QUANTUM 4d GAUGE THEORY AND AN OUTLINE OF THE PROOF

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1. A recollection of R-matrices

1.1. **Warmup: braided monoidal categories.** Suppose that $\mathcal{C}$ is an $\mathcal{E}_2$, or braided, monoidal category. We have seen that we can view an $\mathcal{E}_2$ category as an $\mathcal{E}_1$ object in $\mathcal{E}_1$ monoidal (or just monoidal), categories

$$\mathcal{C} \in \mathrm{Alg}_{\mathcal{E}_2}(\mathcal{C}\mathrm{at}) = \mathrm{Alg}_{\mathcal{E}_1}\left(\mathrm{Alg}_{\mathcal{E}_1}(\mathcal{C}\mathrm{at})\right).$$

Concretely, this means that on $\mathcal{C}$ we have two *compatible* tensor products $\otimes, \boxtimes$. Actually, $\otimes, \boxtimes$ are naturally isomorphic

$$\begin{aligned} V \boxtimes W &\cong (V \otimes 1) \boxtimes (1 \otimes W) \\ &\cong (V \boxtimes 1) \otimes (1 \boxtimes W) \\ &\cong V \otimes W. \end{aligned}$$

All of the juice, therefore, is in the requirement that $\otimes$ and $\boxtimes$ are compatible. The most efficient way to encode this is to require that the bifunctor

$$\boxtimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

determined by $\boxtimes$ is *monoidal*, where we view $\mathcal{C} \times \mathcal{C}, \mathcal{C}$ as monoidal categories via $\otimes$. Spelling out what this means boils down to requiring that there is a natural isomorphism

$$(V \otimes W) \otimes (V' \otimes W') \cong (V \otimes V') \otimes (W \otimes W')$$

for all objects V, V', W, W'. Setting $V, W' = 1$, we see that this isomorphism determines an isomorphism $\sigma_{W,V'} : W \otimes V' \to V' \otimes W$. The resulting structure $(\mathcal{C}, \otimes, \sigma_{V,W})$ defines a *braided* monoidal category, where $\sigma_{V,W}$ are the braiding morphisms.

Conversely, given a braided monoidal category we can define

$$(V \otimes W) \otimes (V' \otimes W') \cong V \otimes (W \otimes V') \otimes W' \overset{\sigma_{W,V'}}{\cong} V \otimes (V' \otimes W) \otimes W' \cong (V \otimes V') \otimes (W \otimes W').$$

In other words, a braided monoidal category determines a monoidal structure on the bifunctor $(V, W) \mapsto V \otimes W$, and vice-versa.

1.2. **Drinfeld's rational R-matrix.** Recall that there is a $\mathbb{C}^\times$-action on $Y(\mathfrak{g})$. We define λ to have $\mathbb{C}^\times$-weight 1.

Theorem 1.1 ([?]). *There is a unique element*

$$R(\lambda) \in Y(\mathfrak{g}) \widehat{\otimes}_{\mathbb{C}[[\hbar]]} Y(\mathfrak{g}) \widehat{\otimes} \mathbb{C}((\lambda))$$

satisfying:

(1) ($\mathbb{C}^\times$-invariance) $R(\lambda)$ is $\mathbb{C}^\times$-invariant.

(2) (Translation equivariance) For all $a \in Y(\mathfrak{g})$

$$(T_\lambda \otimes T_0)\sigma\Delta(a) = R(\lambda)((T_\lambda \otimes T_0)\Delta(a))R(\lambda)^{-1} \in R(\lambda) \in Y(\mathfrak{g}) \widehat{\otimes}_{\mathbb{C}[[\hbar]]} Y(\mathfrak{g}) \widehat{\otimes} \mathbb{C}((\lambda)).$$

(3) (??) $(\Delta \otimes 1)R(\lambda) = R_{13}(\lambda)R_{23}(\lambda)$ and $(1 \otimes \Delta)R(\lambda)$.

Remark 1.2. The R-matrix $R(\lambda)$ satisfies the Yang-Baxter equation with spectral parameter. This equation follows from the three listed properties in the theorem.

1.3. **Interpreting the spectral R-matrix.** Our goal in this section is to interpret the R-matrix conditions in Theorem 1.1 in a more categorical way. The running example we should keep in mind from Section **??** is that to give a braided monoidal structure is equivalent to prescribing monoidality of the tensor product bifunctor. We will see something similar in the case of the R-matrix: the data of a rational R-matrix is equivalent to the monoidality of a certain functor. See Theorem BW: ref

For the remainder of this section, the category $\mathrm{Mod}^{fin}_{Y(g)}$ denotes the category of (non dg) $Y(\mathfrak{g})$-modules that are free and finite dimensional as $\mathbb{C}[[\hbar]]$-modules.

Definition 1.3. If $V \in \mathrm{Mod}^{fin}_{Y(g)}$, define the $Y(\mathfrak{g})((\lambda))$-module

$$T_\lambda V = V \otimes_{Y(\mathfrak{g})} Y(\mathfrak{g})((\lambda)).$$

In the definition, we use the Hopf algebra homomorphism $T_\lambda : Y(\mathfrak{g}) \to Y(\mathfrak{g})((\lambda)) = Y(\mathfrak{g}) \widehat{\otimes} \mathbb{C}((\lambda))$. Similarly, define the $Y(\mathfrak{g})((\lambda))$-module

$$T_0 V = V \widehat{\otimes} \mathbb{C}((\lambda)) = V((\lambda))$$

as the induction of V along the embedding of Hopf algebras $T_0 : Y(\mathfrak{g}) \to Y(\mathfrak{g})((\lambda))$, $a \mapsto a \otimes 1$.

With this notation in place, we can define the following functor that will be relevant for the remainder of the section. Define

$$F_\lambda : \mathrm{Mod}^{fin}_{Y(g)} \times \mathrm{Mod}^{fin}_{Y(g)} \to \mathrm{Mod}_{Y(\mathfrak{g})}((\lambda)) \tag{1}$$

on objects by $(V, W) \mapsto T_0 V \otimes T_\lambda W$. Similarly, we have the functor

$$F'_\lambda : \mathrm{Mod}^{fin}_{Y(g)} \times \mathrm{Mod}^{fin}_{Y(g)} \to \mathrm{Mod}_{Y(\mathfrak{g})}((\lambda))$$

defined on objects by $(V, W) \mapsto T_\lambda V \otimes T_0 W$.

Construction 1.4. Suppose $S(\lambda) \in Y(\mathfrak{g}) \otimes Y(\mathfrak{g})((\lambda))$ is any element, and fix $V, W \in \mathrm{Mod}^{fin}_{Y(g)}$. Note that $V \otimes W = V \otimes_{\mathbb{C}[[\hbar]]} W$ is a module for the Hopf algebra $Y(\mathfrak{g}) \otimes Y(\mathfrak{g})$. Multiplication by $S(\lambda)$ on $V \otimes W((\lambda))$ defines an endomorphism

$$S_{V,W} \in \mathrm{End}_{\mathbb{C}((\lambda))}(V \otimes W((\lambda))).$$

Note that for this endomorphism to be well-defined, we needed V, W to be finite rank as $\mathbb{C}[[\hbar]]$-modules.

Lemma 1.5. *Suppose $S(\lambda) \in Y(\mathfrak{g}) \otimes Y(\mathfrak{g})((\lambda))$ satisfies condition (3) of Theorem 1.1. Then, $S(\lambda)$ defines a natural transformation of functors*

$$\eta_S : F_\lambda \xrightarrow{\cong} F'_\lambda \circ \sigma$$

via the formula

$$\eta_S(V, W) = \sigma \circ S_{V,W} : T_0 V \otimes T_\lambda W \to T_\lambda W \otimes T_0 V.$$

Lemma 1.6. *Suppose $S(\lambda) \in Y(\mathfrak{g}) \otimes Y(\mathfrak{g})((\lambda))$satisfies condition (2) of Theorem 1.1. Then, for all $V_1, V_2, V_3 \in \mathrm{Mod}^{fin}_{Y(g)}$ the following diagrams commute*

We need one last technical lemma before proceeding. Suppose that $\mathcal{C}, \mathcal{D}$ are monoidal categories. We view the tensor products $\otimes_\mathcal{C}$, $\otimes_\mathcal{D}$ as bifunctors

$$\otimes_\mathcal{C} : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \ \ , \ \ \otimes_\mathcal{D} : \mathcal{D} \times \mathcal{D} \to \mathcal{D}.$$

Suppose that $G : \mathcal{C} \to \mathcal{D}$ a functor. A natural transformation making G a monoidal functor is the data of a natural transformation of functors $\mathcal{C} \times \mathcal{C} \to \mathcal{D}$

$$\eta : G \circ \otimes_\mathcal{C} \xrightarrow{\cong} \otimes_\mathcal{D} \circ (G \times G).$$

In other words, we have isomorphisms $G(c \otimes_\mathcal{C} c') \cong G(c) \otimes_\mathcal{D} G(c')$ that are natural in $c, c' \in \mathcal{C}$.

Lemma 1.7. *Let F_λ be the functor from Equation (1). Then, the data of a natural transformation making F a monoidal functor is equivalent to the data of a natural transformation*

$$\eta : F_\lambda \xrightarrow{\cong} F'_\lambda \circ \sigma$$

*such that the diagrams (**??**) and (**??**) commute.*

We can summarize a consequence of the above lemmas in the following way.

Theorem 1.8. *Drinfeld's R-matrix $R(\lambda)$ gives rise to a natural transformation η_R, as in Lemma 1.5, making the functor*

$$F_\lambda : \mathrm{Mod}^{fin}_{Y(g)} \times \mathrm{Mod}^{fin}_{Y(g)} \to \mathrm{Mod}_{Y(\mathfrak{g})((\lambda))}.$$

monoidal.

1.4. **A variation for comodules.**

Define

$$F^{\mathrm{comod}}_\lambda : \mathrm{Comod}_{Y^*(\mathfrak{g})} \times \mathrm{Comod}_{Y^*(\mathfrak{g})} \to \mathrm{Comod}_{Y^*(\mathfrak{g})}((\lambda))$$

by $(V, W) \mapsto T_\lambda V \otimes T_0 W$.

Theorem 1.9. *The following pieces of data are equivalent:*

(1) *An R-matrix with spectral parameter*

$$R(\lambda) \in F^0(Y(\mathfrak{g}) \otimes Y(\mathfrak{g}))((\lambda))$$

satisfying (1),(2), and (3) of Theorem 1.1.

(2) *The data of a natural transformation making F_λ a monoidal functor.*

When such an $R(\lambda)$ is given, we will denote by F_R or $F_{R(\lambda)}$ the associated monoidal functor.

2. The OPE

2.1. **The main result.**

Theorem 2.1 ([?]). *There is a natural equivalence of monoidal functors* $\mathrm{Comod}_{Y^*(\mathfrak{g})} \times \mathrm{Comod}_{Y^*(\mathfrak{g})} \to \mathrm{Comod}_{Y^*(\mathfrak{g})}((\lambda))$

$$F_{OPE} \cong F_{R(\lambda)}$$

where $F(R(\lambda))$ is the monoidal functor corresponding to Drinfeld's R-matrix as in Theorem 1.9.