**SCRATCH**

**Lemma 0.1.** *Let $(k,(\vec{n}))$ be a pair of wheel data such that $k > d+1$. Then the $\epsilon \to 0$ limit of the analytic weight*

$$\lim_{\epsilon\to 0} \widetilde{W}^{k,(n)}_{\epsilon<L} = 0$$

*is identically zero as a distribution on $\mathbb{C}^{dk}$.*

*Proof.* The proof is very similar to the argument we gave in the proof of Lemma **??**, so we will be a bit more concise. First, we make the familiar change of coordinates as in Equations (**??**),(**??**). Using the explicit form the heat kernel and propagator we see that for any $\Phi \in C^\infty_c(\mathbb{C}^{dk})$ the weight is

$$\begin{aligned}
\widetilde{W}^{k,(n)}_{\epsilon<L}(\Phi) \quad &= \int_{w^k\in\mathbb{C}^d} \mathrm{d}^{2d}w^k \int_{(w_1,\ldots,w_{k-1})\in(\mathbb{C}^d)^{k-1}} \left(\prod_{\alpha=1}^{k-1}\mathrm{d}^{2d}w^\alpha\right)\Phi(w^1,\ldots,w^k)\int_{(t_1,\ldots,t_k)\in[\epsilon,L]^{k-1}} \frac{1}{(4\pi\epsilon)^d}\prod_{\alpha=1}^{k-1}\frac{\mathrm{d}t_\alpha}{(4\pi t_\alpha)^d} \\
&\times \sum_{i_1,\ldots,i_{k-1}=1}^{d} \epsilon_{i_1,\ldots,i_d}\left(\frac{\overline{w}^1_{i_1}}{t_1}\frac{(\overline{w}^1)^{n^1}}{t^{|n^1|}}\right)\cdots\left(\frac{\overline{w}^{k-1}_{i_{k-1}}}{t_{k-1}}\frac{(\overline{w}^{k-1})^{n^{k-1}}}{t^{|n^{k-1}|}}\right)\left(\frac{1}{t^{|n^k|}}\left(\sum_{\alpha=1}^{k-1}\overline{w}^\alpha\right)^{n^k}\right) \\
&\times \exp\left(-\sum_{\alpha=1}^{k-1}\frac{|w^\alpha|^2}{t_\alpha}-\frac{1}{\epsilon}\left|\sum_{\alpha=1}^{k-1}w^\alpha\right|^2\right).
\end{aligned}$$

We will integrate by parts to eliminate the factors of $\overline{w}^\alpha_i$.

For each $1 \le \alpha < k$ and $i_\alpha$, define the $\epsilon$ and $t = (t_1,\ldots,t_{k-1})$-dependent holomorphic differential operator

$$D_{\alpha,i_\alpha}(t) := \left(\frac{\partial}{\partial w^\alpha_{i_\alpha}} - \sum_{\beta=1}^{k-1}\frac{t_\beta}{t_1+\cdots+t_{k-1}+\epsilon}\frac{\partial}{\partial w^\beta_{i_\alpha}}\right)\prod_{j=1}^{d}\left(\frac{\partial}{\partial w^\alpha_j} - \sum_{\beta=1}^{k-1}\frac{t_\beta}{t_1+\cdots+t_{k-1}+\epsilon}\frac{\partial}{\partial w^\beta_j}\right)^{n^\alpha_j}.$$

And the $\epsilon, t$-dependent holomorphic differential operator

$$D_k(t) = \prod_{j=1}^{d}\left(\frac{\partial}{\partial w^k_j} - \sum_{\beta=1}^{k-1}\frac{t_\beta}{t_1+\cdots+t_{k-1}+\epsilon}\frac{\partial}{\partial w^\beta_j}\right)^{n^k_j}.$$

By a completely analogous version of Lemma the operators above allow us to integrate by parts and express the weight in the form

$$\begin{aligned}
\widetilde{W}^{k,(n)}_{\epsilon<L}(\Phi) \quad &= \pm\int_{w^k\in\mathbb{C}^d} \mathrm{d}^{2d}w^k \int_{(w_1,\ldots,w_{k-1})\in(\mathbb{C}^d)^{k-1}} \left(\prod_{\alpha=1}^{k-1}\mathrm{d}^{2d}w^\alpha\right)\int_{(t_1,\ldots,t_{k-1})\in[\epsilon,L]^{k-1}} \frac{1}{(4\pi\epsilon)^d}\prod_{\alpha=1}^{k-1}\frac{\mathrm{d}t_\alpha}{(4\pi t_\alpha)^d} \\
&\times\left(\sum_{i_1,\ldots,i_{k-1}}\epsilon_{i_1\cdots,i_d}D_{1,i_1}(t)\cdots D_{k-1,i_{k-1}}(t)D_k(t)\Phi(w^1,\ldots,w^k)\right)\times\exp\left(-\sum_{\alpha=1}^{k-1}\frac{|w^\alpha|^2}{t_\alpha}-\frac{1}{\epsilon}\left|\sum_{\alpha=1}^{k-1}w^\alpha\right|^2\right).
\end{aligned}$$

Observe that the operators $D_{\alpha,i_\alpha}(t), D_k(t)$ are uniformly bounded in $t$. Thus, there exists a constant $C = C(\Phi) > 0$ depending only on the function $\Phi$ such that we can bound the weight as

$$
\begin{aligned}
|\widetilde{W}^{k,(n)}_{\epsilon<L}(\Phi)| \quad \leq \quad & C \int_{(w^1,\ldots,w^{k-1}} \prod_{\alpha=1}^{k-1} \mathrm{d}^{2d} w^\alpha \int_{(t_1,\ldots,t_{k-1})\in[\epsilon,L]^{k-1}} \mathrm{d}t_1 \ldots \mathrm{d}t_k \frac{1}{\epsilon^d t_1^d \cdots t_{k-1}^d} \\
& \times \exp\left( - \sum_{\alpha=1}^{k-1} \frac{|w^\alpha|^2}{t_\alpha} - \frac{1}{\epsilon} \left| \sum_{\alpha=1}^{k-1} w^\alpha \right|^2 \right).
\end{aligned} \tag{1}
$$

Thus, to show that the limit $\lim_{L\to 0} \lim_{\epsilon \to 0} \widetilde{W}^{k,(n)}_{\epsilon<L}(\Phi) = 0$ it suffices to show that the limit of the right-hand side vanishes.

The Gaussian integral over the variables $w_i^\alpha$ contributes the following factor

$$
\int_{(w^1,\ldots,w^{k-1}} \prod_{\alpha=1}^{k-1} \mathrm{d}^{2d} w^\alpha \exp\left( - \sum_{\alpha=1}^{k-1} \frac{|w^\alpha|^2}{t_\alpha} - \frac{1}{\epsilon} \left| \sum_{\alpha=1}^{k-1} w^\alpha \right|^2 \right) = C' \left( \frac{\epsilon t_1 \cdots t_{k-1}}{\epsilon + t_1 + \cdots + t_{k-1}} \right)^d .
$$

Where $C'$ involves factors of 2 and $\pi$. Plugging this back in to the right-hand side of (1) we see that

$$
|\widetilde{W}^{k,(n)}_{\epsilon<L}(\Phi)| \leq CC' \int_{[\epsilon,L]^{k-1}} \frac{\mathrm{d}t_1 \cdots \mathrm{d}t_{k-1}}{(\epsilon + t_1 + \cdots + t_{k-1})^d} \leq CC' \prod_{\alpha=1}^{k-1} \int_{t_\alpha=\epsilon}^{L} \mathrm{d}t_\alpha t_\alpha^{-d/(k-1)}.
$$

In the second inequality we have used the fact that $\epsilon > 0$ and the AM-GM inequality. It is immediate to see that the $\epsilon \to 0$ limit of the above exists provided $k > d+1$, which is the situation we are in, and that the $L \to 0$ limit vanishes. $\square$