Proof of Kevin Costello's Main Theorem

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1. **Introduction.** We will start by recalling some notation and the main theorem of the paper. Throughout the talk, $\mathfrak{g}$ wll be a simple Lie algebra with an invariant pairing $\langle -,-\rangle_{\mathfrak{g}}$. Using the invariant pairing and the residue pairing there are identifications $(\mathfrak{g}[\![z]\!])^\vee \cong \mathfrak{g}^\vee[\partial_z] \cong z^{-1}\mathfrak{g}[z^{-1}]$, where $(-)^\vee$ is the linear topological dual. Consider the theory on $\mathbb{C}_z \times \mathbb{C}_w$ with action given by

$$S(A) = \frac{1}{2\pi i}\int dz CS(A) \tag{1}$$

where

$$CS(A) = \frac{1}{2}\langle A, dA\rangle + \frac{1}{6}\langle A, [A,A]\rangle. \tag{2}$$

The z direction is holomorphic while the w direction is topological. The local L_∞ algebra associated to this theory is given by

$$\mathcal{L}(U) = (\Omega^{0,*}(U,\mathfrak{g}), \partial_z) \xrightarrow{\frac{\partial}{\partial w}} (\Omega^{0,*}(U,\mathfrak{g}), \partial_z)[-1] \tag{3}$$

for each open $U \subseteq \mathbb{C}_z \times \mathbb{C}_w$. Denote the factorization algebra of classical (resp. quantum) observables by Obs^{cl} (resp. Obs^{cl}). Note that the factorization algebra of classical observables given by

$$\mathrm{Obs}^{cl}(U) = C^*(\mathcal{L}(U)) = \mathrm{Sym}(\mathcal{L}(U)^\vee[-1], d_{CE}). \tag{4}$$

The following construction can be applied to both the classical and quantum observables, so the superscript is omitted. Let $\pi : \mathbb{C}^2 \to \mathbb{C}_w$ be the projection. For any open subset $U_z \subseteq \mathbb{C}_z$, let

$$\mathrm{Obs}_{U_z} := \pi_*(\mathrm{Obs}|_{U_z\times\mathbb{C}_w}). \tag{5}$$

Obs_{U_z} is locally constant, hence gives an E_2 algebra. Let $D(z_0,r)$ be the disk of radius r centered at z_0. Then S^1 acts by rotation about z_0 which gives an action on the E_2 algebra $\mathrm{Obs}_{D(z_0,r)}$. Let

$$\mathrm{Obs}_{z_0} = \oplus_{k\in\mathbb{Z}}\mathrm{Obs}^k_{D(z_0,r)}, \tag{6}$$

where $\mathrm{Obs}^k_{D(z_0,r)}$ is the weight k eigenspace of the S^1 action. Obs_{z_0} is then a $\mathbb{Z}$-graded E_2 algebra over $\mathbb{C}[\![\hbar]\!]$ where $\hbar$ has weight 1. This is essentially the same construction that is given in [Costello-Gwilliam] that extracts a vertex algebra from a holomorphic field theory on $\mathbb{C}$.

Recall that $\mathrm{Obs}^q_{z_0}$ has an augmentation map $\mathrm{Obs}^q_{z_0} \to \mathbb{C}[\![\hbar]\!]$ coming from considering $\mathbb{C}_z \times \mathbb{C}_w \subset \mathbb{P}^1_z \times \mathbb{C}_w$ and the fact that $\mathrm{Obs}^q_{\mathbb{P}^1}$ is quasi-isomorphic to $\mathbb{C}[\![\hbar]\!]$. This augmentation is necessary for the Koszul duality employed in this talk.

Definition: For $\mathfrak{g}$ a simple Lie algebra with invariant pairing, make $\mathfrak{g}[\![z]\!]$ into a graded Lie algebra by giving Xz^k degree k $\forall X \in \mathfrak{g}$. The Yangian $Y(\mathfrak{g})$ is the unique topological Hopf algebra flat over the ring $\mathbb{C}[\![\hbar]\!]$ with the following properties.

1) $Y(\mathfrak{g})$ has a grading where the central parameter $\hbar$ has degree 1.

2) There is an isomorphism of Hopf algebras

$$Y(\mathfrak{g}) \otimes_{\mathbb{C}[\![\hbar]\!]} \mathbb{C}[\![\hbar]\!]/(\hbar) \cong \mathcal{U}(\mathfrak{g}[\![z]\!]). \tag{7}$$

3) Any defromation of the Hopf algebra $\mathcal{U}(\mathfrak{g}[\![z]\!])$ gives a Lie bialgebra structure on $\mathfrak{g}[\![z]\!]$. The Lie bialgebra structure that quantizes to the Yangian has the coproduct coming from the map $\mathfrak{g}[\![z]\!] \to \mathfrak{g}[\![z_1]\!] \otimes \mathfrak{g}[\![z_2]\!]$ given by $x \mapsto \frac{[x,c]}{z_1 - z_2}$, where c is the quadratic Casimir of $\mathfrak{g}$.

The dual yangian $Y^*(\mathfrak{g})$ is the topological Hopf algebra that is the continuous linear dual of $Y(\mathfrak{g})$ given by

$$Y^*(\mathfrak{g}) = \mathrm{Hom}_{\mathbb{C}[\![\hbar]\!]}(Y(\mathfrak{g}), \mathbb{C}[\![\hbar]\!]). \tag{8}$$

These maps are not necessarily graded. $Y^*(\mathfrak{g})$ is isomorphic to $\widehat{\mathrm{Sym}}^*(\mathfrak{g}^\vee[\partial_z])[\![\hbar]\!]$ as vector spaces. The product and coproduct are conitnuous $\mathbb{C}[\![\hbar]\!]$-linear maps that are dual to the product and coproduct denoted by

$$m : Y^*(\mathfrak{g}) \otimes_{\mathbb{C}[\![\hbar]\!]} Y^*(\mathfrak{g}) \to Y^*(\mathfrak{g}) \tag{9}$$

$$c : Y^*(\mathfrak{g}) \to Y^*(\mathfrak{g}) \otimes_{\mathbb{C}[\![\hbar]\!]} Y^*(\mathfrak{g}) \tag{10}$$

where $\otimes_{\mathbb{C}[\![\hbar]\!]}$ is the completed projective tensor product.

2. **The Main Theorem.** Now we are ready to state the main theorem of Costello's paper.

Theorem: Let $\mathfrak{g}$ be a simple Lie algebra with invariant pairing $\langle -, - \rangle_{\mathfrak{g}}$, then there is an isomorphism of Hopf algebras

$$Y^*(\mathfrak{g}) \cong H^*(\mathrm{Obs}^{q!}_{z_0}) = H^*(\mathbb{C}[\![\hbar]\!] \otimes^{\mathbb{L}}_{\mathrm{Obs}^q_{z_0}} \mathbb{C}[\![\hbar]\!]). \tag{11}$$

Here is what is needed to verify that $\mathrm{Obs}^{q!}_{z_0}$ satisfies the conditions characterizing the dual Yangian.

1) $\mathrm{Obs}^{q!}_{z_0}$ has a grading where $\hbar$ is central with weight 1.

2) $\mathrm{Obs}^{q!}_{z_0} \mathrm{mod} \hbar \simeq \mathcal{U}(\mathfrak{g}[\![z]\!])^\vee$ (quasi-isomorphic).

3) The Lie bialgebra structure on $(\mathfrak{g}[\![z]\!])^\vee = \mathfrak{g}^\vee[\partial_z]$ that quantizes to $\mathrm{Obs}^{q!}_{z_0}$ is dual to $\mathfrak{g}[\![z]\!] \to \mathfrak{g}[\![z_1]\!] \otimes \mathfrak{g}[\![z_2]\!]$ given by $x \mapsto \frac{[x,c]}{z_1 - z_2}$, where c is the quadratic Casimir of $\mathfrak{g}$.

Note that this is a completed version of the Yangian typically considered in the study of quantum groups. Additionally, the above statement in terms of quasi-siomorphisms are sufficient because taking cohomology of the observables gives the strict isomorphism in the theorem.

Proof. For 1, $\mathrm{Obs}^q_{z_0}$ is a graded E_2 algebra over $\mathbb{C}[\![\hbar]\!]$ so the Koszul dual $\mathrm{Obs}^{q!}_{z_0}$ is a graded Hopf algebra over $\mathbb{C}[\![\hbar]\!]$. For 2, we have a quasi-isomorphism of graded E_2 algebras $\mathrm{Obs}^q_{z_0}\mathrm{mod}\hbar \simeq C^*(\mathfrak{g}[\![z]\!])$. Koszul duality then gives $\mathrm{Obs}^{q!}_{z_0}\mathrm{mod}\hbar \simeq \mathcal{U}(\mathfrak{g}[\![z]\!])^\vee$. Part 3 is simplified by invoking Drinfeld's result, which tells us that the Yangian is completely determined as a Hopf algebra by it's semi-classical ($\mathrm{mod}\hbar^2$) structure. The rest of the talk will focus on showing that the semi-classical structure of $Y^*(\mathfrak{g})$ and $H^*(\mathrm{Obs}^{q!}_{z_0})$ coincide.

The general idea for how to prove 3 is to reduce to a commutator calculation for observables of a 3-dimensional theory. A rough sketch of how this is done is as follows. Suppose we start with a factorization algebra $\mathcal{F}$ on $\mathbb{C}_w = \mathbb{R}^2$ valued in vector spaces (if $\mathcal{F}$ is valued in chain complexes, then the below discussion can be applied to $H^*\mathcal{F}$). This factorization algebra can then restrict to give a factorization algebra on $\mathbb{R}^2\backslash\{0\}$, which can also be considered as a factorization algebra on $S^1 \times \mathbb{R}$. Using the projection $\pi : S^1 \times \mathbb{R} \to \mathbb{R}$, we can consider the pushforward factorization algebra $\pi_*\mathcal{F}$. If $\mathcal{F}$ is locally constant, then so is $\pi_*\mathcal{F}$. Locally constant factorization algebras on $\mathbb{R}$ are equivalent to associative algebras so $\pi_*\mathcal{F}$ gives an associative algebra $A_\mathcal{F}$, called the annular algebra (the preimage of open intervals in $\mathbb{R}$ are open annuli in $S^1\times\mathbb{R}$). This idea will be applied to $\mathcal{F} = \mathrm{Obs}^q_{z_0}$. The goal is then to compute the commutator of linear observables in the semi-classical limit using Feynman diagrams.

3. **Poisson Bracket Construction.** The following poisson bracket construction will be needed to compute the commutator mentioned above. Let A be any filtered associative dg algebra over $\mathbb{C}[\hbar]/\hbar^2$ such that $A^{cl} = A/\hbar$ is commutative and $\hbar F^iA \subset F^{i+2}A$. Note that the filtration is decreasing, i.e. $F^iA \supseteq F^{i+1}A$, so that $F^iA \supseteq F^{i+1}A \supseteq F^{i+2}A \supseteq \hbar F^iA$. Consider elements $\alpha \in H^*\mathrm{Gr}^iA^{cl}$ and $\beta \in H^*\mathrm{Gr}^jA^{cl}$. Let $\widetilde{\alpha} \in F^iA$ and $\widetilde{\beta} \in F^jA$ be lifts of α and β, respectively, which are not necessarily closed. The commutator $[\widetilde{\alpha}, \widetilde{\beta}] \in F^{i+j}A$ is closed in $Gr^{i+j}A$. A^{cl} is commutative, so $H^*(A)\mathrm{mod}\hbar$ is also commutative which implies that $[\widetilde{\alpha}, \widetilde{\beta}] = \hbar\widetilde{\gamma}$ for some $\widetilde{\gamma} \in F^{i+j-2}$. $\hbar\widetilde{\gamma}$ is then closed mod $\hbar F^{i+j-1}$. Taking the cohomology class γ of $\widetilde{\gamma}$ in $H^*\mathrm{Gr}^{i+j-2}A^{cl}$ then defines a Poisson bracket

$$\{-,-\}_{-2} : H^*\mathrm{Gr}^iA^{cl} \times H^*\mathrm{Gr}^jA^{cl} \to H^*\mathrm{Gr}^{i+j-2}A^{cl} \tag{12}$$

gven by $\{\alpha, \beta\}_{-2} = \gamma$. If this Poisson bracket is zero, then $\widetilde{\gamma}$ is exact $\mathrm{mod}F^{i+j-1}$ so we can then consider $\hbar\widetilde{\gamma}' \in \hbar F^{i+j-1}$ which is closed mod $\hbar F^{i+j}$ and such that $d(\hbar\widetilde{\gamma}') = \hbar d(\widetilde{\gamma}') = \hbar\widetilde{\gamma}$. Taking the cohomology class γ' of $\widetilde{\gamma}'$ then gives a second Poisson bracket

$$\{-,-\}_{-1} : H^*\mathrm{Gr}^iA^{cl} \times H^*\mathrm{Gr}^jA^{cl} \to H^*\mathrm{Gr}^{i+j-1}A^{cl} \tag{13}$$

by $\{\alpha, \beta\}_{-1} = \gamma'$.

4. **The Observables for the Computation.** From the classical observables considered above, Koszul duality gives $\mathrm{Obs}^{cl!}_{z_0} = \mathcal{U}(\mathfrak{g}[\![z]\!])^\vee$. We then get an isomorphism of filtered algebras $\mathrm{Hoch}_*(\mathrm{Obs}^{cl}_{z_0}) \cong \mathrm{Hoch}_*(\mathcal{U}(\mathfrak{g}[\![z]\!]))^\vee$ where Hoch_* denotes the Hochshild chains. The Koszul dual symbol doesn't appear on the left because of Morita invariance. Taking cohomology of the associated graded then gives an isomorphism $HH_*(\mathrm{GrObs}^{cl}_{z_0}) \cong HH_*(\mathrm{Sym}(\mathfrak{g}[\![z]\!]))^\vee$. Both of these algebras have a Poisson bracket that describe the semi-classical behavior. The

coproduct of the Yangian determines the Poisson bracket on right side while quantization of the field theory determines the Poisson bracket on the left side. Showing that these Poisson brackets coincide up to a non-zero constant is enough prove part 3 above.

The cocommutator on $Y(\mathfrak{g})\mathrm{mod}\hbar^2$ gives a co-poisson bracket δ on the associated graded $\mathrm{Sym}(\mathfrak{g}[\![z]\!]))$. This is relevant because $HH_0((\mathrm{Sym}(\mathfrak{g}[\![z]\!])) = \mathrm{Sym}(\mathfrak{g}[\![z]\!])$ (which is true for any commutative algebra) so that we only need to consider the Poisson bracket on $HH_0(\mathrm{GrObs}^{cl}_{z_0})$ to determine the semi-classical behavior.

By the Hochschild-Kostant-Rosenberg Theorem, $HH_*(\mathrm{Gr}C^*(\mathfrak{g}[\![z]\!])) = \widehat{\mathrm{Sym}}(\mathfrak{g}[\![z,\delta]\!]^\vee[-1])$ where δ is a parameter of degree 1. Consider the classical observables $\mathrm{Obs}^{cl,\times}_{z_0}$ on $\mathbb{C}_z \times \mathbb{C}^\times_w$ obtained, as above, by taking the direct sum of weight k eigenspaces of the S^1 action of rotation of $\mathbb{C}_z$ about z_0. Then $\mathrm{Obs}^{cl,\times}_{z_0} = C^*(\Omega^*(\mathbb{C}^\times_w)[\![z]\!] \otimes \mathfrak{g})$, which gives $H^0(\mathrm{Obs}^{cl,\times}_{z_0}) = \mathrm{Sym}(\mathfrak{g}[\![z]\!]^\vee \otimes H^1(S^1))$ where $H^1(S^1)$ is in cohomological degree 0. Thus we can make the following identifications

$$HH_0(\mathrm{GrObs}^{cl}_{z_0}) = HH_0(\mathrm{Gr}C^*(\mathfrak{g}[\![z]\!])) = H^0(\mathrm{GrObs}^{cl,\times}_{z_0}). \tag{14}$$

Let's try to give some idea of where this comes from. Locally constant factorization algebras on $\mathbb{R}$ correspond to associative algebras. If we now consider a locally constant factorization algebra $\mathcal{F}$ on S^1, any open interval in S^1 looks like an open interval in $\mathbb{R}$ so the difference between locally constant factorization algebras on S^1 and those on $\mathbb{R}$ can only be seen by looking at the global sections on S^1. If A is the vector space that $\mathcal{F}$ associates to an open interval in S^1, then A has an associative algebra structure. It turns out that (cohomology of) the global sections of $\mathcal{F}$ computes the Hochschild homology of A.

5. **Applying the Bracket Construction.** The quantum observables $\mathrm{Obs}^{q,\times}_{z_0}$ have a homotopy associative product coming from the operator product. By considering the commutator on this space mod$\hbar^2$, the above discussion (where $A = \mathrm{Obs}^{q,\times}_{z_0}/(\hbar^2)$) gives a Poisson bracket on $H^0(\mathrm{GrObs}^{cl,\times}_{z_0})$. The first bracket is homotopically trivial on $\mathrm{GrObs}^{cl,\times}_{z_0}$ because $\mathrm{Obs}^{q,\times}_{z_0}$ is an augmented E_2 algebra. Computing this second Poisson bracket is what is needed to finish the proof of the theorem. This secondary bracket gives a Lie bialgebra structure on $\mathfrak{g}[\![z]\!]$ that is compatible with the given Lie algebra structure. The cobracket is given by

$$z^k\mathfrak{g} \to \Sigma_{r+s=k-1} z^r\mathfrak{g} \otimes z^s\mathfrak{g}. \tag{15}$$

Since $\mathfrak{g}$ is simple, there is only one Lie bialgebra structure on $\mathfrak{g}[\![z]\!]$ with these properties, where the cobracket

$$z\mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g} \tag{16}$$

is dual to the bracket on $\mathfrak{g}$ under the invariant pairing.

For $X \in \mathfrak{g}$ define the linear observable $\oint_{|w|=r}\oint_{|z|=1} z^{-k-1}X : \Omega^1(\mathbb{C}^\times_w)\widehat{\otimes}\mathbb{C}[\![z]\!] \otimes \mathfrak{g} \to \mathbb{C}$ in $\mathrm{Obs}^{cl,\times}_{z_0}$ by

$$\oint_{|w|=r}\oint_{|z|=1} z^{-k-1}X(\alpha \otimes f(z) \otimes Y) = \langle X, Y\rangle_{\mathfrak{g}}(\frac{\partial^k f}{\partial z^k})|_{z=0}\int_{|w|=r}\alpha. \tag{17}$$

This observable is closed and of degree 0 in $\mathrm{GrObs}^{cl,\times}_{z_0}$. To determine the Poisson bracket on $H^0(\mathrm{GrObs}^{cl,\times}_{z_0})$, it is sufficient to consider the case $k=0$, so will then drop it from the notation and denote the corresponding observable $\oint_{|w|=r}\oint_{|z|=1} z^{-1}X$ by $\oint_{|w|=r} X$. Note that under the residue pairing and invariant pairing, $\oint_{|w|=r}\oint_{|z|=1} z^{-k-1}X$ corresponds to $k!Xz^{-k-1}$. So, the lie coalgebra structure on $\mathfrak{g}[\![z]\!]$ is naturaly identified with the natural Lie algebra structure on $z^{-1}\mathfrak{g}[z^{-1}]$. The obsevable defined above can then be thought of as an element in the image of a map $z^{-1}\mathfrak{g}[z^{-1}] \to \mathrm{Obs}^{cl,\times}_{z_0}$ given by $z^{-k-1}X \mapsto \oint_{|w|=r}\oint_{|z|=1} z^{-k-1}X$.

Before making this computation explicit, we first change coordinates by identifying $\mathbb{C}^\times_w$ with $S^1\times\mathbb{R}$ and letting $z_0=0$. Let $S^1_r = 0\times S^1\times r \subset \mathbb{C}_z\times S^1\times\mathbb{R}$. The observable $\oint_{|w|=r} X$ then can be expressed as

$$\oint_{|w|=r} X : \phi \mapsto \oint_{S^1_{|w|=r}} \langle X, \phi\rangle_{\mathfrak{g}} \tag{18}$$

which only depends on the component of ϕ in $\Omega^0(\mathbb{C})\widehat{\otimes}\Omega^1(S^1)\widehat{\otimes}\Omega^0(\mathbb{R})\otimes\mathfrak{g}$. The observable integrates around a circle of radius r and evaluates at $z=0$ and $w=r$. $\mathbb{R}$ can be thought of as a parametrizating the radius of the circle that we are integrating over. The cohomology class of the observable is independent of $r\in\mathbb{R}$, hence we can denote the cohomology class as $\oint_{S^1_w}\oint_{|z|=1} z^{-k-1}X$ for general k, and $\oint_{S^1_w} X$ for $k=0$.

6. **Computing the Bracket via Feynman Diagrams.** To compute the Poisson bracket of two observables, we will first compute the commutator of lifts of the observables in $\mathrm{Obs}^{q,\times}_0 \mathrm{mod}\hbar^2$. Denote such a lift of $\oint_{|w|=r} X$ by $\oint_{|w|=r}\widetilde{X}$. The commutator of two observables will be written as

$$\left[\oint_{|w|=r}\widetilde{X}, \oint_{|w|=s}\widetilde{Y}\right], \text{ for } r<s\in\mathbb{R}. \tag{19}$$

Note that this is computed in the quantum observables, hence will depend on the length scale L corresponding to the infrared cutoff. Considering $\mathrm{Obs}^{q,\times}_0 \mathrm{mod}\hbar^2$ as A in the Poisson bracket construction above, this commutator descends to a Poisson bracket $\{-,-\}_{-1}$ on $H^*(\mathrm{GrObs}^{cl,\times}_0)$. Computing the bracket is done by computing the coefficient of $\hbar$ in the commutator and then taking the cohomology class of the coefficient. This is where the Feynman diagram calculation comes in.

It is sufficient to compute this commutator on the field zZ for $Z\in\mathfrak{g}$ because it vanishes on terms with the order of z not equal to 1. This is because the bracket in $z^{-1}\mathfrak{g}[z^{-1}]$ is $[z^{-1}X, z^{-1}Y] = z^{-2}[X,Y]$, so the commutator in the observables has a $k=2$ component corresponding to $\frac{\partial}{\partial z}|_{z=0}$.

Before we explicitly write out the commutator, we need to pick a gauge fixing condition to define a propogator. Let $Q^{GF} = \bar\partial^*_z + d^*_w$ and let the corresponding propagator be $P\in\bar{\mathcal{E}}\widehat{\otimes}\bar{\mathcal{E}}$. P can be expressed in terms of the heat kernel

$$K_t = K^{scalar}_t d(\bar z_1 - \bar z_2)d(\bar w_1 - \bar w_2)d(w_1-w_2) \in \mathcal{E}\otimes\mathcal{E} \tag{20}$$

where K^{scalar} is the scalar heat kernel for the Laplacian on $\mathbb{C}_z\times\mathbb{C}^\times_w$. The analytic component of the propagator $P(0,L)$ with infrared cutoff (i.e. length scale L) is then given by

$$\int_{t=0}^{L}((\bar{\partial}_z^* + d_w^*)\otimes 1)K_t dt. \tag{21}$$

Note that P can be written as $c\otimes P_0$ where c is the quadratic Casimir corresponding to the Lie algebra component and P_0 is the analytic integral component. The component of $\left[\oint_{|w|=r}\widetilde{X},\oint_{|w|=s}\widetilde{Y}\right][L](zZ)$ that is the coefficient of $\hbar$ then becomes

$$\langle [X,Y],Z\rangle_{\mathfrak{g}}\left\{\oint_{|w_0|=r,z_0=0}\oint_{|w_2|=s,z_0=0} - \oint_{|w_0|=s,z_0=0}\oint_{|w_2|=r,z_0=0}\right\}\int_{z_1,w_1\in\mathbb{C}_z\times\mathbb{C}_w^\times} \tag{22}$$
$$P(0,L)((z_0,w_0),(z_1,w_1))z_1dz_1P(0,L)((z_1,w_1),(z_2,w_2)).$$

Only the cohomology of the observable matters for this calculation, so we can consider the case $r=0$ without loss of generality. The cohomology class of the product $\oint_{|w|=0}\widetilde{X}\oint_{|w|=s}\widetilde{Y}$ then does not change as $s>0$ varies. This describes a family of observables on $\mathbb{R}\backslash\{0\}$. The commutator then computes the jump of this observable as s crosses 0.

The above integral is computed using the propagator for a theory with space of fields

$$\Omega^{0,*}(\mathbb{C}_z)\widehat{\otimes}\Omega^*(\mathbb{R})\widehat{\otimes}\Omega^*(S^1)\otimes\mathfrak{g}[1] \tag{23}$$

X_s and Y_0 are zero when evaluated on fields that have the $\Omega^*(S^1)=\mathcal{H}(S^1)\oplus\mathcal{H}(S^1)^\perp$ component in $\mathcal{H}(S^1)^\perp$. Hence we can project the above propagator onto $\{\Omega^{0,*}(\mathbb{C}_z)\widehat{\otimes}\Omega^*(\mathbb{R})\widehat{\otimes}\mathcal{H}(S^1)\}^{\otimes 2}$ without changing the value of the observable. This is evidently the propagator for a 3-dimensional theory with space of fields given by

$$\Omega^{0,*}(\mathbb{C}_z)\widehat{\otimes}\Omega^*(\mathbb{R})\otimes\mathfrak{g}[\delta][1] \tag{24}$$

where δ corresponds to the class $\frac{d\theta}{2\pi}$ in $H^1(S^1)$. This field theory traditionally has partial connections

$$A = A_w dw + A_z d\bar{z} \tag{25}$$

and

$$B\in s\Omega^0(\mathbb{C}_z\times\mathbb{R}_w) \tag{26}$$

with actional functional

$$\int dz\langle B,F(A)\rangle_{\mathfrak{g}}. \tag{27}$$

The Lie algebra of the gauge group is $\Omega^0(\mathbb{C}_z\times\mathbb{R}_w)\otimes\mathfrak{g}$. Note that the variable w is now a real variable. Now let $P=c\otimes P_0$ be the propagator for this new theory. We are then computing

$$\int_{z_1,w_1} P_0((0,0),(z_1,w_1))z_1dz_1P_0((z_1,w_1),(0,s)) \tag{28}$$

We want to show that this is a Heaviside step function in s. To show this, we first note that the propagator for this theory is

$$Q^{GF} = 2\bar{\partial}_z^* + d_w^* \tag{29}$$

The factor of 2 is a convenient normalization because $[\bar{\partial}, 2\bar{\partial}^*]$ is the usual Laplacian. Let

$$\Delta = -\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} - \frac{\partial^2}{\partial w^2} \tag{30}$$

be the Laplacian. The propagator above is the kernel for the operator $Q^{GF}\Delta^{-1}$, which acts on $\Omega^{0,*}(\mathbb{C})\widehat{\otimes}\Omega^*(\mathbb{R})$. The above integral is then the kernel for the operator

$$Q^{GF}\Delta^{-1}zQ^{GF}\Delta^{-1}. \tag{31}$$

The identity

$$Q^{GF}\Delta^{-1}zQ^{GF}\Delta^{-1} = -Q^{GF}\frac{\partial}{\partial d\bar{z}}(\Delta^{-1})^2 \tag{32}$$

follows from the fact that $(Q^{GF})^2 = 0$ and $[z, Q^{GF}] = 2[z, \bar{\partial}_z^*] = -\frac{\partial}{\partial d\bar{z}}$. The following identity

$$-Q^{GF}\frac{\partial}{\partial d\bar{z}} = \frac{\partial}{\partial d\bar{z}}d_w^* \tag{33}$$

The kernel for $(\Delta^{-1})^2$ is

$$\alpha\sqrt{|z|^2 + w^2}d\bar{z}dw \tag{34}$$

where $\alpha \neq 0$ is a constant, so the kernel for $Q^{GF}\Delta^{-1}zQ^{GF}\Delta^{-1}$ is

$$\alpha\frac{d}{dw}\sqrt{|z|^2 + w^2} = \alpha\frac{w}{\sqrt{|z|^2 + w^2}}. \tag{35}$$

Upon setting $z = 0$, we see that the above integral is a Heaviside step function in s with value α^{-1} when $w > 0$ and $-\alpha^{-1}$ when $w < 0$. The commutator we are computing is in terms of the jump as we corss the origin, so we are done.

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