RANDOM ANALYSIS

1. The higher OPE and descent

In this section we provide a partial analysis of the higher operator product expansion present in holomorphically translation invariant quantum field theories.

1.1. **The observables on the d-disk.**

In this section we give a description of the observables of the $\beta\gamma$ system supported on a d-disk inside $\mathbb{C}^d$. For now, we will only consider the free $\beta\gamma$ system with target a complex vector space V. Thus the observables are..BW: finish

We would like to study a class of local operators...

1.1.1. *Primaries of a CFT.*

In ordinary chiral conformal field theory, there is a collection of operators that, in some sense, generate all other operators. These are called "primary operators" (or primary fields), and are defined by those operators that are killed by the positive part of the Virasoro algebra [**?**], that is, the "lowering operators". To obtain all of the operators one considers the descendants of the primary operators which are obtained by applying the negative part of the Virasoro algebra, or the "raising operators", to the primaries. For example, in the $d = 1$ $\beta\gamma$ system, there are two primary operators:

$$\mathcal{O}_{\gamma,0}(w) : \gamma \mapsto \gamma(w) = \int_{z \in C_w} \frac{\gamma(z)}{z - w} \mathrm{d}z$$
$$\mathcal{O}_{\beta,-1}(w) : \beta \mathrm{d}z \mapsto \beta(w) = \int_{z \in C_w} \frac{\beta(z)}{z - w} \mathrm{d}z,$$

where C_w is any closed contour surrounding w. (The indices $0, -1$ are to indicate the conformal weight.) Consider the operators placed at $w = 0$. We notice that each of these operators are annihilated by the positive half of the Virasoro $L_n = z^{n+1}\partial_z$, $n \geq 0$. The descendants are obtained by iteratively applying the raising operator $L_{-1} = \partial_z$, which in this case is just the infinitesimal translations. Indeed, for each $n \geq 0$ we obtain

$$\mathcal{O}_{\gamma,-n}(w) = \frac{1}{n!}\partial^n \mathcal{O}_{\gamma,0}(w) : \gamma \mapsto \partial_z^n \gamma(z = w)$$
$$\mathcal{O}_{\beta,-n-1}(w) = \frac{1}{n!}\partial^n \mathcal{O}_{\beta,1}(w) : \beta \mathrm{d}z \mapsto \partial_z^n \beta(z = w).$$

There is an S^1 action on $\mathbb{C}$ given by rotations, and this extends to an S^1 action on the $\beta\gamma$ system. In terms of the Virasoro algebra, the infinitesimal action of S^1 is given by the Euler vector field $L_0 = z\partial_z$. There is an induced grading on the factorization algebra of the one-dimensional free $\beta\gamma$ system by the eigenvalues of this S^1 action. Applied to the disk, or local, observables this is precisely the $\mathbb{Z}_{\geq 0}$ conformal weight grading of the chiral CFT. For instance, the operators $\mathcal{O}_{\gamma,-n}(w), \mathcal{O}_{\beta,-n}$ lie in the weight n subspace of the factorization algebra applied to $D(w,r)$ (for any $r > 0$). We will see a similar grading in the higher dimensional holomorphic case.

1.1.2. The $\beta\gamma$ system on $\mathbb{C}^d$ has a symmetry by the group unitary group $U(d)$. Indeed, the fields of the $\beta\gamma$ system are built from sections of certain natural holomorphic vector bundles on $\mathbb{C}^d$. The group $U(d)$ acts by automorphisms on every holomorphic vector bundle, hence it acts on sections via the pull-back.

There is another symmetry that we wish to contemplate. Introduce an action of $U(1)$ on the fields of the theory such that V has weight $q_f \in \mathbb{Z}$ and V^* has weight $-q_f$. The value of the fields γ lie in the vector space V, so these fields are of weight q_f. Conversely, the fields β lie in V^*, so have weight $-q_f$. Since the pairing defining the free theory is only non-zero between a single γ and single β field, the theory is invariant under this symmetry. In the physics literature, this is a so-called "flavor symmetry" of the theory, and so to distinguish it from the other symmetry we will denote this group by $U(1)_f$. This symmetry will be especially relevant when we compute the character of the $\beta\gamma$ system.

Lemma 1.1. *The symmetry by $U(d) \times U(1)_f$ on the classical $\beta\gamma$ system with values in the complex vector space V extends to a symmetry of the factorization algebra of quantum observables Obs^q.*

Proof. The differential on the factorization algebra is of the form $\bar{\partial} + \hbar\Delta$. The operator $\bar{\partial}$ is manifestly equivariant for the action of $U(d)$. Since $U(1)_f$ does not act on spacetime, $\bar{\partial}$ trivially commutes with its action. Further, the action of $U(d)$ is through linear automorphisms, and since the BV Laplacian Δ is a second order differential operator, it certainly commutes with the action of $U(d)$. Likewise, since $U(1)_f$ is compatible with the (-1)-symplectic pairing, it automatically is compatible with Δ. □

We will use the action of $U(d)$ to organize the class of operators we are interested in. The eigenvectors of $U(d)$ are labeled by the eigenvectors of a maximal torus, which we will take to be given by the subgroup

$$T^d = \{\mathrm{diag}(q_1, \ldots, q_d) \mid |q_i| = 1\} \subset U(d).$$

Here, $q_i \in S^1 \subset \mathbb{C}^\times$ are complex numbers of unit modulus. We say that an element v of the factorization algebra has weight $(n_1, \ldots, n_k)$ if $(q_1, \ldots, q_d) \cdot v = q_1^{n_1} \cdots q_d^{n_d} v$. We will use the shorthand $\vec{n} = (n_1, \ldots, n_d)$.

Definition 1.2. (1) Let $w \in \mathbb{C}^d$ and $r > 0$. For any vector of non-negative integers $\vec{n} = (n_1, \ldots, n_d)$ denote by

$$\mathrm{Obs}^{\mathrm{q}}_V(r)^{(\vec{n})} \subset \mathrm{Obs}^{\mathrm{q}}_V(D(w,r))$$

the subcomplex of weight $\vec{n}$ elements.

(2) Let

$$\mathrm{Obs}^{\mathrm{q}}_V(r) := \bigoplus_{\vec{n}} \mathrm{Obs}^{\mathrm{q}}_V(r)^{(\vec{n})}$$

where the direct sum is over all vectors of non-negative integers.

BW: should I make separate dfn for classical and quantum

Remark 1.3. Note that we have excluded $w \in \mathbb{C}^d$ from the notation above. This is because the $\beta\gamma$ system is translation invariant. BW: say more?

We now introduce the following operators that will be of most relevance for our study of the operator product expansion.

Definition 1.4. Let $w \in \mathbb{C}^d$ and $r > 0$. Define the following linear observables supported on $D(w,r)$.

(1) For $n_i \in \mathbb{Z}_{\geq 0}, i = 1,\dots d$, and $v^* \in V^*$ define

$$\mathcal{O}_{\gamma,-\vec{n}}(w;v^*) : \gamma \in \Omega^{0,*}(D(w,r)) \mapsto \left\langle v^*, \left(\frac{\partial^{n_1}}{\partial z_1^{n_1}} \cdots \frac{\partial^{n_d}}{\partial z_d^{n_d}} \gamma(z,\bar{z}) \Big|_{z=w} \right) \right\rangle.$$

Here, the brackets denote the evaluation pairing between V^* and V.

(2) For $m_i \in \mathbb{Z}_{\geq 1}, i = 1,\dots d$, and $v \in V$ define

$$\mathcal{O}_{\beta,-\vec{m}}(w;v) : \beta \mathrm{d}^d z \in \Omega^{d,*}(D(w,r)) \mapsto \left\langle v, \left(\frac{\partial^{m_1-1}}{\partial z_1^{m_1-1}} \cdots \frac{\partial^{m_d-1}}{\partial z_d^{m_d-1}} \beta(z,\bar{z}) \Big|_{z=w} \right) \right\rangle.$$

Our convention is that the evaluation of a Dolbeualt form is zero $\mathrm{d}\bar{z}_i|_{z=w} = 0$. Thus, the above observables are only nonzero when $\gamma \in \Omega^{0,0}(D(w,r))$ and $\beta \mathrm{d}^d z \in \Omega^{d,0}(D(w,r))$. In particular, this implies that these operators are of the following homogenous cohomological degree:

$$\deg(\mathcal{O}_{\gamma,-\vec{n}}(w;v^*)) = 0$$
$$\deg(\mathcal{O}_{\beta,-\vec{m}}(w;v)) = d-1.$$

The minus sign in $\mathcal{O}_{\gamma,-\vec{n}}(w)$ is purely conventional, and meant to match up with the physics and vertex algebra literature BW: ref. One reason for using this convention is to note that for any d-disk $D(0,r)$ there is an embedding of topological vector spaces

$$z_1^{-1} \cdots z_d^{-1} \mathbb{C}[z_1^{-1}, \cdots, z_d^{-1}] \to \left(\Omega^{0,*}(D(w,r)) \right)^\vee$$

that sends a Laurent polynomial $f(z)$ functional

$$\gamma \in \Omega^{0,*}(D(w,r)) \mapsto \oint_{z \in S^{2d-1}} f(z-w) \gamma(z,\bar{z}) \mathrm{d}^d z \wedge \omega_{BM}(z-w, \bar{z}-\bar{w}),$$

where ω_{BM} is the Bochner-Martinelli form of type $(0,d-1)$, and S^{2d-1} is the sphere of radius r around w.

Remark 1.5. This is the analog of the fact that when $d = 1$ there is an embedding

$$z^{-1}\mathbb{C}[z^{-1}] \to \left(\mathcal{O}^{hol}(D) \right)^\vee$$

sending $f(z)$ to the residue functional $g \mapsto \mathrm{Res}_z(f(z)g(z)\mathrm{d}z)$.

Similarly, there is an embedding $z_1^{-1} \cdots z_d^{-1} \mathbb{C}[z_1^{-1}, \dots, z_d^{-1}] \to \left(\Omega^{d,*}(D(w,r)) \right)^\vee$ sending $f(z)$ to the functional

$$\beta \in \Omega^{d,*}(D(w,r)) \mapsto \oint_{S^{2d-1}} f(z-w) \beta(z,\bar{z}) \wedge \omega_{BM}(z-w,\bar{z}-\bar{w}).$$

These embeddings determine a linear map

$$i_D : z_1^{-1} \cdots z_d^{-1} \mathbb{C}[z_1^{-1}, \dots, z_d^{-1}] \otimes (V^* \oplus V[-d+1]) \to \left(\Omega^{0,*}(D(w,r)) \otimes V \oplus \Omega^{d,*}(D(w,r)) \otimes V^*[d-1] \right)^\vee.$$

The right-hand side is simply the linear observables sitting inside of $\mathrm{Obs}^{\mathrm{cl}}_V(D(w,r))$. It follows from the higher dimensional residue formula that, for $n_i \geq 0$, the image of

$$z_1^{-n_1} \cdots z_d^{-n_d} \otimes v^* \in \mathbb{C}[z_1^{-1}, \ldots, z_d^{-1}] \otimes V^*$$

under this map is precisely $\mathcal{O}_{\gamma,-\vec{n}}(w; v^*)$, where $\vec{n} = (n_1, \ldots, n_d)$. Similarly, for $m_i \geq 1$, the image of

$$z_1^{-m_1+1} \cdots z_d^{-m_d+1} \otimes v \in \mathbb{C}[z_1^{-1}, \ldots, z_d^{-1}] \otimes V[-d+1]$$

under this map is $\mathcal{O}_{\beta,\vec{m}}(w; v)$. This motivates the following general definition.

Definition 1.6. For any $f(z) \in \mathbb{C}[z_1^{-1}, \ldots, z_1^{-1}]$, denote by $\mathcal{O}_{\gamma,f}(w; v^*)$ the image of $f \otimes v^*$ under the linear map i_D

$$\mathcal{O}_{\gamma,f}(w; v^*) := i_D(f \otimes v^*)$$

In particular, note $\mathcal{O}_{\gamma, z_1^{-n_1} \cdots z_d^{-n_d}}(w; v^*) = \mathcal{O}_{\gamma,-\vec{n}}(w; v^*)$. Similarly, if $g \in z_1^{-1} \cdots z_d^{-1} \mathbb{C}[z_1^{-1}, \ldots, z_d^{-1}]$, let

$$\mathcal{O}_{\beta,g}(w; v) := i_D(z_1 \cdots z_d g(z) \otimes v).$$

Lemma 1.7. *Let $r < s$. Then, the factorization structure map for including disks $D(0,r) \subset D(0,s)$ induces a diagram*

$$\begin{tikzcd}
\mathrm{Obs}^{\mathrm{q}}_V(D(0,r)) \arrow[r] & \mathrm{Obs}^{\mathrm{q}}_V(D(0,s)) \\
\mathrm{Obs}^{\mathrm{q}}_V(r) \arrow[u] \arrow[r, "\simeq"] & \mathrm{Obs}^{\mathrm{q}}_V(s) \arrow[u]
\end{tikzcd}$$

Further, the bottom horizontal map is a quasi-isomorphism.

Proof. The two vertical maps are the inclusions of the $U(d)$-eigenspaces of the observables supported on disks of radius r and s respectively. It follows from Lemma 1.1 that the factorization algebra is $U(d)$-equivariant, so in particular the factorization algebra structure map for the inclusion of disks $D(0,r) \hookrightarrow D(0,s)$ is a map of $U(d)$-representations. Hence, the map restricts to each of the eigenspaces, yielding the diagram.

In **[?]** it is shown in Corollary 5.3.6.4 that for the one-dimensional $\beta\gamma$ system, the lower map above is a quasi-isomorphism. In fact, a similar argument applies to the $\beta\gamma$ system on $\mathbb{C}^d$. Indeed, consider the collection

$$\{\mathcal{O}_{\gamma,-\vec{n}_1}(0; v_1^*) \cdot \mathcal{O}_{\gamma,-\vec{n}_k}(0; v_k^*) \cdot \mathcal{O}_{\beta,-\vec{m}_1}(0; v_1) \cdots \mathcal{O}_{\beta,-\vec{m}_l}(0; v_l)\}.$$

The collection runs over non-negative integers k, l and sequences $\vec{n}_i = (n_{i,1}, \ldots, n_{i,d})$, $n_{i,j} \geq 0$ and $\vec{m}_i = (m_{i,1}, \ldots, m_{i,d})$, $m_{i,1} \geq 1$. It also runs over vectors v_i, v_j^* in V and V^*, respectively. Now, it follows from Lemma 5.3.6.2 of **[?]** that the above collection form a basis for the cohomology

$$H^* \mathrm{Obs}^{\mathrm{cl}}_V(r)^{(\vec{N})} \subset H^* \mathrm{Obs}^{\mathrm{cl}}(D(0,r))$$

for any r, where $\vec{N} = (N_1, \ldots, N_d)$

$$N_j = \left(n_{1,j} + \cdots + n_{k,j}\right) + \left(m_{1,j} + \cdots + m_{l,j}\right).$$

The result for the quantum observables follows from the spectral sequence BW: finish $\square$

We will denote $\mathcal{V}_V = \mathrm{Obs}^{\mathrm{q}}_V(r)$, which is well-defined up to quasi-isomorphism by the preceding proposition. This is the "state space" of the higher dimensional holomorphic theory.

1.2. **The sphere observables.** We now provide a description of the value of the factorization algebra of observables of the $\beta\gamma$ system applied to another important class of open sets in $\mathbb{C}^d$: neighborhoods of the $(2d-1)$-sphere $S^{2d-1} \subset \mathbb{C}^d$.

We first describe the precise open neighborhoods of the $(2d-1)$-sphere that we will consider. Denote the closed d-disk centered at w of radius r by

$$\bar{D}(w,r) = \left\{ (z_1, z_2) \in \mathbb{C}^2 \mid |z - w| \leq r^2 \right\}.$$

As above, the open disk is denoted $D(w,r)$. Let $\epsilon, r > 0$ be such that $0 < \epsilon < r$, and consider the open submanifold

$$N_{r,\epsilon}(w) := D(w, r+\epsilon) \setminus \bar{D}(w, r-\epsilon) \subset \mathbb{C}^d \setminus \{w\}.$$

For any $\epsilon > 0$, the open set $N_{r,\epsilon}$ is a neighborhood of the closed submanifold given by the sphere of radius r centered at w, $S^{2d-1}_r(w) \subset \mathbb{C}^d \setminus \{w\}$. Note that when $d = 1$, $N_{r,\epsilon}$ is simply an annulus centered at w.

Like in the case of a disk, it is convenient to get our hands on a class of simple observables supported on $N_{r,\epsilon}(w)$. First, we describe linear functionals on the Dolbeualt complex of $N_{r,\epsilon}(w)$. This will lead naturally to a description of linear observables supported on this neighborhood.

BW: describe the complex A_d

Lemma 1.8. *For any neighborhood $N_{r,\epsilon}(w)$ as above, the residue along the $(2d-1)$-sphere centered at w of radius r, $S^{2d-1}_r(w)$, determines an embedding of topological dg vector spaces*

$$i_{S^{2d-1}} : A_d[d-1] \to \left(\Omega^{0,*}(N_{r,\epsilon}(w)) \right)^\vee$$

sending $\alpha \in A_d$ to the functional

$$i_{S^{2d-1}}(\alpha) : \omega \in \Omega^{0,*}(N_{r,\epsilon}(w)) \mapsto \oint_{S^{2d-1}_r(w)} \alpha \wedge \mathrm{d}^d z \wedge \omega.$$

Proof. This is a consequence of Stokes theorem. Suppose $\alpha = \bar{\partial}\alpha'$. Then, for any $\omega \in \Omega^{0,*}(N_{r,\epsilon}(w))$ we have

$$\oint_{S^{2d-1}} (\bar{\partial}\alpha') \wedge \mathrm{d}^d z \wedge \omega = \oint_{S^{2d-1}} \alpha' \wedge \mathrm{d}^d z \wedge \bar{\partial}\omega.$$

The right-hand side is simply $(\bar{\partial} i_N)(\omega) = i_N(\bar{\partial}\omega)$. □

Similarly, there is an embedding $A_d[d-1] \to \left(\Omega^{d,*}(N_{r,\epsilon}(w)) \right)^\vee$ sending $\alpha \in A_d[d-1]$ to the functional

$$\eta \in \Omega^{d,*}(N_{r,\epsilon}(w)) \mapsto \int_{S^{2d-1}_r(w)} \alpha \wedge \eta.$$

These two embeddings allow us to provide a succinct description of the class of linear operators on $N_{r,\epsilon}(w)$ we will be mostly interested in. Indeed, the linear embeddings above, determine a cochain map (that we proceed to denote by the same symbol):

$$i_{S^{2d-1}} : A_d \otimes (V^*[d-1] \oplus V) \to \left(\Omega^{0,*}(N_{r,\epsilon}(w)) \otimes V \oplus \Omega^{d,*}(N_{r,\epsilon}(w)) \otimes V^*[d-1] \right)^\vee \subset \mathrm{Obs}^{\mathrm{cl}}_V(N_{r,\epsilon}(w)).$$

Definition 1.9. Let $\alpha \in A_d$ and $v^* \in V^*$. Define the linear observable

$$\mathcal{O}_{\gamma,\alpha}(w; v^*) := i_{S^{2d-1}}(\alpha \otimes v^*) \in \mathrm{Obs}^{\mathrm{cl}}(N_{r,\epsilon}(w)).$$

Likewise, for $v \in V$, define

$$\mathcal{O}_{\beta, z_1^{-1}\cdots z_d^{-1}\alpha}(w; v) := i_{S^{2d-1}}(\alpha \otimes v).$$

Definition 1.10. Define the dg vector space of *classical sphere observables* to be

$$\mathcal{A}_V^{\mathrm{cl}} := \mathrm{Sym}\left(A_d \otimes (V^*[d-1] \oplus V)\right)$$

equipped with the differential coming from A_d.

Note that A_d has the structure of a commutative dg algebra, but we are not using the multiplication here. The same construction above, applied now to symmetric products of linear operators, determines a cochain map $i_{S^{2d-1}} : \mathcal{A}_V^{\mathrm{cl}} \to \mathrm{Obs}^{\mathrm{cl}}(N_{r,\epsilon}(w))$.

Let $\mathcal{A}_V = \mathcal{A}_V^{\mathrm{cl}}[\hbar]$. Then, since $\Delta|_{\mathcal{A}_V} = 0$, we see that $i_{S^{2d-1}}$ extends to a cochain map

$$i_{S^{2d-1}} : \mathcal{A}_V \to \mathrm{Obs}_V^{\mathrm{q}}(N_{r,\epsilon}(w)).$$

We will refer to $\mathcal{A}_V$ as the *quantum sphere observables*, or when there is no confusion, the sphere observables.

The cohomology of A_d is concentrated in degrees 0 and $d-1$. Explicitly, one can represent the zeroeth cohomology as

$$H^0(A_d) = \mathbb{C}[z_1, \ldots, z_d].$$

Now, let $\omega_{BM}(z, \bar{z})$ be the Bochner-Martinelli kernel of type $(0, d-1)$ from above. We can express the $(d-1)$st cohomology of A_d as

$$H^{d-1}(A_d) = \mathbb{C}[\partial_{z_1}, \cdots, \partial_{z_d}] \cdot \omega_{BM}$$

That is, every element of $H^{d-1}(A_d)$ can be written as a holomorphic polynomial differential operator acting on ω_{BM}. Further, it is convenient to make the $U(d)$-equivariant identification

$$\mathbb{C}[\partial_{z_1}, \cdots, \partial_{z_d}]\omega_{BM} \cong z_1^{-1} \cdots z_d^{-1}\mathbb{C}[z_1^{-1}, \ldots, z_d^{-1}], \tag{1}$$

which makes sense since ω_{BM} has $T^d \subset U(d)$-weight $(-1, \ldots, -1)$.

1.2.1. *The associative algebra associated to the sphere.* So far we have not discussed the structure that the factorization product puts on the dg vector space $\mathcal{A}_V$. To recover this structure, we will only be concerned with open sets that are neighborhoods of spheres, as in the previous section. The configurations of open sets we consider are given by nesting the neighborhoods of the form $N_{r,\epsilon}(w)$, where w is a fixed center.

For simplicity, we assume that our spheres and neighborhoods are all centered at $w = 0$. For $\epsilon < r$ we have defined the open neighborhood $N_{r,\epsilon} = N_{r,\epsilon}(0)$ of the sphere S_r^{2d-1} centered at zero. Pick positive numbers $0 < \epsilon_i < r_i$ such that $r_1 < r < r_2$, $\epsilon_1 < r - r_1$, and $\epsilon_2 < r_2 - r$. Finally, suppose $r > \epsilon > \max\{r - r_1 + \epsilon_1, r_2 - r + \epsilon_2\}$. We consider the factorization product structure map for $\mathrm{Obs}_V^{\mathrm{q}}$ corresponding to the following embedding of open sets

$$N_{r_1,\epsilon_1} \sqcup N_{r_2,\epsilon_2} \hookrightarrow N_{r,\epsilon}, \tag{2}$$

shown schematically in Figure BW: figure. The factorization structure map for this embedding of disjoint open sets is of the form

$$\mathrm{Obs}^{\mathrm{q}}_V(N_{r_1,\epsilon_1}) \otimes \mathrm{Obs}^{\mathrm{q}}_V(N_{r_2,\epsilon_2}) \to \mathrm{Obs}^{\mathrm{q}}_V(N_{r,\epsilon}). \tag{3}$$

Lemma 1.11. *The factorization structure map in (3) restricts to the subspace of sphere observables. That is, there is a commutative diagram*

$$\begin{tikzcd} \mathrm{Obs}^{\mathrm{q}}_V(N_{r_1,\epsilon_1}) \otimes \mathrm{Obs}^{\mathrm{q}}_V(N_{r_2,\epsilon_2}) \arrow[r] & \mathrm{Obs}^{\mathrm{q}}_V \\ \mathcal{A}_V \otimes \mathcal{A}_V \arrow[r, "\mu_2"] \arrow[u] & \mathcal{A}_V \arrow[u] \end{tikzcd}$$

where the top line is the map in (3). The same holds for an arbitrary number of nested neighborhoods of the form $N_{r,\epsilon}$. That is, for any $k \geq 0$ the factorization product restricts to a linear map

$$\mu_k : \mathcal{A}_V^{\otimes k} \to \mathcal{A}_V.$$

Each of the neighborhoods $N_{r,\epsilon}$ are contained in the open submanifold $\mathbb{C}^d \setminus \{0\}$. Note that there is a homeomorphism $\mathbb{C}^d \setminus \{0\} \cong S^{2d-1} \times \mathbb{R}_{>0}$. Further, we have the radial projection map

$$\pi : \mathbb{C}^d \setminus \{0\} = S^{2d-1} \times \mathbb{R}_{>0} \to \mathbb{R}_{>0}$$

that sends $z = (z_1, \ldots, z_d) \mapsto |z| = \sqrt{|z_1|^2 + \cdots + |z_d|^2}$.

A fundamental feature of factorization algebras is that they push forward along smooth maps. We can thus push forward the factorization algebra $\mathrm{Obs}^{\mathrm{q}}_V$ on $\mathbb{C}^d \setminus \{0\}$ along π to obtain a factorization algebra on $\mathbb{R}_{>0}$. To an open interval of the form $(r-\epsilon, r+\epsilon) \subset \mathbb{R}_{>0}$ the factorization algebra assigns precisely the observables supported on $N_{r,\epsilon}$.

Lemma 1.11 implies that the inclusion $\mathcal{A}_V \hookrightarrow \mathrm{Obs}^{\mathrm{q}}(N_{r,\epsilon})$ induces a map of factorization algebras on $\mathbb{R}_{>0}$:

$$\mathcal{F}_{\mathcal{A}_V} \to \pi_*(\mathrm{Obs}^{\mathrm{q}}_V)$$

where $\mathcal{F}_{\mathcal{A}_V}$ assigns, to every interval, the dg vector space $\mathcal{A}_V$. In particular $\mathcal{F}_{\mathcal{A}_V}$ is locally constant, and hence determines the structure of an associative (BW: really A_∞) dg algebra on $\mathcal{A}_V$.

We would now like to identify this algebra structure. To do this, note that we can view $\mathcal{A}_V$ as the symmetric algebra on the following dg vector space

$$A_d \otimes (V^*[d-1] \otimes V) \oplus \mathbb{C} \cdot \hbar.$$

This dg vector space has the structure of a dg Lie algebra, with bracket given by

$$[\alpha \otimes v^*, \alpha \otimes v] = \hbar \langle v^*, v \rangle \oint_{S^{2d-1}} \mathrm{d}^d z \, \alpha \wedge \alpha'.$$

All other brackets are determined by graded anti-symmetry. Moreover, the parameter $\hbar$ is central. Denote this dg Lie algebra by $\mathcal{H}_V$.

Proposition 1.12. *There is a quasi-isomorphism of associative dg algebras*

$$\mathcal{A}_V \simeq U(\mathcal{H}_V)$$

where $U(-)$ is the universal enveloping algebra. The associative dg algebra structure on the left-hand side is determined by the factorization product as in Lemma 1.11 and the discussion above.

Remark 1.13. If $(\mathfrak{g}, \mathrm{d}, [-,-])$ is a dg Lie algebra its universal enveloping algebra is defined explicitly by

$$U(\mathfrak{g}) = \mathrm{Tens}(\mathfrak{g}) / (x \otimes y - (-1)^{|x||y|} y \otimes x - [x,y]).$$

It is immediate to check that the differential d descends to one on $U(\mathfrak{g})$, giving $U(\mathfrak{g})$ the structure of an associative dg algebra.

Proof. We have written down a map of factorization algebras $\mathcal{F}_{\mathcal{A}_V} \to \pi_*(\mathrm{Obs}^{\mathrm{q}}_V)$, where $\mathcal{F}_{\mathcal{A}_V}$ is the locally constant factorization algebra that assigns the cochain complex $\mathcal{A}_V$ to every interval, and whose factorization product is induced from $\pi_*(\mathrm{Obs}^{\mathrm{q}}_V)$.

Let $\underline{U(\mathcal{H}_V)}$ be the locally constant factorization algebra on $\mathbb{R}_{>0}$ based on the associative algebra $U(\mathcal{H}_V)$. We will write down an explicit quasi-isomorphism of locally constant factorization algebras

$$\Phi : \underline{U(\mathcal{H}_V)} \to \mathcal{F}_{\mathcal{A}_V},$$

implying the result.

By Poincaré-Birkoff-Witt, the dg vector spaces $U(\mathcal{H}_V)$ and $\mathcal{A}_V$ are isomorphic. Therefore, if $I \subset \mathbb{R}_{>0}$ is an interval, we define $\Phi(I)$ to be the identity map. We also need to take into account a higher operation.

If I_1, I_2 are two intervals, consider their disjoint union $I_1 \sqcup I_2 \subset \mathbb{R}_{>0}$. We define

$$\Phi(I_1 \sqcup I_2) : U(\mathcal{H}_V) \otimes U(\mathcal{H}_V) \to \mathcal{F}_{\mathcal{A}_V}(I_1 \sqcup I_2)$$

We must explicitly compute the factorization product for certain observables in $\pi_*(\mathrm{Obs}^{\mathrm{q}}_V)$.

$$[\mathcal{O}_{\gamma,\alpha_1}(0;v^*), \mathcal{O}_{\beta,\alpha_2}(0;v)] = \mathcal{O}_{\gamma,\alpha_1}(0;v^*) \star \mathcal{O}_{\beta,\alpha_2}(0;v) - \mathcal{O}_{\beta,\alpha_2}(0;v) \star \mathcal{O}_{\gamma,\alpha_1}(0;v^*).$$

We will compute the factorization product using the explicit form of the propagator of the $\beta\gamma$ system computed in Section **??**. The full propagator is an element

$$P(z,w) = \lim_{L \to \infty} \lim_{\epsilon \to 0} P_{\epsilon < L}(z,w) \in \bar{\mathcal{E}}_V(\mathbb{C}^d) \widehat{\otimes} \bar{\mathcal{E}}_V(\mathbb{C}^d)$$

where the $\bar{\mathcal{E}}_V(\mathbb{C}^d)$ denotes the space of distributional sections on $\mathbb{C}^d$. Explicitly, we found

$$P(z,w) = C_d \; \omega_{BM}(z,w)$$

where $\omega_{BM}(z,w)$ is the Bochner-Martinelli kernel.

Contraction with P determines a degree zero, order two differential operator

$$\partial_P : \mathrm{Obs}^{\mathrm{cl}}_V(U) \to \mathrm{Obs}^{\mathrm{cl}}_V(U)$$

for any open set $U \subset \mathbb{C}^d$. Recall that the classical observables on U are simply given by a symmetric algebra on the continuous dual of $\mathcal{E}_V(U)$. Since $\bar{\mathcal{E}}^\vee = \mathcal{E}^!_c$, we can view the propagator as an symmetric smooth linear map

$$P^\vee : \mathcal{E}^!_{V,c}(\mathbb{C}^d) \widehat{\otimes} \mathcal{E}^!_{V,c}(\mathbb{C}^d) \to \mathbb{C}.$$

The contraction operator ∂_P is determined by declaring it vanishes on $\mathrm{Sym}^{\leq 1}$, and on Sym^2 is given by the linear map $P^\vee$.

To compute the factorization product we use the isomorphism

$$\begin{array}{cccc} W_0^\infty : & \mathrm{Obs}^{\mathrm{cl}}_V(U)[\hbar] & \to & \mathrm{Obs}^{\mathrm{q}}_V(U) \\ & \mathcal{O} & \mapsto & e^{\hbar \partial_P} \mathcal{O} \end{array}$$

that makes sense for any open set U. This is an isomorphism of cochain complexes, with inverse given by $(W_0^\infty)^{-1} = e^{-\hbar \partial_P}$. By **??** it determines the following formula for the factorization product. If $\mathcal{O}, \mathcal{O}'$ are observables supported on disjoint opens U, U', and V is and open set containing U, U', then the factorization structure map is given by

$$\mathcal{O} \star \mathcal{O}' = e^{-\hbar\partial_P}\left(\left(e^{\hbar\partial_P}\mathcal{O}\right)\cdot\left(e^{\hbar\partial_P}\mathcal{O}'\right)\right) \in \mathrm{Obs}^{\mathrm{q}}(V).$$

Here, the $\cdot$ refers to the symmetric product on classical observables.

The calculation of the factorization product relies on the higher dimensional residue formula involving the Bochner-Martinelli form. If f is any any function in $C^\infty(U)$, where U is a domain in $\mathbb{C}^d$, then the residue formula states that for any $z \in D$

$$f(z,\bar{z}) = \int_{w\in\partial U} \mathrm{d}^d w \; f(w)\; \omega_{BM}(z,w) - \int_{w \in D} \mathrm{d}^d w \; (\bar{\partial} f)(w) \wedge \omega_{BM}(z,w).$$

□

1.2.2. *The disk as a module.* We point out the potential confusion with notation for disk observables in Definition 1.6. This is remedied by the following lemma.

Lemma 1.14. *Consider the factorization algebra structure map for the inclusion $N_{r,\epsilon}(w) \hookrightarrow D(w,R)$ where $R > r + \epsilon$:*

$$\mu : \mathrm{Obs}^{\mathrm{q}}_V(N_{r,\epsilon}(w)) \to \mathrm{Obs}^{\mathrm{q}}_V(D(w,R)).$$

Then, in cohomology $H^*(\mu)|_{H^*\mathcal{A}_V}$ is only nonzero on elements in

$$\mathrm{Sym}\left(H^{d-1}(A_d) \otimes (V^*[d-1] \oplus V)\right).$$

On the linear elements inside of this symmetric algebra, the factorization map map satisfies

$$H^*\mu : \mathcal{O}_{\gamma, \partial_z^{\vec{n}}\omega_{BM}} \mapsto \mathcal{O}_{\gamma,-\vec{n}}$$

Proposition 1.15. *The factorization product above gives the cohomology $H^*\mathcal{V}_V$ the structure of a graded module for the associative graded algebra $H^*\mathcal{A}_V$. Moreover, there is an isomorphism of $H^*\mathcal{A}_V$ modules*

$$H^*\mathcal{V} \cong H^*\mathcal{A}_V \otimes_{\mathcal{A}_{V,+}} \mathbb{C}.$$

The tensor product $H^*\mathcal{A}_V \otimes_{\mathcal{A}_{V,+}} \mathbb{C}$ is equal to the induction of the trivial module along the subalgebra $\mathcal{A}_{V,+} \subset H^*\mathcal{A}_V$. In particular, it implies that as a graded vector space

$$H^*\mathcal{V}_V \cong \mathcal{A}_{V,-}[-d+1],$$

which is immediate from our identification (1)

1.3. The colored operad of holomorphic disks.

1.4. Holomorphic descent.

1.4.1. *Topological descent.* BW: review Before jumping in to the construction of operators in holomorphic theories using a descent procedure, we'd like to a review a more familiar topological situation. This concept was introduced by Witten in his introduction of cohomological field theories [**?**]. Expositions of this construction in the context of topological conformal field theory can be found in [**?**, **?**]

Suppose we have a translation invariant theory on $\mathbb{R}^d$ for which all infinitesimal translations are exact for the BRST differential. If Q^{BRST} is the BRST differential this means that for $i = 1, \ldots, d$ there exists operators G_i on the space of fields such that

$$[Q^{BRST}, G_i] = \frac{\partial}{\partial x_i}. \tag{4}$$

Note that since $\partial/\partial x_i$ has BRST degree zero, the operators G_i decrease the BRST degree by one. Here, one thinks of the collection $\{G_i\}$ as providing a homotopy trivialization of the action by infinitesimal translations on the theory. In particular, this means that $\partial/\partial x_i$ acts trivially on the Q^{BRST}-cohomology.

In turn, G_i also acts on the local operators of the theory. Using translation invariance, we can view a local operator $\mathcal{O}$ as a function on space-time $\mathbb{R}^d$. Suppose $\mathcal{O}$ has pure BRST degree k. Using the operator G_i we can consider the function valued operator $G_i\mathcal{O}$ which is of BRST degree $k-1$. Using the frame on $\mathbb{R}^d$ we can then define the *1-form* valued operator

$$\mathcal{O}^{(1)} = \sum_i (G_i\mathcal{O}) \mathrm{d}x_i.$$

By construction, the following relation is satisfied

$$\mathrm{d}_{dR}\mathcal{O} = \sum_i \frac{\partial}{\partial x_i}\mathcal{O}\mathrm{d}x_i = [Q^{BRST}, \mathcal{O}^{(1)}].$$

This is the first so-called *topological descent equation*. In general, we can iterate the above construction to define

$$\mathcal{O}^{(l)} = \sum_{i_1,\ldots i_l} G_{i_1}\cdots G_{i_l}\mathcal{O}\mathrm{d}x_{i_1}\cdots dx_{i_l}.$$

This is an l-form valued operator of BRST degree $k-l$.

The operator $\mathcal{O}^{(l)}$ allows us to define a new class of operators that depend on choosing an l-cycle inside of $\mathbb{R}^d$. Indeed, suppose $Z \subset \mathbb{R}^d$ is a closed l-dimensional submanifold. Define the operator

$$\mathcal{O}_Z = \int_Z \mathcal{O}^{(l)}.$$

The topological descent equations imply that if $\mathcal{O}$ is BRST invariant $Q^{BRST}\mathcal{O} = 0$, then $Q^{BRST}\mathcal{O}_Z = 0$ as well.

Interesting examples of cohomological field theories arise as topological twists of supersymmetric theories. Another class of examples come from topological vertex algebras [**?**, **?**]. In Section BW: ref we will discuss a class of such theories by considering a higher dimensional version of holomorphic gravity.

We know that the local operators of a quantum field theory have the structure of a factorization algebra. In the world of factorization algebras, there is also a notion of being topological: being (homotopically) locally constant. This means that for every embedding of open balls $B \hookrightarrow B'$, the induced factorization structure map $\mathcal{F}(B) \to \mathcal{F}(B')$ is a quasi-isomorphism.

It would be natural to expect that the observables of a topological field theory, in which the infinitesimal translations are BRST exact, should give rise to such a factorization algebra. This is not exactly the case. The relations (4) guarantee a slightly weaker condition on the factorization algebra of observables. Indeed, the resulting action of the operators G_i on the factorization algebra provide us with a sort of "flat connection" on the factorization algebra. The difference between this structure and the locally constant condition is analogous to the discrepancy between D-modules and local systems. It is current work of Elliott and Safranov [] to show how topological twists of supersymmetric theories give rise to such locally constant factorization algebras.

We discuss a more direct way in which we can extract a shadow of a locally constant factorization algebra from a topological field theory using descent. There is an algebraic object associated to any locally constant factorization algebra. Indeed, a famous theorem of Lurie [**?**] states an equivalence of categories

$$\{\text{Locally constant factorization algebras on } \mathbb{R}^d\} \;\simeq\; \{E_d - \text{ algebras}\}.$$

The cohomology of an E_d-algebra has the structure of a P_d-algebra. In the category of cochain complexes, we have the following concrete definition of a P_d-algebra.

Definition 1.16. Let $d \geq 0$. A P_d algebra in cochain complex is a commutative dg algebra (A, d) together with the data of a bracket of degree $1-d$

$$\{-,-\} : A \otimes A \to A[d-1]$$

such that:

(1) the bracket is graded anti-symmetric:

$$\{a,b\} = -(-1)^{|a|+d-1}(-1)^{|b|+d-1};$$

(2) the bracket satisfies graded Jacobi:

$$\{a,\{b,c\}\} = \{\{a,b\},c\} + (-1)^{|a|+d-1}(-1)^{|b|+d-1}\{a,\{b,c\}\};$$

(3) the bracket is a graded bi-derivation for the commutative product:

$$\{a, b\cdot c\} = \{a,b\}\cdot c + (-1)^{|b|(|a|+d-1)} b \cdot \{a,c\}.$$

for all $a, b, c \in A$.

BW: finish

Example 1.17. Topological BF theory. BW: give references For any dg Lie algebra $(\mathfrak{g}, \mathrm{d}_{\mathfrak{g}}, [-,-])$ one can define the following d-dimensional topological field theory. The fields of BF theory with values in $\mathfrak{g}$ consist of

$$(A,B) \in \Omega^*(\mathbb{R}^d; \mathfrak{g}[1]) \oplus \Omega^*(\mathbb{R}^d;\mathfrak{g})[d-2]$$

with action functional

$$S(A,B) = \int \langle B, dA + [A,A]\rangle_{\mathfrak{g}},$$

where $\langle -,-\rangle_{\mathfrak{g}}$ denotes a chosen invariant non-degenerate pairing on $\mathfrak{g}$. The name comes from the fact that $S = \int BF(A)$ where $F(A) = \mathrm{d}A + [A,A]$ is the curvature. The differential is a sum $\mathrm{d} = \mathrm{d}_{dR} + \mathrm{d}_{\mathfrak{g}}$. Note that the theory is translation invariant and has a natural action by the infinitesimal translations $\{\frac{\partial}{\partial x_i}\}$ via Lie derivative.

The class of local operators we consider are defined as

$$\begin{aligned}\mathcal{O}_{A,a}(x) :&A \in \Omega^0(\mathbb{R}^d;\mathfrak{g})[1] \mapsto \langle a, A(x)\rangle_{\mathfrak{g}}\\ \mathcal{O}_{B,a}(x) :&B \in \Omega^0(\mathbb{R}^d,\mathfrak{g})[d-2] \mapsto \langle a, B(x)\rangle.\end{aligned}$$

where $x \in \mathbb{R}^d$ is a fixed point and $a \in \mathfrak{g}$ is a fixed element. Using translation invariance, we view $\mathcal{O}_{A,a}, \mathcal{O}_{B,a}$ as function valued operators on $\mathbb{R}^d$. The total space of local operators can be identified with functions on the shifted tangent bundle to the formal moduli space $B\mathfrak{g}$, $\mathcal{O}(T[d-1]B\mathfrak{g})$. The operator $\mathcal{O}_{A,a}$ corresponds to the linear coordinate on the base of $B\mathfrak{g}$ and $\mathcal{O}_{B,a}$ corresponds to a linear coordinate on the fiber.

We consider the differential operator

$$G_i = \frac{\mathrm{d}}{\mathrm{d}(\mathrm{d}x_i)}$$

that also acts on the space of fields. This operator is equal to the contraction with the vector field $\frac{\partial}{\partial x_i}$. Since G_i commutes with the differential and bracket on the Lie algebra, the Cartan formula implies

$$[Q^{BRST}, G_i] = \left[\mathrm{d}_{dR}, \frac{\mathrm{d}}{\mathrm{d}(\mathrm{d}x_i)}\right] = \frac{\partial}{\partial x_i}.$$

Following the descent procedure above, we go on to define the form valued local operators

$$\mathcal{O}^{(l)}_{A,a} = \sum_{i_1,\ldots,i_l} G_i \mathcal{O}_{A,a} \mathrm{d}x_i$$

and similarly for $\mathcal{O}_{B,a}$. Then, for any l-cycle $Z \subset \mathbb{R}^d$ we obtain operators $\int_Z \mathcal{O}^{(l)}_{A,a}, \int_Z \mathcal{O}^{(l)}_{B,a}$. For example, one can check that the latter operator is of the form

$$\int_Z \mathcal{O}^{(l)}_{B,a} : B \in \Omega^l(\mathbb{R}^d)[d-2-l] \mapsto \int_Z B,$$

which is of degree $-d+2+l$.

To obtain the P_n-bracket via descent we consider the $(d-1)$-sphere $Z = S^{d-1}$, which we assume is centered at the origin. Then, the bracket between the linear operators $\mathcal{O}_{A,a}, \mathcal{O}_{B,a'}$ is computed by the operator product expansion of $\mathcal{O}_{A,a}$ and the descended operator $\int_{S^{d-1}} \mathcal{O}^{(d-1)}_{B,a'}$:

$$\{\mathcal{O}_{A,a}, \mathcal{O}_{B,a'}\}_{P_d} = \mathcal{O}_{A,a}(0) \star \int_{S^{d-1}} \mathcal{O}^{(d-1)}_{B,a'}.$$

A simple OPE calculation BW: finish this

1.4.2. *General theory.* We will now summarize the steps in defining the higher dimensional OPE for holomorphically translation invariant quantum field theories. We note that this is a schematic, and as is usual we will need to regularize at various stages to obtained a well-defined construction.

(1) Suppose $\mathcal{O} \in \mathrm{Obs}_0$ is a local operator supported at $0 \in \mathbb{C}^d$. Let $z \in \mathbb{C}^d$ be another point, and consider the translated operator

$$\mathcal{O}(z) := \tau_z \mathcal{O}.$$

By the property of holomorphic translation invariance, this assignment defines a $\mathcal{O}^{hol}(\mathbb{C}^d)$-valued local operator.

(2) We perform "holomorphic descent" to the function valued operator $\mathcal{O}^{hol}(\mathbb{C}^d)$ to obtain Dolbeualt valued operator

$$\mathcal{O}^{(0,*)}(z) \in \Omega^{0,*}(\mathbb{C}^d) \otimes \mathcal{O}\mathrm{bs}_0.$$

Explicitly,

$$\mathcal{O}^{(0,k)}(z) = \sum_I (\bar{\eta}_I \cdot \mathcal{O}(z)) \mathrm{d}\bar{z}_I$$

where $I = (i_1, \ldots, i_k)$, $1 \leq i_k \leq d$, is a multi-index of length k and $\eta_I = \eta_{i_1 \cdots i_k}$, $\mathrm{d}\bar{z}_I = \mathrm{d}\bar{z}_{i_1} \cdots \mathrm{d}\bar{z}_{i_k}$.

(3) For any $f(z)\mathrm{d}^d z \in \Omega^{d,hol}(\mathbb{C}^d)$, and $w \in \mathbb{C}^d$, define the sphere supported operator

$$\mathcal{O}_f(w, r) := \int_{z \in S^{2d-1}_{w,r}} f(z) \mathrm{d}^d z \mathcal{O}^{(0,d-1)}(z)$$

where $S^3_{w,r}$ is the sphere of radius r centered at w.

(4) If $\mathcal{O}'$ is another local operator supported at zero, we define the f-bracket by

$$\{\mathcal{O}, \mathcal{O}'\}_f := \mathcal{O}_f(0, r) \star \mathcal{O}' \in \mathcal{O}\mathrm{bs}_0$$

where $\star$ denotes the factorization product of a small disk with a small neighborhood of $S^{2d-1}_{0,r}$.

1.4.3. The observables of the $\beta\gamma$ system comes naturally equipped with null-homotopies of the operators $\frac{\partial}{\partial \bar{z}_i}$.

So far, in Section BW: ref we have described the space of local operators on the d-disk of the $\beta\gamma$ system with values in a vector space V. For disks centered at $z \in \mathbb{C}^d$ there are two main classes of operators $O_\gamma(\vec{n}, z; v^*)$ and $O_\beta(\vec{m}, z; v)$ where $\vec{n} = (n_1, \ldots, n_d) \in (\mathbb{Z}_{\geq 0})^d$, $(m_1, \ldots, m_d) \in (\mathbb{Z}_{\geq 1})^d$, $v \in V$, and $v^* \in V$.