



(h)

Morley's Theorem, a Proof

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In Figure 1, the near trisectors of the internal angles at the vertices A, B, and C of a triangle meet in X, Y, and Z. **Morley's theorem** (<http://www.cut-the-knot.org/triangle/Morley/index.shtml>) states that the triangle XYZ is equilateral. We give here a direct Euclidean proof.

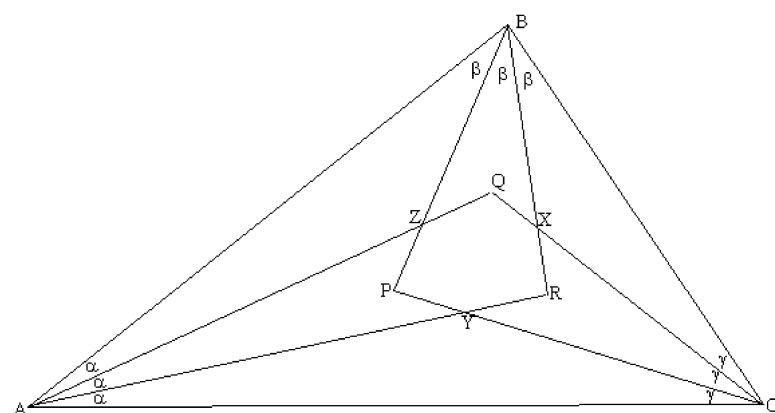


Figure 1

Let the far trisectors meet in P, Q, and R. The following proof that PX, QY, and RZ are concurrent is purely Euclidean; a stronger result can be **proved** (<http://www.cut-the-knot.org/triangle/Morley/Morley.shtml>) using **projective geometry** (<http://www.cut-the-knot.org/triangle/pythpar/Geometries.shtml#projective>). Figure 1 shows the angle values α , β , γ , implying $\alpha + \beta + \gamma = \pi/3$, and $\angle QXR = \pi - \beta - \gamma$. Since Y is the incentre of $\triangle AQC$, it follows that $\angle ZQY = \angle XQY = \beta + \pi/6$. The corresponding angles at R and P are $\gamma + \pi/6$, $\alpha + \pi/6$.

Assume PX, QY and RZ meet in pairs at U, V, and W. By **transitivity** (<http://www.cut-the-knot.org/triangle/remarkable.shtml>), either U, V, W all coincide or are all distinct. We show that the assumption that the three points are distinct leads to a contradiction. Indeed, this might happen in two ways, as illustrated in Figures 2 and 3 (**below**.) We focus on the diagram of Figure 2, the other one being entirely analogous. Sum the angles of quadrilateral QURX, using the above results, to find $\angle WUV = \pi/3$. Hence $\triangle WUV$ is equilateral, with sides $2d$, say. Thus we assume that

$$(1) \quad d \neq 0.$$

Choose X_1 on QX such that UX_1 is parallel to PX. Then the angles $\angle ZUQ$, $\angle QUX_1$, $\angle X_1UV$ all are $\pi/3$. Now choose X_2 and X_3 on PX such that $\angle PX_2X_1 = \pi/2$, and X_1X_3 is parallel to UV. These constructions imply $\angle X_2X_1X = \beta$ (because in $\triangle QWX$, $\angle WQX = \beta + \pi/6$ and $\angle QWX = \pi/3$ hence $\angle QXW = \pi/2 - \beta$) and $X_2X_3 = d$ - from the choice of X_3 .

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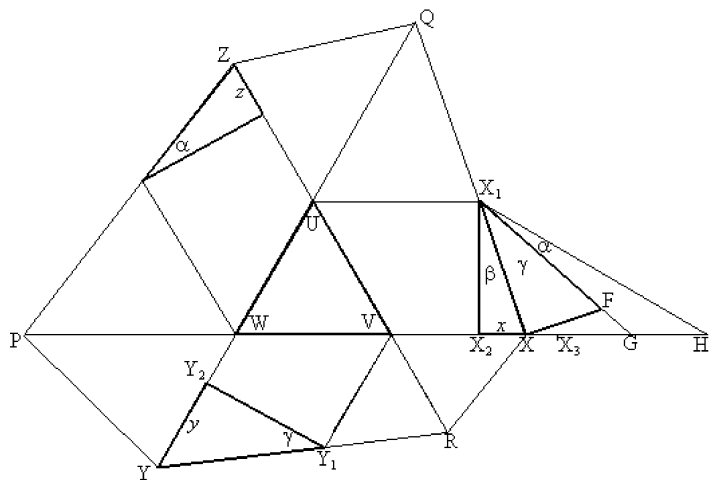


Figure 2

Denote X_2X by x , then $VX = VX_3 - d + x$. But $VX_3 = UX_1 = UZ$, from the congruence of triangles ZUQ and X_1UQ . (These triangles are congruent since they have a common side UQ , the angles at Q are $\beta + \pi/6$, and the angles at U are $\pi/3$.) Hence

$$(2) \quad VX = UZ - d + x.$$

Similarly, defining y and z in the same way as for x ,

$$(3) \quad WY = VX - d + y,$$

$$(4) \quad UZ = WY - d + z.$$

Adding equations (2)-(4) gives

$$(5) \quad x + y + z = 3d.$$

Now extend the line PX to H , such that $\angle X_2X_1H = \pi/3$, and hence $X_2H = 3d$. Indeed, in an equilateral triangle the ratio of the altitude to a half-side equals $\sqrt{3}$. In $\triangle UVW$, this means that $X_1X_2 = d \times \sqrt{3}$. In $\triangle X_1X_2H$ (which is a half of an equilateral triangle), we have $X_2H = X_1X_2 \times \sqrt{3}$. Multiplying through gives

$$\begin{aligned} X_2H &= X_1X_2 \times \sqrt{3} \\ &= (d \times \sqrt{3}) \times \sqrt{3} \\ &= 3d. \end{aligned}$$

Choose G on X_2H such that $\angle XX_1G = \gamma$. Then $\angle GX_1H = \alpha$. Finally, choose F on X_1G such that XF is normal to XX_1 . Then $\triangle XX_1F$ and $\triangle Y_2Y_1Y$ are similar. Since $XX_1 > X_2X_1 = Y_2Y_1$, then $XF > y$. Because $\angle XFG = \pi/2 + \gamma$, it is the greatest angle in $\triangle XFG$. Hence, by *Euclid (I.19)* ([http://www.cut-the-knot.org/pythagoras/EuclidRef_I.shtml#\(I.19\)](http://www.cut-the-knot.org/pythagoras/EuclidRef_I.shtml#(I.19))), XG is the longest side, so $y < XG$. Similarly, $z < GH$. Including $x = X_2X$, it follows that $x + y + z < 3d$. This contradicts (5) and so the assumption $d \neq 0$ was false. That is, PX , QY , and RZ are concurrent, with mutual angles of $\pi/3$.

Now complete the proof of Morley's Theorem. As was observed earlier, in Figure 2, $UZ = UX_1$, so that $\triangle ZUX_1$ is isosceles. Hence YQ , being the bisector at vertex U , is normal to the base ZX_1 . But X_1 and X now the same point, so ZX is normal to YQ . Similarly, XY , YZ are normal to RZ , PX , implying that the $\triangle XYZ$ is equilateral.

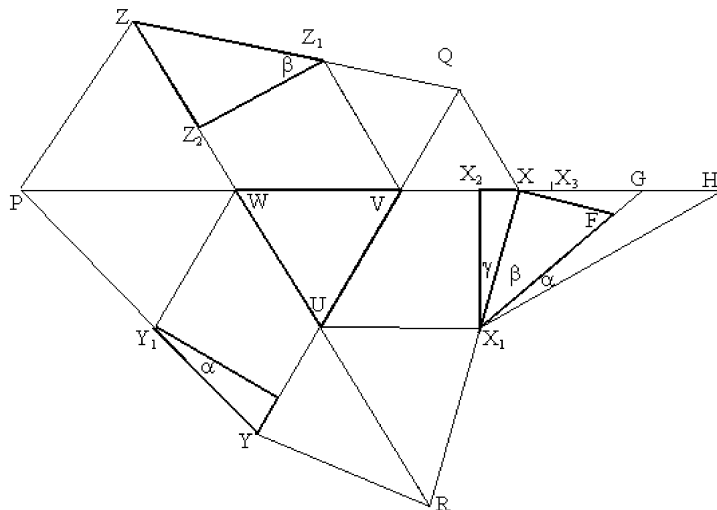


Figure 3

Morley's Miracle (<http://www.cut-the-knot.org/triangle/Morley/index.shtml>)

On Morley and his theorem

1. *Doodling and Miracles* (<http://www.cut-the-knot.org/triangle/Morley/Morley.shtml>)
2. *Morley's Pursuit of Incidence* (<http://www.cut-the-knot.org/triangle/Morley/CenterCircle.shtml>)
3. *Lines, Circles and Beyond* (<http://www.cut-the-knot.org/triangle/Morley/Beyond.shtml>)
4. *On Motivation and Understanding* (<http://www.cut-the-knot.org/triangle/Morley/MorleyFinal.shtml>)
5. *Of Looking and Seeing* (<http://www.cut-the-knot.org/ctk/MorleyConc.shtml>)

Backward proofs

1. *J. Conway's proof* (<http://www.cut-the-knot.org/triangle/Morley/conway.shtml>)
 - *Remarks on J. Conway's proof* (http://www.cut-the-knot.org/triangle/Morley/remarks_c.shtml)
2. *D. J. Newman's proof* (<http://www.cut-the-knot.org/triangle/Morley/newman.shtml>)
3. *B. Bollobás' proof* (<http://www.cut-the-knot.org/triangle/Morley/Bollobas.shtml>)
4. *G. Zsolt Kiss' proof* (<http://www.cut-the-knot.org/triangle/Morley/MorleyZsolt.shtml>)
5. *Backward Proof by B. Stonebridge* (<http://www.cut-the-knot.org/triangle/Morley/sb3.shtml>)
6. *Morley's Equilaterals, Spiridon A. Kuruklis' proof* (<http://www.cut-the-knot.org/m/Geometry/Morley5.shtml>)

Trigonometric proofs

1. *Bankoff's proof* (<http://www.cut-the-knot.org/triangle/Morley/BankoffProof.shtml>)
2. *B. Bollobás' trigonometric proof* (<http://www.cut-the-knot.org/triangle/Morley/BollobasTrig.shtml>)
3. *Proof by R. J. Webster* (<http://www.cut-the-knot.org/triangle/Morley/Webster.shtml>)
4. *A Vector-based Proof of Morley's Trisector Theorem* (<http://www.cut-the-knot.org/triangle/Morley/VectorProof.shtml>)
5. *L. Giugiuc's Proof of Morley's Theorem* (<http://www.cut-the-knot.org/triangle/Morley/Giugiuc.shtml>)
6. *Dijkstra's Proof of Morley's Theorem* (<http://www.cut-the-knot.org/triangle/Morley/Dijkstra.shtml>)

Synthetic proofs

1. *Another proof* (http://www.cut-the-knot.org/triangle/Morley/yours_truly.shtml)
2. *Nikos Dergiades' proof* (<http://www.cut-the-knot.org/triangle/Morley/Dergiades.shtml>)
3. *M. T. Naranengar's proof* (<http://www.cut-the-knot.org/triangle/Morley/Naranengar.shtml>)
4. *An Unexpected Variant* (<http://www.cut-the-knot.org/triangle/Morley/Larry.shtml>)
5. Proof by B. Stonebridge and B. Millar
6. *Proof by B. Stonebridge* (<http://www.cut-the-knot.org/triangle/Morley/sb2.shtml>)
7. *Proof by Roger Smyth* (<http://www.cut-the-knot.org/triangle/Morley/Smyth.shtml>)
8. *Proof by H. D. Grossman* (<http://www.cut-the-knot.org/triangle/Morley/Grossman.shtml>)
9. *Proof by H. Shutrick* (<http://www.cut-the-knot.org/wiki-math/index.php?n=Geometry.MorleysTheorem>)
10. *Original Taylor and Marr's Proof of Morley's Theorem* (<http://www.cut-the-knot.org/m/Geometry/Morley2.shtml>)
11. *Taylor and Marr's Proof - R. A. Johnson's Version* (<http://www.cut-the-knot.org/m/Geometry/Morley.shtml>)

- ## Algebraic proofs

- ## Invalid proofs

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<http://www.cut-the-knot.org/triangle/Morley/sb.shtml> (<http://www.cut-the-knot.org/triangle/Morley/sb.shtml>), Accessed 16 May 2017

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Other solution. ABCD unit square with $N=(1,y)$ and $A'=(1-y,1+y)$: ANA' is right-angled at N...

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