Koszul Duality for E_n Algebras

Yuchen Fu

September 24, 2018

This is the note for a talk given at Joint Northeastern–MIT Graduate Research Seminar "The Yangian and Four-dimensional Gauge Theory" during Fall 2018.

Assumptions We work over a field k of characteristic 0 and implicitly work in the $(\infty, 1)$-setting; so by the word "category" we always mean an $(\infty, 1)$-category. The background "category of categories" is the category of stable presentable $(\infty, 1)$-categories with only colimit-preserving morphisms between them. This category is equipped with a symmetric monoidal structure given by the Lurie tensor product. The category Vect is the $(\infty, 1)$-category of chain complexes over k.

1 Operads, Algebras and Modules

1.1 Operads

We shall use the following version of definition from [FG12]. Let $\mathcal{X}$ be a symmetric monoidal category. Let $\mathcal{X}^\Sigma := \prod_{n\geq 1} \mathrm{Rep}(\Sigma_n)$ be the category of symmetric sequences in $\mathcal{X}$; its objects are collections $\{O(n) \in \mathcal{X}, n \geq 1\}$ such that Σ_n acts on $O(n)$. This category admits a monoidal structure $\star$ such that the following functor is monoidal:

$$\mathcal{X}^\Sigma \to \mathrm{Funct}(\mathcal{X}, \mathcal{X})$$
$$\{O(n)\} \mapsto \left(x \mapsto \bigoplus_{n\geq 1} (O(n) \otimes x^{\otimes n})_{\Sigma_n} \right)$$

Namely it's given by

$$P \star Q = \bigoplus_{n\geq 1} (P(n) \otimes Q^{\odot n})_{\Sigma_n}$$

where $\odot$ is the Day convolution:

$$(P \odot Q)(n) = \bigoplus_{i+j=n} \mathrm{Ind}_{S_i \times S_j}^{S_n} (P(i) \otimes Q(j))$$

The unit object of $(\mathcal{X}^\Sigma, \star)$ is $\mathbf{1}_\star$, given by $\mathbf{1}_\star(1) = \mathbf{1}_\mathcal{X}$ and $\mathbf{1}_\star(n) = 0_\mathcal{X}$ $\forall n > 1$.

We define $\mathrm{Oprd}(\mathcal{X})$, the category of *reduced, augmented* operads over $\mathcal{X}$, to be that of augmented associative algebras in $(\mathcal{X}^\Sigma, \star)$ for which $\mathbf{1}_\mathcal{X} \to \mathcal{O}(1)$ is an isomorphism. The dual notion $\mathrm{coOprd}(\mathcal{X})$ of co-augmented cooperads is defined dually. This means that we have the composition maps

$$O(k) \otimes O(n_1) \otimes \ldots \otimes O(n_k) \to O(n_1 + \ldots + n_k)$$

as well as a unit element in $O(1)$, such that the unital, associative and equivariance laws are satisfied up to coherent homotopy.

Example 1. *The associative operad Ass is given by that $Ass(n) = k[\Sigma_n]$, the regular representation of Σ_n; the operad maps come from substitution. Similarly, the commutative operad Comm is given by $Comm(n) = k$, the trivial representation of Σ_n.*

Linear Dual of Operads Given an operad O such that $O(n)$ has finite dimensional cohomologies, we can define O^* to be $O^*(n) = O(n)^*$, which will be a cooperad.

Shifting Operads For an operad $O \in \mathrm{Oprd(Vect)}$, we use $O[1]$ to denote the operad given on the component level by

$$O[1](n) = \widetilde{O(n)[n-1]}$$

(where the tilde indicates that the Σ_n action needs to be twisted accordingly), such that $c \mapsto c[1]$ gives an equivalence $O[1]\text{-alg(Vect)} \to O\text{-alg(Vect)}$ (see below). The dual notion of suspension of operads is defined analogously; namely, we also require $O[1]\text{-coalg(Vect)} \to O\text{-coalg(Vect)}$ is given by $c \mapsto c[1]$.

1.2 Algebras over Operads

Let $\mathcal{X}$ be as before, and let $\mathcal{C}$ be a commutative algebra object in the category of $\mathcal{X}$-modules. The action

$$(O, c) \mapsto \bigoplus_n (O(n) \otimes c^{\otimes n})_{\Sigma_n}$$

defines the $\star$-action of $\mathcal{X}^\Sigma$ on $\mathcal{C}$. For any operad O and any cooperad O°, define

$$O\text{-alg}(\mathcal{C}) := O\text{-mod}(\mathcal{C}, \star)$$

to be the category of O-algebras in $\mathcal{C}$ and

$$O^\circ\text{-coalg}^{\mathrm{nil}}_{\mathrm{d.p.}}(\mathcal{C}) := O^\circ\text{-comod}(\mathcal{C}, \star).$$

to be the category of O°-coalgebras in $\mathcal{C}$.

Example Algebras over Ass and Comm in a category $\mathcal{C}$ correspond respectively to augmented associative and augmented commutative algebras in $\mathcal{C}$; Similarly, coalgebras over Ass^* and Comm^* in $\mathcal{C}$ correspond to coaugmented coassociative coalgebras and coaugmented cocommutative coalgebras in $\mathcal{C}$.

Remark 1. *Strictly speaking, the augmentation does not come from being a module of the operad, but rather the obvious equivalence of categories $Assoc^{non\text{-}aug}(\mathcal{C}) \simeq Assoc^{aug}(\mathcal{C})$, given by direct sum with **1** / taking the augmentation ideal. To simplify discussion, we'll consider the corresponding algebras as augmented for the rest of this talk.*

1.2.1 Four Types of Comodules

Notice that what we wrote was $O^\circ\text{-coalg}^{\mathrm{nil}}_{\mathrm{d.p.}}(\mathcal{C})$ and not O°-coalg; indeed the former doesn't in general specialize to what we usual call comodules of cooperads (TODO: example). Instead, define the following $*$-action:

$$(O, c) \mapsto \prod_n (O(n) \otimes c^{\otimes n})^{\Sigma_n}$$

and write

$$O^\circ\text{-coalg}(\mathcal{C}) := O^\circ\text{-comod}(\mathcal{C}, *)$$

Then this is the one that specializes to our usual notion.

In addition, define the category $O\text{-coalg}^{\mathrm{nil}}$ to be the one equipped with the action

$$(O, c) \mapsto \bigoplus_n (O(n) \otimes c^{\otimes n})^{\Sigma_n}$$

and $O\text{-coalg}_{\mathrm{d.p.}}$ the one equipped with the action

$$(O, c) \mapsto \prod_n (O(n) \otimes c^{\otimes n})_{\Sigma_n}.$$

We have the averaging functor

$$\mathrm{avg} : O^\circ\text{-coalg}^{\mathrm{nil}}_{\mathrm{d.p.}} \to O^\circ\text{-coalg}^{\mathrm{nil}}$$

which is an isomorphism since we are in characteristic 0. We also have the obvious functor

$$O^\circ\text{-coalg}^{\mathrm{nil}} \to O^\circ\text{-coalg}$$

We compose those two to get a map

$$\mathrm{res} : O^\circ\text{-coalg}^{\mathrm{nil}}_{\mathrm{d.p.}} \to O^\circ\text{-coalg}.$$

This functor commutes with colimits so admit a right adjoint, giving a pair

$$\mathrm{res} : O^\circ\text{-coalg}^{\mathrm{nil}}_{\mathrm{d.p.}} \rightleftarrows O^\circ\text{-coalg} : \mathrm{res}^R$$

Proposition 1 ([FG12])**.** *When $\mathcal{C}$ is pro-nilpotent (as defined in [FG12, Section 4.1]), res is an isomorphism.*

Conjecture 1. *res is always fully faithful.*

1.3 Modules

Let $A \in \mathcal{C}$ be an O-algebra, and let $\mathcal{M}$ be a module category over $\mathcal{C}$ in the category of $\mathcal{X}$-modules. Note that there is a symmetric monoidal category $\mathrm{Sqz}(\mathcal{C}, \mathcal{M})$, the "square zero extension" of $\mathcal{C}$ by $\mathcal{M}$, obtained from $\mathcal{C} \times \mathcal{M}$ by collapsing the $\mathcal{M} \times \mathcal{M} \to \mathcal{M}$ morphisms. We then define the category of A-modules in $\mathcal{M}$, denoted $\mathrm{Mod}_A(\mathcal{M})$, to be the fiber of A under $\pi_1 : O\text{-alg}(\mathrm{Sqz}(\mathcal{C}, \mathcal{M})) \to O\text{-alg}(\mathcal{C})$, which is induced by the projection $\pi_1 : \mathcal{C} \times \mathcal{M} \to \mathcal{C}$. The dually defined comodule category is denoted $\mathrm{Comod}^{\mathrm{nil}}_{A^\vee}(\mathcal{M})$.

Concretely speaking, an A-module structure amounts to an object $M \in \mathcal{M}$ and operation maps

$$O(n) \otimes A^{k-1} \otimes M \otimes A^{n-k} \to M$$

for each $1 \leq k \leq n$, such that all necessary conditions hold.

Left/Right Module For the operad Ass, the notion above recovers the notion of *bimodules* over an associative algebra A. Using colored operads it is also possible to recover the notion of left/right modules, as is detailed in [Lur]. We shall not define those concepts, but for the sake of stating results let us introduce the notation $\mathrm{LMod}_A(\mathcal{M})$ and $\mathrm{RMod}_A(\mathcal{M})$ to denote those two categories.

2 E_n operads

For this talk we shall focus on the case of E_n operads. Namely, for each $n \geq 1$, there is an element $\mathcal{E}_n \in \mathrm{Oprd(Spc)}$ that is realized by the little n-disk or little n-cube operads. The operad in Oprd(Vect) induced by the singular chain functor $C_* : \mathrm{Spc} \to \mathrm{Vect}$ is then called the E_n operad in chain complexes; we will refer to it simply by E_n.

By definition we have $E_1 \simeq \mathrm{Ass}$, so an E_1-algebra is nothing more than an augmented associative algebra. The other extreme is when $n = \infty$, for which we'll write $E_\infty := \mathrm{Comm}$, i.e. E_∞-algebras are augmented commutative algebras. The other E_n cases are interpolations between those two, so can be seen as describing algebras that are "partially commutative". More precisely, there is a sequence of maps between operads

$$E_1 \to E_2 \to \ldots \to E_n \to E_{n+1} \to \ldots \to E_\infty$$

induced from the topological counterpart

$$\mathcal{E}_1 \to \mathcal{E}_2 \to \ldots \to \mathcal{E}_n \to \mathcal{E}_{n+1} \to \ldots \to \mathcal{E}_\infty$$

(where $\mathcal{E}_\infty(n) = *$ for each n) by the standard embedding $\mathbb{R}^n \to \mathbb{R}^{n+1}$.

From now on, Vect will denote the homotopy category of chain complexes. When the category $\mathcal{C}$ is not specified, by E_n-algebras we mean elements of E_n-alg(Vect).

3 Koszul Duality

3.1 Bar Construction for Associative Algebras

Let $\mathcal{A}$ be a monoidal category with limits and colimits, then we have a standard construction of a pair of adjoint functors

$$\mathrm{Bar} : \mathrm{AssocAlg}^{\mathrm{aug}}(\mathcal{A}) \rightleftarrows \mathrm{CoassocCoalg}^{\mathrm{coaug}}(\mathcal{A}) : \mathrm{coBar}$$

where Bar maps R to $\mathbf{1} \otimes_R \mathbf{1}$ ($\mathbf{1}$ is considered as both a left and a right R-module, by means of the augmentation), and coBar defined dually. The comultiplication on $\mathrm{Bar}(R)$ is given by the following:

$$\mathbf{1} \otimes_R \mathbf{1} \simeq \mathbf{1} \otimes_R R \otimes_R \mathbf{1} \to \mathbf{1} \otimes_R \mathbf{1} \otimes_R \mathbf{1} \simeq \mathbf{1} \otimes_R \mathbf{1} \otimes_{\mathbf{1}} \mathbf{1} \otimes_R \mathbf{1} \to (\mathbf{1} \otimes_R \mathbf{1}) \otimes (\mathbf{1} \otimes_R \mathbf{1})$$

The coaugmentation is given by

$$\mathbf{1} \simeq \mathbf{1} \otimes \mathbf{1} \to \mathbf{1} \otimes_R \mathbf{1}$$

It is checked in e.g. [Lur, Theorem 5.2.2.17] that this indeed lands in coassociative algebras.

Now let $\mathcal{A}$ be as above and let $\mathcal{C}$ be an $\mathcal{A}$-module category. Fix some augmented associative algebra $A \in \mathcal{A}$. By general construction we have an adjoint pair

$$\mathrm{Bar}_A : A\text{-mod}(\mathcal{C}) \rightleftarrows \mathcal{C} : \mathrm{triv}_A$$

Namely we have $\mathrm{Bar}_A(M) = M \otimes_A \mathbf{1}$ where $A \to \mathbf{1}$ is the augmentation; by this notation we mean the colimit of the following diagram:

$$\cdots A \otimes M \rightrightarrows M$$

Similarly if A° is an coaugmented coassociative coalgebra and $\mathcal{C}^\circ$ an $\mathcal{A}^\circ$-comodule, then we have an adjoint pair

$$\mathrm{triv}_{A^\circ} : \mathcal{C}^\circ \rightleftarrows A^\circ\text{-mod}(\mathcal{C}^\circ) : \mathrm{coBar}_{A^\circ}$$

3.2 Koszul Duality for Operads

When $\mathcal{A}$ is $\mathcal{X}^\Sigma$ as above, these functors trivially lift to another adjoint pair

$$\mathrm{Bar} : \mathrm{Oprd}(\mathcal{X}) \rightleftarrows \mathrm{coOprd}(\mathcal{X}) : \mathrm{coBar}$$

that are compatible with the obvious forgetful functors. This pair we call the operadic Koszul duality.

Proposition 2 ([FG12, Prop 4.1.2]). *These are mutual equivalences.*

We shall refer to $\mathrm{Bar}(x)$ as the *Koszul dual* of x and write it as $x^\vee$.

Example 2. *The fundamental example in representation theory is $Lie^\vee = Comm^*[1]$, corresponding to the relationship between an Lie algebra and its Chevalley complex.*

3.2.1 Koszul Dual for the E_n Operads

Proposition 3. $E_1^\vee = E_1^*[1]$. *More generally, we have $E_n^\vee \simeq E_n^*[n]$, and the map is compatible with $E_n \to E_{n+1}$.*

The $n = 1$ case is classical and can be found in e.g. [GK94]. (OPTIONAL: do this computation.) For our setting (characteristic 0) the general n case would follow from a corresponding computation in homology operad in [GJ94] plus the formality theorem (see section 5); over $\mathbb{Z}$ this is proven in [Fre11].

3.3 Koszul Duality for Algebras

Now let $\mathcal{C}$ be the same as in section 1.2. For any Koszul pair $(O, O^\vee)$, bar construction for modules gives an adjoint pair:

$$\mathrm{Bar}_O^{\mathrm{naive}} : O\text{-alg}(\mathcal{C}) \rightleftarrows O^\vee\text{-coalg}^{\mathrm{nil}}_{\mathrm{d.p.}}(\mathcal{C}) : \mathrm{coBar}^{\mathrm{naive}}_{O^\vee}$$

Again compatible with the forgetful functors. Now combine with the restriction adjoint pair to get

$$\mathrm{Bar}_O = \mathrm{res} \circ \mathrm{Bar}_O^{\mathrm{naive}} : O\text{-alg}(\mathcal{C}) \rightleftarrows O^\vee\text{-coalg}(\mathcal{C}) : \mathrm{coBar}^{\mathrm{naive}}_{O^\vee} \circ \mathrm{res}^R = \mathrm{cobar}_{O^\vee}$$

This is what we call the algebraic Koszul duality, and we'll write $A^\vee$ for $\mathrm{Bar}_O(A)$ as well. We shall say A is *Koszul* if $A \to (A^\vee)^\vee$ is an isomorphism.

The Two Bar Constructions Agree In the case $O = \mathrm{Ass}$, the Koszul duality above gives a pair of adjunction

$$[1] \circ \mathrm{Bar}_{\mathrm{Ass}} : \mathrm{Assoc}^{\mathrm{aug}}(\mathcal{C}) \rightleftarrows \mathrm{Coassoc}^{\mathrm{coaug}}(\mathcal{C}) : \mathrm{coBar}_{\mathrm{Ass}^*[1]} \circ [-1]$$

This agrees with the bar construction given at the beginning of section 3.

Example 3. *Taking Koszul dual along $Ass^\vee = Ass^*[1]$ gives Hochschild complex; along $Lie^\vee = Comm^*[1]$ gives Chevalley complex; and along $Comm^\vee = Lie^*[1]$ gives Harrison complex.*

3.3.1 Building an Equivalence

Unlike the operadic case, in general we have no reason to expect algebraic Koszul duality to be an equivalence.

Example 4. *Let us consider the pair $Comm^\vee = Lie^*[1]$ over Vect. To show the naive adjunction won't work (namely, the left hand side is "too big"), it suffices to exhibit an object A such that $(A^\vee)^\vee \neq A$. Namely, consider $k[t]$ placed in degree 0. TODO: Finish this.*

Nevertheless, [FG12] proposes a conjecture about how to make this an equivalence. We say an O-algebra A is nilpotent if there exists an N such that $n > N$ implies $O(n) \otimes A^n \to A$ is zero (nulhomotopic), and we define $O\text{-alg}^{\mathrm{nil}}(\mathcal{C})$ to be the subcategory spanned by objects that are limits of nilpotent algebras (we call such objects *pro-nilpotent*).

Observe that the coBar functor lands in this subcategory: write $O^\vee = \lim_k O^{\vee, \leq k}$, where $O^{\vee,\leq k}$ is obtained by erasing $O^\vee(s)$ terms for all $s > k$. For $B \in O^\vee\text{-coalg}^{\mathrm{nil}}_{\mathrm{d.p.}}(\mathcal{C})$ and $A = \mathrm{coBar}_{O^\vee}(B)$, define $A^{\leq k} := \mathrm{coBar}_{O^{\vee,\leq k}}(B)$, then one can check that $A = \lim_{O\text{-alg}(\mathcal{C})}(A^{\leq k})$ and $O(s)$ acts on $A^{\leq k}$ by zero for $s > k$.

So by adjunction, the functor $\mathrm{Bar}_O^{\mathrm{naive}}$ factors as $\overline{\mathrm{Bar}_O^{\mathrm{naive}}} \circ \mathrm{compl}_{\mathcal{O}}$, where the completion functor $\mathrm{compl}_{\mathcal{O}}$ is the left adjoint to the limit-preserving embedding $O\text{-alg}^{\mathrm{nil}}(\mathcal{C}) \to O\text{-alg}(\mathcal{C})$ and $\overline{\mathrm{Bar}_O^{\mathrm{naive}}} : O\text{-alg}^{\mathrm{nil}}(\mathcal{C}) \to O^\vee\text{-coalg}^{\mathrm{nil}}_{\mathrm{d.p.}}(\mathcal{C})$.

Conjecture 2 (FG, 0-connective case by Ching-Harper)**.**

$$\overline{Bar_O^{naive}} : O\text{-}alg^{nil}(\mathcal{C}) \rightleftarrows O^\vee\text{-}coalg^{nil}_{d.p.}(\mathcal{C}) : coBar_{O^\vee}$$

is an equivalence of categories.

As we'll see momentarily, this is a generalization of the classical results in [BGS96] of the auto-equivalence of left finite Koszul algebras.

3.3.2 Koszul Duality for E_1 Algebras

Let us look at E_1-algebras, i.e. the case of associative algebras in Vect.

Theorem 3.1 ([Lur11, Corollary 3.1.15])**.** *Let A be an E_1-algebra. If A is coconnective and locally finite, then A is Koszul.*

Note that coconnective means $\pi_0(A) = k$, $\pi_i(A) = 0$ for $i > 0$, and locally finite means $\dim \pi_i(A) < \infty$ for each i.

Sample Computation Let's do a concrete example with chain complexes. Consider $k[x]$ for x in degree -1, so it is the complex $0 \to k \to k \to 0$ concentrated in degree 0 and -1. Let's compute what the (associative) Koszul dual $k \otimes^L_{k[x]} k$ is. The complex k (concentrated on degree 0) admits the following resolution

$$\ldots \to k[x][2] \to k[x][1] \to k[x] \to k$$

where the maps between complexes are given by

$$\begin{tikzcd} 0 \arrow[r] & k \arrow[r] \arrow[d] & k \arrow[r] \arrow[d, "id"] & 0 \arrow[d] & \\ & 0 \arrow[r] & k \arrow[r] & k \arrow[r] & 0 \end{tikzcd}$$

Thus we can compute the derived tensor product as

$$\mathrm{Tot}(\ldots \to k[2] \to k[1] \to k)$$

which is given by

$$\ldots \to 0 \to k \to 0 \to k$$

i.e. $k[y]$ for y placed in degree -2. Now we compute $\mathrm{coBar}(k[y]) = \mathrm{Hom}_{k[y]\text{-comod}}(k,k) = \mathrm{Hom}_{k[y^*]\text{-mod}}(k,k)$ where y^* is on degree 2. We use the following resolution:

$$0 \to k[y^*][-2] \to k[y^*] \to k \to 0$$

So the derived hom is given by

$$\mathrm{Tot}(k \to k[2] \to 0)$$

which is $k[x]$ again. More generally, if we place a vector space V on degree -1, then the algebra $\mathrm{Sym}(V[1])$ admits the following resolution:

$$\ldots \bigwedge^2 V \otimes (\mathrm{Sym}(V[1])[2]) \to V \otimes (\mathrm{Sym}(V[1])[1]) \to \mathrm{Sym}(V[1]) \to k \to 0$$

From which we can derive that $\mathrm{Sym}(V[1])^\vee = \mathrm{Sym}(V[2])$, considered as a coalgebra.

3.4 Koszul Duality for Modules

Now let $\mathcal{M}$ be the same as in section 1.3. By taking left adjoint to the trivial module functor $\mathcal{M} \to \mathrm{Mod}_A(\mathcal{M})$ we obtain another Bar functor, and similarly a cobar functor. One can check they again lift to an adjoint pair:

$$\mathrm{Bar}_A : \mathrm{Mod}_A(\mathcal{M}) \rightleftarrows \mathrm{Comod}^{\mathrm{nil}}_{A^\vee}(\mathcal{M}) : \mathrm{coBar}_{A^\vee}$$

which we call the *modular Koszul duality.* (Warning: this is slightly different from the one in [FG12, Section 7], where they used what we write as $\mathrm{Bar}^{\mathrm{naive}}_O$. When $\mathcal{C}$ is pro-nilpotent, however, those two notions will agree.)

Even if A is Koszul, there is no guarantee that its modular Koszul duality is an equivalence.

Proposition 4. *When $\mathcal{M}$ is pro-unipotent, these are equivalences.*

3.4.1 The E_1 Case Again

The module/comodule categories for the Koszul pair $\mathrm{Sym}(V[1])^\vee = \mathrm{Sym}(V[2])$ are not equivalent under the Bar/coBar adjunction; what does match up in this case are *finitely generated* $\mathrm{Sym}(V[-1])$ modules and *perfect* $\mathrm{Sym}(V[-2])$ comodules or, in a more recognizable language,

$$\mathrm{IndCoh}(\mathrm{SpecSym}(V[1])) \simeq \mathrm{QCoh}(\mathrm{SpecSym}(V^*[-2]))$$

More generally, we have:

Theorem 3.2 ([Lur11, 3.5.2])**.** *For A a small E_1-algebra (defined in [Lur11, 1.1.11]), there is an equivalence between the category of* ind-coherent *left/right modules (ind-object over small modules, i.e. those whose homotopy groups are finite dimensional) over A and that of left/right comodules over $A^\vee$.*

4 More on E_n Operads

The following, known as *Dunn Additivity*, is the key fact that makes things work:

Theorem 4.1 ([Dun88], [Lur, 5.1.2.2])**.** *For any n, m, we have E_{n+m}-alg$(\mathcal{C}) = E_n$-alg$(E_m$-alg$(\mathcal{C}))$.*

We will not try to prove this theorem, but let us mention that this has a generalization to factorization algebras. Namely it would follow from Lurie's result (locally constant factorization algebras on $\mathbb{R}^n$ are the same as E_n algebras) and the following statement:

Theorem 4.2 (Nick T-5?)**.** *For any manifolds M, N, the factorization algebras on M valued in factorization algebras on N are the same as factorization algebras on $M \times N$.*

In terms of left-right modules, E_k algebra also behave well (everything below would also hold for RMod):

Corollary 1 ([Lur, 4.8.5.20])**.** *For A an E_n-algebra and $\mathcal{M}$ as in section 1.3, $LMod_A(\mathcal{M})$ (where A is viewed as an E_1-algebra) are E_{n-1}-categories.*

In fact something stronger is true:

Corollary 2. *If $\mathcal{M}$ is such that for every $A \in E_n$-alg$(\mathcal{C})$, there exists $M_A \in LMod_A(\mathcal{M})$ such that $A \simeq End_A(M_A)$, then the functor $LMod_\bullet(\mathcal{M})$ (and $RMod_\bullet(\mathcal{M})$) is a fully faithful functor from E_n-alg$(\mathcal{C})$ to E_{n-1}-alg$(\mathcal{C}$-ModCat$)$.*

In particular this is satisfied by $\mathcal{M} = \mathcal{C}$ by taking $M_A = A$. In other words, specifying an E_n-algebra structure on A is equivalent to specifying an E_1-structure on A and an E_{n-1} structure on the representation category $\mathrm{LMod}_{\mathcal{C}}(A)$.

Example 5. *If A is an E_3 algebra, i.e. quasi-triangular Hopf algebra, then its module category is a braided monoidal (E_2) category.*

Now if we have a E_1-algebra A, its left module category would have no monoidal structure; however, its *bimodule* category would again have an E_1 structure. The general statement is the following:

Theorem 4.3 ([Lur, 3.4.4.6])**.** *For $\mathcal{M} = \mathcal{C}$ and $A \in E_n$-alg$(\mathcal{C})$, we have $Mod_A(\mathcal{C}) \in E_n$-alg$(A$-ModCat$)$.*

Remark 2. *The theorem is true more generally for O a coherent operad, as defined in [Lur, 3.3.1]. Also it should be straightforward to separate the exact condition on $\mathcal{M}$ for this to hold.*

4.1 (Co)Hochschild (Co)homology

Notice that when we take $\mathcal{M} = \mathcal{C}$, we have in particular $A \in \mathrm{Mod}_A(\mathcal{C})$, so it makes sense to discuss

$$HH^*_O(A) := \mathrm{End}_{A\text{-bimod}}(A)$$

(where the superscript indicates considering A as an O-algebra) and

$$HH^O_*(A) := A \otimes_{A\text{-bimod}} A.$$

We shall refer to them as the O-Hochschild cohomology/homology of A respectively. The following statement is usually referred to as (higher) *Deligne Conjecture*:

Proposition 5 ([Lur09, 2.5.13], [KS00], [Tam03])**.** *E_n-Hochschild cohomology of an E_n-algebra is an E_{n+1}-algebra.*

Example 6. *For C a monoidal category, its E_1-Hochschild cohomology would be E_2; this is the Drinfeld center.*

5 Koszul Duality for E_2 Algebras and Modules

5.1 Hopf Algebras

Define Bialg($\mathcal{C}$), the category of bialgebras in $\mathcal{C}$, to be

$$E_1\text{-alg}(E_1^*\text{-coalg}(\mathcal{C})) \simeq (E_1^*\text{-coalg}(E_1\text{-alg}(\mathcal{C}))$$

(That these two definitions are equivalent is not obvious and is checked in [GR, IV.2 Appendix C].) Let Hopf($\mathcal{C}$) denote the full subcategory where the monoidal (E_1) structure on the outside is in fact a group structure. This is the category of Hopf algebras that we'll consider.

Let us unpack this definition a bit. The E_1-algebra and the E_1^*-coalgebra structures give us the multiplication and comultiplication maps, and their corresponding augmentation/coaugmentation give the unit/counit maps. Finally, the existence of group structure gives the antipodal map.

Now recall there is a map $E_1 \to E_2$ of operads, which put all segments on the line $\{x = 0\} \subset \mathbb{R}^2$. This induces a map oblv : E_2-alg($\mathcal{C}$) $\to$ E_1-alg($\mathcal{C}$). In this section we explain the following observation by Tamarkin:

Proposition 6. *The shifted E_1 Koszul dual of an E_2 algebra is a Hopf algebra. In other words, we have a functor*

$$F : E_2\text{-}alg(\mathcal{C}) \to Hopf(\mathcal{C})$$

such that the following diagram commutes:

$$\begin{tikzcd}
E_2\text{-}alg(\mathcal{C}) \simeq E_1\text{-}alg(E_1\text{-}alg(\mathcal{C})) \arrow[r, "F"] \arrow[d, "oblv"] & Hopf(\mathcal{C}) \arrow[d, "oblv"] \\
E_1\text{-}alg(\mathcal{C}) \arrow[r, "{[1]\circ\vee}"] & E_1^*\text{-}coalg(\mathcal{C})
\end{tikzcd}$$

where both oblv maps forget about the outer E_1 structure.

Proof. Let Grp(E_1-alg($\mathcal{C}$)) be group objects in the category of E_1-algebras in $\mathcal{C}$. We first claim that the inclusion

$$\text{Grp}(E_1\text{-alg}(\mathcal{C})) \hookrightarrow \text{Monoid}(E_1\text{-alg}(\mathcal{C})) \simeq E_2\text{-alg}(\mathcal{C})$$

is in fact an equivalence. This is true when we replace E_1 with any operad and is proven in [GR, IV.2, Lemma 1.6.2].

So, if the Koszul duality functor $\vee$ is a strict monoidal functor, then $[1] \circ \vee$ would automatically land in Hopf($\mathcal{C}$). By a similar argument as [GR, IV.2, 4.2.6], it boils down to check that for $V, W \in \mathcal{C}$, we have

$$\text{Free}^{\text{gr}}(V \oplus W) = \text{Free}^{\text{gr}}(V) * \text{Free}^{\text{gr}}(W)$$

where $*$ is the free product (pushout) of *associative* algebras and Free^{gr} is the map

$$\mathcal{C} \xrightarrow{\text{deg}=1} \mathcal{C}^{\text{gr},\geq 0} \xrightarrow{\text{Free}} \text{Assoc}(\mathcal{C}^{\text{gr},\geq 0})$$

of first giving an object a grading by placing it on degree 1, then take the free associative algebra. Now it suffices to observe that both sides have the same universal property. ☐

For completeness we attach two more proofs for the case $\mathcal{C} = \text{Vect}$.

Proof by Tannakian Formalism. For any E_2-algebra A, recall that A-mod(Vect) is an E_1-algebra in DGCat, i.e. a monoidal DG category. Now apply modular Koszul to A-mod; in nice cases (TODO), this gives us $A^\vee$-comod for $A^\vee \in E_1^*[1]$-coalg, and by our remark above, the E_1 (monoidal) structure on A-mod gives a monoidal structure on $A^\vee$-comod. Furthermore, by definition, shift by 1 gives an isomorphism $A^\vee$-comod $\simeq$ ($A^\vee[1]$)-comod, equipped with an E_1 structure. Since it also comes with a monoidal forgetful map to the underlying Vect, by general Tannakian formalism we can reconstruct the Hopf algebra $A^\vee[1]$. ☐

Original Proof by Tamarkin. For any given operad $O \in \mathrm{Oprd}(\mathrm{Vect})$, the homology of O (with trivial differential) is again an operad, which we call the *homology operad* of O and denote by HO. The key fact is the following, which is usually referred to as *Kontsevich formality*:

Theorem 5.1 ([Tam03], [Kon97])**.** *$E_n \simeq HE_n$.*

The operad HE_n is P_n, the operad of Poisson n-algebras, that is, Poisson algebras whose brackets has degree $(1-n)$. Next, there is a combinatorially defined operad B_∞, that of the brace algebras.

Proposition 7 ([KS00])**.** *$B_\infty \simeq HB_\infty \simeq P_2$.*

This means that any E_2-algebra is automatically equipped with a B_∞-algebra structure. Finally, an explicit check (e.g. [Foi17]) shows that Bar construction maps B_∞-algebras to Hopf algebras. $\square$

Let us mention in the passing that ideas here also give another proof of the Etingof-Kazhdan quantization theorem, as noted by [Tamarkin07]. Namely, if $\mathfrak{g}$ is a Lie bialgebra, then $\mathrm{Sym}(\mathfrak{g}[-1])$ has, by definition, the structure of an P_2-algebra; then the procedure here would yield a (dg) Hopf algebra. One then checks that the resulting Hopf algebra is concentrated on degree 0, and the degree 0 piece is the quantization $U_\hbar(\mathfrak{g})$ we are looking for.

6 The General Case for E_n

Finally we list some facts about general E_n algebras and modules; note that those are mostly bootstrapped from the E_1 case.

Proposition 8. *Applying Kozdul duality on E_n-algebra is the same thing as applying on it the E_1 Koszul duality n times.*

Proposition 9. *An E_n-module structure is equivalent to n separate E_1-module structures.*

Proof. TODO: Need to supply proof for both facts. $\square$

From both facts we can conclude:

Corollary 3. *For A a small E_n-algebra, the category of ind-coherent left/right modules over A is equivalent to that of left/right comodules over $A^\vee$ (the E_n dual).*

Sketch. Apply E_1 Koszul duality on A n times, each time on a different E_1-algebra structure, to obtain $A_0 = A, A_1, \ldots, A_n = A^\vee$, where $A_j \in E_{n-j}\text{-alg}(E_j\text{-coalg})$ is given as the 1-shifted E_1 Koszul dual of A_{j-1}. Define $\mathcal{C}_j$ to be the category of objects that have $(n-j)$ ind-coherent module structures and j comodule structures over A_j, then repeated application of E_1 duality yields $\mathcal{C}_0 \simeq \ldots \simeq \mathcal{C}_n$. $\square$

Proposition 10 ([Lur11, 4.4.5])**.** *Let A be an E_n-algebra that is n-coconnective (meaning $\pi_i = 0$ for $i \geq n$) and locally finite. Then A is Koszul.*

Proposition 11 ([AF14])**.** *Let $\wedge$ denote the linear dual. Then $HH_*^{E_n}((A^\vee[n])^\wedge) \simeq HH_*^{E_n}(A)^\wedge$ for $A \in E_n$-alg that is $(-n)$-coconnective.*

References

[AF14] David Ayala and John Francis. "Poincaré/Koszul duality". In: (2014).

[BGS96] Alexander Beilinson, Victor Ginzburg, and Wolfgang Soergel. "Koszul Duality Patterns in Representation Theory". In: *Journal of the American Mathematical Society* 9.2 (1996), pp. 473–527. ISSN: 0894-0347. DOI: 10.1090/s0894-0347-96-00192-0. URL: http://dx.doi.org/10.1090/s0894-0347-96-00192-0.

[Dun88] Gerald Dunn. "Tensor product of operads and iterated loop spaces". In: *Journal of Pure and Applied Algebra* 50.3 (1988), pp. 237–258. ISSN: 0022-4049. DOI: 10.1016/0022-4049(88)90103-x. URL: http://dx.doi.org/10.1016/0022-4049(88)90103-x.

[FG12] John Francis and Dennis Gaitsgory. "Chiral Koszul duality". In: *Sel Math* 18.1 (2012), pp. 27–87. ISSN: 1022-1824. DOI: 10.1007/s00029-011-0065-z.

[Foi17] Loïc Foissy. "Algebraic structures associated to operads". In: (2017).

[Fre11] Benoit Fresse. "Koszul duality of En-operads". In: *Sel Math* 17.2 (2011), pp. 363–434. ISSN: 1022-1824. DOI: 10.1007/s00029-010-0047-6.

[GJ94] Ezra Getzler and John DS Jones. "Operads, homotopy algebra and iterated integrals for double loop spaces". In: (1994).

[GK94] Victor Ginzburg and Mikhail Kapranov. "Koszul duality for operads". In: *Duke Mathematical Journal* 76.1 (1994), pp. 203–272. ISSN: 0012-7094. DOI: 10.1215/s0012-7094-94-07608-4. URL: http://dx.doi.org/10.1215/s0012-7094-94-07608-4.

[GR] Dennis Gaitsgory and Nick Rozenblyum. *A Study in Derived Algebraic Geometry*.

[Kel99] Bernhard Keller. "Introduction to A-infinity algebras and modules". In: (1999).

[Kon97] Maxim Kontsevich. "Deformation quantization of Poisson manifolds, I". In: (1997). DOI: 10.1023/b:math.0000027508.00421.bf. URL: http://dx.doi.org/10.1023/b:math.0000027508.00421.bf.

[KS00] Maxim Kontsevich and Yan Soibelman. "Deformations of algebras over operads and Deligne's conjecture". In: (2000).

[LH03] Kenji Lefèvre-Hasegawa. "Sur les A-infini catégories". In: (Mar. 2003).

[Lur] Jacob Lurie. *Higher Algebra*.

[Lur09] Jacob Lurie. "Derived Algebraic Geometry VI: E[k]-Algebras". In: (Oct. 2009).

[Lur11] Jacob Lurie. "Derived Algebraic Geometry X: Formal Moduli Problems". In: (Nov. 2011).

[Tam03] Dmitry E. Tamarkin. "Formality of Chain Operad of Little Discs". In: *Letters in Mathematical Physics* 66.1/2 (Mar. 2003), pp. 65–72. ISSN: 0377-9017. DOI: 10.1023/b:math.0000017651.12703.a1. URL: http://dx.doi.org/10.1023/b:math.0000017651.12703.a1.