RANDOM ANALYSIS

1. The propagator

We will write down the effective propagator for the free $\beta\gamma$ system on flat space $\mathbb{C}^d$. Throughout, we follow the formalism of Costello's renormalizationBW: .... This is a purely analytic calculation, so we will consider the case where the target is one dimensional and hence the fields are

$$\mathcal{E} = \Omega^{0,*}(\mathbb{C}^d) \oplus \Omega^{d,*}(\mathbb{C}^d)[d-1].$$

Choosing the standard flat metric on $\mathbb{C}^d$, we obtain a natural gauge fixing operator

$$Q^{GF} = \bar{\partial}^* : \Omega^{0,*}(\mathbb{C}^d) \to \Omega^{0,*-1}(\mathbb{C}^d)$$

which acts on $(d,*)$ forms in a similar way. Choosing holomorphic coordinates $\{z_i\}$ for $\mathbb{C}^d$ we see that $\bar{\partial}^*$ can be written as

$$\bar{\partial}^* = \sum_{i=1}^d \frac{\partial}{\partial(\mathrm{d}\bar{z}_i)} \frac{\partial}{\partial z_i}.$$

The corresponding operator $[Q, Q^{GF}] = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ is simply the Hodge Laplacian $\Delta_{\bar{\partial}}$.

For $t > 0$, the heat kernel $K_t \in \mathcal{E}(\mathbb{C}^d)\widehat{\otimes}\mathcal{E}(\mathbb{C}^d)$ is characterized by the equation

$$\Delta_{\bar{\partial}} K_t + \frac{\partial}{\partial t} K_t = 0$$

and normalized so that

$$\langle \varphi(x), K_t(x,y) \rangle_x = (e^{-t\Delta_{\bar{\partial}}}\varphi)(y)$$

where $\varphi \in \mathcal{E}$ and $\langle -,- \rangle$ is the (-1)-symplectic pairing. Using the standard formula for the heat kernel for the flat Laplacian on $\mathbb{C}^d$ we have the expression for our heat kernel, including the correct differential form factors

$$K_t(z,w) = \frac{1}{(4\pi t)^d} e^{-|z-w|^2/4t} \left((\mathrm{d}^d z - \mathrm{d}^d w) \wedge \prod_{i=1}^d (\mathrm{d}\bar{z}_i - \mathrm{d}\bar{w}_i) \right)$$

The effective propagator is defined for $0 < \epsilon, L$ and given by

$$P_{\epsilon<L}(z,w) = \int_{t=\epsilon}^L \mathrm{d}t (\bar{\partial}^* \otimes 1) K_t(z,w).$$

We can compute this propagator directly

$$\begin{aligned} P_\epsilon(z,w) &= \frac{1}{(4\pi)^d} \int_{t=\epsilon}^L \mathrm{d}t e^{-|z-w|^2/4t} \frac{1}{(4\pi t)^d} \sum_{j=1}^d (-1)^{j-1} \frac{\bar{z}_j - \bar{w}_j}{4t} (\mathrm{d}^d z - \mathrm{d}^d w) \prod_{i\neq j} (\mathrm{d}\bar{z}_i - \mathrm{d}\mathrm{d}\bar{w}_i) \\ &= \frac{1}{(4\pi)^d} \frac{1}{|z-w|^{2d}} \sum_j (-1)^{j-1} (\bar{z}_j - \bar{w}_j)(\mathrm{d}^d z - \mathrm{d}^d w) \prod_{i\neq j} (\mathrm{d}\bar{z}_i - \mathrm{d}\bar{w}_i) \int_{u=|z-w|^2/L}^{|z-w|^2/\epsilon} \mathrm{d}u u^{d-1} e^{-u}. \end{aligned}$$

In the second line we have made the substitution $u = |z-w|^2/t$. BW: work out factors We see that the differential form part above is precisely the Bochner-Martinelli kernel $\omega_{BM} \in \Omega^*(\mathbb{C}^d \times \mathbb{C}^d \setminus \Delta)$

$$\omega_{BM}(z,w) = C_n \frac{1}{|z-w|^{2d}} \sum_j (-1)^{j-1} (\bar{z}_j - \bar{w}_j)(\mathrm{d}^d z - \mathrm{d}^d w) \prod_{i \neq j} (\mathrm{d}\bar{z}_i - \mathrm{d}\bar{w}_i).$$

Lemma 1.1. *Suppose $z \neq w$. The $\epsilon \to 0, L \to \infty$ limit of the propagator $P_{\epsilon<L}(z,w)$ exists and*

$$\lim_{\epsilon \to 0} \lim_{L \to \infty} P_{\epsilon<L}(z,w) = \omega_{BM}(z,w).$$

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