomology. When $d = 1$, $A_1 = \mathbb{C}[z, z^{-1}]$ and we get algebraic loop space.

Observables

One-dimensional reduction, restricts to a factorization algebra on $\mathbb{R}_{>0} \rightsquigarrow$ dg associative algebra. When $\mathrm{ch}_{d+1}(T_X) = 0$ we get a deformation quantization = "differential operators on the sphere mapping space".

$$\begin{array}{ccc} \mathcal{O}_\hbar\left(T^*\mathrm{Map}(S^{2d-1}, X)\right) & = & : D_\hbar\left(\mathrm{Map}(S^{2d-1}, X)\right) \\ \Big\downarrow{\scriptstyle \hbar \to 0} & & \\ \mathcal{O}\left(T^*\mathrm{Map}(S^{2d-1}, X)\right). & & \end{array}$$

The state space $\mathcal{V}_X$ is equal to the observables supported on the disk in $\mathbb{C}^d$. Factorization product endows $\mathcal{V}_X$ with the structure of a dg module over $D_\hbar(\mathrm{Map}(S^{2d-1}, X))$. It is equal to the "vacuum" module

$$\mathcal{V}_X = D_\hbar \otimes_{D_{\hbar,+}} \mathbb{C}[[\hbar]].$$

Where $D_{\hbar,+} \subset D_\hbar$ is a maximal commutative subalgebra of "positive modes". This plays the role of the Hilbert space in quantum mechanics.

Factorization homology

On a Hopf manifold $Y \cong S^{2d-1} \times S^1$, this algebra appears in the factorization homology.

$$\begin{tikzcd}
\mathbb{C}^d \setminus \{0\} \arrow[r] \arrow[d] & X \arrow[d] \\
\mathbb{R}_{>0} \arrow[r] & S^1.
\end{tikzcd}$$

$$\int_X \mathrm{Obs}^{\mathrm{q}}_{X,\alpha} \simeq \mathrm{Hoch}_*(D_\hbar) \simeq \Omega^{-*}(T^*X)[[\hbar]].$$

The zero modes inside $D_\hbar$ give ordinary differential operators on X, and the Hochschild homology "localizes" to this subalgebra. Reminiscent of Nest-Tsygan's result in deformation quantization.

Conclusions and outlook

- Have not discussed much about "source symmetries" of the holomorphic σ-model. Part of my thesis characterizes symmetries by holomorphic gauge transformations and by holomorphic diffeomorphisms. Lead to higher dimensional Kac-Moody algebras and Virasoro algebras, respectively.
- In particular, there is a dg Lie algebra central extension of holomorphic vector fields on punctured affine space that embeds inside of the associative dg algebra $D_\hbar$. This central extension is parametrized by a higher dimensional version of "central charge" in CFT.
- Gives a sensitive invariant to test dualities in physics (Seiberg duality, mirror symmetry, ...).
- "Gelfand-Kazhdan descent" on the source manifold.