THE THEORY ON HOPF SURFACES

The local theory of the holomorphic σ-model where the source is $\mathbb{C}^d$ has been studied in the last section. We now turn to global aspects of the theory which amounts to putting the theory on more exotic source manifolds. In this section we consider the holomorphic σ-model where the source is a particular compact complex d-fold. In fact, we will allow for holomorphic families of compact complex manifolds as the source and study how the partition function varies in this family.

0.1. **The case of an elliptic curve.** As a warmup we recall a familiar situation

0.2. **Hopf manifolds.** We focus on a family of complex manifolds defined by Hopf in **[?]** defined in every complex dimension d. For $1 \leq i \leq d$ let $q_i \in D(0,1)^\times$ be a nonzero complex number of modulus $|q_i| < 1$. The d-dimensional *Hopf manifold of type* $\mathbf{q} = (q_1, \ldots, q_d)$ is the following quotient of punctured affine space $\mathbb{C}^d \setminus \{0\}$ by the discrete group $\mathbb{Z}^d$:

$$H_{\mathbf{q}} = \left(\mathbb{C}^d \setminus \{0\}\right) \Big/ \left((z_1, \ldots, z_d) \sim (q_1^{2\pi i \mathbb{Z}} z_1, \ldots, q_d^{2\pi i \mathbb{Z}} z_d)\right).$$

Note that in the case $d = 1$ we recover the usual description of an elliptic curve $H_{\mathbf{q}} = E_q = \mathbb{C}^\times / q^{2\pi i \mathbb{Z}}$.

For any d and tuple $(q_1, \ldots, q_d)$ as above, we see that as a smooth manifold there is a diffeomorphism $H_{\mathbf{q}} \cong S^{2d-1} \times S^1$. Indeed, the radial projection map $\mathbb{C}^d \setminus \{0\} \to \mathbb{R}_{>0}$ defines a smooth S^{2d-1}-fibration over $\mathbb{R}_{>0}$. Passing to the quotient, we obtain an S^{2d-1}-fibration

$$H_{\mathbf{q}} \to \mathbb{R}_{>0} / \left(r \sim \lambda^{\mathbb{Z}} \cdot r\right) \cong S^1.$$

Here, $\lambda = (|q_1|^2 + \cdots + |q_d|^2)^{1/2} > 0$. Since there are no non-trivial S^{2d-1} fibrations over S^1 we obtain $H_{\mathbf{q}} = S^{2d-1} \times S^1$.

There is an equivalent description of $H_{\mathbf{q}}$ as a quotient of affine space that we will take advantage of. Consider the expo