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The holomorphic σ-model and its symmetries

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ABSTRACT

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This is the abstract.

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Preface

This is the preface.

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CHAPTER 1

Holomorphic quantum field theory

Our main objective in this chapter is two-fold. First we will define the concept of a holomorphic field theory and set up notation and terminology that we will use throughout the text. Our next goal is more technical, but will provide the backbone for much of the analysis throughout the remainder of this thesis. We will show how certain holomorphic theories are surprisingly well-behaved when it comes to the problem of renormalization.

In [**Cos11**] Costello has provided a mathematical formulation of the Wilsonian approach to quantum field theory. The main takeaway is that to construct a full quantum field theory it suffices to define the theory at each energy (or length) scale and to ask that these descriptions be compatible as we vary the scale. The infamous infinities of quantum field theory arise due to studying behavior of theories at arbitrarily high energies (or small lengths). In physics this is called the ultra-violet (UV) divergence. Classically, a theory is defined by a *local functional* I^{cl} which is a functional on the space of fields obtained by integrating a Lagrangian density. At each scale L a theory is defined by an action functional $I[L]$, which is a function on the space of fields.

Summarizing, there are two main steps to construct a QFT in our formalism.

Renormalization: For each scale L and regulator $\epsilon > 0$ consider the RG flow from scale ϵ to L:

$$W(P_{\epsilon<L}, I). \tag{1.1}$$

In general, the limit $\epsilon \to 0$ will not be defined, but by Costello's main result there exists counterterms $I^{CT}(\epsilon)$ such that the $\epsilon \to 0$ limit of

$$W(P_{\epsilon<L}, I - I^{CT}(\epsilon))$$

is well-defined. Denote this limit by $I[L]$. The family $\{I[L]\}$ defines a prequantization.

Gauge consistency: We then ask if the family $\{I[L]\}$ defines a consistent quantization. For each L we require that $I[L]$ satisfy the scale L quantum master equation....

In this section we are concerned with the first step: renormalization. The complication here is that even very natural field theories can have a very complicated collections of counterterms. For instance, the naive quantization of Chern-Simons theory on a three-manifold has counterterms even at one-loop. For holomorphic theories, however, we will show how the situation becomes much simpler at least at the level of one-loop.

Lemma 1.0.1. *Let $\mathscr{E}$ be a holomorphic theory on $\mathbb{C}^d$ with classical interaction I^{cl}. Then, there exists a one-loop prequantization $\{I[L] \mid L > 0\}$ of I^{cl} involving no counterterms. That is, we can find a propagator $P_{\epsilon<L}$ for which the $\epsilon \to 0$ limit of*

$$W(P_{\epsilon<L}, I) \mod \hbar^2$$

exisits. Moreover, if I is holomorphically translation invariant we can pick the family $\{I[L]\}$ to be holomorphically translation invariant as well.

We will use this result repeatedly throughout this thesis. This result tells us that the analytic difficulties are manageable and that the key focus will be on obstructions to satisfying the quantum master equation. In particular, a corollary of this result will give us a procedure for computing the one-loop obstruction explicitly in terms of Feynman diagrams. We conjecture that an extension of this result should hold to all orders in $\hbar$ which would give a constructive way of analyzing the obstruction theory order by order in $\hbar$. Nevertheless, we will leverage the one-loop behavior to formulate and prove index theorems in the context of holomorphic QFT.

One surprising aspect of this comes from thinking about holomorphic theories in a different way. Any supercharge Q of a supersymmetric theory satisfying $Q^2 = 0$ allows one to construct a "twist". In some cases, where Clifford multiplication with Q spans all translations such a twist becomes a topological theory (in the weak sense). In any case, however, such a Q defines a "holomorphic twist", which results in the type of holomorphic theories we consider. Regularization in supersymmetric theories, especially gauge theories, is notoriously difficult. Our result implies that after twisting the analytic difficulties become much easier to deal with. Consequently, phenomena such as anomalies can be cast in a more algebraic framework. We will see such an example of this in the case of the holomorphic σ-model in the next chapter.

Already, in [**Li**] Li has used a complex one-dimensional version of this fact to all orders in $\hbar$. He uses this to give an elegant interpretation of the quantum master equation for two-dimensional chiral conformal field theories using vertex algebras. We do not make any statements in this work past one-loop quantizations, but the higher loop behavior remains a very interesting and subtle problem that we hope to return to.

1.1. The definition of a quantum field theory

The goal of this section is to review the classical and quantum Batalin-Vilkovisky formalisms. We will also set up the requisite conventions and notations that we will use throughout the thesis.

1.1.1. Classical field theory

Classical field theory is a formalism for describing a physical system in terms of objects called *fields*. Mathematically, the space of fields is a (most often infinite dimensional) vector space $\mathscr{E}$. Classical physics is described by the critical locus of a (usually real or complex valued) linear functional on the space of fields

$$S : \mathscr{E} \to \mathbb{R} \ \text{ or } \ \mathbb{C}, \tag{1.2}$$

called the *action functional*. The critical locus is the locus of fields that have zero variation

$$\mathrm{Crit}(S) := \{\varphi \in \mathscr{E} \mid \mathrm{d}S(\varphi) = 0\}. \tag{1.3}$$

A field φ satisfying the equation $\mathrm{d}S(\varphi) = 0$ is said to be a *solution to the classical equations of motion*.

Even in the finite dimensional case, if the functional S is not sufficiently well-behaved the critical locus can be still be highly singular. The starting point of the *classical Batalin-Vilkovisky* formalism is to instead consider the *derived* critical locus. To get a feel for this, we review the finite dimensional situation. Let M be a manifold, which is our ansatz for $\mathscr{E}$ at the moment, and suppose $S : M \to \mathbb{R}$ is a smooth map. The critical locus is the intersection of the graph of $\mathrm{d}S$ in T^*M with the zero section $0 : M \to T^*M$. Thus, functions on the critical locus are of the form

$$\mathscr{O}(\mathrm{Crit}(S)) = \mathscr{O}(\Gamma(\mathrm{d}S)) \otimes_{\mathscr{O}(T^*M)} \mathscr{O}(M).$$

The derived critical locus is a derived space whose dg ring of functions is

$$\mathscr{O}(\mathrm{Crit}^h(S)) = \mathscr{O}(\Gamma(\mathrm{d}S)) \otimes^{\mathbb{L}}_{\mathscr{O}(T^*M)} \mathscr{O}(M).$$

We have replaced the strict tensor product with the derived one. Using the Koszul resolution of $\mathscr{O}(M)$ as a $\mathscr{O}(T^*M)$-module one can write this derived tensor product as a complex of polyvector fields equipped with some differential:

$$\mathscr{O}(\mathrm{Crit}^h(S)) \simeq \left(\mathrm{PV}^{-*}(M), \iota_{\mathrm{d}S}\right).$$

In cohomological degree $-i$ we have $\mathrm{PV}^{-i}(M) = \Gamma(M, \wedge^i TM)$ and $\iota_{\mathrm{d}S}$ denotes contraction with the one-form $\mathrm{d}S$ (which raises cohomological degree with our regrading convention). With our grading convention we have $\mathscr{O}(T^*[-1]M) = \mathrm{PV}^{-*}(M)$. The space $\mathscr{O}(T^*[-1]M)$ has natural shifted Poisson structure, which takes the form of the familiar Schouten-Nijenhuis bracket of polyvector fields.

The takeaway is that the derived critical locus of a functional $S : M \to \mathbb{R}$ has the structure of a (-1)-symplectic space. This will be the starting point for our definition of a theory in the BV formalism in the general setting.

In all non-trivial examples the space of fields $\mathscr{E}$ is infinite dimensional and we must be careful with what functionals S we allow. The space of fields we consider will always have a natural topology, and we will choose functionals that are continuous with respect to it. We divert for a moment to discuss these issues of infinite dimensional linear algebra.

1.1.1.1. Some functional analysis. Homological algebra will play a paramount role in our approach to perturbative field theory. A problem with this is that the category of topological vector spaces is not an abelian category. It is therefore advantageous to enlarge this to the category of *differentiable vector spaces*. The details of this setup are carried out in the Appendix of [**CG17**], but we will recall some key points.

Let Mfld be the site of smooth manifolds. The covers defining the Grothendieck topology are given by surjective local diffeomorphisms. There is a natural sheaf on this site given by smooth functions $C^\infty : M \mapsto C^\infty(M)$. By definition, a C^∞-module is a module sheaf over C^∞ on Mfld.

For any p the assignment $\Omega^p : M \mapsto \Omega^p(M)$ defines a C^∞-module. Similarly, if F is any C^∞-module we have the C^∞-module of p-forms with values on F defined by the assignment

$$\Omega^1(F) : M \in \mathrm{Mfld} \mapsto \Omega^p(M, F) = \Omega^p(M) \otimes_{C^\infty(M)} F(M).$$

Definition 1.1.1. A *differentiable vector space* is a C^∞-module equipped with a map of sheaves on Mfld

$$\nabla : F \to \Omega^1(F)$$

such that for each M, $\nabla(M)$ defines a flat connection on the $C^\infty(M)$-module $F(M)$. A map of differentiable vector spaces is one of C^∞-modules that intertwines the flat connections. This defines a category that we denote DVS

Our favorite example of differentiable vector spaces are imported directly from geometry.

Example 1.1.2. Suppose E is a vector bundle on a manifold X. Let $\mathscr{E}(X)$ denote the space of smooth global sections. Let $C^\infty(M, \mathscr{E}(X))$ be the space of sections of the bundle $\pi_X^* E$ on $M \times X$ where $\pi_X : M \times X \to X$ is projection. The assignment $M \mapsto C^\infty(M, \mathscr{E}(X))$ is a C^∞-module with flat connection, so defines a differentiable vector space. Similarly, the space of compactly supported sections $\mathscr{E}_c(X)$ is a DVS.

Many familiar categories of topological vector spaces embed inside the category of differentiable vector spaces. Consider the category of locally convex topological vector spaces LCTVS. If V is such a vector space, there is a notion of a smooth map $f : U \subset \mathbb{R}^n \to V$. One can show, Proposition B.3.0.6 of [**CG17**], that this defines a functor $\mathrm{dif}_t : \mathrm{LCTVS} \to \mathrm{DVS}$ sending V to the C^∞-module $M \mapsto C^\infty(M, V)$. If $\mathrm{BVS} \subset \mathrm{LCTVS}$ is the subcategory with the same objects but whose morphisms are bounded linear maps, this functor restricts to embed BVS as a full subcategory $\mathrm{BVS} \subset \mathrm{DVS}$.

There is a notion of completeness that is useful when discussing tensor products. A topological vector space $V \in \mathrm{BVS}$ is *complete* if every smooth map $c : \mathbb{R} \to V$ has an anti-derivative [**KM97**]. There is a full subcategory $\mathrm{CVS} \subset \mathrm{BVS}$ of complete topological vector spaces. The most familiar example of a complete topological vector space will be the smooth sections $\mathscr{E}(X)$ of a vector bundle $E \to X$.

We let Ch(DVS) denote the category of cochain complexes in differentiable vector spaces (we will refer to objects as differentiable vector spaces). It is enriched over the category of differential graded vector spaces in the usual way. We say that a map of differentiable cochain complexes $f : V \to W$ is a quasi-isomorphism if and only if for each M the map $f : C^\infty(M, V) \to C^\infty(M, W)$ is a quasi-isomorphism.

Theorem 1.1.3 (Appendix B [**CG17**])**.** *The full subcategory* $\mathrm{dif}_c : \mathrm{CVS} \subset \mathrm{DVS}$ *is closed under limits, countable coproducts, and sequential colimits of closed embeddings. Furthermore,* CVS *has the structure of a symmetric monoidal category with respect to the completed tensor product* $\widehat{\otimes}_\beta$*.*

We will not define the tensor product $\widehat{\otimes}_\beta$here, but refer the reader the cited reference for a complete exposition. We will recall its key properties below. Often times we will write $\otimes$ for $\widehat{\otimes}_\beta$ where there is no potential conflict of notation. The fundamental property of the tensor product that we use is the following. Suppose that E, F are vector bundles on manifolds X, Y respectively. Then, $\mathscr{E}(X), \mathscr{F}(Y)$ lie in CVS, so it makes sense to take their tensor product using $\widehat{\otimes}_\beta$. There is an isomorphism

$$\mathscr{E}(X)\widehat{\otimes}_\beta\mathscr{F}(Y) \cong \Gamma(X \times Y, E \boxtimes F) \tag{1.4}$$

where $E \boxtimes F$ denotes the external product of bundles, and Γ is smooth sections.

If E is a vector bundle on a manifold X, then the spaces $\mathscr{E}(X), \mathscr{E}_c(X)$ both lie in the subcategory $\mathrm{CVS} \subset \mathrm{DVS}$. The differentiable structure arises from the natural topologies on the spaces of sections.

We will denote by $\overline{\mathscr{E}}(X)$ ($\overline{\mathscr{E}}_c(X)$) the space of (compactly supported) distributional sections. It is useful to bear in mind the following inclusions

$$\begin{tikzcd} \mathscr{E}_c(X) \arrow[r, hook] \arrow[dr, hook] & \overline{\mathscr{E}}_c(X) \arrow[r, hook] & \overline{\mathscr{E}}(X) \\ & \mathscr{E}(X) \arrow[ur, hook] & \end{tikzcd}.$$

Denote by $E^\vee$ the dual vector bundle whose fiber over $x \in X$ is the linear dual of E_x. Let $E^!$ denote the vector bundle $E^\vee \otimes \mathrm{Dens}_X$, where Dens_X is the bundle of densities. In the case X is oriented, Dens_X is isomorphic to the top wedge power of T^*X. Let $\mathscr{E}^!(X)$ denote the space of sections of $E^!$. The natural pairing

$$\mathscr{E}_c(X) \otimes \mathscr{E}^!(X) \to \mathbb{C}$$

that pairs sections of E with the evaluation pairing and integrates the resulting compactly supported top form exhibits $\overline{\mathscr{E}}_c(X)$ as the continuous dual to $\mathscr{E}^!(X)$. Likewise, $\mathscr{E}_c(X)$ is the continuous dual to $\overline{\mathscr{E}}^!(X)$. In this way, the topological vector spaces $\overline{\mathscr{E}}(X)$ and $\overline{\mathscr{E}}_c(X)$ obtain a natural differentiable structure.

If V is any differentiable vector space then we define the space of linear functionals on V to be the space of maps $V^* = \mathrm{Hom}_{\mathrm{DVS}}(V, \mathbb{R})$. Since DVS is enriched over itself this is again a differentiable vector space. Similarly, we can define the polynomial functions of homogeneous degree n to be the space

$$\mathrm{Sym}^n(V^*) = \mathrm{Hom}^{multi}_{\mathrm{DVS}}(V \times \cdots \times V, \mathbb{R})_{S_n}$$

where the hom-space denotes multi-linear maps, and we have taken S_n-coinvariants on the right-hand side. The algebra of functions on V is defined by

$$\mathscr{O}(V) = \prod_n \mathrm{Sym}^n(V^*).$$

As an application of Equation (1.4) we have the following identification.

Lemma 1.1.4. *Let E be a vector bundle on X. Then, there is an isomorphism*

$$\mathscr{O}(\mathscr{E}(X)) \cong \prod_n \mathscr{D}_c(X^n, (E^!)^{\boxtimes n})_{S_n}$$

where $\mathscr{D}_c(X^n, (E^!)^{\boxtimes n})$ is the space of compactly supported distributional sections of the vector bundle $(E^!)^{\boxtimes n}$. Again, we take S_n-coinvariants on the right hand side.

1.1.1.2. Local functionals. In our approach, the space of fields will always be equal to the space of smooth sections of a $\mathbb{Z}$-graded vector bundle $E \to X$ on a manifold $\mathscr{E} = \Gamma(X, E)$. The class of functionals $S : \mathscr{E} \to \mathbb{R}$ defining the classical theories we consider are required to be *local*, or given by the integral of a Lagrangian density. We define this concept now.

Let D_X denote the sheaf of differential operators on X. The ∞-jet bundle $\mathrm{Jet}(E)$ of a vector bundle E is the vector bundle whose fiber over $x \in X$ is the space of formal germs at x of sections of E. It is a standard fact that $\mathrm{Jet}(E)$ is equipped with a flat connection giving its space of sections $J(E) = \Gamma(X, \mathrm{Jet}(E))$ the structure of a D_X-module.

Above, we have defined the algebra of functions $\mathscr{O}(\mathscr{E}(X))$ on the space of sections $\mathscr{E}(X)$. Similarly, let $\mathscr{O}_{red}(\mathscr{E}(X)) = \mathscr{O}(\mathscr{E}(X))/\mathbb{R}$ be the quotient by the constant polynomial functions. The space $\mathscr{O}_{red}(J(E))$ inherits a natural D_X-module structure from $J(E)$.

We refer to $\mathscr{O}_{red}(J(E))$ as the space of *Lagrangians* on the space of sections of the vector bundle E. Indeed, every element $F \in \mathscr{O}_{red}(J(E))$ can be expanded as $F = \sum_n F_n$ where each F_n is an element

$$F_n \in \mathrm{Hom}_{C^\infty_X}(J(E)^{\otimes n}, C^\infty_X)_{S_n} \cong \mathrm{PolyDiff}(\mathscr{E}^{\otimes n}, C^\infty(X))_{S_n}$$

where the right-hand side is the space of polydifferential operators. The proof of the isomorphism on the right-hand side can be found in Chapter 5 of [**Cos11**].

A local functional is given by a Lagrangian densities modulo total derivatives. The mathematical definition is the following.

Definition 1.1.5. Let E be a graded vector bundle on X. Define the sheaf of *local functionals* on X to be

$$\mathscr{O}_{\mathrm{loc}}(\mathscr{E}) = \mathrm{Dens}_X \otimes_{D_X} \mathscr{O}_{red}(J(E)),$$

where we use the natural right D_X-module structure on densities.

Note that we always consider local functionals coming from Lagrangians modulo constants. We will not be concerned with local functions associated to constant Lagrangians.

From the expression for functionals in Lemma 1.1.4 we see that integration defines an inclusion of sheaves

$$i : \mathscr{O}_{\mathrm{loc}}(\mathscr{E}) \hookrightarrow \mathscr{O}_{red}(\mathscr{E}_c). \tag{1.5}$$

Often times when we describe a local functional we will write down its value on test compactly supported sections, then check that it is given by integrating a Lagrangian density, which amounts to lifting the functional along i.

1.1.1.3. The definition of a classical field theory. Before giving the definition, we need to recall what the proper notion of a shifted symplectic structure is in the geometric setting that we work in.

Definition 1.1.6. Let E be a graded vector bundle on X. A *k-shifted symplectic structure* is an isomorphism of graded vector spaces

$$E \cong_\omega E^![k] = (\mathrm{Dens}_X \otimes E^\vee)\,[k]$$

that is graded anti-symmetric.

If ω^* is the formal adjoint of the isomorphism $\omega^* : E \cong E^![k]$, anti-symmetry amounts to the condition $\omega^* = -\omega$. In general, ω does *not* induces a Poisson structure on the space of all functionals $\mathscr{O}(\mathscr{E})$. This is because, as we have seen above, elements of this space are given by distributional sections and hence we cannot pair elements with overlapping support. The symplectic structure does, however, induce a Poisson bracket on *local* functionals. [1] We will denote the bracket induced by a shifted symplectic structure by $\{-,-\}$.

We are now ready to give the precise definition of a classical field theory.

[1]Note that $\mathscr{O}_{\mathrm{loc}}(\mathscr{E})$ is not a shifted Poisson algebra since there is no natural commutative product.

Definition 1.1.7. A *classical field theory* in the BV formalism on a smooth manifold X is a $\mathbb{Z}$-graded vector bundle E equipped with a (-1)-shifted symplectic structure together with a local functional $S \in \mathscr{O}_{\mathrm{loc}}(\mathscr{E})$ such that:

(1) the functional S satisfies the *classical master equation*

$$\{S, S\} = 0;$$

(2) S is at least quadratic, so we can write it (in a unique way) as

$$S(\varphi) = \omega(\varphi, Q\varphi) + I(\varphi)$$

where Q is a linear differential operator such that $Q^2 = 0$, and $I \in \mathscr{O}_{\mathrm{loc}}(\mathscr{E})$ is at least cubic;

(3) the complex $(\mathscr{E}, Q)$ is elliptic.

In the physics literature, the operator Q is known as the linearized BRST operator, and $\{S, -\} = Q + \{I, -\}$ is the full BRST operator. Ellipticity of the complex $(\mathscr{E}, Q)$ is a technical requirement that will be very important in our approach to the issue of renormalization in perturbative quantum field theory. The classical master equation is equivalent to

$$QI + \frac{1}{2}\{I, I\} = 0.$$

A *free theory* is a classical theory with $I = 0$ in the notation above. Thus, a free theory is a simply an elliptic complex equipped with a (-1)-shifted symplectic pairing where the differential in the elliptic complex is graded skew-self adjoint for the pairing.

Although the space $\mathscr{O}(\mathscr{E})$ does not have a well-defined shifted Possoin bracket induced from the symplectic pairing, the operator $\{S, -\} : \mathscr{O}(\mathscr{E}) \to \mathscr{O}(\mathscr{E})[1]$ *is* well-defined since S is local by assumption. By assumption, it is also square zero. The complex of global classical observables of the theory is defined by

$$\mathrm{Obs}^{\mathrm{cl}}_{\mathscr{E}}(X) = (\mathscr{O}(\mathscr{E}(X)), \{S, -\}).$$

This complex is the general replacement for functions on the derived locus from the beginning of this section. Although it does not have a P_0-structure, there is a subspace that does. This is sometimes referred to as the *BRST* complex in the physics literature.

1.1.1.4. A description using L_∞ algebras. There is a completely equivalent way to describe a classical field theory that helps to illuminate the mathematical meaningfulness of the definition given above. The requisite concept we need to introduce is that of a *local Lie algebra* (or local L_∞ algebra).

First, recall that an L_∞ algebra is a modest generalization of a dg Lie algebra where the Jacobi identity is only required to hold up to homotopy. The data of an L_∞ algebra is a graded vector space V with, for each $k \geq 1$, a k-ary bracket

$$\ell_k : V^{\otimes k} \to V[2-k]$$

of cohomological degree $2-k$. These maps are required to satisfy a series of conditions, the first of which says $\ell_1^2 = 0$. The next says that ℓ_2 is a bracket satisfying the Jacobi identity up to a homotopy given by ℓ_3. For a detailed definition see we refer the reader to [**Sta92, Get09**].

We now give the definition of a local L_∞ algebra on a manifold X. This has appeared in Chapter 4 of [**CG**].

Definition 1.1.8. A *local L_∞ algebra* on X is the following data:

(i) a $\mathbb{Z}$-graded vector bundle L on X, whose sheaf of smooth sections we denote $\mathscr{L}^{sh}$, and

(ii) for each positive integer n, a polydifferential operator in n inputs

$$\ell_n : \underbrace{\mathscr{L} \times \cdots \times \mathscr{L}}_{n \text{ times}} \to \mathscr{L}[2-n]$$

such that the collection $\{\ell_n\}_{n \in \mathbb{N}}$ satisfy the conditions of an L_∞ algebra. In particular, $\mathscr{L}$ is a sheaf of L_∞ algebras.

Just as in the case of an ordinary graded vector bundle, we can discuss local functionals on a local Lie algebra L. In this case, the L_∞ structure maps give this the structure of a sheaf of complexes. Indeed, the ∞-jet bundle JL is an L_∞ algebra object in D_X-modules and so we can define the D_X-module of reduced Chevalley-Eilenberg cochains $\mathrm{C}^*_{\mathrm{Lie,red}}(JL)$. Mimicking the definition above, we arrive at the following local version of Lie algebra cohomology that will come up again and again in this thesis.

Definition 1.1.9. Let L be a local Lie algebra. The local Chevalley-Eilenberg cochain complex is the sheaf of cochain complexes

$$\mathrm{C}^*_{\mathrm{loc}}(\mathscr{L}) = \mathrm{Dens}_X \otimes_{D_X} \mathrm{C}^*_{\mathrm{Lie,red}}(L).$$

It turns out that the definition of a classical field theory can be repackaged in terms of certain structures on a local L_∞ algebra. The first piece of data we need to transport to the L_∞ side is that of a symplectic pairing. The underlying data of a local L_∞ algebra L is a graded vector bundle. In Definition 1.1.6 we have already defined a k-shifted symplectic pairing. On the local Lie algebra sign, we ask for $k = -3$ shifted symplectic structures that are also invariant for the L_∞ structure maps.

Also, an important part of a classical field theory is ellipticity. We say a local L_∞ algebra is *elliptic* if the complex $(\mathscr{L}, \mathrm{d} = \ell_1)$ is an elliptic complex.

Proposition 1.1.10. *The following are equivalent:*

(1) a classical field theory in the BV formalism $(\mathscr{E}, \omega, S)$*;*

(2) an elliptic local Lie algebra structure on $L = E[1]$ *equipped with a* (-3)*-shifted symplectic structure.*

Proof. (Sketch) The underlying graded vector bundle of the space of fields $\mathscr{E}$ is E and we obtain the bundle underlying the local L_∞ algebra by shifting this down $L = E[1]$. The (-1)-shifted symplectic structure on E transports to a (-3)-shifted on on L. The L_∞ structure maps for L come from the Taylor components of the action functional S. The exterior derivative of S is a section

$$\mathrm{d}S \in \mathrm{C}^*_{\mathrm{loc}}(\mathscr{L}, \mathscr{L}^![-1]),$$

where on the right-hand side we have zero differential. The Taylor components are of the form $(\mathrm{d}S)_n : \mathscr{L}^{\otimes n} \to \mathscr{L}^![-1]$. Using the shifted symplectic pairing we can identify these Taylor components with maps $(\mathrm{d}S)_n : \mathscr{L}^{\otimes n} \to \mathscr{L}[2]$. Thus, $\mathrm{d}S$ can be viewed as a section of $\mathrm{C}^*_{\mathrm{loc}}(\mathscr{L}, \mathscr{L}[2])$. This is precisely the space controlling deformations of $\mathscr{L}$ as a local Lie algebra. One checks immediately that the classical master equation is equivalent to the fact that $\mathrm{d}S$ is a derivation, hence it determines the structure of a local Lie algebra. The first Taylor component ℓ_1 is precisely the operator Q before, so ellipticity of $(\mathscr{E}, Q)$ is equivalent to ellipticity of $(\mathscr{L}, \ell_1)$. $\square$

1.1.1.5. Moduli problems and Koszul duality. BW: add this

1.1.2. Quantum field theory

We now introduce the notion of a *quantum field theory* in the BV formalism. We follow the effective approach defined by Costello in [**Cos11**].

1.1.2.1. Regularization. We have seen that part of the data of a classical field theory is that of a (-1)-shifted symplectic structure on the space of fields. If E is the graded vector bundle underlying the theory, the symplectic form determined an isomorphism of bundles $E \cong E^![-1]$. We can represent the inclusion $\mathscr{E}_c \hookrightarrow \overline{\mathscr{E}}$ via its integral kernel $K_0 \in \overline{\mathscr{E}} \otimes \overline{\mathscr{E}}^!$. Using the symplectic pairing this is further identified with an element

$$K_0 \in \overline{\mathscr{E}} \otimes \overline{\mathscr{E}}[-1].$$

That is, K_0 is a degree one element in $\overline{\mathscr{E}} \otimes \overline{\mathscr{E}}$. The naïve BV Laplacian $\Delta = \Delta_{K_0}$ is ill-defined acting on functions on $\mathscr{E}$, $\mathscr{O}(\mathscr{E})$. The point of regularization is to find a replacement for this operator.

The first step in regularization is to find a replacement of K_0 as a smooth, i.e. non-distributional, section in the tensor product $\mathscr{E} \otimes \mathscr{E}$. This is a reasonable thing to ask, since by ellipticity we know that the inclusion $\mathscr{E} \otimes \mathscr{E} \hookrightarrow \overline{\mathscr{E}} \otimes \overline{\mathscr{E}}$ is a quasi-isomorphism.

So, we can replace K_0 by such a smooth section up to homotopy. We refer to this as a *regularization* of the kernel.

We use a systematic way of regularization using heat kernels, which can be found in [**Cos11**] or Chapter 8 of [**CG**]. First, we fix the following data, that of a *gauge fixing* operator. This is an operator

$$Q^{GF} : \mathscr{E} \to \mathscr{E}[-1]$$

of cohomological degree -1. We require that $D = [Q, Q^{GF}]$ is a generalized Laplacian acting on sections $\mathscr{E}$ in the sense of [**?**], in addition to other conditions that can be found in Definition 5.4.0.5 in [**CG**].]

The utility of introducing the gauge fixing operator is that it allows us to introduce the operator e^{-tD} which has a kernel that we denote $K_t \in \overline{\mathscr{E}} \otimes \overline{\mathscr{E}}$ for any $t \geq 0$. This kernel satisfies the usual conditions of a heat kernel:

(1) K_t satisfies the heat equation

$$\frac{\partial}{\partial t} K_t + DK_t = 0$$

(2) K_0 is the kernel for the identity operator as above.

Moreover, when $t > 0$ the operator e^{-tD} is *smoothing* so that $K_t \in \mathscr{E} \otimes \mathscr{E} \subset \overline{\mathscr{E}} \otimes \overline{\mathscr{E}}$.

The point of introducing this heat kernel is that it provides a regularization of K_0. Indeed, for any $\epsilon, L \geq 0$ introduce the *propagator*

$$P_{\epsilon<L} = \int_{t=\epsilon}^{L} (Q^{GF} \otimes 1) K_t \mathrm{d}t \in \overline{\mathscr{E}} \otimes \overline{\mathscr{E}}.$$

Then, one immediately checks that

$$K_L - K_\epsilon = QP_{\epsilon<L},$$

so that $P_{\epsilon<L}$ is a homotopy between K_L and K_ϵ. In particular, $P_{0<L}$ provides a homotopy between the identity kernel K_0 and K_L.

Definition 1.1.11. The scale $L > 0$ BV Laplacian is the order two operator

$$\Delta_L = \partial_{K_L} : \mathscr{O}(\mathscr{E}) \to \mathscr{O}(\mathscr{E})$$

given by contraction with the kernel $K_L \in \mathscr{E} \otimes \mathscr{E}$.

We have already mentioned that the bracket $\{-,-\}$ is not defined on the whole space of functionals. The regularized BV operator allows us to define the following *scale L bracket*:

$$\{I, J\}_L := \Delta_L(IJ) - \Delta_L(I)J - (-1)^{|I|} I \Delta_L(J).$$

For $L > 0$ this bracket is defined on all of $\mathscr{O}(\mathscr{E})$, just as Δ_L is.

1.1.2.2. Effective BV quantization. Fix a free BV theory together with a gauge fixing operator. This is the data of an elliptic complex $(\mathscr{E}, Q)$ with a (-1)-shifted symplectic form ω. In addition, let Q^{GF} be a gauge fixing operator so that the regularized heat kernels K_L and propagators $P_{\epsilon<L}$ are defined.

We introduce the formal variable $\hbar$ and consider $\hbar$-dependent functionals $\mathscr{O}(\mathscr{E})[[\hbar]]$. Let $\mathscr{O}^+(\mathscr{E})[[\hbar]] \subset \mathscr{O}(\mathscr{E})[[\hbar]]$ be the subset of functionals that are at least cubic modulo $\hbar$. We define a map

$$W(P_{\epsilon<L}, -) : \mathscr{O}^+(\mathscr{E})[[\hbar]] \to \mathscr{O}^+(\mathscr{E})[[\hbar]],$$

renormalization group flow. Formally, $W(P_{\epsilon<L}, I)$ is defined by the formula

$$e^{W(P_{\epsilon<L},I)/\hbar} = e^{\hbar\partial_{P_{\epsilon<L}}} e^{I/\hbar}.$$

Concretely, $W(P_{\epsilon<L}, I)$ can be written as a sum over graphs Γ

$$W(P_{\epsilon<L}, I) = \sum_{\Gamma} \frac{\hbar^{g(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} W_\Gamma(P_{\epsilon<L}, I),$$

where $W_\Gamma(P_{\epsilon<L}, I)$ is the weight of the graph Γ whose edges are labeled by $P_{\epsilon<L}$ and vertices labeled by I. This is our mathematical definition of the Feynman weight of the graph Γ, and the precise definition can be found in Chapter 2 of [**Cos11**].

In the BV formalism, as developed in [**Cos11, CG17, CG**], one has the following definition of a quantum field theory.

Definition 1.1.12. A *quantum field theory* in the BV formalism consists of a free BV theory $(\mathscr{E}, Q, \omega)$ and an effective family of functionals

$$\{I[L]\}_{L\in(0,\infty)} \subset \mathscr{O}^+_{P,sm}(\mathscr{E})[[\hbar]]$$

that satisfy:

(a) the exact renormalization group (RG) flow equation

$$I[L'] = W(P_{L<L'}, I[L]);$$

(b) the scale L quantum master equation (QME) at every length scale L:

$$(Q + \hbar\Delta_L)e^{I[L]/\hbar} = 0.$$

Equivalently,

$$QI[L] + \hbar\Delta_L I[L] + \frac{1}{2}\{I[L], I[L]\}_L = 0;$$

(c) as $L \to 0$, the functional $S[L]$ has an asymptotic expansion that is local.

The subspace $\mathscr{O}^+_{P,sm}(\mathscr{E})[[\hbar]] \subset \mathscr{O}(\mathscr{E})[[\hbar]]$ is of smooth and proper functionals that are at least cubic modulo $\hbar$. Smooth and properness is a technical condition that we will not delve into in this work, but refer to the original reference of Costello and Gwilliam. The first condition ensures that the scale L action functional $S[L]$ determines the functional at every other scale. The second can be interpreted as saying that we have a proper path integral measure at scale L (i.e., the QME can be seen as a definition of the measure). The third condition implies that the effective action is a quantization of a classical field theory, since a defining property of a classical theory is that its action functional is local. (A full definition is available in Section 8.2 of [**CG**].)

Remark 1.1.13. The length scale is often associated with a choice of Riemannian metric on the underlying manifold, but the formalism of [**Cos11**] keeps track of how the space of quantum BV theories depends upon such a choice (and other choices that might go into issues like renormalization). Hence, when the choices should not be essential — such as with a topological field theory — one can typically show rigorously that different choices give equivalent answers. The length scale is also connected with the use of heat kernels in [**Cos11**], but one can work with more general parametrices (and hence more general notions of "scale"), as explained in Chapter 8 of [**CG**]. We use a natural length scale in this section; when it becomes relevant, in the context of factorization algebras, one must switch to general parametrices.

The locality condition ensures that the limit $I^{cl} = \lim_{L\to 0} I[L] \mod \hbar$ exists and is a local functional. The QME modulo $\hbar$ implies that I^{cl} satisfying the CME, so that $(\mathscr{E}, Q, \omega, I^{cl})$ defines a classical theory in the BV formalism.

1.1.2.3. Deformation theory for quantizations. There is a well-established deformation theory for studying quantizations of a fixed classical field theory. If $(\mathscr{E}, Q, \omega, I)$ is a fixed classical theory, one would like to study the problem of finding quantizations which modulo $\hbar$ are equal to this classical theory.

According to the definition of a QFT there are two main steps.

(1) Find an effective family $\{I[L]\}$ which, modulo $\hbar$, agrees with the classical theory I, and satisfies the RG flow equation. The main result of [**Cos11**] is that this step always has a solution. Naively, the proposed family is of the form $I[L] = W(P_{0<L}, I)$, but since $P_{0<L}$ is distributional this functional may not be well-defined. (This is the problem of UV divergence in QFT) The key fact is that there exists a family of counterterms $I^{CT}(\epsilon) \in \mathscr{O}(\mathscr{E})[[\hbar]]$ such that the limit

$$I[L] = \lim_{\epsilon\to 0} W(P_{\epsilon<L}, I - I^{CT}(\epsilon))$$

does exist. Moreover, it automatically satisfies the RG flow equation.

(2) Once we have the effective family $\{I[L]\}$, the remaining condition to defining a QFT is the quantum master equation. In general this equation is not satisfied, and there may in fact be obstructions to having a solution.

For holomorphic theories we will study both problems above. We will show that the analysis involved in finding counterterms for holomorphic theories is extremely well-behaved. In fact, the counterterms for holomorphic theories on $\mathbb{C}^d$ are all zero. Using this, we will show how solving the QME for holomorphic theories can be done a systematic way.

To study the problem of solving the quantum master equation in general, we work order by order in the formal parameter $\hbar$. Suppose that $I[L]$ is defined modulo $\hbar^{n+2}$ and sastisfies the QME modulo $\hbar^{n+1}$. The obstruction to satisfying the QME at scale L modulo $\hbar^{n+2}$ is the functional

$$\Theta_{n+1}[L] = \hbar^{-n-1}(QI[L] + \frac{1}{2}\{I[L], I[L]\}_L + \hbar\Delta_L).$$

The obstruction $\Theta[L]$ satisfies the classical master equation and hence the limit $\Theta_{n+1} = \lim_{L\to 0}\Theta[L]$ is a local functional and is closed for the differential $Q + \{I, -\}$. It is thus a closed element of degree one of the *deformation complex*

$$\mathrm{Def}_{\mathscr{E}} = \left(\mathscr{O}_{\mathrm{loc}}(\mathscr{E}), Q + \{I, -\}\right).$$

If Θ_{n+1} is cohomologically trivial in $H^1(\mathrm{Def}_{\mathscr{E}})$, the space of possible lifts of $\{I[L]\}$ to a solution of the QME modulo $\hbar^{n+2}$ is a torsor for $H^0(\mathrm{Def}_{\mathscr{E}})$.

1.2. Holomorphic field theories

The goal of this section is to define the notion of a holomorphic field theory. This is a variant of Costello's definition of a BV theory, see the previous section, and we will take for granted that the reader is familiar with the general format. In summary, we modify the definition of a theory by inserting the word "holomorphic" in front of most objects (bundles, differential operators, etc..). By applying the Dolbeault complex in appropriate locations, we will recover Costello's definition of a theory, but with a holomorphic flavor, see Table 1.1.

There are many references in the physics literature to codify the concept of a holomorphic field theory. See, most closely related to our approach, special cases of this in the work of Nekrasov and collaborators in [**Nek96, LMNS97, LMNS96**]. We will discuss in more detail the relationship of our analysis of holomorphic theories to this work in Chapters **??**.

1.2.1. The definition of a holomorphic theory

We give a general definition of a classical holomorphic theory on a general complex manifold X of complex dimension d. We start with the definition of a *free* holomorphic field theory. After that we will go on to define what an interacting holomorphic theory is.

1.2.1.1. Free holomorphic theories. The fields of any theory are always expressed as sections of some $\mathbb{Z}$-graded vector bundle. Here, the $\mathbb{Z}$-grading is the cohomological, or BRST, grading of the theory. For a holomorphic theory we take this graded vector bundle to be holomorphic. By a *holomorphic* $\mathbb{Z}$-graded vector bundle we mean a $\mathbb{Z}$-graded vector bundle $V = \oplus_i V^i$ such that each graded piece V^i is a holomorphic vector bundle. Thus, the data we start with is the following:

(1) a $\mathbb{Z}$-graded holomorphic vector bundle $V^* = \oplus_i V^i[-i]$, so that the finite dimensional holomorphic vector bundle V^i is in cohomological degree i.

A free classical theory is made up of a space of fields as above together with the data of a linearized BRST differential Q^{BRST} and a symplectic pairing. Ordinarily, the BRST operator is a differential operator on the vector bundle defining the fields. For the class of theories we are considering, we want this operator to be holomorphic.

If E and F are two holomorphic vector bundles on X, we can speak of holomorphic differential operators between E and F. First, note that the Hom-bundle $\mathrm{Hom}(E, F)$ inherits a natural holomorphic structure. By definition, a holomorphic differential operator of order m is a linear map

$$D : \Gamma^{hol}(X; E) \to \Gamma^{hol}(X; F)$$

such that, with respect to a holomorphic coordinate chart $\{z_i\}$ on X, D can be written as

$$D|_{\{z_i\}} = \sum_{|I| \leq m} a_I(z) \frac{\partial^{|I|}}{\partial z_I} \tag{1.6}$$

where $a_I(z)$ is a local holomorphic section of $\mathrm{Hom}(E, F)$. Here, the sum is over all multi-indices $I = (i_1, \ldots, i_d)$ and

$$\frac{\partial^{|I|}}{\partial z_I} := \prod_{k=1}^{d} \frac{\partial^{i_k}}{\partial z_k^{i_k}}.$$

The length is defined by $|I| := i_1 + \cdots + i_d$.

Example 1.2.1. The most basic example of a holomorphic differential operator is the ∂ operator for the trivial vector bundle. For each $1 \leq \ell \leq d = \dim_{\mathbb{C}}(X)$, it is a holomorphic differential operator from $E = \wedge^{\ell} T^{1,0*} X$ to $F = \wedge^{\ell+1} T^{1,0*} X$ which on sections is

$$\partial : \Omega^{\ell,hol}(X) \to \Omega^{\ell+1,hol}(X).$$

Locally, of course, it has the form

$$\partial = \sum_{i=1}^{d} (\mathrm{d}z_i \wedge (-)) \frac{\partial}{\partial z_i},$$

where $\mathrm{d}z_i \wedge (-)$ is the vector bundle homomorphism $\wedge^\ell T^{1,0*}X \to \wedge^{\ell+1}T^{1,0*}X$ sending $\alpha \mapsto \mathrm{d}z_i \wedge \alpha$.

The next piece of data we fix is:

(2) a square zero holomorphic differential operator

$$Q^{hol} : \mathscr{V} \to \mathscr{V}[-1]$$

of cohomological degree $+1$. Here $\mathscr{V}$ denotes the holomorphic sections of V.

Finally, to define a free theory we need the data of a symplectic pairing. For reasons to become clear in a moment, we must choose this pairing to have a strange cohomological degree. The last piece of data we fix is:

(3) an invertible bundle map

$$(-,-)_V : V \otimes V \to K_X[d-1]$$

Here, K_X is the canonical bundle on X.

The definition of the fields of an ordinary field theory are the *smooth* sections of the vector bundle V. In our situation this is a silly thing to do since we lose all of the data of the complex structure we used to define the objects above. The more natural thing to do is to take the *holomorphic* sections of the vector bundle V. By construction, the operator Q^{hol} and the pairing $(-,-)_V$ are defined on holomorphic sections, so on the surface this seems reasonable. BW: what should I say the problem is with doing things in the analytic category?

The solution to this problem is in the existence of a resolution for the holomorphic sections of a vector bundle by smooth sections of bundles. Given any holomorphic vector bundle E we can define its *Dolbeault complex* $\Omega^{0,*}(X, E)$ with its Dolbeault operator

$$\overline{\partial} : \Omega^{0,p}(X, E) \to \Omega^{0,p+1}(X, E).$$

Here, $\Omega^{0,p}(X, E)$ denotes smooth sections of the vector bundle $\bigwedge^p T^{0,1*}X \otimes E$.

We now take a graded holomorphic vector bundle V as above, equipped with the differential operator Q^{hol}. We can then define the totalization of the Dolbeault complex with the operator Q^{hol}:

$$\mathscr{E}_V = \left(\Omega^{0,*}(X, E), \overline{\partial} + Q^{hol}\right).$$

The operator $\overline{\partial} + Q^{hol}$ will be the linearized BRST operator of our theory. By assumption, we have $\overline{\partial} Q^{hol} = Q^{hol}\overline{\partial}^*$ so that $(\overline{\partial} + Q^{hol})^2 = 0$ and hence the fields still define a complex. The (-1)-shifted symplectic pairing is obtained by composition of the pairing $(-,-)_V$ with integration on $\Omega^{d,hol}_X$. The thing to observe here is that $(-,-)_V$ extends to the Dolbeault complex in a natural way: we simply combine the wedge product of forms with the pairing on V. The (-1)-shifted pairing ω_V on $\mathscr{E}$ is defined by the diagram

$$\begin{tikzcd}
\mathscr{E}_V \otimes \mathscr{E}_V \arrow[r, "(-,-)_V"] \arrow[dr, dotted, "\omega_V"'] & \Omega^{0,*}(X, K_X)[d-1] \arrow[d, "\int_X"] \\
& \mathbb{C}[-1].
\end{tikzcd}$$

We note that the top Dolbeault forms with values in the canonical bundle K_X are precisely the top forms on the smooth manifold X, so integration makes sense.

We arrive at the following definition.

Definition/Lemma 1. A *free holomorphic theory* on a complex manifold X is the data $(V, Q^{hol}, (-,-)_V)$ as in (1), (2), (3) above such that Q^{hol} is a square zero elliptic differential operator that is graded skew self-adjoint for the pairing $(-,-)_V$. The triple $(\mathscr{E}_V, Q_V = \overline{\partial} + Q^{hol}, \omega_V)$ defines a free BV theory in the usual sense.

The usual prescription for writing down the associated action functional holds in this case. If $\varphi \in \Omega^{0,*}(X, V)$ denotes a field the action is

$$S(\varphi) = \int_X \left(\varphi, (\overline{\partial} + Q^{hol})\varphi\right)_V .$$

The first example we explain is related to the subject of Chapter **??** and will serve as the fundamental example of a holomorphic theory.

Example 1.2.2. *The free $\beta\gamma$ system.* Suppose that

$$V = \underline{\mathbb{C}} \oplus K_X[d-1].$$

Let $(-,-)_V$ be the pairing

$$(\underline{\mathbb{C}} \oplus K_X) \otimes (\underline{\mathbb{C}} \oplus K_X) \to K_X \oplus K_X \to K_X$$

sending $(\lambda, \mu) \otimes (\lambda', \mu') \mapsto (\lambda\mu', \lambda'\mu) \mapsto \lambda\mu' + \lambda'\mu$. In this example we set $Q^{hol} = 0$. One immediately checks that this is a holomorphic free theory as above. The space of fields can be written as

$$\mathscr{E}_V = \Omega^{0,*}(X) \oplus \Omega^{d,*}(X)[d-1].$$

We write $\gamma \in \Omega^{0,*}(X)$ for a field in the first component, and $\beta \in \Omega^{d,*}(X)[d-1]$ for a field in the second component. The action functional reads

$$S(\gamma+\beta, \gamma'+\beta') = \int_X \beta \wedge \overline{\partial}\gamma' + \beta' \wedge \overline{\partial}\gamma.$$

When $d = 1$ this reduces to the ordinary chiral $\beta\gamma$ system from conformal field theory. The $\beta\gamma$ system is a bosonic version of the ghost bc system that appears in the quantization of the bosonic string, see Chapter ?? of [**Pol98**]. We will discuss this higher dimensional version further in Section **??**. For instance, we will see how this theory is the starting block for constructing general holomorphic σ-models.

Of course, there are many variants of the $\beta\gamma$ system that we can consider. For instance, if E is *any* holomorphic vector bundle on X we can take

$$V = E \oplus K_{\mathbb{C}^d} \otimes E^\vee$$

where $E^\vee$ is the linear dual bundle. The pairing is constructed as in the case above where we also use the evaluation pairing between E and $E^\vee$. In thise case, the fields are $\gamma \in \Omega^{0,*}(X, E)$ and $\beta \in \Omega^{d,*}(X, E^\vee)[d-1]$. The action functional is simply

$$S(\gamma, \beta) = \int \mathrm{ev}_E(\beta \wedge \overline{\partial}\gamma).$$

When E is a tensor bundle of type (r,s) this theory is a bosonic version of the bc ghost system of spin (r,s). For a general bundle E we will refer to it as the $\beta\gamma$ system with coefficients in the bundle E.

Example 1.2.3. *The free chiral scalar.* Another basic example is the free chiral scalar. This is a bit outsideBW: finish Let X be a complex manifold with Hermitian metric g. Let $V = \underline{\mathbb{C}}$, the trivial vector bundle. BW: do this

1.2.1.2. Interacting holomorphic theories. We now define what an interacting holomorphic theory is. In general, an interacting field theory on a manifold M is prescribed by the data of a free theory plus a local functional $I \in \mathscr{O}_{\rm loc}(\mathscr{E})$ that satisfies the classical master equation. Recall, the sheaf of local functionals on $\mathscr{E} = \Gamma(E)$ is defined as the sheaf of Lagrangian densities

$$\mathscr{O}_{\rm loc}(\mathscr{E}) = {\rm Dens}_M \otimes_{D_M} \mathscr{O}_{red}(JE).$$

In the expression above JE stands for the sheaf of smooth sections of the ∞-jet bundle ${\rm Jet}(E)$ which has the structure of a D_X-module.

If V is a holomorphic vector bundle we can define the bundle of holomorphic ∞-jets ${\rm Jet}^{hol}(V)$, [**GG80, CW04**]. This is a pro-vector bundle that is holomorphic in a natural way. The fibers of this infinite rank bundle ${\rm Jet}^{hol}(V)$ are isomorphic to

$${\rm Jet}^{hol}(V)|_w = V_w \otimes \mathbb{C}[[z_1, \ldots, z_d]]$$

where $w \in X$ and where $\{z_i\}$ is the choice of a formal coordinate near w. We denote by $J^{hol}V$ denote the sheaf of holomorphic sections of this jet bundle. The sheaf $J^{hol}V$ has the structure of a D_X^{hol}-module, that is, it is equipped with a holomorphic flat connection

∇^{hol}. This is completely analogous to the smooth case. Locally, the holomorphic flat connection is of the form

$$\nabla^{hol}|_w = \sum_{i=1}^{d} \mathrm{d}w_i \left(\frac{\partial}{\partial w_i} - \frac{\partial}{\partial z_i} \right),$$

where $\{w_i\}$ is the local coordinate on X near w and z_i is the fiber coordinate labeling the holomorphic jet expansion. Using holomorphic jets we can make a completely analogous definition in our setting.

Differential operators between holomorphic bundles are the same as bundle maps between the associated jet bundles. Suppose V, W are holomorphic vector bundles with spaces of holomorphic sections given by $\mathscr{V}, \mathscr{W}$ respectively. Then we can express polydifferential operators from V to W as

$$\mathrm{PolyDiff}^{hol}(\mathscr{V} \times \cdots \times \mathscr{V}, \mathscr{W}) \cong \mathrm{Hom}(\mathrm{Jet}^{\mathrm{hol}}(V) \otimes \ldots \otimes \mathrm{Jet}^{\mathrm{hol}}(V), W).$$

Definition 1.2.4. Let V be a vector bundle. The space of *holomorphic Lagrangian densities* on V is

$$\mathscr{O}^{hol}_{red}(V) = \prod_{n>0} \mathrm{Hom}(\mathrm{Jet}^{hol}(V)^{\otimes n}, K_X)_{S_n},$$

where Hom is taken in the category of holomorphic vector bundles.[2] Equivalently, a holomorphic Lagrangian density is of the form $F = \sum_n F_n \in \mathscr{O}^{hol}_{red}(V)$ where each F_n is a holomorphic polydifferential operator

$$F_n : \mathscr{V} \times \cdots \times \mathscr{V} \to \Omega^{d,hol}_X.$$

[2]The holomorphic vector bundle $\mathrm{Jet}^{hol}(V)$ is infinite dimensional and can be expressed as a pro-object in the category of holomorphic vector bundles. We require the bundle maps to be continuous with respect to the natural adic topology.

Suppose that V is part of the data of a free holomorphic theory $(V, Q^{hol}, (-,-)_V)$. The pairing $(-,-)_V$ endows the space of holomorphic Lagrangians with a sort of bracket. Indeed, suppose $F, F' \in \mathscr{O}^{hol}_{red}(V)$. For simplicity suppose F, F' are of homogenous symmetric degree k, k' respectively. Then, their product $F \otimes F'$ is a symmetric element in the homomorphism space

$$\mathrm{Hom}(\mathrm{Jet}^{hol}(V)^{\otimes k+k'}, K_X \otimes K_X).$$

Now, the bundle map $(-,-)_V : V \otimes V \to K_X[d-1]$ is invertible, hence it determines an element $(-,-)_V^{-1} \in V \otimes K_X^*$, where K_X^* is the dual bundle. We can then consider the composition

$$\mathrm{Hom}(\mathrm{Jet}^{hol}(V)^{\otimes n}, K_X \otimes K_X) \xrightarrow{(-,-)_V^{-1}} \mathrm{Hom}(\mathrm{Jet}^{hol}(V)^{\otimes k+k'-2}, K_X^* \otimes K_X \otimes K_X) \longrightarrow \mathrm{Hom}(\mathrm{Jet}^{hol}(V)^{\otimes k+k'}$$

In the first arrow we have evaluated $(-,-)_V^{-1}$ on the first two factors and the second arrow is simply the evaluation pairing. We symmetrize this to obtain an element $\{F, F'\}^{hol} \in \mathrm{Sym}^{k+k'-2}(\mathrm{Jet}^{hol}(V)^{\otimes n}, K_X)_{S_{k+k'-2}}$. In this way, we have produced a map

$$\{-,-\}^{hol} : \mathscr{O}^{hol}_{red}(V) \times \mathscr{O}^{hol}_{red}(V) \to \mathscr{O}^{hol}_{red}(V)[d-1].$$

Note that this bracket is of cohomological degree $-d+1$.

Remark 1.2.5. Note that the above still makes sense if $(-,-)_V$ is a polydifferential operator...BW: should i say more?

We can now state the definition of a classical holomorphic theory.

Definition 1.2.6. A *classical holomorphic theory* on a complex manifold X is the data of a free holomorphic theory $(V, Q^{hol}, (-,-)_V)$ plus a functional

$$I^{hol} \in \prod_{n \geq 3} \mathrm{Hom}(\mathrm{Jet}^{hol}(V)^{\otimes n}, K_X)_{S_n} \subset \mathscr{O}^{hol}_{red}(V)$$

of cohomological degree d such that $Q^{hol} I^{hol} + \{I^{hol}, I^{hol}\}^{hol} = 0$.

Just as in the free case, we see that classical holomorphic theories define ordinary classical BV theories with interactions. The underlying space of fields, as we have already seen is $\mathscr{E}_V = \Omega^{0,*}(X, V)$. We will write $I^{hol} = \sum_k I^{hol}_k$ where I^{hol}_k is symmetric degree k. Now, we know that I^{hol}_k is a $\Omega^{d,hol}_X$-valued functional of the form

$$I^{hol}_k : (\varphi_1, \ldots, \varphi_k) \mapsto D_1(\varphi_1) \cdots D_k(\varphi_k) \in \Omega^{d,hol}_X$$

where $\varphi_i \in \mathscr{V} = \Gamma^{hol}(X, V)$ and D_i is a holomorphic differential operator on $\mathscr{V}$. Every holomorphic differential operator on the holomorphic vector bundle V extends to a differential operator on its Dolbeault complex $\mathscr{E}_V = \Omega^{0,*}(X, V)$. Thus, we can define the functional

$$I^{\Omega^{0,*}}_k : (\varphi_1, \ldots, \varphi_k) \mapsto \int D_1(\varphi_1) \cdots D_k(\varphi_k)$$

where, now φ_i is a section of the Dolbeault complex $\Omega^{0,*}(X, V)$. The symbol $\int$ reminds us that we are working modulo total derivatives, so that the above expression defines an element of $\mathscr{O}_{\mathrm{loc}}(\mathscr{E}_V)$. This defines a linear map $\mathscr{O}^{hol}_{\mathrm{loc}}(V) \to \mathscr{O}_{\mathrm{loc}}(\mathscr{E}_V)$ that we denote $I^{hol} \mapsto I^{\Omega^{0,*}}$. Note that since I^{hol} is cohomological degree d, the local functional $I^{\Omega^{0,*}}$ is degree zero.

Holomorphic theory	BV theory
Holomorphic bundle V	Space of fields $\mathscr{E}_V = \Omega^{0,*}(X, V)$
Holomorphic differential operator Q^{hol}	Linear BRST operator $\overline{\partial} + Q^{hol}$
Non-degenerate pairing $(-,-)_V$	(-1)-symplectic structure ω_V
Holomorphic Lagrangian I^{hol}	Local functional $I^{\Omega^{0,*}} \in \mathscr{O}_{\mathrm{loc}}(\mathscr{E}_V)$

Table 1.1. From holomorphic to BV

Lemma 1.2.7. *Every classical holomorphic theory* $(V, Q^{hol}, (-,-)_V, I^{hol})$ *determines the structure of a classical BV theory. The underlying free BV theory is given in Definition/Lemma 1* $(\mathscr{E}_V, Q, \omega_V)$ *and the interaction is* $I^{\Omega^{0,*}}$.

Proof. We must show that $Q^{hol}I^{hol} + \frac{1}{2}\{I^{hol}, I^{hol}\}^{hol} = 0$ implies the ordinary classical master equation for $I^{\Omega^{0,*}}$:

$$\overline{\partial} I^{\Omega^{0,*}} + Q^{hol} I^{\Omega^{0,*}} + \frac{1}{2}\{I^{\Omega^{0,*}}, I^{\Omega^{0,*}}\} = 0.$$

Since $I^{\Omega^{0,*}}$ is defined using holomorphic differential operators, the first term vanishes. The fact that $Q^{hol}I^{hol} + \frac{1}{2}\{I^{hol}, I^{hol}\}^{hol} = 0$ implies $Q^{hol}I^{\Omega^{0,*}} + \frac{1}{2}\{I^{\Omega^{0,*}}, I^{\Omega^{0,*}}\} = 0$ follows immediately from our definitions. □

Table 1.1 is a useful summary showing how we are producing a BV theory from a holomorphic theory.

Example 1.2.8. *Holomorphic BF-theory* Let $\mathfrak{g}$ be a Lie algebra and X any complex manifold. Consider the following holomorphic vector bundle on X:

$$V = \underline{\mathfrak{g}}_X \oplus K_X \otimes \mathfrak{g}^*[d-1].$$

The pairing $V \otimes V \to K_X[d-1]$ is similar to the pairing for the $\beta\gamma$ system, except we use the evaluation pairing $\langle -.- \rangle_{\mathfrak{g}}$ between $\mathfrak{g}$ and its dual. In this example, $Q^{hol} = 0$. Write $f \in \mathscr{O}_X^{hol}$ and $\beta \in K_X$ and consider

$$I^{hol}(f_1 \otimes X_1, f_2 \otimes X_2, \beta \otimes X^\vee) = f_1 f_2 \beta \langle X^\vee, [X_1, X_2] \rangle + \cdots$$

where the $\cdots$ means that we symmetrize the inputs. This defines an element $I^{hol} \in \mathscr{O}^{hol}_{\mathrm{loc}}(V)^+$ and the Jacobi identity ensures $\{I^{hol}, I^{hol}\}^{hol} = 0$. The fields of the corresponding BV theory are

$$\mathscr{E}_V = \Omega^{0,*}(X, \mathfrak{g}) \oplus \Omega^{d,*}(X, \mathfrak{g}^*)[d-1].$$

The induced local functional $I^{\Omega^{0,*}}$ on $\mathscr{E}_V$ is

$$I^{\Omega^{0,*}}(\alpha, \beta) = \int_X \langle \beta, [\alpha, \alpha] \rangle_{\mathfrak{g}}.$$

The total action is $S(\alpha, \beta) = \int \langle \beta, \overline{\partial} \alpha \rangle + \langle \beta, [\alpha, \alpha] \rangle_{\mathfrak{g}}$. This is formally similar to BF theory (see below) and for that reason we refer to it as *holomorphic* BF theory. In [**Joh94**], or for a more mathematical treatment see [**Cosb**], it is shown that this theory is a twist of $N = 1$ supersymmetric Yang-Mills on $\mathbb{R}^4$.

Example 1.2.9. *Topological BF-theory* BW: do this

In this case $Q^{hol} = \partial$.

When we construct a BV theory from a holomorphic theory $V \rightsquigarrow \mathscr{E}_V$ it is natural to expect that deformations of the theory must come from holomorphic data. In the special case that $Q^{hol} = 0$ we have the following result which relates the deformation complex of the classical theory $\mathscr{E}_V$ to a sheaf built from holomorphic differential operators.

Lemma 1.2.10. *Suppose* $(V, 0, (-,-)_V, I^{hol})$ *is the data of a holomorphic theory with* $Q^{hol} = 0$. *Let* $(\mathscr{E}_V, Q = \overline{\partial}, \omega_V, I)$ *be the corresponding BV theory. Then, there is a quasi-isomorphism of sheaves*

$$\mathrm{Def}_{\mathscr{E}_V} \simeq \Omega_X^{d,hol} \otimes^{\mathbb{L}}_{D_X^{hol}} \mathscr{O}^{hol}_{red}(V)$$

that is compatible with the brackets and $\{-,-\}$ *and* $\{-,-\}^{hol}$ *on both sides.*

Remark 1.2.11. Just as in the ordinary case we can formulate the data of a classical holomorphic theory in terms of sheaves of L_∞ algebras. We will not do that here, but hope the idea of how to do so is clear.

1.2.2. Holomorphically translation invariant theories

When working on affine space $\mathbb{R}^n$ one can ask for a theory to be invariant with respect to translations. In this section, we consider the affine manifold $\mathbb{C}^d = \mathbb{R}^{2d}$ equipped with its standard complex structure and define what a *holomorphically translation invariant* theory is on it. It will be a very special case of a general holomorphic theory as defined above.

Let V be a holomorphic vector bundle on $\mathbb{C}^n$ and suppose we fix an identification of bundles

$$V \cong \mathbb{C}^d \times V_0$$

where V_0 is the fiber of V at $0 \in \mathbb{C}^d$. We want to consider a classical theory with space of fields given by $\Omega^{0,*}(\mathbb{C}^d, V) \cong \Omega^{0,*}(\mathbb{C}^d) \otimes_{\mathbb{C}} V_0$. Moreover, we want this theory to be invariant with respect to the group of translations on $\mathbb{C}^d$. Per usual, it is best to work with the corresponding Lie algebra of translations. Using the complex structure, we choose a presentation for the Lie algebra of all translations given by

$$\mathbb{C}^{2d} \cong \mathrm{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z}_i} \right\}_{1 \leq i \leq d}.$$

To define a theory, we need to fix a non-degenerate pairing on V. Moreover, we want this to be translation invariant. So, suppose

$$(-,-)_V : V \otimes V \to K_{\mathbb{C}^d}[d-1] \tag{1.7}$$

is a skew-symmetric bundle map that is equivariant for the Lie algebra of translations. The shift is so that the resulting pairing on the Dolbeault complex is of the appropriate degree. Here, equivariance means that for sections v, v' we have

$$(\frac{\partial}{\partial z_i} v, v')_V = L_{\partial_{z_i}} (v, v')_V$$

where the right-hand side denotes the Lie derivative applied to $(v, v')_V \in \Omega^{d,hol}_{\mathbb{C}^d}$. There is a similar relation for the anti-holomorphic derivatives. We obtain a $\mathbb{C}$-valued pairing on $\Omega^{0,*}_c(\mathbb{C}^d, V)$ via integration:

$$\int_{\mathbb{C}^d} \circ (-,-)_V : \Omega^{0,*}_c(\mathbb{C}^d, V) \otimes \Omega^{0,*}_c(\mathbb{C}^d, V) \xrightarrow{\wedge \cdot (-,-)_V} \Omega^{d,*}(\mathbb{C}^d) \xrightarrow{\int} \mathbb{C}.$$

The first arrow is the wedge product of forms combined with the pairing on V. The second arrow is only nonzero on forms of type $\Omega^{d,d}$. Clearly, integration is translation invariant, so that the composition is as well.

This pairing $\Omega^{0,*}(\mathbb{C}^d, V)$ together with the differential $\overline{\partial}$ are enough to define a free theory. However, it is convenient to consider a slightly generalized version of this situation. We want to allow deformations of the differential $\overline{\partial}$ on Dolbeault forms of the form

$$Q = \overline{\partial} + Q^{hol}$$

where Q^{hol} is a holomorphic differential operator of the form

$$Q^{hol} = \sum_I \frac{\partial}{\partial z^I} \mu_I \tag{1.8}$$

where I is some multi-index and $\mu_I : V \to V$ is a linear map of cohomological degree $+1$. Note that we have automatically written Q^{hol} in a way that it is translation invariant. Of course, for this differential to define a free theory there needs to be some compatibility with the pairing on V.

We can summarize this in the following definition, which should be viewed as a slight modification of a free theory to this translation invariant holomorphic setting.

Definition 1.2.12. A *holomorphically translation invariant free BV theory* is the data of a holomorphic vector bundle V together with

(1) an identification $V \cong \mathbb{C}^d \times V_0$;

(2) a translation invariant skew-symmetric pairing $(-,-)_V$ as in (1.7);

(3) a holomorphic differential operator Q^{hol} as in (1.8);

such that the following conditions hold

(1) the induced $\mathbb{C}$-valued pairing $\int \circ (-,-)_V$ is non-degenerate;

(2) the operator Q^{hol} satisfies $(\overline{\partial}+Q^{hol})^2 = 0$ and is skew self-adjoint for the pairing:

$$\int (Q^{hol}v, v')_V = \pm \int (v, Q^{hol}v').$$

The first condition is required so that we obtain an actual (-1)-shifted symplectic structure on $\Omega^{0,*}(\mathbb{C}^d, V)$. The second condition implies that the derivation $Q = \overline{\partial} + Q^{hol}$ defines a cochain complex

$$\mathscr{E}_V = \left(\Omega^{0,*}(\mathbb{C}^d, V), \overline{\partial} + Q^{hol}\right),$$

and that Q is skew self-adjoint for the symplectic structure. Thus, in particular, $\mathscr{E}_V$ together with the pairing define a free BV theory in the ordinary sense. In the usual way, we obtain the action functional via

$$S(\varphi) = \int (\varphi, (\overline{\partial} + Q^{hol})\varphi)_V.$$

Before going further, we will give a familiar example from the last section.

Example 1.2.13. *The free $\beta\gamma$ system on $\mathbb{C}^d$.* Consider the $\beta\gamma$ system with coefficients in any holomorphic vector bundle from Example 1.2.2 (and the remarks after it) specialized to the manifold $X = \mathbb{C}^d$. One immediately checks that this is a holomorphically translation invariant free theory.

1.2.2.1. Translation invariant interactions. Let's fix a general free holomorphically translation invariant theory as above. We now define what a holomorphically translation invariant interacting theory is. Recall, translations span a $2d$-dimensional abelian Lie algebra $\mathbb{C}^{2d} = \mathbb{C}\left\{\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z}_i}\right\}$. The first condition that an interaction be holomorphically translation invariant is that it be translation invariant, so invariant for this Lie algebra.

Let $\overline{\eta}_i$ denote the operator on Dolbeault forms given by contraction with the antiholomorphic vector field $\frac{\partial}{\partial \overline{z}_i}$. Note that η_i acts on the Dolbeault complex on $\mathbb{C}^d$ with values in any vector bundle. In particular it acts on the fields of a free holomorphically translation invariant theory as above, in addition to functionals on fields.

Of course, to be holomorphically translation invariant, it is necessary to look at local functionals that are translation invariant. The extra bit of holomorphicity we take into account on local functionals $I \in \mathscr{O}_{\mathrm{loc}}(\mathscr{E}_V)$ is the following.

Definition 1.2.14. A *holomorphically translation invariant* local functional is a translation invariant local functional $I \in \mathscr{O}_{\mathrm{loc}}(\mathscr{E}_V)^{\mathbb{C}^{2d}}$ such that $\eta_i I = 0$ for all $1 \leq i \leq d$.

There is a succinct way of expressing holomorphic translation invariance as the Lie algebra invariants of a certain super Lie algebra. Let the abelian d-dimensional Lie alegebra spanned by the odd elements $\{\eta_i\}$ be denoted by $\mathbb{C}^d[1]$. We want to consider deformations that are invariant for the action by the total dg Lie algebra $\mathbb{C}^{2d|d} = \mathbb{C}^{2d} \oplus \mathbb{C}^d[1]$. The differential sends $\eta_i \mapsto \frac{\partial}{\partial \overline{z}_i}$. The space of holomorphically translation invariant local functionals are denoted by $\mathscr{O}_{\mathrm{loc}}(\mathscr{E}_V)^{\mathbb{C}^{2d|d}}$. The enveloping algebra of $\mathbb{C}^{2d|d}$ is of the form

$$U(\mathbb{C}^{2d|d}) = \mathbb{C}\left[\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z}_i}, \eta_i\right]$$

with differential induced from that in $\mathbb{C}^{2d|d}$. Note that this algebra is quasi-isomorphic to the algebra of constant coefficient holomorphic differential operators $\mathbb{C}[\partial/\partial z_i] \xrightarrow{\simeq} U(\mathbb{C}^{2d|d})$.

Any translation invariant local functional is a sum of functionals of the form

$$\varphi \mapsto \int_{\mathbb{C}^d} F(D_1\varphi, \ldots, D_k\varphi)$$

where $F : V^{\otimes k} \to \mathbb{C} \cdot \mathrm{d}^d z$ is a linear map and each D_α is an operator in the space

$$\mathbb{C}\left[\mathrm{d}\overline{z}_i, \frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z}_i}, \eta_i\right].$$

The condition $\eta_i I = 0$ means that none of the D_i's have any $\mathrm{d}\overline{z}_j$-dependence. Using this description we can exhibit the space of holomorphically translation functionals as follows. Note that if E is any vector bundle on $\mathbb{C}^d$ we can consider the fiber at zero of its jet bundle that we denote $J_0 E$.

Lemma 1.2.15. *Let V be a holomorphic vector bundle on $\mathbb{C}^d$ and denote $\mathscr{E}_V = \Omega^{0,*}(X, V)$. Then*

$$\mathscr{O}_{\mathrm{loc}}(\mathscr{E}_V)^{\mathbb{C}^{2d|d}} \cong \mathbb{C} \cdot \mathrm{d}^d z \otimes_{U(\mathbb{C}^{2d|d})} \mathscr{O}_{red}(J_0 E_V)$$

where E_V is the vector bundle on $\mathbb{C}^d$ such that $\mathscr{E}_V = \Gamma(E_V)$.

This description of holomorphically translation invariant local functionals allows us to give a convenient description of deformations of holomorphically translation invariant theories. Suppose $(V, Q^{hol}, (-,-)_V, I)$ be the data of an interacting holomorphically translation invariant theory on $\mathbb{C}^d$. We have already encountered the space of local functionals $\mathscr{O}_{\mathrm{loc}}(\mathscr{E}_V)$ and the deformation complex of the interacting BV theory is

$$\mathrm{Def}_{\mathscr{E}_V} = \left(\mathscr{O}_{\mathrm{loc}}(\mathscr{E}_V), \overline{\partial} + Q^{hol} + \{I, -\}\right).$$

We'd like to characterize deformations that preserve holomorphically translation invariance.

Recall that in the holomorphic case there is the holomorphic jet bundle $J^{hol}V$. The fiber at zero of this jet bundle may be identified as $J_0^{hol}V = V_0[[z_1, \ldots, z_d]]$ where the z_i's denote the formal jet coordinate.

Corollary 1.2.16. *Suppose that $Q^{hol} = 0$. Then, there is a quasi-isomorphism*

$$(\mathrm{Def}_{\mathscr{E}_V})^{\mathbb{C}^{2d|d}} \simeq \mathbb{C} \cdot \mathrm{d}^d z \otimes^{\mathbb{L}}_{\mathbb{C}[\partial_{z_1}, \ldots, \partial_{z_d}]} \mathscr{O}_{red}(V_0[[z_1, \ldots, z_d]])[d].$$

Equipped with differential $\{I^{hol}, -\}$ where I^{hol} only depends on holomorphic differential operators. Here, $\partial_{z_i} = \frac{\partial}{\partial z_i}$ and $\mathbb{C} \cdot \mathrm{d}^d z$ denotes the trivial right $\mathbb{C}[\partial_{z_i}]$-module.

The local functional I defining the classical holomorphic theory endows $J^{hol}V[-1]$ the structure of a L_∞ algebra in $D_{\mathbb{C}^d}$-modules. Repackaging the statement using Lie algebraic data we can rewrite the equivalence in the lemma as

$$(\mathrm{Def}_{\mathscr{E}_V})^{\mathbb{C}^{2d|d}} \simeq \mathbb{C} \cdot \mathrm{d}^d z \otimes^{\mathbb{L}}_{\mathbb{C}[\partial_{z_1}, \ldots, \partial_{z_d}]} \mathrm{C}^*_{\mathrm{Lie,red}} \left(V_0[[z]][-1] \right)) \, [d].$$

Proof. By Lemma 1.2.15 we have an expression for the holomorphically translation local functionals

$$(\mathrm{Def}_{\mathscr{E}_V})^{\mathbb{C}^{2d|d}} = \left(\mathbb{C} \cdot \mathrm{d}^d z \otimes_{U(\mathbb{C}^{2d|d})} \mathscr{O}_{red}(J_0 E_V)[d], \overline{\partial} + \{I, -\} \right).$$

Since $\mathscr{O}_{red}(J_0 E_V)$ is flat as a $U(\mathbb{C}^{2d|d})$-module, it follows that we can replaces the tensor product by the derived tensor product $\otimes^{\mathbb{L}}$ up to quasi-isomorphism so that

$$(\mathrm{Def}_{\mathscr{E}_V})^{\mathbb{C}^{2d|d}} \simeq \left(\mathbb{C}\cdot \mathrm{d}^d z \otimes^{\mathbb{L}}_{U(\mathbb{C}^{2d|d})} \mathscr{O}_{red}(J_0 E_V)[d], \overline{\partial} + \{I, -\}\right).$$

Consider the complex $\left(\mathscr{O}_{red}(J_0 E_V), \overline{\partial} + \{I, -\}\right)$. This complex is graded by symmetric degree, and the associated spectral sequence has first page the associated graded of $\mathscr{O}_{red}(J_0 E_V)$ equipped with the $\overline{\partial}$ differential. Moreover, at the E_1-page, we have the quasi-isomorphism

$$\left(\mathscr{O}(J_0 E_V), \overline{\partial}\right) = \left(\mathscr{O}_{red}(V_0[[z_i, \overline{z}_i]][\mathrm{d}\overline{z}_i]), \overline{\partial}\right) \simeq \mathscr{O}_{red}(V_0[[z_i]]).$$

Finally, we have already remarked that there is a quasi-isomorphism of algebras $U(\mathbb{C}^{2d|d}) \simeq U(\mathbb{C}^d)$ where the right-hand site is generated by the constant holomorphic vector fields. The proof of the claim follows.

□

1.3. Renormalization of holomorphic theories

In this section we study the renormalization of holomorphically translation invariant field theories on $\mathbb{C}^d$ for any $d \geq 1$. We start with a classical interacting holomorphic theory on $\mathbb{C}^d$ and consider one-loop homotopy RG flow from some finite scale ϵ to scale L. That is, we consider the sum over graphs of genus zero and one where at each vertex we place the holomorphic interaction. To obtain a prequantization of a classical theory one must make sense of the $\epsilon \to 0$ limit of this construction. In general, this involves introducing a family of counterterms. Our main result is that for a holomorphic theory no such counterterms are required, and one obtains a well-defined $\epsilon \to 0$ limit.

We can write the fields of a holomorphic theory on $\mathbb{C}^d$ as

$$\mathscr{E}_V = \left(\Omega^{0,*}(\mathbb{C}^d, V), \overline{\partial} + Q^{hol}\right)$$

where V is a graded holomorphic vector bundle and Q^{hol} is a holomorphic differential operator.

Since the theory is holomorphically translation invariant we have an identification $\Omega^{0,*}(\mathbb{C}^d, V) \cong \Omega^{0,*}(\mathbb{C}^d) \otimes_{\mathbb{C}} V_0$ where V_0 is the fiber of V over $0 \in \mathbb{C}^d$. Further, we can write the (-1)-shifted symplectic structure defining the classical BV theory in the form

$$\omega_V(\alpha \otimes v, \beta \otimes w) = (v, w)_{V_0} \int \mathrm{d}^d z (\alpha \wedge \beta)$$

where $(-,-)_{V_0}$ is a degree $(d-1)$-shifted pairing on the finite dimensional vector space V_0.

We will assume that the holomorphic Lagrangian I^{hol} is also translation invariant and so defines an interaction of the form $I = \sum_k I_k$ where I_k is symmetric degree k and

$$I_k = \int I_k^{hol}(\varphi) = \int D_{k,1}(\varphi) \cdots D_{k,k}(\varphi) \mathrm{d}^d z$$

where each $D_{i,j}$ is a translation invariant holomorphic differential operator.

1.3.1. Holomorphic gauge fixing

To begin the process of renormalization we must fix the data of a gauge fixing operator. Recall, a gauge fixing operator is an operator on fields

$$Q^{GF} : \mathscr{E}_V \to \mathscr{E}_V[-1]$$

of cohomological degree -1 such that $[Q, Q^{GF}]$ is a generalized Laplacian on $\mathscr{E}$ where Q is the linearized BRST operator.

For holomorphic theories there is a convenient choice for a gauge fixing operator. To construct it we fix the standard flat metric on $\mathbb{C}^d$. Doing this, we let $\overline{\partial}^*$ be the adjoint of the operator $\overline{\partial}$. Using the coordinates on $(z_1, \ldots, z_d) \in \mathbb{C}^d$ we can write this operator as

$$\overline{\partial}^* = \sum_{i=1}^{d} \frac{\partial}{\partial(\mathrm{d}\overline{z}_i)} \frac{\partial}{\partial z_i}.$$

Equivalently $\frac{\partial}{\partial(\mathrm{d}\overline{z}_i)}$ is equal to contraction with the anti-holomorphic vector field $\frac{\partial}{\partial \overline{z}_i}$. The operator $\overline{\partial}^*$ extends to the complex of fields via the formula

$$Q^{GF} = \overline{\partial}^* \otimes \mathrm{id}_V : \Omega^{0,*}(X, V) \to \Omega^{0,*-1}(X, V),$$

We claim that this is a gauge fixing operator for our holomorphic theory. Indeed, since Q^{hol} is a translation invariant holomorphic differential operator we have

$$[\overline{\partial} + Q^{hol}, Q^{GF}] = [\overline{\partial}, \overline{\partial}^*] \otimes \mathrm{id}_V.$$

The operator $[\overline{\partial}, \overline{\partial}^*]$ is simply the Dolbeault Laplacian on $\mathbb{C}^d$, which is certainly a generalized Laplacian. In coordinates it is

$$[\overline{\partial}, \overline{\partial}^*] = -\sum_{i=1}^{d} \frac{\partial}{\partial \overline{z}_i} \frac{\partial}{\partial z_i}$$

By definition, the scale $L > 0$ heat kernel $K_L^V \in \mathscr{E}_V(\mathbb{C}^d) \otimes \mathscr{E}_V(\mathbb{C}^d)$ satisfies

$$\omega_V(K_L, \varphi) = e^{-L[Q, Q^{GF}]}\varphi$$

for any field $\varphi \in \mathscr{E}_V$. Pick a basis $\{e_i\}$ of V_0 and let

$$\mathbf{C}_{V_0} = \sum_{i,j} \omega_{ij}(e_i \otimes e_j) \in V_0 \otimes V_0$$

be the quadratic Casimir. Here, (ω_{ij}) is the inverse matrix to the pairing $(-,-)_{V_0}$. We see that for the holomorphic theory we can write this regularized heat kernel as

$$K_L^V(z, w) = K_L^{an}(z, w) \cdot \mathbf{C}_{V_0}$$

where the analytic part is independent of V and equal to

$$K_L^{an}(z, w) = \frac{1}{(4\pi L)^d} e^{-|z-w|^2/4L} \prod_{i=1}^{d} (\mathrm{d}\overline{z}_i - \mathrm{d}\overline{z}_j) \in \Omega^{0,*}(\mathbb{C}^d) \otimes \Omega^{0,*}(\mathbb{C}^d) \cong \Omega^{0,*}(\mathbb{C}^d \times \mathbb{C}^d).$$

The propagator is defined by

$$P_{\epsilon<L}^V(z, w) = \int_{t=\epsilon}^{L} \mathrm{d}t (Q^{GF} \otimes 1) K_L^V(z, w).$$

Since $\mathbf{C}_{V_0}$ is independent of the coordinate on $\mathbb{C}$ this propagator is of the form $P^V_{\epsilon<L}(z,w) = P^{an}_{\epsilon<L}(z,w)\cdot \mathbf{C}_{V_0}$ where

$$\begin{aligned} P^{an}_{\epsilon<L}(z,w) &= \int_{t=\epsilon}^{L} \mathrm{d}t(\overline{\partial}^* \otimes 1)K^V_L(z,w) \\ &= \int_{t=\epsilon}^{L} \mathrm{d}t \frac{1}{(4\pi t)^d} \sum_{j=1}^{d} \left(\frac{z_j - w_j}{4t}\right) e^{-|z-w|^2/4} \prod_{i\neq j}^{d} (\mathrm{d}\overline{z}_i - \mathrm{d}\overline{z}_j). \end{aligned}$$

Our goal in this section is to show that one-loop RG flow produces a prequantization modulo $\hbar^2$ that requires no counterterms. The one-loop RG flow from ϵ to L is defined by the weight expansion

$$W(P^V_{\epsilon<L}, I) = \sum_{\Gamma} \frac{\hbar^{g(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} W_\Gamma(P^V_{\epsilon<L}, I)$$

where the sum is over graphs of genus ≤ 1 and W_Γ is the weight associated to the graph Γ.

For the genus zero graphs, or trees, we do not have any analytic difficulties to worry about. The propagator $P^V_{\epsilon<L}$ is smooth so long as $\epsilon, L > 0$ but when $\epsilon \to 0$ it inherits a singularity along the diagonal $z = w$. But, if Γ is a tree the weight $W_\Gamma(P^V_{0<L}, I)$ only involves multiplication of distributions with transverse singular support, so is well-defined. Thus we have the following.

Lemma 1.3.1. *If Γ is a tree then $\lim_{\epsilon\to 0} W_\Gamma(P_{\epsilon<L}, I)$ exists.*

1.3.2. One-loop weights

The only possible divergences in the $\epsilon \to 0$ limit, then, must come from graphs of genus one. Every graph of genus one is a wheel with some trees protruding from the external edges of the tree. Thus, we can write the weight of a genus one graph as a product of weights associated to trees times the weight associated to a wheel. As we just saw, the weights associated to trees are automatically convergent in the $\epsilon \to 0$ limit, thus it suffices to focus on genus one graphs that are purely wheels with external edges as in Figure BW:fig.

The definition of the weight of the wheel involves placing the propagator at each internal edge and the interaction I at each vertex. The weights are evaluated by placing compactly supported fields $\varphi \in \mathscr{E}_{V,c} = \Omega^{0,*}_c(\mathbb{C}^d, V)$ at each of the external edges. We will make two simplifications:

(1) the only ϵ dependence appears in the analytic part of the propagator $P^{an}_{\epsilon<L}$, so we can forget about the combinatorial factor $\mathbf{C}_{V_0}$ and assume all external edges are labeled by compactly supported Dolbeault forms in $\Omega^{0,*}_c(\mathbb{C}^d)$;

(2) each vertex labeled by I is a sum of interactions of the form

$$\int_{\mathbb{C}^d} D_1(\varphi) \cdots D_k(\varphi) \mathrm{d}^d z$$

where D_i is a translation invariant differential operator. Some of the differential operators will hit the compactly supported Dolbeault forms placed on the external edges of the graph. The remaining operators will hit the internal edges labeled by the propagators. Since a holomorphic differential operator preserves the space of compactly supported Dolbeault forms that is independent of ϵ, we replace each input by an arbitrary compactly supported Dolbeault form.

Thus, for the $\epsilon \to 0$ behavior it suffices to look at weights of wheels with arbitrary compactly supported functions as inputs where each of the internal edges are labeled by some translation invariant holomorphic differential operator

$$D = \sum_{n_1,\dots n_d} \frac{\partial^{n_1}}{\partial z_1^{n_1}} \cdots \frac{\partial^{n_d}}{\partial z_d^{n_d}}$$

applied to the propagator $P^{an}_{\epsilon<L}$. This motivates the following definition.

Definition 1.3.2. Let $\epsilon, L > 0$. In addition, fix the following data.

(a) An integer $k \geq 1$ that will be the number of vertices of the graph.

(b) For each $\alpha = 1, \dots, k$ a sequence of integers

$$\vec{n}^\alpha = (n_1^\alpha, \dots, n_d^\alpha).$$

We denote by $(\vec{n}) = (n_i^j)$ the corresponding $d \times k$ matrix of integers.

The analytic weight associated to the pair $(k, (\vec{n}))$ is the smooth distribution

$$W^{k,(n)}_{\epsilon<L} : C_c^\infty((\mathbb{C}^d)^k) \to \mathbb{C},$$

that sends a smooth compactly supported function $\Phi \in C_c^\infty((\mathbb{C}^d)^k) = C_c^\infty(\mathbb{C}^{dk})$ to

$$W^{k,(n)}_{\epsilon<L}(\Phi) = \int_{(z^1,\dots,z^k)\in(\mathbb{C}^d)^k} \prod_{\alpha=1}^k \mathrm{d}^d z^\alpha \Phi(z^1, \dots, z^\alpha) \prod_{\alpha=1}^k \left(\frac{\partial}{\partial z^i}\right)^{\vec{n}^\alpha} P^{an}_{\epsilon<L}(z^\alpha, z^{\alpha+1}). \tag{1.9}$$

In the above expression, we use the convention that $z^{k+1} = z^1$.

We will refer to the collection of data $(k, (\vec{n}))$ in the definition as *wheel data*. The motivation for this is that the weight $W^{k,(n)}_{\epsilon<L}$ is the analytic part of the full weight $W_\Gamma(P^V_{\epsilon<L}, I)$ where Γ is a wheel with k vertices.

We have reduced the proof of Lemma 1.0.1 to showing that the $\epsilon \to 0$ limit of the analytic weight $W^{k,(n)}_{\epsilon<L}(\Phi)$ exists for any choice of wheel data $(k, (\vec{n}))$. To do this, there are two steps. First, we show a vanishing result that says when $k \geq d$ the weights vanish for purely algebraic reasons. The second part is the most technical aspect of the chapter where we show that for $k > d$ the weights have nice asymptotic behavior as a function of ϵ.

Lemma 1.3.3. *Let $(k, (\vec{n}))$ be a pair of wheel data. If the number of vertices k satisfies $k \leq d$ then*

$$W^{k,(n)}_{\epsilon<L} = 0$$

as a distribution on $\mathbb{C}^{dk}$ for any $\epsilon, L > 0$.

Proof. In the integral expression for the weight (1.9) there is the following factor involving the product over the edges of the propagators:

$$\prod_{\alpha=1}^{k} \left(\frac{\partial}{\partial z^i}\right)^{\vec{n}^\alpha} P^{an}_{\epsilon<L}(z^i, z^{i+1}). \tag{1.10}$$

We will show that this expression is identically zero. To simplify the expression we first make the following change of coordinates on $\mathbb{C}^{dk}$:

$$w^i = z^{\alpha+1} - z^\alpha \quad , \quad 1 \leq \alpha < k \tag{1.11}$$

$$w^k = z^k. \tag{1.12}$$

Introduce the following operators

$$\eta^\alpha = \sum_{i=1}^{d} \overline{w}_i^\alpha \frac{\partial}{\partial(\mathrm{d}\overline{w}_i^\alpha)}$$

acting on differential forms on $\mathbb{C}^{dk}$. The operator η^α lowers the anti-holomorphic Dolbuealt type by one : $\eta : (p,q) \to (p, q-1)$. Equivalently, η^α is contraction with the anti-holomorphic Euler vector field $\overline{w}_i^\alpha \partial/\partial \overline{w}_i^\alpha$.

Once we do this, we see that the expression (1.10) can be written as

$$\left(\left(\sum_{\alpha=1}^{k-1} \eta^\alpha\right) \prod_{i=1}^{d} \left(\sum_{\alpha=1}^{k-1} \mathrm{d}\overline{w}_i^\alpha\right)\right) \prod_{\alpha=1}^{k-1} \left(\eta^\alpha \prod_{i=1}^{d} \mathrm{d}\overline{w}_i^\alpha\right).$$

Note that only the variables $\overline{w}_i^\alpha$ for $i = 1, \ldots, d$ and $\alpha = 1, \ldots, k-1$ appear. Thus we can consider it as a form on $\mathbb{C}^{d(k-1)}$. As such a form it is of Dolbeault type $(0, (d-1) + (k-1)(d-1)) = (0, (d-1)k)$. If $k < d$ then clearly $(d-1)k > d(k-1)$ so the form has greater degree than the dimension of the manifold and hence it vanishes.

The case left to consider is when $k = d$. In this case, the expression in (1.10) can be written as

$$\left(\left(\sum_{\alpha=1}^{d-1} \eta^\alpha\right) \prod_{i=1}^{d} \left(\sum_{\alpha=1}^{d-1} \mathrm{d}\overline{w}_i^\alpha\right)\right) \prod_{\alpha=1}^{d-1} \left(\eta^\alpha \prod_{i=1}^{d} \mathrm{d}\overline{w}_i^\alpha\right). \tag{1.13}$$

Again, since only the variables $\overline{w}_i^\alpha$ for $i = 1, \ldots, d$ and $\alpha = 1, \ldots, d-1$ appear, we can view this as a differential form on $\mathbb{C}^{d(d-1)}$. Furthermore, it is a form of type $(0, d(d-1))$. For any vector field X on $\mathbb{C}^{d(d-1)}$ the interior derivative i_X is a graded derivation. Suppose ω_1, ω_2 are two $(0, *)$ forms on $\mathbb{C}^{d(d-1)}$ such that the sum of their degrees is equal to d^2. Then, $\omega_1 \iota_X \omega_2$ is a top form for any vector field on $\mathbb{C}^{d(d-1)}$. Since $\omega_1\omega_2 = 0$ for form type reasons, we conclude that $\omega_1 \iota_X \omega_2 = \pm(i_X\omega_1)\omega_2$ with sign depending on the dimension d. Applied to the vector field $\overline{z}_i^1 \partial/\partial\overline{w}_i^1$ in (1.13) we see that the expression can be written (up to a sign) as

$$\eta^1 \left(\sum_{\alpha=1}^{d-1} \eta^\alpha \prod_{i=1}^{d} \left(\sum_{\alpha=1}^{d-1} \mathrm{d}\overline{w}_i^\alpha \right) \right) \left(\prod_{i=1}^{d} \mathrm{d}\overline{w}_i^1 \right) \prod_{\alpha=2}^{d-1} \left(\eta^\alpha \prod_{i=1}^{d} \mathrm{d}\overline{w}_i^\alpha \right).$$

Repeating this, for $\alpha = 2, \ldots, k-1$ we can write this expression (up to a sign) as

$$\left(\eta_{k-1} \cdots \eta_2 \eta_1 \sum_{\alpha=1}^{k-1} \eta^\alpha \prod_{i=1}^{d} \left(\sum_{\alpha=1}^{k-1} \mathrm{d}\overline{w}_i^\alpha \right) \right) \prod_{\alpha=1}^{k-1} \prod_{i=1}^{d} \mathrm{d}\overline{w}_i^\alpha$$

The expression inside the parentheses is zero since each term in the sum over α involves a term like $\eta^\beta \eta^\beta = 0$. This completes the proof for $k = d$. $\square$

Lemma 1.3.4. *Let $(k, (\vec{n}))$ be a pair of wheel data such that $k > d$. Then the $\epsilon \to 0$ limit of the analytic weight*

$$\lim_{\epsilon \to 0} W_{\epsilon<L}^{k,(n)}$$

exists as a distribution on $\mathbb{C}dk$.

Proof. We will bound the absolute value of the weight in Equation (1.9) and show that it has a well-defined $\epsilon \to 0$ limit. First, consider the change of coordinates as in Equations (1.11),(1.12). The weight applied to the compactly supported function Φ can be written as

(1.14)

$$\int_{w^k \in \mathbb{C}^d} \mathrm{d}^d w^k \int_{(w_1,\ldots,w_{k-1}) \in (\mathbb{C}^d)^{k-1}} \left(\prod_{\alpha=1}^{k-1} \mathrm{d}^d w^\alpha \right) \Phi(w^1, \ldots, w^k) \left(\prod_{\alpha=1}^{k-1} \left(\frac{\partial}{\partial w^\alpha} \right)^{\vec{n}^\alpha} P_{\epsilon<L}^{an}(w^\alpha) \right) \sum_{\alpha=1}^{k-1} \left(\frac{\partial}{\partial w^\alpha} \right)^{\vec{n}^k} P^{an}$$

For $\alpha = 1, \dots, k-1$ the notation $P^{an}_{\epsilon<L}(w^\alpha)$ makes sense since $P^{an}_{\epsilon<L}(z^\alpha, z^{\alpha+1})$ is only a function of $w^\alpha = z^{\alpha+1} - z^\alpha$. Similarly $P^{an}_{\epsilon<L}(z^{k+1}, z^1)$ is a function of

$$z^k - z^1 = \sum_{\alpha=1}^{k-1} w^\alpha.$$

Expanding out the propagators the weight takes the form

$$\begin{aligned}
&\int_{w^k \in \mathbb{C}^d} \mathrm{d}^{2d} w^k \int_{(w_1,\dots,w_{k-1}) \in (\mathbb{C}^d)^{k-1}} \left(\prod_{\alpha=1}^{k-1} \mathrm{d}^{2d} w^\alpha\right) \Phi(w^1, \dots, w^k) \int_{(t_1,\dots,t_k)\in[\epsilon,L]^k} \prod_{\alpha=1}^{k} \frac{\mathrm{d}t_\alpha}{(4\pi t_\alpha)^d} \\
&\times \sum_{i_1,\dots,i_{k-1}=1}^{d} \left(\frac{\overline{w}^1_{i_1}}{t_1} \frac{(\overline{w}^1)^{n^1}}{t^{|n^1|}}\right) \cdots \left(\frac{\overline{w}^{k-1}_{i_{k-1}}}{t_{k-1}} \frac{(\overline{w}^{k-1})^{n^{k-1}}}{t^{|n^{k-1}|}}\right) \left(\sum_{\alpha=1}^{k-1} \frac{\overline{w}^\alpha_{i_k}}{t_k} \cdot \frac{1}{t^{|n^k|}} \left(\sum_{\alpha=1}^{k-1} \overline{w}^\alpha\right)^{n^k}\right) \\
&\times \exp\left(-\sum_{\alpha=1}^{k-1} \frac{|w^\alpha|^2}{t_\alpha} - \frac{1}{t_k}\left|\sum_{\alpha=1}^{k-1} w^\alpha\right|^2\right)
\end{aligned}$$

The notation used above warrants some explanation. Recall, for each α the vector of integers is defined as $n^\alpha = (n^\alpha_1, \dots, n^\alpha_d)$. We use the notation

$$(\overline{w}^\alpha)^{n^\alpha} = \overline{w}_1^{n_1^\alpha} \cdots \overline{w}_d^{n_d^\alpha}.$$

Furthermore, $|n^\alpha| = n^\alpha_1 + \cdots + n^\alpha_d$. Each factor of the form $\frac{\overline{w}^\alpha_{i_\alpha}}{t_\alpha}$ comes from the application of the operator $\frac{\partial}{\partial z_i}$ in $\overline{\partial}^*$ applied to the propagator. The factor $\frac{(\overline{w}^\alpha)^{n^\alpha}}{t^{|n^\alpha|}}$ comes from applying the operator $\left(\frac{\partial}{\partial w}\right)^{n^\alpha}$ to the propagator. Note that $\overline{\partial}^*$ commutes with any translation invariant holomorphic differential operator, so it doesn't matter which order we do this.

To bound this integral we will recognize each of the factors

$$\frac{\overline{w}^\alpha_{i_\alpha}}{t_\alpha} \frac{(\overline{w}^\alpha)^{n^\alpha}}{t^{|n^\alpha|}}$$

as coming from the application of a certain holomorphic differential operator to the exponential in the last line. We will then integrate by parts to obtain a simple Gaussian integral which will give us the necessary bounds in the t-variables. Let us denote this Gaussian factor by

$$E(w,t) := \exp\left(-\sum_{\alpha=1}^{k-1}\frac{|w^\alpha|^2}{t_\alpha} - \frac{1}{t_k}\left|\sum_{\alpha=1}^{k-1} w^\alpha\right|^2\right)$$

For each α, i_α introduce the $t=(t_1,\ldots,t_k)$-dependent holomorphic differential operator

$$D_{\alpha,i_\alpha}(t) := \left(\frac{\partial}{\partial w^\alpha_{i_\alpha}} - \sum_{\beta=1}^{k-1}\frac{t_\beta}{t_1+\cdots+t_k}\frac{\partial}{\partial w^\beta_{i_\alpha}}\right)\prod_{j=1}^{d}\left(\frac{\partial}{\partial w^\alpha_j} - \sum_{\beta=1}^{k-1}\frac{t_\beta}{t_1+\cdots+t_k}\frac{\partial}{\partial w^\beta_j}\right)^{n^\alpha_j}.$$

The following lemma is an immediate calculation

Lemma 1.3.5. *One has*

$$D_{\alpha,i_\alpha}E(w,t) = \frac{\overline{w}^\alpha_{i_\alpha}}{t_\alpha}\frac{(\overline{w}^\alpha)^{n^\alpha}}{t^{|n^\alpha|}}E(w,t).$$

Note that all of the D_{α,i_α} operators mutually commute. Thus, we can integrate by parts iteratively to obtain the following expression for the weight:

$$\pm\int_{w^k\in\mathbb{C}^d} \mathrm{d}^{2d}w^k \int_{(w_1,\ldots,w_{k-1})\in(\mathbb{C}^d)^{k-1}}\left(\prod_{\alpha=1}^{k-1}\mathrm{d}^{2d}w^\alpha\right)\int_{(t_1,\ldots,t_k)\in[\epsilon,L]^k}\prod_{\alpha=1}^{k}\frac{\mathrm{d}t_\alpha}{(4\pi t_\alpha)^d}$$
$$\times\left(\sum_{i_1,\ldots,i_d} D_{1,i_1}\cdots D_{k-1,i_{k-1}}\sum_{\alpha=1}^{k-1} D_{\alpha,i_k}\Phi(w^1,\ldots,w^k)\right)\times\exp\left(-\sum_{\alpha=1}^{k-1}\frac{|w^\alpha|^2}{t_\alpha} - \frac{1}{t_k}\left|\sum_{\alpha=1}^{k-1}w^\alpha\right|^2\right).$$

BW: all the differential operators D_{α,i_α} are uniformly bounded in t. To make these precise I should find what the uniform bound is.

Thus, the absolute value of the weight is bounded by

(1.15)

$$|W^{k,(n)}_{\epsilon<L}(\Phi)| \leq C \int_{w^k \in \mathbb{C}^d} \mathrm{d}^{2d} w^k \int_{(w^1,\ldots,w^{k-1}} \prod_{\alpha=1}^{k-1} \mathrm{d}^{2d} w^\alpha \Psi(w^1,\ldots,w^{k-1},w^k) \int_{(t_1,\ldots,t_k)\in[\epsilon,L]^k} \mathrm{d}t_1 \ldots \mathrm{d}t_k \frac{1}{(4\pi)^{dk}} \frac{1}{t_1^d \cdots}$$

To compute the right hand side we will perform a Gaussian integration with respect to the variables $(w^1,\ldots,w^{k-1})$. To this end, notice that the exponential can be written as

$$E(w,t) = \exp\left(-\frac{1}{4} M_{\alpha\beta}(w^\alpha, w^\beta)\right)$$

where $(M_{\alpha\beta})$ is the $(k-1)\times(k-1)$ matrix given by

$$\begin{pmatrix} a_1 & b & b & \cdots & b \\ b & a_2 & b & \cdots & b \\ b & b & a_3 & \cdots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \cdots & a_{k-1} \end{pmatrix}$$

where $a_\alpha = t_\alpha^{-1} + t_k^{-1}$ and $b = t_k^{-1}$. The pairing (w^α, w^β) is the usual Hermitian pairing on $\mathbb{C}^d$, $(w^\alpha, w^\beta) = \sum_i w_i^\alpha \overline{w}_i^\beta$. After some straightforward linear algebra we find that

$$\det(M_{\alpha\beta})^{-1} = \frac{t_1 \cdots t_k}{t_1 + \cdots + t_k}.$$

We now perform a Wick expansion for the Gaussian integral in the variables $(w^1,\ldots,w^{k-1})$. For a reference similar to the notation used here see the Appendix of [**EWY18**]. The inequality in (1.15) becomes

(1.16)

$$|W^{k,(n)}_{\epsilon<L}(\Phi)| \leq C' \int_{w^k \in \mathbb{C}^d} \mathrm{d}^{2d} w^k \Psi(0,\ldots,0,w^k) \int_{(t_1,\ldots,t_k)\in[\epsilon,L]^k} \mathrm{d}t_1 \ldots \mathrm{d}t_k \frac{1}{(4\pi)^{dk}} \frac{1}{(t_1 \cdots t_k)^d} \left(\frac{t_1 \cdots t_k}{t_1 + \cdots + t_k} \right)^d +$$

(1.17)

$$= C' \int_{w^k \in \mathbb{C}^d} \mathrm{d}^{2d} w^k \Psi(0,\ldots,0,w^k) \int_{(t_1,\ldots,t_k)\in[\epsilon,L]^k} \mathrm{d}t_1 \ldots \mathrm{d}t_k \frac{1}{(4\pi)^{dk}} \frac{1}{(t_1 + \cdots + t_k)^d} + O(\epsilon).$$

The first term in the Wick expansion is written out explicitly. The $O(\epsilon)$ refers to higher terms in the Wick expansion, which one can show all have order ϵ, so disappear in the $\epsilon \to 0$ limit. The expression $\Psi(0,\ldots,0,w^k)$ means that we have evaluate the function $\Psi(w^1,\ldots,w^k)$ at $w^1 = \ldots = w^{k-1} = 0$ leaving it as a function only of w^k. In the original coordinates this is equivalent to setting $z^1 = \cdots = z^{k-1} = z^k$.

Our goal is to show that $\epsilon \to 0$ limit of the right-hand side exists. The only ϵ dependence on the right hand side of (1.16) is in the integral over the regulation parameters $t_1,\ldots,t_k$. Thus, it suffices to show that the $\epsilon \to 0$ limit of

$$\int_{(t_1,\ldots,t_k)\in[\epsilon,L]^k} \frac{\mathrm{d}t_1 \ldots \mathrm{d}t_k}{(t_1 + \cdots + t_k)^d}$$

exists. By the AM/GM inequality we have $(t_1 + \cdots + t_k)^d \geq (t_1 \cdots t_d)^{d/k}$. So, the integral is bounded by

$$\int_{(t_1,\ldots,t_k)\in[\epsilon,L]^k} \frac{\mathrm{d}t_1 \ldots \mathrm{d}t_k}{(t_1 + \cdots + t_k)^d} \leq \int_{(t_1,\ldots,t_k)\in[\epsilon,L]^k} \frac{\mathrm{d}t_1 \ldots \mathrm{d}t_k}{(t_1 \cdots t_k)^{d/k}} = \frac{1}{(1-d/k)^k} \left(\epsilon^{1-d/k} - L^{1-d/k}\right)^k.$$

By assumption, $d < k$, so the right hand side has a well-defined $\epsilon \to 0$ limit. This concludes the proof.

$\square$

1.3.3. A general fact about chiral anomalies

Once any theory has been renormalized, the next step to constructing a quantization is to solve the quantum master equation. In general, there may be an obstruction to solving this equation. Such obstructions in the physics literature are known as *anomalies*. In general, it may be difficult to characterize such anomalies, but in the case of holomorphic theories on $\mathbb{C}^d$ our result in the previous section makes this problem much easier. Indeed, since there are no counterterms requires, we can plug in the RG flow of the classical action functional and study the quantum master equation directly. As is usual in perturbation theory, we work order by order in $\hbar$ to construct a quantization. In this section we will study the first step, which is to promote a classical theory to a solution of the quantum master equation modulo $\hbar^2$.

As above, $\mathscr{E}$ will be a holomorphically translation invariant theory on $\mathbb{C}^d$ and I will be the holomorphic interaction. The linearized BRST operator is of the form $Q = \overline{\partial} + Q^{hol}$ where Q^{hol} is a holomorphic differential operator. For this section, it will be most convenient to set $Q^{hol} = 0$.

Define $I[L] = W(P_{\epsilon<L}, I) \mod \hbar^2$ as in the last section. Recall, from Section **??** that the regularized quantum master equation at scale L is

$$QI[L] + \hbar\Delta_L I[L] + \frac{1}{2}\{I[L], I[L]\}_L = 0.$$

This is equivalent to the equation $(Q+\hbar\Delta_L)e^{I[L]/\hbar}=0$. Therefore, the obstruction to satisfying the quantum master equation modulo $\hbar^2$ at scale L is

$$\Theta[L]=\hbar^{-1}\left(QI[L]+\hbar\Delta_L I[L]+\frac{1}{2}\{I[L],I[L]\}_L\right),$$

or, equivalently $\Theta[L]=e^{-I[L]/\hbar}(Q+\hbar\Delta_L)e^{I[L]/\hbar}$. By definition, $I[L]=\lim_{\epsilon\to 0}W(P_{\epsilon<L},I)$ which is equivalent to $e^{I[L]/\hbar}=\lim_{\epsilon\to 0}e^{\hbar\partial_{P_{\epsilon<L}}}e^{I/\hbar} \mod \hbar^2$ as a formal series in $\hbar$. Thus, we can rewrite

$$(Q+\hbar\Delta_L)e^{I[L]/\hbar}=\lim_{\epsilon\to 0}(Q+\hbar\Delta_L)\left(e^{\hbar\partial_{P_{\epsilon<L}}}e^{I/\hbar}\right).$$

Now, the operator Q commutes with $e^{\hbar\partial_{P_{\epsilon<L}}}$. Moreover, $\Delta_L e^{\hbar\partial_{P_{\epsilon<L}}}=e^{\hbar\partial_{P_{\epsilon<L}}}\Delta_\epsilon$ acting on functionals. Thus,

$$\lim_{\epsilon\to 0}(Q+\hbar\Delta_L)\left(e^{\hbar\partial_{P_{\epsilon<L}}}e^{I/\hbar}\right)=\lim_{\epsilon\to 0}e^{\hbar\partial_{P_{\epsilon<L}}}(Q+\hbar\Delta_\epsilon)e^{I/\hbar}.$$

Since Δ_ϵ is a BV operator for the bracket $\{-,-\}_\epsilon$, we can rewrite the right-hand side as

$$\frac{1}{\hbar}\lim_{\epsilon\to 0}e^{\hbar P_{\epsilon<L}}(QI+\hbar\Delta_\epsilon I+\frac{1}{2}\{I,I\}_\epsilon)e^{I/\hbar}.$$

For every $\epsilon>0$ we have $\Delta_\epsilon I=0$. Moreover, since I is holomorphic (and since $Q^{hol}=0$) we have $QI=\overline{\partial}I=0$.

We conclude that the one-loop anomaly is

$$\Theta=\lim_{L\to 0}\Theta[L]=\frac{1}{2}\lim_{L\to 0}\lim_{\epsilon\to 0}e^{-I/\hbar}e^{\hbar\partial_{P_{\epsilon<L}}}\left(\{I,I\}_\epsilon e^{I/\hbar}\right)\mod \hbar^2$$

The main result of this section is the following.

Lemma 1.3.6. *The obstruction* $\Theta = \lim_{L\to 0}\Theta[L]$ *to satisfying the one-loop quantum master equation is given by the expression*

$$\Theta = \lim_{L\to 0}\lim_{\epsilon\to 0}\sum_{\Gamma\in \mathrm{Wheel}_{d+1}} W_\Gamma(P_{\epsilon<L}, K_\epsilon, I)$$

where the sum is over all wheels with $(d+1)$*-vertices.*

Proof. Like the proof of the non-existence of counterterms for holomorphic theories, the proof of this result will be the consequence of an explicit calculations and bounds of certain Feynman diagrams.

If we expand the quantity

$$\lim_{\epsilon\to 0} e^{-I/\hbar}e^{\hbar\partial_{P_{\epsilon<L}}}\left(\{I,I\}_\epsilon e^{I/\hbar}\right) \mod \hbar^2 \tag{1.18}$$

over a sum of graphs, we see that the following two types of weights occur. Note that, by assumption, we are only looking at graphs of genus one which look like wheels with possible trees attach. Graphically, the quantity $\{I,I\}_\epsilon$ is the graph of two vertices with a separating edge labeled by the heat kernel K_ϵ. Thus, all weights appearing in the expansion of (1.18) attach the propagator $P_{\epsilon<L}$ to all edges besides a single distinguished edge e, which is labeled by K_ϵ. The two types of weights are the following:

1. the distinguished edge e is separating;
2. the distinguished edge e is *not* separating, and so appears as the internal edge of the wheel portion of the graph.

By the classical master equation, we see that the $\epsilon\to 0$ limit of weights of Type (1) go to zero. Thus, we must only consider the weights of Type (2).

The result will follow from two steps.

(1) If Γ is a wheel with $k < d+1$ vertices, then $W_\Gamma(P_{\epsilon<L}, K_\epsilon, I) = 0$ identically.

(2) If Γ is a wheel with $k > d+1$ vertices, then $\lim_{\epsilon\to 0} W_\Gamma(P_{\epsilon<L}, K_\epsilon, I) = 0$.

The proof of (1) follows a similar to □

1.3.4. Relation to renormalization of topological theories

BW: it's a fun corollary to see that a slight modification of the above actually shows that topological BF theory in any dimension has a one-loop quantization with no counterterms, not sure if I'll put that here.

1.4. Equivariant BV quantization

Equivariant BV quantization is an enhancement of ordinary BV quantization where one takes into account the action of a group or Lie algebra. We will heavily rely on techniques of equivariant BV quantization throughout this thesis, notably in the construction of the holomorphic σ-model in Chapter **??** and in the proof of a local version of the Grothendieck-Riemann-Roch theorem in Chapter **??** using Feynman diagrammatic expansions.

In this section we define the notion of an equivariance in our formalism through the lens of a central result of classical field theory: Noether's theorem. Roughly speaking, this states that symmetries of a theory are encoded by a conserved quantity. For instance, a symmetry by translations gives rise to conservation of energy through the the stress-energy momentum tensor. There is an enhancement of Noether's theorem using the language of factorization algebras proved in [**CG**] that we will not review here, but will recall in

Chapter 3. For us, the manifestation of Noether's theorem will come from a description of a symmetry through a functional satisfying a certain equivariant version of the classical or quantum master equations.

The symmetries of BV theories that we consider is a direct analog of symmetries in ordinary Hamiltonian mechanics, which we briefly recall. Suppose that $\mathfrak{h}$ is a Lie algebra on (M, ω) is an ordinary symplectic manifold. A symplectic action of $\mathfrak{h}$ on X is a map of Lie algebras

$$\rho : \mathfrak{h} \to \mathrm{SympVect}(M)$$

where $\mathrm{SympVect}(M)$ is the Lie algebra of symplectic vector fields, i.e. those vector fields X which preserve the symplectic form $L_X \omega = 0$. On any symplectic manifold, the Poisson algebra of functions admits a Lie algebra map $\mathscr{O}(M) \to \mathrm{SympVect}(M)$ sending a function f to its Hamiltonian vector field $X_f = \{f, -\}$, where $\{-,-\}$ is the Poisson bracket. An action ρ is said to be *inner* if it lifts to a map of Lie algebras $\widetilde{\rho} : \mathfrak{h} \to \mathscr{O}(M)$. Recall that on any symplectic manifold the kernel of $f \mapsto X_f$ is precisely the constant functions.

All of our classical theories arise as (-1)-shifted symplectic formal moduli problems. Hence, suppose we replace the symplectic manifold M by a formal moduli problem $B\mathfrak{g}$, where $\mathfrak{g}$ is some dg Lie (or L_∞ algebra). To give $B\mathfrak{g}$ the structure of a (-1)-shifted symplectic structure is equivalent to having a (-3)-shifted non-degenerate pairing on $\mathfrak{g}$. Functions on $B\mathfrak{g}$ are precisely the Chevalley-Eilenberg cochains $\mathscr{O}(B\mathfrak{g}) = \mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g})$. The (-1)-shifted symplectic structure equips $\mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g})[-1]$ with the structure of a dg Lie algebra. Since all symplectic vector fields are Hamiltonian in this case we see that

$$\mathrm{SympVect}(B\mathfrak{g}) = \mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g})[-1]/\mathbb{C} = \mathrm{C}^*_{\mathrm{Lie,red}}(\mathfrak{g})[-1]$$

where we have taken the quotient by the constants, which by definition is the reduced cochains. We modify the notion of a symplectic action slightly to allow for more general maps of Lie algebras. A symplectic action of $\mathfrak{h}$ on the (-1)-shifted symplectic formal moduli space $B\mathfrak{g}$ is a map of L_∞ algebras, or a homotopy coherent map of dg Lie algebras

$$\rho : \mathfrak{h} \rightsquigarrow \mathrm{C}^*_{\mathrm{Lie,red}}(\mathfrak{g})[-1].$$

Such a map ρ is equivalent to a Maurer-Cartan element in the dg Lie algebra

$$I^{\mathfrak{h}} \in \mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{h}) \otimes \mathrm{C}^*_{\mathrm{Lie,red}}(\mathfrak{g})[-1].$$

This is a cohomological degree $+1$ element $I^{\mathfrak{h}}$ such that $\mathrm{d}I^{\mathfrak{h}} + \frac{1}{2}\{I^{\mathfrak{h}}, I^{\mathfrak{h}}\} = 0$. Here $\{-,-\}$ is the bracket on $\mathrm{C}^*_{\mathrm{Lie,red}}(\mathfrak{g})$ and d is the sum of the Chevalley-Eilenberg differentials on $\mathfrak{h}$ and $\mathfrak{g}$. This is a version of the classical master equation over the base ring $\mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{h})$.

1.4.1. Classical equivariance

We proceed to mimic the above discussion to define the notion of equivariance for a general classical BV theory. Again, let $\mathfrak{h}$ be an L_∞ algebra. A classical field theory, in the BV formalism, is given by an elliptic formal moduli problem satisfying some conditions. In the beginning of this chapter, we saw that this is encoded by a space of fields $\mathscr{E}$, an action functional $S \in \mathscr{O}_{\mathrm{loc}}(\mathscr{E})$, and a (-1)-shifted symplectic structure.

We have just seen that we can express an action of $\mathfrak{h}$ using a Maurer-Cartan element that is a functional of both $\mathfrak{h}$ and $\mathfrak{g}$. The essential difference with this general situation is that we require our functionals to be *local* with respect to their dependence on the fields $\mathscr{E}$. Recall that the shifted symplectic structure induced a P_0-bracket on local functionals

$\mathscr{O}_{\mathrm{loc}}(\mathscr{E})$. Thus, $\mathscr{O}_{\mathrm{loc}}(\mathscr{E})[-1]$ has the structure of a dg Lie algebra with differential given by $\{S, -\}$.

Definition 1.4.1. An action of $\mathfrak{h}$ on a classical theory $(\mathscr{E}, S, \omega)$ is a Maurer-Cartan element of the dg Lie algebra

$$I^{\mathfrak{h}} \in \mathrm{C}^*_{\mathrm{Lie,red}}(\mathfrak{h}) \otimes \mathscr{O}_{\mathrm{loc}}(\mathscr{E})[-1].$$

In other words, $I^{\mathfrak{h}}$ satisfies the *equivariant classical master equation*:

$$\mathrm{d}_{\mathfrak{h}} I^{\mathfrak{h}} + \{S, I^{\mathfrak{h}}\} + \frac{1}{2}\{I^{\mathfrak{h}}, I^{\mathfrak{h}}\} = 0.$$

Analogous to the manipulations above, we see that such an $I^{\mathfrak{h}}$ defines a sequence of maps

$$\mathfrak{h}^{\otimes m} \otimes \mathscr{E}(X)^{\otimes n} \to \mathscr{E}(X)$$

combining to give $\mathscr{E}(X)$ the structure of an L_∞-module over $\mathfrak{h}$. The equivariant classical master equation exhibits $I^{\mathfrak{h}}$ as a conserved quantity encoding the symmetry by the Lie algebra $\mathfrak{h}$. This is the fundamental idea of Noether's theorem.

Remark 1.4.2. There is a natural map $\mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{h}) \to \mathrm{C}^*_{\mathrm{Lie,red}}(\mathfrak{h})$. An *inner* action of $\mathfrak{h}$ on $\mathscr{E}$ is a lift of an action $I^{\mathfrak{h}}$ to an Maurer-Cartan element of the dg Lie algebra $\mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{h}) \otimes \mathscr{O}_{\mathrm{loc}}(\mathscr{E})[-1]$. Note that there is, in general, an obstruction to lifting which lives in the cohomology $H^1_{\mathrm{Lie}}(\mathfrak{h})$. Thus, if $\mathfrak{h}$ is semi-simple we see that actions always lift to inner actions. We will be more interested in this problem in the case that $\mathfrak{h}$ is a *local* Lie algebra, where the obstruction theory is more interesting.

1.4.2. Quantum equivariance

If we start with an $\mathfrak{h}$-equivariant classical BV theory with fields $\mathscr{E}$ with action functional S — so that $\mathfrak{h}$ has an L_∞ action on the fields that preserves the pairing and the action functional S — then we can encode the action of $\mathfrak{h}$ as a Maurer-Cartan element $I^{\mathfrak{h}}$ in $\mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{h}) \otimes \mathscr{O}_{\mathrm{loc}}(\mathscr{E})$. We then view the sum $S + I^{\mathfrak{h}}$ as the *equivariant* action functional: the operator $\{S + I^{\mathfrak{h}}, -\}$ is the twisted differential on $\mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{h}) \otimes \mathscr{O}_{\mathrm{loc}}(\mathscr{E})$ with $I^{\mathfrak{h}}$ as the twisting cocycle, and this operator is square-zero because $\{S + I^{\mathfrak{h}}, S + I^{\mathfrak{h}}\}$ is a "constant" (i.e., lives in $\mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{h})$ and hence is annihilated by the BV bracket).

This perspective suggests the following definition of an equivariant quantum BV theory. The starting data is two-fold: an $\mathfrak{h}$-equivariant classical BV theory with equivariant action functional $S + I^{\mathfrak{h}}$, and a BV quantization $\{S[L]\}$ of the non-equivariant action functional S. Following Costello, it is convenient to write S as $S_{\mathrm{free}} + I$, where the first "free" term is a quadratic functional and the second "interaction" term is cubic and higher. In this situation, the effective action $S[L] = S_{\mathrm{free}} + I[L]$, i.e., only the interaction changes with the length scale.

As in Section **??** we let $\mathscr{O}^+_{P,sm}(\mathscr{E})$ be the functionals that are at least cubic, have proper support, and have smooth first derivative.

Definition 1.4.3. An $\mathfrak{h}$-*equivariant BV quantization* is a collection of effective interactions $\{I^{\mathfrak{h}}[L]\}_{L \in (0\infty)} \subset \mathrm{C}^*_{\mathrm{Lie,red}}(\mathfrak{h}) \otimes \mathscr{O}^+_{P,sm}(\mathscr{E})[[\hbar]]$ satisfying

(a) the RG flow equation

$$W(P^L_\epsilon, I[\epsilon] + I^{\mathfrak{h}}[\epsilon]) = I[L] + I^{\mathfrak{h}}[L]$$

for all $0 < \epsilon < L$,

(b) the equivariant scale L quantum master equation, which is that

$$Q(I[L] + I^{\mathfrak{h}}[L]) + \mathrm{d}_{\mathfrak{h}} I^{\mathfrak{h}}[L] + \frac{1}{2}\{I[L] + I^{\mathfrak{h}}[L], I[L] + I^{\mathfrak{h}}[L]\}_L + \hbar \Delta_L (I[L] + I^{\mathfrak{h}}[L])$$

lives in $\mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{h})$ for every scale L, and

(c) the locality axiom, with the additional condition that as $L \to 0$, we recover the equivariant classical action functional $S + I^{\mathfrak{h}}$ modulo $\hbar$.

In other words, we simply follow the constructions of [**?**] working over the base ring $\mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{h})$. A careful reading of those texts shows that the freedom to work over interesting dg commutative algebras is built into the formalism.

1.4.3. The case of a local Lie algebra

The above formalism works equally well, with some slight modifications, if we replace the Lie algebra $\mathfrak{h}$ by a *local Lie algebra* $\mathscr{L}$ on the manifold where the theory $\mathscr{E}$ lives. This is done in detail in Chapter 11 of [**CG**], and we refer the reader there for more details.

For the classical case, the first thing we must define is where the classical Noether current $I^{\mathfrak{h}} \leftrightarrow I^{\mathscr{L}}$ lives. Naively, we expect this to live in the space

$$\mathrm{C}^*_{\mathrm{Lie,red}}(\mathscr{L}_c(X)) \otimes \mathscr{O}_{\mathrm{loc}}(\mathscr{E})[-1]. \tag{1.19}$$

This is not quite good enough for our purposes since we have not taken into account the *locality* in the Lie algebra direction. Note that (1.19) is a still a dg Lie algebra, just as above. The inclusion (1.5) determines an inclusion of vector spaces

$$\mathscr{O}_{\mathrm{loc}}(\mathscr{L}[1] \oplus \mathscr{E}) \hookrightarrow \mathrm{C}^*_{\mathrm{Lie,red}}(\mathscr{L}_c(X)) \otimes \mathscr{O}_{\mathrm{loc}}(\mathscr{E})$$

We can further quotient this subspace by $\mathscr{O}_{\mathrm{loc}}(\mathscr{L}[1]) \oplus \mathscr{O}_{\mathrm{loc}}(\mathscr{E}) \subset \mathscr{O}_{\mathrm{loc}}(\mathscr{L}[1] \oplus \mathscr{E})$ consisting of those local functionals that depend solely on $\mathscr{L}$ or $\mathscr{E}$ to obtain an inclusion of vector spaces

$$\mathrm{Act}(\mathscr{L}, \mathscr{E}) := \mathscr{O}_{\mathrm{loc}}(\mathscr{L}[1] \oplus \mathscr{E}) / \mathscr{O}_{\mathrm{loc}}(\mathscr{L}[1]) \oplus \mathscr{O}_{\mathrm{loc}}(\mathscr{E}) \hookrightarrow \mathrm{C}^*_{\mathrm{Lie,red}}(\mathscr{L}_c(X)) \otimes \mathscr{O}_{\mathrm{loc}}(\mathscr{E}).$$

Thus, $\mathrm{Act}(\mathscr{L}, \mathscr{E})$ consists of functionals on $\mathscr{L}[1] \oplus \mathscr{E}$ that are local as both a function of $\mathscr{L}[1]$ and $\mathscr{E}$ and do not depend solely on $\mathscr{L}[1]$ and $\mathscr{E}$.

Lemma 1.4.4 (Chapter ?? [**CG**])**.** *The differential and bracket defining the dg Lie algebra* $\mathrm{C}^*_{\mathrm{Lie,red}}(\mathscr{L}_c(X)) \otimes \mathscr{O}_{\mathrm{loc}}(\mathscr{E})[-1]$ *in (1.19) restricts to give a dg Lie algebra structure on the subspace* $\mathrm{Act}(\mathscr{L}, \mathscr{E})[-1]$.

Using this lemma, the following definition is well-posed.

Definition 1.4.5. Let $\mathscr{L}$ be a local Lie algebra and $\mathscr{E}$ a classical field theory. An $\mathscr{L}$ action on $\mathscr{E}$ is a Maurer-Cartan element

$$I^{\mathscr{L}} \in \mathrm{Act}(\mathscr{L}, \mathscr{E})[-1].$$

In other words, $I^{\mathscr{L}}$ satisfies the equivariant classical master equation

$$\mathrm{d}_{\mathscr{L}} I^{\mathscr{L}} + \{S, I^{\mathscr{L}}\} + \frac{1}{2}\{I^{\mathscr{L}}, I^{\mathscr{L}}\} = 0.$$

Remark 1.4.6. Given any L_∞ algebra $\mathfrak{h}$ one can define the local Lie algebra $\Omega^*_X \otimes \mathfrak{h}$ on X. The data of an action of $\mathfrak{h}$ on a theory as in Definition 1.4.1 is equivalent (up to homotopy) to the data of an action of the local Lie algebra $\Omega^*_X \otimes \mathfrak{h}$ in the definition above. In fact, there is an equivalence of dg Lie algebras $\mathrm{C}^*_{\mathrm{Lie,red}}(\mathfrak{h}) \otimes \mathscr{O}_{\mathrm{loc}}(\mathscr{E})[-1] \simeq \mathrm{Act}(\Omega^*_X \otimes \mathfrak{h}, \mathscr{E})$.

1.4.3.1. The quantum story for an action by a local Lie algebra is also similar to the case of an ordinary Lie algebra. There are two spaces of functionals that appear when discussing actions of a local Lie algebra $\mathscr{L}$ on a quantum field theory. We will fix a quantum field theory as in Definition **??**. This is the data of a free BV theory $(\mathscr{E}, Q, \omega)$ together with a family of functionals $\{I[L]\}$ satisfying RG flow and the QME (plus a locality condition).

Definition 1.4.7. An $\mathscr{L}$-action on the quantum field theory $(\mathscr{E}, Q, \omega, \{I[L]\})$ is the data of a family of functionals

$$\{I^{\mathscr{L}}[L]\} \subset \mathscr{O}^+_{P,sm}(\mathscr{L}[1] \oplus \mathscr{E}) / \mathscr{O}_{P,sm}(\mathscr{L}[1])[[\hbar]]$$

satisfying the following properties:

(a) The RG equation $W(P_{L<L'}, I^{\mathscr{L}}[L]) = I^{\mathscr{L}}[L']$;

(b) The equivariant quantum master equation at scale L:

$$\mathrm{d}_{\mathscr{L}} I^{\mathscr{L}}[L] + Q I^{\mathscr{L}}[L] + \frac{1}{2}\{I^{\mathscr{L}}[L], I^{\mathscr{L}}[L]\}_L + \hbar \Delta_L I^{\mathscr{L}}[L] = 0$$

where $\mathrm{d}_{\mathscr{L}}$ is the Chevalley-Eilenberg differential on $\mathscr{L}$;

(c) the locality axiom as in Definition **??**;

(d) under the natural quotient map

$$\mathscr{O}^+_{P,sm}(\mathscr{L}[1] \oplus \mathscr{E})/\mathscr{O}_{P,sm}(\mathscr{L}[1])[[\hbar]] \to \mathscr{O}^+_{P,sm}(\mathscr{E})[[\hbar]]$$

sends $I^{\mathscr{L}}[L] \mapsto I[L]$ for each $L > 0$.

In the definition above we require $I^{\mathscr{L}}[L]$ to be an element in $\mathscr{O}^+_{P,sm}(\mathscr{L}[1] \oplus \mathscr{E})/\mathscr{O}_{P,sm}(\mathscr{L}[1])[[\hbar]]$, which is the space of smooth and proper functionals on $\mathscr{L}[1] \oplus \mathscr{E}$ that are at least cubic modulo $\hbar$ and do not depend solely on $\mathscr{L}$. A stricter definition is that of an *inner* action, where we allow the functionals that depend solely on $\mathscr{L}[1]$.

Definition 1.4.8. An *inner* action of $\mathscr{L}$ on the QFT $(\mathscr{E}, Q, \omega, \{I[L]\})$ is an effective family

$$\{I^{\mathscr{L}}[L]\} \subset \mathscr{O}^+_{P,sm}(\mathscr{L}[1] \oplus \mathscr{E})$$

satisfying conditions (a)-(c) above and under the natural map

$$\mathscr{O}^+_{P,sm}(\mathscr{L}[1] \oplus \mathscr{E})[[\hbar]] \to \mathscr{O}^+_{P,sm}(\mathscr{E})[[\hbar]]$$

we have $I^{\mathscr{L}}[L] \mapsto I[L]$ for each $L > 0$.

Every inner action clearly defines an ordinary action on a QFT. In practice, we will study the problem of *lifting* an ordinary action to an inner action. Just as in the obstruction theory discussed in Section **??** there is a deformation complex controlling this lifting problem. Indeed, suppose $I^{\mathscr{L}}[L] \in \mathscr{O}^+_{P,sm}(\mathscr{L}[1] \oplus \mathscr{E})/\mathscr{O}_{P,sm}(\mathscr{L}[1])[[\hbar]]$ is a family satisfying the condition of having an action by $\mathscr{L}$. We can lift this to a family of functionals

$$\widetilde{I}^{\mathscr{L}}[L] \in \mathscr{O}^+_{P,sm}(\mathscr{L}[1] \oplus \mathscr{E})[[\hbar]]$$

that satisfy RG flow and the locality axioms, but in general they do not satisfy the equivariant quantum master equation. The obstruction is an element

$$\Theta[L] = \mathrm{d}_{\mathscr{L}} \widetilde{I}^{\mathscr{L}}[L] + Q\widetilde{I}^{\mathscr{L}}[L] + \frac{1}{2}\{\widetilde{I}^{\mathscr{L}}[L], \widetilde{I}^{\mathscr{L}}[L]\}_L + \hbar\Delta_L \widetilde{I}^{\mathscr{L}}[L].$$

Since the right-hand side is zero modulo $\mathscr{O}(\mathscr{L}[1])[[\hbar]]$, by assumption, we must have $\Theta[L] \in \mathscr{O}(\mathscr{L}[1])[[\hbar]]$. By homotopy RG flow it suffices to solve this equation at any scale L. Moreover, by the locality axiom the limit $\lim_{L\to 0}\Theta[L]$ exists and is a local functional of $\mathscr{L}[1]$. Thus we arrive at the following.

Lemma 1.4.9. *Suppose $\{I^{\mathscr{L}}[L]\}$ is an effective family defining an action of $\mathscr{L}$ on a QFT. Then, the obstruction to lifting this action to an inner action, that is the anomaly to solving the equvariant quantum master equation, is the degree $+1$ cocycle in $\Theta = \lim_{L\to 0}\Theta[L] \in \mathrm{C}^*_{\mathrm{loc}}(\mathscr{L})$.*

Remark 1.4.10. Equivariant quantization is essentially a version of the background field method in QFT. One treats elements of $\mathscr{L}$ as background fields and the interaction terms $I^{\mathscr{L}}[L]$ encode the variation of the path integral measure with respect to these background fields. (Solving the QME is our definition of well-posedness of the measure.) This should not be confused with *gauging* a theory by $\mathscr{L}$, which involves putting the elements of $\mathscr{L}$ in the theory as propagating fields.

CHAPTER 2

The holomorphic σ-model

This chapter contains a detailed analysis of one of the most fundamental holomorphic field theories: the holomorphic σ-model. This theory is appealing from both the perspective of mathematics and physics. It is an elegant nonlinear σ-model of maps complex d-fold Y into a complex manifold X (of any complex dimension). The equations of motion pick out the holomorphic maps. Thus, from a purely mathematical perspective, it is a compelling example to study because the classical theory naturally involves complex geometry and so must the quantization, although the meaning is less familiar.

From a physical perspective, this class of theories is intimately related to supersymmetric field theories in various dimensions. In complex dimension one this theory is known as the curved $\beta\gamma$ system. It arises naturally as a close cousin of more central theories: it is a half-twist of the $\mathcal{N} = (0,2)$-supersymmetric σ-model [**Wit07**], and it is also the chiral part of the infinite volume limit of the usual (non-supersymmetric) σ-model. In consequence, the curved $\beta\gamma$ system exhibits many features of these theories while enjoying the flavor of complex geometry, rather than super- or Riemannian geometry. In complex dimension two, we will see, in a similar vein, how the holomorphic σ-model arises as a twist of $\mathcal{N} = 1$ supersymmetry in four real dimensions. There is a similar relationship in dimension six.

In complex dimension one, this theory has appeared in a hidden form in the work of Beilinson-Drinfeld and Malikov-Schechtman-Vaintrob [**BD04, MSV99**], and it was subsequently developed by many mathematicians (see [**KV, Che12, Bre07**] among much else). The *chiral differential operators* (CDOs) on a complex n-manifold X are a sheaf of vertex algebras locally resembling a vertex algebra of n free bosons, and the name indicates the analogy with the differential operators, a sheaf of associative algebras on X locally resembling the Weyl algebra for $T^*\mathbb{C}^n$. Unlike the situation for differential operators, which exist on any manifold X, such a sheaf of vertex algebras exists only if $\mathrm{ch}_2(X) = 0$ in $H^2(X, \Omega^2_{cl})$, and each choice of trivialization α of this characteristic class yields a different sheaf $\mathrm{CDO}_{X,\alpha}$. In other words, there is a gerbe of vertex algebras over X, [**GMS00**]. The appearance of this topological obstruction (essentially the first Pontryagin class, but non-integrally) was surprising, and even more surprising was that the character of this vertex algebra was the Witten genus of X, up to a constant depending only on the dimension of X [**BL00**]. These results exhibited the now-familiar rich connections between conformal field theory, geometry, and topology, but arising from a mathematical process rather than a physical argument.

Witten [**Wit07**] explained how CDOs on X arise as the perturbative piece of the chiral algebra of the curved $\beta\gamma$ system, by combining standard methods from physics and mathematics. (In elegant lectures on the curved $\beta\gamma$ system [**Nek**], with a view toward Berkovit's approach to the superstring, Nekrasov also explains this relationship. Kapustin [**Kap**] gave a similar treatment of the closely-related chiral de Rham complex.) This approach also gave a different understanding of the surprising connections with topology, in line with anomalies and elliptic genera as seen from physics. Let us emphasize that only the perturbative sector of the theory appears (i.e., one works near the constant maps from Σ to T^*X, ignoring the nonconstant holomorphic maps); the instanton corrections are more subtle and not captured just by CDOs (see [**?**] for a treatment of the instanton corrections for complex tori).

In this paper we construct mathematically the perturbative sector of the holomorphic σ-model where the source is allowed to have arbitrary complex dimension. We use the approach to quantum field theory developed in [**?**, **?**], thus providing a rigorous construction of the path integral for the holomorphic σ-model. That means we work in the homotopical framework for field theory known as the Batalin-Vilkovisky (BV) formalism, in conjunction with Feynman diagrams and renormalization methods. Just as CDO's have an anomaly we find that the higher dimensional theory admits a quantized action satisfying the quantum master equation only if the target manifold X has $\mathrm{ch}_{d+1}(X) = 0$, where $\mathrm{ch}_{d+1}(X)$ is the $(d+1)$st component of the Chern character.

One key feature of the framework in [**?**] is that every BV theory yields a factorization algebra of observables. (We mean here the version of factorization algebras developed in [**?**], not the version of Beilinson and Drinfeld [**BD04**].) In our situation, locally speaking the theory produces a factorization algebra living on the source manifold $\mathbb{C}^d$. When $d = 1$ the machinery of [**?**] allows one to extract a vertex algebra from this factorization algebra. It is the main result of our work in [**GGW**] that this vertex algebra is precisely the sheaf of CDOs. One can interpret this as showing that in a wholly mathematical setting, one can start with the action functional for the curved $\beta\gamma$ system and recover the sheaf $\mathrm{CDO}_{X,\alpha}$ of vertex algebras on X via the algorithms of [**?**, **?**]. In higher dimensions we take the sheaf on X of factorization algebras on $\mathbb{C}^d$ produced via our work as a definition of higher dimensional chiral differential operators. The higher dimensional theory of vertex algebras has not been fully developed, but we still show how to extract sensitive algebraic objects from this factorization algebras, such as an A_∞-algebra which one can view as a deformation quantization of the mapping space $\mathrm{Map}(S^{2d-1}, X)$.

Let us explain a little about our methods before stating our theorems precisely. The main technical challenge is to encode the nonlinear σ-model in a way so that the BV formalism of [**Cos11**] applies. In [**Cosa**], Costello introduces a sophisticated approach by which he recovers the anomalies and the Witten genus as partition function, but it seems difficult to relate the local operators (e.g. CDO's in dimension one) directly to the factorization algebra of observables of his quantization. Instead, we use formal geometry *à la* Gelfand and Kazhdan [**?**], as applied to the Poisson σ-model by Kontsevich [**?**] and Cattaneo-Felder [**?**]. The basic idea of Gelfand-Kazhdan formal geometry is that every n-manifold X looks, very locally, like the formal n-disk, and so any representation $\mathscr{V}$ of the formal vector fields and formal diffeomorphisms determines a vector bundle $\mathscr{V} \to X$, by a sophisticated variant of the associated bundle construction. (Every tensor bundle arises in this way, for instance.) In particular, the Gelfand-Kazhdan version of characteristic classes for $\mathscr{V}$ live in the Gelfand-Fuks cohomology $H^*_{\mathrm{Lie}}(\mathrm{W}_n)$ and map to the usual characteristic classes for $\mathscr{V}$. There is, for instance, a Gelfand-Fuks version of the Witten class for every tensor bundle.

Thus, we start with the $\beta\gamma$ system on $\mathbb{C}^d$ with target the formal n-disk $\widehat{D}^n = \mathrm{Spec}\, \mathbb{C}[[\mathrm{t}_1, \ldots, \mathrm{t_n}]]$ and examine whether it quantizes *equivariantly* with respect to the actions of formal vector fields W_n and formal diffeomorphisms on the formal n-disk.[1] (These actions are compatible, so that we have a representation of a Harish-Chandra pair.) We call this theory the *equivariant formal $\beta\gamma$ system of rank n*.

[1]In fact, we will see that it is enough to consider formal vector fields along with the finite dimensional Lie group of linear changes of frame GL_n

Theorem 2.0.1. *The* W_n*-equivariant formal* $\beta\gamma$ *system on* $\mathbb{C}^d$ *of rank* n *has an anomaly given by a cocycle* $\mathrm{ch}_{d+1}(\widehat{D}^n)$ *in the Gelfand-Fuks complex* $\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n; \widehat{\Omega}^{d+1}_{n,cl})$*. This cocycle determines an* L_∞ *algebra extension* $\widetilde{\mathrm{W}}_{n,d}$ *of* W_n*. The cocycle is exact in* $\mathrm{C}^*_{\mathrm{Lie}}(\widetilde{\mathrm{W}}_{n,d}; \widehat{\Omega}^{d+1}_{n,cl})$*, and yields a* $\widetilde{\mathrm{W}}_{n,d}$*-equivariant BV quantization, unique up to homotopy. When* $d = 1$*, the partition function of this theory over the moduli of elliptic curves is the formal Witten class in the Gelfand-Fuks complex* $\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n, \bigoplus_k \widehat{\Omega}^k_n[k])[[\hbar]]$*.*

Throughout this paper, $\mathrm{C}^*_{\mathrm{Lie}}$ means the continuous Lie algebra cohomology, thus $\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n, M)$ is the well-known cohomology studied by Gelfand and Fuks [**?**].

Gelfand-Kazhdan formal geometry is used often in deformation quantization. See, for instance, the elegant treatment by Bezrukavnikov-Kaledin [**BK04**]. Here we develop a version suitable for vertex algebras and factorization algebras, which requires allowing homotopical actions of the Lie algebra W_n. (Something like this appears already in [**BD04, KV, Mal08**], but we need a method with the flavor of differential geometry and compatible with Feynman diagrammatics. It would be interesting to relate directly these different approaches.) In consequence, our equivariant theorem implies the following global version.

Theorem 2.0.2. *Let* $d \geq 1$*, and let* X *be a complex manifold. The holomorphic* σ*-model of maps* $\mathbb{C}^d \to X$ *admits a BV quantization if the class*

$$\mathrm{ch}_{d+1}(T^{1,0}X) \in H^{d+1}(X; \Omega^{d+1}_{cl}) \hookrightarrow H^{2d+2}_{dR}(X),$$

vanishes. Moreover, for every choice of trivialization of this class there is a unique (up to homotopy) $U(d)$-invariant, holomorphically translation invariant, cotangent quantization of the holomorphic σ-model.

When $d = 1$ we showed in [**GGW**] how the resulting factorization algebra produced by this result recovers CDO's. Further, when we place the theory on an elliptic curve we recover the Witten genus of the target manifold. In higher dimensions we provide a detailed analysis of the local operators in this theory that is similar in nature to the operators of a chiral CFT. Indeed, we show how the state space is a natural module for the operators on higher dimensional annuli (neighborhoods of spheres). A full theory of higher dimensional vertex algebras has not been fully developed. It is an interesting question to relate our higher dimensional holomorphic factorization algebras to the more algebro-geometric theory of higher dimensional chiral algebras as in Francis-Gaitsgory [**FG12**].

Our techniques for assembling BV theories in families — and their factorization algebras in families — apply to many σ-models already constructed , such as the topological B-model [**LL16**], Rozansky-Witten theory [**CLL**], and topological quantum mechanics [**GG14, GLL17**]. They also allow us to recover quickly nearly all the usual variants on CDOs and structures therein, such as the chiral de Rham complex and the Virasoro actions. In Chapter **??** of this thesis we study the problem of quantizing a higher dimensional version of the Virasoro action. In complex dimension one we recover the usual requirement that the target be Calabi-Yau. In general we get a more sensitive obstruction, which is still satisfied so long as the target admits a flat connection.

2.1. Gelfand-Kazhdan formal geometry

In this section we review the theory of Gelfand-Kazhdan formal geometry and its use in natural constructions in differential geometry, organized in a manner somewhat different from the standard approaches. We emphasize the role of the frame bundle and jet bundles. We conclude with a treatment of the Atiyah class, which may be our only novel addition (although unsurprising) to the formalism. We refer to our treatment of Harish-Chandra pairs and Harish-Chandra geometry given in Part I of [**GGW**].

We remark that from hereon we will work with complex manifolds and holomorphic vector bundles.

2.1.1. A Harish-Chandra pair for the formal disk

Let $\widehat{\mathcal{O}}_n$ denote the algebra of formal power series

$$\mathbb{C}[[t_1, \dots, t_n]],$$

which we view as "functions on the formal n-disk $\widehat{D}^n$." It is filtered by powers of the maximal ideal $\mathfrak{m}_n = (t_1, \dots, t_n)$, and it is the limit of the sequence of artinian algebras

$$\cdots \to \widehat{\mathcal{O}}_n/(t_1, \dots, t_n)^k \to \cdots \widehat{\mathcal{O}}_n/(t_1, \dots, t_n)^2 \to \widehat{\mathcal{O}}_n/(t_1, \dots, t_n) \cong \mathbb{C}.$$

One can use the associated adic topology to interpret many of our constructions, but we will not emphasize that perspective here.

We use W_n to denote the Lie algebra of derivations of $\widehat{\mathcal{O}}_n$, which consists of first-order differential operators with formal power series coefficients:

$$\mathrm{W}_n = \left\{ \sum_{i=1}^{n} f_i \frac{\partial}{\partial t_i} \ : \ f_i \in \widehat{\mathcal{O}}_n \right\}.$$

The group GL_n also acts naturally on $\widehat{\mathcal{O}}_n$: for $M \in \mathrm{GL}_n$ and $f \in \widehat{\mathcal{O}}_n$,

$$(M \cdot f)(t) = f(Mt),$$

where on the right side we view t as an element of $\mathbb{C}^n$ and let M act linearly. In other words, we interpret GL_n as acting "by diffeomorphisms" on $\widehat{D}^n$ and then use the induced pullback action on functions on $\widehat{D}^n$. The actions of both W_n and GL_n intertwine with multiplication of power series, since "the pullback of a product of functions equals the product of the pullbacks."

2.1.1.1. Formal automorphisms. Let Aut_n be the group of filtration-preserving automorphisms of the algebra $\widehat{\mathcal{O}}_n$, which we will see is a pro-algebraic group. Explicitly, such an automorphism ϕ is a map of algebras that preserves the maximal ideal, so ϕ is specified by where it sends the generators $t_1, \ldots, t_n$ of the algebra. In other words, each $\phi \in \mathrm{Aut}_n$ consists of an n-tuple $(\phi_1, \ldots, \phi_n)$ such that each ϕ_i is in the maximal ideal generated by $(t_1, \ldots, t_n)$ and such that there exists an n-tuple $(\psi_1, \ldots, \psi_n)$ where the composite

$$\psi_j(\phi_1(t), \ldots, \phi_n(t)) = t_j$$

for every j (and likewise with ψ and ϕ reversed). This second condition can be replaced by verifying that the Jacobian matrix

$$Jac(\phi) = (\partial\phi_i/\partial t_j) \in \mathrm{Mat}_n(\widehat{\mathcal{O}}_n)$$

is invertible over $\widehat{\mathcal{O}}_n$, by a version of the inverse function theorem.

Note that this group is far from being finite-dimensional, so it does not fit immediately into the setting of Harish-Chandra pairs described above. It is, however, a *pro*-Lie group in the following way. As each $\phi \in \mathrm{Aut}_n$ preserves the filtration on $\widehat{\mathcal{O}}_n$, it induces an automorphism of each partial quotient $\widehat{\mathcal{O}}_n/\mathfrak{m}_n^k$. Let $\mathrm{Aut}_{n,k}$ denote the image of Aut_n in $\mathrm{Aut}(\widehat{\mathcal{O}}_n/\mathfrak{m}_n^k)$; this group $\mathrm{Aut}_{n,k}$ is clearly a quotient of Aut_n. Note, for instance, that $\mathrm{Aut}_{n,1} = \mathrm{GL}_n$. Explicitly, an element ϕ of $\mathrm{Aut}_{n,k}$ is the collection of n-tuples $(\phi_1, \ldots, \phi_n)$ such that each ϕ_i is an element of $\mathfrak{m}_n/\mathfrak{m}_n^k$ and such that the Jacobian matrix $Jac(\phi)$ is invertible in $\widehat{\mathcal{O}}_n/\mathfrak{m}_n^k$. The group $\mathrm{Aut}_{n,k}$ is manifestly a finite dimensional Lie group, as the quotient algebra is a finite-dimensional vector space.

The group of automorphisms Aut_n is the pro-Lie group associated with the natural sequence of Lie groups

$$\cdots \to \mathrm{Aut}_{n,k} \to \mathrm{Aut}_{n,k-1} \to \cdots \to \mathrm{Aut}_{n,1} = \mathrm{GL}_n.$$

Let Aut_n^+ denote the kernel of the map $\mathrm{Aut}_n \to \mathrm{GL}_n$ so that we have a short exact sequence

$$1 \to \mathrm{Aut}_n^+ \to \mathrm{Aut}_n \to \mathrm{GL}_n \to 1.$$

In other words, for an element ϕ of Aut_n^+, each component ϕ_i is of the form $t_i + \mathcal{O}(t^2)$. The group Aut_n^+ is pro-nilpotent, hence contractible.

The Lie algebra of Aut_n is *not* the Lie algebra of formal vector fields W_n. A direct calculation shows that the Lie algebra of Aut_n is the Lie algebra $\mathrm{W}_n^0 \subset \mathrm{W}_n$ of formal vector fields with zero constant coefficient (i.e., that vanish at the origin of $\widehat{D}^n$).

Observe that the group GL_n acts on the Lie algebra W_n by the obvious linear "changes of frame." The Lie algebra $\mathrm{Lie}(\mathrm{GL}_n) = \mathfrak{gl}_n$ sits inside W_n as the linear vector fields

$$\left\{ \sum_{i,j} a_i^j t_i \frac{\partial}{\partial t_j} \ : \ a_j^i \in \mathbb{C} \right\}.$$

We record these compatibilities in the following statement.

Lemma 2.1.1. *The pair* $(\mathrm{W}_n, \mathrm{GL}_n)$ *form a Harish-Chandra pair.*

Proof. The only thing to check is that the derivative of the action of GL_n corresponds with the adjoint action of $\mathfrak{gl}_n \subset \mathrm{W}_n$ on formal vector fields. This is by construction. $\square$

2.1.2. The coordinate bundle

In this section we review the central object in the Gelfand-Kazhdan picture of formal geometry: the coordinate bundle.

2.1.2.1. Given a complex manifold, its *coordinate space* X^{coor} is the (infinite-dimensional) space parametrizing jets of holomorphic coordinates of X. (It is a pro-complex manifold, as we'll see.) Explicitly, a point in X^{coor} consists of a point $x \in X$ together with an ∞-jet class of a local biholomorphism $\phi : U \subset \mathbb{C}^n \to X$ sending a neighborhood U of the origin to a neighborhood of x such that $\phi(0) = x$.

There is a canonical projection map $\pi^{coor} : X^{coor} \to X$ by remembering only the underlying point in X. The group Aut_n acts on X^{coor} by "change of coordinates," i.e., by precomposing a local biholomorphism ϕ with an automorphism of the disk around the origin in $\mathbb{C}^n$. This action identifies π^{coor} as a principal bundle for the pro-Lie group Aut_n.

One way to formalize these ideas is to realize X^{coor} as a limit of finite-dimensional complex manifolds. Let X_k^{coor} be the space consisting of points $(x, [\phi]_k)$, where ϕ is a local biholomorphism as above and $[-]_k$ denotes taking its k-jet equivalence class. Let $\pi_k^{coor} : X_k^{coor} \to X$ be the projection. By construction, the finite-dimensional Lie group $\mathrm{Aut}_{n,k}$ acts on the fibers of the projection freely and transitively so that π_k^{coor} is a principal $\mathrm{Aut}_{n,k}$-bundle. The bundle $X^{coor} \to X$ is the limit of the sequence of principal bundles on X

$$\begin{tikzcd} \cdots \arrow[r] & X_k^{coor} \arrow[r] \arrow[rrrrd, "\pi_k^{coor}"'] & X_{k-1}^{coor} \arrow[r] \arrow[rrrd, "\pi_{k-1}^{coor}"] & \cdots \arrow[r] & X_2^{coor} \arrow[r] \arrow[rd, "\pi_2^{coor}"] & X_1^{coor} \arrow[d, "\pi_1^{coor}"] \\ & & & & & X. \end{tikzcd}$$

In particular, note that the $\mathrm{GL}_n = \mathrm{Aut}_{n,1}$-bundle $\pi_1^{coor} : X_1^{coor} \to X$ is the frame bundle

$$\pi^{fr} : \mathrm{Fr}_X \to X,$$

i.e., the principal bundle associated to the tangent bundle of X.

2.1.2.2. The Grothendieck connection. We can also realize the Lie algebra W_n as an inverse limit. Recall the filtration on W_n by powers of the maximal ideal $\mathfrak{m}_n$ of $\widehat{\mathcal{O}}_n$. Let $\mathrm{W}_{n,k}$ denote the quotient $\mathrm{W}_n/\mathfrak{m}_n^{k+1}\mathrm{W}_n$. For instance, $\mathrm{W}_{n,1} = \mathfrak{aff}_n = \mathbb{C}^n \ltimes \mathfrak{gl}_n$, the Lie algebra of affine transformations of $\mathbb{C}^n$. We have $\mathrm{W}_n = \lim_{k\to\infty} \mathrm{W}_{n,k}$.

The Lie algebra of $\mathrm{Aut}_{n,k}$ is

$$\mathrm{W}^0_{n,k} := \mathfrak{m}_n \cdot \mathrm{W}_n / \mathfrak{m}_n^{k+1} \mathrm{W}^0_n.$$

That is, the Lie algebra of vector fields vanishing at zero modulo the $k+1$ power of the maximal ideal. Thus, the principal $\mathrm{Aut}_{n,k}$-bundle $X^{coor}_k \to X$ induces an exact sequence of tangent spaces

$$\mathrm{W}^0_{n,k} \to T_{(x,[\varphi]_k)} X^{coor} \to T_x X;$$

by using φ, we obtain a canonical isomorphism of tangent spaces $\mathbb{C}^n \cong T_0\mathbb{C}^n \cong T_x X$. Combining these observations, we obtain an isomorphism

$$\mathrm{W}_{n,k} \cong T_{(x,[\varphi]_k)} X^{coor}_k.$$

In the limit $k \to \infty$ we obtain an isomorphism $\mathrm{W}_n \cong T_{(x,[\varphi]_\infty)} X^{coor}$.

Proposition 2.1.2 (Section 5 of [**NT95**], Section 3 of [**CF01**])**.** *There exists a canonical action of* W_n *on* X^{coor} *by holomorphic vector fields, i.e., there is a Lie algebra homomorphism*

$$\theta : \mathrm{W}_n \to \mathcal{X}^{hol}(X^{coor}),$$

where $\mathcal{X}^{hol}(X^{coor})$ *is the Lie algebra of holomorphic vector fields. Moreover, this action induces the isomorphism* $\mathrm{W}_n \cong T_{(x,[\phi]_\infty)} X^{coor}$ *at each point.*

Here, $\mathcal{X}(X^{coor})$ is understood as the inverse limit of the finite-dimensional Lie algebras $\mathcal{X}(X^{coor}_k)$.

The inverse of the map θ provides a connection one-form

$$\omega^{coor} \in \Omega^1_{hol}(X^{coor}; \mathrm{W}_n),$$

which we call the *universal Grothendieck connection* on X. As θ is a Lie algebra homomorphism, ω^{coor} satisfies the Maurer-Cartan equation

$$\partial\omega^{coor} + \frac{1}{2}[\omega^{coor}, \omega^{coor}] = 0. \tag{2.1}$$

Note that the proposition ensures that this connection is universal on all complex manifolds of dimension n and indeed pulls back along local biholomorphisms.

Remark 2.1.3. We can view ω^{coor} as an element of the full de Rham complex $\omega^{coor} \in \Omega^1(X^{coor}; \mathrm{W}_n)$ where the Maurer-Cartan equation reads $\mathrm{d}\omega + \frac{1}{2}[\omega^{coor}, \omega^{coor}] = 0$.

Remark 2.1.4. Both the pair $(\mathrm{W}_n, \mathrm{Aut})$ and the bundle $X^{coor} \to X$ together with ω^{coor} do not fit in the finite dimensional models for Harish-Chandra geometry. They are, however, objects in a larger category of pro-Harish-Chandra pairs and pro-Harish-Chandra bundles, respectively. We do not fully develop this theory here, but it is inherent in the work of [**BK04**]. Indeed, by working with well-behaved representations for the pair $(\mathrm{W}_n, \mathrm{Aut})$, Gelfand, Kazhdan, and others use this universal construction to produce many of the natural constructions in differential geometry. As we remarked earlier, it is a kind of refinement of tensor calculus.

2.1.2.3. A Harish-Chandra structure on the frame bundle. Although the existence of the coordinate bundle X^{coor} is necessary in the remainder of this paper, it is convenient for us to use it in a rather indirect way. Rather, we will work with the frame bundle $\mathrm{Fr}_X \to X$ equipped with the structure of a module for the Harish-Chandra pair $(\mathrm{W}_n, \mathrm{GL}_n)$. The W_n-valued connection on Fr_X is induced from the Grothendieck connection above.

Definition 2.1.5. Let $\mathrm{Exp}(X)$ denote the quotient X^{coor}/GL_n. A holomorphic section of $\mathrm{Exp}(X)$ over X is called a *formal exponential*.

Remark 2.1.6. The space $\mathrm{Exp}(X)$ can be equipped with the structure of a principal Aut_n^+-bundle over X. This structure on $\mathrm{Exp}(X)$ depends on a choice of a section of the short exact sequence

$$1 \to \mathrm{Aut}_n^+ \to \mathrm{Aut}_n \to \mathrm{GL}_n \to 1.$$

It is natural to use the splitting determined by the choice of coordinates on the formal disk.

Note that Aut_n^+ is contractible, and so sections always exist. A formal exponential is useful because it equips the frame bundle with a $(\mathrm{W}_n, \mathrm{GL}_n)$-module structure, as follows.

Proposition 2.1.7. *A formal exponential* σ *pulls back to a* GL_n*-equivariant map* $\tilde{\sigma}$: $\mathrm{Fr}_X \to X^{coor}$*, and hence equips* $(\mathrm{Fr}_x, \sigma^*\omega^{coor})$ *with the structure of a principal* $(\mathrm{W}_n, \mathrm{GL}_n)$*-bundle with flat connection. Moreover, any two choices of formal exponential determine* $(\mathrm{W}_n, \mathrm{GL}_n)$*-structures on* X *that are gauge-equivalent.*

For a full proof, see [**NT95**], [**?**], or [**?**] but the basic idea is easy to explain.

Sketch of proof. The first assertion is tautological, since the data of a section is equivalent to such an equivariant map, but we explicate the underlying geometry. A map

$\rho : \mathrm{Fr}_X \to X^{coor}$ assigns to each pair $(x, \mathbf{y}) \in \mathrm{Fr}_X$, with $x \in X$ and $\mathbf{y} : \mathbb{C}^n \xrightarrow{\cong} T_xX$ a linear frame, an ∞-jet of a biholomorphism $\phi : \mathbb{C}^n \to X$ such that $\phi(0) = x$ and $D\phi(0) = \mathbf{y}$. Being GL_n-equivariant ensures that these biholomorphisms are related by linear changes of coordinates on $\mathbb{C}^n$. In other words, a GL_n-equivariant map $\tilde{\sigma}$ describes how each frame on T_xX exponentiates to a formal coordinate system around x, and so the associated section σ assigns a formal exponential map $\sigma(x)\colon T_xX \to X$ to each point x in X. (Here we see the origin of the name "formal exponential.")

The second assertion would be immediate if X^{coor} were a complex manifold, since the flat bundle structure would pull back, so all issues are about carefully working with pro-manifolds.

The final assertion is also straightforward: the space of sections is contractible since Aut_n^+ is contractible, so one can produce an explicit gauge equivalence. $\square$

Remark 2.1.8. In [?] Willwacher provides a description of the space $\mathrm{Exp}(X)$ of *all* formal exponentials. He shows that it is isomorphic to the space of pairs (∇_0, Φ) where ∇_0 is a torsion-free connection on X for T_X and Φ is a section of the bundle

$$\mathrm{Fr}_X \times_{\mathrm{GL}_n} \mathrm{W}_n^3$$

where $\mathrm{W}_n^3 \subset \mathrm{W}_n$ is the subspace of formal vector fields whose coefficients are at least cubic. In particular, every torsion-free affine connection determines a formal exponential. The familiar case above that produces a formal coordinate from a connection corresponds to choosing the zero vector field.

Definition 2.1.9. A *Gelfand-Kazhdan structure* is a complex manifold X of dimension n together with a formal exponential σ, which makes the frame bundle Fr_X into a flat $(\mathrm{W}_n, \mathrm{GL}_n)$-bundle with connection one-form ω_σ, the pullback of ω^{coor} along the GL_n-equivariant lift $\tilde{\sigma} : \mathrm{Fr}_X \to X^{coor}$.

Example 2.1.10. Consider the case of an open subset $U \subset \mathbb{C}^n$. There are thus natural holomorphic coordinates $\{z_1, \ldots, z_n\}$ on U. These coordinates provides a natural choice of a formal exponential. Moreover, with respect to the isomorphism

$$\Omega^1_{hol}(\mathrm{Fr}_U; \mathrm{W}_n)^{\mathrm{GL}_n} \cong \Omega^1_{hol}(U; \mathrm{W}_n) \cong \mathscr{O}^{hol}(U)[\mathrm{d}z_i] \otimes \mathrm{W}_n,$$

we find that the connection 1-form has the form

$$\omega^{coor} = \sum_{i=1}^{n} \mathrm{d}z_i \otimes \frac{\partial}{\partial t_i},$$

where the $\{t_i\}$ are the coordinates on the formal disk $\widehat{D}^n$.

A Gelfand-Kazhdan structure allows us to apply a version of Harish-Chandra descent, which will be a central tool in our work.

Although we developed Harish-Chandra descent on all flat $(\mathfrak{g}, K)$-bundles, it is natural here to restrict our attention to manifolds of the same dimension, as the notions of coordinate and affine bundle are dimension-dependent. Hence we replace the underlying category of all complex manifolds by a more restrictive setting.

Definition 2.1.11. Let Hol_n denote the category whose objects are complex manifolds of dimension n and whose morphisms are local biholomorphisms. In other words, a map

$f : X \to Y$ in Hol_n is a map of complex manifolds such that each point $x \in X$ admits a neighborhood U on which $f|_U$ is biholomorphic with $f(U)$.

There is a natural inclusion functor $i : \mathrm{Hol}_n \to \mathrm{CplxMan}$ (not fully faithful) and the frame bundle Fr defines a section of the fibered category $i^*\mathrm{VB}$, since the frame bundle pulls back along local biholomorphisms. For similar reasons, the coordinate bundle is a pro-object in $i^*\mathrm{VB}$.

Definition 2.1.12. Let GK_n denote the category fibered over Hol_n whose objects are a Gelfand-Kazhdan structure — that is, a pair (X, σ) of a complex n-manifold and a formal exponential — and whose morphisms are simply local biholomorphisms between the underlying manifolds.

Note that the projection functor from GK_n to Hol_n is an equivalence of categories, since the space of formal exponentials is affine.

2.1.3. The category of formal vector bundles

For most of our purposes, it is convenient and sufficient to work with a small category of $(\mathrm{W}_n, \mathrm{GL}_n)$-modules that is manifestly well-behaved and whose localizations appear throughout geometry in other guises, notably as ∞-jet bundles of vector bundles on complex manifolds. (Although it would undoubtedly be useful, we will not develop here the general theory of modules for the Harish-Chandra pair $(\mathrm{W}_n, \mathrm{GL}_n)$, which would involve subtleties of pro-Lie algebras and their representations.)

We first start by describing the category of $(\mathrm{W}_n, \mathrm{GL}_n)$-modules that correspond to modules over the structure sheaf of a manifold. Note that $\widehat{\mathcal{O}}_n$ is the quintessential example of a commutative algebra object in the symmetric monoidal category of $(\mathrm{W}_n, \mathrm{GL}_n)$-modules, for any natural version of such a category. We consider modules that have actions of both the pair and the algebra $\widehat{\mathcal{O}}_n$ with obvious compatibility restrictions.

Definition 2.1.13. A *formal $\widehat{\mathcal{O}}_n$-module* is a vector space $\mathcal{V}$ equipped with

(i) the structure of a $(\mathrm{W}_n, \mathrm{GL}_n)$-module;

(ii) the structure of a $\widehat{\mathcal{O}}_n$-module;

such that

(1) for all $X \in \mathrm{W}_n$, $f \in \widehat{\mathcal{O}}_n$ and $v \in \mathcal{V}$ we have $X(f \cdot v) = X(f) \cdot v + f \cdot (X \cdot v)$;

(2) for all $A \in \mathrm{GL}_n$ we have $A(f \cdot v) = (A \cdot f) \cdot (A \cdot v)$, where A acts on f by a linear change of frame.

A morphism of formal $\widehat{\mathcal{O}}_n$-modules is a $\widehat{\mathcal{O}}_n$-linear map of $(\mathrm{W}_n, \mathrm{GL}_n)$-modules $f : \mathcal{V} \to \mathcal{V}'$. We denote this category by $\mathrm{Mod}^{\mathcal{O}_n}_{(\mathrm{W}_n, \mathrm{GL}_n)}$.

Just as the category of D-modules is symmetric monoidal via tensor over $\mathcal{O}$, we have the following result.

Lemma 2.1.14. *The category $\mathrm{Mod}^{\mathcal{O}_n}_{(\mathrm{W}_n, \mathrm{GL}_n)}$ is symmetric monoidal with respect to tensor over $\widehat{\mathcal{O}}_n$.*

Proof. The category of $\widehat{\mathcal{O}}_n$-modules is clearly symmetric monoidal by tensoring over $\widehat{\mathcal{O}}_n$. We simply need to verify that the Harish-Chandra module structures extend in a natural way, but this is clear. $\square$

We will often restrict ourselves to considering Harish-Chandra modules as above that are free as underlying $\widehat{\mathcal{O}}_n$-modules. Indeed, let

$$\mathrm{VB}_n \subset \mathrm{Mod}^{\mathcal{O}_n}_{(\mathrm{W}_n, \mathrm{GL}_n)}$$

be the full subcategory spanned by objects that are free and finitely generated as underlying $\widehat{\mathcal{O}}_n$-modules. Upon descent these will correspond to ordinary vector bundles and so we refer to this category as *formal vector bundles*.

The category of formal $\widehat{\mathcal{O}}_n$-modules has a natural symmetric monoidal structure by tensor product over $\widehat{\mathcal{O}}$. The Harish-Chandra action is extended by

$$X \cdot (s \otimes t) = (Xs) \otimes t + s \otimes (Xt).$$

This should not look surprising; it is the same formula for tensoring D-modules over $\mathcal{O}$.

The internal hom $\mathrm{Hom}_{\widehat{\mathcal{O}}}(\mathcal{V}, \mathcal{W})$ also provides a vector bundle on the formal disk, where the Harish-Chandra action is extended by

$$(X \cdot \phi)(v) = X \cdot (\phi(v)) - \phi(X \cdot v).$$

Observe that for any D-module M, we have an isomorphism

$$\mathrm{Hom}_D(\widehat{\mathcal{O}}, M) \cong \mathrm{Hom}_{\mathrm{W}_n}(\mathbb{C}, M)$$

since a map of $\widehat{D}$-modules out of $\widehat{\mathcal{O}}$ is determined by where it sends the constant function 1. Hence we find that there is a quasi-isomorphism

$$\mathbb{R}\mathrm{Hom}_D(\widehat{\mathcal{O}}, \mathcal{V}) \simeq \mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n; \mathcal{V}),$$

or more accurately a zig-zag of quasi-isomorphisms. Here $\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n; \mathcal{V})$ is the continuous cohomology of W_n with coefficients in $\mathcal{V}$. This is known as the *Gelfand-Fuks* cohomology of $\mathcal{V}$ and is what we use for the remainder of the paper.

This relationship extends to the GL_n-equivariant setting as well, giving us the following result.

Lemma 2.1.15. *There is a quasi-isomorphism*

$$\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n, \mathrm{GL}_n; \mathcal{V}) \simeq \mathbb{R}\mathrm{Hom}_D(\widehat{\mathcal{O}}, \mathcal{V})^{\mathrm{GL}_n-\mathrm{eq}},$$

where the superscript $\mathrm{GL}_n - \mathrm{eq}$ *denotes the* GL_n*-equivariant maps.*

Remark 2.1.16. One amusing way to understand this category is as Harish-Chandra descent to the formal n-disk itself. Consider the frame bundle $\widehat{\mathrm{Fr}} = \widehat{D}^n \times \mathrm{GL}_n \to \widehat{D}^n$ of the formal n-disk itself, which possesses a natural flat connection via the Maurer-Cartan form ω_{MC} on GL_n. Let $\rho : \mathrm{GL}_n \to \mathrm{GL}(V)$ be a finite-dimensional representation. Then the subcomplex of $\Omega^*(\widehat{\mathrm{Fr}}) \otimes V$ given by the basic forms is isomorphic to

$$\left(\Omega^*(\widehat{D}^n) \otimes V, \mathrm{d}_{dR} + \rho(\omega_{MC})\right).$$

This equips the associated bundle $\widehat{\mathrm{Fr}} \times^{\mathrm{GL}_n} V$ with a flat connection and hence makes its sheaf of sections a D-module on the formal disk.

Many of the important $\widehat{\mathcal{O}}_n$-modules we will consider simply come from linear tensor representations of GL_n. Given a finite-dimensional GL_n-representation V, we construct a $\widehat{\mathcal{O}}_n$-module $\mathcal{V} \in \mathrm{VB}_n$ as follows.

Consider the decreasing filtration of W_n by vanishing order of jets

$$\cdots \subset \mathfrak{m}_n^2 \cdot \mathrm{W}_n \subset \mathfrak{m}_n^1 \cdot \mathrm{W}_n \subset \mathrm{W}_n.$$

The induced map $\mathfrak{m}_n^1 \cdot \mathrm{W}_n \to \mathfrak{m}_n^1 \cdot \mathrm{W}_n / \mathfrak{m}_n^2 \cdot \mathrm{W}_n \cong \mathfrak{gl}_n$ allows us to restrict V to a $\mathfrak{m}_n^1 \cdot \mathrm{W}_n$-module. We then coinduce this module along the inclusion $\mathfrak{m}^1 \cdot \mathrm{W}_n \subset \mathrm{W}_n$ to get a W_n-module $\mathcal{V} = \mathrm{Hom}_{\mathfrak{m}_n^1 \cdot \mathrm{W}_n}(\mathrm{W}_n, V)$. There is an induced action of GL_n on $\mathcal{V}$. Indeed, as a GL_n-representation one has $\mathcal{V} \cong \widehat{\mathcal{O}}_n \otimes_{\mathbb{C}} V$. Moreover, this action is compatible with the W_n-module structure, so that $\mathcal{V}$ is actually a $(\mathrm{W}_n, \mathrm{GL}_n)$-module. Thus, the construction provides a functor from $\mathrm{Rep}_{\mathrm{GL}_n}$ to VB_n.

Definition 2.1.17. We denote by Tens_n the image of finite-dimensional GL_n-representations in VB_n along this functor. We call it the category of *formal tensor fields*.

As mentioned $\widehat{\mathcal{O}}_n$ is an example, associated to the trivial one-dimensional GL_n representation. Another key example is $\widehat{\mathcal{T}}_n$, the vector fields on the formal disk, which is associated to the defining GL_n representation $\mathbb{C}^n$; it is simply the adjoint representation of W_n. Other examples include $\widehat{\Omega}_n^1$, the 1-forms on the formal disk; it is the correct version of the coadjoint representation, and more generally the space of k-forms on the formal disk $\widehat{\Omega}_n^k$.

The category Tens_n can be interpreted in two other ways, as we will see in subsequent work.

(1) They are the ∞-jet bundles of tensor bundles: for a finite-dimensional GL_n-representation, construct its associated vector bundle along the frame bundle and take its ∞-jets.

(2) They are the flat vector bundles of finite-rank on the formal n-disk that are equivariant with respect to automorphisms of the disk. In other words, they are GL_n-equivariant D-modules whose underlying $\widehat{\mathcal{O}}$-module is finite-rank and free.

It should be no surprise that given a Gelfand-Kazhdan structure on the frame bundle of a non-formal n-manifold X, a formal tensor field descends to the ∞-jet bundle of the corresponding tensor bundle on X. The flat connection on this descent bundle is, of course, the Grothendieck connection on this ∞-jet bundle. (For some discussion, see section 1.3, pages 12-14, of [**Fuk86**].)

Note that the subcategories

$$\mathrm{Tens}_n \hookrightarrow \mathrm{VB}_n \hookrightarrow \mathrm{Mod}^{\mathcal{O}_n}_{(\mathrm{W}_n,\mathrm{GL}_n)}$$

inherit the symmetric monoidal structure constructed above.

2.1.4. Gelfand-Kazhdan descent

We will focus on defining descent for the category VB_n of formal vector bundles.

Fix an n-dimensional manifold X. The main result of this section is that the associated bundle construction along the frame bundle Fr_X,

$$\begin{array}{rccc} \mathrm{Fr}_X \times^{\mathrm{GL}_n} - : & \mathrm{Rep}(\mathrm{GL}_n)^{fin} & \to & \mathrm{VB}(X) \\ & V & \mapsto & \mathrm{Fr}_X \times^{\mathrm{GL}_n} V \end{array},$$

which builds a tensor bundle from a GL_n representation, arises from Harish-Chandra descent for $(\mathrm{W}_n, \mathrm{GL}_n)$. This result allows us to equip tensor bundles with interesting structures (e.g., a vertex algebra structure) by working $(\mathrm{W}_n, \mathrm{GL}_n)$-equivariantly on the formal n-disk. In other words, it reduces the problem of making a universal construction on all n-manifolds to the problem of making an equivariant construction on the formal n-disk, since the descent procedure automates extension from the formal to the global.

Note that every formal vector bundle $\mathcal{V} \in \mathrm{VB}_{(\mathrm{W}_n,\mathrm{GL}_n)}$ is naturally filtered via a filtration inherited from $\widehat{\mathcal{O}}_n$. Explicitly, we see that $\mathcal{V}$ is the limit of the sequence of finite-dimensional vector spaces

$$\cdots \to \widehat{\mathcal{O}}_n/\mathfrak{m}_n^k \otimes V \to \cdots \to \widehat{\mathcal{O}}_n/\mathfrak{m}_n \otimes V \cong V$$

where V is the underlying GL_n-representation. Each quotient $\widehat{\mathcal{O}}_n/\mathfrak{m}_n^k \otimes V$ is a module over $\mathrm{Aut}_{n,k}$, and hence determines a vector bundle on X by the associated bundle construction along X_k^{coor}. In this way, $\mathcal{V}$ produces a natural sequence of vector bundles on X and thus a pro-vector bundle on X.

Given a formal exponential σ on X, we obtain a GL_n-equivariant map from Fr_X to X_k^{coor} for every k, by composing the projection map $X^{coor} \to X_k^{coor}$ with the GL_n-equivariant map from Fr_X to X^{coor}.

Definition 2.1.18. *Gelfand-Kazhdan descent* is the functor

$$\mathrm{desc} : \mathrm{GK}_n^{\mathrm{op}} \times \mathrm{VB}_{(\mathrm{W}_n,\mathrm{GL}_n)} \to \mathrm{Pro}(\mathrm{VB})_{flat}$$

sending (X, σ) — a Gelfand-Kazhdan structure — and a formal vector bundle $\mathcal{V}$ to the pro-vector bundle $\mathrm{Fr}_X \times^{\mathrm{GL}_n} \mathcal{V}$ with flat connection induced by the Grothendieck connection.

When the Gelfand-Kazhdan structure (X, σ) is fixed we will denote the corresponding functor $\mathrm{desc}((X, \sigma), -) : \mathrm{VB}_{(\mathrm{W}_n, \mathrm{GL}_n)} \to \mathrm{Pro}(\mathrm{VB})_{flat}$ by $\mathrm{desc}_{X,\sigma}$.

By Proposition we see that for any two choices of formal exponentials σ, σ' on the same complex manifold X that there is an equivalence of functors

$$\mathrm{desc}((X, \sigma), -) \simeq \mathrm{desc}((X, \sigma'), -) : \mathrm{VB}_{(\mathrm{W}_n, \mathrm{GL}_n)} \to \mathrm{Pro}(\mathrm{VB})_{flat}.$$

Thus, we will often abuse notation and write $\mathrm{desc}_{X,\sigma} = \mathrm{desc}_X$ when a formal exponential is understood.

This functor is, in essence, Harish-Chandra descent, but in a slightly exotic context. It has several nice properties.

Lemma 2.1.19. *For any choice of Gelfand-Kazhdan structure* (X, σ)*, the descent functor* $\mathrm{desc}((X, \sigma), -)$ *is lax symmetric monoidal.*

Proof. For every $\mathcal{V}, \mathcal{W}$ in $\mathrm{VB}_{(\mathrm{W}_n, \mathrm{GL}_n)}$, we have natural maps

$$(\Omega^*(\mathrm{Fr}_X) \otimes \mathcal{V})_{basic} \otimes (\Omega^*(\mathrm{Fr}_X) \otimes \mathcal{W})_{basic} \to (\Omega^*(\mathrm{Fr}_X) \otimes (\mathcal{V} \otimes \mathcal{W}))_{basic} \to (\Omega^*(\mathrm{Fr}_X) \otimes (\mathcal{V} \otimes_{\widehat{\mathcal{O}}_n} \mathcal{W}))_{basic}$$

and the composition provides the natural transformation producing the lax symmetric monoidal structure. $\square$

In particular, we observe that the de Rham complex of $\mathrm{desc}((X, \sigma), \widehat{\mathcal{O}}_n)$ is a commutative algebra object in $\Omega^*(X)$-modules. As every object of $\mathrm{VB}_{(\mathrm{W}_n, \mathrm{GL}_n)}$ is an $\widehat{\mathcal{O}}_n$-module and the morphisms are $\widehat{\mathcal{O}}_n$-linear, we find that descent actually factors through the category of $\mathrm{desc}((\mathrm{Fr}_X,\sigma),\widehat{\mathcal{O}}_n)$-modules. In sum, we have the following.

Lemma 2.1.20. *The descent functor* $\mathrm{desc}((X,\sigma),-)$ *factors as a composite*

$$\mathrm{VB}_n \xrightarrow{\overline{\mathrm{desc}}((X,\sigma),-)} \mathrm{Mod}_{\mathrm{desc}((X,\sigma),\widehat{\mathcal{O}}_n)} \xrightarrow{forget} \mathrm{VB}_{flat}(X)$$

and the functor $\overline{\mathrm{desc}}((X,\sigma),-)$ *is symmetric monoidal.*

As before, we let $\mathscr{D}\mathrm{esc}$ denote the associated local system obtained from desc by taking horizontal sections. This functor is well-known: it recovers the tensor bundles on X.

If $E \to X$ is a holomorphic vector bundle on X we denote by $\mathrm{Jet}^{hol}(E)$ the holomorphic ∞-jet bundle of E. If E_x is the fiber of E over a point $x \in X$, then the fiber of this pro-vector bundle over x can be identified with

$$\mathrm{Jet}^{hol}(E)|_x \cong E_x \times \mathbb{C}[[t_1,\ldots,t_n]].$$

This pro-vector bundle has a canonical flat connection.

Proposition 2.1.21. *For* $\mathcal{V} \in \mathrm{VB}_n$ *corresponding to the* GL_n*-representation* V*, there is a natural isomorphism of flat pro-vector bundles*

$$\mathrm{desc}((X,\sigma),\mathcal{V}) \cong \mathrm{Jet}^{hol}(\mathrm{Fr}_X \times^{\mathrm{GL}_n} V)$$

In other words, the functor of descent along the frame bundle is naturally isomorphic to the functor of taking ∞*-jets of the associated bundle construction.*

As a corollary, we see that the associated sheaf of flat sections is

$$\mathscr{D}\mathrm{esc}((X,\sigma),\mathcal{V}) \cong \Gamma^{hol}(\mathrm{Fr}_X \times^{\mathrm{GL}_n} V)$$

where $\Gamma^{hol}(-)$ denotes holomorphic sections.

In other words, Gelfand-Kazhdan descent produces every tensor bundle. For example, for the defining representation $V = \mathbb{C}^n$ of GL_n, we have $\mathcal{V} = \widehat{\mathcal{T}}_n$, i.e., the vector fields on the formal disk viewed as the adjoint representation of W_n. Under Gelfand-Kazhdan descent, it produces the tangent bundle T on Hol_n.

2.1.5. Formal characteristic classes

2.1.5.1. Recollection. In [**?**], Atiyah examined the obstruction — which now bears his name — to equipping a holomorphic vector bundle with a holomorphic connection from several perspectives. To start, as he does, we take a very structural approach. He begins by constructing the following sequence of vector bundles (see Theorem 1).

Definition 2.1.22. Let G be a complex Lie group. Let $E \to X$ be a holomorphic vector bundle on a complex manifold and $\mathcal{E}$ its sheaf of sections. The *Atiyah sequence* of E is the exact sequence holomorphic vector bundles given by

$$0 \to E \otimes T^*X \to J^1(E) \to E \to 0,$$

where $J^1(E)$ the bundle of *first-order* jets of E The *Atiyah class* is the element $\mathrm{At}(E) \in \mathrm{H}^1(X, \Omega^1_X \otimes \mathrm{End}_{\mathcal{O}_X}(\mathcal{E}))$ associated to the extension above.

Remark 2.1.23. Taking linear duals we see tha above short exact sequence is equivalent to one of the form

$$0 \to \mathrm{End}(E) \to \mathrm{A}(E) \to TX \to 0$$

where $\mathrm{A}(E)$ is the so-called *Atiyah bundle* associated to E.

We should remark that the sheaf $\mathcal{A}(E)$ of holomorphic sections of the Atiyah bundle $\mathrm{A}(E)$ is a Lie algebra by borrowing the Lie bracket on vector fields. By inspection, the Atiyah sequence of sheaves (by taking sections) is a sequence of Lie algebras; in fact, $\mathcal{A}(E)$ is a central example of a Lie algebroid, as the quotient map to vector fields $\mathcal{T}_X$ on X is an anchor map.

Atiyah also examined how this sequence relates to the Chern theory of connections.

Proposition 2.1.24. *A* holomorphic connection *on E is a splitting of the Atiyah sequence (as holomorphic vector bundles).*

Atiyah's first main result in the paper is the following.

Proposition 2.1.25 (Theorem 2, [**?**])**.** *A connection exists on E if and only if the Atiyah class* $\mathrm{At}(E)$ *vanishes.*

He observes immediately after this statement that the construction is functorial in maps of bundles. Later, he finds a direct connection between the Atiyah class and the curvature of a smooth connection. A smooth connections always exists (i.e., the sequence splits as smooth vector bundles, not necessarily holomorphically), and one is free to choose a connection such that the local 1-form only has Dolbeault type $(1, 0)$, i.e., is an element in $\Omega^{1,0}(X;\mathrm{End}(E))$. In that case, the $(1,1)$-component $\Theta^{1,1}$ of the curvature Θ is a 1-cocycle in the Dolbeault complex $(\Omega^{1,*}(X;\mathrm{End}(E)),\overline{\partial})$ for $\mathrm{End}(E)$ and its cohomology class $[\Theta^{1,1}]$ is the Atiyah class $\mathrm{At}(E)$. In consequence, Atiyah deduces the following.

Proposition 2.1.26. *For X a compact Kähler manifold, the kth Chern class $c_k(E)$ of E is given by the cohomology class of $(2\pi i)^{-k}S_k(\mathrm{At}(E))$, where S_k is the kth elementary symmetric polynomial, and hence only depends on the Atiyah class.*

This assertion follows from the degeneracy of the Hodge-to-de Rham spectral sequence. More generally, the term $(2\pi i)^{-k}S_k(\mathrm{At}(E))$ agrees with the image of the kth Chern class in the Hodge cohomology $H^k(X;\Omega^k_{hol})$.

The functoriality of the Atiyah class means that it makes sense not just on a fixed complex manifold, but also on the larger sites Hol_n and GK_n. We thus immediately obtain from Atiyah the following notion.

Definition 2.1.27. For each $V\in\mathrm{VB}(\mathrm{Hol}_n)$, the *Atiyah class* $\mathrm{At}(V)$ is the equivalence class of the extension of the tangent bundle T by $\mathrm{End}(V)$ given by the Atiyah sequence.

Moreover, we have the following.

Lemma 2.1.28. *The cohomology class of $(2\pi i)^{-k}S_k(\mathrm{At}(V))$ provides a section of the sheaf $H^k(X;\Omega^k_{hol})$. On any compact Kähler manifold, it agrees with $c_k(V)$.*

desc

2.1.5.2. The formal Atiyah class. We now wish to show that Gelfand-Kazhdan descent sends an exact sequence in $\mathrm{VB}_{(\mathrm{W}_n,\mathrm{GL}_n)}$ to an exact sequence in $\mathrm{VB}(\mathrm{GK}_n)$ (and hence in $\mathrm{VB}(\mathrm{Hol}_n)$). It will then remain to verify that for each tensor bundle on Hol_n, there is an exact sequence over the formal n-disk that descends to the Atiyah sequence for that tensor bundle.

We will use the notation $\mathrm{desc}(\mathcal{V})$ to denote the functor $\mathrm{desc}(-, \mathcal{V}) : \mathrm{GK}_n^{\mathrm{op}} \to \mathrm{Pro}(\mathrm{VB})_{flat}$, since we want to focus on the sheaf on GK_n (or Hol_n) defined by each formal vector bundle $\mathcal{V}$. Taking flat sections we get an $\mathcal{O}$-module $\mathscr{D}\mathrm{esc}(\mathcal{V})$ which is locally free of finite rank and so determines an object in $\mathrm{VB}(\mathrm{GK}_n)$.

Lemma 2.1.29. *If*

$$\mathcal{A} \to \mathcal{B} \to \mathcal{C}$$

is an exact sequence in $\mathrm{VB}_{(\mathrm{W}_n, \mathrm{GL}_n)}$, *then*

$$\mathscr{D}\mathrm{esc}(\mathcal{A}) \to \mathscr{D}\mathrm{esc}(\mathcal{B}) \to \mathscr{D}\mathrm{esc}(\mathcal{C})$$

is exact in $\mathrm{VB}(\mathrm{GK}_n)$.

Proof. A sequence of vector bundles is exact if and only if the associated sequence of $\mathcal{O}$-modules is exact (i.e., the sheaves of sections of the vector bundles). But a sequence of sheaves is exact if and only if it is exact stalkwise. Observe that there is only one point at which to compute a stalk in the site Hol_n, since every point $x \in X$ has a small neighborhood isomorphic to a small neighborhood of $0 \in \mathbb{C}^n$. As we are working in an analytic setting, the stalk of a $\mathcal{O}$-module at a point x injects into the ∞-jet at x. Hence, it suffices to verifying the exactness of the sequence of ∞-jets. Hence, we consider the ∞-jet at $0 \in \mathbb{C}^n$ of the sequence $\mathrm{desc}(A) \to \mathrm{desc}(B) \to \mathrm{desc}(C)$. But this sequence is simply $A \to B \to C$, which is exact by hypothesis. $\square$

Corollary 2.1.30. *There is a canonical map from* $\mathrm{Ext}^1_{(\mathrm{W}_n, \mathrm{GL}_n)}(\mathcal{B}, \mathcal{A})$ *to* $\mathrm{Ext}^1_{\mathrm{GK}_n}(\mathscr{D}\mathrm{esc}(\mathcal{B}), \mathscr{D}\mathrm{esc}(\mathcal{A}))$.

In particular, once we produce the $(\mathrm{W}_n, \mathrm{GL}_n)$-Atiyah sequence for a formal tensor field $\mathcal{V}$, we will have a very local model for the Atiyah class living in $\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n, \mathrm{GL}_n; \widehat{\Omega}^1_n \otimes_{\widehat{\mathcal{O}}_n} \mathrm{End}_{\widehat{\mathcal{O}}_n}(\mathcal{V}))$.

2.1.5.3. The formal Atiyah sequence. Let $\mathcal{V}$ be a formal vector bundle. We will now construct the "formal" Atiyah sequence associated to $\mathcal{V}$. First, we need to define the $(\mathrm{W}_n, \mathrm{GL}_n)$-module of *first order jets* of $\mathcal{V}$. Let's begin by recalling the construction of jets in ordinary geometry.

If X is a manifold, we have the diagonal embedding $\Delta : X \hookrightarrow X \times X$. Correspondingly, there is the ideal sheaf $\mathcal{I}_\Delta$ on $X \times X$ of functions vanishing along the diagonal. Let $X^{(k)}$ be the ringed space $(X, \mathcal{O}_{X\times X}/\mathcal{I}^k_\Delta)$ describing the kth order neighborhood of the diagonal in $X \times X$. Let $\Delta^{(k)} : X^{(k)} \to X \times X$ denote the natural map of ringed spaces. The projections $\pi_1, \pi_2 : X \times X \to X$ compose with $\Delta^{(k)}$ to define maps $\pi_1^{(k)}, \pi_2^{(k)} : X^{(k)} \to X$. Given an $\mathcal{O}_X$-module $\mathcal{V}$, "push-and-pull" along these projections,

$$J^k_X(\mathcal{V}) = (\pi_1^{(k)})_* (\pi_2^{(k)})^* \mathcal{V},$$

defines the $\mathcal{O}_X$-module of kth order jets of $\mathcal{V}$.

There is a natural adaptation in the formal case. The diagonal map corresponds to an algebra map $\Delta^* : \widehat{\mathcal{O}}_{2n} \to \widehat{\mathcal{O}}_n$. Fix coordinatizations $\widehat{\mathcal{O}}_n = \mathbb{C}[[t_1, \ldots, t_n]]$ and $\widehat{\mathcal{O}}_{2n} = \mathbb{C}[[t'_1, \ldots, t'_n, t''_1, \ldots, t''_n]]$. Then the map is given by $\Delta^*(t'_i) = \Delta^*(t''_i) = t_i$.

Let $\widehat{I}_n = \ker(\Delta^*) \subset \widehat{\mathcal{O}}_{2n}$ be the ideal given by the kernel of Δ^*. For each k there is a quotient map

$$\Delta^{(k)*} : \widehat{\mathcal{O}}_{2n} \to \widehat{\mathcal{O}}_{2n}/\widehat{I}^{k+1}_n,$$

The projection maps have the form

$$\pi_1^{(k)*}, \pi_2^{(k)*} : \widehat{\mathcal{O}}_n \to \widehat{\mathcal{O}}_{2n}/\widehat{I}_n^{k+1},$$

which in coordinates are $\pi_1^*(t_i) = t_i'$ and $\pi_2^*(t_i) = t_i''$.

Definition 2.1.31. Let $\mathcal{V}$ be a formal vector bundle on $\widehat{D}^n$. Consider the $\widehat{\mathcal{O}}_{2n}/\widehat{I}_n^{k+1}$-module $\mathcal{V} \otimes_{\widehat{\mathcal{O}}_n} \left(\widehat{\mathcal{O}}_{2n}/\widehat{I}_n^{k+1}\right)$, where the tensor product uses the $\widehat{\mathcal{O}}_n$-module structure on the quotient $\widehat{\mathcal{O}}_{2n}/\widehat{I}_n^{k+1}$ coming from the map $\pi_2^{(k)*}$. We define the *kth order formal jets of* $\mathcal{V}$, denoted $J^k(\mathcal{V})$, as the restriction of this $\widehat{\mathcal{O}}_{2n}/\widehat{I}_n^{k+1}$-module to a $\widehat{\mathcal{O}}_n$-module using the map $\pi_1^{(k)*} : \widehat{\mathcal{O}}_n \to \widehat{\mathcal{O}}_{2n}/\widehat{I}_n^{k+1}$.

Lemma 2.1.32. *For any $\mathcal{V} \in \mathrm{VB}_n$ the kth order formal jets $J^k(\mathcal{V})$ is an element of VB_n.*

Proof. For $\mathcal{V}$ in VB_n there is an induced action of $(\mathrm{W}_n, \mathrm{GL}_n)$ on the tensor product $\mathcal{V} \otimes_{\widehat{\mathcal{O}}_n} \widehat{\mathcal{O}}_{2n}/\widehat{I}_n^{k+1}$. For fixed k we see that $\widehat{\mathcal{O}}_{2n}/\widehat{I}_n^{k+1}$ is finite rank as a $\widehat{\mathcal{O}}_n$ module. Thus it is immediate that this module satisfies the conditions of a formal vector bundle. $\square$

As a $\mathbb{C}$-linear vector space we have $J^1(\mathcal{V}) = \mathcal{V} \oplus (\mathcal{V} \otimes_{\widehat{\mathcal{O}}_n} \widehat{\Omega}_n^1)$. For $f \in \widehat{\mathcal{O}}_n$ and $(v, \beta) \in \mathcal{V} \oplus (\mathcal{V} \otimes \widehat{\Omega}_n^1)$, the $\widehat{\mathcal{O}}_n$-module structure is given by

$$f \cdot (v, \beta) = (fv, (f\beta + v \otimes \mathrm{d}f)).$$

(This formula is the formal version of Atiyah's description in Section 4 of [**?**], where he uses the notation $\mathcal{D}$.) The following is proved in exact analogy as in the non-formal case which can also be found in Section 4 of [**?**], for instance.

Proposition 2.1.33. *For any* $\mathcal{V} \in \mathrm{VB}_{(\mathrm{W}_n,\mathrm{GL}_n)}$*, the* $\widehat{\mathcal{O}}_n$*-module* $J^1(\mathcal{V})$ *has a compatible action of the pair* $(\mathrm{W}_n, \mathrm{GL}_n)$ *and hence determines an object in* $\mathrm{VB}_{(\mathrm{W}_n,\mathrm{GL}_n)}$*. Moreover, it sits in a short exact sequence of formal vector bundles*

$$\mathcal{V} \otimes \widehat{\Omega}^1_n \to J^1(\mathcal{V}) \to \mathcal{V}. \tag{2.2}$$

Finally, the Gelfand-Kazhdan descent of this short exact sequence is isomorphic to the Atiyah sequence

$$\mathscr{D}\mathrm{esc}_{\mathrm{GK}}(\mathcal{V}) \otimes \Omega^1_{hol} \to J^1\mathscr{D}\mathrm{esc}_{\mathrm{GK}}(\mathcal{V}) \to \mathscr{D}\mathrm{esc}_{\mathrm{GK}}(\mathcal{V}).$$

In particular, $J^1\mathrm{desc}_{\mathrm{GK}}(\mathcal{V}) = \mathrm{desc}_{\mathrm{GK}}(J^1\mathcal{V})$.

We henceforth call the sequence (2.2) *the formal Atiyah sequence* for $\mathcal{V}$.

Remark 2.1.34. Note that $J^1(\mathcal{V})$ is an element of the category VB_n but it is *not* a formal tensor field. That is, it does not come from a linear representation of GL_n via coinduction.

Remark 2.1.35. A choice of a formal coordinate defines a splitting of the first-order jet sequence as $\widehat{\mathcal{O}}_n$-modules. If we write $\mathcal{V} = \widehat{\mathcal{O}}_n \otimes_{\mathbb{C}} V$, then one defines

$$j^1 : \mathcal{V} \to J^1\mathcal{V} \ \ , \ \ f \otimes_{\mathbb{C}} v \mapsto (f \otimes_{\mathbb{C}} v, (1 \otimes_{\mathbb{C}} v) \otimes_{\mathcal{O}} \mathrm{d}f).$$

It is a map of $\widehat{\mathcal{O}}_n$-modules, and it splits the obvious projection $J^1(\mathcal{V}) \to \mathcal{V}$. We stress, however, that it is *not* a splitting of W_n-modules. We will soon see that this is reflected by the existence of a certain characteristic class in Gelfand-Fuks cohomology.

Note the following corollary, which follows from the identification

$$\mathrm{Ext}^1(\mathcal{V} \otimes_{\widehat{\mathcal{O}}_n} \widehat{\Omega}^1_n, \mathcal{V}) \cong \mathrm{C}^1_{\mathrm{Lie}}(\mathrm{W}_n, \mathrm{GL}_n; \widehat{\Omega}^1_n \otimes_{\widehat{\mathcal{O}}_n} \mathrm{End}_{\widehat{\mathcal{O}}_n}(\mathcal{V}))$$

and from the observation that an exact sequence in $\mathrm{VB}(\widehat{D}^n)$ maps to an exact sequence in $\mathrm{VB}(\mathrm{GK}_n)$.

Corollary 2.1.36. *There is a cocycle* $\mathrm{At}^{\mathrm{GF}}(\mathcal{V}) \in \mathrm{C}^1_{\mathrm{Lie}}(\mathrm{W}_n, \mathrm{GL}_n; \widehat{\Omega}^1_n \otimes_{\widehat{\mathcal{O}}_n} \mathrm{End}_{\widehat{\mathcal{O}}_n}(\mathcal{V}))$ *representing the Atiyah class* $\mathrm{At}(\mathrm{desc}(\mathcal{V}))$.

We call this cocycle the Gelfand-Fuks-Atiyah class of $\mathcal{V}$ since it descends to the ordinary Atiyah class for $\mathrm{desc}(\mathcal{V})$ as a sheaf of $\mathcal{O}$-modules.

Definition 2.1.37. The *Gelfand-Fuks-Chern character* is the formal sum $\mathrm{ch}^{\mathrm{GF}}(\mathcal{V}) = \sum_{k \geq 0} \mathrm{ch}^{\mathrm{GF}}_k(\mathcal{V})$, where the kth component

$$\mathrm{ch}^{\mathrm{GF}}_k(\mathcal{V}) := \frac{1}{(-2\pi i)^k k!} \mathrm{Tr}(\mathrm{At}^{\mathrm{GF}}(\mathcal{V})^k)$$

lives in $\mathrm{C}^k_{\mathrm{Lie}}(\mathrm{W}_n, \mathrm{GL}_n; \widehat{\Omega}^k_n)$.

It is a direct calculation to see that $\mathrm{ch}^{\mathrm{GF}}_k(\mathcal{V})$ is closed for the differential on formal differential forms, i.e., it lifts to an element in $\mathrm{C}^k_{\mathrm{Lie}}(\mathrm{W}_n, \mathrm{GL}_n; \widehat{\Omega}^k_{n,cl})$.

2.1.5.4. An explicit formula. In this section we provide an explicit description of the Gelfand-Fuks-Atiyah class

$$\mathrm{At}^{\mathrm{GF}}(\mathcal{V}) \in \mathrm{C}^1_{\mathrm{Lie}}(\mathrm{W}_n, \mathrm{GL}_n; \widehat{\Omega}^1_n \otimes_{\widehat{\mathcal{O}}_n} \mathrm{End}_{\widehat{\mathcal{O}}}(\mathcal{V})).$$

of a formal vector bundle $\mathcal{V}$.

By definition, any formal vector bundle has the form $\mathcal{V} = \widehat{\mathcal{O}}_n \otimes V$, with V a finite-dimensional vector space. We view V as the "constant sections" in $\mathcal{V}$ by the inclusion $i : v \mapsto 1 \otimes v$. This map then determines a connection on $\mathcal{V}$: we define a $\mathbb{C}$-linear map $\nabla : \mathcal{V} \to \widehat{\Omega}^1_n \otimes_{\widehat{\mathcal{O}}_n} \mathcal{V}$ by saying that for any $f \in \widehat{\mathcal{O}}_n$ and $v \in V$,

$$\nabla(fv) = \mathrm{d}_{dR}(f)v,$$

where $\mathrm{d}_{dR} : \widehat{\mathcal{O}}_n \to \widehat{\Omega}^1_n$ denote the de Rham differential on functions. This connection appeared earlier when we defined the splitting of the jet sequence $j^1 = 1 \oplus \nabla$.

The connection ∇ determines an element in $\mathrm{C}^1_{\mathrm{Lie}}(\mathrm{W}_n; \widehat{\Omega}^1_n \otimes_{\widehat{\mathcal{O}}} \mathrm{End}_{\widehat{\mathcal{O}}}(\mathcal{V}))$, as follows. Let

$$\rho_{\mathcal{V}} : \mathrm{W}_n \otimes \mathcal{V} \to \mathcal{V}$$

denote the action of formal vector fields and consider the composition

$$\mathrm{W}_n \otimes V \xrightarrow{\mathrm{id} \otimes i} \mathrm{W}_n \otimes \mathcal{V} \xrightarrow{\rho_{\mathcal{V}}} \mathcal{V} \xrightarrow{\nabla} \widehat{\Omega}^1_n \otimes_{\widehat{\mathcal{O}}} \mathcal{V}.$$

Since V spans $\mathcal{V}$ over $\widehat{\mathcal{O}}_n$, this composite map determines a $\mathbb{C}$-linear map

$$\alpha_{\mathcal{V},\nabla} : \mathrm{W}_n \to \widehat{\Omega}^1_n \otimes_{\widehat{\mathcal{O}}} \mathrm{End}_{\widehat{\mathcal{O}}}(\mathcal{V})$$

by

$$\alpha_{\mathcal{V},\nabla}(X)(fv) = f\nabla(\rho_{\mathcal{V}}(X)(i(v))),$$

with $f \in \widehat{\mathcal{O}}_n$ and $v \in V$.

Proposition 2.1.38. *Let $\mathcal{V}$ be a formal vector bundle. Then $\alpha_{\mathcal{V},\nabla}$ is a representative for the Gelfand-Fuks-Atiyah class* $\mathrm{At}^{\mathrm{GF}}(\mathcal{V})$*.*

Proof. We begin by recalling some general facts about the Gelfand-Fuks-Atiyah class as an extension class of an exact sequence of modules. Viewing $\widehat{\mathcal{O}}_n$ as functions on the formal n-disk, we can ask about the jets of such functions. A choice of formal coordinates corresponds to an identification $\widehat{\mathcal{O}}_n \cong \mathbb{C}[[t_1, \dots, t_n]]$, and that choice provides a trivialization of the jet bundles by providing a preferred frame. This frame identifies, for instance, J^1 with $\widehat{\mathcal{O}}_n \oplus \widehat{\Omega}^1_n$, and the1-jet of a formal function f can be understood as $(f, \mathrm{d}_{dR} f)$.

For a formal vector bundle $\mathcal{V} = \widehat{\mathcal{O}}_n \otimes V$, something similar happens after choosing coordinates. We have $J^1(\mathcal{V}) \cong \mathcal{V} \oplus \widehat{\Omega}^1_n \otimes_{\widehat{\mathcal{O}}_n} \mathcal{V}$ and the 1-jet of an element of $\mathcal{V}$ can be written as

$$\begin{array}{rccc} j^1 : & \mathcal{V} & \to & J^1(\mathcal{V}) \\ & fv & \mapsto & (fv, \mathrm{d}_{dR}(f)v). \end{array}$$

where $f \in \widehat{\mathcal{O}}_n$ and $v \in V$. The projection onto the second summand is precisely the connection ∇ on $\mathcal{V}$ determined by $\mathcal{V} = \widehat{\mathcal{O}}_n \otimes V$, the defining decomposition.

The Gelfand-Fuks-Atiyah class is the failure for this map ∇ to be a map of W_n-modules. Indeed, ∇ determines a map of graded vector spaces

$$1 \otimes \nabla : \mathrm{C}^{\#}_{\mathrm{Lie}}(\mathrm{W}_n; \mathcal{V}) \to \mathrm{C}^{\#}_{\mathrm{Lie}}(\mathrm{W}_n; \widehat{\Omega}^1_n \otimes_{\widehat{\mathcal{O}}} \mathcal{V}).$$

Let $\mathrm{d}_{\mathcal{V}}$ denote the differential on $\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n; \mathcal{V})$ and $\mathrm{d}_{\Omega^1 \otimes \mathcal{V}}$ denote the differential on $\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n; \widehat{\Omega}^1_n \otimes_{\widehat{\Omega}} \mathcal{V})$. The failure for $1 \otimes \nabla$ is precisely the difference

$$(1 \otimes \nabla) \circ \mathrm{d}_{\mathcal{V}} - \mathrm{d}_{\Omega^1 \otimes \mathcal{V}} \circ (1 \otimes \nabla). \tag{2.3}$$

This difference is $\mathrm{C}^{\#}_{\mathrm{Lie}}(\mathrm{W}_n)$ linear and can hence be thought of as a cocycle of degree one in $\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n; \widehat{\Omega}^1 \otimes_{\widehat{\mathcal{O}}} \mathrm{End}_{\widehat{\mathcal{O}}}(\mathcal{V}))$. This is the representative for the Atiyah class.

We proceed to compute this difference. The differential $\mathrm{d}_{\mathcal{V}}$ splits as $\mathrm{d}_{\mathrm{W}_n} \otimes 1_{\mathcal{V}} + \mathrm{d}'$ where $\mathrm{d}_{\mathrm{W}_n}$ is the differential on the complex $\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n)$ and d' encodes the action of W_n on $\mathcal{V}$. Likewise, the differential $\mathrm{d}_{\Omega^1 \otimes \mathcal{V}}$ splits as $\mathrm{d}_{\mathrm{W}_n} \otimes 1_{\Omega^1 \otimes \mathcal{V}} + \mathrm{d}_{\Omega^1} \otimes 1_V + 1_{\Omega^1} \otimes \mathrm{d}'$ where d_{Ω^1} is the differential on the complex $\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n; \widehat{\Omega}^1_n)$.

The de Rham differential clearly commutes with the action of vector fields so that $(1 \otimes \mathrm{d}_{dR}) \circ (\mathrm{d}_{\mathcal{O}} \otimes 1) = (\mathrm{d}_{\mathrm{W}_n} + \mathrm{d}_{\Omega^1}) \circ (1 \otimes \mathrm{d}_{dR})$ so that the the difference in (2.3) reduces to

$$(1 \otimes \nabla) \circ \mathrm{d}' - (1_{\Omega^1} \otimes \mathrm{d}') \circ (1 \otimes \nabla).$$

By definition d' is the piece of the Chevalley-Eilenberg differential that encodes the action of W_n on $\mathcal{V}$, so if we evaluate on an element of the form $1 \in v \in \mathrm{C}^0_{\mathrm{Lie}}(\mathrm{W}_n; V) \subset \mathrm{C}^0_{\mathrm{Lie}}(\mathrm{W}_n; \mathcal{V})$ the only term that survives is the GF 1-cocycle

$$X \mapsto \nabla \mathrm{d}'(1 \otimes v)(X) = \nabla(\rho_{\mathcal{V}}(X)(v)).$$

as desired. $\square$

Corollary 2.1.39. *On the formal vector bundle $\widehat{\mathcal{T}}_n$ encoding formal vector fields, fix the $\widehat{\mathcal{O}}_n$-basis by $\{\partial_j\}$ and the $\widehat{\mathcal{O}}_n$-dual basis of one-forms by $\{\mathrm{d}t^j\}$. The explicit representative for the Atiyah class is given by the Gelfand-Fuks 1-cocycle*

$$f^i \partial_i \mapsto -\mathrm{d}_{dR}(\partial_j f^i)(\mathrm{d}t^j \otimes \partial_i)$$

taking values in $\widehat{\Omega}^1_n \otimes_{\widehat{\mathcal{O}}_n} \mathrm{End}_{\widehat{\mathcal{O}}}(\widehat{\mathcal{T}}_n)$.

Proof. We must compute the action of vector fields on $\widehat{\mathcal{O}}_n$-basis elements of $\widehat{\mathcal{T}}_n$. We fix formal coordinates $\{t_j\}$ and let $\{\partial_j\}$ be the associated constant formal vector fields. Then the structure map is given by the Lie derivative $\rho_{\widehat{\mathcal{T}}}(f^i\partial_i, \partial_j) = -\partial_j f^i$. The formula for the cocycle follows from the Proposition. $\square$

In the above statement, the vector field $f^i\partial_i$ appeared in the Atiyah class through its *Jacobian* $\partial_j f^i$. For any formal vector field $X = f^i\partial_i$ we will use the notation $\mathrm{Jac}(X) = (\partial_j f^i) \in \mathrm{Mat}_n(\widehat{\mathcal{O}}_n)$ for Jacobian. This is an $n \times n$ matrix of formal power power series.

We can use this result to explicitly compute the cocycles representing the Gelfand-Kazhdan Chern characters. For instance, we have the following formulas that will be useful in later sections.

Corollary 2.1.40. *The kth component $\mathrm{ch}_k^{\mathrm{GF}}(\widehat{\mathcal{T}}_n)$ of the universal Chern character of the formal tangent bundle is the cocycle*

$$\frac{1}{(-2\pi i)^k k!}\mathrm{Tr}(\mathrm{At}^{\mathrm{GF}}(\widehat{\mathcal{T}}_n)^{\wedge k}) : (X_1, \ldots, X_k) \mapsto \frac{1}{(-2\pi i)^k k!}\mathrm{Tr}\left(\mathrm{d}_{dR}(\mathrm{Jac}(X_1)) \wedge \cdots \wedge \mathrm{d}_{dR}(\mathrm{Jac}(X_k))\right)$$

in $\mathrm{C}^k_{\mathrm{Lie}}(\mathrm{W}_n, \mathrm{GL}_n; \widehat{\Omega}^k_n)$. *As the de Rham differential* $\mathrm{d}_{dR} : \widehat{\Omega}^{k-1}_n \to \widehat{\Omega}^k_n$ *is* W_n*-equivariant, there is an element* α_{k-1} *in* $\mathrm{C}^k_{\mathrm{Lie}}(\mathrm{W}_n, \mathrm{GL}_n; \widehat{\Omega}^{k-1}_n)$ *such that*

$$\mathrm{ch}_k^{\mathrm{GF}}(\widehat{\mathcal{T}}_n) = \mathrm{d}_{dR}\alpha_{k-1} \tag{2.4}$$

Explicitly:

(2.5)
$$\alpha_k : (f^i_1\partial_i, \ldots, f^i_k\partial_i) \mapsto \frac{1}{(-2\pi i)^k k!}\mathrm{Tr}\left(\mathrm{Jac}(X_1) \wedge \mathrm{d}_{dR}(\mathrm{Jac}(X_2)) \wedge \cdots \wedge \mathrm{d}_{dR}(\mathrm{Jac}(X_k))\right).$$

2.1.6. A family of extended pairs

We will be most interested in the cocycles $\mathrm{ch}_k(\mathcal{V})$ for $k \geq 2$. When $k = 2$ we obtain a 2-cocycle with values in $\widehat{\Omega}^2_{n,cl}$, $\mathrm{ch}_2(\mathcal{V}) \in \mathrm{C}_{\mathrm{Lie}}(\mathrm{W}_n, \mathrm{GL}_n; \widehat{\Omega}^2_{n,cl})$. This 2-cocycle $\mathrm{ch}_2^{\mathrm{GF}}(\mathcal{V})$ determines an abelian extension Lie algebras of W_n by $\widehat{\Omega}^2_{n,cl}$

$$0 \to \widehat{\Omega}^2_{n,cl} \to \widetilde{\mathrm{W}}_{n,\mathcal{V}} \to \mathrm{W}_n \to 0.$$

When $\mathcal{V} = \widehat{\mathcal{T}}_n$, denote this extension by $\widetilde{\mathrm{W}}_{n,\mathcal{V}} = \widetilde{\mathrm{W}}_{n,1}$. (The notation will become clearer momentarily)

We have already discussed the pair $(\mathrm{W}_n, \mathrm{GL}_n)$. We will need that the above extension of Lie algebras fits in to a Harish-Chandra pair as well. The action of GL_n extends to an action on $\widetilde{\mathrm{W}}_{n,1}$ where we declare the action of GL_n on closed two-forms to be the natural one via linear formal automorphisms.

Lemma 2.1.41. *The pair* $(\widetilde{\mathrm{W}}_{n,1}, \mathrm{GL}_n)$ *form a Harish-Chandra pair and fits into an extension of pairs*

$$0 \to \widehat{\Omega}^2_{n,cl} \to (\widetilde{\mathrm{W}}_{n,1}, \mathrm{GL}_n) \to (\mathrm{W}_n, \mathrm{GL}_n) \to 0$$

which is determined by the cocycle $\mathrm{ch}_2^{\mathrm{GF}}(\widehat{\mathcal{T}}_n)$.

One might be worried as to why there is only a non-trivial extension of the Lie algebra in the pair. The choice of a coordinate determines an embedding of linear automorphisms GL_n into formal automorphisms Aut_n. The extension of formal automorphisms Aut_n defined by the group two-cocycle $\mathrm{ch}_2^{\mathrm{GF}}(\widehat{\mathcal{T}}_n)$ is trivial when restricted to GL_n so that it does not get extended.

2.1.6.1. An L_∞ extension. For $k > 2$, it will be useful to think of $\mathrm{ch}_k(\mathcal{V})$ as defining a similar type of extension. For this to make sense, we observe the following interpretation of higher cocycles. Suppose M is a module for a Lie algebra $\mathfrak{g}$, and suppose $c \in \mathrm{C}^k_{\mathrm{Lie}}(\mathfrak{g}; M)$ is a cocycle $\mathrm{d}_{CE}c = 0$. Then, c determines an abelian extension of L_∞-*algebras*

$$0 \to M[k-2] \to \widetilde{\mathfrak{g}} \to \mathfrak{g}$$

As a graded vector space $\widetilde{\mathfrak{g}}$ is $\mathfrak{g} \oplus M[k-2]$ (so that M is placed in degree $2-k$). The L_∞ structure on $\widetilde{\mathfrak{g}}$ is defined by, for $x, y, x_1, \ldots, x_k \in \mathfrak{g}$, $m \in M$:

$$\ell_2(x, y+m) = [x,y] + x \cdot m$$

$$\ell_k(x_1, \ldots, x_k) = c(x_1, \ldots, x_k).$$

Here, $x \cdot m \in M$ uses the module structure.

Thus, for any formal vector bundle $\mathcal{V}$, $\mathrm{ch}_k(\mathcal{V})$ determines an abelian L_∞ extension of W_n by the abelian Lie algebra $\widehat{\Omega}^k_{n,cl}$. The case $\mathcal{V} = \widehat{\mathcal{T}}_n$ will be especially relevant for us.

Definition 2.1.42. Denote by $\widetilde{\mathrm{W}}_{n,d}$ the L_∞ extension of W_n by the module $\widehat{\Omega}^{d+1}_{n,cl}[d-1]$:

$$0 \to \widehat{\Omega}^{d+1}_{n,cl}[d-1] \to \widetilde{\mathrm{W}}_{n,d} \xrightarrow{\pi_{n,d}} \mathrm{W}_n \to 0$$

determined by the $(d+1)$-cocycle $\mathrm{ch}_{d+1}(\widehat{\mathcal{T}}_n) \in \mathrm{C}^{d+1}_{\mathrm{Lie}}(\mathrm{W}_n, \mathrm{GL}_n; \widehat{\Omega}^{d+1}_{n,cl})$.

We would like to have an an analog of Lemma 2.1.40 for $\widetilde{\mathrm{W}}_{n,d}$ and the group GL_n. Indeed, it turns out that $\widetilde{\mathrm{W}}_{n,d}$ is also part of a Harish-Chandra pair. To make this possible, we need to slightly enlarge our category of pairs to include the data of an L_∞ algebra, instead of an ordinary Lie algebra.

2.1.6.2. L_∞ pairs. The concept of an ordinary Harish-Chandra pair involves a Lie group K, a Lie algebra $\mathfrak{g}$ with an action by K, together with an embedding of Lie algebras $\mathrm{Lie}(K) \to \mathfrak{g}$. There is a natural way to relax this to include L_∞ algebras.

Definition 2.1.43. An L_∞ Harish-Chandra pair is a pair $(\mathfrak{g}, K)$ where $\mathfrak{g}$ is an L_∞ algebra and K is a Lie group together with

(1) a linear action of K on $\mathfrak{g}$, $\rho_K : K \to \mathrm{GL}(\mathfrak{g})$;

(2) a map of L_∞ algebras $i : \mathrm{Lie}(K) \rightsquigarrow \mathfrak{g}$;

such that i is compatible with the action ρ_K and the adjoint action of K on $\mathrm{Lie}(K)$.

Remark 2.1.44. A morphism of L_∞ algebras $f : \mathfrak{h} \rightsquigarrow \mathfrak{g}$ is, by definition, a map of the underlying Chevalley-Eilenberg complexes

$$\mathrm{C}_*^{\mathrm{Lie}}(f) : \mathrm{C}^{\mathrm{Lie}}(\mathfrak{h}) \to \mathrm{C}^{\mathrm{Lie}}(\mathfrak{g})$$

as cocoummutative coalgebras. Now, $\mathrm{C}_*^{\mathrm{Lie}}(\mathfrak{g})$, being a free cocoummtative coalgebra, this map is determined by a sequence of maps $f_n : \mathrm{Sym}^n(\mathfrak{h}[1]) \to \mathfrak{g}[1]$ satisfying certain compatibility conditions.

Remark 2.1.45. This is certainly not the most general definition one can imagine for a homotopy enhancement of a Harish-Chandra pair. For instance, we have required that K acts on $\mathfrak{g}$ in a rather strict way. It turns out that this will be enough for our purposes.

The condition that $i : \mathrm{Lie}(K) \to \mathfrak{g}$ be compatible with ρ_K can be stated as follows. The L_∞ map $i : \mathrm{Lie}(K) \rightsquigarrow \mathfrak{g}$ is uniquely determined by a sequence of maps $i_n : \mathrm{Sym}^n(\mathrm{Lie}(K)[1]) \to \mathfrak{g}$, for each $n \geq 1$. We require that for each $n \geq 1$, all $A \in K$, and $x_1, \ldots, x_n \in \mathrm{Lie}(K)$ that

$$\rho_K(A) \cdot i_n(x_1, \ldots, x_n) = i_n\left((\mathrm{Ad}(A) \cdot x_1) \cdots (\mathrm{Ad}(A) \cdot x_n)\right).$$

Here $\mathrm{Ad}(A)$ denotes the adjoint action of $A \in K$ on $\mathrm{Lie}(K)$.

Lemma 2.1.46. *The for any $d \geq 1$ the pair $(\widetilde{\mathrm{W}}_{n,d}, \mathrm{GL}_n)$ has the structure of an L_∞ Harish-Chandra pair.*

Proof. The proof is similar to the case $d = 1$. The linear action of GL_n on $\widetilde{\mathrm{W}}_{n,d}$ comes from the natural one on W_n and $\widehat{\Omega}^{d+1}_{n,cl}$. Now, note that we have an GL_n-equivariant extension

$$\begin{tikzcd} & \widetilde{\mathrm{W}}_{n,d} \arrow[d] \\ \mathfrak{gl}_n \arrow[r] \arrow[ur, dotted] & \mathrm{W}_n \end{tikzcd}$$

since the cocycle $\mathrm{ch}_{d+1}(\widehat{\mathcal{T}}_n)$ vanishes when one of the inputs lies in $\mathfrak{gl}_n$. $\square$

In the next section we will see how the theory of descent for $(\mathrm{W}_n, \mathrm{GL}_n)$ can be extended to the pair $(\widetilde{\mathrm{W}}_{n,d}, \mathrm{GL}_n)$ provided a trivialization of the $(d+1)$st component of the Chern character is trivialized. This will be our main application of this extended pair.

2.2. Descent for extended pairs

2.2.1. General theory of descent for L_∞ pairs

In this section we set up the general theory of descent for L_∞ pairs $(\mathfrak{g}, K)$. Recall, this means that K is still and ordinary Lie group, but $\mathfrak{g}$ is an L_∞ algebra.

Let X be a fixed manifold, for which we are defining descent over. The starting point is the theory of bundles over X for the pair $(\mathfrak{g}, K)$. In the usual context of Harish-Chandra pairs (where $\mathfrak{g}$ is an ordinary Lie algebra), this means that we have a principal K-bundle $P \to X$ equipped with a K-equivariant one-form valued in $\mathfrak{g}$, $\omega \in \Omega^1(P, \mathfrak{g})$ satisfying the flatness condition

$$\mathrm{d}\omega + \frac{1}{2}[\omega, \omega] = 0.$$

In other words, ω is a Maurer-Cartan element of the dg Lie algebra $\Omega^*(P) \otimes \mathfrak{g}$ that is equivariant for the action of K on P and $\mathfrak{g}$.

The theory of Maurer-Cartan forms works just as well in the L_∞ case. First, note that the category of L_∞ algebras is tensored over commutative dg algebras. In other words, if $\mathfrak{g}$ is an L_∞ algebra and A a commutative dg algebra, there is the natural structure of an L_∞ algebra on $A \otimes \mathfrak{g}$. The n-ary brackets are of the form

$$\ell_n^{A \otimes \mathfrak{g}}(a_1 \otimes x_1, \ldots, a_n \otimes x_n) = (a_1 \cdots a_n)\ell_n^{\mathfrak{g}}(x_1, \ldots, x_n)$$

where $\ell_n^{\mathfrak{g}}$ is the n-ary bracket on $\mathfrak{g}$, and where we have used the commutative algebra structure on A.

Definition 2.2.1. Let $(\mathfrak{g}, K)$ be an L_∞ Harish-Chandra pair. A principal $(\mathfrak{g}, K)$-bundle on X is the data:

(1) a principal K-bundle $P \to X$;

(2) a K-invariant element

$$\omega \in \Omega^*(P) \otimes \mathfrak{g}$$

of total degree $+1$;

such that

(1) for all $a_1, \dots, a_n \in \mathrm{Lie}(K)$ we have $\omega(\xi_{a_1}, \cdots, \xi_{a_n}) = i(a_1, \dots, a_n)$ where ξ_{a_i} is the vertical vector field on P determined by a_i, and $i : \mathrm{Lie}(K) \to \mathfrak{g}$ is the L_∞ morphism determining the Harish-Chandra pair;

(2) ω is a Maurer-Cartan element of the L_∞ algebra $\Omega^*(P) \otimes \mathfrak{g}$. In other words,

$$\mathrm{d}\omega + \sum_{n \geq 1} \ell_n(\omega, \dots, \omega) = 0$$

where $\{\ell_n\}$ are the structure maps for $\mathfrak{g}$.

Our main example of a L_∞ Harish-Chandra pair that is not an ordinary pair will be associated to certain natural cohomology classes of formal vector fields. To define descent, we need an appropriate theory of modules for an L_∞ pair $(\mathfrak{g}, K)$.

Definition 2.2.2. A *semi-strict Harish-Chandra module* for the L_∞ pair $(\mathfrak{g}, K)$ is a dg vector space (V, d_V) equipped with

(i) a strict group action ρ_V^K of K, meaning a group map

$$\rho_{V^d}^K : K \to \mathrm{GL}(V^d)$$

for each degree d such that the product map $\prod_d \rho^K_{V^d} : K \to \prod_d \mathrm{GL}(V^d)$ commutes with the differential d_V;

(ii) an L_∞-action of $\mathfrak{g}$ on V, i.e., a map of L_∞-algebras $\rho^{\mathfrak{g}}_V : \mathfrak{g} \rightsquigarrow \mathrm{End}(V)$, such that the composite

$$\mathrm{C}^{\mathrm{Lie}}_*(\rho^{\mathfrak{g}}_V) \circ \mathrm{C}^{\mathrm{Lie}}_*(i) : \mathrm{C}^{\mathrm{Lie}}_*(\mathrm{Lie}(K)) \to \mathrm{C}^{\mathrm{Lie}}_*(\mathrm{End}(V))$$

equals the map

$$\mathrm{C}^{\mathrm{Lie}}_*(D\rho^K_V) : \mathrm{C}^{\mathrm{Lie}}_*(\mathrm{Lie}(K)) \to \mathrm{C}^{\mathrm{Lie}}_*(\mathrm{End}(V)).$$

Here $D\rho^K_V : \mathrm{Lie}(K) \to \mathrm{End}(V)$ is the differential of the strict K-action and $i : \mathrm{Lie}(K) \rightsquigarrow \mathfrak{g}$ is part of the data of the Harish-Chandra pair $(\mathfrak{g}, K)$.

2.2.1.1. Basic forms. Before the construction of descent, we recall a basic object in equivariant differential geometry.

Let V be a finite-dimensional K-representation. Denote by $\underline{V}$ the trivial vector bundle on P with fiber V. Sections of this bundle $\Gamma_P(V)$ have the structure of a K-representation by

$$A \cdot (f \otimes v) := (A \cdot f) \otimes (A \cdot v) \ \ , \ \ A \in K, \ f \in \mathcal{O}(P) \ , v \in V.$$

Every K-invariant section $f : P \to \underline{V}$ induces a section $s(f) : X \to V_X$, where the value of $s(f)$ at $x \in X$ is the K-equivalence class $[(p, f(p)]$, with $p \in \pi^{-1}(x) \cong K$. That is, there is a natural map

$$s : \Gamma_P(\underline{V})^K \to \Gamma_X(V_X)$$

and it is an isomorphism of $\mathcal{O}(X)$-modules. A K-invariant section f of $\underline{V} \to P$ also satisfies the infinitesimal version of invariance:

$$(Y \cdot f) \otimes v + f \otimes \mathrm{Lie}(\rho)(Y) \cdot v = 0$$

for any $Y \in \mathrm{Lie}(K)$.

There is a similiar statement for differential forms with values in the bundle V_X. Let $\Omega^k(P; \underline{V}) = \Omega^k(P) \otimes V$ denote the space of k-forms on P with values in the trivial bundle $\underline{V}$. Given $\alpha \in \Omega^1(X; V_X)$, its pull-back along the projection $\pi : P \to X$ is annihilated by any vertical vector field on P. In general, if $\alpha \in \Omega^k(X; V_X)$, then $i_Y(\pi^*\alpha) = 0$ for all $Y \in \mathrm{Lie}(K)$.

Definition 2.2.3. A k-form $\alpha \in \Omega^k(P; \underline{V})$ is called *basic* if

(1) it is K-invariant: $L_Y\alpha + \rho(Y) \cdot \alpha = 0$ for all $Y \in \mathrm{Lie}(K)$ and

(2) it vanishes on vertical vector fields: $i_Y\alpha = 0$ for all $Y \in \mathrm{Lie}(K)$.

Denote the subspace of basic k-forms by $\Omega^k(P; \underline{V})_{bas}$. Just as with sections, there is a natural isomorphism

$$s : \Omega^k(P; \underline{V})_{bas} \xrightarrow{\cong} \Omega^k(X; V_X)$$

between basic k-forms and k-forms on X with values in the associated bundle. In fact, $\Omega^\#(P; \underline{V})_{bas}$ forms a graded subalgebra of $\Omega^\#(P; \underline{V})$ and the isomorphism s extends to an isomorphism of graded algebras $\Omega^\#(P; \underline{V})_{bas} \cong \Omega^\#(X; V_X)$.

It is manifest that this construction of basic forms is natural in maps of $(\mathfrak{g}, K)$-bundles: basic forms pull back to basic forms along maps of bundles.

2.2.1.2. Semi-strict descent. Starting with the data:

(1) an L_∞ Harish-Chandra pair $(\mathfrak{g}, K)$;

(2) a principal $(\mathfrak{g}, K)$ bundle $(P \to X, \omega)$;

(3) a semi-strict $(\mathfrak{g}, K)$-module V;

we are now ready to define descent along X. It is constructed in the following steps.

(1) Using the linear action of K on V we define the associated vector bundle

$$V_X = P \times^K V$$

on X. Note that the differential forms on X with values in V_X, $\Omega^*(X; V_X)$, is isomorphic, as a dg $\Omega^*(X)$-module, to the complex of basic forms

$$\Omega^*(P; \underline{V})_{bas} \subset \Omega^*(P; \underline{V}).$$

(2) The Maurer-Cartan element $\omega \in \Omega^*(P) \otimes \mathfrak{g}$ allows us to deform the differential on $\Omega^*(P; \underline{V}) = \Omega^*(P) \otimes V$ by the following transfer of Maurer-Cartan elements. By the usual yoga of Koszul duality, the Maurer-Cartan element $\omega \in \Omega^*(P) \otimes \mathfrak{g}$ is equivalent to the data of a map of commutative dg algebras

$$\omega^* : \mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g}) \to \Omega^*(P).$$

We can then use the L_∞ module structure map $\rho_V : \mathfrak{g} \rightsquigarrow \mathrm{End}(V)$ to form the composition

$$\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{End}(V)) \xrightarrow{\mathrm{C}^*_{\mathrm{Lie}}(\rho_V^{\mathfrak{g}})} \mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g}) \xrightarrow{\omega^*} \Omega^*(P).$$

This, in turn, corresponds to a Maurer-Cartan element

$$\omega_V \in \Omega^*(P) \otimes \mathrm{End}(V).$$

We use this element to deform the differential on $\Omega^*(P, \underline{V}) = \Omega^*(P) \otimes V$ via

$$(\Omega^*(P) \otimes V, \mathrm{d} + \omega_V)\,.$$

Here, $\mathrm{d} = \mathrm{d}_{dR} + \mathrm{d}_V$ where d_{dR} is the de Rham differential on P and d_V is the internal differential to V. We can think of $\nabla^V := \mathrm{d} + \omega_V$ as a flat "super-connection" on the trivial bundle $P \times V \to P$. This means that ω_V may contain higher differential forms, not just one-forms. Tracing through the above construction, we see that ω_V actually preserves the subspace of basic forms, so it that ∇^V descends to a flat super-connection on the vector bundle V_X over X. In other words we obtain the $\Omega^*(X)$-module

$$\begin{aligned}\mathbf{desc}\left((P \to X, \omega), V\right) &:= \left(\Omega^*(P, \underline{V})_{bas}, \mathrm{d} + \omega_V\right) \\ &= \left(\Omega^*(X, V_X), \nabla^V\right).\end{aligned}$$

Definition 2.2.4. We will denote the vector bundle V_X equipped with its flat super-connection ∇^V obtained in this way by $\mathrm{desc}((P \to X, \omega), V)$. Its associated de Rham complex is denoted $\mathbf{desc}((P \to X, \omega), V)$.

2.2.2. The flat connection from the extended pair

In Section **??** we have introduced the L_∞ pair $(\widetilde{\mathrm{W}}_{n,d}, \mathrm{GL}_n)$ extending the pair $(\mathrm{W}_n, \mathrm{GL}_n)$. A Gelfand-Kazhdan structure is a natural $(\mathrm{W}_n, \mathrm{GL}_n)$-bundle whose underlying principal bundle is the frame bundle of X, and whose W_n-valued connection comes from the natural flat connection on the coordinate bundle. In this section we define extended Gelfand-Kazhdan structures that are bundles for the pair $(\widetilde{\mathrm{W}}_{n,d}, \mathrm{GL}_n)$.

If $(\mathfrak{f}, f) : (\widetilde{\mathfrak{g}}, \widetilde{K}) \to (\mathfrak{g}, K)$ is a map of pairs, and (P, ω) is a principal $(\mathfrak{g}, K)$-bundle, then a reduction of (P, ω) along $(\mathfrak{f}, f)$ is a principal $(\widetilde{\mathfrak{g}}, \widetilde{K})$-bundle $(\widetilde{P}, \widetilde{\omega})$ together with a map of bundles $\phi : \widetilde{P} \to P$ such that ϕ is a reduction of the principal K-bundle along f and $\mathfrak{f}(\widetilde{\omega}) = \phi^* \omega$.

Definition 2.2.5. Fix a Gelfand-Kazhdan structure (X, σ). A *d-extended Gelfand-Kazhdan structure* extending (X, σ) is a reduction of $(\mathrm{Fr}_X, \omega_\sigma)$ along the map $(\pi_{n,d}, \mathrm{id}) : (\widetilde{\mathrm{W}}_{n,d}, \mathrm{GL}_n) \to (\mathrm{W}_n, \mathrm{GL}_n)$.

Since the map on GL_n is the identity we see that the reduction of the bundle with connection $(P, \widetilde{\omega}_\sigma)$ is necessarily of the form $(\mathrm{Fr}_X, \widetilde{\omega}_\sigma)$ where $\widetilde{\omega}_\sigma \in \Omega^*(\mathrm{Fr}_X) \otimes \widetilde{\mathrm{W}}_{n,d}$ satisfies the generalized Maurer-Cartan equation.

We will show that extended Gelfand-Kazhdan structures are precisely associated to the data of a trivialization of of components of the Chern character $\mathrm{ch}(T_X^{1,0}) \in H^*(X, \Omega^*_{cl})$ and will be important when we discuss descent for the quantization of the holomorphic σ-model in the next section.

Proposition 2.2.6. *Fix an ordinary Gelfand-Kazhdan structure (X, σ). Then, a d-extensed Gelfand-Kazhdan structure exists if and only if $\mathrm{ch}_{d+1}(T_X^{1,0}) = 0$. Moreover, if*

$\mathrm{ch}_{d+1}(T_X^{1,0}) = 0$ *then the equivalence classes of d-extended Gelfand-Kazhdan structures extending (X, σ) is a torsor for the abelian group $H^d(X, \Omega_{cl}^{d+1})$.*

This proposition implies that every trivialization α of the component of the Chern character $\mathrm{ch}_{d+1}(T_X^{1,0})$ determines an extension of the original Gelfand-Kazhdan structure.

Proof. Suppose that we have a d-extension of a Gelfand-Kazhdan structure (X, σ). We will omit the formal exponential in the proof below. We can then use semi-strict descent to define a map in cohomology

$$\widetilde{\mathrm{char}}_X : H^*_{\mathrm{Lie}}(\widetilde{\mathrm{W}}_{n,d}, \mathrm{GL}_n; \widehat{\Omega}_{cl}^{d+1}) \to H^*(X, \Omega_{cl}^{d+1}).$$

This is the characteristic map for the semi-strict descent along the principal $(\widetilde{\mathrm{W}}_{n,d}, \mathrm{GL}_n)$-bundle $(\mathrm{Fr}_X, \widetilde{\omega})$. We are using the $\widetilde{\mathrm{W}}_{n,d}$ module structure on $\widehat{\Omega}_{n,cl}^k$ induced from the map $\pi_{n,d}$. Moreover the ordinary characteristic map $\mathrm{char}_X : H^*_{\mathrm{Lie}}(\mathrm{W}_n, \mathrm{GL}_n; \widehat{\Omega}_{cl}^{d+1}) \to H^*(X, \Omega_{cl}^{d+1})$ factors through this extended characteristic map:

$$\begin{tikzcd} H^*_{\mathrm{Lie}}(\mathrm{W}_n, \mathrm{GL}_n; \widehat{\Omega}_{cl}^{d+1}) \arrow[r, "\pi_{n,d}^*"] \arrow[rr, bend left, "\mathrm{char}_X"] & H^*_{\mathrm{Lie}}(\widetilde{\mathrm{W}}_{n,d}, \mathrm{GL}_n; \widehat{\Omega}_{cl}^{d+1}) \arrow[r, "\widetilde{\mathrm{char}}_X"] & H^*(X, \Omega_{cl}^{d+1}) \end{tikzcd}$$

Now, the image of the Gelfand-Fuks class $\mathrm{ch}_{d+1}^{\mathrm{GF}}(\widehat{\mathcal{T}}_n)$ along char_X is precisely $\mathrm{ch}_{d+1}(T_X^{1,0})$. Notice, however, that the image of $\mathrm{ch}_{d+1}^{\mathrm{GF}}(\widehat{\mathcal{T}}_n)$ in the middle cohomology is trivial. This is because it is the defining cocycle for the L_∞ extension $\widetilde{\mathrm{W}}_{n,d}$. It follows that the component of the Chern character $\mathrm{ch}_{d+1}(T_X^{1,0})$ is trivial in $H^{d+1}(X, \Omega_{cl}^{d+1})$.

Suppose now we fix a trivialization α of the component of the Chern character $\mathrm{ch}_{d+1}(T_X^{1,0}) \in H^{d+1}(X, \Omega_{cl}^{d+1,hol})$. We will resolve Ω_{cl}^{d+1} by holomorphic vector bundles via the complex

$$\Omega_X^{\geq d+1,hol} = \Omega_X^{d+1,hol} \xrightarrow{\widehat{\partial}} \Omega_X^{d+2,hol}[-1] \cdots .$$

For now, we put the $\widehat{\partial}$ for the formal holomorphic de Rham differential to not confuse it with the de Rham differential on X. Suppose we have a trivialization of $\mathrm{ch}_{d+1}(T_X^{1,0})$. We view the trivialization α as a degree d element in $\mathbb{R}\Gamma(X, \Omega_X^{\geq d+1,hol})$.

If $\mathscr{V}$ is any formal vector bundle then Gelfand-Kazhdan descent produces a pro-vector bundle with flat connection $\mathrm{desc}_X(\mathscr{V})$. Flat sections of this vector bundle form a sheaf that we call $\mathscr{D}\mathrm{esc}_X(\mathscr{V})$. Moreover, the de Rham complex is a model for the derived sections of this sheaf:

$$\Omega^*(X, \mathrm{desc}_X(\mathscr{V})) \simeq \mathbb{R}\Gamma(X, \mathscr{D}\mathrm{esc}_X(\mathscr{V})).$$

We consider the complex of formal vector bundles

$$\widehat{\Omega}_n^{\geq d+1} = \widehat{\Omega}_n^{d+1} \xrightarrow{\partial} \widehat{\Omega}_n^{d+2}[-1] \cdots .$$

Descent yields a quasi-isomorphism

$$\Omega^*(X, \mathrm{desc}_X(\widehat{\Omega}_n^{\geq d+1})) \simeq \mathbb{R}\Gamma(X, \Omega_n^{\geq d+1,hol}). \tag{2.6}$$

By construction, the de Rham complex on the left-hand side is of the form

$$\left(\left(\Omega^*(\mathrm{Fr}_X) \otimes \widehat{\Omega}_n^{\geq d+1}\right)_{bas}, \mathrm{d} + \omega_\sigma\right)$$

where ω_σ is the connection one-form defining Gelfand-Kazhdan descent.

Consider the defining exact sequence for the L_∞ algebra $\widetilde{\mathrm{W}}_{n,d}$:

$$0 \to \widehat{\Omega}^{d+1}_{n,cl}[d-1] \to \widetilde{\mathrm{W}}_{n,d} \xrightarrow{\pi_{n,d}} \mathrm{W}_n \to 0.$$

If we resolve $\widehat{\Omega}^{d+1}_{n,cl}[d-1]$ we obtain an extension

$$0 \to \widehat{\Omega}^{\geq d+1}_{n,cl}[d-1] \to \widetilde{\mathrm{W}}'_{n,d} \xrightarrow{\pi'_{n,d}} \mathrm{W}_n \to 0$$

where $\widetilde{\mathrm{W}}'_{n,d}$ is quasi-isomorphic to $\widetilde{\mathrm{W}}_{n,d}$. Let us tensor this exact sequence with the commutative dg algebra $\Omega^*(\mathrm{Fr}_X)$ to obtain an exact sequence of L_∞ algebras

$$0 \to \Omega^*(\mathrm{Fr}_X) \otimes \widehat{\Omega}^{\geq d+1}_{n}[d-1] \to \Omega^*(\mathrm{Fr}_X) \otimes \widetilde{\mathrm{W}}'_{n,d} \to \Omega^*(\mathrm{Fr}_X) \otimes \mathrm{W}_n \to 0.$$

The Gelfand-Kazhdan structure defines a GL_n-invariant Maurer-Cartan element $\omega_\sigma \in \Omega^*(\mathrm{Fr}_X) \otimes \mathrm{W}_n$. Using the quasi-isomorphism (2.6) we see that the trivialization α determines an element of $j_\infty\alpha \in \Omega^*(\mathrm{Fr}_X) \otimes \widehat{\Omega}^{\geq d+1}_{n}[d-1]$. We claim that $\widetilde{\omega}_{\sigma,\alpha} = \omega_\sigma + j_\infty\alpha$ is a GL_n-invariant Maurer-Cartan element in the L_∞ algebra $\Omega^*(\mathrm{Fr}_X) \otimes \widetilde{\mathrm{W}}'_{n,d}$. It is certainly GL_n-invariant, since ω_σ is and $j_\infty\alpha$ is pulled back from X. The Maurer-Cartan equation we must check is of the form

$$\mathrm{d}(\omega_\sigma + j_\infty\alpha) + \widehat{\partial}(\omega_\sigma + j_\infty\alpha) + \sum_{k\geq 2} \frac{1}{k!}\ell_k(\omega_\sigma + j_\infty\alpha) = 0.$$

Where ℓ_k is the k-ary structure map. Rearranging terms on the left-hand side we have

$$\left(\mathrm{d}\omega_\sigma + \frac{1}{2}[\omega_\sigma, \omega_\sigma]\right) + \left(\mathrm{d} + [\omega_\sigma, -]\right) j_\infty\alpha + \sum_{k\geq 3} \frac{1}{k!}\ell_k(\omega_\sigma, \dots, \omega_\sigma).$$

The first term is zero since ω_σ is a flat connection one-form. The term $\mathrm{d} + [\omega_\sigma, -]$ is the differential in the complex (2.6). Thus, by assumption, the second term is equal to $j_\infty \mathrm{ch}_{d+1}(T_X^{1,0})$, the ∞-jet expansion of the Chern character viewed as an element in $\Omega^*(\mathrm{Fr}_X) \otimes \widehat{\Omega}_n^{\geq d+1}[d-1]$. The only nontrivial contribution in the sum appearing in the last term is $k = d+1$, and this is precisely the defining cocycle for the extension $\widetilde{\mathrm{W}}'_{n,1}$ applied to ω_σ. Thus $\ell_{d+1}(\omega_\sigma, \ldots, \omega_\sigma)$ is equal to a multiple of $j_\infty \mathrm{ch}_{d+1}(T_X^{1,0})$. So, up to rescaling α, we see that the MC equation is satisfied.

$\square$

Gelfand-Kazhdan descent is a procedure that produces global objects on arbitrary manifolds from the data of a module for the pair $(\mathrm{W}_n, \mathrm{GL}_n)$. There is a completely analogous theory of modules for the pair $(\widetilde{\mathrm{W}}_{n,d}, \mathrm{GL}_n)$.

Given the data of a d-extension of a GK structure (X, σ), which is prescribed by a trivialization of $\mathrm{ch}_{d+1}(T_X^{1,0})$, we denote the corresponding descent functor by

$$\widetilde{\mathrm{desc}}_{X,\sigma,\alpha} : \mathrm{Mod}_{(\widetilde{\mathrm{W}}_{n,d}, \mathrm{GL}_n)} \to \mathrm{Pro}(VB)^{flat}.$$

The d-extension gives us a $(\widetilde{\mathrm{W}}_{n,d}, \mathrm{GL}_n)$-bundle $(\mathrm{Fr}_X, \widetilde{\omega}_{\sigma,\alpha})$ and hence, in the notation of Definition 2.2.4 we have $\widetilde{\mathrm{desc}}_{X,\sigma,\alpha} = \mathrm{desc}((\mathrm{Fr}_X, \widetilde{\omega}_{\sigma,\alpha}), -)$. When the formal exponential σ is understood, we denote this by $\widetilde{\mathrm{desc}}_{X,\alpha}$. Our main example of a module for the pair $(\widetilde{\mathrm{W}}_{n,d}, \mathrm{GL}_n)$ that is *not* a module for $(\mathrm{W}_n, \mathrm{GL}_n)$ will come from the quantization of the holomorphic d-dimensional σ-model.

2.3. The classical holomorphic σ-model

We will now define the classical field theory whose quantization is the subject of this chapter. We fix two complex manifolds Y and X where Y has complex dimension d. We will mostly be interested in the perturbative theory, but the full theory admits the following concise description. There are two types of fields in the theory:

(1) a map $\gamma : Y \to X$;
(2) an element $\beta \in \Omega^{d,d-1}(Y, \gamma^* T_X^{1,0*})$, i.e. a $(d, d-1)$-form on Y with values in the pull-back of the holomorphic cotangent bundle on X along γ.

For this reason, we will sometimes refer to the theory as the *higher dimensional $\beta\gamma$ system.* The action functional is of the form

$$S(\beta, \gamma) = \int_Y \langle \beta, \overline{\partial}\gamma \rangle_{T^{1,0}X}$$

where $\langle -, - \rangle_{T^{1,0}X}$ denotes the pairing between the holomorphic tangent bundle and its dual. One can immediately read off the equations of motion which state $\overline{\partial}\gamma = 0$ and $\overline{\partial}\beta = 0$. Thus, on-shell the solutions to the equations of motion state the $\gamma : Y \to X$ is a holomorphic map, and β determines an element in the cohomology $H^{d-1}(Y, \Omega^{d,hol} \otimes \gamma^* T_X^{1,0})$. The field β appears linearly in the action functional, and in a way its dynamics are completely determined by γ. In physics terminology it is the conjugate field to γ. In our language we will present the holomorphic σ-model as a cotangent theory and β will be the "fiber" coordinate. Notice that there is a large gauge symmetry present in the theory: for any $\beta' \in \Omega^{d,d-2}(Y, \gamma^* T^{1,0} X)$ the transformation $\beta \mapsto \beta + \overline{\partial}\beta'$ leaves the action invariant.

Our construction will provide a full BV-BRST formulation of the holomorphic σ-model with all gauge symmetries accounted for.

The fundamental approach we take is to construct this theory locally on the target, and then appeal to formal geometry to descend it over any complex manifold. For this reason, we first consider the case of a flat target.

2.3.1. The free $\beta\gamma$ system

In Section **??** we have provided a description, using the language of holomorphic field theories, of the $\beta\gamma$ system. It is not much different to define the $\beta\gamma$ system with target a complex vector space V. The fields together with their linearized BRST operator are

$$\mathscr{E}_V = \Omega^{0,*}(Y, V) \oplus \Omega^{d,*}(Y, V^*)[d-1].$$

We will write fields as (γ, β) to match with the notation above. As usual the notation $[d-1]$ means we shift that copy of the fields down by $d-1$. Note that the elements in degree zero, where the physical fields live, are precisely maps $\gamma : Y \to V$ and sections $\beta \in \Omega^{d,d-1}(Y; V^*)$, just as in the description above. In this flat case the section β has no dependence on γ. The (-1)-shifted symplectic pairing is given by integration along Y combined with the evaluation pairing between V and its dual: $(\gamma, \beta) \mapsto \int_Y \langle \gamma, \beta \rangle_V$. The action functional for this free theory is thus of the form

$$S_V(\beta, \gamma) = \int_Y \langle \beta, \overline{\partial} \gamma \rangle_V.$$

One can immediately check that $\mathscr{E}_V$ arises as the BV theory associated to a free holomorphic theory in the terminology of Chapter **??** where $Q^{hol} = 0$.

Note that the gauge symmetry $\beta \to \beta + \overline{\partial}\beta'$, where $\beta' \in \Omega^{d,d-2}(Y, V^*)$ has naturally been incorporated into our BRST complex (which only consists of a linear operator since the theory is free). Moreover, there are ghosts for ghosts $\beta'' \in \Omega^{d,d-3}(Y, V^*)$, and so on. Together with all of the antifields and antighosts, this makes up our full theory $\mathscr{E}_V$.

The theory $\mathscr{E}_V$ is the cotangent theory to the elliptic moduli problem $\Omega^{0,*}(Y, V)$ which describes holomorphic maps $Y \to V$.

2.3.1.1. The formal $\beta\gamma$ system. In the case that $V = \mathbb{C}^n$ we will see how the free $\beta\gamma$ system is an equivariant BV theory for the Harish-Chandra pair consisting of the group of linear automorphisms and the Lie algebra of formal vector fields on the n-disk. We will refer to this as the *formal* $\beta\gamma$ system, which one should heurstically think of as the $\beta\gamma$ system with target the formal disk $\widehat{D}^n$.

In the remainder of the chapter we will use the notation $\mathscr{E}_{\mathbb{C}^n} = \mathscr{E}_n$ and $S_{\mathbb{C}^n} = S_n$. The group $\mathrm{GL}_n = \mathrm{GL}_n(\mathbb{C})$ acts on $V = \mathbb{C}^n$ in the natural way which extends to an action on the Dolbeualt complex $\Omega^{0,*}(Y, \mathbb{C}^n)$.

Lemma 2.3.1. *The group* GL_n *acts on the theory* $\mathscr{E}_n$*. That is,* GL_n *is a symmetry of the action functional* S_n*.*

Proof. The action of GL_n is induced by the defining representation on $V = \mathbb{C}^n$ and the coadjoint action on $V^* = (\mathbb{C}^n)^*$, so the pairing is preserved by definition. $\square$

This is the first piece of data needed for Gelfand-Kazhdan formal geometry. The next piece is the action by the Lie algebra of formal vector fields. Recall, from Section **??** that to prescribe an action of a Lie algebra $\mathfrak{h}$ on a BV theory $\mathscr{E}$ we must prescribe a Noether current, that is, a Maurer-Cartan element

$$I^{\mathfrak{h}} \in \mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{h}) \otimes \mathscr{O}_{\mathrm{loc}}(\mathscr{E})[-1],$$

which is equivalent to a map of L_∞ algebras $I^{\mathfrak{h}} : \mathfrak{h} \rightsquigarrow \mathscr{O}_{\mathrm{loc}}(\mathscr{E})[-1]$.

Before considering the action of W_n on the field theory, consider first the cotangent bundle of a vector space T^*V. We can write the algebraic functions on T^*V as $\mathscr{O}(T^*V) = \mathscr{O}(V) \otimes \mathrm{Sym}(V)$. The derivations of $\mathscr{O}(V)$, or vector fields on V, have a similar decomposition $\mathrm{Vect}(V) = \mathscr{O}(V) \otimes V$. Note that there is an obvious embedding of vector fields on V inside of functions on T^*V via:

$$\mathrm{Vect}(V) = \mathscr{O}(V) \otimes V \to \mathscr{O}(V) \otimes \mathrm{Sym}(V) = \mathscr{O}(T^*V).$$

This map is compatible with the Lie bracket of vector fields and the standard Poisson bracket on T^*V. Thus, this embedding defines a Hamiltonian action of vector fields on T^*V.

Note that our theory is expressed as a shifted cotangent bundle of an elliptic moduli problem. The construction of our Hamiltonian action is formally similar to the above general construction. Suppose that we have a formal vector field

$$X = \sum_{j=1}^n \sum_{\vec{m}=(m_1,\ldots,m_n)\in\mathbb{N}^n} a_{j,\vec{m}} t_1^{m_1} \cdots t_n^{m_n} \partial_j \in \mathrm{W}_n.$$

Define the local functional $I^{\mathrm{W}}_X \in \mathscr{O}_{\mathrm{loc}}(\mathscr{E}_V)$ via the formula

$$I^{\mathrm{W}}_X(\gamma, \beta) = \sum_{j=1}^n \sum_{\vec{m}\in\mathbb{N}^n} a_{j,\vec{m}} \int_S \gamma_1^{\wedge m_1} \wedge \cdots \wedge \gamma_n^{\wedge m_n} \wedge \beta_j. \tag{2.7}$$

Following definition Definition **??**, the space of local functionals on $\mathscr{E}_n$ is defined by

$$\mathscr{O}_{\mathrm{loc}}(\mathscr{E}_n) = \mathrm{Dens}_Y \otimes_{D_Y} \mathrm{C}^*_{\mathrm{Lie,red}}(J\mathscr{E}_n). \tag{2.8}$$

Here $J\mathscr{E}_n$ denotes the ∞-jet bundle of the graded vector bundle defining $\mathscr{E}_n$. The Dolbeault operator $\overline{\partial}$ defining the classical theory extends to a degree $+1$ operator $\overline{\partial} : \mathscr{O}_{\mathrm{loc}}(\mathscr{E}_n) \to \mathscr{O}_{\mathrm{loc}}(\mathscr{E}_n)[-1]$. The following lemma describes the key properties of the functional I^{W}.

Lemma 2.3.2. *The map $I^{\mathrm{W}} : \mathrm{W}_n \to \mathscr{O}_{\mathrm{loc}}(\mathscr{E}_n)[-1]$ sending $X \mapsto I^{\mathrm{W}}_X$ is a map of dg Lie algebras. Hence,*

$$I^{\mathrm{W}} \in \mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n) \otimes \mathscr{O}_{\mathrm{loc}}(\mathscr{E}_n)[-1]$$

satisfies the equivariant classical master equation

$$(\mathrm{d}_{\mathrm{W}} + \overline{\partial})I^{\mathrm{W}} + \frac{1}{2}\{I^{\mathrm{W}}, I^{\mathrm{W}}\} = 0. \tag{2.9}$$

In particular, I^{W} endows $\mathscr{E}_n$ with the structure of a W_n-equivariant classical BV theory, see Section **??**.

Remark 2.3.3. When restricted to linear vector fields, the action of W_n on $\beta\gamma$ system with target $\widehat{D}^n$ agrees with the action of GL_n described in Lemma **??**. In this sense, we have described an action of the Harish-Chandra pair $(\mathrm{W}_n, \mathrm{GL}_n)$ on the classical $\beta\gamma$ system. This theory can thus be treated by Gelfand-Kazhdan formal geometry. We develop this reasoning more fully in Section **??**. In particular, in the next section we will show that this theory descends to the classical holomorphic σ-model of maps where the target is any complex manifold X. In complex dimension one we this is the theory studied by Costello in [**Cosa**].

The deformation complex of the formal $\beta\gamma$ system is simply the space of local functionals equipped with its linearized BRST differential:

$$\mathrm{Def}_n = \left(\mathscr{O}_{\mathrm{loc}}(\mathscr{E}_n), \overline{\partial}\right).$$

Following the perspective of equivariant BV formalism, the functional I^{W} allows us to define the W_n-equivariant deformation complex

$$\mathrm{Def}_n^{\mathrm{W}} = \left(\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n) \otimes \mathscr{O}_{\mathrm{loc}}(\mathscr{E}_n), \mathrm{d}_{\mathrm{W}} + \overline{\partial} + \{I^{\mathrm{W}}, -\}\right).$$

This is the complex controlling W_n-equivariant deformations of the formal $\beta\gamma$ system on Y with target $\widehat{D}^n$. The fact that the operator $\mathrm{d}_{\mathrm{W}} + \overline{\partial} + \{I^{\mathrm{W}}, -\}$ is square zero is equivalent to the equivariant classical master equation (2.9).

In the next section we will show how the formal $\beta\gamma$ system, which is the theory of holomorphic maps $Y \to \widehat{D}^n$, together with the action of $(\mathrm{W}_n, \mathrm{GL}_n)$ allows us to define a general σ-model of maps $Y \to X$ where X is any complex manifold.

Proposition 2.3.4. *The formal $\beta\gamma$ system $\mathscr{E}_n$ has an action by the Harish-Chandra pair $(\mathrm{W}_n, \mathrm{GL}_n)$. If X is any complex manifold, the Gelfand-Kazhdan descent $\mathrm{desc}_X(\mathscr{E}_n)$ is equivalent to cotangent theory of the formal completion of the derived space of holomorphic maps $Y \to X$ near the constant maps.*

Remark 2.3.5. After setting up the appropriate terminology in the next section, we will refer to the cotangent theory of the formal completion of the derived space of holomorphic maps from $Y \to X$ simply as the *holomorphic σ-model*.

2.3.2. A description using L_∞ spaces

We now give a second description of the holomorphic σ-model. This approach is based on the geometry of L_∞ spaces developed by Costello [**Cosa**] and Gwilliam-Grady [**?, ?**]. We will relate it to our description above using formal geometry.

The language of L_∞ spaces allows one to incorporate many natural geometries in the language of Lie theory. Of course, L_∞ spaces are much more flexible than ordinary manifolds, and so also provide a nice geometric description of stacky-like objects as well. The key aspect of the formalism we will utilize is based on a general result of Costello [**?, Cosa**] that states σ-models in the BV formalism can be represented as maps from a elliptic ringed space to an L_∞ space. An elliptic ringed space is a pair $(Y, \mathscr{A})$ where Y is a manifold and $\mathscr{A}$ is a sheaf of commutative dg algebras defined over Ω^*_Y satisfying some conditions. For a precise definition see Definition ?? of [**?**]. The most important condition for us is that underlying sheaf of cochain complexes is elliptic. For us, the elliptic ringed space representing the source of the σ-model is always of the form

$$Y_{\overline{\partial}} = (Y, \Omega^{0,*}_Y).$$

We refer to this as the *Dolbeualt space* of the complex manifold Y. Of course, $\Omega^{0,*}_Y \simeq \mathscr{O}^{hol}_Y$ as sheaves, but the resolution is necessary since holomorphic functions are not the smooth sections of any vector bundle.

By definition, an L_∞ space is a manifold X together with a sheaf of curved L_∞ algebras $\mathfrak{g}$ defined over the de Rham complex Ω^*_X. The most important L_∞ space for us exists on any complex manifold X. In [**Cosa**] it is shown that there exists an L_∞ space

$(X, \mathfrak{g}_{X_{\overline{\partial}}})$ which is uniquely characterized by the fact that its Chevalley-Eilenberg cochains is isomorphic to the de Rham complex of holomorphic jets of the trivial bundle:

$$\mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g}_{X_{\overline{\partial}}}) \cong_\sigma \Omega^*(X, J^{hol}_X).$$

On the left-hand side the cochains are taken over the ring Ω^*_X, and the isomorphism is as Ω^*_X-modules. The differential on the left hand side is pulled back along an isomorphism of pro-vector bundles $\sigma : \widehat{\mathrm{Sym}}(T^{1,0*}_X) \xrightarrow{\cong} J^{hol}_X$. This isomorphism σ is constructed by fixing a connection on the tangent bundle T_X and using its associated exponential map at each point x to identify the formal neighborhood of x in X with the formal neighborhood of the origin in T_xX. In this way, the ∞-jet of a function at x is identified with a formal power series in T^*_xX, which is the desired isomorphism σ.

But this procedure is precisely how Gelfand-Kazhdan descent works! Once we fix a formal exponential on the frame bundle of X — typically via a choice of connection — we have an isomorphism σ. Moreover, the descent of $\widehat{\mathcal{O}}_n = \mathrm{C}^*_{\mathrm{Lie}}(\mathbb{C}^n[-1])$ using this data is exactly $\Omega^*(X, \widehat{\mathrm{Sym}}(T^{1,0*}_X))$ equipped with the pullback of the Grothendieck connection along σ. In other words, Gelfand-Kazhdan descent recovers Costello's curved L_∞ algebra, once one applies the Koszul duality. We can summarize this in the following way.

Lemma 2.3.6. *Let $\mathfrak{g}_n = \mathbb{C}^n[-1]$. Then, the Gelfand-Kazhdan descent* $\mathrm{desc}_X(\mathfrak{g}_n)$ *has the structure of a curved L_∞ algebra defined over Ω^*_X. Moreover, it is equivalent to Costello's L_∞ algebra $\mathfrak{g}_{X_{\overline{\partial}}}$.*

We'd now like to describe how formal geometry allows us to describe holomorphic σ-models. First, we summarize Costello's approach for characterizing mapping stacks using

L_∞ spaces. We will then apply this to the case that the target is the L_∞ space $\mathfrak{g}_{X_{\overline{\partial}}}$ to obtain a model for the holomorphic σ-model.

By definition, a map from the locally ringed space $(Y, \mathscr{A})$ to the L_∞ space $(X, \mathfrak{g})$ is a smooth map of underlying manifolds $\varphi : Y \to X$ together with the data of a Maurer-Cartan element in

$$\varphi^* \mathfrak{g} = \mathscr{A} \otimes_{\varphi^* \Omega^*_X} \varphi^{-1} \mathfrak{g},$$

The extra data of an elliptic ringed space is an ideal $\mathscr{I} \subset \mathscr{A}$, and we require that this Maurer-Cartan element vanish modulo $\mathscr{I} \subset \mathscr{A}$. If one were to use a functor of points approach to define the L_∞ space $(X, \mathfrak{g})$, this would be the value of $(X, \mathfrak{g})$ on the ringed space $(Y, \mathscr{A})$.

Lemma 2.3.7 ([**Cosa**] Lemma 3.1.1)**.** *Suppose Y, X are complex manifolds. Then, a map*

$$\varphi : Y_{\overline{\partial}} \to (X, \mathfrak{g}_{X_{\overline{\partial}}})$$

is the same as a holomorphic map $\varphi : Y \to X$.

What this says is that the curving in $\mathfrak{g}_{X_{\overline{\partial}}}$ pulls back to zero along φ precisely when the map is holomorphic.

This lemma gives a procedure for describing the formal neighborhood of a fixed holomorphic map in the moduli space of all maps $Y \to X$. If $\varphi : Y \to X$ is holomorphic, the lemma implies that there is an isomorphism of $\Omega^{0,*}_Y$-modules $\varphi^* \mathfrak{g}_{X_{\overline{\partial}}} \cong \Omega^{0,*}(Y, \varphi^* T^{1,0} X[-1])$. Since $\mathrm{C}^*_{\mathrm{Lie}}(\varphi^* \mathfrak{g}_{X_{\overline{\partial}}}) = \Omega^{0,*}(Y, \varphi^* J^{hol}_X)$, a Maurer-Cartan element in this L_∞ algebra with values in the test Artinian dg ring (R, m) is a map of $\Omega_Y^{0,*}$-algebras

$$\Omega^{0,*}(Y, \varphi^* J_X^{hol}) \to \Omega_Y^{0,*} \otimes m.$$

This is precisely a deformation of the holomorphic map φ.

In particular, when φ is a constant map, we see that the curved L_∞ algebra

$$\Omega^{0,*}(Y) \otimes \mathfrak{g}_{X_{\overline{\partial}}}$$

defined over Ω_X^* controls deformations of constant maps inside of all holomorphic maps $Y \to X$. The following formalizes this statement and is proved in detail in [**Cosa**].

Proposition 2.3.8 ([**Cosa**] Proposition 5.0.1)**.** *Let* $\mathrm{MC}_{(X,\mathfrak{g}_{X_{\overline{\partial}}})}(Y, \Omega^{0,*})$ *be the derived space of maps* $(Y, \Omega_Y^{0,*}) \to (X, \mathfrak{g}_{X_{\overline{\partial}}})$*. Then, there is a subspace of* $\widehat{\mathrm{MC}}$ *consisting of those maps whose underlying smooth map of manifolds* $Y \to X$ *is constant. Moreover, this subspace is represented by the* L_∞ *space*

$$(X, \Omega^{0,*}(Y) \otimes \mathfrak{g}_{X_{\overline{\partial}}}).$$

The derived space of maps from a ringed space to an L_∞ space is a huge object, and in general will not be represented by an L_∞ space. What this proposition says that there is a formal completion inside of this mapping space near the constant maps that is described by the L_∞ space $(X, \Omega^{0,*}(Y) \otimes \mathfrak{g}_{X_{\overline{\partial}}})$.

We have the following interpretation of the L_∞ algebra $\Omega^{0,*}(Y) \otimes \mathfrak{g}_X$ via formal geometry. This follows immediately from Lemma 2.3.6 above.

Lemma 2.3.9. *The Gelfand-Kazhdan descent* $\mathrm{desc}_X(\Omega^{0,*}(Y) \otimes \mathfrak{g}_n)$ *has the structure of a curved* L_∞ *algebra over* Ω^*_X *and is equivalent to* $\Omega^{0,*}_Y \otimes \mathfrak{g}_{X_{\overline{\partial}}}$ *as in Proposition 2.3.8.*

As a corollary of Proposition 2.3.8 and this lemma we see that the Gelfand-Kazhdan descent along X of $\Omega^{0,*}_Y \otimes \mathfrak{g}_n$ is the curved L_∞ algebra controlling deformations of constant maps inside of all holomorphic maps $Y \to X$.

Since descent intertwines with the shifted cotangent bundle construction, we see that the descent of the BV theory $\mathscr{E}_n = T^*[-1](\Omega^{0,*}_Y \otimes \mathfrak{g}_n)$ along X is the shifted cotangent bundle of the elliptic moduli problem of deformations of constant maps inside of all holomorphic maps $Y \to X$. Explicitly, the cotangent theory to the moduli problem described by $\Omega^{0,*}(Y) \otimes \mathfrak{g}_{X_{\overline{\partial}}}$ has fields of the form

$$\mathscr{E}_{Y\to X} = \Omega^{0,*}(Y) \otimes \mathfrak{g}_{X_{\overline{\partial}}}[1] \oplus \Omega^{d,*}(Y) \otimes \mathfrak{g}^\vee_{X_{\overline{\partial}}}[-2]$$

where $\mathfrak{g}^\vee_X$ denotes the Ω^*_X-linear dual. The theory is described by some interaction $I_{Y\to X} \in \mathscr{O}_{\mathrm{loc}}(\mathscr{E}_{Y\to X})$. Local functionals $\mathscr{O}_{\mathrm{loc}}(\mathscr{E}_{Y\to X})$ are defined similarly to the usual way, such as Equation (2.8), except the Chevalley-Eilenberg chains $\mathrm{C}^*_{\mathrm{Lie,red}}(J\mathscr{E}_{Y\to X})$ is understood to be taken over the dg ring Ω^*_X.

Definition 2.3.10. The *holomorphic σ-model of maps $Y \to X$* is the classical BV theory, defined over the ring Ω^*_X, with space of fields $\mathscr{E}_{Y\to X}$ and classical interaction $I_{Y\to X}$. This is the cotangent theory of the moduli space of holomorphic maps $Y \to X$ that are infinitesimally close to the constant maps.

Remark 2.3.11. We remark on the abuse of terminology since we are only working around the constant maps throughout this work. It would be very interesting to study the general holomorphic σ-model where one works in perturbation theory around a generic holomorphic map.

As a result of the above discussion we see that the space of fields is exactly the Gelfand-Kazhdan descent of the formal theory $\mathscr{E}_{Y\to X} = \mathrm{desc}_X(\mathscr{E}_n)$. Moreover, under the characteristic map

$$\mathrm{char}_X : \mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n, \mathrm{GL}_n; \mathscr{O}_{\mathrm{loc}}(\mathscr{E}_n)) \to \mathscr{O}_{\mathrm{loc}}(\mathscr{E}_{Y\to X})$$

the interaction $I^{\mathrm{W}} \mapsto I_{Y\to X}$. This proves Proposition 2.3.4.

Remark 2.3.12. Note that using L_∞ spaces one can make sense of the σ-model of maps $(X, \mathfrak{g})$ where $\mathfrak{g}$ is *any* curved L_∞ algebra on X. We will denote this theory by $\mathscr{E}_{Y\to B\mathfrak{g}}$

$$\mathscr{E}_{Y\to B\mathfrak{g}} = \Omega^{0,*}(Y) \otimes \mathfrak{g}[1] \oplus \Omega^{d,*}(Y) \otimes \mathfrak{g}^\vee[-2].$$

The classical interaction defining the theory is $I_{Y\to B\mathfrak{g}} \in \mathscr{O}_{\mathrm{loc}}(\mathscr{E}_{Y\to B\mathfrak{g}})$.

2.4. Deformations of the holomorphic σ-model

We now turn to computing the deformation complex of the holomorphic σ-model. This will be important when we quantize the σ-model, as the deformation complex controls both the obstructions and moduli space of such quantizations.

In this section we allow $\mathfrak{g}$ to be a (possibly) curved L_∞ algebra over a commutative dg ring R and consider the holomorphic σ-model of maps $Y \to B\mathfrak{g}$, where Y is a complex d-fold. This was the most general form of the holomorphic σ-model from the previous section. We will be most interested in the following two cases:

(1) the simplest case where $R = \mathbb{C}$ and $\mathfrak{g} = \mathbb{C}^n[-1]$ is the trivial L_∞ algebra with $\ell_k = 0$ for all $k \geq 0$;

(2) when X is a smooth manifold $R = \Omega^*_X$, and $\mathfrak{g}$ is a curved L_∞ algebra over Ω^*_X. Thus, $\mathfrak{g}$ is part of an L_∞ space $(X, \mathfrak{g})$ over X in the terminology of [**?, ?**].

We have discussed how these two cases are related. Indeed, through Gelfand-Kazhdan descent along a complex manifold we can patch together the case (1) to the situation in (2) where $\mathfrak{g} = \mathfrak{g}_{X_{\overline{\partial}}}$, the curved L_∞ algebra encoding the complex structure.

The theory we are studying is a cotangent theory of the form $T^*[-1](\Omega^{0,*}(Y, \mathfrak{g}[1]))$. In particular, there is an action of the abelian group $\mathbb{C}^\times_{cot}$ which assigns the base direction a weight of zero and the fiber a weight of $+1$. Thus, if $(\gamma, \beta) \in \Omega^{0,*}(Y, \mathfrak{g})[1] \oplus \Omega^{d,*}(Y, \mathfrak{g}^\vee)[d-1]$, then an element $\lambda \in \mathbb{C}^\times_{cot}$ acts by

$$\lambda \cdot (\gamma, \beta) = (\gamma, \lambda\beta).$$

Our first reduction is to restrict ourselves to studying deformations that are compatible with this $\mathbb{C}^\times_{cot}$ action.

Note that the symplectic pairing of the theory, as well as the classical action functional, is of $\mathbb{C}^\times_{cot}$-weight (-1). Our convention is that the parameter $\hbar$ *has* $\mathbb{C}^\times_{cot}$*-weight* (-1) as well. There are two compelling reasons for making this definition. The first deals with studying correlation functions for the theory. If we require the observables of the theory to be equivariant for this rescaling of the cotangent fibers, this means that the factorization product must have $\mathbb{C}^\times_{cot}$ weight zero. In the case that the theory is free, we have seen that the factorization product between two operators of the theory $\mathcal{O}, \mathcal{O}'$ is computed by a Moyal type formula

$$\mathcal{O} \star \mathcal{O}' = e^{-\hbar \partial_P} \left(e^{\hbar \partial_P} \mathcal{O} \cdot e^{\hbar \partial_P} \mathcal{O}' \right).$$

Since the symplectic pairing is $\mathbb{C}^\times_{cot}$-weight (-1) we observe that the propagator is also $\mathbb{C}^\times_{cot}$-weight $(+1)$. [2] For the product to have weight zero we are then forced to take $\hbar$ to have opposite weight to P.

The other, related reason, we choose this weight for $\hbar$ is that we would like to require our BV complex to be equivariant for rescaling the fibers as well. The classical BRST differential is of the form $\{S, -\} = Q + \{I, -\}$. We have already said that the classical action is of weight (-1). Since the symplectic pairing is also degree (-1), this means that the P_0 bracket is degree $+1$. Thus, the classical BRST complex is manifestly equivariant. The quantum BV differential involves deforming this classical differential by $\hbar\Delta$. For the same reason as the Poisson bracket, the BV Laplacian has weight $(+1)$. Thus, we see that in order to have an equivariant differential we are again forced to take $\hbar$ to have weight -1.

In the case of an interacting theory, we have the following restriction on the quantum interactions of the theory as well. We can expand an effective interaction as

$$I[L] = \sum_{g \geq 0} \hbar^g I^{(g)}[L].$$

In order for $I[L]$ to have $\mathbb{C}^\times_{cot}$ weight (-1) we see that $I^{(g)}[L]$ must have weight $g-1$. We are only studying a one-loop quantization of the holomorphic theory, so the effective action has the form $I[L] = I^{(0)} + \hbar I^{(1)}[L]$ and hence $I^{(1)}[L]$ has weight zero.

[2]This actually requires that we also take the gauge fixing operator to be of $\mathbb{C}^\times_{cot}$-weight zero, which is the natural thing to do for cotangent theories.

Thus, all one-loop quantities compatible with the $\mathbb{C}^\times_{cot}$ action also have weight zero, including the one-loop anomaly. For this reason, we will be most concerned with the piece of the deformation complex that is $\mathbb{C}^\times_{cot}$-weight zero. This amounts to looking just at local functionals of the γ-field.

Definition 2.4.1. The *deformation complex for cotangent quantizations* of the holomorphic σ-model of maps $Y \to B\mathfrak{g}$ is the cochain complex

$$\mathrm{Def}^{\mathrm{cot}}_{Y\to B\mathfrak{g}} = \left(\mathscr{O}_{\mathrm{loc}}(\Omega^{0,*}_Y \otimes \mathfrak{g}), \overline{\partial} + \{I_{Y\to B\mathfrak{g}}, -\}\right)$$

Here, $I_{Y\to B\mathfrak{g}}$ is the restriction of the interaction defining the classical theory of maps $Y_{\overline{\partial}} \to B\mathfrak{g}$.

The right-hand side is simply the local cochains of the local Lie algebra $\Omega^{0,*}_Y \otimes \mathfrak{g}$ on Y:

$$\left(\mathscr{O}_{\mathrm{loc}}(\Omega^{0,*}_Y \otimes \mathfrak{g}), \overline{\partial} + \{I_{Y\to B\mathfrak{g}}, -\}\right) = \mathrm{C}^*_{\mathrm{loc}}(\Omega^{0,*}_Y \otimes \mathfrak{g}).$$

We defined local cochains $\mathrm{C}^*_{\mathrm{loc}}(\mathscr{L})$ of a local Lie algebra in Section **??**.

We will be most interested in seeing how both the anomaly and the resulting quantum correction induced by the anomaly are realized inside the complex $\mathrm{Def}^{\mathrm{cot}}_{Y\to Bg}$. Before doing this, we'd like to restrict ourselves to looking at quantizations preserving further symmetries.

We now specialize to the case that the source is d-dimensional affine space $Y = \mathbb{C}^d$. On $\mathbb{C}^d$ there is the natural action of Lie group of translations. This is a real Lie group of real dimension $2d$ whose complexified Lie algebra $\mathbb{C}^{2d}$ is generated by the constant vector fields $\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i}$. In fact, this action lifts to an action of the dg Lie algebra

$$\mathbb{C}^{2d|d} = \mathbb{C}^{2d} \oplus \mathbb{C}^d[1]$$

where the even parts are generated by the constant vector fields and the odd piece is generated by the symbols $\frac{\partial}{\partial(\mathrm{d}\bar{z}_i)}$. The differential sends $\frac{\partial}{\partial(\mathrm{d}\bar{z}_i)} \mapsto \frac{\partial}{\partial \bar{z}_i}$. We encountered this dg Lie algebra in Section **??** when defining holomorphically translation invariant theories.

Lemma 2.4.2. *The holomorphic σ-model of maps $\mathbb{C}^d \to B\mathfrak{g}$ is holomorphically translation invariant. In particular, it has an action by the super Lie algebra $\mathbb{C}^{2d|d}$.*

The deformation complex controlling cotangent quantizations that are holomorphically translation invariant is equal to the subcomplex $\left(\mathrm{Def}^{\mathrm{cot}}_{\mathbb{C}^d \to B\mathfrak{g}}\right)^{\mathbb{C}^{2d|d}} \subset \mathrm{Def}^{\mathrm{cot}}_{\mathbb{C}^d \to B\mathfrak{g}}$.

Finally, there is one more group of symmetries we'd like to consider. The group $U(d)$ acts on $\mathbb{C}^d$ via the defining representation. This extends to an action on any tensor bundle on $\mathbb{C}^d$ by bundle automorphisms, and hence acts on sections via the pull back. In particular, it acts on the elliptic complex $\Omega^{0,*}(\mathbb{C}^d) \otimes \mathfrak{g}$ where $\mathfrak{g}$ is any curved L_∞ algebra defined over some dg ring R. As with a group action on any elliptic moduli problem, this extends to one on the cotangent theory in a way that preserves the (-1)-shifted symplectic pairing, hence it acts on the holomorphic σ-model of maps $\mathbb{C}^d \to B\mathfrak{g}$

In conclusion, we have the following lemma exhibiting the symmetries of the holomorphic σ-model we will take into account.

Lemma 2.4.3. *The classical theory of holomorphic maps $\mathbb{C}^d \to B\mathfrak{g}$ is holomorphically translation invariant. Moreover, it is equivariant for the group $U(d)$. When $\mathfrak{g} = \mathbb{C}^n[-1]$*

the action of translations and $U(d)$ on the formal σ-model $\mathbb{C}^d \to \widehat{D}^n$ is compatible with the action of the Harish-Chandra pair $(\mathrm{W}_n, \mathrm{GL}_n)$.

The second statement of the lemma is immediate since the actions of translations and $U(d)$ and $(\mathrm{W}_n, \mathrm{GL}_n)$ clearly commute. In what follows, we will consider deformations that are also invariant for this group of symmetries.

2.4.1. Forms as local functionals

Before we compute the possible deformations of the holomorphic σ-model, we describe how certain differential forms on the formal stack $B\mathfrak{g}$ yield local functionals of the holomorphic σ-model of maps $Y \to B\mathfrak{g}$. Indeed, we will define a map of cochain complexes

$$J : \Omega^{d+1}_{cl}(B\mathfrak{g}) \to \left(\mathrm{Def}^{\mathrm{cot}}_{\mathbb{C}^d \to B\mathfrak{g}}\right)^{\mathbb{C}^{2d|d}}.$$

Recall, the right-hand side consists of the holomorphically translation invariant deformations. Moreover, for each $\omega \in \Omega^{d+1}_{cl}$ the functional J_ω is $U(d)$-invariant.

The functions on a formal moduli stack $B\mathfrak{g}$ are given by the Chevalley-Eilenberg complex $\mathscr{O}(B\mathfrak{g}) = \mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g})$. By definition, the k-forms on a formal moduli stack $B\mathfrak{g}$ are defined by

$$\Omega^k(B\mathfrak{g}) := \mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g}; \mathrm{Sym}^k \mathfrak{g}^\vee[-k])$$

where $\mathfrak{g}^\vee$ denotes the coadjoint module of $\mathfrak{g}$.

As a simple check, note that in the case $\mathfrak{g} = \mathbb{C}^n[-1]$ the above complex reduces to

$$\Omega^k(B\mathfrak{g}) = \mathbb{C}[t_1, \ldots, t_n] \otimes \wedge^k(t_1^\vee, \cdots, t_n^\vee),$$

where $t_i^\vee$ denotes the dual coordinate. Everything is in cohomological degree zero. If we identify $t_i^\vee \leftrightarrow \mathrm{d}t_i$, this is the usual definition of the algebraic de Rham forms.

Let $\partial : \Omega^k(B\mathfrak{g}) \to \Omega^{k+1}(B\mathfrak{g})$ be the de Rham operator for $B\mathfrak{g}$. We use ∂ to denote the de Rham differential on $B\mathfrak{g}$. This is because our two main examples of $B\mathfrak{g}$ will be the formal holomorphic disk $\widehat{D}^n$ or the formal moduli space associated to any complex manifold X. In each of these cases, the differential above is the holomorphic Dolbeualt operator $\partial : \Omega^k_{hol} \to \Omega^{k+1}_{hol}$. The space of closed k-forms is

$$\widehat{\Omega}^k_{cl}(B\mathfrak{g}) = \left(\Omega^k(B\mathfrak{g}) \xrightarrow{\partial} \Omega^{k+1}(B\mathfrak{g})[-1] \to \cdots\right).$$

With the requisite notation set up we are now ready to define the map J. For now, let Y be any complex manifold. Observe that any function on $B\mathfrak{g}$, $a \in \mathscr{O}(B\mathfrak{g})$ can be extended to a $\Omega^{0,*}(Y)$-valued functional

$$a^Y : \mathrm{Sym}\left(\Omega^{0,*}(Y) \otimes \mathfrak{g}[1]\right) \to \Omega^{0,*}(Y).$$

Suppose a is a homogenous polynomial of degree m. The map a^Y is defined by

$$a^Y : (\gamma_1 \otimes \xi_1) \otimes \cdots \otimes (\gamma_m \otimes \xi_m) \mapsto (\gamma_1 \wedge \cdots \wedge \gamma_m) a(\xi_1, \ldots, \xi_m).$$

We will sometimes write this succinctly as $a^Y(\gamma) \in \Omega^{0,*}(Y)$. Similarly, we can extend any k-form $\omega \in \Omega^k(B\mathfrak{g}) := \mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g}; \mathrm{Sym}^k \mathfrak{g}^\vee[-k])$ on $B\mathfrak{g}$ to a $\Omega^{0,*}(Y) \otimes \mathrm{Sym}^k \mathfrak{g}^\vee[-k]$-valued functional form valued functional

$$\omega^Y : \mathrm{Sym}\left(\Omega^{0,*}(Y) \otimes \mathfrak{g}[1]\right) \to \Omega^{0,*}(Y) \otimes \mathrm{Sym}^k \mathfrak{g}^\vee[-k].$$

Definition 2.4.4. For each k define

$$J^k : \Omega^k(B\mathfrak{g})[k] \to \mathscr{O}_{\mathrm{loc}}(\Omega^{0,*}(Y) \otimes \mathfrak{g}[1]) \ \ , \ \ \omega \mapsto J^k_\omega$$

by the formula

$$J^k_\omega(\gamma) = \int \langle \omega(\gamma), \partial\gamma \cdots \partial\gamma \rangle_{\mathfrak{g}},$$

where $\langle -,- \rangle$ denotes the pairing between $\mathfrak{g}$ and its dual.

Note that J^k_ω is a local functional which is induced from the holomorphic Lagrangian $\gamma \mapsto \langle \gamma, \partial\gamma \cdots \partial\gamma \rangle$.

Next, we introduce the truncated de Rham complex

(2.10)

$$R[1] \longrightarrow \mathscr{O}(B\mathfrak{g}) \xrightarrow{\partial} \Omega^1(B\mathfrak{g})[-1] \longrightarrow \cdots \longrightarrow \Omega^{d-1}(B\mathfrak{g})[-d+1] \xrightarrow{\partial} \Omega^d(B\mathfrak{g})[-d].$$

Here, R is the ring for which $\mathfrak{g}$ is defined over. Now, there is an obvious quotient map $\Omega^*(B\mathfrak{g}) \to$ (2.10), where $\Omega^*(B\mathfrak{g})$ is the full de Rham complex. The kernel is the complex of (shifted) closed $(d+1)$-forms $\Omega^{d+1}_{cl}(B\mathfrak{g})[-d-1]$. It follows that we have an exact sequence

$$\Omega^{d+1}_{cl}(B\mathfrak{g})[-d-1] \to \Omega^*(B\mathfrak{g}) \to (2.10).$$

Since the middle term is acyclic, it follows that the connecting map (which is degree one) is a quasi-isomorphism

$$(2.10) \xrightarrow{\simeq} \Omega^{d+1}_{cl}(B\mathfrak{g})[-d]. \tag{2.11}$$

Lemma 2.4.5. *Let $d = \dim_{\mathbb{C}} Y$. The map J^d determines a map of cochain complexes*

$$J = J^d : \Omega^{d+1}_{cl}(B\mathfrak{g})[d] \to \mathrm{Def}_{Y \to B\mathfrak{g}}.$$

Proof. We will show that J^d determines a cochain map from the truncated de Rham complex in (2.10) to $\mathrm{Def}_{Y\to B\mathfrak{g}}$. Using the quasi-isomorphism in (2.11) we obtain the desired map from closed $(d+1)$-forms.

Thus, it suffices to show that if $\omega = \partial\alpha$, where $\alpha \in \Omega^{d-1}(B\mathfrak{g})$ that $J_\omega(\gamma) = 0$. Notice that J_ω is the local functional obtained from integrating the Lagrangian density

$$\mathbf{J}^d_\omega(\gamma) = \langle \omega(\gamma), \partial\gamma \cdots \partial\gamma \rangle \in \Omega^{d,*}(Y).$$

We will show that as Lagrangian densities $\mathbf{J}_{\partial\alpha} = \partial \mathbf{J}^{d-1}_\alpha$ where $\mathbf{J}^{d-1}_\alpha$ is the $\Omega^{d-1,*}(Y)$-valued function $\mathbf{J}^{d-1}_\alpha(\gamma) = \langle \alpha(\gamma), \partial\gamma\cdots\partial\gamma\rangle$. Then for $\omega = \partial\alpha$, the Lagrangian is a total derivative, hence zero as a local functional.

We prove this by induction in d. For $d = 1$, we must show that $\mathbf{J}^1_{\partial\alpha} = \partial\alpha^Y$. Suppose that $\alpha \in \widehat{\mathcal{O}}_n$ is a linear function $\alpha : \mathfrak{g}[1] \to R$. Then, $\partial\alpha$ is the very simply functional $R \to \mathfrak{g}^\vee[-1]$ corresponding to the dual of α. Thus, $\mathbf{J}^1_{\partial\alpha} = \partial J_\alpha$. To see the claim in general we use the fact that ∂ is a derivation. Indeed, if $\alpha, \alpha' \in \mathscr{O}(B\mathfrak{g})$ then $\partial((\alpha\alpha')^Y) = \partial(\alpha^Y\alpha'^Y) = \partial(\alpha^Y)\alpha'(Y) \pm \alpha^Y\partial(\alpha'^Y)$.

$\square$

2.4.1.1. Computing the deformation complex. In this section we specialize the functional J to the space $Y = \mathbb{C}^d$ and use it to completely characterize the $U(d)$-invariant, holomorphically translation invariant deformation complex.

Proposition 2.4.6. *The map $J : \Omega^{d+1}_{cl}(B\mathfrak{g})[d] \to \mathrm{Def}^{\mathrm{cot}}_{\mathbb{C}^d \to B\mathfrak{g}}$ factors through the holomorphically translation invariant deformation complex:*

$$J : \Omega^{d+1}_{cl}(B\mathfrak{g})[d] \to \left(\mathrm{Def}^{\mathrm{cot}}_{\mathbb{C}^d \to B\mathfrak{g}}\right)^{\mathbb{C}^{2d|d}}.$$

Furthermore, J defines a quasi-isomorphism into the $U(d)$-invariant subcomplex of the right-hand side.

To compute the translation invariant deformation complex we will invoke Proposition BW: hol trans invt def from Section BW: ref. Note that the deformation complex is simply the (reduced) local cochains on the local Lie algebra $\Omega^{0,*}_{\mathbb{C}^d} \otimes \mathfrak{g}$. Thus, in the notation of Section BW: same ref the bundle L is simply the trivial bundle $\mathfrak{g}$. Thus, we see that the translation invariant deformation complex is quasi-isomorphic to the following cochain complex

$$\left(\mathrm{Def}^{\mathrm{cot}}_{Y \to B\mathfrak{g}}\right)^{\mathbb{C}^{2d|d}} \simeq \mathbb{C} \cdot \mathrm{d}^d z \otimes^{\mathbb{L}}_{\mathbb{C}\left[\frac{\partial}{\partial z_i}\right]} \mathrm{C}^*_{\mathrm{Lie,red}}(\mathfrak{g}[[z_1, \ldots, z_d]])[d].$$

We'd like to recast the right-hand side in a more geometric way.

Note that the the algebra $\mathbb{C}\left[\frac{\partial}{\partial z_i}\right]$ is the enveloping algebra of the abelian Lie algebra $\mathbb{C}^d = \mathbb{C}\left\{\frac{\partial}{\partial z_i}\right\}$. Thus, the complex we are computing is of the form

$$\mathbb{C} \cdot \mathrm{d}^d z \otimes^{\mathbb{L}}_{U(\mathbb{C}^d)} \mathrm{C}^*_{\mathrm{Lie,red}}(\mathfrak{g}[[z_1, \ldots, z_d]])[d].$$

Since $\mathbb{C} \cdot \mathrm{d}^d z$ is the trivial module, this is precisely the Chevalley-Eilenberg cochain complex computing Lie algebra homology of $\mathbb{C}^d$ with values in the module $\mathrm{C}^*_{\mathrm{Lie,red}}(\mathfrak{g}[[z_1, \ldots, z_d]])$:

$$\left(\mathrm{Def}^{\mathrm{cot}}_{Y \to B\mathfrak{g}}\right)^{\mathbb{C}^d} \simeq \mathrm{C}^{\mathrm{Lie}}_*\left(\mathbb{C}^d ; \mathrm{C}^*_{\mathrm{Lie,red}}(\mathfrak{g}[[z_1, \ldots, z_d]])\mathrm{d}^d z\right)[d].$$

We will keep $\mathrm{d}^d z$ in the notation since below we are interested in computing the $U(d)$-invariants.

To compute the cohomology of this complex, we will first describe the differential explicitly. There are two components to the differential. The first is the "internal" differential coming from the Lie algebra cohomology of $\mathfrak{g}[[z_1, \ldots, z_d]]$, we will write this as $\mathrm{d}_{\mathfrak{g}}$. The second comes from the $\mathbb{C}^d$-module structure on $\mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g}[[z_1, \ldots, z_n]])$ and is the differential computing the Lie algebra homology, which we denote $\mathrm{d}_{\mathbb{C}^d}$. We will employ a spectral sequence whose first term turns on the $\mathrm{d}_{\mathfrak{g}}$ differential. The next term turns on the differential $\mathrm{d}_{\mathbb{C}^d}$.

As a graded vector space, the cochain complex we are trying to compute has the form

$$\mathrm{Sym}(\mathbb{C}^d[1]) \otimes \mathrm{C}^*_{\mathrm{Lie,red}}\left(\mathfrak{g}[[z_1, \ldots, z_d]]\right) \mathrm{d}^d z[d].$$

The spectral sequence is induced by the increasing filtration of $\mathrm{Sym}(\mathbb{C}^d[1])$ by symmetric powers

$$F^k = \mathrm{Sym}^{\leq k}(\mathbb{C}^d[1]) \otimes \mathrm{C}^*_{\mathrm{Lie,red}}\left(\mathfrak{g}[[z_1, \ldots, z_d]]\right) \mathrm{d}^d z[d].$$

Remark 2.4.7. In the examples we are most interested in (namely $\mathfrak{g} = \mathbb{C}^n[-1]$ and $\mathfrak{g} = \mathfrak{g}_{X_{\bar\partial}}$) we can understand the spectral sequence we are using as a version of the Hodge-to-de Rham spectral sequence.

As above, we write the generators of $\mathbb{C}^d$ by $\frac{\partial}{\partial z_i}$. Also, note that the reduced Chevalley-Eilenberg complex has the form

$$\mathrm{C}^*_{\mathrm{Lie,red}}(\mathfrak{g}[[z_1, \ldots, z_n]]) = \left(\mathrm{Sym}^{\geq 1}\left(\mathfrak{g}^\vee[z_1^\vee, \ldots, z_d^\vee][-1]\right), \mathrm{d}_{\mathfrak{g}}\right),$$

where $z_i^\vee$ is the dual variable to z_i.

Recall, we are only interested in the $U(d)$-invariant subcomplex of this deformation complex. Sitting inside of $U(d)$ we have $S^1 \subset U(d)$ as multiples of the identity. This induces an overall weight grading to the complex. The group $U(d)$ acts in the standard way on $\mathbb{C}^d$. Thus, z_i has weight $(+1)$ and both $z_i^\vee$ and $\frac{\partial}{\partial z_i}$ have S^1-weight (-1). Moreover, the volume element $\mathrm{d}^d z$ has S^1 weight d. It follows that in order to have total S^1-weight that the total number of $\frac{\partial}{\partial z_i}$ and $z_i^\vee$ must add up to d. Thus, as a graded vector space the invariant subcomplex has the following decomposition

$$\bigoplus_k \mathrm{Sym}^k(\mathbb{C}^d[1]) \otimes \left(\bigoplus_{i \leq d-k} \mathrm{Sym}^i \left(\mathfrak{g}^\vee[z_1^\vee, \ldots, z_d^\vee][-1]\right) \right) \mathrm{d}^d z[d].$$

It follows from Schur-Weyl that the space of $U(d)$ invariants of the dth tensor power of the fundamental representation $\mathbb{C}^d$ is one-dimensional, spanned by the top exterior power. Thus, when we pass to the $U(d)$-invariants, only the unique totally antisymmetric tensor involving $\frac{\partial}{\partial z_i}$ and $z_i^\vee$ survives. Thus, for each k, we have

(2.12)
$$\left(\mathrm{Sym}^k(\mathbb{C}^d[1]) \otimes \left(\bigoplus_{i \leq d-k} \mathrm{Sym}^i \left(\mathfrak{g}^\vee[z_1^\vee, \ldots, z_d^\vee][-1]\right) \right) \mathrm{d}^d z \right) \cong \wedge^k \left(\frac{\partial}{\partial z_i} \right) \wedge \wedge^{d-k} \left(z_i^\vee \right) \mathrm{C}^*_{\mathrm{Lie}} \left(\mathfrak{g}, \mathrm{Sym}^{d-k}(\mathfrak{g}^\vee) \right) \mathrm{d}^d$$

Here, $\wedge^k \left(\frac{\partial}{\partial z_i}\right) \wedge \wedge^{d-k} \left(z_i^\vee\right)$ is just a copy of the determinant $U(d)$-representation, but we'd like to keep track of the appearances of the partial derivatives and $z_i^\vee$. Note that for degree reasons, we must have $k \leq d$. When $k = 0$ this complex is the (shifted) space of functions modulo constants on the formal moduli space $B\mathfrak{g}$, $\mathscr{O}_{red}(B\mathfrak{g})[d]$. When $k \geq 1$ this the (shifted) space of k-forms on the formal moduli space $B\mathfrak{g}$, which we write as $\Omega^k(B\mathfrak{g})[d+k]$. Thus, we see that before turning on the differential on the next page, our complex looks like

$$\begin{array}{lcccc} (2.13) & \underline{-2d} & \cdots & \underline{-d-1} & \underline{-d} \\ \\ & \mathscr{O}_{red}(B\mathfrak{g}) & \cdots & \Omega^{d-1}(B\mathfrak{g}) & \Omega^{d}(B\mathfrak{g}). \end{array}$$

We've omitted the extra factors for simplicity.

We now turn on the differential $\mathrm{d}_{\mathbb{C}^d}$ coming from the Lie algebra homology of $\mathbb{C}^d = \mathbb{C}\left\{\frac{\partial}{\partial z_i}\right\}$ with values in the above module. Since this Lie algebra is abelian the differential is completely determined by how the operators $\frac{\partial}{\partial z_i}$ act. We can understand this action explicitly as follows. Note that $\frac{\partial}{\partial z_i} z_j = \delta_{ij}$, thus we may as well think of $z_i^\vee$ as the element $\frac{\partial}{\partial z_i}$. Consider the subspace corresponding to $k = d$ in Equation (2.12):

$$\frac{\partial}{\partial z_1} \cdots \frac{\partial}{\partial z_d} \mathrm{C}^*_{\mathrm{Lie,red}}(\mathfrak{g}) \mathrm{d}^d z.$$

Then, if $x \in \mathfrak{g}^\vee[-1] \subset \mathrm{C}^*_{\mathrm{Lie,red}}(\mathfrak{g})$ we observe that

$$\mathrm{d}_{\mathbb{C}^d}\left(\frac{\partial}{\partial z_1} \cdots \frac{\partial}{\partial z_d} \otimes f \otimes \mathrm{d}^d z\right) = \det(\partial_i, z_j^\vee) \otimes 1 \otimes x \otimes \mathrm{d}^d z \in \wedge^{d-1}\left(\frac{\partial}{\partial z_i}\right) \wedge \mathbb{C}\{z_i^\vee\} \mathrm{C}^*_{\mathrm{Lie}}\left(\mathfrak{g}, \mathfrak{g}^\vee\right) \mathrm{d}^d z.$$

This follows from the fact that the action of $\frac{\partial}{\partial z_i}$ on $x = x \otimes 1 \in \mathfrak{g}^\vee \otimes \mathbb{C}[z_i^\vee]$ is given by

$$\frac{\partial}{\partial z_i} \cdot (x \otimes 1) = 1 \otimes x \otimes z_i^\vee \in \mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g}, \mathfrak{g}^\vee) z_i^\vee.$$

By the Leibniz rule we can extend this to get the formula for general elements $f \in \mathrm{C}^*_{\mathrm{Lie,red}}(\mathfrak{g})$. We find that getting rid of all the factors of z_i we recover precisely the de

Rham differential

$$\begin{tikzcd} \mathrm{C}^*_{\mathrm{Lie,red}}(\mathfrak{g})[2d] \arrow[r, "\mathrm{d}_{\mathbb{C}^d}"] \arrow[d, equal] & \mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g}, \mathfrak{g}^\vee)[2d-1] \arrow[d, equal] \\ \mathscr{O}_{red}(B\mathfrak{g}) \arrow[r, "\partial"] & \Omega^1(B\mathfrak{g}). \end{tikzcd}$$

A similar argument shows that $\mathrm{d}_{\mathbb{C}^d}$ agrees with the de Rham differential on each $\Omega^k(B\mathfrak{g})$.

We conclude that the E_2 page of this spectral sequence is quasi-isomorphic to the following truncated de Rham complex.

$$\begin{array}{ccccccccc} \underline{-2d} & & \underline{-2d+1} & & \cdots & & \underline{-d-1} & & \underline{-d} \\ & & & & & & & & \\ \mathscr{O}_{red}(B\mathfrak{g}) & \xrightarrow{\partial} & \Omega^1(B\mathfrak{g}) & \longrightarrow & \cdots & \longrightarrow & \Omega^{d-1}(B\mathfrak{g}) & \xrightarrow{\partial} & \Omega^d(B\mathfrak{g}). \end{array} \tag{2.14}$$

This is precisely a shifted version of the complex we had in (2.10). We saw that it was quasi-isomorphic, through the de Rham differential, to $\Omega^{d+1}_{cl}[d]$. This completes the proof.

We can apply this general result to the case $\mathfrak{g} = \mathbb{C}^n[-1]$. Doing this we have the following corollary.

Corollary 2.4.8. *Let* Def_n *be the deformation complex of the formal* $\beta\gamma$ *system with target* $\widehat{D}^n$*. There is a* $(\mathrm{W}_n, \mathrm{GL}_n)$*-equivariant quasi-isomorphism*

$$J : \widehat{\Omega}^{d+1}_{n,cl}[d] \xrightarrow{\simeq} \left(\left(\mathrm{Def}_n^{\mathrm{cot}}\right)^{\mathbb{C}^{2d|d}}\right)^{U(d)} \subset \mathrm{Def}_n.$$

This induces a quasi-isomorphism into the $(\mathrm{W}_n, \mathrm{GL}_n)$-equivariant deformation complex

$$J^{\mathrm{W}} : \mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n, \mathrm{GL}_n; \widehat{\Omega}^{d+1}_{n,cl}) \xrightarrow{\simeq} \left(\left(\mathrm{Def}_n^{\mathrm{W,cot}}\right)^{\mathbb{C}^{2d|d}}\right)^{U(d)} \subset \mathrm{Def}_n^{\mathrm{W}}. \tag{2.15}$$

Moreover, upon performing Gelfand-Kazhdan descent, it implies that on any complex manifold X we can use J to identify the deformation complex for the holomorphic σ-model of maps $\mathbb{C}^d \to X$:

$$J^X : \Omega^{d+1}_{X,cl}[d] \xrightarrow{\simeq} \left(\left(\mathrm{Def}^{\mathrm{cot}}_{\mathbb{C}^d \to X}\right)^{\mathbb{C}^{2d|d}}\right)^{U(d)}.$$

2.5. BV quantization of the holomorphic σ-model

As we have already discussed, the formalism of BV quantization of any theory consists of two steps: I) renormalization, and II) solving the quantum master equation. For holomorphic theories, as the one we are studying in this section, we have proved a general result about the one-loop renormalization theory on flat space $\mathbb{C}^d$. We will leverage this result to turn the problem of quantization to studying solutions of the quantum master equation.

The formal $\beta\gamma$ system $\mathscr{E}_n$ is a free BV theory and hence admits a natural quantization. (See Chapter 6 of [**?**] for an extensive development.) To study the general holomorphic σ-model we want to quantize *equivariantly* with respect to the action of W_n. We will find that there is an obstruction to quantizing equivariantly, given by the Gelfand-Fuks Chern class $\mathrm{ch}^{\mathrm{GF}}_{d+1}(\widehat{\mathcal{T}}_n)$ defined in Section **??**. This obstruction is a very local avatar of the anomaly described by Witten and Nekrasov [**Wit07, Nek**] in the complex one-dimensional holomorphic σ-model. We will refer to Chapter **??** for notations and terminology of equivariant BV quantization.

The section splits up into two main parts, first we study the W_n-equivariant quantization of the formal $\beta\gamma$ system. Then we show how Gelfand-Kazhdan formal geometry intertwines with BV quantization to define the quantization general target complex manifold.

2.5.1. The W_n-equivariant quantization

In this section we construct the prequantization of the holomorphic σ-model.

2.5.1.1. A reminder of the propagator. We wrote down the general propagator for translation invariant holomorphic theories on $\mathbb{C}^d$ in Section **??**. In this section we recall the construction of the propagator for the theory we consider of holomorphic maps $\mathbb{C}^d \to B\mathfrak{g}$.

The propagator is of the form $P_{\epsilon<L} = P^{an}_{\epsilon<L}\mathrm{Cas}_{\mathfrak{g}}$ where $\mathrm{Cas}_{\mathfrak{g}} \in \mathfrak{g} \otimes \mathfrak{g}^* \oplus \mathfrak{g}^* \otimes \mathfrak{g}$ is the quadratic Casimir of the L_∞ algebra $\mathfrak{g}$. The analytic piece of the propagator is the one associated to the theory whose target is one-dimensional $\mathbb{C}$ that we denote by

$$\mathscr{E} = \Omega^{0,*}(\mathbb{C}^d) \oplus \Omega^{d,*}(\mathbb{C}^d)[d-1].$$

Choosing the standard flat metric on $\mathbb{C}^d$, we obtain a natural gauge fixing operator

$$Q^{GF} = \overline{\partial}^* : \Omega^{0,*}(\mathbb{C}^d) \to \Omega^{0,*-1}(\mathbb{C}^d)$$

which acts on $(d,*)$ forms in a similar way. The corresponding operator $[Q, Q^{GF}] = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ is simply the Hodge Laplacian $\Delta_{\overline{\partial}}$.

For $t > 0$, the heat kernel $K^{an}_t \in \mathscr{E}(\mathbb{C}^d)\widehat{\otimes}\mathscr{E}(\mathbb{C}^d)$ is characterized by the equation

$$\Delta_{\overline{\partial}} K^{an}_t + \frac{\partial}{\partial t} K^{an}_t = 0$$

and normalized so that

$$\langle \varphi(x), K_t(x,y)\rangle_x = (e^{-t\Delta_{\overline{\partial}}}\varphi)(y)$$

where $\varphi \in \mathscr{E}$ and $\langle -, -\rangle$ is the (-1)-symplectic pairing. Using the standard formula for the heat kernel for the flat Laplacian on $\mathbb{C}^d$ we have the expression for our heat kernel, including the correct differential form factors

$$K_t(z,w) = \frac{1}{(4\pi t)^d} e^{-|z-w|^2/4t}\left((\mathrm{d}^d z - \mathrm{d}^d w)\wedge \prod_{i=1}^d (\mathrm{d}\overline{z}_i - \mathrm{d}\overline{w}_i)\right)$$

The effective propagator is defined for $0 < \epsilon, L$ and given by

$$P_{\epsilon<L}(z,w) = \int_{t=\epsilon}^{L} \mathrm{d}t(\overline{\partial}^* \otimes 1)K_t(z,w).$$

We can compute this propagator directly

$$\begin{aligned} P_\epsilon(z,w) &= \frac{1}{(4\pi)^d}\int_{t=\epsilon}^{L} \mathrm{d}t e^{-|z-w|^2/4t}\frac{1}{(4\pi t)^d}\sum_{j=1}^{d}(-1)^{j-1}\frac{\overline{z}_j - \overline{w}_j}{4t}(\mathrm{d}^d z - \mathrm{d}^d w)\prod_{i\neq j}(\mathrm{d}\overline{z}_i - \mathrm{d}\mathrm{d}\overline{w}_i) \\ &= \frac{1}{(4\pi)^d}\frac{1}{|z-w|^{2d}}\sum_j (-1)^{j-1}(\overline{z}_j - \overline{w}_j)(\mathrm{d}^d z - \mathrm{d}^d w)\prod_{i\neq j}(\mathrm{d}\overline{z}_i - \mathrm{d}\overline{w}_i)\int_{u=|z-w|^2/L}^{|z-w|^2/\epsilon} \mathrm{d}u u^{d-1}e^{-u}. \end{aligned}$$

In the second line we have made the substitution $u = |z-w|^2/t$. BW: work out factors We see that the differential form part above is precisely the Bochner-Martinelli kernel $\omega_{BM} \in \Omega^*(\mathbb{C}^d \times \mathbb{C}^d \setminus \Delta)$

$$\omega_{BM}(z,w) = C_n \frac{1}{|z-w|^{2d}}\sum_j (-1)^{j-1}(\overline{z}_j - \overline{w}_j)(\mathrm{d}^d z - \mathrm{d}^d w)\prod_{i\neq j}(\mathrm{d}\overline{z}_i - \mathrm{d}\overline{w}_i).$$

A simple corollary of the above calculation is the following fact that we will use later on in Section **??**.

Lemma 2.5.1. *Suppose $z \neq w$. The $\epsilon \to 0, L \to \infty$ limit of the propagator $P_{\epsilon<L}(z,w)$ exists and*

$$\lim_{\epsilon \to 0} \lim_{L \to \infty} P_{\epsilon<L}(z,w) = \omega_{BM}(z,w).$$

2.5.1.2. The prequantization. Our first step is to construct an equivariant effective prequantization. (i.e., effective actions satisfying the locality and RG flow conditions but not necessarily the QME condition) for the W_n-equivariant formal $\beta\gamma$ system. We have already reviewed what a prequantization is in Section **??**, but we briefly recall the main elements here. Essentially, we try to run the RG flow from the classical theory by naively guessing

$$I^{\mathrm{W}}[L] = \lim_{\epsilon \to 0} W(P_{\epsilon<L}, I^{\mathrm{W}}) \tag{2.16}$$

and then adding counterterms to deal with singularities that prevent this limit from existing. (One of the main theorems of [**?**] guarantees that we can construct such a prequantization.)

In general, the limit Equation (2.16) may be ill-defined and counterterms would be necessary. The key in our situation is that the equivariant $\beta\gamma$ system is a holomorphic theory on $\mathbb{C}^d$ so that we can apply Lemma **??**. The existence of the holomorphic gauge fixing operator $\overline{\partial}^*$ was the crucial tool in proving this well-behaved analyticity. The form of the propagators for the $\beta\gamma$ system are special cases of those used in Section **??**, and we recall their exact form below.

As an immediate corollary of Lemma **??**, the following definition is well-defined.

Definition 2.5.2. For $L > 0$, let

$$I^{\mathrm{W}}[L] := \lim_{\epsilon \to 0} W(P_{\epsilon<L}, I^{\mathrm{W}}) = \lim_{\epsilon \to 0} \sum_{\Gamma} \frac{\hbar^{g(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} W_{\Gamma}(P_{\epsilon<L}, I^{\mathrm{W}}).$$

Here the sum is over all isomorphism classes of stabled connected graphs, but only graphs of genus ≤ 1 contribute nontrivially. By construction, the collection satisfies the RG flow equation and its tree-level $L \to 0$ limit is manifestly I^{W}. Hence $\{I^{\mathrm{W}}[L]\}_{L \in (0,\infty)}$ is a W_n*-equivariant prequantization* of the W_n-equivariant classical formal $\beta\gamma$ system.

Organizing the sums by genus of the graphs, we write the interaction as a sum $I^{\mathrm{W}}[L] = I^{\mathrm{W},0}[L] + \hbar I^{\mathrm{W},1}[L]$ where

$$\begin{aligned} I^{\mathrm{W},0}[L] &= \sum_{\Gamma \in \text{ Trees}} \frac{1}{|\mathrm{Aut}(\Gamma)|} W_{\Gamma}(P_{\epsilon<L}, I^{\mathrm{W}}), \\ I^{\mathrm{W},1}[L] &= \sum_{\Gamma \in \text{ 1-loop}} \frac{1}{|\mathrm{Aut}(\Gamma)|} W_{\Gamma}(P_{\epsilon<L}, I^{\mathrm{W}}). \end{aligned}$$

With these technicalities out of the way, we can now turn to studying the obstruction to satisfying the equivariant quantum master equation.

2.5.1.3. The one-loop anomaly. We now move on to calculating the one-loop anomaly of the equivariant theory.

Proposition 2.5.3. *There is an obstruction to a* W_n*-equivariant quantization of the formal* $\beta\gamma$ *system on* $\mathbb{C}^d$ *that preserves the symmetry by the group* $U(d) \ltimes \mathbb{C}^d$*. It is represented by a non-trivial cocycle of degree one*

$$\Theta_{d,n} \in \mathrm{Def}_n^{\mathrm{W}}$$

such that

$$\Theta_{d,n} = aJ^{\mathrm{W}}(\mathrm{ch}^{\mathrm{GF}}_{d+1}(\widehat{\mathcal{T}}_n))$$

for some non-zero number a, where J^{W} is the quasi-isomorphism of Equation (2.15) and $\mathrm{ch}^{\mathrm{GF}}_{d+1}(\widehat{\mathcal{T}}_n)$ *is the component of the Gelfand-Fuks-Chern character living in* $\mathrm{C}^{d+1}_{\mathrm{Lie}}(\mathrm{W}_n, \mathrm{GL}_n; \widehat{\Omega}^{d+1}_{n,cl})$.

By definition, the scale L *obstruction cocycle* $\Theta_{d,n}[L]$ is the failure for the interaction $I^{\mathrm{W}}[L]$ to satisfy the scale L equivariant quantum master equation. Explicitly, one has

$$\hbar\Theta_{d,n}[L] = (\mathrm{d}_{\mathrm{W}_n} + Q)I^{\mathrm{W}}[L] + \hbar\Delta_L I^{\mathrm{W}}[L] + \{I^{\mathrm{W}}[L], I^{\mathrm{W}}[L]\}_L,$$

where the right hand side is divisible by $\hbar$ since $I^{\mathrm{W},0}$ satisfies the classical master equation so that the $\hbar^0$ component vanishes. Moreover, the right hand side has no components weighted by $\hbar^2$ or higher powers, because the BV Laplacian Δ_L vanishes on $I^{\mathrm{W},1}[L]$ as it is only a function of γ and a vector field X. Thus, we have

$$\hbar\Theta_{d,n}[L] = (\mathrm{d}_{\mathrm{W}_n} + Q)I^{\mathrm{W},1}[L] + \hbar\Delta_L I^{\mathrm{W},0}[L] + 2\{I^{\mathrm{W},0}[L], I^{\mathrm{W},1}[L]\}_L,$$

and so $\Theta_{d,n}[L]$ only depends on γ and hence is a degree one element of $\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n; \mathscr{O}(\mathscr{E}_n))$.

The first lemma we state is a consequence of the general characterization of anomalies for holomorphic theories on flat space $\mathbb{C}^d$ proved in Section **??** in Chapter **??**. It reduces the calculation of the anomaly to a Feynman diagrams that are wheels with certain edges that are marked.

Lemma 2.5.4. *The limit* $\Theta_{d,n} := \lim_{L\to 0} \Theta_{d,n}[L]$ *exists and is an element of degree one in* $\mathrm{Def}_n^{\mathrm{W}}$*. Moreover, it is given by*

$$\lim_{\epsilon\to 0} \sum_{\substack{\Gamma\in(d+1)\text{-vertex wheels}\\ e\in\mathrm{Edge}(\Gamma)}} W_{\Gamma,e}(P_{\epsilon<1}, K_\epsilon, I^{\mathrm{W}}[\epsilon]),$$

where the sum is over wheels Γ *with* $(d+1)$ *vertices and a distinguished inner edge* e*.*

Remark 2.5.5. In the lemma above, the notation $W_{\Gamma,e}(P_{\epsilon<1}, K_\epsilon, I^{\mathrm{W}}[\epsilon])$ denotes a variation on the usual weight associated to a graph. As usual, we attach the interaction term $I^{\mathrm{W}}[\epsilon]$ to each vertex. To the distinguished internal edge labeled e, we attach the heat kernel K_ϵ, but we attach the propagator $P_{\epsilon<1}$ to every other internal edge.

Proof. By Corollary 16.0.5 of [**Cosa**] we see that the scale L obstruction is given by a sum over wheels. That is

$$\Theta[L] = \sum_{\substack{\Gamma\in\mathrm{Wheels}\\ e\in\mathrm{Edge}(\Gamma)}} W_{\Gamma,e}(P_{\epsilon<1}, K_\epsilon, I^{\mathrm{W}}[\epsilon]).$$

Thus, to prove the lemma we must show that only the $(d+1)$-vertex wheels contribute to the $\epsilon\to 0$ limit. Both of these First, we have It is here that we must recall the explicit form of the heat kernel and propagator:

$\square$

We now turn to the proof of Proposition 2.5.3. We must construct the obstruction cocycle $\Theta_{d,n}$ by the techniques of perturbative field theory. In the end, we want to recognize it as the local functional $J^{\mathrm{W}}(\mathrm{ch}_{d+1}^{\mathrm{GF}}(\widehat{\mathcal{T}}_n))$.

In the below calculation we write $\Theta = \Theta_{d,n}$ as the dimensions d, n will be fixed. The limit in Lemma **??** can be moved inside the summation, i.e., the weight for each 2-vertex wheel Γ with edge e has an $\epsilon \to 0$ limit. We denote this summand by

$$\Theta_{\Gamma,e} = \lim_{\epsilon \to 0} W_{\Gamma,e}(P^1_\epsilon, K_\epsilon, I^{\mathrm{W}}[\epsilon]).$$

By the nature of the graph, this functional is of the form

$$\Theta_{\Gamma,e} : \mathrm{W}_n^{\otimes(d+1)} \otimes \mathrm{Sym}(\Omega_c^{0,*} \otimes \mathfrak{g}_n[1]) \to \mathbb{C}.$$

Given formal vector fields $X_1, \ldots, X_d$, let $\Theta_{\Gamma,e}(X_1, \ldots, X_d)$ denote the associated local functional in $\mathscr{O}_{\mathrm{loc}}(\mathscr{E}_n)$.

Due to linear dependence on the vector fields, it suffices to assume that X_α are of the form $X_\alpha = a^i_\alpha \partial_i$, for $\alpha = 1, \ldots, d+1$, where the coefficient $a^i_\alpha \in \widehat{\mathcal{O}}_n$ is homogeneous of degrees k_α. In this case, up to permutations of vertices there is only one graph Γ whose functional $\Theta_{\Gamma,e}(X_1, \ldots, X_{d+1})$ is nonzero. Choose an ordering of the vertices $v_1, \ldots, v_{d+1}$. The vertex v_α has valency $k_\alpha + 1$, namely BW: picture For this graph, the functional $\Theta_{\Gamma,e}(X_1, \ldots, X_{d+1})$ is homogeneous of degree $(\sum_\alpha k_\alpha) - d - 1$:

$$\Theta_{\Gamma,e}(X_1, \ldots, X_{d+1}) : \mathrm{Sym}^{(\sum_\alpha k_\alpha) - d - 1}(\Omega_c^{0,*}(\mathbb{C}) \otimes \mathbb{C}^n) \to \mathbb{C}.$$

By describing this functional explicitly, we will complete the proof of Proposition 2.5.3, as it will agree on the nose with $J^{\mathrm{W}}(\mathrm{ch}^{\mathrm{GF}}_{d+1}(\widehat{\mathcal{T}}_n))$.

Lemma 2.5.6. *For $\alpha = 0, \ldots, d$, let $X_\alpha = a^i_\alpha \partial_i \in \mathrm{W}_n$ be homogeneous of degree k^α. Let Γ be the $(d+1)$-vertex wheel with ordered vertices of valencies $k^0 + 1, \ldots, k^d + 1$, and* *mark one internal edge as distinguished. Then, we have an identification* $\Theta_{\Gamma,e}(X_0,\ldots,X_d) = aJ^{\mathrm{W}}_{\mathrm{ch}^{\mathrm{GF}}_{d+1}(\widehat{\mathcal{T}}_n)}(X_1,\ldots,X_{d+1})$ *for some nonzero number* a.

Proof. Let us introduce the following notation. Recall, if $X = a^i\partial_i$ is a formal vector field, we have defined its Jacobian matrx $\mathrm{Jac}(X) = (\partial_j a^i) \in \mathrm{Mat}_n(\widehat{\mathcal{O}}_n)$. Also, given any formal power series $a \in \widehat{\mathcal{O}}_n$ we have seen how to extend to to a functional

$$a : \mathrm{Sym}(\Omega^{0,*}(\mathbb{C}^d) \otimes \mathbb{C}^n) \to \Omega^{0,*}(\mathbb{C}^d) \ \ , \ \ \gamma \mapsto a(\gamma).$$

Given a formal vector field X, we will use $\mathrm{Jac}(X)(\gamma)$ to denote the matrix of Dolbeualt forms by applying this to each entry in the Jacobian.

Ignoring the analytic factors momentarily, we observe that in computing the weight of the graph Γ, we contract β legs with γ legs. In our case, the X_α-vertex contributes a single β leg, which then contracts with the k^α different γ legs from the Y-vertex. This contributes a factor of the Jacobian $\mathrm{Jac}(X_\alpha)(\gamma)$ at each vertex. Since we are computing a wheel, the total contribution is the trace of the product of the Jacobians. Putting in the analytic factors we see that the weight of the diagram is of the form

$$\Theta_{\Gamma,e}(X_\alpha)(\gamma) = \lim_{\epsilon\to 0}\int_{(z^1,\ldots,z^{d+1})\in(\mathbb{C}^d)^{d+1}} \left(\prod_{\alpha=0}^{d} \mathrm{d}^d z^\alpha\right) \mathrm{Tr}\left(\mathrm{Jac}(X_0)(\gamma)(z_0)\cdots\mathrm{Jac}(X_{d+1})(\gamma)(z_{d+1})\right) \times$$
$$K^{an}_\epsilon(z_0,z_d)\prod_{\alpha=1}^{d} P^{an}_{\epsilon<L}(z^{\alpha-1},z^\alpha)$$

We now turn to actually computing this weight. The method is very similar to our estimate of the weight of a diagram in a general holomorphically translation invariant theory on $\mathbb{C}^d$ in Section **??**. First, we simplify the expression above with some notation.

Write

$$\Phi(z_1,\ldots,z_d)=\mathrm{Tr}\left(\mathrm{Jac}(X_1)(\gamma)(z_1)\cdots\mathrm{Jac}(X_{d+1})(\gamma)(z_{d+1})\right)\in\Omega^{0,*}(\mathbb{C}^d\times\cdots\times\mathbb{C}^d).$$

We perform the usual change of coordinates

$$\begin{aligned} w^\alpha &= z^\alpha - z^{\alpha-1} \quad ,\alpha=1,\ldots,d \\ w^0 &= z^0. \end{aligned}$$

Notice that the product of the heat kernel and the propagator is of the form

$$\begin{aligned} K^{an}_{\epsilon<L}\left(\sum_{\alpha=1}^d w^\alpha\right)\prod_{\alpha=1}^d P^{an}_{\epsilon<L}(w^\alpha) &= \pm\frac{1}{(4\pi\epsilon)^d}\int_{(t_1,\ldots,t_k)\in[\epsilon,L]^d}\prod_{\alpha=1}^d\frac{\mathrm{d}t_\alpha}{(4\pi t_\alpha)^d}\times \\ & \sum_{i_1,\ldots,i_d}\epsilon_{i_1,\ldots,i_d}\left(\prod_{\alpha=1}^d\frac{\overline{w}^\alpha_{i_\alpha}}{4t_\alpha}\right)\exp\left(-\sum_{\alpha=1}^d\frac{|w^\alpha|^2}{4t_\alpha}-\frac{1}{4\epsilon}\left|\sum_{\alpha=1}^d w^\alpha\right|^2\right)\prod_{\alpha,i=1}^d\mathrm{d}\overline{w}^\alpha_i. \end{aligned}$$

Here $\epsilon_{i_1,\ldots,i_d}$ is totally antisymmetric tensor and the above expression is proportional to the top anti-holomorphic form in the variables w^α. It follows that in the product $\Phi K^{an}_\epsilon P^{an}_{\epsilon<L}$ the only term in the expansion of Φ that contributes is

$$\sum_I\Phi(w^0,\ldots,w^d)_I\mathrm{d}\overline{w}^0_I$$

where the sum is over the multi-index $I=(i_1,\ldots,i_d)$ and $\Phi(w^0,\ldots,w^d)_I\in C^\infty(\mathbb{C}^d\times\cdots\times\mathbb{C}^d)$.

Thus, it suffices to compute, for a fixed compactly supported function $\Psi \in C^\infty(\mathbb{C}^d \times \cdots \times \mathbb{C}^d)$ the weight

$$\begin{aligned}\Theta(\epsilon) \quad := \quad & \int_{(w^0,\dots,w^d)\in(\mathbb{C}^d)^{d+1}} \left(\prod_{\alpha=0}^{d} \mathrm{d}^{2d} w^\alpha\right) \Psi(w^0,\dots,w^d) \frac{1}{(4\pi\epsilon)^d} \int_{(t_1,\dots,t_k)\in[\epsilon,L]^d} \prod_{\alpha=1}^{d} \frac{\mathrm{d}t_\alpha}{(4\pi t_\alpha)^d} \\ & \times \sum_{i_1,\dots,i_d} \epsilon_{i_1,\dots,i_d} \left(\prod_{\alpha=1}^{d} \frac{\overline{w}^\alpha_{i_\alpha}}{4t_\alpha}\right) \exp\left(-\sum_{\alpha=1}^{d} \frac{|w^\alpha|^2}{4t_\alpha} - \frac{1}{4\epsilon}\left|\sum_{\alpha=1}^{d} w^\alpha\right|^2\right).\end{aligned}$$

We will plug in the expressions $\Phi(w^0,\dots,w^d)_I$ at the end. We proceed in a similar way as in the calculation of weights for general holomorphic theories: first we will perform an integration by parts to put the integral in a Gaussian form, then we will compute this Gaussian integral over the variables $w^1,\dots w^d$. We will then be left with, in the $\epsilon \to 0$ limit, an expression for the local functional that we claimed is given by $J(\mathrm{ch}^{\mathrm{GF}}_{d+1}(\widehat{\mathcal{T}}_n))(X_\alpha)$.

Let

$$E(w,t) := \exp\left(-\sum_{\alpha=1}^{d} \frac{|w^\alpha|^2}{4t_\alpha} - \frac{1}{4\epsilon}\left|\sum_{\alpha=1}^{d} w^\alpha\right|^2\right),$$

which we can write as $\exp\left(-\frac{1}{4}M_{\alpha\beta}(w^\alpha, w^\beta)\right)$ where $(M_{\alpha\beta})$ is the symmetric $d\times d$ matrix with $M_{\alpha\alpha} = t_\alpha^{-1} + \epsilon^{-1}$ and $M_{\alpha\beta} = \epsilon^{-1}$ for $\alpha \neq \beta$. Here, (w^α, w^β) is the Hermitian inner product.

Introduce the holomorphic t-dependent differential operators

$$\begin{aligned} D_{\alpha,i_\alpha}(t) &= \frac{1}{t_\alpha} \sum_{\beta=1}^{d} M^{-1}_{\alpha\beta} \frac{\partial}{\partial w^\beta_{i_\alpha}} \\ &= \frac{\partial}{\partial w^\alpha_{i_\alpha}} - \sum_{\beta=1}^{d} \frac{t_\beta}{\epsilon + t_1 + \cdots + t_d} \frac{\partial}{\partial w^\beta_{i_\alpha}} \end{aligned}$$

Analogously to Lemma 1.3.5 one has

$$D_{\alpha,i_\alpha}(t)E(w,t) = \frac{\overline{w}^{\alpha}_{i_\alpha}}{t_\alpha}E(w,t).$$

Since each of the $D_{\alpha,i_\alpha}(t)$ commute we can iteratively perform an integration by parts to write the weight as

$$\begin{aligned}\Theta(\epsilon) \quad := \quad & \int_{(w^0,\dots,w^d)\in(\mathbb{C}^d)^{d+1}} \left(\prod_{\alpha=0}^{d} \mathrm{d}^{2d}w^\alpha\right) \frac{1}{(4\pi\epsilon)^d} \int_{(t_1,\dots,t_k)\in[\epsilon,L]^d} \prod_{\alpha=1}^{d} \frac{\mathrm{d}t_\alpha}{(4\pi t_\alpha)^d} \\ & \times \sum_{i_1,\dots,i_d} \epsilon_{i_1,\dots,i_d} \left(\prod_{\alpha=1}^{d} D_{\alpha,i_\alpha}(t)\Psi(w^0,\dots,w^d)\right) E(w,t).\end{aligned}$$

We now perform the Wick integration over the variables $w^1,\dots,w^d$. The leading term is of the form

(2.17)

$$\int_{w^0\in\mathbb{C}^d} \mathrm{d}^{2d}w^0 \frac{1}{(4\pi\epsilon)^d} \int_{(t_1,\dots,t_k)\in[\epsilon,L]^d} \prod_{\alpha=1}^{d} \frac{\mathrm{d}t_\alpha}{t_\alpha^d} \sum_{i_1,\dots,i_d} \epsilon_{i_1,\dots,i_d} \left(\prod_{\alpha=1}^{d} \frac{\partial}{\partial w^\alpha_{i_\alpha}}\Psi\right)\Bigg|_{w^1=\dots=w^d=0} \frac{1}{t_1\cdots t_d}\det(M)^{-1}\det(M)$$

We have used the expression for the determinant as a sum over indices $i_1,\dots,i_d$: $\det(A) = \sum_{i_1,\dots,i_d} \epsilon_{i_1,\dots,i_d} A_{1i_1}\cdots A_{di_d}$ hence:

$$\sum_{i_1,\dots,i_d} \epsilon_{i_1,\dots,i_d} \left(\prod_{\alpha=1}^{d} \frac{1}{t_\alpha}\sum_{\beta=1}^{d} M^{-1}_{\alpha\beta}\frac{\partial}{\partial w^\beta_{i_\alpha}}\Psi\right)\Bigg|_{w^1=\dots=w^d=0} = \sum_{i_1,\dots,i_d} \epsilon_{i_1,\dots,i_d} \left(\prod_{\alpha=1}^{d} \frac{\partial}{\partial w^\alpha_{i_\alpha}}\Psi\right)\Bigg|_{w^1=\dots=w^d=0} \frac{1}{t_1\cdots t_d}\det(M$$

The term $\det(M)^{-d}$ comes from performing the d-dimensional Gaussian integral. A calculation we performed in Section **??** shows that

$$\det(M_{\alpha\beta}) = \frac{\epsilon + t_1 + \dots + t_d}{\epsilon t_1\cdots t_d}$$

Hence, we can write the first term in the Wick expansion (2.17) as

$$\int_{w^0\in\mathbb{C}^d} \mathrm{d}^{2d}w^0 \sum_{i_1,\ldots,i_d} \epsilon_{i_1,\ldots,i_d} \left(\prod_{\alpha=1}^{d} \frac{\partial}{\partial w^{\alpha}_{i_\alpha}} \Psi \right) \Bigg|_{w^1=\cdots=w^d=0} \frac{1}{(4\pi)^d} \int_{(t_1,\ldots,t_k)\in[\epsilon,L]^d} \frac{\epsilon}{(\epsilon+t_1+\cdots+t_d)^{d+1}} \mathrm{d}t_1\cdots\mathrm{d}t_d.$$

The t-integral is easily seen to be convergent as $\epsilon\to 0$. Finally, plugging back in $\Psi = \sum_I \Phi_I$ we see that the obstruction can be written as

$$\Theta_{\Gamma,e}(X_\alpha) = \lim_{\epsilon\to 0}\Theta(\epsilon) = C\int_{\mathbb{C}^d} \mathrm{Tr}(\mathrm{Jac}(X_0)(\gamma)\partial\mathrm{Jac}(X_1)(\gamma)\cdots\partial\mathrm{Jac}(X_d)(\gamma)). \tag{2.18}$$

where C is some nonzero constant.

We have expressed the components $\mathrm{ch}^{\mathrm{GF}}_{d+1}(\widehat{\mathcal{T}}_n)\in \mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n;\widehat{\Omega}^{d+1}_{n,cl})$ of the Gelfand-Fuks-Chern character in Section **??**. Since these classes are valued in closed $(d+1)$-forms, we can express them as images under the de Rham operator of $\widehat{\Omega}^d_n$ valued classes. Indeed, we did this in Equation (2.4) where we found the class

$$\alpha_d : (X_0,\ldots,X_d)\mapsto \frac{1}{(-2\pi i)^{d+1}(d+1)!}\mathrm{Tr}\left(\mathrm{Jac}(X_0)\wedge\partial(\mathrm{Jac}(X_1))\wedge\cdots\wedge\partial(\mathrm{Jac}(X_d))\right)\in\widehat{\Omega}^d_n$$

satisfies $\partial\alpha_d = \mathrm{ch}^{\mathrm{GF}}_{d+1}(\widehat{\mathcal{T}}_n)$. Finally we note that (2.18) is a nonzero multiple of the local functional $J^{\mathrm{W}}_{\alpha_d}(X_0,\ldots,X_d)\in\mathscr{O}_{\mathrm{loc}}(\mathscr{E}_n)$, so we are done.

□

Remark 2.5.7. Note that when restricted to *linear* vector fields $\mathfrak{gl}_n\hookrightarrow \mathrm{W}_n$, the entire obstruction Θ vanishes. This vanishing means that there is no obstruction to quantizing equivariantly for the Lie algebra $\mathfrak{gl}_n$. This result is just the Lie algebra-level version of an earlier observation: the action of the group GL_n lifts $\hbar$-linearly to an action on the quantization.

2.5.1.4. The extended theory. We have just seen that there is a one-loop anomaly to quantizing the formal $\beta\gamma$ system in a way that is W_n-equivariant. This says that Gelfand-Kazhdan formal geometry does not allow us to descend the theory to an arbitrary complex manifold. In this section we use the calculation of the anomaly cocycle in the last section to build a theory that is equivariant for a bigger Lie algebra, which will allow us to do an extended version of descent as we discussed in Section **??**.

The Gelfand-Fuks-Chern character determines an extension of L_∞ algebras

$$0 \to \widehat{\Omega}^{d+1}_{n,cl}[d-1] \to \widetilde{\mathrm{W}}_{n,d} \xrightarrow{\pi_{n,d}} \mathrm{W}_n \to 0.$$

We have already seen that there is a map of cochain complexes

$$J : \widehat{\Omega}^{d+1}_{n,cl}[d] \to \mathrm{Def}_n = \mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n) \otimes \mathrm{Def}_n.$$

We will view J as an element $\widetilde{J} \in \mathrm{C}^1_{\mathrm{Lie}}(\widehat{\Omega}^{d+1}_{n,cl} \otimes \mathrm{Def}_n \subset \mathrm{C}^*_{\mathrm{Lie}}(\widetilde{\mathrm{W}}_{n,d}) \otimes \mathrm{Def}_n$.

Our main result of this section is the following.

Theorem 2.5.8. *The effective family $\{I^{\mathrm{W}}[L] + \hbar\widetilde{J}[L]\}_{L>0}$ defines a $\widetilde{\mathrm{W}}_{n,d}$-equivariant quantization of the n-dimensional formal $\beta\gamma$ system on $\mathbb{C}^d$ such that:*

1. *in addition, it is equivariant for the group GL_n in a way that is compatible with the Lie algebra map $\mathrm{Lie}(\mathrm{GL}_n) = \mathfrak{gl}_n \hookrightarrow \widetilde{\mathrm{W}}_{n,d}$;*
2. *this quantization is both holomorphically translation invariant and invariant for the group $U(d)$.*

Item (1) implies that the quantization of the formal $\beta\gamma$ system on $\mathbb{C}^d$ with target $\widehat{D}^n$ is equivariant for the pair $(\widetilde{\mathrm{W}}_{n,d}, \mathrm{GL}_n)$. We will use this, combined with the construction of extended descent, to produce the global holomorphic σ-model. Item (2) implies that the only moduli of the theory on a general target manifold is in the choice of an extended Gelfand-Kazhdan structure. We'll expound upon this in more detail in the next section.

First, we see that I^{W} defines a classical $\widetilde{\mathrm{W}}_{n,d}$-equivariant theory. The extended deformation complex is defined by

$$\widetilde{\mathrm{Def}}^{\mathrm{W}} = \mathrm{C}^*_{\mathrm{Lie}}(\widetilde{\mathrm{W}}_{n,d}) \otimes \mathrm{Def}_n.$$

The map $\pi_{n,d} : \widetilde{\mathrm{W}}_{n,d} \to \mathrm{W}_n$ defines a map of dg Lie algebras

$$\pi^*_{n,d} : \mathrm{Def}^{\mathrm{W}}_n[-1] \to \widetilde{\mathrm{Def}}^{\mathrm{W}}_n[-1].$$

Hence the Maurer-Cartan element $I^{\mathrm{W}} \in \mathrm{Def}^{\mathrm{W}}_n[-1]$ defining the W_n-theory defines a $\widetilde{\mathrm{W}}_{n,d}$-theory via $\pi^*_{n,d} I^{\mathrm{W}}$.

We can run homotopy RG flow to $\pi^*_{n,d} I^{\mathrm{W}}$ to obtain a prequantization just as in the non-extended case. Since everything is natural under maps of the dg Lie algebra defining the classical theory, we obtain the following relationship between the anomaly for the extended theory and the non-extended theory.

Lemma 2.5.9. *The effective family* $\{\pi^*_{n,d} I^{\mathrm{W}}[L] \mod \hbar^2\}$ *determines a one-loop prequantization of the* $\widetilde{\mathrm{W}}_{n,d}$*-equivariant classical theory. The obstruction to satisfying the scale* L $\widetilde{\mathrm{W}}_{n,d}$*-equivariant classical master equation is*

$$\widetilde{\Theta}_{n,d}[L] = \pi^*_{n,d} \Theta_{n,d}[L].$$

In particular $\lim_{L\to 0} \widetilde{\Theta}_{n,d}[L] = \widetilde{\Theta} \in \widetilde{\mathrm{Def}}^{\mathrm{W}}_n$ *exists and is equal to* $\pi^*_{n,d}\Theta_{n,d}$.

The key difference in the extended case is that this anomaly is *cohomologically* trivial. The idea is based on the following elementary fact about Lie algebras. Let $\mathfrak{h}$ be a Lie algebra and V a module for $\mathfrak{h}$. Moreover, suppose $\alpha \in \mathrm{C}^{2+k}_{\mathrm{Lie}}(\mathfrak{h}; V)$ is a cocycle. Then, we can form the L_∞ extension

$$0 \to V[k] \to \widetilde{\mathfrak{h}} \xrightarrow{\pi} \mathfrak{h} \to 0.$$

The brackets between in $\widetilde{\mathfrak{h}}$ are defined by $\ell_2(x, y) := [x, y]_{\mathfrak{h}}$ and $\ell_{2-k}(x, \ldots) = \alpha(x, \ldots)$ where $[-, -]_{\mathfrak{h}}$ is the bracket in the original Lie algebra. The bracket between $x \in \mathfrak{h}$ and $v \in V$ is $[x, v]_{\widetilde{\mathfrak{h}}} = x \cdot v$. We can pull back the cocycle $\pi^*\alpha \in \mathrm{C}^*_{\mathrm{Lie}}(\widetilde{\mathfrak{h}}; V)$. In this situation, this pullback cocycle is automatically trivial. An explicit trivializing element is $\mathrm{id}_V : V \to V$ viewed as an element of the Chevalley-Eilenberg complex $\mathrm{C}^*_{\mathrm{Lie}}(\widetilde{\mathfrak{h}}; V)$.

We have already mentioned that $J : \widehat{\Omega}^{d+1}_{n,cl}[d] \to \mathrm{Def}_n$ can be viewed as an element $\widetilde{J}$ in $\widetilde{\mathrm{Def}}_n^{\mathrm{W}}$. The following lemma follows the same logic as the above paragraph.

Lemma 2.5.10. *The local functional* $\widetilde{J} \in \widetilde{\mathrm{Def}}_n^{\mathrm{W}}$ *trivializes* $\pi^*_{n,d}\Theta_{n,d}$ *in the extended deformation complex:*

$$\left(\overline{\partial} + \mathrm{d}_{\widetilde{\mathrm{W}}_{n,d}}\right) \widetilde{J} + \{\pi^*_{n,d} I^{\mathrm{W}}, \widetilde{J}\} = \pi^*_{n,d}\Theta_{n,d}.$$

Proof. The functional J is the image of $\mathrm{id}_{\Omega^{d+1}}$ under the map

$$\mathrm{C}^*_{\mathrm{Lie}}(\widetilde{\mathrm{W}}_{n,1}; \widehat{\Omega}^{d+1}_{n,cl}[d]) = \mathrm{C}^*_{\mathrm{Lie}}(\widetilde{\mathrm{W}}_{n,1}) \otimes_{\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n)} \mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n; \widehat{\Omega}^{d+1}_{n,cl}) \xrightarrow{\mathrm{id} \otimes J} \mathrm{C}^*_{\mathrm{Lie}}(\widetilde{\mathrm{W}}_{n,1}) \otimes_{\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n)} \mathrm{Def}_n^{\mathrm{W}} = \widetilde{\mathrm{Def}}_n^{\mathrm{W}}.$$

Denote this composition by $J^{\widetilde{W}}$, so that $\widetilde{J} = J^{\widetilde{W}}(\mathrm{id}_{\Omega^{d+1}})$. The composition above is a map of cochains, so for any $\varphi \in \mathrm{C}^*_{\mathrm{Lie}}(\widetilde{\mathrm{W}}_{n,1}; \widehat{\Omega}^{d+1}_{n,cl}[d])$ we have

$$J^{\widetilde{\mathrm{W}}}(\mathrm{d}_{\widetilde{\mathrm{W}}_{n,d}}\varphi) = \overline{\partial} J^{\widetilde{\mathrm{W}}}(\varphi) + \{\pi^*_{n,d} I^{\mathrm{W}}, J^{\widetilde{\mathrm{W}}}(\varphi)\}$$

In particular, for $\varphi = \mathrm{id}_{\Omega^2}$ we have

$$J^{\widetilde{\mathrm{W}}}(\pi^*_{n,d}\mathrm{ch}^{\mathrm{GF}}_{d+1}(\widehat{\mathcal{T}}_n)) = \overline{\partial}\widetilde{J} + \{\widetilde{I}^{\mathrm{W}}, \widetilde{J}\}.$$

We have already seen that the image of $\pi^*\mathrm{ch}^{\mathrm{GF}}_{d+1}(\widehat{\mathcal{T}}_n)$ under $J^{\widetilde{\mathrm{W}}}$ is the obstruction cocycle $\pi^*_{n,d}\Theta$, and this is what we wanted to show. □

The fact that this trivialization at the level of the local deformation complex allows us to define a one-loop quantization follows from the following general result. To state it, suppose that $\mathscr{E}$ is a general theory with classical interaction $I \in \mathscr{O}_{\mathrm{loc}}(\mathscr{E})$.

Lemma 2.5.11 (Lemma 3.33 of [**LL16**])**.** *Suppose I^{qc} and $O_1 \in \mathscr{O}_{\mathrm{loc}}(\mathscr{E})$ satisfy*

$$QI^{qc} + \{I, I^{qc}\} = O_1.$$

Then, for each L, the functional

$$I^{qc}[L] = \lim_{\epsilon \to 0} \sum_{\substack{\Gamma \in \mathrm{Trees} \\ v \in V(\Gamma)}} W_{\Gamma,v}(P_{\epsilon<L}, I, I^{qc})$$

satisfies

$$QI^{qc}[L] + \{I^{(0)}[L], I^{qc}[L]\}_L = O_1[L]. \tag{2.19}$$

Proof. For the non-equivariant case, see the referenced Lemma in [**LL16**]. The equivariant case is an immediate consequence. $\square$

In the lemma I^{qc} stands for "quantum correction", since deforming the action functional by it allows us to produce a solution to the QME. We can now finish the proof of Theorem 2.5.8. For simplicity, we will drop $\pi^*_{n,d}$ from the notation and just view $\pi^*_{n,d} I^{\mathrm{W}} = I^{\mathrm{W}}$ as a $\widetilde{\mathrm{W}}_{n,1}$-equivariant functional. We consider the effective family

$$\{I^{\mathrm{W}}[L] + \hbar \widetilde{J}[L]\}.$$

As a consequence of the Lemmas 2.5.10 and 2.5.11 the scale L, $\widetilde{\mathrm{W}}_{n,d}$-equivariant quantum master equation for the functional $I[L] + \hbar\widetilde{J}[L]$ is satisfied:

$$(\mathrm{d}_{\widetilde{\mathrm{W}}} + \overline{\partial})(I^{\mathrm{W}}[L] + \hbar\widetilde{J}[L]) + \frac{1}{2}\{I^{\mathrm{W}}[L] + \hbar\widetilde{J}[L], I^{\mathrm{W}}[L] + \hbar\widetilde{J}[L]\}_L + \hbar\Delta_L(I^{\mathrm{W}}[L] + \hbar\widetilde{J}[L]) = 0.$$

The functional J is GL_n-invariant. Moreover, the original non-extended prequantization $I^{\mathrm{W}}[L]$ is GL_n-equivariant, this quantization is as well. Now, the map $J : \widehat{\Omega}^{d+1}_{n,cl}[d] \to \mathrm{Def}_n$ is $U(d) \ltimes \mathbb{C}^d$-invariant. Thus, the effective family above is as well. The moduli of cotangent quantizations that are holomorphically translation invariant, and invariant for the group $U(d)$, is controlled by the extended deformation complex

$$\left(\left(\widetilde{\mathrm{Def}}_n^{\mathrm{W,cot}}\right)^{\mathbb{C}^{2d|d}}\right)^{U(d)} \tag{2.20}$$

We have already seen that the non-extended version of this complex is quasi-isomorphic to $\widehat{\Omega}^{d+1}_{n,cl}[d]$ in Corollary 2.4.8. Since this quasi-isomorphism is $\widetilde{\mathrm{W}}_{n,d}$-equivariant we see that (2.20) is quasi-isomorphic to

$$\mathrm{C}^*_{\mathrm{Lie}}(\widetilde{\mathrm{W}}_{n,d}, \mathrm{GL}_n; \widehat{\Omega}^{d+1}_{n,cl}[d]).$$

In cohomology, deformations live in H^0 of this complex which is $H^d(\widetilde{\mathrm{W}}_{n,d}, \mathrm{GL}_n; \widehat{\Omega}^{d+1}_{n,cl})$.

Lemma 2.5.12. *The cohomology $H^d(\widetilde{\mathrm{W}}_{n,d}, \mathrm{GL}_n; \widehat{\Omega}^{d+1}_{n,cl})$ is trivial.*

Proof. There is a spectral sequence computing $H^*(\widetilde{\mathrm{W}}_{n,d}, \mathrm{GL}_n; \widehat{\Omega}^{d+1}_{n,cl})$ from filtering cochains on $\widetilde{\mathrm{W}}_{n,d}$ by the number of $\widehat{\Omega}^{d+1}_{n,cl}$-inputs. The E_1-page of the spectral sequence is

$$H^*_{\mathrm{Lie}}(\mathrm{W}_n \ltimes \widehat{\Omega}^{d+1}_{n,cl}[d-1], \mathrm{GL}_n; \widehat{\Omega}^{d+1}_{n,cl}),$$

where the semi-direct product uses the natural W_n-module structure on forms.

□

This completes the proof of Theorem 2.5.8.

2.5.2. Quantization on general manifolds via formal geometry

We now show how our results in the last section allow us to construct the quantization of the holomorphic σ-model on general target complex manifolds satisfying the condition $\mathrm{ch}_{d+1}(T^{1,0}_X) = 0$.

We have already seen how formal geometry allows us to descend the classical $(\mathrm{W}_n, \mathrm{GL}_n)$-equivariant formal $\beta\gamma$ system $\mathscr{E}_n$ to the holomorphic σ-model with arbitrary complex target X. The global holomorphic σ-model of maps $\mathbb{C}^d \to X$ infinitesimally close to the constant maps is described by the (curved) elliptic L_∞ algebra $\mathscr{E}_{\mathbb{C}^d \to X}[-1]$, which is defined over the de Rham complex Ω^*_X. In terms of the formal $\beta\gamma$ system we saw that

$$\mathscr{E}_{\mathbb{C}^d\to X}[-1] = \mathbf{desc}_X(\mathscr{E}_n[-1]) \tag{2.21}$$

as elliptic L_∞ algebras defined over Ω^*_X. Likewise, there is a relationship between the deformation complexes $\mathrm{Def}_{\mathbb{C}^d\to X} = \mathrm{desc}_X(\mathrm{Def}_n)$. The characteristic map is of the form

$$\mathrm{char}_X : \mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n, \mathrm{GL}_n; \mathrm{Def}_n) \to \mathrm{Def}_{\mathbb{C}^d\to X}$$

Note that $\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n, \mathrm{GL}_n; \mathrm{Def}_n) \subset \mathrm{Def}_n^{\mathrm{W}}$ and I^{W} lies in this subcomplex. Under the characteristic map, we obtain the functional $I_X = \mathrm{char}_X(I^{\mathrm{W}}) \in \mathrm{Def}_{\mathbb{C}^d\to X}$ that solves the Ω^*_X-linear classical master equation. This is equivalent to the data involved in the identification (2.21).

When we quantize, we found that there is an obstruction to having a $(\mathrm{W}_n, \mathrm{GL}_n)$-equivariance, but we have an action by the bigger L_∞ pair $(\widetilde{\mathrm{W}}_{n,d}, \mathrm{GL}_n)$. Equivalently, the naive RG flow of I^{W} does not satisfy the quantum master equation, but we can found a modification of it that does.

Every complex manifold X admits a bundle of coordinates, a Gelfand-Kazhdan structure, which allows us to construct global objects using the data of a $(\mathrm{W}_n, \mathrm{GL}_n)$-module. We saw in Section **??** that not every complex manifold admits bundle of coordinates with a $(\widetilde{\mathrm{W}}_{n,d}, \mathrm{GL}_n)$-action. However, for every trivialization of the characteristic class $\mathrm{ch}_{d+1}(T_X^{1,0})$ we found that there did exist an extended Gelfand-Kazhdan structure, which one can think of as a reduction of the original bundle of coordinates to the pair $(\widetilde{\mathrm{W}}_{n,d}, \mathrm{GL}_n)$.

Let us fix an extended Gelfand-Kazhdan structure as in Section **??**. This was given by an ordinary GK structure, so a manifold X and a formal exponential σ, together with a trivialization α of $\mathrm{ch}_{d+1}(T_X^{1,0})$. As usual, we will omit the data of a formal exponential in the below. We denote the de Rham complex of the corresponding descent functor by

$$\widetilde{\mathbf{desc}}_{X,\alpha} : \mathrm{Mod}_{(\widetilde{\mathrm{W}}_{n,d}, \mathrm{GL}_n)} \to \mathrm{Mod}_{\Omega^*_X}.$$

Just as in ordinary descent, there is a characteristic map of the form

$$\widetilde{\mathrm{char}}_{X,\alpha} : \mathrm{C}^*_{\mathrm{Lie}}(\widetilde{\mathrm{W}}_{n,d}, \mathrm{GL}_n; \mathscr{O}(\mathscr{E}_n)[[\hbar]]) \to \mathscr{O}(\mathscr{E}_{\mathbb{C}^d \to X})[[\hbar]].$$

The extended family $\{I^{\mathrm{W}}[L] + \hbar\widetilde{J}[L]\}$ determines a family $\{I_{X,\alpha}[L]\}$ where, for each L,

$$I_{X,\alpha}[L] = \widetilde{\mathrm{char}}_{X,\alpha}\left(I^{\mathrm{W}}[L] + \hbar\widetilde{J}[L]\right).$$

An immediate corollary of our main result in the previous section, Theorem 2.5.8, is that this family solves the Ω^*_X-linear quantum master equation. Hence, it determines a quantization of the holomorphic σ-model of maps $\mathbb{C}^d \to X$.

Theorem 2.5.13. *Let α be a trivialization of $\mathrm{ch}_{d+1}(T_X^{1,0})$. Then, the family $\{I_{X,\alpha}[L]\}_{L>0}$ where*

$$I_{X,\alpha}[L] = \widetilde{\mathrm{char}}_{X,\alpha}\left(I^{\mathrm{W}}[L] + \hbar\widetilde{J}[L]\right) \in \mathscr{O}(\mathscr{E}_{\mathbb{C}^d \to X})[[\hbar]]$$

defines a holomorphically translation invariant, $U(d)$-invariant, cotangent quantization of the holomorphic σ-model of maps $\mathbb{C}^d \to X$.

Since Gelfand-Kazhdan descent is completely dependent on the target, it is compatible with all of the source symmetries we mentioned in the statement of the formal quantization in Theorem 2.5.8. Thus, the family $\{I_{X,\alpha}\}$ defines a $U(d) \ltimes \mathbb{C}^d$ equivariant cotangent quantization of $I_{\mathbb{C}^d \to X}$. The final part of the main theorem, Theorem 2.0.2, concerns identifying the moduli of quantizations respecting holomorphic translation invariance and the action of $U(d)$. We have shown that formally, the extended quantization is unique up to homotopy. Thus, the only moduli for the theory comes from the choice of an extended Gelfand-Kazhdan structure. We showed in Proposition 2.2.6 that the space of extended structures, when they exist, is a torsor for $H^d(X, \Omega^{d+1}_{cl})$. This completes the proof of the main theorem.

2.6. The local operators

In this section we analyze the local operators of the holomorphic σ-model. Our partial goal is exhibit the similarities present in the local operators of this higher dimensional holomorphic theory with the local operators of two-dimensional chiral conformal field theory.

It is the main result of [**CG**] that the observables of a quantum field theory have the structure of a factorization algebra. By definition, the observables supported on an open set U are equal to the completed symmetric algebra of functions on the fields supported on U. Throughout this section we will focus on the holomorphic σ-model of maps $\mathbb{C}^d \to V$ where V is a vector space. This theory is free, and its quantum observables admit a minimal description in terms of compactly supported functions (and Dolbeault forms) on $\mathbb{C}^d$, which we will recall momentarily.

In ordinary chiral conformal field theory, there is a collection of operators that, in some sense, generate all other operators. These are called "primary operators" (or primary fields), and are defined by those operators that are killed by the positive part of the

Virasoro algebra [**?**], that is, the "lowering operators". To obtain all of the operators one considers the descendants of the primary operators which are obtained by applying the negative part of the Virasoro algebra, or the "raising operators", to the primaries. For example, in the $d = 1$ $\beta\gamma$ system, there are two primary operators:

$$\mathcal{O}_{\gamma,0}(w) : \gamma \mapsto \gamma(w) = \int_{z \in C_w} \frac{\gamma(z)}{z-w} \mathrm{d}z$$
$$\mathcal{O}_{\beta,-1}(w) : \beta \mathrm{d}z \mapsto \beta(w) = \int_{z \in C_w} \frac{\beta(z)}{z-w} \mathrm{d}z,$$

where C_w is any closed contour surrounding w. (The indices $0, -1$ are to indicate the conformal weight.) Consider the operators placed at $w = 0$. We notice that each of these operators are annihilated by the positive half of the Virasoro $L_n = z^{n+1}\partial_z$, $n \geq 0$. The descendants are obtained by iteratively applying the raising operator $L_{-1} = \partial_z$, which in this case is just the infinitesimal translations. Indeed, for each $n \geq 0$ we obtain

$$\mathcal{O}_{\gamma,-n}(w) = \frac{1}{n!}\partial^n \mathcal{O}_{\gamma,0}(w) : \gamma \mapsto \partial_z^n \gamma(z = w)$$
$$\mathcal{O}_{\beta,-n-1}(w) = \frac{1}{n!}\partial^n \mathcal{O}_{\beta,1}(w) : \beta \mathrm{d}z \mapsto \partial_z^n \beta(z = w).$$

There is an S^1 action on $\mathbb{C}$ given by rotations, and this extends to an S^1 action on the $\beta\gamma$ system. In terms of the Virasoro algebra, the infinitesimal action of S^1 is given by the Euler vector field $L_0 = z\partial_z$. There is an induced grading on the factorization algebra of the one-dimensional free $\beta\gamma$ system by the eigenvalues of this S^1 action. Applied to the disk, or local, observables this is precisely the $\mathbb{Z}_{\geq 0}$ conformal weight grading of the chiral CFT. For instance, the operators $\mathcal{O}_{\gamma,-n}(w), \mathcal{O}_{\beta,-n}$ lie in the weight n subspace of the factorization algebra applied to $D(w,r)$ (for any $r>0$). We will see a similar grading in the higher dimensional holomorphic case.

2.6.1. The factorization algebra of observables

We work the the holomorphic σ-model of maps $\mathbb{C}^d \to V$ where V is a vector space. This is simply the $\beta\gamma$ system with values in V and the fields have the form

$$\mathscr{E}_V = \Omega^{0,*}(\mathbb{C}^d, V) \oplus \Omega^{d,*}(\mathbb{C}^d, V^*)[d-1].$$

We begin by defining the factorization algebra of classical observables.

2.6.1.1. The classical observables.

Definition 2.6.1. The *classical observables* supported on $U \subset \mathbb{C}^d$, $\overline{\mathrm{Obs}}_V^{\mathrm{cl}}(U)$, is the algebra of functions on the space of fields $\mathscr{E}_V(U)$ equipped with the differential given by extending $\overline{\partial}$ as a derivation.

Remark 2.6.2. We reserve the unbarred notation for the smeared classical observables to be introduced below.

Explicitly, the underlying graded algebra is

$$\mathrm{Sym}(\overline{\Omega}_c^{d,*}(U,V^*)[d] \oplus \overline{\Omega}_c^{0,*}(U,V)[1]).$$

The differential can be understood explicitly as follows. For some n-fold tensor product of linear functionals on the fields

$$a = \alpha_1 \otimes \cdots \otimes \alpha_n,$$

we have

$$\overline{\partial}(a) = (\overline{\partial}\alpha_1) \otimes \cdots \otimes \alpha_n \pm \alpha_1 \otimes (\overline{\partial}\alpha_1) \otimes \cdots \otimes \alpha_n + \cdots \pm \alpha_1 \otimes \cdots \otimes (\overline{\partial}\alpha_n).$$

This differential is equivariant with respect to the permutation action of the symmetric group S_n and hence induces a differential on the nth symmetric power.

It is manifest that these observables are natural with respect to holomorphic embeddings. That is, given a holomorphic embedding $i : U \hookrightarrow V$, there is a natural extension map

$$i_* : \mathrm{Obs}^{\mathrm{cl}}_n(U) \to \mathrm{Obs}^{\mathrm{cl}}_n(V)$$

that is naturally induced by the restriction map of fields

$$i^* : \mathscr{E}_V(U') \to \mathscr{E}_V(U).$$

Indeed, we have a factorization algebra on $\mathbb{C}^d$ by Theorem 5.2.1 of [**CG17**].

Definition 2.6.3. Let $\mathrm{Obs}^{\mathrm{cl}}_V$ denote the factorization algebra on $\mathbb{C}^d$ of classical observables for the holomorphic σ-model of maps $\mathbb{C}^d \to V$.

We remark that as $\mathrm{GL}(V)$ acts naturally on the factorization algebra of classical observables, extending the action of $\mathrm{GL}(V)$ on the fields. This action manifestly respects the differential $\overline{\partial}$, which only depends on the source $\mathbb{C}^d$ and not on the target V.

2.6.1.2. The quantum observables. The BV formalism suggests that the quantum observables on S should arise by

(a) tensoring the underlying graded vector space of $\mathrm{Obs}^{\mathrm{cl}}_n$ with $\mathbb{C}[[\hbar]]$ and

(b) modifying the differential to $\overline{\partial} + \hbar\Delta$, where Δ is the BV Laplacian.

This suggestion does not work because Δ is not defined on all of the observables; the naive formula involves an ill-defined pairing of distributions. There are two ways to circumvent this difficulty. First, one can work with a smaller class of observables — such as those arising from smooth functionals, not distributional ones — and this approach is developed in detail for the free $\beta\gamma$ system in Chapter 5, Section 3 of [**CG17**]. (We discuss this approach in Section **??**, where we also show the two approaches agree.) Second, one can mollify Δ instead. This approach is developed in a very broad context of [**CG**], and we have encountered it already in the scale L BV Laplacians Δ_L. These two approaches provide quasi-isomorphic factorization algebras, as we show in Proposition ?? of [**GGW**]. For analyzing the free theory of holomorphic maps $\mathbb{C}^d \to V$ it is most convenient to use the first approach, which we do here.

A classical result of Atiyah-Bott, Proposition 6.1 in [**?**], implies that for any complex manifold U the subcomplex

$$\Omega^{p,*}_c(U) \subset \overline{\Omega}^{p,*}(U)$$

is quasi-isomorphic to the full complex of distributional forms. This follows from ellipticity of the Dolbeault complex. Consequently we can introduce the qausi-isomorphic subcomplex

$$\mathrm{Obs}^{\mathrm{cl}}_V(U) := \left(\mathrm{Sym}(\Omega^{d,*}_c(U,V^*)[d] \oplus \Omega^{0,*}_c(U,V)[1]), \overline{\partial}\right) \xrightarrow{\simeq} \left(\mathrm{Sym}(\overline{\Omega}^{d,*}_c(U,V^*)[d] \oplus \overline{\Omega}^{0,*}_c(U,V)[1]), \overline{\partial}\right) =$$

Just as in the case above, it is easy to see that the assignment $U \mapsto \mathrm{Obs}^{\mathrm{cl}}_V(U)$ defines a factorization algebra on $\mathbb{C}^d$.

Definition 2.6.4. The quantum observables supported on $U \subset \mathbb{C}^d$ is the cochain complex

$$\mathrm{Obs}^{\mathrm{q}}_V(U) = \left(\mathrm{Sym}(\Omega^{d,*}_c(U, V^*)[d] \oplus \Omega^{0,*}_c(U, V)[1]), \overline{\partial} + \hbar\Delta\right).$$

By Theorem 5.3.10 of [**?**] the assignment $U \mapsto \mathrm{Obs}^{\mathrm{q}}_V(U)$ defines a factorization algebra on $\mathbb{C}^d$. This will be our main object of study for the remainder of this section.

2.6.2. The observables on the d-disk

In this section we give a description of the observables of the holomorphic σ-model supported on a d-disk inside $\mathbb{C}^d$.

2.6.2.1. The cohomology of the observables.

Lemma 2.6.5. *For any d-dimensional disk in $\mathbb{C}^d$ there is an isomorphism*

$$H^*\left(\mathrm{Obs}^{\mathrm{q}}_V(D(w,r))\right) \cong \mathrm{Sym}\left(\left(\mathscr{O}^{hol}(D(w,r)\right)^\vee \otimes V^* \oplus \left(\Omega^{d,hol}(D(w,r))\right)^\vee \otimes V[-d+1]\right)[\hbar]$$

where the $(-)^\vee$ is the topological dual.

2.6.2.2. An explicit characterization. The $\beta\gamma$ system on $\mathbb{C}^d$ has a symmetry by the unitary group $U(d)$, which we have already encountered when studying the quantization of the general holomorphic σ-model of maps $\mathbb{C}^d \to B\mathfrak{g}$. Indeed, the fields of the $\beta\gamma$ system are built from sections of certain natural holomorphic vector bundles on $\mathbb{C}^d$. The group $U(d)$ acts by automorphisms on every holomorphic vector bundle, hence it acts on sections via the pull-back.

There is another symmetry that will be relevant later on when we exhibit a calculation of the character for the local operators. Introduce an action of $U(1)$ on the fields of the theory such that V has weight $q_f \in \mathbb{Z}$ and V^* has weight $-q_f$. The value of the fields γ lie in the vector space V, so these fields are of weight q_f. Conversely, the fields β lie in V^*, so have weight $-q_f$. Since the pairing defining the free theory is only non-zero between a single γ and single β field, the theory is invariant under this symmetry. In the physics literature, this is a so-called "flavor symmetry" of the theory, and so to distinguish it from the other symmetry we will denote this group by $U(1)_f$. This symmetry will be especially relevant when we compute the character of the $\beta\gamma$ system.

Lemma 2.6.6. *The symmetry by $U(d) \times U(1)_f$ on the classical $\beta\gamma$ system with values in the complex vector space V extends to a symmetry of the factorization algebra of smoothed quantum observables* $\mathrm{Obs}^{\mathrm{q}}_V$.

Proof. The differential on the factorization algebra is of the form $\overline{\partial} + \hbar\Delta$. The operator $\overline{\partial}$ is manifestly equivariant for the action of $U(d)$. Since $U(1)_f$ does not act on spacetime, $\overline{\partial}$ trivially commutes with its action. Further, the action of $U(d)$ is through linear automorphisms, and since the BV Laplacian Δ is a second order differential operator, it certainly commutes with the action of $U(d)$. Likewise, since $U(1)_f$ is compatible with the (-1)-symplectic pairing, it automatically is compatible with Δ. $\square$

We will use the action of $U(d)$ to organize the class of operators we are interested in. The eigenvectors of $U(d)$ are labeled by the eigenvectors of a maximal torus, which we will take to be given by the subgroup

$$T^d = \{\mathrm{diag}(q_1, \ldots, q_d) \mid |q_i| = 1\} \subset U(d).$$

Here, $q_i \in S^1 \subset \mathbb{C}^\times$ are complex numbers of unit modulus. We say that an element v of the factorization algebra has weight $(n_1, \ldots, n_k)$ if $(q_1, \ldots, q_d) \cdot v = q_1^{n_1} \cdots q_d^{n_d} v$. We will use the shorthand $\vec{n} = (n_1, \ldots, n_d)$.

Definition 2.6.7. (1) Let $w \in \mathbb{C}^d$ and $r > 0$. For any vector of non-negative integers $\vec{n} = (n_1, \ldots, n_d)$ denote by

$$\mathrm{Obs}^{\mathrm{q}}_V(r)^{(\vec{n})} \subset \mathrm{Obs}^{\mathrm{q}}_V(D(w,r))$$

the subcomplex of weight $\vec{n}$ elements.

(2) Let

$$\mathrm{Obs}^{\mathrm{q}}_V(r) := \bigoplus_{\vec{n}} \mathrm{Obs}^{\mathrm{q}}_V(r)^{(\vec{n})}$$

where the direct sum is over all vectors of non-negative integers.

By setting $\hbar = 0$ this also induces weight spaces for the classical observables.

Remark 2.6.8. Note that we have excluded $w \in \mathbb{C}^d$ from the notation above. This is because the $\beta\gamma$ system, as we have already pointed out, is a translation invariant factorization algebra (in fact, it's holomorphically translation invariant). In particular if z, w are any points then translation by z induces an isomorphism

$$\tau_z : \mathrm{Obs}^{\mathrm{q}}_V(D(w,r)) \cong \mathrm{Obs}^{\mathrm{q}}_V(D(w-z,r)).$$

Translation clearly preserves the action by $U(d)$, so this isomorphism restricts to the weight spaces defined above.

We now introduce the following operators that will be of most relevance for our study of the operator product expansion.

Definition 2.6.9. Let $w \in \mathbb{C}^d$ and $r > 0$. Define the following linear observables supported on $D(w, r)$.

(1) For $n_i \in \mathbb{Z}_{\geq 0}, i = 1, \ldots d$, and $v^* \in V^*$ define

$$\mathcal{O}_{\gamma,-\vec{n}}(w; v^*) : \gamma \in \Omega^{0,*}(D(w,r)) \mapsto \left\langle v^*, \left(\frac{\partial^{n_1}}{\partial z_1^{n_1}} \cdots \frac{\partial^{n_d}}{\partial z_d^{n_d}} \gamma(z, \overline{z}) \bigg|_{z=w} \right) \right\rangle_V .$$

Here, the brackets denote the evaluation pairing between V^* and V.

(2) For $m_i \in \mathbb{Z}_{\geq 1}, i = 1, \ldots d$, and $v \in V$ define

$$\mathcal{O}_{\beta,-\vec{m}}(w; v) : \beta \mathrm{d}^d z \in \Omega^{d,*}(D(w,r)) \mapsto \left\langle v, \left(\frac{\partial^{m_1 - 1}}{\partial z_1^{m_1 - 1}} \cdots \frac{\partial^{m_d - 1}}{\partial z_d^{m_d - 1}} \beta(z, \overline{z}) \bigg|_{z=w} \right) \right\rangle_V .$$

The braces $\langle -, - \rangle_V$ denotes the evaluation pairing for the vector space V and its dual.

Our convention is that the evaluation of a Dolbeualt form is zero $\mathrm{d}\overline{z}_i|_{z=w} = 0$. Thus, the above observables are only nonzero when $\gamma \in \Omega^{0,0}(D(w,r))$ and $\beta \mathrm{d}^d z \in \Omega^{d,0}(D(w,r))$. In particular, this implies that these operators are of the following homogenous cohomological degree:

$$\deg(\mathcal{O}_{\gamma,-\vec{n}}(w; v^*)) = 0$$

$$\deg(\mathcal{O}_{\beta,-\vec{m}}(w; v)) = d - 1.$$

Remark 2.6.10. The minus sign in $\mathcal{O}_{\gamma,-\vec{n}}(w; v^*)$ is purely conventional, and meant to match up with the physics and vertex algebra literature see Chapter ?? of [**?**], for instance.

One reason for using this convention is motivated by the state-operator correspondence by realizing the above operators as coming from residues over higher dimensional spheres. Note that for any d-disk $D(0,r)$ there is an embedding of topological vector spaces

$$z_1^{-1}\cdots z_d^{-1}\mathbb{C}[z_1^{-1},\cdots,z_d^{-1}] \to \left(\Omega^{0,*}(D(w,r))\right)^{\vee}$$

that sends a Laurent polynomial $f(z)$ functional

$$\gamma \in \Omega^{0,*}(D(w,r)) \mapsto \oint_{z\in S^{2d-1}} f(z-w)\gamma(z,\overline{z}) \wedge \left(\mathrm{d}^d z \wedge \omega^{BM}(z-w,\overline{z}-\overline{w})\right),$$

where ω_{BM} is the Bochner-Martinelli form of type $(0,d-1)$, and S^{2d-1} is the sphere of radius r around w. The operator $\mathcal{O}_{\gamma,-\vec{n}}(w;v^*)$ corresponds to the Laurent polynomial $f(z) = z^{-n_1}\cdots z^{-n_d}$. We will elaborate more on these types of sphere operators in the next section.

Lemma 2.6.11. *Let $r < s$. Then, the factorization structure map for including disks $D(0,r) \subset D(0,s)$ induces a diagram*

$$\begin{tikzcd} \mathrm{Obs}^{\mathrm{q}}_V(D(0,r)) \arrow[r] & \mathrm{Obs}^{\mathrm{q}}_V(D(0,s)) \\ \mathrm{Obs}^{\mathrm{q}}_V(r) \arrow[u] \arrow[r, "\simeq"] & \mathrm{Obs}^{\mathrm{q}}_V(s) \arrow[u] \end{tikzcd}$$

Further, the bottom horizontal map is a quasi-isomorphism.

Proof. The two vertical maps are the inclusions of the $U(d)$-eigenspaces of the observables supported on disks of radius r and s respectively. It follows from Lemma

2.6.6 that the factorization algebra is $U(d)$-equivariant, so in particular the factorization algebra structure map for the inclusion of disks $D(0,r) \hookrightarrow D(0,s)$ is a map of $U(d)$-representations. Hence, the map restricts to each of the eigenspaces, yielding the diagram.

In [?] it is shown in Corollary 5.3.6.4 that for the one-dimensional $\beta\gamma$ system, the lower map above is a quasi-isomorphism. A completely similar argument applies to the $\beta\gamma$ system on $\mathbb{C}^d$. Indeed, consider the collection

$$\{\mathcal{O}_{\gamma,-\vec{n}_1}(0;v_1^*)\cdot\mathcal{O}_{\gamma,-\vec{n}_k}(0;v_k^*)\cdot\mathcal{O}_{\beta,-\vec{m}_1}(0;v_1)\cdots\mathcal{O}_{\beta,-\vec{m}_l}(0;v_l)\}.$$

The collection runs over non-negative integers k,l and sequences $\vec{n}_i = (n_{i,1},\ldots,n_{i,d})$, $n_{i,j} \geq 0$ and $\vec{m}_i = (m_{i,1},\ldots,m_{i,d})$, $m_{i,1} \geq 1$. It also runs over vectors v_i, v_j^* in V and V^*, respectively. Now, it follows from Lemma 5.3.6.2 of [?] that the above collection form a basis for the cohomology

$$H^*\mathrm{Obs}_V^{\mathrm{cl}}(r)^{(\vec{N})} \subset H^*\mathrm{Obs}^{\mathrm{cl}}(D(0,r))$$

for any r, where $\vec{N} = (N_1,\ldots,N_d)$

$$N_j = (n_{1,j}+\cdots+n_{k,j}) + (m_{1,j}+\cdots+m_{l,j})\,.$$

The result for the quantum observables follows from the spectral sequence induced by the $\hbar$-filtration. $\square$

We will denote $\mathcal{V}_V = \mathrm{Obs}^{\mathrm{q}}_V(r)$, which is well-defined up to quasi-isomorphism by the preceding proposition. This is the "state space" of the higher dimensional holomorphic theory. We will elaborate more on its structure later on in this section.

2.6.3. The sphere observables

We turn to providing a description of the value of the factorization algebra of observables of the $\beta\gamma$ system applied to another important class of open sets in $\mathbb{C}^d$: neighborhoods of the $(2d-1)$-sphere $S^{2d-1} \subset \mathbb{C}^d$. We then study the algebraic structure that the factorization product endows the collection of sphere operators with.

Heuristically speaking, the operators we will consider are supported on $(2d-1)$ sphere. Since the factorization algebra only takes values on open sets, we need to fix small neighborhoods of the spheres in order to define the observables precisely. Let us explain the exact open neighborhoods of the $(2d-1)$-sphere that we will consider. Denote the closed d-disk centered at w of radius r by

$$\overline{D}(w,r) = \left\{(z_1,\ldots,z_d) \in \mathbb{C}^d \mid |z-w| \leq r\right\}.$$

As above, the open disk is denoted $D(w,r)$. Let $\epsilon, r > 0$ be such that $0 < \epsilon < r$, and consider the open submanifold

$$N_{r,\epsilon}(w) := D(w, r+\epsilon) \setminus \overline{D}(w, r-\epsilon) \subset \mathbb{C}^d \setminus \{w\}.$$

For any $\epsilon > 0$, the open set $N_{r,\epsilon}$ is a neighborhood of the closed submanifold given by the sphere of radius r centered at w, $S^{2d-1}_r(w) \subset \mathbb{C}^d \setminus \{w\}$. Note that when $d = 1$, $N_{r,\epsilon}$ is simply an annulus centered at w.

Like in the case of a disk, it is convenient to get our hands on a class of simple observables supported on $N_{r,\epsilon}(w)$. To describe these particular observables we introduce the dg algebra A_d discussed in the Appendix of Chapter **??**. This algebra is a dg model for the derived space of sections of algebraic functions on the punctured affine space $\mathbb{A}^{d\times}$:

$$A_d \simeq \mathbb{R}\Gamma(\mathbb{A}^{d\times}, \mathscr{O}^{alg}).$$

We refer the reader to the Appendix for a more detailed discussion. What we will use at the moment is the existence of a linear embedding of cochain complexes $A_d \hookrightarrow \Omega^{0,*}(\mathbb{C}^d \setminus \{0\})$, which is dense at the level of cohomology. In particular, A_d embedds inside the Dolbeault complex of any of the spherical shells $N_{\epsilon,r}$ we have just introduced.

We have the following general fact about linear functionals on the Dolbeualt complex of $N_{r,\epsilon}(w)$. This lemma will allow us to describe linear observables supported on these neighborhoods.

Lemma 2.6.12. *For any neighborhood $N_{r,\epsilon}(w)$ as above, the residue along the $(2d-1)$-sphere centered at w of radius r, $S_r^{2d-1}(w)$, determines an embedding of topological dg vector spaces*

$$i_{S^{2d-1}} : A_d[d-1] \to \left(\Omega^{0,*}(N_{r,\epsilon}(w))\right)^\vee$$

sending $\alpha \in A_d$ to the functional

$$i_{S^{2d-1}}(\alpha) : \omega \in \Omega^{0,*}(N_{r,\epsilon}(w)) \mapsto \oint_{S_r^{2d-1}(w)} \alpha \wedge \mathrm{d}^d z \wedge \omega.$$

Proof. This is a consequence of Stokes' theorem. Suppose $\alpha = \overline{\partial}\alpha'$. Then, for any $\omega \in \Omega^{0,*}(N_{r,\epsilon}(w))$ we have

$$\oint_{S^{2d-1}} (\overline{\partial}\alpha') \wedge \mathrm{d}^d z \wedge \omega = \oint_{S^{2d-1}} \alpha' \wedge \mathrm{d}^d z \wedge \overline{\partial}\omega.$$

The right-hand side is simply $(\overline{\partial} i_N)(\omega) = i_N(\overline{\partial}\omega)$. □

Similarly, there is an embedding $A_d[d-1] \to \left(\Omega^{d,*}(N_{r,\epsilon}(w))\right)^\vee$ sending $\alpha \in A_d[d-1]$ to the functional

$$\eta \in \Omega^{d,*}(N_{r,\epsilon}(w)) \mapsto \int_{S_r^{2d-1}(w)} \alpha \wedge \eta.$$

These two embeddings allow us to provide a succinct description of the class of linear operators on $N_{r,\epsilon}(w)$ we are interested in. Indeed they determine a cochain map (that we proceed to denote by the same symbol):

$$i_{S^{2d-1}} : A_d \otimes (V^*[d-1] \oplus V) \to \left(\Omega^{0,*}(N_{r,\epsilon}(w)) \otimes V \oplus \Omega^{d,*}(N_{r,\epsilon}(w)) \otimes V^*[d-1]\right)^\vee \subset \mathrm{Obs}^{\mathrm{cl}}_V\left(N_{r,\epsilon}(w)\right).$$

Definition 2.6.13. Let $\alpha \in A_d$ and $v^* \in V^*$. Define the linear observable

$$\mathcal{O}_{\gamma,\alpha}(w; v^*) := i_{S^{2d-1}}(\alpha \otimes v^*) \in \mathrm{Obs}^{\mathrm{cl}}(N_{r,\epsilon}(w)).$$

Likewise, for $v \in V$, define

$$\mathcal{O}_{\beta, z_1^{-1} \cdots z_d^{-1} \alpha}(w; v) := i_{S^{2d-1}}(\alpha \otimes v)$$

Definition 2.6.14. Define the *classical sphere observables* to be the commutative dg algebra

$$\mathscr{A}_V^{\mathrm{cl}} := \mathrm{Sym}\left(A_d \otimes (V^*[d-1] \oplus V)\right)$$

equipped with the differential coming from A_d.

Note that A_d has the structure of a commutative dg algebra, but we are not using the multiplication here. The same construction above, applied now to symmetric products of linear operators, determines a cochain map $i_{S^{2d-1}} : \mathscr{A}_V^{\mathrm{cl}} \to \mathrm{Obs}^{\mathrm{cl}}(N_{r,\epsilon}(w))$.

Let $\mathscr{A}_V = \mathscr{A}_V^{\mathrm{cl}}[\hbar]$. Then, since $\Delta|_{\mathscr{A}_V} = 0$, we see that $i_{S^{2d-1}}$ extends to a cochain map

$$i_{S^{2d-1}} : \mathscr{A}_V \to \mathrm{Obs}_V^{\mathrm{q}}(N_{r,\epsilon}(w)).$$

We will refer to $\mathscr{A}_V$ as the *quantum sphere observables*, or when there is no confusion, the sphere observables.

2.6.3.1. Nesting spherical shells. We now discuss what happens when we study the factorization product between the observables supported on spheres. This will endow the cochain complex $\mathscr{A}_V$ with the structure of an associative (really A_∞) algebra. To recover this structure, we will only be concerned with open sets that are neighborhoods of spheres, as in the previous section. The factorization product is defined for any disjoint configurations of open sets. The configurations of open sets we consider are given by nesting the neighborhoods of the form $N_{r,\epsilon}(w)$, where w is a fixed center.

For simplicity, we assume that our spheres and neighborhoods are all centered at $w = 0$. For $x\epsilon < r$ we have defined the open neighborhood $N_{r,\epsilon} = N_{r,\epsilon}(0)$ of the sphere S_r^{2d-1} centered at zero. Pick positive numbers $0 < \epsilon_i < r_i$ such that $r_1 < r < r_2$,

$\epsilon_1 < r - r_1$, and $\epsilon_2 < r_2 - r$. Finally, suppose $r > \epsilon > \max\{r - r_1 + \epsilon_1, r_2 - r + \epsilon_2\}$. We consider the factorization product structure map for $\mathrm{Obs}^{\mathrm{q}}_V$ corresponding to the following embedding of open sets

$$N_{r_1,\epsilon_1} \sqcup N_{r_2,\epsilon_2} \hookrightarrow N_{r,\epsilon}, \tag{2.22}$$

shown schematically in Figure BW: figure. The factorization structure map for this embedding of disjoint open sets is of the form

$$\mathrm{Obs}^{\mathrm{q}}_V(N_{r_1,\epsilon_1}) \otimes \mathrm{Obs}^{\mathrm{q}}_V(N_{r_2,\epsilon_2}) \to \mathrm{Obs}^{\mathrm{q}}_V(N_{r,\epsilon}). \tag{2.23}$$

Lemma 2.6.15. *The factorization structure map in (2.23) restricts to the subspace of sphere observables. That is, there is a commutative diagram*

$$\begin{tikzcd} \mathrm{Obs}^{\mathrm{q}}_V(N_{r_1,\epsilon_1}) \otimes \mathrm{Obs}^{\mathrm{q}}_V(N_{r_2,\epsilon_2}) \arrow[r] & \mathrm{Obs}^{\mathrm{q}}_V \\ \mathscr{A}_V \otimes \mathscr{A}_V \arrow[u] \arrow[r, "\mu_2"] & \mathscr{A}_V \arrow[u] \end{tikzcd}$$

where the top line is the map in (2.23). The same holds for an arbitrary number of nested neighborhoods of the form $N_{r,\epsilon}$. That is, for any $k \geq 0$ the factorization product restricts to a linear map

$$\mu_k : \mathscr{A}_V^{\otimes k} \to \mathscr{A}_V .$$

Each of the neighborhoods $N_{r,\epsilon}$ are contained in the open submanifold $\mathbb{C}^d \setminus \{0\}$. Note that there is a homeomorphism $\mathbb{C}^d \setminus \{0\} \cong S^{2d-1} \times \mathbb{R}_{>0}$. Further, we have the radial projection map

$$\pi : \mathbb{C}^d \setminus \{0\} = S^{2d-1} \times \mathbb{R}_{>0} \to \mathbb{R}_{>0}$$

that sends $z = (z_1, \ldots, z_d) \mapsto |z| = \sqrt{|z_1|^2 + \cdots + |z_d|^2}$.

A fundamental feature of factorization algebras is that they push forward along smooth maps. We can thus push forward the factorization algebra $\mathrm{Obs}^{\mathrm{q}}_V$ on $\mathbb{C}^d \setminus \{0\}$ along π to obtain a factorization algebra on $\mathbb{R}_{>0}$. To an open interval of the form $(r-\epsilon, r+\epsilon) \subset \mathbb{R}_{>0}$ the factorization algebra assigns precisely the observables supported on $N_{r,\epsilon}$.

Lemma 2.6.15 implies that there is a factorization algebra $\mathscr{F}_{\mathscr{A}_V}$ associated to $\mathscr{A}_V$ and that the inclusion $\mathscr{A}_V \hookrightarrow \mathrm{Obs}^{\mathrm{q}}(N_{r,\epsilon})$ induces a map of factorization algebras on $\mathbb{R}_{>0}$:

$$\mathscr{F}_{\mathscr{A}_V} \to \pi_*(\mathrm{Obs}^{\mathrm{q}}_V)$$

The factorization algebra $\mathscr{F}_{\mathscr{A}_V}$ assigns to every interval the dg vector space $\mathscr{A}_V$. In particular $\mathscr{F}_{\mathscr{A}_V}$ is locally constant, and hence determines the structure of an A_∞ algebra on $\mathscr{A}_V$. We would now like to identify this algebra structure.

We will proceed in two ways. First, we will use the Moyal formula of Section **??** as well as the explicit form of the propagator from Section **??** to deduce the operator product expansion between cohomology classes of operators corresponding to $\mathscr{A}_V$. This will tell us what the algebra structure is on the cohomology $H^*(\mathscr{A}_V)$. Second, we will use the smoothed description of the observables as a factorization enveloping algebra to nail down the precise algebra structure at the cochain level.

Note that we can view $\mathscr{A}_V$ as the symmetric algebra on the following cochain complex

$$A_d \otimes (V^*[d-1] \otimes V) \oplus \mathbb{C} \cdot \hbar.$$

This complex has the structure of a dg Lie algebra, with bracket given by

$$[\alpha \otimes v^*, \alpha \otimes v] = \hbar \langle v^*, v\rangle \oint_{S^{2d-1}} \alpha \wedge \alpha' \mathrm{d}^d z. \tag{2.24}$$

All other brackets are determined by graded anti-symmetry and declaring the parameter $\hbar$ is central. Denote this dg Lie algebra by $\mathscr{H}_V$.

Our main result is that the dg algebra structure on $\mathscr{A}_V$ endowed by the factorization product is equivalent to the universal enveloping algebra $U(\mathscr{H}_V)$ of the dg Lie algebra $\mathscr{H}_V$.

Remark 2.6.16. If $(\mathfrak{g}, \mathrm{d}, [-,-])$ is a dg Lie algebra its universal enveloping algebra is defined explicitly by

$$U(\mathfrak{g}) = \mathrm{Tens}(\mathfrak{g})/(x \otimes y - (-1)^{|x||y|} y \otimes x - [x,y]).$$

It is immediate to check that the differential d descends to one on $U(\mathfrak{g})$, giving $U(\mathfrak{g})$ the structure of an associative dg algebra.

2.6.3.2. Using the Moyal formula. As eluded to before, we now identify the algebra structure on the cohomology of $\mathscr{A}_V$ induced by the map of factorization algebras $\mathscr{F}_{\mathscr{A}_V} \to \pi_*(\mathrm{Obs}^{\mathrm{q}}_V)$, where $\mathscr{F}_{\mathscr{A}_V}$ is the locally constant factorization algebra that assigns the cochain complex $\mathscr{A}_V$ to every interval.

Let $\underline{U(\mathscr{H}_V)}$ be the locally constant factorization algebra on $\mathbb{R}_{>0}$ based on the associative algebra $U(\mathscr{H}_V)$. We will write down an explicit isomorphism of locally constant factorization algebras

$$\Phi : \underline{U(H^* \mathscr{H}_V)} \to H^* \mathscr{F}_{\mathscr{A}_V},$$

implying the result.

By Poincaré-Birkoff-Witt, the dg vector spaces $U(\mathscr{H}_V)$ and $\mathscr{A}_V$ are isomorphic. Therefore, if $I \subset \mathbb{R}_{>0}$ is an interval, we define $\Phi(I)$ to be the identity map. Thus, it suffices to show that the associative algebra structure on the spherical observables agrees with that of $U(\mathscr{H}_V)$ in cohomology.

We turn to an explicit calculation of factorization product for observables in $\pi_*(\mathrm{Obs}^{\mathrm{q}}_V)$. If $\mathscr{O}, \mathscr{O}' \in U(\mathscr{H}_V)$ then we can compute the commutator $[\mathscr{O}, \mathscr{O}']$ in the factorization algebra as follows. For $i = 1, 2, 3$ let $\epsilon_i, r_i > 0$ be such that

$$\epsilon \leq \epsilon_1 < r_1 \leq \epsilon_2 < r_2 \leq \epsilon_3 < r_3 \leq r$$

and consider the configurations

$$i_{12} : N_{r_1,\epsilon_1} \sqcup N_{r_2,\epsilon_2} \hookrightarrow N_{r,\epsilon}$$

and

$$i_{23} : N_{r_2,\epsilon_2} \sqcup N_{r_3,\epsilon_3} \hookrightarrow N_{r,\epsilon}$$

in $\mathbb{C}^{\mathrm{d}} \backslash \{0\}$. If $I_i = (r_i - \epsilon_i, r_i + \epsilon$ and $I = (r - \epsilon, r + \epsilon)$, these correspond to the configurations $i_{12} : I_1 \sqcup I_2 \hookrightarrow I$ and $i_{23} : I_2 \sqcup I_3 \hookrightarrow I$ in $\mathbb{R}_{>0}$, respectively. The induced factorization structure maps are

$$\begin{aligned} \star_{12} \quad &: \quad \mathrm{Obs}^{\mathrm{q}}_V(N_{r_1,\epsilon_1}) \otimes \mathrm{Obs}^{\mathrm{q}}_V(N_{r_2,\epsilon_2}) \to \mathrm{Obs}^{\mathrm{q}}_V(N_{r,\epsilon}) \\ \star_{23} \quad &: \quad \mathrm{Obs}^{\mathrm{q}}_V(N_{r_2,\epsilon_2}) \otimes \mathrm{Obs}^{\mathrm{q}}_V(N_{r_3,\epsilon_3}) \to \mathrm{Obs}^{\mathrm{q}}_V(N_{r,\epsilon}). \end{aligned} \tag{2.25}$$

The commutator $[\mathscr{O}, \mathscr{O}']$ is computed via the formula

$$\mathscr{O} \star_{12} \mathscr{O}' - \mathscr{O}' \star_{23} \mathscr{O}. \tag{2.26}$$

In the notation $\mathscr{O} \star_{12} \mathscr{O}'$ we view $\mathscr{O}$ as having support in N_{r_1,ϵ_1} and $\mathscr{O}'$ as having support in N_{r_2,ϵ_2}.

We compute this commutator at the level of cohomology. The cohomology of A_d is concentrated in degrees 0 and $d-1$. Explicitly, one can represent the zeroeth cohomology as

$$H^0(A_d) = \mathbb{C}[z_1, \ldots, z_d].$$

Now, let $\omega_{BM}(z, \overline{z})$ be the Bochner-Martinelli kernel of type $(0, d-1)$ from above. We can express the $(d-1)$st cohomology of A_d as

$$H^{d-1}(A_d) = \mathbb{C}[\partial_{z_1}, \cdots, \partial_{z_d}] \cdot \omega_{BM}$$

That is, every element of $H^{d-1}(A_d)$ can be written as a holomorphic polynomial differential operator acting on ω_{BM}. Further, it is convenient to make the $U(d)$-equivariant identification

$$\mathbb{C}[\partial_{z_1}, \cdots, \partial_{z_d}]\omega_{BM} \cong z_1^{-1} \cdots z_d^{-1} \mathbb{C}[z_1^{-1}, \ldots, z_d^{-1}], \tag{2.27}$$

which makes sense since ω_{BM} has $T^d \subset U(d)$-weight $(-1, \ldots, -1)$.

Recall that $\mathscr{H}_V = A_d \otimes (V^*[d-1] \oplus V)$. It follows from above that the cohomology of $\mathscr{H}_V$ is concentrated in degrees $-(d-1), 0, d-1$. The non-trivial Lie algebra structure on $\mathscr{H}_V$ comes from the ordinary symplectic pairing on this space, as we've already discussed.

Suppose v, v^* are in V, V^*, respectively and $\alpha, \alpha' \in A_d$. The corresponding classical observables $\mathcal{O}_{\gamma,\alpha}(0; v^*)$ and $\mathcal{O}_{\beta, z_1^{-1}\cdots z_d^{-1}\alpha'}(0; v)$ have cohomological degrees

$$\deg\left(\mathcal{O}_{\gamma,\alpha}(0; v^*)\right) = |\alpha| - d + 1$$
$$\deg\left(\mathcal{O}_{\beta, z_1^{-1}\cdots z_d^{-1}\alpha'}(0; v)\right) = |\alpha'|,$$

where $|\alpha|$ denotes the differential form degree. In cohomology the only nontrivial form degrees of α, α' that survive are $0, d-1$. Suppose that $|\alpha| = 0$. Then, the only way we could obtain a nontrivial commutator between the operators above is if $|\alpha'| = d - 1$.

We will compute the factorization product in (2.26) using our explicit formula for the propagator of the $\beta\gamma$ system computed in Lemma 2.5.1. We diverge a moment to recall how this construction works. The main idea is that the propagator allows us to promote a classical observable to a quantum observable. Recall, the full propagator is an element

$$P(z, w) = \lim_{L\to\infty} \lim_{\epsilon\to 0} P_{\epsilon<L}(z, w) \in \overline{\mathscr{E}}_V(\mathbb{C}^d) \widehat{\otimes} \overline{\mathscr{E}}_V(\mathbb{C}^d)$$

where the $\overline{\mathscr{E}}_V(\mathbb{C}^d)$ denotes the space of distributional sections on $\mathbb{C}^d$. Explicitly, we showed that

$$P(z, w) = C_d\, \omega_{BM}(z, w)$$

where $\omega_{BM}(z, w)$ is the Bochner-Martinelli kernel.

Contraction with P determines a degree zero, order two differential operator

$$\partial_P : \mathrm{Obs}^{\mathrm{cl}}_V(U) \to \mathrm{Obs}^{\mathrm{cl}}_V(U)$$

for any open set $U \subset \mathbb{C}^d$. Recall that the classical observables on U are simply given by a symmetric algebra on the continuous dual of $\mathscr{E}_V(U)$. Since $\overline{\mathscr{E}}^\vee = \mathscr{E}_c^!$, we can view the propagator as an symmetric smooth linear map

$$P^\vee : \mathscr{E}_{V,c}^!(\mathbb{C}^d) \widehat{\otimes} \mathscr{E}_{V,c}^!(\mathbb{C}^d) \to \mathbb{C}.$$

The contraction operator ∂_P is determined by declaring it vanishes on $\mathrm{Sym}^{\leq 1}$, and on Sym^2 is given by the linear map $P^\vee$.

To compute the factorization product we use the isomorphism

$$\begin{array}{rccc} W_0^\infty : & \mathrm{Obs}_V^{\mathrm{cl}}(U)[\hbar] & \to & \mathrm{Obs}_V^{\mathrm{q}}(U) \\ & \mathcal{O} & \mapsto & e^{\hbar \partial_P} \mathcal{O} \end{array}$$

that makes sense for any open set U. This is an isomorphism of cochain complexes, with inverse given by $(W_0^\infty)^{-1} = e^{-\hbar\partial_P}$. By **??** it determines the following formula for the factorization product. If $\mathcal{O}, \mathcal{O}'$ are observables supported on disjoint opens U, U', and V is and open set containing U, U', then the factorization structure map is given by

$$\mathcal{O} \star \mathcal{O}' = e^{-\hbar\partial_P} \left(\left(e^{\hbar\partial_P} \mathcal{O} \right) \cdot \left(e^{\hbar\partial_P} \mathcal{O}' \right) \right) \in \mathrm{Obs}^{\mathrm{q}}(V).$$

Here, the $\cdot$ refers to the symmetric product on classical observables.

The calculation of the factorization product relies on the higher dimensional residue formula involving the Bochner-Martinelli form. If f is any any function in $C^\infty(U)$, where U is a domain in $\mathbb{C}^d$, then the residue formula states that for any $z \in D$

$$f(z, \overline{z}) = \int_{w \in \partial U} \mathrm{d}^d w \; f(w) \, \omega_{BM}(z, w) - \int_{w \in D} \mathrm{d}^d w \; (\overline{\partial} f)(w) \wedge \omega_{BM}(z, w).$$

In particular, if $f(z,\overline{z})$ is holomorphic the second term drops out and we get the familiar expression for the higher dimensional residue.

We can now perform the main calculation. Recall, we have fixed observables $\mathcal{O}_{\gamma,\alpha}(0;v^*)$ and $\mathcal{O}_{\beta,z_1^{-1}\cdots z_d^{-1}\alpha'}(0;v)$. In the notation of Equation (2.25), we have

$$\begin{aligned}
\mathcal{O}_{\gamma,\alpha}(0;v^*)\star_{12}\mathcal{O}_{\beta,z_1^{-1}\cdots z_d^{-1}\alpha'}(0;v) &= \mathcal{O}_{\gamma,\alpha}(0;v^*)\cdot\mathcal{O}_{\beta,z_1^{-1}\cdots z_d^{-1}\alpha'}(0;v) \\
&\quad +\hbar\langle v,v^*\rangle\oint_{|z^1|=r_1}\oint_{|z^2|=r_2}\alpha(z^1)\mathrm{d}^dz^1\alpha'(z^2)P(z^1,z^2) \\
&= \mathcal{O}_{\gamma,\alpha}(0;v^*)\cdot\mathcal{O}_{\beta,z_1^{-1}\cdots z_d^{-1}\alpha'}(0;v) \\
&\quad +\hbar\langle v,v^*\rangle\oint_{|z^1|=r_1}\oint_{|z^2|=r_2}\alpha(z^1)\alpha'(z^2)\mathrm{d}^dz^1\omega_{BM}(z^1,z^2) \\
&= \mathcal{O}_{\gamma,\alpha}(0;v^*)\cdot\mathcal{O}_{\beta,z_1^{-1}\cdots z_d^{-1}\alpha'}(0;v)+\hbar\langle v,v^*\rangle\oint_{|z|=r_1}\alpha(z)\alpha'(z)\mathrm{d}^dz \\
&\quad +\hbar\langle v,v^*\rangle\oint_{|z^1|=r_1}\int_{z^2\in D(0,r_2)}\alpha(z^1)(\overline{\partial}\alpha')(z^2)\omega_{BM}(z^1,z^2).
\end{aligned}$$

In the first line we have used the Moyal formula. In the second line we have used the explicit form of the propagator. In the third line we have used the higher residue formula. Finally, since we are only interested in the cohomology class of the product, we can assume that α,α' are both holomorphic. In particular, the third term in the last line vanishes. The calculation for the $\star_{23}$ product is similar. We conclude that in cohomology the commutator between the quantum observables $\mathcal{O}_{\gamma,\alpha}(0;v^*)$ and $\mathcal{O}_{\beta,z_1^{-1}\cdots z_d^{-1}\alpha'}(0;v)$ is precisely

$$\#\hbar\langle v,v^*\rangle\oint_{|z|=r_1}\alpha(z)\alpha'(z)\mathrm{d}^dz.$$

This agrees with the commutator (2.24) in $\mathscr{H}_V$. The extension to commutators between non-linear observables is completely analogous. Thus, we conclude that as associative graded algebras one as

$$U(H^*\mathscr{H}_V) \cong H^*\mathscr{A}_V.$$

2.6.3.3. Using smoothed observables. We now provide a refined description of the algebra of sphere operators, yet this approach may seem more indirect. It relies on interpreting the observables of the $\beta\gamma$ system as the *factorization envelope* of a certain sheaf of Lie algebras.

The linear smoothed observables, equipped with the linearized BRST differential, on any $U \subset \mathbb{C}^d$ form the subcomplex

$$\Omega_c^{d,*}(U) \otimes V^*[d] \oplus \Omega^{0,*}(U) \otimes V[1] \subset \mathrm{Obs}_V^{\mathrm{cl}}(U).$$

Using the P_0 bracket restricted to the linear observables, we can form the central extension of dg Lie algebras

$$0 \to \mathbb{C}[-1] \cdot \hbar \to \mathscr{H}_V'(U) \to \Omega_c^{d,*}(U) \otimes V^*[d] \oplus \Omega^{0,*}(U) \to 0.$$

This is similar to the construction of the ordinary Heisenberg algebra (such as $\mathscr{H}_V$ above). For classical linear observables the Lie bracket is defined by $[\mathcal{O}, \mathcal{O}'] = \hbar\{\mathcal{O}, \mathcal{O}'\}$, where $\{-,-\}$ is the P_0 bracket. Since the P_0 bracket is degree $+1$ to make this a dg Lie algebra we must put $\hbar$ in degree $+1$ as well. Note that this construction works well as we vary the open set U. Namely, $U \mapsto \mathscr{H}_V'(U)$ is a cosheaf of Lie algebras on $\mathbb{C}^d$. An elementary observation identifies the smoothed quantum observables with the factorization enveloping algebra of $\widetilde{\mathscr{H}_V}$:

$$\mathrm{Obs}_V^{\mathrm{q}} \cong \mathbb{U}(\mathscr{H}_V').$$

Indeed, the right hand side assigns to each open U the cochain complex $\mathrm{C}_*^{\mathrm{Lie}}(\widetilde{\mathscr{H}_V}(U)) = \left(\mathrm{Sym}(\mathscr{H}_V'(U)), \overline{\partial} + \mathrm{d}_{CE}\right)$. One checks directly that d_{CE} is precisely the BV Laplacian $\hbar\Delta$.

Proposition 2.6.17. *There is a locally constant factorization algebra $\mathscr{F}_V$ on $\mathbb{R}_{>0}$ with the following properties:*

(1) $\mathscr{F}_V$ admits a map of factorization algebras

$$\mathscr{F}_V \to \rho_*(\mathrm{Obs}_V^{\mathrm{q}})$$

that is dense at the level of cohomology.

(2) As a locally constant one-dimensional factorization algebra $\mathscr{F}_V$ is equivalent to the dg algebra $U(\mathscr{H}_V)$.

Proof. We will write down the factorization algebra $\mathscr{F}_V$ and then prove the above two properties we claim it satisfies. Consider the local Lie algebra on $\mathbb{R}_{>0}$ whose compactly supported sections are $\Omega^*_{\mathbb{R}_{>0},c} \otimes \mathscr{H}_V$. The Lie bracket is encoded by the Lie bracket on $\mathscr{H}_V$ combined with the wedge product of forms on $\mathbb{R}_{>0}$. Now, we define $\mathscr{F}_V$ as the factorization envelope of this local Lie algebra

$$\mathscr{F}_V = \mathbb{U}\left(\Omega^*_{\mathbb{R}_{>0},c} \otimes \mathscr{H}_V\right).$$

We have just expressed $\mathrm{Obs}_V^{\mathrm{q}}$ as a factorization enveloping algebra as well. Since the pushforward commutes with the functor $\mathbb{U}(-)$, to construct the map in (1) it suffices to provide a map of factorization Lie algebras

$$\Phi : \Omega^*_{\mathbb{R}_{>0},c} \otimes \mathscr{H}_V \to \rho_* \mathscr{H}_V'.$$

Recall that as a vector space $\widetilde{\mathscr{H}_V} = A_d \otimes (V^*[d-1] \oplus V)$. Let $I \subset \mathbb{R}_{>0}$ be an open subset, we will describe the map $\Phi(I)$. There is the natural map $\rho^* : \Omega_c^*(I) \to \Omega_c^*(\rho^{-1}(I))$ given by the pull back of differential forms. We can post compose this with the natural projection $\mathrm{pr}_{\Omega^{0,*}} : \Omega_c^* \to \Omega_c^{0,*}$ to obtain a map of commutative algebras $\mathrm{pr}_{\Omega^{0,*}} \circ \rho^* : \Omega_c^*(I) \to \Omega_c^{0,*}(\rho^{-1}(I))$. The map j from Proposition **??** determines a map of dg commutative algebras $j : A_d \to \Omega^{0,*}(\rho^{-1}(I))$. Thus, we obtain a map

$$\begin{aligned} \Phi(I) = (\mathrm{pr}_{\Omega^{0,*}} \circ \rho^*) \otimes j \otimes \mathrm{id}_V : \quad \Omega_c^*(I) \otimes A_d \otimes V &\to \quad \Omega_c^{0,*}\left((\rho^{-1}(I)\right) \otimes V \\ \varphi \otimes a \otimes v \quad &\mapsto \quad (((\mathrm{pr}_{\Omega^{0,*}} \circ \rho^*)\varphi) \wedge j(a)) \otimes v \end{aligned}$$

Note that since the map j is a dense map in cohomology so is $\Phi(I)$ for each $I \subset \mathbb{R}_{>0}$. The map on the $A_d \otimes V^*[d-1]$ component of $\mathscr{H}_V$ is defined similarly. Moreover, on the central factor $\hbar\Omega_c^*(I) \subset \Omega^*_{\mathbb{R}_{>0},c} \otimes \mathscr{H}_V$ we define

$$\Phi(I)(\hbar\varphi) = \hbar \int_I \varphi.$$

To show that this is a map of cosheaves of dg Lie algebras we must show that the differentials and brackets are compatible. The differential on $\mathscr{H}_V$ is $\mathrm{d}_{dR,\mathbb{R}} + \overline{\partial}$ where $\overline{\partial}$ is the differential on A_d. Let $\varphi \otimes a \otimes v^*$ be an element in $\Omega^*(I) \otimes A_d \otimes V^*[d-1]$. The differential applied to this element is

$$\frac{\partial \varphi}{\partial r} \mathrm{d}r \otimes a \otimes v^* + \varphi \otimes \overline{\partial} a \otimes v^*.$$

Under $\Phi(I)$ this element gets mapped to

$$\sum_i \frac{\partial \varphi}{\partial r} \frac{z_i}{2r} \mathrm{d}\overline{z}_i \wedge a(z, \overline{z}) \otimes v^* + \varphi(r) \wedge \overline{\partial} a(z, \overline{z}) \otimes v^*.$$

To see that the differentials are compatible, we note that when acting on functions $\varphi(r)$ that only depend on the radius, one has $\frac{\partial\varphi}{\partial\bar{z}_i} = \frac{z_i}{2r}\frac{\partial\varphi}{\partial r}$. The fact that the differentials are compatible follows immediately.

Now, suppose $\varphi \otimes a \otimes v^* \in \Omega^*_c(I) \otimes A_d \otimes V^*[d-1]$ and $\psi \otimes b \otimes v \in \Omega^*_c(I) \otimes A_d \otimes V$. The Lie bracket in $\mathscr{H}_V$ of these elements is

$$[\varphi \otimes a \otimes v^*, \psi \otimes b \otimes v]_{\mathscr{H}_V} = \hbar\langle v, v^*\rangle \int_I \varphi\psi \oint ab\mathrm{d}^d z. \tag{2.28}$$

Now, using the definition of the (-1)-shifted symplectic structure defining the free $\beta\gamma$ system, we have

$$\begin{aligned}
[\Phi(I)(\varphi \otimes a \otimes v^*), \Phi(I)(\psi \otimes b \otimes v)]_{\mathscr{H}'_V} &= \hbar\langle v, v^*\rangle \int_{\rho^{-1}(I)} \phi(r)a(z,\overline{z})\psi(r)b(z,\overline{z})\mathrm{d}^d z \\
&= \hbar\langle v, v*\rangle \int_{r\in I} \phi(r)\psi(r) \oint_{S_r^{2d-1}} a(z,\overline{z})b(z,\overline{z})\mathrm{d}^d z.
\end{aligned}$$

This is precisely the image of the right hand side of (2.28) under $\Phi(I)$. Thus, Φ determines a map of cosheaves of Lie algebras. By functoriality of the enveloping factorization algebra together with compatibility under pushforward $\mathbb{U}(\rho_*\mathscr{F}) \cong \rho_*\mathbb{U}(\mathscr{F})$, we obtain a map of factorization algebras

$$\Phi : \mathscr{F}_V = \mathbb{U}\left(\Omega^*_{\mathbb{R}_{>0},c} \otimes \mathscr{H}_V\right) \to \rho_*\mathbb{U}(\mathscr{H}'_V) = \rho_*\mathrm{Obs}^{\mathrm{q}}_V.$$

$\square$

2.6.4. The disk as a module

In the beginning of this section we extracted a subspace of the cohomology of the observables on the d-dimensional disk

$$\mathscr{V}_V \subset \mathrm{Obs}^{\mathrm{q}}_V(D(0,r))$$

by looking at the $U(d)$ weight spaces. We have also seen how the factorization product endows a subspace of the observables supported on neighborhoods of spheres $S^{2d-1} \subset N_{\epsilon,r}$

$$\mathscr{A}_V \subset \mathrm{Obs}^{\mathrm{q}}_V(N_{\epsilon,r})$$

with the structure of a dg associative algebra. In this section we study a different piece of the factorization algebra that equips $\mathscr{V}_V$ with the structure of a module over $\mathscr{A}_V$. Moreover, we will identify this module structure in a way that is reminiscent of the state space of a vertex algebra in the world of chiral CFT.

First, we describe the factorization structure map for a very simple configuration of open sets. Suppose $R > r + \epsilon$ and consider the inclusion

$$N_{r,\epsilon} \hookrightarrow D(0,R). \tag{2.29}$$

This configuration induces the following composition

$$\mathscr{A}_V \hookrightarrow \mathrm{Obs}^{\mathrm{q}}_V(N_{r,\epsilon}) \to \mathrm{Obs}^{\mathrm{q}}_V(D(0,R)) \xrightarrow{H^*(-)} H^*\left(\mathrm{Obs}^{\mathrm{q}}_V(D(0,R))\right). \tag{2.30}$$

The first arrow is just the inclusion of the sphere algebra. The middle arrow is the factorization structure map associated to (2.29). The map $H^*(-)$ is projection onto cohomology. Usually this does not exist, but in the case of the observables on a disk the cohomology is concentrated in the top degree so that the map makes sense. Recall, the state space $\mathscr{V}_V$ embeds inside the cohomology of the observables on a disk, we will see in the next lemma that the map above factors through $\mathscr{V}_V$, hence we get a map $\mathscr{A}_V \to \mathscr{V}_V$.

To state the lemma, recall the presentation for the cohomology of the commutative dg algebra A_d in terms of the Bochner-Martinelli kernel. One has a $U(d)$-equivariant presentation

$$H^{d-1}(A_d) = \mathbb{C}\left[\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_d}\right]\omega^{BM},$$

where, on the right hand side we take the cohomology class.

Lemma 2.6.18. *The above composition (**??**) factors through the cohomology of the state space $H^*\mathscr{V}_V$ to define a map $\pi_- : \mathscr{A}_V \to H^*\mathscr{V}_V$. This is a map of symmetric algebras, further on linear elements $a \otimes v^*, b \otimes v \in A_d \otimes (V^*[d-1] \oplus V) \subset \mathscr{A}_V$ the map is*

$$\pi_-(a \otimes v^*) = \begin{cases} \mathcal{O}_{\gamma,-\vec{n}}(0; v^*), & \text{if } |a| = d-1 \\ 0, & \text{otherwise.} \end{cases}$$

and

$$\pi_-(b \otimes v) = \begin{cases} \mathcal{O}_{\beta,-\vec{m}}(0; v), & \text{if } |b| = d-1 \\ 0, & \text{otherwise.} \end{cases}$$

Where, $a = (\frac{\partial}{\partial z})^{\vec{n}}\omega^{BM} \in A_d^{d-1}$ and $b = (\frac{\partial}{\partial z})^{\vec{m}}\omega^{BM} \in A_d^{d-1}$.

The notation π_- will become apparent momentarily.

Proof. For degree reasons it is automatic that in the composition in (**??**) is only nonzero on $a \otimes v^*, b \otimes v \in A_d \otimes (V^*[d-1] \oplus V)$ if $|a| = |b| = d-1$. Since ω^{BM} is $U(d)$ invariant, it is clear that the element $a = \partial^{\vec{n}}\omega^{BM}$ it lives in the weight $-\vec{n}$ subspace and defines the observable

$$\gamma \otimes v \mapsto \langle v, v^* \rangle \oint_{S^{2d-1}} \gamma(z, \overline{z})(\frac{\partial}{\partial z})^{\vec{n}}\omega^{BM} \mathrm{d}^d z.$$

Since we are only interested in the cohomology class, we can assume that γ is holomorphic. In this case, the residue formula implies that this is precisely the observable $\mathcal{O}_{\gamma;-\vec{n}}(0; v^*)$. The argument for $b \otimes v$ is similar. □

We consider the configuration of open sets of a small d-disk enclosed by a neighborhood $N_{\epsilon,r}$. Concretely, suppose $r_1 < r_2 - \epsilon < r_2 + \epsilon < R$ and consider the inclusion of opens

$$D(0, r_1) \sqcup N_{\epsilon, r_2} \hookrightarrow D(0, R). \tag{2.31}$$

Consider the following diagram

$$\begin{tikzcd}
\mathrm{Obs}^{\mathrm{q}}_V(D(0,r_1)) \otimes \mathrm{Obs}^{\mathrm{q}}_V(N_{\epsilon,r_2}) \arrow[r, "\mu"] & \mathrm{Obs}^{\mathrm{q}}_V(D(0,R)) \\
\mathrm{Obs}^{\mathrm{q}}_V(D) \otimes \mathscr{A}_V \arrow[u] & \\
\mathscr{V}_V \otimes \mathscr{A}_V \arrow[u] \arrow[r, dotted] & \mathscr{V}_V \arrow[uu]
\end{tikzcd}$$

The top horizontal line μ is the factorization structure map coming from the configuration in (2.31).

All of the upward pointing vertical arrows are the inclusions of $\mathscr{A}_V, \mathscr{V}_V$ into the sphere and disk observables, respectively. We claim that the bottom horizontal arrow exists; that is, the restricted factorization product factors through $\mathscr{V}$. This follows from the fact that the factorization structure map preserves the $U(d)$-eigenspaces.

We have seen that the commutative dg algebra A_d has cohomology concentrated in degrees 0 and $d-1$. Since the complex is concentrated in degrees $0, \ldots, d-1$ there exists a quotient map $q : A_d \to H^{d-1}(A_d)$. In the remainder of the section we use the notation $A_{d,-} := H^{d-1}(A_d)$. In addition, let $A_{d,+}$ denote the kernel of this map $A_{d,+} = \ker(q) \subset A_d$.

Correspondingly, there is an abelian dg Lie subalgebra

$$\mathscr{H}_{V,+} = A_{d,+} \otimes (V^*[d-1] \oplus V) \subset \mathscr{A}_V$$

and a commutative subalgebra $\mathscr{A}_{V,+} = U(\mathscr{H}_{V,+}) \subset \mathscr{A}_V$. In fact, this is a maximal commutative subalgebra of $\mathscr{A}_V$. Using $A_{d,-}$ we can similarly define the cochain complex $\mathscr{H}_{V,-} = A_{d,-} \otimes (V^*[d-1] \oplus V)$. As cochain complexes there is a splitting $\mathscr{H}_V = \mathscr{H}_{V,+} \oplus \mathscr{H}_{V,-}$. Hence, by the PBW theorem there is a splitting $\mathscr{A}_V = \mathscr{A}_{V,+} \otimes \mathscr{A}_{V,-}$ as cochain complexes.

Proposition 2.6.19. *The factorization product corresponding disks enclosed by the neighborhoods $N_{r,\epsilon}$ endows the state space $\mathscr{V}_V$ the the structure of a module over the dg algebra $\mathscr{A}_V$. Moreover, as $\mathscr{A}_V$-modules there is a quasi-isomorphism*

$$\mathscr{V}_V \simeq \mathscr{A}_V \otimes_{\mathscr{A}_{V,+}} \mathbb{C}.$$

Remark 2.6.20. The subalgebra of sphere operators $\mathscr{A}_{V,+}$ is the higher dimensional generalization of "annihilation operators" in the context of CFT. Repeated application of these operators kills any vector in $\mathscr{V}_V$. Similarly, the quotient $\mathscr{A}_{V,-}$ is the collection of "creation operators".

2.6.5. A formula for the character

In this section we compute the character of the action of $U(d) \times U(1)_f$ on the local observables of the free $\beta\gamma$ system with values in V. We have already seen that the quantum theory is equivariant for this group, so it makes sense to compute such a character. By definition, the character is conjugation invariant, so it is completely determined by its value on the subgroup $T^d \times U(1)_f \subset U(d) \times U(1)_f$. Choose the following basis for the maximal torus of $U(d)$:

$$T^d = \{\mathrm{diag}(q_1, \ldots, q_d) \mid |q_i| = 1\} \subset U(d).$$

We label the generator of $U(1)_f$ by u. We view the character as an element in the power series ring $\mathbb{C}[[q_i^{\pm}, u^{\pm q_f}]]$.

We will perform the detailed calculation in the case that the complex dimension $d = 2$, with an aim to compare to the formula for the character of the $\mathcal{N} = 1$ supersymmetric chiral multiplet on $\mathbb{R}^4$. The higher dimensional calculation is similar, and the result is given following the two dimensional calculation.

The local operators of the theory we are those supported on the disk $D^2 \subset \mathbb{C}^2$. Since the theory is translation invariant it suffices to consider a disk centered at the origin $0 \in \mathbb{C}^2$. When $d = 2$ we use Proposition **??** to read off the cohomology of the disk observables $H^*\mathrm{Obs}^{\mathrm{q}}_V(D^2)$:

$$\mathrm{Sym}\left((\mathscr{O}^{hol}(D^2)\otimes V)^\vee\right)\otimes \mathrm{Sym}\left((\Omega^{2,hol}(D^2)\otimes V^*)^\vee[-1]\right).$$

Proposition 2.6.21. *The $U(2)\times U(1)_f$ character of the local operators of the $\beta\gamma$ system on $\mathbb{C}^2$ is equal to the elliptic Γ-function*

$$\Gamma_{ell}(u;q_1,q_2)=\prod_{n_1,n_2\geq 0}\frac{1-u^{q_f}q_1^{n_1-1}q_2^{n_2-1}}{1-u^{-q_f}q_1^{n_1}q_2^{n_2}}\in\mathbb{C}[[q_1^{\pm},q_2^{\pm},u^{\pm q_f}]].$$

For an introduction to the elliptic Γ-function and other related hypergeometric series we refer to the reference [**?**].

Proof. We recall the basis for a $U(2)$-eigenspaces of the observables on a 2-disk that we described in Section **??**. Fix non-negative integers $n_1,n_2\geq 0$ and elements $v\in V$, $v^*\in V^*$ consider the following linear observables on the 2-disk:

$$\begin{array}{ccccccc} O_\gamma(n_1,n_2;v^*) & : & \gamma\otimes w & \in\mathscr{O}^{hol}(D^2)\otimes V & \mapsto & \mathrm{ev}(v^*,w)\frac{\partial^{n_1}}{\partial z_1^{n_1}}\frac{\partial^{n_2}}{\partial z_2^{n_2}}\gamma(0) \\ O_\beta(n_1+1,n_2+1;v) & : & \beta\mathrm{d}z_1\mathrm{d}z_2\otimes w^* & \in\Omega^{2,hol}(D^2)\otimes V^* & \mapsto & \mathrm{ev}(w^*,v)\frac{\partial^{n_1}}{\partial z_1^{n_1}}\frac{\partial^{n_2}}{\partial z_2^{n_2}}\beta(0). \end{array}$$

For fixed $n_1,n_2\geq 0$, let $V^*_{n_1,n_2}$ denote the linear span of operators $O_\gamma(n_1,n_2;v^*)$. As a vector space $V^*_{n_1,n_2}\cong V^*$, but we want to remember the weights under $U(2)$. Likewise, for $n_1,n_2>0$, let $V_{n_1,n_2}\cong V$ be the linear span of the operators $O_\beta(n_1,n_2;v)$.

There is an injective map of graded vector spaces

$$\mathrm{Sym}\left(\left(\bigoplus_{n_1,n_2\geq 0}V^*_{n_1,n_2}\right)\oplus\left(\bigoplus_{n_1,n_2>0}V_{n_1,n_2}[-1]\right)\right)\to \mathrm{Sym}\left(\left(\mathscr{O}^{hol}(D^2)\otimes V\right)^\vee\oplus\left(\Omega^{2,hol}(D^2)\otimes V^*\right)^\vee[-1]\right),$$

where the right-hand side is the cohomology of the observables on D^2 and the left-hand side is the cohomology of the state space that we denoted $H^*\mathscr{V}_V$ in Section 2.6.2.

Thus, to compute the character of the local operators it suffices to compute it on the vector space

$$\mathrm{Sym}\left(\left(\bigoplus_{n_1,n_2\geq 0} V^*_{n_1,n_2}\right)\oplus\left(\bigoplus_{n_1,n_2>0}\oplus V_{n_1,n_2}[-1]\right)\right)\cong \mathrm{Sym}\left(\bigoplus_{n_1,n_2\geq 0} V^*_{n_1,n_2}\right)\otimes\bigwedge\left(\bigoplus_{n_1,n_2>0} V_{n_1,n_2}\right).$$

We have used the convention that as (ungraded) vector spaces the symmetric algebra of a vector space in odd degree is the exterior algebra. For instance, $\mathrm{Sym}(W[-1])=\bigwedge(W)$ as ungraded vector spaces. We can further simplify the right-hand side as

$$\bigotimes_{n_1,n_2\geq 0}\left(\mathrm{Sym}(V^*_{n_1,n_2})\right)\bigotimes\bigotimes_{n_1,n_2>0}\left(\bigwedge(V_{n_1,n_2})\right).$$

The character of the symmetric algebra $\mathrm{Sym}(V^*_{n_1,n_2})$ is equal to $(1-u^{-q_f}q_1^{n_1}q_2^{n_2})^{-1}$ and the character of $\bigwedge(V_{n_1,n_2})$ is equal to $(1-u^{q_f}q_1^{n_1}q_2^{n_2})$. The formula for character in the statement of the proposition follows from the fact that the character of a tensor product is the product of the characters. $\square$

We have seen in Proposition BW: ref that when the complex dimension $d=2$, the free $\beta\gamma$ system is equivalent to the holomorphic twist of the free $\mathcal{N}=1$ chiral multiplet in four dimensions. In [?] Equation 5.58 the index for the $\mathcal{N}=1$ chiral multiplet is computed, and our answer is easily seen to agree with theirs. We conclude that in this instance that under the holomorphic twist the superconformal index was sent to the character of the local observables of the holomorphic theory. We will see BW: ref that this is a general fact about superconformal indices.

Without much more difficulty, one can obtain the formula for the character of the holomorphic σ-model of maps $\mathbb{C}^d \to V$ for any d.

Proposition 2.6.22. *The $U(d) \times U(1)_f$ character of the local operators of the holomorphic σ-model of maps $\mathbb{C}^2 \to V$ is equal to the formal series*

$$\prod_{n_1,\ldots,n_d \geq 0} \frac{1 - u^{q_f} q_1^{n_1 - 1} \cdots q_d^{n_d - 1}}{1 - u^{-q_f} q_1^{n_1} \cdots q_d^{n_d}} \in \mathbb{C}[[q_1^{\pm}, \ldots, q_d^{\pm}, u^{\pm q_f}]].$$

CHAPTER 3

Local symmetries of holomorphic theories

In this chapter we investigate the symmetries that generic holomorphic quantum field theories possess. Our overarching goal is to develop tools for understanding such symmetries that provide a systematic generalization of methods used in chiral conformal field theory on Riemann surfaces, especially for the Kac-Moody and Virasoro vertex algebras. We will utilize the tools of BV quantization and factorization algebras that has already heavily percolated this thesis.

We will focus on two main types of symmetries: holomorphic gauge symmetries and symmetries by holomorphic diffeomorphisms. An ordinary gauge symmetry is characterized as being local on the spacetime manifold. Each of the types of symmetries we consider share this characteristic, but they also enjoy an additional structure: they are holomorphic (up to homotopy) on the spacetime manifold. This means that they are specific to the type of theories we consider. Moreover, they store more interesting information about the geometry of the underlying manifold as compared to the smooth version of such symmetries.

Infinitesimally speaking, a symmetry is encoded by the action of a Lie algebra. For the holomorphic gauge symmetry this will become a sort of current algebra which is equivalent to holomorphic functions on the complex manifold with values in a Lie algebra. For the holomorphic diffeomorphisms this Lie algebra is that of holomorphic vector fields. Locality implies that this actually extends to a symmetry by a sheafy version of a Lie algebra. The precise sheafy version we mean is called a *local Lie algebra*, which we will recall in the main body of the text. To every local Lie algebra we can assign a factorization algebra through the so-called factorization enveloping algebra:

$$\mathbb{U} : \mathrm{Lie}_X \to \mathrm{Fact}_X .$$

Here, Lie_X is the category of local Lie algebras which we will recall in the main body of the text. By this construction, we see that the symmetries themselves of field theories give rise to factorization algebras.

One compelling reason for constructing a factorization algebra model for Lie algebras encoding the symmetries of a theory is that it allows one to consider universal versions of such objects. In the case of the symmetry by a current algebra of a Lie algebra in chiral conformal field theory this has been spelled out in the book [**?**]. For the case of conformal symmetry our work in [**Wil17**] provides a factorization algebra lift of the ordinary Virasoro vertex algebra that exists uniformly on the site of Riemann surfaces. In this chapter, we extend each of these objects to arbitrary complex dimensions. Our formulation lends itself to an explicit computation of the factorization homology along certain complex manifolds, for which we will focus on several examples.

Studying such local symmetries involves rich geometric input even at the classical level, but the skeptical mathematician may view this as a repackaging of already familiar objects in complex geometry. The main advantage of working with factorization algebra analogs of such symmetries is in their relationship to studying quantizations of field theories. A similar obstruction deformation theory for studying quantizations of classical field theories also allows us to study the problem of *quantizing* local symmetries of a field theory. Moreover, we already know that factorization algebras describe the operator product expansion of the observables of a QFT. A formulation of Noether's theorem in [?] makes the relationship between the associated factorization algebra of a symmetry and the factorization algebra of observables of a theory.

Of course, quantizing a symmetry of a field theory may not always exist. In fact, this failure sheds light into subtle field theoretic phenomena of the underlying system. For example, in the case of conformal symmetries of a conformal field theory, the failure is exactly measured by the *central charge* of the theory. It is well established that the central charge is a very important characterization of a conformal field theory. At the Lie theoretic level, this failure is measured by a cocycle which in turn defines a central extension of the Lie algebra. It is this central extension that acts on the theory.

For this reason, an essential aspect of studying the local symmetries of holomorphic field theories we mentioned above is to characterize the possible cocycles that give rise to central extensions. As we have already mentioned, for vector fields in complex dimension one this is related to the central charge and the central extension of the Witt algebra (vector fields on the circle) known as the Virasoro Lie algebra. In the case of a current algebra associated to a Lie algebra, central extensions are related to the *level* and the corresponding central extensions are called affine algebras.

Theorem 3.0.1. *The following is true about the local Lie algebras associated to holomorphic diffeomorphisms and holomorphic gauge symmetries.*

(1) Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{g}^X$ is associated current algebra defined on any complex manifold X. There is an embedding of the cohomology $H^*_{Lie}(\mathfrak{g}, \mathrm{Sym}^{d+1} g^\vee[-d-1])$ inside of the local cohomology of $\mathfrak{g}^X$.

(2) There is an isomorphism between the local cohomology of holomorphic vector fields on any complex manifold X of dimension d and $H^*_{dR}(X) \otimes H^*_{GF}(\mathrm{W}_d)[2d]$, where $H^*_{GF}(\mathrm{W}_d)$ is the Gelfand-Fuks cohomology of vector fields on the formal disk.

The central extensions we are interested in come from classes of degree $+1$ of the above local Lie algebras. In the case of holomorphic vector fields the result above implies that all such extensions are parametrized by $H^{2d+1}(\mathrm{W}_d)$. It is a classical result of Fuks [**Fuk86**] that this cohomology is isomorphic to $H^{2d+2}(BU(d))$. In complex dimension one this cohomology is one dimensions corresponding to the class c_1^2. In general we obtain new classes, which are shown to agree with calculations in the physics literature in dimensions four and six.

In general, any of these cohomology classes define factorization algebras by twisting the factorization enveloping algebra. We especially focus on this construction in the case that the complex d-fold is equal to affine space $\mathbb{C}^d$, or some of its natural submanifolds. In the case of the current algebra, our result is compatible with recent work of Kapranov et. al. in [**FHK**] where they study higher dimensional versions of affine algebras, and their relationship to the (derived) moduli space of G-bundles in an analogous way that affine algebras are related to the moduli of bundles on curves via Kac-Moody uniformization. Our second main result shows how to recover these higher affine algebras from our factorization algebra on punctured affine space $\mathbb{C}^d \setminus \{0\}$, see Theorem **??**.

The extensions of part (1) of Theorem 3.0.1 are related to cohomology classes in the moduli of G-bundles on complex d-folds. We will show how techniques in equivariant BV quantization lead to natural families of QFTs defined over formal neighborhoods in the moduli space of G-bundles. Our techniques allow us to study quantizations of such families, in particular there are anomalies to quantization. An explicit analysis of Feynman diagrams leads to a computation of certain classes in the local cohomology which we relate to Chern classes of natural line bundles on $\mathrm{Bun}_G(X)$. This leads us to our next main result which is to prove a version of the Grothendieck-Riemann-Roch (GRR) theorem using the aforementioned methods of BV quantization, see Theorem 3.4.1.

3.1. The current algebra on complex manifolds

3.1.1. Definitions

We recall some definitions that we will use throughout the paper. The first concept we introduce is that of a *local Lie algebra*. This is the central object needed to discuss symmetries of field theories that are local on the spacetime manifold.

Throughout this paper we will use L_∞ algebras. This is a modest generalization of a dg Lie algebra where the Jacobi identity is only required to hold up to homotopy. The data of an L_∞ algebra is a graded vector space V with, for each $k \geq 1$, a k-ary bracket

$$\ell_k : V^{\otimes k} \to V[2-k]$$

of cohomological degree $2-k$. These maps are required to satisfy a series of conditions, the first of which says $\ell_1^2 = 0$. The next says that ℓ_2 is a bracket satisfying the Jacobi identity up to a homotopy given by ℓ_3. For a detailed definition see we refer the reader to [**Sta92, ?**].

We now give the definition of a local L_∞ algebra on a manifold X. This has appeared in Chapter 4 of [**CG**].

Definition 3.1.1. A *local L_∞ algebra* on X is the following data:

(i) a $\mathbb{Z}$-graded vector bundle L on X, whose sheaf of smooth sections we denote $\mathscr{L}^{sh}$, and

(ii) for each positive integer n, a polydifferential operator in n inputs

$$\ell_n : \underbrace{\mathscr{L}^{sh} \times \cdots \times \mathscr{L}^{sh}}_{n \text{ times}} \to \mathscr{L}[2-n]$$

such that the collection $\{\ell_n\}_{n\in\mathbb{N}}$ satisfy the conditions of an L_∞ algebra. Thus, $\mathscr{L}^{sh}$ is a sheaf of L_∞ algebras.

In practice, we prefer to work with the compactly supported sections of L, for which we reserve the more succinct notation $\mathscr{L}$.

Definition 3.1.2. Given a local L_∞ algebra $(L, \{\ell_n\})$ on X, let $\mathscr{L}$ denote the precosheaf of L_∞ algebras that assigns compactly supported sections of L to each open of X.

We typically refer to the local L_∞ algebra $(L, \{\ell_n\})$ by $\mathscr{L}$. We will often use local *Lie* algebra, especially if $\mathscr{L}$ is a precosheaf of dg Lie algebras and hence has trivial $\ell_{n\geq 3}$.

Example 3.1.3. Let $P \to X$ be a principal G-bundle. The adjoint bundle is a bundle of Lie algebras that we denote $\mathrm{ad}(P) \to X$. We will hereafter use $\mathscr{A}\mathrm{d}(P)$ to denote the *cosheaf* of compactly supported sections of Dolbeault complex of $\mathrm{ad}(P)$

$$\mathscr{A}\mathrm{d}(P)(U) = \Omega_c^{0,*}(U; \mathrm{ad}(P)).$$

In keeping with our conventions, $\mathscr{A}\mathrm{d}(P)^{sh}$ will denote the corresponding *sheaf* of sections of the Dolbeault complex

$$\mathscr{A}\mathrm{d}(P)^{sh}(U) = \Omega^{0,*}(U; \mathrm{ad}(P)).$$

The Dolbeualt differential $\overline{\partial}$ and the fiberwise Lie bracket on $\mathrm{ad}(P)$ endow $\mathscr{A}\mathrm{d}(P)^{sh}$ with the structure of a sheaf of dg Lie algebras on X.

The following lemma follows from tracing through definitions.

Lemma 3.1.4. *For any holomorphic principal bundle $P \to X$, the Dolbeualt complex of forms with values in* $\mathrm{ad}(P)$ *is a local Lie algebra.*

Example 3.1.5. Another key local Lie algebra makes sense on an arbitrary complex d-fold. Let $\mathfrak{g}$ be an ordinary Lie algebra, such as $\mathfrak{sl}_n$. There is a natural assignment

$$\mathscr{G}^{sh} : X \mapsto \Omega^{0,*}(X) \otimes \mathfrak{g},$$

where X is a complex d-fold. In fact, this assignment defines a sheaf of dg Lie algebras on the category of complex d-folds and local biholomorphisms,[1] and $\mathscr{G}$ to denote the cosheaf of compactly supported sections $\Omega^{0,*}_c \otimes \mathfrak{g}$. For any $\mathfrak{g}$, $\mathscr{G}$ defines a local Lie algebra on the category of d-folds, though we don't elaborate on the requisite categorical machinery to make this precise. We use $\mathscr{G}_X$ to denote the restriction of $\mathscr{G}$ to a fixed complex d-fold X. This defines a local Lie algebra whose associated cosheaf of sections is $U \subset X \mapsto$

[1] A biholomorphism is a map $\phi : X \to Y$ that is biijective and both ϕ and ϕ^{-1} are holomorphic. A *local* biholomorphism means a map $\phi : X \to Y$ such that for every point $x \in X$ has a neighborhood on which ϕ is a biholomorphism.

$\Omega_c^{0,*}(U) \otimes \mathfrak{g}$. Note that in the case of the trivial holomorphic principal G-bundle on X one as $\mathscr{G}_X = \mathscr{A}\mathrm{d}(\mathrm{triv})$.

3.1.2. Factorization Lie algebras

A factorization Lie algebra is a useful concept that we will utilize to make the connection between local Lie algebras on factorization algebras. Ordinarily, when we discuss factorization algebras we mean a symmetric monoidal functor from the category of opens on a fixed manifold, with monoidal product given by disjoint union, to the category of chain complexes, with monoidal product given by tensor product. However, a factorization algebra can be defined with an arbitrary symmetric monoidal category as the target. This definition has appeared in multiple sources, such as [**CG17**], [**?**], or [**?**].

Definition 3.1.6. Let $\mathscr{C}^{\otimes}$ be a symmetric monoidal category and X a space. A *prefactorization algebra* on X with values in $\mathscr{C}$ is a functor of symmetric monoidal categories

$$\mathscr{F} : \mathrm{Disj}_X^{\sqcup} \to \mathscr{C}^{\otimes}.$$

A *strict factorization algebra* with values in $\mathscr{C}$ is a prefactorization algebra $\mathscr{F}$ such that:

(1) $\mathscr{F}$ is a cosheaf with respect to the Weiss topology;
(2) for any disjoint open sets $U, V \subset X$ the structure map $\mathscr{F}(U) \otimes \mathscr{F}(V) \to \mathscr{F}(U \sqcup V)$ is an isomorphism.

There are two important symmetric monoidal categories we will be most interested in as the target of a factorization algebra. The first is the category of chain complexes $\mathrm{Ch}^{\otimes}$ (over $\mathbb{C}, \mathbb{R}$) with symmetric monoidal product given by the tensor product. The next is the category of dg Lie algebras $\mathrm{dgLie}^{\oplus}$ with symmetric monoidal structure given by the direct sum.

In both of these categories there is the notion of a quasi-isomorphism, which allows us to weaken the above definition slightly.

Definition 3.1.7. Let $\mathscr{F}$ be a prefactorization algebra on X with values in $\mathscr{C} = \mathrm{Ch}^{\otimes}$ or $\mathrm{dgLie}^{\oplus}$. Then, $\mathscr{F}$ is a *homotopy factorization algebra* if

(1) $\mathscr{F}$ is a homotopy cosheaf with respect to the Weiss topology;
(2) for any disjoint open sets $U, V \subset X$ the structure map $\mathscr{F}(U) \otimes \mathscr{F}(V) \to \mathscr{F}(U \sqcup V)$ is a quasi-isomorphism.

Not surprisingly, there is a version of the definition for homotopy factorization algebras with values in an arbitrary symmetric monoidal ∞-category, but we do not wish to dive into the general formalism here but refer to the source references [**?, ?, ?**] for a complete treatment.

For the remainder of this paper we will only discuss factorization algebras valued in these categories $\mathrm{Ch}^{\otimes}, \mathrm{dgLie}^{\oplus}$. When we do not say otherwise, a *factorization algebra* will mean a homotopy factorization algebra with values in Ch. Likewise, a *factorization Lie algebra* will mean a homotopy factorization algebra with values in dgLie. Note that the direct sum is *not* the coproduct for Lie algebras, so a prefactorization Lie algebra is different than just a precosheaf of Lie algebras.

We have already encountered a modest extension of the category of dg Lie algebras to the category of L_∞ algebras $L_\infty\mathrm{Alg}$ which will come up in our discussion below. This category is also symmetric monoidal using the direct sum, and we will also refer to homotopy factorization algebras with values in $L_\infty\mathrm{Alg}$ as factorization Lie algebras.

The primary appearance of factorization Lie algebras, for us, comes from local Lie algebras.

Lemma 3.1.8. *Suppose $(L, \{\ell_n\})$ is a local Lie algebra on X. Then, the compactly supported sections $\mathscr{L}$ has the structure of a factorization Lie algebra.*

Proof. By the cosheaf property, we know that $\mathscr{L}(U \sqcup V) \cong \mathscr{L}(U) \oplus \mathscr{L}(V)$. This is an isomorphism of L_∞ algebras since any element of $\mathscr{L}(U)$ commutes with $\mathscr{L}(V)$ inside of $\mathscr{L}(U \sqcup V)$. Similarly, if $\{U_i\}$ is a disjoint collection of opens in X and $\sqcup_i U_i \subset W$, then we define the factorization structure map by

$$\oplus_i \mathscr{L}(U_i) \cong \mathscr{L}(\sqcup_i U_i) \to \mathscr{L}(W)$$

where the second map is the structure map for the cosheaf. These structure maps exhibit $\mathscr{L}$ as a prefactorization Lie algebra (i.e. a prefactorization algebra valued in the category of L_∞ algebras). $\square$

There is a functor from dg Lie algebras to cochain complexes

$$\mathrm{C}_*^{\mathrm{Lie}} : \mathrm{dgLie} \to \mathrm{Ch}$$

sending $(\mathfrak{g}, \mathrm{d}, [-,-])$ to the complex

$$\mathrm{C}_*^{\mathrm{Lie}}(\mathfrak{g}) = (\mathrm{Sym}(\mathfrak{g}[1]), \mathrm{d} + \mathrm{d}_{CE})\,.$$

Here, d denotes the extension of the differential on $\mathfrak{g}$ to the symmetric algebra by the Leibniz rule, and d_{CE} encodes the Lie bracket. There is a completely similar functor from L_∞ algebras to chain complexes that we denote by the same name.

The functor $\mathrm{C}_*^{\mathrm{Lie}}$ is symmetric monoidal with respect to the direct sum of Lie algebras and the tensor product of cochain complexes $\mathrm{C}_*^{\mathrm{Lie}}(\mathfrak{g} \oplus \mathfrak{h}) = \mathrm{C}_*^{\mathrm{Lie}}(\mathfrak{g}) \otimes \mathrm{C}_*^{\mathrm{Lie}}(\mathfrak{h})$. Since a factorization Lie algebra uses the direct sum monoidal structure, the following definition makes sense.

Definition/Lemma 2. Suppose $\mathscr{G}$ is a factorization Lie algebra on a manifold X then, $\mathrm{C}_*^{\mathrm{Lie}}(\mathscr{G})$ has the structure of a factorization algebra (valued in cochain complexes with tensor product).

We have already seen that every local Lie algebra gives rise to a factorization Lie algebra. By the construction above, we obtain the following composition of functors.

$$\mathrm{Lie}_X \to \mathrm{Fact}_X^{\mathrm{Lie}} \to \mathrm{Fact}_X$$

Here Lie_X is the category of local Lie algebras on X. If $\mathscr{L}$ is a local Lie algebra we let $\mathbb{U}(\mathscr{L})$ be the image under this composition, and call it the *factorization enveloping algebra* of $\mathscr{L}$.

3.1.3. Local cohomology

In this section we study the cohomology of the local Lie algebra $\mathscr{A}\mathrm{d}(P)$. As we have already encountered many times in this thesis, the cohomology we are interested in consists of those functionals on the local Lie algebra that are *local*. From the perspective of local Lie algebras, one appealing aspect of this class of functionals is that they give rise to local Lie algebra extensions of the current algebra. These extensions will be appear when we quantize holomorphic gauge symmetries.

In Section **??** we have discussed local cohomology of a local Lie algebra, but we briefly recall it here. The basic idea is that a local cochain is a functional on the local Lie algebra obtained by integrating a polydifferential operator applied to an element in the local Lie algebra. If L is a graded vector bundle, let JL denote the corresponding ∞-jet bundle. If L is the underlying vector bundle of a local Lie algebra then JL has the structure of a bundle of Lie algebras. Thus, we may consider its reduced Chevalley-Eilenberg cochain complex $\mathrm{C}^*_{\mathrm{Lie,red}}(JL)$[2]. For any vector bundle JL has the structure of a D_X-module. In the case of a local Lie algebra, JL is a Lie algebra object in D_X-modules. Thus, $\mathrm{C}^*_{\mathrm{Lie}}(JL)$ is a commutative dg algebra in D_X-modules. The local cochain complex is obtained by tensoring the right D_X-modules of densities on X over D_X with this D_X-module.

Definition 3.1.9. Let $\mathscr{L}$ be a local Lie algebra on X. The local cohomology of $\mathscr{L}$ is defined as

$$\mathrm{C}^*_{\mathrm{loc}}(\mathscr{L}) = \Omega^{d,d}_X \otimes_{D_X} \mathrm{C}^*_{\mathrm{Lie,red}}(JL).$$

[2] A local functional will always be defined modulo constants, hence we look at reduced cochains.

This is a sheaf of cochain complexes on X whose global sections we will denote by $\mathrm{C}^*_{\mathrm{loc}}(\mathscr{L}(X))$.

We note that the cochain complex of local functionals is a subcomplex of $\mathrm{C}^*_{\mathrm{Lie,red}}(\mathscr{L}(X))$, the reduced Lie algebra cochains of the global sections $\mathscr{L}(X)$. The differential on local functionals is, in essence, just precomposition with the polydifferentials defining the brackets of $\mathscr{L}$. Altogether $\mathrm{C}^*_{\mathrm{loc}}(\mathscr{L})$ is just a version of diagonal Gelfand-Fuks cohomology [**?**] for this kind of Lie algebra. We will discuss this further when we approach the local Lie algebra of holomorphic vector fields.

In ordinary Lie theory, central extensions are parametrized by cocycles on the Lie algebra valued in the trivial module. Similarly, local cocycles define central extensions of local Lie algebras.

Definition 3.1.10. A cocycle Θ of degree $2+k$ in $\mathrm{C}^*_{\mathrm{loc}}(\mathscr{L})$ determines a *k-shifted central extension*

$$0 \to \mathbb{C}[k] \to \widehat{\mathscr{L}_\theta} \to \mathscr{L} \to 0 \tag{3.1}$$

of precosheaves of L_∞ algebras, where the L_∞ structure maps are defined by

$$\widehat{\ell}_n(x_1, \ldots, x_n) = (\ell_n(x_1, \ldots, x_n), \Theta(x_1, \ldots, x_n)).$$

Cohomologous cocycles determine quasi-isomorphic extensions of precosheaves of Lie algebras. Much of the rest of the section is devoted to constructing and analyzing various cocycles and the resulting extensions.

Local cocycles give a direct way of deforming the factorization enveloping algebra of a local Lie algebra. Suppose that we have a local cocycle $\Theta \in \mathrm{C}^*_{\mathrm{loc}}(\mathscr{L})$ is of cohomological degree $+1$. We define the *twisted factorization enveloping algebra* to be the factorization algebra sending the open set $U \subset X$ to the cochain complex

$$\begin{aligned}\mathbb{U}_\Theta(\mathscr{L})(U) &= (\mathrm{Sym}(\mathscr{L}(U)[1] \oplus \mathbb{C} \cdot K), \mathrm{d}_{\mathscr{L}} + K \cdot \Theta) \\ &= (\mathrm{Sym}(\mathscr{L}(U)[1])[K], \mathrm{d}_{\mathscr{L}} + K \cdot \Theta)\end{aligned}$$

where $\mathrm{d}_{\mathscr{L}}$ denotes the differential on the untwisted factorization enveloping algebra applied to U and Θ is the operator on the symmetric algebra extending the cocycle $\Theta : \mathrm{Sym}(\mathscr{L}(U)[1]) \to \mathbb{C} \cdot K$ by demanding that it is a graded derivation. Here, K is a formal algebraic parameter. We denote this twisted factorization enveloping algebra by $\mathbb{U}_\Theta(\mathscr{L})$. We will consider this as a factorization algebra valued in the symmetric monoidal category of chain complexes that are linear over the commutative ring $\mathbb{C}[K]$. Specialization at certain values of K yields an ordinary factorization algebra.

3.1.3.1. The J functional. There is a particular family of local cocycles that has special importance in studying symmetries of higher dimensional holomorphic gauge theories.

Let us recall the familiar complex one-dimensional case that we wish to extend. Let Σ be a Riemann surface, and let $\mathfrak{g}$ be a simple Lie algebra with Killing form κ. Consider the local Lie algebra $\mathscr{G}_\Sigma = \Omega^{0,*}_c(\Sigma) \otimes \mathfrak{g}$ on Σ. There is a natural cocycle depending precisely on two inputs:

$$\theta(\alpha \otimes M, \beta \otimes N) = \kappa(M, N) \int_\Sigma \alpha \wedge \partial \beta,$$

where $\alpha, \beta \in \Omega^{0,*}_c(\Sigma)$ and $M, N \in \mathfrak{g}$. In Chapter 5 of [**CG17**] it is shown how the twisted factorization envelope of $\mathscr{G}_X$ via this cocycle recovers the Kac-Moody vertex algebra and the affine algebra extending $L\mathfrak{g} = \mathfrak{g}[z, z^{-1}]$.

We are interested in a generalization of this construction in arbitrary dimensions. Let θ be an invariant polynomial on $\mathfrak{g}$ of homogenous degree $d+1$. That is, θ is an element of $\mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$. For any complex d-fold X we can extend θ to a functional $J_X(\theta)$ on the Dolbeault complex $\Omega^{0,*}_c(X) \otimes \mathfrak{g}$ by the formula

$$J_X(\theta)(\omega_0 \otimes Y_0, \ldots, \omega_d \otimes Y_d) = \theta(Y_0, \ldots, Y_d) \int_X \omega_0 \wedge \partial\omega_1 \cdots \wedge \partial\omega_d. \tag{3.2}$$

Note that we use d copies of the holomorphic derivative $\partial : \Omega^{0,*} \to \Omega^{1,*}$ to obtain an element of $\Omega^{d,*}_c$ in the integrand (and hence something that has a chance of being integrated). If we extend θ to a functional on the Dolbeault complex in the natural way

$$\theta : \Omega^{*,*}(X)^{\otimes d+1} \to \Omega^{*,*}(X)$$

then we can write the cocycle more succinctly as $J_X(\theta)(\alpha_0, \ldots, \alpha_d) = \int_X \theta(\alpha_0, \partial\alpha_1, \ldots, \partial\alpha_d)$.

This formula clearly makes sense for any complex d-fold X, and since integration is local on X, it intertwines nicely with the structure maps of $\mathscr{G}_X$.

Proposition 3.1.11. *For any complex d-fold X and invariant polynomial $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$, the functional $J_X(\theta)$ is a local functional in $\mathrm{C}^*_{\mathrm{loc}}(\mathscr{G}_X)$. In fact, the assignment*

$$J_X : \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}[-1] \to \mathrm{C}^*_{\mathrm{loc}}(\mathscr{G}_X) \quad , \quad \theta \mapsto J_X(\theta)$$

is an cochain map.

Proof. The functional $J_X(\theta)$ is local as it is expressed as the integral of a multilinear map composed with a product of differential operators. We need to show that $J_X(\theta)$ is closed for the differential on $\mathrm{C}^*_{\mathrm{loc}}(\mathscr{G}_X)$. The total differential splits as a sum $\overline{\partial} + \mathrm{d}_{\mathfrak{g}}$ where $\overline{\partial}$ denotes the induced $\overline{\partial}$ differential on functionals and d_{CE} is constructed from the Lie bracket on $\mathfrak{g}$. We observe that

$$\overline{\partial} J_X(\theta) = 0$$

$$\mathrm{d}_{CE} J_X(\theta) = 0.$$

The first line follows from the fact that $\overline{\partial}$ and ∂ are graded commutative. The differential d_{CE} is obtained from the differential in the Chevalley-Eilenberg complex of $\mathfrak{g}$ in a natural way. The second line follows from the fact that the homogenous polynomial $\theta : \mathfrak{g} \times \cdots \times \mathfrak{g} \to \mathbb{C}$ is closed in the Chevalley-Eilenberg complex for $\mathfrak{g}$. $\square$

Having the fundamental construction of the cocycle down, we discuss two modest extensions of the construction. First is to consider an arbitrary G-bundle P on X. Suppose $\mathrm{ad}(P)$ is trivialized over an open set $U \subset X$. On this open set, we can write an element $\alpha \in \mathscr{A}\mathrm{d}(P)(U) = \Omega^{0,*}_c(U, \mathrm{ad}(P))$ as $\alpha = \omega \otimes X$ where $\omega \in \Omega^{0,*}(X)$ and $X \in \mathfrak{g}$. Thus, the formula above for $J_X(\theta)$ still makes sense on $\mathscr{A}\mathrm{d}(P)(U)$. Since the expression for the cocycle is clearly independent of the choice of a coordinate it glues to define a global section. Thus, for any principal bundle we have a cochain map

$$J_X^P : \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}[-1] \to \mathrm{C}^*_{\mathrm{loc}}(\mathscr{A}\mathrm{d}(P)(X))$$

given by the same formula as in (3.2).

If $\mathfrak{g}$ is the Lie algebra of a group G, there is an interpretation of the space of extensions $\mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$ in terms of G.

Proposition 3.1.12. [?] *Let G be an affine algebraic group scheme (such as $\mathrm{GL}_n(\mathbb{C})$). Then, there is an isomorphism*

$$H^i(BG, \Omega^j) \cong H^{i-j}(G, \mathrm{Sym}^j(\mathfrak{g}^*)).$$

In the case that $i = j = d+1$ we find that $H^{d+1}(BG, \Omega^{d+1}) = \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$. Thus the central extensions have a familiar interpretation in terms of the Dolbeualt cohomology of BG.

Remark 3.1.13. When $\mathfrak{g}$ is a an arbitrary dg Lie algebra, or more generally an L_∞ algebra, we have encountered a version of $J_X(\theta)$ in Section 2.4.1. We showed that for any L_∞ algebra there is a map of cochain complexes $J : \Omega^{d+1}_{cl}(B\mathfrak{g})[d] \to \mathrm{C}^*_{\mathrm{loc}}(\mathscr{G}_X)$. The expression for J_X in (3.2) is an explicit formula for this construction in the case that $\mathfrak{g}$ is an ordinary Lie algebra. Indeed, when $\mathfrak{g}$ is an ordinary Lie algebra we have $H^{d+1}(\Omega^{d+1}_{cl}(B\mathfrak{g})) = \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$, so the construction in Section 2.4.1.

On $X = \mathbb{C}^d$ the functional $J_{\mathbb{C}^d}$ gives us the following complete description of a natural subcomplex of local cochains. On $\mathbb{C}^d$ exists a natural action by the group $U(d)$, where $U(d)$ acts in the defining way on $\mathbb{C}^d$. Moreover, there since $\mathscr{G}_{\mathbb{C}^d}$ is built from the Dolbeault complex on $\mathbb{C}^d$ there is an action of the dg Lie algebra $\mathbb{C}^{2d|d}$ controling holomorphic translations, see Section **??**. For each $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$ the functional $J_{\mathbb{C}^d}(\theta)$ is invariant for $U(d)$ and $\mathbb{C}^{2d|d}$. In fact, this describes up to quasi-isomorphism all such functionals.

Proposition 3.1.14. *The map* $J_{\mathbb{C}^d} : \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}[-1] \to \mathrm{C}^*_{\mathrm{loc}}(\mathscr{G}_{\mathbb{C}^d})$ *factors through the subcomplex of local cochains that are holomorphically translation invariant and invariant for the group* $U(d)$ *to define a quasi-isomorphism*

$$J_{\mathbb{C}^d} : \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}[-1] \xrightarrow{\simeq} \left(\mathrm{C}^*_{\mathrm{loc}}(\mathscr{G}(\mathbb{C}^d))^{\mathbb{C}^{2d|d}}\right)^{U(d)}$$

This is a special case of Theorem **??** in the case that $\mathfrak{g}$ is an ordinary Lie algebra. We refer the reader to that section for details.

3.1.4. The Kac-Moody factorization algebra

Finally, we can define the central object of this paper.

Definition 3.1.15. Let X be any complex manifold of dimension d equipped with a principal G-bundle P. Moreover, suppose $\Theta \in \mathrm{C}^*_{\mathrm{loc}}(\mathscr{A}\mathrm{d}(P))$ is a local cocycle of degree $+1$. The *Kac-Moody factorization algebra on* X *of type* Θ is the twisted factorization envelope

$$\mathbb{U}_\Theta(\mathscr{A}\mathrm{d}(P)) : U \subset X \mapsto \left(\mathrm{Sym}\left(\Omega^{0,*}_c(U, \mathrm{ad}(P))[1]\right)[K], \overline{\partial} + \mathrm{d}_{CE} + \Theta\right).$$

When $\Theta = J^P_X(\theta)$ for $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$ we denote this by $\mathbb{U}_\theta(\mathscr{A}\mathrm{d}(P)) = \mathbb{U}_{J^P_X(\theta)}(\mathscr{A}\mathrm{d}(P))$.

As in the definition of twisted factorization enveloping algebras above, the factorization algebras $\mathbb{U}_\Theta(\mathscr{A}\mathrm{d}(P))$ take values in dg modules for the ring $\mathbb{C}[K]$. In keeping with conventions above, when P is the trivial bundle on X we will denote the Kac-Moody factorization algebra by $\mathbb{U}_\Theta(\mathscr{G}_X)$.

Remark 3.1.16. For fixed θ the cocycle $J_X(\theta)$ is more-or-less independent of the complex manifold X. In this way, the factorization algebra $\mathbb{U}_\theta(\mathscr{G})$ actually defines a factorization algebra on the entire *category* of complex manifolds of a fixed dimension. We will not explore this type of *universal* factorization algebra here, but leave it to future work.

3.2. Local structures of the Kac-Moody factorization algebra

The theory of factorization algebras we study here, and whose foundations have been layed out in [**CG17**], is largely motivated by the study of chiral algebras due to Beilinson and Drinfeld [**BD04**]. Part of their original goal was to develop a geometric counterpart to the algebraic theory of vertex algebras. In [**CG17**] the relationship between factorization algebras on vertex algebras has been made completely explicit.

Every holomorphically translation invariant factorization algebra on the complex manifold $X = \mathbb{C}$ determines the structure of a vertex algebra. The underlying vector space, or state space, of the vertex algebra is given by the value of the factorization algebra assigns to a disk $D \subset \mathbb{C}$. The operator product expansion is encoded by the factorization product of configurations of disjoint disks inside of larger disks. It is shown that the Kac-Moody factorization algebra $\mathbb{U}_\kappa(\mathscr{G}_\mathbb{C})$ on $\mathbb{C}$, where κ is a symmetric invariant bilinear form, recovers the Kac-Moody vertex algebra in this way.

The fundamental structure we want contemplate comes from considering the factorization product for a different flavor of configurations of open sets. Suppose that $\mathscr{F}$ is a holomorphically translation invariant factorization algebra on $\mathbb{C}$. Consider a disk D that is completely encircled by an annulus Ann. Further, these disjoint open sets embed inside of a bigger disk D_{big} of the same center as in Figure BW: fig. The structure map is of the form

$$\mathscr{F}(\mathrm{Ann}) \otimes \mathscr{F}(D) \to \mathscr{F}(D_{big}).$$

If the ... BW: finish

The annular algebra $\mathscr{A}$ that we just discussed in the complex one-dimensional case has a generalization to arbitrary dimensions.The higher dimensional versions of annuli we consider are given by open sets equal to neighborhoods of $(2d-1)$-spheres. In this section we describe the higher dimensional version of this annular algebra for the Kac-Moody factorization algebra. This amounts to specializing the factorization algebra to the complex manifold $X = \mathbb{C}^d \setminus \{0\}$ and extracting the data of an A_∞-algebra from the factorization product in the radial direction. The reduction of the factorization algebra along $S^{2d-1} \subset \mathbb{C}^d \setminus \{0\}$ produces a one-dimensional factorization algebra via pushing forward along the radial projection map $\mathbb{C}^d \setminus \{0\} \to \mathbb{R}_{>0}$. Embedded inside of this factorization algebra is a locally constant factorization algebra, which will define for us our A_∞-algebra. Furthermore, we show how the factorization product of disks with higher dimensional annuli provide the structure a (A_∞-)module on the value of the factorization algebra on the disk.

We will recognize this A_∞-algebra as the universal enveloping algebra of an L_∞ algebra which is obtained as a central extension of an algebraic version of the sphere algebra

$$\mathrm{Map}(S^{2d-1}, \mathfrak{g}). \tag{3.3}$$

When $d = 1$ there is an embedding $\mathfrak{g}[z, z^{-1}] \hookrightarrow C^\infty(S^1) \otimes \mathfrak{g} = \mathrm{Map}(S^1, \mathfrak{g})$, induced by the embedding of algebraic functions on punctured affine line inside of smooth functions on S^1. The affine algebras are given by extensions of algebraic loop algebra $\mathscr{O}^{alg}(\mathbb{A}^{1\times}) = \mathfrak{g}[z, z^{-1}]$ prescribed by a 2-cocycle involving the algebraic residue pairing. Note that this cocycle is *not* pulled back from any cocycle on $\mathscr{O}^{alg}(\mathbb{A}^1) \otimes \mathfrak{g} = \mathfrak{g}[z]$.

When $d > 1$ Hartog's theorem implies that the space of holomorphic functions on punctured affine space is the same as the space of holomorphic functions on affine space. The same holds for algebraic functions, so that $\mathscr{O}^{alg}(\mathbb{A}^{d\times}) \otimes \mathfrak{g} = \mathscr{O}^{alg}(\mathbb{A}^d) \otimes \mathfrak{g}$. In particular, the naive algebraic replacement $\mathscr{O}^{alg}(\mathbb{A}^{d\times}) \otimes \mathfrak{g}$ of (3.3) has no interesting central extensions. However, as opposed to the punctured line, the punctured affine space $\mathbb{A}^{d\times}$ has interesting higher cohomology.

The key idea is that we replace the commutative algebra $\mathscr{O}^{alg}(\mathbb{A}^{d\times})$ by the derived space of sections $\mathbb{R}\Gamma(\mathbb{A}^{d\times}, \mathscr{O})$. This complex has interesting cohomology and leads to nontrivial extensions of the dg Lie algebra $\mathbb{R}\Gamma(\mathbb{A}^{d\times}, \mathscr{O}) \otimes \mathfrak{g}$. Concretely, we will use a dg model A_d for $\mathbb{R}\Gamma(\mathbb{A}^{d\times}, \mathscr{O})$ due to [**FHK**] that is an algebraic analog of the tangential Dolbeault complex of the $(2d-1)$-sphere inside of the Dolbeault complex of $\mathbb{C}^d \setminus \{0\}$:

$$\Omega_b^{0,*}(S^{2d-1}) \subset \Omega^{0,*}(\mathbb{C}^d \setminus \{0\}).$$

See [**?**] for details on the definition of $\Omega_b^{0,*}(S^{2d-1})$. The degree zero part of $\Omega_b^{0,*}(S^{2d-1})$ is $C^\infty(S^{2d-1})$, so we can view $A_d \otimes \mathfrak{g}$ as a derived enhancement of the mapping space in (3.3).

The model A_d, by definition, has cohomology equal to the cohomology of $\mathbb{R}\Gamma(\mathbb{A}^{d\times}, \mathscr{O})$. In [**FHK**] they have studied a class of cocycles associated to elements $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$

that are algebraic analogs of the local cocycles we introduced in the previous section. The cocycle is of total cohomological degree +2 and so determines a central extension of $A_d \otimes \mathfrak{g}$ that we denote $\widehat{\mathfrak{g}}_{d,\theta}$. Our first main result is that our "higher annular algebra" of the Kac-Moody factorization algebra from the discussion above recovers this Lie algebra extension.

Theorem 3.2.1. *Let $\mathscr{F}_{1d}$ be the one-dimensional factorization algebra obtained by the reduction of the Kac-Moody factorization algebra $\mathbb{U}_\alpha\left(\mathscr{G}_{\mathbb{C}^d\setminus\{0\}}\right)$ along the sphere $S^{2d-1} \subset \mathbb{C}^d \setminus \{0\}$. There is a dense subfactorization algebra $\mathscr{F}^{lc}_{1d} \subset \mathscr{F}_{1d}$ that is locally constant. As a one-dimensional locally constant factorization algebra, $\mathscr{F}^{lc}_{1d}$ is equivalent to the A_∞-algebra $U(\widehat{\mathfrak{g}}_{d,\theta})$ of* [**FHK**].

In the final part of this section we specialize to the manifold $X = (\mathbb{C}\setminus\{0\})^d$. Note that when $d = 1$ this is the same as above the annular algebra, but for $d > 1$ this factorization algebra has a different flavor. We will show how to extract the data of an E_d-algebra from this configuration, and discuss its role in the theory of higher dimensional vertex algebras.

3.2.1. The higher sphere algebras

The affine algebra associated to a Lie algebra $\mathfrak{g}$ together with an invariant pairing κ is defined as a central extension of the loop algebra of $\mathfrak{g}$

$$\mathbb{C} \to \widehat{\mathfrak{g}}_\kappa \to Lg$$

where we use the algebraic loop algebra $L\mathfrak{g} = g[z, z^{-1}]$. The central extension is determined by the cocycle

$$(f \otimes X, g \otimes Y) \mapsto \oint f \mathrm{d}g \kappa(X, Y).$$

A natural generalization of the loop algebra is to generalize the circle S^1, which is equal to the units in $\mathbb{C}$, by the sphere S^{2d-1}, which is equal to the units in $\mathbb{C}^d$. That is, we work with a "sphere algebra" of maps from S^{2d-1} into $\mathfrak{g}$. For topologists, this direction might seem natural, but it may not seem too natural from the perspective of algebraic geometry. In particular, an algebro-geometric sphere is given by a punctured affine d-space $\mathbb{A}^{d\times} = \mathbb{A}^d \setminus \{0\}$ or a punctured formal d-disk, but every map from these spaces to $\mathfrak{g}$ extends to a map from $\mathbb{A}^d$ or the formal d-disk into $\mathfrak{g}$ (essentially, by Hartog's lemma). Thus, this direction seems fruitless, since naively there would be no interesting central extensions. The key to evading this issue is to work with the *derived* space of maps. Indeed, the sheaf cohomology of $\mathscr{O}$ on the punctured affine d-space is interesting.

This fact ought not to be too surprising: as a smooth manifold, punctured affine d-space is equivalent to $\mathbb{R}_{>0} \times S^{2d-1}$, and this equivalence manifests itself in the cohomology of the structure sheaf. Explicitly,

$$H^*(\mathbb{A}^{d\times}, \mathscr{O}^{alg}) = \begin{cases} 0, & * \neq 0, d-1 \\ \mathbb{C}[z_1, \dots, z_d], & * = 0 \\ \mathbb{C}[z_1^{-1}, \dots, z_d^{-1}]\frac{1}{z_1 \cdots z_d}, & * = d-1 \end{cases}$$

as one can show by direct computation (e.g., use the cover by the affine opens of the form $\mathbb{A}^d \setminus \{z_i = 0\}$). When $d = 1$, this recovers the usual Laurent series; and it is natural to view the above as the higher-dimensional analogue of the Laurent series, with the polar part now in degree $d-1$.

Hence, the derived global sections $\mathbb{R}\Gamma(\mathbb{A}^{d\times}, \mathscr{O})$ of $\mathscr{O}$ provide a homotopy-commutative algebra, and thus one obtains a homotopy-Lie algebra by tensoring with $\mathfrak{g}$, which we will call the sphere Lie algebra by analogy with the loop Lie algebra. One can then study central extensions of this homotopy-Lie algebra, which are analogous to the affine Kac-Moody Lie algebras. For explicit constructions, it is convenient to have a commutative dg algebra that models the derived global sections. It should be no surprise that we like to work with the Dolbeault complex. We will use this approach to relate the sphere Lie algebra and its extensions to the current algebras that we've already introduced.

An explicit dg model A_d for the derived global sections has been written down in [**FHK**] based on the Jouanolou method for resolving singularities. We have recalled its definition in the Appendix of this chapter.

We are interested in the dg Lie algebra $A_d \otimes \mathfrak{g}$. For any d and symmetric function $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$, in [**FHK**] they define the cocycle

$$\theta_{FHK} : (A_d \otimes \mathfrak{g})^{\otimes(d+1)} \to \mathbb{C} \quad , \quad a_0 \cdots a_d \mapsto \mathrm{Res}_{z=0}\theta(a_0, \mathrm{d}a_1, \ldots, \mathrm{d}a_d),$$

where d is the algebraic de Rham differential. It is immediate that this cocycle has cohomological degree $+2$ and so determines a(n) (unshifted) dg Lie algebra central extension of $A_d \otimes \mathfrak{g}$:

$$\mathbb{C} \to \widehat{\mathfrak{g}}_{d,\theta} \to A_d \otimes \mathfrak{g}. \tag{3.4}$$

Our aim is to show how the Kac-Moody factorization algebra is related to this dg Lie algebra.

3.2.2. The strategy

We consider the restriction of the factorization algebra $\mathbb{U}_\theta(\mathscr{G})$ on $\mathbb{C}^d \setminus \{0\}$ to the collection of open sets diffeomorphic to spherical shells. This restriction has the structure of a one-dimensional factorization algebra corresponding to the iterated nesting of spherical shells. We show that there is a dense subfactorization algebra that is locally constant, hence corresponds to an E_1 algebra. We conclude by identifying an A_∞ model for this algebra as the universal enveloping algebra of a certain L_∞ algebra, that agree with the higher dimensional affine algebras of [**FHK**]

Introduce the radial projection map

$$\rho : \mathbb{C}^d \setminus 0 \to \mathbb{R}_{>0}$$

sending $z = (z_1, \ldots, z_d)$ to $|z| = \sqrt{|z_1|^2 + \cdots + |z_d|^2}$. We will restrict our factorization algebra to spherical shells by pushing forward the factorization algebra along this map. Indeed, the preimage of an open interval is such a spherical shell, and the factorization product on the line is equivalent to the nesting of shells.

3.2.2.1. The case of zero level. First we will consider the higher Kac-Moody factorization algebra on $\mathbb{C}^d \setminus \{0\}$ "at level zero". That is, the factorization algebra $\mathbb{U}(\mathscr{G}_{\mathbb{C}^d \setminus \{0\}}$. In this section we will omit $\mathbb{C}^d \setminus \{0\}$ from the notation, and simply refer to the factorization algebra by $\mathbb{U}(\mathscr{G})$.

Let $\rho_*(\mathbb{U}\mathscr{G})$ be the factorization algebra on $\mathbb{R}_{>0}$ obtained by pushing forward along the radial projection map. Explicitly, to an open set $I \subset \mathbb{R}_{>0}$ this factorization algebra assigns the dg vector space

$$\mathrm{C}_*^{\mathrm{Lie}}\left(\Omega_c^{0,*}(\rho^{-1}(I)) \otimes \mathfrak{g}\right).$$

Let $I \subset \mathbb{R}_{>0}$ be an open subset. There is the natural map $\rho^* : \Omega_c^*(I) \to \Omega_c^*(\rho^{-1}(I))$ given by the pull back of differential forms. We can post compose this with the natural projection $\mathrm{pr}_{\Omega^{0,*}} : \Omega_c^* \to \Omega_c^{0,*}$ to obtain a map of commutative algebras $\mathrm{pr}_{\Omega^{0,*}} \circ \rho^* : \Omega_c^*(I) \to \Omega_c^{0,*}(\rho^{-1}(I))$. The map j from Proposition .1.2 determines a map of dg commutative algebras $j : A_d \to \Omega^{0,*}(\rho^{-1}(I))$. Thus, we obtain a map

$$\begin{aligned} \Phi(I) = (\mathrm{pr}_{\Omega^{0,*}} \circ \rho^*) \otimes j : \quad \Omega_c^*(I) \otimes A_d &\to \Omega_c^{0,*}((\rho^{-1}(I)) \\ \varphi \otimes a &\mapsto ((\mathrm{pr}_{\Omega^{0,*}} \circ \rho^*)\varphi) \wedge j(a) \end{aligned} \tag{3.5}$$

Since this is a map of commutative dg algebras it defines a map of dg Lie algebras

$$\Phi(I) \otimes \mathrm{id}_{\mathfrak{g}} : (\Omega_c^*(I) \otimes A_d) \otimes \mathfrak{g} = \Omega_c^*(I) \otimes (A_d \otimes \mathfrak{g}) \to \Omega^{0,*}(\rho^{-1}(I)) \otimes \mathfrak{g}$$

which maps $(\varphi \otimes a) \otimes X \mapsto \Phi(\varphi \otimes a) \otimes X$. We will drop the $\mathrm{id}_{\mathfrak{g}}$ from the notation and will denote this map simply by $\Phi(I)$. Note that $\Phi(I)$ is compatible with inclusions of open sets, hence extends to a map of cosheaves of dg Lie algebras that we will call Φ.

We can summarize the results as follows.

Proposition 3.2.2. *The map Φ extends to a map of factorization Lie algebras*

$$\Phi : \Omega_{\mathbb{R}_{>0},c}^* \otimes (A_d \otimes \mathfrak{g}) \to \rho_*\mathscr{G}.$$

Hence, it defines a map of factorization algebras

$$\mathrm{C}_*(\Phi) : U^{fact}\left(\Omega^*_{\mathbb{R}_{>0}} \otimes (A_d \otimes \mathfrak{g})\right) \to \rho_*\left(\mathbb{U}\mathscr{G}\right).$$

The fact that we obtain a map of factorization algebras follows from applying the functor $\mathrm{C}_*^{\mathrm{Lie}}(-)$ to Φ. It is immediate to see that this functor commutes with push-forward.

3.2.2.2. The case of non-zero level. We now proceed to the proof of Theorem . The dg Lie algebra $\mathfrak{g}_{d,\theta}$ determines a dg associative algebra via its universal enveloping algebra $U(\mathfrak{g}_{d,\theta})$. This dg algebra determines a factorization algebra on the one-manifold $\mathbb{R}_{>0}$ that assigns to every open interval $I \subset \mathbb{R}_{>0}$ the dg vector space $U(A_d \otimes \mathfrak{g})$. The factorization product is uniquely determined by the algebra structure. Henceforth, we denote this factorization algebra by $U(\mathfrak{g}_{d,\theta})^{fact}$.

To prove the theorem we will construct a sequence of maps of factorization Lie algebras on $\mathbb{R}_{>0}$:

$$\begin{tikzcd} & \mathscr{G}_1 \arrow[dr, "\Phi_1"] & & \mathscr{G}_2 \\ \mathscr{G}_0 \arrow[ur, "\simeq", "\Phi_0"'] & & \mathscr{G}_1' \arrow[ur, "\Phi_2"'] & . \end{tikzcd}$$

The factorization envelope of $\mathscr{G}_0$ is equivalent to the factorization algebra $U(\widehat{\mathfrak{g}}_{d,\theta})^{fact}$. Moreover, the factorization envelope of $\mathscr{G}_2$ is the push-forward of of the higher Kac-Moody factorization algebra $\rho_*\mathbb{U}\mathscr{G}$. Hence, the desired map of factorization algebras is produced by applying the factorization envelope functor to the above composition of factorization Lie algebras.

First, we introduce the factorization Lie algebra $\mathscr{G}_0$. To an open set $I \subset \mathbb{R}$, it assigns the dg Lie algebra $\mathscr{G}_0(I) = \Omega^*_c(I) \otimes \widehat{\mathfrak{g}}_{d,\theta}$, where $\widehat{\mathfrak{g}}_{d,\theta}$ is the central extension from Equation (3.4). The differential and Lie bracket are determined by the fact that we are tensoring a commutative dg algebra with a dg Lie algebra. A slight variant of Proposition 3.4.0.1 in [**CG17**], which shows that the one-dimensional factorization envelope of an ordinary Lie algebra produces its ordinary universal enveloping algebra, shows that there is a quasi-isomorphism of factorization algebras on $\mathbb{R}$,

$$(U\widehat{\mathfrak{g}}_{d,\theta})^{fact} \xrightarrow{\simeq} \mathrm{C}^{\mathrm{Lie}}_*(\mathscr{G}_0).$$

The factorization Lie algebra $\mathscr{G}_0$ is a central extension of the factorization Lie algebra $\Omega^*_{\mathbb{R},c} \otimes (A_d \otimes \mathfrak{g})$ by the trivial module $\Omega^*_c \oplus \mathbb{C} \cdot K$. Indeed, the cocycle determining the central extension is given by

$$\theta_0(\varphi_0\alpha_0, \ldots, \varphi_d\alpha_d) = (\varphi_0 \wedge \cdots \wedge \varphi_d)\theta_{A_d}(\alpha_1, \ldots, \alpha_d).$$

The factorization Lie algebra $\Omega^*_{\mathbb{R},c} \otimes (A_d \otimes \mathfrak{g})$ is the compactly supported sections of the local Lie algebra $\Omega^*_{\mathbb{R}} \otimes (A_d \otimes \mathfrak{g})$ and this cocycle determining the extension is a local cocycle.

Next, we define the factorization dg Lie algebra $\mathscr{G}_1$ on $\mathbb{R}$. This is also obtained as a central extension of the factorization Lie algebra $\Omega^*_{\mathbb{R},c} \otimes (A_d \otimes \mathfrak{g})$:

$$0 \to \mathbb{C} \cdot K[-1] \to \mathscr{G}_1 \to \Omega^*_{\mathbb{R},c} \otimes (A_d \otimes \mathfrak{g}) \to 0$$

determined by the following cocycle. For an open interval I write $\varphi_i \in \Omega^*_c(I)$, $\alpha_i \in A_d \otimes \mathfrak{g}$. The cocycle is defined by

$$\theta_1(\varphi_0\alpha_0, \ldots, \varphi_d\alpha_d) = \left(\int_I \varphi_0 \wedge \cdots \varphi_d\right) \theta_{\mathrm{FHK}}(\alpha_0, \ldots, \alpha_d) \tag{3.6}$$

where θ_{FHK} was defined in Definition **??**.

The functional θ_1 determines a local cocycle in $\mathrm{C}^*_{\mathrm{loc}}\left(\Omega^*_{\mathbb{R}} \otimes (A_d \otimes \mathfrak{g})\right)$ of degree one.

We now define a map of factorization Lie algebras $\Phi_0 : \mathscr{G}_0 \to \mathscr{G}_1$. On and open set $I \subset \mathbb{R}$, we define the map $\Phi_0(I) : \mathscr{G}_0(I) \to \mathscr{G}_1(I)$ by

$$\Phi_0(I)(\varphi\alpha, \psi K) = \left(\varphi\alpha, \int \psi \cdot K\right).$$

For a fixed open set $I \subset \mathbb{R}$, the map Φ_0 fits into the commutative diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \Omega^*_c(I) \otimes \mathbb{C} \cdot K \arrow[r] \arrow[d, "\int", "\simeq"'] & \mathscr{G}_0(I) \arrow[r] \arrow[d, "\Phi_0(I)"] & \Omega^*_c(I) \otimes (A_d \otimes \mathfrak{g}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{C} \cdot K[-1] \arrow[r] & \mathscr{G}_1(I) \arrow[r] & \Omega^*_c(I) \otimes (A_d \otimes \mathfrak{g}) \arrow[r] & 0.
\end{tikzcd}$$

To see that $\Phi_0(I)$ is a map of dg Lie algebras we simply observe that the cocycles determining the central extensions are related by $\theta_1 = \int \circ\, \theta_0$, where $\int : \Omega^*_c(I) \to \mathbb{C}$ as in the diagram above. Since $\int$ is a quasi-isomorphism, the map $\Phi_0(I)$ is as well. It is clear that as we vary the interval I we obtain a quasi-isomorphism of factorization Lie algebras $\Phi_0 : \mathscr{G}_0 \xrightarrow{\simeq} \mathscr{G}_1$.

We now define the factorization dg Lie algebra $\mathscr{G}_1'$. Like $\mathscr{G}_0$ and $\mathscr{G}_0$, it is a central extension of $\Omega^*_{\mathbb{R},c} \otimes (A_d \otimes \mathfrak{g})$. The cocycle determining the central extension is defined by

$$\theta_1'(\varphi_0 a_0 X_0, \ldots, \ldots, \varphi_d a_d X_d) = \theta_1(\varphi_0 a_0 X_0, \ldots, \ldots, \varphi_d a_d X_d) + \widetilde{\theta}_1(\varphi_0 a_0 X_0, \ldots, \ldots, \varphi_d a_d X_d)$$

where θ_1 was defined in Equation (3.6). Before writing down the explicit formula for $\widetilde{\theta}_1$ we introduce some notation. Set

$$E = r\frac{\partial}{\partial r},$$
$$\mathrm{d}\vartheta = \sum_i \frac{\mathrm{d}z_i}{z_i}.$$

We view E as a vector field on $\mathbb{R}_{>0}$ and $\mathrm{d}\vartheta$ as a $(1,0)$-form on $\mathbb{C}^d \setminus 0$. Define the functional

$$\widetilde{\theta}_1(\varphi_0 a_0 X_0, \ldots, \varphi_d a_d X_d) = \frac{1}{2}\sum_{i=1}^d \left(\int_I \varphi_0 (E\cdot\varphi_i)\varphi_1 \cdots \widehat{\varphi_i} \cdots \varphi_d\right)\left(\oint (a_0 a_i \mathrm{d}\vartheta)\, \partial a_1 \cdots \widehat{\partial a_i} \cdots \partial a_d\right)\theta(X_0, \ldots$$

The functional $\widetilde{\theta}$ defines a local functional in $\mathrm{C}^*_{\mathrm{loc}}\left(\Omega^*_{\mathbb{R}_{>0}} \otimes (A_d \otimes \mathfrak{g})\right)$ of cohomological degree one. One immediately checks that it is a cocycle. This completes the definition of the factorization Lie algebra $\mathscr{G}_1'$.

The factorization Lie algebras $\mathscr{G}_1$ and $\mathscr{G}_1'$ are identical as precosheaves of vector spaces. In fact, if we put a filtration on $\mathscr{G}_1$ and $\mathscr{G}_1'$ where the central element K has filtration degree one, then the associated graded factorization Lie algebras Gr $\mathscr{G}_1$ and Gr $\mathscr{G}_1'$ are also identified. The only difference in the Lie algebra structures comes from the deformation of the cocycle determining the extension of $\mathscr{G}_1'$ given by $\widetilde{\theta}_1$.

In fact, we will show that $\widetilde{\theta}_1$ is actually an exact cocycle via the cobounding element $\eta \in \mathrm{C}^*_{\mathrm{loc}}\left(\Omega^*_{\mathbb{R}_{>0}} \otimes (A_d \otimes \mathfrak{g})\right)$ defined by

$$\eta(\varphi_0 a_0 X_0, \ldots, \varphi_d a_d X_d) = \sum_{i=1}^{d} \left(\int_I \varphi_0 \left(\iota_E \varphi_i\right) \varphi_1 \cdots \widehat{\varphi_i} \cdots \varphi_d \right) \left(\oint (a_0 a_i \mathrm{d}\vartheta)\, \partial a_1 \cdots \widehat{\partial a_i} \cdots \partial a_d \right) \theta(X_0, \ldots, X$$

Lemma 3.2.3. *One has* $\mathrm{d}\eta = \widetilde{\theta}_1$*, where* d *is the differential for the cochain complex* $\mathrm{C}^*_{\mathrm{loc}}(\Omega^*_{\mathbb{R}_{>0}} \otimes (A_d \otimes \mathfrak{g}))$*. In particular, the factorization Lie algebras* $\mathscr{G}_1$ *and* $\mathscr{G}_1'$ *are quasi-isomorphic (as* L_∞ *algebras). An explicit quasi-isomorphism is given by the* L_∞ *map* $\Phi_1 : \mathscr{G}_1 \to \mathscr{G}_1'$ *that sends the central element* K *to itself and an element* $(\varphi_0 a_0 X_0, \ldots, \varphi_d a_d X_d) \in \mathrm{Sym}^{d+1}(\Omega^*_c \otimes (A_d \otimes \mathfrak{g})$ *to*

$$(\varphi_0 a_0 X_0, \ldots, \varphi_d a_d X_d) + \eta(\varphi_0 a_0 X_0, \ldots, \varphi_d a_d X_d) \cdot K \in \mathrm{Sym}^{d+1}(\Omega^*_c \otimes (A_d \otimes \mathfrak{g})) \oplus \mathbb{C} \cdot K.$$

Finally, we define the factorization Lie algebra $\mathscr{G}_2$. We have already seen that the local cocycle $J(\theta) \in \mathrm{C}^*_{\mathrm{loc}}(\mathfrak{g}^{\mathbb{C}^d})$ determines a central extension of factorization Lie algebras

$$0 \to \mathbb{C} \cdot K[-1] \to \mathscr{G}_{J(\theta)} \to \Omega^{0,*}_{\mathbb{C}^d, c} \otimes \mathfrak{g} \to 0.$$

Of course, we can restrict $\mathscr{G}_{J(\theta)}$ to a factorization algebra on $\mathbb{C}^d \setminus 0$. The factorization algebra $\mathscr{G}_2$ is defined as the pushforward of this restriction along the radial projection: $\mathscr{G}_2 := \rho_* \left(\mathscr{G}_{J(\theta)}|_{\mathbb{C}^d \setminus 0} \right)$.

Recall the map $\Phi : \Omega^*_{\mathbb{R}_{>0}, c} \otimes (A_d \otimes \mathfrak{g}) \to \rho_*(\Omega^{0,*}_{\mathbb{C}^d \setminus 0, c} \otimes \mathfrak{g})$ defined in Equation (3.5). On each open set $I \subset \mathbb{R}_{>0}$ we can extend Φ by the identity on the central element to a linear map $\Phi_2 : \mathscr{G}_1'(I) \to \mathscr{G}_2(I)$.

Lemma 3.2.4. *The map $\Phi_2 : \mathscr{G}_1'(I) \to \mathscr{G}_2(I)$ is a map of dg Lie algebras. Moreover, it extends to a map of factorization Lie algebras $\Phi_2 : \mathscr{G}_1' \to \mathscr{G}_2$.*

Proof. Modulo the central element Φ_2 reduces to the map Φ, which we have already seen is a map of factorization Lie algebras in Proposition 3.2.2. Thus, to show that Φ_2 is a map of factorization Lie algebras we need to show that it is compatible with the cocycles determing the respective central extensions. That is, we need to show that

$$\theta_1'(\varphi_0 a_0 X_0, \ldots, \varphi_d a_d X_d) = \theta_2(\Phi(\varphi_0 a_0 X_0), \ldots, \Phi(\varphi_d a_d X_d)) \tag{3.7}$$

for all $\varphi_i a_i X_i \in \Omega^*_c(I) \otimes (A_d \otimes \mathfrak{g})$. The cocycle θ_1' is only nonzero if one of the φ_i inputs is a 1-form. We evaluate the left-hand side on the $(d+1)$-tuple $(\varphi_0 \mathrm{d}r a_0 X_0, \varphi_1 a_1 X_1, \ldots, \varphi_d a_d X_d)$ where $\varphi_i \in C^\infty_c(I)$, $a_i \in A_d$, $X_i \in \mathfrak{g}$ for $i = 0, \ldots, d$. The result is

$$\left(\int_I \varphi_0 \cdots \varphi_d \mathrm{d}r\right)\left(\oint a_0 \partial a_1 \cdots \partial a_d\right)\theta(X_0, \ldots, X_d) \tag{3.8}$$

$$\frac{1}{2}\sum_{i=1}^d \left(\int_I \varphi_0 (E \cdot \varphi_i) \varphi_1 \cdots \widehat{\varphi_i} \cdots \varphi_d \mathrm{d}r\right)\left(\oint (a_0 a_i \mathrm{d}\vartheta)\, \partial a_1 \cdots \widehat{\partial a_i} \cdots \partial a_d\right)\theta(X_0, \ldots, X_d) \tag{3.9}$$

We wish to compare this to the right-hand side of Equation (3.7). Recall that $\Phi(\varphi_0 \mathrm{d}r a_0 X_0) = \varphi(r)\mathrm{d}r a_0(z) X_0$ and $\Phi(\varphi_i a_i X_i) = \varphi(r) a_i(z) X_i$. Plugging this into the explicit formula for the cocycle θ_2 we see the right-hand side of (3.7) is

$$\left(\int_{\rho^{-1}(I)} \varphi_0(r)\mathrm{d}r a_0(z) \partial(\varphi_1(r) a_1(z)) \cdots \partial(\varphi_d(r) a_d(z))\right)\theta(X_0, \ldots, X_d). \tag{3.10}$$

We pick out the term in (3.10) in which the ∂ operators only act on the elements $a_i(z)$, $i = 1, \ldots, d$. This term is of the form

$$\int_{\rho^{-1}(I)} \varphi_0(r) \cdots \varphi_d(r) \mathrm{d}r a_0(z) \partial(a_1(z)) \cdots \partial(a_d(z)) \theta(X_0, \ldots, X_d).$$

Separating variables we find that this is precisely the first term (3.8) in the expansion of the left-hand side of (3.7).

Now, note that we can rewrite the ∂-operator in terms of the radius r as

$$\partial = \sum_{i=1}^{d} \mathrm{d}z_i \frac{\partial}{\partial z_i} = \sum_{i=1}^{d} \mathrm{d}z_i \overline{z}_i \frac{\partial}{\partial(r^2)} = \sum_{i=1}^{d} \mathrm{d}z_i \frac{r^2}{2z_i} \frac{\partial}{\partial r}.$$

The remaining terms in (3.10) correspond to the expansion of

$$\partial(\varphi_1(r) a_1(z)) \cdots \partial(\varphi_d(r) a_d(z)),$$

using the Leibniz rule, for which the ∂ operators act on at least one of the functions $\varphi_1, \ldots, \varphi_d$. In fact, only terms in which ∂ acts on precisely one of the functions $\varphi_1, \ldots, \varphi_d$ will be nonzero. For instance, consider the term

$$(\partial \varphi_1) a_1(z) (\partial \varphi_2) a_2(z) \partial(\varphi_3(z) a_3(z)) \cdots \partial(\varphi_d(z) a_d(z)). \tag{3.11}$$

Now, $\partial \varphi_i(r) = \omega \frac{\partial \varphi}{\partial r}$ where ω is the one-form $\sum_i (r^2/2z_i) \mathrm{d}z_i$. Thus, (3.11) is equal to

$$\left(\omega \frac{\partial \varphi_1}{\partial r}\right) a_1(z) \left(\omega \frac{\partial \varphi_2}{\partial r}\right) a_2(z) \partial(\varphi_3(z) a_3(z)) \cdots \partial(\varphi_d(z) a_d(z),$$

which is clearly zero as ω appears twice.

We observe that terms in the expansion of (3.10) for which ∂ acts on precisely one of the functions $\varphi_1, \ldots, \varphi_d$ can be written as

$$\sum_{i=1}^d \int_{\rho^{-1}(I)} \varphi_0(r) \left(r \frac{\partial}{\partial r} \varphi_i(r) \right) \varphi_1(r) \cdots \widehat{\varphi_i(r)} \cdots \varphi_d(r) \mathrm{d}r \frac{r}{2 z_i} \mathrm{d}z_i a_0(z) a_i(z) \partial a_1(z) \cdots \widehat{\partial a_i(z)} \cdots \partial a_d(z).$$

Finally, notice that the function $z_i/2r$ is independent of the radius r. Thus, separating variables we find the integral can be written as

$$\frac{1}{2} \sum_{i=1}^d \left(\int_I \varphi_0 \left(r \frac{\partial}{\partial r} \varphi_i \right) \varphi_1 \cdots \widehat{\varphi_i} \cdots \varphi_d \mathrm{d}r \right) \left(\oint \frac{\mathrm{d}z_i}{z_i} a_0 a_i \partial a_2 \cdots \widehat{\partial a_i} \cdots \partial a_d \right).$$

This is precisely equal to the second term (3.9) above. Hence, the cocycles are compatible and the proof is complete.

□

3.2.3. An E_d algebra from tori

There is another direction that one may look to extend the notion of affine algebras to higher dimensions. The affine algebra is a central extension of the loop algebra on $\mathfrak{g}$. Instead of looking at higher dimensional sphere algebras, one can consider higher *torus* algebras; or iterated loop algebras:

$$L^d \mathfrak{g} = \mathbb{C}[z_1^{\pm}, \cdots, z_d^{\pm}] \otimes \mathfrak{g}.$$

These iterated loop algebras are algebraic versions of the torus mapping space $\mathrm{Map}(S^1 \times \cdots \times S^1, \mathfrak{g})$. In this section we show what information the Kac-Moody vertex algebra implies about extensions of such iterated loop algebras.

To do this we specialize the Kac-Moody factorization algebra to the complex manifold $(\mathbb{C}^\times)^d$, which is homotopy equivalent to the topologists torus $(S^1)^{\times d}$. We show, in a similar way as above, how to extract the structure of an E_d algebra from considering the nesting of "polyannuli" in $(\mathbb{C}^\times)^d$. These E_d-algebras are related to interesting extensions of the Lie algebra $L^d\mathfrak{g}$.

When $d = 1$, we have seen that the nesting of ordinary annuli give rise to the structure of an associative algebra. For $d > 1$, a polyannulus is a complex submanifold of the form $\mathrm{Ann}_1 \times \cdots \times \mathrm{Ann}_d \subset (\mathbb{C}^\times)^d$ where each $\mathrm{Ann}_i \subset \mathbb{C}^\times$ is an ordinary annulus. Equivalently, a polyannulus is the complement of a closed polydisk inside of a larger open polydisk. We will see how the nesting of annuli in each component gives rise to the structure of a locally constant factorization algebra in d *real* dimensions, and hence defines an E_d algebra.

A result of Knudsen [**BK04**], which we recall in Section **??**, states that every dg Lie algebra determines an E_d-algebra, for any $d > 1$, called the universal E_d enveloping algebra. To state its properties properly, we need to be in the context of ∞-categories.

Theorem 3.2.5 ([**BK04**]). *Let $\mathscr{C}$ be a stable, $\mathbb{C}$-linear, presentable, symmetric monoidal ∞-category. There is an adjunction*

$$U^{E^d} : \mathrm{LieAlg}(\mathscr{C}) \leftrightarrows E_d\mathrm{Alg}(\mathscr{C}) : F$$

such that for any object $X \in \mathscr{C}$ one has $\mathrm{Free}_{E_d}(X) \simeq U^{E_d}\mathrm{Free}_{Lie}(\Sigma^{d-1}X)$.

We are most interested in the case $\mathscr{C}$ is the category of chain complexes with tensor product $\mathrm{Ch}^\otimes$. In this situation, the enveloping algebra U^{E^d} agrees with the ordinary universal enveloping algebra when $d = 1$. When the twisting cocycle defining the Kac-Moody factorization algebra is zero we will see that the E_d algebra coming from the product of polyannuli is equivalent to $U^{E_d}(L^d\mathfrak{g})$. When we turn on a twisting cocycle we will find the E_d-enveloping algebra of a central extension of the iterated loop algebra.

The Kac-Moody factorization algebra on the d-fold $(\mathbb{C}^\times)^d$ determines a real d-dimensional factorization algebra by considering the radius in each complex direction. We denote this factorization algebra on $\mathbb{R}^d$ by $\vec{\rho}_* (\mathscr{G}_{\mathbb{C}^{\times d}})$.

To state our result precisely, let $\vec{\rho} : (\mathbb{C}^\times)^d \to (\mathbb{R}_{>0})^d$ be the projection $(z_1, \ldots, z_d) \mapsto (|z_1|, \cdots, |z_d|)$. We will pushforward the Kac-Moody factorization algebra along $\vec{\rho}$.

On the Lie algebra side, we see that the following formula defines a cocycle on $L^d\mathfrak{g}$ of degree $d + 1$:

$$\begin{array}{lccl} L^d\theta : & (L^d\mathfrak{g})^{\otimes d+1} & \to & \mathbb{C} \\ & (f_0 \otimes X_0) \otimes \cdots \otimes (f_d \otimes X_d) & \mapsto & \theta(X_0, \ldots, X_d) \oint_{|z_1|=1} \cdots \oint_{|z_d|=1} f_0 \mathrm{d}f_1 \cdots \mathrm{d}f_d. \end{array}$$

Here $f_i \otimes X_i \in L^d\mathfrak{g} = \mathbb{C}[z_1^{\pm}, \cdots, z_d^{\pm}] \otimes \mathfrak{g}$. The above is just an iterated version of the usual residue pairing. This cocycle determines a shifted Lie algebra extension of the iterated loop algebra

$$\mathbb{C}[d-1] \to \widehat{L^d\mathfrak{g}}_\theta \to L^d\mathfrak{g},$$

that appears in the theorem below.

The following can be proved in exact analogy as the above result for sphere algebras and we omit the proof here.

Proposition 3.2.6. *Fix* $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$ *and let* $\vec{\rho}_* \mathbb{U}_\theta \mathscr{G}_{(\mathbb{C}^\times)^d}$ *be the factorization algebra on* $(\mathbb{R}_{>0})^d$ *obtained by reducing the Kac-Moody factorization algebra along the d-torus. There exists a dense d-dimensional subfactorization algebra* $\mathscr{F}^{lc}$ *that is locally constant and is equivalent, as* E_d*-algebras, to*

$$U^{E_d}\left(\widehat{L^d \mathfrak{g}_\theta}\right).$$

3.2.4. The disk as a module

3.3. The Kac-Moody factorization algebra on general manifolds

In this section we explore global properties of the Kac-Moody factorization algebra on complex manifolds.

3.3.1. The P_0-structure

Every associative algebra determines a Lie algebra via the commutator. There is a left adjoint to this forgetful functor given by the enveloping algebra of a Lie algebra. Given a Lie algebra $\mathfrak{g}$, this enveloping algebra $U\mathfrak{g}$ can also be thought of as a quantization of a certain Poisson algebra. The Poincaré-Birkoff-Witt theorem says that the associated graded Gr $U\mathfrak{g}$ by the filtration given by symmetric degree is precisely $\mathbb{C}[\mathfrak{g}^*]$. It is a classical fact that the linear dual $\mathfrak{g}^*$ of a Lie algebra has the structure of a Poisson manifold. The Poisson bracket on $\mathbb{C}[\mathfrak{g}^*] = \mathrm{Sym}(\mathfrak{g})$ is defined by extending the Lie bracket on the quadratic functions by the Leibniz rule.

In a completely analogous way, the factorization enveloping algebra of a local Lie algebra has a "classical limit" given by a P_0 factorization algebra. Recall, the factorization enveloping algebra of a local Lie algebra $\mathscr{L}$ evaluated on an open set U is given by the Chevalley-Eilenberg complex of the compactly supported sections on U

$$\mathbb{U}(\mathscr{L})(U) = \mathrm{C}_*^{\mathrm{Lie}}(\mathscr{L}(U)) = (\mathrm{Sym}^*(\mathscr{L}(U)[1]), \mathrm{d}_{\mathscr{L}} + \mathrm{d}_{CE}) .$$

There is a filtration of this complex defined by $F^k = \mathrm{Sym}^{\geq k}(\mathscr{L}(U)[1])$. Moreover, this defines a filtration of the factorization algebra $\mathbb{U}(\mathscr{L})$.

Lemma 3.3.1. *Let $\mathscr{L}$ be a local Lie algebra. Then, the associated graded factorization algebra* $\mathrm{Gr}\ \mathbb{U}(\mathscr{L})$ *has the structure of a P_0 factorization algebra. Similarly, if $\alpha \in \mathrm{C}^*_{\mathrm{loc}}(\mathscr{L})$ is a cocycle of cohomological degree one then* $\mathrm{Gr}\ \mathbb{U}_\alpha(\mathscr{L})$ *has the structure of a P_0 factorization algebra.*

Up to issues of functional analysis, one should think of the P_0 algebra $\mathrm{Gr}\ \mathbb{U}(\mathscr{L})$ as the algebra of functions on the sheaf of dg vector spaces $\mathscr{L}^\vee[-1]$ with differential induced from that on $\mathscr{L}$. The P_0 algebra $\mathrm{Gr}\ \mathbb{U}_\alpha(\mathscr{L})$ is equal to functions on the same sheaf of dg vector spaces but with bracket modified by α.

Corollary 3.3.2. *For any principal G-bundle $P \to X$ consider the associated graded factorization algebra*

$$\mathrm{Gr}\ \mathbb{U}(\mathscr{A}\mathrm{d}(P)) : U \mapsto \left(\mathrm{Sym}^*(\Omega^{0,*}_c(U)[1]), \overline{\partial}\right) .$$

Then, any element $\alpha \in H^1_{\mathrm{loc}}(\mathscr{A}\mathrm{d}(P))$ determines the structure of a P_0 factorization algebra on $\mathrm{Gr}\ \mathbb{U}(\mathscr{A}\mathrm{d}(P))$.

In the case that $\alpha = J_X(\theta)$ is the local cocycle corresponding to a symmetric polynomial $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$ the Poisson structure can be described explicitly as follows. The Poisson tensor is of the form $\Pi = \Pi_{[-,-]} + \Pi_\theta$ where

$$\Pi_{[-,-]} = \wedge \otimes [-,-] : \left(\Omega_X^{d,*} \otimes \mathfrak{g}\right) \otimes \left(\Omega_X^{0,*} \otimes \mathfrak{g}\right) \to \Omega_X^{d,*} \otimes \mathfrak{g}$$

and

$$\Pi_\theta : \left(\Omega_X^{0,*} \otimes \mathfrak{g}\right)^{\otimes d} \to \Omega_X^{d,*} \otimes \mathfrak{g}$$

sends $\alpha_1 \otimes \cdots \otimes \alpha_d \mapsto \partial\alpha_1 \wedge \cdots \wedge \partial\alpha_d$.

3.3.2. Factorization homology along Hopf surfaces

Proposition 3.3.3. *Let X be a Hopf manifold and suppose $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$ is any $\mathfrak{g}$-invariant polynomial of degree $(d+1)$. Then, there is a quasi-isomorphism*

$$\int_X \mathbb{U}_\theta(\mathscr{G}_X) \simeq \mathrm{Hoch}_*(U\mathfrak{g})[K]$$

where K is the central parameter of cohomological degree zero.

Proof. Let's first consider the untwisted case. In this case, we must show $\int_X \mathbb{U}(\mathscr{G}_X) \simeq \mathrm{Hoch}_*(U\mathfrak{g})$. The factorization homology on the left hand side is computed by

$$\int_X \mathbb{U}(\mathscr{G}_X) = \mathrm{C}_*^{\mathrm{Lie}}(\Omega^{0,*}(X) \otimes \mathfrak{g}).$$

We have already seen in Section ?? that every Hopf manifold is Dolbeualt formal. Thus, there is a quasi-isomorphism

$$\left(H^{0,*}(X), 0\right) \simeq \left(\Omega^{0,*}(X), \overline{\partial}\right).$$

In fact, we have written down a preferred presentation for the cohomology ring of X given by $H^{0,*}(X) = \mathbb{C}[\delta]$ where $|\delta| = 1$. A particular Dolbeault representative for δ given by

$$\overline{\partial}(\log |z|^2) = \sum_i \frac{z_i \mathrm{d}\overline{z}_i}{|z|^2}$$

where $z = (z_1, \ldots, z_d)$ is the coordinate on $\mathbb{C}^d \setminus \{0\}$.

Applied to the global sections of the Kac-Moody we see that there is a quasi-isomorphism

$$\int_X \mathbb{U}(\mathscr{G}_X) \simeq \mathrm{C}_*^{\mathrm{Lie}}(\mathbb{C}[\delta] \otimes \mathfrak{g}).$$

Now, note that $\mathrm{C}_*^{\mathrm{Lie}}(\mathbb{C}[\delta] \otimes \mathfrak{g}) = \mathrm{C}_*^{\mathrm{Lie}}(\mathfrak{g} \oplus \mathfrak{g}[-1]) = \mathrm{C}_*^{\mathrm{Lie}}(\mathfrak{g}, \mathrm{Sym}(\mathfrak{g}))$, where $\mathrm{Sym}(\mathfrak{g})$ is the symmetric product of the adjoint action of $\mathfrak{g}$ on itself. By Poincaré-Birkoff-Witt there is an isomorphism of vector spaces $\mathrm{Sym}(\mathfrak{g}) = U\mathfrak{g}$, so we can write this as $\mathrm{C}_*^{\mathrm{Lie}}(\mathfrak{g}, \mathrm{Sym}(\mathfrak{g}))$.

Now, any $U(\mathfrak{g})$-bimodule M is automatically a module for the Lie algebra $\mathfrak{g}$ by the formula $x \cdot m = xm - mx$ where $x \in \mathfrak{g}$ and $m \in M$. Moreover, for any such bimodule there is a quasi-isomorphism of cochain complexes

$$\mathrm{C}_*^{\mathrm{Lie}}(\mathfrak{g}, M) \simeq \mathrm{Hoch}_*(U\mathfrak{g}, M).$$

This is proved, for instance, in Section 2.3 of [?]. Applied to the bimodule $M = U\mathfrak{g}$ itself we obtain $\mathrm{C}^{\mathrm{Lie}}(\mathfrak{g}, U\mathfrak{g}) \simeq \mathrm{Hoch}(U\mathfrak{g})$.

The twisted case is similar. Let θ be as in the statement. Then, the factorization homology is equal to

$$\int_X \mathbb{U}_\theta(\mathscr{G}_X) = \left(\mathrm{Sym}(\Omega^{0,*}(X)\otimes\mathfrak{g})[K], \overline{\partial} + \mathrm{d}_{CE} + \mathrm{d}_\theta\right).$$

Applying Dolbeualt formality again we see that this is quasi-isomorphic to the cochain complex

$$(\mathrm{Sym}(\mathfrak{g}[\delta])[K], \mathrm{d}_{CE} + \mathrm{d}_\theta)\,. \tag{3.12}$$

We note that d_θ is identically zero on $\mathrm{Sym}(\mathfrak{g}[\delta])$. Indeed, for degree reasons, at least one of the inputs must be from $\mathfrak{g} \hookrightarrow \mathfrak{g}[\delta] = \mathfrak{g} \oplus \mathfrak{g}[-1]$, which consists of constant functions on X with values in the Lie algebra $\mathfrak{g}$. In the formula for the local cocycle (**??**) associated to θ it is clear that if any one of the inputs is constant the cocycle vanishes. Indeed, one can integrate by parts to put it in the form $\int \partial\alpha\cdots\partial\alpha$, which is the integral of a total derivative, hence zero since X has no boundary. Thus (3.12) just becomes the Chevalley-Eilenberg complex with values in the trivial module $\mathbb{C}[K]$. By the same argument as in the untwisted case, we conclude that in this case the factorization homology is quasi-isomorphic to $\mathrm{Hoch}_*(U\mathfrak{g})[K]$ as desired. $\square$

There is an interesting consequence of this calculation to the Hochschild homology for the A_∞ algebra $U(\widehat{\mathfrak{g}}_{d,\theta})$. It is easiest to state this when X is a Hopf manifold of the form $(\mathbb{C}^d \setminus \{0\})/q^{\mathbb{Z}}$ for a single $q \in D(0,1)^\times$ where the quotient is by the relation $(z_1,\ldots,z_d) \simeq (q^{\mathbb{Z}}z_1,\ldots,q^{\mathbb{Z}})$. Let $p_q : \mathbb{C}^d \setminus \{0\} \to X$ be the quotient map. Consider the following diagram

$$\begin{tikzcd} \mathbb{C}^d \setminus \{0\} \arrow[r, "p_q"] \arrow[d, "\rho"] & X \arrow[d, "\overline{\rho}"] \\ \mathbb{R}_{>0} \arrow[r, "\overline{p}_q"] & S^1 \end{tikzcd}$$

Here, ρ is the radial projection map and $\overline{\rho}$ is the induced map defined by the quotient. The action of $\mathbb{Z}$ on $\mathbb{C}^d \setminus \{0\}$ gives $\mathscr{G}_{\mathbb{C}^d \setminus \{0\}}$ the structure of a $\mathbb{Z}$-equivariant factorization algebra. In turn, this determines an action of $\mathbb{Z}$ on pushforward factorization algebra $\rho_* \mathscr{G}_{\mathbb{C}^d \setminus \{0\}}$. We have seen that there is a dense locally constant subfactorization algebra on $\mathbb{R}_{>0}$ of the pushforward that is equivalent as an E_1 algebra to $U(\widehat{\mathfrak{g}}_{d,\theta})$. A consequence of excision for factorization homology, see Lemma 3.18 [**?**], implies that there is a quasi-isomorphism

$$\mathrm{Hoch}_*(U(\widehat{\mathfrak{g}}_{d,\theta}), q) \simeq \int_{S^1} \overline{\rho}_* \mathbb{U}_\alpha(\mathscr{G}_X),$$

where the right-hand side is the Hochschild homology of the algebra $U\widehat{\mathfrak{g}}_{d,\theta}$ with coefficients in the bimodule $U\widehat{\mathfrak{g}}_{d,\theta}$ with the ordinary left-module structure and right-module structure given by twisting the ordinary action by the automorphism corresponding to the element $1 \in \mathbb{Z}$ on the algebra.

Moreover, by the push-forward for factorization homology, Proposition 3.23 [**?**], there is an equivalence

$$\int_{S^1} \overline{\rho}_* \mathbb{U}_\alpha(\mathscr{G}_X) \xrightarrow{\simeq} \int_X \mathbb{U}_\alpha(\mathscr{G}_X).$$

We have just shown that the factorization homology of $\mathscr{G}_X$ is equal to the Hochschild homology of $U\mathfrak{g}$ so that

$$\mathrm{Hoch}_*(U(\mathfrak{g}_{d,\theta}), q) \simeq \mathrm{Hoch}_*(U\mathfrak{g})[K].$$

This statement is purely algebraic as the dependence on the manifold for which the Kac-Moody lives has dropped out. It may be easiest to understand in the case $d = 1$ and $\theta = 0$. Then $\mathfrak{g}_{d,\theta}$ is simply the loop algebra $L\mathfrak{g} = g[z, z^{-1}]$. The action of $\mathbb{Z}$ on $L\mathfrak{g}$ simply rotates the loop parameter: for $z^n \otimes \mathfrak{g} \in L\mathfrak{g} = \mathbb{C}[z, z^{-1}] \otimes \mathfrak{g}$ the action of $1 \in \mathbb{Z}$ is $1 \cdot (z^n \otimes \mathfrak{g}) = q^n z^n \otimes \mathfrak{g}$. In turn, the bimodule structure of $U(\mathfrak{g}[z, z^{-1}])$ on itself, which we denote $U(\mathfrak{g}[z, z^{-1}])_q$ is the ordinary one on the left and on the right is given by twisting by the automorphism corresponding to $1 \in \mathbb{Z}$. The complex $\mathrm{Hoch}_*(U(g[z, z^{-1}]), q)$ is the Hochschild homology of $U(\mathfrak{g}[z, z^{-1}])$ with values in this bimodule so the statement implies

$$\mathrm{Hoch}_* \left(U(\mathfrak{g}[z, z^{-1}]), U(\mathfrak{g}[z, z^{-1}])_q\right) \simeq \mathrm{Hoch}(U\mathfrak{g}).$$

3.3.3. A variant of the factorization algebra

So far we have mostly restricted ourselves to studying the Kac-Moody factorization algebra corresponding to local cocycles of type $J_X(\theta)$ where $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$. There is another class of local cocycles that appear when studying symmetries of holomorphic theories. Unlike the cocycle $J_X(\theta)$, which in some sense did not depend on the manifold X, this class of cocycles is more dependent on the manifold for which the current algebra lives.

Let X be a complex manifold of dimension d and suppose ω is a (k, k) form on X. Fix, in addition, a form $\theta_{d+1-k} \in \mathrm{Sym}(\mathfrak{g}^*)^{\mathfrak{g}}$. Then, we may consider the cochain on $\mathscr{G}(X)$:

$$\begin{array}{rccc} \phi_{\theta,\omega} : & \mathscr{G}(X)^{\otimes d+1-k} & \to & \mathbb{C} \\ & \alpha_0 \otimes \cdots \otimes \alpha_{d-k} & \mapsto & \displaystyle\int_X \omega \wedge \theta_{d+1-k}(\alpha_0, \partial\alpha_1, \ldots, \partial\alpha_{d-k}) \end{array}$$

It is clear that $\phi_{\theta,\omega}$ is a local cochain on $\mathscr{G}(X)$.

Lemma 3.3.4. *Let $\theta \in \mathrm{Sym}^{d+1-k}(\mathfrak{g}^*)^{\mathfrak{g}}$ and suppose $\omega \in \Omega^{k,k}(X)$ satisfies $\overline{\partial}\omega = 0$ and $\partial\omega = 0$. Then, $\phi_{\theta,\omega} \in \mathrm{C}^*_{\mathrm{loc}}(\mathscr{G}_X)$ is a local cocycle. Moreover, for fixed θ the cohomology class $[\phi_{\theta,\omega}] \in H^1_{\mathrm{loc}}(\mathscr{G}_X)$ only depends on the cohomology class*

$$[\omega] \in H^k(X, \Omega^k_{cl}).$$

Note that when $\omega = 1$ it trivially satisfies the conditions of the lemma. In this case $\phi_{\theta,1} = J_X(\theta)$ in the notation of the last section.

This class of cocycles is related to the ordinary Kac-Moody factorization and vertex algebra on Riemann surfaces in a natural way. We consider the complex manifold $X = \Sigma \times \mathbb{P}^{d-1}$ where Σ is a Riemann surface and $\mathbb{P}^{d-1}$ is $(d-1)$-dimensional complex projective space. Suppose that $\omega \in \Omega^{d-1,d-1}(\mathbb{P}^{d-1})$ is the natural volume form, this clearly satisfies the conditions of Lemma 3.3.4 and so determines a degree one cocycle $\phi_{\kappa,\omega} \in \mathrm{C}^*_{\mathrm{loc}}(\mathscr{G}_{\Sigma\times\mathbb{P}^{d-1}})$ where κ is some $\mathfrak{g}$-invariant bilinear form $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$. We can then consider the twisted factorization enveloping algebra of $\mathscr{G}_{\Sigma\times\mathbb{P}^{d-1}}$ by the cocycle $\phi_{\kappa,\omega}$.

Recall that if $p : X \to Y$ and $\mathscr{F}$ is a factorization algebra on X, then the pushforward $p_*\mathscr{F}$ on Y is defined on opens by $p_*\mathscr{F} : U \subset Y \mapsto \mathscr{F}(p^{-1}U)$.

Proposition 3.3.5. *Let $\pi : \Sigma \times \mathbb{P}^{d-1} \to \Sigma$ be the projection. Then, there is an isomorphism of factorization algebras on Σ from the pushforward along π of the Kac-Moody factorization algebra on $\Sigma \times \mathbb{P}^{d-1}$ of type $\phi_{\kappa,\omega}$ and the Kac-Moody*

$$\pi_*\mathbb{U}_{\phi_{\kappa,\theta}}(\mathscr{G}_{\Sigma\times\mathbb{P}^{d-1}}) \simeq \mathbb{U}_{\mathrm{vol}(\omega)\kappa}(\mathscr{G}_\Sigma)$$

The twisted factorization envelope on the right-hand side is the familiar Kac-Moody factorization alegbra on Riemann surfaces associated to a multiple of the pairing κ. The twisting $\mathrm{vol}(\omega)\kappa$ corresponds to a cocycle of the type in the previous section

$$J(\mathrm{vol}(\omega)\kappa) = \mathrm{vol}(\omega) \int_\Sigma \kappa(\alpha, \partial\beta)$$

where $\mathrm{vol}(\omega) = \int_{\mathbb{P}^{d-1}} \omega$.

Proof. Suppose that $U \subset \Sigma$ is open. Then, the factorization algebra $\pi_* \mathbb{U}_{\phi_{\kappa,\theta}} (\mathscr{G}_{\Sigma \times \mathbb{P}^{d-1}})$ assigns to U the cochain complex

$$\left(\mathrm{Sym}\left(\Omega^{0,*}(U \times \mathbb{P}^{d-1})\right)[1][K], \overline{\partial} + K\phi_{\kappa,\omega}|_{U\times\mathbb{P}^{d-1}}\right), \tag{3.13}$$

where $\phi_{\kappa,\omega}|_{U\times\mathbb{P}^{d-1}}$ is the restriction of the cocycle to the open set $U \times \mathbb{P}^{d-1}$. Since projective space is Dolbeault formal its Dolbeault complex is quasi-isomorphic to its cohomology. Thus, we have

$$\Omega^{0,*}(U \times \mathbb{P}^{d-1}) = \Omega^{0,*}(U) \otimes \Omega^{0,*}(\mathbb{P}^{d-1}) \simeq \Omega^{0,*}(U) \otimes H^*(\mathbb{P}^{d-1}, \mathscr{O}) \cong \Omega^{0,*}(U).$$

Under this quasi-isomorphism, the restricted cocycle has the form

$$\phi_{\kappa,\omega}|_{U\times\mathbb{P}^{d-1}}(\alpha \otimes 1, \beta \otimes 1) = \int_U \kappa(\alpha, \partial\beta) \int_{\mathbb{P}^{n-1}} \omega$$

where $\alpha, \beta \in \Omega^{0,*}(U)$ and 1 denotes the unit constant function on $\mathbb{P}^{d-1}$. This is precisely the value of the local functional $\mathrm{vol}(\omega) J_\Sigma(\kappa)$ on the open set $U \subset \Sigma$. Thus, the cochain complex (3.13) is quasi-isomorphic to

$$\left(\mathrm{Sym}\left(\Omega^{0,*}(U)\right)[1][K], \overline{\partial} + K\mathrm{vol}(\omega)J_\Sigma(\kappa)\right). \tag{3.14}$$

We recognize this as the value of the Kac-Moody factorization algebra on Σ of type $\mathrm{vol}(\omega)J_\Sigma(\kappa)$. It is immediate to see that identifications above are natural with respect to maps of opens, so that the factorization structure maps are the desired ones. This completes the proof. □

BW: relate to nekrasov

3.4. Universal Grothendieck-Riemann-Roch from BV quantization

The main goal of the BV formalism developed in [?] is to rigorously construct quantum field theories using a combination of homological methods and a rigorous model for renormalization. A particular nicety of this approach is the ability to study *families* of field theories, which we will turn into an equivariant version of BV quantization, see Section **??**. In this section we will consider a family of QFT's parametrized by the moduli space of principal G-bundles. Our main result is to interpret a certain anomaly coming from BV quantization as a families index over $\mathrm{Bun}_G(X)$. This anomaly is computed via an explicit Feynman diagrammatic calculation and is related to a local cocycle of the current algebra discussed in Section 3.1.3.1.

We interpret this result as a formal universal version of the Grothendieck-Riemann-Roch theorem over the moduli space of bundles.

We will arrive at the result in a way that is local-to-global on spacetime which we formulate in terms of factorization algebras. The main them of Costello and Gwilliam's approach to QFT is that the observables of a QFT determine a factorization algebra. We study the associated family of factorization algebras associated to the family of QFT's over the moduli space of G-bundles. We will recollect a formulation of Noether's theorem for symmetries of a theory in terms of factorization algebras developed in Chapter 11 of [**CG**]. The central object in this discussion is a "local index" which describes how the Kac-Moody factorization algebra acts on the observables of the QFT. Locally on spacetime we see how Noether's theorem provides a *free field realization* of the Kac-Moody factorization algebra generalizing that of the Kac-Moody vertex algebra in chiral conformal field theory [**?**].

We now give a brief summary of the results, with a background for the situation we consider. Suppose that P is a fixed holomorphic G-bundle on a complex manifold X. We have already seen how to express the formal deformation space of P inside of the moduli of G-bundles using the dg Lie algebra $\mathscr{A}\mathrm{d}(P)(X) = \Omega^{0,*}(X, \mathrm{ad}(P))$. In particular, any Maurer-Cartan element of $\mathscr{A}\mathrm{d}(P)(X)$ defines a deformation of P. We have seen that there is a refinement of this dg Lie algebra to a local Lie algebra $\mathscr{A}\mathrm{d}(P)$ whose factorization envelope defines the higher Kac-Moody factorization algebra above. To any G-representation V we will define a holomorphic theory with fields $\mathscr{E}_V$ that is equivariant for this local Lie algebra. Equivalently, we can think of $\mathscr{E}_V$ as defining a family of theories over the formal completion of P in the moduli of G-bundles

$$\begin{tikzcd} \mathscr{E}_V|_{P'} \arrow[r] \arrow[d] & \mathscr{E}_V \arrow[d] \\ \{P'\} \arrow[r] & \mathrm{Bun}_G(X)^{\wedge}_P. \end{tikzcd}$$

Over each fiber P' the theory $\mathscr{E}_V|_{P'}$ is a *free* theory, so admits a canonical BV quantization. Our formulation of equivariant BV quantization is codification of the problem of gluing together these quantizations in a compatible way. We will show how this presents itself in the failure of the BV quantization to be a *flat* family. Our main result is the following.

Theorem 3.4.1. *Let P be any principal G-bundle over a compact affine complex manifold X of dimension d. Suppose V is a G-representation. Then, the factorization homology $\int_X \mathrm{Obs}^{\mathrm{q}}_V$ defines a line bundle over the formal neighborhood of P inside of the moduli of G-bundles. Moreover, its first Chern class is*

$$c_1\left(\int_X \mathrm{Obs}^{\mathrm{q}}_V\right) = C\mathrm{ch}^{\mathfrak{g}}_{d+1}(V)$$

under the identification of $\mathrm{ch}^{\mathfrak{g}}_{d+1}(V)$ as a cohomology class on the formal neighborhood of P inside of the moduli of G-bundles in Equation (3.18) explained below. Here, C is some nonzero complex number.

There is an elucidating geometric description of how the classes $\mathrm{ch}_{d+1}(V)$ appear: they describe curvatures of line bundles over the moduli of G-bundles. Let $\mathrm{Bun}_G(X)$ denote the moduli space of G-bundles on the complex d-fold X. [3] Over the space $\mathrm{Bun}_G(X) \times X$ there is the *universal* G-bundle. If $P \to X$ is a G-bundle, the fiber over the point $\{[P]\} \times X$ is precisely the G-bundle $P \to X$. This universal G-bundle is classified by a

[3] For $d > 1$ [**FHK**] have constructed a global smooth derived realization of this space, but its full structure will not be used in this discussion.

map $f : \mathrm{Bun}_G(X) \times X \to BG$. Consider the following diagram

$$\begin{tikzcd} & \mathrm{Bun}_G(X) \times X \arrow[dl, "\pi"'] \arrow[dr, "f"] & \\ \mathrm{Bun}_G(X) & & BG \end{tikzcd}$$

where $\pi : \mathrm{Bun}_G(X) \times X \to \mathrm{Bun}_G(X)$ denotes the projection. If $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}} \cong H^{d+1}(G, \Omega^{d+1}) \subset H^{2d+2}(BG)$ then we obtain via push-pull in the diagram above

$$\int_\pi \circ f^*\theta \in H^2(\mathrm{Bun}_G(X)).$$

Let $\mathscr{P}$ denote the universal principal G-bundle. This is the G-bundle over $\mathrm{Bun}_G(X) \times X$ whose fiber over $\{[P \to X]\} \times X$ is the principal G-bundle $P \to X$ itself. Given any representation V we can define the vector bundle

$$\mathscr{V} = \mathscr{P} \times^G V$$

over $\mathrm{Bun}_G(X) \times X$. The fiber of this bundle over $\{[P \to X]\} \times X$ is the associated vector bundle $P \times^G V$. We take the determinant of the derived pushforward of $\mathscr{V}$ along π to obtain a line bundle $\det(\mathbb{R}\pi_*\mathscr{V})$ on $\mathrm{Bun}_G(X)$. We will see how the global observables $\int_X \mathrm{Obs}^{\mathrm{q}}_{P,V}$ provide a formal version of this line bundle near a fixed principal bundle P. Moreover, if we naively apply the Grothendieck-Riemann-Roch theorem in this universal context one finds

$$c_1(\det(\mathbb{R}\pi_*\mathscr{V})) = \int_\pi \mathrm{Td}_X \cdot \mathrm{ch}(\mathscr{V}) \in H^2(\mathrm{Bun}_G(X)).$$

In the case that X is affine, so that $\mathrm{Td}_X = 1$, our theorem provides a proof of this formula using methods of perturbative QFT. To prove the theorem on a general complex manifold we need to take into account the action of holomorphic vector fields, which is the content of the next section.

3.4.1. The classical family

In this section, we consider a BV theory that is equivariant for the local Lie algebra $\mathscr{A}\mathrm{d}(P)$ in the language of Section **??**. Let V be any G-representation, and define the associated vector bundle $\mathscr{V}_P = P \times^G V$ on X. The holomorphic theory we consider is based on the graded holomorphic vector bundle $\mathscr{V}_P \oplus K_X \otimes \mathscr{V}_P^*[d-1]$, where $\mathscr{V}_P^*$ is the linear dual bundle. The fields of the associated free BV theory are

$$\mathscr{E}_{P,V} = \Omega^{0,*}(X, \mathscr{V}_P) \oplus \Omega^{d,*}(X, \mathscr{V}_P^*)[d-1].$$

This is simply the $\beta\gamma$ system on X twisted by the vector bundle $\mathscr{V}_P$. The action functional is $\int \langle \beta, \overline{\partial}\gamma \rangle_V$ where the pairing is between V and its dual. In particular, the theory $\mathscr{E}_V$ is free. Let $\mathrm{Obs}^{\mathrm{q}}_{P,V}$ denote the corresponding factorization algebra of quantum observables.

The action of $\mathfrak{g}$ on V extends to an action of the local Lie algebra $\mathscr{A}\mathrm{d}(P)$ on this classical BV theory. To define this equivariance we need to presribe a Noether current.

Lemma 3.4.2. *The local Noether current $I^{\mathfrak{g}} \in \mathrm{C}^*_{\mathrm{loc}}(\mathscr{A}\mathrm{d}(P)) \otimes \mathscr{O}_{\mathrm{loc}}(\mathscr{E}_{P,V})$ defined by*

$$I^{\mathfrak{g}}(\alpha, \gamma, \beta) = \int_X \langle \beta, \alpha \cdot \gamma \rangle_V$$

satisfies the equivariant classical master equation

$$(\mathrm{d}_{\mathfrak{g}} + \overline{\partial})I^{\mathfrak{g}} + \frac{1}{2}\{I^{\mathfrak{g}}, I^{\mathfrak{g}}\} = 0,$$

where $\mathrm{d}_{\mathfrak{g}}$ *encodes the Lie algebra structure on* $\mathscr{A}\mathrm{d}(P)$*. Hence,* $I^{\mathfrak{g}}$ *gives* $\mathscr{E}_V$ *the structure of a classical* $\mathscr{A}\mathrm{d}(P)$*-equivariant theory.*

Proof. If α is an element in $\mathscr{A}\mathrm{d}(P)$ and $\gamma \in \Omega^{0,*}(X, \mathscr{V}_P)$ we define $\alpha \cdot \gamma$ through the $\mathfrak{g}$-module structure of $\mathfrak{g}$ on V combined with the wedge product of Dolbeault forms. Note that $I^{\mathfrak{g}}$ is arises from holomorphic differential operators so that $\overline{\partial} I^{\mathfrak{g}} = 0$. From the definition of the bracket we see that for α_1, α_2 one has $\{\int \langle \beta, \alpha_1 \cdot \gamma \rangle, \int \langle \beta, \alpha_2 \cdot \gamma \rangle\} = \int \langle \beta, [\alpha_1, \alpha_2] \cdot \gamma \rangle$ which cancels the term coming from $\mathrm{d}_{\mathfrak{g}}$. □

3.4.2. BV quantization in families

The main technique we employ is equivariant BV quantization, which we have reviewed in Section **??**. Our main result holds for a compact affine manifold, which we will view as coming from a quotient of an open set in affine space $\mathbb{C}^d$. Thus, we will mostly work with the theory defined on $\mathbb{C}^d$ and afterwards deduce our main result on the quotient via descent. Thus, we will work with the $\beta\gamma$ system

$$\mathscr{E}_V = \Omega^{0,*}(\mathbb{C}^d, V) \oplus \Omega^{d,*}(\mathbb{C}^d, V^*)[d-1]$$

where V is some $\mathfrak{g}$-module. The local Lie algebra which acts on this theory is $\mathscr{G} = \Omega^{0,*}(\mathbb{C}^d, \mathfrak{g})$.

Our first step is to construct an equivariant effective prequantization. for the $\mathscr{G}$-equivariant theory. As has been the case over and over again in this thesis, our situation for constructing the prequantization is vastly simplified since our theory comes from holomorphic data. Indeed, the equivariant $\beta\gamma$ system is a holomorphic theory on $\mathbb{C}^d$ so that we can apply Lemma **??**. As an immediate corollary, the following definition is well-defined.

Definition 3.4.3. For $L > 0$, let

$$I^{\mathfrak{g}}[L] := \lim_{\epsilon \to 0} W(P_{\epsilon<L}, I^{\mathfrak{g}}) = \lim_{\epsilon \to 0} \sum_{\Gamma} \frac{\hbar^{g(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} W_\Gamma(P_{\epsilon<L}, I^{\mathfrak{g}}).$$

Here the sum is over all isomorphism classes of stabled connected graphs, but only graphs of genus ≤ 1 contribute nontrivially. By construction, the collection satisfies the RG flow equation and its tree-level $L \to 0$ limit is manifestly $I^{\mathfrak{g}}$. Hence $\{I^{\mathfrak{g}}[L]\}_{L\in(0,\infty)}$ is a *$\mathscr{G}$-equivariant prequantization.*

Our next step is to compute the obstruction to quantization of the $\mathscr{G}$-equivariant theory. By definition, the scale L *obstruction cocycle* $\Theta_V[L]$ is the failure for the interaction $I^{\mathfrak{g}}[L]$ to satisfy the scale L equivariant quantum master equation. Explicitly, one has

$$\hbar\Theta_V[L] = (\mathrm{d}_{\mathfrak{g}} + Q)I^{\mathfrak{g}}[L] + \hbar\Delta_L I^{\mathfrak{g}}[L] + \{I^{\mathfrak{g}}[L], I^{\mathfrak{g}}[L]\}_L.$$

A completely analogous argument as in Corollary 16.0.5 of [**Cosa**] we see that the scale L obstruction is given by a sum over wheels.

Lemma 3.4.4. *Only wheels contribute to the anomaly cocycle $\Theta_V[L]$. Moreover, one has*

$$\Theta_V[L] = \sum_{\substack{\Gamma \in \mathrm{Wheels} \\ e \in \mathrm{Edge}(\Gamma)}} W_{\Gamma,e}(P_{\epsilon<1}, K_\epsilon, I^{\mathfrak{g}}[\epsilon]),$$

where the sum is over wheels and distinguished edges. The notation $W_{\Gamma,e}(P_{\epsilon<1}, K_\epsilon, I^{\mathfrak{g}}[\epsilon])$ means we place the propagator at all edges besides the distinguished one, where we place K_ϵ.

The only fields that propagate are the $\beta\gamma$ fields with values in V. Since all vertices are trivalent we see that the anomaly cocycle is only a functional of the background fields $\mathscr{G}$, see Figure BW: ref. In particular, there is no obstruction to having an *action* by $\mathscr{G}$, only an obstruction to having an *inner action.* Concretely, the external edges of any closed wheel occurring in the expansion of the anomaly must be labeled by $\mathscr{G}$. As an immediate consequence we have the following.

Lemma 3.4.5. *The effective family $\{I^{\mathfrak{g}}[L]\}$ defines a one-loop exact $\mathscr{G}$-equivariant quantum field theory. In other words, it satisfies the $\mathscr{G}$-equivariant quantum master equation modulo functionals purely of the background fields $\mathscr{G}$.*

It follows that the anomaly $\{\Theta[L]\}$ measures the obstruction to $\{I^{\mathfrak{g}}[L]\}$ to defining an *inner* action.

3.4.2.1. The anomaly calculation. We now perform the main technical calculation of the anomaly cocycle.

Proposition 3.4.6. *The $L \to 0$ limit of the anomaly cocycle $\Theta = \lim_{L\to 0} \Theta_V[L] \in \mathrm{C}^*_{\mathrm{loc}}(\mathscr{G})$ is of the form*

$$\Theta_V = C \cdot J_{\mathbb{C}^d}(\mathrm{ch}^{\mathfrak{g}}_{d+1}(V)),$$

where $\mathrm{ch}^{\mathfrak{g}}_{d+1}(V) \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$ and where $J_{\mathbb{C}^d} : \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}} \to \mathrm{C}^*_{\mathrm{loc}}(\mathscr{G})$ is the map of Lemma 3.1.11 and where C some constant only depending on the dimension d.

To compute the anomaly we refer to the following result about the expression for the anomaly cocycle in terms of the Feynman diagram expansion. This is proved in direct analogy to Lemma 2.5.4. We have already seen in Lemma **??** that only wheels contribute to the anomaly cocycle. Then an explicit analysis of the analytic behavior shows that in the $\epsilon \to 0$ limit only the wheel with $(d+1)$-vertices contributes.

Lemma 3.4.7. *The limit $\Theta_V := \lim_{L\to 0} \Theta_V[L]$ exists and is an element of degree one in $\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n, \mathrm{C}^*_{\mathrm{loc}}(\mathfrak{g}^{\mathbb{C}}_n))$. Moreover, it is given by*

$$\Theta_V = \lim_{\epsilon\to 0} \sum_{\substack{\Gamma \in (d+1)\text{-vertex wheels} \\ e \in \mathrm{Edge}(\Gamma)}} W_{\Gamma,e}(P_{\epsilon<1}, K_\epsilon, I^{\mathfrak{g}}[\epsilon]),$$

where the sum is over wheels Γ with two vertices and a distinguished inner edge e.

The lemma implies that we only need to consider the wheel with $d+1$ vertices. Each trivalent vertex is labeled by both an analytic factor and Lie algebraic factor. The Lie algebraic part of each vertex can be thought of as the defining map of the representation $\rho : \mathfrak{g} \to \mathrm{End}(V)$. The diagrammitcs of the wheel amounts to taking the trace of the symmetric $(d+1)$st power of this Lie algebra factor. Thus, the Lie algebraic factor of the weight of the wheel is the $(d+1)$st component of the character of the representation

$$\mathrm{ch}^{\mathfrak{g}}_{d+1}(V) = \frac{1}{(d+1)!}\mathrm{Tr}\left(\rho(X)^{d+1}\right) \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*).$$

To finish the calculation we must compute the analytic weight of the wheel with $d+1$ vertices. Recall, our goal is to identify the anomaly Θ with the image of $\mathrm{ch}^{\mathfrak{g}}_{d+1}(V)$ under the map

$$J : \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}} \to \mathrm{C}^*_{\mathrm{loc}}(\Omega^{0,*}(\mathbb{C}^d) \otimes \mathfrak{g})$$

that sends an element θ to the local functional $\int \theta(\alpha\partial\alpha \cdots \partial\alpha)$. We have just seen that the Lie algebra factor in local functional representing the anomaly agrees with the $(d+1)$st Chern character. Thus, to finish we must show the following.

Lemma 3.4.8. *As a functional on the abelian dg Lie algebra $\Omega^{0,*}(\mathbb{C}^d)$, the analytic factor of the weight $\lim_{L\to 0}\lim_{\epsilon\to 0} W_{\Gamma,e}(P_{\epsilon<L}, K_\epsilon, I^{\mathfrak{g}})$ is equal to a multiple of the local functional*

$$\int \alpha\partial\alpha\cdots\partial\alpha \in \mathrm{C}^*_{\mathrm{loc}}(\Omega^{0,*}(\mathbb{C}^d)).$$

Proof. Let's fix some notation. We enumerate the vertices by integers $a = 0, \ldots, d$. Label the coordinate at the ith vertex by $z^{(a)} = (z^{(a)}_1, \ldots, z^{(a)}_d)$. The incoming edges of the wheel will be denoted by homogeneous Dolbeault forms

$$\alpha^{(a)} = \sum_J A^{(a)}_J \mathrm{d}\overline{z}^{(a)}_J \in \Omega^{0,*}_c(\mathbb{C}^d).$$

where the sum is over the multiindex $J = (j_1, \ldots, j_k)$ where $j_a = 1, \ldots, d$ and $(0, k)$ is the homogenous Dolbeault form type. For instance, if α is a $(0,2)$ form we would write

$$\alpha = \sum_{j_1 < j_2} A_{(j_1, j_2)} \mathrm{d}\overline{z}_{j_1} \mathrm{d}\overline{z}_{j_2}.$$

Denote the functional obtained as the $\epsilon \to 0$ weight of the wheel with $(d+1)$ vertices from Lemma **??** by W_L. The $L \to 0$ limit of W_L is the local functional representing the one-loop anomaly Θ.

The weight has the form

$$W_L(\alpha^{(0)}, \ldots, \alpha^{(d)}) = \pm \lim_{\epsilon \to 0} \int_{\mathbb{C}^{d(d+1)}} \left(\alpha^{(0)}(z^{(0)}) \cdots \alpha^{(d)}(z^{(d)})\right) K_\epsilon(z^{(0)}, z^{(d)}) \prod_{a=1}^{d} P_{\epsilon, L}(z^{(a-1)}, z^{(a)}).$$

We introduce coordinates

$$\begin{aligned} w^{(0)} &= z^{(0)} \\ w^{(a)} &= z^{(a)} - z^{(a-1)} \quad 1 \leq a \leq d. \end{aligned}$$

The heat kernel and propagator part of the integral is of the form

$$\begin{aligned} K_\epsilon(w^{(0)}, w^{(d)}) \prod_{a=1}^{d} P_{\epsilon, L}(w^{(a-1)}, w^{(a)}) \quad = \quad & \frac{1}{(4\pi\epsilon)^d} \int_{t_1, \ldots, t_d = \epsilon}^{L} \frac{\mathrm{d}t_1 \cdots \mathrm{d}t_d}{(4\pi t_1)^d \cdots (4\pi t_d)^d} \frac{1}{t_1 \cdots t_d} \\ & \times \mathrm{d}^d w^{(0)} \prod_{i=1}^{d} (\mathrm{d}\overline{w}_i^{(1)} + \cdots + \mathrm{d}\overline{w}_i^{(d)}) \prod_{a=1}^{d} \mathrm{d}^d w^{(a)} \left(\sum_{i=1}^{d} \overline{w}_i^{(a)} \prod_{j \neq i} \mathrm{d}\overline{w}_j^{(a)} \right) \\ & \times e^{-\sum_{a,b=1}^{d} M_{ab} w^{(a)} \cdot \overline{w}^{(b)}}. \end{aligned}$$

Here, M_{ab} is the $d \times d$ square matrix satisfying

$$\sum_{a,b=1}^{d} M_{ab} w^{(a)} \cdot \overline{w}^{(b)} = |\sum_{a=1}^{d} w^{(a)}|^2/\epsilon + \sum_{a=1}^{d} |w^{(a)}|^2/t_a.$$

Note that

$$\prod_{i=1}^{d} (\mathrm{d}\overline{w}_i^{(1)} + \cdots + \mathrm{d}\overline{w}_i^{(d)}) \prod_{a=1}^{d} \left(\sum_{i=1}^{d} \overline{w}_i^{(a)} \prod_{j \neq i} \mathrm{d}\overline{w}_j^{(a)} \right) = \left(\sum_{i_1,\ldots i_d} \epsilon_{i_1 \cdots i_d} \prod_{a=1}^{d} \overline{w}_{i_a}^{(a)} \right) \prod_{a=1}^{d} \mathrm{d}^d \overline{w}^{(a)}.$$

In particular, only the $\mathrm{d}w_i^{(0)}$ components of $\alpha^{(0)} \cdots \alpha^{(d)}$ can contribute to the weight.

Let $\Phi =$ BW: some contraction of $\alpha^{(0)} \cdots \alpha^{(d)}$ by a antiholomorphic vector field. Then, the weight can be written as

$$\begin{aligned} W(\alpha^{(0)}, \ldots, \alpha^{(d)}) &= \lim_{\epsilon \to 0} \int_{\mathbb{C}^{d(d+1)}} \left(\prod_{a=0}^{d} \mathrm{d}^d w^{(a)} \mathrm{d}^d \overline{w}^{(a)} \right) \Phi \\ &\times \frac{1}{(4\pi\epsilon)^d} \int_{t_1,\ldots,t_d=\epsilon}^{L} \frac{\mathrm{d}t_1 \cdots \mathrm{d}t_d}{(4\pi t_1)^d \cdots (4\pi t_d)^d} \frac{1}{t_1 \cdots t_d} \sum_{i_1,\ldots,i_d} \epsilon_{i_1 \cdots i_d} \overline{w}_{i_1}^{(1)} \cdots \overline{w}_{i_d}^{(d)} e^{-\sum_{a,b=1}^{d} M_{ab} w^{(a)} \cdot \overline{w}^{(b)}} \end{aligned}$$

Applying Wick's lemma in the variables $w^{(1)}, \ldots, w^{(d)}$, together with some elementary analytic bounds, we find that the weight above becomes to the following integral over $\mathbb{C}^d$

$$f(L) \int_{w^{(0)} \in \mathbb{C}^d} \mathrm{d}^d w^{(0)} \mathrm{d}^d \overline{w}^{(0)} \sum_{i_1,\ldots,i_d} \epsilon_{i_1 \cdots i_d} \left(\frac{\partial}{\partial w_{i_1}^{(1)}} \cdots \frac{\partial}{\partial w_{i_d}^{(d)}} \Phi \right) |_{w^{(1)} = \cdots = w^{(d)} = 0}$$

where

$$f(L) = \lim_{\epsilon \to 0} \int_{t_1,\ldots,t_d=\epsilon}^{L} \frac{\epsilon}{(\epsilon + t_1 + \cdots + t_d)^{d+1}} \mathrm{d}^d t.$$

In fact, $f(L)$ is independent of L and is equal to some nonzero constant $C \neq 0$. Finally, plugging in the forms $\alpha^{(0)}, \ldots, \alpha^{(d)}$, we observe that the integral over $w^{(0)} \in \mathbb{C}^d$ simplifies to

$$C \int_{\mathbb{C}^d} \alpha^{(0)} \partial \alpha^{(1)} \cdots \partial \alpha^{(d)}$$

as desired. $\square$

This completes the proof of Proposition **??**.

3.4.3. Local to global

In this section we finish the proof of our main result Theorem 3.4.1 by showing how our local calculation above implies the formula for the anomaly on a general compact affine manifold X. By an complex affine manifold, we mean a quotient

$$q : U \subset \mathbb{C}^d \to X$$

of an open subset $U \subset \mathbb{C}^d$ by a free and proper action of a discrete subgroup of the affine group $U(d) \ltimes \mathbb{C}^d$. We consider affine manifolds that are also compact. To deduce our main theorem we will show that the theory and the anomaly above also exhibit equivariance for the affine group on $\mathbb{C}^d$, thus it will descend to any affine manifold.

We have stated the main result for an arbitrary principal G-bundle P on the affine manifold X. Suppose the discrete subgroup $\Gamma \leq U(d) \ltimes \mathbb{C}^d$ defines the affine manifold $q : U \to X = U/\Gamma$ as above. Then, principal G-bundles on X are equivalent to Γ-equivariant principal G-bundles on U.

Let $\mathscr{E}$ be an arbitrary elliptic complex on X, and suppose the Lie algebra $\mathfrak{h}$ acts on $\mathscr{E}$. Since X is compact, the cohomology $H^*(\mathscr{E}(X))$ is finite dimensional. It therefore makes sense to define the character of the action of $\mathfrak{h}$ on $H^*(\mathscr{E}(X))$.

$$\chi_{\mathscr{E}} : \mathfrak{h} \to \mathbb{C} \ \ , \ \ M \in \mathfrak{h} \mapsto \mathrm{STr}_{H^*(\mathscr{E}(X))}(M). \tag{3.15}$$

Here, STr denotes the supertrace. The character factors through the determinant of the representation. For the graded character above, we must use the superdeterminant which we denote by $\det(H^*(\mathscr{E}(X)))$. Free BV quantization gives a natural field theoretic interpretation of this determinant.

Proposition 3.4.9 ([?] Lemma 12.7.0.1)**.** *Let $\mathscr{E}$ be any elliptic complex on a compact manifold X and let $T^*[-1]\mathscr{E}$ be the corresponding free BV theory given by the shifted cotangent bundle. Let $\mathrm{Obs}^{\mathrm{q}}_{\mathscr{E}}$ be the factorization algebra of quantum observables of this theory. Then, there is an isomorphism*

$$H^*\left(\mathrm{Obs}^{\mathrm{q}}_{\mathscr{E}}(X)\right) \cong \det H^*(\mathscr{E}(X))[n]$$

where n is the Euler characteristic of $\mathscr{E}(X)$ modulo 2.

Notice that the classical free theory $\mathscr{E}_V$ is equivariant for the affine group $U(d) \ltimes \mathbb{C}^d$. Thus, it defines a classical theory on any affine manifold X. This theory is free and of the form

$$\mathscr{E}_V(X) = T^*[-1](\Omega^{0,*}(X, V))$$

where $T^*[-1]$ denotes the shifted cotangent bundle. Thus, the global quantum observables satisfy

$$H^*(\mathrm{Obs}^{\mathrm{q}}_V(X)) = \det\left(H^*(X, \mathscr{O}^{hol}) \otimes V\right) [??] \tag{3.16}$$

In Section 3.4.1 we have showed how the classical theory $\mathscr{E}_V$ has an an action by the local Lie algebra $\mathscr{G}_X$. This arose from an action of $\mathscr{G}(X) = \Omega^{0,*}(X, \mathfrak{g})$ on the elliptic complex $\Omega^{0,*}(X, V)$. At the level of cohomology we have an action of $H^*(\mathscr{G}(X))$ on $H^*(\Omega^{0,*}(X, V))$ and hence a character χ_V as in Equation (3.15) which is an element in $H^*_{red}(\mathscr{G}(X))$.

The local Lie algebra cohomology of any local Lie algebra embeds inside its ordinary (reduced) Lie algebra cohomology of global sections $\mathrm{C}^*_{\mathrm{loc}}(\mathscr{L}(X)) \subset \mathrm{C}^*_{\mathrm{Lie,red}}(\mathscr{L}(X))$. The character (3.15) is an element in $H^*_{red}(\mathscr{L}(X))$. As an immediate corollary of [?] Theorem 12.6.0.1 we have the following relationship between the anomaly cocycle and the character.

Proposition 3.4.10. *Suppose $\mathscr{L}$ is a local Lie algebra that acts on the elliptic complex $\mathscr{E}$ on a compact manifold X. Let $\Theta_{\mathscr{E}} \in \mathrm{C}^*_{\mathrm{loc}}(\mathscr{L})$ be the local cocycle measuring the failure to satisfy the $\mathscr{L}$-equivariant classical master equation (that is, the obstruction to having an inner action). Then, its global cohomology class satisfies $[\Theta_{\mathscr{E}}(X)] = \chi_{\mathscr{E}} \in \mathrm{C}^*_{\mathrm{Lie,red}}(\mathscr{L}(X))$ where $\chi_{\mathscr{E}}$ is the trace of the action of $H^*(\mathscr{L}(X))$ on $H^*(\mathscr{E}(X))$.*

For the case of $\mathscr{L} = \mathscr{G}_X$ we have an embedding of cochain complexes

$$\mathrm{C}^*_{\mathrm{loc}}(\mathscr{G}(X)) \hookrightarrow \mathrm{C}^*_{\mathrm{Lie,red}}(\mathscr{G}(X)) = \mathrm{C}^*_{\mathrm{Lie,red}}(\Omega^{0,*}(X) \otimes \mathfrak{g}).$$

By Kodaira-Spencer theory have already seen that the global sections of the local Lie algebra $\mathscr{G}(X)$ is a model for the formal neighborhood of the trivial G-bundle inside of G-bundles. In particular, the $\mathscr{G}(X)$-module of quantum observables defines a line bundle $\int_X \mathrm{Obs}^{\mathrm{q}}_{\mathscr{E}_V}$ over this formal neighborhood. Its character as a $\mathscr{G}(X)$-module is identified with the first Chern class of the corresponding line bundle $\chi_{\mathscr{E}}(\mathrm{Obs}^{\mathrm{q}}_V(X)) = c_1(\int_X \mathrm{Obs}^{\mathrm{q}}_V)$.

Now, notice that the one-loop quantization we constructed in the previous section, as well as the anomaly cocycle $\Theta_V \in \mathrm{C}^*_{\mathrm{loc}}(\mathscr{G}_{\mathbb{C}^d})$ are equivariant for the group $U(d) \ltimes \mathbb{C}^d$. Thus, they descend to the global sections of $\mathrm{C}^*_{\mathrm{loc}}(\mathscr{G}_X)$ for any affine manifold X. Explicitly, if $\Gamma \subset U(d) \ltimes \mathbb{C}^d$ is the discrete subgroup such that $X = U/\Gamma$ where $U \subset \mathbb{C}^d$, then under the isomorphism

$$\mathrm{C}^*_{\mathrm{loc}}(\mathscr{G}(X)) \cong \mathrm{C}^*_{\mathrm{loc}}(\mathscr{G}(U))^{\Gamma}$$

we have $\Theta_V(X) \leftrightarrow \Theta_V(U)$.

Further, we have an identification

$$\mathrm{C}^*_{\mathrm{Lie,red}}(\Omega^{0,*}(X) \otimes \mathfrak{g}) = \mathscr{O}_{red}\left(\mathrm{Bun}_G(X)^{\wedge}_{triv}\right) \cong \Omega^1_{cl}\left(\mathrm{Bun}_G(X)^{\wedge}_{triv}\right)$$

where we have used the equivalence of reduced functions and closed one-forms which makes sense on any formal moduli space. At the level of H^1 we have the composition composition

$$\mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}} \xrightarrow{J^X} H^1_{\mathrm{loc}}(\mathscr{G}(X)) \to H^1(\Omega^1_{cl}\left(\mathrm{Bun}_G(X)^{\wedge}_{triv}\right). \tag{3.17}$$

As a corollary of Proposition 3.4.10 and our calculation of the local anomaly cocycle we see that the image of $\mathrm{ch}^{\mathfrak{g}}_{d+1}(V)$ is equal to $[\Theta_V(X)] = [c_1(\int_X \mathrm{Obs}^{\mathfrak{q}}_V)]$.

The same holds when we work around any holomorphic principal bundle P on X, so that we have an embedding of cochain complexes

$$\mathrm{C}^*_{\mathrm{loc}}(\mathscr{A}\mathrm{d}(P)(X)) \hookrightarrow \Omega^1_{cl}\left(\mathrm{Bun}_G(X)^{\wedge}_P\right).$$

which determines a composition

$$\mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}} \xrightarrow{J_P^X} H^1_{\mathrm{loc}}(\mathscr{A}\mathrm{d}(P)(X)) \to H^1(\Omega^1_{cl}(\mathrm{Bun}_G(X)^\wedge_P)\,. \tag{3.18}$$

Since every principal G-bundle P on X is trivial when we pull it back to $U \subset \mathbb{C}^d$, the above local anomaly calculation proves that $[c_1(\int_X \mathrm{Obs}^{\mathrm{q}}_{P,V})] = C\mathrm{ch}^{\mathfrak{g}}_{d+1}(V)$ in this case as well. This completes the proof of Theorem 3.4.1.

3.4.4. A module for the higher Kac-Moody

The last part of this section we diverge to deduce a consequence of the quantum Noether theorem using our analysis above by exhibiting a module for the higher affine algebras from the previous section. For convenience, we fix the trivial $\mathfrak{g}$-bundle $P = \mathrm{triv}$ so that $\mathscr{A}\mathrm{d}(P) = \mathscr{G}_X$.

On any manifold X, the quantum Noether theorem BW: reference provides a map of factorization algebras

$$\Phi_X : \mathbb{U}_\alpha(\mathscr{G}_X) \to \mathrm{Obs}^{\mathrm{q}}_V,$$

for some $\alpha \in H^1_{\mathrm{loc}}(\mathscr{G}_X)$. The factorization algebra $\mathrm{Obs}^{\mathrm{q}}_V$ is the quantum observables of the $\beta\gamma$ system on X with values in the $\mathfrak{g}$-module V. This is a free field theory, thus the above map has the flavor of a *free field realization* of the Kac-Moody factorization algebra. In particular when $X = \mathbb{C}^d$, or any affine manifold, the calculation above shows that there is a map of factorization algebras

$$\Phi_{\mathbb{C}^d} : \mathbb{U}_{\mathrm{ch}_{d+1}(V)}(\mathscr{G}_{\mathbb{C}^d}) \to \mathrm{Obs}^{\mathrm{q}}_V.$$

Next, consider the case $X = \mathbb{C}^d \setminus \{0\}$. By functoriality of pushforwards, the quantum Noether theorem produces a map of one-dimensional factorization algebras

$$\rho_* \Phi : \rho_* \mathbb{U}_{\mathrm{ch}_{d+1}(V)}(\mathscr{G}_{\mathbb{C}^d \setminus \{0\}}) \to \rho_* \mathrm{Obs}^{\mathrm{q}}_V .$$

We have exhibited a locally constant dense subfactorization algebra $\mathscr{F}^{lc}_{1d}$ of $\rho_* \mathbb{U}_\alpha(\mathscr{G}_X)$ which is equivalent, as an E_1-algebra, to $U\widehat{\mathfrak{g}}_{d,\mathrm{ch}_{d+1}(V)}$. Similarly, in Section **??** we have shown that there is a locally constant dense subfactorization algebra that is equivalent to the dg algebra $\mathscr{A}_V$.

The map $\rho_*\Phi$ restricts to these dense subfactorization algebras and so defines a map of E_1 algebras

$$\rho_* \Phi : U\widehat{\mathfrak{g}}_{d,\mathrm{ch}_{d+1}(V)} \to \mathscr{A}_V .$$

Also, in Section **??** we have shown how the disk operators $\mathscr{V}_V$ form a module, through the factorization product, for the dg algebra $\mathscr{A}_V$. This is essentially the Fock module of the algebra $\mathscr{A}_V$, thus we should view the above map $\rho_*\Phi$ as being a higher dimensional analog of the "free field realization" for the higher dimensional affine algebras.

Further, by induction along the map $\rho_*\Phi$, we obtain the following.

Proposition 3.4.11. *The map $\rho_*\Phi$ endows the space $\mathscr{V}_V$ with the structure of a module over the E_1-algebra $U\widehat{\mathfrak{g}}_{d,\mathrm{ch}_{d+1}(V)}$. Equivalently, $\mathscr{V}_V$ is an A_∞-module for $U\widehat{\mathfrak{g}}_{d,\mathrm{ch}_{d+1}(V)}$.*

The module $\mathscr{V}_V$ is the prototype for a higher dimensional version of a Verma module for ordinary affine algebras.

3.5. Holomorphic diffeomorphisms

The next type of symmetry we consider is that of holomorphic reparametrizations, or holomorphic diffeomorphisms. A holomorphic diffeomorphism $f : X \to Y$ between complex manifolds is a bijective holomorphic map whose inverse is also holomorphic. Under composition, holomorphic diffeomorphisms from X to itself combine to form a Lie group $\mathrm{Diff}^{hol}(X)$. We study theories that have an action of holomorphic diffeomorphisms which leave the action functional invariant. An example of such a theory is one for which the action functional can be written down in a holomorphically covariant way (e.g. one that only uses universal constructions in complex geometry). Morally speaking, the Lie algebra of holomorphic diffeomorphisms from X to itself is equal to holomorphic vector fields on X. Of course, some care must be taken to make this precise as $\mathrm{Diff}^{hol}(X)$ is not a finite dimensional manifold. We will not be concerned with these functional analytic issues since we will take as a starting point theories that have symmetries by the Lie algebra of holomorphic vector fields. It is an interesting question if our constructions lift to the level of the Lie group, but we will not address that here.

3.5.1. Holomorphic vector fields

Covariance in field theory is usually reserved for theories that can be written in a way that uses only natural constructions in differential geometry and so is independent of a choice of a local coordinate. This is precisely the condition that the theory is invariant with respect to the group of diffeomorphisms The obvious holomorphic analog of this makes sense for theories defined on complex manifolds. In this section we introduce a local-to-global Lie algebraic version of holomorphic covariance using a natural local Lie algebra associated to holomorphic diffeomorphisms.

In the two-dimensional chiral case we will see that a holomorphic covariant theory is the same thing as a chiral conformal field theory.

3.5.1.1. The local Lie algebra. Just as in the case of holomorphic gauge symmetries, there is a local Lie algebra associated to holomorphic vector fields. For any complex manifold X, the holomorphic tangent bundle $T^{1,0}X$ is a holomorphic vector bundle and hence admits a Dolbeault complex

$$\mathscr{T}(X) := \Omega^{0,*}(X, T^{1,0}X).$$

Together with the $\overline{\partial}$ operator, there is a natural extension of the Lie bracket of holomorphic vector fields that gives this complex the structure of a dg Lie algebra. The underlying graded vector space of $\mathscr{T}(X)$ is clearly the global sections of a smooth manifold. Moreover, the differential and Lie bracket are differential and bidifferential operators respectively. Thus:

Lemma 3.5.1. *For any complex manifold X, $\mathscr{T}(X) = \Omega^{0,*}(X, T^{1,0}X)$ has the structure of a local Lie algebra.*

In keeping with the conventions above, when we want to stress the sheaf-like nature of this local Lie algebra we use the notation $\mathscr{T}_X^{sh}$. This is a sheaf of dg Lie algebras which assigns to an open set $U \subset X$ the dg Lie algebra $\Omega^{0,*}(U, T^{1,0}U)$. We will use the notation $\mathscr{T}_X$ to denote the associated cosheaf $U \mapsto \Omega^{0,*}_c(U, T^{1,0}U)$.

3.5.1.2. Holomorphically covariant theories. The local Lie algebra $\mathscr{T}_X$ allows us to define the following stronger notion of a holomorphic theory. Recall the definition of a holomorphic theory on a complex manifold X from Section 1.2. This consists of the data of a holomorphic vector bundle $V \to X$, a holomorphic differential operator $Q^{hol} : V \to V[1]$, a shifted symplectic pairing $(-,-)_V$ on V, and a holomorphic Lagrangian density I^{hol}.

Definition 3.5.2. A holomorphic theory $(V, Q^{hol}, \omega, (-,-)_V, I^{hol})$ is *holomorphically covariant* if the associated BV theory admits an action by the local Lie algebra $\mathscr{T}_X$.

Many of the holomorphic theories we have encountered are, in addition, holomorphically covariant. Recall that the data of an action of a local Lie algebra $\mathscr{L}$ on a theory $\mathscr{E}$ is given by a Maurer-Cartan element in the the dg Lie algebra $\mathrm{Act}(\mathscr{L}, \mathscr{E})[-1]$ from Section 1.4.3. This is a sub dg Lie algebra of $\mathrm{C}^*_{\mathrm{Lie,red}}(\mathscr{L}(X)) \otimes \mathscr{O}_{\mathrm{loc}}(\mathscr{E})[-1]$ where the dependence on the local Lie algebra must also be local.

Example 3.5.3. Consider the $\beta\gamma$ system on a complex manifold X with values in a vector space V. We consider this theory extensively in Section **??**. The fields $\mathscr{E}_V$ consist of elements $\gamma \in \Omega^{0,*}(X, V)$ together with their conjugates. The dg Lie algebra $\mathscr{T}(X)$ acts on the cochain complex $\Omega^{0,*}(X, V)$ via Lie derivative: if $\xi \in \mathscr{T}(X)$ of degree k and $\gamma \in \Omega^{0,l}(X)$ then $L_\xi \gamma \in \Omega^{0,k+l}(X, V)$ is defined. It is immediate to see that this is compatible with the $\overline{\partial}$ operator. The classical Noether current defining the classical action is

$$I^{\mathscr{T}}(\xi, \beta, \gamma) = \int \langle \beta, L_\xi \gamma \rangle_V,$$

where $\langle -, - \rangle$ is, as usual, the pairing between V and its dual. This functional defines a Maurer-Cartan element in

$$I^{\mathscr{T}} \in \mathrm{Act}(\mathscr{T}, \mathscr{E}_V)[-1] \subset \mathrm{C}^*_{\mathrm{Lie,red}}(\mathscr{T}(X)) \otimes \mathscr{O}_{\mathrm{loc}}(\mathscr{E}_V)[-1]$$

and hence we have a classical action of $\mathscr{T}_X$ on $\mathscr{E}_V$.

There is a variation of this example that plays an important role for us. The holomorphic tensor bundle of type (r, s) on a manifold X is the holomorphic vector bundle

$$V_r^s = \underbrace{T^{1,0}X \otimes \cdots \otimes T^{1,0}X}_{r \text{ copies}} \otimes \underbrace{T^{*1,0}X \otimes \cdots \otimes T^{*1,0}X}_{s \text{ copies}}.$$

Similarly, there are anti-holomorphic versions. The local Lie algebra $\mathscr{T}_X$ acts on any holomorphic tensor bundle on X via Lie derivative. This extends to a map

$$L : \mathscr{T}_X \times \Omega^{0,*}(X, V_r^s) \to \Omega^{0,*}(X, V_r^s) \ \ , \ \ (\xi, \gamma) \mapsto L_\xi \gamma,$$

giving the Dolbeault complex $\Omega^{0,*}(X, V_r^s)$ the structure of a dg module for $\mathscr{T}(X)$. In a completely analogous way to the lemma above, we have the following.

Lemma 3.5.4. *The $\beta\gamma$ system twisted by the tensor bundle of type (r, s) has an action by the local Lie algebra $\mathscr{T}_X$ given by the local functional*

$$I^{\mathscr{T}}(\xi, \gamma, \beta) = \int \langle \beta, L_\xi \gamma \rangle_{V_r^s}.$$

Hence, it is a holomorphically covariant theory.

3.5.1.3. Higher central charges. Just as in the case of the current algebra, we can apply the factorization enveloping algebra to $\mathscr{T}_X$ to obtain a factorization algebra $\mathbb{U}(\mathscr{T}_X)$ on any complex manifold X. The interesting deformations of this factorization algebra come from local cocycles on $\mathscr{T}_X$ which define the twisted enveloping algebras.

Definition 3.5.5. Let $\alpha \in H^1_{loc}(\mathscr{T}_X)$. The *Virasoro factorization algebra* on X of central charge α is the twisted factorization enveloping algebra $\mathbb{U}_\alpha(\mathscr{T}_X)$.

The motivation for the term *central charge* will become clear momentarily. For a complex manifold of dimension one, we have shown in [**?**] that $H^*_{loc}(\mathscr{T}_\Sigma) = \Omega^*(\Sigma)[1]$. Thus, on a connected Riemann surface there is a unique, up to scale, local cohomology class of degree one that we normalize by $H^1_{loc}(\mathscr{T}_\Sigma) = \mathbb{C} \cdot \omega_{Vir}$. This cocycle ω_{Vir}, which we will recall below, is related to the cocycle defining the usual extension of the one-dimensional Witt algebra. Moreover, in [**?**], we have shown that locally this twisted factorization envelope recovers the Virasoro vertex algebra. Implicit in the statement below is the relationship between one-dimensional holomorphic factorization algebras and vertex algebras that we recalled at the beginning of Section 3.2.

Theorem 3.5.6 ([**?**])**.** *Let $c \in \mathbb{C}$. The factorization envelope $\mathbb{U}_{c\cdot\omega_{Vir}}(\mathscr{T}_\mathbb{C})$ is a holomorphically translation invariant factorization algebra and its cohomology defines a vertex algebra $\mathbb{V}\mathrm{ert}(\mathbb{U}_{c\cdot\omega_{Vir}}(\mathscr{T}_\mathbb{C}))$. Moreover, this vertex algebra is isomorphic to the Virasoro vertex algebra of charge c:*

$$\mathbb{V}\mathrm{ert}(\mathbb{U}_{c\cdot\omega_{Vir}}(\mathscr{T}_\mathbb{C})) \cong \mathrm{Vir}_c.$$

We will see how this twisted factorization enveloping algebra appears when studying quantization of holomorphically covariant theories in any dimension.

The definition of a quantum field theory that is holomorphically covariant is similar to the classical case. We refer again to Section 1.4.3 for the definition of an action of a local Lie algebra on a quantum field theory. Recall, there were essentially two separate notions of a quantum symmetry: that of an action by a local Lie algebra $\mathscr{L}$, and that of an *inner action*. To have an action of a local Lie algebra, one must prescribe a family of $\mathscr{L}$-dependent functionals $\{I^{\mathscr{L}}[L]\}$ satisfying the renormalized quantum master equation modulo functionals that dependent solely on $\mathscr{L}$. To have an inner action, the quantum master equation must be satisfied on the nose. We have discussed a deformation theory for lifting an action to an inner action; in particular, there is an obstruction that lives in $H^1_{\rm loc}(\mathscr{L})$ to lifting an action to an inner action.

Definition 3.5.7. A quantum field theory is holomorphically covariant if it admits an action by the local Lie algebra $\mathscr{T}_X$. The *central charge* of a holomorphically covariant quantum field theory is the obstruction to lift this action to an inner action. This is an element

$$\mathfrak{c}_{\mathscr{E}} \in H^1_{\rm loc}(\mathscr{T}_X)[[\hbar]].$$

In complex dimension one, this definition agrees with the usual definition of the central charge in chiral conformal field theory.

3.5.1.4. Chiral conformal field theory. In complex dimension one there is an intimate relationship between complex and Riemannian structures. Every Riemann surface admits a natural Riemannian metric and hence a conformal structure. Conversely, a conformal class of a metric defines a complex structures. It is well-known that the moduli of Riemann surfaces is equivalent to the moduli of conformal structures.

We can see this at the level of local Lie algebras as follows. Fix a Riemann surface Σ and denote by g_0 the associated Riemannian metric. Define the *Riemannian local Lie algebra* as follows. Using the fixed metric g_0 define the two-term complex

$$\mathrm{Riem}(\Sigma, g_0) = \Gamma(\Sigma, T\Sigma) \xrightarrow{Lg_0} \mathrm{Sym}^2(T^*\Sigma)[-1]$$

where the differential sends a vector field X to $L_X g_0$, the Lie derivative of g_0 with respect to X. The Lie bracket of vector fields gives this complex the structure of a dg Lie algebra. Better yet, it is immediate to see that it is a local Lie algebra. The dg Lie algebra $\mathrm{Riem}(\Sigma, g_0)$ is the derived replacement for the one-shifted tangent space of the moduli space of Riemannian structures on Σ at g_0.

There is a natural map of local Lie algebras $\mathscr{T}_\Sigma \to \mathrm{Riem}(\Sigma, g_0)$. In degree zero this is just the inclusion of the holomorphic tangent bundle inside of the full tangent bundle. In degree one, note that the metric g_0 defines an inclusion

$$T^{*0,1}\Sigma \otimes T^{1,0}\Sigma \cong_{g_0} T^{*0,1}\Sigma \otimes T^{*1,0}\Sigma \subset \mathrm{Sym}^2(T^*\Sigma).$$

Elements of degree one are sections of the bundle on the left-hand side. The map in degree one is the inclusion above.

Next, we define the *conformal local Lie algebra*. This is similar to the Riemannian local Lie algebra where we take into account conformal equivalences of metrics. Define the two-term complex

$$\mathrm{Conf}(\Sigma, g_0) = \Gamma(\Sigma, T\Sigma) \oplus C^\infty(\Sigma) \xrightarrow{D_{g_0}} \mathrm{Sym}^2(T^*\Sigma)[-1].$$

The differential is defined by $D_{g_0}(X, f) = L_X g_0 + f g_0$. The second term encodes the infinitesimal action of the conformal group. The Lie bracket of vector fields combined with the obvious action of vector fields on functions gives the above complex the structure of a local Lie algebra. Of course, every Riemannian structure defines a conformal structure, so there is a map of local Lie algebras $\mathrm{Riem}(\Sigma, g_0) \to \mathrm{Conf}(\Sigma, g_0)$.

Thus, we obtain a composition

$$\mathscr{T}_\Sigma \to \mathrm{Riem}(\Sigma, g_0) \to \mathrm{Conf}(\Sigma, g_0). \tag{3.19}$$

Every conformal field theory (in perturbation theory around the metric g_0) is hence a holomorphically covariant theory in our sense.

For conformal field theories, the Weyl, or trace, anomaly is the quantity that measures the central charge. At the level of the Lie algebra $\mathrm{Conf}(\Sigma, g_0)$ the Weyl anomaly is represented by the local 1-cocycle

$$\phi^{g_0}_{\mathrm{Weyl}}(X, f, \alpha) = \int_\Sigma f R_{g_0+\alpha} \mathrm{dvol}_{g_0+\alpha} + \int_\Sigma \mathrm{Jac}(X) R_{g_0+\alpha} \mathrm{dvol}_{g_0+\alpha},$$

where $R_{g_0+\alpha}$ is the scalar curvature of the metric $g_0 + \alpha$ expanded formally in the variable α. One immediately checks that $\phi^{g_0}_{\mathrm{Weyl}} \in \mathrm{C}^*_{\mathrm{loc}}(\mathrm{Conf}(\Sigma, g_0))$ is a cocycle of cohomological degree one. Moreover, under the map of local Lie algebras, one checks that $\phi^{g_0}_{\mathrm{Weyl}}$ pulls back to the cocycle $\omega_{\mathrm{Vir}} \in \mathrm{C}^*_{\mathrm{loc}}(\mathscr{T}_\Sigma)$ defined by

$$\omega_{\mathrm{Vir}}(\xi_1, \xi_2 \mathrm{d}\bar{z}) = \int_\Sigma \mathrm{Jac}(\xi_1) \partial(\mathrm{Jac}(\xi_2)) \mathrm{d}\bar{z}$$

We have already remarked that ω_{Vir} is the generator of the cohomology $H^1_{\mathrm{loc}}(\mathscr{T}_\Sigma)$ for any Riemann surface. Thus, in complex dimension one we see that our definition of central charge agrees with the usual one from CFT. We have verified an explicit calculation of the central charge in the BV formalism in Section **??** of [?].

3.5.2. Gelfand-Fuks cohomology

Our aim is to classify the space of central charges of a holomorphically covariant quantum field theory in any dimension. The description we give will be in terms of a certain cohomology of vector fields on the disk, called *Gelfand-Fuks* cohomology. In this section we recall some facts about the Lie algebra cohomology of formal vector fields W_d on the d-disk with values in certain non-trivial modules. We refer to Section **??** for the requisite notation for objects living on the formal disk.

In Section 2.1.5.3 we have constructed the formal Atiyah class for any formal vector bundle $\mathcal{V}$ on $\widehat{D}^n$. It is an element of the relative Gelfand-Fuks cohomology

$$\mathrm{At}^{\mathrm{GF}}(\mathcal{V}) \in \mathrm{C}^1_{\mathrm{Lie}}(\mathrm{W}_d, \mathrm{GL}_d; \widehat{\Omega}^1_d \otimes_{\widehat{\mathcal{O}}_d} \mathrm{End}_{\widehat{\mathcal{O}}_d}(\mathcal{V})).$$

From the Atiyah class we have built the formal Chern character using the usual formula

$$\mathrm{ch}^{\mathrm{GF}}(\mathcal{V}) = \mathrm{Tr}\left(\exp\left(\frac{1}{2\pi i}\mathrm{At}^{\mathrm{GF}}(\mathcal{V})\right)\right),$$

and have studied how components of this formal Chern character give rise to L_∞ extensions of W_d that appear as natural universal symmetries of quantizations of higher dimensional holomorphic σ models with target $\widehat{D}^d$.

In this section we arrive at the Lie algebra of formal vector fields, and its cohomology, from a different perspective. Instead of using formal geometry to construct universal objects on the *target* of a σ model, we will see how Gelfand-Fuks classes characterize holomorphic symmetries on the higher *world-sheet*, or source manifold.

The symmetry is that of holomorphic reparametrizations. Infinitesimally, this is described by the Lie algebra of holomorphic vector fields. We have already seen BW: ref that classical theories on a complex manifold X with such a symmetry by holomorphic reparametrizations admit an action by the local Lie algebra $\mathscr{T}_X = \Omega^{0,*}(X, T_X^{1,0})$.

The Gelfand-Fuks classes we will consider in this section appear as anomalies for quantizing an action by the local Lie algebra $\mathscr{T}_X$. In other words, these classes parametrize shifted central extensions of $\mathscr{T}_X$, just as the classes $\theta \in \mathrm{Sym}^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$ defined central extensions of the current algebra $\mathfrak{g}^X$. By our usual yoga of studying equivariant quantizations, we know such anomalies live in the local cohomology complex $\mathrm{C}^*_{\mathrm{loc}}(\mathscr{T}_X)$.

Definition/Lemma 3. Consider the following two classes of cocycles on W_d.

Chern type: For $1 \leq k \leq d$, let $\tau_k \in \mathrm{C}^k_{\mathrm{Lie}}(\mathrm{W}_d; \widehat{\Omega}^k_d)$ be the cocycle

$$\tau_k = s_k\left(\mathrm{At}^{\mathrm{GF}}(\widehat{\mathcal{T}}_d)\right),$$

where $s_k(\mathrm{At}^{\mathrm{GF}}(\widehat{\mathcal{T}}_d))$ is the homogeneous degree k piece of the characteristic polynomial defined by $\det(I + t\mathrm{At}^{\mathrm{GF}}(\widehat{\mathcal{T}}_d))$.

GL type: For $1 \leq i \leq d$ let $a_i \in \mathrm{C}^{2i-1}_{\mathrm{Lie}}(\mathrm{W}_d; \widehat{\mathcal{O}}_d)$ be the cocycle

$$a_i : (\xi_1, \ldots, \xi_{2i-1}) \mapsto \sum_{\sigma \in S_{2i-1}} \mathrm{sign}(\sigma)\mathrm{Tr}(\mathrm{Jac}(\xi_{\sigma(1)}) \cdots \mathrm{Jac}(\xi_{\sigma(2i-1)}).$$

We will use the notation $\widehat{\Omega}^{\#}_d = \oplus_k \widehat{\Omega}^k_d[-k]$ to denote the graded W_d-module with $\widehat{\Omega}^k_d$ sitting in degree k. The wedge product of forms endows this W_d-module with the structure of a graded commutative algebra.

If V is a graded vector space then we use the notation $\mathbb{C}[V]$ to denote the free graded $\mathbb{C}$-algebra on V. If V is spanned by vectors $\{v_i\}$ we will use the shorthand $\mathbb{C}[v_i]$ for this graded algebra.

Theorem 3.5.8 ([**?**])**.** *The bigraded commutative algebra $H^*(\mathrm{W}_d; \widehat{\Omega}^{\#}_d)$ is isomorphic to the bigraded commutative algebra*

$$(\mathbb{C}[a_1, \ldots, a_{2d-1}, \tau_1, \ldots, \tau_d])/\left(\tau_1^{j_1} \cdots \tau_d^{j_d}\right),$$

where the quotient is over all indices $\{j_1, \ldots, j_d\}$ that satisfiy $j_1 + 2j_2 + \cdots + dj_d > d$. Here a_{2i-1} is in bidegree $(2i-1, 0)$ and τ_j is in bidegree (j, j).

In the above result we have not turned on the de Rham differential $\mathrm{d}_{dR} : \widehat{\Omega}^k_d \to \widehat{\Omega}^{k+1}_d$. This endows $\widehat{\Omega}^*_d = (\widehat{\Omega}^{\#}_d, \mathrm{d}_{dR})$ with the structure of a dg commutative algebra in W_d-modules. The formal Poincaré lemma asserts that the inclusion of the trivial W_d-module

$$\mathbb{C} \xrightarrow{\simeq} \widehat{\Omega}^*_d$$

is a quasi-isomorphism. In turn, we obtain a quasi-isomorphism of Chevalley-Eilenberg complexes

$$\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_d) \xrightarrow{\simeq} \mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_d; \widehat{\Omega}^*_d).$$

We may think of the cochain complex $\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_d; \widehat{\Omega}^*_d)$ as the total complex of the double complex with vertical differential given by the W_d Chevalley-Eilenberg differential for the graded module $\widehat{\Omega}^{\#}_d$, and horizontal differential equal to the de Rham differential.

To any double complex there is a spectral sequence abutting to the cohomology of the total complex. The E_1 page of this spectral sequence is given by the cohomology of the vertical differential. Moreover, if the double complex is a bigraded algebra so are each of the pages. In this case, the E_1 page is precisely the bigraded algebra of Theorem 3.5.8 and we have a spectral sequence

$$E_2^{p,q} = \left(H^q(\mathrm{W}_d; \widehat{\Omega}^p_d), \mathrm{d}_{dR}\right) \implies H^*(\mathrm{W}_d; \widehat{\Omega}^*_d) \cong H^*(\mathrm{W}_d). \tag{3.20}$$

Example 3.5.9. For the case $d = 1$ the spectral sequence collapses at the E_2 page. The only nontrivial cohomology is $\mathbb{C}$ in bidegree $(0,0)$ and $a_1 \cdot \tau_1$ in bidgree $(1,2)$. The 1-cocycle valued in formal power series a_1 is given by $a_1(f_i\partial_i) = \partial_i f_i \in \widehat{\mathcal{O}}_n$. The 1-cocycle valued in formal 1-forms τ_1 is given by $\tau_1(g_j\partial_j) = \mathrm{d}_{dR}(\partial_j g_j)$. To obtain the generator of $H^3(\mathrm{W}_1)$ we perform the following zig-zag:

$$\begin{tikzcd}
\mathrm{C}^3_{\mathrm{Lie}}(\mathrm{W}_1) \arrow[r] & \mathrm{C}^3_{\mathrm{Lie}}(\mathrm{W}_1; \widehat{\mathcal{O}}_1) & \\
& \mathrm{C}^2_{\mathrm{Lie}}(\mathrm{W}_1; \widehat{\mathcal{O}}_1) \arrow[u, "\mathrm{d}_{CE}"] \arrow[r, "\mathrm{d}_{dR}"] & \mathrm{C}^2_{\mathrm{Lie}}(\mathrm{W}_1; \widehat{\Omega}^1_1).
\end{tikzcd}$$

The de Rham differential kills $a_1 \cdot \tau_1$, so there exists an $\alpha \in \mathrm{C}^2_{\mathrm{Lie}}(\mathrm{W}_1; \widehat{\mathcal{O}}_1)$ such that $\mathrm{d}_{dR}\alpha = -a_1 \cdot \tau_1$. Now, the class $\mathrm{d}^{\widehat{\mathcal{O}}}_{CE}\alpha \in \mathrm{C}^3_{\mathrm{Lie}}(\mathrm{W}_1; \widehat{\mathcal{O}}_n)$ satisfies

$$\begin{aligned} \mathrm{d}_{dR}(\mathrm{d}^{\widehat{\mathcal{O}}}_{CE}\alpha) &= -\mathrm{d}_{CE}(a_1\tau_1) = 0 \\ \mathrm{d}_{CE}\mathrm{d}^{\widehat{\mathcal{O}}}_{CE}\alpha &= 0. \end{aligned}$$

Here, $\mathrm{d}^{\widehat{\mathcal{O}}}_{CE}$ denote the Chevalley-Eilenberg differential for $\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_1; \widehat{\mathcal{O}}_1)$ and d_{CE} is the restriction of this Chevalley-Eilenberg differential to $\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_1)$. The first line says that $\mathrm{d}_{CE}\alpha$ lifts to $\mathrm{C}^3_{\mathrm{Lie}}(\mathrm{W}_1)$, and the second line says that it is a cocycle for the absolute cohomology. Finally, note that $(\mathrm{d}^{\widehat{\mathcal{O}}}_{CE} + \mathrm{d}_{dR})\alpha = \mathrm{d}^{\widehat{\mathcal{O}}}_{CE}\alpha - a_1\tau_1$. Thus, in the total complex $\mathrm{d}^{\widehat{\mathcal{O}}}_{CE}\alpha$ is homotopic to $a_1\tau_1$, and so $[\mathrm{d}^{\widehat{\mathcal{O}}}_{CE}\alpha]$ is the generator of $H^3(\mathrm{W}_1)$.

For general $d \geq 1$, one can apply this spectral sequence to understand the cohomology $H^*(\mathrm{W}_d)$. To describe it, we introduce the following topological space. Let $\mathrm{Gr}(d,n)$ be the complex Grassmannian of d-planes in $\mathbb{C}^n$. Denote by $\mathrm{Gr}(d,\infty)$ the colimit of the natural sequence

$$\mathrm{Gr}(d,d) \to \mathrm{Gr}(d,d+1) \to \cdots .$$

It is a standard fact that $\mathrm{Gr}(d,\infty)$ is a model for the classifying space $B\mathrm{U}(d)$ of principal $\mathrm{U}(d)$-bundles. Let $E\mathrm{U}(d) \to B\mathrm{U}(d)$ be the universal principal $\mathrm{U}(d)$-bundle. Using the colimit description above, we have a natural skeletal filtration of $B\mathrm{U}(d)$ by

$$\mathrm{sk}_k B\mathrm{U}(d) = \mathrm{Gr}(d,k).$$

Let X_d denote the restriction of $E\mathrm{U}(d)$ over the $2d$-skeleton:

$$\begin{tikzcd} X_d \arrow[r] \arrow[d] & E\mathrm{U}(d) \arrow[d] \\ \mathrm{sk}_{2d} B\mathrm{U}(d) \arrow[r] & B\mathrm{U}(d). \end{tikzcd}$$

Remark 3.5.10. Though not the way the Gelfand and Fuks originally proved the result, one can use the computation of the cohomology of W_d with coefficients in $\widehat{\Omega}^k_d$ together with the spectral sequence (3.20) to prove this description of $H^*(\mathrm{W}_d)$. Indeed, the spectral sequence (3.20) is isomorphic, up to regradings, to the Serre spectral sequence for the principal $\mathrm{U}(d)$-bundle $X_d \to \mathrm{sk}_{2d} B\mathrm{U}(d)$. In other words, the formal de Rham differential on $\widehat{\Omega}^*_d$ is exactly the E_2 differential for the Serre spectral sequence.

Theorem 3.5.11 ([?] Theorem 2.2.4)**.** *There is an isomorphism of graded vector spaces*

$$H^*(\mathrm{W}_d) \cong H^*_{dR}(X_d).$$

Moreover, the commutative algebra structure on $H^*(\mathrm{W}_d)$ is trivial.

As a simple example, note that when $d = 1$ we have $\mathrm{sk}_2 B\mathrm{U}(1) = \mathbb{P}^1 \subset \mathbb{P}^\infty = B\mathrm{U}(1)$. Moreover, the restriction of the universal bundle is Hopf fibration $U(1) \to S^3 \to \mathbb{P}^1$. In particular, one has $X_1 = S^3$.

3.5.3. The local cohomology of holomorphic vector fields

Our main result in this section is a complete classification of the local cohomology of the sheaf of Dolbeualt complex of holomorphic vector fields $\mathscr{T}_X = \Omega^{0,*}(X; T_X^{1,0})$ on any complex manifold. This description involves the Gelfand-Fuks cohomology of formal vector fields that we have just discussed and will give a classifications of the higher dimensional central charges for holomorphically covariant field theories.

Theorem 3.5.12. *Let X be a complex d-fold. There is a quasi-isomorphism of sheaves of cochain complexes*

$$\mathrm{C}^*_{\mathrm{loc}}(\mathscr{T}_X) \ \simeq \ \Omega^*_X \otimes \mathrm{C}^*_{\mathrm{Lie,red}}(\mathrm{W}_d)[2d]$$

where Ω^*_X is the sheaf of de Rham forms on X.

The core of the argument is in interpretting the local Lie algebra cohomology as the cohomology of vector fields on the formal disk through the process of *Gelfand-Kazhdan descent* that we introduced in Chapter **??** Before moving on to the proof, we have the immediate cohomological interpretation of the calculation. Recall that when we study classical BV theories equivariant for a local Lie algebra $\mathscr{L}$, the space the failure for quantizing the BV theory in a way that is equivariant for the Lie algebra is measured by an anomaly class in the local cohomology. For holomorphic diffeomorphisms, we obtain the following.

Corollary 3.5.13. *For X any complex manifold of complex dimension d one has at the level of cohomology*

$$H^k_{\mathrm{loc}}(\mathscr{T}_X) \;\cong\; \bigoplus_{i=0}^{2d} H^i_{dR}(X) \otimes H^{2d+k-i}_{\mathrm{Lie,red}}(\mathrm{W}_d).$$

In particular, if the manifold is connected the space of anomalies for holomorphic diffeomorphisms for a theory defined on X is:

$$H^1_{\mathrm{loc}}(\mathscr{T}_X) = H^{2d+1}_{\mathrm{Lie}}(\mathrm{W}_d) = H^{2d+2}(BU(d)),$$

which is independent of the complex manifold.

The corollary implies that the cohomology $H^{2d+1}_{\mathrm{Lie}}(\mathrm{W}_d)$ deserves to be thought of as the space of "higher dimensional central charges" of a classically holomorphic diffeomorphism invariant theory. After the proof of our main result we will how this relates to the central extensions of holomorphic vector fields and the role of these extensions in quantum field theory.

Proof. We recall a description of the local cohomology complex using D-modules given in Section 4.5 of [**CG**]. Let $\mathscr{L}$ be any local Lie algebra on X with associated graded vector bundle L. The local Lie algebra cohomology is defined as the sheaf

$$\mathrm{C}^*_{\mathrm{loc}}(\mathscr{L}) = \Omega^{d,d}_X \otimes_{D_X} \mathrm{C}^*_{\mathrm{Lie,red}}(JL)$$

where JL is the D_X-module given by taking the ∞-jets of the underlying vector bundle of $\mathscr{L}$. In [**?**] it was shown that $\mathrm{C}^*_{\mathrm{Lie,red}}(JL)$ is flat as a D_X-module, thus we can replace the tensor product above by a left-derived tensor product

$$\Omega^{d,d}_X \otimes_{D_X} \mathrm{C}^*_{\mathrm{Lie,red}}(JL) \simeq \Omega^{d,d}_X \otimes^{\mathbb{L}}_{D_X} \mathrm{C}^*_{\mathrm{Lie,red}}(JL). \tag{3.21}$$

The Spenser resolution is a free resolution of $\Omega^{d,d}_X$ as a right D_X-module (by D_X we mean smooth differential operators) given by

$$M^* = \left(\cdots \to \Omega^{2d-1} \otimes_{C^\infty_X} D_X \xrightarrow{\nabla_D} \Omega^{d,d} \otimes_{C^\infty_X} D_X\right)$$

The differential ∇_D is determined by the natural flat connection on D_X. This complex M^* is concentrated in degree $-2d, \ldots, 0$. Via this resolution, we see that (3.21) is quasi-isomorphic to

$$M^* \otimes_{D_X} \mathrm{C}^*_{\mathrm{Lie,red}}(JL) \simeq \left(\cdots \to \Omega^{2d-1} \otimes_{C^\infty_X} \mathrm{C}^*_{\mathrm{Lie,red}}(JL) \xrightarrow{\nabla_D} \Omega^{d,d} \otimes_{C^\infty_X} \mathrm{C}^*_{\mathrm{Lie,red}}(JL)\right).$$

The right-hand side is, by definition, the shifted de Rham complex of the D_X-module $\mathrm{C}^*_{\mathrm{Lie,red}}(JL)$ so we obtain

$$\mathrm{C}^*_{\mathrm{loc}}(\mathscr{L}) \simeq \Omega^*(X, \mathrm{C}^*_{\mathrm{Lie,red}}(JL))[2d]. \tag{3.22}$$

Now, suppose that $\mathscr{L}$ is a holomorphic local Lie algebra of the form $\Omega^{0,*}(X, L^{hol})$ where L^{hol} is a holomorphic vector bundle. In the above notation, the underlying smooth vector bundle is $L = \bigwedge^* T^{0,1^*}X \otimes L^{hol}$.

We have used the notation JE to denote the smooth sections of the infinite rank vector bundle $\mathrm{Jet}(E)$. If E is a holomorphic vector bundle let $\mathrm{Jet}^{hol}(E)$ denote the infinite rank holomorphic vector bundle of holomorphic jets. Similarly, let $J^{hol}E$ be the holomorphic sections of this bundle. This is a D_X^{hol}-module where D_X^{hol} is the sheaf of holomorphic differential operators. Equivalently, a D_X^{hol}-module is a holomorphic vector bundle with a holomorphic flat connection. Of course, any D_X^{hol}-module E forgets to a smooth D_X-module that we denote E^{C^∞}.

Lemma 3.5.14. *Let L, L^{hol} be as above. There is a quasi-isomorphism of D_X-modules $JL \simeq (J^{hol}L^{hol})^{C^\infty}$.*

Proof. Let $\mathscr{L}$ be the sheaf of sections of L. The Dolbeualt complex is a resolution of the sheaf of holomorphic sections; thus there is a quasi-isomorphism $\mathscr{L} \simeq \mathscr{L}^{hol}$ of $\mathscr{O}_X^{hol}$-modules. $\square$

This means that we can further reduce the expression for the local cohomology in (3.22) to

$$\mathrm{C}^*_{\mathrm{loc}}(\mathscr{L}) \simeq \Omega^*(X, \mathrm{C}^*_{\mathrm{Lie,red}}(J^{hol}L^{hol}))[2d]. \tag{3.23}$$

We have dropped the notation $(-)^{C^\infty}$ for convenience.

We now turn to the local Lie algebra in question, namely $\mathscr{T}_X$. This is, of course, a holomorphic local Lie algebra as it is given by $\mathscr{T}_X = \Omega^{0,*}(X, TX)$. The underlying holomorphic vector bundle is the holomorphic tangent bundle $T^{1,0}X$.

Suppose that $\mathscr{V}$ is any $(\mathrm{W}_d, \mathrm{GL}_d)$-module. Then, Gelfand-Kazhdan descent along the complex manifold X yields the D_X-module $\mathrm{desc}_X(\mathscr{V})$. In the case that $\mathscr{V} = \widehat{\mathcal{T}}_d$ we have seen that the D_X-module $\mathrm{desc}_X(\widehat{\mathcal{T}}_d)$ is equivalent to the D_X-module $J^{hol}T^{1,0}X$.

Lemma 3.5.15. *Gelfand-Kazhdan descent is symmetric monoidal. That is, if $\mathscr{V}, \mathscr{V}'$ are two $(\mathrm{W}_d, \mathrm{GL}_d)$-modules, then*

$$\mathscr{V} \otimes_{\widehat{\mathcal{O}}_n} \mathscr{V}' \simeq \mathrm{desc}_X(\mathscr{V}) \otimes_{J^{hol}\mathscr{O}_X^{hol}} \mathrm{desc}_X(\mathscr{V}').$$

This implies that there is a string of isomorphisms of D_X-modules

$$J^{hol}\mathrm{C}^*_{\mathrm{Lie,red}}(T^{1,0}X) = \mathrm{desc}(\mathrm{C}^*_{\mathrm{Lie,red}}(\mathrm{W}_d)) \cong \mathrm{C}^*_{\mathrm{Lie,red}}(\mathrm{desc}(\mathrm{W}_d)) = \mathrm{C}^*_{\mathrm{Lie,red}}(J^{hol}T^{1,0}X).$$

Equivalently, by **??** we know that the functor of jets is symmetric monoidal, so the same result follows.

To summarize we see that the Gelfand-Kazhdan descent of the $(\mathrm{W}_n, \mathrm{GL}_d)$-module $\mathrm{C}_{\mathrm{Lie,red}}(\mathrm{W}_d)$ is equal to the D_X-module $\mathrm{C}^*_{\mathrm{Lie,red}}(J^{hol}T^{1,0}_X)$. This is precisely the D_X-module present in the definition of the local cohomology of $\mathscr{T}_X$. Indeed, by Lemma 3.5.14 we have

$$\mathrm{C}^*_{\mathrm{loc}}(\mathscr{T}_X) \simeq \Omega^*\left(X, \mathrm{C}^*_{\mathrm{Lie,red}}(J^{hol}T^{1,0}_X)\right).$$

Thus, the de Rham complex of the D_X-module given by descent is precisely the local cohomology

$$\mathrm{C}^*_{\mathrm{loc}}(\mathscr{T}_X) \simeq \Omega^*\left(X, \mathrm{desc}_X(\mathrm{C}^*_{\mathrm{Lie,red}}(\mathrm{W}_d))\right)$$

The interpretation via descent will allow us to describe this de Rham complex explicitly. Suppose that $\mathfrak{g}$ is any Lie algebra. Then $\mathfrak{g}$ acts on itself (and its dual) via the adjoint action. This extends to an action of $\mathfrak{g}$ on its Chevalley-Eilenberg complex $\mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g}; M)$, where M is any $\mathfrak{g}$-module via the formula

$$(x \cdot \varphi)(x_1, \dots, x_k) = \sum_i \varphi(x_1, \dots, [x, x_i], \dots, x_k) - x \cdot \varphi(x_1, \dots, x_k)$$

Here, $x, x_i \in \mathfrak{g}$ and φ is a k-cochain with values in M. The $[-,-]$ denotes adjoint action, and the $\cdot$ is the $\mathfrak{g}$-module structure on M. The following lemma is well-known. The same formula holds for the reduced cochains.

Lemma 3.5.16. *The $\mathfrak{g}$-module structure on the cochain complexes $\mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g})$ and $\mathrm{C}^*_{\mathrm{Lie,red}}(\mathfrak{g})$ is homotopically trivial.*

For the case of an infinite dimensional Lie algebra, such as W_d, the same result holds when we use the continuous, or Gelfand-Fuks, Lie algebra cohomology. Thus, W_d acts homotopically trivial on $\mathrm{C}^*_{\mathrm{Lie,red}}(\mathrm{W}_d)$.

This implies that the descent $\mathrm{desc}_X(\mathrm{W}_d)$ has a homotopically trivial D_X-module structure. Equivalently, this means that the flat connection on $\mathrm{C}^*_{\mathrm{Lie,red}}(J^{hol}T^{1,0}X)$ is gauge equivalent to the trivial connection. Thus, there is a quasi-isomorphism of de Rham complexes

$$\Omega^*\left(X, \mathrm{C}^*_{\mathrm{Lie,red}}(J^{hol}T^{1,0}_X)\right) \simeq \Omega^*(X) \otimes_{C^\infty_X} \underline{\mathrm{C}^*_{\mathrm{Lie,red}}(\mathrm{W}_d)}_X$$

where the underline denotes the sections of the trivial bundle with fiber $\mathrm{C}^*_{\mathrm{Lie,red}}(\mathrm{W}_d)$. We have identified the left hand side with the local cohomology complex, so we are done. $\square$

3.5.3.1. An explicit description of the local cocycles. We'd like to leverage our knowledge of the the Gelfand-Fuks cohomology of formal vector fields to provide an explicit description of the local cocycles. The theorem in the previous section gives a very general equivalence of the local cohomology on any complex manifold with the Gelfand-Fuks cohomology, but writing down the form of the local cocycle from the description on a formal disk is not so obvious.

For instance, consider the case $d = 1$ and we work on $X = \mathbb{C}$. The cohomology $H^*_{\mathrm{Lie,red}}(\mathrm{W}_1)$ is one-dimensional concentrated in degree 3. We'd like to describe the local cocycle corresponding to the generator of $H^3(\mathrm{W}_1) \cong H^1_{\mathrm{loc}}(\mathscr{T}_{\mathbb{C}})$ explicitly. Recall, using the formal Hodge-to-de Rham spectral sequence we saw that the generator of $H^3(\mathrm{W}_1)$ came from the element $a_1\tau_1 \in H^2_{\mathrm{Lie}}(\mathrm{W}_1; \widehat{\Omega}^1_1)$ on the E_2 page of the spectral sequence (3.20).

Now, the 1-cocycles a_1, τ_1 can both be interpreted as functionals on the Dolbeault complex $\Omega^{0,*}(\mathbb{C}, T^{1,0}\mathbb{C})$. Indeed, if $\xi = \alpha(z,\overline{z})\partial_z$ is an element of the Dolbeault complex we can define

$$\widetilde{a}_1(\xi) = \partial_z \alpha(z,\overline{z}) \in \Omega^{0,*}(\mathbb{C})$$

$$\widetilde{\tau}_1(\xi) = \partial(\partial_z \alpha(z,\overline{z})) \in \Omega^{1,*}(\mathbb{C}).$$

Each of these cocycles clearly only depends on the jet of the vector field $\alpha\partial_z$. Similarly, the product is $\widetilde{a}_1\widetilde{\tau}_1$ is the bilinear functional on jets of $\mathscr{T}_{\mathbb{C}}$:

$$\widetilde{a}_1\widetilde{\tau}_1(\xi_1,\xi_2) = \partial_z\alpha_1(z,\overline{z})\partial(\partial_z\alpha_2(z,\overline{z})) \in \Omega^{1,*}(\mathbb{C})$$

This is a density precisely when $|\alpha_1|+|\alpha_2| = 1$. Thus, $\widetilde{a}_1\widetilde{\tau}_1$ determines a degree $+1$ density valued cochain on $J\mathscr{T}_{\mathbb{C}}$; in other other words an element of $\mathrm{C}^*_{\mathrm{loc}}(\mathscr{T}_{\mathbb{C}})$ that we write as

$$\int_{\mathbb{C}} \partial_z\alpha_1(z,\overline{z})\partial(\partial_z\alpha_2(z,\overline{z})),$$

which is the local cocycle we denoted ω_{Vir} above. If we integrate by parts, we can put this local functional in the form $\int f\partial_z^3 g \mathrm{d}z\mathrm{d}\overline{z}$. If one restricts this local functional to the annulus and performs the radial integration, we recover the usual formula for the generator of $H^2(\mathrm{Vect}(S^1))$ BW: citation defining the central extension of the Virasoro Lie algebra.

This can be generalized to arbitrary dimensions in a natural way.

Our first goal is to construct, from a Gelfand-Fuks classes in $\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_d)$ and $\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n;\widehat{\Omega}^*_d)$, a local functional on $\mathscr{T}_X$. We have seen that the cochain complex $\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_d;\widehat{\Omega}^*_d)$, equipped with the total differential $\mathrm{d}_{CE}+\mathrm{d}_{dR}$, computes the absolute Gelfand-Fuks cohomology $H^*(\mathrm{W}_d)$ through the formal Hodge to de Rham spectral sequence. We will use this property to represent elements of $H^*(\mathrm{W}_d)$ by local functionals on $\mathscr{T}_X$ by first representing elements in $\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n;\widehat{\Omega}^*_d)$ by local functionals.

We can decompose an element $\alpha \in \mathrm{C}^k_{\mathrm{Lie}}(\mathrm{W}_d;\widehat{\Omega}^*_d)$ as

$$\alpha = f^I \mathrm{d}t_I.$$

The sum is over the multi-index $I=(i_1,\ldots,i_k)$, where $1 \leq i_j \leq d$. For each I, f^I is a k multi-linear symmetric functional on W_d valued in $\widehat{\mathcal{O}}_d$

$$f^I : \mathrm{Sym}^k(\mathrm{W}_d[1]) \to \widehat{\mathcal{O}}_d.$$

We extend f^I to a functional on the Dolbeault complex $\Omega^{0,*}(\mathbb{C}^d;T^{1,0}\mathbb{C}^d)$ as follows. Using the framing on $\mathbb{C}^d$, every element of the Dolbeualt complex can be written as

$$X^J(z,\overline{z})\mathrm{d}\overline{z}_J$$

where $J = (j_1, \ldots, j_l)$ is a multi-index and X^J is an ordinary holomorphic vector field on $\mathbb{C}^d$. We extend f^I to a Dolbeualt valued functional $\Omega^{0,*}(\mathbb{C}^d; T^{1,0}\mathbb{C}^d)$ via the formula

$$\begin{array}{cccc} f^I_{\Omega^{0,*}} : & \mathrm{Sym}^k\left(\Omega^{0,*}(\mathbb{C}^d; T^{1,0})\right) & \to & \Omega^{0,*}(\mathbb{C}^d) \\ & \left(X_1^{J(1)}(z,\overline{z})\mathrm{d}\overline{z}_{J(1)}, \ldots, X_k^{J(k)}(z,\overline{z})\mathrm{d}\overline{z}_{J(k)}\right) & \mapsto & f^I(X_1^{J(1)}, \ldots, X_k^{J(k)})\mathrm{d}\overline{z}_{J(1)} \wedge \cdots \mathrm{d}\overline{z}_{J(k)} \end{array}$$

The local functional corresponding to the original element $\alpha = f^I \mathrm{d}t_I \in \mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n; \widehat{\Omega}^*_d)$ is defined by the k-multi-linear functional

$$(\xi_1, \ldots, \xi_k) \mapsto \int_{\mathbb{C}^d} f^I_{\Omega^{0,*}}(\xi_1, \ldots, \xi_k)\mathrm{d}z_I.$$

Denote this functional by $J^{GF}(\alpha)$. Note that it is only nonzero when the multi-index I is a permutation of $(1, \ldots, d)$. Since it is given by the integral of a some multi-differential operators against a density it is manifestly a local functional.

3.5.3.2. A comparison to other cocycles. We have already seen in Section **??** that when $d = 1$ the unique local cocycle associated to the generator in $H^3(\mathrm{W}_1)$ agrees with the Weyl anomaly in CFT. There has been an extensive effort in both the physics and math community to classify cocycles pertaining to conformal anomalies in any dimension. In real dimensions 4 and 6 see [**?**, **?**] and for conjectures in higher dimensions see [**?**]. Our results above classify holomorphic versions of these conformal anomalies in arbitrary even dimensions.

In complex dimensions 2 and 3 the dimension of the space of the degree one local cohomology $H^1_{\mathrm{loc}}(\mathscr{T}_X)$ agrees with dimension of the space of non-equivalent cocycles for the higher conformal anomalies studied in [**?**, **?**]. Specifically, in complex dimension 2 (so real dimension 4) this space is 2-dimensional corresponding to the characteristic classes

$c_1^3, c_1c_2 \in H^6(BU(2))$. In complex dimension 3 this space is 4-dimensional corresponding to the classes $c_1^4, c_1^2c_2, c_1c_3, c_2^2$. It would be interesting to directly relate our local cocycles to theirs.

These references describe two classes of cocycles: Type a and type b-cocycles. In any dimension, there is a single type a-cocycle, and the remaining cocycles are of type b. Using the isomorphism $H^{2d+1}(\mathrm{W}_d) \cong H^{2d+2}(BU(d))$ we conjecture that the type a-cocycle corresponds to the characteristic class $c_1c_d \in H^{2d+2}(BU(d))$.

3.5.4. The holomorphic anomaly for σ-models

.1. The dg model for punctured affine space

In this section we review a dg model for the derived space of sections of the structure sheaf on punctured affine space in any dimensions. We will be mostly concerned with the sheaf of algebraic functions. This model has appeared in the work of [**FHK**], based on the Jouanolou resolution of singularities.

Let $\mathbb{A}^d$ be algebraic affine space with sheaf of functions given by $\mathscr{O}^{alg}(\mathbb{A}^d) = \mathbb{C}[z_1, \ldots, z_d]$. Denote $\mathbb{A}^{d\times} = \mathbb{A}^d \setminus \{0\}$. When $d = 1$ the punctured space $\mathbb{A}^{1\times}$ is an affine scheme with $H^0(\mathbb{A}^{1\times}, \mathscr{O}^{alg}) = \mathbb{C}[z^{\pm}]$. When $d > 1$ the punctured space $\mathbb{A}^{d\times}$ is no longer affine. In fact, the cohomology is

$$H^*(\mathbb{A}^{d\times}, \mathscr{O}^{alg}) = \begin{cases} 0, & * \neq 0, d-1 \\ \mathbb{C}[z_1, \ldots, z_d], & * = 0 \\ \mathbb{C}[z_1^{-1}, \ldots, z_d^{-1}]\frac{1}{z_1 \cdots z_d}, & * = d-1 \end{cases}.$$

The dg commutative algebra $\mathbb{R}(\mathbb{A}^{d\times}, \mathscr{O}^{alg})$ is well-defined up to quasi-isomorphism. We will recall the construction of an explicit model.

Definition .1.1. Let $A_d = \oplus_{p=0}^{d} \oplus_{q=0}^{d} A_d^{p,q}$ be the bigraded commutative algebra generated by elements

$$z_1, \ldots, z_d, z_1^*, \ldots, z_d^*, (zz^*)^{-1}$$

in bidegree $(0,0)$, where $zz^* = \sum_i z_i z_i^*$, elements

$$\mathrm{d}z_1, \ldots, \mathrm{d}z_d$$

in bidegree $(1,0)$, and

$$\mathrm{d}z_1^*, \ldots, \mathrm{d}z_d^*$$

in bidegree $(0,1)$. Introduce a $*$-weight, so that $z_i^*, \mathrm{d}z_i^*$ have $*$-weight $+1$ and $(z_i^*)^{-1}$ has $*$-weight -1. We require that:

(i) every element is of total $*$-weight zero and

(ii) the contraction of every element with the Euler vector field $\sum_i z_i^* \partial_{z_i^*}$ vanishes.

There is a map $\overline{\partial} : A_d^{p,q} \to A_d^{p,q+1}$ of bidegree $(0,1)$ defined formally by

$$\overline{\partial} = \sum_i \mathrm{d}z_i^* \frac{\partial}{\partial z_i^*}$$

and a map of bidegree $(1,0)$ defined by

$$\partial = \sum_i \mathrm{d}z_i \frac{\partial}{\partial z_i}.$$

This differentials commute $\overline{\partial}\partial = \partial\overline{\partial}$ and each square to zero.

When $p = 0$ we see that the resulting complex $(A_d, \overline{\partial}) = (\oplus_q A_d^q[-q], \overline{\partial})$ has the structure of a commutative dg algebra. This commutative dg algebra is model for $\mathbb{R}(\mathbb{A}^{d\times}, \mathscr{O}^{alg})$. Note that by conditions (i),(ii) this complex is concentrated in degrees $0, 1, \ldots, d-1$.

For each p, the complex $A_d^{p,*} = (\oplus_q A^{p,q}[-q], \overline{\partial})$ is a model for the $\mathbb{R}\Gamma(\mathbb{A}^{d\times}, \mathscr{O}^{alg})$-module given by the derived space of sections of holomorphic p-forms $\mathbb{R}\Gamma(\mathbb{A}^{d\times}, \Omega^{p,alg})$. We will denote the resulting bigraded algebra by

$$A_d^{*,*} = \oplus_{p=0} A_d^{p,*}[-p] = \oplus_{p=0} \oplus_{q=0} A_d^{p,q}[-p-q].$$

It is immediate to check that the formula for the ordinary Bochner-Martinelli kernel makes sense in the algebra A_d. That is, we define

$$\omega_{BM}^{alg}(z, z^*) = \frac{(d-1)!}{(2\pi i)^d} \frac{1}{(zz^*)^d} \sum_{i=1}^{d} (-1)^{i-1} z_i^* \mathrm{d}z_1^* \wedge \cdots \wedge \widehat{\mathrm{d}z_i^*} \wedge \cdots \wedge \mathrm{d}z_d^*,$$

which is an element of A_d^{d-1}.

The key properties of the dg algebra A_d and its dg modules $A_d^{p,*}$ we will utilize are summarized in the following result of [**FHK**].

Proposition .1.2 ([**FHK**] Proposition 1.3.1)**.**

(1) The commutative dg algebra $(A_d, \overline{\partial})$ is a model for $\mathbb{R}\Gamma(A^{d\times}, \mathscr{O}^{alg})$

$$A_d \simeq \mathbb{R}\Gamma(\mathbb{A}^{d\times}, \mathscr{O}^{alg}).$$

Similarly, $(A_d^{p,*}, \overline{\partial}) \simeq \mathbb{R}\Gamma(\mathbb{A}^{d\times}, \Omega^{p,alg})$.

(2) There is a dense map of commutative bigraded algebras

$$j : A_d^{*,*} \to \Omega^{*,*}(\mathbb{C}^d \setminus \{0\})$$

sending $z_i \mapsto z_i$, $z_i^* \mapsto \overline{z}_i$, *and* $\mathrm{d}z_i^* \mapsto \mathrm{d}\overline{z}_i$ *that is compatible with the* $\overline{\partial}$ *and* ∂ *differentials on both sides.*

(3) Finally, there is a unique GL_n*-equivariant residue map*

$$\mathrm{Res}_{z=0} : A_d^{d,d-1} \to \mathbb{C}$$

that satisfies

$$\mathrm{Res}_{z=0}\left(f(z)\omega_{BM}^{alg}(z,z^*)\mathrm{d}z_1 \cdots \mathrm{d}z_d\right) = f(0)$$

where $f(z) \in \mathbb{C}[z_1,\ldots,z_d]$*. In particular, for any* $\omega \in A_d^{d,d-1}$ *one has*

$$\mathrm{Res}_{z=0}(\omega) = \oint_{S^{2d-1}} j(\omega)$$

where S^{2d-1} *is any sphere centered at the origin in* $\mathbb{C}^d$*.*

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APPENDIX A

Title of Appendix

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