Solutions for HW4

4.24 **(a)** From Eq. (4-59),

$$F(\mu,\nu) = \Im[f(t,z)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,z)e^{-j2\pi(\mu t+vz)}dtdz$$

From Eq. (2-23), we know that the Fourier transform is a linear operator if

$$\Im\left[a_1 f_1(t,z) + a_2 f_2(t,z)\right] = a_1\Im\left[f_1(t,z)\right] + a_2\Im\left[f_2(t,z)\right]$$

Substituting into the definition of the Fourier transform yields

$$\begin{aligned}\Im\left[a_1 f_1(t,z) + a_2 f_2(t,z)\right] &= \textstyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[a_1 f_1(t,z) + a_2 f_2(t,z)\right] \\ &\quad \times e^{-j2\pi(\mu t+vz)}dtdz \\ &= a_1 \textstyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,z)e^{-j2\pi(\mu t+vz)}dtdz \\ &\quad + a_2 \textstyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_2(t,z)e^{-j2\pi(\mu t+vz)}dtdz \\ &= a_1\Im\left[f_1(t,z)\right] + a_2\Im\left[f_2(t,z)\right]\end{aligned}$$

where the second step follows from the distributive property of the integral. The linearity of the inverse transform is proved in exactly the same way.

(b) The linearity of the discrete case is demonstrated in the same way:

$$\begin{aligned}\Im\left[a_1 f_1(x,y) + a_2 f_2(x,y)\right] &= \textstyle\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}\left[a_1 f_1(x,y) + a_2 f_2(x,y)\right]e^{-j2\pi(ux/M+vy/N)} \\ &= a_1 \textstyle\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f_1(x,y)e^{-j2\pi(ux/M+vy/N)} \\ &\quad + a_2 \textstyle\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f_2(x,y)e^{-j2\pi(ux/M+vy/N)} \\ &= a_1\Im\left[f_1(x,y)\right] + a_2\Im\left[f_2(x,y)\right]\end{aligned}$$

The linearity of the inverse transform is proved in exactly the same way.

4.28 **(a)** We solve the problem by direct substitution into Eq. (4-67):

$$\begin{aligned}\Im[\delta(x,y)] &= \textstyle\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}\delta(x,y)e^{-j2\pi(ux/M+vy/N)}\\ &= \textstyle\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}e^{-j2\pi(ux/M+vy/N)}\delta(x,y)\\ &= e^{-j2\pi(u[0]/M+v[0]/N)}\\ &= 1\end{aligned}$$

where the third step follows from the sifting property of the 2-D impulse, Eq. (4-58). Because we used direct substitution into Eq. (4-67), and this equation and Eq. (4-68) are a Fourier transform pair, it must follow that the left side of the double arrow is the IDFT of the right: $\Im^{-1}[1] = \delta(x,y)$.

(b) We solve this problem by starting with the inverse DFT, Eq. (4-68):

$$\begin{aligned}\Im^{-1}[MN\delta(u,v)] &= \tfrac{MN}{MN}\textstyle\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}\delta(u,v)e^{j2\pi(ux/M+vy/N)}\\ &= e^{j2\pi(0x/M+0y/N)}\\ &= 1\end{aligned}$$

where the second step follows from the sifting property of the 2-D impulse, Eq. (4-58). Because we used direct substitution into Eq. (4-68), and this equation and Eq. (4-67) are a Fourier transform pair, it must follow that the right side of the double arrow is the DFT of the left: $\Im[1] = MN\delta(x,y)$.

(c) We solve the problem by direct substitution into the forward DFT, Eq. (4-67):

$$\begin{aligned}\Im\left[\delta\left(x-x_0,y-y_0\right)\right] &= \textstyle\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}\delta\left(x-x_0,y-y_0\right)e^{-j2\pi(ux/M+vy/N)}\\ &= \textstyle\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}e^{-j2\pi(ux/M+vy/N)}\delta\left(x-x_0,y-y_0\right)\\ &= e^{-j2\pi(ux_0/M+vy_0/N)}\end{aligned}$$

where the last step follows from the sifting property of the 2-D discrete impulse, Eq. (4-58). Because we used direct substitution into Eq. (4-67), and this equation and Eq. (4-68) are a Fourier transform pair, it must follow that the left side of the double arrow is the IDFT of the right: $\delta\left(x-x_0,y-y_0\right) = \Im^{-1}\left[e^{-j2\pi(ux_0/M+vy_0/N)}\right]$.

We could have solved this problem directly with the result of Problem 4.27(b) by letting $f\left(x-x_0,y-y_0\right) = \delta\left(x-x_0,y-y_0\right)$, and recognizing that $F(u,v)$ would be 1 (see part (a)).

(d) We solve the problem by direct substitution into the inverse DFT, Eq. (4-68):

$$\begin{aligned}\mathfrak{I}^{-1}\left[MN\delta\left(u-u_0,v-v_0\right)\right] &= \tfrac{MN}{MN}\textstyle\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}\delta\left(u-u_0,v-v_0\right)e^{j2\pi(ux/M+vy/N)}\\ &= \textstyle\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}e^{j2\pi(ux/M+vy/N)}\delta\left(u-u_0,v-v_0\right)\\ &= e^{j2\pi(u_0x/M+v_0y/N)}\end{aligned}$$

where the last step follows from the sifting property of the 2-D discrete impulse, Eq. (4-58). Because we used direct substitution into Eq. (4-68), and this equation and Eq. (4-67) are a Fourier transform pair, it must follow that the right side of the double arrow is the DFT of the left: $\mathfrak{I}\left[e^{j2\pi(u_0x/M+v_0y/N)}\right] = MN\delta\left(u-u_0,v-v_0\right)$.

We could have solved this problem directly with the result of Problem 4.27(a) by letting $f(x,y)=1$, and recognizing that when $f=1$, then $F=\delta$ (see part (b)).

(e) We solve this problem by direct substitution into the forward DFT, Eq. (4-67), and using Euler's formula to express the cosine in terms of exponentials:

$$\mathfrak{I}\left[\cos\left(\tfrac{2\pi\mu_0x}{M}+\tfrac{2\pi v_0y}{N}\right)\right] = \textstyle\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}\left[\frac{e^{j2\pi\left(\frac{\mu_0x}{M}+\frac{v_0y}{N}\right)}+e^{-j2\pi\left(\frac{\mu_0x}{M}+\frac{v_0y}{N}\right)}}{2}\right]e^{-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

$$\begin{aligned}&= \tfrac{1}{2}\textstyle\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}e^{j2\pi\left(\frac{u_0x}{M}+\frac{v_0y}{N}\right)}e^{-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}+\tfrac{1}{2}\textstyle\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}e^{j2\pi\left(-\frac{\mu_0x}{M}-\frac{v_0y}{N}\right)}e^{-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}\\ &= \tfrac{MN}{2}\left[\delta\left(u-u_0,v-v_0\right)+\delta\left(u+u_0,v+v_0\right)\right]\\ &= \tfrac{MN}{2}\left[\delta\left(u+u_0,v+v_0\right)+\delta\left(u-u_0,v-v_0\right)\right]\end{aligned}$$

where the second step puts the expressions in the form of Fourier transforms of exponentials. The results of these transforms are given in the statement of part (d), as the third step shows.

(f) We solve this problem by direct substitution into the forward DFT, Eq. (4-67), and using Euler's formula to express the sine in terms of exponentials:

$$\mathfrak{I}\left[sin\left(2\pi\mu_0x/M+2\pi v_0y/N\right)\right] = \textstyle\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}\left[\frac{e^{j2\pi(\mu_0x/M+v_0y/N)}-e^{-j2\pi(\mu_0x/M+v_0y/N)}}{2j}\right]e^{-j2\pi(ux/M+vy/N)}$$

$$\begin{aligned}&= \tfrac{1}{2j}\textstyle\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}e^{j2\pi(\mu_0x/M+v_0y/N)}e^{-j2\pi(ux/M+vy/N)}\\ &-\tfrac{1}{2j}\textstyle\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}e^{j2\pi(-\mu_0x/M-v_0y/N)}e^{-j2\pi(ux/M+vy/N)}\\ &= \tfrac{MN}{2j}\left[\delta\left(u-u_0,v-v_0\right)-\delta\left(u+u_0,v+v_0\right)\right]\\ &= \tfrac{jMN}{2}\left[\delta\left(u+u_0,v+v_0\right)-\delta\left(u-u_0,v-v_0\right)\right]\end{aligned}$$

where the second step puts the expressions in the form of Fourier transforms of exponentials. The results of these transforms are given in the statement of part (d), as the third step shows.

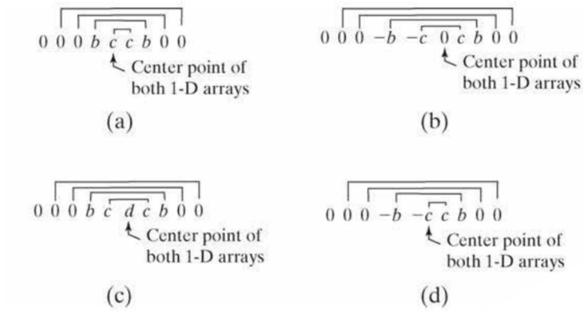

Figure P4.32

4.32 **(a)** The symmetry of this sequence is even, so the first element is arbitrary. In order to embed the sequence in a field of 0 s, we set the first element to 0 . The length of the sequence is 5 elements, so its center is at $M=2$, which is the $3rd$ point in the sequence (remember, we start counting at 0). The center of a 1-D field of 0 's nine elements long is at 4 , which is the 5 th point from the left. Thus, the sequence to embed is now $\{0, b, c, c, b\}$ which, when embedded (with coinciding centers) in a field of 9 zeros, looks like Fig. P4.32(a). The fact that all paired elements have the same values proves that this is an even sequence. As usual, the first element of an even sequence can have any value.

(b)-(d) Following the same reasoning as in (a), (b) is an odd array; (c) is even, and (d) is odd. See Figs. P4.32(b) through (d).

4.48 We want to show that

$$\Im^{-1}\left[A e^{-\left(\mu^2+v^2\right)/2\sigma^2}\right]=A 2\pi\sigma^2 e^{-2\pi^2\sigma^2\left(t^2+z^2\right)}.$$

The explanation will be clearer if we start with one variable. We want to show that, if

$$H(\mu)=e^{-\mu^2/2\sigma^2}$$

then

$$\begin{aligned} h(t) &= \Im^{-1}[H(\mu)] \\ &= \int_{-\infty}^{\infty} e^{-\mu^2/2\sigma^2} e^{j2\pi\mu t} d\mu \\ &= \sqrt{2\pi}\sigma^{-2\pi^2\sigma^2 t^2} \end{aligned}$$

We can express the integral in the preceding equations as

$$h(t)=\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left[\mu^2-j4\pi\sigma^2\mu t\right]}d\mu$$

Making use of the identity

$$e^{-\frac{(2\pi)^2\sigma^2t^2}{2}}e^{\frac{(2\pi)^2\sigma^2t^2}{2}}=1$$

in the preceding integral yields

$$\begin{aligned} h(t) \quad &= e^{-\frac{(2\pi)^2\sigma^2t^2}{2}} \int_{-\infty}^{\infty} e^{\frac{1}{2\sigma^2}\left[\mu^2-j4\pi\sigma^2\mu t-(2\pi)^2\sigma^4t^2\right]} \ d\mu \\ &= e^{-\frac{(2\pi)^2\sigma^2t^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left[\mu-j2\pi\sigma^2t\right]^2}d\mu \end{aligned}$$

Next, we make the change of variables $r=\mu-j2\pi\sigma^2t$. Then, $dr=d\mu$ and the preceding integral becomes

$$h(t)=e^{-\frac{(2\pi)^2\sigma^2t^2}{2}}\int_{-\infty}^{\infty} e^{-\frac{r^2}{2\sigma^2}}dr$$

Finally, we multiply and divide the right side of this equation by $\sqrt{2\pi}\sigma$ and obtain

$$h(t)=\sqrt{2\pi}\sigma e^{-\frac{(2\pi)^2\sigma^2t^2}{2}}\left[\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{r^2}{2\sigma^2}}dr\right]$$

The expression inside the brackets is recognized as the Gaussian probability density function whose value from $-\infty$ to ∞ is 1 . Therefore,

$$h(t)=\sqrt{2\pi}\sigma e^{-2\pi^2\sigma^2t^2}$$

With the preceding results as background, we are now ready to show that

$$\begin{aligned} h(t,z) \quad &= \Im^{-1}\left[Ae^{-\left(\mu^2+\nu^2\right)/2\sigma^2}\right] \\ &= A2\pi\sigma^2e^{-2\pi^2\sigma^2\left(t^2+z^2\right)} \end{aligned}$$

By substituting directly into the definition of the inverse Fourier transform we have:

$$\begin{aligned} h(t,z) \qquad &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Ae^{-\left(\mu^2+v^2\right)/2\sigma^2}e^{j2\pi(\mu t+vz)}d\mu dv \\ &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} Ae^{\left(-\frac{\mu^2}{2\sigma^2}+j2\pi\mu t\right)}d\mu\right]e^{\left(-\frac{v^2}{2\sigma^2}+j2\pi vz\right)}dv \end{aligned}$$

The integral inside the brackets is recognized from the previous discussion to be equal to $A\sqrt{2\pi}\sigma e^{-2\pi^2\sigma^2t^2}$. Then, the preceding integral becomes

$$h(t,z) = A\sqrt{2\pi}\sigma e^{-2\pi^2\sigma^2 t^2} \int_{-\infty}^{\infty} e^{\left(-\frac{v^2}{2\sigma^2}+j2\pi v z\right)} dv$$

We recognize the remaining integral to be equal to $\sqrt{2\pi}\sigma e^{-2\pi^2\sigma^2 z^2}$, from which we have the final result:

$$\begin{aligned} h(t,z) &= \left(A\sqrt{2\pi}\sigma e^{-2\pi^2\sigma^2 t^2}\right)\left(\sqrt{2\pi}\sigma e^{-2\pi^2\sigma^2 z^2}\right) \\ &= A2\pi\sigma^2 e^{-2\pi^2\sigma^2\left(t^2+z^2\right)}. \end{aligned}$$

4.52 **(a)** From Chapter 3

$$\nabla^2 f(t,z) = \frac{\partial^2 f(t,z)}{\partial t^2} + \frac{\partial^2 f(t,z)}{\partial z^2}$$

From entry 12 in Table 4.4,

$$\frac{\partial^2 f(t,z)}{\partial t^2} \Leftrightarrow (j2\pi\mu)^2 F(\mu,\nu) \text{ and } \frac{\partial^2 f(t,z)}{\partial z^2} \Leftrightarrow (j2\pi\nu)^2 F(\mu,\nu)$$

From which it follows that

$$\nabla^2 f(t,z) \Leftrightarrow -4\pi^2\left(\mu^2+v^2\right)F(\mu,v)$$

(b) Generate a $P \times Q$ array centered on $[P/2, Q/2]$:

$$H(u,v) = -4\pi^2\left([u-P/2]^2 + [v-Q/2]^2\right)$$

for $u = 0,1,2,\ldots,P-1$ and $v = 0,1,2,\ldots,Q-1$ where P and Q are sizes to which the input image is padded prior to filtering. Then use $H(u,v)$ as any other filter transfer function.

(c) The Laplacian operator is isotropic, a property that is captured well by the way the Laplacian transfer function is generated in the frequency domain. The Laplacian kernel with a -8 in the center and 1 's surrounding it is a closer approximation to an isotropic kernel than the kernel with a -4 in the center and only four 1’s surrounding it. Thus, we would expect the kernel with the -8 to give a result that is closer to the result obtained using the Laplacian transfer function in the frequency domain.

5.18 A linear, space invariant system is completely characterized by its impulse response, so the manager does not need to do anything else to complete the characterization of the system. The function given is $H(u,v)$, the system transfer function (Fourier transform of the impulse response). Thus, the output of the system to any image input, $f(x,y)$, can be computed in the frequency domain by

$$G(u,v) = H(u,v)F(u,v)$$

where $F(u,v)$ is the Fourier transform of the input image, and we are assuming negligible noise. The output image is then

$$g(x,y) = \mathcal{F}^{-1}[G(u,v)]$$

Regarding the function performed by the system, we see that the transfer function is the sum of Gaussian lowpass filter and a Gaussian highpass filter. Thus, H is a bandpass filter. (You can confirm this by plotting a cross section of the function.)

5.26 Based on the solution to Problems (5.23) and (5.25), we know that we can divide the problem into two components of motion. The first component of motion is in the x-direction only, for a time interval T. We know from the previous two problems that $x_0(t) = at/T$, with corresponding transfer function,

$$H_1(u,v) = \frac{T}{\pi ua}\sin(\pi ua)e^{-j\pi ua}$$

The second component of motion is for the same time interval, and the only difference is that motion is in the opposite direction: $x_0(t) = -at/T$. Thus, the form of the transfer function is identical to the first, with a replaced by $-a$:

$$H_2(u,v) = \frac{T}{-\pi ua}\sin(-\pi ua)e^{j\pi ua} = \frac{T}{\pi ua}\sin(\pi ua)e^{j\pi ua}$$

where the second step follows from the fact that the sine is an odd function. Combining the two terms we obtain

$$\begin{aligned} H(u,v) &= \frac{T}{\pi ua}\sin(\pi ua)e^{-j\pi ua} + \frac{T}{\pi ua}\sin(\pi ua)e^{j\pi ua} \\ &= \frac{T}{\pi ua}\sin(\pi ua)\left[e^{-j\pi ua} + e^{j\pi ua}\right] \end{aligned}$$

Thus, we conclude that the image will be blurred by this composite function and will not equal the original image. Because we are dealing with a linear degradation, the net result of applying this composite function to an input image would be the same as blurring the image in the positive x-direction with the first transfer function, and then blurring the resulting image with the second transfer function.

5.27 From Eq. (5.74),

$$H(u,v) = \int_0^T e^{j2\pi u x_0(t)} dt = \int_0^T e^{-j2\pi\left(\frac{at^2}{2}\right)u} dt$$

$$= \int_0^T e^{-j\pi u a t^2} dt$$

Using Euler's formula we obtain

$$\int_0^T e^{-j\pi u a t^2} dt = \int_0^T \left[\cos(\pi u a t^2) - j\sin(\pi u a t^2)\right] dt$$

$$= \sqrt{\frac{T^2}{2\pi u a T^2}} \left[C(\sqrt{\pi u a}T) - jS(\sqrt{\pi u a}T)\right]$$

where the forms

$$C(z) = \int_0^z \cos t^2 dt$$

and

$$S(z) = \int_0^z \sin t^2 dt$$

are Fresnel cosine and sine integrals.

6.13 **(a)**

Red: Gray level 0

Green: Gray level 2

Blue: Gray level 5

Magenta: Gray level 6

Cyan: Gray level 4

Yellow: Gray level 1

White: Gray level 0

Black: Gray level 0

(b)

Red: Gray level 7

Green: Gray level 7

Blue: Gray level 7

Magenta: Gray level 7

Cyan: Gray level 7

Yellow: Gray level 7

White: Gray level 0

Black: Gray level 0

(c)

Red: Gray level 2

Green: Gray level 2

Blue: Gray level 2

Magenta: Gray level 5

Cyan: Gray level 5

Yellow: Gray level 5

White: Gray level 7

Black: Gray level 0

6.14 **(a)** It is given that the colors in Fig. 6.14(a) are primary spectrum colors. It also is given that the gray-level images in the problem statement are 8-bit images. The latter condition means that hue (angle) can only be divided into a maximum number of 256 values. Because hue values are represented in the interval from $0°$ to $360°$, this means that for an 8-bit image the increments between contiguous hue values are now $360/255$. Another way of looking at this is that the entire $[0, 360]$ hue scale is compressed to the range $[0, 255]$. Thus, for example, yellow (the first primary color we encounter), which is $60°$ now becomes 43 (the closest integer) in the integer scale of the 8-bit image shown in the problem statement. Similarly, green, which is $120°$, becomes 85 in this image. From this we easily compute the values of the other two regions as being 170 and 213. The region in the middle is pure white [equal proportions of red, green, and blue in Fig. 6.14(a)] so its hue by definition is 0. This also is true of the black background.

(b) The colors are spectrum colors, so they are fully saturated. Therefore, the values 255 shown apply to all circle regions. The region in the center of the color image is white, so its saturation is 0.

(c) The key to getting the values in this figure is to realize that the center portion of the color image is white, which means equal intensities of fully saturated red, green, and blue. Therefore, the value of both darker gray regions in the intensity image have value 85 (i.e., the same value as the other corresponding region). Similarly, equal proportions of the secondaries yellow, cyan, and magenta produce white, so the two lighter gray regions have the same value (170) as the region shown in the figure. The center of the image is white, so its value is 255.

6.20 The complement of a color is the color opposite it on the color circle of Fig. 6.30. The hue component is the angle from red in a counterclockwise direction normalized by 360 degrees. For a color on the top half of the circle (i.e., $0 \leq H \leq 0.5$), the hue of the complementary color is $H + 0.5$. For a color on the bottom half of the circle (i.e., for $0.5 \leq H \leq 1$), the hue of the complement is $H - 0.5$.

6.25 (a)

85	0
170	85

(a) Hue

255	255
255	255

(b) Saturation

85	85
85	85

(c) intensity

Assume that the component image values of the HSI image are in the range [0,255]. Call the component images H (hue), S (saturation), and I (intensity).

It is given that the image is fully saturated, so image S will be constant with value 255. Similarly, all the squares are at their maximum value so, from Eq. (7-19), the intensity image also will be constant, with value 85 . Recall from Fig. 6.12 that the value of hue is an angle. Because the range of values of H is normalized to [0, 255], we see from that figure, for example, that as we go around the circle in the counterclockwise direction a hue value of 0 corresponds to red, a value of 85 to green, and a value of 170 to blue.

(b) The saturation image is constant, so smoothing it will produce the same constant value.

85	$0 \leftrightarrow 85$	0
$85 \leftrightarrow 170$	$0 \leftrightarrow 170$	$0 \leftrightarrow 85$
170	$85 \leftrightarrow 170$	85

(c) When the averaging mask is fully contained in a square, there is no blurring because the value of each square is constant. When the mask contains portions of two or more squares, the value produced at the center of the mask will be between the values of the two squares, and will depend on the relative proportions of the squares occupied by the mask. We know from (a) that the value of the red point is 0 and the value of the green point is 85. Thus, the values in the blurred band between red and green vary from 0 to 85 because averaging is a linear operation.