LOCAL SYMMETRIES OF HOLOMORPHIC THEORIES

CONTENTS

In this chapter we investigate the symmetries that generic holomorphic quantum field theories possess. Our overarching goal is to develop tools for understanding such symmetries that provide a systematic generalization of methods used in chiral conformal field theory on Riemann surfaces, especially for the Kac-Moody and Virasoro vertex algebras. We will utilize the tools of BV quantization and factorization algebras that has already heavily percolated this thesis.

We will focus on two main types of symmetries: holomorphic gauge symmetries and symmetries by holomorphic diffeomorphisms. An ordinary gauge symmetry is characterized as being local on the spacetime manifold. Each of the types of symmetries we consider share this characteristic, but they also enjoy an additional structure: they are holomorphic (up to homotopy) on the spacetime manifold. This means that they are specific to the type of theories we consider. Moreover, they store more interesting information about the geometry of the underlying manifold as compared to the smooth version of such symmetries.

Infinitesimally speaking, a symmetry is encoded by the action of a Lie algebra. For the holomorphic gauge symmetry this will become a sort of current algebra which is equivalent to holomorphic functions on the complex manifold with values in a Lie algebra. For the holomorphic diffeomorphisms this Lie algebra is that of holomorphic vector fields. Locality implies that this actually extends to a symmetry by a sheafy version of a Lie algebra. The precise sheafy version we mean is called a *local Lie algebra*, which we will recall in the main body of the text. To every local Lie algebra we can assign a factorization algebra through the so-called factorization enveloping algebra:

$$\mathbb{U} : \mathrm{Lie}_X \to \mathrm{Fact}_X.$$

Here, Lie_X is the category of local Lie algebras which we will recall in the main body of the text. By this construction, we see that the symmetries themselves of field theories give rise to factorization algebras.

One compelling reason for constructing a factorization algebra model for Lie algebras encoding the symmetries of a theory is that it allows one to consider universal versions of such objects. In the case of the symmetry by a current algebra of a Lie algebra in chiral conformal field theory this has been spelled out in the book [**?**]. For the case of conformal symmetry our work in [**?**] provides a factorization algebra lift of the ordinary Virasoro vertex algebra that exists uniformly on the site of Riemann surfaces. In this chapter, we extend each of these objects to arbitrary complex dimensions. Our formulation lends itself to an explicit computation of the factorization homology along certain complex manifolds, for which we will focus on several examples.

Studying such local symmetries involves rich geometric input even at the classical level, but the skeptical mathematician may view this as a repackaging of already familiar objects in complex geometry. The main advantage of working with factorization algebra analogs of such symmetries is in their relationship to studying quantizations of field theories. A similar obstruction deformation theory for studying quantizations of classical field theories also allows us to study the problem of *quantizing* local symmetries of a field theory. Moreover, we already know that factorization algebras describe the operator product expansion of the observables of a QFT. A formulation of Noether's theorem in [**?**] makes the relationship between the associated factorization algebra of a symmetry and the factorization algebra of observables of a theory.

Of course, quantizing a symmetry of a field theory may not always exist. In fact, this failure sheds light into subtle field theoretic phenomena of the underlying system. For example, in the case of conformal symmetries of a conformal field theory, the failure is exactly measured by the *central charge* of the theory. It is well established that the central charge is a very important characterization of a conformal field theory. At the Lie theoretic level, this failure is measured by a cocycle which in turn defines a central extension of the Lie algebra. It is this central extension that acts on the theory.

For this reason, an essential aspect of studying the local symmetries of holomorphic field theories we mentioned above is to characterize the possible cocycles that give rise to central extensions. As we have already mentioned, for vector fields in complex dimension one this is related to the central charge and the central extension of the Witt algebra (vector fields on the circle) known as the Virasoro Lie algebra. In the case of a current algebra associated to a Lie algebra, central extensions are related to the *level* and the corresponding central extensions are called affine algebras.

Theorem 0.1. *The following is true about the local Lie algebras associated to holomorphic diffeomorphisms and holomorphic gauge symmetries.*

(1) Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{g}^X$ is associated current algebra defined on any complex manifold X. There is an embedding of the cohomology $H^*_{Lie}(\mathfrak{g}, \mathrm{Sym}^{d+1} g^\vee[-d-1])$ inside of the local cohomology of $\mathfrak{g}^X$.
(2) There is an isomorphism between the local cohomology of holomorphic vector fields on any complex manifold X of dimension d and $H^*_{dR}(X) \otimes H^*_{GF}(\mathrm{W}_d)[2d]$, where $H^*_{GF}(\mathrm{W}_d)$ is the Gelfand-Fuks cohomology of vector fields on the formal disk.

The central extensions we are interested in come from classes of degree $+1$ of the above local Lie algebras. In the case of holomorphic vector fields the result above implies that all such extensions are parametrized by $H^{2d+1}(\mathrm{W}_d)$. It is a classical result of Fuks [**?**] that this cohomology is isomorphic to $H^{2d+2}(BU(d))$. In complex dimension one this cohomology is one dimensions corresponding to the class c_1^2. In general we obtain new classes, which are shown to agree with calculations in the physics literature in dimensions four and six.

In general, any of these cohomology classes define factorization algebras by twisting the factorization enveloping algebra. We especially focus on this construction in the case that the complex d-fold is equal to affine space $\mathbb{C}^d$, or some of its natural submanifolds. In the case of the current algebra, our result is compatible with recent work of Kapranov et. al. in [**?**] where they study higher dimensional versions of affine algebras, and their relationship to the (derived) moduli space of G-bundles in an analogous way that affine algebras are related to the moduli of bundles on curves via Kac-Moody uniformization. Our second main result shows how to recover these higher affine algebras from our factorization algebra on punctured affine space $\mathbb{C}^d \setminus \{0\}$, see Theorem **??**.

The extensions of part (1) of Theorem 0.1 are related to cohomology classes in the moduli of G-bundles on complex d-folds. We will show how techniques in equivariant BV quantization lead to natural families of QFTs defined over formal neighborhoods in the moduli space of G-bundles. Our techniques allow us to study quantizations of such families, in particular there are anomalies to quantization. An explicit analysis of Feynman diagrams leads to a computation of certain classes in the local cohomology which we relate to Chern classes of natural line bundles on $\mathrm{Bun}_G(X)$. This leads us to our next main result which is to prove a version of the Grothendieck-Riemann-Roch (GRR) theorem using the aforementioned methods of BV quantization, see Theorem 4.1.

1. THE CURRENT ALGEBRA ON COMPLEX MANIFOLDS

1.1. **Definitions.** We recall some definitions that we will use throughout the paper. The first concept we introduce is that of a *local Lie algebra*. This is the central object needed to discuss symmetries of field theories that are local on the spacetime manifold.

Throughout this paper we will use L_∞ algebras. This is a modest generalization of a dg Lie algebra where the Jacobi identity is only required to hold up to homotopy. The data of an L_∞ algebra is a graded vector space V with, for each $k \geq 1$, a k-ary bracket

$$\ell_k : V^{\otimes k} \to V[2-k]$$

of cohomological degree $2-k$. These maps are required to satisfy a series of conditions, the first of which says $\ell_1^2 = 0$. The next says that ℓ_2 is a bracket satisfying the Jacobi identity up to a homotopy given by ℓ_3. For a detailed definition see we refer the reader to [**?**, **?**].

We now give the definition of a local L_∞ algebra on a manifold X. This has appeared in Chapter 4 of [**?**].

Definition 1.1. A *local L_∞ algebra* on X is the following data:

(i) a $\mathbb{Z}$-graded vector bundle L on X, whose sheaf of smooth sections we denote $\mathcal{L}^{sh}$, and
(ii) for each positive integer n, a polydifferential operator in n inputs

$$\ell_n : \underbrace{\mathcal{L}^{sh} \times \cdots \times \mathcal{L}^{sh}}_{n \text{ times}} \to \mathcal{L}[2-n]$$

such that the collection $\{\ell_n\}_{n\in\mathbb{N}}$ satisfy the conditions of an L_∞ algebra. Thus, $\mathcal{L}^{sh}$ is a sheaf of L_∞ algebras.

In practice, we prefer to work with the compactly supported sections of L, for which we reserve the more succinct notation $\mathcal{L}$.

Definition 1.2. Given a local L_∞ algebra $(L, \{\ell_n\})$ on X, let $\mathcal{L}$ denote the precosheaf of L_∞ algebras that assigns compactly supported sections of L to each open of X.

We typically refer to the local L_∞ algebra $(L, \{\ell_n\})$ by $\mathcal{L}$. We will often use local *Lie* algebra, especially if $\mathcal{L}$ is a precosheaf of dg Lie algebras and hence has trivial $\ell_{n\geq 3}$.

Example 1.3. Let $P \to X$ be a principal G-bundle. The adjoint bundle is a bundle of Lie algebras that we denote $\mathrm{ad}(P) \to X$. We will hereafter use $\mathcal{A}\mathrm{d}(P)$ to denote the *cosheaf* of compactly supported sections of Dolbeault complex of $\mathrm{ad}(P)$

$$\mathcal{A}\mathrm{d}(P)(U) = \Omega^{0,*}_c(U; \mathrm{ad}(P)).$$

In keeping with our conventions, $\mathcal{A}\mathrm{d}(P)^{sh}$ will denote the corresponding *sheaf* of sections of the Dolbeault complex

$$\mathcal{A}\mathrm{d}(P)^{sh}(U) = \Omega^{0,*}(U; \mathrm{ad}(P)).$$

The Dolbeualt differential $\bar{\partial}$ and the fiberwise Lie bracket on $\mathrm{ad}(P)$ endow $\mathcal{A}\mathrm{d}(P)^{sh}$ with the structure of a sheaf of dg Lie algebras on X.

The following lemma follows from tracing through definitions.

Lemma 1.4. *For any holomorphic principal bundle $P \to X$, the Dolbeualt complex of forms with values in $\mathrm{ad}(P)$ is a local Lie algebra.*

Example 1.5. Another key local Lie algebra makes sense on an arbitrary complex d-fold. Let $\mathfrak{g}$ be an ordinary Lie algebra, such as $\mathfrak{sl}_n$. There is a natural assignment

$$\mathcal{G}^{sh} : X \mapsto \Omega^{0,*}(X) \otimes \mathfrak{g},$$

where X is a complex d-fold. In fact, this assignment defines a sheaf of dg Lie algebras on the category of complex d-folds and local biholomorphisms,[1] and $\mathcal{G}$ to denote the cosheaf of compactly supported sections $\Omega_c^{0,*} \otimes \mathfrak{g}$. For any $\mathfrak{g}$, $\mathcal{G}$ defines a local Lie algebra on the category of d-folds, though we don't elaborate on the requisite categorical machinery to make this precise. We use $\mathcal{G}_X$ to denote the restriction of $\mathcal{G}$ to a fixed complex d-fold X. This defines a local Lie algebra whose associated cosheaf of sections is $U \subset X \mapsto \Omega_c^{0,*}(U) \otimes \mathfrak{g}$. Note that in the case of the trivial holomorphic principal G-bundle on X one as $\mathcal{G}_X = \mathcal{A}\mathrm{d}(\mathrm{triv})$.

1.2. **Factorization Lie algebras.** A factorization Lie algebra is a useful concept that we will utilize to make the connection between local Lie algebras on factorization algebras. Ordinarily, when we discuss factorization algebras we mean a symmetric monoidal functor from the category of opens on a fixed manifold, with monoidal product given by disjoint union, to the category of chain complexes, with monoidal product given by tensor product. However, a factorization algebra can be defined with an arbitrary symmetric monoidal category as the target. This definition has appeared in multiple sources, such as [**?**], [**?**], or [**?**].

Definition 1.6. Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal category and X a space. A *prefactorization algebra* on X with values in $\mathcal{C}$ is a functor of symmetric monoidal categories

$$\mathcal{F} : \mathrm{Disj}_X^{\sqcup} \to \mathcal{C}^{\otimes}.$$

A *strict factorization algebra* with values in $\mathcal{C}$ is a prefactorization algebra $\mathcal{F}$ such that:

(1) $\mathcal{F}$ is a cosheaf with respect to the Weiss topology;
(2) for any disjoint open sets $U, V \subset X$ the structure map $\mathcal{F}(U) \otimes \mathcal{F}(V) \to \mathcal{F}(U \sqcup V)$ is an isomorphism.

There are two important symmetric monoidal categories we will be most interested in as the target of a factorization algebra. The first is the category of chain complexes $\mathrm{Ch}^{\otimes}$ (over $\mathbb{C}, \mathbb{R}$) with symmetric monoidal product given by the tensor product. The next is the category of dg Lie algebras $\mathrm{dgLie}^{\oplus}$ with symmetric monoidal structure given by the direct sum.

In both of these categories there is the notion of a quasi-isomorphism, which allows us to weaken the above definition slightly.

Definition 1.7. Let $\mathcal{F}$ be a prefactorization algebra on X with values in $\mathcal{C} = \mathrm{Ch}^{\otimes}$ or $\mathrm{dgLie}^{\oplus}$. Then, $\mathcal{F}$ is a *homotopy factorization algebra* if

(1) $\mathcal{F}$ is a homotopy cosheaf with respect to the Weiss topology;

[1] A biholomorphism is a map $\phi : X \to Y$ that is biijective and both ϕ and ϕ^{-1} are holomorphic. A *local* biholomorphism means a map $\phi : X \to Y$ such that for every point $x \in X$ has a neighborhood on which ϕ is a biholomorphism.

(2) for any disjoint open sets $U, V \subset X$ the structure map $\mathcal{F}(U) \otimes \mathcal{F}(V) \to \mathcal{F}(U \sqcup V)$ is a quasi-isomorphism.

Not surprisingly, there is a version of the definition for homotopy factorization algebras with values in an arbitrary symmetric monoidal ∞-category, but we do not wish to dive into the general formalism here but refer to the source references [**?**, **?**, **?**] for a complete treatment.

For the remainder of this paper we will only discuss factorization algebras valued in these categories $\mathrm{Ch}^{\otimes}$, $\mathrm{dgLie}^{\oplus}$. When we do not say otherwise, a *factorization algebra* will mean a homotopy factorization algebra with values in Ch. Likewise, a *factorization Lie algebra* will mean a homotopy factorization algebra with values in dgLie. Note that the direct sum is *not* the coproduct for Lie algebras, so a prefactorization Lie algebra is different than just a precosheaf of Lie algebras.

We have already encountered a modest extension of the category of dg Lie algebras to the category of L_∞ algebras $L_\infty\mathrm{Alg}$ which will come up in our discussion below. This category is also symmetric monoidal using the direct sum, and we will also refer to homotopy factorization algebras with values in $L_\infty\mathrm{Alg}$ as factorization Lie algebras.

The primary appearance of factorization Lie algebras, for us, comes from local Lie algebras.

Lemma 1.8. *Suppose $(L, \{\ell_n\})$ is a local Lie algebra on X. Then, the compactly supported sections $\mathcal{L}$ has the structure of a factorization Lie algebra.*

Proof. By the cosheaf property, we know that $\mathcal{L}(U \sqcup V) \cong \mathcal{L}(U) \oplus \mathcal{L}(V)$. This is an isomorphism of L_∞ algebras since any element of $\mathcal{L}(U)$ commutes with $\mathcal{L}(V)$ inside of $\mathcal{L}(U \sqcup V)$. Similarly, if $\{U_i\}$ is a disjoint collection of opens in X and $\sqcup_i U_i \subset W$, then we define the factorization structure map by

$$\oplus_i \mathcal{L}(U_i) \cong \mathcal{L}(\sqcup_i U_i) \to \mathcal{L}(W)$$

where the second map is the structure map for the cosheaf. These structure maps exhibit $\mathcal{L}$ as a prefactorization Lie algebra (i.e. a prefactorization algebra valued in the category of L_∞ algebras). □

There is a functor from dg Lie algebras to cochain complexes

$$\mathrm{C}_*^{\mathrm{Lie}} : \mathrm{dgLie} \to \mathrm{Ch}$$

sending $(\mathfrak{g}, \mathrm{d}, [-,-])$ to the complex

$$\mathrm{C}_*^{\mathrm{Lie}}(\mathfrak{g}) = (\mathrm{Sym}(\mathfrak{g}[1]), \mathrm{d} + \mathrm{d}_{CE}) \, .$$

Here, d denotes the extension of the differential on $\mathfrak{g}$ to the symmetric algebra by the Leibniz rule, and d_{CE} encodes the Lie bracket. There is a completely similar functor from L_∞ algebras to chain complexes that we denote by the same name.

The functor $\mathrm{C}_*^{\mathrm{Lie}}$ is symmetric monoidal with respect to the direct sum of Lie algebras and the tensor product of cochain complexes $\mathrm{C}_*^{\mathrm{Lie}}(\mathfrak{g} \oplus \mathfrak{h}) = \mathrm{C}_*^{\mathrm{Lie}}(\mathfrak{g}) \otimes \mathrm{C}_*^{\mathrm{Lie}}(\mathfrak{h})$. Since a factorization Lie algebra uses the direct sum monoidal structure, the following definition makes sense.

Definition/Lemma 1. Suppose $\mathcal{G}$ is a factorization Lie algebra on a manifold X then, $\mathrm{C}_*^{\mathrm{Lie}}(\mathcal{G})$ has the structure of a factorization algebra (valued in cochain complexes with tensor product).

We have already seen that every local Lie algebra gives rise to a factorization Lie algebra. By the construction above, we obtain the following composition of functors.

$$\mathrm{Lie}_X \to \mathrm{Fact}_X^{\mathrm{Lie}} \to \mathrm{Fact}_X$$

Here Lie_X is the category of local Lie algebras on X. If $\mathcal{L}$ is a local Lie algebra we let $\mathbb{U}(\mathcal{L})$ be the image under this composition, and call it the *factorization enveloping algebra* of $\mathcal{L}$.

1.3. **Local cohomology.** In this section we study the cohomology of the local Lie algebra $\mathcal{A}\mathrm{d}(P)$. As we have already encountered many times in this thesis, the cohomology we are interested in consists of those functionals on the local Lie algebra that are *local*. From the perspective of local Lie algebras, one appealing aspect of this class of functionals is that they give rise to local Lie algebra extensions of the current algebra. These extensions will be appear when we quantize holomorphic gauge symmetries.

In Section **??** we have discussed local cohomology of a local Lie algebra, but we briefly recall it here. The basic idea is that a local cochain is a functional on the local Lie algebra obtained by integrating a polydifferential operator applied to an element in the local Lie algebra. If L is a graded vector bundle, let JL denote the corresponding ∞-jet bundle. If L is the underlying vector bundle of a local Lie algebra then JL has the structure of a bundle of Lie algebras. Thus, we may consider its reduced Chevalley-Eilenberg cochain complex $\mathrm{C}^*_{\mathrm{Lie,red}}(JL)$[2]. For any vector bundle JL has the structure of a D_X-module. In the case of a local Lie algebra, JL is a Lie algebra object in D_X-modules. Thus, $\mathrm{C}^*_{\mathrm{Lie}}(JL)$ is a commutative dg algebra in D_X-modules. The local cochain complex is obtained by tensoring the right D_X-modules of densities on X over D_X with this D_X-module.

Definition 1.9. Let $\mathcal{L}$ be a local Lie algebra on X. The local cohomology of $\mathcal{L}$ is defined as

$$\mathrm{C}^*_{\mathrm{loc}}(\mathcal{L}) = \Omega^{d,d}_X \otimes_{D_X} \mathrm{C}^*_{\mathrm{Lie,red}}(JL).$$

This is a sheaf of cochain complexes on X whose global sections we will denote by $\mathrm{C}^*_{\mathrm{loc}}(\mathcal{L}(X))$.

We note that the cochain complex of local functionals is a subcomplex of $\mathrm{C}^*_{\mathrm{Lie,red}}(\mathcal{L}(X))$, the reduced Lie algebra cochains of the global sections $\mathcal{L}(X)$. The differential on local functionals is, in essence, just precomposition with the polydifferentials defining the brackets of $\mathcal{L}$. Altogether $\mathrm{C}^*_{\mathrm{loc}}(\mathcal{L})$ is just a version of diagonal Gelfand-Fuks cohomology **[?]** for this kind of Lie algebra. We will discuss this further when we approach the local Lie algebra of holomorphic vector fields.

In ordinary Lie theory, central extensions are parametrized by cocycles on the Lie algebra valued in the trivial module. Similarly, local cocycles define central extensions of local Lie algebras.

Definition 1.10. A cocycle Θ of degree $2+k$ in $\mathrm{C}^*_{\mathrm{loc}}(\mathcal{L})$ determines a *k-shifted central extension*

$$0 \to \mathbb{C}[k] \to \widehat{\mathcal{L}}_\theta \to \mathcal{L} \to 0 \tag{1}$$

of precosheaves of L_∞ algebras, where the L_∞ structure maps are defined by

$$\widehat{\ell}_n(x_1, \ldots, x_n) = (\ell_n(x_1, \ldots, x_n), \Theta(x_1, \ldots, x_n)).$$

[2]A local functional will always be defined modulo constants, hence we look at reduced cochains.

Cohomologous cocycles determine quasi-isomorphic extensions of precosheaves of Lie algebras. Much of the rest of the section is devoted to constructing and analyzing various cocycles and the resulting extensions.

Local cocycles give a direct way of deforming the factorization enveloping algebra of a local Lie algebra. Suppose that we have a local cocycle $\Theta \in C^*_{\text{loc}}(\mathcal{L})$ is of cohomological degree $+1$. We define the *twisted factorization enveloping algebra* to be the factorization algebra sending the open set $U \subset X$ to the cochain complex

$$\begin{aligned}\mathbb{U}_\Theta(\mathcal{L})(U) &= (\text{Sym}(\mathcal{L}(U)[1] \oplus \mathbb{C}\cdot K), \mathrm{d}_{\mathcal{L}} + K\cdot\Theta) \\ &= (\text{Sym}(\mathcal{L}(U)[1])[K], \mathrm{d}_{\mathcal{L}} + K\cdot\Theta)\end{aligned}$$

where $\mathrm{d}_{\mathcal{L}}$ denotes the differential on the untwisted factorization enveloping algebra applied to U and Θ is the operator on the symmetric algebra extending the cocycle $\Theta : \text{Sym}(\mathcal{L}(U)[1]) \to \mathbb{C}\cdot K$ by demanding that it is a graded derivation. Here, K is a formal algebraic parameter. We denote this twisted factorization enveloping algebra by $\mathbb{U}_\Theta(\mathcal{L})$. We will consider this as a factorization algebra valued in the symmetric monoidal category of chain complexes that are linear over the commutative ring $\mathbb{C}[K]$. Specialization at certain values of K yields an ordinary factorization algebra.

1.3.1. *The J functional.* There is a particular family of local cocycles that has special importance in studying symmetries of higher dimensional holomorphic gauge theories.

Let us recall the familiar complex one-dimensional case that we wish to extend. Let Σ be a Riemann surface, and let $\mathfrak{g}$ be a simple Lie algebra with Killing form κ. Consider the local Lie algebra $\mathcal{G}_\Sigma = \Omega^{0,*}_c(\Sigma) \otimes \mathfrak{g}$ on Σ. There is a natural cocycle depending precisely on two inputs:

$$\theta(\alpha \otimes M, \beta \otimes N) = \kappa(M,N) \int_\Sigma \alpha \wedge \partial\beta,$$

where $\alpha, \beta \in \Omega^{0,*}_c(\Sigma)$ and $M, N \in \mathfrak{g}$. In Chapter 5 of [?] it is shown how the twisted factorization envelope of $\mathcal{G}_X$ via this cocycle recovers the Kac-Moody vertex algebra and the affine algebra extending $L\mathfrak{g} = \mathfrak{g}[z, z^{-1}]$.

We are interested in a generalization of this construction in arbitrary dimensions. Let θ be an invariant polynomial on $\mathfrak{g}$ of homogenous degree $d+1$. That is, θ is an element of $\text{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$. For any complex d-fold X we can extend θ to a functional $J_X(\theta)$ on the Dolbeault complex $\Omega^{0,*}_c(X) \otimes \mathfrak{g}$ by the formula

$$J_X(\theta)(\omega_0 \otimes Y_0, \ldots, \omega_d \otimes Y_d) = \theta(Y_0, \ldots, Y_d) \int_X \omega_0 \wedge \partial\omega_1 \cdots \wedge \partial\omega_d. \tag{2}$$

Note that we use d copies of the holomorphic derivative $\partial : \Omega^{0,*} \to \Omega^{1,*}$ to obtain an element of $\Omega^{d,*}_c$ in the integrand (and hence something that has a chance of being integrated). If we extend θ to a functional on the Dolbeault complex in the natural way

$$\theta : \Omega^{*,*}(X)^{\otimes d+1} \to \Omega^{*,*}(X)$$

then we can write the cocycle more succinctly as $J_X(\theta)(\alpha_0, \ldots, \alpha_d) = \int_X \theta(\alpha_0, \partial\alpha_1, \ldots, \partial\alpha_d)$.

This formula clearly makes sense for any complex d-fold X, and since integration is local on X, it intertwines nicely with the structure maps of $\mathcal{G}_X$.

Proposition 1.11. *For any complex d-fold X and invariant polynomial $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$, the functional $J_X(\theta)$ is a local functional in $\mathrm{C}^*_{\mathrm{loc}}(\mathcal{G}_X)$. In fact, the assignment*

$$J_X : \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}[-1] \to \mathrm{C}^*_{\mathrm{loc}}(\mathcal{G}_X) \quad , \quad \theta \mapsto J_X(\theta)$$

is an cochain map.

Proof. The functional $J_X(\theta)$ is local as it is expressed as the integral of a multilinear map composed with a product of differential operators. We need to show that $J_X(\theta)$ is closed for the differential on $\mathrm{C}^*_{\mathrm{loc}}(\mathcal{G}_X)$. The total differential splits as a sum $\bar{\partial} + \mathrm{d}_{\mathfrak{g}}$ where $\bar{\partial}$ denotes the induced $\bar{\partial}$ differential on functionals and d_{CE} is constructed from the Lie bracket on $\mathfrak{g}$. We observe that

$$\bar{\partial} J_X(\theta) = 0$$
$$\mathrm{d}_{CE} J_X(\theta) = 0.$$

The first line follows from the fact that $\bar{\partial}$ and ∂ are graded commutative. The differential d_{CE} is obtained from the differential in the Chevalley–Eilenberg complex of $\mathfrak{g}$ in a natural way. The second line follows from the fact that the homogenous polynomial $\theta : \mathfrak{g} \times \cdots \times \mathfrak{g} \to \mathbb{C}$ is closed in the Chevalley–Eilenberg complex for $\mathfrak{g}$. □

Having the fundamental construction of the cocycle down, we discuss two modest extensions of the construction. First is to consider an arbitrary G-bundle P on X. Suppose $\mathrm{ad}(P)$ is trivialized over an open set $U \subset X$. On this open set, we can write an element $\alpha \in \mathcal{A}\mathrm{d}(P)(U) = \Omega^{0,*}_c(U, \mathrm{ad}(P))$ as $\alpha = \omega \otimes X$ where $\omega \in \Omega^{0,*}(X)$ and $X \in \mathfrak{g}$. Thus, the formula above for $J_X(\theta)$ still makes sense on $\mathcal{A}\mathrm{d}(P)(U)$. Since the expression for the cocycle is clearly independent of the choice of a coordinate it glues to define a global section. Thus, for any principal bundle we have a cochain map

$$J_X^P : \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}[-1] \to \mathrm{C}^*_{\mathrm{loc}}(\mathcal{A}\mathrm{d}(P)(X))$$

given by the same formula as in (2).

If $\mathfrak{g}$ is the Lie algebra of a group G, there is an interpretation of the space of extensions $\mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$ in terms of G.

Proposition 1.12. **[?]** *Let G be an affine algebraic group scheme (such as $\mathrm{GL}_n(\mathbb{C})$). Then, there is an isomorphism*

$$H^i(BG, \Omega^j) \cong H^{i-j}(G, \mathrm{Sym}^j(\mathfrak{g}^*)).$$

In the case that $i = j = d+1$ we find that $H^{d+1}(BG, \Omega^{d+1}) = \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$. Thus the central extensions have a familiar interpretation in terms of the Dolbeualt cohomology of BG.

Remark 1.13. When $\mathfrak{g}$ is a an arbitrary dg Lie algebra, or more generally an L_∞ algebra, we have encountered a version of $J_X(\theta)$ in Section **??**. We showed that for any L_∞ algebra there is a map of cochain complexes $J : \Omega^{d+1}_{cl}(B\mathfrak{g})[d] \to \mathrm{C}^*_{\mathrm{loc}}(\mathcal{G}_X)$. The expression for J_X in (2) is an explicit formula for this construction in the case that $\mathfrak{g}$ is an ordinary Lie algebra. Indeed, when $\mathfrak{g}$ is an ordinary Lie algebra we have $H^{d+1}(\Omega^{d+1}_{cl}(B\mathfrak{g})) = \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$, so the construction in Section **??**.

On $X = \mathbb{C}^d$ the functional $J_{\mathbb{C}^d}$ gives us the following complete description of a natural subcomplex of local cochains. On $\mathbb{C}^d$ exists a natural action by the group $U(d) \ltimes \mathbb{C}^d$, where $\mathbb{C}^d$ acts by translations and $U(d)$ acts in the defining way on $\mathbb{C}^d$. For each $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$ the functional $J_{\mathbb{C}^d}(\theta)$ is invariant for this group. In fact, this describes up to quasi-isomorphism all such functionals.

Proposition 1.14. *The map $J_{\mathbb{C}^d} : \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}[-1] \to \mathrm{C}^*_{\mathrm{loc}}(\mathcal{G}_{\mathbb{C}^d})$ factors through the subcomplex of local cochains that are invariant for the group $U(d) \ltimes \mathbb{C}^d$ to define a quasi-isomorphism*

$$J_{\mathbb{C}^d} : \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}[-1] \xrightarrow{\simeq} \mathrm{C}^*_{\mathrm{loc}}(\mathcal{G}(\mathbb{C}^d))^{U(d)\ltimes\mathbb{C}^d}$$

This is a special case of Theorem **??** in the case that $\mathfrak{g}$ is an ordinary Lie algebra. We refer the reader to that section for details.

1.4. **The Kac-Moody factorization algebra.** Finally, we can define the central object of this paper.

Definition 1.15. Let X be any complex manifold of dimension d equipped with a principal G-bundle P. Moreover, suppose $\Theta \in \mathrm{C}^*_{\mathrm{loc}}(\mathcal{A}\mathrm{d}(P))$ is a local cocycle of degree $+1$. The *Kac-Moody factorization algebra on X of type* Θ is the twisted factorization envelope

$$\mathbb{U}_\Theta(\mathcal{A}\mathrm{d}(P)) : U \subset X \mapsto \left(\mathrm{Sym}\left(\Omega^{0,*}_c(U, \mathrm{ad}(P))[1]\right)[K], \bar\partial + \mathrm{d}_{CE} + \Theta\right).$$

When $\Theta = J^P_X(\theta)$ for $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$ we denote this by $\mathbb{U}_\theta(\mathcal{A}\mathrm{d}(P)) = \mathbb{U}_{J^P_X(\theta)}(\mathcal{A}\mathrm{d}(P))$.

As in the definition of twisted factorization enveloping algebras above, the factorization algebras $\mathbb{U}_\Theta(\mathcal{A}\mathrm{d}(P))$ take values in dg modules for the ring $\mathbb{C}[K]$. In keeping with conventions above, when P is the trivial bundle on X we will denote the Kac-Moody factorization algebra by $\mathbb{U}_\Theta(\mathcal{G}_X)$.

Remark 1.16. For fixed θ the cocycle $J_X(\theta)$ is more-or-less independent of the complex manifold X. In this way, the factorization algebra $\mathbb{U}_\theta(\mathcal{G})$ actually defines a factorization algebra on the entire *category* of complex manifolds of a fixed dimension. We will not explore this type of *universal* factorization algebra here, but leave it to future work.

2. Local structures of the Kac-Moody factorization algebra

The theory of factorization algebras we study here, and whose foundations have been layed out in [**?**], is largely motivated by the study of chiral algebras due to Beilinson and Drinfeld [**?**]. Part of their original goal was to develop a geometric counterpart to the algebraic theory of vertex algebras. In [**?**] the relationship between factorization algebras on vertex algebras has been made completely explicit.

Every holomorphically translation invariant factorization algebra on the complex manifold $X = \mathbb{C}$ determines the structure of a vertex algebra. The underlying vector space, or state space, of the vertex algebra is given by the value of the factorization algebra assigns to a disk $D \subset \mathbb{C}$. The operator product expansion is encoded by the factorization product of configurations of disjoint disks inside of larger disks. It is shown that the Kac-Moody factorization algebra $\mathbb{U}_\kappa(\mathcal{G}_{\mathbb{C}})$ on $\mathbb{C}$, where κ is a symmetric invariant bilinear form, recovers the Kac-Moody vertex algebra in this way.

The fundamental structure we want contemplate comes from considering the factorization product for a different flavor of configurations of open sets. Suppose that $\mathcal{F}$ is a holomorphically translation invariant factorization algebra on $\mathbb{C}$. Consider a disk D that is completely encircled by an annulus Ann. Further, these disjoint open sets embed inside of a bigger disk D_{big} of the same center as in Figure BW: fig. The structure map is of the form

$$\mathcal{F}(\mathrm{Ann}) \otimes \mathcal{F}(D) \to \mathcal{F}(D_{big}).$$

If the ... BW: finish

The annular algebra $\mathcal{A}$ that we just discussed in the complex one-dimensional case has a generalization to arbitrary dimensions.The higher dimensional versions of annuli we consider are given by open sets equal to neighborhoods of $(2d-1)$-spheres. In this section we describe the higher dimensional version of this annular algebra for the Kac-Moody factorization algebra. This amounts to specializing the factorization algebra to the complex manifold $X = \mathbb{C}^d \setminus \{0\}$ and extracting the data of an A_∞-algebra from the factorization product in the radial direction. The reduction of the factorization algebra along $S^{2d-1} \subset \mathbb{C}^d \setminus \{0\}$ produces a one-dimensional factorization algebra via pushing forward along the radial projection map $\mathbb{C}^d \setminus \{0\} \to \mathbb{R}_{>0}$. Embedded inside of this factorization algebra is a locally constant factorization algebra, which will define for us our A_∞-algebra. Furthermore, we show how the factorization product of disks with higher dimensional annuli provide the structure a (A_∞-)module on the value of the factorization algebra on the disk.

We will recognize this A_∞-algebra as the universal enveloping algebra of an L_∞ algebra which is obtained as a central extension of an algebraic version of the sphere algebra

$$\mathrm{Map}(S^{2d-1}, \mathfrak{g}). \tag{3}$$

When $d = 1$ there is an embedding $\mathfrak{g}[z, z^{-1}] \hookrightarrow C^\infty(S^1) \otimes \mathfrak{g} = \mathrm{Map}(S^1, \mathfrak{g})$, induced by the embedding of algebraic functions on punctured affine line inside of smooth functions on S^1. The affine algebras are given by extensions of algebraic loop algebra $\mathcal{O}^{alg}(\mathbb{A}^{1\times}) = \mathfrak{g}[z, z^{-1}]$ prescribed by a 2-cocycle involving the algebraic residue pairing. Note that this cocycle is *not* pulled back from any cocycle on $\mathcal{O}^{alg}(\mathbb{A}^1) \otimes \mathfrak{g} = \mathfrak{g}[z]$.

When $d > 1$ Hartog's theorem implies that the space of holomorphic functions on punctured affine space is the same as the space of holomorphic functions on affine space. The same holds for algebraic functions, so that $\mathcal{O}^{alg}(\mathbb{A}^{d\times}) \otimes \mathfrak{g} = \mathcal{O}^{alg}(\mathbb{A}^d) \otimes \mathfrak{g}$. In particular, the naive algebraic replacement $\mathcal{O}^{alg}(\mathbb{A}^{d\times}) \otimes \mathfrak{g}$ of (3) has no interesting central extensions. However, as opposed to the punctured line, the punctured affine space $\mathbb{A}^{d\times}$ has interesting higher cohomology.

The key idea is that we replace the commutative algebra $\mathcal{O}^{alg}(\mathbb{A}^{d\times})$ by the derived space of sections $\mathbb{R}\Gamma(\mathbb{A}^{d\times}, \mathcal{O})$. This complex has interesting cohomology and leads to nontrivial extensions of the dg Lie algebra $\mathbb{R}\Gamma(\mathbb{A}^{d\times}, \mathcal{O}) \otimes \mathfrak{g}$. Concretely, we will use a dg model A_d for $\mathbb{R}\Gamma(\mathbb{A}^{d\times}, \mathcal{O})$ due to **[?]** that is an algebraic analog of the tangential Dolbeault complex of the $(2d-1)$-sphere inside of the Dolbeault complex of $\mathbb{C}^d \setminus \{0\}$:

$$\Omega^{0,*}_b(S^{2d-1}) \subset \Omega^{0,*}(\mathbb{C}^d \setminus \{0\}).$$

See **[?]** for details on the definition of $\Omega^{0,*}_b(S^{2d-1})$. The degree zero part of $\Omega^{0,*}_b(S^{2d-1})$ is $C^\infty(S^{2d-1})$, so we can view $A_d \otimes \mathfrak{g}$ as a derived enhancement of the mapping space in (3).

The model A_d, by definition, has cohomology equal to the cohomology of $\mathbb{R}\Gamma(\mathbb{A}^{d\times}, \mathcal{O})$. In **[?]** they have studied a class of cocycles associated to elements $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$ that are algebraic analogs of the local cocycles we introduced in the previous section. The cocycle is of total cohomological degree $+2$ and so determines a central extension of $A_d \otimes \mathfrak{g}$ that we denote $\mathfrak{g}_{d,\theta}$. Our first main result is that our "higher annular algebra" of the Kac-Moody factorization algebra from the discussion above recovers this Lie algebra extension.

Theorem 2.1. *Let $\mathcal{F}_{1d}$ be the one-dimensional factorization algebra obtained by the reduction of the Kac-Moody factorization algebra $\mathbb{U}_\alpha\left(\mathcal{G}_{\mathbb{C}^d\setminus\{0\}}\right)$ along the sphere $S^{2d-1} \subset \mathbb{C}^d \setminus \{0\}$. There is a dense subfactorization algebra $\mathcal{F}_{1d}^{lc} \subset \mathcal{F}_{1d}$ that is locally constant. As a one-dimensional locally constant factorization algebra, $\mathcal{F}_{1d}^{lc}$ is equivalent to the A_∞-algebra $U(\mathfrak{g}_{d,\theta})$ of* **[?]**.

In the final part of this section we specialize to the manifold $X = (\mathbb{C} \setminus \{0\})^d$. Note that when $d = 1$ this is the same as above the annular algebra, but for $d > 1$ this factorization algebra has a different flavor. We will show how to extract the data of an E_d-algebra from this configuration, and discuss its role in the theory of higher dimensional vertex algebras.

2.1. **The higher sphere algebras.** The affine algebra associated to a Lie algebra $\mathfrak{g}$ together with an invariant pairing κ is defined as a central extension of the loop algebra of $\mathfrak{g}$

$$\mathbb{C} \to \widehat{\mathfrak{g}}_\kappa \to Lg$$

where we use the algebraic loop algebra $L\mathfrak{g} = g[z, z^{-1}]$. The central extension is determined by the cocycle

$$(f \otimes X, g \otimes Y) \mapsto \oint f \mathrm{d}g \kappa(X, Y).$$

A natural generalization of the loop algebra is to generalize the circle S^1, which is equal to the units in $\mathbb{C}$, by the sphere S^{2d-1}, which is equal to the units in $\mathbb{C}^d$. That is, we work with a "sphere algebra" of maps from S^{2d-1} into $\mathfrak{g}$. For topologists, this direction might seem natural, but it may not seem too natural from the perspective of algebraic geometry. In particular, an algebro-geometric sphere is given by a punctured affine d-space $\mathbb{A}^{d\times} = \mathbb{A}^d \setminus \{0\}$ or a punctured formal d-disk, but every map from these spaces to $\mathfrak{g}$ extends to a map from $\mathbb{A}^d$ or the formal d-disk into $\mathfrak{g}$ (essentially, by Hartog's lemma). Thus, this direction seems fruitless, since naively there would be no interesting central extensions. The key to evading this issue is to work with the *derived* space of maps. Indeed, the sheaf cohomology of $\mathcal{O}$ on the punctured affine d-space is interesting.

This fact ought not to be too surprising: as a smooth manifold, punctured affine d-space is equivalent to $\mathbb{R}_{>0} \times S^{2d-1}$, and this equivalence manifests itself in the cohomology of the structure sheaf. Explicitly,

$$H^*(\mathbb{A}^{d\times}, \mathcal{O}^{alg}) = \begin{cases} 0, & * \neq 0, d-1 \\ \mathbb{C}[z_1, \ldots, z_d], & * = 0 \\ \mathbb{C}[z_1^{-1}, \ldots, z_d^{-1}]\frac{1}{z_1 \cdots z_d}, & * = d-1 \end{cases}$$

as one can show by direct computation (e.g., use the cover by the affine opens of the form $\mathbb{A}^d \setminus \{z_i = 0\}$). When $d = 1$, this recovers the usual Laurent series; and it is natural to view the above as the higher-dimensional analogue of the Laurent series, with the polar part now in degree $d-1$.

Hence, the derived global sections $\mathbb{R}\Gamma(\mathbb{A}^{d\times}, \mathcal{O})$ of $\mathcal{O}$ provide a homotopy-commutative algebra, and thus one obtains a homotopy-Lie algebra by tensoring with $\mathfrak{g}$, which we will call the sphere Lie algebra by analogy with the loop Lie algebra. One can then study central extensions of this homotopy-Lie algebra, which are analogous to the affine Kac-Moody Lie algebras. For explicit constructions, it is convenient to have a commutative dg algebra that models the derived global sections. It should be no surprise that we like to work with the Dolbeault complex. We will use this approach to relate the sphere Lie algebra and its extensions to the current algebras that we've already introduced.

An explicit dg model A_d for the derived global sections has been written down in [**?**] based on the Jouanolou method for resolving singularities. We have recalled its definition in the Appendix of this chapter.

We are interested in the dg Lie algebra $A_d \otimes \mathfrak{g}$. For any d and symmetric function $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$, in [**?**] they define the cocycle

$$\theta_{FHK} : (A_d \otimes \mathfrak{g})^{\otimes(d+1)} \to \mathbb{C} \ \ , \ \ a_0 \cdots a_d \mapsto \mathrm{Res}_{z=0}\theta(a_0, \mathrm{d}a_1, \ldots, \mathrm{d}a_d),$$

where d is the algebraic de Rham differential. It is immediate that this cocycle has cohomological degree $+2$ and so determines a(n) (unshifted) dg Lie algebra central extension of $A_d \otimes \mathfrak{g}$:

$$\mathbb{C} \to \widehat{\mathfrak{g}}_{d,\theta} \to A_d \otimes \mathfrak{g}. \tag{4}$$

Our aim is to show how the Kac-Moody factorization algebra is related to this dg Lie algebra.

2.2. **The strategy.** We consider the restriction of the factorization algebra $\mathbb{U}_\theta(\mathcal{G})$ on $\mathbb{C}^d \setminus \{0\}$ to the collection of open sets diffeomorphic to spherical shells. This restriction has the structure of a one-dimensional factorization algebra corresponding to the iterated nesting of spherical shells. We show that there is a dense subfactorization algebra that is locally constant, hence corresponds to an E_1 algebra. We conclude by identifying an A_∞ model for this algebra as the universal enveloping algebra of a certain L_∞ algebra, that agree with the higher dimensional affine algebras of [**?**]

Introduce the radial projection map

$$\rho : \mathbb{C}^d \setminus 0 \to \mathbb{R}_{>0}$$

sending $z = (z_1, \ldots, z_d)$ to $|z| = \sqrt{|z_1|^2 + \cdots + |z_d|^2}$. We will restrict our factorization algebra to spherical shells by pushing forward the factorization algebra along this map. Indeed, the preimage of an open interval is such a spherical shell, and the factorization product on the line is equivalent to the nesting of shells.

2.2.1. *The case of zero level.* First we will consider the higher Kac-Moody factorization algebra on $\mathbb{C}^d \setminus \{0\}$ "at level zero". That is, the factorization algebra $\mathbb{U}(\mathcal{G}_{\mathbb{C}^d \setminus \{0\}}$. In this section we will omit $\mathbb{C}^d \setminus \{0\}$ from the notation, and simply refer to the factorization algebra by $\mathbb{U}(\mathcal{G})$.

Let $\rho_* (\mathbb{U}\mathcal{G})$ be the factorization algebra on $\mathbb{R}_{>0}$ obtained by pushing forward along the radial projection map. Explicitly, to an open set $I \subset \mathbb{R}_{>0}$ this factorization algebra assigns the dg vector space

$$\mathrm{C}_*^{\mathrm{Lie}} \left(\Omega_c^{0,*}(\rho^{-1}(I)) \otimes \mathfrak{g} \right).$$

Let $I \subset \mathbb{R}_{>0}$ be an open subset. There is the natural map $\rho^* : \Omega_c^*(I) \to \Omega_c^*(\rho^{-1}(I))$ given by the pull back of differential forms. We can post compose this with the natural projection $\mathrm{pr}_{\Omega^{0,*}} : \Omega_c^* \to \Omega_c^{0,*}$ to obtain a map of commutative algebras $\mathrm{pr}_{\Omega^{0,*}} \circ \rho^* : \Omega_c^*(I) \to \Omega_c^{0,*}(\rho^{-1}(I))$. The map j from Proposition A.2 determines a map of dg commutative algebras $j : A_d \to \Omega^{0,*}(\rho^{-1}(I))$. Thus, we obtain a map

$$\Phi(I) = (\mathrm{pr}_{\Omega^{0,*}} \circ \rho^*) \otimes j : \begin{array}{ccc} \Omega^*_c(I) \otimes A_d & \to & \Omega^{0,*}_c\left((\rho^{-1}(I)\right) \\ \varphi \otimes a & \mapsto & ((\mathrm{pr}_{\Omega^{0,*}} \circ \rho^*)\varphi) \wedge j(a) \end{array} \tag{5}$$

Since this is a map of commutative dg algebras it defines a map of dg Lie algebras

$$\Phi(I) \otimes \mathrm{id}_{\mathfrak{g}} : (\Omega^*_c(I) \otimes A_d) \otimes \mathfrak{g} = \Omega^*_c(I) \otimes (A_d \otimes \mathfrak{g}) \to \Omega^{0,*}(\rho^{-1}(I)) \otimes \mathfrak{g}$$

which maps $(\varphi \otimes a) \otimes X \mapsto \Phi(\varphi \otimes a) \otimes X$. We will drop the $\mathrm{id}_{\mathfrak{g}}$ from the notation and will denote this map simply by $\Phi(I)$. Note that $\Phi(I)$ is compatible with inclusions of open sets, hence extends to a map of cosheaves of dg Lie algebras that we will call Φ.

We can summarize the results as follows.

Proposition 2.2. *The map Φ extends to a map of factorization Lie algebras*

$$\Phi : \Omega^*_{\mathbb{R}_{>0},c} \otimes (A_d \otimes \mathfrak{g}) \to \rho_* \mathcal{G}.$$

Hence, it defines a map of factorization algebras

$$\mathrm{C}_*(\Phi) : U^{fact}\left(\Omega^*_{\mathbb{R}_{>0}} \otimes (A_d \otimes \mathfrak{g})\right) \to \rho_*\left(\mathbb{U}\mathcal{G}\right).$$

The fact that we obtain a map of factorization algebras follows from applying the functor $\mathrm{C}^{\mathrm{Lie}}_*(-)$ to Φ. It is immediate to see that this functor commutes with push-forward.

2.2.2. *The case of non-zero level.* We now proceed to the proof of Theorem . The dg Lie algebra $\mathfrak{g}_{d,\theta}$ determines a dg associative algebra via its universal enveloping algebra $U(\mathfrak{g}_{d,\theta})$. This dg algebra determines a factorization algebra on the one-manifold $\mathbb{R}_{>0}$ that assigns to every open interval $I \subset \mathbb{R}_{>0}$ the dg vector space $U(A_d \otimes \mathfrak{g})$. The factorization product is uniquely determined by the algebra structure. Henceforth, we denote this factorization algebra by $U(\mathfrak{g}_{d,\theta})^{fact}$.

To prove the theorem we will construct a sequence of maps of factorization Lie algebras on $\mathbb{R}_{>0}$:

$$\begin{tikzcd} & \mathcal{G}_1 \arrow[dr, "\Phi_1"] & & \mathcal{G}_2 \\ \mathcal{G}_0 \arrow[ur, "\simeq", "\Phi_0"'] & & \mathcal{G}_1' \arrow[ur, "\Phi_2"'] & \end{tikzcd}.$$

The factorization envelope of $\mathcal{G}_0$ is equivalent to the factorization algebra $U(\widehat{\mathfrak{g}}_{d,\theta})^{fact}$. Moreover, the factorization envelope of $\mathcal{G}_2$ is the push-forward of of the higher Kac–Moody factorization algebra $\rho_*\mathbb{U}\mathcal{G}$. Hence, the desired map of factorization algebras is produced by applying the factorization envelope functor to the above composition of factorization Lie algebras.

First, we introduce the factorization Lie algebra $\mathcal{G}_0$. To an open set $I \subset \mathbb{R}$, it assigns the dg Lie algebra $\mathcal{G}_0(I) = \Omega^*_c(I) \otimes \widehat{\mathfrak{g}}_{d,\theta}$, where $\widehat{\mathfrak{g}}_{d,\theta}$ is the central extension from Equation (4). The differential and Lie bracket are determined by the fact that we are tensoring a commutative dg algebra with a dg Lie algebra. A slight variant of Proposition 3.4.0.1 in [?], which shows that the one-dimensional factorization envelope of an ordinary Lie algebra produces its ordinary universal enveloping algebra, shows that there is a quasi-isomorphism of factorization algebras on $\mathbb{R}$,

$$(U\widehat{\mathfrak{g}}_{d,\theta})^{fact} \xrightarrow{\simeq} \mathrm{C}^{\mathrm{Lie}}_*(\mathcal{G}_0).$$

The factorization Lie algebra $\mathcal{G}_0$ is a central extension of the factorization Lie algebra $\Omega^*_{\mathbb{R},c} \otimes (A_d \otimes \mathfrak{g})$ by the trivial module $\Omega^*_c \oplus \mathbb{C} \cdot K$. Indeed, the cocycle determining the central extension is given by

$$\theta_0(\varphi_0\alpha_0, \ldots, \varphi_d\alpha_d) = (\varphi_0 \wedge \cdots \wedge \varphi_d)\theta_{A_d}(\alpha_1, \ldots, \alpha_d).$$

The factorization Lie algebra $\Omega^*_{\mathbb{R},c} \otimes (A_d \otimes \mathfrak{g})$ is the compactly supported sections of the local Lie algebra $\Omega^*_{\mathbb{R}} \otimes (A_d \otimes \mathfrak{g})$ and this cocycle determining the extension is a local cocycle.

Next, we define the factorization dg Lie algebra $\mathcal{G}_1$ on $\mathbb{R}$. This is also obtained as a central extension of the factorization Lie algebra $\Omega^*_{\mathbb{R},c} \otimes (A_d \otimes \mathfrak{g})$:

$$0 \to \mathbb{C} \cdot K[-1] \to \mathcal{G}_1 \to \Omega^*_{\mathbb{R},c} \otimes (A_d \otimes \mathfrak{g}) \to 0$$

determined by the following cocycle. For an open interval I write $\varphi_i \in \Omega^*_c(I)$, $\alpha_i \in A_d \otimes \mathfrak{g}$. The cocycle is defined by

$$\theta_1(\varphi_0\alpha_0, \ldots, \varphi_d\alpha_d) = \left(\int_I \varphi_0 \wedge \cdots \varphi_d\right)\theta_{\mathrm{FHK}}(\alpha_0, \ldots, \alpha_d) \tag{6}$$

where θ_{FHK} was defined in Definition **??**.

The functional θ_1 determines a local cocycle in $\mathrm{C}^*_{\mathrm{loc}}\left(\Omega^*_{\mathbb{R}} \otimes (A_d \otimes \mathfrak{g})\right)$ of degree one.

We now define a map of factorization Lie algebras $\Phi_0 : \mathcal{G}_0 \to \mathcal{G}_1$. On and open set $I \subset \mathbb{R}$, we define the map $\Phi_0(I) : \mathcal{G}_0(I) \to \mathcal{G}_1(I)$ by

$$\Phi_0(I)(\varphi\alpha, \psi K) = \left(\varphi\alpha, \int \psi \cdot K\right).$$

For a fixed open set $I \subset \mathbb{R}$, the map Φ_0 fits into the commutative diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \Omega^*_c(I) \otimes \mathbb{C} \cdot K \arrow[r] \arrow[d, "\int", "\simeq"'] & \mathcal{G}_0(I) \arrow[r] \arrow[d, "\Phi_0(I)"] & \Omega^*_c(I) \otimes (A_d \otimes \mathfrak{g}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{C} \cdot K[-1] \arrow[r] & \mathcal{G}_1(I) \arrow[r] & \Omega^*_c(I) \otimes (A_d \otimes \mathfrak{g}) \arrow[r] & 0.
\end{tikzcd}$$

To see that $\Phi_0(I)$ is a map of dg Lie algebras we simply observe that the cocycles determining the central extensions are related by $\theta_1 = \int \circ\, \theta_0$, where $\int : \Omega^*_c(I) \to \mathbb{C}$ as in the diagram above. Since $\int$ is a quasi-isomorphism, the map $\Phi_0(I)$ is as well. It is clear that as we vary the interval I we obtain a quasi-isomorphism of factorization Lie algebras $\Phi_0 : \mathcal{G}_0 \xrightarrow{\simeq} \mathcal{G}_1$.

We now define the factorization dg Lie algebra $\mathcal{G}'_1$. Like $\mathcal{G}_0$ and $\mathcal{G}_0$, it is a central extension of $\Omega^*_{\mathbb{R},c} \otimes (A_d \otimes \mathfrak{g})$. The cocycle determining the central extension is defined by

$$\theta'_1(\varphi_0 a_0 X_0, \ldots, \ldots, \varphi_d a_d X_d) = \theta_1(\varphi_0 a_0 X_0, \ldots, \ldots, \varphi_d a_d X_d) + \widetilde{\theta}_1(\varphi_0 a_0 X_0, \ldots, \ldots, \varphi_d a_d X_d)$$

where θ_1 was defined in Equation (6). Before writing down the explicit formula for $\widetilde{\theta}_1$ we introduce some notation. Set

$$E = r\frac{\partial}{\partial r},$$
$$\mathrm{d}\vartheta = \sum_i \frac{\mathrm{d}z_i}{z_i}.$$

We view E as a vector field on $\mathbb{R}_{>0}$ and $\mathrm{d}\vartheta$ as a $(1,0)$-form on $\mathbb{C}^d \setminus 0$. Define the functional

$$\widetilde{\theta}_1(\varphi_0 a_0 X_0, \ldots, \varphi_d a_d X_d) = \frac{1}{2}\sum_{i=1}^{d}\left(\int_I \varphi_0 (E\cdot\varphi_i)\varphi_1\cdots\widehat{\varphi_i}\cdots\varphi_d\right)\left(\oint (a_0 a_i \mathrm{d}\vartheta)\,\partial a_1\cdots\widehat{\partial a_i}\cdots\partial a_d\right)\theta(X_0,\ldots,X_d).$$

The functional $\widetilde{\theta}$ defines a local functional in $\mathrm{C}^*_{\mathrm{loc}}\left(\Omega^*_{\mathbb{R}_{>0}}\otimes(A_d\otimes\mathfrak{g})\right)$ of cohomological degree one. One immediately checks that it is a cocycle. This completes the definition of the factorization Lie algebra $\mathcal{G}_1'$.

The factorization Lie algebras $\mathcal{G}_1$ and $\mathcal{G}_1'$ are identical as precosheaves of vector spaces. In fact, if we put a filtration on $\mathcal{G}_1$ and $\mathcal{G}_1'$ where the central element K has filtration degree one, then the associated graded factorization Lie algebras Gr $\mathcal{G}_1$ and Gr $\mathcal{G}_1'$ are also identified. The only difference in the Lie algebra structures comes from the deformation of the cocycle determining the extension of $\mathcal{G}_1'$ given by $\widetilde{\theta}_1$.

In fact, we will show that $\widetilde{\theta}_1$ is actually an exact cocycle via the cobounding element $\eta \in \mathrm{C}^*_{\mathrm{loc}}\left(\Omega^*_{\mathbb{R}_{>0}}\otimes(A_d\otimes\mathfrak{g})\right)$ defined by

$$\eta(\varphi_0 a_0 X_0, \ldots, \varphi_d a_d X_d) = \sum_{i=1}^{d}\left(\int_I \varphi_0\left(\iota_E\varphi_i\right)\varphi_1\cdots\widehat{\varphi_i}\cdots\varphi_d\right)\left(\oint (a_0 a_i \mathrm{d}\vartheta)\,\partial a_1\cdots\widehat{\partial a_i}\cdots\partial a_d\right)\theta(X_0,\ldots,X_d).$$

Lemma 2.3. *One has $\mathrm{d}\eta = \widetilde{\theta}_1$, where d is the differential for the cochain complex $\mathrm{C}^*_{\mathrm{loc}}(\Omega^*_{\mathbb{R}_{>0}}\otimes(A_d\otimes\mathfrak{g}))$. In particular, the factorization Lie algebras $\mathcal{G}_1$ and $\mathcal{G}_1'$ are quasi-isomorphic (as L_∞ algebras). An explicit quasi-isomorphism is given by the L_∞ map $\Phi_1 : \mathcal{G}_1 \to \mathcal{G}_1'$ that sends the central element K to itself and an element $(\varphi_0 a_0 X_0, \ldots, \varphi_d a_d X_d) \in \mathrm{Sym}^{d+1}(\Omega^*_c\otimes(A_d\otimes\mathfrak{g})$ to*

$$(\varphi_0 a_0 X_0, \ldots, \varphi_d a_d X_d) + \eta(\varphi_0 a_0 X_0, \ldots, \varphi_d a_d X_d)\cdot K \in \mathrm{Sym}^{d+1}(\Omega^*_c\otimes(A_d\otimes\mathfrak{g}))\oplus\mathbb{C}\cdot K.$$

Finally, we define the factorization Lie algebra $\mathcal{G}_2$. We have already seen that the local cocycle $J(\theta)\in\mathrm{C}^*_{\mathrm{loc}}(\mathfrak{g}^{\mathbb{C}^d})$ determines a central extension of factorization Lie algebras

$$0\to\mathbb{C}\cdot K[-1]\to\mathcal{G}_{J(\theta)}\to\Omega^{0,*}_{\mathbb{C}^d,c}\otimes\mathfrak{g}\to 0.$$

Of course, we can restrict $\mathcal{G}_{J(\theta)}$ to a factorization algebra on $\mathbb{C}^d\setminus 0$. The factorization algebra $\mathcal{G}_2$ is defined as the pushforward of this restriction along the radial projection: $\mathcal{G}_2 := \rho_*\left(\mathcal{G}_{J(\theta)}|_{\mathbb{C}^d\setminus 0}\right)$.

Recall the map $\Phi : \Omega^*_{\mathbb{R}_{>0},c}\otimes(A_d\otimes\mathfrak{g})\to\rho_*(\Omega^{0,*}_{\mathbb{C}^d\setminus 0,c}\otimes\mathfrak{g})$ defined in Equation (5). On each open set $I\subset\mathbb{R}_{>0}$ we can extend Φ by the identity on the central element to a linear map $\Phi_2 : \mathcal{G}_1'(I)\to\mathcal{G}_2(I)$.

Lemma 2.4. *The map $\Phi_2 : \mathcal{G}_1'(I)\to\mathcal{G}_2(I)$ is a map of dg Lie algebras. Moreover, it extends to a map of factorization Lie algebras $\Phi_2 : \mathcal{G}_1'\to\mathcal{G}_2$.*

Proof. Modulo the central element Φ_2 reduces to the map Φ, which we have already seen is a map of factorization Lie algebras in Proposition 2.2. Thus, to show that Φ_2 is a map of factorization Lie algebras we need to show that it is compatible with the cocycles determing the respective central extensions. That is, we need to show that

(7) $$\theta_1'(\varphi_0 a_0 X_0,\ldots,\varphi_d a_d X_d) = \theta_2(\Phi(\varphi_0 a_0 X_0),\ldots,\Phi(\varphi_d a_d X_d))$$

for all $\varphi_i a_i X_i \in \Omega^*_c(I) \otimes (A_d \otimes \mathfrak{g})$. The cocycle θ'_1 is only nonzero if one of the φ_i inputs is a 1-form. We evaluate the left-hand side on the $(d+1)$-tuple $(\varphi_0 \mathrm{d}r a_0 X_0, \varphi_1 a_1 X_1, \ldots, \varphi_d a_d X_d)$ where $\varphi_i \in C^\infty_c(I)$, $a_i \in A_d$, $X_i \in \mathfrak{g}$ for $i = 0, \ldots, d$. The result is

$$\left(\int_I \varphi_0 \cdots \varphi_d \mathrm{d}r\right) \left(\oint a_0 \partial a_1 \cdots \partial a_d\right) \theta(X_0, \ldots, X_d) \tag{8}$$

$$+ \quad \frac{1}{2} \sum_{i=1}^d \left(\int_I \varphi_0 (E \cdot \varphi_i) \varphi_1 \cdots \widehat{\varphi_i} \cdots \varphi_d \mathrm{d}r\right) \left(\oint (a_0 a_i \mathrm{d}\vartheta)\, \partial a_1 \cdots \widehat{\partial a_i} \cdots \partial a_d\right) \theta(X_0, \ldots, X_d) \tag{9}$$

We wish to compare this to the right-hand side of Equation (7). Recall that $\Phi(\varphi_0 \mathrm{d}r a_0 X_0) = \varphi(r) \mathrm{d}r a_0(z) X_0$ and $\Phi(\varphi_i a_i X_i) = \varphi(r) a_i(z) X_i$. Plugging this into the explicit formula for the cocycle θ_2 we see the right-hand side of (7) is

$$\left(\int_{\rho^{-1}(I)} \varphi_0(r) \mathrm{d}r a_0(z) \partial(\varphi_1(r) a_1(z)) \cdots \partial(\varphi_d(r) a_d(z))\right) \theta(X_0, \ldots, X_d). \tag{10}$$

We pick out the term in (10) in which the ∂ operators only act on the elements $a_i(z)$, $i = 1, \ldots, d$. This term is of the form

$$\int_{\rho^{-1}(I)} \varphi_0(r) \cdots \varphi_d(r) \mathrm{d}r a_0(z) \partial(a_1(z)) \cdots \partial(a_d(z)) \theta(X_0, \ldots, X_d).$$

Separating variables we find that this is precisely the first term (8) in the expansion of the left-hand side of (7).

Now, note that we can rewrite the ∂-operator in terms of the radius r as

$$\partial = \sum_{i=1}^d \mathrm{d}z_i \frac{\partial}{\partial z_i} = \sum_{i=1}^d \mathrm{d}z_i \bar{z}_i \frac{\partial}{\partial (r^2)} = \sum_{i=1}^d \mathrm{d}z_i \frac{r^2}{2 z_i} \frac{\partial}{\partial r}.$$

The remaining terms in (10) correspond to the expansion of

$$\partial(\varphi_1(r) a_1(z)) \cdots \partial(\varphi_d(r) a_d(z)),$$

using the Leibniz rule, for which the ∂ operators act on at least one of the functions $\varphi_1, \ldots, \varphi_d$. In fact, only terms in which ∂ acts on precisely one of the functions $\varphi_1, \ldots, \varphi_d$ will be nonzero. For instance, consider the term

$$(\partial \varphi_1) a_1(z) (\partial \varphi_2) a_2(z) \partial(\varphi_3(z) a_3(z)) \cdots \partial(\varphi_d(z) a_d(z)). \tag{11}$$

Now, $\partial \varphi_i(r) = \omega \frac{\partial \varphi}{\partial r}$ where ω is the one-form $\sum_i (r^2/2z_i) \mathrm{d}z_i$. Thus, (11) is equal to

$$\left(\omega \frac{\partial \varphi_1}{\partial r}\right) a_1(z) \left(\omega \frac{\partial \varphi_2}{\partial r}\right) a_2(z) \partial(\varphi_3(z) a_3(z)) \cdots \partial(\varphi_d(z) a_d(z),$$

which is clearly zero as ω appears twice.

We observe that terms in the expansion of (10) for which ∂ acts on precisely one of the functions $\varphi_1, \ldots, \varphi_d$ can be written as

$$\sum_{i=1}^d \int_{\rho^{-1}(I)} \varphi_0(r) \left(r \frac{\partial}{\partial r} \varphi_i(r)\right) \varphi_1(r) \cdots \widehat{\varphi_i(r)} \cdots \varphi_d(r) \mathrm{d}r \frac{r}{2 z_i} \mathrm{d}z_i a_0(z) a_i(z) \partial a_1(z) \cdots \widehat{\partial a_i(z)} \cdots \partial a_d(z).$$

Finally, notice that the function $z_i/2r$ is independent of the radius r. Thus, separating variables we find the integral can be written as

$$\frac{1}{2} \sum_{i=1}^d \left(\int_I \varphi_0 \left(r \frac{\partial}{\partial r} \varphi_i\right) \varphi_1 \cdots \widehat{\varphi}_i \cdots \varphi_d \mathrm{d}r\right) \left(\oint \frac{\mathrm{d}z_i}{z_i} a_0 a_i \partial a_2 \cdots \widehat{\partial a_i} \cdots \partial a_d\right).$$

This is precisely equal to the second term (9) above. Hence, the cocycles are compatible and the proof is complete.

□

2.3. **An E_d algebra from tori.** There is another direction that one may look to extend the notion of affine algebras to higher dimensions. The affine algebra is a central extension of the loop algebra on $\mathfrak{g}$. Instead of looking at higher dimensional sphere algebras, one can consider higher *torus* algebras; or iterated loop algebras:

$$L^d\mathfrak{g} = \mathbb{C}[z_1^{\pm}, \cdots, z_d^{\pm}] \otimes \mathfrak{g}.$$

These iterated loop algebras are algebraic versions of the torus mapping space $\mathrm{Map}(S^1 \times \cdots \times S^1, \mathfrak{g})$. In this section we show what information the Kac-Moody vertex algebra implies about extensions of such iterated loop algebras.

To do this we specialize the Kac-Moody factorization algebra to the complex manifold $(\mathbb{C}^\times)^d$, which is homotopy equivalent to the topologists torus $(S^1)^{\times d}$. We show, in a similar way as above, how to extract the structure of an E_d algebra from considering the nesting of "polyannuli" in $(\mathbb{C}^\times)^d$. These E_d-algebras are related to interesting extensions of the Lie algebra $L^d\mathfrak{g}$.

When $d = 1$, we have seen that the nesting of ordinary annuli give rise to the structure of an associative algebra. For $d > 1$, a polyannulus is a complex submanifold of the form $\mathrm{Ann}_1 \times \cdots \times \mathrm{Ann}_d \subset (\mathbb{C}^\times)^d$ where each $\mathrm{Ann}_i \subset \mathbb{C}^\times$ is an ordinary annulus. Equivalently, a polyannulus is the complement of a closed polydisk inside of a larger open polydisk. We will see how the nesting of annuli in each component gives rise to the structure of a locally constant factorization algebra in d *real* dimensions, and hence defines an E_d algebra.

A result of Knudsen [**?**], which we recall in Section **??**, states that every dg Lie algebra determines an E_d-algebra, for any $d > 1$, called the universal E_d enveloping algebra. To state its properties properly, we need to be in the context of ∞-categories.

Theorem 2.5 ([**?**]). *Let $\mathcal{C}$ be a stable, $\mathbb{C}$-linear, presentable, symmetric monoidal ∞-category. There is an adjunction*

$$U^{E^d} : \mathrm{LieAlg}(\mathcal{C}) \leftrightarrows E_d\mathrm{Alg}(\mathcal{C}) : F$$

such that for any object $X \in \mathcal{C}$ one has $\mathrm{Free}_{E_d}(X) \simeq U^{E_d}\mathrm{Free}_{Lie}(\Sigma^{d-1}X)$.

We are most interested in the case $\mathcal{C}$ is the category of chain complexes with tensor product $\mathrm{Ch}^{\otimes}$. In this situation, the enveloping algebra U^{E^d} agrees with the ordinary universal enveloping algebra when $d = 1$. When the twisting cocycle defining the Kac-Moody factorization algebra is zero we will see that the E_d algebra coming from the product of polyannuli is equivalent to $U^{E_d}(L^d\mathfrak{g})$. When we turn on a twisting cocycle we will find the E_d-enveloping algebra of a central extension of the iterated loop algebra.

The Kac–Moody factorization algebra on the d-fold $(\mathbb{C}^\times)^d$ determines a real d-dimensional factorization algebra by considering the radius in each complex direction. We denote this factorization algebra on $\mathbb{R}^d$ by $\vec{\rho}_*(\mathcal{G}_{\mathbb{C}^{\times d}})$.

To state our result precisely, let $\vec{\rho} : (\mathbb{C}^\times)^d \to (\mathbb{R}_{>0})^d$ be the projection $(z_1, \ldots, z_d) \mapsto (|z_1|, \cdots, |z_d|)$. We will pushforward the Kac-Moody factorization algebra along $\vec{\rho}$.

On the Lie algebra side, we see that the following formula defines a cocycle on $L^d\mathfrak{g}$ of degree $d+1$:

$$\begin{array}{cccc} L^d\theta : & (L^d\mathfrak{g})^{\otimes d+1} & \to & \mathbb{C} \\ & (f_0 \otimes X_0) \otimes \cdots \otimes (f_d \otimes X_d) & \mapsto & \theta(X_0, \ldots, X_d) \oint_{|z_1|=1} \cdots \oint_{|z_d|=1} f_0 \mathrm{d} f_1 \cdots \mathrm{d} f_d. \end{array}$$

Here $f_i \otimes X_i \in L^d\mathfrak{g} = \mathbb{C}[z_1^{\pm}, \cdots, z_d^{\pm}] \otimes \mathfrak{g}$. The above is just an iterated version of the usual residue pairing. This cocycle determines a shifted Lie algebra extension of the iterated loop algebra

$$\mathbb{C}[d-1] \to \widehat{L^d\mathfrak{g}}_\theta \to L^d\mathfrak{g},$$

that appears in the theorem below.

Theorem 2.6. *Fix $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$ and let $\vec{\rho}_* \mathbb{U}_\theta \mathcal{G}_{(\mathbb{C}^\times)^d}$ be the factorization algebra on $(\mathbb{R}_{>0})^d$ obtained by reducing the Kac-Moody factorization algebra along the d-torus. There exists a dense d-dimensional subfactorization algebra $\mathcal{F}^{lc}$ that is locally constant and is equivalent, as E_d-algebras, to*

$$U^{E_d}\left(\widehat{L^d\mathfrak{g}}_\theta\right).$$

2.4. The disk as a module.

3. The Kac-Moody factorization algebra on general manifolds

In this section we explore global properties of the Kac-Moody factorization algebra on complex manifolds.

3.1. The P_0-structure. Every associative algebra determines a Lie algebra via the commutator. There is a left adjoint to this forgetful functor given by the enveloping algebra of a Lie algebra. Given a Lie algebra $\mathfrak{g}$, this enveloping algebra $U\mathfrak{g}$ can also be thought of as a quantization of a certain Poisson algebra. The Poincaré-Birkoff-Witt theorem says that the associated graded Gr $U\mathfrak{g}$ by the filtration given by symmetric degree is precisely $\mathbb{C}[\mathfrak{g}^*]$. It is a classical fact that the linear dual $\mathfrak{g}^*$ of a Lie algebra has the structure of a Poisson manifold. The Poisson bracket on $\mathbb{C}[\mathfrak{g}^*] = \mathrm{Sym}(\mathfrak{g})$ is defined by extending the Lie bracket on the quadratic functions by the Leibniz rule.

In a completely analogous way, the factorization enveloping algebra of a local Lie algebra has a "classical limit" given by a P_0 factorization algebra. Recall, the factorization enveloping algebra of a local Lie algebra $\mathcal{L}$ evaluated on an open set U is given by the Chevalley-Eilenberg complex of the compactly supported sections on U

$$\mathbb{U}(\mathcal{L})(U) = C_*^{\mathrm{Lie}}(\mathcal{L}(U)) = \left(\mathrm{Sym}^*(\mathcal{L}(U)[1]), \mathrm{d}_{\mathcal{L}} + \mathrm{d}_{CE}\right).$$

There is a filtration of this complex defined by $F^k = \mathrm{Sym}^{\geq k}(\mathcal{L}(U)[1])$. Moreover, this defines a filtration of the factorization algebra $\mathbb{U}(\mathcal{L})$.

Lemma 3.1. *Let $\mathcal{L}$ be a local Lie algebra. Then, the associated graded factorization algebra* Gr $\mathbb{U}(\mathcal{L})$ *has the structure of a P_0 factorization algebra. Similarly, if $\alpha \in C^*_{\mathrm{loc}}(\mathcal{L})$ is a cocycle of cohomological degree one then* Gr $\mathbb{U}_\alpha(\mathcal{L})$ *has the structure of a P_0 factorization algebra.*

Up to issues of functional analysis, one should think of the P_0 algebra Gr $\mathbb{U}(\mathcal{L})$ as the algebra of functions on the sheaf of dg vector spaces $\mathcal{L}^\vee[-1]$ with differential induced from that on $\mathcal{L}$. The P_0 algebra Gr $\mathbb{U}_\alpha(\mathcal{L})$ is equal to functions on the same sheaf of dg vector spaces but with bracket modified by α.

Corollary 3.2. *For any principal G-bundle $P \to X$ consider the associated graded factorization algebra*

$$\mathrm{Gr}\ \mathbb{U}(\mathcal{A}\mathrm{d}(P)) : U \mapsto \left(\mathrm{Sym}^*(\Omega^{0,*}_c(U)[1]), \bar{\partial}\right).$$

Then, any element $\alpha \in H^1_{\mathrm{loc}}(\mathcal{A}\mathrm{d}(P))$ determines the structure of a P_0 factorization algebra on Gr $\mathbb{U}(\mathcal{A}\mathrm{d}(P))$.

In the case that $\alpha = J_X(\theta)$ is the local cocycle corresponding to a symmetric polynomial $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$ the Poisson structure can be described explicitly as follows. The Poisson tensor is of the form $\Pi = \Pi_{[-,-]} + \Pi_\theta$ where

$$\Pi_{[-,-]} = \wedge \otimes [-,-] : \left(\Omega^{d,*}_X \otimes \mathfrak{g}\right) \otimes \left(\Omega^{0,*}_X \otimes \mathfrak{g}\right) \to \Omega^{d,*}_X \otimes \mathfrak{g}$$

and

$$\Pi_\theta : \left(\Omega^{0,*}_X \otimes \mathfrak{g}\right)^{\otimes d} \to \Omega^{d,*}_X \otimes \mathfrak{g}$$

sends $\alpha_1 \otimes \cdots \otimes \alpha_d \mapsto \partial\alpha_1 \wedge \cdots \wedge \partial\alpha_d$.

3.2. Factorization homology along Hopf surfaces.

Proposition 3.3. *Let X be a Hopf manifold and suppose $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$ is any $\mathfrak{g}$-invariant polynomial of degree $(d+1)$. Then, there is a quasi-isomorphism*

$$\int_X \mathbb{U}_\theta(\mathcal{G}_X) \simeq \mathrm{Hoch}_*(U\mathfrak{g})[K]$$

where K is the central parameter of cohomological degree zero.

Proof. Let's first consider the untwisted case. In this case, we must show $\int_X \mathbb{U}(\mathcal{G}_X) \simeq \mathrm{Hoch}_*(U\mathfrak{g})$. The factorization homology on the left hand side is computed by

$$\int_X \mathbb{U}(\mathcal{G}_X) = \mathrm{C}^{\mathrm{Lie}}_*(\Omega^{0,*}(X) \otimes \mathfrak{g}).$$

We have already seen in Section **??** that every Hopf manifold is Dolbeualt formal. Thus, there is a quasi-isomorphism

$$\left(H^{0,*}(X), 0\right) \simeq \left(\Omega^{0,*}(X), \bar{\partial}\right).$$

In fact, we have written down a preferred presentation for the cohomology ring of X given by $H^{0,*}(X) = \mathbb{C}[\delta]$ where $|\delta| = 1$. A particular Dolbeault representative for δ given by

$$\bar{\partial}(\log|z|^2) = \sum_i \frac{z_i \mathrm{d}\bar{z}_i}{|z|^2}$$

where $z = (z_1, \ldots, z_d)$ is the coordinate on $\mathbb{C}^d \setminus \{0\}$.

Applied to the global sections of the Kac-Moody we see that there is a quasi-isomorphism

$$\int_X \mathbb{U}(\mathcal{G}_X) \simeq \mathrm{C}^{\mathrm{Lie}}_*(\mathbb{C}[\delta] \otimes \mathfrak{g}).$$

Now, note that $\mathrm{C}^{\mathrm{Lie}}_*(\mathbb{C}[\delta] \otimes \mathfrak{g}) = \mathrm{C}^{\mathrm{Lie}}_*(\mathfrak{g} \oplus \mathfrak{g}[-1]) = \mathrm{C}^{\mathrm{Lie}}_*(\mathfrak{g}, \mathrm{Sym}(\mathfrak{g}))$, where $\mathrm{Sym}(\mathfrak{g})$ is the symmetric product of the adjoint action of $\mathfrak{g}$ on itself. By Poincaré-Birkoff-Witt there is an isomorphism of vector spaces $\mathrm{Sym}(\mathfrak{g}) = U\mathfrak{g}$, so we can write this as $\mathrm{C}^{\mathrm{Lie}}_*(\mathfrak{g}, \mathrm{Sym}(\mathfrak{g}))$.

Now, any $U(\mathfrak{g})$-bimodule M is automatically a module for the Lie algebra $\mathfrak{g}$ by the formula $x \cdot m = xm - mx$ where $x \in \mathfrak{g}$ and $m \in M$. Moreover, for any such bimodule there is a quasi-isomorphism of cochain complexes

$$\mathrm{C}^{\mathrm{Lie}}_*(\mathfrak{g}, M) \simeq \mathrm{Hoch}_*(U\mathfrak{g}, M).$$

This is proved, for instance, in Section 2.3 of [?]. Applied to the bimodule $M = U\mathfrak{g}$ itself we obtain $\mathrm{C}^{\mathrm{Lie}}(\mathfrak{g}, U\mathfrak{g}) \simeq \mathrm{Hoch}(U\mathfrak{g})$.

The twisted case is similar. Let θ be as in the statement. Then, the factorization homology is equal to

$$\int_X \mathbb{U}_\theta(\mathcal{G}_X) = \left(\mathrm{Sym}(\Omega^{0,*}(X) \otimes \mathfrak{g})[K], \bar{\partial} + \mathrm{d}_{CE} + \mathrm{d}_\theta\right).$$

Applying Dolbeualt formality again we see that this is quasi-isomorphic to the cochain complex

$$(\mathrm{Sym}(\mathfrak{g}[\delta])[K], \mathrm{d}_{CE} + \mathrm{d}_\theta)\,. \tag{12}$$

We note that d_θ is identically zero on $\mathrm{Sym}(\mathfrak{g}[\delta])$. Indeed, for degree reasons, at least one of the inputs must be from $\mathfrak{g} \hookrightarrow \mathfrak{g}[\delta] = \mathfrak{g} \oplus \mathfrak{g}[-1]$, which consists of constant functions on X with values in the Lie algebra $\mathfrak{g}$. In the formula for the local cocycle (**??**) associated to θ it is clear that if any one of the inputs is constant the cocycle vanishes. Indeed, one can integrate by parts to put it in the form $\int \partial\alpha \cdots \partial\alpha$, which is the integral of a total derivative, hence zero since X has no boundary. Thus (12) just becomes the Chevalley-Eilenberg complex with values in the trivial module $\mathbb{C}[K]$. By the same argument as in the untwisted case, we conclude that in this case the factorization homology is quasi-isomorphic to $\mathrm{Hoch}_*(U\mathfrak{g})[K]$ as desired. $\square$

There is an interesting consequence of this calculation to the Hochschild homology for the A_∞ algebra $U(\widehat{\mathfrak{g}}_{d,\theta})$. It is easiest to state this when X is a Hopf manifold of the form $(\mathbb{C}^d \setminus \{0\})/q^{\mathbb{Z}}$ for a single $q \in D(0,1)^\times$ where the quotient is by the relation $(z_1, \ldots, z_d) \simeq (q^{\mathbb{Z}} z_1, \ldots, q^{\mathbb{Z}})$. Let $p_q : \mathbb{C}^d \setminus \{0\} \to X$ be the quotient map. Consider the following diagram

$$\begin{array}{ccc} \mathbb{C}^d \setminus \{0\} & \xrightarrow{p_q} & X \\ \downarrow \rho & & \downarrow \bar{\rho} \\ \mathbb{R}_{>0} & \xrightarrow{\bar{p}_q} & S^1 \end{array}$$

Here, ρ is the radial projection map and $\bar{\rho}$ is the induced map defined by the quotient. The action of $\mathbb{Z}$ on $\mathbb{C}^d \setminus \{0\}$ gives $\mathcal{G}_{\mathbb{C}^d \setminus \{0\}}$ the structure of a $\mathbb{Z}$-equivariant factorization algebra. In turn, this determines an action of $\mathbb{Z}$ on pushforward factorization algebra $\rho_* \mathcal{G}_{\mathbb{C}^d \setminus \{0\}}$. We have seen that there is a dense locally constant subfactorization algebra on $\mathbb{R}_{>0}$ of the pushforward that is equivalent as an E_1 algebra to $U(\widehat{\mathfrak{g}}_{d,\theta})$. A consequence of excision for factorization homology, see Lemma 3.18 [?], implies that there is a quasi-isomorphism

$$\mathrm{Hoch}_*(U(\widehat{\mathfrak{g}}_{d,\theta}), q) \simeq \int_{S^1} \bar{\rho}_* \mathbb{U}_\alpha(\mathcal{G}_X),$$

where the right-hand side is the Hochschild homology of the algebra $U\widehat{\mathfrak{g}}_{d,\theta}$ with coefficients in the bimodule $U\widehat{\mathfrak{g}}_{d,\theta}$ with the ordinary left-module structure and right-module structure given by twisting the ordinary action by the automorphism corresponding to the element $1 \in \mathbb{Z}$ on the algebra.

Moreover, by the push-forward for factorization homology, Proposition 3.23 **[?]**, there is an equivalence

$$\int_{S^1} \bar{\rho}_* \mathbb{U}_\alpha(\mathcal{G}_X) \xrightarrow{\simeq} \int_X \mathbb{U}_\alpha(\mathcal{G}_X).$$

We have just shown that the factorization homology of $\mathcal{G}_X$ is equal to the Hochschild homology of $U\mathfrak{g}$ so that

$$\mathrm{Hoch}_*(U(\mathfrak{g}_{d,\theta}), q) \simeq \mathrm{Hoch}_*(U\mathfrak{g})[K].$$

This statement is purely algebraic as the dependence on the manifold for which the Kac-Moody lives has dropped out. It may be easiest to understand in the case $d = 1$ and $\theta = 0$. Then $\mathfrak{g}_{d,\theta}$ is simply the loop algebra $L\mathfrak{g} = g[z, z^{-1}]$. The action of $\mathbb{Z}$ on $L\mathfrak{g}$ simply rotates the loop parameter: for $z^n \otimes \mathfrak{g} \in L\mathfrak{g} = \mathbb{C}[z, z^{-1}] \otimes \mathfrak{g}$ the action of $1 \in \mathbb{Z}$ is $1 \cdot (z^n \otimes \mathfrak{g}) = q^n z^n \otimes \mathfrak{g}$. In turn, the bimodule structure of $U(\mathfrak{g}[z, z^{-1}])$ on itself, which we denote $U(\mathfrak{g}[z, z^{-1}])_q$ is the ordinary one on the left and on the right is given by twisting by the automorphism corresponding to $1 \in \mathbb{Z}$. The complex $\mathrm{Hoch}_*(U(g[z, z^{-1}]), q)$ is the Hochschild homology of $U(\mathfrak{g}[z, z^{-1}])$ with values in this bimodule so the statement implies

$$\mathrm{Hoch}_*\left(U(\mathfrak{g}[z, z^{-1}]), U(\mathfrak{g}[z, z^{-1}])_q\right) \simeq \mathrm{Hoch}(U\mathfrak{g}).$$

3.3. **A variant of the factorization algebra.** So far we have mostly restricted ourselves to studying the Kac-Moody factorization algebra corresponding to local cocycles of type $J_X(\theta)$ where $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$. There is another class of local cocycles that appear when studying symmetries of holomorphic theories. Unlike the cocycle $J_X(\theta)$, which in some sense did not depend on the manifold X, this class of cocycles is more dependent on the manifold for which the current algebra lives.

Let X be a complex manifold of dimension d and suppose ω is a (k, k) form on X. Fix, in addition, a form $\theta_{d+1-k} \in \mathrm{Sym}(\mathfrak{g}^*)^{\mathfrak{g}}$. Then, we may consider the cochain on $\mathcal{G}(X)$:

$$\begin{array}{rccc} \phi_{\theta,\omega} : & \mathcal{G}(X)^{\otimes d+1-k} & \to & \mathbb{C} \\ & \alpha_0 \otimes \cdots \otimes \alpha_{d-k} & \mapsto & \displaystyle\int_X \omega \wedge \theta_{d+1-k}(\alpha_0, \partial\alpha_1, \ldots, \partial\alpha_{d-k}) \end{array}$$

It is clear that $\phi_{\theta,\omega}$ is a local cochain on $\mathcal{G}(X)$.

Lemma 3.4. *Let $\theta \in \mathrm{Sym}^{d+1-k}(\mathfrak{g}^*)^{\mathfrak{g}}$ and suppose $\omega \in \Omega^{k,k}(X)$ satisfies $\bar{\partial}\omega = 0$ and $\partial\omega = 0$. Then, $\phi_{\theta,\omega} \in \mathrm{C}^*_{\mathrm{loc}}(\mathcal{G}_X)$ is a local cocycle. Moreover, for fixed θ the cohomology class $[\phi_{\theta,\omega}] \in H^1_{\mathrm{loc}}(\mathcal{G}_X)$ only depends on the cohomology class*

$$[\omega] \in H^k(X, \Omega^k_{cl}).$$

Note that when $\omega = 1$ it trivially satisfies the conditions of the lemma. In this case $\phi_{\theta,1} = J_X(\theta)$ in the notation of the last section.

This class of cocycles is related to the ordinary Kac-Moody factorization and vertex algebra on Riemann surfaces in a natural way. We consider the complex manifold $X = \Sigma \times \mathbb{P}^{d-1}$ where Σ is a Riemann surface and $\mathbb{P}^{d-1}$ is $(d-1)$-dimensional complex projective space. Suppose that $\omega \in \Omega^{d-1,d-1}(\mathbb{P}^{d-1})$ is the natural volume form, this clearly satisfies the conditions of Lemma 3.4 and so determines a degree one cocycle $\phi_{\kappa,\omega} \in \mathrm{C}^*_{\mathrm{loc}}(\mathcal{G}_{\Sigma \times \mathbb{P}^{d-1}})$ where κ is some $\mathfrak{g}$-invariant bilinear form $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$. We can then consider the twisted factorization enveloping algebra of $\mathcal{G}_{\Sigma \times \mathbb{P}^{d-1}}$ by the cocycle $\phi_{\kappa,\omega}$.

Recall that if $p : X \to Y$ and $\mathcal{F}$ is a factorization algebra on X, then the pushforward $p_*\mathcal{F}$ on Y is defined on opens by $p_*\mathcal{F} : U \subset Y \mapsto \mathcal{F}(p^{-1}U)$.

Proposition 3.5. *Let $\pi : \Sigma \times \mathbb{P}^{d-1} \to \Sigma$ be the projection. Then, there is an isomorphism of factorization algebras on Σ from the pushforward along π of the Kac-Moody factorization algebra on $\Sigma \times \mathbb{P}^{d-1}$ of type $\phi_{\kappa,\omega}$ and the Kac-Moody*

$$\pi_* \mathbb{U}_{\phi_{\kappa,\theta}} \left(\mathcal{G}_{\Sigma\times\mathbb{P}^{d-1}}\right) \simeq \mathbb{U}_{\mathrm{vol}(\omega)\kappa}(\mathcal{G}_\Sigma)$$

The twisted factorization envelope on the right-hand side is the familiar Kac-Moody factorization alegbra on Riemann surfaces associated to a multiple of the pairing κ. The twisting $\mathrm{vol}(\omega)\kappa$ corresponds to a cocycle of the type in the previous section

$$J(\mathrm{vol}(\omega)\kappa) = \mathrm{vol}(\omega) \int_\Sigma \kappa(\alpha, \partial\beta)$$

where $\mathrm{vol}(\omega) = \int_{\mathbb{P}^{d-1}} \omega$.

Proof. Suppose that $U \subset \Sigma$ is open. Then, the factorization algebra $\pi_* \mathbb{U}_{\phi_{\kappa,\theta}} \left(\mathcal{G}_{\Sigma\times\mathbb{P}^{d-1}}\right)$ assigns to U the cochain complex

$$\left(\mathrm{Sym}\left(\Omega^{0,*}(U \times \mathbb{P}^{d-1})\right)[1][K], \bar\partial + K\phi_{\kappa,\omega}|_{U\times\mathbb{P}^{d-1}}\right), \tag{13}$$

where $\phi_{\kappa,\omega}|_{U\times\mathbb{P}^{d-1}}$ is the restriction of the cocycle to the open set $U \times \mathbb{P}^{d-1}$. Since projective space is Dolbeault formal its Dolbeault complex is quasi-isomorphic to its cohomology. Thus, we have

$$\Omega^{0,*}(U \times \mathbb{P}^{d-1}) = \Omega^{0,*}(U) \otimes \Omega^{0,*}(\mathbb{P}^{d-1}) \simeq \Omega^{0,*}(U) \otimes H^*(\mathbb{P}^{d-1}, \mathcal{O}) \cong \Omega^{0,*}(U).$$

Under this quasi-isomorphism, the restricted cocycle has the form

$$\phi_{\kappa,\omega}|_{U\times\mathbb{P}^{d-1}}(\alpha \otimes 1, \beta \otimes 1) = \int_U \kappa(\alpha, \partial\beta) \int_{\mathbb{P}^{n-1}} \omega$$

where $\alpha, \beta \in \Omega^{0,*}(U)$ and 1 denotes the unit constant function on $\mathbb{P}^{d-1}$. This is precisely the value of the local functional $\mathrm{vol}(\omega) J_\Sigma(\kappa)$ on the open set $U \subset \Sigma$. Thus, the cochain complex (13) is quasi-isomorphic to

$$\left(\mathrm{Sym}\left(\Omega^{0,*}(U)\right)[1][K], \bar\partial + K\mathrm{vol}(\omega) J_\Sigma(\kappa)\right). \tag{14}$$

We recognize this as the value of the Kac-Moody factorization algebra on Σ of type $\mathrm{vol}(\omega) J_\Sigma(\kappa)$. It is immediate to see that identifications above are natural with respect to maps of opens, so that the factorization structure maps are the desired ones. This completes the proof. □

4. Universal Grothendieck-Riemann-Roch from BV quantization

The main goal of the BV formalism developed in [**?**] is to rigorously construct quantum field theories using a combination of homological methods and a rigorous model for renormalization. A particular nicety of this approach is the ability to study *families* of field theories, which we will turn into an equivariant version of BV quantization, see Section **??**. In this section we will consider a family of QFT's parametrized by the moduli space of principal G-bundles. Our main result is to interpret a certain anomaly coming from BV quantization as a families index over $\mathrm{Bun}_G(X)$. This anomaly is computed via an explicit Feynman diagrammatic calculation and is related to a local cocycle of the current algebra discussed in Section 1.3.1.

We will interpret this as a formal universal version of the Grothendieck-Riemann-Roch theorem over the moduli space of bundles.

We will arrive at the result in a way that is local-to-global on spacetime which we formulate in terms of factorization algebras. The main them of Costello and Gwilliam's approach to QFT is that the observables of a QFT determine a factorization algebra. We study the associated family of factorization algebras associated to the family of QFT's over the moduli space of G-bundles. We will recollect a formulation of Noether's theorem for symmetries of a theory in terms of factorization algebras developed in Chapter 11 of [**?**]. The central object in this discussion is a "local index" which describes how the Kac–Moody factorization algebra acts on the observables of the QFT. Locally on spacetime we see how Noether's theorem provides a *free field realization* of the Kac-Moody factorization algebra generalizing that of the Kac-Moody vertex algebra in chiral conformal field theory [**?**].

We now give a brief summary of the results, with a background for the situation we consider. Suppose that P is a fixed holomorphic G-bundle on a complex manifold X. We have already seen how to express the formal deformation space of P inside of the moduli of G-bundles using the dg Lie algebra $\mathcal{A}\mathrm{d}(P)(X) = \Omega^{0,*}(X, \mathrm{ad}(P))$. In particular, any Maurer-Cartan element of $\mathcal{A}\mathrm{d}(P)(X)$ defines a deformation of P. We have seen that there is a refinement of this dg Lie algebra to a local Lie algebra $\mathcal{A}\mathrm{d}(P)$ whose factorization envelope defines the higher Kac-Moody factorization algebra above. To any G-representation V we will define a holomorphic theory with fields $\mathcal{E}_V$ that is equivariant for this local Lie algebra. Equivalently, we can think of $\mathcal{E}_V$ as defining a family of theories over the formal completion of P in the moduli of G-bundles

$$
\begin{tikzcd}
\mathcal{E}_V|_{P'} \arrow[r] \arrow[d] & \mathcal{E}_V \arrow[d] \\
\{P'\} \arrow[r] & \mathrm{Bun}_G(X)^\wedge_P .
\end{tikzcd}
$$

Over each fiber P' the theory $\mathcal{E}_V|_{P'}$ is a *free* theory, so admits a canonical BV quantization. Our formulation of equivariant BV quantization is codification of the problem of gluing together these quantizations in a compatible way. We will show how this presents itself in the failure of the BV quantization to be a *flat* family. Our main result is the following.

Theorem 4.1. *Let P be any principal G-bundle over a compact affine complex manifold X of dimension d. Suppose V is a G-representation. Then, the factorization homology $\int_X \mathrm{Obs}_V^{\mathfrak{q}}$ defines a line bundle over the formal neighborhood of P inside of the moduli of G-bundles. Moreover, its first Chern class is*

$$
c_1\left(\int_X \mathrm{Obs}_V^{\mathfrak{q}}\right) = C\mathrm{ch}_{d+1}^{\mathfrak{g}}(V)
$$

under the identification of $\mathrm{ch}_{d+1}^{\mathfrak{g}}(V)$ as a cohomology class on the formal neighborhood of P inside of the moduli of G-bundles in Equation (18) explained below. Here, C is some nonzero complex number.

There is an elucidating geometric description of how the classes $\mathrm{ch}_{d+1}(V)$ appear: they describe curvatures of line bundles over the moduli of G-bundles. Let $\mathrm{Bun}_G(X)$ denote the moduli space of G-bundles on the complex d-fold X. [3] Over the space $\mathrm{Bun}_G(X) \times X$ there is the *universal* G-bundle. If $P \to X$ is a G-bundle, the fiber over the point $\{[P]\} \times X$ is precisely the G-bundle $P \to X$. This universal G-bundle is classified by a map $f : \mathrm{Bun}_G(X) \times X \to BG$. Consider the following diagram

$$\begin{tikzcd} & \mathrm{Bun}_G(X) \times X \arrow[dl, "\pi"'] \arrow[dr, "f"] & \\ \mathrm{Bun}_G(X) & & BG \end{tikzcd}$$

where $\pi : \mathrm{Bun}_G(X) \times X \to \mathrm{Bun}_G(X)$ denotes the projection. If $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}} \cong H^{d+1}(G, \Omega^{d+1}) \subset H^{2d+2}(BG)$ then we obtain via push-pull in the diagram above

$$\int_\pi \circ f^* \theta \in H^2(\mathrm{Bun}_G(X)).$$

Let $\mathcal{P}$ denote the universal principal G-bundle. This is the G-bundle over $\mathrm{Bun}_G(X) \times X$ whose fiber over $\{[P \to X]\} \times X$ is the principal G-bundle $P \to X$ itself. Given any representation V we can define the vector bundle

$$\mathcal{V} = \mathcal{P} \times^G V$$

over $\mathrm{Bun}_G(X) \times X$. The fiber of this bundle over $\{[P \to X]\} \times X$ is the associated vector bundle $P \times^G V$. We take the determinant of the derived pushforward of $\mathcal{V}$ along π to obtain a line bundle $\det(\mathbb{R}\pi_* \mathcal{V})$ on $\mathrm{Bun}_G(X)$. We will see how the global observables $\int_X \mathrm{Obs}^{\mathrm{q}}_{P,V}$ provide a formal version of this line bundle near a fixed principal bundle P. Moreover, if we naively apply the Grothendieck-Riemann-Roch theorem in this universal context one finds

$$c_1(\det(\mathbb{R}\pi_* \mathcal{V})) = \int_\pi \mathrm{Td}_X \cdot \mathrm{ch}(\mathcal{V}) \in H^2(\mathrm{Bun}_G(X)).$$

In the case that X is affine, so that $\mathrm{Td}_X = 1$, our theorem provides a proof of this formula using methods of perturbative QFT. To prove the theorem on a general complex manifold we need to take into account the action of holomorphic vector fields, which is the content of the next section.

4.1. **The classical family.** In this section, we consider a BV theory that is equivariant for the local Lie algebra $\mathcal{A}\mathrm{d}(P)$ in the language of Section **??**. Let V be any G-representation, and define the associated vector bundle $\mathcal{V}_P = P \times^G V$ on X. The holomorphic theory we consider is based on the graded holomorphic vector bundle $\mathcal{V}_P \oplus K_X \otimes \mathcal{V}_P^*[d-1]$, where $\mathcal{V}_P^*$ is the linear dual bundle. The fields of the associated free BV theory are

$$\mathcal{E}_{P,V} = \Omega^{0,*}(X, \mathcal{V}_P) \oplus \Omega^{d,*}(X, \mathcal{V}_P^*)[d-1].$$

This is simply the $\beta\gamma$ system on X twisted by the vector bundle $\mathcal{V}_P$. The action functional is $\int \langle \beta, \bar{\partial}\gamma \rangle_V$ where the pairing is between V and its dual. In particular, the theory $\mathcal{E}_V$ is free. Let $\mathrm{Obs}^{\mathrm{q}}_{P,V}$ denote the corresponding factorization algebra of quantum observables.

The action of $\mathfrak{g}$ on V extends to an action of the local Lie algebra $\mathcal{A}\mathrm{d}(P)$ on this classical BV theory. To define this equivariance we need to presribe a Noether current.

[3] For $d > 1$ [**?**] have constructed a global smooth derived realization of this space, but its full structure will not be used in this discussion.

Lemma 4.2. *The local Noether current $I^{\mathfrak{g}} \in C^*_{\rm loc}(\mathcal{A}\mathrm{d}(P)) \otimes \mathcal{O}_{\rm loc}(\mathcal{E}_{P,V})$ defined by*

$$I^{\mathfrak{g}}(\alpha, \gamma, \beta) = \int_X \langle \beta, \alpha \cdot \gamma \rangle_V$$

satisfies the equivariant classical master equation

$$(\mathrm{d}_{\mathfrak{g}} + \bar{\partial}) I^{\mathfrak{g}} + \frac{1}{2}\{I^{\mathfrak{g}}, I^{\mathfrak{g}}\} = 0,$$

where $\mathrm{d}_{\mathfrak{g}}$ encodes the Lie algebra structure on $\mathcal{A}\mathrm{d}(P)$. Hence, $I^{\mathfrak{g}}$ gives $\mathcal{E}_V$ the structure of a classical $\mathcal{A}\mathrm{d}(P)$-equivariant theory.

Proof. If α is an element in $\mathcal{A}\mathrm{d}(P)$ and $\gamma \in \Omega^{0,*}(X, \mathcal{V}_P)$ we define $\alpha \cdot \gamma$ through the $\mathfrak{g}$-module structure of $\mathfrak{g}$ on V combined with the wedge product of Dolbeault forms. Note that $I^{\mathfrak{g}}$ is arises from holomorphic differential operators so that $\bar{\partial} I^{\mathfrak{g}} = 0$. From the definition of the bracket we see that for α_1, α_2 one has $\{\int \langle \beta, \alpha_1 \cdot \gamma \rangle, \int \langle \beta, \alpha_2 \cdot \gamma \rangle\} = \int \langle \beta, [\alpha_1, \alpha_2] \cdot \gamma \rangle$ which cancels the term coming from $\mathrm{d}_{\mathfrak{g}}$. □

4.2. **BV quantization in families.** The main technique we employ is equivariant BV quantization, which we have reviewed in Section **??**. Our main result holds for a compact affine manifold, which we will view as coming from a quotient of an open set in affine space $\mathbb{C}^d$. Thus, we will mostly work with the theory defined on $\mathbb{C}^d$ and afterwards deduce our main result on the quotient via descent. Thus, we will work with the $\beta\gamma$ system

$$\mathcal{E}_V = \Omega^{0,*}(\mathbb{C}^d, V) \oplus \Omega^{d,*}(\mathbb{C}^d, V^*)[d-1]$$

where V is some $\mathfrak{g}$-module. The local Lie algebra which acts on this theory is $\mathcal{G} = \Omega^{0,*}(\mathbb{C}^d, \mathfrak{g})$.

Our first step is to construct an equivariant effective prequantization. for the $\mathcal{G}$-equivariant theory. As has been the case over and over again in this thesis, our situation for constructing the prequantization is vastly simplified since our theory comes from holomorphic data. Indeed, the equivariant $\beta\gamma$ system is a holomorphic theory on $\mathbb{C}^d$ so that we can apply Lemma **??**. As an immediate corollary, the following definition is well-defined.

Definition 4.3. For $L > 0$, let

$$I^{\mathfrak{g}}[L] := \lim_{\epsilon \to 0} W(P_{\epsilon < L}, I^{\mathfrak{g}}) = \lim_{\epsilon \to 0} \sum_{\Gamma} \frac{\hbar^{g(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} W_\Gamma(P_{\epsilon < L}, I^{\mathfrak{g}}).$$

Here the sum is over all isomorphism classes of stabled connected graphs, but only graphs of genus ≤ 1 contribute nontrivially. By construction, the collection satisfies the RG flow equation and its tree-level $L \to 0$ limit is manifestly $I^{\mathfrak{g}}$. Hence $\{I^{\mathfrak{g}}[L]\}_{L \in (0,\infty)}$ is a $\mathcal{G}$*-equivariant prequantization*.

Our next step is to compute the obstruction to quantization of the $\mathcal{G}$-equivariant theory. By definition, the scale L *obstruction cocycle* $\Theta_V[L]$ is the failure for the interaction $I^{\mathfrak{g}}[L]$ to satisfy the scale L equivariant quantum master equation. Explicitly, one has

$$\hbar \Theta_V[L] = (\mathrm{d}_{\mathfrak{g}} + Q) I^{\mathfrak{g}}[L] + \hbar \Delta_L I^{\mathfrak{g}}[L] + \{I^{\mathfrak{g}}[L], I^{\mathfrak{g}}[L]\}_L.$$

A completely analogous argument as in Corollary 16.0.5 of **[?]** we see that the scale L obstruction is given by a sum over wheels.

Lemma 4.4. *Only wheels contribute to the anomaly cocycle $\Theta_V[L]$. Moreover, one has*

$$\Theta_V[L] = \sum_{\substack{\Gamma \in \text{Wheels} \\ e \in \text{Edge}(\Gamma)}} W_{\Gamma,e}(P_{\epsilon<1}, K_\epsilon, I^{\mathfrak{g}}[\epsilon]),$$

where the sum is over wheels and distinguished edges. The notation $W_{\Gamma,e}(P_{\epsilon<1}, K_\epsilon, I^{\mathfrak{g}}[\epsilon])$ means we place the propagator at all edges besides the distinguished one, where we place K_ϵ.

The only fields that propagate are the $\beta\gamma$ fields with values in V. Since all vertices are trivalent we see that the anomaly cocycle is only a functional of the background fields $\mathcal{G}$, see Figure BW: ref. In particular, there is no obstruction to having an *action* by $\mathcal{G}$, only an obstruction to having an *inner action*. Concretely, the external edges of any closed wheel occurring in the expansion of the anomaly must be labeled by $\mathcal{G}$. As an immediate consequence we have the following.

Lemma 4.5. *The effective family $\{I^{\mathfrak{g}}[L]\}$ defines a one-loop exact $\mathcal{G}$-equivariant quantum field theory. In other words, it satisfies the $\mathcal{G}$-equivariant quantum master equation modulo functionals purely of the background fields $\mathcal{G}$.*

It follows that the anomaly $\{\Theta[L]\}$ measures the obstruction to $\{I^{\mathfrak{g}}[L]\}$ to defining an *inner* action.

4.2.1. *The anomaly calculation.* We now perform the main technical calculation of the anomaly cocycle.

Proposition 4.6. *The $L \to 0$ limit of the anomaly cocycle $\Theta = \lim_{L\to 0} \Theta_V[L] \in \mathrm{C}^*_{\mathrm{loc}}(\mathcal{G})$ is of the form*

$$\Theta_V = C \cdot J_{\mathbb{C}^d}(\mathrm{ch}^{\mathfrak{g}}_{d+1}(V)),$$

where $\mathrm{ch}^{\mathfrak{g}}_{d+1}(V) \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$ and where $J_{\mathbb{C}^d} : \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}} \to \mathrm{C}^*_{\mathrm{loc}}(\mathcal{G})$ is the map of Lemma 1.11 and where C some constant only depending on the dimension d.

To compute the anomaly we refer to the following result about the expression for the anomaly cocycle in terms of the Feynman diagram expansion. This is proved in direct analogy to Lemma **??**. We have already seen in Lemma **??** that only wheels contribute to the anomaly cocycle. Then an explicit analysis of the analytic behavior shows that in the $\epsilon \to 0$ limit only the wheel with $(d+1)$-vertices contributes.

Lemma 4.7. *The limit $\Theta_V := \lim_{L\to 0} \Theta_V[L]$ exists and is an element of degree one in $\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n, \mathrm{C}^*_{\mathrm{loc}}(\mathfrak{g}_n^{\mathbb{C}}))$. Moreover, it is given by*

$$\Theta_V = \lim_{\epsilon \to 0} \sum_{\substack{\Gamma \in (d+1)\text{-vertex wheels} \\ e \in \text{Edge}(\Gamma)}} W_{\Gamma,e}(P_{\epsilon<1}, K_\epsilon, I^{\mathfrak{g}}[\epsilon]),$$

where the sum is over wheels Γ with two vertices and a distinguished inner edge e.

The lemma implies that we only need to consider the wheel with $d+1$ vertices. Each trivalent vertex is labeled by both an analytic factor and Lie algebraic factor. The Lie algebraic part of each vertex can be thought of as the defining map of the representation $\rho : \mathfrak{g} \to \mathrm{End}(V)$. The diagrammitcs of the wheel amounts to taking the trace of the symmetric $(d+1)$st power of this Lie algebra factor. Thus, the Lie algebraic factor of the weight of the wheel is the $(d+1)$st component of the character of the representation

$$\mathrm{ch}^{\mathfrak{g}}_{d+1}(V) = \frac{1}{(d+1)!}\mathrm{Tr}\left(\rho(X)^{d+1}\right) \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*).$$

To finish the calculation we must compute the analytic weight of the wheel with $d+1$ vertices. Recall, our goal is to identify the anomaly Θ with the image of $\mathrm{ch}^{\mathfrak{g}}_{d+1}(V)$ under the map

$$J : \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}} \to \mathrm{C}^*_{\mathrm{loc}}(\Omega^{0,*}(\mathbb{C}^d) \otimes \mathfrak{g})$$

that sends an element θ to the local functional $\int \theta(\alpha\partial\alpha \cdots \partial\alpha)$. We have just seen that the Lie algebra factor in local functional representing the anomaly agrees with the $(d+1)$st Chern character. Thus, to finish we must show the following.

Lemma 4.8. *As a functional on the abelian dg Lie algebra $\Omega^{0,*}(\mathbb{C}^d)$, the analytic factor of the weight $\lim_{L\to 0}\lim_{\epsilon\to 0} W_{\Gamma,e}(P_{\epsilon<L}, K_\epsilon, I^{\mathfrak{g}})$ is equal to a multiple of the local functional*

$$\int \alpha\partial\alpha \cdots \partial\alpha \in \mathrm{C}^*_{\mathrm{loc}}(\Omega^{0,*}(\mathbb{C}^d)).$$

Proof. Let's fix some notation. We enumerate the vertices by integers $a = 0, \ldots, d$. Label the coordinate at the ith vertex by $z^{(a)} = (z^{(a)}_1, \ldots, z^{(a)}_d)$. The incoming edges of the wheel will be denoted by homogeneous Dolbeault forms

$$\alpha^{(a)} = \sum_J A^{(a)}_J \mathrm{d}\bar{z}^{(a)}_J \in \Omega^{0,*}_c(\mathbb{C}^d).$$

where the sum is over the multiindex $J = (j_1, \ldots, j_k)$ where $j_a = 1, \ldots, d$ and $(0,k)$ is the homogenous Dolbeault form type. For instance, if α is a $(0,2)$ form we would write

$$\alpha = \sum_{j_1<j_2} A_{(j_1,j_2)}\mathrm{d}\bar{z}_{j_1}\mathrm{d}\bar{z}_{j_2}.$$

Denote the functional obtained as the $\epsilon \to 0$ weight of the wheel with $(d+1)$ vertices from Lemma **??** by W_L. The $L \to 0$ limit of W_L is the local functional representing the one-loop anomaly Θ.

The weight has the form

$$W_L(\alpha^{(0)}, \ldots, \alpha^{(d)}) = \pm \lim_{\epsilon\to 0}\int_{\mathbb{C}^{d(d+1)}} \left(\alpha^{(0)}(z^{(0)}) \cdots \alpha^{(d)}(z^{(d)})\right) K_\epsilon(z^{(0)}, z^{(d)}) \prod_{a=1}^{d} P_{\epsilon,L}(z^{(a-1)}, z^{(a)}).$$

We introduce coordinates

$$\begin{aligned} w^{(0)} &= z^{(0)} \\ w^{(a)} &= z^{(a)} - z^{(a-1)} \quad 1 \le a \le d. \end{aligned}$$

The heat kernel and propagator part of the integral is of the form

$$\begin{aligned} K_\epsilon(w^{(0)}, w^{(d)}) \prod_{a=1}^{d} P_{\epsilon,L}(w^{(a-1)}, w^{(a)}) \quad = \quad & \frac{1}{(4\pi\epsilon)^d}\int_{t_1,\ldots,t_d=\epsilon}^{L} \frac{\mathrm{d}t_1 \cdots \mathrm{d}t_d}{(4\pi t_1)^d \cdots (4\pi t_d)^d}\frac{1}{t_1 \cdots t_d} \\ & \times \mathrm{d}^d w^{(0)} \prod_{i=1}^{d}(\mathrm{d}\bar{w}^{(1)}_i + \cdots + \mathrm{d}\bar{w}^{(d)}_i) \prod_{a=1}^{d} \mathrm{d}^d w^{(a)} \left(\sum_{i=1}^{d} \bar{w}^{(a)}_i \prod_{j\neq i} \mathrm{d}\bar{w}^{(a)}_j\right) \\ & \times e^{-\sum_{a,b=1}^{d} M_{ab} w^{(a)}\cdot\bar{w}^{(b)}}. \end{aligned}$$

Here, M_{ab} is the $d \times d$ square matrix satisfying

$$\sum_{a,b=1}^{d} M_{ab} w^{(a)} \cdot \bar{w}^{(b)} = |\sum_{a=1}^{d} w^{(a)}|^2/\epsilon + \sum_{a=1}^{d} |w^{(a)}|^2/t_a.$$

Note that

$$\prod_{i=1}^{d}(\mathrm{d}\bar{w}_i^{(1)} + \cdots + \mathrm{d}\bar{w}_i^{(d)}) \prod_{a=1}^{d} \left(\sum_{i=1}^{d} \bar{w}_i^{(a)} \prod_{j \neq i} \mathrm{d}\bar{w}_j^{(a)} \right) = \left(\sum_{i_1,\ldots i_d} \epsilon_{i_1 \cdots i_d} \prod_{a=1}^{d} \bar{w}_{i_a}^{(a)} \right) \prod_{a=1}^{d} \mathrm{d}^d \bar{w}^{(a)}.$$

In particular, only the $\mathrm{d}w_i^{(0)}$ components of $\alpha^{(0)} \cdots \alpha^{(d)}$ can contribute to the weight.

Let Φ = BW: some contraction of $\alpha^{(0)} \cdots \alpha^{(d)}$ by a antiholomorphic vector field. Then, the weight can be written as

$$\begin{aligned} W(\alpha^{(0)}, \ldots, \alpha^{(d)}) &= \lim_{\epsilon \to 0} \int_{\mathbb{C}^{d(d+1)}} \left(\prod_{a=0}^{d} \mathrm{d}^d w^{(a)} \mathrm{d}^d \bar{w}^{(a)} \right) \Phi \\ &\times \frac{1}{(4\pi\epsilon)^d} \int_{t_1,\ldots,t_d=\epsilon}^{L} \frac{\mathrm{d}t_1 \cdots \mathrm{d}t_d}{(4\pi t_1)^d \cdots (4\pi t_d)^d} \frac{1}{t_1 \cdots t_d} \sum_{i_1,\ldots,i_d} \epsilon_{i_1 \cdots i_d} \bar{w}_{i_1}^{(1)} \cdots \bar{w}_{i_d}^{(d)} e^{-\sum_{a,b=1}^{d} M_{ab} w^{(a)} \cdot \bar{w}^{(b)}} \end{aligned}$$

Applying Wick's lemma in the variables $w^{(1)}, \ldots, w^{(d)}$, together with some elementary analytic bounds, we find that the weight above becomes to the following integral over $\mathbb{C}^d$

$$f(L) \int_{w^{(0)} \in \mathbb{C}^d} \mathrm{d}^d w^{(0)} \mathrm{d}^d \bar{w}^{(0)} \sum_{i_1,\ldots,i_d} \epsilon_{i_1 \cdots i_d} \left(\frac{\partial}{\partial w_{i_1}^{(1)}} \cdots \frac{\partial}{\partial w_{i_d}^{(d)}} \Phi \right) |_{w^{(1)} = \cdots = w^{(d)} = 0}$$

where

$$f(L) = \lim_{\epsilon \to 0} \int_{t_1,\ldots,t_d=\epsilon}^{L} \frac{\epsilon}{(\epsilon + t_1 + \cdots + t_d)^{d+1}} \mathrm{d}^d t.$$

In fact, $f(L)$ is independent of L and is equal to some nonzero constant $C \neq 0$. Finally, plugging in the forms $\alpha^{(0)}, \ldots, \alpha^{(d)}$, we observe that the integral over $w^{(0)} \in \mathbb{C}^d$ simplifies to

$$C \int_{\mathbb{C}^d} \alpha^{(0)} \partial \alpha^{(1)} \cdots \partial \alpha^{(d)}$$

as desired. □

This completes the proof of Proposition **??**.

4.3. **Local to global.** In this section we finish the proof of our main result Theorem 4.1 by showing how our local calculation above implies the formula for the anomaly on a general compact affine manifold X. By an complex affine manifold, we mean a quotient

$$q : U \subset \mathbb{C}^d \to X$$

of an open subset $U \subset \mathbb{C}^d$ by a free and proper action of a discrete subgroup of the affine group $U(d) \ltimes \mathbb{C}^d$. We consider affine manifolds that are also compact. To deduce our main theorem we will show that the theory and the anomaly above also exhibit equivariance for the affine group on $\mathbb{C}^d$, thus it will descend to any affine manifold.

We have stated the main result for an arbitrary principal G-bundle P on the affine manifold X. Suppose the discrete subgroup $\Gamma \leq U(d) \ltimes \mathbb{C}^d$ defines the affine manifold $q : U \to X = U/\Gamma$ as above. Then, principal G-bundles on X are equivalent to Γ-equivariant principal G-bundles on U.

Let $\mathcal{E}$ be an arbitrary elliptic complex on X, and suppose the Lie algebra $\mathfrak{h}$ acts on $\mathcal{E}$. Since X is compact, the cohomology $H^*(\mathcal{E}(X))$ is finite dimensional. It therefore makes sense to define the character of the action of $\mathfrak{h}$ on $H^*(\mathcal{E}(X))$.

$$\chi_{\mathcal{E}} : \mathfrak{h} \to \mathbb{C} \ \ , \ \ M \in \mathfrak{h} \mapsto \mathrm{STr}_{H^*(\mathcal{E}(X))}(M). \tag{15}$$

Here, STr denotes the supertrace. The character factors through the determinant of the representation. For the graded character above, we must use the superdeterminant which we denote by $\det(H^*(\mathcal{E}(X)))$. Free BV quantization gives a natural field theoretic interpretation of this determinant.

Proposition 4.9 (**[?]** Lemma 12.7.0.1)**.** *Let $\mathcal{E}$ be any elliptic complex on a compact manifold X and let $T^*[-1]\mathcal{E}$ be the corresponding free BV theory given by the shifted cotangent bundle. Let $\mathrm{Obs}^{\mathrm{q}}_{\mathcal{E}}$ be the factorization algebra of quantum observables of this theory. Then, there is an isomorphism*

$$H^*\left(\mathrm{Obs}^{\mathrm{q}}_{\mathcal{E}}(X)\right) \cong \det H^*(\mathcal{E}(X))[n]$$

where n is the Euler characteristic of $\mathcal{E}(X)$ modulo 2.

Notice that the classical free theory $\mathcal{E}_V$ is equivariant for the affine group $U(d) \ltimes \mathbb{C}^d$. Thus, it defines a classical theory on any affine manifold X. This theory is free and of the form

$$\mathcal{E}_V(X) = T^*[-1](\Omega^{0,*}(X,V))$$

where $T^*[-1]$ denotes the shifted cotangent bundle. Thus, the global quantum observables satisfy

$$H^*(\mathrm{Obs}^{\mathrm{q}}_V(X)) = \det\left(H^*(X,\mathcal{O}^{hol})\otimes V\right)[??] \tag{16}$$

In Section 4.1 we have showed how the classical theory $\mathcal{E}_V$ has an an action by the local Lie algebra $\mathcal{G}_X$. This arose from an action of $\mathcal{G}(X) = \Omega^{0,*}(X,\mathfrak{g})$ on the elliptic complex $\Omega^{0,*}(X,V)$. At the level of cohomology we have an action of $H^*(\mathcal{G}(X))$ on $H^*(\Omega^{0,*}(X,V))$ and hence a character χ_V as in Equation (15) which is an element in $H^*_{red}(\mathcal{G}(X))$.

The local Lie algebra cohomology of any local Lie algebra embeds inside its ordinary (reduced) Lie algebra cohomology of global sections $\mathrm{C}^*_{\mathrm{loc}}(\mathcal{L}(X)) \subset \mathrm{C}^*_{\mathrm{Lie,red}}(\mathcal{L}(X))$. The character (15) is an element in $H^*_{red}(\mathcal{L}(X))$. As an immediate corollary of **[?]** Theorem 12.6.0.1 we have the following relationship between the anomaly cocycle and the character.

Proposition 4.10. *Suppose $\mathcal{L}$ is a local Lie algebra that acts on the elliptic complex $\mathcal{E}$ on a compact manifold X. Let $\Theta_{\mathcal{E}} \in \mathrm{C}^*_{\mathrm{loc}}(\mathcal{L})$ be the local cocycle measuring the failure to satisfy the $\mathcal{L}$-equivariant classical master equation (that is, the obstruction to having an inner action). Then, its global cohomology class satisfies $[\Theta_{\mathcal{E}}(X)] = \chi_{\mathcal{E}} \in \mathrm{C}^*_{\mathrm{Lie,red}}(\mathcal{L}(X))$ where $\chi_{\mathcal{E}}$ is the trace of the action of $H^*(\mathcal{L}(X))$ on $H^*(\mathcal{E}(X))$.*

For the case of $\mathcal{L} = \mathcal{G}_X$ we have an embedding of cochain complexes

$$\mathrm{C}^*_{\mathrm{loc}}(\mathcal{G}(X)) \hookrightarrow \mathrm{C}^*_{\mathrm{Lie,red}}(\mathcal{G}(X)) = \mathrm{C}^*_{\mathrm{Lie,red}}(\Omega^{0,*}(X)\otimes\mathfrak{g}).$$

By Kodaira-Spencer theory have already seen that the global sections of the local Lie algebra $\mathcal{G}(X)$ is a model for the formal neighborhood of the trivial G-bundle inside of G-bundles. In particular, the $\mathcal{G}(X)$-module of quantum observables defines a line bundle $\int_X \mathrm{Obs}^{\mathrm{q}}_{\mathcal{E}_V}$ over this formal neighborhood. Its character as a $\mathcal{G}(X)$-module is identified with the first Chern class of the corresponding line bundle $\chi_{\mathcal{E}}(\mathrm{Obs}^{\mathfrak{q}}_V(X)) = c_1(\int_X \mathrm{Obs}^{\mathfrak{q}}_V)$.

Now, notice that the one-loop quantization we constructed in the previous section, as well as the anomaly cocycle $\Theta_V \in C^*_{\mathrm{loc}}(\mathcal{G}_{\mathbb{C}^d})$ are equivariant for the group $U(d) \ltimes \mathbb{C}^d$. Thus, they descend to the global sections of $C^*_{\mathrm{loc}}(\mathcal{G}_X)$ for any affine manifold X. Explicitly, if $\Gamma \subset U(d) \ltimes \mathbb{C}^d$ is the discrete subgroup such that $X = U/\Gamma$ where $U \subset \mathbb{C}^d$, then under the isomorphism

$$C^*_{\mathrm{loc}}(\mathcal{G}(X)) \cong C^*_{\mathrm{loc}}(\mathcal{G}(U))^{\Gamma}$$

we have $\Theta_V(X) \leftrightarrow \Theta_V(U)$.

Further, we have an identification

$$C^*_{\mathrm{Lie,red}}(\Omega^{0,*}(X) \otimes \mathfrak{g}) = \mathcal{O}_{red}\left(\mathrm{Bun}_G(X)^{\wedge}_{triv}\right) \cong \Omega^1_{cl}\left(\mathrm{Bun}_G(X)^{\wedge}_{triv}\right)$$

where we have used the equivalence of reduced functions and closed one-forms which makes sense on any formal moduli space. At the level of H^1 we have the composition composition

$$\mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}} \xrightarrow{J^X} H^1_{\mathrm{loc}}(\mathcal{G}(X)) \to H^1(\Omega^1_{cl}\left(\mathrm{Bun}_G(X)^{\wedge}_{triv}\right). \tag{17}$$

As a corollary of Proposition 4.10 and our calculation of the local anomaly cocycle we see that the image of $\mathrm{ch}^{\mathfrak{g}}_{d+1}(V)$ is equal to $[\Theta_V(X)] = [c_1(\int_X \mathrm{Obs}^{\mathfrak{q}}_V)]$.

The same holds when we work around any holomorphic principal bundle P on X, so that we have an embedding of cochain complexes

$$C^*_{\mathrm{loc}}(\mathcal{A}\mathrm{d}(P)(X)) \hookrightarrow \Omega^1_{cl}\left(\mathrm{Bun}_G(X)^{\wedge}_P\right).$$

which determines a composition

$$\mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}} \xrightarrow{J^X_P} H^1_{\mathrm{loc}}(\mathcal{A}\mathrm{d}(P)(X)) \to H^1(\Omega^1_{cl}\left(\mathrm{Bun}_G(X)^{\wedge}_P\right). \tag{18}$$

Since every principal G-bundle P on X is trivial when we pull it back to $U \subset \mathbb{C}^d$, the above local anomaly calculation proves that $[c_1(\int_X \mathrm{Obs}^{\mathfrak{q}}_{P,V})] = C\mathrm{ch}^{\mathfrak{g}}_{d+1}(V)$ in this case as well. This completes the proof of Theorem 4.1.

4.4. **A module for the higher Kac-Moody.** The last part of this section we diverge to deduce a consequence of the quantum Noether theorem using our analysis above by exhibiting a module for the higher affine algebras from the previous section. For convenience, we fix the trivial $\mathfrak{g}$-bundle $P = \mathrm{triv}$ so that $\mathcal{A}\mathrm{d}(P) = \mathcal{G}_X$.

On any manifold X, the quantum Noether theorem BW: reference provides a map of factorization algebras

$$\Phi_X : \mathbb{U}_{\alpha}(\mathcal{G}_X) \to \mathrm{Obs}^{\mathfrak{q}}_V,$$

for some $\alpha \in H^1_{\mathrm{loc}}(\mathcal{G}_X)$. The factorization algebra $\mathrm{Obs}^{\mathfrak{q}}_V$ is the quantum observables of the $\beta\gamma$ system on X with values in the $\mathfrak{g}$-module V. This is a free field theory, thus the above map has the flavor of a *free field realization* of the Kac-Moody factorization algebra. In particular when $X = \mathbb{C}^d$, or any affine manifold, the calculation above shows that there is a map of factorization algebras

$$\Phi_{\mathbb{C}^d} : \mathbb{U}_{\mathrm{ch}_{d+1}(V)}(\mathcal{G}_{\mathbb{C}^d}) \to \mathrm{Obs}^{\mathfrak{q}}_V.$$

Next, consider the case $X = \mathbb{C}^d \setminus \{0\}$. By functoriality of pushforwards, the quantum Noether theorem produces a map of one-dimensional factorization algebras

$$\rho_* \Phi : \rho_* \mathbb{U}_{\mathrm{ch}_{d+1}(V)}(\mathcal{G}_{\mathbb{C}^d \setminus \{0\}}) \to \rho_* \mathrm{Obs}^{\mathrm{q}}_V.$$

We have exhibited a locally constant dense subfactorization algebra $\mathcal{F}^{lc}_{1d}$ of $\rho_* \mathbb{U}_\alpha(\mathcal{G}_X)$ which is equivalent, as an E_1-algebra, to $U\widehat{\mathfrak{g}}_{d,\mathrm{ch}_{d+1}(V)}$. Similarly, in Section **??** we have shown that there is a locally constant dense subfactorization algebra that is equivalent to the dg algebra $\mathcal{A}_V$.

The map $\rho_* \Phi$ restricts to these dense subfactorization algebras and so defines a map of E_1 algebras

$$\rho_* \Phi : U\widehat{\mathfrak{g}}_{d,\mathrm{ch}_{d+1}(V)} \to \mathcal{A}_V.$$

Also, in Section **??** we have shown how the disk operators $\mathcal{V}_V$ form a module, through the factorization product, for the dg algebra $\mathcal{A}_V$. This is essentially the Fock module of the algebra $\mathcal{A}_V$, thus we should view the above map $\rho_* \Phi$ as being a higher dimensional analog of the "free field realization" for the higher dimensional affine algebras.

Further, by induction along the map $\rho_* \Phi$, we obtain the following.

Proposition 4.11. *The map $\rho_* \Phi$ endows the space $\mathcal{V}_V$ with the structure of a module over the E_1-algebra $U\widehat{\mathfrak{g}}_{d,\mathrm{ch}_{d+1}(V)}$. Equivalently, $\mathcal{V}_V$ is an A_∞-module for $U\widehat{\mathfrak{g}}_{d,\mathrm{ch}_{d+1}(V)}$.*

The module $\mathcal{V}_V$ is the prototype for a higher dimensional version of a Verma module for ordinary affine algebras.

5. Holomorphic diffeomorphisms

The next type of symmetry we consider is that of holomorphic reparametrizations, or holomorphic diffeomorphisms. A holomorphic diffeomorphism $f : X \to Y$ between complex manifolds is a bijective holomorphic map whose inverse is also holomorphic. Under composition, holomorphic diffeomorphisms from X to itself combine to form a Lie group $\mathrm{Diff}^{hol}(X)$. We study theories that have an action of holomorphic diffeomorphisms which leave the action functional invariant. An example of such a theory is one for which the action functional can be written down in a holomorphically covariant way (e.g. one that only uses universal constructions in complex geometry). Morally speaking, the Lie algebra of holomorphic diffeomorphisms from X to itself is equal to holomorphic vector fields on X. Of course, some care must be taken to make this precise as $\mathrm{Diff}^{hol}(X)$ is not a finite dimensional manifold. We will not be concerned with these functional analytic issues since we will take as a starting point theories that have symmetries by the Lie algebra of holomorphic vector fields. It is an interesting question if our constructions lift to the level of the Lie group, but we will not address that here.

5.1. **Holomorphic vector fields.** Covariance in field theory is usually reserved for theories that can be written in a way that uses only natural constructions in differential geometry and so is independent of a choice of a local coordinate. This is precisely the condition that the theory is invariant with respect to the group of diffeomorphisms The obvious holomorphic analog of this makes sense for theories defined on complex manifolds. In this section we introduce a local-to-global Lie algebraic version of holomorphic covariance using a natural local Lie algebra associated to holomorphic diffeomorphisms.

In the two-dimensional chiral case we will see that a holomorphic covariant theory is the same thing as a chiral conformal field theory.

5.1.1. *The local Lie algebra.* Just as in the case of holomorphic gauge symmetries, there is a local Lie algebra associated to holomorphic vector fields. For any complex manifold X, the holomorphic tangent bundle $T^{1,0}X$ is a holomorphic vector bundle and hence admits a Dolbeault complex

$$\mathcal{T}(X) := \Omega^{0,*}(X, T^{1,0}X).$$

Together with the $\bar{\partial}$ operator, there is a natural extension of the Lie bracket of holomorphic vector fields that gives this complex the structure of a dg Lie algebra. The underlying graded vector space of $\mathcal{T}(X)$ is clearly the global sections of a smooth manifold. Moreover, the differential and Lie bracket are differential and bidifferential operators respectively. Thus:

Lemma 5.1. *For any complex manifold X, $\mathcal{T}(X) = \Omega^{0,*}(X, T^{1,0}X)$ has the structure of a local Lie algebra.*

In keeping with the conventions above, when we want to stress the sheaf-like nature of this local Lie algebra we use the notation $\mathcal{T}_X^{sh}$. This is a sheaf of dg Lie algebras which assigns to an open set $U \subset X$ the dg Lie algebra $\Omega^{0,*}(U, T^{1,0}U)$. We will use the notation $\mathcal{T}_X$ to denote the associated cosheaf $U \mapsto \Omega_c^{0,*}(U, T^{1,0}U)$.

5.1.2. *Holomorphically covariant theories.* The local Lie algebra $\mathcal{T}_X$ allows us to define the following stronger notion of a holomorphic theory. Recall the definition of a holomorphic theory on a complex manifold X from Section **??**. This consists of the data of a holomorphic vector bundle $V \to X$, a holomorphic differential operator $Q^{hol} : V \to V[1]$, a shifted symplectic pairing $(-,-)_V$ on V, and a holomorphic Lagrangian density I^{hol}.

Definition 5.2. A holomorphic theory $(V, Q^{hol}, \omega, (-,-)_V, I^{hol})$ is *holomorphically covariant* if the associated BV theory admits an action by the local Lie algebra $\mathcal{T}_X$.

Many of the holomorphic theories we have encountered are, in addition, holomorphically covariant. Recall that the data of an action of a local Lie algebra $\mathcal{L}$ on a theory $\mathcal{E}$ is given by a Maurer-Cartan element in the the dg Lie algebra $\mathrm{Act}(\mathcal{L}, \mathcal{E})[-1]$ from Section **??**. This is a sub dg Lie algebra of $\mathrm{C}^*_{\mathrm{Lie,red}}(\mathcal{L}(X)) \otimes \mathcal{O}_{\mathrm{loc}}(\mathcal{E})[-1]$ where the dependence on the local Lie algebra must also be local.

Example 5.3. Consider the $\beta\gamma$ system on a complex manifold X with values in a vector space W. The fields $\mathcal{E}_W$ consist of elements $\gamma \in \Omega^{0,*}(X, W)$ together with their conjugates. The dg Lie algebra $\mathcal{T}(X)$ acts on the cochain complex $\Omega^{0,*}(X, W)$ via Lie derivative: if $\xi \in \mathcal{T}(X)$ of degree k and $\gamma \in \Omega^{0,l}(X)$ then $L_\xi \gamma \in \Omega^{0,k+l}(X, W)$ is defined. It is immediate to see that this is compatible with the $\bar{\partial}$ operator. The classical Noether current defining the classical action is

$$I^{\mathcal{T}}(\xi, \beta, \gamma) = \int \langle \beta, L_\xi \gamma \rangle_W,$$

where $\langle -,- \rangle$ is, as usual, the pairing between W and its dual. This functional defines a Maurer-Cartan element in

$$I^{\mathcal{T}} \in \mathrm{Act}(\mathcal{T}, \mathcal{E}_W)[-1] \subset \mathrm{C}^*_{\mathrm{Lie,red}}(\mathcal{T}(X)) \otimes \mathcal{O}_{\mathrm{loc}}(\mathcal{E}_W)[-1]$$

and hence we have a classical action of $\mathcal{T}_X$ on $\mathcal{E}_W$.

There is a variation of this example that plays an important role for us. The holomorphic tensor bundle of type (r,s) on a manifold X is the holomorphic vector bundle

$$V_r^s = \underbrace{T^{1,0}X \otimes \cdots \otimes T^{1,0}X}_{r \text{ copies}} \otimes \underbrace{T^{*1,0}X \otimes \cdots \otimes T^{*1,0}X}_{s \text{ copies}}.$$

Similarly, there are anti-holomorphic versions. The local Lie algebra $\mathcal{T}_X$ acts on any holomorphic tensor bundle on X via Lie derivative. This extends to a map

$$L : \mathcal{T}_X \times \Omega^{0,*}(X, V_r^s) \to \Omega^{0,*}(X, V_r^s) \ \ , \ \ (\xi, \gamma) \mapsto L_\xi \gamma,$$

giving the Dolbeault complex $\Omega^{0,*}(X, V_r^s)$ the structure of a dg module for $\mathcal{T}(X)$. In a completely analogous way to the lemma above, we have the following.

Lemma 5.4. *The $\beta\gamma$ system twisted by the tensor bundle of type (r,s) has an action by the local Lie algebra $\mathcal{T}_X$ given by the local functional*

$$I^{\mathcal{T}}(\xi, \gamma, \beta) = \int \langle \beta, L_\xi \gamma \rangle_{V_r^s}.$$

Hence, it is a holomorphically covariant theory.

5.1.3. *Higher central charges.* Just as in the case of the current algebra, we can apply the factorization enveloping algebra to $\mathcal{T}_X$ to obtain a factorization algebra $\mathbb{U}(\mathcal{T}_X)$ on any complex manifold X. The interesting deformations of this factorization algebra come from local cocycles on $\mathcal{T}_X$ which allow us to define the twisted enveloping algebra.

Definition 5.5. Let $\alpha \in H^1_{loc}(\mathcal{T}_X)$. The *Virasoro factorization algebra* on X of central charge α is the twisted factorization enveloping algebra $\mathbb{U}_\alpha(\mathcal{T}_X)$.

The motivation for the term *central charge* will become clear momentarily.

For a complex manifold of dimension one, we have shown in **[?]** that $H^*_{loc}(\mathcal{T}_\Sigma) = \Omega^*(\Sigma)[1]$. Thus, on a connected Riemann surface there is a unique, up to scale, local cohomology class of degree one that we normalize by $H^1_{loc}(\mathcal{T}_\Sigma) = \mathbb{C} \cdot \omega_{Vir}$. This cocycle ω_{Vir}, which we will recall below, is related to the cocycle defining the usual extension of the one-dimensional Witt algebra. Moreover, in [**?**], we have shown that locally this twisted factorization envelope recovers the Virasoro vertex algebra. Implicit in the statement below is the relationship between one-dimensional holomorphic factorization algebras and vertex algebras that we recalled at the beginning of Section **??**.

Theorem 5.6 ([**?**])**.** *Let $c \in \mathbb{C}$. The factorization envelope $\mathbb{U}_{c\cdot\omega_{Vir}}(\mathcal{T}_{\mathbb{C}})$ is a holomorphically translation invariant factorization algebra and its cohomology defines a vertex algebra $\mathbb{V}\mathrm{ert}(\mathbb{U}_{c\cdot\omega_{Vir}}(\mathcal{T}_{\mathbb{C}}))$. Moreover, this vertex algebra is isomorphic to the Virasoro vertex algebra of charge c:*

$$\mathbb{V}\mathrm{ert}(\mathbb{U}_{c\cdot\omega_{Vir}}(\mathcal{T}_{\mathbb{C}})) \cong \mathrm{Vir}_c.$$

We will see how this twisted factorization enveloping algebra appears when studying quantization of holomorphically covariant theories in any dimension.

The definition of a quantum field theory that is holomorphically covariant is similar to the classical case. We refer again to Section **??** for the definition of an action of a local Lie algebra on a quantum field theory. Recall, there were essentially two separate notions of a quantum symmetry: that of an action by a local Lie algebra $\mathcal{L}$, and that of an *inner action*. To have an action of a local Lie algebra, one must prescribe a family of $\mathcal{L}$-dependent functionals $\{I^{\mathcal{L}}[L]\}$ satisfying the renormalized quantum master equation modulo functionals that dependent solely on $\mathcal{L}$. To have an inner action, the quantum master equation must be satisfied on the nose. We have discussed a deformation theory for lifting an action to an inner action; in particular, there is an obstruction that lives in $H^1_{\text{loc}}(\mathcal{L})$ to lifting an action to an inner action.

Definition 5.7. A quantum field theory is holomorphically covariant if it admits an action by the local Lie algebra $\mathcal{T}_X$. The *central charge* of a holomorphically covariant quantum field theory is the obstruction to lift this action to an inner action. This is an element

$$\mathfrak{c}_{\mathcal{E}} \in H^1_{\text{loc}}(\mathcal{T}_X)[[\hbar]].$$

5.1.4. *Chiral conformal field theory.* In complex dimension one there is an intimate relationship between complex and Riemannian structures. Every Riemann surface admits a natural Riemannian metric and hence a conformal structure. Conversely, a conformal class of a metric defines a complex structures. It is well-known that the moduli of Riemann surfaces is equivalent to the moduli of conformal structures.

We can see this at the level of local Lie algebras as follows. Fix a Riemann surface Σ and denote by g_0 the associated Riemannian metric. Define the *Riemannian local Lie algebra* as follows. Using the fixed metric g_0 define the two-term complex

$$\text{Riem}(\Sigma, g_0) = \Gamma(\Sigma, T\Sigma) \xrightarrow{L g_0} \text{Sym}^2(T^*\Sigma)[-1]$$

where the differential sends a vector field X to $L_X g_0$, the Lie derivative of g_0 with respect to X. The Lie bracket of vector fields gives this complex the structure of a dg Lie algebra. Better yet, it is immediate to see that it is a local Lie algebra. The dg Lie algebra $\text{Riem}(\Sigma, g_0)$ is the derived replacement for the one-shifted tangent space of the moduli space of Riemannian structures on Σ at g_0.

There is a natural map of local Lie algebras $\mathcal{T}_\Sigma \to \text{Riem}(\Sigma, g_0)$. In degree zero this is just the inclusion of the holomorphic tangent bundle inside of the full tangent bundle. In degree one, note that the metric g_0 defines an inclusion

$$T^{*0,1}\Sigma \otimes T^{1,0}\Sigma \cong_{g_0} T^{*0,1}\Sigma \otimes T^{*1,0}\Sigma \subset \text{Sym}^2(T^*\Sigma).$$

Elements of degree one are sections of the bundle on the left-hand side. The map in degree one is the inclusion above.

Next, we define the *conformal local Lie algebra*. This is similar to the Riemannian local Lie algebra where we take into account conformal equivalences of metrics. Define the two-term complex

$$\text{Conf}(\Sigma, g_0) = \Gamma(\Sigma, T\Sigma) \oplus C^\infty(\Sigma) \xrightarrow{D_{g_0}} \text{Sym}^2(T^*\Sigma)[-1].$$

The differential is defined by

$$D_{g_0}(X, f) = L_X g_0 + f g_0.$$

The second term encodes the infinitesimal action of the conformal group. The Lie bracket of vector fields combined with the obvious action of vector fields on functions gives the above complex the structure of a local Lie algebra. Of course, every Riemannian structure defines a conformal structure, so there is a map of local Lie algebras $\mathrm{Riem}(\Sigma, g_0) \to \mathrm{Conf}(\Sigma, g_0)$.

Thus, we obtain a composition

$$\mathcal{T}_\Sigma \to \mathrm{Riem}(\Sigma, g_0) \to \mathrm{Conf}(\Sigma, g_0).$$

5.2. **Gelfand-Fuks cohomology.** Our aim is to classify the space of central charges of a holomorphically covariant quantum field theory. The description we give will be in terms of a certain cohomology of vector fields on the disk, called *Gelfand-Fuks* cohomology. In this section we recall some facts about the Lie algebra cohomology of formal vector fields W_d on the d-disk with values in certain non-trivial modules. We refer to Section **??** for the requisite notation for objects living on the formal disk.

In Section **??** we have constructed the formal Atiyah class for any formal vector bundle $\mathcal{V}$ on $\widehat{D}^n$. It is an element of the relative Gelfand-Fuks cohomology

$$\mathrm{At}^{\mathrm{GF}}(\mathcal{V}) \in \mathrm{C}^1_{\mathrm{Lie}}(\mathrm{W}_d, \mathrm{GL}_d; \widehat{\Omega}^1_d \otimes_{\widehat{\mathcal{O}}_d} \mathrm{End}_{\widehat{\mathcal{O}}_d}(\mathcal{V})).$$

From the Atiyah class we have built the formal Chern character using the usual formula

$$\mathrm{ch}^{\mathrm{GF}}(\mathcal{V}) = \mathrm{Tr}\left(\exp\left(\frac{1}{2\pi i}\mathrm{At}^{\mathrm{GF}}(\mathcal{V})\right)\right),$$

and have studied how components of this formal Chern character give rise to L_∞ extensions of W_d that appear as natural universal symmetries of quantizations of higher dimensional holomorphic σ models with target $\widehat{D}^d$.

In this section we arrive at the Lie algebra of formal vector fields, and its cohomology, from a different perspective. Instead of using formal geometry to construct universal objects on the *target* of a σ model, we will see how Gelfand-Fuks classes characterize holomorphic symmetries on the higher *world-sheet*, or source manifold.

The symmetry is that of holomorphic reparametrizations. Infinitesimally, this is described by the Lie algebra of holomorphic vector fields. We have already seen BW: ref that classical theories on a complex manifold X with such a symmetry by holomorphic reparametrizations admit an action by the local Lie algebra $\mathcal{T}_X = \Omega^{0,*}(X, T^{1,0}_X)$.

The Gelfand-Fuks classes we will consider in this section appear as anomalies for quantizing an action by the local Lie algebra $\mathcal{T}_X$. In other words, these classes parametrize shifted central extensions of $\mathcal{T}_X$, just as the classes $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$ defined central extensions of the current algebra $\mathfrak{g}^X$. By our usual yoga of studying equivariant quantizations, we know such anomalies live in the local cohomology complex $\mathrm{C}^*_{\mathrm{loc}}(\mathcal{T}_X)$.

Definition/Lemma 2. Consider the following two classes of cocycles on W_d.

Chern type: For $1 \leq k \leq d$, let $\tau_k \in \mathrm{C}^k_{\mathrm{Lie}}(\mathrm{W}_d; \widehat{\Omega}^k_d)$ be the cocycle

$$\tau_k = s_k\left(\mathrm{At}^{\mathrm{GF}}(\widehat{\mathcal{T}}_d)\right),$$

where $s_k(\mathrm{At}^{\mathrm{GF}}(\widehat{\mathcal{T}}_d))$ is the homogeneous degree k piece of the characteristic polynomial defined by $\det(I + t\mathrm{At}^{\mathrm{GF}}(\widehat{\mathcal{T}}_d))$.

GL type: For $1 \leq i \leq d$ let $a_i \in \mathrm{C}^{2i-1}_{\mathrm{Lie}}(\mathrm{W}_d; \widehat{\mathcal{O}}_d)$ be the cocycle

$$a_i : (\xi_1, \ldots, \xi_{2i-1}) \mapsto \sum_{\sigma \in S_{2i-1}} \mathrm{sign}(\sigma) \mathrm{Tr}(\mathrm{Jac}(\xi_{\sigma(1)}) \cdots \mathrm{Jac}(\xi_{\sigma(2i-1)}).$$

We will use the notation $\widehat{\Omega}^{\#}_d = \oplus_k \widehat{\Omega}^k_d[-k]$ to denote the graded W_d-module with $\widehat{\Omega}^k_d$ sitting in degree k. The wedge product of forms endows this W_d-module with the structure of a graded commutative algebra.

If V is a graded vector space then we use the notation $\mathbb{C}[V]$ to denote the free graded $\mathbb{C}$-algebra on V. If V is spanned by vectors $\{v_i\}$ we will use the shorthand $\mathbb{C}[v_i]$ for this graded algebra.

Theorem 5.8 ([?]). *The bigraded commutative algebra $H^*(\mathrm{W}_d; \widehat{\Omega}^{\#}_d)$ is isomorphic to the bigraded commutative algebra*

$$(\mathbb{C}[a_1, \ldots, a_{2d-1}, \tau_1, \ldots, \tau_d]) / \left(c_1^{j_1} \cdots c_d^{j_d}\right),$$

where the quotient is over all indices $\{j_1, \ldots, j_d\}$ that satisfiy $j_1 + 2j_2 + \cdots + dj_d > d$. Here a_{2i-1} is in bidegree $(2i-1, 0)$ and τ_j is in bidegree (j, j).

In the above result we have not turned on the de Rham differential $\mathrm{d}_{dR} : \widehat{\Omega}^k_d \to \widehat{\Omega}^{k+1}_d$. This endows $\widehat{\Omega}^*_d = (\widehat{\Omega}^{\#}_d, \mathrm{d}_{dR})$ with the structure of a dg commutative algebra in W_d-modules. The formal Poincaré lemma asserts that the inclusion of the trivial W_d-module

$$\mathbb{C} \xrightarrow{\simeq} \widehat{\Omega}^*_d$$

is a quasi-isomorphism. In turn, we obtain a quasi-isomorphism of Chevalley-Eilenberg complexes

$$\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_d) \xrightarrow{\simeq} \mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_d; \widehat{\Omega}^*_d).$$

We may think of the cochain complex $\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_d; \widehat{\Omega}^*_d)$ as the total complex of the double complex with vertical differential given by the W_d Chevalley-Eilenberg differential for the graded module $\widehat{\Omega}^{\#}_d$, and horizontal differential equal to the de Rham differential.

To any double complex there is a spectral sequence abutting to the cohomology of the total complex. The E_1 page of this spectral sequence is given by the cohomology of the vertical differential. Moreover, if the double complex is a bigraded algebra so are each of the pages. In this case, the E_1 page is precisely the bigraded algebra of Theorem 5.8 and we have a spectral sequence

$$E_2^{p,q} = \left(H^q(\mathrm{W}_d; \widehat{\Omega}^p_d), \mathrm{d}_{dR}\right) \implies H^*(\mathrm{W}_d; \widehat{\Omega}^*_d) \cong H^*(\mathrm{W}_d). \tag{19}$$

Example 5.9. For the case $d = 1$ the spectral sequence collapses at the E_2 page. The only nontrivial cohomology is $\mathbb{C}$ in bidegree $(0, 0)$ and $a_1 \cdot \tau_1$ in bidgree $(1, 2)$. The 1-cocycle valued in formal power series a_1 is given by $a_1(f_i \partial_i) = \partial_i f_i \in \widehat{\mathcal{O}}_n$. The 1-cocycle valued in formal 1-forms τ_1 is given by $\tau_1(g_j \partial_j) = \mathrm{d}_{dR}(\partial_j g_j)$. To obtain the generator of $H^3(\mathrm{W}_1)$ we perform the following zig-zag:

$$\begin{tikzcd}
\mathrm{C}^3_{\mathrm{Lie}}(\mathrm{W}_1) \arrow[r] & \mathrm{C}^3_{\mathrm{Lie}}(\mathrm{W}_1; \widehat{\mathcal{O}}_1) & \\
& \mathrm{C}^2_{\mathrm{Lie}}(\mathrm{W}_1; \widehat{\mathcal{O}}_1) \arrow[u, "\mathrm{d}_{CE}"] \arrow[r, "\mathrm{d}_{dR}"] & \mathrm{C}^2_{\mathrm{Lie}}(\mathrm{W}_1; \widehat{\Omega}^1_1).
\end{tikzcd}$$

The de Rham differential kills $a_1 \cdot \tau_1$, so there exists an $\alpha \in \mathrm{C}^2_{\mathrm{Lie}}(\mathrm{W}_1; \widehat{\mathcal{O}}_1)$ such that $\mathrm{d}_{dR}\alpha = -a_1 \cdot \tau_1$. Now, the class $\mathrm{d}^{\widehat{\mathcal{O}}}_{CE}\alpha \in \mathrm{C}^3_{\mathrm{Lie}}(\mathrm{W}_1; \widehat{\mathcal{O}}_n)$ satisfies

$$\begin{aligned} \mathrm{d}_{dR}(\mathrm{d}^{\widehat{\mathcal{O}}}_{CE}\alpha) &= -\mathrm{d}_{CE}(a_1\tau_1) = 0 \\ \mathrm{d}_{CE}\mathrm{d}^{\widehat{\mathcal{O}}}_{CE}\alpha &= 0. \end{aligned}$$

Here, $\mathrm{d}^{\widehat{\mathcal{O}}}_{CE}$ denote the Chevalley-Eilenberg differential for $\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_1; \widehat{\mathcal{O}}_1)$ and d_{CE} is the restriction of this Chevalley-Eilenberg differential to $\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_1)$. The first line says that $\mathrm{d}_{CE}\alpha$ lifts to $\mathrm{C}^3_{\mathrm{Lie}}(\mathrm{W}_1)$, and the second line says that it is a cocycle for the absolute cohomology. Finally, note that $(\mathrm{d}^{\widehat{\mathcal{O}}}_{CE} + \mathrm{d}_{dR})\alpha = \mathrm{d}^{\widehat{\mathcal{O}}}_{CE}\alpha - a_1\tau_1$. Thus, in the total complex $\mathrm{d}^{\widehat{\mathcal{O}}}_{CE}\alpha$ is homotopic to $a_1\tau_1$, and so $[\mathrm{d}^{\widehat{\mathcal{O}}}_{CE}\alpha]$ is the generator of $H^3(\mathrm{W}_1)$.

For general $d \geq 1$, one can apply this spectral sequence to understand the cohomology $H^*(\mathrm{W}_d)$. To describe it, we introduce the following topological space. Let $\mathrm{Gr}(d,n)$ be the complex Grassmannian of d-planes in $\mathbb{C}^n$. Denote by $\mathrm{Gr}(d,\infty)$ the colimit of the natural sequence

$$\mathrm{Gr}(d,d) \to \mathrm{Gr}(d,d+1) \to \cdots .$$

It is a standard fact that $\mathrm{Gr}(d,\infty)$ is a model for the classifying space $B\mathrm{U}(d)$ of principal $\mathrm{U}(d)$-bundles. Let $E\mathrm{U}(d) \to B\mathrm{U}(d)$ be the universal principal $\mathrm{U}(d)$-bundle. Using the colimit description above, we have a natural skeletal filtration of $B\mathrm{U}(d)$ by

$$\mathrm{sk}_k B\mathrm{U}(d) = \mathrm{Gr}(d,k).$$

Let X_d denote the restriction of $E\mathrm{U}(d)$ over the $2d$-skeleton:

$$\begin{tikzcd} X_d \arrow[r] \arrow[d] & E\mathrm{U}(d) \arrow[d] \\ \mathrm{sk}_{2d}B\mathrm{U}(d) \arrow[r] & B\mathrm{U}(d). \end{tikzcd}$$

Remark 5.10. Though not the way the Gelfand and Fuks originally proved the result, one can use the computation of the cohomology of W_d with coefficients in $\widehat{\Omega}^k_d$ together with the spectral sequence (19) to prove this description of $H^*(\mathrm{W}_d)$. Indeed, the spectral sequence (19) is isomorphic, up to regradings, to the Serre spectral sequence for the principal $\mathrm{U}(d)$-bundle $X_d \to \mathrm{sk}_{2d}B\mathrm{U}(d)$. In other words, the formal de Rham differential on $\widehat{\Omega}^*_d$ is exactly the E_2 differential for the Serre spectral sequence.

Theorem 5.11 ([?] Theorem 2.2.4)**.** *There is an isomorphism of graded vector spaces*

$$H^*(\mathrm{W}_d) \cong H^*_{dR}(X_d).$$

Moreover, the commutative algebra structure on $H^*(\mathrm{W}_d)$ is trivial.

As a simple example, note that when $d = 1$ we have $\mathrm{sk}_2 B\mathrm{U}(1) = \mathbb{P}^1 \subset \mathbb{P}^\infty = B\mathrm{U}(1)$. Moreover, the restriction of the universal bundle is Hopf fibration $U(1) \to S^3 \to \mathbb{P}^1$. In particular, one has $X_1 = S^3$.

5.3. **The local cohomology of holomorphic vector fields.** Our main result in this section is a complete classification of the local cohomology of the sheaf of Dolbeualt complex of holomorphic vector fields $\mathcal{T}_X = \Omega^{0,*}(X; T_X^{1,0})$ on any complex manifold. This description involves the Gelfand-Fuks cohomology of formal vector fields that we have just discussed and will give a classifications of the higher dimensional central charges for holomorphically covariant field theories.

Theorem 5.12. *Let X be a complex d-fold. There is a quasi-isomorphism of sheaves of cochain complexes*

$$C^*_{\rm loc}(\mathcal{T}_X) \;\simeq\; \Omega^*_X \otimes C^*_{\rm Lie,red}(W_d)[2d]$$

where Ω^*_X is the sheaf of de Rham forms on X.

The core of the argument is in interpretting the local Lie algebra cohomology as the cohomology of vector fields on the formal disk through the process of *Gelfand-Kazhdan descent* that we introduced in Chapter **??** Before moving on to the proof, we have the immediate cohomological interpretation of the calculation. Recall that when we study classical BV theories equivariant for a local Lie algebra $\mathcal{L}$, the space the failure for quantizing the BV theory in a way that is equivariant for the Lie algebra is measured by an anomaly class in the local cohomology. For holomorphic diffeomorphisms, we obtain the following.

Corollary 5.13. *For X any complex manifold of complex dimension d one has at the level of cohomology*

$$H^k_{\rm loc}(\mathcal{T}_X) \;\cong\; \bigoplus_{i=0}^{2d} H^i_{dR}(X) \otimes H^{2d+k-i}_{\rm Lie,red}(W_d).$$

In particular, if the manifold is connected the space of anomalies for holomorphic diffeomorphisms for a theory defined on X is:

$$H^1_{\rm loc}(\mathcal{T}_X) = H^{2d+1}_{\rm Lie}(W_d),$$

which is independent of the complex manifold.

The corollary implies that the cohomology $H^{2d+1}_{\rm Lie}(W_d)$ deserves to be thought of as the space of "higher dimensional central charges" of a classically holomorphic diffeomorphism invariant theory. After the proof of our main result we will how this relates to the central extensions of holomorphic vector fields and the role of these extensions in quantum field theory.

Proof. We recall a description of the local cohomology complex using D-modules given in Section 4.5 of [**?**]. Let $\mathcal{L}$ be any local Lie algebra on X with associated graded vector bundle L. The local Lie algebra cohomology is defined as the sheaf

$$C^*_{\rm loc}(\mathcal{L}) = \Omega^{d,d}_X \otimes_{D_X} C^*_{\rm Lie,red}(JL)$$

where JL is the D_X-module given by taking the ∞-jets of the underlying vector bundle of $\mathcal{L}$. In [**?**] it was shown that $C^*_{\rm Lie,red}(JL)$ is flat as a D_X-module, thus we can replace the tensor product above by a left-derived tensor product

$$\Omega^{d,d}_X \otimes_{D_X} C^*_{\rm Lie,red}(JL) \simeq \Omega^{d,d}_X \otimes^{\mathbb{L}}_{D_X} C^*_{\rm Lie,red}(JL). \tag{20}$$

The Spenser resolution is a free resolution of $\Omega^{d,d}_X$ as a right D_X-module (by D_X we mean smooth differential operators) given by

$$M^* = \left(\cdots \to \Omega^{2d-1} \otimes_{C^\infty_X} D_X \xrightarrow{\nabla_D} \Omega^{d,d} \otimes_{C^\infty_X} D_X \right)$$

The differential ∇_D is determined by the natural flat connection on D_X. This complex M^* is concentrated in degree $-2d, \ldots, 0$. Via this resolution, we see that (20) is quasi-isomorphic to

$$M^* \otimes_{D_X} \mathrm{C}^*_{\mathrm{Lie,red}}(JL) \simeq \left(\cdots \to \Omega^{2d-1} \otimes_{C^\infty_X} \mathrm{C}^*_{\mathrm{Lie,red}}(JL) \xrightarrow{\nabla_D} \Omega^{d,d} \otimes_{C^\infty_X} \mathrm{C}^*_{\mathrm{Lie,red}}(JL) \right).$$

The right-hand side is, by definition, the shifted de Rham complex of the D_X-module $\mathrm{C}^*_{\mathrm{Lie,red}}(JL)$ so we obtain

$$\mathrm{C}^*_{\mathrm{loc}}(\mathcal{L}) \simeq \Omega^*(X, \mathrm{C}^*_{\mathrm{Lie,red}}(JL))[2d]. \tag{21}$$

Now, suppose that $\mathcal{L}$ is a holomorphic local Lie algebra of the form $\Omega^{0,*}(X, L^{hol})$ where L^{hol} is a holomorphic vector bundle. In the above notation, the underlying smooth vector bundle is $L = \bigwedge^* T^{0,1^*} X \otimes L^{hol}$.

We have used the notation JE to denote the smooth sections of the infinite rank vector bundle $\mathrm{Jet}(E)$. If E is a holomorphic vector bundle let $\mathrm{Jet}^{hol}(E)$ denote the infinite rank holomorphic vector bundle of holomorphic jets. Similarly, let $J^{hol}E$ be the holomorphic sections of this bundle. This is a D_X^{hol}-module where D_X^{hol} is the sheaf of holomorphic differential operators. Equivalently, a D_X^{hol}-module is a holomorphic vector bundle with a holomorphic flat connection. Of course, any D_X^{hol}-module E forgets to a smooth D_X-module that we denote E^{C^∞}.

Lemma 5.14. *Let L, L^{hol} be as above. There is a quasi-isomorphism of D_X-modules $JL \simeq (J^{hol}L^{hol})^{C^\infty}$.*

Proof. Let $\mathcal{L}$ be the sheaf of sections of L. The Dolbeualt complex is a resolution of the sheaf of holomorphic sections; thus there is a quasi-isomorphism $\mathcal{L} \simeq \mathcal{L}^{hol}$ of $\mathcal{O}_X^{hol}$-modules. □

This means that we can further reduce the expression for the local cohomology in (21) to

$$\mathrm{C}^*_{\mathrm{loc}}(\mathcal{L}) \simeq \Omega^*(X, \mathrm{C}^*_{\mathrm{Lie,red}}(J^{hol}L^{hol}))[2d]. \tag{22}$$

We have dropped the notation $(-)^{C^\infty}$ for convenience.

We now turn to the local Lie algebra in question, namely $\mathcal{T}_X$. This is, of course, a holomorphic local Lie algebra as it is given by $\mathcal{T}_X = \Omega^{0,*}(X, TX)$. The underlying holomorphic vector bundle is the holomorphic tangent bundle $T^{1,0}X$.

Suppose that $\mathcal{V}$ is any $(\mathrm{W}_d, \mathrm{GL}_d)$-module. Then, Gelfand-Kazhdan descent along the complex manifold X yields the D_X-module $\mathrm{desc}_X(\mathcal{V})$. In the case that $\mathcal{V} = \widehat{\mathcal{T}}_d$ we have seen that the D_X-module $\mathrm{desc}_X(\widehat{\mathcal{T}}_d)$ is equivalent to the D_X-module $J^{hol}T^{1,0}X$.

Lemma 5.15. *Gelfand-Kazhdan descent is symmetric monoidal. That is, if $\mathcal{V}, \mathcal{V}'$ are two $(\mathrm{W}_d, \mathrm{GL}_d)$-modules, then*

$$\mathcal{V} \otimes_{\widehat{\mathcal{O}}_n} \mathcal{V}' \simeq \mathrm{desc}_X(\mathcal{V}) \otimes_{J^{hol}\mathcal{O}_X^{hol}} \mathrm{desc}_X(\mathcal{V}').$$

This implies that there is a string of isomorphisms of D_X-modules

$$J^{hol}\mathrm{C}^*_{\mathrm{Lie,red}}(T^{1,0}X) = \mathrm{desc}(\mathrm{C}^*_{\mathrm{Lie,red}}(\mathrm{W}_d)) \cong \mathrm{C}^*_{\mathrm{Lie,red}}(\mathrm{desc}(\mathrm{W}_d)) = \mathrm{C}^*_{\mathrm{Lie,red}}(J^{hol}T^{1,0}X).$$

Equivalently, by **??** we know that the functor of jets is symmetric monoidal, so the same result follows.

To summarize we see that the Gelfand-Kazhdan descent of the $(\mathrm{W}_n, \mathrm{GL}_d)$-module $\mathrm{C}_{\mathrm{Lie,red}}(\mathrm{W}_d)$ is equal to the D_X-module $\mathrm{C}^*_{\mathrm{Lie,red}}(J^{hol}T^{1,0}_X)$. This is precisely the D_X-module present in the definition of the local cohomology of $\mathcal{T}_X$. Indeed, by Lemma 5.14 we have

$$\mathrm{C}^*_{\mathrm{loc}}(\mathcal{T}_X) \simeq \Omega^*\left(X, \mathrm{C}^*_{\mathrm{Lie,red}}(J^{hol}T^{1,0}_X)\right).$$

Thus, the de Rham complex of the D_X-module given by descent is precisely the local cohomology

$$\mathrm{C}^*_{\mathrm{loc}}(\mathcal{T}_X) \simeq \Omega^*\left(X, \mathrm{desc}_X(\mathrm{C}^*_{\mathrm{Lie,red}}(\mathrm{W}_d))\right)$$

The interpretation via descent will allow us to describe this de Rham complex explicitly. Suppose that $\mathfrak{g}$ is any Lie algebra. Then $\mathfrak{g}$ acts on itself (and its dual) via the adjoint action. This extends to an action of $\mathfrak{g}$ on its Chevalley-Eilenberg complex $\mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g}; M)$, where M is any $\mathfrak{g}$-module via the formula

$$(x \cdot \varphi)(x_1, \ldots, x_k) = \sum_i \varphi(x_1, \ldots, [x, x_i], \ldots, x_k) - x \cdot \varphi(x_1, \ldots, x_k)$$

Here, $x, x_i \in \mathfrak{g}$ and φ is a k-cochain with values in M. The $[-,-]$ denotes adjoint action, and the $\cdot$ is the $\mathfrak{g}$-module structure on M. The following lemma is well-known. The same formula holds for the reduced cochains.

Lemma 5.16. *The $\mathfrak{g}$-module structure on the cochain complexes $\mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g})$ and $\mathrm{C}^*_{\mathrm{Lie,red}}(\mathfrak{g})$ is homotopically trivial.*

For the case of an infinite dimensional Lie algebra, such as W_d, the same result holds when we use the continuous, or Gelfand-Fuks, Lie algebra cohomology. Thus, W_d acts homotopically trivial on $\mathrm{C}^*_{\mathrm{Lie,red}}(\mathrm{W}_d)$.

This implies that the descent $\mathrm{desc}_X(\mathrm{W}_d)$ has a homotopically trivial D_X-module structure. Equivalently, this means that the flat connection on $\mathrm{C}^*_{\mathrm{Lie,red}}(J^{hol}T^{1,0}X)$ is gauge equivalent to the trivial connection. Thus, there is a quasi-isomorphism of de Rham complexes

$$\Omega^*\left(X, \mathrm{C}^*_{\mathrm{Lie,red}}(J^{hol}T^{1,0}_X)\right) \simeq \Omega^*(X) \otimes_{C^\infty_X} \underline{\mathrm{C}^*_{\mathrm{Lie,red}}(\mathrm{W}_d)}_X$$

where the underline denotes the sections of the trivial bundle with fiber $\mathrm{C}^*_{\mathrm{Lie,red}}(\mathrm{W}_d)$. We have identified the left hand side with the local cohomology complex, so we are done. □

5.3.1. *An explicit description of the local cocycles.* We'd like to leverage our knowledge of the the Gelfand-Fuks cohomology of formal vector fields to provide an explicit description of the local cocycles. The theorem in the previous section gives a very general equivalence of the local cohomology on any complex manifold with the Gelfand-Fuks cohomology, but writing down the form of the local cocycle from the description on a formal disk is not so obvious.

For instance, consider the case $d = 1$ and we work on $X = \mathbb{C}$. The cohomology $H^*_{\mathrm{Lie,red}}(\mathrm{W}_1)$ is one-dimensional concentrated in degree 3. We'd like to describe the local cocycle corresponding to the generator of $H^3(\mathrm{W}_1) \cong H^1_{\mathrm{loc}}(\mathcal{T}_{\mathbb{C}})$ explicitly. Recall, using the formal Hodge-to-de

Rham spectral sequence we saw that the generator of $H^3(\mathrm{W}_1)$ came from the element $a_1\tau_1 \in H^2_{\mathrm{Lie}}(\mathrm{W}_1; \widehat{\Omega}^1_1)$ on the E_2 page of the spectral sequence (19).

Now, the 1-cocycles a_1, τ_1 can both be interpreted as functionals on the Dolbeault complex $\Omega^{0,*}(\mathbb{C}, T^{1,0}\mathbb{C})$. Indeed, if $\xi = \alpha(z,\bar{z})\partial_z$ is an element of the Dolbeault complex we can define

$$\widetilde{a}_1(\xi) = \partial_z \alpha(z,\bar{z}) \in \Omega^{0,*}(\mathbb{C})$$
$$\widetilde{\tau}_1(\xi) = \partial(\partial_z\alpha(z,\bar{z})) \in \Omega^{1,*}(\mathbb{C}).$$

Each of these cocycles clearly only depends on the jet of the vector field $\alpha\partial_z$. Similarly, the product is $\widetilde{a}_1\widetilde{\tau}_1$ is the bilinear functional on jets of $\mathcal{T}_{\mathbb{C}}$:

$$\widetilde{a}_1\widetilde{\tau}_1(\xi_1, \xi_2) = \partial_z\alpha_1(z,\bar{z})\partial(\partial_z\alpha_2(z,\bar{z})) \in \Omega^{1,*}(\mathbb{C})$$

This is a density precisely when $|\alpha_1| + |\alpha_2| = 1$. Thus, $\widetilde{a}_1\widetilde{\tau}_1$ determines a degree $+1$ density valued cochain on $J\mathcal{T}_{\mathbb{C}}$; in other other words an element of $\mathrm{C}^*_{\mathrm{loc}}(\mathcal{T}_{\mathbb{C}})$ that we write as

$$\int_{\mathbb{C}} \partial_z\alpha_1(z,\bar{z})\partial(\partial_z\alpha_2(z,\bar{z})).$$

If we integrate by parts, we can put this local functional in the form $\int f\partial_z^3 g \mathrm{d}z\mathrm{d}\bar{z}$. If one restricts this local functional to the annulus and performs the radial integration, we recover the usual formula for the generator of $H^2(\mathrm{Vect}(S^1))$ BW: citation defining the central extension of the Virasoro Lie algebra. In fact, in **[?]** BW: finish

This can be generalized to arbitrary dimensions in a natural way.

Our first goal is to construct, from a Gelfand-Fuks classes in $\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_d)$ and $\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n; \widehat{\Omega}^*_d)$, a local functional on $\mathcal{T}_X$. We have seen that the cochain complex $\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_d; \widehat{\Omega}^*_d)$, equipped with the total differential $\mathrm{d}_{CE} + \mathrm{d}_{dR}$, computes the absolute Gelfand-Fuks cohomology $H^*(\mathrm{W}_d)$ through the formal Hodge to de Rham spectral sequence. We will use this property to represent elements of $H^*(\mathrm{W}_d)$ by local functionals on $\mathcal{T}_X$ by first representing elements in $\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n; \widehat{\Omega}^*_d)$ by local functionals.

We can decompose an element $\alpha \in \mathrm{C}^k_{\mathrm{Lie}}(\mathrm{W}_d; \widehat{\Omega}^*_d)$ as

$$\alpha = f^I \mathrm{d}t_I.$$

The sum is over the multi-index $I = (i_1, \ldots, i_k)$, where $1 \leq i_j \leq d$. For each I, f^I is a k multi-linear symmetric functional on W_d valued in $\widehat{\mathcal{O}}_d$

$$f^I : \mathrm{Sym}^k(\mathrm{W}_d[1]) \to \widehat{\mathcal{O}}_d.$$

We extend f^I to a functional on the Dolbeault complex $\Omega^{0,*}(\mathbb{C}^d; T^{1,0}\mathbb{C}^d)$ as follows. Using the framing on $\mathbb{C}^d$, every element of the Dolbeualt complex can be written as

$$X^J(z,\bar{z})\mathrm{d}\bar{z}_J$$

where $J = (j_1, \ldots, j_l)$ is a multi-index and X^J is an ordinary holomorphic vector field on $\mathbb{C}^d$. We extend f^I to a Dolbeualt valued functional $\Omega^{0,*}(\mathbb{C}^d; T^{1,0}\mathbb{C}^d)$ via the formula

$$\begin{array}{cccc} f^I_{\Omega^{0,*}} : & \mathrm{Sym}^k\left(\Omega^{0,*}(\mathbb{C}^d; T^{1,0})\right) & \to & \Omega^{0,*}(\mathbb{C}^d) \\ & \left(X_1^{J(1)}(z,\bar{z})\mathrm{d}\bar{z}_{J(1)}, \ldots, X_k^{J(k)}(z,\bar{z})\mathrm{d}\bar{z}_{J(k)}\right) & \mapsto & f^I(X_1^{J(1)}, \ldots, X_k^{J(k)})\mathrm{d}\bar{z}_{J(1)} \wedge \cdots \mathrm{d}\bar{z}_{J(k)} \end{array}$$

The local functional corresponding to the original element $\alpha = f^I \mathrm{d}t_I \in \mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n; \widehat{\Omega}^*_d)$ is defined by the k-multi-linear functional

$$(\xi_1, \ldots, \xi_k) \mapsto \int_{\mathbb{C}^d} f^I_{\Omega^{0,*}}(\xi_1, \ldots, \xi_k) \mathrm{d}z_I.$$

Denote this functional by $J^{GF}(\alpha)$. Note that it is only nonzero when the multi-index I is a permutation of $(1, \ldots, d)$. Since it is given by the integral of a some multi-differential operators against a density it is manifestly a local functional.

5.3.2. *A comparison to other cocycles.*

5.4. **The holomorphic anomaly for σ-models.**

APPENDIX A. THE DG MODEL FOR PUNCTURED AFFINE SPACE

In this section we review a dg model for the derived space of sections of the structure sheaf on punctured affine space in any dimensions. We will be mostly concerned with the sheaf of algebraic functions. This model has appeared in the work of **[?]**, based on the Jouanolou resolution of singularities.

Let $\mathbb{A}^d$ be algebraic affine space with sheaf of functions given by $\mathcal{O}^{alg}(\mathbb{A}^d) = \mathbb{C}[z_1, \ldots, z_d]$. Denote $\mathbb{A}^{d\times} = \mathbb{A}^d \setminus \{0\}$. When $d = 1$ the punctured space $\mathbb{A}^{1\times}$ is an affine scheme with $H^0(\mathbb{A}^{1\times}, \mathcal{O}^{alg}) = \mathbb{C}[z^{\pm}]$. When $d > 1$ the punctured space $\mathbb{A}^{d\times}$ is no longer affine. In fact, the cohomology is

$$H^*(\mathbb{A}^{d\times}, \mathcal{O}^{alg}) = \begin{cases} 0, & * \neq 0, d-1 \\ \mathbb{C}[z_1, \ldots, z_d], & * = 0 \\ \mathbb{C}[z_1^{-1}, \ldots, z_d^{-1}]\frac{1}{z_1 \cdots z_d}, & * = d-1 \end{cases}.$$

The dg commutative algebra $\mathbb{R}(\mathbb{A}^{d\times}, \mathcal{O}^{alg})$ is well-defined up to quasi-isomorphism. We will recall the construction of an explicit model.

Definition A.1. Let $A_d = \oplus_{p=0}^d \oplus_{q=0}^d A_d^{p,q}$ be the bigraded commutative algebra generated by elements

$$z_1, \ldots, z_d, z_1^*, \ldots, z_d^*, (zz^*)^{-1}$$

in bidegree $(0,0)$, where $zz^* = \sum_i z_i z_i^*$, elements

$$\mathrm{d}z_1, \ldots, \mathrm{d}z_d$$

in bidegree $(1,0)$, and

$$\mathrm{d}z_1^*, \ldots, \mathrm{d}z_d^*$$

in bidegree $(0,1)$. Introduce a $*$-weight, so that $z_i^*, \mathrm{d}z_i^*$ have $*$-weight $+1$ and $(z_i^*)^{-1}$ has $*$-weight -1. We require that:

(i) every element is of total $*$-weight zero and
(ii) the contraction of every element with the Euler vector field $\sum_i z_i^* \partial_{z_i^*}$ vanishes.

There is a map $\bar{\partial} : A_d^{p,q} \to A_d^{p,q+1}$ of bidegree $(0,1)$ defined formally by

$$\bar{\partial} = \sum_i \mathrm{d}z_i^* \frac{\partial}{\partial z_i^*}$$

and a map of bidegree $(1,0)$ defined by

$$\partial = \sum_i \mathrm{d}z_i \frac{\partial}{\partial z_i}.$$

This differentials commute $\bar{\partial}\partial = \partial\bar{\partial}$ and each square to zero.

When $p = 0$ we see that the resulting complex $(A_d, \bar{\partial}) = (\oplus_q A_d^q[-q], \bar{\partial})$ has the structure of a commutative dg algebra. This commutative dg algebra is model for $\mathbb{R}(\mathbb{A}^{d\times}, \mathcal{O}^{alg})$. Note that by conditions (i),(ii) this complex is concentrated in degrees $0, 1, \ldots, d-1$.

For each p, the complex $A_d^{p,*} = (\oplus_q A^{p,q}[-q], \bar{\partial})$ is a model for the $\mathbb{R}\Gamma(\mathbb{A}^{d\times}, \mathcal{O}^{alg})$-module given by the derived space of sections of holomorphic p-forms $\mathbb{R}\Gamma(\mathbb{A}^{d\times}, \Omega^{p,alg})$. We will denote the resulting bigraded algebra by

$$A_d^{*,*} = \oplus_{p=0} A_d^{p,*}[-p] = \oplus_{p=0} \oplus_{q=0} A_d^{p,q}[-p-q].$$

It is immediate to check that the formula for the ordinary Bochner-Martinelli kernel makes sense in the algebra A_d. That is, we define

$$\omega_{BM}^{alg}(z, z^*) = \frac{(d-1)!}{(2\pi i)^d} \frac{1}{(zz^*)^d} \sum_{i=1}^{d} (-1)^{i-1} z_i^* \mathrm{d}z_1^* \wedge \cdots \wedge \widehat{\mathrm{d}z_i^*} \wedge \cdots \wedge \mathrm{d}z_d^*,$$

which is an element of A_d^{d-1}.

The key properties of the dg algebra A_d and its dg modules $A_d^{p,*}$ we will utilize are summarized in the following result of [**?**].

Proposition A.2 ([**?**] Proposition 1.3.1)**.**

(1) The commutative dg algebra $(A_d, \bar{\partial})$ is a model for $\mathbb{R}\Gamma(A^{d\times}, \mathcal{O}^{alg})$

$$A_d \simeq \mathbb{R}\Gamma(\mathbb{A}^{d\times}, \mathcal{O}^{alg}).$$

Similarly, $(A_d^{p,*}, \bar{\partial}) \simeq \mathbb{R}\Gamma(\mathbb{A}^{d\times}, \Omega^{p,alg})$.

(2) There is a dense map of commutative bigraded algebras

$$j : A_d^{*,*} \to \Omega^{*,*}(\mathbb{C}^d \setminus \{0\})$$

sending $z_i \mapsto z_i$, $z_i^* \mapsto \bar{z}_i$, and $\mathrm{d}z_i^* \mapsto \mathrm{d}\bar{z}_i$ that is compatible with the $\bar{\partial}$ and ∂ differentials on both sides.

(3) Finally, there is a unique GL_n-equivariant residue map

$$\mathrm{Res}_{z=0} : A_d^{d,d-1} \to \mathbb{C}$$

that satisfies

$$\mathrm{Res}_{z=0}\left(f(z)\omega_{BM}^{alg}(z, z^*)\mathrm{d}z_1 \cdots \mathrm{d}z_d\right) = f(0)$$

where $f(z) \in \mathbb{C}[z_1, \ldots, z_d]$. In particular, for any $\omega \in A_d^{d,d-1}$ one has

$$\mathrm{Res}_{z=0}(\omega) = \oint_{S^{2d-1}} j(\omega)$$

where S^{2d-1} is any sphere centered at the origin in $\mathbb{C}^d$.