

FILTERED KOSZUL DUALITY

DMYTRO MATVIEIEVSKYI

ABSTRACT. In this talk we define the filtered Koszul dual $A^!$ of a complete filtered augmented dg-algebra A . We will construct a quasi-equivalences between the categories $A\text{-mod}$ and $A^!\text{-comod}$ of correctly defined modules and comodules correspondingly. For an E_2 -algebra A with some additional assumptions we show that $H^0(A^!)$ is a Hopf algebra. The main reference for this talk is [Cos13, Chapters 7, 8].

1. FILTERED KOSZUL DUALITY

1.1. Classic Koszul duality. Let A be an augmented E_1 -algebra. We define the Koszul dual coalgebra $A^!$ as the derived functor $\mathbb{C} \otimes_A^{\mathbb{L}} \mathbb{C}$. That is a coaugmented dg-coalgebra that can be computed by the bar resolution. Please, note that in the standard definition, the Koszul dual of A should be an algebra. This algebra is linearly dual to the coalgebra $A^!$.

The following is an important theorem of Tamarkin. The goal of this talk is to prove a slightly generalized version of this theorem.

Theorem 1.1. *Let A be an E_2 -algebra which is augmented as an E_1 -algebra. Then $A^!$ has a natural structure of a Hopf algebra (up to coherent homotopy).*

The following example is especially interesting for us. Let \mathfrak{g} be a nilpotent Lie algebra concentrated in cohomological degree 0, and $U(\mathfrak{g})$ be its universal enveloping algebra. We denote the Chevalley-Eilenberg complex of \mathfrak{g} by $CE(\mathfrak{g})$. The following lemma is well-known and will be proved later in a slightly bigger generality.

Lemma 1.2. *We have an isomorphism of topological Hopf algebras $(CE(\mathfrak{g}))^! \simeq U(\mathfrak{g})^\vee$, where the latter one is the linear dual to $U(\mathfrak{g})$ with the dual Hopf algebra structure.*

Remark 1.3. *In this talk we always assume an existence of topology on all vector spaces we have. For example, we can talk about dual Hopf algebra structures without any additional assumptions.*

1.2. Filtered Koszul duality. We want to establish Lemma 1.2 for a general Lie algebra \mathfrak{g} . For this reason we will use the following approach.

Definition 1.4. *A complete filtered cochain complex is a cochain complex V with a decreasing filtration $V = F^0V \supset F^1V \supset \dots$ by sub-cochain complexes such that $V = \varprojlim V/F^iV$. Morphisms between complete filtered cochain complexes are the maps of complexes that preserve filtrations. A morphism is said to be a quasi-isomorphism, if the induced map on the associated graded is a quasi-isomorphism. We denote the category of complete filtered vector spaces by \mathbf{FVect} , and the category of complete filtered cochain complexes by $\mathbf{dgFVect}$.*

For two complete filtered cochain complexes V and W we define their tensor product $V \otimes W$ by $\varprojlim (V/F^iV) \otimes (W/F^jW)$ and endow it with a natural filtration. For a collection $V_i \in \mathbf{dgFVect}$ we define their completed direct sum by $\oplus V_i = \varprojlim (\oplus V_i/F^kV_i)$. Note that $\oplus V_i$ is the coproduct of the collection V_i . These two operations make $\mathbf{dgFVect}$ a symmetric monoidal category.

Consider $A \in \mathbf{Alg}(\mathbf{dgFVect})$, i.e. a complete filtered cochain complex together with an associative multiplication map $A \otimes A \rightarrow A$ such that $F^iA \cdot F^jA \subset F^{i+j}A$. We are particularly interested in the following two examples.

Example 1.5. If \mathfrak{g} is a dg Lie algebra, then $CE(\mathfrak{g}) \in \text{Alg}(\text{dgFVect})$ with the filtration given by $F^i CE(\mathfrak{g}) = \text{Sym}^{\geq i}(\mathfrak{g}^*[-1])$.

We will use this example to show that this definition of dgFVect carries much more structure than just taking the associated graded. For a \mathfrak{g} -module M we set $CE(\mathfrak{g}, M) = CE(\mathfrak{g}) \otimes M$ to be a module over dg algebra $CE(\mathfrak{g})$. We have an induced filtration $F^i CE(\mathfrak{g}, M) = F^i CE(\mathfrak{g}) \otimes M$.

Lemma 1.6. Let M, N be two \mathfrak{g} -modules. Then $CE(\mathfrak{g}, M)$ and $CE(\mathfrak{g}, N)$ are quasi-isomorphic as complete filtered modules over dg-algebra $CE(\mathfrak{g})$ if and only if M is quasi-isomorphic to N as \mathfrak{g} -modules.

Proof. Let $f : CE(\mathfrak{g}, M) \rightarrow CE(\mathfrak{g}, N)$ be a quasi-isomorphism. By definition, we have a collection of quasi-isomorphisms $\text{gr}^k CE(\mathfrak{g}, M) = \Lambda^k \mathfrak{g}^* \otimes M \rightarrow \Lambda^k \mathfrak{g}^* \otimes N = \text{gr}^k CE(\mathfrak{g}, N)$. For $k = 0$ we have a quasi-isomorphism $\underline{f} : M \rightarrow N$. For $k = 1$ we have a quasi-isomorphism $\mathfrak{g}^* \otimes M \simeq \mathfrak{g}^* \otimes N$, that shows that \underline{f} is compatible with a \mathfrak{g} -module structure.

In the opposite direction the proof is straightforward. \square

Example 1.7. For any Lie algebra the universal enveloping algebra $U(\mathfrak{g})$ has a natural filtration such that $F_i U(\mathfrak{g})$ is spanned by words of length $\leq i$ in the generators \mathfrak{g} . Dually, we get an increasing filtration $F^i U(\mathfrak{g})^\vee = (U(\mathfrak{g})/F_i U(\mathfrak{g}))^\vee$, and therefore $U(\mathfrak{g})^\vee \in \text{Alg}(\text{FVect})$.

Now suppose that A is augmented. We can define the filtered Koszul dual of A by $A^! = \varprojlim (\mathbb{C} \otimes_{A/F^i A}^\mathbb{L} \mathbb{C})$. Analogously to the classical case, this derived tensor product can be computed by the bar complex, that is defined in the following way.

For a dg-algebra A we set $\overline{B}A = \bigoplus_n A^{\otimes n}[n]$ with the differential $d = d_A + d_m$, where $d_A(a_1 \otimes a_2 \otimes \dots \otimes a_n) = \pm a_1 \otimes \dots \otimes da_i \otimes \dots \otimes a_n$ and $d_m(a_1 \otimes a_2 \otimes \dots \otimes a_n) = \pm (-1)^i a_1 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_n$. The signs here are defined by Koszul rule.

Proposition 1.8. Let \mathfrak{g} be any Lie algebra. Then, $CE(\mathfrak{g})^! \simeq U(\mathfrak{g})^\vee$ as Hopf algebras.

Proof. The filtration from Example 1.5 extends to a decreasing filtration on $CE(\mathfrak{g})^!$. To compute $\varprojlim (\bigoplus_n (CE(\mathfrak{g})/F^m CE(\mathfrak{g}))^{\otimes n}[n])$ we consider the spectral sequence E_{pq}^\bullet of the filtration. For $q < m$ we have $E_{pq}^0 = \bigoplus_{k_1+k_2+\dots+k_{-q}=p} \Lambda^{k_1}(\mathfrak{g}^*)[k_1] \otimes \dots \otimes \Lambda^{k_{-q}}(\mathfrak{g}^*)[k_{-q}]$. Note that this spectral sequence is bounded and therefore converges $E_{pq}^\infty \rightarrow H^{p+q}((CE(\mathfrak{g})/F^m CE(\mathfrak{g}))^!)$. We have $E_{p\bullet}^0$ is the complex $\Lambda^1(\mathfrak{g}^*) \otimes \Lambda^1(\mathfrak{g}^*) \otimes \dots \otimes \Lambda^1(\mathfrak{g}^*) \rightarrow \dots \rightarrow \Lambda^{-q}(\mathfrak{g}^*)$, so $E_{pq}^1 = S^p(\mathfrak{g}^*)$ if $q = -p$ and 0 else. All further differentials in the spectral sequence are trivial, so taking the limit we get an isomorphism of associative algebras $H^0(CE(\mathfrak{g})^!) \simeq \hat{S}(\mathfrak{g}^*)$, where the latter one stands for the completed symmetric algebra of \mathfrak{g}^* . We have an isomorphism of commutative algebras $U(\mathfrak{g})^\vee \rightarrow \hat{S}(\mathfrak{g}^*)$ induced from the PBW map of coalgebras $\phi : S(\mathfrak{g}) \rightarrow U(\mathfrak{g})$. Therefore $CE(\mathfrak{g})^! \simeq U(\mathfrak{g})^\vee$.

Both coproducts are induced from the coproduct in the tensor coalgebra, so this is an isomorphism of Hopf algebras. \square

2. FILTERED KOSZUL DUALITY FOR FVect_h

2.1. Generalities and the main results. We want to show that Koszul duality from Section 1 is actually a duality. To be more precise, we have an equivalence of categories $A\text{-mod} \simeq A^!\text{-comod}$ for correctly defined subcategories of modules and comodules respectively. We will prove it in a more general setting. Let FVect_h be the category of free complete filtered modules over $\mathbb{C}[[h]]$, i.e. objects in FVect_h are of the form $V \otimes \mathbb{C}[[h]]$, where $V \in \text{FVect}$. Moreover, for $V \in \text{FVect}_h$ we want $hF^k V \subset F^{k+1} V$. We define morphisms in FVect_h as maps of $\mathbb{C}[[h]]$ -modules that respect filtration. Let $\mathcal{C} = K(\text{FVect}_h)$ be the dg category of cochain complexes of objects in FVect_h , and let $\mathcal{C}^b \subset \mathcal{C}$ be a full sub dg category of objects, such that each graded piece is bounded above and below as a cochain complex.

Note that dg categories \mathcal{C} and \mathcal{C}^b are enriched in itself. For $V, W \in \mathcal{C}$ we define the filtration $F^k \text{Hom}(V, W)$ on the cochain complex of morphisms by $F^k \text{Hom}(V, W) = \{f \in \text{Hom}(V, W), f(F^i V) \subset F^{i+k} W \text{ for all } i\}$.

Then $hF^k \text{Hom}(V, W) \subset F^{k+1} \text{Hom}(V, W)$, and $\text{Hom}(V, W) = \varprojlim \text{Hom}(V, W)/F^k \text{Hom}(V, W)$, so $\text{Hom}(V, W) \in \mathcal{C}$.

For an algebra $A \in \text{Alg}(\mathcal{C}^b)$ we always demand it to have an isomorphism $A/F^1 A \simeq \mathbb{C}$. For a coalgebra $C \in \text{Coalg}(\mathcal{C}^b)$ we demand an isomorphism $C/F^1 C \simeq \mathbb{C}$. Let us make some definitions that will be used throughout the talk.

Definition 2.1. Let A be as above. We say that an A -module M is free, if it is of the form $M = A \otimes V$ for an object $V \in \mathcal{C}^b$. We say that M is quasi-free, if it admits an increasing filtration $0 = G^0 M \subset G^1 M \subset \dots$ by sub A -modules, such that $M = \varinjlim G^i M$, and $G^i M/G^{i-1} M$ is a free A -module. We denote the dg category of quasi-free A -modules by $A\text{-mod}$.

Definition 2.2. Let C be as above. We say that a C -comodule M is cofree, if it is of the form $M = C \otimes V$ for an object $V \in \mathcal{C}^b$. We say that M is quasi-cofree, if it admits a decreasing filtration $M = G^0 M \supset G^1 M \supset \dots$ by sub A -comodules, such that $M = \varprojlim M/G^i M$, and $G^i M/G^{i+1} M$ is a cofree A -comodule. We denote the dg category of quasi-cofree A -comodules by $C\text{-comod}$.

Definition 2.3. We define an $A_\infty A$ -module as $V \in \mathcal{C}^b$, with a collection of maps $\mu_n : A^{\otimes n} \otimes V \rightarrow V[1-n]$ for $n \geq 2$ satisfying the standard A_∞ -identity, that is $\sum (-1)^{r+st} m_u(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) = 0$. Note that this is equivalent to giving the differential on $\overline{B}A \otimes V$ that is compatible with the comodule structure over the coalgebra $\overline{B}A$. We denote the dg category of $A_\infty A$ -modules by $A_\infty A\text{-mod}$, where the morphisms are A_∞ -maps.

The following is the plan of this talk and simultaneously the list of the main results.

1) We construct the Koszul dual $A^! = \mathbb{C}[[h]] \otimes_A^L \mathbb{C}[[h]]$, and a quasi-equivalence of the categories $A\text{-mod} \simeq A^!\text{-comod}$.

2) Let $\text{Fin}(A^!\text{-comod})$ be the full subcategory of $A^!\text{-comod}$ consisting of $M \in A^!\text{-comod}$, such that $H^*(\text{gr } M/(h))$ is of finite total dimension. Set $\text{Perf}(A)$ to be the full subcategory of perfect modules in $A\text{-mod}$. Then the previous equivalence restricts to a quasi-equivalence $\text{Perf}(A) \simeq \text{Fin}(A^!\text{-comod})$.

3) Suppose that A is an E_2 -algebra, $H^i(A^!) = 0$ if $i \neq 0$, and $H^0(A^!)$ is $\mathbb{C}[[h]]$ -flat. Then there is a Hopf algebra structure on $H^0(A^!)$, and a quasi-equivalence of monoidal dg categories $A\text{-mod} \simeq H^0(A^!)\text{-comod}$.

2.2. Quasi-free modules. In this subsection we want to explain why we want to work with quasi-free modules. We have the following proposition.

Proposition 2.4. Let A be an algebra in \mathcal{C}^b . Then, every quasi-isomorphism $f : M \rightarrow N$ of quasi-free A -modules is a homotopy equivalence. The analogous statement holds for quasi-cofree comodules.

Proof. We will prove the proposition for quasi-free modules over an algebra. The coalgebra case is analogous. We need the following lemma.

Lemma 2.5. Let N, M_1, M_2 be quasi-free A -modules, and $f : M_1 \rightarrow M_2$ be a quasi-isomorphism. Then the map $\text{Hom}(N, M_1) \rightarrow \text{Hom}(N, M_2)$ is a quasi-isomorphism.

Proof. For any module M we have $\text{Hom}(N, M) = \text{Hom}(\varinjlim G^k N, M) = \varprojlim \text{Hom}(G^k N, M)$. We will show that the map $\text{Hom}(G^k N, M_1) \rightarrow \text{Hom}(G^k N, M_2)$ is a quasi-isomorphism using induction on k . Since $G^k N/G^{k-1} N$ is free, the natural map $\text{Hom}(G^k N, M) \rightarrow \text{Hom}(G^{k-1} N, M)$ is surjective.

Therefore we have the following commutative diagram.

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Hom}(G^k N / G^{k-1} N, M_1) & \longrightarrow & \mathrm{Hom}(G^k N, M_1) & \longrightarrow & \mathrm{Hom}(G^{k-1} N, M_1) \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Hom}(G^k N / G^{k-1} N, M_2) & \longrightarrow & \mathrm{Hom}(G^k N, M_2) & \longrightarrow & \mathrm{Hom}(G^{k-1} N, M_2) \longrightarrow 0
\end{array}$$

It induces a map between the long exact sequences of cohomologies, where 5-lemma gives us the desired result. \square

To deduce the proposition we look at the quasi-isomorphism $\mathrm{Hom}(N, M) \xrightarrow{f \circ} \mathrm{Hom}(N, N)$. For $\mathrm{Id}_N \in \mathrm{Hom}(N, N)$ we have a unique up to homotopy $g \in \mathrm{Hom}(N, M)$, such that $f \circ g = \mathrm{Id}_N$. That is a right homotopy inverse to f , and therefore quasi-isomorphism. Considering the quasi-isomorphism $\mathrm{Hom}(M, N) \xrightarrow{g \circ} \mathrm{Hom}(M, M)$ we find a right homotopy inverse to g , which must be homotopic to f . \square

2.3. The main property of the Koszul duality. For a dg algebra A over \mathbb{C} it is well-known that under some assumptions the categories $A\text{-mod}$ of A -modules and $A^1\text{-comod}$ of A^1 -comodules are quasi-equivalent. In this section we will prove an analogue of this theorem for $A \in \mathrm{Alg}(\mathcal{C}^b)$.

Theorem 2.6. *There are quasi-equivalences of dg categories $A\text{-mod} \simeq A_\infty A\text{-mod} \simeq A^1\text{-comod}$.*

Proof. Let $\mathbb{1}$ stand for $\mathbb{C}[[h]]$, the unit object in \mathcal{C}^b . Recall that A is augmented as $\mathbb{C}[[h]]$ -algebra, i.e. we have $A = \mathbb{1} \oplus \bar{A}$ for some non-unital algebra \bar{A} . The reduced bar complex $BA := \bigoplus_n \bar{A}^{\otimes n}[n] = \prod_n \bar{A}^{\otimes n}[n]$ gives a model for A^1 . For every m , the corresponding graded piece is bounded, so $A^1 \in \mathcal{C}^b$.

Let us first construct a quasi-equivalence $A_\infty A\text{-mod} \simeq A^1\text{-comod}$. For an $A_\infty A$ -module N we set $N^! = A^1 \otimes N = \bigoplus_n \bar{A}^{\otimes n}[n] \otimes N$. The $A_\infty A$ -module structure on N gives a differential $d_{N^!}$ on $N^!$, so $N^!$ is a dg A^1 -comodule. We have to check that $N^!$ is a quasi-free A^1 -module. Since $N \in \mathcal{C}^b$, we have a decreasing filtration $F^i N$, and $N = \varprojlim N / F^i N$. We set $G^i N^! = A^1 \otimes F^i N$ to be a decreasing filtration of $N^!$ by sub- A^1 -comodules. By construction, $N^! = \varprojlim N^! / G^i N^!$, and $G^i N^! / G^{i+1} N^!$ is a free A^1 -comodule. Therefore $N^! \in A^1\text{-comod}$.

To finish the proof of the second quasi-equivalence of Theorem, we want to show that every quasi-cofree A^1 -comodule is homotopy equivalent to a one obtained as above. For $M \in A^1\text{-comod}$ we consider its free resolution of A^1 -comodules. That is a A^1 -comodule of the form $A^1 \otimes V$ for $V \in \mathcal{C}^b$ that is quasi-isomorphic to M . It is enough to show that $A^1 \otimes V$ can be obtained from an $A_\infty A$ -module or, equivalently, endow V with the structure of $A_\infty A$ -module. The differential d on $A^1 \otimes V$ is defined by the collection of maps $d^k : \bar{A}^{\otimes k} \otimes V \rightarrow V$. This collection of maps endows V with a structure of $A_\infty A$ -module.

Now we focus on the quasi-equivalence $A\text{-mod} \simeq A_\infty A\text{-mod}$. Every A -module M has a natural structure of $A_\infty A$ -module. We have to show that for any quasi-free A -modules M and N the natural map $\mathrm{Hom}_{A\text{-mod}}(M, N) \rightarrow \mathrm{Hom}_{A_\infty A\text{-mod}}(M, N)$ is a quasi-isomorphism, and that any $A_\infty A\text{-mod}$ is homotopy equivalent to a strict A -module.

We define the standard bar resolution $BM = \bigoplus_{n \geq 0} A \otimes \bar{A}^{\otimes n} \otimes M[n]$. Note that we have a natural quasi-isomorphism $BM \rightarrow M$. An argument similar the one for $N^!$ above shows that BM is a quasi-free A -module. Therefore, by Lemma 2.5 $\mathrm{Hom}_{A\text{-mod}}(BM, N) \rightarrow \mathrm{Hom}_{A\text{-mod}}(M, N)$ is a quasi-isomorphism. Note that $\mathrm{Hom}_{A\text{-mod}}(BM, N) = \mathrm{Hom}_{A_\infty A\text{-mod}}(M, N)$ by the definition of a morphism of $A_\infty A$ -modules, so $\mathrm{Hom}_{A\text{-mod}}(M, N) \rightarrow \mathrm{Hom}_{A_\infty A\text{-mod}}(M, N)$ is a quasi-isomorphism.

Now let M be a $A_\infty A$ -module. Then BM is a quasi-free A -module, and there is a homotopy equivalence between BM and M . \square

2.4. Quasi-equivalence between categories of perfect modules and finite comodules. Let us introduce some new notations. We say that $V \in \mathcal{C}^b$ is *finite* if $H^*(\text{gr } V/(h))$ is of finite total dimension. Let $M = V \otimes_{\mathbb{C}[[h]]} A$ for $V \in \mathcal{C}^b$ be a free A -module. We say that M is of *finite rank* if V is finite as an object in \mathcal{C}^b .

An A -module M is *perfect* if the $\text{gr } A$ -module $\text{gr } M$ is quasi-isomorphic to the one obtained from free $\text{gr } A$ modules of finite rank by a finite number of iterated cones. We denote the full sub dg-category of perfect A -modules by $\text{Perf}(A)$. As in the classical case, $\text{Perf}(A)$ is equivalent to the full subcategory of dualizable objects.

We say that an A^1 -comodule N is *finite* if it is finite as an object in \mathcal{C}^b . We denote the full sub dg-category of finite A^1 -comodules by $\text{Fin}(A^1)$.

Proposition 2.7. *The quasi-equivalence of Theorem 2.6 restricts to a quasi-equivalence $\text{Perf}(A) \simeq \text{Fin}(A^1)$.*

Proof. Consider $M \in \text{Perf}(A)$. We have to check that $M^! = 1 \otimes_A^{\mathbb{L}} M$ is finite. From the construction of the filtration on $M^!$ we have $\text{gr } M^! = 1 \otimes_{\text{gr } A}^{\mathbb{L}} \text{gr } M$. The functor $1 \otimes_A^{\mathbb{L}}$ sends cones to cones, and the cone of two finite comodules is a finite comodule. Therefore it is enough to show that for $M = V \otimes_{\mathbb{C}[[h]]} A$ we have $M^! \in A^1\text{-comod}$ if V is finite.

We have $\text{gr } M = \text{gr } A \otimes_{\mathbb{C}[[h]]} \text{gr } V$. Therefore, $\text{gr } M^!/(h) = (1 \otimes_{\text{gr } A}^{\mathbb{L}} \text{gr } M)/(h) = \text{gr } V/(h)$, so $M^!$ is finite. It is easy to see that we can obtain any finite A^1 -comodule in this way.

Now suppose that $1 \otimes_{\mathbb{C}[[h]]}^{\mathbb{L}} M$ is finite for some $M \in A\text{-mod}$. We want to show that M is perfect. Let us consider a free resolution of $\text{gr } M$ as a $\text{gr } A$ -module. It gives a quasi-isomorphism $\text{gr } A \otimes_{\mathbb{C}} V \simeq \text{gr } M$. Note that $\text{gr}(1 \otimes_A^{\mathbb{L}} M)/(h) = \text{gr } V/(h)$, so V is finite.

Let V^i be the i -th graded component of V . Note that $V^i = 0$ for $i < 0$, and $H^*(V^i) = 0$ for $i \gg 0$. The differential on $V \otimes \text{gr } A$ is given by the collection of maps $d_i : V^i \rightarrow V^i \oplus \bigoplus_{0 \leq j < i} V^j \otimes \text{gr}^{i-j} A$. We define the increasing filtration $F^i(V \otimes \text{gr } A) = \bigoplus_{j \leq i} V^j \otimes \text{gr } A$. Note that this is a filtration by dg $\text{gr } A$ modules. Since $H^*(V^i) = 0$ for $i \gg 0$, we have $F^k(V \otimes \text{gr } A) = V \otimes \text{gr } A$ for k big enough. Therefore we have a finite filtration of $\text{gr } M$, such that the associated graded parts are free $\text{gr } A$ -modules. So M is perfect. \square

3. KOSZUL DUAL OF E_2 ALGEBRA IS A HOPF ALGEBRA

3.1. Equivalence between C -comod and $H^0(C)$ -comod. The main goal of this section is to prove the following proposition.

Proposition 3.1. *Let $C \in \text{Coalg}(\mathcal{C}^b)$ be a coalgebra, such that $H^i(C) = 0$ for $i \neq 0$, and $H^0(C)$ is a flat $\mathbb{C}[[h]]$ -module. Then there is a equivalence of dg categories $C\text{-comod} \simeq H^0(C)\text{-comod}$.*

Proof. The proof consists of two steps. We will first produce a coalgebra $D \in \text{Coalg}(\mathcal{C}^b)$, and coalgebra maps $C \rightarrow D$ and $H^0(C) \rightarrow D$ that are quasi-isomorphisms. After that we show that if a coalgebra map $C \rightarrow D$ is a quasi-isomorphism, then there is a quasi-equivalence $C\text{-comod} \rightarrow D\text{-comod}$.

Let d stand for the differential in C , and C^i for the piece in i -th cohomological degree. We define a cochain complex D by $D^i = C^i$ for $i > 0$, $D^i = 0$ for $i < 0$, and $D^0 = C^0/\text{Im } d$. We have a natural map $C \rightarrow D$. Note that the kernel is a coideal in C , so this map endows D with a correctly defined coalgebra structure. Since $H^0(C)$ is $\mathbb{C}[[h]]$ -flat, D is $\mathbb{C}[[h]]$ -flat, so $D \in \text{Coalg}(\mathcal{C}^b)$. The natural embedding $H^0(C) \rightarrow D$ is a coalgebra map, so we are done with the first step.

Now let C, D be coalgebras in \mathcal{C}^b , such that $C/F^1 C = \mathbb{C} = D/F^1 D$, and $\Phi : C \rightarrow D$ be a quasi-isomorphism. Let \bar{C} and \bar{D} be the kernels of the corresponding counit maps. We define the functors $\Phi_* : C\text{-comod} \rightarrow D\text{-comod}$ and $\Phi^* : D\text{-comod} \rightarrow C\text{-comod}$ in the following way.

Recall that for a left comodule M and a right comodule N over a coalgebra C we can define the cotensor product $M \odot_C N$ in the following way. Let Δ_M and Δ_N be the comodule structures.

Then we have two maps $\Delta_M \otimes \text{id}, \text{id} \otimes \Delta_N : M \otimes N \rightarrow M \otimes C \otimes N$. We set $M \odot_C N$ to be the equalizer.

For $M \in C\text{-comod}$ we set $\Phi_*(M) = M \odot_D^{\mathbb{R}} D = \prod (M \otimes \bar{D}^{\otimes n} \otimes D)[-n]$ to be the cobar resolution of M as a D -comodule. As for the bar construction, the differential here is the sum of two pieces. The first one is applying the differentials of M and D to every multiplier. The second one is using the coalgebra structure.

Dually, for $N \in D\text{-comod}$ we set $\Phi^*(N) = N \odot_D^{\mathbb{R}} C$.

We want to show that there are natural quasi-isomorphisms $\text{Id}_{C\text{-comod}} \simeq \Phi^* \Phi_*$ and $\Phi_* \Phi^* \simeq \text{Id}_{D\text{-comod}}$. We will construct the first one, the second is analogous.

Let M be a C -comodule. We have natural quasi-isomorphisms $M \simeq M \odot_C^{\mathbb{R}} C$ as C -comodules and $M \simeq \Phi_*(M) = M \odot_D^{\mathbb{R}} D$ as D -comodules. Applying the functor Φ^* we get a quasi-isomorphism $M \odot_D^{\mathbb{R}} C \simeq \Phi^* \Phi_* M$. It remains to construct a quasi-isomorphism $M \odot_C^{\mathbb{R}} C \rightarrow M \odot_D^{\mathbb{R}} C$.

Let $M \odot_C^{\Delta} C$ be a cosimplicial object, whose n -simplices are $M \otimes \bar{C}^{\otimes n} \odot C$, and face and degeneracy morphisms are the classical maps on cobar construction. Note that $M \odot_C^{\mathbb{R}} C$ is the totalization of $M \odot_C^{\Delta} C$. Similarly, let $M \odot_D^{\Delta} C$ be a cosimplicial object, whose n -simplices are $M \otimes \bar{D}^{\otimes n} \odot C$. The quasi-isomorphism $\Phi : C \rightarrow D$ induces a map of cosimplicial complexes $M \odot_C^{\Delta} C \rightarrow M \odot_D^{\Delta} C$ that is a levelwise quasi-isomorphism. Therefore the map on totalizations is a quasi-isomorphism. \square

3.2. Koszul dual of E_2 -algebra. In this section we will prove the main theorem of the talk. We need the following proposition.

Proposition 3.2. *Let $A \in \text{Alg}(C^b)$ be an augmented E_2 -algebra. Then the category $A\text{-mod}$ is monoidal up to homotopy.*

Proof. In this proof we follow [CG16, Section 8.2]. We aim to explain, why this fact should be true and how the proof should look like instead of giving the formal proof, so we omit details about ∞ -issues that arise here.

Let Cat be the $(\infty, 2)$ -category of all ∞ -categories. For any monoidal category C we have a full subcategory $\text{Alg}_{E_1}(C)$ of algebra objects up to homotopy. We have a functor $\text{LMod}_{\bullet} : \text{Alg}_{E_1}(C) \rightarrow \text{LMod}_C(\text{Cat})$ that sends an E_1 -algebra $A \in C$ in the category of left modules over A .

We have the following proposition due to Lurie.

Proposition 3.3. *$[Lur] \text{Alg}_{E_n}(C) \simeq \text{Alg}_{E_{n-1}}(\text{Alg}_{E_1}(C))$.*

Let us apply the functor Alg_{E_1} to the functor LMod_{\bullet} . Then we get a functor $\text{LMod}_{\bullet}^2 : \text{Alg}_{E_2}(C) \rightarrow \text{Alg}_{E_1}(\text{LMod}_C(\text{Cat}))$. We have a natural forgetful functor $\text{Alg}_{E_2}(C) \rightarrow \text{Alg}_{E_1}(C)$. It induces the following commutative diagram.

$$\begin{array}{ccc} \text{Alg}_{E_2}(C) & \xrightarrow{\text{Alg}_{E_1}(\text{LMod}_{\bullet})} & \text{Alg}_{E_1}(\text{LMod}_C(\text{Cat})) \\ \downarrow & & \downarrow \\ \text{Alg}_{E_1}(C) & \xrightarrow{\text{LMod}_{\bullet}} & \text{LMod}_C(\text{Cat}) \end{array}$$

Therefore the category $\text{LMod}(A)$ of the left A -modules has a monoidal structure (up to homotopy) coming from the multiplication in $\text{Alg}_{E_1}(\text{LMod}_{\bullet})(A)$. \square

Suppose that $H^i(A^!) = 0$ for $i \neq 0$, and $H^0(A^!)$ is $\mathbb{C}[[h]]$ -flat. Then by Theorem 2.6 and Proposition 3.1 we have a quasi-equivalence $A\text{-mod} \simeq H^0(A^!)\text{-comod}$, so the former one induces a monoidal structure from the one on $A\text{-mod}$. We want to show that it induces a Hopf algebra structure on $H^0(A^!)$.

Theorem 3.4. *Let A be as above. Then $H^0(A^!)$ is a Hopf algebra.*

We need the standard Tannaka-Krein formalism to deduce the theorem. Let us recall its framework following [Eti+16].

Let \mathcal{C} be a monoidal category. Let \mathbf{Vect} be the monoidal category of vector spaces. Suppose that we have a fiber functor $F : \mathcal{C} \rightarrow \mathbf{Vect}$, i.e. an exact faithful functor endowed with a natural isomorphism $J_{X,Y} : F(X) \otimes F(Y) \simeq F(X \otimes Y)$ for every pair X, Y of objects in \mathcal{C} .

For the pair (\mathcal{C}, F) we can define a coalgebra $\mathrm{Coend}(F) := (\oplus_{X \in \mathcal{C}} F(X)^* \otimes F(X))/E$, where E is generated by elements of the form $y^* \otimes F(f)x - F(f)y^* \otimes x$ for $x \in F(X)$, $y^* \in F(Y)^*$ and $f \in \mathrm{Hom}(X, Y)$. If \mathcal{C} is monoidal, $\mathrm{Coend}(F)$ has an algebra structure. Indeed, $\mathrm{Coend}(F) = \mathrm{End}(F)^\vee$. On $\mathrm{End}(F)$ we have a coalgebra structure given in the following way.

We have a functor $F \times F : \mathcal{C} \times \mathcal{C} \rightarrow \mathbf{Vect}$, and a universal category $\mathcal{C} \boxtimes \mathcal{C}$, such that the product of any two right exact functors $\mathcal{C} \rightarrow \mathbf{Vect}$ uniquely factors through $\mathcal{C} \boxtimes \mathcal{C}$. Let $F \boxtimes F : \mathcal{C} \boxtimes \mathcal{C} \rightarrow \mathbf{Vect}$ be the corresponding functor. Note that we have $\mathrm{End}(F \boxtimes F) = \mathrm{End}(F) \otimes \mathrm{End}(F)$. We set $\Delta(a)_{X,Y} = J_{X,Y}^{-1} a_{X \otimes Y} J_{X,Y}$. That is a correctly defined coproduct, that induces an algebra structure on $\mathrm{Coend}(F)$. Moreover, if \mathcal{C} is rigid, then $\mathrm{Coend}(F)$ admits an antipode.

Now let H be a Hopf algebra. We define the category H -comod of left H -comodules. It has a monoidal structure coming from the product on H . The natural forgetful functor to \mathbf{Vect} is a fiber functor.

Theorem 3.5. [Eti+16, Theorem 5.4.1]

The assignments $(\mathcal{C}, F) \rightarrow H = \mathrm{Coend}(F)$, $H \rightarrow (H\text{-comod}, \mathrm{Forget})$ are mutually inverse bijections between the set of rigid monoidal categories with a fiber functor F , up to monoidal equivalence and isomorphisms of monoidal functors, and the set of Hopf algebras up to isomorphism.

Proof of Theorem 3.4. We will use this formalism in slightly bigger generality. Composing Section 1 and Proposition 3.1 we get the quasi-equivalence $A\text{-mod} \simeq H^0(A^!)\text{-comod}$ that induces a monoidal up to homotopy structure on $H^0(A^!)\text{-comod}$. Let $H^0(A^!)\text{-comod}^0$ be the category of free non-dg $H^0(A^!)$ -modules. Since any quasi-free module is a limit of finite extensions of free modules, the monoidal structure on $H^0(A^!)\text{-comod}$ can be constructed from the one on $H^0(A^!)\text{-comod}^0$. Costello's idea is to show that this monoidal structure coincides with the original one, so the monoidal up to homotopy structure on $H^0(A^!)\text{-comod}$ is actually a strict monoidal structure.

Since $A \rightarrow \mathbb{C}[[h]]$ is a map of E_2 -algebras, the corresponding forgetful functor $A\text{-mod} \rightarrow \mathcal{C}^b$ is a monoidal functor. Therefore we have an exact faithful monoidal functor $H^0(A^!)\text{-comod} \rightarrow \mathcal{C}^b$. Applying Theorem 3.5 we get a bialgebra structure on $H^0(A^!)$. Restricting additionally to the category $\mathrm{Fin}(H^0(A^!)) \simeq \mathrm{Perf}(A)$, we have a left dual for every object, so $H^0(A^!)$ admits an antipode. \square

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