# COHOMOLOGY OF CONFORMAL ALGEBRAS

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*To Bertram Kostant on his seventieth birthday*

## CONTENTS

## INTRODUCTION

The notion of a conformal algebra encodes an axiomatic description of the operator product expansion of chiral fields in conformal field theory. On the other hand, it is an adequate tool for the study of infinite-dimensional Lie algebras satisfying the locality property [K2]–[K4], [DK]. Likewise, conformal modules over a conformal algebra $A$ correspond to conformal modules over the associated Lie algebra Lie $A$ [CK]. The main examples of Lie algebras Lie $A$ are the Lie algebras "based" on the punctured complex plane $\mathbb{C}^\times$, namely the Lie algebra $\mathcal{V}ect\,\mathbb{C}^\times$ of vector fields on $\mathbb{C}^\times$ (= Virasoro algebra) and the Lie algebra of maps of $\mathbb{C}^\times$ to a finite-dimensional Lie algebra (= loop algebra). Their irreducible conformal modules are the spaces of densities on $\mathbb{C}^\times$ and loop modules, respectively, [CK]. Since complete reducibility does not hold in this case (cf. [F], [CKW]), one may expect that their cohomology theory is very interesting.

In the present paper we develop a cohomology theory of conformal algebras with coefficients in an arbitrary module. We introduce the basic and the reduced complexes, the latter being a quotient of the former. The basic complex turns out to be isomorphic to the Lie algebra complex for the so-called annihilation subalgebra (Lie $A)_-$ of Lie $A$. For the main examples the annihilation subalgebra turns out to be its complex-plane counterpart (*i.e.*, $\mathbb{C}^\times$ is replaced by $\mathbb{C}$). The cohomology of these Lie algebras has been extensively studied in [GF1, GF2, FF, Fe1, F, Fe2]. This allows us to compute the cohomology of the conformal algebra $A$, which in its turn captures main features of the cohomology of the Lie algebra Lie $A$. As a byproduct of our considerations, we compute the cohomology of a current Lie algebra on $\mathbb{C}$ with values in an irreducible highest-weight module (see Theorem 8.2), which has been known only when the module is trivial [Fe1].

The first cohomology theory in the context of operator product expansion was the cohomology theory of vertex algebras and conformal field theories introduced in [KV]. The cohomology theory of the present paper relates to the cohomology theory of [KV] as much as Chevalley–Eilenberg cohomology of Lie algebras relates to Hochschild (or more exactly, Harrison) cohomology of commutative associative algebras. The two theories possess standard properties of cohomology theories. For example, the cohomology of [KV] describes deformations of vertex algebras, and the cohomology of this paper describes same of conformal algebras. However, the cohomology of [KV] is hard to compute,

whereas this paper offers the computation of cohomology in most of the important examples.

The paper is organized as follows. In Section 1 we recall the definition of a conformal algebra and of a (conformal) module over it and describe their relation to formal distribution Lie algebras and conformal modules.

In Section 2 we construct the *basic* complex $\widetilde{C}^\bullet(A, M)$ and its quotient, the *reduced* complex $C^\bullet(A, M)$, for a module $M$ over a conformal algebra $A$. These complexes define the basic and reduced cohomology of a conformal algebra $A$.

In Section 3 we show that this cohomology parameterizes $A$-module extensions, abelian conformal-algebra extensions, first-order deformations, etc. (Theorem 3.1).

In Section 4 we construct the dual, homology complexes. In Section 5 we define the exterior multiplication, contraction and module structure for the basic complex.

In Section 6 we prove that the basic complex is isomorphic to the Lie algebra complex of the annihilation algebra (Theorem 6.1). Along with Proposition 1.1 this implies, in particular, that basic cohomology can be defined via a derived functor. Apparently this is not the case for the reduced complex.

In Section 7 we compute the cohomology with trivial coefficients of the Virasoro conformal algebra Vir both for the basic and reduced complexes (Theorem 7.1). As one could expect, the calculation and the result are closely related to Gelfand–Fuchs's calculation of the cohomology of $\mathcal{V}ect\,\mathbb{C}^\times$ [GF1]. We also compute both cohomologies of Vir with coefficients in the modules of densities (Theorem 7.2). This result is closely related to the work of Feigin and Fuchs [FF, F].

In Section 8 we compute the cohomology of the current conformal algebras both with trivial coefficients (Theorem 8.1) and with coefficients in current modules (Theorem 8.2). This allows us, in particular, to classify abelian extensions of current algebras (Remark 8.1). Of course, abelian extensions of Vir can be classified by making use of Theorem 7.2. This problem has been solved earlier by M. Wakimoto and one of the authors of the present paper by a lengthy but direct calculation; however, in the case of current algebras the direct calculation is all but impossible.

In Section 9 we briefly discuss the analogues of Hochschild and cyclic cohomology for associative conformal algebras and of Leibniz cohomology.

In Section 10 we indicate how to generalize our cohomology theory to the case of conformal algebras in several indeterminates and discuss its relation to cohomology of Cartan's filtered Lie algebras.

In Section 11 we introduce anticommuting higher differentials which may be useful for computing the cohomology of the basic complex with non-trivial coefficients.

In Section 12 we briefly discuss the relation of our cohomology theory to Lie algebras in a general pseudo-tensor category introduced in [BD].

In the last Section 13 we list several open questions.

Unless otherwise specified, all vector spaces, linear maps and tensor products are considered over the field $\mathbb{C}$ of complex numbers. We will use the divided-powers notation $\lambda^{(m)} = \lambda^m/m!$, $m \in \mathbb{Z}_+$, where $\mathbb{Z}_+$ is the set of non-negative integers.

*Acknowledgment.* The second author is grateful to Jean-Louis Loday for inspiring discussions on Leibniz algebras, whose conformal version, see Section 9.3, seems to be an essential notion in the case of nonlocal fields.

## 1. Preliminaries on conformal algebras and modules

**Definition 1.1.** A (*Lie*) *conformal algebra* is a $\mathbb{C}[\partial]$-module $A$ endowed with a $\lambda$-bracket $[a_\lambda b]$ which defines a linear map $A \otimes A \to A[\lambda]$, where $A[\lambda] = \mathbb{C}[\lambda] \otimes A$, subject to the following axioms:

**Conformal sesquilinearity:** $[\partial a_\lambda b] = -\lambda[a_\lambda b]$, $[a_\lambda \partial b] = (\partial + \lambda)[a_\lambda b]$;

**Skew-symmetry:** $[a_\lambda b] = -[b_{-\lambda-\partial} a]$;

**Jacobi identity:** $[a_\lambda[b_\mu c]] = [[a_\lambda b]_{\lambda+\mu} c] + [b_\mu[a_\lambda c]]$.

Conformal algebras appear naturally in the context of formal distribution Lie algebras as follows. Let $\mathfrak{g}$ be a vector space. A $\mathfrak{g}$-*valued formal distribution* is a series of the form $a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}$, where $a_n \in \mathfrak{g}$ and $z$ is an indeterminate. We denote the space of such distributions by $\mathfrak{g}[[z, z^{-1}]]$ and the operator $\partial_z$ on this space by $\partial$.

Let $\mathfrak{g}$ be a Lie algebra. Two $\mathfrak{g}$-valued formal distributions are called *local* if

$$(z-w)^N[a(z), b(w)] = 0 \text{ for } N \gg 0.$$

This is equivalent to saying that one has an expansion of the form [K2]:

$$(1.1) \qquad [a(z), b(w)] = \sum_{j=0}^{N-1} \big(a(w)_{(j)} b(w)\big) \partial_w^{(j)} \delta(z-w),$$

where

$$(1.2) \qquad a(w)_{(j)}b(w) = \text{Res}_z(z-w)^j[a(z),b(w)]$$

and

$$\delta(z-w) = \sum_{n\in\mathbb{Z}} z^{-n-1}w^n.$$

Let $\mathcal{F}$ be a family of pairwise local $\mathfrak{g}$-valued formal distributions such that the coefficients of all distributions from $\mathcal{F}$ span $\mathfrak{g}$. Then the pair $(\mathfrak{g},\mathcal{F})$ is called a *formal distribution Lie algebra*.

Let $\overline{\mathcal{F}}$ denote the minimal subspace of $\mathfrak{g}[[z,z^{-1}]]$ containing $\mathcal{F}$ which is closed under all $j$-th products (1.2) and $\partial$-invariant. One knows that $\overline{\mathcal{F}}$ still consists of pairwise local distributions [K2]. Letting

$$[a_\lambda b] = \sum_{n\in\mathbb{Z}_+} \lambda^{(n)}a_{(n)}b,$$

one endows $\overline{\mathcal{F}}$ with the structure of a conformal algebra, which is denoted by $\text{Conf}(\mathfrak{g},\mathcal{F})$ [DK, K2].

Conversely, given a conformal algebra $A$, one associates to it the *maximal* formal distribution Lie algebra (Lie $A$, $A$) as follows.

Let Lie $A = A[t,t^{-1}]/(\partial+\partial_t)A[t,t^{-1}]$ and let $a_n$ denote the image of $at^n$ in Lie $A$. Then the formula $(a,b\in A, m,n\in\mathbb{Z})$:

$$(1.3) \qquad [a_m,b_n] = \sum_{j\in\mathbb{Z}_+} \binom{m}{j}(a_{(j)}b)_{m+n-j}$$

gives a well defined bracket making Lie $A$ a Lie algebra. It forms a formal distribution Lie algebra with the family of pairwise local distributions $\mathcal{F} = \{a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}\}_{a\in A}$. We have: $\text{Conf}(\text{Lie } A, \mathcal{F}) \simeq A$ via the map $a \mapsto a(z)$ [K2].

The Lie algebra Lie $A$ carries a derivation $T$ induced by $-\partial_t$:

$$(1.4) \qquad T(a_n) = -na_{n-1}.$$

It is clear from (1.3) that the $\mathbb{C}$-span of the $a_n$ with $n\in\mathbb{Z}_+, a\in A$, is a $T$-invariant subalgebra of the Lie algebra Lie $A$. This subalgebra is denoted by $(\text{Lie } A)_-$ and is called the *annihilation Lie algebra* of $A$. The semidirect sum $(\text{Lie } A)^- = \mathbb{C}T + (\text{Lie } A)_-$ is called the *extended annihilation Lie algebra*.

If one drops the skew-symmetry in the definition of a Lie algebra $\mathfrak{g}$, but keeps the Leibniz version of the Jacobi identity $[a,[b,c]] = [[a,b],c] + [b,[a,c]]$, then $\mathfrak{g}$ is called a (left) *Leibniz algebra*, see [L1]. If one also drops the condition of locality on $\mathcal{F}$, then $(\mathfrak{g},\mathcal{F})$ is called a *formal distribution Leibniz algebra*. In this case $\text{Conf}(\mathfrak{g},\mathcal{F})$ is a *Leibniz*

*conformal algebra*, *i.e.*, the skew-symmetry axiom in the definition of
a Lie conformal algebra is dropped.

**Definition 1.2.** A *module* $M$ over a Lie conformal algebra $A$ is a $\mathbb{C}[\partial]$-module endowed with the $\lambda$-action $a_\lambda v$ which defines a map $A \otimes M \to M[[\lambda]]$ such that

$$(1.5) \qquad a_\lambda(b_\mu v) - b_\mu(a_\lambda v) = [a_\lambda b]_{\lambda+\mu} v,$$

$$(1.6) \qquad (\partial a)_\lambda v = -\lambda a_\lambda v, \quad a_\lambda(\partial v) = (\partial + \lambda) a_\lambda v.$$

If $a_\lambda v \in M[\lambda]$ for all $a \in A$, $v \in M$, then the $A$-module $M$ is called
*conformal*. If $M$ is finitely generated over $\mathbb{C}[\partial]$, $M$ is simply called
*finite*.

**Definition 1.3** ([DK]). A *conformal linear map* from an $A$-module $M$
to an $A$-module $N$ is a $\mathbb{C}$-linear map $f \colon M \to N[\lambda]$, denoted $f_\lambda \colon M \to N$, such that $f_\lambda \partial = (\partial + \lambda) f_\lambda$. The space of such maps is denoted
$\mathrm{Chom}(M, N)$. It has canonical structures of a $\mathbb{C}[\partial]$- and an $A$-module:

$$(\partial f)_\lambda = -\lambda f_\lambda,$$

$$(a_\mu f)_\lambda m = a_\mu(f_{\lambda-\mu} m) - f_{\lambda-\mu}(a_\mu m),$$

where $a \in A$, $m \in M$, and $f \in \mathrm{Chom}(M, N)$. When the two modules
$M$ and $N$ are conformal and finite, the module $\mathrm{Chom}(M, N)$ will also
be conformal.

For a finite module $M$, let $\mathrm{Cend}\, M = \mathrm{Chom}(M, M)$ denote the space
of conformal linear endomorphisms of $M$. Besides the $A$-module structure, $\mathrm{Cend}\, M$ carries the natural structure

$$(f_\lambda g)_\mu m = f_\lambda(g_{\mu-\lambda} m), \qquad f, g \in \mathrm{Cend}\, M, m \in M,$$

of an associative conformal algebra in the sense of the following definition, see [K4].

**Definition 1.4.** An *associative conformal algebra* is a $\mathbb{C}[\partial]$-module
$A$ endowed with a $\lambda$-multiplication $a_\lambda b$ which defines a linear map
$A \otimes A \to A[\lambda]$ subject to the following axioms:

    **Conformal sesquilinearity:** $(\partial a)_\lambda b = -\lambda a_\lambda b, a_\lambda \partial b = (\partial + \lambda) a_\lambda b$;
    **Associativity:** $a_\lambda(b_\mu c) = (a_\lambda b)_{\lambda+\mu} c$.

The $\lambda$-bracket $[a_\lambda b] = a_\lambda b - b_{-\lambda-\partial} a$ makes an associative conformal
algebra, in particular, $\mathrm{Cend}\, M$, a Lie conformal algebra. $\mathrm{Cend}\, M$ with
this structure is denoted $\mathrm{gc}\, M$ and called the *general Lie conformal
algebra of a module* $M$ [DK, K4].

Given an associative conformal algebra $A$, a *left* (or *right*) *module*
$M$ over it may be defined naturally, for example, like in Definition 1.2.

A *bimodule* may be defined by adding the axiom $a_\lambda(m_\mu b) = (a_\lambda m)_{\lambda+\mu} b$ to the list of those for a left and right module. A (bi)module is called *conformal*, provided the action(s) satisfy the usual polynomiality conditions. The structure of a conformal bimodule on $M$ is equivalent to an extension of the associative conformal algebra structure to the space $A \oplus \epsilon M$, where $\epsilon^2 = 0$.

We will be working with Lie conformal algebras and modules over them throughout the paper, except when we discuss Hochschild cohomology in Section 9.1. We will therefore usually shorten the term "Lie conformal algebra" to "conformal algebra".

Conformal modules over conformal algebras appear naturally in the context of conformal modules over formal distribution Lie algebras as follows. Let $(\mathfrak{g}, \mathcal{F})$ be a formal distribution Lie algebra and let $V$ be a $\mathfrak{g}$-module. Suppose that $\mathcal{E}$ is a family of $V$-valued formal distributions which spans $V$ and such that any $a(z) \in \mathcal{F}$ and $v(z) \in \mathcal{E}$ form a local pair, *i.e.*,

$$(z-w)^N a(z)v(w) = 0 \quad \text{for } N \gg 0.$$

Then $(V, \mathcal{E})$ is called a *conformal* $(\mathfrak{g}, \mathcal{F})$-module. As before, we have:

$$(1.7) \qquad a(z)v(w) = \sum_{j=0}^{N-1} \big(a(w)_{(j)}v(w)\big)\partial_w^{(j)}\delta(z-w),$$

where

$$(1.8) \qquad a(w)_{(j)}v(w) = \operatorname{Res}_z(z-w)^j a(z)v(w).$$

Let $\overline{\mathcal{E}}$ denote the minimal subspace of $V[[z, z^{-1}]]$ containing $\mathcal{E}$ which is closed under all $j$-th actions (1.8) and is $\partial$-invariant. One knows that all pairs $a(z) \in \overline{\mathcal{F}}$ and $v(z) \in \overline{\mathcal{E}}$ are still local [K2, K4]. Letting

$$a_\lambda v = \sum_{n \in \mathbb{Z}_+} \lambda^{(n)} a_{(n)} v,$$

one endows $\overline{\mathcal{E}}$ with the structure of a conformal $\overline{\mathcal{F}}$-module [K2, K4].

Conversely, given a conformal $A$-module $M$, one associates to it the *maximal* conformal (Lie $A$, $A$)-module $(V(M), M)$ in a way similar to the one the Lie algebra Lie $A$ has been constructed. We let $V(M) = M[t, t^{-1}]/(\partial + \partial_t)M[t, t^{-1}]$, with the well-defined Lie $A$-action

$$(1.9) \qquad a_m v_n = \sum_{j \in \mathbb{Z}_+} \binom{m}{j} (a_{(j)}v)_{m+n-j},$$

where, as before, $v_n$ stands for the image of $vt^n$ in $V(M)$ [K2]. As before, we denote by $V(M)_-$ the $\mathbb{C}$-span of the $v_n$, where $v \in M$,

$n \in \mathbb{Z}_+$. It is clear from (1.9) that $V(M)_-$ is a $(\text{Lie } A)^-$- and a $(\text{Lie } A)_-$-submodule of $V(M)$.

The following obvious observation plays a key role in representation theory of conformal algebras [CK].

**Proposition 1.1.** *A module $M$ over a conformal algebra $A$ carries the natural structure of a module over the extended annihilation Lie algebra $(\text{Lie } A)^-$. This correspondence establishes an equivalence of the category of $A$-modules and that of $(\text{Lie } A)^-$-modules. The $A$-module $M$ is conformal, iff as a $(\text{Lie } A)^-$-module it satisfies the condition*

$$(1.10) \qquad a_n v = 0 \quad \text{for } a \in A, \quad v \in V, \quad n \gg 0.$$

*Remark 1.1.* As a $(\text{Lie } A)^-$-module, a conformal $A$-module $M$ is isomorphic to the module $V(M)/V(M)_-$.

*Remark 1.2.* [K2] One can show that the map $A \mapsto (\text{Lie } A, A)$ (respectively, $M \mapsto (V(M), M)$) establishes a bijection between isomorphism classes of conformal algebras (respectively, of conformal modules over conformal algebras) and equivalence classes of formal distribution Lie algebras $(\mathfrak{g}, \mathcal{F})$ (respectively, of conformal modules over $(\text{Lie } A, A)$). By definition, all formal distribution Lie algebras $((\text{Lie } A)/I, \mathcal{F})$, where $I$ is an ideal of $\text{Lie } A$ having trivial intersection with $A$, and $\overline{\mathcal{F}} = A$ are *equivalent* (and similarly for modules).

**Example 1.1.** Let $\mathfrak{g}$ be a Lie algebra and let $\tilde{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}]$ be the associated loop (= current) algebra (with the obvious bracket: $[at^m, bt^n] = [a, b]t^{m+n}, a, b \in \mathfrak{g}, m, n \in \mathbb{Z}$). For $a \in \mathfrak{g}$ let $a(z) = \sum_{m \in \mathbb{Z}}(at^m)z^{-m-1} \in \tilde{\mathfrak{g}}[[z, z^{-1}]]$. Then

$$[a(z), b(w)] = [a, b](w)\delta(z - w),$$

hence the family $\mathcal{F} = \{a(z) | a \in \mathfrak{g}\}$ consists of pairwise local formal distributions and $(\tilde{\mathfrak{g}}, \mathcal{F})$ is a formal distribution Lie algebra. Note that

$$\overline{\mathcal{F}} = \mathbb{C}[\partial]\mathcal{F} \simeq \mathbb{C}[\partial] \otimes \mathfrak{g}$$

is a conformal algebra with the $\lambda$-bracket

$$[a_\lambda b] = [a, b], \quad a, b \in \mathfrak{g}.$$

This conformal algebra is called the *current conformal algebra* associated to $\mathfrak{g}$ and is denoted by $\text{Cur } \mathfrak{g}$. Note that $\text{Lie}(\text{Cur } \mathfrak{g}, \mathcal{F}) \simeq \tilde{\mathfrak{g}}$, hence $\tilde{\mathfrak{g}}$ is the maximal formal distribution algebra. The corresponding annihilation algebra is $\tilde{\mathfrak{g}}_- = \mathfrak{g}[t]$ and the extended annihilation algebra is $\mathbb{C}\partial_t + \mathfrak{g}[t]$.

Given a $\mathfrak{g}$-module $U$, one may associate the conformal $\tilde{\mathfrak{g}}$-module $\tilde{U} = U[t, t^{-1}]$ with the obvious action of $\tilde{\mathfrak{g}}$, and the formal Cur $\mathfrak{g}$-module $M_U = \mathbb{C}[\partial] \otimes U$ defined by

$$a_\lambda u = au \, , \quad a \in \mathfrak{g} \, , u \in U \, .$$

We have: $V(M_U) \simeq \tilde{U}$ as $\tilde{\mathfrak{g}}$-modules.

It is known that, provided that $\mathfrak{g}$ is finite-dimensional semisimple, the Cur $\mathfrak{g}$-modules $M_U$, where $U$ is a finite-dimensional irreducible $\mathfrak{g}$-module, exhaust all finite irreducible non-trivial Cur $\mathfrak{g}$-modules [CK].

**Example 1.2.** Let $\mathcal{V}ect\,\mathbb{C}^\times$ denote the Lie algebra of all regular vector fields on $\mathbb{C}^\times$. The vector fields $t^n \partial_t$ ($n \in \mathbb{Z}$) form a basis of $\mathcal{V}ect\,\mathbb{C}^\times$ and the formal distribution $L(z) = -\sum_{n \in \mathbb{Z}} (t^n \partial_t) z^{-n-1}$ is local (with respect to itself), since

$$[L(z), L(w)] = \partial_w L(w)\delta(z-w) + 2L(w)\delta'_w(z-w) \, .$$

Hence $(\mathcal{V}ect\,\mathbb{C}^\times, \{L\})$ is a formal distribution Lie algebra. The associated conformal algebra

$$\mathrm{Vir} = \mathbb{C}[\partial]L \, , \quad [L_\lambda L] = (\partial + 2\lambda)L$$

is called the *Virasoro conformal algebra*.

Note that $\mathrm{Lie}\,(\mathrm{Vir}, \{L\}) \simeq \mathcal{V}ect\,\mathbb{C}^\times$, hence $\mathcal{V}ect\,\mathbb{C}^\times$ is the maximal formal distribution algebra. The corresponding annihilation algebra $(\mathcal{V}ect\,\mathbb{C}^\times)_- = \mathcal{V}ect\,\mathbb{C}$, the Lie algebra of regular vector fields on $\mathbb{C}$, and $(\mathcal{V}ect\,\mathbb{C}^\times)^-$ is isomorphic to the direct sum of $(\mathcal{V}ect\,\mathbb{C}^\times)_-$ and the 1-dimensional Lie algebra.

It is known that all free non-trivial Vir-modules of rank 1 over $\mathbb{C}[\partial]$ are the following ones ($\Delta, \alpha \in \mathbb{C}$):

$$M_{\Delta,\alpha} = \mathbb{C}[\partial]v \, , \quad L_\lambda v = (\partial + \alpha + \Delta\lambda)v \, .$$

We have: $V(M_{\Delta,\alpha}) \simeq \mathbb{C}[t, t^{-1}]e^{-\alpha t}(dt)^{1-\Delta}$ as $\mathcal{V}ect\,\mathbb{C}$-modules. The module $M_{\Delta,\alpha}$ is irreducible, iff $\Delta \neq 0$. The module $M_{0,\alpha}$ contains a unique non-trivial submodule $(\partial + \alpha)M_{0,\alpha}$ isomorphic to $M_{1,\alpha}$. It is known that the modules $M_{\Delta,\alpha}$ with $\Delta \neq 0$ exhaust all finite irreducible non-trivial Vir-modules [CK].

It is known [DK] that the conformal algebras Cur $\mathfrak{g}$, where $\mathfrak{g}$ is a finite-dimensional simple Lie algebra, and Vir exhaust all finite simple conformal algebras. For that reason we shall discuss mainly these two examples in what follows.

## 2. BASIC DEFINITIONS

**Definition 2.1.** An *n-cochain* ($n \in \mathbb{Z}_+$) of a conformal algebra $A$ with coefficients in a module $M$ over it is a $\mathbb{C}$-linear map

$$\gamma : A^{\otimes n} \to M[\lambda_1, \ldots, \lambda_n]$$
$$a_1 \otimes \cdots \otimes a_n \mapsto \gamma_{\lambda_1, \ldots, \lambda_n}(a_1, \ldots, a_n),$$

where $M[\lambda_1, \ldots, \lambda_n]$ denotes the space of polynomials with coefficients in $M$, satisfying the following conditions:

**Conformal antilinearity:** $\gamma_{\lambda_1, \ldots, \lambda_n}(a_1, \ldots, \partial a_i, \ldots, a_n)$
= $-\lambda_i \gamma_{\lambda_1, \ldots, \lambda_n}(a_1, \ldots, a_i, \ldots, a_n)$ for all $i$;

**Skew-symmetry:** $\gamma$ is skew-symmetric with respect to simultaneous permutations of $a_i$'s and $\lambda_i$'s.

We let $A^{\otimes 0} = \mathbb{C}$, as usually, so that a 0-cochain $\gamma$ is an element of $M$. Sometimes, when the module $M$ is not conformal, one may consider formal power series instead of polynomials in this definition.

We define a differential $d$ of a cochain $\gamma$ as follows:

$$(d\gamma)_{\lambda_1, \ldots, \lambda_{n+1}}(a_1, \ldots, a_{n+1})$$
$$= \sum_{i=1}^{n+1} (-1)^{i+1} a_{i\lambda_i} \gamma_{\lambda_1, \ldots, \widehat{\lambda_i}, \ldots, \lambda_{n+1}}(a_1, \ldots, \widehat{a_i}, \ldots, a_{n+1})$$
$$+ \sum_{\substack{i,j=1 \\ i<j}}^{n+1} (-1)^{i+j} \gamma_{\lambda_i + \lambda_j, \lambda_1, \ldots, \widehat{\lambda_i}, \ldots, \widehat{\lambda_j}, \ldots, \lambda_{n+1}}([a_{i\lambda_i} a_j], a_1, \ldots, \widehat{a_i}, \ldots, \widehat{a_j},$$
$$\ldots, a_{n+1}),$$

where $\gamma$ is extended linearly over the polynomials in $\lambda_i$. In particular, if $\gamma \in M$ is a 0-cochain, then $(d\gamma)_\lambda(a) = a_\lambda \gamma$.

*Remark* 2.1. Conformal antilinearity implies the following relation for an *n*-cochain $\gamma$:

$$\gamma_{\lambda+\mu, \lambda_1, \ldots}([a_\lambda b], a_1, \ldots) = \gamma_{\lambda+\mu, \lambda_1, \ldots}([a_{-\partial - \mu} b], a_1, \ldots).$$

**Lemma 2.1.** 1. *The operator $d$ preserves the space of cochains;*
2. $d^2 = 0$.

*Proof.* 1. The only non-trivial point in checking the skew-symmetry of $d\gamma$ amounts to the equation

$$\gamma_{\lambda+\mu, \lambda_1, \ldots, \lambda_{n-1}}([a_\lambda b], a_1, \ldots, a_{n-1}) = -\gamma_{\lambda+\mu, \lambda_1, \ldots, \lambda_{n-1}}([b_\mu a], a_1, \ldots, a_{n-1}),$$

which follows from Remark 2.1 and the skew-symmetry of $[a_\lambda b]$.

2. To check that $d^2 = 0$, we will compute $d^2\gamma$ for an $n$-cochain $\gamma$.

$$(d^2\gamma)_{\lambda_1,\ldots,\lambda_{n+2}}(a_1,\ldots,a_{n+2})$$

$$= \sum_{i=1}^{n+2} (-1)^{i+1} a_{i\lambda_i}(d\gamma)_{\lambda_1,\ldots,\widehat{\lambda_i},\ldots,\lambda_{n+2}}(a_1,\ldots,\widehat{a_i},\ldots,a_{n+2})$$

$$+ \sum_{\substack{i,j=1 \\ i<j}}^{n+2} (-1)^{i+j}(d\gamma)_{\lambda_i+\lambda_j,\lambda_1,\ldots,\widehat{\lambda}_{i,j},\ldots,\lambda_{n+2}}([a_{i\lambda_i}a_j],a_1,\ldots,\widehat{a}_{i,j},\ldots,a_{n+2})$$

$$= \sum_{\substack{i,j=1 \\ i\neq j}}^{n+2} (-1)^{i+j}\,\mathrm{sign}\{j,i\}a_{i\lambda_i}\big(a_{j\lambda_j}\gamma_{\lambda_1,\ldots,\widehat{\lambda}_{i,j},\ldots,\lambda_{n+2}}(a_1,\ldots,\widehat{a}_{i,j},\ldots,a_{n+2})\big)$$

$$+ \sum_{\substack{i,j,k=1 \\ j<k,i\neq j,k}}^{n+2} (-1)^{i+j+k+1}\,\mathrm{sign}\{j,k,i\}a_{i\lambda_i}\gamma_{\lambda_j+\lambda_k,\lambda_1,\ldots,\widehat{\lambda}_{i,j,k},\ldots,\lambda_{n+2}}([a_{j\lambda_j}a_k],$$
$$a_1,\ldots,\widehat{a}_{i,j,k},\ldots,a_{n+2})$$

$$+ \sum_{\substack{i,j,k=1 \\ i<j,k\neq i,j}}^{n+2} (-1)^{i+j+k}\,\mathrm{sign}\{k,i,j\}a_{k\lambda_k}\gamma_{\lambda_i+\lambda_j,\lambda_1,\ldots,\widehat{\lambda}_{i,j,k},\ldots,\lambda_{n+2}}([a_{i\lambda_i}a_j],$$
$$a_1,\ldots,\widehat{a}_{i,j,k},\ldots,a_{n+2})$$

$$+ \sum_{\substack{i,j=1 \\ i<j}}^{n+2} (-1)^{i+j}[a_{i\lambda_i}a_j]_{\lambda_i+\lambda_j}\gamma_{\lambda_1,\ldots,\widehat{\lambda}_{i,j},\ldots,\lambda_{n+2}}(a_1,\ldots,\widehat{a}_{i,j},\ldots,a_{n+2})$$

$$+ \sum_{\substack{\text{distinct } i,j,k,l=1 \\ i<j,k<l}}^{n+2} (-1)^{i+j+k+l}\,\mathrm{sign}\{i,j,k,l\}$$
$$\times \gamma_{\lambda_k+\lambda_l,\lambda_i+\lambda_j,\lambda_1,\ldots,\widehat{\lambda}_{i,j,k,l},\ldots,\lambda_{n+2}}([a_{k\lambda_k}a_l],[a_{i\lambda_i}a_j],a_1,\ldots,\widehat{a}_{i,j,k,l},\ldots,a_{n+2})$$

$$+ \sum_{\substack{i,j,k=1 \\ i<j,k\neq i,j}}^{n+2} (-1)^{i+j+k+1}\,\mathrm{sign}\{i,j,k\}$$
$$\times \gamma_{\lambda_i+\lambda_j+\lambda_k,\lambda_1,\ldots,\widehat{\lambda}_{i,j,k},\ldots,\lambda_{n+2}}([[a_{i\lambda_i}a_j]_{\lambda_i+\lambda_j}a_k],a_1,\ldots,\widehat{a}_{i,j,k},\ldots,a_{n+2}),$$

where $\mathrm{sign}\{i_1,\ldots,i_p\}$ is the sign of the permutation putting the indices in the increasing order and $\widehat{a}_{i,j,\ldots}$ means that $a_i,a_j,\ldots$ are omitted. Notice that each term in the summation over $i,j,k,l$ is skew with respect

to the permutation $\begin{pmatrix} i & j & k & l \\ k & l & i & j \end{pmatrix}$. Therefore, the terms of that summation will cancel pairwise. The first and the forth summations cancel each other, because $M$ is a conformal algebra module:

$$-a_{i\lambda_i}(a_{j\lambda_j}m) + a_{j\lambda_j}(a_{i\lambda_i}m) + [a_{i\lambda_i}a_j]_{\lambda_i+\lambda_j}m = 0.$$

The second summation becomes equal to the third one after the substitution $(ikj)$, except that they differ by a sign. Thus, they cancel each other, as well. Finally, the sixth summation can be rewritten as a summation over $i < j < k$ of the sum of three permutations of the initial summand. Precisely, in the first entry of $\gamma$, we will have

$$(2.1) \qquad [[a_{i\lambda_i}a_j]_{\lambda_i+\lambda_j}a_k] - [[a_{i\lambda_i}a_k]_{\lambda_i+\lambda_k}a_j] + [[a_{j\lambda_j}a_k]_{\lambda_j+\lambda_k}a_i].$$

Using Remark 2.1, we can transform the sum (2.1) inside $\gamma$ into

$$[[a_{i\lambda_i}a_j]_{\lambda_i+\lambda_j}a_k] - [[a_{i\lambda_i}a_k]_{-\partial-\lambda_j}a_j] + [[a_{j\lambda_j}a_k]_{-\partial-\lambda_i}a_i],$$

which vanishes by the Jacobi identity and skew-symmetry in $A$. Thus, we see that all of the terms in $d^2\gamma$ cancel. $\qquad\blacksquare$

Thus the cochains of a conformal algebra $A$ with coefficients in a module $M$ form a complex, which will be denoted

$$\widetilde{C}^\bullet = \widetilde{C}^\bullet(A, M) = \bigoplus_{n\in\mathbb{Z}_+} \widetilde{C}^n(A, M).$$

This complex is called the *basic complex* for the $A$-module $M$. This is not yet the complex defining the right cohomology of a conformal algebra: we need to consider a certain quotient complex.

Define the structure of a (left) $\mathbb{C}[\partial]$-module on $\widetilde{C}^\bullet(A, M)$ by letting

$$(2.2) \qquad (\partial \cdot \gamma)_{\lambda_1,\ldots,\lambda_n}(a_1,\ldots,a_n) = \left(\partial_M + \sum_{i=1}^n \lambda_i\right)\gamma_{\lambda_1,\ldots,\lambda_n}(a_1,\ldots,a_n),$$

where $\partial_M$ denotes the action of $\partial$ on $M$.

**Lemma 2.2.** $d\partial = \partial d$, and therefore the graded subspace $\partial\widetilde{C}^\bullet \subset \widetilde{C}^\bullet$ forms a subcomplex.

*Proof.* The first summation in the differential transforms the factor $\partial_M + \sum_{i=1}^n \lambda_i$ into $\partial_M + \sum_{i=1}^{n+1} \lambda_i$, because of the conformal sesquilinearity of the $\lambda$-bracket. The second summation does the same for more obvious reasons. $\qquad\blacksquare$

Define the quotient complex

$$C^\bullet(A, M) = \widetilde{C}^\bullet(A, M)/\partial\widetilde{C}^\bullet(A, M) = \bigoplus_{n\in\mathbb{Z}_+} C^n(A, M),$$

called the *reduced complex*.

**Definition 2.2.** *The basic cohomology* $\widetilde{\mathrm{H}}^\bullet(A, M)$ *of a conformal algebra $A$ with coefficients in a module $M$ is the cohomology of the basic complex $\widetilde{C}^\bullet$. The (reduced) cohomology $\mathrm{H}^\bullet(A, M)$ is the cohomology of the reduced complex $C^\bullet = C^\bullet(A, M) = \widetilde{C}^\bullet / \partial \widetilde{C}^\bullet$.*

*Remark 2.2.* The basic cohomology $\widetilde{\mathrm{H}}^\bullet(A, M)$ is naturally a $\mathbb{C}[\partial]$-module, whereas the reduced cohomology $\mathrm{H}^\bullet(A, M)$ is a complex vector space.

*Remark 2.3.* The exact sequence $0 \to \partial \widetilde{C}^\bullet \to \widetilde{C}^\bullet \to C^\bullet \to 0$ gives the long exact sequence of cohomology:

$$(2.3) \qquad 0 \to \mathrm{H}^0(\partial \widetilde{C}^\bullet) \to \widetilde{\mathrm{H}}^0(A, M) \to \mathrm{H}^0(A, M) \to$$
$$\to \mathrm{H}^1(\partial \widetilde{C}^\bullet) \to \widetilde{\mathrm{H}}^1(A, M) \to \mathrm{H}^1(A, M) \to$$
$$\to \mathrm{H}^2(\partial \widetilde{C}^\bullet) \to \widetilde{\mathrm{H}}^2(A, M) \to \mathrm{H}^2(A, M) \to \cdots$$

**Proposition 2.1.** *In degrees $\geq 1$, the complexes $\widetilde{C}^\bullet$ and $\partial \widetilde{C}^\bullet$ are isomorphic under the map*

$$(2.4) \qquad\qquad \widetilde{C}^\bullet \to \partial \widetilde{C}^\bullet, \qquad \gamma \mapsto \partial \cdot \gamma.$$

*Therefore, $\mathrm{H}^q(\partial \widetilde{C}^\bullet) \simeq \widetilde{\mathrm{H}}^q(A, M)$ for $q \geq 1$, and the natural sequence $0 \to \mathrm{Ker} \, \partial[0] \to \widetilde{\mathrm{H}}^0(A, M) \to \mathrm{H}^0(\partial \widetilde{C}^\bullet) \to 0$, where $\mathrm{Ker} \, \partial[0]$ is the subcomplex $\mathrm{Ker} \, \partial$ of $\widetilde{C}^\bullet$, in fact concentrated in degree zero, is exact. When the module $M$ is $\mathbb{C}[\partial]$-free, the above isomorphisms take place in all degrees $\geq 0$.*

*Proof.* Indeed, the modules $\widetilde{C}^n(A, M)$, $n \geq 1$, are free over $\mathbb{C}[\partial]$, because they are free over $\mathbb{C}[\lambda_i]$. Lemma 2.2 shows that the map (2.4) is a morphism of complexes. When $M$ is $\mathbb{C}[\partial]$-free, this argument extends over to $n = 0$. $\qquad \blacksquare$

*Remark 2.4.* This proposition does not imply that in the long exact sequence (2.3), the maps $\mathrm{H}^q(\partial \widetilde{C}^\bullet) \to \widetilde{\mathrm{H}}^q(A, M)$ induced by the embedding $\partial \widetilde{C}^\bullet \subset \widetilde{C}^\bullet$ are isomorphisms.

## 3. EXTENSIONS AND DEFORMATIONS

Our cohomology theory describes extensions and deformations, just as any cohomology theory.

**Theorem 3.1.**   1. $\widetilde{\mathrm{H}}^0(A, M) = M^A = \{m \in M \mid a_\lambda m = 0 \ \forall a \in A\}$.
   2. *The isomorphism classes of extensions*

$$0 \to M \to E \to \mathbb{C} \to 0$$

   *of the trivial $A$-module $\mathbb{C}$ ($\partial$ and $A$ act by zero) by a conformal $A$-module $M$ correspond bijectively to $\mathrm{H}^0(A, M)$.*
   3. *The isomorphism classes of $\mathbb{C}[\partial]$-split extensions*

$$0 \to M \to E \to N \to 0$$

   *of conformal modules over a conformal algebra $A$ correspond bijectively to $\mathrm{H}^1(A, \mathrm{Chom}(N, M))$, where $M$ and $N$ are assumed to be finite and $\mathrm{Chom}(N, M)$ is the $A$-module of conformal linear maps from $N$ to $M$. If, in particular, $N = \mathbb{C}$ is the trivial module, then there exist no non-trivial $\mathbb{C}[\partial]$-split extensions.*
   4. *Let $C$ be a conformal $A$-module, considered as a conformal algebra with respect to the zero $\lambda$-bracket. Then the equivalence classes of $\mathbb{C}[\partial]$-split "abelian" extensions*

$$0 \to C \to \widetilde{A} \to A \to 0$$

   *of the conformal algebra $A$ correspond bijectively to $\mathrm{H}^2(A, C)$.*
   5. *The equivalence classes of first-order deformations of a conformal algebra $A$ (leaving the $\mathbb{C}[\partial]$-action intact) correspond bijectively to $\mathrm{H}^2(A, A)$.*

*Proof.* 1. The computation of $\widetilde{\mathrm{H}}^0(A, M)$ follows directly from the definition: for $m \in M = \widetilde{C}^0(A, M)$ and $a \in A$, $(dm)_\lambda(a) = a_\lambda m$.

2. Given an extension

$$0 \to M \to E \to \mathbb{C} \to 0$$

of modules over a conformal algebra $A$, pick a splitting of this short exact sequence over $\mathbb{C}$, *i.e.*, assume that as a complex vector space, $E \simeq M \oplus \mathbb{C} = \{(m, n) \mid m \in M, n \in \mathbb{C}\}$. Define $f \in M$ by writing down the action of $\partial$ on the pair $(m, 1) \in E$:

$$(3.1) \qquad \partial(m, 1) = (\partial m + f, 0).$$

We claim that $f \in M = \widetilde{C}^0(A, M)$ defines a zero-cocycle in the reduced complex $C^\bullet(A, M)$ and thereby a class in $\mathrm{H}^0(A, M)$.

To see that, define a one-cochain $\gamma \in \widetilde{C}^1(A, M)$ using the action of $A$ on $E$:

$$(3.2) \qquad a_\lambda(m, 1) = (a_\lambda m + \gamma_\lambda(a), 0).$$

for $a \in A$. The conformal antilinearity of $\gamma$: $\gamma_\lambda(\partial a) = -\lambda\gamma_\lambda(a)$, follows from the fact that $(\partial a)_\lambda(m, 1) = -\lambda(a_\lambda(m, 1))$. The property $a_\lambda(\partial(m, 1)) = (\lambda + \partial)(a_\lambda(m, 1))$ of the action of $A$ on $E$ expands as

$$(3.3) \qquad (df)_\lambda = (\partial\gamma)_\lambda,$$

which means that $df = 0$ in the reduced complex.

If we choose another splitting $(m, n)'$ of the extension $E$, it will differ by an element $g \in M$:

$$(m, 1)' = (m + g, 1),$$

so that the new zero-cocycle becomes $f' = f + \partial g$, therefore defining the same cochain in the reduced complex.

If we have two isomorphic extensions and choose a compatible splitting over $\mathbb{C}$, we will get exactly the same zero-cocycles $f$ corresponding to them. This proves that isomorphism classes of extensions give rise to elements of $H^0(A, M)$.

Conversely, given a cocycle in $C^0(A, M)$, we can choose a representative $f \in M$ of it to alter the natural $\mathbb{C}[\partial]$-module structure on $M \oplus \mathbb{C}$ by adding $f$ to the action of $\partial$ on $M \oplus \mathbb{C}$ as in (3.1). This will obviously extend to an action of the free commutative algebra $\mathbb{C}[\partial]$. We can also alter the natural $A$-module structure by adding $\gamma$ to the action of $a \in A$ as in (3.2), where $\gamma$ is a solution of Equation (3.3), which means that $f$ is a cocycle in the reduced complex. This action will be conformally linear in $(m, n)$, because of (3.3), and antilinear in $A$, because of the conformal antilinearity of $\gamma$. This action will define an $A$-module structure on $M \oplus \mathbb{C}$, because $d\gamma = 0$, which follows from (3.3) and the fact that $\mathbb{C}[\partial]$ acts freely on basic two-cochains.

By construction the natural mappings $M \to M \oplus \mathbb{C}$ and $M \oplus \mathbb{C} \to \mathbb{C}$ will be morphisms of $\mathbb{C}[\partial]$- and $A$-modules.

This construction of a new conformal module structure on $M \oplus \mathbb{C}$ involved a number of choices. The choice of a different representative $f' = f + \partial g$ defines an isomorphism of the two $\mathbb{C}[\partial]$-module structures on $M \oplus \mathbb{C}$, which automatically becomes an isomorphism of the corresponding $A$-module structures, because the corresponding $\gamma$'s are unique. The one-cochain $\gamma$ is uniquely determined by $f$, because $\mathbb{C}[\partial]$ acts freely on the space $\widetilde{C}^1(A, M)$ of basic one-cochains.

3. We will adjust the proof of Part 2 to the new situation. Given a $\mathbb{C}[\partial]$-split extension

$$0 \to M \to E \to N \to 0$$

of modules over a conformal algebra $A$, pick a splitting of the short exact sequence over $\mathbb{C}[\partial]$, *i.e.*, assume that as a $\mathbb{C}[\partial]$-module, $E \simeq$

$M \oplus N = \{(m, n) \mid m \in M, n \in N\}$. We are going to construct a reduced one-cochain with coefficients in $\mathrm{Chom}(N, M)$ out of this data. Note that such cochains are linear maps $\gamma = \gamma_\lambda(a)_\mu$ from $A \otimes N$ to $M$ depending on two variables $\lambda$ and $\mu$, considered modulo $\lambda - \mu$. Note that $\gamma_\lambda(a)_\mu \mod (\lambda - \mu)$ is fully determined by the restriction $\gamma_\lambda(a)_\lambda$ to the diagonal $\lambda = \mu$. Define a one-cochain $\gamma \in C^1(A, \mathrm{Chom}(N, M))$ using the action of $A$ on $E$:

$$(3.4) \qquad a_\lambda(m, n) = (a_\lambda m + \gamma_\lambda(a)_\lambda n, a_\lambda n)$$

for $a \in A$. The conformal antilinearity of $\gamma$: $\gamma_\lambda(\partial a)_\lambda = -\lambda \gamma_\lambda(a)_\lambda$, follows from the fact that $(\partial a)_\lambda(m, n) = -\lambda(a_\lambda(m, n))$. The property $a_\lambda(\partial(m, n)) = (\lambda + \partial)(a_\lambda(m, n))$ of the action of $A$ on $E$ expands as

$$(3.5) \qquad (\partial \gamma)_\lambda = 0,$$

which means that $\gamma_\lambda(a)_\lambda$ is a conformal linear map $N \to M$. Finally, the module property (1.5) for elements in $E$ implies that $d\gamma = 0$.

If we choose another $\mathbb{C}[\partial]$-splitting $(m, n)'$ of the extension $E$, it will differ by an element $\beta \in \mathrm{Hom}_{\mathbb{C}[\partial]}(N, M)$:

$$(m, n)' = (m + \beta(n), n).$$

$\mathrm{Hom}_{\mathbb{C}[\partial]}(N, M)$ may be identified with the degree-zero part of $\mathrm{Chom}(N, M)$, so that the new one-cocycle becomes $\gamma' = \gamma + d\beta$, therefore defining the same cohomology class.

If we have two isomorphic extensions and choose a compatible splitting over $\mathbb{C}[\partial]$, we will have exactly the same one-cocycles $\gamma$ corresponding to them. This proves that isomorphism classes of extensions give rise to elements of $\mathrm{H}^1(A, \mathrm{Chom}(N, M))$.

Conversely, given a cohomology class in $\mathrm{H}^1(A, \mathrm{Chom}(N, M))$, we can choose a representative $\gamma \in C^1(A, \mathrm{Chom}(N, M))$ of it to alter the natural $A$-module structure on $M \oplus N$ by adding $\gamma$ to the action of $A$ on $M \oplus N$ as in (3.4). This action will be conformally linear in $(m, n)$, because of (3.5), and antilinear in $A$, because of the conformal antilinearity of $\gamma$. This action will define an $A$-module structure on $M \oplus N$, because $d\gamma = 0$ after the restriction to $\mu = \lambda_1 + \lambda_2$ in $\widetilde{C}^2(A, \mathrm{Chom}(N, M))$.

By construction the natural mappings $M \to M \oplus N$ and $M \oplus N \to N$ will be morphisms of $\mathbb{C}[\partial]$- and $A$-modules.

This construction of a new conformal module structure on $M \oplus N$ is independent on the choice of a different representative $\gamma' = \gamma + d\beta$, because it defines an isomorphic structure of an $A$-module on $M \oplus N$.

Finally, if $N = \mathbb{C}$, then $\mathrm{Chom}(\mathbb{C}, M) = 0$, and therefore, there are no split extensions.

4. Given a $\mathbb{C}[\partial]$-split extension of a conformal algebra $A$ by a module $C$, choose a splitting $\widetilde{A} = C \oplus A$ thereof. Then the bracket in $\widetilde{A}$

$$[(0, a)_\lambda (0, b)] = (c_\lambda(a, b), a_\lambda b) \qquad \text{for } a, b \in A$$

defines a sesquilinear map $c\colon A \otimes A \to C[\lambda]$, which we may combine with the natural mapping

$$C[\lambda] \to C[\lambda_1, \lambda_2]/(\partial + \lambda_1 + \lambda_2),$$
$$p(\lambda) \mapsto p(\lambda_1),$$

to get the composite mapping, denoted $c_{\lambda_1, \lambda_2}$. It defines a two-cochain, because it is obviously skew and $(c_\lambda(\partial a, b), -\lambda_a b) = [(0, \partial a)_\lambda(0, b)] = [\partial(0, a)_\lambda(0, b)] = -\lambda[(0, a)_\lambda(0, b)] = (-\lambda c_\lambda(a, b), -\lambda a_\lambda b)$, which implies $c_{\lambda_1, \lambda_2}(\partial a, b) = -\lambda_1 c_{\lambda_1, \lambda_2}(a, b)$, and similarly, $c_{\lambda_1, \lambda_2}(a, \partial b) = -\lambda_2 c_{\lambda_1, \lambda_2}(a, b) \mod (\partial + \lambda_1 + \lambda_2)$. In fact, this two-cochain $c$ is a cocycle:

$$\begin{aligned}
dc = {}& a_{\lambda_1} c_{\lambda_2, \lambda_3}(b, c) - b_{\lambda_2} c_{\lambda_1, \lambda_3}(a, c) + c_{\lambda_3} c_{\lambda_1, \lambda_2}(a, b) \\
& - c_{\lambda_1 + \lambda_2, \lambda_3}(a_{\lambda_1} b, c) + c_{\lambda_1 + \lambda_3, \lambda_2}(a_{\lambda_1} c, b) - c_{\lambda_2 + \lambda_3, \lambda_1}(b_{\lambda_2} c, a) = 0.
\end{aligned}$$

This is just because the corresponding three-term relation, the Jacobi relation, is satisfied in $\widetilde{A}$.

The construction of $c$ assumed the choice of a splitting $\widetilde{A} = C \oplus A$. A different splitting would differ by a mapping $f\colon A \to C$, which can be thought of as $f\colon A \to C[\lambda]/(\partial + \lambda)$, which would contribute by $df$ to $c$.

Thus, any extension determines a cohomology class in $\mathrm{H}^2(A, C)$. The above arguments can be traced back to show that a class in the cohomology group defines an extension.

5. Let $D = \mathbb{C}[\epsilon]/(\epsilon^2)$ be the algebra of *dual numbers*. Then a *first-order deformation* of a conformal algebra $A$ is the structure of a conformal algebra over $D$ on $A \otimes D$, so that the map $A \otimes D \to A$, $a \otimes p(\epsilon) \mapsto p(0) \cdot a$, is a morphism of conformal algebras and the action of $\partial$ on $A \otimes D$ is induced from that on the first factor. This means classes of first-order deformations are in bijection with classes of $\mathbb{C}[\partial]$-split abelian extensions of $A$ with the $A$-module $A$ in the sense of Part 2 of this theorem. Therefore, they are classified by $\mathrm{H}^2(A, A)$.  $\square$

## 4. Homology

Dualizing the cohomology theory we have defined above, the space $\widetilde{C}_n(A, M)$ of *n-chains of a conformal algebra $A$ with coefficients in a conformal module $M$ over it* is defined as the quotient of

$$A^{\otimes n} \otimes \operatorname{Hom}(\mathbb{C}[\lambda_1, \dots, \lambda_n], M),$$

where $\mathrm{Hom}(\mathbb{C}[\lambda_1, \ldots, \lambda_n], M)$ is the space of $\mathbb{C}$-linear maps from the space of polynomials to the module $M$, by the following relations:

1. $a_1 \otimes \cdots \otimes \partial a_i \otimes \cdots \otimes a_n \otimes \phi = -a_1 \otimes \cdots \otimes a_i \otimes \cdots \otimes a_n \otimes T_i\phi$, where $(T_i\phi)(f) = \phi(\lambda_i f)$;
2. $a_1 \otimes \cdots \otimes a_i \otimes \cdots \otimes a_j \otimes \cdots \otimes a_n \otimes \phi = -a_1 \otimes \cdots \otimes a_j \otimes \cdots \otimes a_i \otimes \cdots \otimes a_n \otimes \tau^*_{ij}\phi$, where $(\tau^*_{ij}\phi)(f(\lambda_1, \ldots, \lambda_i, \ldots, \lambda_j, \ldots, \lambda_n)) = \phi(f(\lambda_1, \ldots, \lambda_j, \ldots, \lambda_i, \ldots, \lambda_n))$.

One can also define a differential which takes $n$-chains to $(n-1)$-chains as follows:

$$
\begin{aligned}
&\delta(a_1 \otimes \cdots \otimes a_n \otimes \phi) \\
&= \sum_{i=1}^{n}(-1)^{i+1}p_i(a_1 \otimes \cdots \otimes \widehat{a}_i \otimes \cdots \otimes a_n \otimes a_{i\lambda_i}\phi) \\
&+ \sum_{\substack{i,j=1 \\ i<j}}^{n}(-1)^{i+j}p_{ij}([a_{i\lambda_i}a_j] \otimes a_1 \otimes \cdots \otimes \widehat{a}_i \otimes \cdots \otimes \widehat{a}_j \otimes \cdots \otimes a_n \otimes \phi),
\end{aligned}
$$

where $p_i$ is the natural pairing map $\mathbb{C}[\lambda_i] \otimes \mathrm{Hom}(\mathbb{C}[\lambda_1, \ldots, \lambda_n], M) \to \mathrm{Hom}(\mathbb{C}[\lambda_1, \ldots, \widehat{\lambda}_i, \ldots, \lambda_n], M)$ and $p_{ij}$ is the pairing $\mathbb{C}[\lambda_i] \otimes \mathrm{Hom}(\mathbb{C}[\lambda_1, \ldots, \lambda_n], M) \to \mathrm{Hom}(\mathbb{C}[\lambda_i + \lambda_j, \lambda_1, \ldots, \widehat{\lambda}_i, \ldots, \widehat{\lambda}_j, \ldots, \lambda_n], M)$. Similar computations to those in the cochain case show that the operator $\delta$ is well-defined and $\delta^2 = 0$.

One can define *basic homology* $\widetilde{\mathrm{H}}_{\bullet}(A, M)$ as the homology of the chain complex and *reduced homology* as the homology of the subcomplex $C_{\bullet}(A, M)$ of $\partial$-invariant chains, where $\partial$ acts as

$$
\partial(a_1 \otimes \cdots \otimes a_n \otimes \phi) = a_1 \otimes \cdots \otimes a_n \otimes \Big(\partial\phi - \sum_{i=1}^{n} T_i\phi\Big),
$$

where $(\partial\phi)(f) = \partial(\phi(f))$, $f \in \mathbb{C}[\lambda_1, \ldots \lambda_n]$. There are obviously natural pairings $\widetilde{\mathrm{H}}_q(A, M^*) \otimes \widetilde{\mathrm{H}}^q(A, M) \to \mathbb{C}$ and $\mathrm{H}_q(A, M^*) \otimes \mathrm{H}^q(A, M) \to \mathbb{C}$ for $q \geq 0$, where $M^* = \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})$ is the linear dual space with a natural structure of an $A$-module:

$$
\begin{aligned}
(\partial f)(m) &= -f(\partial m), \\
(a_\lambda f)(m) &= -f(a_\lambda m)
\end{aligned}
$$

for $f \in M^*$, $m \in M$, and $a \in A$. One expects these pairings to be perfect, when, for instance, either of the (co)homology spaces is finite-dimensional.

## 5. Exterior multiplication, contraction, and module structure

For any $u \in \widetilde{C}^m(A, \mathbb{C})$, where $\mathbb{C}$ is the one-dimensional space with the zero action of $A$, let $\epsilon(u)$ be the operator of *exterior multiplication* on $\widetilde{C}^\bullet(A, M)$:

$$(\epsilon(u)\gamma)_{\lambda_1, \dots, \lambda_{m+n}}(a_1, \dots, a_{m+n})$$
$$= \sum_{\pi \in S_{m+n}} \frac{\operatorname{sign} \pi}{m! \, n!} \, u_{\lambda_{\pi(1)}, \dots, \lambda_{\pi(m)}}\big(a_{\pi(1)}, \dots, a_{\pi(m)}\big)$$
$$\times \gamma_{\lambda_{\pi(m+1)}, \dots, \lambda_{\pi(m+n)}}\big(a_{\pi(m+1)}, \dots, a_{\pi(m+n)}\big).$$

Define also a wedge product $u \wedge v = \epsilon(u)\gamma$ on $\widetilde{C}^\bullet(A, \mathbb{C})$. It is clear that $\epsilon(u \wedge v) = \epsilon(u)\epsilon(v)$ for any $u, v \in V$, therefore, we have a graded commutative associative algebra structure on $\widetilde{C}^\bullet(A, \mathbb{C})$, along with a $\widetilde{C}^\bullet(A, \mathbb{C})$-module structure on $\widetilde{C}^\bullet(A, M)$.

Similarly, for any chain $v = a_1 \otimes \cdots \otimes a_n \otimes \phi \in \widetilde{C}_n(A, \mathbb{C})$, let $\iota(v)$ be the following *contraction* operator $\widetilde{C}^m(A, M) \to \widetilde{C}^{m-n}(A, M)$, for $m \geq n$:

$$(\iota(v)\gamma)_{\lambda_{n+1}, \dots, \lambda_m}(a_{n+1}, \dots, a_m) = p\big(\phi \otimes \gamma_{\lambda_1, \dots, \lambda_m}(a_1, \dots, a_m)\big),$$

where $p$ is the natural pairing $\mathbb{C}[\lambda_1, \dots, \lambda_n]^* \otimes \mathbb{C}[\lambda_1, \dots, \lambda_m] \to \mathbb{C}[\lambda_{n+1}, \dots, \lambda_m]$. Note that for any $u \in \widetilde{C}^1(A, \mathbb{C})$ and $v \in \widetilde{C}_1(A, \mathbb{C})$,

$$\epsilon(u)\iota(v) + \iota(v)\epsilon(u) = \iota(v)u.$$

Furthermore for any $a \in A$, define the following structure of a *module over the conformal algebra $A$* on $\widetilde{C}^\bullet(A, M)$:

$$(\theta_\lambda(a)\gamma)_{\lambda_1, \dots, \lambda_n}(a_1, \dots, a_n)$$
$$= a_\lambda \gamma_{\lambda_1, \dots, \lambda_n}(a_1, \dots, a_n) - \sum_{i=1}^n \gamma_{\lambda_1, \dots, \lambda+\lambda_i, \dots, \lambda_n}(a_1, \dots, [a_\lambda a_i], \dots, a_n).$$

Define $\iota_\lambda(a)$ in a similar fashion:

$$(\iota_\lambda(a)\gamma)_{\lambda_1, \dots, \lambda_{n-1}}(a_1, \dots, a_{n-1}) = \gamma_{\lambda, \lambda_1, \dots, \lambda_{n-1}}(a, a_1, \dots, a_{n-1}).$$

Note that every $a \in A$ defines naturally a one-chain $a \otimes \gamma_{\lambda_0} \in \widetilde{C}_1(A, \mathbb{C})$ depending on a parameter $\lambda_0$, where $\gamma_{\lambda_0}(f(\lambda)) = f(\lambda_0)$. Then we have $\iota_\lambda(a) = \iota(a \otimes \gamma_\lambda)$. The fundamental identity

$$d\iota_\lambda + \iota_\lambda d = \theta_\lambda$$

of classical Lie theory is also valid in the context of conformal algebras. It also implies $d\theta_\lambda = \theta_\lambda d$. As in the Lie algebra case, the induced action of $A$ on $\overset{\bullet}{\mathrm{H}}(A, M)$ is trivial, cf. Remark 6.2.

## 6. COHOMOLOGY OF CONFORMAL ALGEBRAS AND THEIR ANNIHILATION LIE ALGEBRAS

6.1. **Cohomology of the basic complex.** Let $A$ be a conformal algebra and $M$ a conformal module over it. Then $M$ is a module over the annihilation Lie algebra $\mathfrak{g}_- = (\mathrm{Lie}\, A)_-$, see Section 1. Let $C^\bullet(\mathfrak{g}_-, M)$ be the Chevalley–Eilenberg complex defining the cohomology of $\mathfrak{g}_-$ with coefficients in $M$. Recall that, by definition (see, *e.g.*, [F]), $C^n(\mathfrak{g}_-, M)$ is the space of skew-symmetric linear functionals $\gamma\colon (\mathfrak{g}_-)^{\otimes n} \to M$ which are *continuous, i.e.,*

$$\gamma(a_{1_{m_1}} \otimes \cdots \otimes a_{n_{m_n}}) = 0$$

for all but a finite number of $m_1, \ldots, m_n \in \mathbb{Z}_+$, where $a_1, \ldots, a_n \in A$, and $a_{i_{m_i}} \in \mathfrak{g}_- = (\mathrm{Lie}\, A)_- = A[t]/(\partial + \partial_t)A[t]$ is the image of the element $a_i t^{m_i}$.

$C^\bullet(\mathfrak{g}_-, M)$ has the following structure of a $\mathbb{C}[\partial]$-module:

$$(6.1) \quad (\partial\gamma)(a_1 \otimes \cdots \otimes a_n)$$
$$= \partial\big(\gamma(a_1 \otimes \cdots \otimes a_n)\big) - \sum_{i=1}^n \gamma(a_1 \otimes \cdots \otimes \partial a_i \otimes \cdots \otimes a_n),$$
$$\gamma \in C^n(\mathfrak{g}_-, M).$$

**Theorem 6.1.** *There is a canonical isomorphism of complexes $\widetilde{C}^\bullet(A, M)$ and $C^\bullet(\mathfrak{g}_-, M)$, compatible with the action of $\mathbb{C}[\partial]$. Consequently, the complex $C^\bullet(A, M)$ is isomorphic to $C^\bullet(\mathfrak{g}_-, M)/\partial C^\bullet(\mathfrak{g}_-, M)$.*

*Proof.* For a cochain $\alpha \in \widetilde{C}^n(A, M)$, we write

$$\alpha_{\lambda_1, \ldots, \lambda_n}(a_1, \ldots, a_n) = \sum_{m_1, \ldots, m_n \in \mathbb{Z}_+} \lambda_1^{(m_1)} \cdots \lambda_n^{(m_n)} \alpha_{(m_1, \ldots, m_n)}(a_1, \ldots, a_n).$$

In terms of the linear maps

$$\alpha_{(m_1, \ldots, m_n)}\colon A^{\otimes n} \to M,$$
$$a_1 \otimes \cdots \otimes a_n \mapsto \alpha_{(m_1, \ldots, m_n)}(a_1, \ldots, a_n),$$

the definition of $\widetilde{C}^\bullet(A, M)$ translates as follows.

1. For any $a_1, \ldots, a_n \in A$, $\alpha_{(m_1,\ldots,m_n)}(a_1, \ldots, a_n)$ is non-zero for only a finite number of $(m_1, \ldots, m_n)$.
2. $\alpha_{(m_1,\ldots,m_i,\ldots,m_n)}(a_1, \ldots, \partial a_i, \ldots, a_n)$
   $= -m_i \alpha_{(m_1,\ldots,m_i-1,\ldots,m_n)}(a_1, \ldots, a_i, \ldots, a_n)$.
3. $\alpha$ is skew-symmetric with respect to simultaneous permutations of $a_i$'s and $m_i$'s.

The differential is given by:

$$(d\gamma)_{(m_1,\ldots,m_{n+1})}(a_1, \ldots, a_{n+1})$$
$$= \sum_{i=1}^{n+1} (-1)^{i+1} a_{i(m_i)} \gamma_{(m_1,\ldots,\widehat{m_i},\ldots,m_{n+1})}(a_1, \ldots, \widehat{a_i}, \ldots, a_{n+1})$$
$$+ \sum_{\substack{i,j=1 \\ i<j}}^{n+1} \sum_{k=0}^{m_i} (-1)^{i+j} \binom{m_i}{k} \gamma_{(m_i+m_j-k,m_1,\ldots,\widehat{m_i},\ldots,\widehat{m_j},\ldots,m_{n+1})}(a_{i(k)}a_j, a_1,$$
$$\ldots, \widehat{a_i}, \ldots, \widehat{a_j}, \ldots, a_{n+1}).$$

Define linear maps $\phi^n \colon \widetilde{C}^n(A, M) \to C^n(\mathfrak{g}_-, M)$ by the formula

$$(\phi^n \alpha)(a_{1_{m_1}} \otimes \cdots \otimes a_{n_{m_n}}) = \alpha_{(m_1,\ldots,m_n)}(a_1, \ldots, a_n).$$

They are well-defined due to above condition 2. Clearly, $\phi^n$ are bijective and, using (1.3), it is easy to see that $\phi^{n+1} \circ d = d \circ \phi^n$. Moreover, $\phi^n \circ \partial = \partial \circ \phi^n$, where $\partial$ acts on $\widetilde{C}^\bullet(A, M)$ via (2.2) and on $C^\bullet(\mathfrak{g}_-, M)$ via (6.1). $\qquad\square$

**Corollary 6.1.** $\widetilde{\mathrm{H}}^\bullet(A, M) \simeq \mathrm{H}^\bullet(\mathfrak{g}_-, M)$.

*Remark 6.1.* Similar results hold for homology. To a chain $a_1 \otimes \cdots \otimes a_n \otimes \phi \in \widetilde{C}_n(A, M)$ ($a_i \in A$, $\phi \in \mathrm{Hom}(\mathbb{C}[\lambda_1, \ldots, \lambda_n], M)$) we associate the chain

$$\langle \phi, a_{1_{\lambda_1}} \otimes \cdots \otimes a_{n_{\lambda_n}} \rangle \in C_n(\mathfrak{g}_-, M).$$

In other words, $a_1 \otimes \cdots \otimes a_n \otimes \left(\partial_{\lambda_1}^{(m_1)} \cdots \partial_{\lambda_n}^{(m_n)}|_{\lambda_1=\cdots=\lambda_n=0}\right)$ corresponds to $a_{1_{m_1}} \otimes \cdots \otimes a_{n_{m_n}}$.

*Remark 6.2.* One can easily see that the exterior multiplication, contraction, module structure, etc., of Section 5 are equivalent to the corresponding notions for the annihilation Lie algebra $\mathfrak{g}_-$. For example, if $\theta(a_m)$ denotes the action of $a_m \in \mathfrak{g}_-$ on $C^\bullet(\mathfrak{g}_-, M) \simeq \widetilde{C}^\bullet(A, M)$, then

$$\theta_\lambda(a) = \sum_{m \in \mathbb{Z}_+} \lambda^{(m)} \theta(a_m).$$

In particular, the action of $A$ on $\widetilde{H}^\bullet(A, M)$ is trivial.

6.2. **Cohomology of the reduced complex.** Now we assume that $M$ is a free $\mathbb{C}[\partial]$-module: $M = \mathbb{C}[\partial] \otimes_{\mathbb{C}} U$ for some vector space $U$. Then the $\mathfrak{g}_-$-module $V_- = V(M)_-$ is just $U[t]$ with

$$a_m(ut^n) = \sum_{j=0}^{m} \binom{m}{j} (a_{(j)}u) \, t^{m+n-j}, \quad \partial(ut^n) = -nut^{n-1},$$

for $u \in U$, $a \in A$, see Section 1. In terms of the usual generating series $a_\lambda = \sum_{m \geq 0} \lambda^{(m)} a_m$, this can be rewritten as

$$a_\lambda(ut^n) = (a_\lambda u) \, t^n e^{t\lambda}.$$

**Theorem 6.2.** *If $A$ is a conformal algebra and $M$ a conformal module which is free as a $\mathbb{C}[\partial]$-module, then the complex $C^\bullet(A, M)$ is isomorphic to the subcomplex $C^\bullet(\mathfrak{g}_-, V_-)^\partial$ of $\partial$-invariant cochains in $C^\bullet(\mathfrak{g}_-, V_-)$.*

*Proof.* Let $\beta \in C^n(\mathfrak{g}_-, V_-)$. As in the proof of Theorem 6.1, consider the generating series

$$(6.2) \quad \beta_{\lambda_1, \dots, \lambda_n; t}(a_1, \dots, a_n)$$
$$= \sum_{m_1, \dots, m_n \in \mathbb{Z}_+} \lambda_1^{(m_1)} \cdots \lambda_n^{(m_n)} \beta(a_{1m_1} \otimes \cdots \otimes a_{nm_n}).$$

By Equation (6.1), $\partial$ acts on $\beta_{\lambda_1, \dots, \lambda_n; t}$ as $-\partial_t + \sum \lambda_i$. Hence $\beta$ is $\partial$-invariant, iff

$$(6.3) \quad \beta_{\lambda_1, \dots, \lambda_n; t}(a_1, \dots, a_n) = \gamma_{\lambda_1, \dots, \lambda_n}(a_1, \dots, a_n) \, e^{t \sum \lambda_i}$$

where $\gamma_{\lambda_1, \dots, \lambda_n} = \beta_{\lambda_1, \dots, \lambda_n; t}|_{t=0}$ takes values in $U$. Identifying $U$ with $1 \otimes U \subset M$, we can consider $\gamma$ as an element of $\widetilde{C}^n(A, M)$. It is easy to check that $\beta \mapsto \overline{\gamma} := \gamma \mod (\partial + \sum \lambda_i)$ is a chain map from $C^\bullet(\mathfrak{g}_-, V_-)$ to $C^\bullet(A, M)$.

Conversely, for $\overline{\gamma} \in C^n(A, M)$ choose a representative $\gamma \in \widetilde{C}^n(A, M)$ such that $\overline{\gamma} = \gamma \mod (\partial + \sum \lambda_i)$. Define $\beta \in C^n(\mathfrak{g}_-, V_-)^\partial$ by (6.2, 6.3) with $\partial$ substituted by $-\partial_t$ in $\gamma_{\lambda_1, \dots, \lambda_n}(a_1, \dots, a_n) \in M = U[\partial]$. Then clearly, $\beta$ is independent of the choice of $\gamma$.

The correspondence $\beta \leftrightarrow \overline{\gamma}$ establishes an isomorphism between $C^\bullet(\mathfrak{g}_-, V_-)^\partial$ and $C^\bullet(A, M)$. $\qquad \blacksquare$

*Remark 6.3.* Identifying $C^\bullet(A, M)$ with $C^\bullet(\mathfrak{g}_-, M)/\partial C^\bullet(\mathfrak{g}_-, M)$, we can rewrite (6.3) as

$$\beta(a_{1m_1} \otimes \cdots \otimes a_{nm_n})$$

$$= \sum_{k_1,\ldots,k_n \in \mathbb{Z}_+} \binom{m_1}{k_1} \cdots \binom{m_n}{k_n} \gamma(a_{1k_1} \otimes \cdots \otimes a_{nk_n}) \, t^{\sum m_i - \sum k_i},$$

$(a_i \in A, m_i \in \mathbb{Z}_+)$.

**6.3. Cohomology of conformal algebras and formal distribution Lie algebras.** Let $\mathfrak{g}$ be the maximal formal distribution Lie algebra corresponding to a conformal algebra $A$ (see Section 1). Suppose $\gamma \in C^n(A, M)$. The following formula defines an $n$-cochain $\widetilde{\gamma}$ on the Lie algebra $\mathfrak{g}$:

$$\widetilde{\gamma}(a_1 f_1(t), \ldots, a_n f_n(t))$$
$$= \mathrm{Res}_{\lambda_1,\ldots,\lambda_n} \gamma_{\partial_1,\ldots,\partial_n}(a_1, \ldots, a_n) \delta(\lambda_1 - \lambda_2) \ldots \delta(\lambda_1 - \lambda_n) f_1(\lambda_1) \ldots f_n(\lambda_n),$$

where $a_i \in A$, $f_i \in \mathbb{C}[t]$, $\partial_i = \partial/\partial \lambda_i$, and when substituting $\partial$ into a polynomial, one has to use the divided powers $\partial^{(k)} = \partial^k/k!$. This formula is equivalent to the one from Remark 6.3, where $m_i$'s are now allowed to take negative values. This correspondence defines a morphism of complexes and, therefore, cohomology.

## 7. Cohomology of the Virasoro conformal algebra

The conformal algebra with one free generator $L$ as a $\mathbb{C}[\partial]$-module and $\lambda$-bracket

$$[L_\lambda L] = (\partial + 2\lambda)L$$

is called the *Virasoro conformal algebra* Vir, cf. Example 1.2.

**7.1. Cohomology of Vir with trivial coefficients.** Here we will compute the cohomology of Vir with trivial coefficients $\mathbb{C}$, where both $\partial$ and $L$ act by zero.

**Theorem 7.1.** *For the Virasoro conformal algebra* Vir,

$$\dim \widetilde{\mathrm{H}}^q(\mathrm{Vir}, \mathbb{C}) = \begin{cases} 1 & \text{if } q = 0 \text{ or } 3, \\ 0 & \text{otherwise,} \end{cases}$$

*and*

$$\dim \mathrm{H}^q(\mathrm{Vir}, \mathbb{C}) = \begin{cases} 1 & \text{if } q = 0, \, 2, \text{ or } 3, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* Let us first identify the cohomology complex. An $n$-cochain $\gamma$ in this case is determined by its value on $L^{\otimes n}$:

$$P(\lambda_1, \ldots, \lambda_n) = \gamma_{\lambda_1,\ldots,\lambda_n}(L, \ldots, L).$$

Obviously, $P(\lambda_1, \ldots, \lambda_n)$ is a skew-symmetric polynomial with values in $\mathbb{C}$. The differential is then determined by the following formula:

$$(dP)(\lambda_1, \ldots, \lambda_{n+1})$$
$$= \sum_{\substack{i,j=1 \\ i<j}}^{n+1} (-1)^{i+j}(\lambda_i - \lambda_j) P(\lambda_i + \lambda_j, \lambda_1, \ldots, \widehat{\lambda}_i, \ldots, \widehat{\lambda}_j, \ldots, \lambda_{n+1}).$$

This describes the complex $\widetilde{C}^\bullet$. The complex $C^\bullet$ producing the cohomology of Vir is nothing but the quotient of $\widetilde{C}^\bullet$ by the ideal spanned by $\sum_{i=1}^n \lambda_i$ in each degree $n$. In other words, $C^n$ is the space of regular (polynomial) functions on the hyperplane $\sum_{i=1}^n \lambda_i = 0$ in $\mathbb{C}^n$ which are skew in the variables $\lambda_1, \ldots, \lambda_n$. This complex appeared as an intermediate step in Gelfand–Fuchs's 1968 computation [GF1] of the cohomology of the Virasoro Lie algebra, and the cohomology of $C^\bullet$ was computed therein.

Consider the following homotopy operator $\widetilde{C}^q \to \widetilde{C}^{q-1}$

$$k(P) = (-1)^q \frac{\partial P}{\partial \lambda_q}\bigg|_{\lambda_q=0}.$$

A straightforward computation shows that $(dk+kd)P = (\deg P - q)P$ for $P \in \widetilde{C}^q$, where $\deg P$ is the total degree of $P$ in $\lambda_1, \ldots, \lambda_q$. Thus, only those homogeneous cochains whose degree as a polynomial is equal to their degree as a cochain contribute to the cohomology of $\widetilde{C}^\bullet$. These polynomials must be skew and therefore divisible by $\Lambda_q = \prod_{i<j}(\lambda_i - \lambda_j)$, whose polynomial degree is $q(q-1)/2$. The quadratic inequality $q(q-1)/2 \leq q$ has $q = 0, 1, 2$, and 3 as the only integral solutions. For $q = 0$, the whole $\widetilde{C}^0 = \mathbb{C}$ contributes to $\mathrm{H}^0(\widetilde{C}^\bullet)$. For $q = 1$, the only polynomial of degree 1 is $\lambda_1$, up to a constant factor. $d\lambda_1 = \lambda_2^2 - \lambda_1^2$, which is the only skew polynomial of degree 2 in two variables. This shows that $\widetilde{\mathrm{H}}^1 = \widetilde{\mathrm{H}}^2 = 0$. Finally, for $q = 3$, the only skew polynomial of degree 3 in 3 variables is $\Lambda_3$, up to a constant. It is easy to see that this polynomial represents a non-trivial class in the cohomology. Indeed, it is closed, because a skew-symmetric function in four variables has a degree at least 6, which is greater than $\deg(d\Lambda_3) = 4$. And $\Lambda_3$ is not a coboundary, because it can be the coboundary of a two-cochain of degree 2, which must be a constant factor of $\lambda_2^2 - \lambda_1^2 = d\lambda_1$, whose coboundary is zero.

The computation of the cohomology of the quotient complex $C^\bullet$ is based on the short exact sequence

$$(7.1) \qquad\qquad 0 \to \partial\widetilde{C}^\bullet \to \widetilde{C}^\bullet \to C^\bullet \to 0.$$

By definition, $\partial \widetilde{C}^0 = 0$. To find the cohomology of $\partial \widetilde{C}^\bullet$, define a homotopy $k_1 \colon \partial \widetilde{C}^q \to \partial \widetilde{C}^{q-1}$ as $k_1(\partial P) = \partial k(P)$, where $\partial = \sum_i \lambda_i$ and $P \in \widetilde{C}^q$. Then $(dk_1 + k_1 d)\partial P = (\deg P - q)\partial P$. As in the previous paragraph, this implies that $\deg P = q = 0$, 1, 2, or 3. Up to constant factors, the only polynomials in $\partial \widetilde{C}^\bullet$ with this property are $P_1 = \lambda_1^2$ for $q = 1$, $P_2 = (\lambda_1 + \lambda_2)(\lambda_1^2 - \lambda_2^2)$ for $q = 2$, and $P_3 = (\lambda_1 + \lambda_2 + \lambda_3)\Lambda_3$ for $q = 3$. One computes: $dP_1 = -P_2$ and $dP_3 = 0$. Therefore $\mathrm{H}^q(\partial \widetilde{C}^\bullet) = 0$ for all $q$ but $q = 3$, where it is one-dimensional with the generator $P_3$.

Thus, the long exact sequence of cohomology associated with (7.1) looks as follows:

$$
\begin{array}{ccccccc}
0 & \longrightarrow & 0 & \longrightarrow & \mathbb{C} & \longrightarrow & \mathrm{H}^0(\mathrm{Vir}, \mathbb{C}) \longrightarrow \\
 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathrm{H}^1(\mathrm{Vir}, \mathbb{C}) \longrightarrow \\
 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathrm{H}^2(\mathrm{Vir}, \mathbb{C}) \longrightarrow \\
 & \longrightarrow & \mathbb{C}P_3 & \longrightarrow & \mathbb{C}\Lambda_3 & \longrightarrow & \mathrm{H}^3(\mathrm{Vir}, \mathbb{C}) \longrightarrow \\
 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathrm{H}^4(\mathrm{Vir}, \mathbb{C}) \longrightarrow \\
 & \longrightarrow & 0 & \longrightarrow & \dots
\end{array}
$$

We see that $\mathrm{H}^0(\mathrm{Vir}, \mathbb{C}) = \mathbb{C}$ and $\mathrm{H}^q(\mathrm{Vir}, \mathbb{C}) = 0$ for $q = 1, 4, 5, 6, \dots$ and $\mathrm{H}^3(\mathrm{Vir}, \mathbb{C}) = \mathbb{C}\Lambda_3$ and $\mathrm{H}^2(\mathrm{Vir}, \mathbb{C}) = \mathbb{C}(\lambda_1^3 - \lambda_2^3)$, because $d(\lambda_1^3 - \lambda_2^3) = P_3$. $\qquad\blacksquare$

*Remark* 7.1. In fact, this computation shows that the cohomology of the Virasoro conformal algebra is the primitive part of the cohomology ring of the Virasoro Lie algebra, in addition to $\mathbb{C}$ in degree 0. The reduction of the basic cohomology to the computation of Gelfand and Fuchs [GF1] might be made using Corollary 6.1, but we preferred to use a direct argument in the proof.

*Remark* 7.2. Instead of the trivial Vir-module $\mathbb{C}$, consider the module $\mathbb{C}_a$, which is the one-dimensional vector space $\mathbb{C}$ on which all elements of Vir act by zero, and $\partial v = av$ for $v \in \mathbb{C}_a$, $a \neq 0$ being a given complex constant. Then Proposition 2.1 shows that $\mathrm{H}^q(\partial \widetilde{C}^\bullet) \simeq \widetilde{\mathrm{H}}^q(\mathrm{Vir}, \mathbb{C}_a)$ for $q \geq 0$, and the long exact sequence (2.3) combined with the computation of $\widetilde{\mathrm{H}}^\bullet(\mathrm{Vir}, \mathbb{C}_a)$, which is obviously isomorphic to $\widetilde{\mathrm{H}}^\bullet(\mathrm{Vir}, \mathbb{C})$, provided by Theorem 7.1, shows that $\mathrm{H}^q(\mathrm{Vir}, \mathbb{C}_a) = 0$ for all $q$.

7.2. **Cohomology of** Vir **with coefficients in** $M_{\Delta,\alpha}$. Recall (Example 1.2) that $M_{\Delta,\alpha}$ ($\Delta, \alpha \in \mathbb{C}$) is the following Vir-module

$$M_{\Delta,\alpha} = \mathbb{C}[\partial]v, \quad L_\lambda v = (\partial + \alpha + \Delta\lambda)v.$$

As in the previous subsection, we identify the space of $n$-cochains $C^n(\text{Vir}, M_{\Delta,\alpha})$ with the space of all $\mathbb{C}$-valued skew-symmetric polynomials in $n$ variables: for any $\gamma \in C^n(\text{Vir}, M_{\Delta,\alpha})$, there is a unique polynomial $P(\lambda_1, \ldots, \lambda_n)$ such that

$$\gamma_{\lambda_1, \ldots, \lambda_n}(L, \ldots, L) = P(\lambda_1, \ldots, \lambda_n)v \mod (\partial + \lambda_1 + \cdots + \lambda_n).$$

Then the differential is given by the formula

$$(dP)(\lambda_1, \ldots, \lambda_{n+1}) =$$
$$= \sum_{i=1}^{n+1} (-1)^{i+1} \Big(\alpha - \sum_{j=1}^{n+1} \lambda_j + \Delta\lambda_i\Big) P(\lambda_1, \ldots, \widehat{\lambda_i}, \ldots, \lambda_{n+1})$$
$$+ \sum_{\substack{i,j=1 \\ i<j}}^{n+1} (-1)^{i+j}(\lambda_i - \lambda_j) P(\lambda_i + \lambda_j, \lambda_1, \ldots, \widehat{\lambda_i}, \ldots, \widehat{\lambda_j}, \ldots, \lambda_{n+1}).$$

Now we interpret this in terms of the Lie algebra $\mathcal{V}ect\mathbb{C}$ of regular vector fields on $\mathbb{C}$, which is the annihilation algebra of Vir, see Section 1. To $\gamma \in C^n(\text{Vir}, M_{\Delta,\alpha})$ we associate a linear map $\beta \colon \bigwedge^n \mathcal{V}ect\mathbb{C} \to \mathbb{C}$ by the formula

$$\sum_{m_1, \ldots, m_n \in \mathbb{Z}_+} \lambda_1^{(m_1)} \cdots \lambda_n^{(m_n)} \beta(L_{(m_1)} \wedge \cdots \wedge L_{(m_n)}) = P(\lambda_1, \ldots, \lambda_n),$$

where $L_{(m)} = -t^m \partial_t$.

Then the differential is

$$(d\beta)(L_{(m_1)} \wedge \cdots \wedge L_{(m_{n+1})})$$
$$= \sum_{i=1}^{n+1} (-1)^{i+1} \big(\alpha \delta_{m_i,0} + (\Delta - 1)\delta_{m_i,1}\big) \beta(L_{(m_1)} \wedge \cdots \wedge \widehat{L_{(m_i)}} \wedge \cdots$$
$$\cdots \wedge L_{(m_{n+1})}\big)$$
$$+ \sum_{\substack{i,j=1 \\ i<j}}^{n+1} (-1)^{i+j}(1 - \delta_{m_i,0})(1 - \delta_{m_j,0}) \beta\big([L_{(m_i)}, L_{(m_j)}], L_{(m_1)} \wedge \cdots$$
$$\cdots \wedge \widehat{L_{(m_i)}} \wedge \cdots \wedge \widehat{L_{(m_j)}} \wedge \ldots \wedge L_{(m_{n+1})}\big).$$

Let $\mathcal{V}ect_0\mathbb{C}$ be the subalgebra of $\mathcal{V}ect\mathbb{C}$ of vector fields that vanish at the origin. It is spanned by the elements $L_{(m)} = -t^m\partial_t$, $m \geq 1$. Let $U_\Delta$ be a 1-dimensional $\mathcal{V}ect_0\mathbb{C}$-module on which $L_{(m)}$ acts as 0 for $m \geq 2$ and $L_{(1)}$ acts as a multiplication by $\Delta$.

**Theorem 7.2.**     1. $\mathrm{H}^\bullet(\text{Vir}, M_{\Delta,\alpha}) = 0$ if $\alpha \neq 0$.

2. $H^q(\text{Vir}, M_{\Delta,0}) \simeq H^q(\mathcal{V}ect_0\mathbb{C}, U_{\Delta-1}) \oplus H^{q-1}(\mathcal{V}ect_0\mathbb{C}, U_{\Delta-1})$ for any $q$ ($H^q = 0$ for $q < 0$ by definition).

3. $\dim H^q(\text{Vir}, M_{\Delta,0}) = \dim H^q(\mathcal{V}ect\mathbb{C}, \mathbb{C}[t, t^{-1}](dt)^{1-\Delta})$. Explicitly:

$$\dim H^q(\text{Vir}, M_{1-(3r^2\pm r)/2, 0}) = \begin{cases} 2 & \text{for } q = r+1, \\ 1 & \text{for } q = r, r+2, \\ 0 & \text{otherwise}, \end{cases}$$

and $H^q(\text{Vir}, M_{\Delta,0}) = 0$ if $\Delta \neq 1 - (3r^2 \pm r)/2$ for any $r \in \mathbb{Z}_+$.

*Proof.* We have seen that the complex $C^\bullet(\text{Vir}, M_{\Delta,\alpha})$ is isomorphic to $(\bigwedge^\bullet \mathcal{V}ect\mathbb{C})^*$ endowed with the above non-standard differential. Let $\pi\colon (\bigwedge^q \mathcal{V}ect\mathbb{C})^* \to (\bigwedge^q \mathcal{V}ect_0\mathbb{C})^*$ be the restriction map. It is easy to see that in fact $\pi$ is a chain map from $C^q(\text{Vir}, M_{\Delta,\alpha})$ to $C^q(\mathcal{V}ect_0\mathbb{C}, U_{\Delta-1})$, where we identify $U_{\Delta-1} = \mathbb{C}$ as a vector space. Define another map $\iota\colon (\bigwedge^{q-1} \mathcal{V}ect\mathbb{C})^* \to (\bigwedge^q \mathcal{V}ect_0\mathbb{C})^*$ by the formula

$$(\iota\beta)(L_{(m_1)} \wedge \cdots \wedge L_{(m_q)})$$
$$= \sum_{i=1}^q (-1)^{i+1} \delta_{m_i,0} \, \beta\big(L_{(m_1)} \wedge \cdots \wedge \widehat{L_{(m_i)}} \wedge \cdots \wedge L_{(m_q)}\big).$$

Then $\iota$ is a chain map from $C^{q-1}(\mathcal{V}ect_0\mathbb{C}, U_{\Delta-1})$ to $C^q(\text{Vir}, M_{\Delta,\alpha})$.

We have a short exact sequence of complexes

$$0 \to C^{q-1}(\mathcal{V}ect_0\mathbb{C}, U_{\Delta-1}) \xrightarrow{\iota} C^q(\text{Vir}, M_{\Delta,\alpha}) \xrightarrow{\pi} C^q(\mathcal{V}ect_0\mathbb{C}, U_{\Delta-1}) \to 0.$$

A splitting $\phi\colon C^q(\mathcal{V}ect_0\mathbb{C}, U_{\Delta-1}) \to (\bigwedge^q \mathcal{V}ect\mathbb{C})^*$ is given by the formula

$$(\phi\beta)(L_{(m_1)} \wedge \cdots \wedge L_{(m_q)}) = \begin{cases} \beta(L_{(m_1)} \wedge \cdots \wedge L_{(m_q)}) & \text{if all } m_i \geq 1, \\ 0 & \text{otherwise}. \end{cases}$$

One checks that if $d\beta = 0$ then $d\phi\beta = \alpha\iota\beta$.

Hence, the cohomology long exact sequence associated to the above short exact sequence of complexes looks as follows:

$$\xrightarrow{\alpha\,\text{id}} H^{q-1}(\mathcal{V}ect_0\mathbb{C}, U_{\Delta-1}) \xrightarrow{\iota} H^q(\text{Vir}, M_{\Delta,\alpha}) \xrightarrow{\pi} H^q(\mathcal{V}ect_0\mathbb{C}, U_{\Delta-1})$$
$$\xrightarrow{\alpha\,\text{id}} H^q(\mathcal{V}ect_0\mathbb{C}, U_{\Delta-1}) \xrightarrow{\iota} \cdots$$

This proves Parts 1 and 2.

Part 3 follows from Part 2 and the results of Feigin and Fuchs, see [F, §2.3]. (Note that our $U_\Delta$ is exactly their $E_{-\Delta}$.) □

### 8. Cohomology of current conformal algebras

8.1. **Cohomology with trivial coefficients.** Here we will compute the cohomology of a current conformal algebra Cur $\mathfrak{g}$ with trivial coefficients for a finite-dimensional semisimple Lie algebra $\mathfrak{g}$. Recall from Example 1.1 that the current conformal algebra Cur $\mathfrak{g}$ is $\mathbb{C}[\partial] \otimes \mathfrak{g}$ with the $\lambda$-bracket

$$[a_\lambda b] = [a, b] \qquad \text{for } a, b \in \mathfrak{g}.$$

The basic complex in this case becomes bigraded, the second grading given by the total degree in $\lambda_i$, which we will call the $\lambda$-*degree*, of the restriction of the cochain to the subspace $\mathfrak{g}$ of generators of Cur $\mathfrak{g}$. The differential respects the $\lambda$-degree, and therefore the complex splits into the direct sum of its graded subcomplexes. Let $\widetilde{C}_0^\bullet \subset \widetilde{C}^\bullet$ be the subcomplex of zero $\lambda$-degree. This subcomplex is obviously isomorphic to the Chevalley–Eilenberg complex $C^\bullet(\mathfrak{g}, \mathbb{C})$ of the Lie algebra $\mathfrak{g}$.

**Theorem 8.1.**   1. *The embedding* $C^\bullet(\mathfrak{g}, \mathbb{C}) \subset \widetilde{C}^\bullet$ *is a quasi-isomorphism, i.e., it induces an isomorphism on cohomology. Therefore,*

$$\widetilde{H}^\bullet(\mathrm{Cur}\,\mathfrak{g}, \mathbb{C}) \simeq \mathrm{H}^\bullet(\mathfrak{g}, \mathbb{C}) \simeq \left(\bigwedge{}^\bullet \mathfrak{g}^*\right)^{\mathfrak{g}}.$$

2. *For* $q \geq 0$

$$\mathrm{H}^q(\mathrm{Cur}\,\mathfrak{g}, \mathbb{C}) \simeq \mathrm{H}^q(\mathfrak{g}, \mathbb{C}) \oplus \mathrm{H}^{q+1}(\mathfrak{g}, \mathbb{C}).$$

*Proof.* 1. According to Theorem 6.1, the complexes $\widetilde{C}^\bullet(\mathrm{Cur}\,\mathfrak{g}, \mathbb{C})$ and $C^\bullet(\mathfrak{g}[t], \mathbb{C})$ are isomorphic, because $\mathfrak{g}[t]$ is the annihilation subalgebra of Cur $\mathfrak{g}$, see Example 1.1. Moreover, the part of $\lambda$-degree zero maps isomorphically to the Chevalley–Eilenberg complex $C^\bullet(\mathfrak{g}, \mathbb{C})$, which is the subcomplex of $C^\bullet(\mathfrak{g}[t], \mathbb{C})$ of cochains vanishing on $t\mathfrak{g}[t]$. Thus, $\widetilde{H}^\bullet(\mathrm{Cur}\,\mathfrak{g}, \mathbb{C}) \simeq \mathrm{H}^\bullet(\mathfrak{g}[t], \mathbb{C})$, which is isomorphic to $\mathrm{H}^\bullet(\mathfrak{g}, \mathbb{C})$ via the subcomplex of cochains vanishing on $t\mathfrak{g}[t]$ by a result of Feigin [Fe1, Fe2]; see a different proof of Feigin's result in Section 8.2, which covers the case of non-trivial coefficients as well. The computation of $\mathrm{H}^\bullet(\mathfrak{g}, \mathbb{C})$ via the invariants of the dual exterior algebra is standard, see *e.g.*, [Fe2].

2. Consider the long exact sequence (2.3). The mapping $\mathrm{H}^q(\partial \widetilde{C}^\bullet) \to \mathrm{H}^q(\widetilde{C})$ for $q \geq 1$ is zero, because the cohomology of $\mathrm{H}^q(\widetilde{C})$ is concentrated in $\lambda$-degree zero (see the first statement of the Theorem) and the cohomology of $\mathrm{H}^q(\partial \widetilde{C}^\bullet)$ is concentrated in $\lambda$-degree one (see Proposition 2.1). The same is true even for $q = 0$, because $\partial \widetilde{C}^0 = 0$ and the degree-zero differential $d \colon \widetilde{C}^0 \to \widetilde{C}^1$ is zero. Thus (2.3) splits into the short exact sequences

$$0 \to \mathrm{H}^q(\mathfrak{g}, \mathbb{C}) \to \mathrm{H}^q(\mathrm{Cur}\,\mathfrak{g}, \mathbb{C}) \to \mathrm{H}^{q+1}(\mathfrak{g}, \mathbb{C}) \to 0$$

for each $q \geq 0$.                                                            □

*Remark* 8.1. The same argument as in Remark 7.2 shows that $\mathrm{H}^\bullet(\mathrm{Cur}\,\mathfrak{g}, \mathbb{C}_a) = 0$, where $\mathbb{C}_a$ is the one-dimensional Cur $\mathfrak{g}$-module on which Cur $\mathfrak{g}$ acts trivially and $\partial$ acts by a multiplication by $a \neq 0$.

**8.2. Cohomology with coefficients in a current module.** Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra, and $U$ a $\mathfrak{g}$-module. Recall (Example 1.1) that the current module $M_U$ over Cur $\mathfrak{g}$ is defined as $M_U = \mathbb{C}[\partial] \otimes U$ with
$$a_\lambda u = au \qquad \text{for } a \in \mathfrak{g},\ u \in U.$$

**Proposition 8.1.** $\mathrm{H}^\bullet(\mathrm{Cur}\,\mathfrak{g}, M_U) \simeq \mathrm{H}^\bullet(\mathfrak{g}[t], U)$ *where the Lie algebra* $\mathfrak{g}[t]$ *acts on the* $\mathfrak{g}$-*module* $U$ *by evaluation at* $t = 0$.

This can be deduced from Theorem 6.2 but we will give a more direct argument.

*Proof.* Since $M_U$ is free over $\mathbb{C}[\partial]$, any cochain $\alpha \in C^n(\mathrm{Cur}\,\mathfrak{g}, M_U)$ has a unique representative mod $(\partial + \lambda_1 + \cdots + \lambda_n)$ independent of $\partial$. Explicitly, there is a unique $\beta \colon \mathfrak{g}^{\otimes n} \to \mathbb{C}[\lambda_1, \ldots, \lambda_m] \otimes U$ such that
$$\alpha_{\lambda_1, \ldots, \lambda_n}(a_1, \ldots, a_n)$$
$$= \beta_{\lambda_1, \ldots, \lambda_n}(a_1 \otimes \cdots \otimes a_n) \mod (\partial + \lambda_1 + \cdots + \lambda_n)$$
for $a_1, \ldots, a_n \in \mathfrak{g}$. Now writing
$$\beta_{\lambda_1, \ldots, \lambda_n}(a_1 \otimes \cdots \otimes a_n)$$
$$= \sum_{m_1, \ldots, m_n \in \mathbb{Z}_+} \lambda_1^{(m_1)} \cdots \lambda_n^{(m_n)} \beta(t^{m_1} a_1 \wedge \cdots \wedge t^{m_n} a_n)$$
we can interpret $\beta$ as a cochain $\bigwedge^n \mathfrak{g}[t] \to U$, as in the proof of Theorem 6.1.                                                                  □

To compute $\mathrm{H}^\bullet(\mathfrak{g}[t], U)$, we apply the Hochschild–Serre spectral sequence (see, *e.g.*, [F, §1.5.1]) for the ideal $t\mathfrak{g}[t]$ of $\mathfrak{g}[t]$. The $E_2$ term is

(8.1)     $E_2^{p,q} \simeq \mathrm{H}^p(\mathfrak{g}, \mathrm{H}^q(t\mathfrak{g}[t], U)) \simeq \mathrm{H}^p(\mathfrak{g}) \otimes \mathrm{H}^q(t\mathfrak{g}[t], U)^\mathfrak{g}$
$$\simeq \mathrm{H}^p(\mathfrak{g}) \otimes \big(\mathrm{H}^q(t\mathfrak{g}[t]) \otimes U\big)^\mathfrak{g}.$$

We used that $U$ is a trivial $t\mathfrak{g}[t]$-module and that $\mathrm{H}^p(\mathfrak{g}, U) \simeq \mathrm{H}^p(\mathfrak{g}) \otimes U^\mathfrak{g}$ for any module $U$ over a simple Lie algebra $\mathfrak{g}$.

Of course, $\mathrm{H}^p(\mathfrak{g})$ is well-known (cf. Theorem 8.1), so we only need $\mathrm{H}^q(t\mathfrak{g}[t])$. The latter can be deduced from a famous result of Kostant [Ko] (generalized to the affine Kac–Moody case).

First, we need some notation from [K1]. Fix a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. Let $W$, $\Delta$, $\Delta_+$, $\Delta_l$, $\rho$, $\theta$, $h^\vee$ be respectively the Weyl group, the set of roots, the set of positive roots, the set of long roots, the half sum of positive roots, the highest root, and the dual Coxeter number of $\mathfrak{g}$. Let $\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] + \mathbb{C}K + \mathbb{C}d$ be the affine Kac–Moody algebra associated to $\mathfrak{g}$. The corresponding objects for $\widehat{\mathfrak{g}}$ will be hatted. For example, $\widehat{\mathfrak{g}} = \widehat{\mathfrak{n}}_- \oplus \widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{n}}_+$, where $\widehat{\mathfrak{n}}_\pm = t^{\pm 1}\mathfrak{g}[t^{\pm 1}] + \mathfrak{n}_\pm$, $\widehat{\mathfrak{h}} = \mathfrak{h} + \mathbb{C}K + \mathbb{C}d$. Denote by $\delta$ and $\Lambda_0$ the elements of $\widehat{\mathfrak{h}}^*$ that correspond to $K$ and $d$ via the isomorphism $\widehat{\mathfrak{h}}^* \simeq \widehat{\mathfrak{h}}$ given by the invariant bilinear form $(\cdot | \cdot)$ of $\widehat{\mathfrak{g}}$, normalized by $(\theta | \theta) = 2$. Recall that the simple roots of $\widehat{\mathfrak{g}}$ are $\widehat{\alpha}_0 = \delta - \theta$, $\widehat{\alpha}_i = \alpha_i$ ($1 \leq i \leq l := \operatorname{rank} \mathfrak{g}$), where $\alpha_i$ are the simple roots of $\mathfrak{g}$. The element $\widehat{\rho} \in \widehat{\mathfrak{h}}^*$ is defined by the property $\langle \widehat{\rho}, \widehat{\alpha}_i \rangle = 1$ ($0 \leq i \leq l$), i.e., $\widehat{\rho} = \rho + h^\vee \Lambda_0$. We denote by bar the projection from $\widehat{\mathfrak{h}}^*$ onto $\mathfrak{h}^*$. Also recall that $\widehat{W} = W \ltimes T$, where $T$ is the group of translations $t_\gamma$ ($\gamma \in \mathbb{Z}\Delta_l$) such that $\overline{t_\gamma(\lambda)} = \lambda + \langle \lambda, K \rangle \gamma$ for $\lambda \in \mathfrak{h}^*$, ($w \in W$ acts on $t_\gamma$ by $w t_\gamma w^{-1} = t_{w(\gamma)}$). For $\widehat{w} \in \widehat{W}$ we denote its length by $\ell(\widehat{w})$. Finally, if $\Lambda \in \mathfrak{h}^*$ is a dominant weight, we denote by $V(\Lambda)$ the irreducible $\mathfrak{g}$-module with highest weight $\Lambda$.

Now we can state

**Lemma 8.1.** 1. *As a $\mathfrak{g}$-module*

$$\mathrm{H}^q(t\mathfrak{g}[t]) \simeq \bigoplus_{\widehat{w} \in \widehat{W}^1, \ell(\widehat{w}) = q} V(\overline{\widehat{w}(\widehat{\rho}) - \widehat{\rho}})$$

*where* $\widehat{W}^1 := \{\widehat{w} \in \widehat{W} \,|\, \widehat{w}^{-1}\Delta_+ \subset \widehat{\Delta}_+\}$.
2. *Equivalently,*

$$\mathrm{H}^q(t\mathfrak{g}[t]) \simeq \bigoplus_{(w,\gamma) \in WT^1, \ell(t_\gamma w) = q} V(w(\rho) - \rho + h^\vee \gamma)$$

*where* $WT^1 := \{(w, \gamma) \in W \ltimes \mathbb{Z}\Delta_l \,|\, (\gamma | \alpha) \geq 0 \;\forall \alpha \in \Delta_+, \;\; (\gamma | \alpha) > 0 \;\forall \alpha \in \Delta_+ \cap w\Delta_-\}$.

*Proof.* Part 1 is a special case of Theorem 5.14 of Kostant [Ko] (generalized to the affine Kac–Moody case). His Lie algebra $\mathfrak{g}$ will be the affine Kac–Moody algebra $\widehat{\mathfrak{g}}$. We take the parabolic subalgebra $\mathfrak{u} = \mathfrak{g}[t] + \mathbb{C}K + \mathbb{C}d$ of $\widehat{\mathfrak{g}}$, then $\mathfrak{n} = t\mathfrak{g}[t]$, $\mathfrak{g}_1 = \mathfrak{g} + \mathbb{C}K + \mathbb{C}d$.

Part 2 is standard, using that $\widehat{W} = W \ltimes T$ (see [K1]). $\qquad\blacksquare$

**Lemma 8.2.** *For any $\widehat{w} \in \widehat{W}$, we have:*
1. $\widehat{\rho} - \widehat{w}(\widehat{\rho}) = \sum_{\beta \in \widehat{\Delta}_+ \cap \widehat{w}\widehat{\Delta}_-} \beta$.
2. $\ell(\widehat{w}) = \left| \widehat{\Delta}_+ \cap \widehat{w}\widehat{\Delta}_- \right|$.

3. $\widehat{\rho} - \widehat{w}(\widehat{\rho}) \in \mathbb{Z}\delta$, iff $\widehat{w} = 1$.

*Proof.* Parts 1 and 2 are exercises from [K1, Chap. 3] and left to the reader.

Suppose $\widehat{\rho} - \widehat{w}(\widehat{\rho}) = n\delta$, $n \in \mathbb{Z}$. Then by Part 1, $n\delta \in \mathbb{Z}_+\widehat{\Delta}_+$. Since $\widehat{w}^{-1}(\delta) = \delta$, applying $\widehat{w}^{-1}$ to Part 1, we get $n\delta \in \mathbb{Z}_+\widehat{\Delta}_-$. Hence $n = 0$. But then Parts 1 and 2 imply that $\ell(\widehat{w}) = 0$, *i.e.*, $\widehat{w} = 1$. $\qquad\blacksquare$

It follows from Part 3 of the lemma that for any $\Lambda \in \mathfrak{h}^*$ there is at most one $\widehat{w} \in \widehat{W}$ such that $\Lambda = \overline{\widehat{w}(\rho)} - \overline{\rho}$. Define $\ell(\Lambda)$ to be the length of this $\widehat{w}$ if it exists, and $+\infty$ otherwise. Then we can restate Lemma 8.1 as follows:

$$(8.2) \qquad \mathrm{H}^q(t\mathfrak{g}[t]) \simeq \bigoplus_{\Lambda \in \mathfrak{h}^*, \, \ell(\Lambda)=q} V(\Lambda),$$

where $V(\Lambda)$ is a finite-dimensional representation of highest weight $\Lambda$.

**Theorem 8.2.** *Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra with a fixed Cartan subalgebra $\mathfrak{h}$. Let $U$ be an irreducible $\mathfrak{g}$-module. Then*

$$\mathrm{H}^n(\mathrm{Cur}\,\mathfrak{g}, M_U) \simeq \mathrm{H}^n(\mathfrak{g}[t], U) \simeq \mathrm{H}^{n-\ell^*(U)}(\mathfrak{g}).$$

*Here $\ell^*(U) = +\infty$ whenever $U$ is infinite-dimensional, $\ell^*(U) = \ell(\Lambda^*)$ whenever $U = V(\Lambda)$ is a finite-dimensional irreducible module with a highest weight $\Lambda$, $\Lambda^*$ is the highest weight of the contragredient module $V(\Lambda)^*$, $\ell(\Lambda)$ is as above, and we agree that $\mathrm{H}^n = 0$ for $n < 0$ (including $n = -\infty$).*

*Proof.* The first isomorphism in the theorem is from Proposition 8.1. To compute $\mathrm{H}^\bullet(\mathfrak{g}[t], V(\Lambda))$, we apply the Hochschild–Serre spectral sequence for the Lie algebra $\mathfrak{g}[t]$, its module $U$, and its ideal $t\mathfrak{g}[t]$.

If $U = V(\Lambda)$, then $U^* \simeq V(\Lambda^*)$ and Equations (8.1, 8.2) imply that the $E_2$ term is

$$E_2^{p,q} = \begin{cases} \mathrm{H}^p(\mathfrak{g}) & \text{for } q = \ell(\Lambda^*) < +\infty, \\ 0 & \text{otherwise.} \end{cases}$$

Hence the spectral sequence degenerates at $E_2$ and $\mathrm{H}^n(\mathfrak{g}[t], V(\Lambda)) \simeq \mathrm{H}^{n-\ell(\Lambda^*)}(\mathfrak{g})$.

If $U$ is infinite-dimensional, then again by (8.1, 8.2), we have $E_2^{p,q} = 0$. $\qquad\blacksquare$

**Corollary 8.1.** [Fe1, Fe2]. $\mathrm{H}^\bullet(\mathfrak{g}[t]) \simeq \mathrm{H}^\bullet(\mathfrak{g})$ *where the isomorphism is induced from evaluation at $t = 0$.*

**Corollary 8.2.** *For any semisimple $\mathfrak{g}$-module $U$:*

1. $\mathrm{H}^1(\mathrm{Cur}\,\mathfrak{g}, M_U) \simeq \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, U)$. *Explicitly, the isomorphism is given by:*

$$\alpha_\lambda(a) = \lambda\,\varphi(a) \mod (\partial + \lambda)$$

   *for* $a \in \mathfrak{g}$, $\varphi \in \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, U)$.

2. $\mathrm{H}^2(\mathrm{Cur}\,\mathfrak{g}, M_U) \simeq \mathrm{Hom}_{\mathfrak{g}}(\bigwedge^2 \mathfrak{g}/\mathfrak{g}, U)$, *provided that* $\mathfrak{g} \not\simeq \mathfrak{sl}_2$. *Explicitly, the isomorphism is given by:*

$$\alpha_{\lambda_1,\lambda_2}(a_1, a_2) = \lambda_1 \lambda_2\,\varphi(a_1, a_2) \mod (\partial + \lambda_1 + \lambda_2)$$

   *for* $a_1, a_2 \in \mathfrak{g}$, $\varphi \in \mathrm{Hom}_{\mathfrak{g}}(\bigwedge^2 \mathfrak{g}/\mathfrak{g}, U)$.

*Proof.* It is easy to check that the above formulas indeed give cocycles. In fact, 2. gives a cocycle for any $\varphi \in \mathrm{Hom}_{\mathfrak{g}}(\bigwedge^2 \mathfrak{g}, U)$; however, any $\varphi \in \mathrm{Hom}_{\mathfrak{g}}(\bigwedge^2 \mathfrak{g}, \mathfrak{g})$ gives a coboundary. Next, we use Lemma 8.1 and the fact that $\mathrm{H}^n(\mathrm{Cur}\,\mathfrak{g}, M_U) \simeq \mathrm{Hom}_{\mathfrak{g}}(\mathrm{H}^n(t\mathfrak{g}[t]), U)$ for $n = 1, 2$.

1. The only element of $\widehat{W}^1$ of length 1 is the simple reflection $r_{\widehat{\alpha}_0}$ with respect to the root $\widehat{\alpha}_0$. Then $r_{\widehat{\alpha}_0}(\widehat{\rho}) - \widehat{\rho} = -\widehat{\alpha}_0 = \theta - \delta$. Hence $\mathrm{H}^1(t\mathfrak{g}[t]) \simeq V(\theta) \simeq \mathfrak{g}$ as a $\mathfrak{g}$-module.

2. All elements of $\widehat{W}^1$ of length 2 are of the form $r_{\widehat{\alpha}_0} r_{\widehat{\alpha}_i}$, where $i$ is such that $\langle \widehat{\alpha}_i, \widehat{\alpha}_0^\vee \rangle \neq 0$. Then $r_{\widehat{\alpha}_0} r_{\widehat{\alpha}_i}(\widehat{\rho}) - \widehat{\rho} = -\widehat{\alpha}_0 - \widehat{\alpha}_i + \langle \widehat{\alpha}_i, \widehat{\alpha}_0^\vee \rangle \widehat{\alpha}_0$.

   When $\mathfrak{g} = \mathfrak{sl}_2$ we get $\mathrm{H}^2(t\mathfrak{g}[t]) \simeq V(2\alpha_1)$, see the next example. When $\mathfrak{g} = \mathfrak{sl}_{l+1}$, $l \geq 2$, there are two possibilities for $i$: either $i = 1$ or $i = l$; then $\mathrm{H}^2(t\mathfrak{g}[t]) \simeq V(2\theta - \alpha_1) \oplus V(2\theta - \alpha_l)$. For $\mathfrak{g} \neq \mathfrak{sl}_{l+1}$ there is a unique possibility for $i$ and $\mathrm{H}^2(t\mathfrak{g}[t]) \simeq V(2\theta - \alpha_i)$.

   In all cases, except for $\mathfrak{sl}_2$, one can check that $\mathrm{H}^2(t\mathfrak{g}[t]) \simeq \bigwedge^2 \mathfrak{g}/\mathfrak{g}$. $\quad\square$

**Example 8.1.** Let $V(m)$ be the unique irreducible $\mathfrak{sl}_2$-module of dimension $m+1$. Then $\dim \mathrm{H}^n(\mathrm{Cur}\,\mathfrak{sl}_2, M_{V(m)}) = 1$ for $m = 2n, 2(n-3)$, and $= 0$ otherwise.

Let $\{e, f, h\}$ be the standard basis of $\mathfrak{sl}_2$. Then the module $V(2n)$ is isomorphic to $\mathrm{S}^n \mathfrak{sl}_2/(h^2 - 4ef)$. Note that $\mathrm{S}^\bullet \mathfrak{sl}_2/(h^2 - 4ef)$ is the coordinate ring of the nilpotent cone of $\mathfrak{sl}_2$. This description of $V(2n)$ allows us to give an explicit formula for the cocycles that represent $\mathrm{H}^n(\mathrm{Cur}\,\mathfrak{sl}_2, M_U)$ for any $\mathfrak{sl}_2$-module $U$. Namely, $\mathrm{H}^n(\mathrm{Cur}\,\mathfrak{sl}_2, M_U) \simeq \mathrm{Hom}_{\mathfrak{sl}_2}(\mathrm{S}^n \mathfrak{sl}_2/(h^2 - 4ef), U) \oplus \mathrm{Hom}_{\mathfrak{sl}_2}(\mathrm{S}^{n-3} \mathfrak{sl}_2/(h^2 - 4ef), U)$. The cocycle $\alpha \in C^n(\mathrm{Cur}\,\mathfrak{sl}_2, M_U)$ that corresponds to $(\varphi_n, \varphi_{n-3})$ is

$$\begin{aligned}
\alpha_{\lambda_1,\dots,\lambda_n}&(a_1, \dots, a_n) \\
&= \Pi(\lambda_1, \dots, \lambda_n)\,\varphi_n(a_1, \dots, a_n) \\
&+ \sum_{1 \leq i < j < k \leq n} c_3(a_i, a_j, a_k)\,\Pi(\lambda_1, \dots, \widehat{\lambda}_i, \dots, \widehat{\lambda}_j, \dots, \widehat{\lambda}_k, \dots, \lambda_n) \\
&\hspace{4cm} \times \varphi_{n-3}(a_1, \dots, \widehat{a}_i, \dots, \widehat{a}_j, \dots, \widehat{a}_k, \dots, a_n),
\end{aligned}$$

where $\Pi(\lambda_1, \ldots, \lambda_n) = \lambda_1 \cdots \lambda_n \prod_{1 \leq r < s \leq n} (\lambda_r - \lambda_s)$ and $c_3(a_1, a_2, a_3) = (a_1 \wedge a_2 \wedge a_3)/(e \wedge f \wedge h)$ is the generator of $H^3(\mathfrak{sl}_2) \simeq \mathbb{C}$.

*Remark* 8.2. Corollary 8.2 in light of Theorem 3.1 implies the following explicit description of the two-cocycles $c_\lambda(a, b)$ corresponding to abelian extensions

$$0 \to M_U \to \widetilde{A} \to A \to 0$$

of a current conformal algebra $A = \text{Cur } \mathfrak{g}$ by a current module $M_U$. (See the proof of Theorem 3.1, Part 4 for the notation.)

When $\mathfrak{g} \neq \mathfrak{sl}_2$, abelian extensions are parameterized by elements $\varphi \in \text{Hom}_{\mathfrak{g}}(\bigwedge^2 \mathfrak{g}/\mathfrak{g}, U)$ and the corresponding cocycle is $c_\lambda(a, b) = \lambda(\partial + \lambda)\varphi(a, b)$.

When $\mathfrak{g} = \mathfrak{sl}_2$, abelian extensions are parameterized by elements $\varphi \in \text{Hom}_{\mathfrak{sl}_2}(S^2 \mathfrak{sl}_2/(h^2 - 4ef), U) = \text{Hom}_{\mathfrak{sl}_2}(V(4), U)$ and $c_\lambda(a, b) = \lambda(\partial + \lambda)(\partial + 2\lambda)\varphi(a, b)$.

## 9. Hochschild, cyclic, and Leibniz cohomology

9.1. **Hochschild cohomology.** We can similarly define the notion of Hochschild cohomology by considering the following analogues of the basic and reduced complexes for an associative conformal algebra $A$ and a conformal bimodule $M$ over it, see Definition 1.4.

**Definition 9.1.** A *Hochschild n-cochain* ($n \in \mathbb{Z}_+$) *of an associative conformal algebra $A$ with coefficients in a conformal bimodule $M$ over it* is a $\mathbb{C}$-linear operator

$$\gamma : A^{\otimes n} \to M[\lambda_1, \ldots, \lambda_n]$$
$$a_1 \otimes \cdots \otimes a_n \mapsto \gamma_{\lambda_1, \ldots, \lambda_n}(a_1, \ldots, a_n),$$

satisfying the following condition:

    **Conformal antilinearity:** $\gamma_{\lambda_1, \ldots, \lambda_n}(a_1, \ldots, \partial a_i, \ldots, a_n)$
    $= -\lambda_i \gamma_{\lambda_1, \ldots, \lambda_n}(a_1, \ldots, a_i, \ldots, a_n)$ for all $i$.

The differential $d$ of a cochain $\gamma$ is defined as follows:

$(d\gamma)_{\lambda_1, \ldots, \lambda_{n+1}}(a_1, \ldots, a_{n+1})$

$= a_{1\lambda_1} \gamma_{\lambda_2, \ldots, \lambda_{n+1}}(a_2, \ldots, a_{n+1})$

$+ \sum_{i=1}^{n} (-1)^i \gamma_{\lambda_1, \ldots, \lambda_{i-1}, \lambda_i + \lambda_{i+1}, \lambda_{i+2}, \ldots, \lambda_{n+1}}(a_1, \ldots, a_{i-1}, a_{i\lambda_i} a_{i+1},$

$$a_{i+2}, \ldots, a_{n+1})$$

$+ (-1)^{n+1} \gamma_{\lambda_1, \ldots, \lambda_n}(a_1, \ldots, a_n)_{-\partial - \lambda_{n+1}} a_{n+1}.$

One can verify that the operator $d$ preserves the space of cochains and $d^2 = 0$. The cochains of an associative conformal algebra $A$ with coefficients in a bimodule $M$ form a complex $\widetilde{C}^\bullet = \widetilde{C}^\bullet(A, M)$, called the *basic Hochschild complex*. As in the Lie conformal algebra case, $\widetilde{C}^\bullet(A, M)$ carries the structure of a (left) $\mathbb{C}[\partial]$-module:

$$(9.1) \quad (\partial \cdot \gamma)_{\lambda_1, \ldots, \lambda_n}(a_1, \ldots, a_n) = \Big(\partial_M + \sum_{i=1}^n \lambda_i\Big)\gamma_{\lambda_1, \ldots, \lambda_n}(a_1, \ldots, a_n).$$

A straightforward computation shows that $d$ commutes with $\partial$. The quotient complex

$$C^\bullet(A, M) = \widetilde{C}^\bullet(A, M)/\partial\widetilde{C}^\bullet(A, M)$$

is called the *reduced Hochschild complex*, and its cohomology is called the *reduced Hochschild cohomology* $\mathrm{H}^\bullet(A, M)$, as opposed to the *basic Hochschild cohomology* $\widetilde{\mathrm{H}}^\bullet(A, M)$, which is the cohomology of the basic Hochschild complex $\widetilde{C}^\bullet$. Low-degree Hochschild cohomology groups can be interpreted along the lines of Section 3, *e.g.*, $\widetilde{\mathrm{H}}^0(A, M) = \{m \in M \mid a_\lambda m = m_{-\partial - \lambda}a \ \ \forall a \in A\}$.

*Remark 9.1.* One has obvious analogues of Theorems 6.1, 6.2, and Proposition 8.1 for Hochschild cohomology.

For a current conformal algebra $\mathrm{Cur}\, A$, where $A$ is a $\mathbb{C}$-algebra, the reduced Hochschild cohomology $\mathrm{H}^\bullet(\mathrm{Cur}\, A, \mathrm{Cur}\, A) \simeq \mathrm{H}^\bullet(A[t], A[t])$, by the analogue of Proposition 8.1. By the Hochschild–Kostant–Rosenberg Theorem [HKR], when $A$ is the algebra of regular functions on an affine nonsingular scheme $\mathrm{Spec}\, A$ over $\mathbb{C}$, the latter cohomology is isomorphic to the space of polyvector fields $\bigwedge_A T_A \otimes_\mathbb{C} \bigwedge_{\mathbb{C}[t]} \mathbb{C}[t]\partial_t$ on the product $\mathrm{Spec}\, A \times \mathbb{A}^1$, where $T_A = \mathrm{Der}(A, A)$ is the left module of vector fields on $\mathrm{Spec}\, A$.

*Remark 9.2.* For a commutative associative conformal algebra [K4], one can define the analogue of the Harrison cohomology by placing the symmetry condition on Hochschild cochains. This cohomology is the closest analogue of the one introduced in [KV] in the context of vertex algebras.

## 9.2. Cyclic cohomology.

In this section we define an analogue of cyclic cohomology, see [C, L1, Ts], for an associative conformal algebra $A$. Define its *basic cyclic cohomology* $\widetilde{\mathrm{HC}}^\bullet(A)$ as the cohomology of the complex $\widetilde{\mathrm{CC}}^\bullet$, where $\widetilde{\mathrm{CC}}^n$, $n \in \mathbb{Z}_+$, is the space of $\mathbb{C}$-linear operators

$$\gamma : A^{\otimes(n+1)} \to \mathbb{C}[\lambda_0, \ldots, \lambda_n]$$

$$a_0 \otimes \cdots \otimes a_n \mapsto \gamma_{\lambda_0,\ldots,\lambda_n}(a_0,\ldots,a_n),$$

satisfying the following conditions:

**Conformal antilinearity:** $\gamma_{\lambda_0,\ldots,\lambda_n}(a_0,\ldots,\partial a_i,\ldots,a_n)$
$= -\lambda_i \gamma_{\lambda_0,\ldots,\lambda_n}(a_0,\ldots,a_i,\ldots,a_n);$
**Cyclic invariance:** $\gamma_{\lambda_1,\ldots,\lambda_n,\lambda_0}(a_1,\ldots,a_n,a_0)$
$= (-1)^n \gamma_{\lambda_0,\ldots,\lambda_n}(a_0,\ldots,a_n).$

The differential $d$ of a cochain $\gamma$ is defined as follows:

$$(d\gamma)_{\lambda_0,\ldots,\lambda_{n+1}}(a_0,\ldots,a_{n+1})$$
$$= \sum_{i=0}^{n}(-1)^i \gamma_{\lambda_0,\ldots,\lambda_{i-1},\lambda_i+\lambda_{i+1},\lambda_{i+2},\ldots,\lambda_{n+1}}(a_0,\ldots,a_{i-1},a_{i\lambda_i}a_{i+1},$$
$$a_{i+2},\ldots,a_{n+1})$$
$$+ (-1)^{n+1}\gamma_{\lambda_{n+1}+\lambda_0,\ldots,\lambda_n}(a_{n+1\lambda_{n+1}}a_0,\ldots,a_n).$$

The *reduced cyclic cohomology* $\mathrm{HC}^\bullet(A)$ may be defined as the cohomology of the quotient complex by the action of $\partial$, as in the Hochschild case.

9.3. **Leibniz cohomology.** Nonlocal collections of formal distributions lead to the notion of a Leibniz conformal algebra, see Section 1:

**Definition 9.2.** A *Leibniz conformal algebra* is a $\mathbb{C}[\partial]$-module $A$ endowed with a $\lambda$-bracket $[a_\lambda b]$ which defines a conformally sesquilinear map $A \otimes A \to A[[\lambda]]$ satisfying the Jacobi identity as in Definition 1.1.

The difference from Definition 1.1 of a Lie conformal algebra is that the skew-symmetry axiom is omitted and formal power series in $\lambda$ are allowed. For a Leibniz conformal algebra $A$, the definition of a (*left*) *module* $M$ *over it* is the same as that for Lie conformal algebras, see Definition 1.2. The *space* $C^n(A,M)$ *of $n$-cochains of a Leibniz algebra $A$ with values in a module $M$* is the space of $\mathbb{C}$-linear operators

$$\gamma : A^{\otimes n} \to M[[\lambda_1,\ldots,\lambda_n]]$$
$$a_1 \otimes \cdots \otimes a_n \mapsto \gamma_{\lambda_1,\ldots,\lambda_n}(a_1,\ldots,a_n),$$

which are conformally antilinear:

$$\gamma_{\lambda_1,\ldots,\lambda_n}(a_1,\ldots,\partial a_i,\ldots,a_n) = -\lambda_i \gamma_{\lambda_1,\ldots,\lambda_n}(a_1,\ldots,a_i,\ldots,a_n).$$

The differential $d$ of a cochain $\gamma$ is defined as follows:

$$(d\gamma)_{\lambda_1,\ldots,\lambda_{n+1}}(a_1,\ldots,a_{n+1})$$

$$= \sum_{i=1}^{n+1} (-1)^{i+1} a_{i\,\lambda_i} \gamma_{\lambda_1,\ldots,\widehat{\lambda}_i,\ldots,\lambda_{n+1}}(a_1,\ldots,\widehat{a}_i,\ldots,a_{n+1})$$

$$+ \sum_{1 \le i < j \le n+1} (-1)^i \gamma_{\lambda_1,\ldots,\widehat{\lambda}_i,\ldots,\lambda_{j-1},\lambda_i+\lambda_j,\lambda_{j+1},\ldots,\lambda_{n+1}}(a_1,\ldots,\widehat{a}_i,$$

$$\ldots,a_{j-1},[a_{i\,\lambda_i}a_j],a_{j+1},\ldots,a_{n+1}),$$

where $\gamma$ is extended linearly over the polynomials in $\lambda_i$. One can verify that the operator $d$ preserves the space of cochains and $d^2 = 0$. The $n$-cochains, $n \in \mathbb{Z}_+$, of a Leibniz conformal algebra $A$ with coefficients in a module $M$ form a complex $\widetilde{C}^\bullet = \widetilde{C}^\bullet(A, M)$, called the *basic Leibniz complex*.

Equation (9.1) defines the structure of a left $\mathbb{C}[\partial]$-module on $\widetilde{C}^\bullet(A, M)$, which commutes with $d$. The quotient complex

$$C^\bullet(A, M) = \widetilde{C}^\bullet(A, M)/\partial\widetilde{C}^\bullet(A, M)$$

is called the *reduced Leibniz complex*. Its cohomology is called the *reduced Leibniz cohomology* $\mathrm{H}^\bullet(A, M)$, as opposed to the *basic Leibniz cohomology* $\widetilde{\mathrm{H}}^\bullet(A, M)$, which is the cohomology of the basic Leibniz complex $\widetilde{C}^\bullet$. These are conformal analogues of cohomology of Leibniz algebras, see [Cu, L1, L2].

## 10. Generalization to conformal algebras in higher dimensions

The theory of conformal algebras, their representations and cohomology has a straightforward generalization to the case when $\lambda$ is a vector.

Let us fix a natural number $r$. We replace a single indeterminate $\lambda$ by the vector $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r)$ and $\partial$ by $\boldsymbol{\partial} = (\partial_1, \ldots, \partial_r)$, and use the multi-index notation like $\boldsymbol{\lambda^{(m)}} = \lambda_1^{(m_1)} \cdots \lambda_r^{(m_r)}$ for $\boldsymbol{m} \in \mathbb{Z}^r$, $\delta(\boldsymbol{z} - \boldsymbol{w}) = \prod_i \delta(z_i - w_i)$, etc. Then everything from Sections 1–6 and 9 holds.

Examples of conformal algebras in $r$ indeterminates are provided by $r$-dimensional current algebras, cf. Examples 1.1 and 11.2. Other important examples are the Cartan algebras of vector fields. The structure theory of higher dimensional conformal algebras, including a classification of the simple ones, is currently being developed [BDK].

**Example 10.1.** The Lie algebra $W_r = \mathrm{Der}\,\mathbb{C}[x_1, x_1^{-1}, \ldots, x_r, x_r^{-1}]$ is spanned by the coefficients of the formal distributions

$$L^i(\boldsymbol{z}) = -\delta(\boldsymbol{z} - \boldsymbol{x})\partial_{x_i} \left( = -\sum_{\boldsymbol{m} \in \mathbb{Z}^r} x_1^{m_1} \cdots x_r^{m_r} \partial_{x_i} z_1^{-m_1-1} \cdots z_r^{-m_r-1} \right).$$

They are pairwise local, since

$$[L^i(\boldsymbol{z}), L^j(\boldsymbol{w})] = \partial_{w_i}\big(L^j(\boldsymbol{w})\delta(\boldsymbol{z}-\boldsymbol{w})\big) - \partial_{z_j}\big(L^i(\boldsymbol{w})\delta(\boldsymbol{z}-\boldsymbol{w})\big)$$
$$= \partial_{w_i}L^j(\boldsymbol{w})\delta(\boldsymbol{z}-\boldsymbol{w}) + L^j(\boldsymbol{w})\partial_{w_i}\delta(\boldsymbol{z}-\boldsymbol{w}) + L^i(\boldsymbol{w})\partial_{w_j}\delta(\boldsymbol{z}-\boldsymbol{w}).$$

The corresponding conformal algebra is $A = \bigoplus_{i=1}^{r}\mathbb{C}[\boldsymbol{\partial}]L^i$ with $\boldsymbol{\lambda}$-brackets

$$(10.1) \qquad [L^i{}_{\boldsymbol{\lambda}}L^j] = \partial_i L^j + \lambda_i L^j + \lambda_j L^i.$$

Its annihilation algebra is $W_{r-} = \mathrm{Der}\,\mathbb{C}[x_1,\ldots,x_r]$. For $r=1$ $A$ is the Virasoro conformal algebra Vir, see Example 1.2.

By Corollary 6.1, the cohomology of $W_{r-}$ with trivial coefficients is the same as the cohomology of the complex $\widetilde{C}^\bullet(A,\mathbb{C})$. The latter can be described as follows. Let $V$ be the vector space $\bigoplus_{i=1}^{r}\mathbb{C}L^i$. Every cochain $\alpha \in \widetilde{C}^n(A,\mathbb{C})$ is uniquely determined by its values on $V^{\otimes n}$:

$$\alpha\colon V^{\otimes n} \to \mathbb{C}[\boldsymbol{\lambda}_1,\ldots,\boldsymbol{\lambda}_n], \quad L^{k_1}\otimes\ldots\otimes L^{k_n} \mapsto \alpha^{k_1,\ldots,k_n}_{\boldsymbol{\lambda}_1,\ldots,\boldsymbol{\lambda}_n}.$$

The differential is given by the formula

$$(d\alpha)^{k_1,\ldots,k_{n+1}}_{\boldsymbol{\lambda}_1,\ldots,\boldsymbol{\lambda}_{n+1}} = \sum_{\substack{i,j=1\\i<j}}^{n+1}(-1)^{i+j}\lambda_{i,k_j}\alpha^{k_i,k_1,\ldots,\widehat{k_i},\ldots,\widehat{k_j},\ldots,k_{n+1}}_{\boldsymbol{\lambda}_i+\boldsymbol{\lambda}_j,\boldsymbol{\lambda}_1,\ldots,\widehat{\boldsymbol{\lambda}_i},\ldots,\widehat{\boldsymbol{\lambda}_j},\ldots,\boldsymbol{\lambda}_{n+1}}$$
$$- \sum_{\substack{i,j=1\\i<j}}^{n+1}(-1)^{i+j}\lambda_{j,k_i}\alpha^{k_j,k_1,\ldots,\widehat{k_i},\ldots,\widehat{k_j},\ldots,k_{n+1}}_{\boldsymbol{\lambda}_i+\boldsymbol{\lambda}_j,\boldsymbol{\lambda}_1,\ldots,\widehat{\boldsymbol{\lambda}_i},\ldots,\widehat{\boldsymbol{\lambda}_j},\ldots,\boldsymbol{\lambda}_{n+1}},$$

where $\lambda_{i,k}$ is the $k$th coordinate of the vector $\boldsymbol{\lambda}_i$.

The cohomology of the Lie algebra $W_{r-}$ with trivial coefficients was computed by Gelfand and Fuchs [GF2] (see also [F, §2.2.2]).

**Example 10.2.** The subalgebra of divergence 0 derivations is a formal distribution subalgebra of $W_r$. The corresponding conformal algebra is the following subalgebra of the algebra in Example 10.1: $\{\sum_i P_i(\boldsymbol{\partial})L^i \mid \sum_i P_i(\boldsymbol{\partial})\partial_i = 0\}$.

**Example 10.3.** The subalgebra $H_r$, $r = 2s$, of Hamiltonian derivations is a formal distribution subalgebra of $W_r$. The corresponding conformal algebra is of rank one: $A = \mathbb{C}[\boldsymbol{\partial}]L$ with $\boldsymbol{\lambda}$-bracket

$$[L_{\boldsymbol{\lambda}}L] = \sum_{i=1}^{s}(\lambda_{s+i}\partial_i L - \lambda_i \partial_{s+i}L).$$

Its annihilation algebra $H_{r-}$ is the Lie algebra of Hamiltonian derivations of $\mathbb{C}[x_1,\ldots,x_r]$.

The $n$th term of the complex $\widetilde{C}^\bullet(A, \mathbb{C})$, whose cohomology is $\mathrm{H}^\bullet(H_{r-})$, can be identified with the space of skew-symmetric polynomials in $\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_n$. The differential is given by the formula

$$(dP)(\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_{n+1})$$
$$= \sum_{\substack{i,j=1 \\ i<j}}^{n+1} (-1)^{i+j} (\boldsymbol{\lambda}_i | \boldsymbol{\lambda}_j) P(\boldsymbol{\lambda}_i + \boldsymbol{\lambda}_j, \boldsymbol{\lambda}_1, \dots, \widehat{\boldsymbol{\lambda}}_i, \dots, \widehat{\boldsymbol{\lambda}}_j, \dots, \boldsymbol{\lambda}_{n+1})$$

where $(\boldsymbol{\lambda}|\boldsymbol{\mu}) = \sum_{k=1}^{s} (\lambda_k \mu_{s+k} - \lambda_{s+k} \mu_k)$.

For $r = 2$ this complex has been known for quite a long time, but the computation of its cohomology is still an open problem (see [F, §2.2.7]).

**Example 10.4.** The subalgebra $K_r$, $r = 2s+1$, of contact derivations is also a formal distribution subalgebra of $W_r$, but the corresponding conformal algebra is of infinite rank. It is better viewed as a Lie* algebra of rank 1, see Section 12.

## 11. HIGHER DIFFERENTIALS

For the computation of the cohomology with non-trivial coefficients of the Lie algebras of vector fields, it is useful to know the cohomology of their subalgebras of vector fields which have a zero of certain order at the origin (see [F]). The argument of Theorem 6.1 can be generalized to give a complex which produces this cohomology.

Let $A$ be a conformal algebra in $r$ indeterminates which is a free $\mathbb{C}[\boldsymbol{\partial}]$-module: $A = \bigoplus_{i \in I} \mathbb{C}[\boldsymbol{\partial}] L^i$. For fixed $\boldsymbol{N} \in \mathbb{Z}_+^r$, we define $\mathfrak{g}_{\boldsymbol{N}} \equiv (\mathrm{Lie}\, A)_{\boldsymbol{N}}$ to be the subspace of the annihilation algebra $\mathfrak{g}_- = (\mathrm{Lie}\, A)_-$, spanned by $L_{\boldsymbol{m}}^i$, $i \in I$, $\boldsymbol{m} \geq \boldsymbol{N}$ (meaning that $m_i \geq N_i$ for each $i$). We are interested in the case when $\mathfrak{g}_{\boldsymbol{N}}$ is a *Lie subalgebra* of $\mathfrak{g}_-$. Note that this is always true when the entries of $\boldsymbol{N}$ are large enough. Indeed, we can write

$$(11.1) \qquad [L_{\boldsymbol{\lambda}}^i L^j] = \sum_{k \in I} C_{ij}^k (\boldsymbol{\lambda}, \boldsymbol{\partial}) L^k$$

for some uniquely determined polynomials $C_{ij}^k$. Then

$$(11.2) \qquad [L_{\boldsymbol{\lambda}}^i, L_{\boldsymbol{\mu}}^j] = \sum_{k \in I} C_{ij}^k (\boldsymbol{\lambda}, -\boldsymbol{\lambda} - \boldsymbol{\mu}) L_{\boldsymbol{\lambda}+\boldsymbol{\mu}}^k.$$

It follows that for large $\boldsymbol{N}$ the commutator $[\partial_{\boldsymbol{\lambda}}^{\boldsymbol{N}} L_{\boldsymbol{\lambda}}^i, \partial_{\boldsymbol{\mu}}^{\boldsymbol{N}} L_{\boldsymbol{\mu}}^j]$ can be expressed in terms of $\partial_{\boldsymbol{\lambda}+\boldsymbol{\mu}}^{\boldsymbol{N}} L_{\boldsymbol{\lambda}+\boldsymbol{\mu}}^k$. Since $\partial_{\boldsymbol{\lambda}}^{\boldsymbol{N}} L_{\boldsymbol{\lambda}}^i = \sum_{\boldsymbol{m} \geq \boldsymbol{N}} L_{\boldsymbol{m}}^i \boldsymbol{\lambda}^{(\boldsymbol{m}-\boldsymbol{N})}$, this shows that $\mathfrak{g}_{\boldsymbol{N}}$ is a Lie subalgebra of $\mathfrak{g}_-$.

Let $M$ be a module over the conformal algebra $A$. Then $M$ is a $\mathfrak{g}_-$-module and hence also a $\mathfrak{g}_{\boldsymbol{N}}$-module. Let $V$ be the vector space $\bigoplus_{i \in I} \mathbb{C}L^i$. As in Section 6, the $n$th term of the complex $C^\bullet(\mathfrak{g}_{\boldsymbol{N}}, M)$ can be identified with the space of linear maps

$$
(11.3) \qquad \begin{aligned}
\alpha : V^{\otimes n} &\to \mathbb{C}[\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_n] \otimes_{\mathbb{C}} M, \\
L^{k_1} \otimes \cdots \otimes L^{k_n} &\mapsto \alpha_{\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_n}^{k_1, \ldots, k_n},
\end{aligned}
$$

which are skew-symmetric with respect to simultaneous permutations of $k_i$'s and $\boldsymbol{\lambda}_i$'s. Using (11.2), one can easily write its differential $d_{\boldsymbol{N}}$.

**Example 11.1.** Let $A$ be the conformal algebra associated to the Lie algebra $W_r$ of vector fields (Example 10.1). Then for $\boldsymbol{N} \in \mathbb{Z}_+^r$, $W_{r,\boldsymbol{N}}$ is the Lie algebra of vector fields $\sum P_i(\boldsymbol{x})\partial_{x_i}$ such that all $P_i(\boldsymbol{x})$ are divisible by $\boldsymbol{x}^{\boldsymbol{N}}$. Equation (10.1) implies

$$
[\partial_{\boldsymbol{\lambda}}^{\boldsymbol{N}} L_{\boldsymbol{\lambda}}^i, \partial_{\boldsymbol{\mu}}^{\boldsymbol{N}} L_{\boldsymbol{\mu}}^j] = \partial_{\boldsymbol{\lambda}}^{\boldsymbol{N}} \lambda_j \partial_{\boldsymbol{\lambda}+\boldsymbol{\mu}}^{\boldsymbol{N}} L_{\boldsymbol{\lambda}+\boldsymbol{\mu}}^i - \partial_{\boldsymbol{\mu}}^{\boldsymbol{N}} \mu_i \partial_{\boldsymbol{\lambda}+\boldsymbol{\mu}}^{\boldsymbol{N}} L_{\boldsymbol{\lambda}+\boldsymbol{\mu}}^j.
$$

The differential $d_{\boldsymbol{N}}$ of the complex (11.3) is given by the formula

$$
\begin{aligned}
(d_{\boldsymbol{N}}\alpha)_{\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_{n+1}}^{k_1, \ldots, k_{n+1}} = &\sum_{i=1}^{n+1} (-1)^{i+1} \partial_{\boldsymbol{\lambda}_i}^{\boldsymbol{N}} L^{k_i} \alpha_{\boldsymbol{\lambda}_1, \ldots, \widehat{\boldsymbol{\lambda}}_i, \ldots, \boldsymbol{\lambda}_{n+1}}^{k_1, \ldots, \widehat{k}_i, \ldots, k_{n+1}} \\
&+ \sum_{\substack{i,j=1 \\ i<j}}^{n+1} (-1)^{i+j} \partial_{\boldsymbol{\lambda}_i}^{\boldsymbol{N}} \lambda_{i,k_j} \alpha_{\boldsymbol{\lambda}_i+\boldsymbol{\lambda}_j, \boldsymbol{\lambda}_1, \ldots, \widehat{\boldsymbol{\lambda}}_i, \ldots, \widehat{\boldsymbol{\lambda}}_j, \ldots, \boldsymbol{\lambda}_{n+1}}^{k_i, k_1, \ldots, \widehat{k}_i, \ldots, \widehat{k}_j, \ldots, k_{n+1}} \\
&- \sum_{\substack{i,j=1 \\ i<j}}^{n+1} (-1)^{i+j} \partial_{\boldsymbol{\lambda}_j}^{\boldsymbol{N}} \lambda_{j,k_i} \alpha_{\boldsymbol{\lambda}_i+\boldsymbol{\lambda}_j, \boldsymbol{\lambda}_1, \ldots, \widehat{\boldsymbol{\lambda}}_i, \ldots, \widehat{\boldsymbol{\lambda}}_j, \ldots, \boldsymbol{\lambda}_{n+1}}^{k_j, k_1, \ldots, \widehat{k}_i, \ldots, \widehat{k}_j, \ldots, k_{n+1}}.
\end{aligned}
$$

**Example 11.2.** Let $\mathfrak{g}$ be a Lie algebra. Then the current algebra $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[x_1, x_1^{-1}, \ldots, x_r, x_r^{-1}]$ is spanned by the coefficients of the pairwise local formal distributions $a(\boldsymbol{z}) := a \otimes \delta(\boldsymbol{x} - \boldsymbol{z})$, $a \in \mathfrak{g}$. They satisfy $[a(\boldsymbol{z}), b(\boldsymbol{w})] = [a, b](\boldsymbol{w})\delta(\boldsymbol{z} - \boldsymbol{w})$. The corresponding conformal algebra is $A = \mathbb{C}[\boldsymbol{\partial}] \otimes_{\mathbb{C}} \mathfrak{g}$ with $\boldsymbol{\lambda}$-brackets determined by

$$
[a_{\boldsymbol{\lambda}}b] = [a, b] \qquad \text{for} \quad a, b \in \mathfrak{g}.
$$

The annihilation algebra of $A$ is $\tilde{\mathfrak{g}}_- = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[\boldsymbol{x}]$ and for $\boldsymbol{N} \in \mathbb{Z}_+^r$ $\tilde{\mathfrak{g}}_{\boldsymbol{N}} = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[\boldsymbol{x}]\boldsymbol{x}^{\boldsymbol{N}}$. Now $C^n(\tilde{\mathfrak{g}}_{\boldsymbol{N}}, M)$ consists of all

$$
\begin{aligned}
\alpha : \mathfrak{g}^{\otimes n} &\to M[\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_n], \\
a_1 \otimes \cdots \otimes a_n &\mapsto \alpha_{\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_n}(a_1, \ldots, a_n),
\end{aligned}
$$

skew-symmetric with respect to simultaneous permutations of $a_i$'s and $\boldsymbol{\lambda}_i$'s. The differential $d_{\boldsymbol{N}}$ is given by

$$(d_{\boldsymbol{N}}\alpha)_{\boldsymbol{\lambda}_1,\ldots,\boldsymbol{\lambda}_{n+1}}(a_1,\ldots,a_{n+1})$$

$$= \sum_{i=1}^{n+1}(-1)^{i+1}\partial^{\boldsymbol{N}}_{\boldsymbol{\lambda}_i}a_{i\boldsymbol{\lambda}_i}\alpha_{\boldsymbol{\lambda}_1,\ldots,\widehat{\boldsymbol{\lambda}}_i,\ldots,\boldsymbol{\lambda}_{n+1}}(a_1,\ldots,\widehat{a_i},\ldots,a_{n+1})$$

$$+ \sum_{\substack{i,j=1\\i<j}}^{n+1}(-1)^{i+j}\,\partial^{\boldsymbol{N}}_{\boldsymbol{\lambda}_i}\alpha_{\boldsymbol{\lambda}_i+\boldsymbol{\lambda}_j,\boldsymbol{\lambda}_1,\ldots,\widehat{\boldsymbol{\lambda}}_i,\ldots,\widehat{\boldsymbol{\lambda}}_j,\ldots,\boldsymbol{\lambda}_{n+1}}\big([a_i,a_j],a_1,\ldots,\widehat{a_i},$$

$$\ldots,\widehat{a_j},\ldots,a_{n+1}\big).$$

*Remark* 11.1. It is easy to see that in the above examples the differentials satisfy $d_{\boldsymbol{N}}d_{\boldsymbol{N}'}+d_{\boldsymbol{N}'}d_{\boldsymbol{N}}=0$.

## 12. RELATION TO LIE* ALGEBRAS

The theory of conformal algebras is in many ways analogous to the theory of Lie algebras. The reason is that in fact conformal algebras can be considered as Lie algebras in certain pseudo-tensor categories, instead of the category of vector spaces. A pseudo-tensor category [BD] is a category equipped with "polylinear maps" and a way to compose them. This is enough to define the notions of Lie algebra, representations, cohomology, etc.

As an example, consider first the category $\mathcal{V}ec$ of vector spaces (over $\mathbb{C}$). For a finite non-empty set $I$ and a collection of vector spaces $\{L_i\}_{i\in I}$, $M$, we can define *polylinear maps* from $\{L_i\}_{i\in I}$ to $M$:

$$P_I(\{L_i\}_{i\in I}, M) := \mathrm{Hom}(\otimes_{i\in I}L_i, M).$$

This is a vector space with an action of the symmetric group $S_I$ on it.

For any surjection of finite sets $J \xrightarrow{\pi} I$ and a collection $\{K_j\}_{j\in J}$, we have the obvious compositions of polylinear maps

$$(12.1)\quad P_I(\{L_i\}_{i\in I}, M)\otimes\bigotimes_{i\in I}P_{J_i}(\{K_j\}_{j\in J_i}, L_i)\to P_J(\{K_j\}_{j\in J}, M),$$

$$(12.2)\quad \phi\times\{\psi_i\}_{i\in I}\mapsto\phi\circ(\otimes_{i\in I}\psi_i)\equiv\phi(\{\psi_i\}_{i\in I}),$$

where $J_i:=\pi^{-1}(i)$ for $i\in I$.

The compositions have the following properties:

**Associativity:** If $H \twoheadrightarrow J$, $\{F_h\}_{h\in H}$ is a family of objects and $\chi_j \in P_{H_j}(\{F_h\}_{h\in H_j}, K_j)$, then $\phi\big(\{\psi_i(\{\chi_j\}_{j\in J_i})\}_{i\in I}\big) = \big(\phi(\{\psi_i\}_{i\in I})\big)(\{\chi_j\}_{j\in J}) \in P_H(\{F_h\}_{h\in H}, M)$.

**Unit:** For any object $M$ there is an element $\mathrm{id}_M \in P_1(\{M\}, M)$ such that for any $\phi \in P_I(\{L_i\}_{i\in I}, M)$ one has $\mathrm{id}_M(\phi) = \phi(\{\mathrm{id}_{L_i}\}_{i\in I}) = \phi$.

**Equivariance:** The compositions (12.1) are equivariant with respect to the natural action of the symmetric group.

**Definition 12.1.** [BD]. A *pseudo-tensor category* is a class of objects $\mathcal{M}$ together with vector spaces $P_I(\{L_i\}_{i \in I}, M)$ on which the symmetric group $S_I$ acts, and composition maps (12.1), satisfying the above three properties.

*Remark* 12.1. For a pseudo-tensor category $\mathcal{M}$ and objects $L, M \in \mathcal{M}$, let $\mathrm{Hom}(L, M) = P_1(\{L\}, M)$. This gives a structure of an ordinary (additive) category on $\mathcal{M}$ and all $P_I$ are functors $(\mathcal{M}^\circ)^I \times \mathcal{M} \to \mathcal{V}ec$. (Here $\mathcal{M}^\circ$ denotes the dual category of $\mathcal{M}$.)

*Remark* 12.2. The notion of pseudo-tensor category is a straightforward generalization of the notion of operad. By definition, an *operad* is a pseudo-tensor category with only one object.

It is instructive to think of a polylinear map $\phi \in P_n(\{L_i\}_{i=1}^n, M)$ as an operation with $n$ inputs and 1 output, represented by the figure
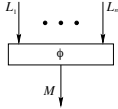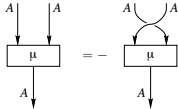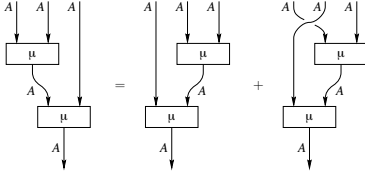


**Definition 12.2.** A *Lie algebra in a pseudo-tensor category* $\mathcal{M}$ is an object $A$ and $\mu \in P_2(\{A, A\}, A)$ with the following properties.

**Skew-symmetry:** $\mu = -\sigma_{12}\,\mu$, where $\sigma_{12} = (12) \in S_2$.

**Jacobi identity:** $\mu(\mu(\cdot, \cdot), \cdot) = \mu(\cdot, \mu(\cdot, \cdot)) - \sigma_{12}\,\mu(\cdot, \mu(\cdot, \cdot))$, where now $\sigma_{12} = (12)$ is viewed as an element of $S_3$.

Pictorially, the skew-symmetry and the Jacobi identity for a Lie algebra $(A, \mu)$ look as follows:
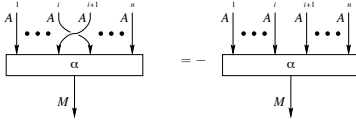
**Definition 12.3.** A *representation* of a Lie algebra $(A, \mu)$ is an object $M$ together with $\rho \in P_2(\{A, M\}, M)$ satisfying
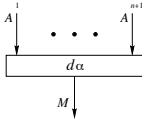
$$\rho(\mu(\cdot, \cdot), \cdot) = \rho(\cdot, \rho(\cdot, \cdot)) - \sigma_{12}\, \rho(\cdot, \rho(\cdot, \cdot)).$$

**Definition 12.4.** An *n-cochain* of a Lie algebra $(A, \mu)$ with coefficients in a module $(M, \rho)$ over it is a polylinear operation $\alpha \in P_n(\{A, \ldots, A\}, M)$ which is skew-symmetric, *i.e.*, satisfying



for all $i = 1, \ldots, n$.

The differential of a cochain is defined as follows:

The same computation as in the ordinary Lie algebra case shows that $d^2 = 0$. The cohomology of the resulting complex is called the (*reduced*) *cohomology of $A$ with coefficients in $M$* and is denoted by $\mathrm{H}^\bullet(A, M)$.

*Remark* 12.3. One can also define the notions of associative algebra or commutative algebra in a pseudo-tensor category, their representations and analogues of the Hochschild, cyclic, or Harrison cohomology.

**Example 12.1.** A Lie algebra in the category of vector spaces $\mathcal{V}ec$ is just an ordinary Lie algebra. The same is true for representations and cohomology.

**Example 12.2.** Let $D$ be a cocommutative bialgebra with comultiplication $\Delta$ and counit $\varepsilon$. Then the category $\mathcal{M}^l(D)$ of left $D$-modules is a symmetric tensor category. Hence, $\mathcal{M}^l(D)$ is a pseudo-tensor category with polylinear maps

(12.3) $$P_I(\{L_i\}_{i \in I}, M) := \mathrm{Hom}_D(\otimes_{i \in I} L_i, M).$$

A Lie algebra in the category $\mathcal{M}^l(D)$ is an ordinary Lie algebra which is also a left $D$-module and such that its bracket is a homomorphism of $D$-modules.

**Example 12.3.** Let $D$ be as in Example 12.2. We introduce a pseudo-tensor category $\mathcal{M}^*(D)$ with the same objects as $\mathcal{M}^l(D)$ but with another pseudo-tensor structure [BD]

$$(12.4) \qquad P_I(\{L_i\}_{i \in I}, M) := \mathrm{Hom}_{D^{\otimes I}}(\boxtimes_{i \in I} L_i, D^{\otimes I} \otimes_D M).$$

Here $\boxtimes_{i \in I}$ is the tensor product functor $\mathcal{M}^l(D)^I \to \mathcal{M}^l(D^{\otimes I})$. For $J \overset{\pi}{\twoheadrightarrow} I$ the composition of polylinear maps is defined as follows:

$$(12.5) \qquad \phi(\{\psi_i\}_{i \in I}) := \Delta^{(\pi)}(\phi) \circ (\boxtimes_{i \in I} \psi_i).$$

Here $\Delta^{(\pi)}$ is the functor $\mathcal{M}^l(D^{\otimes I}) \to \mathcal{M}^l(D^{\otimes J})$, $M \mapsto D^{\otimes J} \otimes_{D^{\otimes I}} M$ where $D^{\otimes I}$ acts on $D^{\otimes J}$ via the iterated comultiplication determined by $\pi$. The symmetric group $S_I$ acts on $P_I(\{L_i\}_{i \in I}, M)$ by simultaneously permuting the factors in $\boxtimes_{i \in I} L_i$ and $D^{\otimes I}$.

**Definition 12.5.** A *Lie$^*$ algebra* is a Lie algebra in the pseudo-tensor category $\mathcal{M}^*(D)$ defined above.

The following examples of Lie$^*$ algebras are important:

1. When $D = \mathbb{C}$ we recover Example 12.1.
2. For $D = \mathbb{C}[\partial]$ (with $\Delta(\partial) = \partial \otimes 1 + 1 \otimes \partial$, $\varepsilon(\partial) = 0$) we get exactly the notions of conformal algebras, conformal modules over them and the reduced cohomology theory introduced in this paper.
3. For $D = \mathbb{C}[\partial_1, \ldots, \partial_r]$ we get conformal algebras in $r$ indeterminates, see Section 10.
4. When $D = \mathbb{C}[\Gamma]$ is the group algebra of a group $\Gamma$, one obtains the $\Gamma$-conformal algebras studied in [GK].
5. Let $\Gamma$ be a subgroup of $\mathbb{C}^*$ and let $D = \mathbb{C}[\partial] \rtimes \mathbb{C}[\Gamma] = \bigoplus_{m \in \mathbb{Z}_+, \alpha \in \Gamma} \mathbb{C} \partial^m T_\alpha$ with multiplication $T_\alpha T_\beta = T_{\alpha\beta}$, $T_1 = 1$, $T_\alpha \partial T_\alpha^{-1} = \alpha\partial$ and comultiplication $\Delta(\partial) = \partial \otimes 1 + 1 \otimes \partial$, $\Delta(T_\alpha) = T_\alpha \otimes T_\alpha$. Then we get the $\Gamma$-conformal algebras studied in [BDK] (cf. [K4]).
6. Let now $D = \mathbb{C}[\partial] \times F(\Gamma)$, where $F(\Gamma)$ is the function algebra of a commutative group $\Gamma$. In other words, $D = \bigoplus_{m \in \mathbb{Z}_+, \alpha \in \Gamma} \mathbb{C} \partial^m \pi_\alpha$ with multiplication $\pi_\alpha \pi_\beta = \delta_{\alpha,\beta} \pi_\alpha$, $\partial \pi_\alpha = \pi_\alpha \partial$ and comultiplication $\Delta(\partial) = \partial \otimes 1 + 1 \otimes \partial$, $\Delta(\pi_\alpha) = \sum_{\gamma \in \Gamma} \pi_{\alpha\gamma^{-1}} \otimes \pi_\gamma$. Then one gets the notion of $\Gamma$-twisted conformal algebra [BDK] (cf. [K4]).
7. Let $D = U(\mathfrak{h})$ be the universal enveloping algebra of the Heisenberg Lie algebra $\mathfrak{h}$ with generators $a_i$, $b_i$, $c$ and the only non-zero commutation relations $[a_i, b_i] = c$ ($1 \leq i \leq s$). Let $A = DL$ be a free left

$D$-module of rank one. Define $\mu \in P_2(\{A, A\}, A)$ by the formula

$$\mu(L \boxtimes L) = \Big( \sum_{i=1}^{s} (a_i \otimes b_i - b_i \otimes a_i) + c \otimes 1 - 1 \otimes c \Big) \otimes_D L.$$

Then $(A, \mu)$ is a Lie algebra in the category $\mathcal{M}^*(D)$ with annihilation algebra $K_{r-}$, $r = 2s + 1$, cf. Example 10.2.

## 13. Open problems

There are a number of interesting problems which we left beyond the scope of this paper.

1. Compute the cohomology of $\operatorname{Cur} \mathfrak{g}$ with coefficients in $\operatorname{Chom}(M, N)$, where $M$ and $N$ are current modules. The same for the Virasoro conformal algebra, where $M$ and $N$ are modules of densities. Only $\mathrm{H}^1$ is known (see [CKW]), and the result is highly nontrivial.

2. Compute the cohomology of the general conformal algebra $\mathrm{gc}_N$ and its infinite-rank subalgebras, see [K4], with trivial coefficients. Is it true that $\mathrm{H}^\bullet(\mathrm{gc}_N, \mathbb{C}[\partial]^N)$ is trivial?

3. Study the relationship between $\mathrm{H}^\bullet(A, M)$ and $\mathrm{H}^\bullet(\operatorname{Lie} A, V(M))$. A mapping between the two is given in Section 6.3. Our computations show that in the case of a current or the Virasoro conformal algebra $A$, the image of $\mathrm{H}^\bullet(A, \mathbb{C})$ contains all generators of $\mathrm{H}^\bullet(\operatorname{Lie} A, \mathbb{C})$.

4. Compute the cohomology of conformal algebras in several indeterminates.

5. Compute the Hochschild and cyclic conformal cohomology of $\operatorname{Cend}(M)$. These problems are apparently related to 2.

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