The Yangian and four dimensional gauge theory

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There is a deep connection between gauge theory and quantum groups. Perhaps the most ubiquitous is the relationship between the category of line operators in Chern-Simons theory on three manifolds and the braided monoidal category of representations of the quantum group. In the seminal work, Costello [Cosa, Cosb] has shown how certain infinite dimensional quantum groups, Yangians, arise from the algebra of operators of a *four* dimensional (supersymmetric) gauge theory. The key to this result relies on the formalism of Costello-Gwilliam that the factorization algebras of a quantum field theory form a factorization algebra. This is an enhancement of the well-known description of a conformal field theory in terms of vertex, or chiral, algebras. The primary goal of this seminar is to outline Costello's construction which starts from the factorization algebra description of the observables of a four dimensional gauge theory and recovers the Yangian quantum group of the gauge Lie algebra. Time permitting, we can f ocus on more concrete and computational sides of the program began in the works [CWYa, CWYb].

Some keywords: Factorization algebras, Koszul duality, quantum groups, Yangians.

Informal prerequisites: Basic category theory, homological algebra, and representation theory for Lie algebras. Knowledge of vector bundles, connections, and differential forms.

Week 1, Aug 27–31   *An introduction to factorization algebras*. The definition of a factorization algebra with values in a symmetric monoidal category. Lurie's result that locally constant factorization algebras on $\mathbb{R}^n$ are equivalent to E_n-algebras. Hochschild homology as a special case of factorization homology. [CG17, Lur, AF15].

Week 2, Sep 4–7   *Holomorphic factorization algebras and vertex algebras*. Background on vertex algebras. Describe functor from the category of holomorphic factorization algebras to vertex algebras. Relationship to *chiral algebras* of Beilinson-Drinfeld. [CG17, BD04].

Week 3, Sep 10–14   *Koszul duality for E_n-algebras*. Categories of (co)modules for E_n-algebras. Koszul duality for associative algebras and its generalization for augmented E_n-algebras (with special attention to the case $n = 2$). Outline result of Tamarkin that the Koszul dual of an E_2-algebra is a Hopf algebra. Interplay between Hochschild homology and Koszul duality. [Cosb, Tam03].

Week 4, Sep 17–21   *Quantum groups and the Yangian*. BW: Not sure exactly how to structure this, some input would be greatly appreciated! Drinfeld's universal R-matrix. Relationship to integrable systems and lattice models. [ES02, CP94]

Week 5, Sep 24–28 *Quantum groups and the Yangian (cont.).*

Week 6, Oct 1–5 *"Four-dimensional Chern-Simons theory".* The moduli space of holomorphic, partially flat, connections on a complex surface.[1] The moduli space of multiplicative Higgs bundles on a Riemann surface.

Week 7, Oct 9–12 *Batalin-Vilkovisky quantization and renormalization.* An introduction to BV quantization for gauge theories. BW: More details

Week 8, Oct 15–19 *Batalin-Vilkovisky quantization and renormalization (cont.).*

Week 9, Oct 22–26 *Observables of the four-dimensional gauge theory.* Deformation of functions on the classical moduli space defined by BV quantization to small orders in $\hbar$. Line operators and a generalized version of "conformal blocks" from CFT. [Cos11, CG, Cosb]

Week 10, Oct 29–Nov 2 *Costello's main result.*

Theorem 0.1. *The Koszul dual of the E_2-algebra of quantum observables of four-dimensional Chern-Simons theory on $\mathbb{C}_z \times \mathbb{R}^2_w$ (restricted to a factorization algebra on $\{z = 0\} \times \mathbb{R}^2_w$) is Koszul dual to the Yangian Hopf algebra.*

[Cos11, CG, Cosb, CWYa, CWYb]

Week 11, Nov 5–9 *Holomorphic factorization and the universal R-matrix* The quantum OPE as a map of E_2-algebras. Hochschild homology for categories. [Cosb].

Week 12, Nov 13–16 *The quantum Yang-Baxter equation* How the quantum master equation for BV quantization implies the quantum Yang-Baxter equation. [CWYa, CWYb].

Weeks ?? Enhancements and variations of the construction. Realizing various spin systems by tweaking the input data. Coupling to surface operators.

References

[AF15] David Ayala and John Francis. Factorization homology of topological manifolds. *J. Topol.*, 8(4):1045–1084, 2015.

[BD04] Alexander Beilinson and Vladimir Drinfeld. *Chiral algebras*, volume 51 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2004.

[CG] Kevin Costello and Owen Gwilliam. Factorization algebras in quantum field theory. Vol. 2. available at `http://people.mpim-bonn.mpg.de/gwilliam`.

[CG17] Kevin Costello and Owen Gwilliam. *Factorization algebras in quantum field theory. Vol. 1*, volume 31 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2017.

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[1]Roughly, a holomorphic partially flat connecton on a complex manifold of the form $X \times Y$ is a holomorphic connection that is holomorphically flat in the Y-direction.

[Cosa]   Kevin Costello. Integrable lattice models from four-dimensional field theories. Available at https://arxiv.org/abs/1308.0370.

[Cosb]   Kevin Costello. Supersymmetric gauge theory and the Yangian. Available at http://arxiv.org/abs/1303.2632.

[Cos11] Kevin Costello. *Renormalization and effective field theory*, volume 170 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2011.

[CP94]   Vyjayanthi Chari and Andrew Pressley. *A guide to quantum groups*. Cambridge University Press, Cambridge, 1994.

[CWYa]   Kevin Costello, Edward Witten, and Masahito Yamazaki. Gauge Theory and Integrability, i. Available at https://arxiv.org/abs/1709.09993.

[CWYb]   Kevin Costello, Edward Witten, and Masahito Yamazaki. Gauge Theory and Integrability, ii. Available at https://arxiv.org/abs/1802.01579.

[ES02]   Pavel Etingof and Olivier Schiffmann. *Lectures on quantum groups*. Lectures in Mathematical Physics. International Press, Somerville, MA, second edition, 2002.

[Lur]   Jacob Lurie. Higher Algebra. available at http://www.math.harvard.edu/~lurie/papers/higheralgebra.pdf.

[Tam03] Dmitry E. Tamarkin. Formality of chain operad of little discs. *Lett. Math. Phys.*, 66(1-2):65–72, 2003.