

MA1512 Cheat Sheet

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Separable Equations

$$M(x)dx = N(y)dy$$

Homogeneous Functions

If $y' = f(x, y)$ and $f(tx, ty) = t^n f(x, y)$ for some n , then:

$$\frac{dz}{f(1, z) - z} = \frac{dx}{x}, \quad \text{where } z = \frac{y}{x}.$$

Linear Change of Variable

If $y' = f(ax + by + c)$, set $u = ax + by + c$ and solve.

Exact Equations

If $M(x, y)dx + N(x, y)dy = 0$ and $M_y = N_x$ (by the Mixed Derivatives Theorem), then let $f_x = M(x, y)$ and $f_y = N(x, y)$.

Solve $f(x, y)$. Alternatively, use integrating factor $R(x) = e^{\int g(x)dx}$, where $g(x) = \frac{M_y - N_x}{N}$.

Linear First Order ODE

$$y' + p(x)y = q(x)$$

$$u = e^{\int p(x)dx}$$

$$u(y' + py) = uq \implies (uy)' = uq$$

Reduction of Order

If $f(x, y', y'') = 0$, set $y' = p$ and $y'' = p'$

If $f(x, y', y'') = 0$, set $y' = p$ and $y'' = pp'$

Bernoulli's Equation

If an equation has the form $y' + p(x)y = q(x)y^n$, divide by y^n and let $z = y^{1-n}$:

$$z' + (1 - n)p(x)z = (1 - n)q(x).$$

Homogeneous ODE

$$y'' + py' + qy = 0.$$

Characteristic equation: $r^2 + pr + q = 0$.

Case 1: Real distinct roots: $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$.

Case 2: Distinct Complex roots: If solution is $a \pm ib$, then $y = e^{ax}(c_1 \cos(bx) + c_2 \sin(bx))$.

Case 3: Equal real roots: $y = c_1 e^{r_1 x} + c_2 x e^{r_1 x}$.

Method of Undetermined Coefficients

The solution to the equation

$$y'' + py' + qy = R(x)$$

is of the form

$$y = y_g + y_p$$

where y_g is the general solution found by letting $R(x)$ to be 0 and y_p is the particular solution that is to be determined

Case 1: $R(x) = P(x)e^{kx}$, Substitute $y_p = u(x)e^{kx}$

Case 2: $R(x)$ is a trigonometric function with angular frequency b . Substitute $y_p = u(x)e^{a+ib}$ and take the real or imaginary component

of the resultant solution based on whether $R(x)$ has sin or cos.

Alternatively, let $y_p = u(x)(A \sin bx + B \cos bx)$

Case 3: $R(x)$ is a polynomial. Substitute $y_p = A_0 + A_1 x + \dots A_n x^n$, $x(A_0 + A_1 x + \dots + A_n x^n)$, etc where the degree of the polynomial is the degree of $R(x)$.

Method of Variation of Parameters

$$y'' + p(x)y' + q(x)y = r(x)$$

Let $W(y_1, y_2) = y_1 y_2' - y_1' y_2$. Then:

$$u = - \int \frac{y_2 r(x)}{W(y_1, y_2)} dx, \quad v = \int \frac{y_1 r(x)}{W(y_1, y_2)} dx.$$

Solution: $y_p = u y_1 + v y_2$.

Determining One Solution from Another

If y_1 is a solution to a homogeneous second-order differential equation, then $y_2 = v y_1$ where

$$v = \int \frac{1}{y_1^2} e^{-\int P dx} dx.$$

Superposition

If y_1 is the solution to the equation

$$y'' + p(x)y' + q(x)y = g(x)$$

and y_2 is the solution to the equation

$$y'' + p(x)y' + q(x)y = h(x)$$

then for all constants C_1 and C_2 , the function $y = C_1 y_1 + C_2 y_2$ is a solution to the equation

$$y'' + p(x)y' + q(x)y = g(x) + h(x)$$

Laplace Transforms

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t) dt, \quad f(t) = \mathcal{L}^{-1}[F(s)]$$

$$1 \rightarrow \frac{1}{s}, \quad e^{at} \rightarrow \frac{1}{s-a}, \quad t^n \rightarrow \frac{n!}{s^{n+1}}, \quad \sqrt{t} \rightarrow \frac{\sqrt{\pi}}{s^{3/2}},$$

$$\cos(at) \rightarrow \frac{s}{s^2 + a^2}, \quad \sin(at) \rightarrow \frac{a}{s^2 + a^2},$$

$$\cosh(at) \rightarrow \frac{s}{s^2 - a^2}, \quad \sinh(at) \rightarrow \frac{a}{s^2 - a^2},$$

$$t \cos(at) \rightarrow \frac{s^2 - a^2}{(s^2 + a^2)^2}, \quad t \sin(at) \rightarrow \frac{2as}{s^2 + a^2},$$

$$\sin(at + b) \rightarrow \frac{s \sin b + a \cos b}{s^2 + a^2}, \quad \cos(at + b) \rightarrow \frac{s \cos b - a \sin b}{s^2 + a^2},$$

$$f(ct) \rightarrow \frac{1}{c} \mathcal{L}(f(s-c)), \quad u(t-c) \rightarrow \frac{e^{-cs}}{s}, \quad \delta(t-c) \rightarrow e^{-cs}$$

1. Given $f(t)$, $g(t)$ and $a, b \in \mathbb{R}$,

$$\mathcal{L}[af(t) + bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)].$$

2. If $\mathcal{L}[f(t)] = F(s)$, then $\mathcal{L}[t \cdot f(t)] = -\frac{d}{ds} F(s)$.

3. If $\mathcal{L}[f(t)] = F(s)$, then $\mathcal{L}[e^{at} \cdot f(t)] = F(s-a)$.

4. Given $y(t)$, $\mathcal{L}[y'(t)] = s\mathcal{L}[y] - y(0)$,

$$\text{Similarly, } \mathcal{L}[y''(t)] = s\mathcal{L}[y'] - y'(0) = s^2 \mathcal{L}[y] - sy(0) - y'(0),$$

$$\mathcal{L}\left(\int_0^t y(\tau) d\tau\right) \rightarrow \frac{1}{s} \mathcal{L}(y).$$

5. Given $\mathcal{L}[f(t)] = F(s)$,

$$\mathcal{L}[f(t-c)u(t-c)] = e^{-sc} F(s).$$

Dirac Delta

$$\delta(t) \rightarrow 1, \quad \delta(t-a) \rightarrow e^{-as}, \quad \int_0^\infty \delta(t) dt = 1$$

Malthus Model

$$\frac{dN}{dt} = BN - DN = kN, \quad N(t) = \hat{N} e^{kt}.$$

Logistic Model

$$\frac{dN}{dt} = BN - DN = BN - (sN)N = BN - sN^2,$$

$$N_\infty = \frac{B}{s}, \quad N(t) = N_\infty \quad (\hat{N} = N_\infty)$$

$$N(t) = \frac{N_\infty}{1 + \left(\frac{N_\infty}{\hat{N}} - 1\right) e^{-Bt}}, \quad (\hat{N} < N_\infty), \quad y(t) = \frac{y_\infty}{1 + \left(\frac{y_\infty}{y_0} - 1\right) e^{-kt}},$$

$$N(t) = \frac{N_\infty}{1 - \left(1 - \frac{\hat{N}}{N_\infty}\right) e^{-Bt}}, \quad (\hat{N} > N_\infty).$$

Harvesting Model

$$\frac{dN}{dt} = (B - sN)N - E, \quad \frac{dy}{dt} = -\frac{k}{y_\infty} y^2 + ky - E$$

The quadratic curve has no solution when $E > \frac{B^2}{4s}$. This means the derivative will always be negative and population would dwindle to zero.

The quadratic curve has one solution when $E = \frac{B^2}{4s}$. This means there is one unstable equilibrium at $\frac{B}{2s}$.

In the last case, there is a stable and unstable equilibrium at two roots of the equation when the derivative is zero.

Wave Equations

$$c^2 y_{xx} = y_{tt}, \quad y(t, 0) = 0, \quad y(t, \pi) = 0, \quad y(0, x) = f(x)$$

$$y_t(0, x) = 0, \quad y(t, x) = \frac{1}{2} [f(x+ct) + f(x-ct)]$$

Heat Equations

$$u_t = c^2 u_{xx}, \quad u(L, t) = 0, \quad u(0, t) = 0$$

$$u_n(x, t) = e^{(-c^2 \pi^2 n^2 t)/l^2} (\beta_n \sin \frac{\pi n}{l} x)$$

Boundary is from 0 to l , to determine β_n : use $u(x, 0) = f(x)$

Trigonometric Integration

$$\begin{aligned}\frac{d}{dx} \sin^{-1} x &= \frac{1}{\sqrt{1-x^2}}, & \csc x &= \frac{1}{\sin x} \\ \frac{d}{dx} \cos^{-1} x &= \frac{-1}{\sqrt{1-x^2}}, & \sec x &= \frac{1}{\cos x} \\ \frac{d}{dx} \tan^{-1} x &= \frac{1}{1+x^2}, & \cot x &= \frac{1}{\tan x}\end{aligned}$$

Integration Rules

$$\begin{aligned}\int uv &= u \int v \, dx - \int \frac{du}{dx} \int v \, dx \, dx, & \frac{d}{dx} \left(\frac{u}{v} \right) &= \frac{u'v - uv'}{v^2} \\ \frac{d}{dx}(uv) &= u \frac{dv}{dx} + v \frac{du}{dx}, & \int u \, dv &= uv - \int v \, du \\ u &= g(x), du = g'(x) \, dx, & \int f(g(x))g'(x) \, dx &= \int f(u) \, du\end{aligned}$$

Simple Harmonic Motion

$$k = mg/\Delta l, \quad \omega = \sqrt{k/m}, \quad y(t) = R \cos(\omega t - \delta)$$

Trigonometric Identities

$$\begin{aligned}1 + \tan^2 u &= \sec^2 u, & 1 + \cot^2 u &= \csc^2 u \\ \sin(-x) &= -\sin x, & \cos(-x) &= \cos x, & \tan(-x) &= -\tan x \\ \sin\left(\frac{\pi}{2} - x\right) &= \cos x, & \cos\left(\frac{\pi}{2} - x\right) &= \sin x \\ \tan\left(\frac{\pi}{2} - x\right) &= \cot x, & \cot\left(\frac{\pi}{2} - x\right) &= \tan x \\ \sec\left(\frac{\pi}{2} - x\right) &= \csc x, & \csc\left(\frac{\pi}{2} - x\right) &= \sec x \\ \sin(x \pm y) &= \sin x \cos y \pm \cos x \sin y \\ \cos(x \pm y) &= \cos x \cos y \mp \sin x \sin y \\ \tan(x \pm y) &= \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}, & \cos^2 x + \sin^2 x &= 1 \\ \cos(2x) &= \cos^2 x - \sin^2 x = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x \\ \sin(2x) &= 2 \sin x \cos x, & \tan(2x) &= \frac{2 \tan x}{1 - \tan^2 x} \\ \sin\left(\frac{x}{2}\right) &= \pm \sqrt{\frac{1 - \cos x}{2}}, & \cos\left(\frac{x}{2}\right) &= \pm \sqrt{\frac{1 + \cos x}{2}} \\ \tan\left(\frac{x}{2}\right) &= \frac{\sin x}{1 + \cos x} = \frac{1 - \cos x}{\sin x} \\ \sin^2 x &= \frac{1 - \cos(2x)}{2}, & \cos^2 x &= \frac{1 + \cos(2x)}{2} \\ \tan^2 x &= \frac{1 - \cos(2x)}{1 + \cos(2x)} \\ \sin x \sin y &= \frac{1}{2} [\cos(x - y) - \cos(x + y)] \\ \cos x \cos y &= \frac{1}{2} [\cos(x - y) + \cos(x + y)] \\ \sin x \cos y &= \frac{1}{2} [\sin(x + y) + \sin(x - y)]\end{aligned}$$

$$\begin{aligned}\tan x \tan y &= \frac{\tan x + \tan y}{\cot x + \cot y}, & \tan x \cot y &= \frac{\tan x + \cot y}{\cot x + \tan y} \\ \sin x + \sin y &= 2 \sin\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right) \\ \sin x - \sin y &= 2 \cos\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right) \\ \cos x + \cos y &= 2 \cos\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right) \\ \cos x - \cos y &= -2 \sin\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right) \\ \tan x \pm \tan y &= \frac{\sin(x \pm y)}{\cos x \cos y} \\ a \cos \theta \pm b \sin \theta &= R \cos(\theta \mp \alpha) \\ a \sin \theta \pm b \cos \theta &= R \sin(\theta \pm \alpha) \\ \alpha &= \arctan\left(\frac{b}{a}\right), & R &= \sqrt{a^2 + b^2}\end{aligned}$$

Hyperbolic Functions

$$\begin{aligned}\cosh t &= \frac{e^t + e^{-t}}{2}, & \sinh t &= \frac{e^t - e^{-t}}{2}, & \tanh t &= \frac{\sinh t}{\cosh t} \\ \cosh^2 t - \sinh^2 t &= 1, & (\sinh x)' &= \cosh x \\ (\cosh x)' &= \sinh x, & (\tanh x)' &= \operatorname{sech}^2 x, & (\sinh^{-1} x)' &= \frac{1}{\sqrt{1+x^2}} \\ (\cosh^{-1} x)' &= \frac{1}{\sqrt{x^2-1}}, & (\tanh^{-1} x)' &= \frac{1}{1-x^2}\end{aligned}$$

Derivative Rules

$$\begin{aligned}\frac{d}{dx}(x^n) &= nx^{n-1}, & \frac{d}{dx}(\ln x) &= \frac{1}{x}, & \frac{d}{dx}(a^x) &= a^x \ln a \\ \frac{d}{dx}(e^x) &= e^x, & \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx}, & v' &= \frac{d}{dx}(y^{-1}) = -y^{-2} \cdot y' \\ \frac{d}{dx}(uv) &= \frac{du}{dx}v + u \frac{dv}{dx}, & \frac{d}{dx}\left(\frac{u}{v}\right) &= \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} \\ \frac{d}{dx}(\sin x) &= \cos x, & \frac{d}{dx}(\cos x) &= -\sin x \\ \frac{d}{dx}(\tan x) &= \sec^2 x, & \frac{d}{dx}(\cot x) &= -\csc^2 x \\ \frac{d}{dx}(\sec x) &= \sec x \tan x, & \frac{d}{dx}(\csc x) &= -\csc x \cot x \\ \frac{d}{dx}(\sin^{-1} x) &= \frac{1}{\sqrt{1-x^2}}, & \frac{d}{dx}(\cos^{-1} x) &= -\frac{1}{\sqrt{1-x^2}} \\ \frac{d}{dx}(\tan^{-1} x) &= \frac{1}{x^2+1}, & \frac{d}{dx}(\sin^2 x) &= 2 \sin x \cos x = \sin(2x) \\ \frac{d}{dx}(\cos^2 x) &= 2 \cos x (-\sin x) = -\sin(2x) \\ \frac{d}{dx}(\tan^2 x) &= 2 \tan x \sec^2 x, & \frac{d}{dx}(\sin^3 x) &= 3 \sin^2 x \cdot \cos x \\ \frac{d}{dx}(\cot^2 x) &= 2 \cot x (-\csc^2 x) = -2 \cot x \csc^2 x\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}(\sec^2 x) &= 2 \sec x \sec x \tan x = 2 \sec^2 x \tan x \\ \frac{d}{dx}(\csc^2 x) &= 2 \csc x (-\csc x \cot x) = -2 \csc^2 x \cot x \\ \frac{d}{dx}(\cos^4 x) &= 4 \cos^3 x \cdot (-\sin x) = -4 \cos^3 x \sin x\end{aligned}$$

Integral Formulas

$$\begin{aligned}\int x^n \, dx &= \frac{x^{n+1}}{n+1} + C, & \int 1 \, dx &= x + C, & \int x \ln x \, dx &= \frac{x^2}{2} \ln x - \frac{x^2}{4} + C \\ \int \sin(kx) \, dx &= -\frac{\cos(kx)}{k} + C, & \int \cos(kx) \, dx &= \frac{\sin(kx)}{k} + C \\ \int \sec^2(x) \, dx &= \tan(x) + C, & \int \csc^2(x) \, dx &= -\cot(x) + C \\ \int \sec(x) \tan(x) \, dx &= \sec(x) + C, & \int \csc(x) \cot(x) \, dx &= -\csc(x) + C \\ \int \frac{1}{x} \, dx &= \ln|x| + C, & \int a^x \, dx &= \frac{a^x}{\ln a} + C, & \int e^x \, dx &= e^x + C \\ \int \sin^2(x) \, dx &= \int \frac{1 - \cos(2x)}{2} \, dx = -\frac{\sin(2x) - 2x}{4} + C \\ \int \cos^2(x) \, dx &= \int \frac{1 + \cos(2x)}{2} \, dx = \frac{x - \frac{\sin(2x)}{2}}{2} + C\end{aligned}$$

Standard Integrals

$$\begin{aligned}\int (ax+b)^n \, dx &= \frac{(ax+b)^{n+1}}{(n+1)a} + C, & \int \ln x \, dx &= x \ln x - x \\ \int \frac{1}{ax+b} \, dx &= \frac{1}{a} \ln|ax+b| + C, & \int e^{ax+b} \, dx &= \frac{1}{a} e^{ax+b} + C \\ \int \sin(ax+b) \, dx &= -\frac{1}{a} \cos(ax+b) + C, \\ \int \cos(ax+b) \, dx &= \frac{1}{a} \sin(ax+b) + C, \\ \int \tan(ax+b) \, dx &= \frac{1}{a} \ln|\sec(ax+b)| + C, \\ \int \sec(ax+b) \, dx &= \frac{1}{a} \ln|\sec(ax+b) + \tan(ax+b)| + C, \\ \int \csc(ax+b) \, dx &= -\frac{1}{a} \ln|\csc(ax+b) + \cot(ax+b)| + C, \\ \int \cot(ax+b) \, dx &= -\frac{1}{a} \ln|\csc(ax+b)| + C, \\ \int \sec^2(ax+b) \, dx &= \frac{1}{a} \tan(ax+b) + C, \\ \int \csc^2(ax+b) \, dx &= -\frac{1}{a} \cot(ax+b) + C, \\ \int \sec(ax+b) \tan(ax+b) \, dx &= \frac{1}{a} \sec(ax+b) + C, \\ \int \csc(ax+b) \cot(ax+b) \, dx &= -\frac{1}{a} \csc(ax+b) + C\end{aligned}$$