DEFORMATIONS OF HIGHER CDOS

1. Deformations of the holomorphic σ-model

In this section we allow $\mathfrak{g}$ to be a curved L_∞ algebra over a commutative dg ring R and consider the holomorphic σ-model of maps $Y \to B\mathfrak{g}$, where Y is a complex d-fold. We will be most interested in the following two cases:

(1) the simplest case where $R = \mathbb{C}$ and $\mathfrak{g} = \mathbb{C}^n[-1]$ is the trivial L_∞ algebra with $\ell_k = 0$ for all $k \geq 0$;

(2) when X is a smooth manifold $R = \Omega^*_X$, and $\mathfrak{g}$ is a curved L_∞ algebra over Ω^*_X. Thus, $\mathfrak{g}$ is part of an L_∞ space $(X, \mathfrak{g})$ over X in the terminology of [**?**, **?**].

We have discussed how these two cases are related. Indeed, through Gelfand-Kazhdan descent along a complex manifold we can patch together the case (1) to the situation in (2) where $\mathfrak{g} = \mathfrak{g}_{X_{\bar\partial}}$, the curved L_∞ algebra encoding the complex structure.

The theory we are studying is a cotangent theory of the form $T^*[-1](\Omega^{0,*}(Y, \mathfrak{g}[1]))$. In particular, there is an action of the abelian group $\mathbb{C}^\times_{cot}$ which assigns the base direction a weight of zero and the fiber a weight of $+1$. Thus, if $(\gamma, \beta) \in \Omega^{0,*}(Y, \mathfrak{g})[1] \oplus \Omega^{d,*}(Y, \mathfrak{g}^\vee)[d-1]$, then an element $\lambda \in \mathbb{C}^\times_{cot}$ acts by

$$\lambda \cdot (\gamma, \beta) = (\gamma, \lambda\beta).$$

Our first reduction is to restrict ourselves to studying deformations that are compatible with this $\mathbb{C}^\times_{cot}$ action.

Note that the symplectic pairing of the theory, as well as the classical action functional, is of $\mathbb{C}^\times_{cot}$-weight (-1). Our convention is that the parameter $\hbar$ *has* $\mathbb{C}^\times_{cot}$*-weight* (-1) as well. There are two compelling reasons for making this definition. The first deals with studying correlation functions for the theory. If we require the observables of the theory to be equivariant for this rescaling of the cotangent fibers, this means that the factorization product must have $\mathbb{C}^\times_{cot}$ weight zero. In the case that the theory is free, we have seen that the factorization product between two operators of the theory $\mathcal{O}, \mathcal{O}'$ is computed by a Moyal type formula

$$\mathcal{O} \star \mathcal{O}' = e^{-\hbar\partial_P}\left(e^{\hbar\partial_P}\mathcal{O} \cdot e^{\hbar\partial_P}\mathcal{O}'\right).$$

Since the symplectic pairing is $\mathbb{C}^\times_{cot}$-weight (-1) we observe that the propagator is also $\mathbb{C}^\times_{cot}$-weight $(+1)$. [1] For the product to have weight zero we are then forced to take $\hbar$ to have opposite weight to P.

The other, related reason, we choose this weight for $\hbar$ is that we would like to require our BV complex to be equivariant for rescaling the fibers as well. The classical BRST differential is of the form $\{S, -\} = Q + \{I, -\}$. We have already said that the classical action is of weight (-1). Since the symplectic pairing is also degree (-1), this means that the P_0 bracket is degree

[1] This actually requires that we also take the gauge fixing operator to be of $\mathbb{C}^\times_{cot}$-weight zero, which is the natural thing to do for cotangent theories.

$+1$. Thus, the classical BRST complex is manifestly equivariant. The quantum BV differential involves deforming this classical differential by $\hbar\Delta$. For the same reason as the Poisson bracket, the BV Laplacian has weight $(+1)$. Thus, we see that in order to have an equivariant differential we are again forced to take $\hbar$ to have weight -1.

In the case of an interacting theory, we have the following restriction on the quantum interactions of the theory as well. We can expand an effective interaction as

$$I[L] = \sum_{g \geq 0} \hbar^g I^{(g)}[L].$$

In order for $I[L]$ to have $\mathbb{C}^\times_{cot}$ weight (-1) we see that $I^{(g)}[L]$ must have weight $g-1$. We are only studying a one-loop quantization of the holomorphic theory, so the effective action has the form $I[L] = I^{(0)} + \hbar I^{(1)}[L]$ and hence $I^{(1)}[L]$ has weight zero.

Thus, all one-loop quantities compatible with the $\mathbb{C}^\times_{cot}$ action also have weight zero, including the one-loop anomaly. For this reason, we will be most concerned with the piece of the deformation complex that is $\mathbb{C}^\times_{cot}$-weight zero. This amounts to looking just at local functionals of the γ-field.

Definition 1.1. The *deformation complex for cotangent quantizations* of the holomorphic σ-model of maps $Y \to B\mathfrak{g}$ is the cochain complex

$$\mathrm{Def}^{\mathrm{cot}}_{Y \to B\mathfrak{g}} = \mathrm{C}^*_{\mathrm{loc}}(\Omega^{0,*}_Y \otimes \mathfrak{g}).$$

This is simply the local cochains of the local Lie algebra $\Omega^{0,*}_Y \otimes \mathfrak{g}$ on Y.

We will be most interested in seeing how both the anomaly and the resulting quantum correction induced by the anomaly are realized inside the complex $\mathrm{Def}^{\mathrm{cot}}_{Y \to Bg}$. Before doing this, we'd like to restrict ourselves to looking at quantizations preserving further symmetries.

BW: do this

1.1. **Forms as local functionals.** Before we compute the possible deformations of the holomorphic σ-model, we describe how certain differential forms on the formal stack $B\mathfrak{g}$ yield local functionals of the holomorphic σ-model of maps $Y \to B\mathfrak{g}$. Indeed, we will define a map of cochain complexes

$$J : \Omega^{d+1}_{cl}(B\mathfrak{g}) \xrightarrow{\simeq} \left(\mathrm{Def}_{\mathbb{C}^d \to B\mathfrak{g}}\right)^{\mathbb{C}^d \ltimes U(d)}.$$

The functions on a formal moduli stack $B\mathfrak{g}$ are given by the Chevalley-Eilenberg complex $\mathcal{O}(B\mathfrak{g}) = \mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g})$. By definition, the k-forms on a formal moduli stack $B\mathfrak{g}$ are defined by

$$\Omega^k(B\mathfrak{g}) := \mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g}; \mathrm{Sym}^k \mathfrak{g}^\vee[-k])$$

where $\mathfrak{g}^\vee$ denotes the coadjoint module of $\mathfrak{g}$.

As a simple check, note that in the case $\mathfrak{g} = \mathbb{C}^n[-1]$ the above complex reduces to

$$\Omega^k(B\mathfrak{g}) = \mathbb{C}[t_1, \ldots, t_n] \otimes \wedge^k(t_1^\vee, \cdots, t_n^\vee)$$

where $t_i^\vee$ denotes the dual coordinate. Everything is in cohomological degree zero. If we identify $t_i^\vee \leftrightarrow \mathrm{d}t_i$, this is the usual definition of the algebraic de Rham forms.

BW: finish. define de Rham operator, closed forms, J map, geometric interpretation..

Remark 1.2. We use ∂ to denote the de Rham differential on $B\mathfrak{g}$. This is because our two main examples of $B\mathfrak{g}$ will be the formal holomorphic disk $\widehat{D}^n$ or the formal moduli space associated to any complex manifold X. In each of these cases, the differential above is the holomorphic Dolbeualt operator $\partial : \Omega^k_{hol} \to \Omega^{k+1}_{hol}$.

1.1.1.

Theorem 1.3. *Consider the deformation complex for cotangent quantizations of the holomorphic σ-model of maps $\mathbb{C}^d \to B\mathfrak{g}$. There is a quasi-isomorphism of the $\mathbb{C}^d \ltimes U(d)$ invariant subcomplex with the complex of closed $(d+1)$-forms on $B\mathfrak{g}$:*

$$J : \Omega^{d+1}_{cl}(B\mathfrak{g}) \xrightarrow{\simeq} \left(\mathrm{Def}_{\mathbb{C}^d \to B\mathfrak{g}}\right)^{\mathbb{C}^d \ltimes U(d)}.$$

To compute the translation invariant deformation complex we will invoke Proposition BW: hol trans invt def from Section BW: ref. Note that the deformation complex is simply the (reduced) local cochains on the local Lie algebra $\Omega^{0,*}_{\mathbb{C}^d} \otimes \mathfrak{g}$. Thus, in the notation of Section BW: same ref the bundle L is simply the trivial bundle $\mathfrak{g}$. Thus, we see that the translation invariant deformation complex is quasi-isomorphic to the following cochain complex

$$\left(\mathrm{Def}^{\mathrm{cot}}_{Y \to B\mathfrak{g}}\right)^{\mathbb{C}^d} \simeq \mathbb{C} \cdot \mathrm{d}^d z \otimes^{\mathbb{L}}_{\mathbb{C}\left[\frac{\partial}{\partial z_i}\right]} \mathrm{C}^*_{\mathrm{Lie,red}}(\mathfrak{g}[[z_1, \ldots, z_d]]).$$

We'd like to recast the right-hand side in a more algebraic way.

Note that the the algebra $\mathbb{C}\left[\frac{\partial}{\partial z_i}\right]$ is the enveloping algebra of the abelian Lie algebra $\mathbb{C}^d = \mathbb{C}\left\{\frac{\partial}{\partial z_i}\right\}$. Thus, the complex we are computing is of the form

$$\mathbb{C} \cdot \mathrm{d}^d z \otimes^{\mathbb{L}}_{U(\mathbb{C}^d)} \mathrm{C}^*_{\mathrm{Lie,red}}(\mathfrak{g}[[z_1, \ldots, z_d]]).$$

Since $\mathbb{C} \cdot \mathrm{d}^d z$ is the trivial module, this is precisely the Chevalley-Eilenberg cochain complex computing Lie algebra homology of $\mathbb{C}^d$ with values in the module $\mathrm{C}^*_{\mathrm{Lie,red}}(\mathfrak{g}[[z_1, \ldots, z_d]])$:

$$\left(\mathrm{Def}^{\mathrm{cot}}_{Y \to B\mathfrak{g}}\right)^{\mathbb{C}^d} \simeq \mathrm{C}^{\mathrm{Lie}}_*\left(\mathbb{C}^d ; \mathrm{C}^*_{\mathrm{Lie,red}}(\mathfrak{g}[[z_1, \ldots, z_d]])\mathrm{d}^d z\right).$$

We will keep $\mathrm{d}^d z$ in the notation since below we are interested in computing the $U(d)$-invariants.

To compute the cohomology of this complex, we will first describe the differential explicitly. There are two components to the differential. The first is the "internal" differential coming from the Lie algebra cohomology of $\mathfrak{g}[[z_1, \ldots, z_d]]$, we will write this as $\mathrm{d}_{\mathfrak{g}}$. The second comes from the $\mathbb{C}^d$-module structure on $\mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g}[[z_1, \ldots, z_n]])$ and is the differential computing the Lie algebra homology, which we denote $\mathrm{d}_{\mathbb{C}^d}$. We will employ a spectral sequence whose first term turns on the $\mathrm{d}_{\mathfrak{g}}$ differential. The next term turns on the differential $\mathrm{d}_{\mathbb{C}^d}$.

As a graded vector space, the cochain complex we are trying to compute has the form

$$\mathrm{Sym}(\mathbb{C}^d[1]) \otimes \mathrm{C}^*_{\mathrm{Lie,red}}\left(\mathfrak{g}[[z_1, \ldots, z_d]]\right)\right) \mathrm{d}^d z.$$

The spectral sequence is induced by the increasing filtration of $\mathrm{Sym}(\mathbb{C}^d[1])$ by symmetric powers

$$F^k = \mathrm{Sym}^{\leq k}(\mathbb{C}^d[1]) \otimes \mathrm{C}^*_{\mathrm{Lie,red}}\left(\mathfrak{g}[[z_1, \ldots, z_d]]\right)\right) \mathrm{d}^d z.$$

Remark 1.4. In the examples we are most interested in we can understand the spectral sequence we are using as a version of the Hodge to de Rham spectral sequence.

As above, we write the generators of $\mathbb{C}^d$ by $\frac{\partial}{\partial z_i}$. Also, note that the reduced Chevalley-Eilenberg complex has the form

$$C^*_{\text{Lie,red}}(\mathfrak{g}[[z_1,\ldots,z_n]]) = \left(\text{Sym}^{\geq 1}\left(\mathfrak{g}^\vee[z_1^\vee,\ldots,z_d^\vee][-1]\right), \mathrm{d}_\mathfrak{g}\right),$$

where $z_i^\vee$ is the dual variable to z_i.

Recall, we are only interested in the $U(d)$-invariant subcomplex of this deformation complex. Sitting inside of $U(d)$ we have $S^1 \subset U(d)$ as multiples of the identity. This induces an overall weight grading to the complex. The group $U(d)$ acts in the standard way on $\mathbb{C}^d$. Thus, z_i has weight $(+1)$ and both $z_i^\vee$ and $\frac{\partial}{\partial z_i}$ have S^1-weight (-1). Moreover, the volume element $\mathrm{d}^d z$ has S^1 weight d. It follows that in order to have total S^1-weight that the total number of $\frac{\partial}{\partial z_i}$ and $z_i^\vee$ must add up to d. Thus, as a graded vector space the invariant subcomplex has the following decomposition

$$\bigoplus_k \text{Sym}^k(\mathbb{C}^d[1]) \otimes \left(\bigoplus_{i\leq d-k} \text{Sym}^i\left(\mathfrak{g}^\vee[z_1^\vee,\ldots,z_d^\vee][-1]\right)\right)\mathrm{d}^d z.$$

It follows from Schur-Weyl that the space of $U(d)$ invariants of the dth tensor power of the fundamental representation $\mathbb{C}^d$ is one-dimensional, spanned by the top exterior power. Thus, when we pass to the $U(d)$-invariants, only the unique totally antisymmetric tensor involving $\frac{\partial}{\partial z_i}$ and $z_i^\vee$ survives. Thus, for each k, we have

(1)
$$\left(\text{Sym}^k(\mathbb{C}^d[1]) \otimes \left(\bigoplus_{i\leq d-k} \text{Sym}^i\left(\mathfrak{g}^\vee[z_1^\vee,\ldots,z_d^\vee][-1]\right)\right)\mathrm{d}^d z\right) \cong \wedge^k\left(\frac{\partial}{\partial z_i}\right)\wedge\wedge^{d-k}\left(z_i^\vee\right) C^*_{\text{Lie}}\left(\mathfrak{g}, \text{Sym}^{d-k}(\mathfrak{g}^\vee)\right)\mathrm{d}^d z.$$

Here, $\wedge^k\left(\frac{\partial}{\partial z_i}\right)\wedge\wedge^{d-k}\left(z_i^\vee\right)$ is just a copy of the determinant $U(d)$-representation, but we'd like to keep track of the appearances of the partial derivatives and $z_i^\vee$. Note that for degree reasons, we must have $k \leq d$. When $k = 0$ this complex is the (shifted) space of functions modulo constants on the formal moduli space $B\mathfrak{g}$, $\mathcal{O}_{red}(B\mathfrak{g})[d]$. When $k \geq 1$ this the (shifted) space of k-forms on the formal moduli space $B\mathfrak{g}$, which we write as $\Omega^k(B\mathfrak{g})[d+k]$. Thus, we see that before turning on the differential on the next page, our complex looks like

(2)
$$\begin{array}{cccc} \underline{-2d} & \cdots & \underline{-d-1} & \underline{-d} \\ & & & \\ \mathcal{O}_{red}(B\mathfrak{g}) & \cdots & \Omega^{d-1}(B\mathfrak{g}) & \Omega^d(B\mathfrak{g}). \end{array}$$

We've omitted the extra factors for simplicity.

We now turn on the differential $\mathrm{d}_{\mathbb{C}^d}$ coming from the Lie algebra homology of $\mathbb{C}^d = \mathbb{C}\left\{\frac{\partial}{\partial z_i}\right\}$ with values in the above module. Since this Lie algebra is abelian the differential is completely determined by how the operators $\frac{\partial}{\partial z_i}$ act. We can understand this action explicitly as follows. Note that $\frac{\partial}{\partial z_i} z_j = \delta_{ij}$, thus we may as well think of $z_i^\vee$ as the element $\frac{\partial}{\partial z_i}$. Consider the subspace corresponding to $k = d$ in Equation (1):

$$\frac{\partial}{\partial z_1}\cdots\frac{\partial}{\partial z_d} C^*_{\text{Lie,red}}(\mathfrak{g})\mathrm{d}^d z.$$

Then, if $x \in \mathfrak{g}^\vee[-1] \subset \mathrm{C}^*_{\mathrm{Lie,red}}(\mathfrak{g})$ we observe that

$$\mathrm{d}_{\mathbb{C}^d}\left(\frac{\partial}{\partial z_1}\cdots\frac{\partial}{\partial z_d}\otimes f\otimes \mathrm{d}^d z\right) = \det(\partial_i, z_j^\vee)\otimes 1\otimes x\otimes \mathrm{d}^d z \in \wedge^{d-1}\left(\frac{\partial}{\partial z_i}\right)\wedge \mathbb{C}\{z_i^\vee\}\mathrm{C}^*_{\mathrm{Lie}}\left(\mathfrak{g},\mathfrak{g}^\vee\right)\mathrm{d}^d z.$$

This follows from the fact that the action of $\frac{\partial}{\partial z_i}$ on $x = x\otimes 1 \in \mathfrak{g}^\vee\otimes\mathbb{C}[z_i^\vee]$ is given by

$$\frac{\partial}{\partial z_i}\cdot(x\otimes 1) = 1\otimes x\otimes z_i^\vee \in \mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g},\mathfrak{g}^\vee)z_i^\vee.$$

By the Leibniz rule we can extend this to get the formula for general elements $f \in \mathrm{C}^*_{\mathrm{Lie,red}}(\mathfrak{g})$. We find that getting rid of all the factors of z_i we recover precisely the de Rham differential

$$\begin{tikzcd}
\mathrm{C}^*_{\mathrm{Lie,red}}(\mathfrak{g})[2d] \arrow[r, "\mathrm{d}_{\mathbb{C}^d}"] \arrow[d, equal] & \mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g},\mathfrak{g}^\vee)[2d-1] \arrow[d, equal] \\
\mathcal{O}_{red}(B\mathfrak{g}) \arrow[r, "\partial"] & \Omega^1(B\mathfrak{g}).
\end{tikzcd}$$

A similar argument shows that $\mathrm{d}_{\mathbb{C}^d}$ agrees with the de Rham differential on each $\Omega^k(B\mathfrak{g})$.

We conclude that the E_2 page of this spectral sequence is quasi-isomorphic to the following truncated de Rham complex.

$$\begin{array}{cccccc}
(3) & \underline{-2d} & \underline{-2d+1} & \cdots & \underline{-d-1} & \underline{-d} \\
\end{array}$$

$$\mathcal{O}_{red}(B\mathfrak{g}) \xrightarrow{\partial} \Omega^1(B\mathfrak{g}) \longrightarrow \cdots \longrightarrow \Omega^{d-1}(B\mathfrak{g}) \xrightarrow{\partial} \Omega^d(B\mathfrak{g}).$$

For now, denote this complex by (3).

Consider the full de Rham complex

$$\begin{aligned}
\Omega^*(B\mathfrak{g}) \quad &= \quad R[1] \longrightarrow \mathcal{O}(B\mathfrak{g}) \xrightarrow{\partial} \Omega^1(B\mathfrak{g})[-1] \longrightarrow \cdots \\
&= \quad \mathcal{O}_{red}(B\mathfrak{g}) \xrightarrow{\partial} \Omega^1(B\mathfrak{g}) \longrightarrow \cdots .
\end{aligned}$$

The second line follows from the definition of reduced Chevalley-Eilenberg cochains $\mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g}) = \mathrm{coker}(R \to \mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g}))$. Now, there is an obvious quotient map $\Omega^*(B\mathfrak{g})[2d] \to$ (3) whose kernel is the complex of (shifted) closed $(d+1)$-forms

$$\Omega^{d+1}_{cl}(B\mathfrak{g})[d-1] = \Omega^{d+1}(B\mathfrak{g})[d-1] \xrightarrow{\partial} \Omega^{d+1}(B\mathfrak{g})[d-2] \to \cdots .$$

It follows that we have an exact sequence

$$\Omega^{d+1}(B\mathfrak{g})[d-1] \to \Omega^*(B\mathfrak{g}) \to (3).$$

Since the middle term is acyclic, it follows that the connecting map (which is degree one) is a quasi-isomorphism (3) $\xrightarrow{\simeq} \Omega^{d+1}_{cl}(B\mathfrak{g})[d]$. This completes the proof.