R-MATRICES FROM 4-DIMENSIONAL FACTORIZATION ALGEBRAS

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1. A recollection of R-matrices

1.1. **Warmup: braided monoidal categories.** Suppose that $\mathcal{C}$ is an $\mathcal{E}_2$, or braided, monoidal category. We have seen that we can view an $\mathcal{E}_2$ category as an $\mathcal{E}_1$ object in $\mathcal{E}_1$ monoidal (or just monoidal), categories

$$\mathcal{C} \in \mathrm{Alg}_{\mathcal{E}_2}(\mathcal{C}\mathrm{at}) = \mathrm{Alg}_{\mathcal{E}_1}\left(\mathrm{Alg}_{\mathcal{E}_1}(\mathcal{C}\mathrm{at})\right).$$

Concretely, this means that on $\mathcal{C}$ we have two *compatible* tensor products $\otimes, \boxtimes$. Actually, $\otimes, \boxtimes$ are naturally isomorphic

$$\begin{aligned} V \boxtimes W &\cong (V \otimes 1) \boxtimes (1 \otimes W) \\ &\cong (V \boxtimes 1) \otimes (1 \boxtimes W) \\ &\cong V \otimes W. \end{aligned}$$

All of the juice, therefore, is in the requirement that $\otimes$ and $\boxtimes$ are compatible. The most efficient way to encode this is to require that the bifunctor

$$\boxtimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

determined by $\boxtimes$ is *monoidal*, where we view $\mathcal{C} \times \mathcal{C}, \mathcal{C}$ as monoidal categories via $\otimes$. Spelling out what this means boils down to requiring that there is a natural isomorphism

$$(V \otimes W) \otimes (V' \otimes W') \cong (V \otimes V') \otimes (W \otimes W')$$

for all objects V, V', W, W'. Setting $V, W' = 1$, we see that this isomorphism determines an isomorphism $\sigma_{W,V'} : W \otimes V' \to V' \otimes W$. The resulting structure $(\mathcal{C}, \otimes, \sigma_{V,W})$ defines a *braided* monoidal category, where $\sigma_{V,W}$ are the braiding morphisms.

Conversely, given a braided monoidal category we can define

$$(V \otimes W) \otimes (V' \otimes W') \cong V \otimes (W \otimes V') \otimes W' \overset{\sigma_{W,V'}}{\cong} V \otimes (V' \otimes W) \otimes W' \cong (V \otimes V') \otimes (W \otimes W').$$

In other words, a braided monoidal category determines a monoidal structure on the bifunctor $(V, W) \mapsto V \otimes W$, and vice-versa.

1.2. **Drinfeld's rational R-matrix.** We have been working with topological algebras / coalgebras, and modules / comodules over them that have an additional topological structure. We arrived at this topological structure through the data of a filtration.

In Week ??, the category of filtered, bounded, free, dg $\mathbb{C}[[\hbar]]$-modules $\mathcal{C}^b$ was introduced. An object $V \in \mathcal{C}^b$ is a cochain complex of the form

$$V_0 \otimes \mathbb{C}[[\hbar]]$$

where V_0 is a cochain complex equipped with a decreasing filtration

$$V = F^0 V \supset F^1 V \supset \cdots$$

and satisfies $V \cong \lim V/F^i V$. The category $\mathcal{C}^b$ is symmetric monoidal. The tensor product of two objects $V, W \in \mathcal{C}^b$ is the completed one, where we take into account the filtrations in the following way

$$V \otimes_{\mathbb{C}[[\hbar]]} W = \lim_{i,j} \left((V/F^i V) \otimes_{\mathbb{C}[[\hbar]]} W/F^j W \right) \in \mathcal{C}^b.$$

Introduce the formal Laurent series ring $\mathbb{C}((\lambda))$. We will need to consider a Hopf algebra built from $Y(\mathfrak{g})$ by adjoining the parameter λ. To do things properly, we need to do this in the following completed way. For V an object in $\mathcal{C}^b$, define the completed tensor product

$$V \widehat{\otimes} \mathbb{C}((\lambda)) = \lim_i \operatorname{colim}_k \lim_j \lambda^{-k} (V/F^i V)[\lambda]/\lambda^j.$$

We will often simply denote this by $V((\lambda))$.

Remark 1.1. This formula for the completed tensor product arises from viewing $\mathbb{C}((\lambda))$ as an ind-pro object

$$\mathbb{C}((\lambda)) = \operatorname{colim}_k \lambda^{-k} \mathbb{C}[[\lambda]] = \operatorname{colim}_k \lim_j \lambda^{-k} \mathbb{C}[\lambda]/\lambda^j.$$

Remark 1.2. Throughout, we let $\otimes_{alg}$ denote the algebraic tensor product. Note that $Y(\mathfrak{g})((\lambda))$ and $Y(\mathfrak{g}) \otimes_{alg} \mathbb{C}((\lambda))$ are *different* Hopf algebras, but we have an inclusion of Hopf algebras

$$Y(\mathfrak{g}) \otimes_{alg} \mathbb{C}((\lambda)) \hookrightarrow Y(\mathfrak{g})((\lambda)).$$

The category of $Y(\mathfrak{g})((\lambda))$-modules is defined similarly as in in the plain $Y(\mathfrak{g})$ case. In particular, we focus on modules that are quasi-free, which means they admit a filtration whose assoociated graded pieces are of the form $V \otimes_{alg} (A((\lambda)))$.

Recall that $Y(\mathfrak{g})$ is a topological Hopf algebra over the ring $\mathbb{C}[[\hbar]]$. Moreover, there is a $\mathbb{C}^\times$-action on $Y(\mathfrak{g})$ inducing a grading where $\hbar$ has weight 1. We define λ to have $\mathbb{C}^\times$-weight 1.

Theorem 1.3 ([?]). *There is a unique element*

$$R(\lambda) \in Y(\mathfrak{g})\widehat{\otimes}_{\mathbb{C}[[\hbar]]}Y(\mathfrak{g})\widehat{\otimes}\mathbb{C}((\lambda))$$

satisfying:

(1) *($\mathbb{C}^\times$-invariance) $R(\lambda)$ is $\mathbb{C}^\times$-invariant.*

(2) *(Translation equivariance) For all $a \in Y(\mathfrak{g})$*

$$(T_\lambda \otimes T_0)\sigma\Delta(a) = R(\lambda)((T_\lambda \otimes T_0)\Delta(a))R(\lambda)^{-1} \in R(\lambda) \in Y(\mathfrak{g})\widehat{\otimes}_{\mathbb{C}[[\hbar]]}Y(\mathfrak{g})\widehat{\otimes}\mathbb{C}((\lambda)).$$

(3) *(??) $(\Delta \otimes 1)R(\lambda) = R_{13}(\lambda)R_{23}(\lambda)$ and $(1 \otimes \Delta)R(\lambda)$.*

Remark 1.4. The R-matrix $R(\lambda)$ satisfies the Yang-Baxter equation with spectral parameter. This equation follows from the three listed properties in the theorem.

1.3. **Interpreting the spectral R-matrix.** Our goal in this section is to interpret the R-matrix conditions in Theorem 1.3 in a more categorical way. The running example we should keep in mind from Section **??** is that to give a braided monoidal structure is equivalent to prescribing monoidality of the tensor product bifunctor. We will see something similar in the case of the R-matrix: the data of a rational R-matrix is equivalent to the monoidality of a certain functor. See Theorems 1.10 and 1.11.

For the remainder of this section, the category $\mathrm{Mod}^{fin}_{Y(g)}$ denotes the category of (non dg) $Y(\mathfrak{g})$-modules that are free and finite dimensional as $\mathbb{C}[[\hbar]]$-modules.

Definition 1.5. If $V \in \mathrm{Mod}^{fin}_{Y(g)}$, define the $Y(\mathfrak{g})((\lambda))$-module

$$T_\lambda V = V \otimes_{Y(\mathfrak{g})} Y(\mathfrak{g})((\lambda)).$$

In the definition, we use the Hopf algebra homomorphism $T_\lambda : Y(\mathfrak{g}) \to Y(\mathfrak{g})((\lambda)) = Y(\mathfrak{g})\widehat{\otimes}\mathbb{C}((\lambda))$. Similarly, define the $Y(\mathfrak{g})((\lambda))$-module

$$T_0 V = V\widehat{\otimes}\mathbb{C}((\lambda)) = V((\lambda))$$

as the induction of V along the embedding of Hopf algebras $T_0 : Y(\mathfrak{g}) \to Y(\mathfrak{g})((\lambda))$, $a \mapsto a \otimes 1$.

With this notation in place, we can define the following functor that will be relevant for the remainder of the section. Define

$$F_\lambda : \mathrm{Mod}^{fin}_{Y(g)} \times \mathrm{Mod}^{fin}_{Y(g)} \to \mathrm{Mod}_{Y(\mathfrak{g})}((\lambda)) \tag{1}$$

on objects by $(V, W) \mapsto T_0 V \otimes T_\lambda W$. Similarly, we have the functor

$$F'_\lambda : \mathrm{Mod}^{fin}_{Y(g)} \times \mathrm{Mod}^{fin}_{Y(g)} \to \mathrm{Mod}_{Y(\mathfrak{g})}((\lambda))$$

defined on objects by $(V, W) \mapsto T_\lambda V \otimes T_0 W$.

Construction 1.6. Suppose $S(\lambda) \in Y(\mathfrak{g}) \otimes Y(\mathfrak{g})((\lambda))$ is any element, and fix $V, W \in \mathrm{Mod}^{fin}_{Y(g)}$. Note that $V \otimes W = V \otimes_{\mathbb{C}[[\hbar]]} W$ is a module for the Hopf algebra $Y(\mathfrak{g}) \otimes Y(\mathfrak{g})$. Multiplication by $S(\lambda)$ on $V \otimes W((\lambda))$ defines an endomorphism

$$S_{V,W} \in \mathrm{End}_{\mathbb{C}((\lambda))}(V \otimes W((\lambda))).$$

Note that for this endomorphism to be well-defined, we needed V, W to be finite rank as $\mathbb{C}[[\hbar]]$-modules.

Lemma 1.7. *Suppose $S(\lambda) \in Y(\mathfrak{g}) \otimes Y(\mathfrak{g})((\lambda))$ satisfies condition (3) of Theorem 1.3. Then, $S(\lambda)$ defines a natural transformation of functors*

$$\eta_S : F_\lambda \xrightarrow{\cong} F'_\lambda \circ \sigma$$

via the formula

$$\eta_S(V, W) = \sigma \circ S_{V,W} : T_0 V \otimes T_\lambda W \to T_\lambda W \otimes T_0 V.$$

Lemma 1.8. *Suppose $S(\lambda) \in Y(\mathfrak{g}) \otimes Y(\mathfrak{g})((\lambda))$satisfies condition (2) of Theorem 1.3. Then, for all $V_1, V_2, V_3 \in \mathrm{Mod}^{fin}_{Y(g)}$ the following diagrams commute*

We need one last technical lemma before proceeding. Suppose that $\mathcal{C}, \mathcal{D}$ are monoidal categories. We view the tensor products $\otimes_{\mathcal{C}}$, $\otimes_{\mathcal{D}}$ as bifunctors

$$\otimes_{\mathcal{C}} : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \ \ , \ \ \otimes_{\mathcal{D}} : \mathcal{D} \times \mathcal{D} \to \mathcal{D}.$$

Suppose that $G : \mathcal{C} \to \mathcal{D}$ a functor. A natural transformation making G a monoidal functor is the data of a natural transformation of functors $\mathcal{C} \times \mathcal{C} \to \mathcal{D}$

$$\eta : G \circ \otimes_{\mathcal{C}} \xrightarrow{\cong} \otimes_{\mathcal{D}} \circ (G \times G).$$

In other words, we have isomorphisms $G(c \otimes_{\mathcal{C}} c') \cong G(c) \otimes_{\mathcal{D}} G(c')$ that are natural in $c, c' \in \mathcal{C}$.

Lemma 1.9. *Let F_λ be the functor from Equation (1). Then, the data of a natural transformation making F a monoidal functor is equivalent to the data of a natural transformation*

$$\eta : F_\lambda \xrightarrow{\cong} F'_\lambda \circ \sigma$$

*such that the diagrams (**??**) and (**??**) commute.*

We can summarize a consequence of the above lemmas in the following way.

Theorem 1.10. *Drinfeld's R-matrix $R(\lambda)$ gives rise to a natural transformation η_R, as in Lemma 1.7, making the functor*

$$F_\lambda : \mathrm{Mod}^{fin}_{Y(g)} \times \mathrm{Mod}^{fin}_{Y(g)} \to \mathrm{Mod}_{Y(\mathfrak{g})}((\lambda)).$$

monoidal.

1.4. **A variation for comodules.**

1.4.1. *An asymmetry between modules and comodules.* Thus far, we have used the fact that any $Y(\mathfrak{g})$-module induces a $Y(\mathfrak{g})((\lambda)) = Y(\mathfrak{g})\widehat{\otimes}\mathbb{C}((\lambda))$-module. In the comodule case, this does not work. The reason for this is the following.

Suppose A is any algebra in $\mathcal{C}^b$. Then, we obtain a $\mathbb{C}((\lambda))$ algebra $A((\lambda)) = A\widehat{\otimes}\mathbb{C}((\lambda))$ as the following composition

$$A((\lambda)) \otimes_{\mathbb{C}((\lambda)),alg} A((\lambda)) \to A((\lambda))\widehat{\otimes}_{\mathbb{C}((\lambda))} A((\lambda)) = (A \otimes A)\widehat{\otimes}\mathbb{C}((\lambda)) \xrightarrow{m_A} A\widehat{\otimes}\mathbb{C}((\lambda)) = A((\lambda)).$$

The first arrow is the embedding of the algebraic tensor product inside of the completed tensor product. This says that every algebra A induces a $\mathbb{C}((\lambda))$-linear algebra $A((\lambda))$.

The same is not true for coalgebras. Suppose C is a coalgebra in $\mathcal{C}^b$. The coproduct extends $\mathbb{C}((\lambda))$-linearly to a map

$$C((\lambda)) \to (C \otimes C)((\lambda)) = (C \otimes C)\widehat{\otimes}\mathbb{C}((\lambda)).$$

Note that the right hand side *differs* from the $\mathbb{C}((\lambda))$-algebraic tensor product $C((\lambda)) \otimes_{\mathbb{C}((\lambda)),alg} C((\lambda))$. There is only a map from this algebraic tensor product into the right hand side above. Thus, C does not induce a coalgebra structure on $C((\lambda))$. Moreover, this shows that the notion of cofree comodules does not exist for $C((\lambda))$.

1.4.2. Define

$$F_\lambda^{\text{comod}} : \text{Comod}_{Y^*(\mathfrak{g})} \times \text{Comod}_{Y^*(\mathfrak{g})} \to \text{Comod}_{Y^*(\mathfrak{g})}((\lambda))$$

by $(V, W) \mapsto T_\lambda V \otimes T_0 W$.

Theorem 1.11. *The following pieces of data are equivalent:*

(i) *An R-matrix with spectral parameter*

$$R(\lambda) \in F^0(Y(\mathfrak{g}) \otimes Y(\mathfrak{g}))((\lambda))$$

satisfying (1),(2), and (3) of Theorem 1.3.

(ii) *The data of a natural transformation making F_λ a monoidal functor.*

Proof. By Lemma 1.9, the data (ii) of making F_λ monoidal is equivalent to a natural isomorphism

$$\eta_{V,W} : T_0(V) \otimes T_\lambda(W) \cong T_\lambda(W) \otimes T_0(V),$$

satisfying conditions BW: from R matrix lemmas. Thus, we must show that an R-matrix as in (i) is equivalent to this data.

Let V, W be $Y^*(\mathfrak{g})$ comodules BW: what does this mean, and let Δ_V, Δ_W denote the coaction maps. Any element $\alpha \in Y(\mathfrak{g}) \otimes Y(\mathfrak{g})$ defines a composition

$$V \otimes W \xrightarrow{\Delta_V \otimes \Delta_W} (Y^*(\mathfrak{g}) \otimes V) \otimes (Y^*(\mathfrak{g}) \otimes W) \xrightarrow{\alpha_{13}} V \otimes W.$$

Since Drinfeld's $R(\lambda) \in Y(\mathfrak{g}) \otimes Y(\mathfrak{g})((\lambda))$ is $\mathbb{C}^\times$-weight zero, it determines a filtration preserving map

$$R_{V,W}(\lambda) : V \otimes W \to V$$

□

When such an $R(\lambda)$ is given, we will denote by F_R or $F_{R(\lambda)}$ the associated monoidal functor.

2. The OPE

Proposition 2.1. *The map of $\mathcal{E}_2$ algebras arising from the factorization product of disks in the z-plane*

$$m : \mathrm{Obs}_0 \otimes \mathrm{Obs}_0 \to \mathrm{Obs}_{D(0,s)} \otimes \Omega^{0,*}(A)$$

determines a map

$$m_{OPE} : \mathrm{Obs}_0 \otimes \mathrm{Obs}_0 \to \mathrm{Obs}_0 \otimes \mathbb{C}((z))$$

that fits into the commuting square of $\mathcal{E}_2$-algebras

$$\begin{tikzcd} \mathrm{Obs}_0 \otimes \mathrm{Obs}_0 \arrow[r, "m_{OPE}"] \arrow[d, "m"] & \mathrm{Obs}_0 \otimes \mathbb{C}((z)) \arrow[d] \\ \mathrm{Obs}_{D(0,s)} \otimes \Omega^{0,*}(D(0,r))[z^{-1}] \arrow[r] & \mathrm{Obs}_{D(0,s)} \otimes \mathbb{C}((z)). \end{tikzcd}$$

2.1. The main result.

Theorem 2.2 ([?])**.** *There is a natural equivalence of monoidal functors*

$$F_{OPE} \cong F_{R(\lambda)} : \mathrm{Comod}_{Y^*(\mathfrak{g})} \times \mathrm{Comod}_{Y^*(\mathfrak{g})} \to \mathrm{Comod}_{Y^*(\mathfrak{g})}((\lambda))$$

where $F(R(\lambda))$ is the monoidal functor corresponding to Drinfeld's R-matrix as in Theorem 1.11.

Proof. First, we need to show that there is an equivalence of functors (forgetting monoidality).

Since F_{OPE} is monoidal, there is an isomorphism

$$F_{OPE}(V \times W) = F_{OPE}((V \times 1) \otimes (1 \otimes W)) = F_{OPE}(V \times 1) \otimes F_{OPE}(1 \times W).$$

Here, $1 = \mathbb{C}[[\hbar]]$ is the trivial $\mathbb{C}[[\hbar]]$-linear $Y(\mathfrak{g})$-module.

It suffices to show that there is a natural quasi-isomorphism

$$F_{OPE}(V \times 1) \otimes F_{OPE}(1 \times W) \cong T_0(V) \otimes T_\lambda(W).$$

Recall, F_{OPE} was induced from the map of E_2 algebras

$$m_{OPE}(\lambda) : \mathrm{Obs}_0 \otimes \mathrm{Obs}_0 \to \mathrm{Obs}_0((\lambda)).$$

When restricted to $\mathrm{Obs}_0 \otimes \{1\}$, this map is simply the $\mathbb{C}((\lambda))$-linear extension of the identity map. Thus, $F_{OPE}(V \times 1) = T_0(V)$.

We thus need to show that $F_{OPE}(1 \times W) = T_\lambda(W)$. For this, consider the derivation $\partial_z = \frac{\partial}{\partial z}$ acting on Obs_0. Define

$$\widetilde{T}_\lambda : \mathrm{Obs}_0 \to \mathrm{Obs}_0[[\lambda]]$$

via $\widetilde{T}_\lambda(O) = e^{\lambda \partial_z} O \in \mathrm{Obs}_0[[\lambda]]$.

Lemma 2.3. *The restriction of m'_{OPE}*

$$m'_{OPE}(1 \otimes -) : \mathrm{Obs}_0 \to \mathrm{Obs}_0((\lambda))$$

agrees with $\widetilde{T}_{-\lambda}$.

Proof. This is almost by construction. By Proposition **??**, this map factors as

$$\mathrm{Obs}_0 \to \Omega^{0,*}(D(0,r))[z^{-1}] \to \Omega^{0,*}(\widehat{D}) \simeq \mathbb{C}((z))$$

for any $r > 0$. The middle map is power series expansion, which lands in the Dolbeault complex of the formal disk. Since we are identifying $z \leftrightarrow -\lambda$, the lemma follows. □

The lemma implies that the functor

$$F(1 \times -) : \mathrm{Mod}_{\mathrm{Obs}_0} \to \mathrm{Mod}_{\mathrm{Obs}_0}((\lambda))$$

sends

$$V \mapsto \widetilde{T}_{-\lambda}(V) = V \otimes_{\mathrm{Obs}_0} {}_1(\mathrm{Obs}_0)_{-\lambda}$$

Here, ${}_1(\mathrm{Obs}_0)_{-\lambda}$ denotes the $\mathrm{Obs}_0 - \mathrm{Obs}_0$ bimodule for which an element $O \in \mathrm{Obs}_0$ acts by itself on the left and by $\widetilde{T}_{-\lambda}(O)$ on the right.

The main result of this seminar is that the Koszul dual of Obs_0 is the dual Yangian $Y^*(\mathfrak{g})$. This Koszul duality trades modules over Obs_0 for comodules over $Y^*(\mathfrak{g})$. The endofunctor $\widetilde{T}_{-\lambda}$ on modules for Obs_0 induces an endofunctor on the category of comodules for $Y^*(\mathfrak{g})$.

In terms of λ, there is a sign flip under this duality. That is, if W is a $Y^*(\mathfrak{g})$-comodule, define $\widetilde{T}_\lambda(W)$ to be the $Y^*(\mathfrak{g})$-comodule $W \otimes_{Y^*(\mathfrak{g})} {}_1Y^*(\mathfrak{g})_\lambda$.

The claim is that the functor $\widetilde{T}_{-\lambda}$ on Obs_0-modules intertwines with the functor $\widetilde{T}_\lambda$ on $Y^*(\mathfrak{g})$-comodules under Koszul duality.

What we have just seen is that when we view F_{OPE} as a bifunctor

$$F_{OPE} : \mathrm{Comod}_{Y^*(\mathfrak{g})} \times \mathrm{Comod}_{Y^*(\mathfrak{g})} \to \mathrm{Comod}_{Y^*(\mathfrak{g})}((\lambda))$$

that $F_{OPE}(1 \times W) \simeq \widetilde{T}_\lambda(W)$. The last thing we need to show is that $\widetilde{T}_\lambda$ and T_λ agree. The result follows from the following lemma.

Lemma 2.4. *There is a unique homomorphism of Hopf algebras $T : Y(\mathfrak{g}) \to Y(\mathfrak{g})[[\lambda]]$ such that*

(1) Classically, $T|_{\hbar=0} : U(\mathfrak{g}[[z]]) \to U(\mathfrak{g}[[z]])[[\lambda]]$ arises from the map of Lie algebras

$$\mathfrak{g}[[z]] \to \mathfrak{g}[[z, \lambda]] \quad , \quad z \mapsto z + \lambda.$$

(2) T is $\mathbb{C}^\times$-equivariant.

(3) $T \mod \lambda = \mathrm{id}$.

From this lemma, the result is immediate. We have just shown that $F_{OPE}(V \times W) \cong T_0(V) \otimes T_\lambda(W)$. Thus, F_{OPE} agrees with $F_{R(\lambda)}$.

□