Chapter 1 Sequences and Series

1. Arithmetic Progression (A.P.)

$(a) n^{th}\ term, a_n = a + (n-1)d, for\ all\ n \in Z^+$

$(b) common\ difference, d = a_n - a_{n-1}$

$(c) sum\ of\ first\ n\ terms, S_n = \frac{n}{2}[2a + (n-1)d]$

$(d) sum\ of\ first\ n\ terms, S_n = \frac{n}{2}[a_1 + a_n]$

$(e) n^{th}\ term, n = \frac{a_n - a_1}{d} + 1$

2. Geometric Progression (G.P.)

$(a) n^{th}\ term, a_n = ar^{n-1}$

$(b) common\ ratio, r = \frac{a_n}{a_{n-1}}$

$(c) sum\ of\ first\ n\ terms, S_n = \frac{a}{1-r}[1 - r^n] (if\ r \neq 1)$

$(d) sum\ of\ first\ n\ terms, S_n = na\ (if\ r = 1)$

$(e) sum\ to\ infinity, S_\infty = \frac{a}{1-r}\ (if -1 < r < 1)$

3. Binomial Theorem

$(a) (a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}a^1 b^{n-1} + \binom{n}{n}a^0 b^n$

$where \binom{n}{r} = \frac{n!}{r!(n-r)!}$ [just input nCr in calculator]

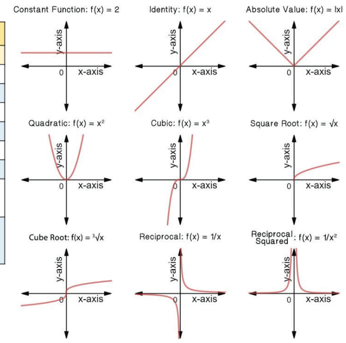
$(b) (1+a)^n = 1 + na + \frac{n(n-1)}{2!}a^2 + \frac{n(n-1)(n-2)}{3!}a^3 + \cdots$

4. Telescoping Sum

$(a) \sum_{i=1}^{n} u_i = u_1 + u_2 + u_3 + \cdots + u_n = a_1 - a_{n+1}$

$(a) \sum_{i=m}^{n} u_i = u_m + u_{m+1} + u_{m+2} + \cdots + u_n = a_m - a_{n+1}$

Transformation Rules for Functions		
Function Notation	Type of Transformation	Change to Coordinate Point
f(x) + d	Vertical translation up d units	(x, y) → (x, y + d)
f(x) − d	Vertical translation down d units	(x, y) → (x, y − d)
f(x + c)	Horizontal translation left c units	(x, y) → (x − c, y)
f(x − c)	Horizontal translation right c units	(x, y) → (x + c, y)
−f(x)	Reflection over x-axis	(x, y) → (x, −y)
f(−x)	Reflection over y-axis	(x, y) → (−x, y)
af(x)	Vertical stretch for |a| > 1 Vertical compression for 0 < |a| < 1	(x, y) → (x, ay)
f(bx)	Horizontal compression for |b| > 1 Horizontal stretch for 0 < |b| < 1	$(x, y) \to \left(\frac{x}{b}, y\right)$



Use the method of linear approximation to estimate the following numbers.

(a) $\sqrt[3]{8.01}$

$y = f(x) = \sqrt[3]{x}$ at $x = 8$

$\frac{dy}{dx} = \frac{d}{dx}\left(x^{\frac{1}{3}}\right) = \frac{1}{3}x^{-\frac{2}{3}} \to \frac{dy}{dx}(8) = \frac{1}{12}$

$y = \frac{dy}{dx}(8)(8.01 - 8) + 2 = \frac{1}{12}(0.01) + 2 = \frac{2401}{1200}$

Express 0.321321321321 as a rational number.

Solution:

0.321321321321... = 0.321 + 0.000321 + ...

$S_\infty = \frac{a}{1-r} = \frac{0.321}{1-0.001} = \frac{0.321}{0.999} = \frac{107}{333}$

Evaluate the following telescoping sums.

(a) $\sum_{r=1}^{99} \lg\left(\frac{r+1}{r}\right)$

$\sum_{r=1}^{99} \lg(r+1) - \sum_{r=1}^{99} \lg(r)$

$= (\lg 2 + \lg 3 + \ldots + \lg 100) - (\lg 1 + \lg 2 + \ldots + \lg 99) = \lg 100 - \lg 1 = 2$

Chapter 2 Derivatives

1. Derivative Rules

Function	Derivative
$(f(x))^n$	$nf'(x)(f(x))^{n-1}$
$\cos(f(x))$	$-f'(x) \cdot \sin f(x)$
$\sin(f(x))$	$f'(x) \cdot \cos(f(x))$
$\tan(f(x))$	$f'(x) \cdot \sec^2(f(x))$
$\sec(f(x))$	$f'(x) \cdot \sec(f(x))\tan(f(x))$
$\csc(f(x))$	$-f'(x) \cdot \csc(f(x))\cot(f(x))$
$\cot(f(x))$	$-f'(x) \cdot \csc^2(f(x))$
$e^{f(x)}$	$f'(x) \cdot e^{f(x)}$
$\ln(f(x))$	$\frac{f'(x)}{f(x)}$
$\sin^{-1}(f(x))$	$\frac{f'(x)}{\sqrt{1-(f(x))^2}}$
$\cos^{-1}(f(x))$	$-\frac{f'(x)}{\sqrt{1-(f(x))^2}}$
$\tan^{-1}(f(x))$	$\frac{f'(x)}{1+(f(x))^2}$

2. Specific Rules

$(a) Product\ Rule, \frac{d}{dx}(uv) = u'v + uv'$

$(b) Quotient\ Rule, \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - uv'}{v^2}$

$(c) Chain\ Rule, \frac{dy}{dx} = \frac{dy}{du} x \frac{du}{dx}$

3. Implicit Differentiation

$(a) \frac{d}{dx}(y^n) = ny^{n-1}\frac{dy}{dx}$

$(b) \frac{d}{dx}(\sin y) = \cos y \frac{dy}{dx}$

$(c) \frac{d}{dx}(e^y) = e^y \frac{dy}{dx}$

$(d) \frac{d}{dx}(\ln y) = \frac{1}{y}\frac{dy}{dx}$

4. Parametric Differentiation

$\begin{cases} y = u(t) \\ x = v(t) \end{cases}; \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{u'(t)}{v'(t)}$

Chapter 2 Derivatives

1. Tangent and normal

$(a)\ m_{tangent}\ x\ m_{normal} = -1$

$(b)\ Tangent\ line: y - y_0 = m(x - x_0)$

$(c)\ Normal\ line: y - y_0 = -\frac{1}{m}(x - x_0)$

$(d)\ if\ tangent\ line\ parallel\ to\ x-axis, \frac{dy}{dx} = 0; if\ parallel\ to\ y-axis, \frac{dy}{dx} = \pm\infty$

2. First and second derivative test (minimum and maximum points)

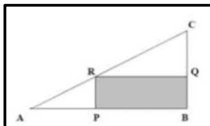
$(a) f'(x) > 0\ for\ x \in (a,c)\ and\ f'(x) < 0\ for\ x \in (c,b) \to f(x)$ [local maximum]

$(b)\ f'(x) < 0\ for\ x \in (a,c)\ and\ f'(x) > 0\ for\ x \in (c,b) \to f(x)$ [local minimum]

$(c)\ f'(x) = 0$ [saddle point]

$(d)\ f''(x) < 0$ [maximum point]; $f''(x) > 0$ [minimum point]

$(e)\ f''(x) < 0$ [concave down]; $f''(x) > 0$ [concave up]



ABC is a right-angled triangle with B = 90°, AC = 34 cm and BC = 16 cm. Find the largest rectangle PBQR that can be inscribed in triangle ABC.

$AB^2 + BC^2 = AC^2 \to AB^2 + 16^2 = 34^2$

$\to AB = 30$

Let PB = x, AB = 30 − x, QB = y, CQ = 16 − y

$\frac{CB}{AB} = \frac{RP}{AP} \to \frac{16}{30} = \frac{y}{30-x}; 30y = 16(30-x); y = \frac{8}{15}(30-x)$

$Area = xy = x\left(\frac{8}{15}(30-x)\right) = \frac{8}{15}(30x - x^2)$

$\frac{dA}{dx} == \frac{8}{15}(30 - 2x) \to Let \frac{dA}{dx} = 0 \to x = 15$

$\frac{d^2A}{dx^2} = \frac{8}{15}(-2) = -\frac{16}{15} > 0$ [maximum value]

$y = \frac{8}{15}(30 - x) = \frac{8}{15}(30 - 15) = 8$

$Area = xy = (15)(8) = 120\ cm^2$

Trigonometric Identities + Hemisphere Formulas

1. $\sin x \cos x = \frac{1}{2}\sin 2x$
2. $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
3. $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
4. $\sin x \cdot \cos y = \frac{1}{2}[\sin(x+y) + \sin(x-y)]$
5. $\cos x \cdot \sin y = \frac{1}{2}[\sin(x+y) - \sin(x-y)]$
6. $\cos x \cdot \cos y = \frac{1}{2}[\cos(x+y) + \cos(x-y)]$
7. $\sin x \cdot \sin y = -\frac{1}{2}[\cos(x+y) - \cos(x-y)]$

1. $\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$
2. $\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$
3. $\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$
4. $\cos x - \cos y = 2 - \sin\frac{x+y}{2}\sin\frac{x-y}{2}$

1. $\frac{1}{\cos x} = \sec x$
2. $\frac{1}{\sin x} = \csc x$
3. $\frac{\sin x}{\cos x} = \tan x$
4. $\frac{\cos x}{\sin x} = \cot x$

1. $\sin^2 x + \cos^2 x = 1$
2. $1 + \tan^2 x = \sec^2 x$
3. $1 + \cot^2 x = \csc^2 x$

1. $\sin 2x = 2\sin x \cos x$

   $\cos 2x = 2\cos^2 x - \sin^2 x$

2. $= 2\cos^2 x - 1$

   $= 1 - 2\sin^2 x$

3. $\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$

1. $\sin^2 x = \frac{1 - \cos 2x}{2}$
2. $\cos^2 x = \frac{1 + \cos 2x}{2}$

1. $\sin(x+y) = \sin x\cos y + \cos x \sin y$
2. $\sin(x-y) = \sin x\cos y - \cos x \sin y$
3. $\cos(x+y) = \cos x\cos y - \sin x \sin y$
4. $\cos(x-y) = \cos x\cos y + \sin x \sin y$
5. $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$
6. $\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$

(20) $\sin\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1 - \cos x}{2}}$

(21) $\cos\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1 + \cos x}{2}}$

(22) $\tan\left(\frac{x}{2}\right) = \pm\frac{1 + \cos x}{2}$

(23) $\sin\left(\frac{\pi}{2} - x\right) = \cos x$

(24) $\cos\left(\frac{\pi}{2} - x\right) = \sin x$

(25) $\tan\left(\frac{\pi}{2} - x\right) = \sin x$

(26) $\cot\left(\frac{\pi}{2} - x\right) = \tan x$

(27) $\csc\left(\frac{\pi}{2} - x\right) = \sec x$

(28) $\sec\left(\frac{\pi}{2} - x\right) = \csc x$

(29) $\sin(x \pm 2\pi) = \sin x$

(30) $\cos(x \pm 2\pi) = \cos x$

(31) $\tan(x \pm 2\pi) = \tan x$

(32) $\cot(x \pm 2\pi) = \cot x$

(33) $\csc(x \pm 2\pi) = \csc x$

(34) $\sec(x \pm 2\pi) = \sec x$

(35) $\sin^2 x = \frac{1 - \cos 2x}{2}$

(36) $\cos^2 x = \frac{1 + \cos 2x}{2}$

Sphere

$V = \frac{4}{3}\pi r^3 \quad SA = 4\pi r^2$

Cone $V = \frac{1}{3}\pi r^2 h$

$SA = \pi r(r + \sqrt{r^2 + h^2})$

Cube

$V = s^3 \quad SA = 6s^2$

Cylinder

$V = \pi r^2 h$

$SA = 2\pi r(r + h)$

Rect Prism

$V = lwh$

$SA = 2lw + 2lh + 2wh$

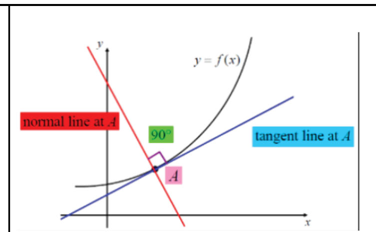
Pyramid

$V = \frac{1}{3}Bh$

where B is the area of the base

$SA = B + \frac{1}{2}Pl$

where P is the perimeter of the base and l is the slant height



Result:

(gradient of **tangent** line) × (gradient of **normal** line) = -1

$m_{tangent} \times m_{normal} = -1$

$m_{normal} = -\frac{1}{m_{tangent}}$

Equation of Line:

$y - y_1 = m(x - x_1)$

Chapter 3 Integrals

1. $\int x^n\, dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$

   $\int 1\, dx = \int dx = x + C, \quad$ (Special case, $n = 0$)
2. $\int \sin(kx)\, dx = -\frac{\cos(kx)}{k} + C$
3. $\int \cos(kx)\, dx = \frac{\sin(kx)}{k} + C$
4. $\int \sec^2(x)\, dx = \tan(x) + C$
5. $\int \csc^2(x)\, dx = -\cot(x) + C$
6. $\int \sec(x)\tan(x)\, dx = \sec(x) + C$
7. $\int \csc(x)\cot(x)\, dx = -\csc(x) + C$
8. $\int \frac{1}{x}\, dx = \ln|x| + C$
9. $\int a^x\, dx = \frac{a^x}{\ln a} + C$
10. $\int e^x\, dx = e^x + C$

1. $\int (ax+b)^n\, dx = \frac{(ax+b)^{n+1}}{(n+1)a} + C \quad (n \neq -1)$
2. $\int \frac{1}{ax+b}\, dx = \frac{1}{a}\ln|ax+b| + C$
3. $\int e^{ax+b}\, dx = \frac{1}{a}e^{ax+b} + C$
4. $\int \sin(ax+b)\, dx = -\frac{1}{a}\cos(ax+b) + C$
5. $\int \cos(ax+b)\, dx = \frac{1}{a}\sin(ax+b) + C$
6. $\int \tan(ax+b)\, dx = \frac{1}{a}\ln|\sec(ax+b)| + C$
7. $\int \sec(ax+b)\, dx = \frac{1}{a}\ln|\sec(ax+b) + \tan(ax+b)| + C$
8. $\int \csc(ax+b)\, dx = -\frac{1}{a}\ln|\csc(ax+b) + \cot(ax+b)| + C$
9. $\int \cot(ax+b)\, dx = -\frac{1}{a}\ln|\csc(ax+b)| + C$
10. $\int \sec^2(ax+b)\, dx = \frac{1}{a}\tan(ax+b) + C$
11. $\int \csc^2(ax+b)\, dx = -\frac{1}{a}\cot(ax+b) + C$

12. $\int \sec(ax+b) \cdot \tan(ax+b)\, dx = \frac{1}{a}\sec(ax+b) + C$
13. $\int \csc(ax+b) \cdot \cot(ax+b)\, dx = -\frac{1}{a}\csc(ax+b) + C$
14. $\int \frac{1}{a^2 + (x+b)^2}\, dx = \frac{1}{a}\tan^{-1}\left(\frac{x+b}{a}\right) + C$
15. $\int \frac{1}{\sqrt{a^2 - (x+b)^2}}\, dx = \sin^{-1}\left(\frac{x+b}{a}\right) + C$
16. $\int \frac{-1}{\sqrt{a^2 - (x+b)^2}}\, dx = \cos^{-1}\left(\frac{x+b}{a}\right) + C$
17. $\int \frac{1}{a^2 - (x+b)^2}\, dx = \frac{1}{2a}\ln\left|\left(\frac{x+b+a}{x+b-a}\right)\right| + C$
18. $\int \frac{1}{(x+b)^2 - a^2}\, dx = \frac{1}{2a}\ln\left|\left(\frac{x+b-a}{x+b+a}\right)\right| + C$
19. $\int \frac{1}{\sqrt{(x+b)^2 + a^2}}\, dx = \ln\left|(x+b) + \sqrt{(x+b)^2 + a^2}\right| + C$
20. $\int \frac{1}{\sqrt{(x+b)^2 - a^2}}\, dx = \ln\left|(x+b) + \sqrt{(x+b)^2 - a^2}\right| + C$

Evaluate $\int \sin^4(x)\cos(x)dx$

Let $u = \sin(x)$

Then $\frac{du}{dx} = \cos(x)$

$du = \cos(x)dx$

$\int \sin^4(x)\cos(x)dx = \int u^4 du$

$= \frac{1}{5}u^5 + C$

$= \frac{1}{5}\sin^5(x) + C$

Evaluate $\int (x^2 + 2x - 3)^2(x+1)\, dx$

Let $u = x^2 + 2x - 3$

Then $\frac{du}{dx} = 2(x+1)$

$du = 2(x+1)dx$

$\frac{1}{2}du = (x+1)dx$

$\int (x^2 + 2x - 3)^2(x+1)\, dx = \int u^2 \frac{1}{2}\, du$

$= \frac{1}{6}u^3 + C$

$= \frac{1}{6}(x^2 + 2x - 3)^3 + C$

To evaluate: $\int f'(x)g(f(x))\, dx$

Method: Let $u = f(x)$

(31) $\int_a^a f(x)dx = 0$

(32) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

(33) $\int_a^b kf(x)dx = k\int_a^b f(x)dx$

(34) $\int_a^b [f(x) \pm g(x)]dx$

$= \int_a^b f(x)\, dx \pm \int_a^b g(x)\, dx$

(35) $\int u\, dv = uv - \int v\, du$

(36) $A = \int_a^b y_2 - y_1\, dx$

(37) $V = \pi\int_a^b y^2\, dx$ [rotate x − axis]

(38) $V = \pi\int_a^b x^2\, dy$ [rotate y − axis]

Rules of Integration by Parts

Logarithmic Function	$\ln(ax+b)$ *or its higher powers*	*Make the substitution* $u = ax + b$ *to simplify the integral*
Inverse Trigonometric Function	$\sin^{-1}(ax+b), \cos^{-1}(ax+b), \tan^{-1}(ax+b)$	
Algebraic Function	*power functions* x^a, *polynomials*	
Trigonometric Function	$\sin(ax+b), \cos(ax+b), \tan(ax+b), \cot(ax+b), \csc(ax+b), \sec(ax+b)$	
Exponential Function	e^{ax+b}	

Chapter 4 Vectors

(1) Magnitude of a vector a, $|a| = \|a\| =$ length of vector $a = \sqrt{x_1^2 + y_1^2}$

(2) Let λ be a scalar. λa is the vector that is parallel to a and has magnitude $|\lambda||a|$. $\lambda > 0$ where a and λa are in the same direction.

(3) Given $A(x_1, y_1)$ and $B(x_2, y_2)$; then $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} - \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}$

(4) For any point $A(x_1, y_1, z_1)$, the vector $\overrightarrow{OA}$ = position vector of A with respect to O. $|\overrightarrow{OA}| = \sqrt{x_1^2 + y_1^2 + z_1^2}$

$\overrightarrow{OA} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = x_1 i + y_1 j + z_1 k$, where $i = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, j = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, k = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

(5) For two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$, the length of P_1P_2 is $|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

(6) Let $u = x_1 i + y_1 j + z_1 k$ and $v = x_2 i + y_2 j + z_2 k$, then $u \cdot v = x_1x_2 + y_1y_2 + z_1z_2$, where the angle between these two vectors, θ

$\cos\theta = \frac{a \cdot b}{|a||b|} = \frac{x_1x_2 + y_1y_2 + z_1z_2}{\sqrt{x_1^2 + y_1^2 + z_1^2} \cdot \sqrt{x_1^2 + y_1^2 + z_1^2}}$

(7) Vector product: Definition and properties

(i) Vector product of a and b, denoted by $a \times b$ is defined as follows: $a \times b$ is perpendicular to both a and b; direction of $a \times b$ is given by the right − hand rule; $|a x b| = |a||b| \sin\theta$, where θ = angle between a and b.

(8) Vector product (Cross product): Method 1

Let $v_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$, then their vector product is $v_1 x\, v_2 = \begin{vmatrix} i & j & k \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = (y_1z_2 - y_2z_1)i - (x_1z_2 - x_2z_1)j + x_1y_2 - x_2y_1)k$

(9) Lines in Three − Dimensional Space

(i) The line L passes through a point A and is parallel to a vector u has vector equation, $f = a + \lambda u$; a = point, u = direction vector

(ii) point (x_0, y_0, z_0), direction vector $= ai + bj + ck$, $\lambda = t \longrightarrow$ vector equation, $r(t) = (x_0 i + y_0 j + z_0 k) + t(ai + bj + ck) = r_0 + tv$

(10) Given $r = xi + yj + zk = (x_0 i + y_0 j + z_0 k) + t(ai + bj + ck)$; passing through $A(x_o, y_0, z_0)$ and parallel to $u = di + ej + fk$

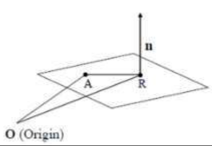
$r = xi + yj + zk = (x_0 i + y_0 j + z_0 k) + \lambda(di + ej + fk) = (x_0 + \lambda d)i + (y_0 + \lambda e)j + (z_0 + \lambda f)k$

$\begin{cases} x = x_0 + at \\ y = y_0 + bt \longrightarrow r = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \lambda \begin{pmatrix} d \\ e \\ f \end{pmatrix} \\ z = z_0 + ct \end{cases}$

(11) Intersecting Lines and Skew Lines

Given 2 lines, $r = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} + \lambda \begin{pmatrix} r_1 \\ s_1 \\ t_1 \end{pmatrix}$ and $r = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} + \lambda \begin{pmatrix} r_2 \\ s_2 \\ t_2 \end{pmatrix} \rightarrow$ coincident or identical, parallel and not coincident, non − parallel and intersecting; non − parallel and non intersecting (skew lines)

(12) Planes in Three − Dimensional Space



Vector perpendicular to a given plane denoted by n. It is a normal vector to the plane. Fix a point A on the plane and let P be any point on the plane. Let the position vectors of A and P be a and r. Then the vector $\overrightarrow{AP}$ is perpendicular to the normal vector n. Hence, $(r - a) \cdot n = 0 \rightarrow r \cdot n = a \cdot n$

(a) Two vectors a and b are parallel, where $a = \lambda b$ for some scalar $\lambda \neq 0$.
(b) A, B and C are collinear if and only if $\overrightarrow{AB}$ parallel $\overrightarrow{AC}$ and $\overrightarrow{BC}$.
(c) Unit vector, $\hat{v} = \frac{1}{\|v\|} v$
(d) $a \cdot b = |a||b| \cos\theta$
(e) $a \cdot b = b \cdot a$; $a \cdot a = |a|^2$
(f) $\lambda(a \cdot b) = (\lambda a) \cdot b = a \cdot (\lambda b)$
(g) $a \cdot (b + c) = a \cdot b + a \cdot c$
(h) $a \cdot b = 0$ if and only if $b \perp a$

(a) $a \times b = -b \times a$
(b) $a \times a = 0$
(c) $i \times j = k, j \times k = i, k \times i = j$
(d) $\lambda(a \times b) = (\lambda a) x b = a x (\lambda b)$
(e) $a \times (b + c) = a \times b + a \times c$

(a) Acute angle between planes:
n_1, n_2 be normal vectors to the planes
$\cos\theta = \left|\frac{n_1 \cdot n_2}{|n_1||n_2|}\right|$
(b) Acute angle between line and plane:
u: direction vector of line
v: normal vector of plane
$\sin\theta = \left|\frac{u \cdot n}{|u||n|}\right|$
(c) Intersection of two planes
$v_1: r \cdot n_1 = d_1$ and $v_2: r \cdot n_2 = d_2$
Vector equation of $L: r = a + \lambda(n_1 \times n_2)$

- If $x \approx a$, then $f(x) \approx f(a)$
  - $101 \approx 100 \Rightarrow \sqrt{101} \approx \sqrt{100} = 10$
    $\sqrt{101} = 10.04988....$ Error $\approx 0.5\%$
  - $101 \approx 100 \Rightarrow 101^2 \approx 100^2 = 100000$
    $101^2 = 10201$. Error $\approx 2\%$
- $f'(a)$ is the rate of change of $y = f(x)$ at $x = a$
  - $x \approx a \Rightarrow \frac{f(x) - f(a)}{x - a} \approx f'(a) = \frac{dy}{dx}|_{x=a}$
  - $f(x) \approx f'(a)(x - a) + f(a)$
  - $y = f'(a)(x - a) + f(a)$ is the **Tangent Line** of $y = f(x)$ at $x = a$

Ordinary differentiation	Implicit differentiation
$\frac{d}{dx}(x^{2}) = 2x$	$\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$
$\frac{d}{dx} = nx^{n-1}$	$\frac{d}{dx}(y^n) = ny^{n-1}\frac{dy}{dx}$
$\frac{d}{dx}(\sin x) = \cos$	$\frac{d}{dx}(\sin y) = \cos y\frac{dy}{dx}$
$\frac{d}{dx}(e^x) = e^x$	$\frac{d}{dx}(e^y) = e^y\frac{dy}{dx}$
$\frac{d}{dx}(\ln x) = \frac{1}{x}$	$\frac{d}{dx}(\ln y) = \frac{1}{y}\frac{dy}{dx}$
$\frac{d}{dx}(x) = 1$	$\frac{d}{dx}(y) = 1\frac{dy}{dx}$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$a^2 - b^2 = (a+b)(a-b)$
$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$
$(a+b)^2 = a^2 + 2ab + b^2$
$(a-b)^2 = a^2 - 2ab + b^2$

- $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{x^2+1}$

- $\frac{d}{dx}(x^n) = nx^{n-1}$
- $\frac{d}{dx}(\ln x) = \frac{1}{x}$
- $\frac{d}{dx}(a^x) = a^x \ln a$
- $\frac{d}{dx}(e^x) = e^x$
- $\frac{d}{dx}(\sin x) = \cos x$
- $\frac{d}{dx}(\cos x) = (-\sin x)$
- $\frac{d}{dx}(\tan x) = \sec^2 x$
- $\frac{d}{dx}(\cot x) = -\csc^2 x$
- $\frac{d}{dx}(\sec x) = \sec x \tan x$
- $\frac{d}{dx}(\csc x) = -\csc x \cot x$
- $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

Find the foot of perpendicular from the given point A to the plane Π, and calculate the distance from A to Π.
(a) $A(5, -3, 4)$, Π: $3x - 4y + z = 5$
point Q be $(3\lambda + 5, -4\lambda - 5, \lambda + 4)$
$3x - 4y + z = 5$ [Substitute corresponding]
$9\lambda + 15 + 16\lambda + 12 + \lambda + 4 = 5 \rightarrow \lambda = -1$
coordinates, Q $(2,1,3)$
$D = \sqrt{(2-5)^2 + (1+3)^2 + (3-4)^2} = \sqrt{26}$

Relative to the origin O, the point A has a position vector $2i + 9j - 6k$ and the point B has position vector $6i + 3j + 6k$. The point C is such that $\overrightarrow{OC} = 2\overrightarrow{OA}$ and D is the midpoint of segment AB.
(a) Find the position vectors of C and D.
$\overrightarrow{OC} = 2\overrightarrow{OA} = 2(2, 9, -6) = (4, 18, -12)$
$\overrightarrow{OD} = \frac{1}{2}\overrightarrow{AB} = \frac{1}{2}(6 + 2, 3 + 9, 6 - 6) = (4, 6, 0)$
(b) Find a vector equivalent of the line L through C and D.
$\overrightarrow{DC} = \overrightarrow{OC} - \overrightarrow{OD} = 12(0, 1, -1)$
$r = 4i + 6j + \lambda(j - k)$
(c) Find the point at which L intersects the line through O and B.
$(4, 6, 0) + \lambda(0, 1, -1) = t(6, 3, 6) \rightarrow 6 = 6t \rightarrow t = \frac{2}{3}$
intersection point: $\frac{2}{3}(6, 3, 6) = (4, 2, 4)$

Let $A(1, -3, 2)$, $B(0, -4, 5)$ and $C(5, 0, -3)$ be points in $\mathbb{R}^3$.
(a) Find a vector equation of the line L through A and B.
$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (0, -4, 5) - (1, -3, 2) = (-1, -1, 3)$
$r = i - 3j + 2k + \lambda(-i - j + 3k)$
(b) Show that C does not lie on the line L.
$(5, 0, -3) = (1, -3, 2) + \lambda(-1, -1, 3)$
$5 = 1 - \lambda \rightarrow \lambda = -4$; $0 = -3 - \lambda = -3$; $-3 = 2 + 3\lambda \rightarrow \lambda = -\frac{5}{3}$ [λ not consistent]
(c) Find the foot of perpendicular from C to L and hence determine the image of C under a reflection with respect to L.
$r = (1, -3, 2) + \lambda(-1, -1, 3)$; $\overrightarrow{CQ} = (1 - \lambda, -3 - \lambda, 2 + 3\lambda) - (5, 0, -3)$
$\overrightarrow{CQ} = (-4 - \lambda, -3 - \lambda, 5 + 3\lambda) \cdot (-1, -1, 3) = 0 \rightarrow 4 + \lambda + 3 + \lambda + 15 + 9\lambda = 0$
$\lambda = -2$; $\overrightarrow{OQ} = (1, -3, 2) + (-2)(-1, -1, 3) = (3, -1, -4)$
$C' = (x, y, z) = \left(\frac{5 + x}{2}, \frac{0 + y}{2}, \frac{-3 + z}{2}\right) = (3, -1, 4) \rightarrow C'(x, y, z) = (1, -2, -5)$
(d) Find the vector equation of the image of the lne through A and C under a reflection with respect to L.
$\overrightarrow{AC'} = \overrightarrow{OC'} - \overrightarrow{OA'} = (1, -2, -5) - (1, -3, 2) = (0, 1, -7)$
$A(1, -3, 2)$ and $C\ (0, 1, -7) \rightarrow i - 3j + 2k + \lambda(j - 7k)$
(e) Find the distance between C and L.
$CQ = \sqrt{(3-5)^2 + (-1-0)^2 + (-4+3)^2} = \sqrt{6}$

Consider the planes $r \cdot (i - j) = 3$ and $r \cdot (j + k) = 1$.
(a) Find the acute angle between the two planes.
$\cos\theta = \frac{a \cdot b}{|a||b|} = \frac{x_1x_2 + y_1y_2 + z_1z_2}{\sqrt{x_1^2 + y_1^2 + z_1^2} \cdot \sqrt{x_1^2 + y_1^2 + z_1^2}}$
(b) Find a vector equation of the line of intersection.
$n_1 = (1, -1, 0)$; $n_2 = (0, 1, 1) \rightarrow n_1n_2 = (1, -1, 0)\ x\ (0, 1, 1)$
$\begin{vmatrix} i & j & k \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = (-1 - 0)i - (1 - 0)j + (1 - 0)k = -i - j + k\ (-1 - 1, 1)$
$x - y = 3$ − − − (1) and $y + z = 1$ − − − (2)
(1) + (2), $x - y + y + z = 4 \rightarrow x + z = 4$
Let $z = 0$, $x = 4$; hence $y = 1$
equation: $4i + j + \lambda(-i - j + k)$

Find the equation, in the form of $r \cdot n = d$, of the plane which
(i) is perpendicular to the vector $4i + 3j + 5k$ which contains the point $(2, -2, 0)$
$r \cdot n = n \cdot a \rightarrow r \cdot \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix} \rightarrow r \cdot \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix} = 8 - 6 + 0 \rightarrow r \cdot n = 2 \rightarrow r \cdot \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix} = 2$
(ii) passes through $A(1, 2, 3)$ and $B(2, 2, -1)$ and $C(0, 0, 1)$
$\overrightarrow{AB} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}$; $\overrightarrow{AC} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix}$
$n = u\ x\ v = \begin{vmatrix} i & j & k \\ 1 & 0 & -4 \\ -1 & -2 & -2 \end{vmatrix} = \begin{pmatrix} -8 \\ -6 \\ -2 \end{pmatrix} = 2\begin{pmatrix} -4 \\ -3 \\ -1 \end{pmatrix}$
$r \cdot n = a \cdot n \rightarrow r \begin{pmatrix} -4 \\ -3 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ -3 \\ -1 \end{pmatrix} \rightarrow r \cdot \begin{pmatrix} -4 \\ -3 \\ -1 \end{pmatrix} = -13 \rightarrow r \cdot \begin{pmatrix} -4 \\ -3 \\ -1 \end{pmatrix} = -13$
(iii) contains $A(3, 4, 5)$ and line L: $r = 4i + 3j + 5k + \mu(-1 + 2j - 3k)$
$r = \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix} + \mu\begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix} \rightarrow \overrightarrow{AQ} = \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \rightarrow n = \begin{vmatrix} i & j & k \\ 1 & -1 & 0 \\ -1 & 2 & -3 \end{vmatrix} \rightarrow r\begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} = 26$

Two lines L1 and L2 have vector equations given respectively by
$r = i + j + k + \lambda(2i + j + k)$ and
$r = 4i + j + 10k + \mu(i + 3k)$.
(a) Show that L1 and L2 intersects, and find the point of intersection.
$(1,1,1) + \lambda(2,1,1) = (4,1,10) + \mu(1,0,3)$
$1 + 2\lambda = 4 + \mu$; $1 + \lambda = 1$; $1 + \lambda = 10 + 3\mu$
$\lambda = 0$; $\mu = -3 \rightarrow$ all $\mu = -3$
Intersection point: $(4,1,10) - 3(1,0,3)$
(b) Find the acute angle between L1 and L2.
$\cos\theta = \left|\frac{n_1 \cdot n_2}{|n_1||n_2|}\right|$

Find an equation of the plane which is parallel to the vectors $i + 2k$ and $3i + j + k$, and contains the point $(0, -1, -2)$.
$(1, 0, 2,)\ x\ (3, 1, 1)$
$\begin{vmatrix} i & j & k \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{vmatrix} = -2i + 5j + k \rightarrow (-2, 5, 1)$
$a(x - x_0) + 6(y - y_0) + c(z - z_0)$
$-2(x - 0) + 5(y + 1) + (z + 2) = 0$
$-2x + 5y + z = 0$

Let R be the region bounded by the graphs of $y = \frac{16}{x^2}$ and $y = \frac{1}{2}x - 1$ and the line $x = 2$
Denote the area of the region R by A. Let V be the volume of the solid formed by rot − ating R completely about the x − axis and W be the volume of the solid formed by rotating R completely about the y − axis.
(i) Find the value of A.
$\frac{16}{x^2} = \frac{1}{2}x - 1 \rightarrow x = 4$ and [line $x = 2$]
$A = \int_2^4 \frac{16}{x^2} - \frac{1}{2}x + 1\, dx = 3$ (units^2)
(ii) Find the value of V.
$V = \int_2^4 \pi\left(\frac{16}{x^2}\right)^2 dx - \frac{1}{3}\pi r^2 h = \frac{26}{3}\pi$ (units^3)
(iii) Find the value of W.
$y = \frac{16}{x^2} \rightarrow x^2 = \frac{16}{y}$ AND $y = \frac{1}{2}x - 1$, find x^2
$W = \left[\int_1^4 \pi\left(\frac{16}{y}\right) dy - \pi r^2 h\right] +$
$\left[\int_0^1 \pi(4y^2 + 8y + 4) dy - \pi r^2 h\right]$
$= (16 \ln 4)\pi - \frac{20}{3}\pi$ (units^3)