HOLOMORPHIC THEORIES AND RENORMALIZATION

CONTENTS

Our main objective in this chapter is two-fold. First we will define the concept of a holomorphic field theory and set up notation and terminology that we will use throughout the text. Our next goal is more technical, but will provide the backbone for much of the analysis throughout the remainder of this thesis. We will show how certain holomorphic theories are surprisingly well-behaved when it comes to the problem of renormalization.

In **[?]** Costello has provided a mathematical formulation of the Wilsonian approach to quantum field theory. The main takeaway is that to construct a full quantum field theory it suffices to define the theory at each energy (or length) scale and to ask that these descriptions be compatible as we vary the scale. The infamous infinities of quantum field theory arise due to studying behavior of theories at arbitrarily high energies (or small lengths). In physics this is called the ultra-violet (UV) divergence.

A classical theory is described by a local functional I on the space of fields....BW: not sure how much to review. This might go in the overview section of the thesis.

Summarizing, there are two main steps to construct a QFT in our formalism.

Renormalization: For each scale L and regulator $\epsilon > 0$ consider the RG flow from scale ϵ to L:

$$W(P_{\epsilon<L}, I). \tag{1}$$

In general, the limit $\epsilon \to 0$ will not be defined, but by Costello's main result there exists counterterms $I^{CT}(\epsilon)$ such that the $\epsilon \to 0$ limit of

$$W(P_{\epsilon<L}, I - I^{CT}(\epsilon))$$

is well-defined. Denote this limit by $I[L]$. The family $\{I[L]\}$ defines a prequantization.

Gauge consistency: We then ask if the family $\{I[L]\}$ defines a consistent quantization. For each L we require that $I[L]$ satisfy the scale L quantum master equation....

In this section we are concerned with the first step, that of renormalization. The complication here is that even very natural field theories can have a very complicated collections of counterterms. For instance, the naive quantization of Chern-Simons theory on a three-manifold has counterterms even at one-loop. For holomorphic theories, however, we will show how the situation becomes much simpler at least at the level of one-loop.

Theorem 0.1. *Let $\mathcal{E}$ be the fields of a holomorphic theory on $\mathbb{C}^d$ with classical interaction $I \in \mathcal{O}_{\text{loc}}(\mathcal{E})$. Then, there exists a one-loop prequantization $\{I[L] \mid L > 0\}$ of I involving no counterterms. That is, we can find a propagator $P_{\epsilon<L}$ for which the $\epsilon \to 0$ limit of*

$$W(P_{\epsilon<L}, I) \mod \hbar^2$$

exisits. Moreover, if I is holomorphically translation invariant we can pick the family $\{I[L]\}$ to be holomorphically translation invariant as well.

We will use this result repeatedly throughout this thesis. For instance, knowing the one-loop behavior of a field theory is enough to study the possible anomalies to quantization. We will leverage this to formulate and prove index theorems in the context of holomorphic QFT.

One surprising aspect of this comes from thinking about holomorphic theories in a different way. Any supercharge Q of a supersymmetric theory satisfying $Q^2 = 0$ allows one to construct a "twist". In some cases, where clifford multiplication with Q spans all translations such a twist becomes a topological theory (in the weak sense). In any case, however, such a Q defines a "holomorphic twist", which results in the type of holomorphic theories we consider. Regularization in supersymmetric theories, especially gauge theories, is notoriously difficult. Our result implies that after twisting the analytic difficulties become much easier to deal with. Consequently, phenomena such as anomalies can be cast in a more algebraic framework. We will see such an example of this in the case of the holomorphic σ-model in the next chapter.

Already, in **[?]** Li has used a complex one-dimensional version of this fact to all orders in $\hbar$. He uses this to give an elegant interpretation of the quantum master equation for two-dimensional chiral conformal field theories using vertex algebras. We do not make any statements in this work past one-loop quantizations, but the higher loop behavior remains a very interesting and subtle problem that we hope to return to.

1. THE DEFINITION OF A QUANTUM FIELD THEORY

1.1. Classical field theory.

1.2. **Renormalization.** In the BV formalism, as developed in [**?**, **?**], a quantum BV theory consists of a space of fields and an effective action functional $\{S[L]\}_{L\in(0,\infty)}$, which is a family of non-local functionals on the fields that are parametrized by a length scale L and satisfy

(a) an exact renormalization group (RG) flow equation,
(b) the scale L quantum master equation (QME) at every length scale L, and
(c) as $L \to 0$, the functional $S[L]$ has an asymptotic expansion that is local.

The first condition ensures that the scale L action functional $S[L]$ determines the functional at every other scale. The second can be interpreted as saying that we have a proper path integral measure at scale L (i.e., the QME can be seen as a definition of the measure). The third condition implies that the effective action is a quantization of a classical field theory, since a defining property of a classical theory is that its action functional is local. (A full definition is available in Section 8.2 of [**?**].)

Remark 1.1. The length scale is associated with a choice of Riemannian metric on the underlying manifold, but the formalism of [**?**] keeps track of how the space of quantum BV theories depends upon such a choice (and other choices that might go into issues like renormalization). Hence, when the choices should not be essential — such as with a topological field theory — one can typically show rigorously that different choices give equivalent answers. The length scale is also connected with the use of heat kernels in [**?**], but one can work with more general parametrices (and hence more general notions of "scale"), as explained in Chapter 8 of [**?**]. We use a natural length scale in this section; when it becomes relevant, in the context of factorization algebras, we switch to general parametrices.

1.3. **The quantum master equation.**

2. Holomorphic field theories

The goal of this section is to define the notion of a holomorphic field theory.

2.1. **The definition of a holomorphic theory.** We give a general definition of a classical holomorphic theory on a general complex manifold X of complex dimension d. We start with the definition of a *free* holomorphic field theory. After that we will go on to define what an interacting holomorphic theory is.

2.1.1. *Free holomorphic theories.* The fields of any theory are always expressed as sections of some $\mathbb{Z}$-graded vector bundle. Here, the $\mathbb{Z}$-grading is the cohomological, or BRST, grading of the theory. For a holomorphic theory we take this graded vector bundle to be holomorphic. By a *holomorphic* $\mathbb{Z}$-graded vector bundle we mean a $\mathbb{Z}$-graded vector bundle $V = \oplus_i V^i$ such that each graded piece V^i is a holomorphic vector bundle. Thus, the data we start with is the following:

(1) a $\mathbb{Z}$-graded holomorphic vector bundle $V^* = \oplus_i V^i[-i]$, so that the finite dimensional holomorphic vector bundle V^i is in cohomological degree i.

A free classical theory is made up of a space of fields as above together with the data of a linearized BRST differential Q^{BRST} and a symplectic pairing. Ordinarily, the BRST operator is a differential operator on the vector bundle defining the fields. For the class of theories we are considering, we want this operator to be holomorphic.

If E and F are two holomorphic vector bundles on X, we can speak of holomorphic differential operators between E and F. First, note that the Hom-bundle $\mathrm{Hom}(E, F)$ inherits a natural holomorphic structure. By definition, a holomorphic differential operator of order m is a linear map

$$D : \Gamma^{hol}(X; E) \to \Gamma^{hol}(X; F)$$

such that, with respect to a holomorphic coordinate chart $\{z_i\}$ on X, D can be written as

$$D|_{\{z_i\}} = \sum_{|I| \leq m} a_I(z) \frac{\partial^{|I|}}{\partial z_I} \tag{2}$$

where $a_I(z)$ is a local holomorphic section of $\mathrm{Hom}(E, F)$. Here, the sum is over all multi-indices $I = (i_1, \ldots, i_d)$ and

$$\frac{\partial^{|I|}}{\partial z_I} := \frac{\partial^{i_k}}{\partial z_k^{i_k}}.$$

The length is defined by $|I| := i_1 + \cdots + i_d$.

The most basic example of a holomorphic differential operator is the ∂ operator which, for each $1 \leq \ell \leq d$, is a holomorphic differential operator from $E = \wedge^\ell T^{1,0*}X$ to $F = \wedge^{\ell+1} T^{1,0*}X$. Locally, of course, it has the form

$$\partial = \sum_{i=1}^{d} (\mathrm{d}z_i \wedge (-)) \frac{\partial}{\partial z_i},$$

where $\mathrm{d}z_i \wedge (-)$ is the vector bundle homomorphism $\wedge^\ell T^{1,0*}X \to \wedge^{\ell+1} T^{1,0*}X$ sending $\alpha \mapsto \mathrm{d}z_i \wedge \alpha$.

The next piece of data we fix is:

(2) a square zero holomorphic differential operator

$$Q^{hol} : V \to V[-1]$$

of cohomological degree $+1$.

Finally, to define a free theory we need the data of a symplectic pairing. For reasons to become clear in a moment, we must choose this pairing to have a strange cohomological degree. The last piece of data we fix is:

(3) an invertible bundle map

$$(-, -)_V : V \times V \to \Omega^{d,hol}_X[d-1]$$

Here, $\Omega^{d,hol}_X$ is the holomorphic canonical bundle on X.

The definition of the fields of an ordinary field theory are the *smooth* sections of the vector bundle V. In our situation this is a silly thing to do since we lose all of the data of the complex structure we used to define the objects above. The more natural thing to do is take the *holomorphic* sections of the vector bundle V. By construction, the operator Q^{hol} and the pairing $(-, -)_V$ are defined on holomorphic sections, so on the surface this seems reasonable. BW: what should I say the problem is with doing things in the analytic category?

The solution to this problem is in the existence of a resolution for the holomorphic sections of a vector bundle by smooth sections of bundles. Given any holomorphic vector bundle E we can define its *Dolbeualt complex* $\Omega^{0,*}(X,E)$ with it's Dolbeualt operator

$$\bar{\partial} : \Omega^{0,p}(X,E) \to \Omega^{0,p+1}(X,E).$$

Here, $\Omega^{0,p}(X,E)$ denotes smooth sections of the vector bundle

$$\bigwedge^{p} T^{0,1*}X \otimes E$$

and $\bar{\partial}$ is defined in the usual way BW: recall this?.

Using this construction, we take the fields of our free theory to be the complex

$$\mathcal{E}_V = \left(\Omega^{0,*}(X,E), \bar{\partial} + Q^{hol}\right).$$

The operator $\bar{\partial} + Q^{hol}$ is the total linearized BRST operator. By assumption we have $\bar{\partial}Q^{hol} = Q^{hol}\bar{\partial}^*$ so that $(\bar{\partial}+Q^{hol})^2 = 0$ and so the fields still define a complex. The (-1)-shifted symplectic pairing is obtained by composition of the pairing $(-,-)_V$ with integration on $\Omega_X^{d,hol}$. The thing to observe here is that $(-,-)_V$ extends to the Dolbeualt complex in a natural way: we simply combine the wedge product of forms with the pairing on V. The (-1)-shifted pairing ω on $\mathcal{E}$ is defined by the diagram

$$\begin{tikzcd}
\mathcal{E}_V \otimes \mathcal{E}_V \arrow[r, "(-,-)_V"] \arrow[dr, dotted, "\omega_V"'] & \Omega^{0,*}(X, \Omega_X^{d,hol})[d-1] \arrow[d, "\int_X"] \\
& \mathbb{C}[-1].
\end{tikzcd}$$

We arrive at the following definition.

Definition/Lemma 1. A *free holomorphic theory* on a complex manifold X is the data $(V, Q^{hol}, (-,-)_V)$ as in (1), (2), (3) above such that Q^{hol} is a square zero elliptic differential operator that is graded skew self-adjoint for the pairing $(-,-)_V$. The triple $(\mathcal{E}_V, Q_V = \bar{\partial} + Q^{hol}, \omega_V)$ defines a free BV theory in the usual sense.

The usual prescription for writing down the associated action functional holds in this case. If $\varphi \in \Omega^{0,*}(X,V)$ denotes a field the action is

$$S(\varphi) = \int_X \left(\varphi, (\bar{\partial} + Q^{hol})\varphi\right)_V.$$

The first example we explain is related to the subject of Chapter **??** and will serve as the fundamental example of a holomorphic theory.

Example 2.1. *The free $\beta\gamma$ system*. Suppose that

$$V = \underline{\mathbb{C}} \oplus \Omega_X^{d,hol}[d-1].$$

Let $(-,-)_V$ be the pairing

$$(\underline{\mathbb{C}} \oplus \Omega_X^{d,hol}) \otimes (\underline{\mathbb{C}} \oplus \Omega_X^{d,hol}) \to \Omega_X^{d,hol} \oplus \Omega_X^{d,hol} \to \Omega_X^{d,hol}$$

sending $(\lambda, \mu) \otimes (\lambda', \mu') \mapsto (\lambda\mu', \lambda'\mu) \mapsto \lambda\mu' + \lambda'\mu$. In this example we set $Q^{hol} = 0$. One immediately checks that this is a holomorphic free theory as above. The space of fields can be written as

$$\mathcal{E}_V = \Omega^{0,*}(X) \oplus \Omega^{d,*}(X)[d-1].$$

We write $\gamma \in \Omega^{0,*}(X)$ for a field in the first component, and $\beta \in \Omega^{d,*}(X)[d-1]$ for a field in the second component. The action functional reads

$$S(\gamma + \beta, \gamma' + \beta') = \int_X \beta \wedge \bar{\partial}\gamma' + \beta' \wedge \bar{\partial}\gamma.$$

When $d = 1$ this reduces to the ordinary chiral $\beta\gamma$ system from conformal field theory BW: ref. We will discuss this higher dimensional version further in Section BW: . For instance, we will see how this theory is the starting block for constructing general holomorphic σ-models.

Example 2.2. *The free chiral scalar*. Another basic example is the free chiral scalar. Let X be a complex manifold with Hermitian metric g. Let $V = \underline{\mathbb{C}}$, the trivial vector bundle. BW: do this

2.1.2. *Interacting holomorphic theories.* We now define what an interacting holomorphic theory is. In general, an interacting field theory on a manifold M is prescribed by the data of a free theory plus a local functional $I \in \mathcal{O}_{\text{loc}}(\mathcal{E})$ that satisfies the classical master equation. Recall, the sheaf of local functionals on $\mathcal{E} = \Gamma(E)$ is defined as the sheaf of Lagrangian densities

$$\mathcal{O}_{\text{loc}}(\mathcal{E}) = \text{Dens}_M \otimes_{D_M} \mathcal{O}_{red}(JE).$$

In the expression above JE stands for the sheaf of smooth sections of the ∞-jet bundle $\text{Jet}(E)$ which has the structure of a D_X-module.

If V is a holomorphic vector bundle let $J^{hol}E$ denote the sheaf of holomorphic sections of the holomorphic bundle of holomorphic ∞-jets $\text{Jet}^{hol}(V)$. The fibers of the infinite rank vector bundle $\text{Jet}^{hol}(V)$ are isomorphic to

$$\text{Jet}^{hol}(V)|_w = V_w \times \mathbb{C}[[z_1, \ldots, z_d]]$$

where $w \in X$. The sheaf $J^{hol}V$ has the natural structure of a D_X^{hol}-module, that is, it is equipped with a holomorphic flat connection ∇^{hol}. This is completely analogous to the smooth case. Locally, the holomorphic flat connection is of the form

$$\nabla^{hol}|_w = \sum_{i=1}^{d} \mathrm{d}w_i \left(\frac{\partial}{\partial w_i} - \frac{\partial}{\partial z_i} \right),$$

where $\{w_i\}$ is the local coordinate on X near w and z_i is the fiber coordinate labeling the holomorphic jet expansion. Using holomorphic jets we can make a completely analogous definition in our setting.

Definition 2.3. Let V be a vector bundle. The space of *holomorphic Lagrangian densities* on V is

$$\mathcal{O}^{hol}_{red}(V) = \prod_{n>0} \text{Hom}(\text{Jet}(V)^{\times n}, K_X).$$

Equivalently, a holomorphic Lagrangian density is of the form $F = \sum_n F_n \in \mathcal{O}^{hol}_{red}(V)$ where each F_n is a holomorphic polydifferential operator

$$F_n : V \otimes \cdots \otimes V \to K_X.$$

Definition 2.4. Suppose V is a graded holomorphic vector bundle. We define the sheaf of *holomorphic* local functionals on V by

$$\mathcal{O}_{\mathrm{loc}}^{hol}(V) = \Omega_X^{d,hol} \otimes_{D_X^{hol}} \mathcal{O}_{red}(J^{hol}V)[d]$$

Suppose that V is part of the data of a free holomorphic theory $(V, Q^{hol}, (-,-)_V)$. The pairing $(-,-)_V$ endows the space of local functionals with a bracket of cohomological degree $+1$ that we denote by $\{-,-\}^{hol}$. We can now state the definition of a classical holomorphic theory.

Definition 2.5. A *classical holomorphic theory* on a complex manifold X is the data of a free holomorphic theory $(V, Q^{hol}, (-,-)_V)$ plus a functional

$$I^{hol} \in \mathcal{O}_{\mathrm{loc}}^{hol}(V)^+$$

of cohomological degree zero such that $Q^{hol} I^{hol} + \{I^{hol}, I^{hol}\}^{hol} = 0$.

Definition/Lemma 2. Let $(V, Q^{hol}, (-,-)_V, I^{hol})$ be the data of an interacting holomorphic theory. Then $Q^{hol} + \{I^{hol}, -\}$ equips $\mathcal{O}_{\mathrm{loc}}^{hol}(V)$ with the structure of a sheaf of cochain complexes that we will denote

$$\mathrm{Def}_V^{hol} := \left(\mathcal{O}_{\mathrm{loc}}^{hol}(V), Q^{hol} + \{I^{hol}, -\}^{hol}\right).$$

Just as in the free case, we see that classical holomorphic theories are special cases of ordinary classical BV theories. The underlying space of fields, as we have already seen is $\mathcal{E}_V = \Omega^{0,*}(X, V)$. Now, we know that I^{hol} is a $\Omega_X^{d,hol}$-valued functional that is a linear combination of functionals of the form

$$(\varphi_1, \ldots, \varphi_k) \mapsto D_1(\varphi_1) \cdots D_k(\varphi_k) \in \Omega_X^{d,hol}$$

where φ_i is a section of V^{hol} and D_i is a holomorphic differential operator on V. Now, every holomorphic differential operator on the holomorphic vector bundle V extends to a differential operator on its Dolbeualt complex $\mathcal{E}_V = \Omega^{0,*}(X, V)$. Thus, we can define the functional

$$(\alpha_1 \otimes \varphi_1, \ldots, \alpha_k \otimes \varphi_k) \mapsto \int_X D_1(\alpha_1 \otimes \varphi_1) \cdots D_k(\alpha_k \otimes \varphi_k)$$

where $\alpha_i \otimes \varphi_i \in \Omega^{0,*}(X, V)$. The symbol $\int_X$ reminds us that we are working modulo total derivatives, so that the above expression defines an element of $\mathcal{O}_{\mathrm{loc}}(\mathcal{E}_V)$. This defines a linear map $\mathcal{O}_{\mathrm{loc}}^{hol}(V) \to \mathcal{O}_{\mathrm{loc}}(\mathcal{E}_V)$ that we denote $I^{hol} \mapsto I^{\Omega^{0,*}}$.

Lemma 2.6. *Every classical holomorphic theory $(V, Q^{hol}, (-,-)_V, I^{hol})$ determines the structure of a classical BV theory. The underlying free BV theory is given in Definition/Lemma 1 $(\mathcal{E}_V, Q, \omega_V)$ and the interaction is $I^{\Omega^{0,*}}$.*

Proof. We must show that $Q^{hol} I^{hol} + \frac{1}{2}\{I^{hol}, I^{hol}\}^{hol} = 0$ implies

$$\bar{\partial} I^{\Omega^{0,*}} + Q^{hol} I^{\Omega^{0,*}} + \frac{1}{2}\{I^{\Omega^{0,*}}, I^{\Omega^{0,*}}\} = 0.$$

□

Example 2.7. *Holomorphic BF-theory* Let $\mathfrak{g}$ be a Lie algebra and X any complex manifold. Consider the following holomorphic vector bundle on X:

$$V = \underline{\mathfrak{g}}_X \oplus \Omega_X^{d,hol} \otimes \mathfrak{g}^*[d-1].$$

The pairing $V \otimes V \to \Omega^d_{hol}[d-1]$ is similar to the pairing for the $\beta\gamma$ system, except we use the evaluation pairing $\langle -.- \rangle_{\mathfrak{g}}$ between $\mathfrak{g}$ and its dual. In this example, $Q^{hol} = 0$. Write $f \in \mathcal{O}^{hol}_X$ and $\beta \in \Omega^{d,hol}_X$ and consider

$$I^{hol}(f_1 \otimes X_1, f_2 \otimes X_2, \beta \otimes X^\vee) = f_1 f_2 \beta \langle X^\vee, [X_1, X_2] \rangle + \cdots$$

where the $\cdots$ means that we symmetrize the inputs. This defines an element $I^{hol} \in \mathcal{O}^{hol}_{\mathrm{loc}}(V)^+$ and the Jacobi identity ensures $\{I^{hol}, I^{hol}\}^{hol} = 0$. The fields of the corresponding BV theory are

$$\mathcal{E}_V = \Omega^{0,*}(X, \mathfrak{g}) \oplus \Omega^{d,*}(X, \mathfrak{g}^*)[d-1].$$

The induced local functional $I^{\Omega^{0,*}}$ on $\mathcal{E}_V$ is

$$I^{\Omega^{0,*}}(\alpha, \beta) = \int_X \langle \beta, [\alpha, \alpha] \rangle_{\mathfrak{g}}.$$

Example 2.8. *Topological BF-theory*

Lemma 2.9. *Suppose $(V, 0, (-,-)_V, I^{hol})$ is the data of a holomorphic theory with $Q^{hol} = 0$. Let $(\mathcal{E}_V, Q = \bar{\partial}, \omega_V, I)$ be the corresponding BV theory. Then, there is a quasi-isomorphism of sheaves*

$$\mathrm{Def}^{hol}_V \simeq \mathrm{Def}_{\mathcal{E}_V}$$

compatible with the brackets $\{-,-\}^{hol}$ and $\{-,-\}$ on both sides.

Remark 2.10. Just as in the ordinary case we can formulate the data of a classical holomorphic theory in terms of sheaves of L_∞ algebras. We will not do that here, but hope the idea of how to do so is clear.

2.2. **Holomorphically translation invariant theories.** When working on affine space $\mathbb{R}^n$ one can ask for a theory to be invariant with respect to translations. In this section, we consider the affine manifold $\mathbb{C}^d = \mathbb{R}^{2d}$ equipped with its standard complex structure and define what a *holomorphically translation invariant* theory is on it. It will be a very special case of a general holomorphic theory as defined above.

Let V be a holomorphic vector bundle on $\mathbb{C}^n$ and suppose we fix an identification of bundles

$$V \cong \mathbb{C}^d \times V_0$$

where V_0 is the fiber of V at $0 \in \mathbb{C}^d$. We want to consider a classical theory with space of fields given by $\Omega^{0,*}(\mathbb{C}^d, V) \cong \Omega^{0,*}(\mathbb{C}^d) \otimes_{\mathbb{C}} V_0$. Moreover, we want this theory to be invariant with respect to the group of holomorphic translations on $\mathbb{C}^d$. Per usual, it is best to work with the corresponding Lie algebra of translations. Using the complex structure, we choose a presentation for the Lie algebra of all translations given by

$$\mathbb{C}^{2d} \cong \mathrm{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i} \right\}_{1 \leq i \leq d}.$$

To define a theory, we need to fix a non-degenerate pairing on V. Moreover, we want this to be translation invariant. So, suppose

$$\langle \ , \ \rangle_V : V \otimes V \to \Omega^{d,hol}_{\mathbb{C}^d}[d-1] \tag{3}$$

is a skew-symmetric bundle map that is equivariant for the Lie algebra of translations. The shift is so that the resulting pairing on the Dolbeualt complex is of the appropriate degree. Here, equivariance means that for sections v, v' we have

$$\langle \frac{\partial}{\partial z_i} v, v' \rangle_V = L_{\partial_{z_i}} \langle v, v' \rangle_V$$

where the right-hand side denotes the Lie derivative applied to $(v, v')_V \in \Omega^{d,hol}_{\mathbb{C}^d}$. There is a similar relation for the anti-holomorphic derivatives. We obtain a $\mathbb{C}$-valued pairing on $\Omega^{0,*}_c(\mathbb{C}^d, V)$ via integration:

$$\int_{\mathbb{C}^d} \circ (-,-)_V : \Omega^{0,*}_c(\mathbb{C}^d, V) \otimes \Omega^{0,*}_c(\mathbb{C}^d, V) \xrightarrow{\wedge \cdot (-,-)_V} \Omega^{d,*}(\mathbb{C}^d) \xrightarrow{\int} \mathbb{C}.$$

The first arrow is the wedge product of forms combined with the pairing on V. The second arrow is only nonzero on forms of type $\Omega^{d,d}$. Clearly, integration is translation invariant, so that the composition is as well.

This pairing $\Omega^{0,*}(\mathbb{C}^d, V)$ together with the differential $\bar{\partial}$ are enough to define a free theory. However, it is convenient to consider a slightly generalized version of this situation. We want to allow deformations of the differential $\bar{\partial}$ on Dolbeault forms of the form

$$Q = \bar{\partial} + Q^{hol}$$

where Q^{hol} is a holomorphic differential operator of the form

$$Q^{hol} = \sum_I \frac{\partial}{\partial z^I} \mu_I \tag{4}$$

where I is some multi-index and $\mu_I : V \to V$ is a linear map of cohomological degree $+1$. Note that we have automatically written Q^{hol} in a way that it is translation invariant. Of course, for this differential to define a free theory there needs to be some compatibility with the pairing on V. We can summarize this in the following definition, which should be viewed as a slight modification of a free theory to this translation invariant holomorphic setting.

Definition 2.11. A *holomorphically translation invariant free BV theory* is the data of a holomorphic vector bundle V together with

(1) an identification $V \cong \mathbb{C}^d \times V_0$;
(2) a translation invariant skew-symmetric pairing $\langle -,- \rangle_V$ as in (3);
(3) a holomorphic differential operator Q^{hol} as in (4);

such that the following conditions hold

(1) the induced $\mathbb{C}$-valued pairing $\int \circ \langle -,- \rangle_V$ is non-degenerate;
(2) the operator Q^{hol} satisfies $(\bar{\partial} + Q^{hol})^2 = 0$ and

$$\int \langle Q^{hol} v, v' \rangle_V = \pm \int \langle v, Q^{hol} v' \rangle.$$

The first condition is required so that we obtain an actual (-1)-shifted symplectic structure on $\Omega^{0,*}(\mathbb{C}^d, V)$. The second condition implies that the derivation $Q = \bar{\partial} + Q^{hol}$ defines a cochain complex

$$\mathcal{E}_V = \left(\Omega^{0,*}(\mathbb{C}^d, V), \bar{\partial} + Q^{hol} \right),$$

and that Q is skew self-adjoint for the symplectic structure. Thus, in particular, $\mathcal{E}_V$ together with the pairing define a free BV theory in the ordinary sense. In the usual way, we obtain the action functional via

$$S(\varphi) = \int \langle \varphi, (\bar{\partial} + Q^{hol})\varphi \rangle_V.$$

Before going further, we will list some familiar examples.

Example 2.12. *The free $\beta\gamma$ system on $\mathbb{C}^d$.* Suppose that

$$V = \underline{\mathbb{C}} \oplus K_{\mathbb{C}^d}[d-1].$$

Let $\langle\ ,\ \rangle_V$ be the pairing

$$(\underline{\mathbb{C}} \oplus K_{\mathbb{C}^d}) \otimes (\underline{\mathbb{C}} \oplus K_{\mathbb{C}^d}) \to K_{\mathbb{C}^d} \oplus K_{\mathbb{C}^d} \to K_{\mathbb{C}^d}$$

sending $(\lambda, \mu) \otimes (\lambda', \mu') \mapsto (\lambda\mu', \lambda'\mu) \mapsto \lambda\mu' + \lambda'\mu$. Finally, let $Q^{hol} = 0$. One immediately checks that this is a holomorphically translation invariant free theory as above. The space of fields can be written as

$$\Omega^{0,*}(\mathbb{C}^d) \oplus \Omega^{d,*}(\mathbb{C}^d)[d-1].$$

We write γ for a field in the first component, and β for a field in the second component. The action functional reads

$$S(\gamma + \beta, \gamma' + \beta') = \int_{\mathbb{C}^d} \beta \wedge \bar{\partial}\gamma' + \beta' \wedge \bar{\partial}\gamma.$$

When $d = 1$ this reduces to the ordinary chiral $\beta\gamma$ system from conformal field theory BW: ref. We will discuss this higher dimensional version further in Section BW: . For instance, we will see how this theory is the starting block for constructing general holomorphic σ-models.

Of course, there are many variants of the $\beta\gamma$ system that we can consider. For instance, if E is *any* holomorphic vector bundle we can take

$$V = E \oplus K_{\mathbb{C}^d} \otimes E^\vee$$

where $E^\vee$ is the linear dual bundle. The pairing is constructed as in the case above where we also use the evaluation pairing between E and $E^\vee$, $\mathrm{ev}_E : E \otimes E^\vee \to \mathbb{C}$. In thise case, the fields are $\gamma \in \Omega^{0,*}(\mathbb{C}^d, E)$ and $\beta \in \Omega^{d,*}(\mathbb{C}^d, E^\vee)[d-1]$. The action functional is simply

$$S(\gamma, \beta) = \int_{\mathbb{C}^d} \mathrm{ev}_E(\beta \wedge \bar{\partial}\gamma).$$

Example 2.13. *Topological ...* Consider the above example with $Q^{hol} = \partial$..BW: finish.

2.2.1. *Holomorphic translation invariant deformations.* Any local Lie algebra on a manifold endows the structure of an L_∞ algebra on its fibers. In particular, if L the graded vector bundle associated to local Lie algebra on $\mathbb{C}^d$, its fiber over 0, L_0, is equipped with the structure of an L_∞ algebra.

Suppose $\mathcal{L}$ is a holomorphically translation invariant local Lie algebra on $\mathbb{C}^d$ of the form $\Omega^{0,*}(\mathbb{C}^d, L)$ where L is a graded holomorphic vector bundle. In this situation, we are interested in studying the local cochains of $\mathcal{L}$ that are translation invariant. We will use the following result over and over again throughout this work.

Proposition 2.14. *Suppose $\mathcal{L}$ is a holomorphically translation invariant local Lie algebra on $\mathbb{C}^d$ such that $\ell_1 = \bar{\partial}$. Then, one has*

$$C^*_{\rm loc}(\mathcal{L})^{\mathbb{C}^d} \simeq \mathbb{C} \cdot \mathrm{d}^d z \otimes^{\mathbb{L}}_{\mathbb{C}[\frac{\partial}{\partial z_i}]} C^*_{\rm Lie, red}(L_0[[z_1, \ldots, z_d]])[d].$$

For instance, if $L = \underline{\mathfrak{g}}$ is the constant bundle on $\mathbb{C}^d$ where $\mathfrak{g}$ is an ordinary Lie (or L_∞) algebra one has $L_0 = \mathfrak{g}$ so that

$$C^*_{\rm loc}(\Omega^{0,*}(\mathbb{C}^d, \mathfrak{g}))^{\mathbb{C}^d} \simeq \mathbb{C} \cdot \mathrm{d}^d z \otimes^{\mathbb{L}}_{\mathbb{C}[\frac{\partial}{\partial z_i}]} C^*_{\rm Lie, red}(\mathfrak{g}[[z_1, \ldots, z_d]]).$$

2.2.2. *Holomorphic deformations on an arbitrary complex manifold.* BW: fix and move this to above There is a more general result that holds on an arbitrary complex manifold. Recall, that on a complex manifold X we have introduced the notion of a holomorphic local Lie algebra L. Let $\mathcal{L} = \mathrm{Dol}(L) = \Omega^{0,*}(X, L)$ be its associated local Lie algebra. Ordinarily, the local Lie algebra cohomology of a local Lie algebra is computed in terms of the Lie algebra cohomology of the associated jet bundle. With holomorphicity and some mild assumption, we can, up to quasi-isomorphism, exhibit a smaller complex computing this local cohomology.

Proposition 2.15. *Suppose L is a holomorphic local Lie algebra with $\ell_1 = 0$, and let $\mathcal{L} = \Omega^{0,*}(X, L)$ be its associated local Lie algebra. There is a quasi-isomorphisms of sheaves of cochain complexes*

$$C^*_{\rm loc}(\mathcal{L}) \simeq \Omega^{d,hol}_X \otimes^{\mathbb{L}}_{D_X^{hol}} C^*_{\rm Lie}(J\mathcal{L}^{hol}).$$

Proof. Recall, the definition of $C^*_{\rm loc}(\mathcal{L})$ is given in terms of D-module data by

$$C^*_{\rm loc}(\mathcal{L}) = \Omega^{d,d}_X \otimes^{\mathbb{L}}_{D_X} C^*_{\rm Lie}(J\mathcal{L})$$

where $J\mathcal{L}$ denotes the ∞-jet bundle of $\Omega^{0,*}(X, L)$. Of course, $J\mathcal{L}$ is a bundle equipped with a natural flat connection, and hence the structure of a D_X-module. The Chevalley-Eilenberg complex $C^*_{\rm Lie}(J\mathcal{L})$ inherits this D_X-module structure.

On the other hand, if we view L as a holomorphic vector bundle, it makes sense to look at the *holomorphic* jet bundle $J^{hol}L$. This holomorphic vector bundle is equipped with a holomorphic flat connection, and hence is a module for the sheaf of holomorphic differential operators D^{hol}_X.

Lemma 2.16. *Let $(J^{hol}L)^{C^\infty}$ be the D^{hol}_X-module $J^{hol}L$ viewed, via the forgetful functor, as a D_X-module. Then, there is a quasi-isomorphism of dg D_X-modules $(J^{hol}L)^{C^\infty} \simeq J\Omega^{0,*}(X, L)$. Furthermore, this quasi-isomorphism is compatible with the L_∞ structures, so that we obtain a quasi-isomorphism of dg D_X-modules $C^*_{\rm Lie}(J^{hol}L)^{C^\infty} \simeq C^*_{\rm Lie}(J\mathcal{L})$.*

Proof. For any holomorphic vector bundle E, the Dolbeualt complex of E is a resolution of the sheaf of holomorphic sections of E. Thus, there is an equivalence of sheaves on X

$$\Omega^{0,*}_X(E) \simeq \Gamma^{hol}_X(E).$$

Thus, there is an equivalence of sheaves on X:

$$J^{hol}L \simeq J\Omega^{0,*}(X, L).$$

We need to see that this equivalence respects the D_X-module structure present on both sides.... □

To finish the proof, we verify the following general lemma.

Lemma 2.17. *Suppose V is a D_X^{hol}-module, and let V^{C^∞} denote its underlying D_X-module. Then, there is a quasi-isomorphism of sheaves of cochain complexes*

$$\Omega_X^{d,hol} \otimes^{\mathbb{L}}_{D_X^{hol}} V[d] \simeq \Omega_X^{d,d} \otimes^{\mathbb{L}}_{D_X} V^{C^\infty}.$$

□

3. Renormalization of holomorphic theories

In this section we study the renormalization of holomorphically translation invariant field theories on $\mathbb{C}^d$ for any $d \geq 1$. We start with a classical interacting holomorphic theory on $\mathbb{C}^d$ and consider one-loop homotopy RG flow from some finite scale ϵ to scale L. That is, we consider the sum over graphs of genus zero and one where at each vertex we place the holomorphic interaction. To obtain a prequantization of a classical theory one must make sense of the $\epsilon \to 0$ limit of this construction. In general, this involves introducing a family of counterterms. Our main result is that for a holomorphic theory no such counterterms are required, and one obtains a well-defined $\epsilon \to 0$ limit.

We can write the fields of a holomorphic theory on $\mathbb{C}^d$ as

$$\mathcal{E} = \left(\Omega^{0,*}(\mathbb{C}^d, V), \bar\partial + Q^{hol}\right)$$

where V is a graded holomorphic vector bundle and Q^{hol} is a holomorphic differential operator.

Since the theory is holomorphically translation invariant we have an identification $\Omega^{0,*}(\mathbb{C}^d, V) \cong \Omega^{0,*}(\mathbb{C}^d) \otimes_{\mathbb{C}} V_0$ where V_0 is the fiber of V over $0 \in \mathbb{C}^d$. Further, we can write the (-1)-shifted symplectic structure defining the classical BV theory in the form

$$\omega(\alpha \otimes v, \beta \otimes w) = (v, w)_{V_0} \int \mathrm{d}^d z (\alpha \wedge \beta)$$

where $(-,-)_{V_0}$ is a degree $(d-1)$-shifted BW: check pairing on the finite dimensional vector space V_0.

3.1. Holomorphic gauge fixing.

To begin the process of renormalization we must fix the data of a gauge fixing operator. A gauge fixing operator is an operator on fields

$$Q^{GF} : \mathcal{E} \to \mathcal{E}[1]$$

of cohomological degree -1 such that $[Q, Q^{GF}]$ is a generalized Laplacian on $\mathcal{E}$ where Q is the linearized BRST operator. For a full definition of this see Definition ?? **??**.

For holomorphic theories there is a convenient choice for a gauge fixing operator. To construct it we fix the standard flat metric on $\mathbb{C}^d$. Doing this, we let $\bar\partial^*$ be the adjoint of the operator $\bar\partial$. Using the coordinates on $(z_1, \ldots, z_d) \in \mathbb{C}^d$ we can write this operator as

$$\bar\partial^* = \sum_{i=1}^d \frac{\partial}{\partial(\mathrm{d}\bar z_i)} \frac{\partial}{\partial z_i}.$$

Equivalently $\frac{\partial}{\partial(\mathrm{d}\bar z_i)}$ is equal to contraction with the anti-holomorphic vector field $\frac{\partial}{\partial \bar z_i}$. The operator $\bar\partial^*$ extends to the complex of fields via the formula

$$Q^{GF} = \bar\partial^* \otimes \mathrm{id}_V : \Omega^{0,*}(X, V) \to \Omega^{0,*-1}(X, V),$$

We claim that this is a gauge fixing operator for our holomorphic theory. Indeed, since Q^{hol} is a translation invariant holomorphic differential operator we have

$$[\bar{\partial} + Q^{hol}, Q^{GF}] = [\bar{\partial}, \bar{\partial}^*] \otimes \mathrm{id}_V.$$

The operator $[\bar{\partial}, \bar{\partial}^*]$ is simply the Dolbeualt Laplacian on $\mathbb{C}^d$, which is certainly a generalized Laplacian. In coordinates it is

$$[\bar{\partial}, \bar{\partial}^*] = -\sum_{i=1}^{d} \frac{\partial}{\partial \bar{z}_i} \frac{\partial}{\partial z_i}$$

By definition, the heat kernel is the dual BW: check factor

Pick a basis $\{e_i\}$ of V_0 and let

$$\mathbf{C}_{V_0} = \sum_{i,j} \omega_{ij} (e_i \otimes e_j) \in V_0 \otimes V_0$$

be the quadratic Casimir. Here, (ω_{ij}) is the inverse matrix to the pairing $(-,-)_{V_0}$. The regularized heat kernel then takes the form

$$K_\epsilon(z,w) = K^{an}(z,w) \cdot \mathbf{C}_{V_0}$$

Lemma 3.1. *If Γ is a tree then $\lim_{\epsilon \to 0} W_\Gamma(P_{\epsilon<L}, I)$ exists.*

3.2. **One-loop weights.**

Definition 3.2. Let $\epsilon, L > 0$. In addition, fix the following data.

(1) An integer $k \geq 1$ that will be the number of vertices of the graph.
(2) For each $\alpha = 1, \dots, k$ a sequence of integers

$$\vec{n}^\alpha = (n_1^\alpha, \dots, n_d^\alpha).$$

We denote by $(\vec{n}) = (n_i^j)$ the corresponding $d \times k$ matrix of integers.
(3) A smooth compactly supported function $\Phi \in C_c^\infty((\mathbb{C}^d)^k) = C_c^\infty(\mathbb{C}^{dk})$.

The analytic weight associated to the triple $(k, (\vec{n}), \Phi)$ is

$$W^\Phi_{k,(\vec{n})}(\epsilon, L) = \int_{(z^1,\dots,z^k) \in (\mathbb{C}^d)^k} \prod_{\alpha=1}^{k} \mathrm{d}^d z^\alpha \Phi(z^1, \dots, z^\alpha) \prod_{\alpha=1}^{k} \left(\frac{\partial}{\partial z^i}\right)^{\vec{n}^\alpha} P^{an}_{\epsilon<L}(z^i, z^{i+1}). \tag{5}$$

In the above expression, we use the convention that $z^{k+1} = z^1$.

We will refer to the collection of data $(k, (\vec{n}), \Phi)$ in the definition as *wheel data*. The motivation for this is that the weight $W^\Phi_{k,(\vec{n})}(\epsilon, L)$ is the analytic part of the full weight $W_\Gamma(P_{\epsilon<L}, I)$ where Γ is a wheel with k vertices.

We have reduced the proof of Theorem **??** to showing that the $\epsilon \to 0$ limit of the analytic weight $W^\Phi_{k,(\vec{n})}(\epsilon, L)$ exists for any tripe of wheel data $(k, (\vec{n}), \Phi)$. To do this, there are two steps. First, we show a vanishing result that says when $k \geq d$ the weights vanish for purely algebraic reasons. The second part is the most technical aspect of the chapter where we show that for $k > d$ the weights have nice asymptotic behavior as a function of ϵ.

Lemma 3.3. *Let $(k, (\vec{n}), \Phi)$ be a triple of wheel data. If the number of vertices k satisfies $k \leq d$ then*

$$W^\Phi_{k,(\vec{n})}(\epsilon, L) = 0$$

for any $\epsilon, L > 0$.

Proof. In the integral expression for the weight (5) there is the following factor involving the product over the edges of the propagators:

$$\prod_{\alpha=1}^{k}\left(\frac{\partial}{\partial z^i}\right)^{\vec{n}^\alpha} P^{an}_{\epsilon<L}(z^i, z^{i+1}). \tag{6}$$

We will show that this expression is identically zero. To simplify the expression we first make the following change of coordinates on $\mathbb{C}^{dk}$:

$$w^i = z^{\alpha+1} - z^\alpha \quad , \quad 1 \le \alpha < k \tag{7}$$

$$w^k = z^k. \tag{8}$$

Introduce the following operators

$$\eta^\alpha = \sum_{i=1}^{d} \bar{w}_i^\alpha \frac{\partial}{\partial(\mathrm{d}\bar{w}_i^\alpha)}$$

acting on differential forms on $\mathbb{C}^{dk}$. The operator η^α lowers the anti-holomorphic Dolbuealt type by one : $\eta : (p,q) \to (p, q-1)$. Equivalently, η^α is contraction with the anti-holomorphic Euler vector field $\bar{w}_i^\alpha \partial/\partial\bar{w}_i^\alpha$.

Once we do this, we see that the expression (6) can be written as

$$\left(\left(\sum_{\alpha=1}^{k-1}\eta^\alpha\right)\prod_{i=1}^{d}\left(\sum_{\alpha=1}^{k-1}\mathrm{d}\bar{w}_i^\alpha\right)\right)\prod_{\alpha=1}^{k-1}\left(\eta^\alpha\prod_{i=1}^{d}\mathrm{d}\bar{w}_i^\alpha\right).$$

Note that only the variables $\bar{w}_i^\alpha$ for $i = 1, \ldots, d$ and $\alpha = 1, \ldots, k-1$ appear. Thus we can consider it as a form on $\mathbb{C}^{d(k-1)}$. As such a form it is of Dolbeualt type $(0, (d-1) + (k-1)(d-1)) = (0, (d-1)k)$. If $k < d$ then clearly $(d-1)k > d(k-1)$ so the form has greater degree than the dimension of the manifold and hence it vanishes.

The case left to consider is when $k = d$. In this case, the expression in (6) can be written as

$$\left(\left(\sum_{\alpha=1}^{d-1}\eta^\alpha\right)\prod_{i=1}^{d}\left(\sum_{\alpha=1}^{d-1}\mathrm{d}\bar{w}_i^\alpha\right)\right)\prod_{\alpha=1}^{d-1}\left(\eta^\alpha\prod_{i=1}^{d}\mathrm{d}\bar{w}_i^\alpha\right). \tag{9}$$

Again, since only the variables $\bar{w}_i^\alpha$ for $i = 1, \ldots, d$ and $\alpha = 1, \ldots, d-1$ appear, we can view this as a differential form on $\mathbb{C}^{d(d-1)}$. Furthermore, it is a form of type $(0, d(d-1))$. For any vector field X on $\mathbb{C}^{d(d-1)}$ the interior derivative i_X is a graded derivation. Suppose ω_1, ω_2 are two $(0,*)$ forms on $\mathbb{C}^{d(d-1)}$ such that the sum of their degrees is equal to d^2. Then, $\omega_1 \iota_X \omega_2$ is a top form for any vector field on $\mathbb{C}^{d(d-1)}$. Since $\omega_1\omega_2 = 0$ for form type reasons, we conclude that $\omega_1 \iota_X \omega_2 = \pm(i_X\omega_1)\omega_2$ with sign depending on the dimension d. Applied to the vector field $\bar{z}_i^1 \partial/\partial\bar{w}_i^1$ in ([**?**]) we see that the expression can be written (up to a sign) as

$$\eta^1\left(\sum_{\alpha=1}^{d-1}\eta^\alpha\prod_{i=1}^{d}\left(\sum_{\alpha=1}^{d-1}\mathrm{d}\bar{w}_i^\alpha\right)\right)\left(\prod_{i=1}^{d}\mathrm{d}\bar{w}_i^1\right)\prod_{\alpha=2}^{d-1}\left(\eta^\alpha\prod_{i=1}^{d}\mathrm{d}\bar{w}_i^\alpha\right).$$

Repeating this, for $\alpha = 2, \ldots, k-1$ we can write this expression (up to a sign) as

$$\left(\eta_{k-1}\cdots\eta_2\eta_1\sum_{\alpha=1}^{k-1}\eta^\alpha\prod_{i=1}^{d}\left(\sum_{\alpha=1}^{k-1}\mathrm{d}\bar{w}_i^\alpha\right)\right)\prod_{\alpha=1}^{k-1}\prod_{i=1}^{d}\mathrm{d}\bar{w}_i^\alpha$$

The expression inside the parentheses is zero since each term in the sum over α involves a term like $\eta^\beta\eta^\beta = 0$. This completes the proof for $k = d$. $\square$

Lemma 3.4. *Let $(k, (\vec{n}), \Phi)$ be a triple of wheel data such that $k > d$. Then the $\epsilon \to 0$ limit of the analytic weight*

$$\lim_{\epsilon \to 0} W^{\Phi}_{k,(\vec{n})}(\epsilon, L)$$

exists.

Proof. We will bound the absolute value of the weight in Equation (5) and show that it has a well-defined $\epsilon \to 0$ limit. First, consider the change of coordinates as in Equations (7),(8). The weight can be written as

(10)

$$\int_{w^k \in \mathbb{C}^d} \mathrm{d}^d w^k \int_{(w_1,\ldots,w_{k-1}) \in (\mathbb{C}^d)^{k-1}} \left(\prod_{\alpha=1}^{k-1} \mathrm{d}^d w^\alpha\right) \Phi(w^1, \ldots, w^k) \left(\prod_{\alpha=1}^{k-1} \left(\frac{\partial}{\partial w^\alpha}\right)^{\vec{n}^\alpha} P^{an}_{\epsilon<L}(w^\alpha)\right) \sum_{\alpha=1}^{k-1} \left(\frac{\partial}{\partial w^\alpha}\right)^{\vec{n}^k} P^{an}\left(\sum_{\alpha=1}^{k-1} w^\alpha\right).$$

For $\alpha = 1, \ldots, k-1$ the notation

$$P^{an}_{\epsilon<L}(w^\alpha) = \int_{t_\alpha=\epsilon}^{L} \frac{\mathrm{d}t_\alpha}{4\pi t_\alpha} \bar{\partial}^* \mathit{BW:FINISH}$$

makes sense since $P^{an}_{\epsilon<L}(z^\alpha, z^{\alpha+1})$ is only a function of $w^\alpha = z^{\alpha+1} - z^\alpha$. Similarly $P^{an}_{\epsilon<L}(z^{k+1}, z^1)$ is a function of

$$z^k - z^1 = \sum_{\alpha=1}^{k-1} w^\alpha.$$

Expanding out the propagators the weight takes the form

$$\begin{aligned}
&\int_{w^k \in \mathbb{C}^d} \mathrm{d}^{2d} w^k \int_{(w_1,\ldots,w_{k-1}) \in (\mathbb{C}^d)^{k-1}} \left(\prod_{\alpha=1}^{k-1} \mathrm{d}^{2d} w^\alpha\right) \Phi(w^1, \ldots, w^k) \int_{(t_1,\ldots,t_k) \in [\epsilon, L]^k} \prod_{\alpha=1}^{k} \frac{\mathrm{d}t_\alpha}{4\pi t_\alpha} \\
&\times \sum_{i_1,\ldots,i_{k-1}=1}^{d} \left(\frac{\bar{w}^1_{i_1}}{t_1} \frac{(\bar{w}^1)^{n^1}}{t^{|n^1|}}\right) \cdots \left(\frac{\bar{w}^{k-1}_{i_{k-1}}}{t_{k-1}} \frac{(\bar{w}^{k-1})^{n^{k-1}}}{t^{|n^{k-1}|}}\right) \left(\sum_{\alpha=1}^{k-1} \frac{\bar{w}^\alpha_{i_k}}{t_k} \cdot \frac{1}{t^{|n^k|}} \left(\sum_{\alpha=1}^{k-1} \bar{w}^\alpha\right)^{n^k}\right) \\
&\times \exp\left(-\sum_{\alpha=1}^{k-1} \frac{|w^\alpha|^2}{t_\alpha} - \frac{1}{t_k} \left|\sum_{\alpha=1}^{k-1} w^\alpha\right|^2\right)
\end{aligned}$$

The notation used above warrants some explanation. Recall, for each α the vector of integers is defined as $n^\alpha = (n^\alpha_1, \ldots, n^\alpha_d)$. We use the notation

$$(\bar{w}^\alpha)^{n^\alpha} = \bar{w}_1^{n_1^\alpha} \cdots \bar{w}_d^{n_d^\alpha}.$$

Furthermore, $|n^\alpha| = n^\alpha_1 + \cdots + n^\alpha_d$. Each factor of the form $\frac{\bar{w}^\alpha_{i_\alpha}}{t_\alpha}$ comes from the application of the operator $\frac{\partial}{\partial z_i}$ in $\bar{\partial}^*$ applied to the propagator. The factor $\frac{(\bar{w}^\alpha)^{n^\alpha}}{t^{|n^\alpha|}}$ comes from applying the operator $\left(\frac{\partial}{\partial w}\right)^{n^\alpha}$ to the propagator. Note that $\bar{\partial}^*$ commutes with any translation invariant holomorphic differential operator, so it doesn't matter which order we do this.

To bound this integral we will recognize each of the factors

$$\frac{\bar{w}^\alpha_{i_\alpha}}{t_\alpha} \frac{(\bar{w}^\alpha)^{n^\alpha}}{t^{|n^\alpha|}}$$

as coming from the application of a certain holomorphic differential operator to the exponential in the last line. We will then integrate by parts to obtain a simple Gaussian integral which will give us the necessary bounds in the t-variables. Let us denote this Gaussian factor by

$$E(w,t) := \exp\left(-\sum_{\alpha=1}^{k-1}\frac{|w^\alpha|^2}{t_\alpha} - \frac{1}{t_k}\left|\sum_{\alpha=1}^{k-1} w^\alpha\right|^2\right)$$

For each α, i_α introduce the $t = (t_1,\ldots,t_k)$-dependent holomorphic differential operator

$$D_{\alpha,i_\alpha}(t) := \left(\frac{\partial}{\partial w^\alpha_{i_\alpha}} - \sum_{\beta=1}^{k-1}\frac{t_\beta}{t_1+\cdots+t_k}\frac{\partial}{\partial w^\beta_{i_\alpha}}\right)\prod_{j=1}^{d}\left(\frac{\partial}{\partial w^\alpha_j} - \sum_{\beta=1}^{k-1}\frac{t_\beta}{t_1+\cdots+t_k}\frac{\partial}{\partial w^\beta_j}\right)^{n^\alpha_j}.$$

The following lemma is an immediate calculation

Lemma 3.5. *One has*

$$D_{\alpha,i_\alpha}E(w,t) = \frac{\bar{w}^\alpha_{i_\alpha}}{t_\alpha}\frac{(\bar{w}^\alpha)^{n^\alpha}}{t^{|n^\alpha|}}E(w,t).$$

Note that all of the D_{α,i_α} operators mutually commute. Thus, we can integrate by parts iteratively to obtain the following expression for the weight:

$$\pm\int_{w^k\in\mathbb{C}^d} \mathrm{d}^{2d}w^k \int_{(w_1,\ldots,w_{k-1})\in(\mathbb{C}^d)^{k-1}}\left(\prod_{\alpha=1}^{k-1}\mathrm{d}^{2d}w^\alpha\right)\int_{(t_1,\ldots,t_k)\in[\epsilon,L]^k}\prod_{\alpha=1}^{k}\frac{\mathrm{d}t_\alpha}{4\pi t_\alpha}$$
$$\times\left(\sum_{i_1,\ldots,i_d} D_{1,i_1}\cdots D_{k-1,i_{k-1}}\sum_{\alpha=1}^{k-1}D_{\alpha,i_k}\Phi(w^1,\ldots,w^k)\right)\times\exp\left(-\sum_{\alpha=1}^{k-1}\frac{|w^\alpha|^2}{t_\alpha} - \frac{1}{t_k}\left|\sum_{\alpha=1}^{k-1}w^\alpha\right|^2\right).$$

BW: all the differential operators D_{α,i_α} are uniformly bounded in t. To make these precise I should find what the uniform bound is.

Thus, the absolute value of the weight is bounded by

(11)
$$|W^\Phi_{k,(\vec{n})}(\epsilon,L)| \leq C\int_{w^k\in\mathbb{C}^d}\mathrm{d}^{2d}w^k\int_{(w^1,\ldots,w^{k-1}}\prod_{\alpha=1}^{k-1}\mathrm{d}^{2d}w^\alpha\Psi(w^1,\ldots,w^{k-1},w^k)\int_{(t_1,\ldots,t_k)\in[\epsilon,L]^k}\mathrm{d}t_1\ldots\mathrm{d}t_k\frac{1}{(4\pi)^{dk}}\frac{1}{t_1^d\cdots t_k^d}\times E(w,t$$

To compute the right hand side we will perform a Gaussian integration with respect to the variables $(w^1,\ldots,w^{k-1})$. To this end, notice that the exponential can be written as

$$E(w,t) = \exp\left(-M_{\alpha\beta}(w^\alpha,w^\beta)\right)$$

where $(M_{\alpha\beta})$ is the $(k-1)\times(k-1)$ matrix given by

$$\begin{pmatrix} a_1 & b & b & \cdots & b\\ b & a_2 & b & \cdots & b\\ b & b & a_3 & \cdots & b\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ b & b & b & \cdots & a_{k-1}\end{pmatrix}$$

where $a_\alpha = t_\alpha^{-1} + t_k^{-1}$ and $b = t_k^{-1}$. The pairing (w^α, w^β) is the usual Hermitian pairing on $\mathbb{C}^d$, $(w^\alpha,w^\beta) = \sum_i w^\alpha_i \bar{w}^\beta_i$. After some straightforward linear algebra we find that

$$\det(M_{\alpha\beta})^{-1} = \frac{t_1\cdots t_k}{t_1+\cdots+t_k}.$$

Thus, after performing the Gaussian integration over $(w^1, \ldots, w^{k-1})$ the inequality in (11) becomes

(12)
$$|W^{\Phi}_{k,(\vec{n})}(\epsilon, L)| \leq C' \int_{w^k \in \mathbb{C}^d} \mathrm{d}^{2d} w^k \Psi(0, \ldots, 0, w^k) \int_{(t_1, \ldots, t_k) \in [\epsilon, L]^k} \mathrm{d}t_1 \ldots \mathrm{d}t_k \frac{1}{(4\pi)^{dk}} \frac{1}{(t_1 \cdots t_k)^d} \left(\frac{t_1 \cdots t_k}{t_1 + \cdots + t_k} \right)^d + O(\epsilon)$$
(13)
$$= C' \int_{w^k \in \mathbb{C}^d} \mathrm{d}^{2d} w^k \Psi(0, \ldots, 0, w^k) \int_{(t_1, \ldots, t_k) \in [\epsilon, L]^k} \mathrm{d}t_1 \ldots \mathrm{d}t_k \frac{1}{(4\pi)^{dk}} \frac{1}{(t_1 + \cdots + t_k)^d}$$

The expression $\Psi(0, \ldots, 0, w^k)$ means that we have evaluate the function $\Psi(w^1, \ldots, w^k)$ at $w^1 = \ldots = w^{k-1} = 0$ leaving it as a function only of w^k. In the original coordinates this is equivalent to setting $z^1 = \cdots = z^{k-1} = z^k$.

Our goal is to show that $\epsilon \to 0$ limit exists. The only ϵ dependence on the right hand side of (12) is in the integral over the regulation parameters $t_1, \ldots, t_k$. Thus, it suffices to show that the $\epsilon \to 0$ limit of
$$\int_{(t_1, \ldots, t_k) \in [\epsilon, L]^k} \frac{\mathrm{d}t_1 \ldots \mathrm{d}t_k}{(t_1 + \cdots + t_k)^d}$$
exists. By the AM/GM inequality we have $(t_1 + \cdots + t_k)^d \geq (t_1 \cdots t_d)^{d/k}$. So, the integral is bounded by
$$\int_{(t_1, \ldots, t_k) \in [\epsilon, L]^k} \frac{\mathrm{d}t_1 \ldots \mathrm{d}t_k}{(t_1 + \cdots + t_k)^d} \leq \int_{(t_1, \ldots, t_k) \in [\epsilon, L]^k} \frac{\mathrm{d}t_1 \ldots \mathrm{d}t_k}{(t_1 \cdots t_k)^{d/k}} = \frac{1}{(1 - d/k)^k} \left(\epsilon^{1-d/k} - L^{1-d/k} \right)^k.$$
By assumption, $d < k$, so the right hand side has a well-defined $\epsilon \to 0$ limit. This concludes the proof.

□

4. Equivariant BV quantization

Equivariant BV quantization is an enhancement of ordinary BV quantization where one takes into account the action of a group or Lie algebra. We will heavily rely on techniques of equivariant BV quantization throughout this thesis, notably in the construction of the holomorphic σ-model in Chapter **??** and in the proof of a local version of the Grothendieck-Riemann-Roch theorem in Chapter **??** using Feynman diagrammatic expansions.

4.1. **Classical equivariance.** The equivariance that we consider is a direct analog of symmetries in ordinary Hamiltonian mechanics, which we briefly recall. Suppose that $\mathfrak{h}$ is a Lie algebra on (M, ω) is an ordinary symplectic manifold. A symplectic action of $\mathfrak{h}$ on X is a map of Lie algebras
$$\rho : \mathfrak{h} \to \mathrm{SympVect}(M)$$
where $\mathrm{SympVect}(M)$ is the Lie algebra of symplectic vector fields, i.e. those vector fields X which preserve the symplectic form $L_X \omega = 0$. On any symplectic manifold, the Poisson algebra of functions admits a Lie algebra map $\mathcal{O}(M) \to \mathrm{SympVect}(M)$ sending a function f to its Hamiltonian vector field $X_f = \{f, -\}$, where $\{-, -\}$ is the Poisson bracket. An action ρ is said to be *inner* if it lifts to a map of Lie algebras $\widetilde{\rho} : \mathfrak{h} \to \mathcal{O}(M)$. Recall that on any symplectic manifold the kernel of $f \mapsto X_f$ is precisely the constant functions.

All of our classical theories arise as (-1)-shifted symplectic formal moduli problems. Hence, suppose we replace the symplectic manifold M by a formal moduli problem $B\mathfrak{g}$, where $\mathfrak{g}$ is some dg Lie (or L_∞ algebra). To give $B\mathfrak{g}$ the structure of a (-1)-shifted symplectic structure is equivalent to having a (-3)-shifted non-degenerate pairing on $\mathfrak{g}$. Functions on $B\mathfrak{g}$ are precisely the Chevalley-Eilenberg cochains $\mathcal{O}(B\mathfrak{g}) = C^*_{\mathrm{Lie}}(\mathfrak{g})$. The (-1)-shifted symplectic structure equips $C^*_{\mathrm{Lie}}(\mathfrak{g})[-1]$ with the structure of a dg Lie algebra. Since all symplectic vector fields are Hamiltonian in this case we see that

$$\mathrm{SympVect}(B\mathfrak{g}) = C^*_{\mathrm{Lie}}(\mathfrak{g})[-1]/\mathbb{C} = C^*_{\mathrm{Lie,red}}(\mathfrak{g})[-1]$$

where we have taken the quotient by the constants, which by definition is the reduced cochains. We modify the notion of a symplectic action slightly to allow for more general maps of Lie algebras. A symplectic action of $\mathfrak{h}$ on the (-1)-shifted symplectic formal moduli space $B\mathfrak{g}$ is a map of L_∞ algebras, or a homotopy coherent map of dg Lie algebras

$$\rho : \mathfrak{h} \rightsquigarrow C^*_{\mathrm{Lie,red}}(\mathfrak{g})[-1].$$

Such a map ρ is equivalent to a Maurer-Cartan element in the dg Lie algebra

$$C^*_{\mathrm{Lie}}(\mathfrak{h}) \otimes C^*_{\mathrm{Lie,red}}(\mathfrak{g})[-1].$$

This is a cohomological degree $+1$ element $I^{\mathfrak{h}}$ such that $\mathrm{d}I^{\mathfrak{h}} + \frac{1}{2}\{I^{\mathfrak{h}}, I^{\mathfrak{h}}\} = 0$. Here $\{-,-\}$ is the bracket on $C^*_{\mathrm{Lie,red}}(\mathfrak{g})$ and d is the sum of the Chevalley-Eilenberg differentials on $\mathfrak{h}$ and $\mathfrak{g}$.

4.2. **Quantum equivariance.** If we start with an $\mathfrak{h}$-equivariant classical BV theory with fields $\mathcal{E}$ with action functional S — so that $\mathfrak{h}$ has an L_∞ action on the fields that preserves the pairing and the action functional S — then we can encode the action of $\mathfrak{h}$ as a Maurer-Cartan element $I^{\mathfrak{h}}$ in $C^*_{\mathrm{Lie}}(\mathfrak{h}) \otimes \mathcal{O}_{\mathrm{loc}}(\mathcal{E})$. (For the formal $\beta\gamma$ system, we did this in Lemma **??**.) We then view the sum $S + I^{\mathfrak{h}}$ as the *equivariant* action functional: the operator $\{S + I^{\mathfrak{h}}, -\}$ is the twisted differential on $C^*_{\mathrm{Lie}}(\mathfrak{h}) \otimes \mathcal{O}_{\mathrm{loc}}(\mathcal{E})$ with $I^{\mathfrak{h}}$ as the twisting cocycle, and this operator is square-zero because $\{S + I^{\mathfrak{h}}, S + I^{\mathfrak{h}}\}$ is a "constant" (i.e., lives in $C^*_{\mathrm{Lie}}(\mathfrak{h})$ and hence is annihilated by the BV bracket).

This perspective suggests the following definition of an equivariant quantum BV theory. The starting data is two-fold: an $\mathfrak{h}$-equivariant classical BV theory with equivariant action functional $S + I^{\mathfrak{h}}$, and a BV quantization $\{S[L]\}$ of the non-equivariant action functional S. Following Costello, it is convenient to write S as $S_{\mathrm{free}} + I$, where the first "free" term is a quadratic functional and the second "interaction" term is cubic and higher. In this situation, the effective action $S[L] = S_{\mathrm{free}} + I[L]$, i.e., only the interaction changes with the length scale.

Definition 4.1. An $\mathfrak{h}$-*equivariant BV quantization* is a collection of effective interactions $\{I^{\mathfrak{h}}[L]\}_{L\in(0\infty)}$ satisfying

(a) the RG flow equation

$$W(P_\epsilon^L, I[\epsilon] + I^{\mathfrak{h}}[\epsilon]) = I[L] + I^{\mathfrak{h}}[L]$$

for all $0 < \epsilon < L$,

(b) the equivariant scale L quantum master equation, which is that

$$Q(I[L] + I^{\mathfrak{h}}[L]) + \mathrm{d}_{\mathfrak{h}} I^{\mathfrak{h}}[L] + \frac{1}{2}\{I[L] + I^{\mathfrak{h}}[L], I[L] + I^{\mathfrak{h}}[L]\}_L + \hbar\Delta_L(I[L] + I^{\mathfrak{h}}[L])$$

lives in $C^*_{\mathrm{Lie}}(\mathfrak{h})$ for every scale L, and

(c) the locality axiom, with the additional condition that as $L \to 0$, we recover the equivariant classical action functional $S + I^{\mathfrak{h}}$ modulo $\hbar$.

In other words, we simply follow the constructions of **[?, ?]** working over the base ring $C^*_{\mathrm{Lie}}(\mathfrak{h})$. A careful reading of those texts shows that the freedom to work over interesting dg commutative algebras is built into the formalism.

4.3. **The case of a local Lie algebra.** The above formalism works equally well, with some slight modifications, if we replace the Lie algebra $\mathfrak{h}$ by a *local Lie algebra* on the manifold where the theory lives.

BW: do this

Lemma 4.2. *Suppose $\{I^{\mathcal{L}}[L]\}$ is an effective family satisfying the $\mathcal{L}$-equivariant quantum master equation modulo $C^*_{\mathrm{loc}}(\mathcal{L})$. Then, the obstruction to lifting this action to an inner action, that is the anomaly to solving the quantum master equation, is a degree $+1$ cocycle in $C^*_{\mathrm{loc}}(\mathcal{L})$.*

Remark 4.3. Equivariant quantization is essentially a version of the background field method in QFT. One treats elements of $\mathcal{L}$ as background fields and the interaction terms $I^{\mathcal{L}}[L]$ encode the variation of the path integral measure with respect to these background fields. (Solving the QME is our definition of well-posedness of the measure.) This should not be confused with *gauging* a theory by $\mathcal{L}$, which involves putting the elements of $\mathcal{L}$ in the theory as propagating fields.