Morley's Theorem, a Proof

Brian Stonebridge (University of Bristol, UK)
Bill Millar

In Figure 1, the near trisectors of the internal angles at the vertices A, B, and C of a triangle meet in X, Y, and Z. ***Morley's theorem (http://www.cut-the-knot.org/triangle/Morley/index.shtml)*** states that the triangle XYZ is equilateral. We give here a direct Euclidean proof.

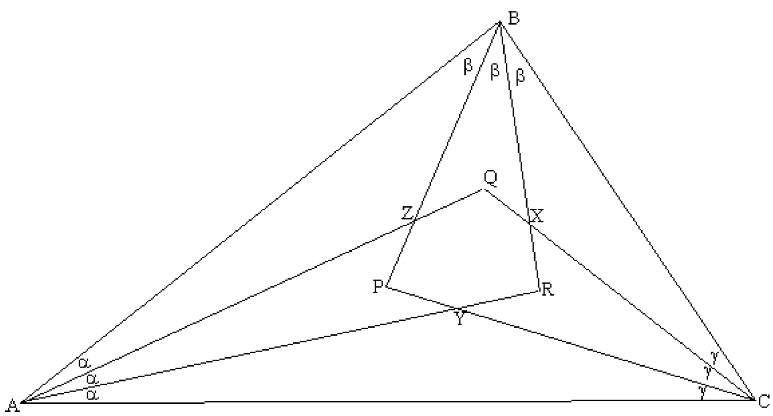

Figure 1

Let the far trisectors meet in P, Q, and R. The following proof that PX, QY, and RZ are concurrent is purely Euclidean; a stronger result can be ***proved (http://www.cut-the-knot.org/triangle/Morley/Morley.shtml)*** using ***projective geometry (http://www.cut-the-knot.org/triangle/pythpar/Geometries.shtml#projective)***. Figure 1 shows the angle values α, β, γ, implying $\alpha + \beta + \gamma = \pi/3$, and $\angle QXR = \pi - \beta - \gamma$. Since Y is the incentre of ΔAQC, it follows that $\angle ZQY = \angle XQY = \beta + \pi/6$. The corresponding angles at R and P are $\gamma + \pi/6$, $\alpha + \pi/6$.

Assume PX, QY and RZ meet in pairs at U, V, and W. By ***transitivity (http://www.cut-the-knot.org/triangle/remarkable.shtml)***, either U, V, W all coincide or are all distinct. We show that the assumption that the three points are distinct leads to a contradiction. Indeed, this might happen in two ways, as illustrated in Figures 2 and 3 (***below***.) We focus on the diagram of Figure 2, the other one being entirely analogous. Sum the angles of quadrilateral QURX, using the above results, to find $\angle WUV = \pi/3$. Hence ΔWUV is equilateral, with sides $2d$, say. Thus we assume that

(1) $\quad d \ne 0$.

Choose X_1 on QX such that UX_1 is parallel to PX. Then the angles ZUQ, QUX_1, X_1UV all are $\pi/3$. Now choose X_2 and X_3 on PX such that $\angle PX_2X_1 = \pi/2$, and X_1X_3 is parallel to UV. These constructions imply $\angle X_2X_1X = \beta$ (because in ΔQWX, $\angle WQX = \beta + \pi/6$ and $\angle QWX = \pi/3$ hence $\angle QXW = \pi/2 - \beta$) and $X_2X_3 = d$ - from the choice of X_3.

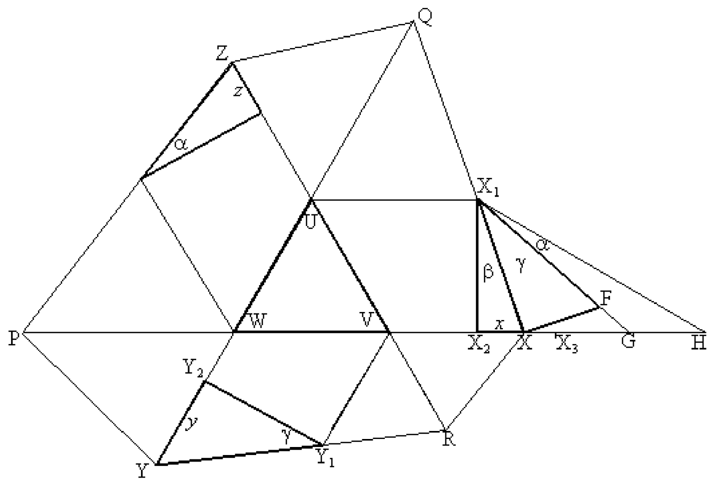

Figure 2

Denote X_2X by x, then $VX = VX_3 - d + x$. But $VX_3 = UX_1 = UZ$, from the congruence of triangles ZUQ and X_1UQ. (These triangles are congruent since they have a common side UQ, the angles at Q are $\beta + \pi/6$, and the angles at U are $\pi/3$.) Hence

(2) $VX = UZ - d + x$.

Similarly, defining y and z in the same way as for x,

(3) $WY = VX - d + y$,

(4) $UZ = WY - d + z$.

Adding equations (2)-(4) gives

(5) $x + y + z = 3d$.

Now extend the line PX to H, such that $\angle X_2X_1H = \pi/3$, and hence $X_2H = 3d$. Indeed, in an equilateral triangle the ratio of the altitude to a half-side equals $\sqrt{3}$. In ΔUVW, this means that $X_1X_2 = d \times \sqrt{3}$. In ΔX_1X_2H (which is a half of an equilateral triangle), we have $X_2H = X_1X_2 \times \sqrt{3}$. Multiplying through gives

$$\begin{aligned} X_2H &= X_1X_2 \times \sqrt{3} \\ &= (d \times \sqrt{3}) \times \sqrt{3} \\ &= 3d. \end{aligned}$$

Choose G on X_2H such that $\angle XX_1G = \gamma$. Then $\angle GX_1H = \alpha$. Finally, choose F on X_1G such that XF is normal to XX_1. Then ΔXX_1F and ΔY_2Y_1Y are similar. Since $XX_1 > X_2X_1 = Y_2Y_1$, then $XF > y$. Because $\angle XFG = \pi/2 + \gamma$, it is the greatest angle in ΔXFG. Hence, by ***Euclid (I.19) (http://www.cut-the-knot.org/pythagoras/EuclidRef_I.shtml#(I.19))***, XG is the longest side, so $y < XG$. Similarly, $z < GH$. Including $x = X_2X$, it follows that $x + y + z < 3d$. This contradicts ***(5)*** and so the assumption ***d ≠ 0*** was false. That is, PX, QY, and RZ are concurrent, with mutual angles of $\pi/3$.

Now complete the proof of Morley's Theorem. As was observed earlier, in Figure 2, $UZ = UX_1$, so that ΔZUX_1 is isosceles. Hence YQ, being the bisector at vertex U, is normal to the base ZX_1. But X_1 and X now the same point, so ZX is normal to YQ. Similarly, XY, YZ are normal to RZ, PX, implying that the ΔXYZ is equilateral.

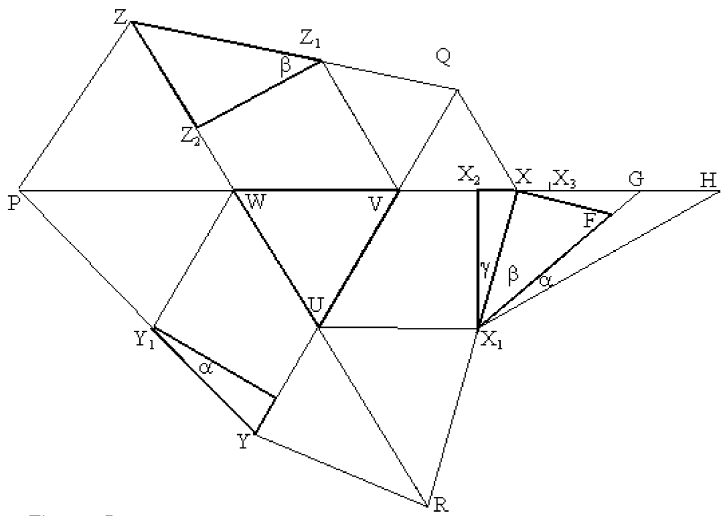

Figure 3

Morley's Miracle (http://www.cut-the-knot.org/triangle/Morley/index.shtml)

12. ***Morley's Theorem: Second Proof by Roger Smyth (http://www.cut-the-knot.org/triangle/Morley/Smyth2.shtml)***
13. ***Proof by A. Robson (http://www.cut-the-knot.org/triangle/Morley/Robson.shtml)***

Algebraic proofs

1. ***Morley's Redux and More, Alain Connes' proof (http://www.cut-the-knot.org/ctk/MorleysRedux.shtml)***

Invalid proofs

1. ***Bankoff's conundrum (http://www.cut-the-knot.org/triangle/Morley/bankoff.shtml)***
2. ***Proof by Nolan L Aljaddou (http://www.cut-the-knot.org/triangle/Morley/Aljaddou.shtml)***
3. ***Morley's Theorem: A Proof That Needs Fixing (http://www.cut-the-knot.org/triangle/Morley/MorleyFalse.shtml)***

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Alexander Bogomolny, Proof by B. Stonebridge and B. Millar *from Interactive Mathematics Miscellany and Puzzles*
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5d

The problem and all variants can be solved by explicit coordinates of the involved points, using co...

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The simpler, the better! There is in fact almost nothing to prove. The vector MN can be translate...

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Other solution. ABCD unit square with N=(1,y) and A'=(1-y,1+y): ANA' is right-angled at N...

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