LOCAL FUNCTIONALS FROM FORMAL VECTOR FIELDS

0.1. **Gelfand-Fuks cohomology.** The formal disk $\widehat{D}^d$ has ring of functions given by formal power series in d-variables $\widehat{\mathcal{O}}_d = \mathbb{C}[[t_1, \ldots, t_d]]$. Let W_n be the Lie algebra of formal vector fields. In other words, the Lie algebra of derivations of $\widehat{\mathcal{O}}_d$.

As with any Lie algebra, there is the adjoint module that we denote by $\widehat{\mathcal{T}}_d$.

Definition/Lemma 1. Consider the following two classes of cocycles on W_d.

Chern type: For $1 \leq k \leq n$, let $\tau_k \in \mathrm{C}^k_{\mathrm{Lie}}(\mathrm{W}_d; \widehat{\Omega}^k_d)$ be the cocycle

$$\tau_k = \sigma_k \left(\mathrm{At}(\widehat{\mathcal{T}}_d)\right) ...finish...$$

GL type: For $1 \leq i \leq d$ let $\xi_i \in \mathrm{C}^{2i-1}_{\mathrm{Lie}}(\mathrm{W}_d; \widehat{\mathcal{O}}_d)$ be the cocycle

$$\xi_i : (f_{i_1}\partial_{i_1}, \ldots, f_{i_{2i-1}}\partial_{i_{2i-1}}) \mapsto \sum ..$$

We will use the notation $\widehat{\Omega}^\#_d = \oplus_k \widehat{\Omega}^k_d[-k]$ to denote the graded W_d-module with $\widehat{\Omega}^k_d$ sitting in degree k. The wedge product of forms endows this W_d-module with the structure of a graded commutative algebra.

If V is a graded vector space then we use the notation $\mathbb{C}[V]$ to denote the free graded $\mathbb{C}$-algebra on V. If V is spanned by vectors $\{v_i\}$ we will use the shorthand $\mathbb{C}[v_i]$ for this graded algebra.

Theorem 0.1 ([?]). *The bigraded commutative algebra $H^*(\mathrm{W}_d; \widehat{\Omega}^\#_d)$ is isomorphic to the bigraded commutative algebra*

$$\left(\mathbb{C}[\xi_1, \ldots, \xi_{2d-1}, \tau_1, \ldots, \tau_d]\right) / \left(c_1^{j_1} \cdots c_d^{j_d}\right),$$

where the quotient is over all indices $\{j_1, \ldots, j_d\}$ that satisfiy $j_1 + 2j_2 + \cdots + dj_d > d$. Here ξ_{2i-1} is in bidegree $(2i-1, 0)$ and τ_j is in bidegree (j, j).

In the above result we have not turned on the de Rham differential $\mathrm{d}_{dR} : \widehat{\Omega}^k_d \to \widehat{\Omega}^{k+1}_d$. This endows $\widehat{\Omega}^*_d = (\widehat{\Omega}^\#_d, \mathrm{d}_{dR})$ with the structure of a dg commutative algebra in W_d-modules. The formal Poincaré lemma asserts that the inclusion of the trivial W_d-module

$$\mathbb{C} \xrightarrow{\simeq} \widehat{\Omega}^*_d$$

is a quasi-isomorphism. In turn, we obtain a quasi-isomorphism of Chevalley-Eilenberg complexes

$$\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_d) \xrightarrow{\simeq} \mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_d; \widehat{\Omega}^*_d).$$

We may think of the cochain complex $\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_d; \widehat{\Omega}^*_d)$ as the total complex of the double complex with vertical differential given by the W_d Chevalley-Eilenberg differential for the graded module $\widehat{\Omega}^\#_d$, and horizontal differential equal to the de Rham differential.

To any double complex there is a spectral sequence abutting to the cohomology of the total complex. The E_1 page of this spectral sequence is given by the cohomology of the vertical differential. Moreover, if the double complex is a bigraded algebra so are each of the pages. In this case, the E_1 page is precisely the bigraded algebra of Theorem 0.1 and we have a spectral sequence

$$E_2^{p,q} = \left(H^q(\mathrm{W}_d; \widehat{\Omega}^p_d), \mathrm{d}_{dR}\right) \implies H^*(\mathrm{W}_d; \widehat{\Omega}^*_d) \cong H^*(\mathrm{W}_d). \tag{1}$$

Example 0.2. For the case $d = 1$ the spectral sequence collapses at the E_2 page. The only nontrivial cohomology is $\mathbb{C}$ in bidegree $(0,0)$ and $\xi_1 \cdot \tau_1$ in bidgree $(1,2)$. The 1-cocycle valued in formal power series ξ_1 is given by $\xi_1(f_i \partial_i) = \partial_i f_i \in \widehat{\mathcal{O}}_n$. The 1-cocycle valued in formal 1-forms τ_1 is given by $\tau_1(g_j \partial_j) = \mathrm{d}_{dR}(\partial_j g_j)$. To obtain the generator of $H^3(\mathrm{W}_1)$ we perform the following zig-zag:

$$\begin{tikzcd}
\mathrm{C}^3_{\mathrm{Lie}}(\mathrm{W}_1) \arrow[r] & \mathrm{C}^3_{\mathrm{Lie}}(\mathrm{W}_1; \widehat{\mathcal{O}}_1) & \\
& \mathrm{C}^2_{\mathrm{Lie}}(\mathrm{W}_1; \widehat{\mathcal{O}}_1) \arrow[u, "\mathrm{d}_{CE}"] \arrow[r, "\mathrm{d}_{dR}"] & \mathrm{C}^2_{\mathrm{Lie}}(\mathrm{W}_1; \widehat{\Omega}^1_1).
\end{tikzcd}$$

The de Rham differential kills $\xi_1 \cdot \tau_1$, so there exists an $\alpha \in \mathrm{C}^2_{\mathrm{Lie}}(\mathrm{W}_1; \widehat{\mathcal{O}}_1)$ such that $\mathrm{d}_{dR}\alpha = -\xi_1 \cdot \tau_1$. Now, the class $\mathrm{d}^{\widehat{\mathcal{O}}}_{CE}\alpha \in \mathrm{C}^3_{\mathrm{Lie}}(\mathrm{W}_1; \widehat{\mathcal{O}}_n)$ satisfies

$$\begin{aligned}
\mathrm{d}_{dR}(\mathrm{d}^{\widehat{\mathcal{O}}}_{CE}\alpha) &= -\mathrm{d}_{CE}(\xi_1\tau_1) = 0 \\
\mathrm{d}_{CE}\mathrm{d}^{\widehat{\mathcal{O}}}_{CE}\alpha &= 0.
\end{aligned}$$

Here, $\mathrm{d}^{\widehat{\mathcal{O}}}_{CE}$ denote the Chevalley-Eilenberg differential for $\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_1; \widehat{\mathcal{O}}_1)$ and d_{CE} is the restriction of this Chevalley-Eilenberg differential to $\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_1)$. The first line says that $\mathrm{d}_{CE}\alpha$ lifts to $\mathrm{C}^3_{\mathrm{Lie}}(\mathrm{W}_1)$, and the second line says that it is a cocycle for the absolute cohomology. Finally, note that $(\mathrm{d}^{\widehat{\mathcal{O}}}_{CE} + \mathrm{d}_{dR})\alpha = \mathrm{d}^{\widehat{\mathcal{O}}}_{CE}\alpha - \xi_1\tau_1$. Thus, in the total complex $\mathrm{d}^{\widehat{\mathcal{O}}}_{CE}\alpha$ is homotopic to $\xi_1\tau_1$, and so $[\mathrm{d}^{\widehat{\mathcal{O}}}_{CE}\alpha]$ is the generator of $H^3(\mathrm{W}_1)$.

For general $d \geq 1$, one can apply this spectral sequence to understand the cohomology $H^*(\mathrm{W}_d)$. To describe it, we introduce the following topological space. Let $\mathrm{Gr}(d,n)$ be the complex Grassmannian of d-planes in $\mathbb{C}^n$. Denote by $\mathrm{Gr}(d,\infty)$ the colimit of the natural sequence

$$\mathrm{Gr}(d,d) \to \mathrm{Gr}(d,d+1) \to \cdots .$$

It is a standard fact that $\mathrm{Gr}(d,\infty)$ is a model for the classifying space $B\mathrm{U}(d)$ of principal $\mathrm{U}(d)$-bundles. Let $E\mathrm{U}(d) \to B\mathrm{U}(d)$ be the universal principal $\mathrm{U}(d)$-bundle. Using the colimit description above, we have a natural skeletal filtration of $B\mathrm{U}(d)$ by

$$\mathrm{sk}_k B\mathrm{U}(d) = \mathrm{Gr}(d,k).$$

Let X_d denote the restriction of $E\mathrm{U}(d)$ over the $2d$-skeleton:

$$\begin{tikzcd}
X_d \arrow[r] \arrow[d] & E\mathrm{U}(d) \arrow[d] \\
\mathrm{sk}_{2d} B\mathrm{U}(d) \arrow[r] & B\mathrm{U}(d).
\end{tikzcd}$$

Remark 0.3. Though not the way the Gelfand and Fuks originally proved the result, one can use the computation of the cohomology of W_d with coefficients in $\widehat{\Omega}^k_d$ together with the spectral sequence (1) to prove this description of $H^*(\mathrm{W}_d)$. Indeed, the spectral sequence (1) is isomorphic, up to regradings, to the Serre spectral sequence for the principal $\mathrm{U}(d)$-bundle $X_d \to \mathrm{sk}_{2d} B\mathrm{U}(d)$.

In other words, the formal de Rham differential on $\widehat{\Omega}^*_d$ is exactly the E_2 differential for the Serre spectral sequence.

Theorem 0.4 ([**?**] Theorem 2.2.4)**.** *There is an isomorphism of graded vector spaces*

$$H^*(\mathrm{W}_d) \cong H^*_{dR}(X_d).$$

Moreover, the commutative algebra structure on $H^*(\mathrm{W}_d)$ is trivial.

As a simple example, note that when $d = 1$ we have $\mathrm{sk}_2 B\mathrm{U}(1) = \mathbb{P}^1 \subset \mathbb{P}^\infty = B\mathrm{U}(1)$. Moreover, the restriction of the universal bundle is Hopf fibration $U(1) \to S^3 \to \mathbb{P}^1$. In particular, one has $X_1 = S^3$.

0.2. **Local cocycles on holomorphic vector fields.** We now turn to a description of local central extensions of the local Lie algebra of holomorphic vector fields $\mathcal{T}_X = \Omega^{0,*}(X; T^{1,0}_X)$ for any complex d-fold X. Recall, such a central extension is determined by a cocycle in complex of local functionals $\mathrm{C}^*_{\mathrm{loc}}(\mathcal{T}_X)$. Our main result is to identify such local cocycles with Gelfand-Fuks cocycles we have just discussed.

Our first goal is to construct, from a Gelfand-Fuks class in $\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_d)$, a local functional on $\mathcal{T}_X$. We have seen that the cochain complex $\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_d; \widehat{\Omega}^*_d)$, equipped with the total differential $\mathrm{d}_{CE} + \mathrm{d}_{dR}$, computes the absolute Gelfand-Fuks cohomology $H^*(\mathrm{W}_d)$. We will use this property to represent elements of $H^*(\mathrm{W}_d)$ by local cocycles on $\mathcal{T}_X$.

Using the natural framing on the formal disk, we can decompose a class $\alpha \in \mathrm{C}^k_{\mathrm{Lie}}(\mathrm{W}_d; \widehat{\Omega}^*_d)$ as

$$\alpha = f^I \mathrm{d}t_I$$

where the sum is over the multi-index $I = (i_1, \ldots, i_k)$ where $1 \leq i_j \leq d$, and for each I, f^I is a k multi-linear symmetric functional on W_d valued in $\widehat{\mathcal{O}}_d$

$$f^I : \mathrm{Sym}^k(\mathrm{W}_d[1]) \to \widehat{\mathcal{O}}_d.$$

We extend f^I to a functional on the Dolbeault complex $\Omega^{0,*}(\mathbb{C}^d; T^{1,0}\mathbb{C}^d)$ as follows. Using the framing on $\mathbb{C}^d$, every element of the Dolbeualt complex can be written as

$$X^J(z, \bar{z})\mathrm{d}\bar{z}_J$$

where $J = (j_1, \ldots, j_l)$ is a multi-index and X^J is an ordinary holomorphic vector field on $\mathbb{C}^d$. We extend f^I to a Dolbeualt valued functional $\Omega^{0,*}(\mathbb{C}^d; T^{1,0}\mathbb{C}^d)$ via the formula

$$\begin{array}{cccc} f^I_{\Omega^{0,*}} : & \mathrm{Sym}^k\left(\Omega^{0,*}(\mathbb{C}^d; T^{1,0})\right) & \to & \Omega^{0,*}(\mathbb{C}^d) \\ & \left(X_1^{J(1)}(z,\bar{z})\mathrm{d}\bar{z}_{J(1)}, \ldots, X_k^{J(k)}(z,\bar{z})\mathrm{d}\bar{z}_{J(k)}\right) & \mapsto & f^I(X_1^{J(1)}, \ldots, X_k^{J(k)})\mathrm{d}\bar{z}_{J(1)} \wedge \cdots \mathrm{d}\bar{z}_{J(k)} \end{array}$$

The local functional corresponding to the original class $\alpha = f^I \mathrm{d}t_I \in \mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n; \widehat{\Omega}^*_d)$ is defined by the k-multi-linear functional

$$(\xi_1, \ldots, \xi_k) \mapsto \int_{\mathbb{C}^d} f^I_{\Omega^{0,*}}(\xi_1, \ldots, \xi_k)\mathrm{d}z_I.$$

Denote this functional by $J^{GF}(\alpha)$. Note that it is only nonzero when the multi-index I is a permutation of $(1, \ldots, d)$. Since it is given by the integral of a some multi-differential operators against a density it is manifestly a local functional.

Proposition 0.5. *Let $\mathrm{C}^*_{\mathrm{loc}}(\mathcal{T}_{\mathbb{C}^d})$ be the local functionals of $\mathcal{T}_{\mathbb{C}^d}$ on $\mathbb{C}^d$. The map*

$$J^{GF} : \mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_d; \widehat{\Omega}^*_n)[2d] \to \mathrm{C}^*_{\mathrm{loc}}(\mathcal{T}_{\mathbb{C}^d})$$

sending $\alpha \mapsto J^{GF}(\alpha)$ is a map of cochain complexes. Moreover, it is a quasi-isomorphism.

Theorem 0.6. *Let X be a complex d-fold. Then, the map*

$$J^{GF} : \underline{\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_d; \widehat{\Omega}^*_n)}[2d] \to \mathrm{C}^*_{\mathrm{loc}}(\mathcal{T}_X)$$

is a quasi-isomorphism of sheaves. In particular, there is an isomorphism of graded vector spaces

$$H^{*+2d}(\mathrm{W}_n) \cong H^*(X, \mathrm{C}^*_{\mathrm{loc}}(\mathcal{T}_X)), \tag{2}$$

where the right-hand side denotes the hypercohomology.

Example 0.7. Again, take the case $d = 1$. We can describe the local cocycle corresponding to the generator $H^3(\mathrm{W}_1) \cong H^1(\mathcal{T}_{\mathbb{C}})$ explicitly. Recall, the generator of $H^3(\mathrm{W}_1)$ came from the element $\xi_1\tau_1 \in \mathrm{C}^2_{\mathrm{Lie}}(\mathrm{W}_1; \widehat{\Omega}^1_1)$ on the E_2 page of the spectral sequence (1). Using the formulas for ξ_1, τ_1 above, we see that the local functional $J^{GF}(\xi_1\tau_1)$ is given by

$$\left(f(z,\bar{z})\frac{\partial}{\partial z}, g(z,\bar{z})\mathrm{d}\bar{z}\frac{\partial}{\partial z}\right) \mapsto \int_{\mathbb{C}} \left(\frac{\partial}{\partial z} f\right) \partial \left(\frac{\partial}{\partial z} g\right) \mathrm{d}\bar{z}.$$

For instance, the linear functional $\tau_1 : g(t)\frac{\partial}{\partial t} \mapsto \mathrm{d}_{dR}(\partial_t g(t))$ is mapped to the functional on the Dolbeualt complex of holomorphic vector fields given by $g(z,\bar{z})\frac{\partial}{\partial z} \mapsto \partial(\partial_z g(z,\bar{z}))$.

If we integrate by parts, we can put $J^{GF}(\xi_1\tau_1)$ in the form $\int f\partial_z^3 g \mathrm{d}z\mathrm{d}\bar{z}$. If one restricts this local functional to the annulus and performs the radial integration, we recover the usual formula for the generator of $H^2(\mathrm{Vect}(S^1))$ BW: citation defining the central extension of the Virasoro Lie algebra. In fact, in **[?]** BW: finish