National University of Singapore

MA1511 Engineering Calculus

Semester 1 (2023–2024)

Time allowed: 1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

1. Please write your Student Number only. Do not write your name.
2. This examination paper contains **TEN** questions and comprises **FIVE** pages. Answer **ALL** questions.
3. Students are to write the answers for each question on a new page.
4. The total mark for this paper is **ONE HUNDRED.**
5. This is a **CLOSED BOOK** (with authorized material) examination. Students are only allowed to bring into the examination hall one A4 double side help sheet.
6. You may use any calculator. However, you should lay out systematically the various steps in the calculations.

Question 1 [10 marks]

Find $\dfrac{\partial f}{\partial x}$ if $f(x,y,z) = z\ln(x^2 y\cos z) + x\sin(xyz)$, where $x^2 y\cos z > 0$.

Question 2 [10 marks]

Use the method of Lagrange multipliers to find the points on the sphere $x^2 + y^2 + z^2 = 36$ that are closest to and farthest from the point $(1,2,2)$.

Suggestion: Let $f(x,y,z) = (x-1)^2 + (y-2)^2 + (z-2)^2$.

(Zero marks will be awarded if the method of Lagrange multipliers is not used.)

Question 3 [10 marks]

Find the exact value of the iterated integral $\displaystyle\int_0^4 \int_{\sqrt{y}}^2 \frac{1}{1+2x^3}\, dxdy$.

Question 4 [10 marks]

Find the exact value of the iterated integral $\displaystyle\int_{-1}^0 \int_{-\sqrt{1-x^2}}^0 \frac{2}{1+\sqrt{x^2+y^2}}\, dydx$.

Question 5 [10 marks]

Let C be the curve of intersection of the sphere $x^2 + y^2 + z^2 = 8$ and the cone $z^2 = x^2 + y^2, z \geq 0$. Find the exact value of the line integral

$$\int_C (2x^2 + 2y^2 + z + y)ds.$$

Question 6 [10 marks]

Let the parametric surface S be

$$\mathrm{r}(u,v) = u\mathrm{i} + 2v^2\mathrm{j} + (u^2 + v)\mathrm{k}.$$

Suppose the equation of the tangent plane to the parametric surface S at $(x,y,z) = (2,2,3)$ is $Ax - y + Cz = D$. Find the values of A, C and D.

Question 7 [10 marks]

Let $P(x,y)=\dfrac{-y}{x^2+4y^2}$, $\mathrm{Q}(x,y)=\dfrac{x}{x^2+4y^2}$,

where $x^2+4y^2\neq 0$.

It is known that $\dfrac{\partial P(x,y)}{\partial y}=\dfrac{\partial Q(x,y)}{\partial x}$.

Find the exact value of the line integral

$$\oint_C P(x,y)dx+Q(x,y)dy,$$

(a) if C is the closed curve $x^2+4y^2=4$, taken in a counterclockwise direction,

(b) if C is the rectangular curve with vertices $(4,-3),(4,3),(-4,3)$ and $(-4,-3)$, taken in a counterclockwise direction.

(Hint: In (a), if (x,y) is a point on C, then $x^2+4y^2=4$. Hence the line integral can be simplified. The area enclosed by the closed curve $x^2+4y^2=4$ is 2π.)

Question 8 [10 marks]

It is known that the vector field

$$F(x,y,z)=\left(\frac{1}{y}-\frac{2y}{x^3}\right)\mathrm{i}+\left(\frac{1}{x^2}-\frac{x}{y^2}\right)\mathrm{j}+2z^2\mathrm{k}$$

is a conservative field.

Use the fundamental theorem of line integrals to find the exact value of the line integral

$$\int_C F\cdot dr,$$

where C is the curve $\mathrm{r}(t)=t\ \mathrm{i}+t^2\ \mathrm{j}+(t-1)\ \mathrm{k}, 1\leq t\leq 2$.

(Zero marks will be awarded if the fundamental theorem of line integrals is not used.)

Question 9 [10 marks]

(a) Find the exact value of $\lim_{n\to\infty} \frac{(n+1)^{n+1}}{n^n} \frac{1}{(n+1)^2}$.

(b) Let $g(x) = (1+x^2)\cos(x^3)$. Find the exact value of $g^{(14)}(0)$.
Give your answer in terms of factorials.

Question 10 [10 marks]

(a) Find the radius of convergence for the power series

$$\sum_{n=1}^{\infty} \frac{1}{3^n + (-2)^n} \frac{(5x+1)^{2n+1}}{n}.$$

(b) Let $f(x) = \sum_{n=0}^{\infty} \frac{4n+1}{n!} x^{4n} = \sum_{n=0}^{\infty} \left(\frac{4n}{n!} + \frac{1}{n!} \right) x^{4n}$. Find the exact value of
$f\left((\ln 2)^{\frac{1}{4}} \right)$.

A table of the Maclaurin Series is given on page 5.

Maclaurin Series

$$\frac{1}{1-x}=1+x+x^2+...+x^n+... \text{ for } -1<x<1.$$

$$\frac{1}{1+x}=1-x+x^2-...+(-x)^n+... \text{ for } -1<x<1.$$

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+... \text{ for all } x.$$

$$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+... \text{ for all } x.$$

$$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+... \text{ for all } x.$$

$$\ln(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+... \text{ for } -1<x\le 1.$$

$$\tan^{-1} x = x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+... \text{ for } -1\le x\le 1.$$

END OF PAPER