GELFAND-KAZHDAN FORMAL GEOMETRY

1. Gelfand-Kazhdan formal geometry

In this section we review the theory of Gelfand-Kazhdan formal geometry and its use in natural constructions in differential geometry, organized in a manner somewhat different from the standard approaches. We emphasize the role of the frame bundle and jet bundles. We conclude with a treatment of the Atiyah class, which may be our only novel addition (although unsurprising) to the formalism.

We remark that from hereon we will work with complex manifolds and holomorphic vector bundles.

1.1. **A Harish-Chandra pair for the formal disk.** Let $\widehat{\mathcal{O}}_n$ denote the algebra of formal power series

$$\mathbb{C}[[t_1, \ldots, t_n]],$$

which we view as "functions on the formal n-disk $\widehat{D}^n$." It is filtered by powers of the maximal ideal $\mathfrak{m}_n = (t_1, \ldots, t_n)$, and it is the limit of the sequence of artinian algebras

$$\cdots \to \widehat{\mathcal{O}}_n/(t_1, \ldots, t_n)^k \to \cdots \widehat{\mathcal{O}}_n/(t_1, \ldots, t_n)^2 \to \widehat{\mathcal{O}}_n/(t_1, \ldots, t_n) \cong \mathbb{C}.$$

One can use the associated adic topology to interpret many of our constructions, but we will not emphasize that perspective here.

We use W_n to denote the Lie algebra of derivations of $\widehat{\mathcal{O}}_n$, which consists of first-order differential operators with formal power series coefficients:

$$\mathrm{W}_n = \left\{ \sum_{i=1}^{n} f_i \frac{\partial}{\partial t_i} \;:\; f_i \in \widehat{\mathcal{O}}_n \right\}.$$

The group GL_n also acts naturally on $\widehat{\mathcal{O}}_n$: for $M \in \mathrm{GL}_n$ and $f \in \widehat{\mathcal{O}}_n$,

$$(M \cdot f)(t) = f(Mt),$$

where on the right side we view t as an element of $\mathbb{C}^n$ and let M act linearly. In other words, we interpret GL_n as acting "by diffeomorphisms" on $\widehat{D}^n$ and then use the induced pullback action on functions on $\widehat{D}^n$. The actions of both W_n and GL_n intertwine with multiplication of power series, since "the pullback of a product of functions equals the product of the pullbacks."

1.1.1. *Formal automorphisms.* Let Aut_n be the group of filtration-preserving automorphisms of the algebra $\widehat{\mathcal{O}}_n$, which we will see is a pro-algebraic group. Explicitly, such an automorphism ϕ is a map of algebras that preserves the maximal ideal, so ϕ is specified by where it sends the generators $t_1, \ldots, t_n$ of the algebra. In other words, each $\phi \in \mathrm{Aut}_n$ consists of an n-tuple $(\phi_1, \ldots, \phi_n)$ such that each ϕ_i is in the maximal ideal generated by $(t_1, \ldots, t_n)$ and such that there exists an n-tuple $(\psi_1, \ldots, \psi_n)$ where the composite

$$\psi_j(\phi_1(t), \ldots, \phi_n(t)) = t_j$$

for every j (and likewise with ψ and ϕ reversed). This second condition can be replaced by verifying that the Jacobian matrix

$$Jac(\phi) = (\partial \phi_i / \partial t_j) \in \mathrm{Mat}_n(\widehat{\mathcal{O}}_n)$$

is invertible over $\widehat{\mathcal{O}}_n$, by a version of the inverse function theorem.

Note that this group is far from being finite-dimensional, so it does not fit immediately into the setting of HC-pairs described above. It is, however, a *pro*-Lie group in the following way. As each $\phi \in \mathrm{Aut}_n$ preserves the filtration on $\widehat{\mathcal{O}}_n$, it induces an automorphism of each partial quotient $\widehat{\mathcal{O}}_n / \mathfrak{m}_n^k$. Let $\mathrm{Aut}_{n,k}$ denote the image of Aut_n in $\mathrm{Aut}(\widehat{\mathcal{O}}_n / \mathfrak{m}_n^k)$; this group $\mathrm{Aut}_{n,k}$ is clearly a quotient of Aut_n. Note, for instance, that $\mathrm{Aut}_{n,1} = \mathrm{GL}_n$. Explicitly, an element ϕ of $\mathrm{Aut}_{n,k}$ is the collection of n-tuples $(\phi_1, \ldots, \phi_n)$ such that each ϕ_i is an element of $\mathfrak{m}_n / \mathfrak{m}_n^k$ and such that the Jacobian matrix $Jac(\phi)$ is invertible in $\widehat{\mathcal{O}}_n / \mathfrak{m}_n^k$. The group $\mathrm{Aut}_{n,k}$ is manifestly a finite dimensional Lie group, as the quotient algebra is a finite-dimensional vector space.

The group of automorphisms Aut_n is the pro-Lie group associated with the natural sequence of Lie groups

$$\cdots \to \mathrm{Aut}_{n,k} \to \mathrm{Aut}_{n,k-1} \to \cdots \to \mathrm{Aut}_{n,1} = \mathrm{GL}_n.$$

Let Aut_n^+ denote the kernel of the map $\mathrm{Aut}_n \to \mathrm{GL}_n$ so that we have a short exact sequence

$$1 \to \mathrm{Aut}_n^+ \to \mathrm{Aut}_n \to \mathrm{GL}_n \to 1.$$

In other words, for an element ϕ of Aut_n^+, each component ϕ_i is of the form $t_i + \mathcal{O}(t^2)$. The group Aut_n^+ is pro-nilpotent, hence contractible.

The Lie algebra of Aut_n is *not* the Lie algebra of formal vector fields W_n. A direct calculation shows that the Lie algebra of Aut_n is the Lie algebra $\mathrm{W}_n^0 \subset \mathrm{W}_n$ of formal vector fields with zero constant coefficient (i.e., that vanish at the origin of $\widehat{D}^n$).

Observe that the group GL_n acts on the Lie algebra W_n by the obvious linear "changes of frame." The Lie algebra $\mathrm{Lie}(\mathrm{GL}_n) = \mathfrak{gl}_n$ sits inside W_n as the linear vector fields

$$\left\{ \sum_{i,j} a_i^j t_i \frac{\partial}{\partial t_j} \ : \ a_j^i \in \mathbb{C} \right\}.$$

We record these compatibilities in the following statement.

Lemma 1.1. *The pair* $(\mathrm{W}_n, \mathrm{GL}_n)$ *form a Harish-Chandra pair.*

Proof. The only thing to check is that the differential of the action of GL_n corresponds with the adjoint action of $\mathfrak{gl}_n \subset \mathrm{W}_n$ on formal vector fields. This is by construction. □

1.2. **The coordinate bundle.** In this section we review the central object in the Gelfand-Kazhdan picture of formal geometry: the coordinate bundle.

1.2.1. Given a complex manifold, its *coordinate space* X^{coor} is the (infinite-dimensional) space parametrizing jets of holomorphic coordinates of X. (It is a pro-complex manifold, as we'll see.) Explicitly, a point in X^{coor} consists of a point $x \in X$ together with an ∞-jet class of a local biholomorphism $\phi : U \subset \mathbb{C}^n \to X$ sending a neighborhood U of the origin to a neighborhood of x such that $\phi(0) = x$.

There is a canonical projection map $\pi^{coor} : X^{coor} \to X$ by remembering only the underlying point in X. The group Aut_n acts on X^{coor} by "change of coordinates," i.e., by precomposing a local biholomorphism ϕ with an automorphism of the disk around the origin in $\mathbb{C}^n$. This action identifies π^{coor} as a principal bundle for the pro-Lie group Aut_n.

One way to formalize these ideas is to realize X^{coor} as a limit of finite-dimensional complex manifolds. Let X_k^{coor} be the space consisting of points $(x, [\phi]_k)$, where ϕ is a local biholomorphism as above and $[-]_k$ denotes taking its k-jet equivalence class. Let $\pi_k^{coor} : X_k^{coor} \to X$ be the projection. By construction, the finite-dimensional Lie group $\mathrm{Aut}_{n,k}$ acts on the fibers of the projection freely and transitively so that π_k^{coor} is a principal $\mathrm{Aut}_{n,k}$-bundle. The bundle $X^{coor} \to X$ is the limit of the sequence of principal bundles on X

$$\begin{tikzcd}
\cdots \arrow[r] & X_k^{coor} \arrow[r] \arrow[drrrr, "\pi_k^{coor}"'] & X_{k-1}^{coor} \arrow[r] \arrow[drrr, "\pi_{k-1}^{coor}"] & \cdots \arrow[r] & X_2^{coor} \arrow[r] \arrow[dr, "\pi_2^{coor}"] & X_1^{coor} \arrow[d, "\pi_1^{coor}"] \\
 & & & & & X.
\end{tikzcd}$$

In particular, note that the $\mathrm{GL}_n = \mathrm{Aut}_{n,1}$-bundle $\pi_1^{coor} : X_1^{coor} \to X$ is the frame bundle

$$\pi^{fr} : \mathrm{Fr}_X \to X,$$

i.e., the principal bundle associated to the tangent bundle of X.

1.2.2. *The Grothendieck connection.* We can also realize the Lie algebra W_n as an inverse limit. Recall the filtration on W_n by powers of the maximal ideal $\mathfrak{m}_n$ of $\widehat{\mathcal{O}}_n$. Let $\mathrm{W}_{n,k}$ denote the quotient $\mathrm{W}_n / \mathfrak{m}_n^{k+1} \mathrm{W}_n$. For instance, $\mathrm{W}_{n,1} = \mathfrak{aff}_n = \mathbb{C}^n \ltimes \mathfrak{gl}_n$, the Lie algebra of affine transformations of $\mathbb{C}^n$. We have $\mathrm{W}_n = \lim_{k \to \infty} \mathrm{W}_{n,k}$.

The Lie algebra of $\mathrm{Aut}_{n,k}$ is

$$\mathrm{W}_{n,k}^0 := \mathfrak{m}_n \cdot \mathrm{W}_n / \mathfrak{m}_n^{k+1} \mathrm{W}_n^0.$$

That is, the Lie algebra of vector fields vanishing at zero modulo the $k+1$ power of the maximal ideal. Thus, the principal $\mathrm{Aut}_{n,k}$-bundle $X_k^{coor} \to X$ induces an exact sequence of tangent spaces

$$\mathrm{W}_{n,k}^0 \to T_{(x,[\varphi]_k)} X^{coor} \to T_x X;$$

by using φ, we obtain a canonical isomorphism of tangent spaces $\mathbb{C}^n \cong T_0 \mathbb{C}^n \cong T_x X$. Combining these observations, we obtain an isomorphism

$$\mathrm{W}_{n,k} \cong T_{(x,[\varphi]_k)} X_k^{coor}.$$

In the limit $k \to \infty$ we obtain an isomorphism $\mathrm{W}_n \cong T_{(x,[\varphi]_\infty)} X^{coor}$.

Proposition 1.2 (Section 5 of [**?**], Section 3 of [**?**])**.** *There exists a canonical action of* W_n *on* X^{coor} *by holomorphic vector fields, i.e., there is a Lie algebra homomorphism*

$$\theta : \mathrm{W}_n \to \mathcal{X}^{hol}(X^{coor}).$$

Moreover, this action induces the isomorphism $\mathrm{W}_n \cong T_{(x,[\phi]_\infty)} X^{coor}$ *at each point.*

Here, $\mathcal{X}(X^{coor})$ is understood as the inverse limit of the finite-dimensional Lie algebras $\mathcal{X}(X_k^{coor})$.

The inverse of the map θ provides a connection one-form

$$\omega^{coor} \in \Omega^1_{hol}(X^{coor}; \mathrm{W}_n),$$

which we call the *universal Grothendieck connection* on X. As θ is a Lie algebra homomorphism, ω^{coor} satisfies the Maurer-Cartan equation

$$\partial\omega^{coor} + \frac{1}{2}[\omega^{coor}, \omega^{coor}] = 0. \tag{1}$$

Note that the proposition ensures that this connection is universal on all complex manifolds of dimension n and indeed pulls back along local biholomorphisms.

Remark 1.3. Both the pair $(\mathrm{W}_n, \mathrm{Aut})$ and the bundle $X^{coor} \to X$ together with ω^{coor} do not fit in our model for general Harish-Chandra descent above. They are, however, objects in a larger category of pro-Harish-Chandra pairs and pro-Harish-Chandra bundles, respectively. We do not develop this theory here, but it is inherent in the work of [**?**]. Indeed, by working with well-behaved representations for the pair $(\mathrm{W}_n, \mathrm{Aut})$, Gelfand, Kazhdan, and others use this universal construction to produce many of the natural constructions in differential geometry. As we remarked earlier, it is a kind of refinement of tensor calculus.

1.2.3. *A Harish-Chandra structure on the frame bundle.* Although the existence of the coordinate bundle X^{coor} is necessary in the remainder of this paper, it is convenient for us to use it in a rather indirect way. Rather, we will work with the frame bundle $\mathrm{Fr}_X \to X$ equipped with the structure of a module for the Harish-Chandra pair $(\mathrm{W}_n, \mathrm{GL}_n)$. The W_n-valued connection on Fr_X is induced from the Grothendieck connection above.

Definition 1.4. Let $\mathrm{Exp}(X)$ denote the quotient X^{coor}/GL_n. A holomorphic section of $\mathrm{Exp}(X)$ over X is called a *formal exponential*.

Remark 1.5. The space $\mathrm{Exp}(X)$ can be equipped with the structure of a principal Aut_n^+-bundle over X. This structure on $\mathrm{Exp}(X)$ depends on a choice of a section of the short exact sequence

$$1 \to \mathrm{Aut}_n^+ \to \mathrm{Aut}_n \to \mathrm{GL}_n \to 1.$$

It is natural to use the splitting determined by the choice of coordinates on the formal disk.

Note that Aut_n^+ is contractible, and so sections always exist. A formal exponential is useful because it equips the frame bundle with a $(\mathrm{W}_n, \mathrm{GL}_n)$-module structure, as follows.

Proposition 1.6. *A formal exponential σ pulls back to a GL_n-equivariant map $\tilde{\sigma} : \mathrm{Fr}_X \to X^{coor}$, and hence equips $(\mathrm{Fr}_x, \sigma^*\omega^{coor})$ with the structure of a principal $(\mathrm{W}_n, \mathrm{GL}_n)$-bundle with flat connection. Moreover, any two choices of formal exponential determine $(\mathrm{W}_n, \mathrm{GL}_n)$-structures on X that are gauge-equivalent.*

For a full proof, see [**?**], [**?**], or [**?**] but the basic idea is easy to explain.

Sketch of proof. The first assertion is tautological, since the data of a section is equivalent to such an equivariant map, but we explicate the underlying geometry. A map $\rho : \mathrm{Fr}_X \to X^{coor}$ assigns to each pair $(x, \mathbf{y}) \in \mathrm{Fr}_X$, with $x \in X$ and $\mathbf{y} : \mathbb{C}^n \xrightarrow{\cong} T_x X$ a linear frame, an ∞-jet of a biholomorphism $\phi : \mathbb{C}^n \to X$ such that $\phi(0) = x$ and $D\phi(0) = \mathbf{y}$. Being GL_n-equivariant ensures that these biholomorphisms are related by linear changes of coordinates on $\mathbb{C}^n$. In other words, a GL_n-equivariant map $\tilde{\sigma}$ describes how each frame on $T_x X$ exponentiates to a formal coordinate system around x, and so the associated section σ assigns a formal exponential map $\sigma(x)\colon T_xX \to X$ to each point x in X. (Here we see the origin of the name "formal exponential.")

The second assertion would be immediate if X^{coor} were a complex manifold, since the flat bundle structure would pull back, so all issues are about carefully working with pro-manifolds.

The final assertion is also straightforward: the space of sections is contractible since Aut_n^+ is contractible, so one can produce an explicit gauge equivalence. □

Remark 1.7. In **[?]** Willwacher provides a description of the space $\mathrm{Exp}(X)$ of *all* formal exponentials. He shows that it is isomorphic to the space of pairs (∇_0, Φ) where ∇_0 is a torsion-free connection on X for T_X and Φ is a section of the bundle

$$\mathrm{Fr}_X \times_{\mathrm{GL}_n} \mathrm{W}_n^3$$

where $\mathrm{W}_n^3 \subset \mathrm{W}_n$ is the subspace of formal vector fields whose coefficients are at least cubic. In particular, every torsion-free affine connection determines a formal exponential. The familiar case above that produces a formal coordinate from a connection corresponds to choosing the zero vector field.

Definition 1.8. A *Gelfand-Kazhdan structure* on the frame bundle $\mathrm{Fr}_X \to X$ of a complex manifold X of dimension n is a formal exponential σ, which makes Fr_X into a flat $(\mathrm{W}_n, \mathrm{GL}_n)$-bundle with connection one-form ω^σ, the pullback of ω^{coor} along the GL_n-equivariant lift $\tilde{\sigma} : \mathrm{Fr}_X \to X^{coor}$.

Example 1.9. Consider the case of an open subset $U \subset \mathbb{C}^n$. There are thus natural holomorphic coordinates $\{z_1, \ldots, z_n\}$ on U. These coordinates provides a natural choice of a formal exponential. Moreover, with respect to the isomorphism

$$\Omega^1_{hol}(\mathrm{Fr}_U; \mathrm{W}_n)^{\mathrm{GL}_n} \cong \Omega^1_{hol}(U; \mathrm{W}_n),$$

we find that the connection 1-form has the form

$$\omega^{coor} = \sum_{i=1}^n \mathrm{d}z_i \otimes \frac{\partial}{\partial t_i},$$

where the $\{t_i\}$ are the coordinates on the formal disk $\widehat{D}^n$.

A Gelfand-Kazhdan structure allows us to apply a version of Harish-Chandra descent, which will be a central tool in our work.

Although we developed Harish-Chandra descent on all flat $(\mathfrak{g}, K)$-bundles, it is natural here to restrict our attention to manifolds of the same dimension, as the notions of coordinate and affine bundle are dimension-dependent. Hence we replace the underlying category of all complex manifolds by a more restrictive setting.

Definition 1.10. Let Hol_n denote the category whose objects are complex manifolds of dimension n and whose morphisms are local biholomorphisms. In other words, a map $f : X \to Y$ in Hol_n is a map of complex manifolds such that each point $x \in X$ admits a neighborhood U on which $f|_U$ is biholomorphic with $f(U)$.

There is a natural inclusion functor $i : \mathrm{Hol}_n \to \mathrm{CplxMan}$ (not fully faithful) and the frame bundle Fr defines a section of the fibered category $i^*\mathrm{VB}$, since the frame bundle pulls back along local biholomorphisms. For similar reasons, the coordinate bundle is a pro-object in $i^*\mathrm{VB}$.

Definition 1.11. Let GK_n denote the category fibered over Hol_n whose objects are a Gelfand-Kazhdan structure — that is, a pair (X, σ) of a complex n-manifold and a formal exponential — and whose morphisms are simply local biholomorphisms between the underlying manifolds.

Note that the projection functor from GK_n to Hol_n is an equivalence of categories, since the space of formal exponentials is affine.

1.3. **The category of formal vector bundles.** For most of our purposes, it is convenient and sufficient to work with a small category of $(\mathrm{W}_n, \mathrm{GL}_n)$-modules that is manifestly well-behaved and whose localizations appear throughout geometry in other guises, notably as ∞-jet bundles of vector bundles on complex manifolds. (Although it would undoubtedly be useful, we will not develop here the general theory of modules for the Harish-Chandra pair $(\mathrm{W}_n, \mathrm{GL}_n)$, which would involve subtleties of pro-Lie algebras and their representations.)

We first start by describing the category of $(\mathrm{W}_n, \mathrm{GL}_n)$-modules that correspond to modules over the structure sheaf of a manifold. Note that $\widehat{\mathcal{O}}_n$ is the quintessential example of a commutative algebra object in the symmetric monoidal category of $(\mathrm{W}_n, \mathrm{GL}_n)$-modules, for any natural version of such a category. We consider modules that have actions of both the pair and the algebra $\widehat{\mathcal{O}}_n$ with obvious compatibility restrictions.

Definition 1.12. A *formal $\widehat{\mathcal{O}}_n$-module* is a vector space $\mathcal{V}$ equipped with

(i) the structure of a $(\mathrm{W}_n, \mathrm{GL}_n)$-module;
(ii) the structure of a $\widehat{\mathcal{O}}_n$-module;

such that

(1) for all $X \in \mathrm{W}_n$, $f \in \widehat{\mathcal{O}}_n$ and $v \in \mathcal{V}$ we have $X(f \cdot v) = X(f) \cdot v + f \cdot (X \cdot v)$;
(2) for all $A \in \mathrm{GL}_n$ we have $A(f \cdot v) = (A \cdot f) \cdot (A \cdot v)$, where A acts on f by a linear change of frame.

A morphism of formal $\widehat{\mathcal{O}}_n$-modules is a $\widehat{\mathcal{O}}_n$-linear map of $(\mathrm{W}_n, \mathrm{GL}_n)$-modules $f : \mathcal{V} \to \mathcal{V}'$. We denote this category by $\mathrm{Mod}^{\mathcal{O}_n}_{(\mathrm{W}_n, \mathrm{GL}_n)}$.

Just as the category of D-modules is symmetric monoidal via tensor over $\mathcal{O}$, we have the following result.

Lemma 1.13. *The category $\mathrm{Mod}^{\mathcal{O}_n}_{(\mathrm{W}_n, \mathrm{GL}_n)}$ is symmetric monoidal with respect to tensor over $\widehat{\mathcal{O}}_n$.*

Proof. The category of $\widehat{\mathcal{O}}_n$-modules is clearly symmetric monoidal by tensoring over $\widehat{\mathcal{O}}_n$. We simply need to verify that the Harish-Chandra module structures extend in a natural way, but this is clear. $\square$

We will often restrict ourselves to considering Harish-Chandra modules as above that are free as underlying $\widehat{\mathcal{O}}_n$-modules. Indeed, let

$$\mathrm{VB}_n \subset \mathrm{Mod}^{\mathcal{O}_n}_{(\mathrm{W}_n, \mathrm{GL}_n)}$$

be the full subcategory spanned by objects that are free and finitely generated as underlying $\widehat{\mathcal{O}}_n$-modules. Upon descent these will correspond to ordinary vector bundles and so we refer to this category as *formal vector bundles*.

The category of formal $\widehat{\mathcal{O}}_n$-modules has a natural symmetric monoidal structure by tensor product over $\widehat{\mathcal{O}}$. The Harish-Chandra action is extended by

$$X \cdot (s \otimes t) = (Xs) \otimes t + s \otimes (Xt).$$

This should not look surprising; it is the same formula for tensoring D-modules over $\mathcal{O}$.

The internal hom $\mathrm{Hom}_{\widehat{\mathcal{O}}}(\mathcal{V}, \mathcal{W})$ also provides a vector bundle on the formal disk, where the Harish-Chandra action is extended by

$$(X \cdot \phi)(v) = X \cdot (\phi(v)) - \phi(X \cdot v).$$

Observe that for any D-module M, we have an isomorphism

$$\mathrm{Hom}_D(\widehat{\mathcal{O}}, M) \cong \mathrm{Hom}_{\mathrm{W}_n}(\mathbb{C}, M)$$

since a map of $\widehat{D}$-modules out of $\widehat{\mathcal{O}}$ is determined by where it sends the constant function 1. Hence we find that there is a quasi-isomorphism

$$\mathbb{R}\mathrm{Hom}_D(\widehat{\mathcal{O}}, \mathcal{V}) \simeq \mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n; \mathcal{V}),$$

or more accurately a zig-zag of quasi-isomorphisms. Here $\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n; \mathcal{V})$ is the continuous cohomology of W_n with coefficients in $\mathcal{V}$. This is known as the *Gelfand-Fuks* cohomology of $\mathcal{V}$ and is what we use for the remainder of the paper.

This relationship extends to the GL_n-equivariant setting as well, giving us the following result.

Lemma 1.14. *There is a quasi-isomorphism*

$$\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n, \mathrm{GL}_n; \mathcal{V}) \simeq \mathbb{R}\mathrm{Hom}_D(\widehat{\mathcal{O}}, \mathcal{V})^{\mathrm{GL}_n - \mathrm{eq}},$$

where the superscript $\mathrm{GL}_n - \mathrm{eq}$ *denotes the* GL_n*-equivariant maps.*

Remark 1.15. One amusing way to understand this category is as Harish-Chandra descent to the formal n-disk itself. Consider the frame bundle $\widehat{\mathrm{Fr}} = \widehat{D}^n \times \mathrm{GL}_n \to \widehat{D}^n$ of the formal n-disk itself, which possesses a natural flat connection via the Maurer-Cartan form ω_{MC} on GL_n. Let $\rho : \mathrm{GL}_n \to \mathrm{GL}(V)$ be a finite-dimensional representation. Then the subcomplex of $\Omega^*(\widehat{\mathrm{Fr}}) \otimes V$ given by the basic forms is isomorphic to

$$\left(\Omega^*(\widehat{D}^n) \otimes V, \mathrm{d}_{dR} + \rho(\omega_{MC})\right).$$

This equips the associated bundle $\widehat{\mathrm{Fr}} \times^{\mathrm{GL}_n} V$ with a flat connection and hence makes its sheaf of sections a D-module on the formal disk.

Many of the important $\widehat{\mathcal{O}}_n$-modules we will consider simply come from linear tensor representations of GL_n. Given a finite-dimensional GL_n-representation V, we construct a $\widehat{\mathcal{O}}_n$-module $\mathcal{V} \in \mathrm{VB}_n$ as follows.

Consider the decreasing filtration of W_n by vanishing order of jets

$$\cdots \subset \mathfrak{m}_n^2 \cdot \mathrm{W}_n \subset \mathfrak{m}_n^1 \cdot \mathrm{W}_n \subset \mathrm{W}_n.$$

The induced map $\mathfrak{m}_n^1 \cdot \mathrm{W}_n \to \mathfrak{m}_n^1 \cdot \mathrm{W}_n / \mathfrak{m}_n^2 \cdot \mathrm{W}_n \cong \mathfrak{gl}_n$ allows us to restrict V to a $\mathfrak{m}_n^1 \cdot \mathrm{W}_n$-module. We then coinduce this module along the inclusion $\mathfrak{m}^1 \cdot \mathrm{W}_n \subset \mathrm{W}_n$ to get a W_n-module $\mathcal{V} = \mathrm{Hom}_{\mathfrak{m}_n^1 \cdot \mathrm{W}_n}(\mathrm{W}_n, V)$. There is an induced action of GL_n on $\mathcal{V}$. Indeed, as a GL_n-representation one has $\mathcal{V} \cong \widehat{\mathcal{O}}_n \otimes_{\mathbb{C}} V$. Moreover, this action is compatible with the W_n-module structure, so that $\mathcal{V}$ is actually a $(\mathrm{W}_n, \mathrm{GL}_n)$-module. Thus, the construction provides a functor from $\mathrm{Rep}_{\mathrm{GL}_n}$ to VB_n.

Definition 1.16. We denote by Tens_n the image of finite-dimensional GL_n-representations in VB_n along this functor. We call it the category of *formal tensor fields*.

As mentioned $\widehat{\mathcal{O}}_n$ is an example, associated to the trivial one-dimensional GL_n representation. Another key example is $\widehat{\mathcal{T}}_n$, the vector fields on the formal disk, which is associated to the defining GL_n representation $\mathbb{C}^n$; it is simply the adjoint representation of W_n. Other examples include $\widehat{\Omega}^1_n$, the 1-forms on the formal disk; it is the correct version of the coadjoint representation, and more generally the space of k-forms on the formal disk $\widehat{\Omega}^k_n$.

The category Tens_n can be interpreted in two other ways, as we will see in subsequent work.

(1) They are the ∞-jet bundles of tensor bundles: for a finite-dimensional GL_n-representation, construct its associated vector bundle along the frame bundle and take its ∞-jets.
(2) They are the flat vector bundles of finite-rank on the formal n-disk that are equivariant with respect to automorphisms of the disk. In other words, they are GL_n-equivariant D-modules whose underlying $\widehat{\mathcal{O}}$-module is finite-rank and free.

It should be no surprise that given a Gelfand-Kazhdan structure on the frame bundle of a non-formal n-manifold X, a formal tensor field descends to the ∞-jet bundle of the corresponding tensor bundle on X. The flat connection on this descent bundle is, of course, the Grothendieck connection on this ∞-jet bundle. (For some discussion, see section 1.3, pages 12-14, of [**?**].)

Note that the subcategories

$$\mathrm{Tens}_n \hookrightarrow \mathrm{VB}_n \hookrightarrow \mathrm{Mod}^{\mathcal{O}_n}_{(\mathrm{W}_n, \mathrm{GL}_n)}$$

inherit the symmetric monoidal structure constructed above.

1.4. **Gelfand-Kazhdan descent.** We will focus on defining descent for the category VB_n of formal vector bundles.

Fix an n-dimensional manifold X. The main result of this section is that the associated bundle construction along the frame bundle Fr_X,

$$\begin{array}{rccc} \mathrm{Fr}_X \times^{\mathrm{GL}_n} - : & \mathrm{Rep}(\mathrm{GL}_n)^{fin} & \to & \mathrm{VB}(X) \\ & V & \mapsto & \mathrm{Fr}_X \times^{\mathrm{GL}_n} V \end{array},$$

which builds a tensor bundle from a GL_n representation, arises from Harish-Chandra descent for $(\mathrm{W}_n, \mathrm{GL}_n)$. This result allows us to equip tensor bundles with interesting structures (e.g., a vertex algebra structure) by working $(\mathrm{W}_n, \mathrm{GL}_n)$-equivariantly on the formal n-disk. In other words, it reduces the problem of making a universal construction on all n-manifolds to the problem of making an equivariant construction on the formal n-disk, since the descent procedure automates extension from the formal to the global.

Note that every formal vector bundle $\mathcal{V} \in \mathrm{VB}_{(\mathrm{W}_n, \mathrm{GL}_n)}$ is naturally filtered via a filtration inherited from $\widehat{\mathcal{O}}_n$. Explicitly, we see that $\mathcal{V}$ is the limit of the sequence of finite-dimensional vector spaces

$$\cdots \to \widehat{\mathcal{O}}_n / \mathfrak{m}_n^k \otimes V \to \cdots \to \widehat{\mathcal{O}}_n / \mathfrak{m}_n \otimes V \cong V$$

where V is the underlying GL_n-representation. Each quotient $\widehat{\mathcal{O}}_n/\mathfrak{m}_n^k \otimes V$ is a module over $\mathrm{Aut}_{n,k}$, and hence determines a vector bundle on X by the associated bundle construction along X_k^{coor}. In this way, $\mathcal{V}$ produces a natural sequence of vector bundles on X and thus a pro-vector bundle on X.

Given a formal exponential σ on X, we obtain a GL_n-equivariant map from Fr_X to X_k^{coor} for every k, by composing the projection map $X^{coor} \to X_k^{coor}$ with the GL_n-equivariant map from Fr_X to X^{coor}.

Definition 1.17. *Gelfand-Kazhdan descent* is the functor

$$\mathrm{desc}_{\mathrm{GK}} : \mathrm{GK}_n^{\mathrm{op}} \times \mathrm{VB}_{(\mathrm{W}_n, \mathrm{GL}_n)} \to \mathrm{Pro}(\mathrm{VB})_{flat}$$

sending (Fr_X, σ) — a frame bundle with formal exponential — and a formal vector bundle $\mathcal{V}$ to the pro-vector bundle $\mathrm{Fr}_X \times^{\mathrm{GL}_n} \mathcal{V}$ with flat connection induced by the Grothendieck connection.

This functor is, in essence, Harish-Chandra descent, but in a slightly exotic context. It has several nice properties.

Lemma 1.18. *For any choice of Gelfand-Kazhdan structure* (Fr_X, σ)*, the descent functor* $\mathrm{desc}_{\mathrm{GK}}((\mathrm{Fr}_X, \sigma), -)$ *is lax symmetric monoidal.*

Proof. For every $\mathcal{V}, \mathcal{W}$ in $\mathrm{VB}_{(\mathrm{W}_n, \mathrm{GL}_n)}$, we have natural maps

$$(\Omega^*(\mathrm{Fr}_X) \otimes \mathcal{V})_{basic} \otimes (\Omega^*(\mathrm{Fr}_X) \otimes \mathcal{W})_{basic} \to (\Omega^*(\mathrm{Fr}_X) \otimes (\mathcal{V} \otimes \mathcal{W}))_{basic} \to (\Omega^*(\mathrm{Fr}_X) \otimes (\mathcal{V} \otimes_{\widehat{\mathcal{O}}_n} \mathcal{W}))_{basic}$$

and the composition provides the natural transformation producing the lax symmetric monoidal structure. □

In particular, we observe that the de Rham complex of $\mathrm{desc}_{\mathrm{GK}}((\mathrm{Fr}_X, \sigma), \widehat{\mathcal{O}}_n)$ is a commutative algebra object in $\Omega^*(X)$-modules. As every object of $\mathrm{VB}_{(\mathrm{W}_n, \mathrm{GL}_n)}$ is an $\widehat{\mathcal{O}}_n$-module and the morphisms are $\widehat{\mathcal{O}}_n$-linear, we find that descent actually factors through the category of $\mathrm{desc}_{\mathrm{GK}}((\mathrm{Fr}_X, \sigma), \widehat{\mathcal{O}}_n)$-modules. In sum, we have the following.

Lemma 1.19. *The descent functor* $\mathrm{desc}_{\mathrm{GK}}((\mathrm{Fr}_X, \sigma), -)$ *factors as a composite*

$$\mathrm{VB}_n \xrightarrow{\widetilde{\mathrm{desc}}_{\mathrm{GK}}((\mathrm{Fr}_X, \sigma), -)} \mathrm{Mod}_{\mathrm{desc}_{\mathrm{GK}}((\mathrm{Fr}_X, \sigma), \widehat{\mathcal{O}}_n)} \xrightarrow{forget} \mathrm{VB}_{flat}(X)$$

and the functor $\widetilde{\mathrm{desc}}_{\mathrm{GK}}((\mathrm{Fr}_X, \sigma), -)$ *is symmetric monoidal.*

As before, we let $\mathcal{D}\mathrm{esc}_{\mathrm{GK}}$ denote the associated local system obtained from $\mathrm{desc}_{\mathrm{GK}}$ by taking horizontal sections. This functor is well-known: it recovers the tensor bundles on X.

If $E \to X$ is a holomorphic vector bundle on X we denote by $J_{hol}^{\infty}(E)$ the holomorphic ∞-jet bundle of E. If E_0 is the fiber of E over a point $x \in X$, then the fiber of this pro-vector bundle over x can be identified with

$$J_{hol}^{\infty}(E)|_x \cong E_0 \times \mathbb{C}[[t_1, \ldots, t_n]].$$

This pro-vector bundle has a canonical flat connection.

Proposition 1.20. *For $\mathcal{V} \in \mathrm{VB}_n$ corresponding to the GL_n-representation V, there is a natural isomorphism of flat pro-vector bundles*

$$\mathrm{desc}_{\mathrm{GK}}((\mathrm{Fr}(X), \omega^\sigma), \mathcal{V}) \cong J^\infty_{hol}(\mathrm{Fr}_X \times^{\mathrm{GL}_n} V)$$

In other words, the functor of descent along the frame bundle is naturally isomorphic to the functor of taking ∞-jets of the associated bundle construction.

As a corollary, we see that the associated sheaf of flat sections is

$$\mathcal{D}\mathrm{esc}_{\mathrm{GK}}(\omega^\sigma, \mathcal{V}) \cong \Gamma_{hol}(\mathrm{Fr}_X \times^{\mathrm{GL}_n} V)$$

where $\Gamma_{hol}(-)$ denotes the space of holomorphic sections.

In other words, Gelfand-Kazhdan descent produces every tensor bundle. For example, for the defining representation $V = \mathbb{C}^n$ of GL_n, we have $\mathcal{V} = \widehat{\mathcal{T}}_n$, i.e., the vector fields on the formal disk viewed as the adjoint representation of W_n. Under Gelfand-Kazhdan descent, it produces the tangent bundle T on Hol_n.

1.5. **Formal characteristic classes.**

1.5.1. *Recollection.* In [**?**], Atiyah examined the obstruction — which now bears his name — to equipping a holomorphic vector bundle with a holomorphic connection from several perspectives. To start, as he does, we take a very structural approach. He begins by constructing the following sequence of vector bundles (see Theorem 1).

Definition 1.21. Let G be a complex Lie group. Let $E \to X$ be a holomorphic vector bundle on a complex manifold and $\mathcal{E}$ its sheaf of sections. The *Atiyah sequence* of E is the exact sequence holomorphic vector bundles given by

$$0 \to E \otimes T^*X \to J^1(E) \to E \to 0,$$

where $J^1(E)$ the bundle of *first-order* jets of E The *Atiyah class* is the element $\mathrm{At}(E) \in \mathrm{H}^1(X, \Omega^1_X \otimes \mathrm{End}_{\mathcal{O}_X}(\mathcal{E}))$ associated to the extension above.

Remark 1.22. Taking linear duals we see tha above short exact sequence is equivalent to one of the form

$$0 \to \mathrm{End}(E) \to \mathrm{A}(E) \to TX \to 0$$

where $\mathrm{A}(E)$ is the so-called *Atiyah bundle* associated to E.

We should remark that the sheaf $\mathcal{A}(E)$ of holomorphic sections of the Atiyah bundle $\mathrm{A}(E)$ is a Lie algebra by borrowing the Lie bracket on vector fields. By inspection, the Atiyah sequence of sheaves (by taking sections) is a sequence of Lie algebras; in fact, $\mathcal{A}(E)$ is a central example of a Lie algebroid, as the quotient map to vector fields $\mathcal{T}_X$ on X is an anchor map.

Atiyah also examined how this sequence relates to the Chern theory of connections.

Proposition 1.23. *A* holomorphic connection *on E is a splitting of the Atiyah sequence (as holomorphic vector bundles).*

Atiyah's first main result in the paper is the following.

Proposition 1.24 (Theorem 2, [**?**]). *A connection exists on E if and only if the Atiyah class* $\mathrm{At}(E)$ *vanishes.*

He observes immediately after this statement that the construction is functorial in maps of bundles. Later, he finds a direct connection between the Atiyah class and the curvature of a smooth connection. A smooth connections always exists (i.e., the sequence splits as smooth vector bundles, not necessarily holomorphically), and one is free to choose a connection such that the local 1-form only has Dolbeault type $(1,0)$, i.e., is an element in $\Omega^{1,0}(X;\mathrm{End}(E))$. In that case, the $(1,1)$-component $\Theta^{1,1}$ of the curvature Θ is a 1-cocycle in the Dolbeault complex $(\Omega^{1,*}(X;\mathrm{End}(E)),\overline{\partial})$ for $\mathrm{End}(E)$ and its cohomology class $[\Theta^{1,1}]$ is the Atiyah class $\mathrm{At}(E)$. In consequence, Atiyah deduces the following.

Proposition 1.25. *For X a compact Kähler manifold, the kth Chern class $c_k(E)$ of E is given by the cohomology class of $(2\pi i)^{-k}S_k(\mathrm{At}(E))$, where S_k is the kth elementary symmetric polynomial, and hence only depends on the Atiyah class.*

This assertion follows from the degeneracy of the Hodge-to-de Rham spectral sequence. More generally, the term $(2\pi i)^{-k}S_k(\mathrm{At}(E))$ agrees with the image of the kth Chern class in the Hodge cohomology $H^k(X;\Omega^k_{hol})$.

The functoriality of the Atiyah class means that it makes sense not just on a fixed complex manifold, but also on the larger sites Hol_n and GK_n. We thus immediately obtain from Atiyah the following notion.

Definition 1.26. For each $V \in \mathrm{VB}(\mathrm{Hol}_n)$, the *Atiyah class* $\mathrm{At}(V)$ is the equivalence class of the extension of the tangent bundle T by $\mathrm{End}(V)$ given by the Atiyah sequence.

Moreover, we have the following.

Lemma 1.27. *The cohomology class of $(2\pi i)^{-k}S_k(\mathrm{At}(V))$ provides a section of the sheaf $H^k(X;\Omega^k_{hol})$. On any compact Kähler manifold, it agrees with $c_k(V)$.*

1.5.2. *The formal Atiyah class.* We now wish to show that Gelfand-Kazhdan descent sends an exact sequence in $\mathrm{VB}_{(\mathrm{W}_n,\mathrm{GL}_n)}$ to an exact sequence in $\mathrm{VB}(\mathrm{GK}_n)$ (and hence in $\mathrm{VB}(\mathrm{Hol}_n)$). It will then remain to verify that for each tensor bundle on Hol_n, there is an exact sequence over the formal n-disk that descends to the Atiyah sequence for that tensor bundle.

We will use the notation $\mathrm{desc}_{\mathrm{GK}}(\mathcal{V})$ to denote the functor $\mathrm{desc}_{\mathrm{GK}}(-,\mathcal{V}) : \mathrm{GK}_n^{\mathrm{op}} \to \mathrm{Pro}(\mathrm{VB})_{flat}$, since we want to focus on the sheaf on GK_n (or Hol_n) defined by each formal vector bundle $\mathcal{V}$. Taking flat sections we get an $\mathcal{O}$-module $\mathcal{D}\mathrm{esc}_{\mathrm{GK}}(\mathcal{V})$ which is locally free of finite rank and so determines an object in $\mathrm{VB}(\mathrm{GK}_n)$.

Lemma 1.28. *If*

$$\mathcal{A} \to \mathcal{B} \to \mathcal{C}$$

is an exact sequence in $\mathrm{VB}_{(\mathrm{W}_n,\mathrm{GL}_n)}$*, then*

$$\mathcal{D}\mathrm{esc}_{\mathrm{GK}}(\mathcal{A}) \to \mathcal{D}\mathrm{esc}_{\mathrm{GK}}(\mathcal{B}) \to \mathcal{D}\mathrm{esc}_{\mathrm{GK}}(\mathcal{C})$$

is exact in $\mathrm{VB}(\mathrm{GK}_n)$.

Proof. A sequence of vector bundles is exact if and only if the associated sequence of $\mathcal{O}$-modules is exact (i.e., the sheaves of sections of the vector bundles). But a sequence of sheaves is exact if and only if it is exact stalkwise. Observe that there is only one point at which to compute a stalk in the site Hol_n, since every point $x \in X$ has a small neighborhood isomorphic to a small neighborhood of $0 \in \mathbb{C}^n$. As we are working in an analytic setting, the stalk of a $\mathcal{O}$-module at a point x injects into the ∞-jet at x. Hence, it suffices to verifying the exactness of the sequence of ∞-jets. Hence, we consider the ∞-jet at $0 \in \mathbb{C}^n$ of the sequence $\mathrm{desc}_{\mathrm{GK}}(A) \to \mathrm{desc}_{\mathrm{GK}}(B) \to \mathrm{desc}_{\mathrm{GK}}(C)$. But this sequence is simply $A \to B \to C$, which is exact by hypothesis. $\square$

Corollary 1.29. *There is a canonical map from* $\mathrm{Ext}^1_{(\mathrm{W}_n,\mathrm{GL}_n)}(\mathcal{B},\mathcal{A})$ *to* $\mathrm{Ext}^1_{\mathrm{GK}_n}(\mathcal{D}\mathrm{esc}_{\mathrm{GK}}(\mathcal{B}),\mathcal{D}\mathrm{esc}_{\mathrm{GK}}(\mathcal{A}))$.

In particular, once we produce the $(\mathrm{W}_n,\mathrm{GL}_n)$-Atiyah sequence for a formal tensor field $\mathcal{V}$, we will have a very local model for the Atiyah class living in $\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n,\mathrm{GL}_n;\widehat{\Omega}^1_n \otimes_{\widehat{\mathcal{O}}_n} \mathrm{End}_{\widehat{\mathcal{O}}_n}(\mathcal{V}))$.

1.5.3. *The formal Atiyah sequence.* Let $\mathcal{V}$ be a formal vector bundle. We will now construct the "formal" Atiyah sequence associated to $\mathcal{V}$. First, we need to define the $(\mathrm{W}_n,\mathrm{GL}_n)$-module of *first order jets* of $\mathcal{V}$. Let's begin by recalling the construction of jets in ordinary geometry.

If X is a manifold, we have the diagonal embedding $\Delta : X \hookrightarrow X \times X$. Correspondingly, there is the ideal sheaf $\mathcal{I}_\Delta$ on $X \times X$ of functions vanishing along the diagonal. Let $X^{(k)}$ be the ringed space $(X,\mathcal{O}_{X\times X}/\mathcal{I}^k_\Delta)$ describing the kth order neighborhood of the diagonal in $X \times X$. Let $\Delta^{(k)} : X^{(k)} \to X \times X$ denote the natural map of ringed spaces. The projections $\pi_1,\pi_2 : X \times X \to X$ compose with $\Delta^{(k)}$ to define maps $\pi_1^{(k)},\pi_2^{(k)} : X^{(k)} \to X$. Given an $\mathcal{O}_X$-module $\mathcal{V}$, "push-and-pull" along these projections,

$$J^k_X(\mathcal{V}) = (\pi_1^{(k)})_*(\pi_2^{(k)})^*\mathcal{V},$$

defines the $\mathcal{O}_X$-module of kth order jets of $\mathcal{V}$.

There is a natural adaptation in the formal case. The diagonal map corresponds to an algebra map $\Delta^* : \widehat{\mathcal{O}}_{2n} \to \widehat{\mathcal{O}}_n$. Fix coordinatizations $\widehat{\mathcal{O}}_n = \mathbb{C}[[t_1,\ldots,t_n]]$ and $\widehat{\mathcal{O}}_{2n} = \mathbb{C}[[t'_1,\ldots,t'_n,t''_1,\ldots,t''_n]]$. Then the map is given by $\Delta^*(t'_i) = \Delta^*(t''_i) = t_i$.

Let $\widehat{I}_n = \ker(\Delta^*) \subset \widehat{\mathcal{O}}_{2n}$ be the ideal given by the kernel of Δ^*. For each k there is a quotient map

$$\Delta^{(k)*} : \widehat{\mathcal{O}}_{2n} \to \widehat{\mathcal{O}}_{2n}/\widehat{I}^{k+1}_n,$$

The projection maps have the form

$$\pi_1^{(k)*},\pi_2^{(k)*} : \widehat{\mathcal{O}}_n \to \widehat{\mathcal{O}}_{2n}/\widehat{I}^{k+1}_n,$$

which in coordinates are $\pi_1^*(t_i) = t'_i$ and $\pi_2^*(t_i) = t''_i$.

Definition 1.30. Let $\mathcal{V}$ be a formal vector bundle on $\widehat{D}^n$. Consider the $\widehat{\mathcal{O}}_{2n}/\widehat{I}^{k+1}_n$-module $\mathcal{V} \otimes_{\widehat{\mathcal{O}}_n} \left(\widehat{\mathcal{O}}_{2n}/\widehat{I}^{k+1}_n\right)$, where the tensor product uses the $\widehat{\mathcal{O}}_n$-module structure on the quotient $\widehat{\mathcal{O}}_{2n}/\widehat{I}^{k+1}_n$ coming from the map $\pi_2^{(k)*}$. We define the *kth order formal jets of* $\mathcal{V}$, denoted $J^k(\mathcal{V})$, as the restriction of this $\widehat{\mathcal{O}}_{2n}/\widehat{I}^{k+1}_n$-module to a $\widehat{\mathcal{O}}_n$-module using the map $\pi_1^{(k)*} : \widehat{\mathcal{O}}_n \to \widehat{\mathcal{O}}_{2n}/\widehat{I}^{k+1}_n$.

Lemma 1.31. *For any* $\mathcal{V} \in \mathrm{VB}_n$ *the kth order formal jets* $J^k(\mathcal{V})$ *is an element of* VB_n.

Proof. For $\mathcal{V}$ in VB_n there is an induced action of $(\mathrm{W}_n, \mathrm{GL}_n)$ on the tensor product $\mathcal{V} \otimes_{\widehat{\mathcal{O}}_n} \widehat{\mathcal{O}}_{2n}/\widehat{I}_n^{k+1}$. For fixed k we see that $\widehat{\mathcal{O}}_{2n}/\widehat{I}_n^{k+1}$ is finite rank as a $\widehat{\mathcal{O}}_n$ module. Thus it is immediate that this module satisfies the conditions of a formal vector bundle. □

As a $\mathbb{C}$-linear vector space we have $J^1(\mathcal{V}) = \mathcal{V} \oplus (\mathcal{V} \otimes_{\widehat{\mathcal{O}}_n} \widehat{\Omega}^1_n)$. For $f \in \widehat{\mathcal{O}}_n$ and $(v, \beta) \in \mathcal{V} \oplus (\mathcal{V} \otimes \widehat{\Omega}^1_n)$, the $\widehat{\mathcal{O}}_n$-module structure is given by

$$f \cdot (v, \beta) = (fv, (f\beta + v \otimes \mathrm{d}f)).$$

(This formula is the formal version of Atiyah's description in Section 4 of **[?]**, where he uses the notation $\mathcal{D}$.) The following is proved in exact analogy as in the non-formal case which can also be found in Section 4 of **[?]**, for instance.

Proposition 1.32. *For any $\mathcal{V} \in \mathrm{VB}_{(\mathrm{W}_n, \mathrm{GL}_n)}$, the $\widehat{\mathcal{O}}_n$-module $J^1(\mathcal{V})$ has a compatible action of the pair $(\mathrm{W}_n, \mathrm{GL}_n)$ and hence determines an object in $\mathrm{VB}_{(\mathrm{W}_n, \mathrm{GL}_n)}$. Moreover, it sits in a short exact sequence of formal vector bundles*

$$\mathcal{V} \otimes \widehat{\Omega}^1_n \to J^1(\mathcal{V}) \to \mathcal{V}. \tag{2}$$

Finally, the Gelfand-Kazhdan descent of this short exact sequence is isomorphic to the Atiyah sequence

$$\mathcal{D}\mathrm{esc}_{\mathrm{GK}}(\mathcal{V}) \otimes \Omega^1_{hol} \to J^1 \mathcal{D}\mathrm{esc}_{\mathrm{GK}}(\mathcal{V}) \to \mathcal{D}\mathrm{esc}_{\mathrm{GK}}(\mathcal{V}).$$

In particular, $J^1 \mathrm{desc}_{\mathrm{GK}}(\mathcal{V}) = \mathrm{desc}_{\mathrm{GK}}(J^1\mathcal{V})$.

We henceforth call the sequence (2) *the formal Atiyah sequence* for $\mathcal{V}$.

Remark 1.33. Note that $J^1(\mathcal{V})$ is an element of the category VB_n but it is *not* a formal tensor field. That is, it does not come from a linear representation of GL_n via coinduction.

Remark 1.34. A choice of a formal coordinate defines a splitting of the first-order jet sequence as $\widehat{\mathcal{O}}_n$-modules. If we write $\mathcal{V} = \widehat{\mathcal{O}}_n \otimes_{\mathbb{C}} V$, then one defines

$$j^1 : \mathcal{V} \to J^1\mathcal{V} \ \ , \ \ f \otimes_{\mathbb{C}} v \mapsto (f \otimes_{\mathbb{C}} v, (1 \otimes_{\mathbb{C}} v) \otimes_{\mathcal{O}} \mathrm{d}f).$$

It is a map of $\widehat{\mathcal{O}}_n$-modules, and it splits the obvious projection $J^1(\mathcal{V}) \to \mathcal{V}$. We stress, however, that it is *not* a splitting of W_n-modules. We will soon see that this is reflected by the existence of a certain characteristic class in Gelfand-Fuks cohomology.

Note the following corollary, which follows from the identification

$$\mathrm{Ext}^1(\mathcal{V} \otimes_{\widehat{\mathcal{O}}_n} \widehat{\Omega}^1_n, \mathcal{V}) \cong \mathrm{C}^1_{\mathrm{Lie}}(\mathrm{W}_n, \mathrm{GL}_n; \widehat{\Omega}^1_n \otimes_{\widehat{\mathcal{O}}_n} \mathrm{End}_{\widehat{\mathcal{O}}_n}(\mathcal{V}))$$

and from the observation that an exact sequence in $\mathrm{VB}(\widehat{D}^n)$ maps to an exact sequence in $\mathrm{VB}(\mathrm{GK}_n)$.

Corollary 1.35. *There is a cocycle $\mathrm{At}^{\mathrm{GF}}(\mathcal{V}) \in \mathrm{C}^1_{\mathrm{Lie}}(\mathrm{W}_n, \mathrm{GL}_n; \widehat{\Omega}^1_n \otimes_{\widehat{\mathcal{O}}_n} \mathrm{End}_{\widehat{\mathcal{O}}_n}(\mathcal{V}))$ representing the Atiyah class $\mathrm{At}(\mathrm{desc}_{\mathrm{GK}}(\mathcal{V}))$.*

We call this cocycle the Gelfand-Fuks-Atiyah class of $\mathcal{V}$ since it descends to the ordinary Atiyah class for $\mathrm{desc}(\mathcal{V})$ as a sheaf of $\mathcal{O}$-modules.

Definition 1.36. The *Gelfand-Fuks-Chern character* is the formal sum $\mathrm{ch}^{\mathrm{GF}}(\mathcal{V}) = \sum_{k\geq 0} \mathrm{ch}_k^{\mathrm{GF}}(\mathcal{V})$, where the kth component

$$\mathrm{ch}_k^{\mathrm{GF}}(\mathcal{V}) := \frac{1}{(-2\pi i)^k k!}\mathrm{Tr}(\mathrm{At}^{\mathrm{GF}}(\mathcal{V})^k)$$

lives in $\mathrm{C}^k_{\mathrm{Lie}}(\mathrm{W}_n, \mathrm{GL}_n; \widehat{\Omega}^k_n)$.

It is a direct calculation to see that $\mathrm{ch}_k^{\mathrm{GF}}(\mathcal{V})$ is closed for the differential on formal differential forms, i.e., it lifts to an element in $\mathrm{C}^k_{\mathrm{Lie}}(\mathrm{W}_n, \mathrm{GL}_n; \widehat{\Omega}^k_{n,cl})$.

1.5.4. *An explicit formula.* In this section we provide an explicit description of the Gelfand-Fuks-Atiyah class

$$\mathrm{At}^{\mathrm{GF}}(\mathcal{V}) \in \mathrm{C}^1_{\mathrm{Lie}}(\mathrm{W}_n, \mathrm{GL}_n; \widehat{\Omega}^1_n \otimes_{\widehat{\mathcal{O}}_n} \mathrm{End}_{\widehat{\mathcal{O}}}(\mathcal{V})).$$

of a formal vector bundle $\mathcal{V}$.

By definition, any formal vector bundle has the form $\mathcal{V} = \widehat{\mathcal{O}}_n \otimes V$, with V a finite-dimensional vector space. We view V as the "constant sections" in $\mathcal{V}$ by the inclusion $i : v \mapsto 1 \otimes v$. This map then determines a connection on $\mathcal{V}$: we define a $\mathbb{C}$-linear map $\nabla : \mathcal{V} \to \widehat{\Omega}^1_n \otimes_{\widehat{\mathcal{O}}_n} \mathcal{V}$ by saying that for any $f \in \widehat{\mathcal{O}}_n$ and $v \in V$,

$$\nabla(fv) = \mathrm{d}_{dR}(f)v,$$

where $\mathrm{d}_{dR} : \widehat{\mathcal{O}}_n \to \widehat{\Omega}^1_n$ denote the de Rham differential on functions. This connection appeared earlier when we defined the splitting of the jet sequence $j^1 = 1 \oplus \nabla$.

The connection ∇ determines an element in $\mathrm{C}^1_{\mathrm{Lie}}(\mathrm{W}_n; \widehat{\Omega}^1_n \otimes_{\widehat{\mathcal{O}}} \mathrm{End}_{\widehat{\mathcal{O}}}(\mathcal{V}))$, as follows. Let

$$\rho_{\mathcal{V}} : \mathrm{W}_n \otimes \mathcal{V} \to \mathcal{V}$$

denote the action of formal vector fields and consider the composition

$$\mathrm{W}_n \otimes V \xrightarrow{\mathrm{id}\otimes i} \mathrm{W}_n \otimes \mathcal{V} \xrightarrow{\rho_{\mathcal{V}}} \mathcal{V} \xrightarrow{\nabla} \widehat{\Omega}^1_n \otimes_{\widehat{\mathcal{O}}} \mathcal{V}.$$

Since V spans $\mathcal{V}$ over $\widehat{\mathcal{O}}_n$, this composite map determines a $\mathbb{C}$-linear map

$$\alpha_{\mathcal{V},\nabla} : \mathrm{W}_n \to \widehat{\Omega}^1_n \otimes_{\widehat{\mathcal{O}}} \mathrm{End}_{\widehat{\mathcal{O}}}(\mathcal{V})$$

by

$$\alpha_{\mathcal{V},\nabla}(X)(fv) = f\nabla(\rho_{\mathcal{V}}(X)(i(v))),$$

with $f \in \widehat{\mathcal{O}}_n$ and $v \in V$.

Proposition 1.37. *Let $\mathcal{V}$ be a formal vector bundle. Then $\alpha_{\mathcal{V},\nabla}$ is a representative for the Gelfand-Fuks-Atiyah class* $\mathrm{At}^{\mathrm{GF}}(\mathcal{V})$.

Proof. We begin by recalling some general facts about the Gelfand-Fuks-Atiyah class as an extension class of an exact sequence of modules. Viewing $\widehat{\mathcal{O}}_n$ as functions on the formal n-disk, we can ask about the jets of such functions. A choice of formal coordinates corresponds to an identification $\widehat{\mathcal{O}}_n \cong \mathbb{C}[[t_1, \ldots, t_n]]$, and that choice provides a trivialization of the jet bundles by providing a preferred frame. This frame identifies, for instance, J^1 with $\widehat{\mathcal{O}}_n \oplus \widehat{\Omega}^1_n$, and the1-jet of a formal function f can be understood as $(f, \mathrm{d}_{dR} f)$.

For a formal vector bundle $\mathcal{V} = \widehat{\mathcal{O}}_n \otimes V$, something similar happens after choosing coordinates. We have $J^1(\mathcal{V}) \cong \mathcal{V} \oplus \widehat{\Omega}^1_n \otimes_{\widehat{\mathcal{O}}_n} \mathcal{V}$ and the 1-jet of an element of $\mathcal{V}$ can be written as

$$\begin{array}{rccc} j^1: & \mathcal{V} & \to & J^1(\mathcal{V}) \\ & fv & \mapsto & (fv, \mathrm{d}_{dR}(f)v). \end{array}$$

where $f \in \widehat{\mathcal{O}}_n$ and $v \in V$. The projection onto the second summand is precisely the connection ∇ on $\mathcal{V}$ determined by $\mathcal{V} = \widehat{\mathcal{O}}_n \otimes V$, the defining decomposition.

The Gelfand-Fuks-Atiyah class is the failure for this map ∇ to be a map of W_n-modules. Indeed, ∇ determines a map of graded vector spaces

$$1 \otimes \nabla : \mathrm{C}^{\#}_{\mathrm{Lie}}(\mathrm{W}_n; \mathcal{V}) \to \mathrm{C}^{\#}_{\mathrm{Lie}}(\mathrm{W}_n; \widehat{\Omega}^1_n \otimes_{\widehat{\mathcal{O}}} \mathcal{V}).$$

Let $\mathrm{d}_{\mathcal{V}}$ denote the differential on $\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n; \mathcal{V})$ and $\mathrm{d}_{\Omega^1 \otimes \mathcal{V}}$ denote the differential on $\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n; \widehat{\Omega}^1_n \otimes_{\widehat{\Omega}} \mathcal{V})$. The failure for $1 \otimes \nabla$ is precisely the difference

$$(1 \otimes \nabla) \circ \mathrm{d}_{\mathcal{V}} - \mathrm{d}_{\Omega^1 \otimes \mathcal{V}} \circ (1 \otimes \nabla). \tag{3}$$

This difference is $\mathrm{C}^{\#}_{\mathrm{Lie}}(\mathrm{W}_n)$ linear and can hence be thought of as a cocycle of degree one in $\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n; \widehat{\Omega}^1 \otimes_{\widehat{\mathcal{O}}} \mathrm{End}_{\widehat{\mathcal{O}}}(\mathcal{V}))$. This is the representative for the Atiyah class.

We proceed to compute this difference. The differential $\mathrm{d}_{\mathcal{V}}$ splits as $\mathrm{d}_{\mathrm{W}_n} \otimes 1_{\mathcal{V}} + \mathrm{d}'$ where $\mathrm{d}_{\mathrm{W}_n}$ is the differential on the complex $\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n)$ and d' encodes the action of W_n on $\mathcal{V}$. Likewise, the differential $\mathrm{d}_{\Omega^1 \otimes \mathcal{V}}$ splits as $\mathrm{d}_{\mathrm{W}_n} \otimes 1_{\Omega^1 \otimes \mathcal{V}} + \mathrm{d}_{\Omega^1} \otimes 1_V + 1_{\Omega^1} \otimes \mathrm{d}'$ where d_{Ω^1} is the differential on the complex $\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{W}_n; \widehat{\Omega}^1_n)$.

The de Rham differential clearly commutes with the action of vector fields so that $(1 \otimes \mathrm{d}_{dR}) \circ (\mathrm{d}_{\mathcal{O}} \otimes 1) = (\mathrm{d}_{\mathrm{W}_n} + \mathrm{d}_{\Omega^1}) \circ (1 \otimes \mathrm{d}_{dR})$ so that the the difference in (3) reduces to

$$(1 \otimes \nabla) \circ \mathrm{d}' - (1_{\Omega^1} \otimes \mathrm{d}') \circ (1 \otimes \nabla).$$

By definition d' is the piece of the Chevalley-Eilenberg differential that encodes the action of W_n on $\mathcal{V}$, so if we evaluate on an element of the form $1 \in v \in \mathrm{C}^0_{\mathrm{Lie}}(\mathrm{W}_n; V) \subset \mathrm{C}^0_{\mathrm{Lie}}(\mathrm{W}_n; \mathcal{V})$ the only term that survives is the GF 1-cocycle

$$X \mapsto \nabla \mathrm{d}'(1 \otimes v)(X) = \nabla(\rho_{\mathcal{V}}(X)(v)).$$

as desired. $\square$

Corollary 1.38. *On the formal vector bundle $\widehat{\mathcal{T}}_n$ encoding formal vector fields, fix the $\widehat{\mathcal{O}}_n$-basis by $\{\partial_j\}$ and the $\widehat{\mathcal{O}}_n$-dual basis of one-forms by $\{\mathrm{d}t^j\}$. The explicit representative for the Atiyah class is given by the Gelfand-Fuks 1-cocycle*

$$f^i \partial_i \mapsto -\mathrm{d}_{dR}(\partial_j f^i)(\mathrm{d}t^j \otimes \partial_i)$$

taking values in $\widehat{\Omega}^1_n \otimes_{\widehat{\mathcal{O}}_n} \mathrm{End}_{\widehat{\mathcal{O}}}(\widehat{\mathcal{T}}_n)$.

Proof. We must compute the action of vector fields on $\widehat{\mathcal{O}}_n$-basis elements of $\widehat{\mathcal{T}}_n$. We fix formal coordinates $\{t_j\}$ and let $\{\partial_j\}$ be the associated constant formal vector fields. Then the structure map is given by the Lie derivative $\rho_{\widehat{\mathcal{T}}}(f^i \partial_i, \partial_j) = -\partial_j f^i$. The formula for the cocycle follows from the Proposition. $\square$

We can use this result to explicitly compute the cocycles representing the Gelfand-Kazhdan Chern characters. For instance, we have the following formulas that will be useful in later sections.

Corollary 1.39. *The kth component* $\mathrm{ch}_k^{\mathrm{GF}}(\widehat{\mathcal{T}}_n)$ *of the universal Chern character of the formal tangent bundle is the cocycle*

$$\frac{1}{(-2\pi i)^k k!}\mathrm{Tr}(\mathrm{At}^{\mathrm{GF}}(\widehat{\mathcal{T}}_n)^{\wedge k}) : (f_1^i\partial_i, \ldots, f_k^i\partial_i) \mapsto \frac{1}{(-2\pi i)^k k!}\mathrm{Tr}\left(\mathrm{d}_{dR}(\mathrm{Jac}(f_1)) \wedge \cdots \wedge \mathrm{d}_{dR}(\mathrm{Jac}(f_k))\right)$$

in $\mathrm{C}^k_{\mathrm{Lie}}(\mathrm{W}_n, \mathrm{GL}_n; \widehat{\Omega}^k_n)$. *Here,* $\mathrm{Jac}(f)$ *is the* $n \times n$ *matrix valued in* $\widehat{\mathcal{O}}_n$ *with* (ij) *entry given by* $\partial_j f_i$. *As the de Rham differential* $\mathrm{d}_{dR} : \widehat{\Omega}^{k-1}_n \to \widehat{\Omega}^k_n$ *is* W_n*-equivariant, there is an element* α_{k-1} *in* $\mathrm{C}^k_{\mathrm{Lie}}(\mathrm{W}_n, \mathrm{GL}_n; \widehat{\Omega}^{k-1}_n)$ *such that*

$$\mathrm{ch}_k^{\mathrm{GF}}(\widehat{\mathcal{T}}_n) = \mathrm{d}_{dR}\alpha_{k-1}$$

Explicitly:

$$\alpha_k : (f_1^i\partial_i, \ldots, f_k^i\partial_i) \mapsto \frac{1}{(-2\pi i)^k k!}\mathrm{Tr}\left(\mathrm{Jac}(f_1) \wedge \mathrm{d}_{dR}(\mathrm{Jac}(f_2)) \wedge \cdots \wedge \mathrm{d}_{dR}(\mathrm{Jac}(f_k))\right).$$

1.6. **A family of extended pairs.** We will be most interested in the cocycles $\mathrm{ch}_k(\mathcal{V})$ for $k \geq 2$. When $k = 2$ we obtain a 2-cocycle with values in $\widehat{\Omega}^2_{n,cl}$, $\mathrm{ch}_2(\mathcal{V}) \in \mathrm{C}_{\mathrm{Lie}}(\mathrm{W}_n, \mathrm{GL}_n; \widehat{\Omega}^2_{n,cl})$. This 2-cocycle $\mathrm{ch}_2^{\mathrm{GF}}(\mathcal{V})$ determines an abelian extension Lie algebras of W_n by $\widehat{\Omega}^2_{n,cl}$

$$0 \to \widehat{\Omega}^2_{n,cl} \to \widetilde{\mathrm{W}}_{n,\mathcal{V}} \to \mathrm{W}_n \to 0.$$

When $\mathcal{V} = \widehat{\mathcal{T}}_n$, denote this extension by $\widetilde{\mathrm{W}}_{n,\mathcal{V}} = \widetilde{\mathrm{W}}_{n,1}$. (The notation will become clearer momentarily)

We have already discussed the pair $(\mathrm{W}_n, \mathrm{GL}_n)$. We will need that the above extension of Lie algebras fits in to a Harish-Chandra pair as well. The action of GL_n extends to an action on $\widetilde{\mathrm{W}}_{n,1}$ where we declare the action of GL_n on closed two-forms to be the natural one via linear formal automorphisms.

Lemma 1.40. *The pair* $(\widetilde{\mathrm{W}}_{n,1}, \mathrm{GL}_n)$ *form a Harish-Chandra pair and fits into an extension of pairs*

$$0 \to \widehat{\Omega}^2_{n,cl} \to (\widetilde{\mathrm{W}}_{n,1}, \mathrm{GL}_n) \to (\mathrm{W}_n, \mathrm{GL}_n) \to 0$$

which is determined by the cocycle $\mathrm{ch}_2^{\mathrm{GF}}(\widehat{\mathcal{T}}_n)$.

One might be worried as to why there is only a non-trivial extension of the Lie algebra in the pair. The choice of a coordinate determines an embedding of linear automorphisms GL_n into formal automorphisms Aut_n. The extension of formal automorphisms Aut_n defined by the group two-cocycle $\mathrm{ch}_2^{\mathrm{GF}}(\widehat{\mathcal{T}}_n)$ is trivial when restricted to GL_n so that it does not get extended.

1.6.1. *An L_∞ extension.* For $k > 2$, it will be useful to think of $\mathrm{ch}_k(\mathcal{V})$ as defining a similar type of extension. For this to make sense, we observe the following phenomena for higher cocycles. Suppose M is a module for a Lie algebra $\mathfrak{g}$, and suppose $c \in \mathrm{C}^k_{\mathrm{Lie}}(\mathfrak{g}; M)$ is a cocycle $\mathrm{d}_{CE}c = 0$. Then, c determines an abelian extension of *L_∞-algebras*

$$0 \to M[k-2] \to \widetilde{\mathfrak{g}} \to \mathfrak{g}$$

As a graded vector space $\widetilde{\mathfrak{g}}$ is $\mathfrak{g} \oplus M[k-2]$ (so that M is placed in degree $2-k$). The L_∞ structure on $\widetilde{\mathfrak{g}}$ is defined by, for $x, y, x_1, \ldots, x_k \in \mathfrak{g}$, $m \in M$:

$$\ell_2(x, y+m) = [x,y] + x \cdot m$$
$$\ell_k(x_1, \ldots, x_k) = c(x_1, \ldots, x_k).$$

Here, $x \cdot m \in M$ uses the module structure.

Thus, for any formal vector bundle $\mathcal{V}$, $\mathrm{ch}_k(\mathcal{V})$ determines an abelian L_∞ extension of W_n by the abelian Lie algebra $\widehat{\Omega}^k_{n,cl}$. The case $\mathcal{V} = \widehat{\mathcal{T}}_n$ will be especially relevant for us.

Definition 1.41. Denote by $\widetilde{\mathrm{W}}_{n,d}$ the L_∞ extension of W_n by the module $\widehat{\Omega}^{d+1}_{n,cl}[d-1]$:

$$0 \to \widehat{\Omega}^{d+1}_{n,cl}[d-1] \to \widetilde{\mathrm{W}}_{n,d} \to \mathrm{W}_n \to 0$$

determined by the $(d+1)$-cocycle $\mathrm{ch}_{d+1}(\widehat{\mathcal{T}}_n) \in \mathrm{C}^{d+1}_{\mathrm{Lie}}(\mathrm{W}_n, \mathrm{GL}_n; \widehat{\Omega}^{d+1}_{n,cl})$.

We would like to have an an analog of Lemma 1.40 for $\widetilde{\mathrm{W}}_{n,d}$ and the group GL_n. To make this possible, we need to slightly enlarge our category of Harish-Chandra pairs to include the data of an L_∞ algebra, instead of an ordinary Lie algebra.

1.6.2. *L_∞ pairs.* The concept of an ordinary Harish-Chandra pair involves a Lie group K, a Lie algebra $\mathfrak{g}$ with an action by K, together with an embedding of Lie algebras $\mathrm{Lie}(K) \to \mathfrak{g}$. There is a natural way to relax this to include L_∞ algebras.

Definition 1.42. An L_∞ Harish-Chandra pair is a pair $(\mathfrak{g}, K)$ where $\mathfrak{g}$ is an L_∞ algebra and K is a Lie group together with

(1) a linear action of K on $\mathfrak{g}$, $\rho_K : K \to \mathrm{GL}(\mathfrak{g})$;
(2) a map of L_∞ algebras $i : \mathrm{Lie}(K) \rightsquigarrow \mathfrak{g}$;

such that i is compatible with the action ρ_K and the adjoint action of K on $\mathrm{Lie}(K)$.

Remark 1.43. A morphism of L_∞ algebras $f : \mathfrak{h} \rightsquigarrow \mathfrak{g}$ is, by definition, a map of the underlying Chevalley-Eilenberg complexes

$$\mathrm{C}^{\mathrm{Lie}}_*(f) : \mathrm{C}^{\mathrm{Lie}}(\mathfrak{h}) \to \mathrm{C}^{\mathrm{Lie}}(\mathfrak{g})$$

as cocoummutative coalgebras. Now, $\mathrm{C}^{\mathrm{Lie}}_*(\mathfrak{g})$, being a free cocoummtative coalgebra, this map is determined by a sequence of maps $f_n : \mathrm{Sym}^n(\mathfrak{h}[1]) \to \mathfrak{g}[1]$ satisfying certain compatibility conditions.

Remark 1.44. This is certainly not the most general definition one can imagine for a homotopy enhancement of a Harish-Chandra pair. For instance, we have required that K acts on $\mathfrak{g}$ in a rather strict way. It turns out that this will be enough for our purposes.

The condition that $i : \mathrm{Lie}(K) \to \mathfrak{g}$ be compatible with ρ_K can be stated as follows. The L_∞ map $i : \mathrm{Lie}(K) \rightsquigarrow \mathfrak{g}$ is uniquely determined by a sequence of maps $i_n : \mathrm{Sym}^n(\mathrm{Lie}(K)[1]) \to \mathfrak{g}$, for each $n \geq 1$. We require that for each $n \geq 1$, all $A \in K$, and $x_1, \ldots, x_n \in \mathrm{Lie}(K)$ that

$$\rho_K(A) \cdot i_n(x_1, \ldots, x_n) = i_n\left((\mathrm{Ad}(A) \cdot x_1) \cdots (\mathrm{Ad}(A) \cdot x_n)\right).$$

Here $\mathrm{Ad}(A)$ denotes the adjoint action of $A \in K$ on $\mathrm{Lie}(K)$.

Lemma 1.45. *The for any $d \geq 1$ the pair $(\widetilde{\mathrm{W}}_{n,d}, \mathrm{GL}_n)$ has the structure of an L_∞ Harish-Chandra pair.*

Proof. The proof is similar to the case $d = 1$. The linear action of GL_n on $\widetilde{\mathrm{W}}_{n,d}$ comes from the natural one on W_n and $\widehat{\Omega}^{d+1}_{n,cl}$. Now, note that we have an GL_n-equivariant extension

$$\begin{tikzcd} & \widetilde{\mathrm{W}}_{n,d} \arrow[d] \\ \mathfrak{gl}_n \arrow[r] \arrow[ur, dotted] & \mathrm{W}_n \end{tikzcd}$$

since the cocycle $\mathrm{ch}_{d+1}(\widehat{\mathcal{T}}_n)$ vanishes when one of the inputs lies in $\mathfrak{gl}_n$. $\square$

In the next section we will see how the theory of descent for $(\mathrm{W}_n, \mathrm{GL}_n)$ can be extended to the pair $(\widetilde{\mathrm{W}}_{n,d}, \mathrm{GL}_n)$ provided a trivialization of the $(d+1)$st component of the Chern character is trivialized. This will be our main application of this extended pair.

2. Descent for extended pairs

2.1. **General theory of descent for L_∞ pairs.** In this section we set up the general theory of descent for L_∞ pairs $(\mathfrak{g}, K)$. Recall, this means that K is still and ordinary Lie group, but $\mathfrak{g}$ is an L_∞ algebra.

Let X be a fixed manifold, for which we are defining descent over. The starting point is the theory of bundles over X for the pair $(\mathfrak{g}, K)$. In the usual context of Harish-Chandra pairs (where $\mathfrak{g}$ is an ordinary Lie algebra), this means that we have a principal K-bundle $P \to X$ equipped with a K-equivariant one-form valued in $\mathfrak{g}$, $\omega \in \Omega^1(P, \mathfrak{g})$ satisfying the flatness condition

$$\mathrm{d}\omega + \frac{1}{2}[\omega, \omega] = 0.$$

In other words, ω is a Maurer-Cartan element of the dg Lie algebra $\Omega^*(P) \otimes \mathfrak{g}$ that is equivariant for the action of K on P and $\mathfrak{g}$.

The theory of Maurer-Cartan forms works just as well in the L_∞ case. First, note that the category of L_∞ algebras is tensored over commutative dg algebras. In other words, if $\mathfrak{g}$ is an L_∞ algebra and A a commutative dg algebra, there is the natural structure of an L_∞ algebra on $A \otimes \mathfrak{g}$. The n-ary brackets are of the form

$$\ell_n^{A \otimes \mathfrak{g}}(a_1 \otimes x_1, \ldots, a_n \otimes x_n) = (a_1 \cdots a_n)\ell_n^{\mathfrak{g}}(x_1, \ldots, x_n)$$

where $\ell_n^{\mathfrak{g}}$ is the n-ary bracket on $\mathfrak{g}$, and where we have used the commutative algebra structure on A.

Definition 2.1. Let $(\mathfrak{g}, K)$ be an L_∞ Harish-Chandra pair. A principal $(\mathfrak{g}, K)$-bundle on X is the data:

(1) a principal K-bundle $P \to X$;
(2) a K-invariant element

$$\omega \in \Omega^*(P) \otimes \mathfrak{g}$$

of total degree $+1$;

such that

(1) for all $a_1, \dots, a_n \in \mathrm{Lie}(K)$ we have $\omega(\xi_{a_1}, \cdots, \xi_{a_n}) = i(a_1, \dots, a_n)$ where ξ_{a_i} is the vertical vector field on P determined by a_i, and $i : \mathrm{Lie}(K) \to \mathfrak{g}$ is the L_∞ morphism determining the Harish-Chandra pair;

(2) ω is a Maurer-Cartan element of the L_∞ algebra $\Omega^*(P) \otimes \mathfrak{g}$. In other words,

$$\mathrm{d}\omega + \sum_{n \geq 1} \ell_n(\omega, \dots, \omega) = 0$$

where $\{\ell_n\}$ are the structure maps for $\mathfrak{g}$.

To define descent, we need an appropriate theory of modules for an L_∞ pair $(\mathfrak{g}, K)$.

Definition 2.2. A *semi-strict Harish-Chandra module* for the L_∞ pair $(\mathfrak{g}, K)$ is a dg vector space (V, d_V) equipped with

(i) a strict group action ρ_V^K of K, meaning a group map

$$\rho_{V^d}^K : K \to \mathrm{GL}(V^d)$$

for each degree d such that the product map $\prod_d \rho_{V^d}^K : K \to \prod_d \mathrm{GL}(V^d)$ commutes with the differential d_V;

(ii) an L_∞-action of $\mathfrak{g}$ on V, i.e., a map of L_∞-algebras $\rho_V^{\mathfrak{g}} : \mathfrak{g} \rightsquigarrow \mathrm{End}(V)$, such that the composite

$$\mathrm{C}_*^{\mathrm{Lie}}(\rho_V^{\mathfrak{g}}) \circ \mathrm{C}_*^{\mathrm{Lie}}(i) : \mathrm{C}_*^{\mathrm{Lie}}(\mathrm{Lie}(K)) \to \mathrm{C}_*^{\mathrm{Lie}}(\mathrm{End}(V))$$

equals the map

$$\mathrm{C}_*^{\mathrm{Lie}}(D\rho_V^K) : \mathrm{C}_*^{\mathrm{Lie}}(\mathrm{Lie}(K)) \to \mathrm{C}_*^{\mathrm{Lie}}(\mathrm{End}(V)).$$

Here $D\rho_V^K : \mathrm{Lie}(K) \to \mathrm{End}(V)$ is the differential of the strict K-action and $i : \mathrm{Lie}(K) \rightsquigarrow \mathfrak{g}$ is part of the data of the Harish-Chandra pair $(\mathfrak{g}, K)$.

2.1.1. *Basic forms.* Before the construction of descent, we recall a basic object in equivariant differential geometry.

Let V be a finite-dimensional K-representation. Denote by $\underline{V}$ the trivial vector bundle on P with fiber V. Sections of this bundle $\Gamma_P(V)$ have the structure of a K-representation by

$$A \cdot (f \otimes v) := (A \cdot f) \otimes (A \cdot v) \ \ , \ \ A \in K,\ f \in \mathcal{O}(P)\ , v \in V.$$

Every K-invariant section $f : P \to \underline{V}$ induces a section $s(f) : X \to V_X$, where the value of $s(f)$ at $x \in X$ is the K-equivalence class $[(p, f(p)]$, with $p \in \pi^{-1}(x) \cong K$. That is, there is a natural map

$$s : \Gamma_P(\underline{V})^K \to \Gamma_X(V_X)$$

and it is an isomorphism of $\mathcal{O}(X)$-modules. A K-invariant section f of $\underline{V} \to P$ also satisfies the infinitesimal version of invariance:

$$(Y \cdot f) \otimes v + f \otimes \mathrm{Lie}(\rho)(Y) \cdot v = 0$$

for any $Y \in \mathrm{Lie}(K)$.

There is a similiar statement for differential forms with values in the bundle V_X. Let $\Omega^k(P; \underline{V}) = \Omega^k(P) \otimes V$ denote the space of k-forms on P with values in the trivial bundle $\underline{V}$. Given $\alpha \in \Omega^1(X; V_X)$, its pull-back along the projection $\pi : P \to X$ is annihilated by any vertical vector field on P. In general, if $\alpha \in \Omega^k(X; V_X)$, then $i_Y(\pi^*\alpha) = 0$ for all $Y \in \mathrm{Lie}(K)$.

Definition 2.3. A k-form $\alpha \in \Omega^k(P; \underline{V})$ is called *basic* if

(1) it is K-invariant: $L_Y \alpha + \rho(Y) \cdot \alpha = 0$ for all $Y \in \mathrm{Lie}(K)$ and
(2) it vanishes on vertical vector fields: $i_Y \alpha = 0$ for all $Y \in \mathrm{Lie}(K)$.

Denote the subspace of basic k-forms by $\Omega^k(P; \underline{V})_{bas}$. Just as with sections, there is a natural isomorphism

$$s : \Omega^k(P; \underline{V})_{bas} \xrightarrow{\cong} \Omega^k(X; V_X)$$

between basic k-forms and k-forms on X with values in the associated bundle. In fact, $\Omega^\#(P; \underline{V})_{bas}$ forms a graded subalgebra of $\Omega^\#(P; \underline{V})$ and the isomorphism s extends to an isomorphism of graded algebras $\Omega^\#(P; \underline{V})_{bas} \cong \Omega^\#(X; V_X)$.

It is manifest that this construction of basic forms is natural in maps of $(\mathfrak{g}, K)$-bundles: basic forms pull back to basic forms along maps of bundles.

2.1.2. Starting with the data:

(1) an L_∞ Harish-Chandra pair $(\mathfrak{g}, K)$;
(2) a principal $(\mathfrak{g}, K)$ bundle $(P \to X, \omega)$;
(3) a semi-strict $(\mathfrak{g}, K)$-module V;

we are now ready to define descent along X. It is constructed in the following steps.

(1) Using the linear action of K on V we define the associated vector bundle

$$V_X = P \times^K V$$

on X. Note that the differential forms on X with values in V_X, $\Omega^*(X; V_X)$, is isomorphic, as a dg $\Omega^*(X)$-module, to the complex of basic forms

$$\Omega^*(P; \underline{V})_{bas} \subset \Omega^*(P; \underline{V}).$$

(2) The Maurer-Cartan element $\omega \in \Omega^*(P) \otimes \mathfrak{g}$ allows us to deform the differential on $\Omega^*(P; \underline{V}) = \Omega^*(P) \otimes V$ by the following transfer of Maurer-Cartan elements. By the usual yoga of Koszul duality, the Maurer-Cartan element $\omega \in \Omega^*(P) \otimes \mathfrak{g}$ is equivalent to the data of a map of commutative dg algebras

$$\omega^* : \mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g}) \to \Omega^*(P).$$

We can then use the L_∞ module structure map $\rho_V : \mathfrak{g} \rightsquigarrow \mathrm{End}(V)$ to form the composition

$$\mathrm{C}^*_{\mathrm{Lie}}(\mathrm{End}(V)) \xrightarrow{\mathrm{C}^*_{\mathrm{Lie}}(\rho_V^{\mathfrak{g}})} \mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g}) \xrightarrow{\omega^*} \Omega^*(P).$$

This, in turn, corresponds to a Maurer-Cartan element

$$\omega_V \in \Omega^*(P) \otimes \mathrm{End}(V).$$

We use this element to deform the differential on $\Omega^*(P, \underline{V}) = \Omega^*(P) \otimes V$ via

$$(\Omega^*(P) \otimes V, \mathrm{d} + \omega_V).$$

Here, $\mathrm{d} = \mathrm{d}_{dR} + \mathrm{d}_V$ where d_{dR} is the de Rham differential on P and d_V is the internal differential to V. We can think of $\nabla^V := \mathrm{d} + \omega_V$ as a flat "super-connection" on the trivial bundle $P \times V \to P$. This means that ω_V may contain higher differential forms, not just one-forms. Tracing through the above construction, we see that ω_V actually preserves the subspace of basic forms, so it that ∇^V descends to a flat super-connection on the vector bundle V_X over X. In other words we obtain the $\Omega^*(X)$-module

$$\begin{aligned}\mathbf{desc}\left((P \to X, \omega), V\right) &:= \left(\Omega^*(P, \underline{V})_{bas}, \mathrm{d} + \omega_V\right) \\ &= \left(\Omega^*(X, V_X), \nabla^V\right).\end{aligned}$$

Definition 2.4. We will denote the vector bundle V_X equipped with its flat superconnection ∇^V obtained in this way by $\mathrm{desc}((P \to X, \omega), V)$. Its associated de Rham complex is denoted $\mathbf{desc}((P \to X, \omega), V)$.

Remark 2.5. This construction of descent enjoys a number of nice functorial properties. BW: ..finish.

2.2. **The flat connection from the extended pair.**