The Yangian and four dimensional gauge theory

The purpose of this course is to explore a connection between a certain class of gauge theories defined on complex surfaces and deformations of infinite dimensional Lie algebras. Mathematically, a gauge theory involves studying certain properties of connections defined on bundles over a smooth manifold. Often these theories are motivated from physics (such as supersymmetric gauge theories), but often have neat and concise mathematical descriptions. The type of infinite dimensional Lie algebras we study are of the form $\mathfrak{g}[[z]]$ where z is a formal variable. The *Yangian* deformations are a type of quantum group, and are realized as deformations of the universal enveloping algebra $U(\mathfrak{g}[[z]])$. These deformations have appeared in various mathematical contexts, most notably in the theory of integrable systems and statistical mechanics.

The gauge theory we study is defined on manifolds of the form $\Sigma \times S$ where Σ is Riemann surface and S is a real two-dimensional manifold. Mathematically, the class of connections defining the gauge theory are, in a precise sense, *holomorphic* in the direction of Σ and *flat* in the direction of S. The connection between gauge theory and quantum groups we will study is similar in spirit to perhaps a more well-studied relationship between Chern-Simons theory, the study of flat connections on three-manifolds, and quantum groups.

We follow the seminal work of Costello [Cosa, Cosb] showing how Yangians arise from the algebra of operators of the four-dimensional gauge theory. The key to this result relies on the formalism of Costello-Gwilliam [CG17, CG] that the algebra of operators of a quantum field theory form a *factorization algebra*. This is a vast generalization of the description of algebras of operators in conformal field theory as vertex algebras. Factorization algebras simultaneously generalize the notion of a vertex algebra and algebras over more familiar operads, such as the operad of little disks. The primary goal of this seminar is to study Costello's construction which starts from the factorization algebra description of the operators of a four dimensional gauge theory and recovers the Yangian quantum group of the gauge Lie algebra. Time permitting, we can focus on more concrete and computational sides of the program began in the works [CWYa, CWYb].

Some keywords: Factorization algebras, Koszul duality, quantum groups, Yangians, complex geometry.

Informal prerequisites: Basic category theory, homological algebra, and Lie algebras. Knowledge of some basic differential geometry including vector bundles, connections, and differential forms.

Instructors: Chris Beasley, Valerio Toledano-Laredo, Brian Williams BW: more?

Detailed (approximate) syllabus

Week 1, Sep 4–7 *An introduction to factorization algebras*. The definition of a factorization algebra with values in a symmetric monoidal category. Lurie's result that locally constant factorization algebras on $\mathbb{R}^n$ are equivalent to E_n-algebras. Hochschild homology as a special case of factorization homology. [CG17, Lur, AF15].

Week 2, Sep 10–14 *Holomorphic factorization algebras and vertex algebras*. Background on vertex algebras. Describe functor from the category of holomorphic factorization algebras to vertex algebras. Relationship to *chiral algebras* of Beilinson-Drinfeld. [CG17, BD04].

Week 3, Sep 17–21 *Koszul duality for E_n-algebras*. Categories of (co)modules for E_n-algebras. Koszul duality for associative algebras and its generalization for augmented E_n-algebras (with special attention to the case $n = 2$). Outline result of Tamarkin that the Koszul dual of an E_2-algebra is a Hopf algebra. Interplay between Hochschild homology and Koszul duality. [Cosb, Tam03].

Week 4, Sep 24–28 *Quantum groups and the Yangian*. BW: Not sure exactly how to structure this, some input would be greatly appreciated! Drinfeld's universal R-matrix. Relationship to integrable systems and lattice models. [ES02, CP94]

Week 5, Oct 1–5 *"Four-dimensional Chern-Simons theory"*. The moduli space of holomorphic, partially flat, connections on a complex surface.[1] The moduli space of multiplicative Higgs bundles on a Riemann surface.

Week 6, Oct 9–12 *Batalin-Vilkovisky quantization and renormalization*. An introduction to BV quantization for gauge theories.

Week 7, Oct 15–19 *Batalin-Vilkovisky quantization and renormalization (cont.)*.

Week 8, Oct 22–26 *Observables of the four-dimensional gauge theory*. Deformation of functions on the classical moduli space defined by BV quantization to small orders in $\hbar$. Line operators and a generalized version of "conformal blocks" from CFT. [Cos11, CG, Cosb]

Week 9, Oct 29–Nov 2 *Costello's main result*.

Theorem 0.1. *The Koszul dual of the E_2-algebra of quantum observables of four-dimensional Chern-Simons theory on $\mathbb{C}_z \times \mathbb{R}^2_w$ (restricted to a factorization algebra on $\{z = 0\} \times \mathbb{R}^2_w$) is Koszul dual to the Yangian Hopf algebra.*

[Cos11, CG, Cosb, CWYa, CWYb]

Week 10, Nov 5–9 *Holomorphic factorization and the universal R-matrix* The quantum OPE as a map of E_2-algebras. Hochschild homology for categories. [Cosb].

Week 11, Nov 13–16 *The quantum Yang-Baxter equation* How the quantum master equation for BV quantization implies the quantum Yang-Baxter equation. [CWYa, CWYb].

[1] Roughly, a holomorphic partially flat connecton on a complex manifold of the form $X \times Y$ is a holomorphic connection that is holomorphically flat in the Y-direction.

Weeks ?? Enhancements and variations of the construction. Realizing various spin systems by tweaking the input data. Coupling to surface operators.

References

[AF15] David Ayala and John Francis. Factorization homology of topological manifolds. *J. Topol.*, 8(4):1045–1084, 2015.

[BD04] Alexander Beilinson and Vladimir Drinfeld. *Chiral algebras*, volume 51 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2004.

[CG] Kevin Costello and Owen Gwilliam. Factorization algebras in quantum field theory. Vol. 2. available at http://people.mpim-bonn.mpg.de/gwilliam.

[CG17] Kevin Costello and Owen Gwilliam. *Factorization algebras in quantum field theory. Vol. 1*, volume 31 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2017.

[Cosa] Kevin Costello. Integrable lattice models from four-dimensional field theories. Available at https://arxiv.org/abs/1308.0370.

[Cosb] Kevin Costello. Supersymmetric gauge theory and the Yangian. Available at http://arxiv.org/abs/1303.2632.

[Cos11] Kevin Costello. *Renormalization and effective field theory*, volume 170 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2011.

[CP94] Vyjayanthi Chari and Andrew Pressley. *A guide to quantum groups*. Cambridge University Press, Cambridge, 1994.

[CWYa] Kevin Costello, Edward Witten, and Masahito Yamazaki. Gauge Theory and Integrability, i. Available at https://arxiv.org/abs/1709.09993.

[CWYb] Kevin Costello, Edward Witten, and Masahito Yamazaki. Gauge Theory and Integrability, ii. Available at https://arxiv.org/abs/1802.01579.

[ES02] Pavel Etingof and Olivier Schiffmann. *Lectures on quantum groups*. Lectures in Mathematical Physics. International Press, Somerville, MA, second edition, 2002.

[Lur] Jacob Lurie. Higher Algebra. available at http://www.math.harvard.edu/~lurie/papers/higheralgebra.pdf.

[Tam03] Dmitry E. Tamarkin. Formality of chain operad of little discs. *Lett. Math. Phys.*, 66(1-2):65–72, 2003.