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COHOMOLOGY OF THE LIE ALGEBRA OF FORMAL VECTOR FIELDS

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Abstract. We calculate the cohomology of the Lie algebra of formal vector fields at the origin in a euclidean space. The results are applied to the investigation of the Lie algebra of tangent vector fields on a smooth manifold.

Introduction

0.1. By a *formal vector field* at the point $0 = (0, \cdots, 0)$ of the space $\mathbf{R}^n$ we mean a linear combination of the form $\sum_{i=1}^n p_i(x_1, \cdots, x_n)e_i$, where $e_1, \cdots, e_n$ are the vectors of the standard basis of $\mathbf{R}^n$ and the $p_i(x_1, \cdots, x_n)$ are formal power series with real coefficients in the coordinates $x_1, \cdots, x_n$ of the space. The set of formal vector fields can also be defined as the inverse limit of the system $\{S_r, \pi_r\}$, where S_r is the space of r-sets of vector fields of class C^r at the point $0 = (0, \cdots, 0)$ and π_r: $S_r \to S_{r-1}$ is the natural projection. From this definition we see that the formal vector fields constitute a linear topological space.

The commutator $[\xi', \xi'']$ of the formal vector fields

$$\xi' = \sum_{i=1}^n p_i'(x_1, \ldots, x_n)e_i, \quad \xi'' = \sum_{i=1}^n p_i''(x_1, \ldots, x_n)e_i$$

is defined by the formula

$$[\xi', \xi''] = \sum_{i=1}^n \left[\sum_{j=1}^n \left(p_j' \frac{\partial p_i''}{\partial x_j} - p_j'' \frac{\partial p_i'}{\partial x_j}\right)\right] e_i.$$

With respect to commutation, the formal vector fields at 0 constitute a topological Lie algebra, denoted by W_n. The subject of this paper is the calculation of the cohomology of this topological Lie algebra with coefficients in the identity real representation (i.e., in the field $\mathbf{R}$ with trivial action of the algebra W_n).

0.2. In order to formulate the final result, we have to describe certain auxiliary topological spaces X_n $(n = 1, 2, \cdots)$. Suppose $N \geq 2n$, and let $(E(N, n), p, G(N, n))$ be the canonical $U(n)$-bundle over the (complex) Grassmann manifold $G(N, n)$. The usual cell decomposition of the manifold $G(N, n)$ (cf., e.g., [1], Russian p. 89) has the property that the $2n$th skeleton $(G(N, n))_{2n}$ for $N \geq 2n$ is independent of N. The inverse image of the set $(G(N, n))_{2n}$ under the mapping p we denote by X_n.

The space X_1 is a three-dimensional sphere. The remaining spaces X_i do not have such a simple geometric interpretation.

0.3. The central result of the paper is the following:

Theorem. *For all q we have an isomorphism:*

$$H^q(W_n; \mathbf{R}) = H^q(X_n; \mathbf{R}).$$

Multiplication in the ring $H^*(W_n; \mathbf{R})$ and in the ring $H^*(X_n; \mathbf{R})$ is trivial, i.e., the product of any two elements of positive dimension is zero.

The cohomology of the space X_n can be found by using standard methods of topology. Considerable information is obtained about this cohomology in the process of proving Theorem 0.3 (see §5). For example, it is trivial for $0 < q \leq 2n$ and for $q > n(n+2)$.

0.4. The results of this paper are closely connected with our results on the cohomology of Lie algebras of vector fields on manifolds [2]. Specifically, the standard cochain complex of the algebra W_n is, as we shall see in §1 (cf. 1.6), just the complex $\{\bigoplus_m \tilde{\breve{P}}^q_m, \tilde{\breve{\nabla}}\}$ considered in [2] (cf. [2], §§4–6), and so the spaces $H^q(W_n; \mathbf{R})$ coincides with the spaces $\tilde{\breve{D}}^q$ of homology of this complex. This makes it possible to obtain some new information from Propositions 7.5 and 8.5 of [2]. For example, we prove that the spaces $\mathfrak{H}^q(M)$ of cohomology of dimension q of the Lie algebra of smooth vector fields on an n-dimensional orientable compact manifold M with coefficients in $\mathbf{R}$ are trivial for $0 < q \leq n$ (Theorem 6.3). Of other applications of Theorem 0.3, we note the recently published calculation [3] of the rings $\mathfrak{H}^*(T^n) = \Sigma_q \mathfrak{H}^q(T^n)$, where T^n is a torus of dimension n.

0.5. With the exception of §6, this paper can be read independently of [2]. However, §1.6 would have to be omitted and Proposition 1.8 taken for granted (or re-proved).

0.6. An important point in the proof of Theorem 0.3 is the fact that the cohomology of the unitary group $U(n)$ occurs in it as the cohomology of the Lie algebra $\mathfrak{gl}(n; \mathbf{R})$ of real matrices of order n. For this observation we are indebted to M. V. Losik, to whom we are glad to express our sincere gratitude.

§1. The standard complex

1.1. We recall that the cohomology of a topological Lie algebra $\mathfrak{g}$ with coefficients in a $\mathfrak{g}$-module M is defined as the homology of the complex $\{C^q(\mathfrak{g}; M), d^q(\mathfrak{g}; M)\}$, where $C^q(\mathfrak{g}; M)$ is the space of continuous skew-symmetric q-linear functionals on $\mathfrak{g}$ with values in M, and the differential $d^q(\mathfrak{g}; M)$: $C^q(\mathfrak{g}; M) \to C^{q+1}(\mathfrak{g}; M)$ is defined by the formula

$$[d^q(\mathfrak{g}; M)P](\xi_1, \ldots, \xi_{q+1}) =$$

$$= \sum_{1 \leqslant s < t \leqslant q+1} (-1)^{s+t-1} P([\xi_s, \xi_t], \xi_1, \ldots, \hat{\xi}_s, \ldots, \hat{\xi}_t, \ldots, \xi_{q+1}) + \sum_{1 \leqslant s \leqslant q+1} (-1)^s \xi_s P(\xi_1, \ldots, \hat{\xi}_s, \ldots, \xi_{q+1}).$$

Here $\xi_i \in \mathfrak{g}$, and $\xi_s P(\xi_1, \cdots, \hat{\xi}_s, \cdots, \xi_{q+1})$ denotes the result of the action of the element $\xi_s \in \mathfrak{g}$ on $P(\xi_1, \cdots, \hat{\xi}_s, \cdots, \xi_{q+1})$ as on an element of the $\mathfrak{g}$-module M.

The complex $\{C^q(\mathfrak{g}; M), d^q(\mathfrak{g}; M)\}$ is called the *standard cochain complex of the algebra* $\mathfrak{g}$ *with values in* M.

1.2. For brevity we denote the space $\mathbf{R}^n$ by T and the Lie algebra W_n by W. The standard cochain complex of W with values in $\mathbf{R}$ we shall denote simply by $\{C^q, d^q\}$. Thus, C^q is the space of continuous skew-symmetric q-linear real functionals on W, and $d^q\colon C^q \to C^{q+1}$ is the homomorphism defined by

$$(d^q P)(\xi_1, \ldots, \xi_{q+1}) = \sum_{1 \leqslant s < t \leqslant q+1} (-1)^{s+t-1} P([\xi_s, \xi_t], \xi_1, \ldots, \hat{\xi}_s, \ldots, \hat{\xi}_t, \ldots, \xi_{q+1}). \tag{1}$$

Every continuous linear functional on W can be represented uniquely as a finite sum of functionals of the form

$$\Sigma_i p_i e_i \to \nu_0 (D (\Sigma_i \alpha(e_i) p_i)),$$

where α is a functional on T, D is a differential operator, and ν_0 is the functional assigning to every power series its free term. The space of differential operators in the space of power series is just $S^* T = \oplus_m S^m T$—the sum of all symmetric powers of the space T. Thus, the space W' conjugate to W is canonically isomorphic with $S^* T \otimes T'$. Finally, the space of continuous skew-symmetric q-linear functionals on W is obviously $\Lambda^q(W')$—the qth exterior power of the space W'. In short, we have:

$$C^q = \Lambda^q (S^* T \otimes T'). \tag{2}$$

1.3. We shall sometimes regard the elements of the space C^q as functionals on the space conjugate to C^q, i.e., as functions of $2q$ vector arguments $\alpha_1, \cdots, \alpha_q \in T'$, $\beta_1, \cdots, \beta_q \in T$, polynomially dependent on the components of the covectors $\alpha_1, \cdots, \alpha_q$, multilinear in $\beta_1, \cdots, \beta_q$, and changing sign under simultaneous interchange of α_i with α_j and β_i with β_j. In this notation the differential $d = d^q\colon C^q \to C^{q+1}$ is given by the formula

$$(dP)(\alpha_1, \ldots, \alpha_{q+1}, \beta_1, \ldots, \beta_{q+1}) = \sum_{1 \leqslant s < t \leqslant q+1} (-1)^{s+t-1} P(\alpha_s + \alpha_t, \alpha_1, \ldots, \hat{\alpha}_s, \ldots, \hat{\alpha}_t, \ldots, \alpha_{q+1}; \beta(s, t), \beta_1, \ldots, \hat{\beta}_s, \ldots, \hat{\beta}_t, \ldots, \beta_{q+1}), \tag{3}$$

where $\beta(s, t) = (\alpha_t, \beta_s)\beta_t - (\alpha_s, \beta_t)\beta_s$.

1.4. The standard cochain complex of a Lie algebra $\mathfrak{g}$ with values in $\mathbf{R}$ has a canonical multiplicative structure: the multiplication $C^q(\mathfrak{g}; \mathbf{R}) \otimes C^r(\mathfrak{g}; \mathbf{R}) \to C^{q+r}(\mathfrak{g}; \mathbf{R})$ is defined by the formula

$$(PQ)(\xi_1, \ldots, \xi_{q+r}) = \sum (-1)^{i_1+\cdots+i_q-q(q+1)/2} P(\xi_{i_1}, \ldots, \xi_{i_q}) Q(\xi_{j_1}, \ldots, \xi_{j_r}),$$

where the summation is over all partitions of the set $\{1, \cdots, q+r\}$ into disjoint subsets $\{i_1, \cdots, i_q\}$, $\{j_1, \cdots, j_r\}$ with $i_1 < \cdots < i_q$, $j_1 < \cdots < j_r$.

This multiplication is connected with the differential $d = d(\mathfrak{g}; \mathbf{R})$ through the usual formula $d(PQ) = (dP)Q + (-1)^{\dim P} P(dQ)$ and defines in the space $H^*(\mathfrak{g}; \mathbf{R}) = \sum_q H^q(\mathfrak{g}; \mathbf{R})$ the structure of an algebra over the field $\mathbf{R}$.

When $\mathfrak{g} = W$, cochain multiplication is expressed in the language of 1.3 by the formula

$$\begin{aligned}(PQ)&(\alpha_1, \ldots, \alpha_{q+r}; \beta_1, \ldots, \beta_{q+r}) \\ &= \sum (-1)^{i_1+\cdots+i_q-q(q+1)/2} P(\alpha_{i_1}, \ldots, \alpha_{i_q}; \beta_{i_1}, \ldots, \beta_{i_q}) \\ &\qquad \times Q(\alpha_{j_1}, \ldots, \alpha_{j_r}; \beta_{j_1}, \ldots, \beta_{j_r}).\end{aligned} \tag{4}$$

1.5. As is clear from (2), there are canonical representations of the group $GL(n, \mathbf{R})$ in the spaces C^q. In the language of 1.3 these representations are given by the formula

$$(gP)(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_q) = P(\alpha_1 g', \ldots, \alpha_q g'; g\beta_1, \ldots, g\beta_q). \tag{5}$$

We see from (3) and (4) that the differential d and multiplication $C^q \otimes C^r \to C^{q+r}$ are compatible with these representations.

The above representations induce representations of the Lie algebra $\mathfrak{gl}(n, \mathbf{R})$ in the spaces C^q. The formulas describing these representations can be obtained by using the standard basis $\{A_{kl}\}$ in $\mathfrak{gl}(n, \mathbf{R})$ (here $A_{kl} = \|a_{ij}(k, l)\|$, where $a_{ij}(k, l) = \delta_{ik}\delta_{jl}$). The element $A_{kl} \in \mathfrak{gl}(n, \mathbf{R})$ acts in C^q according to the formula

$$\begin{aligned}(A_{kl}P)&(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_q) \\ &= \sum_{i=1}^{q} \Big[P(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_{i-1}, A_{kl}\beta_i, \beta_{i+1}, \ldots, \beta_q) \\ &\qquad - x_{ik} \frac{\partial P}{\partial x_{il}} (\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_q)\Big];\end{aligned} \tag{6}$$

here $x_{ij} = (\alpha_i, e_j)$ is the jth component of the covector α_i.

1.6. Let F be an element of C^q, and τ a set of nonnegative integers $\tau_1, \cdots, \tau_n$ with $\tau_1 + \cdots + \tau_n = q$. The formula

$$P^\tau(\alpha_1, \ldots, \alpha_q) = P(\alpha_1, \ldots, \alpha_q; \underbrace{e_1, \ldots, e_1}_{\tau_1}, \ldots, \underbrace{e_n, \ldots, e_n}_{\tau_n})$$

defines a polynomial P^τ in the components of the covectors $\alpha_1, \cdots, \alpha_q$ such that if $\tau_1 + \cdots + \tau_{s-1} < i < j \leq \tau_1 + \cdots + \tau_s$, where $1 \leq s \leq n$, then

$$P^{\tau}(\alpha_1, \ldots, \alpha_q) = -P^{\tau}(\alpha_1, \ldots, \alpha_{i-1}, \alpha_j, \alpha_{i+1}, \ldots, \alpha_{j-1}, \alpha_i, \alpha_{j+1}, \ldots, \alpha_q).$$

Clearly, specifying an element $P \subset C^q$ is equivalent to specifying for all τ polynomials P^{τ} with the indicated property, i.e., the space C^q coincides with the space $\oplus_m \widetilde{P}^q_m$ of [2] (cf. [2], 4.1). It is easy to see that the differential d: $C^q \to C^{q+1}$ also coincides with the mapping $\widetilde{\nabla}: \oplus_m \widetilde{P}^q_m \to \oplus_m \widetilde{P}^{q+1}_m$ defined in §5.1 of [2] and so the spaces $\widetilde{D}^q = \oplus_m \widetilde{D}^q_m$ of [2] coincide with the spaces $H^q(W_n; \mathbf{R})$ under investigation.

1.7. The element $P = P(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_q)$ is a polynomial in the variables x_{ij}, y_{ij} $(1 \le i \le q, 1 \le j \le n)$, where $x_{i1}, \cdots, x_{in}$ are the components of the covector α_i and $y_{i1}, \cdots, y_{in}$ components of the vector β_i. If P is homogeneous of degree m in the variables x_{ij}, then, as is clear from (3), dP is homogeneous of degree $m+1$ in the variables x_{ij}. Hence the complex $\{C^q, d^q\}$ splits into the direct sum of subcomplexes $\{C^q_k, d^q\}$ $(k = \cdots, -1, 0, 1, \cdots)$, where C^q_k is the subspace of C^q composed of the polynomials $P(x_{ij}, y_{ij})$ homogeneous of degree $q+k$ in the x_{ij}. Clearly this decomposition is invariant with respect to the action of the group $GL(n, \mathbf{R})$, and if $\alpha \in C^q_k$, $\beta \in C^r_l$, then $\alpha\beta \in C^{q+r}_{k+l}$. In particular, $\{C^q_0, d^q\}$ is a multiplicative subcomplex of the complex $\{C^q, d^q\}$, invariant with respect to the action of $GL(n, \mathbf{R})$.

1.8. *The inclusion of the complex $\{C^q_0, d^q\}$ in the complex $\{C^q, d^q\}$ induces an isomorphism of homology.*

To prove this it is sufficient to establish that the complexes $\{C^q_k, d^q\}$ with $k \neq 0$ have zero homology. But this follows from Proposition 6.2 of [2].

Remark. The cited Proposition 6.2 lets us select from the complex $\{C^q, d^q\}$ a smaller subcomplex with the same homology (the subcomplex of "elements of zero polydegree"), as done in [2]. But this subcomplex is inconvenient here since it is not invariant with respect to the action of $GL(n, \mathbf{R})$.

1.9. *If $0 < q < n$, then $H^q(W_n; \mathbf{R}) = 0$.*

This follows from Proposition 6.6 of [2]. Nevertheless, we give the proof.

Proof. For each element $Q \in C^q$ define the elements $\sigma_i Q \in C^q$ $(i = 1, \cdots, n)$ by the equality

$$\sigma_i Q(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_q)$$
$$= [Q(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_q)](\alpha_1 + \ldots + \alpha_q, e_i).$$

It is easy to show that if $Q_1, \cdots, Q_k \in C^q$ and $\sigma_{i_1} Q_1 + \cdots + \sigma_{i_k} Q_k = 0$ $(i_1, \cdots, i_k$ distinct), then Q_k can be put in the form $\sigma_{i_1} R_1 + \cdots + \sigma_{i_{k}-1} R_{k-1}$, where $R_1, \cdots, R_{k-1} \in C^q$ (this follows from the independence of the elements $x_{1i} + \cdots + x_{qi}$ $(i = 1, \cdots, n)$ in the ring of polynomials in $\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_q$). It is also clear that $d(\sigma_i Q) = \sigma_i dQ$, and if $Q \in C^q_k$, then $\sigma_i Q \in C^q_{k+1}$.

Let $P \in C^q_0$, $q < n$ and $dP = 0$. We show that $P \in \operatorname{Im} d$.

Since $d(\sigma_i P) = 0$ for all i and $\sigma_i P \in C^q_1$, there are elements $P_i \in C^{q-1}_1$ with

$dP_i = \sigma_i P$ (cf. 1.8). Similarly, for $s > 1$, elements $P_{i_1 \cdots i_s} \in C_s^{q+s}$ with $1 \le i_1 < \cdots < i_s \le n$ are defined (inductively with respect to s) by the equalities

$$dP_{i_1 \dots i_s} = \sum_{k=1}^{s} (-1)^{k-1} \sigma_{i_k} P_{i_1 \dots \hat{i}_k \dots i_s}. \tag{7}$$

Of course, these elements are not defined uniquely: if the elements $P_{j_1 \cdots j_{s-1}}$ for all sets $j_1 \cdots j_{s-1}$, have already been chosen, then $P_{i_1 \cdots i_s}$ is defined up to terms of the form dR with $R \in C_s^{q-s-1}$. Since $P_{1 \cdots q+1} \in C^{-1} = 0$, we have $P_{1 \cdots q+1} = 0$, and so

$$\sum_{k=1}^{q+1} (-1)^{k-1} \sigma_k P_{1 \dots \hat{k} \dots q+1} = 0.$$

Hence $P_{1 \cdots q} = \Sigma_{k=1}^{q} \sigma_k R_k$, where $R_k \in C^0$. Replace $P_{1 \cdots \hat{k} \cdots q}$ by $P_{1 \cdots \hat{k} \cdots q} + (-1)^k dR_k$. Since

$$\sum_{k=1}^{q} (-1)^{k-1} \sigma_k [P_{1 \dots \hat{k} \dots q} + (-1)^k dR_k] = d\left(P_{1 \dots q} - \sum_{k=1}^{q} \sigma_k R_k \right) = 0,$$

we can then take $P_{1 \cdots q}$ to be 0. Thus, the elements $P_{i_1 \cdots i_s}$ satisfying (7) can be chosen so that $P_{1 \cdots q} = 0$. Repeating the same argument we arrive at a system of elements $P_{i_1 \cdots i_s}$ satisfying conditions (7) and such that $P_{12} = 0$. Then $\sigma_1 P_2 - \sigma_2 P_1 = dP_{12} = 0$, so that $P_1 = \sigma_1 R$ for some $R \in C^q$ and $\sigma_1 (P - dR) = dP_1 - \sigma_1 dR = 0$. Hence $P = dR$.

§2. The filtration

The aim of this section is to define in the complex $\{C_0^q, d^q\}$ a filtration of which the corresponding spectral sequence will be used in what follows to compute the homology of the complex. [1)]

2.1. Each monomial in the polynomial $P = P(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_q) = P(x_{ij}, y_{ij}) \in C_0^q$ has degrees m_{ij} with respect to the variables x_{ij}. The numbers $m_i = \Sigma_{j=1}^{n} m_{ij}$ $(i = 1, \cdots, q)$ we shall call simply the *degrees* of the monomial. Clearly, $m_1, \cdots, m_q$ are nonnegative integers, and $m_1 + \cdots + m_q = q$.

We shall say that the element P has filtration $\ge u$ if among the degrees of each of its monomials there are at least u different from unity. The elements of filtration $\ge u$ form a subspace of the space C_0^q, denoted by $F^u C_0^q$. Clearly, $F^0 C_0^q = C_0^q$,

1) The filtration defined here does not differ essentially from the Serre-Hochschild filtration [5], constructed relative to the subalgebra $\mathfrak{gl}(n, \mathbf{R})$ of the algebra W_n (the imbedding of $\mathfrak{gl}(n, \mathbf{R})$ into W_n is induced in a natural manner by the imbedding of the group of linear transformations into the group of diffeomorphisms).

$F^u C_0^q \supset F^{u'} C_0^q$ for $u \le u'$, and the intersection $\bigcap_{u=0}^{\infty} F^u C_0^q$ consists of zero alone.

2.2. *The filtration F is compatible with the differential d.*

Indeed, in passing from the element P to the element dP, a monomial with degrees $m_1, \cdots, m_q$ is transformed, as is clear from (3), into a sum of monomials with degrees $m_2, \cdots, m_s, \alpha, m_{s+1}, \cdots, m_{t-1}, \beta, m_t, \cdots, m_q$, where $1 \le s < t \le q+1$, $\alpha + \beta = m_1 + 1$. The number of degrees different from unity is not reduced by this transformation: if $m_1 \neq 1$, then $m_1 + 1 \neq 2$, and so either $\alpha \neq 1$ or $\beta \neq 1$. Hence $d(F^u C_0^q) \subset F^u C_0^{q+1}$.

2.3. *The filtration F is multiplicative, i.e. if $P \in F^u C_0^q$, $Q \in F^{u'} C_0^{q'}$, then $PQ \in F^{u+u'} C_0^{q+q'}$.*

2.4. *The filtration F is compatible with the action of the group $GL(n, \mathbf{R})$, i.e., $g(F^u C_0^q) \subset F^u C_0^q$ for $g \in GL(n, \mathbf{R})$, $0 \le u < \infty$.*

2.5. Put $E_0^{u,v} = (F^u C_0^{u+v})/(F^{u+1} C_0^{u+v})$ and denote by $d_0^{u,v}$ the homomorphism $E_0^{u,v} \to E_0^{u,v+1}$ obtained by factorization from d^{u+v}. The bigraded complex $\{E_0^{u,v}, d_0^{u,v}\}$ is the zero term of the multiplicative spectral sequence $\{E_r^{u,v}, d_r^{u,v}\colon E_r^{u,v} \to E_r^{u+r,v-r+1}\}$, whose limit term is the ring associated with the homology ring of the complex $\{C_0^q, d^q\}$, i.e., with $H^*(W_n; \mathbf{R})$. The group $GL(n, \mathbf{R})$ and the algebra $\mathfrak{gl}(n, \mathbf{R})$ act on each of the spaces $E_r^{u,v}$, and the differentials $d_r^{u,v}$ are compatible with these actions (cf. 1.4, 2.4).

We turn now to the study of the spectral sequence $\{E_r^{u,v}, d_r^{u,v}\}$.

§3. The zero term in the spectral sequence

The results of this section can also be derived from §2 of [5]. We prefer, however, an independent derivation, since direct references would be difficult.

3.1. The space $E_0^{u,v}$ can be identified with the space of polynomials in $C_0^{u,v}$ comprised of those monomials that have v degrees equal to unity and u different from unity. Each such monomial $p(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_q)$ can be represented uniquely as a product $p'(\alpha_{i_1}, \cdots, \alpha_{i_v}; \beta_{i_1}, \cdots, \beta_{i_v}) \cdot p''(\alpha_{j_1}, \cdots, \alpha_{j_u}; \beta_{j_1}, \cdots, \beta_{j_u})$, in which the first factor is a multilinear function of all its arguments. If we split up in this way each of the monomials comprising the polynomial $P \in E_0^{u,v}$, and group terms obtained from each other by simultaneous interchange of α_i with α_j and β_i with β_j, we obtain a representation of the element P as a finite sum $\Sigma_k P'_k P''_k$, where $P'_k \in E_0^{0,v}$, $P''_k \in E_0^{u,0}$ (multiplication is in the sense of 1.4). Clearly, this representation defines an isomorphism

$$E_0^{u,v} = E_0^{0,v} \otimes E_0^{u,0}.$$

We emphasize that the ring structure in E_0 is compatible with this decomposition, i.e., if $P \in E_0^{0,v}$, $Q \in E_0^{u,0}$, then the product PQ is equal to the element $P \otimes Q \in E_0^{u,v}$.

3.2. The following proposition describes the "zero column" of the term E_0.

The complex $\{E_0^{0,q}, d_0^{0,q}\}$ is isomorphic with the standard cochain complex of the algebra $\mathfrak{gl}(n, \mathbf{R})$ with values in $\mathbf{R}$ and trivial action of the algebra.

Proof. By 3.1, $E_0^{0,q}$ is the space of functions of covectors $\alpha_1, \cdots, \alpha_q$ and vectors $\beta_1, \cdots, \beta_q$, which are linear in all their arguments and change sign under simultaneous interchange of α_i with α_j and β_i with β_j, i.e., $E_0^{0,q} = \Lambda^q(T \otimes T')$. The space $(T \otimes T')' = T' \otimes T$ can be identified with the space $\mathfrak{gl}(n, \mathbf{R})$ of linear operators in the space T (the element $\alpha \otimes \beta$ acts in T according to the formula $x \to \alpha(x)\beta$), and so $E_0^{0,q} = \Lambda^q(\mathfrak{gl}(n, \mathbf{R})')$. Let $P = P(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_q) \in E_0^{0,q}$; using the multilinearity of F, we can rewrite (3) as

$$dP(\alpha_1, \ldots, \alpha_{q+1}; \beta_1, \ldots, \beta_{q+1})$$
$$= \sum_{1 \leqslant s < t \leqslant q+1} (-1)^{s+t-1} [P(\alpha_s, \alpha_1, \ldots, \hat{\alpha}_s, \ldots, \hat{\alpha}_t, \ldots, \alpha_{q+1}; (\alpha_t, \beta_s)\beta_t, \beta_1, \ldots$$
$$\ldots, \hat{\beta}_s, \ldots, \hat{\beta}_t, \ldots, \beta_{q+1}) - P(\alpha_t, \alpha_1, \ldots, \hat{\alpha}_s, \ldots, \hat{\alpha}_t, \ldots, \alpha_{q+1};$$
$$(\alpha_s, \beta_t)\beta_s, \beta_1, \ldots, \hat{\beta}_s, \ldots, \hat{\beta}_t, \ldots, \beta_{q+1})]$$
$$+ \sum_{1 \leqslant s < t \leqslant q+1} (-1)^{s+t-1} [P(\alpha_t, \alpha_1, \ldots, \hat{\alpha}_s, \ldots, \hat{\alpha}_t, \ldots, \alpha_{q+1};$$
$$(\alpha_t, \beta_s)\beta_t, \beta_1, \ldots, \hat{\beta}_s, \ldots, \hat{\beta}_t, \ldots, \beta_{q+1}) - P(\alpha_s, \alpha_1, \ldots, \hat{\alpha}_s, \ldots, \hat{\alpha}_t, \ldots, \alpha_{q+1};$$
$$(\alpha_s, \beta_t)\beta_s, \beta_1, \ldots, \hat{\beta}_s, \ldots, \hat{\beta}_t, \ldots, \beta_{q+1})].$$

The first sum on the right-hand side is a linear function of all its arguments, and the second sum, by contrast, consists of terms, each of which is not linearly dependent on each of the covectors $\alpha_1, \cdots, \alpha_q$. Hence the second sum belongs to $F^1C_0^q$, and $d_0^{0,q}P$ coincides with the first sum. Since the commutator of the elements $\alpha \otimes \beta$, $\alpha' \otimes \beta' \in \mathfrak{gl}(n, \mathbf{R})$, where $\alpha, \alpha' \in T'$, $\beta, \beta' \in T$, is obviously equal to $(\alpha, \beta')\alpha' \otimes \beta - (\alpha', \beta)\alpha \otimes \beta'$, the differential $d_0^{0,q}$ is just the homomorphism $d\colon \Lambda^q(\mathfrak{gl}(n, \mathbf{R})') \to \Lambda^{q+1}(\mathfrak{gl}(n, \mathbf{R})')$, acting according to the formula

$$dP(\xi_1, \ldots, \xi_{q+1})$$
$$= \sum_{1 \leqslant s < t \leqslant q+1} (-1)^{s+t-1} P([\xi_s, \xi_t], \xi_1, \ldots, \hat{\xi}_s, \ldots, \hat{\xi}_t, \ldots, \xi_{q+1}),$$

i.e., it coincides with the differential of the standard complex of the algebra $\mathfrak{gl}(n, \mathbf{R})$.

3.3. Now we examine the action of the differential d_0 on the elements of the "zero row", i.e., on the elements of the spaces $E_0^{q,0}$. Let $\{A'_{kl}\}$ be the basis in $\mathfrak{gl}(n, \mathbf{R})' = T \otimes T' = E_0^{0,1}$ conjugate to the basis $\{A_{kl}\}$ in $\mathfrak{gl}(n, \mathbf{R}) = T' \otimes T$ (note that $A_{kl} = e'_l \otimes e_k$, $A'_{kl} = e_l \otimes e'_k$, where $\{e'_i\}$ is the basis in T' conjugate to the basis $\{e_i\}$ in T).

For any element $P \in E_0^{q,0}$ we have the equality

$$d_0^{q,0}P = -\sum\nolimits_{k,l} A'_{kl} \otimes (A_{kl}P) \tag{8}$$

(where $A_{kl}P$ is the image of P under the action of the element $A_{kl} \in \mathfrak{gl}(n, \mathbf{R})$; cf. 1.5).

Proof. By definition, $d_0^{q,0}P$ is the part of the polynomial dP comprised of monomials linearly dependent on one of the covectors α, i.e.,

$$(d_0^{q,0}P)(x_{ij}, y_{ij})$$

$$= \sum_{i=1}^{q+1} \sum_{l=1}^{n} x_{il} \left[\frac{\partial}{\partial x_{il}} dP(\alpha_1, \ldots, \alpha_{q+1}; \beta_1, \ldots, \beta_{q+1}) \right]_{\alpha_i = 0}$$

$$= \sum_{i=1}^{q+1} \sum_{l=1}^{n} \sum_{k=1}^{n} x_{il} y_{ik} \left[\frac{\partial}{\partial x_{il}} dP(\alpha_1, \ldots, \alpha_{q+1}; \beta_1, \ldots, \beta_{q+1}) \right]_{\alpha_i = 0,\ \beta_i = e_k}.$$

From this we see that

$$d_0^{q,0}P = \sum\nolimits_{k,l} A'_{k,l} \otimes P_{k,l},$$

where

$$P_{k,l} = (-1)^q \left[\frac{\partial}{\partial x_{q+1,l}} dP(\alpha_1, \ldots, \alpha_{q+1}; \beta_1, \ldots, \beta_q, e_k) \right]_{\alpha_{q+1} = 0}.$$

Replace dP in accordance with (3); in doing so we can drop the terms with $t \neq q+1$, since the degrees of the monomials in each of these terms in the components of the covector α_{q+1} are different from unity, and after differentiating with respect to $x_{q+1,l}$ and equating α_{q+1} to zero they become zero. We obtain:

$$P_{k,l} = \frac{\partial}{\partial x_{q+1,l}} \Big[\sum_{1 \leqslant s \leqslant q} (-1)^s P(\alpha_s + \alpha_{q+1}, \alpha_1, \ldots, \hat{\alpha}_s, \ldots$$

$$\ldots, \alpha_q; (\alpha_{q+1}, \beta_s) e_k - x_{s\kappa} \beta_s, \beta_1, \ldots, \hat{\beta}_s, \ldots, \beta_q) \Big]_{\alpha_{q+1} = 0}$$

$$= \sum_{1 \leqslant s \leqslant q} \Big[x_{sq} \frac{\partial}{\partial x_{sl}} P(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_q)$$

$$- P(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_{s-1}, y_{sl} e_k, \beta_{s+1}, \ldots, \beta_q) \Big].$$

It remains only to observe that $y_{sl} e_k = A_{kl} \beta_s$ and use formula (6) of 1.4.

3.4. From Propositions 3.2 and 3.3 we can obtain a complete description of the term E_0.

For each u and v there is a canonical isomorphism of the space $E_0^{u,v} = E_0^{0,v} \otimes E_0^{u,0}$ onto the space $\mathrm{Hom}((E_0^{0,v})', E_0^{u,0}) = \mathrm{Hom}(\Lambda^v(\mathfrak{gl}(n, \mathbf{R})), E_0^{u,0})$, i.e., a canonical isomorphism $E_0^{u,v} \to C^v(\mathfrak{gl}(n, \mathbf{R}); E_0^{u,0})$ (cf. 1.1). We denote it by $\eta^{u,v}$.

The isomorphisms $\eta^{u,v}$ define for each u isomorphisms of the complex $\{E_0^{u,v}, d_0^{u,v}\}_{0 \leq v < \infty}$ onto the complex $\{C^v(\mathfrak{gl}(n, \mathbf{R}); E_0^{u,0}), d^v(\mathfrak{gl}(n, \mathbf{R}); E_0^{u,0})\}$.

Proof. We must prove that for any elements $P \in E_0^{0,v}$, $Q \in E_0^{u,0}$ the element $P \otimes Q \in E_0^{u,v}$ has identical images under the homomorphisms

$$\eta^{u,v+1} \circ d_0^{u,v}, \quad [d^v(\mathfrak{gl}(n, \mathbf{R}); E_0^{u,0})] \circ \eta^{u,v} : E_0^{u,v} \to C^{v+1}(\mathfrak{gl}(n, \mathbf{R}); E_0^{u,0})$$

(for $u = 0$ this has already been proved; cf. 3.2). By the definition of the homomorphism $\eta^{u,v}$, the value of the cochain $\eta^{u,v}(P \otimes Q) \in C^v(\mathfrak{gl}(n, \mathbf{R}); E_0^{u,}\ )$ on the elements $\xi_1, \cdots, \xi_v \in \mathfrak{gl}(n, \mathbf{R})$ is equal to $[(\eta^{0,v}P)(\xi_1, \cdots, \xi_v)]Q \in E_0^{u,0}$. Hence for any $\xi_1, \cdots, \xi_{v+1} \in \mathfrak{gl}(n, \mathbf{R})$ we have:

$$\{[d^v(\mathfrak{gl}(n, \mathbf{R}); E_0^{u,0})](\eta^{u,v}(P \otimes Q))\}(\xi_1, \ldots, \xi_{v+1})$$

$$= \sum_{1 \leqslant s < t \leqslant v+1} (-1)^{s+t-1} [(\eta^{0,v}P)([\xi_s, \xi_t], \xi_1, \ldots, \hat{\xi}_s, \ldots, \hat{\xi}_t, \ldots, \xi_{v+1})] Q$$

$$+ \sum_{1 \leqslant s \leqslant v+1} (-1)^s [(\eta^{0,v}P)(\xi_1, \ldots, \hat{\xi}_s, \ldots, \xi_{v+1})](\xi_s Q);$$

$$[\eta^{u,v+1}(d_0^{u,v}(P \otimes Q))](\xi_1, \ldots, \xi_{v+1})$$

$$= [\eta^{u,v+1}(d_0^{0,v}P \otimes Q + (-1)^v P \otimes d_0^{u,0}Q)](\xi_1, \ldots, \xi_{v+1})$$

$$= \sum_{1 \leqslant s < t \leqslant v+1} (-1)^{s+t-1} [(\eta^{0,v}P)([\xi_s, \xi_t], \xi_1, \ldots, \hat{\xi}_s, \ldots, \hat{\xi}_t, \ldots, \xi_{v+1})] Q$$

$$- \sum_{1 \leqslant s \leqslant v+1} (-1)^v (-1)^{v-s} \textstyle\sum_{k,l} (\eta^{0,v}P)(\xi_1, \ldots, \hat{\xi}_s, \ldots, \xi_{v+1}) A'_{kl}(\xi_s) A_{kl} Q.$$

It remains only to observe that since the bases $\{A'_{kl}\}$, $\{A_{kl}\}$ are conjugate, we have

$$\sum_{k,l} A'_{k,l}(\xi_s) A_{k,l} = \xi_s.$$

3.5. *For all u and v the space $E_1^{u,v}$ is canonically isomorphic to the space $H^v(\mathfrak{gl}(n, \mathbf{R}); E_0^{u,0})$ of cohomology of dimension v of the Lie algebra $\mathfrak{gl}(n, \mathbf{R})$ with coefficients in the $\mathfrak{gl}(n, \mathbf{R})$-module $E_0^{u,0}$.*

This follows from 3.4.

§4. The first term of the spectral sequence

The problem of describing the term E_1 of our spectral sequence has thus been reduced to that of examining the cohomology of the Lie algebra $\mathfrak{gl}(n, \mathbf{R})$ with coefficients in various representations; this is the subject of the present section.

We start with two well-known facts from the theory of Lie algebras.

4.1. Let M be an arbitrary finite-dimensional $\mathfrak{gl}(n, \mathbf{R})$-module. Denote by M_0 the subspace of M consisting of those elements m such that $Am = 0$ for any matrix $A \in \mathfrak{gl}(n, \mathbf{R})$ with $\operatorname{Tr} A = 0$. Obviously M_0 is a submodule of the $\mathfrak{gl}(n, \mathbf{R})$-module M: if $m \in M_0$ and $B \in \mathfrak{gl}(n, \mathbf{R})$, then for any matrix $A \in \mathfrak{gl}(n, \mathbf{R})$ with $\operatorname{Tr} A = 0$ we have the equality $ABm = A(B - (\operatorname{Tr} B)E)m + (\operatorname{Tr} B)Am = 0$, so that $Bm \in M_0$.

The inclusion $M_0 \to M$ induces an isomorphism $H^*(\mathfrak{gl}(n, \mathbf{R}), M_0) = H^*(\mathfrak{gl}(n, \mathbf{R}); M)$.

Proof. The direct decomposition $\mathfrak{gl}(n, \mathbf{R}) = \mathbf{R} \oplus \mathfrak{sl}(n, \mathbf{R})$ induces canonical isomorphisms $H^*(\mathfrak{gl}(n, \mathbf{R}); M_0) = H^*(\mathbf{R}; H^*(\mathfrak{sl}(n, \mathbf{R}); M_0))$, and $H^*(\mathfrak{gl}(n, \mathbf{R}); M) = H^*(\mathbf{R}; H^*(\mathfrak{sl}(n, \mathbf{R}; M))$; the homomorphism

$$H^*(\mathfrak{sl}(n, \mathbf{R}); M_0) \to H^*(\mathfrak{sl}(n, \mathbf{R}); M),$$

induced by the inclusion $M_0 \to M$ is likewise an isomorphism, in view of the simplicity of the algebra $\mathfrak{sl}(n, \mathbf{R})$ ([4], Russian p. 84).

4.2. *There exists a ring isomorphism* $H^*(\mathfrak{gl}(n, R); R) = H^*(U(n); R)$. *In other words, there exist elements* $\phi_i = \phi_i(n) \in H^*(\mathfrak{gl}(n,R); R)$, $i = 1, \cdots, n$, *such that* $\dim \phi_i(n) = 2i-1$ *and* $H^*(\mathfrak{gl}(n, R); R)$ *is the exterior algebra in* $\phi_1(n), \cdots, \phi_n(n)$ *over* R ([6], *Russian* p. 191). *The generators* $\phi_i(n)$ *can be chosen so that the cohomology isomorphism induced by the usual inclusion* $\mathfrak{gl}(n, R) \to \mathfrak{gl}(m, R)$ $(m \geq n)$ *takes* $\phi_i(m)$ *into* $\phi_i(n)$ *for* $i \leq n$ *and into zero for* $i > n$.

Proof. The complexifications of the Lie algebras $\mathfrak{gl}(n, R)$ and $\mathfrak{u}(n)$ are canonically isomorphic (both coincide with $\mathfrak{gl}(n, C)$), so that $H^*(\mathfrak{gl}(n, R); R) = H^*(\mathfrak{u}(n); R)$. In turn, a canonical isomorphism $H^*(\mathfrak{u}(n); R) \to H^*(U(n); R)$ is induced, since $U(n)$ is compact, by the natural imbedding of the standard complex of the algebra $\mathfrak{u}(n)$ into the de Rham complex of the group $U(n)$. The last assertion of the proposition is obvious.

Remark. For the generators $\phi_i(n)$ of the exterior algebra $H^*(\mathfrak{gl}(n, R); R)$ we can take the cohomology classes of the cocycles $\Phi_i \in C^{2i-1}(\mathfrak{gl}(n, R); R)$, where

$$\Phi_i(\xi_1, \ldots, \xi_{2i-1}) = \sum \varepsilon(k_1, \ldots, k_{2i-1}) \operatorname{Tr}(\xi_{k_1}, \ldots, \xi_{k_{2i-1}}),$$

with the summation over all permutations $(k_1, \cdots, k_{2i-1})$ of the numbers $1, \cdots, 2i-1$, and $\epsilon(k_1, \cdots, k_{2i-1}) = 1$ or -1 according to the parity of the permutation $(k_1, \cdots, k_{2i-1})$. This observation will not be used, however, in what follows.

4.3. We proceed to examination of the term E_1. Let K^u be the subspace of $E_0^{u,0}$ consisting of those elements on which the action of the algebra $\mathfrak{sl}(n, R)$ is trivial. Clearly $K^* = \Sigma_{u=0}^{\infty} K^u$ is a subring of the ring $E_0^{*,0} = \Sigma_{u=0}^{\infty} E_0^{u,0}$.

We construct *a canonical isomorphism of the bigraded rings* $E_1 = \Sigma_{u,v} E_1^{u,v}$ *and* $K^* \otimes H^*(U(n); R)$.

Observe first of all that the algebra $\mathfrak{gl}(n, R)$ also acts trivially on K^*. This follows from the fact that the equality $e(P) = 0$, where $e \in \mathfrak{gl}(n, R)$ is the identity matrix, holds for all $P \in C_0^q$ (which in turn follows easily from (6), using the fact that the degree of each monomial in the polynomial P for the totality of the variables x_{ij} is equal to q). By 4.1, the inclusion $K^u \to E_0^{u,0}$ induces an isomorphism $H^*(\mathfrak{gl}(n, R); K^u) = H^*(\mathfrak{gl}(n, R); E_0^{u,0})$. As for the equality $H^*(\mathfrak{gl}(n, R); K^*) = K^* \otimes H^*(U(n); R)$, it follows, since $\mathfrak{gl}(n, R)$ acts trivially on K^*, from 4.2.

4.4. *Every element of* C_0^q *invariant with respect to the action of the group* $SL(n, R)$ *is a linear combination of polynomials in* $\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_q$ *of the form*

$$(\alpha_{s_1}, \beta_1) \ldots (\alpha_{s_q}, \beta_q) \tag{9}$$

(*equality allowed among the indices* $s_1, \cdots, s_q$). *In particular, this holds for the elements of the space* K^*.

Proof. Since the degrees of the monomials (see 2.1) are invariant with respect to the action of the group $SL(n, R)$, it suffices to prove this for invariant cochains $P \in C_0^q$ that are homogeneous in the sense that all the monomials of the polynomial P have, up to order, the same degree $m_1, \cdots, m_q$. More than that, it suffices to prove that we already have a sum of polynomials of the form (9) in the polynomial $P_0(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_q)$, comprised of those monomials of P that have degree m_1 in α_1, degree m_2 in $\alpha_2, \cdots$, degree m_q in α_q, where $m_1 \geq m_2 \geq \cdots \geq m_q$.

The polynomial P_0 can be regarded as a tensor. More precisely, there exists a unique tensor

$$\tau = \tau(\gamma_1, \ldots, \gamma_q; \beta_1, \ldots, \beta_q) \in [(\otimes^q T') \otimes (\otimes^q T)]'$$

such that

$$P(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_q) = \tau(\underbrace{\alpha_1, \ldots, \alpha_1}_{m_1}, \ldots, \underbrace{\alpha_q, \ldots, \alpha_q}_{m_q}; \beta_1, \ldots, \beta_q) \tag{10}$$

and symmetric with respect to the arguments $\gamma_{m_1+\cdots+m_{s-1}+1}, \cdots, \gamma_{m_1+\cdots+m_s}$ for $1 \leq s \leq q$. Clearly the tensor τ is also invariant with respect to the group $SL(n, \mathrm{R})$.

By a classical theorem of the theory of invariants, the algebra of tensors invariant with respect to the group $SL(n, \mathrm{R})$ is generated by tensors of the form:

1) (α, β), where $\alpha \in T'$, $\beta \in T$;

2) $\det(\alpha_1, \cdots, \alpha_n)$, where $\alpha_1, \cdots, \alpha_n \in T'$;

3) $\det(\beta_1, \cdots, \beta_n)$, where $\beta_1, \cdots, \beta_n \in T$ ([7], Theorem II 6.A).

Since $\det(\alpha_1, \cdots, \alpha_n) \cdot \det(\beta_1, \cdots, \beta_n) = \det \|(\alpha_i, \beta_j)\|$, invariant tensors depending on the same number of vectors and covectors are generated by generators of only the first form. It follows that τ is a linear combination of tensors of the form $(\alpha_{i_1}, \beta_1) \cdots (\alpha_{i_q}, \beta_q)$ (where the indices $i_1, \cdots, i_q$ are distinct), and application of (10) yields our assertion.

4.5. *Let ψ_k be the element of $E_0^{2k,0}$ defined by the equality*

$$\psi_k(\alpha_1, \ldots, \alpha_{2k}; \beta_1, \ldots, \beta_{2k})$$
$$= \Sigma_{(i_1,\ldots,i_{2k})} \varepsilon(i_1, \ldots, i_{2k})(\alpha_{i_k}, \beta_{i_1})(\alpha_{i_1}, \beta_{i_2}) \ldots (\alpha_{i_{k-1}}, \beta_{i_k})$$
$$\times (\alpha_{i_1}, \beta_{i_{k+1}}) \ldots (\alpha_{i_k}, \beta_{i_{2k}})$$

(summation over all permutations $(i_1, \cdots, i_{2k})$ of the numbers $1, 2, \cdots, 2k$). Then the algebra K^* is generated by the elements $\psi_1, \cdots, \psi_n$. If $s_1 + 2s_2 + \cdots + ns_n > n$, then $\psi_1^{s_1}\psi_2^{s_2} \cdots \psi_n^{s_n} = 0$; and these relations constitute a basis in the space of relations between monomials in $\psi_1, \cdots, \psi_n$.

Proof. *Take $P \in K^q$.* By 4.4 we have:

$$P(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_q) = \sum_{s_1,\ldots,s_q} a_{s_1,\ldots,s_q}(\alpha_{s_1}, \beta_1) \ldots (\alpha_{s_q}, \beta_q).$$

Suppose a coefficient $a_{s_1,\cdots,s_q}$ is different from zero. Since $P \in E_0^{q,0}$, i.e., since the filtration of P is equal to q, the expression $a_{s_1,\dots,s_q}(\alpha_{s_1}, \beta_1)\cdots(\alpha_{s_q}, \beta_q)$ is not linear in any α, and therefore among the numbers $s_1, \cdots, s_q$ none is encountered only once; in particular, among them at most $[q/2]$ are distinct. Let $j_1, \cdots, j_r$ be all the natural numbers not exceeding q and not encountered among $s_1, \cdots, s_q$. By the preceding remark we have $r \geq q/2$. But the numbers $s_{j_1}, \cdots, s_{j_r}$ are distinct: if $s_{j_k} = s_{j_l}$, then the product $(\alpha_{s_1}, \beta_1)\cdots(\alpha_{s_q}, \beta_q)$ goes into itself under simultaneous interchange of α_{j_k} with α_{j_l} and β_{j_k} with β_{j_l}, so that $a_{s_1\cdots s_q} = -a_{s_1\cdots s_q}$; consequently, we have also $r \leq q/2$; i.e., $r = q/2$. We conclude that q is even, and that among the indices $s_1, \cdots, s_q$ exactly half the numbers $1, \cdots, q$ are not encountered at all, and the rest exactly twice each. Using skew-symmetry, we find that P is a linear combination of polynomials

$$P_\tau(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_q)$$
$$= \Sigma_{(i_1\ldots i_q)}\varepsilon(i_1\ldots i_q)(\alpha_{\tau(i_1)}, \beta_{i_1})\ldots(\alpha_{\tau(i_r)}, \beta_{i_r})(\alpha_{i_1}, \beta_{i_{r+1}})\ldots(\alpha_{i_r}, \beta_{i_{2r}}), \tag{11}$$

where $r = q/2$ and τ is a permutation of the indices $i_1, \cdots, i_r$. Since every permutation factors into a product of cyclic permutations, the element P_τ is representable as a sum of products of the elements ψ_k.

The part of the sum (11) corresponding to fixed values of the indices $i_1, \cdots, i_r$ is multilinear and skew-symmetric in $\beta_{i_{r+1}}, \cdots, \beta_{i_{2r}}$; that is to say, in r vectors of an n-dimensional space. Hence if $r = q/2 > n$, then $P_\tau = 0$. This means that $\psi_1^{s_1}\psi_2^{s_2}\cdots\cdots\psi_N^{s_N} = 0$ for $s_1 + 2s_2 + \cdots + Ns_N > n$, and in particular, $\psi_i = 0$ for $i > n$.

Finally, for $r \leq n$ the elements P_τ are linearly independent. Indeed, if $\Sigma a_\tau P_\tau = 0$, then putting $\alpha_1 = e_1, \cdots, \alpha_r = e_r, \alpha_{r+1} = \cdots = \alpha_{2r} = 0, \beta_1 = e_{\tau(1)}, \cdots, \beta_r = e_{\tau(r)}, \beta_{r+1} = e_1, \cdots, \beta_{2r} = e_r$, we obtain that $r! \cdot a_\tau = 0$.

4.6. Proposition 4.5 together with the isomorphism 4.3 allows us to give a complete description of the term E_1.

The bigraded ring E_1 *is isomorphic to the tensor product* $\Lambda(\phi_1, \cdots, \phi_n) \otimes \{\mathrm{R}[\psi_1, \cdots, \psi_n]/I\}$, *where* $\phi_k \in E_1^{0,2k-1}$, $\psi_k \in E_1^{2k,0}$, *and* I *is the ideal in the ring* $\mathrm{R}[\psi_1, \cdots, \psi_n]$ *generated by the monomials* $\psi_1^{s_1}\psi_2^{s_2}\cdots\psi_n^{s_n}$ *with* $s_1 + 2s_1 + \cdots \cdots + ns_n > n$.

4.7. As follows from 4.6, $E_1^{u,v} = 0$ for odd u. Since $d_1^{u,v}$ is a homomorphism from the space $E_1^{u,v}$ to the space $E_1^{u+1,v}$, it follows that $d_1 \equiv 0$. We arrive therefore at the following proposition.

The term E_2 *of our spectral sequence is isomorphic to the term* E_1.

§5. The remaining terms of the spectral sequence

Throughout this section we shall denote by the same letter an element of $\operatorname{Ker} d_r \subset E_r$ and the element it determines in E_{r+1}. For example, if $\alpha \in E_2$ and $d_2\alpha = 0$, there is defined an element $d_3\alpha \in E_3$; if $d_3\alpha = 0$, there is defined an element $d_4\alpha \in E_4$; etc.

5.1. *If* $1 \leq k \leq n$, *then* $d_i^{0,2k-1}\phi_k = 0$ *for* $i = 2, \cdots, 2k-1$, *and* $d_{2k}^{0,2k-1}\phi_k = \psi_k$.

Proof. The inclusion $W_n \to W_m$ induces a homomorphism of the spectral sequence $\{'E_r^{u,v}, 'd_r^{u,v}\}$, corresponding to the algebra W_m into the given spectral sequence $\{E_r^{u,v}, d_r^{u,v}\}$. The ring $'E_r$ is generated by generators $\phi_1', \cdots, \phi_m', \psi_1', \cdots, \psi_m'$ analogous to $\phi_1, \cdots, \phi_n, \psi_1, \cdots, \psi_n$, and the homomorphism $'E_r \to E_r$ takes $\phi_1', \cdots, \phi_n', \psi_1', \cdots, \psi_n'$ into $\phi_1, \cdots, \phi_n, \psi_1, \cdots, \psi_n$ (for ψ this is obvious from the definition in 4.5, and for ϕ it follows from Proposition 4.2). By Proposition 1.9, $H^q(W_m; \mathbf{R}) = 0$ for $0 < q < m$ and therefore $'E_\infty^{u,v} = 0$ for $0 < u + v < m$. Now apply to the spectral sequence $\{'E, 'd\}$ the theorem of Borel ([6], Theorem 13.1). We obtain that if m is sufficiently large, then $'d_i^{0,2k-1}\phi_k' = 0$ and $'d_{2k}^{0,2k-1}\phi_k' = \psi_k'$ for $i = 2, \cdots, 2k-1$ and $k = 1, \cdots, n$. This implies our assertion.

5.2. Proposition 5.1 and the formulas connecting the differential of the spectral sequence with multiplication allow us to give a complete description of our spectral sequence. We obtain the following.

Consider an element $P = \phi_{i_1}, \phi_{i_k}\psi_{j_1}^{s_1} \cdots \psi_{j_l}^{s_l} \in E_2^{2(j_1 s_1 + \cdots + j_l s_l),\, 2(i_1 + \cdots + i_k) - k}$, *where* $i_1 < \cdots < i_k$, $j_1 < \cdots < j_l$ *and all the* s_i *are different from zero. If* $k = 0$, *or* $i_1 > j_1$, *or* $i_1 + j_1 s_1 + \cdots + j_l s_l > n$, *then* $d_r P = 0$ *for all* r. *Otherwise,* $d_r P = 0$ *for* $r > 2i_1$, *and* $d_{2i_1} P = \phi_{i_2} \cdots \phi_{i_k}\psi_{i_1}\psi_{j_1}^{s_1} \cdots \psi_{j_l}^{s_l}$.

This proposition yields an algorithm for determining the spaces $E_\infty^{u,v}$, and thereby the dimensions of the spaces $H^q(W_n; \mathbf{R})$.

5.3. *If* $u \leq n$ *and* $u + v > 0$, *then* $E_\infty^{u,v} = 0$. *If* $0 < u + v \leq 2n$, *then* $E_\infty^{u,v} = 0$.

Proof. Take $P = \phi_{i_1} \cdots \phi_{i_k}\psi_{j_1}^{s_1} \cdots \psi_{j_l}^{s_l} \in E_2^{u,v}$, and suppose $u + v > 0$ and either $u \leq n$ or $u + v \leq 2n$. There are two possibilities. The first is that $i_1 > j_1$ (or $k = 0$), and then $2j_1 \leq n$; in this case P is the image of the element $\phi_{j_1}\phi_{i_1} \cdots$ $\cdots \phi_{i_k}\psi_{j_1}^{s_1-1}\psi_{j_2}^{s_2} \cdots \psi_{j_l}^{s_l} \in E_2$ under the differential $d_2^{u-2j_1, v+2j_1-1}$. The second possibility is that $i_1 \leq j_1$ (or $l = 0$), and then $i_1 + u/2 \leq n$; in this case $d_{2i_1} P \neq 0$. In either case, P does not remain in E_∞.

5.4. **Corollary.** *If* $0 < q \leq 2n$, *then* $H^q(W_n; \mathbf{R}) = 0$.

5.5. It is easy to find the first nontrivial cohomology space for the algebra W_n. *This is the space* $H^{2n+1}(W_n; \mathbf{R})$; *its dimension* ρ_n *is one less than the number of representations of the integer* $n+1$ *as a sum of nonnegative integers (representations differing only in the order of summands being regarded as the same).* The proof is left to the reader.

5.6. *Multiplication in the ring* $H^*(W_n; \mathbf{R})$ *is trivial.*

Proof. Proposition 5.3 implies that every element of $H^*(W_n, \mathbf{R})$, of positive dimension is represented by a cocycle of filtration $> n$, and therefore the product of two such elements is represented by a cocycle of filtration $> 2n$. Since $E_\infty^{u,v} = 0$ for $u > 2n$, the product is zero.

5.7. We can now prove Theorem 0.3 of the Introduction. The second term of the real cohomology spectral sequence of the bundle $(X_n, p, (G(N, n))_{2n})$ described in the Introduction is $H^*((G(N, n))_{2n}; \mathbf{R}) \otimes H^*(U(n); \mathbf{R})$. From Theorem 19.6 of [6] it follows that this tensor product is isomorphic as a bigraded algebra to the term E_2 of our spectral sequence, and also that in the two spectral sequences the actions of the differentials are the same. This means that the limits of the spectral sequences are the same, and thus $H^q(W_n; \mathbf{R}) = H^q(X_n, \mathbf{R})$ for all q. This equality together with 5.6 is the content of Theorem 0.3.

5.8. We note in conclusion that Proposition 5.1 can be proved by direct computation of differentials, without reference to Proposition 1.9. For this one can use the Remark in §4.2.

§6. Applications to the Lie algebras of smooth vector fields on manifolds

6.1. Let $\mathfrak{A}(M)$ be the Lie algebra of vector fields of class C^∞ on a compact orientable n-dimensional manifold M of class C^∞. In the cochain complex $\mathcal{C} = \mathcal{C}(M)$ of this algebra with values in $\mathbf{R}$ consider the subcomplex $\mathcal{C}_d = \mathcal{C}_d(M)$ consisting of those cochains $P \in \mathcal{C}^q$ such that whenever the supports of the vector fields $\xi_1, \cdots, \xi_q$ have (taken together) void intersection, then $P(\xi_1, \cdots, \xi_q) = 0$. In [2] we called this subcomplex diagonal. To compute its cohomology (to which in a certain sense we reduced the problem of finding the cohomology of the algebra $\mathfrak{A}(M)$) a spectral sequence was constructed. The main proposition concerning this spectral sequence ([2], Proposition 7.5) can be formulated as follows, using the results of the present paper (viz., Theorem 0.3 and Propositions 1.6 and 5.4):

There exists a spectral sequence $\{E_r^{p,q}, d_r^{p,q}\}$ *which converges to the cohomology of the complex* $\mathcal{C}_d$, *such that* $E_2^{p,q} = H_{-p}(M; \mathbf{R}) \otimes H^q(X_n; \mathbf{R})$. *In particular,* $E_2^{p,q} = 0$ *for* $0 < p + q \leq n$.

A similar sharpening can be made of Proposition 8.5 of [2]. Without recalling the relevent definitions, we remark only that for the spaces ${}^{(k)}E_2^{p,q}$ considered there it follows from the results of the present paper that ${}^{(k)}E_2^{p,q} = 0$ for $0 < u + v \leq kn + k - 1$.

6.2. *The first nontrivial cohomology space for the diagonal complex is* $H_{n+1}(\mathcal{C}_d)$. *Its dimension (over* $\mathbf{R}$*) is* ρ_n (see 5.5).

This follows from the preceding remarks and §5.5.

6.3. *The q-dimensional cohomology space of the Lie algebra of smooth vector fields of a compact orientable n-dimensional manifold M coincides for $q \leq 2n$ with the q-dimensional cohomology space of the complex $\mathcal{C}_d(M)$. In particular,* $H^q(\mathfrak{A}(M); \mathbf{R}) = 0$ *for* $q \leq n$, *and* $\dim H^{n+1}(\mathfrak{A}(M); \mathbf{R}) = \rho_n$.

This follows from 6.2 and Proposition 8.5 of [2].

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