A REMARK ON BEILINSON-DRINFELD ALGEBRAS

BRIAN R. WILLIAMS

Throughout these notes, $\hbar$ is a formal parameter.

Definition 0.1. A *Beilinson-Drinfeld* (BD) algebra is:

(i) a dg commutative algebra (A, Δ_{BD}) (here, Δ_{BD} is the differential) defined over the ring $\mathbb{C}[[\hbar]]$;

(ii) together with a Poisson bracket $\{-,-\} : A \otimes_{\mathbb{C}[[\hbar]]} A \to A$ of degree $+1$;

such that Δ is a derivation for the Poisson bracket and

$$\Delta_{BD}(a \cdot b) = (\Delta_{BD} a) \cdot b + (-1)^{|a|} a \cdot (\Delta_{BD} b) + \hbar (-1)^{|a|} \{a, b\} \tag{1}$$

for all $a, b \in A$.

A BD algebra is an algebra over a $\mathbb{C}[[\hbar]]$-linear operad that we call the *BD operad*. We think about this operad as living over the formal scheme $\mathrm{Spf}(\mathbb{C}[[\hbar]])$. Over 0, the operad is precisely the P_0-operad. That is, if A is a BD algebra, the commutative dg algebra

$$A^{\mathrm{cl}} = A \otimes_{\mathbb{C}[[\hbar]]} \mathbb{C}_{\hbar=0}$$

inherits the bracket $\{-,-\}$. Moreover, the last term in (1) goes away, so that the differential is now a derivation for the product as well. Thus A^{cl} has the structure of a P_0-algebra.

Generically, away from 0, the BD operad naively looks very close to the Batalin-Vilkovisky (BV) operad. Recall, the BV operad is equivalent to the homology of the *framed* little 2-disks operad. Thus, a dg algebra over the BV operad consists of: (1) a differential d of degree $+1$, (2) a commutative product of degree zero, (3) a Poisson bracket $\{-,-\}$ of degree (-1), and (4) an operator Δ_{BV} of degree -1. The BV operator satisfies the formula

$$\Delta_{BV}(a \cdot b) = (\Delta_{BV} a) \cdot b + (-1)^{|a|} a \cdot (\Delta_{BV} b) + (-1)^{|a|} \{a, b\}.$$

Besides degree issues, there are important differences between BD algebras and BV algebras. On one hand, the operator Δ_{BD} in the definition of a BD algebra is the *differential*, whereas Δ_{BD} comes from the unary operation in the BV operad. In particular, when $\hbar$ is generic, the equation (1) implies that the bracket is actually homotopically trivialized by $\hbar^{-1}\Delta_{BD}$. Thus, when we specialize a BD algebra to a generic value of $\hbar$ we just obtain a pointed cochain complex.

Lemma 0.2. *The BD operad is an operad over the formal scheme* $\mathrm{Spf}(\mathbb{C}[[\hbar]])$*. Over* 0 *it is equivalent to the* P_0 *operad. Away from* 0 *it is equivalent to the* E_0*-operad.*