HOLOMORPHIC DESCENT

0.1. **The colored operad of holomorphic disks.**

0.2. **Holomorphic descent.**

0.2.1. *Topological descent.* BW: review Before jumping in to the construction of operators in holomorphic theories using a descent procedure, we'd like to a review a more familiar topological situation. This concept was introduced by Witten in his introduction of cohomological field theories [**?**]. Expositions of this construction in the context of topological conformal field theory can be found in [**?**, **?**]

Suppose we have a translation invariant theory on $\mathbb{R}^d$ for which all infinitesimal translations are exact for the BRST differential. If Q^{BRST} is the BRST differential this means that for $i = 1, \ldots, d$ there exists operators G_i on the space of fields such that

$$[Q^{BRST}, G_i] = \frac{\partial}{\partial x_i}. \tag{1}$$

Note that since $\partial/\partial x_i$ has BRST degree zero, the operators G_i decrease the BRST degree by one. Here, one thinks of the collection $\{G_i\}$ as providing a homotopy trivialization of the action by infinitesimal translations on the theory. In particular, this means that $\partial/\partial x_i$ acts trivially on the Q^{BRST}-cohomology.

In turn, G_i also acts on the local operators of the theory. Using translation invariance, we can view a local operator $\mathcal{O}$ as a function on space-time $\mathbb{R}^d$. Suppose $\mathcal{O}$ has pure BRST degree k. Using the operator G_i we can consider the function valued operator $G_i\mathcal{O}$ which is of BRST degree $k-1$. Using the frame on $\mathbb{R}^d$ we can then define the *1-form* valued operator

$$\mathcal{O}^{(1)} = \sum_i (G_i\mathcal{O})\mathrm{d}x_i.$$

By construction, the following relation is satisfied

$$\mathrm{d}_{dR}\mathcal{O} = \sum_i \frac{\partial}{\partial x_i}\mathcal{O}\mathrm{d}x_i = [Q^{BRST}, \mathcal{O}^{(1)}].$$

This is the first so-called *topological descent equation*. In general, we can iterate the above construction to define

$$\mathcal{O}^{(l)} = \sum_{i_1,\ldots i_l} G_{i_1}\cdots G_{i_l}\mathcal{O}\mathrm{d}x_{i_1}\cdots dx_{i_l}.$$

This is an l-form valued operator of BRST degree $k-l$.

The operator $\mathcal{O}^{(l)}$ allows us to define a new class of operators that depend on choosing an l-cycle inside of $\mathbb{R}^d$. Indeed, suppose $Z \subset \mathbb{R}^d$ is a closed l-dimensional submanifold. Define the operator

$$\mathcal{O}_Z = \int_Z \mathcal{O}^{(l)}.$$

The topological descent equations imply that if $\mathcal{O}$ is BRST invariant $Q^{BRST}\mathcal{O} = 0$, then $Q^{BRST}\mathcal{O}_Z = 0$ as well.

Interesting examples of cohomological field theories arise as topological twists of supersymmetric theories. Another class of examples come from topological vertex algebras [**?**, **?**]. In Section BW: ref we will discuss a class of such theories by considering a higher dimensional version of holomorphic gravity.

We know that the local operators of a quantum field theory have the structure of a factorization algebra. In the world of factorization algebras, there is also a notion of being topological: being (homotopically) locally constant. This means that for every embedding of open balls $B \hookrightarrow B'$, the induced factorization structure map $\mathcal{F}(B) \to \mathcal{F}(B')$ is a quasi-isomorphism.

It would be natural to expect that the observables of a topological field theory, in which the infinitesimal translations are BRST exact, should give rise to such a factorization algebra. This is not exactly the case. The relations (**??**) guarantee a slightly weaker condition on the factorization algebra of observables. Indeed, the resulting action of the operators G_i on the factorization algebra provide us with a sort of "flat connection" on the factorization algebra. The difference between this structure and the locally constant condition is analogous to the discrepancy between D-modules and local systems. It is current work of Elliott and Safranov [] to show how topological twists of supersymmetric theories give rise to such locally constant factorization algebras.

We discuss a more direct way in which we can extract a shadow of a locally constant factorization algebra from a topological field theory using descent. There is an algebraic object associated to any locally constant factorization algebra. Indeed, a famous theorem of Lurie [**?**] states an equivalence of categories

$$\{\text{Locally constant factorization algebras on } \mathbb{R}^d\} \;\simeq\; \{E_d - \text{ algebras}\}.$$

The cohomology of an E_d-algebra has the structure of a P_d-algebra. In the category of cochain complexes, we have the following concrete definition of a P_d-algebra.

Definition 0.1. Let $d \geq 0$. A P_d algebra in cochain complex is a commutative dg algebra (A, d) together with the data of a bracket of degree $1-d$

$$\{-,-\} : A \otimes A \to A[d-1]$$

such that:

(1) the bracket is graded anti-symmetric:

$$\{a, b\} = -(-1)^{|a|+d-1}(-1)^{|b|+d-1};$$

(2) the bracket satisfies graded Jacobi:

$$\{a, \{b, c\}\} = \{\{a, b\}, c\} + (-1)^{|a|+d-1}(-1)^{|b|+d-1}\{a, \{b, c\}\};$$

(3) the bracket is a graded bi-derivation for the commutative product:

$$\{a, b \cdot c\} = \{a, b\} \cdot c + (-1)^{|b|(|a|+d-1)} b \cdot \{a, c\}.$$

for all $a, b, c \in A$.

BW: finish

Example 0.2. Topological BF theory. BW: give references For any dg Lie algebra $(\mathfrak{g}, \mathrm{d}_{\mathfrak{g}}, [-,-])$ one can define the following d-dimensional topological field theory. The fields of BF theory with values in $\mathfrak{g}$ consist of

$$(A, B) \in \Omega^*(\mathbb{R}^d; \mathfrak{g}[1]) \oplus \Omega^*(\mathbb{R}^d; \mathfrak{g})[d-2]$$

with action functional

$$S(A, B) = \int \langle B, dA + [A, A] \rangle_{\mathfrak{g}},$$

where $\langle -, - \rangle_{\mathfrak{g}}$ denotes a chosen invariant non-degenerate pairing on $\mathfrak{g}$. The name comes from the fact that $S = \int BF(A)$ where $F(A) = \mathrm{d}A + [A, A]$ is the curvature. The differential is a sum $\mathrm{d} = \mathrm{d}_{dR} + \mathrm{d}_{\mathfrak{g}}$. Note that the theory is translation invariant and has a natural action by the infinitesimal translations $\{\frac{\partial}{\partial x_i}\}$ via Lie derivative.

The class of local operators we consider are defined as

$$\begin{aligned} \mathcal{O}_{A,a}(x) &: A \in \Omega^0(\mathbb{R}^d; \mathfrak{g})[1] \mapsto \langle a, A(x) \rangle_{\mathfrak{g}} \\ \mathcal{O}_{B,a}(x) &: B \in \Omega^0(\mathbb{R}^d, \mathfrak{g})[d-2] \mapsto \langle a, B(x) \rangle. \end{aligned}$$

where $x \in \mathbb{R}^d$ is a fixed point and $a \in \mathfrak{g}$ is a fixed element. Using translation invariance, we view $\mathcal{O}_{A,a}, \mathcal{O}_{B,a}$ as function valued operators on $\mathbb{R}^d$. The total space of local operators can be identified with functions on the shifted tangent bundle to the formal moduli space $B\mathfrak{g}$, $\mathcal{O}(T[d-1]B\mathfrak{g})$. The operator $\mathcal{O}_{A,a}$ corresponds to the linear coordinate on the base of $B\mathfrak{g}$ and $\mathcal{O}_{B,a}$ corresponds to a linear coordinate on the fiber.

We consider the differential operator

$$G_i = \frac{\mathrm{d}}{\mathrm{d}(\mathrm{d}x_i)}$$

that also acts on the space of fields. This operator is equal to the contraction with the vector field $\frac{\partial}{\partial x_i}$. Since G_i commutes with the differential and bracket on the Lie algebra, the Cartan formula implies

$$[Q^{BRST}, G_i] = \left[\mathrm{d}_{dR}, \frac{\mathrm{d}}{\mathrm{d}(\mathrm{d}x_i)}\right] = \frac{\partial}{\partial x_i}.$$

Following the descent procedure above, we go on to define the form valued local operators

$$\mathcal{O}^{(l)}_{A,a} = \sum_{i_1,\ldots,i_l} G_i \mathcal{O}_{A,a} \mathrm{d}x_i$$

and similarly for $\mathcal{O}_{B,a}$. Then, for any l-cycle $Z \subset \mathbb{R}^d$ we obtain operators $\int_Z \mathcal{O}^{(l)}_{A,a}, \int_Z \mathcal{O}^{(l)}_{B,a}$. For example, one can check that the latter operator is of the form

$$\int_Z \mathcal{O}^{(l)}_{B,a} : B \in \Omega^l(\mathbb{R}^d)[d-2-l] \mapsto \int_Z B,$$

which is of degree $-d+2+l$.

To obtain the P_n-bracket via descent we consider the $(d-1)$-sphere $Z = S^{d-1}$, which we assume is centered at the origin. Then, the bracket between the linear operators $\mathcal{O}_{A,a}, \mathcal{O}_{B,a'}$ is computed by the operator product expansion of $\mathcal{O}_{A,a}$ and the descended operator $\int_{S^{d-1}} \mathcal{O}^{(d-1)}_{B,a'}$:

$$\{\mathcal{O}_{A,a}, \mathcal{O}_{B,a'}\}_{P_d} = \mathcal{O}_{A,a}(0) \star \int_{S^{d-1}} \mathcal{O}^{(d-1)}_{B,a'}.$$

A simple OPE calculation BW: finish this

0.2.2. *General theory.* We will now summarize the steps in defining the higher dimensional OPE for holomorphically translation invariant quantum field theories. We note that this is a schematic, and as is usual we will need to regularize at various stages to obtained a well-defined construction.

(1) Suppose $\mathcal{O} \in \mathcal{O}bs_0$ is a local operator supported at $0 \in \mathbb{C}^d$. Let $z \in \mathbb{C}^d$ be another point, and consider the translated operator

$$\mathcal{O}(z) := \tau_z \mathcal{O}.$$

By the property of holomorphic translation invariance, this assignment defines a $\mathcal{O}^{hol}(\mathbb{C}^d)$-valued local operator.

(2) We perform "holomorphic descent" to the function valued operator $\mathcal{O}^{hol}(\mathbb{C}^d)$ to obtain Dolbeualt valued operator

$$\mathcal{O}^{(0,*)}(z) \in \Omega^{0,*}(\mathbb{C}^d) \otimes \mathcal{O}bs_0.$$

Explicitly,

$$\mathcal{O}^{(0,k)}(z) = \sum_I (\bar{\eta}_I \cdot \mathcal{O}(z)) \mathrm{d}\bar{z}_I$$

where $I = (i_1, \ldots, i_k)$, $1 \leq i_k \leq d$, is a multi-index of length k and $\eta_I = \eta_{i_1 \cdots i_k}$, $\mathrm{d}\bar{z}_I = \mathrm{d}\bar{z}_{i_1} \cdots \mathrm{d}\bar{z}_{i_k}$.

(3) For any $f(z)\mathrm{d}^d z \in \Omega^{d,hol}(\mathbb{C}^d)$, and $w \in \mathbb{C}^d$, define the sphere supported operator

$$\mathcal{O}_f(w,r) := \int_{z \in S^{2d-1}_{w,r}} f(z) \mathrm{d}^d z \mathcal{O}^{(0,d-1)}(z)$$

where $S^3_{w,r}$ is the sphere of radius r centered at w.

(4) If $\mathcal{O}'$ is another local operator supported at zero, we define the f-bracket by

$$\{\mathcal{O}, \mathcal{O}'\}_f := \mathcal{O}_f(0,r) \star \mathcal{O}' \in \mathcal{O}bs_0$$

where $\star$ denotes the factorization product of a small disk with a small neighborhood of $S^{2d-1}_{0,r}$.

0.2.3. The observables of the $\beta\gamma$ system comes naturally equipped with null-homotopies of the operators $\frac{\partial}{\partial \bar{z}_i}$.

So far, in Section BW: ref we have described the space of local operators on the d-disk of the $\beta\gamma$ system with values in a vector space V. For disks centered at $z \in \mathbb{C}^d$ there are two main classes of operators $O_\gamma(\vec{n}, z; v^*)$ and $O_\beta(\vec{m}, z; v)$ where $\vec{n} = (n_1, \ldots, n_d) \in (\mathbb{Z}_{\geq 0})^d$, $(m_1, \ldots, m_d) \in (\mathbb{Z}_{\geq 1})^d$, $v \in V$, and $v^* \in V$.