The holomorphic σ-model and its symmetries

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Outline of this talk

1. Rapid overview of the BV formalism.
2. Holomorphic theories, in general. One-loop finiteness and a formula for the general chiral anomaly.
3. The holomorphic σ-model and its factorization algebra.

The BV formalism

The Batalin-Vilkovisky formalism is a technique used to study quantization in field theory. A generalization of the usual problem of *deformation quantization*.

$$\mathrm{SympMfld} \xrightarrow{\mathcal{O}} \mathrm{Alg}_{\mathrm{Poiss}} \xleftarrow{\hbar \to 0} \mathrm{Alg}_{\mathbb{C}[[\hbar]]}$$

$$(M, \omega) \longmapsto (\mathcal{O}(M), \Pi_\omega) \longleftarrow (\mathcal{O}(M)[[\hbar]], \star)\,.$$

In field theory, one works on a smooth manifold X (the spacetime).

$$\mathrm{BV}-\mathrm{Theory}(X) \xrightarrow{\mathrm{Obs}} \mathrm{FactAlg}(X)_{P_0} \xleftarrow{\hbar \to 0} \mathrm{FactAlg}(X)_{\mathrm{BV}}.$$

Given a classical BV theory we study lifts of the P_0 factorization algebra of classical observables to the *BV* factorization algebra of quantum observables.
In the one-dimensional case $X = \mathbb{R}$ there exists a classical BV theory associated to a symplectic manifold (M, ω). In this case, BV quantization recovers ordinary deformation quantization.

The BV formalism (cont.)

In QFT, BV algebras provide a mathematical model for the path integral (see Costello's book on renormalization).

Definition

A BV algebra is a triple (A, Q, Δ) where (A, Q) is a commutative dg algebra, and $\Delta : A \to A$ is a degee one linear map such that

(a) $\Delta^2 = [\Delta, Q] = 0$;

(b) the degree one bilinear map

$$\{a, b\} := \Delta(ab) - \Delta(a)b \pm a\Delta(b)$$

satisfies graded Jacobi, and is a graded biderivation with respect to the commutative product.

Thus $\{-, -\}$ behaves like a Poisson bracket, except with a weird shift. We say an element $I = I_0 + \hbar I_1 + \cdots \in A[[\hbar]]$ satisfies the *quantum master equation* (QME) if

$$\boxed{(Q + \hbar\Delta)e^{I/\hbar} = 0.}$$

We call $\hbar$ the *perturbation* parameter.

The BV formalism (cont.)

When we set $\hbar = 0$, the QME reduces to condition

$$\boxed{QI_0 + \frac{1}{2}\{I, I\} = 0}.$$

We call this the classical master equation (CME).

Example

Suppose $A = \mathcal{O}(V) = \mathrm{Sym}(V^*)$ for some graded vector space V. Then a functional I_0 satisfying the CME is equivalent to an data of an L_∞ structure on the graded vector space $V[-1]$.

Most important example of BV algebras in QFT come from (-1)-shifted geometry. Suppose (V, ω) is a (-1)-shifted symplectic vector space. Then, the symmetric tensor $K_0 := \omega^{-1} \in \mathrm{Sym}^2(V)$ defines an operator (of order two)

$$\Delta_0 = \partial_{K_0} : \mathcal{O}(V) \to \mathcal{O}(V)$$

by contraction. This operator defines a BV algebra $(\mathcal{O}(V), Q, \Delta_0)$, where Q is the internal differential of V.

The BV formalism (cont.)

Suppose that $P \in \mathrm{Sym}^2(V)$ is a symmetric tensor of degree zero, and define $K_P = K_0 + QP$. One checks that K_P defines another BV algebra based on $\mathcal{O}(V)$.
Given $I \in \mathcal{O}^+(V)$ (at least cubic), define $W(P, I) \in \mathcal{O}(V)[[\hbar]]$ formally by

$$\boxed{e^{W(P,I)/\hbar} = e^{\hbar\partial_P} e^{I/\hbar}}.$$

Lemma

The functional I satisfies the QME relative to K_0 if and only if $W(P, I)$ satisfies the QME relative to K_P.

The functional $W(P, I)$ decomposes as a sum over connected graphs

$$W(P, I) = \sum_{\Gamma} \frac{\hbar^{g(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} W_\Gamma(P, I),$$

where W_Γ is the *weight* of the graph Γ.

Field theory

A classical field theory on a smooth manifold M is:

(i) a graded vector bundle E whose sections we denote $\mathcal{E}$;

(ii) a differential operator $Q : \mathcal{E} \to \mathcal{E}$ of degree one;

(iii) a graded antisymmetric bundle map $(-,-)_E : E \otimes E \to \mathrm{Dens}_X$ of degree (-1) that is fiberwise nondegenerate.

(iv) a *local functional* $I_0 \in \mathcal{O}_{\mathrm{loc}}(\mathcal{E})$ satisfying the CME.

We require that $(\mathcal{E}, Q)$ is an elliptic complex. The pairing $(-,-)_E$ defines a (-1)-shifted symplectic structure via integration

$$\omega = \int_X \circ (-,-)_E .$$

The sheaf of sections $\mathcal{E}$ evaluated on an open set U returns the graded space $\mathcal{E}(U)$ which we refer to as the space of fields supported on U.

Holomorphic field theory

In the world of complex geometry we have the following definition of a *holomorphic* field theory on a complex manifold X:

(i) a graded holomorphic vector bundle V on X whose sheaf of holomorphic sections we denote $\mathcal{V}^{hol}$;

(ii) a holomorphic differential operator $Q^{hol} : \mathcal{V}^{hol} \to \mathcal{V}^{hol}$ of degree one;

(iii) a graded antisymmetric bundle map $(-,-)_V : V \otimes V \to K_X$ of degree $(d-1)$ that is fiberwise nondegenerate.

(iv) a holomorphic Lagrangian $\mathcal{I}_0^{hol}$ satisfying the CME.

Holomorphic theory	BV theory
Holomorphic bundle V	Space of fields $\mathcal{E}_V = \Omega^{0,*}(X, V)$
Holomorphic differential operator Q^{hol}	Linear BRST operator $\overline{\partial} + Q^{hol}$
Non-degenerate pairing $(-,-)_V$	(-1)-symplectic structure ω_V
Holomorphic Lagrangian $\mathcal{I}_0^{hol}$	Local functional $I_0^{\Omega^{0,*}} \in \mathcal{O}_{\mathrm{loc}}(\mathcal{E}_V)$

Table: From holomorphic to BV

Regularization

Let $(\mathcal{E}, Q, \omega, I_0)$ be a classical BV theory. The first thing to do is define the BV operator $\Delta_0 = \omega^{-1}$.

- **Problem:** The tensor ω^{-1} is *distributional*, thus Δ_0 is not well-defined on functionals.

The solution is to find a homotopy replacement for K_0

$$\widetilde{K} = K_0 + QP,$$

so that its BV operator is well-defined. (By elliptic regularity, one always exists). Such a regularization is parametrized by a length scale $L > 0$. For each $L < L'$ a regularization scheme prescribes a *propagator* $P_{L<L'}$ such that

$$K_{L'} = K_L + QP_{L<L'}$$

where $K_L, K_{L'}$ are both smooth and $\lim_{L \to 0} K_L = K_0$.

The definition of a QFT

By definition, a quantization is a family of functionals $\{I[L]\}$ satisfying the following two conditions:

1. the collection of functionals $\{I[L]\}$ are related by *renormalization group flow*
$$I[L'] = W(P_{L<L'}, I[L]).$$
2. for each L, the functional solves the *scale L* quantum master equation
$$(Q + \hbar\Delta_L)e^{I[L]/\hbar} = 0.$$

For abstract reasons, proved by Costello, one can always find a family such that (1) is satisfied. In general, the answer is not constructive and involves choosing counterterms with respect to a renormalization scheme. There may be unavoidable obstructions to solving problem (2).

The naïve definition of $I[L]$ is to apply the operator $P_{0<L}$ to the classical interaction

$$I[L] = W(P_{0<L}, I_0)$$

The problem is that the right-hand side is rarely well-defined (same issue as above). A solution to this, which always exists, is to find counterterms.

Theorem

There is a regularization scheme for **holomorphic theories** *on $\mathbb{C}^d$ such that the limit*

$$I[L] = \lim_{\epsilon \to 0} W(P_{\epsilon<L}, I_0) \mod \hbar^2$$

exists. In other words, holomorphic theories on $\mathbb{C}^d$ are one-loop finite.

The main ingredient is in the existence of the *gauge fixing operator* $\overline{\partial}^*$.

- Studying the quantizations of holomorphic theories on $\mathbb{C}^d$ reduces to solving the quantum master equation. This is essentially an algebraic problem.

A corollary of this result is a characterization of the *anomaly*, or obstruction, for a holomorphic theory to solve the QME.

Corollary

The obstruction for a classical holomorphic theory on $\mathbb{C}^d$ to admit a one-loop quantization is given by the following expression:

$$\Theta = \lim_{\epsilon, L \to 0} \sum_{\Gamma \in \mathrm{Wheel}_{d+1}} W_\Gamma(P_{\epsilon < L}, K_\epsilon, I_0).$$

Pictorially PICTURE

This gives a holomorphic characterization, and generalization, of the Adler-Bell-Jackiw anomaly for four-dimensional gauge theory.

The holomorphic σ-model

The holomorphic σ-model is a prototypical holomorphic theory. Let X, Y be complex manifolds and consider the mapping space:

$$\mathrm{Map}^{hol}(Y,X) = \{f : Y \to X \text{ holomorphic}\}.$$

There are a few issues:

1. a classical theory involves a shifted symplectic pairing. The theory we study is of the form

   $$T^*[-1]\left(\mathrm{Map}^{hol}(Y,X)\right).$$

   In degree zero, the fields consist of a map $\gamma : Y \to X$ together with a class $\beta \in \Omega^{d,d-1}(Y, \gamma^* T^{*1,0}X)$. The action functional is

   $$S(\beta,\gamma) = \int_Y \beta \wedge \bar{\partial}\gamma.$$

   Notice when we vary γ, β we obtain $\bar{\partial}\gamma = 0 = \bar{\partial}\beta$.
2. To make this into a BV theory, we must perturb around a fixed holomorphic map; we look at the formal neighborhood of constant maps $\mathrm{Map}(Y,X)^{\wedge}_{const}$.

Local-to-global

Our construction of the holomorphic σ-model is local-to-global on the target manifold. We phrase the theory in the style of *formal geometry* due to Gelfand, Kazhdan, Fuks. To every n-dimensional manifold X (smooth, complex, symplectic, etc..) there exists a universal bundle of coordinates:

$$
\begin{tikzcd}
\mathrm{W}_n \circlearrowright X^{coor} \arrow[dr, "\simeq"] \arrow[dd] & \\
& \mathrm{Fr}_X \arrow[ul, bend right, "\sigma"'] \arrow[dl] \\
X &
\end{tikzcd}
.
$$

X^{coor} is a principal Aut_n-bundle together with a transitive action of the Lie algebra of *formal vector fields* in n-dimensions W_n. There is

$$\omega^{coor} \in \Omega^1(X^{coor}, \mathrm{W}_n)^{\mathrm{Aut}_n} \xrightarrow{\sigma^*} \Omega^1(\mathrm{Fr}_X, \mathrm{W}_n)^{\mathrm{GL}_n}$$

satisfying the Maurer-Cartan equation $\mathrm{d}\omega^{coor} + \frac{1}{2}[\omega^{coor}, \omega^{coor}] = 0$.

Gelfand-Kazhdan descent

Define a category of "formal vector bundles" on the formal n-disk. In particular, these are $(\mathrm{W}_n, \mathrm{GL}_n)$-modules. For each X, there is a functor

$$\begin{tikzcd}
\mathcal{V} \arrow[rr, mapsto] \arrow[d, phantom, "\cap" rotate=90] & & \left(\mathrm{Fr}_X \times^{\mathrm{GL}_n} \mathcal{V}, \nabla^{coor}\right) \arrow[d, phantom, "\cap" rotate=90] \\
\mathrm{VB}_{\widehat{D}^n} \arrow[rr, "\mathrm{desc}_X"] & & \mathrm{VB}_X^{flat} \\
\mathrm{Mod}_{(\mathrm{W}_n, \mathrm{GL}_n)} \arrow[u, hook] & &
\end{tikzcd}$$

Moreover, there are "formal characteristic classes" that live in the Gelfand-Fuks cohomology. The descent functor determines a transformation of cohomology theories and hence a map of complexes

$$\mathrm{char}_X : \mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n, \mathrm{GL}_n; \mathcal{V}) \to \Omega^*(X, \mathrm{desc}_X(\mathcal{V})).$$

When $\mathcal{V} = \widehat{\mathcal{O}}_n$, $\mathrm{desc}_X(\widehat{\mathcal{O}}_n) = J^\infty \mathcal{O}_X$ equipped with its natural flat connection. Recover all natural bundles in this way.