MA1511 2324 S1 Solutions

Q 1 $f(x,y,z) = z\ln(x^2 y\cos z) + x\sin(xyz)$

$$\frac{\partial f}{\partial x} = z\frac{1}{x^2 y\cos z}(2x)y\cos z + \sin(xyz) + x\cos(xyz)(yz)$$

$$= (z)\left(\frac{2}{x}\right) + \sin(xyz) + x\cos(xyz)(yz)$$

or $f(x,y,z) = z(2\ln x + \ln y + \ln\cos z) + x\sin(xyz)$

$$\frac{\partial f}{\partial x} = (z)\left(\frac{2}{x}\right) + \sin(xyz) + x\cos(xyz)(yz)$$

Q 2 let $f(x,y,z) = (x-1)^2 + (y-2)^2 + (z-2)^2$

$$g(x,y,z) = x^2+y^2+z^2-36$$

Extreme occurs at (x,y,z) if $\exists\,\lambda$ such that

$$\begin{pmatrix} f_x(x,y,z) \\ f_y(x,y,z) \\ f_z(x,y,z) \end{pmatrix} = \lambda \begin{pmatrix} g_x(x,y,z) \\ g_y(x,y,z) \\ g_z(x,y,z) \end{pmatrix}$$

$$\begin{pmatrix} 2(x-1) \\ 2(y-2) \\ 2(z-2) \end{pmatrix} = \lambda \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$$

$\therefore \quad x-1=\lambda x \quad \cdots (1)$

$y-2=\lambda y \quad \cdots (2)$

$z-2=\lambda z \quad \cdots (3)$

Now get rid of λ, from (1), (2) and (3)

get $\lambda = \dfrac{x-1}{x} = \dfrac{y-2}{y} = \dfrac{z-2}{z}$

$\therefore \quad y=2x \quad , \quad z=2x$ subst. into $x^2+y^2+z^2=36$

get $x^2+4x^2+4x^2=36$

$9x^2=36 \quad \therefore x=\pm 2$

when $x=2, y=4, z=4$

$f(2,4,4)=(2-1)^2+(4-2)^2+(4-2)^2$

$=1^2+2^2+2^2=9$

when $x=-2, y=-4, z=-4$

$f(-2,-4,-4)=(-2-1)^2+(-4-2)^2+(-4-2)^2$

$=9+36+36=81$

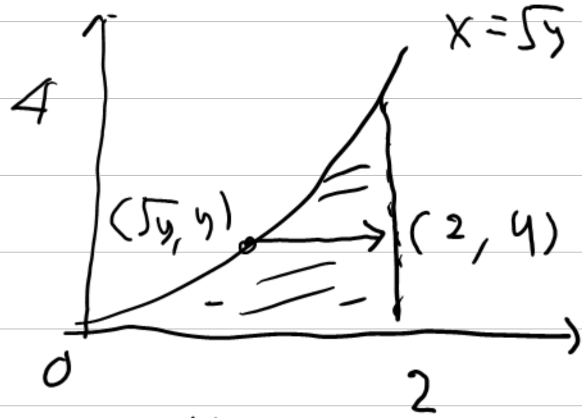
The point $(2,4,4)$ on the sphere is closest to the point $(1,2,2)$.

The point $(-2,-4,-4)$ on the sphere is farthest from the point $(1,2,2)$

Q3

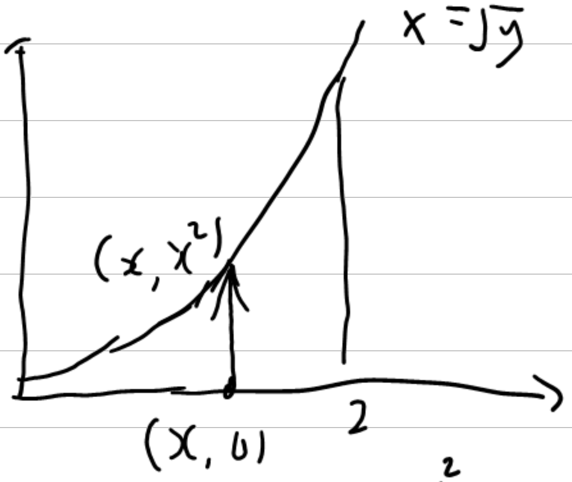
$$\int_0^4 \int_{\sqrt{y}}^2 \frac{1}{1+2x^3}\, dx\, dy$$

First sketch the region of the above integral.



Now change the order of the integration

$x = \sqrt{y}$ $\therefore$ $x^2 = y$



$$\int_0^4 \int_{\sqrt{y}}^2 \frac{1}{1+2x^3}\, dx\, dy = \int_0^2 \left[\int_0^{x^2} \frac{1}{1+2x^3}\, dy \right] dx$$

$$= \int_0^2 \frac{1}{1+2x^3} \left[\int_0^{x^2} dy \right] dx = \int_0^2 \frac{1}{1+2x^3}\, x^2\, dx$$

Let $u = 2x^3$. Then $\frac{du}{dx} = 6x^2$ $\therefore \frac{1}{6}du = x^2 dx$

when $x=0$, then $u=0$
$x=2$, then $u=16$

$$\int_0^2 \frac{x^2}{1+2x^3}\,dx = \frac{1}{6}\int_0^{16} \frac{1}{1+u}\,du$$

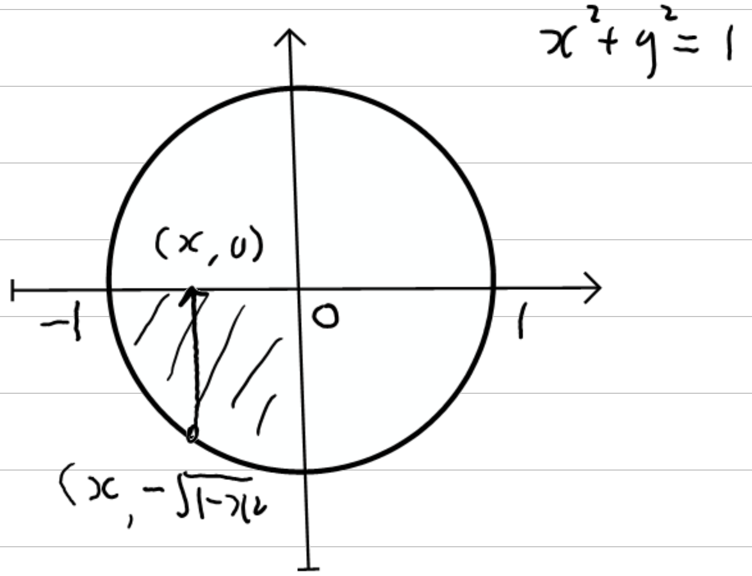
$$= \frac{1}{6}\Big[\ln(1+u)\Big]_0^{16} = \frac{1}{6}\big[\ln(17) - \ln 1\big]$$

$$= \frac{1}{6}\ln(17)$$

$$\therefore \int_0^4 \int_{\sqrt{y}}^2 \frac{1}{1+2x^3}\,dx\,dy = \frac{1}{6}\ln(17)$$

Q 4 $\displaystyle\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{0} \frac{2}{1+\sqrt{x^2+y^2}}\,dy\,dx$

First sketch the region of the above integral.



Use polar coordinate

Let $x = r\cos\theta$, $y = r\sin\theta$. $dxdy = r\,dr\,d\theta$

$$\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{0} \frac{2}{1+\sqrt{x^2+y^2}}\,dy\,dx$$

$$= \int_{\pi}^{\frac{3\pi}{2}}\left[\int_{0}^{1} \frac{2}{1+r}\, r\,dr\right]d\theta$$

$$= \int_{\pi}^{\frac{3\pi}{2}} 2\left[\int_0^1 \frac{r}{1+r}\,dr\right] d\theta$$

$$= \int_{\pi}^{\frac{3\pi}{2}} 2\left[\int_0^1 \left(1-\frac{1}{1+r}\right)dr\right] d\theta$$

$$= \int_{\pi}^{\frac{3\pi}{2}} 2\left[r-\ln(1+r)\right]_0^1 d\theta$$

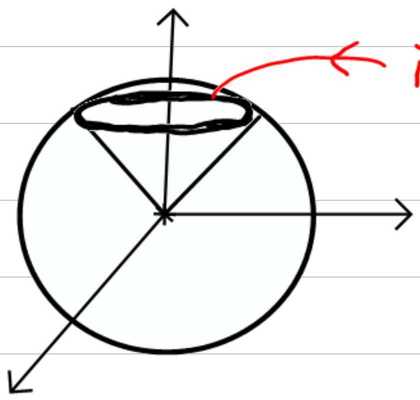
$$= \int_{\pi}^{\frac{3\pi}{2}} 2\left(1-\ln 2\right) d\theta$$

$$= 2(1-\ln 2)\left(\frac{3\pi}{2}-\pi\right)$$

$$= (2)\left(\frac{\pi}{2}\right)(1-\ln 2)$$

$$= (1-\ln 2)\,\pi$$

Q 5



Solve $x^2+y^2+z^2=8$

$z^2=x^2+y^2, \quad z>0$

get $2(x^2+y^2)=8$

$x^2+y^2=4$

$\therefore$ Intersection cure is $x^2+y^2=4$ at $z=2$

Let $r(t) = 2\cos t\, i + 2\sin t\, j + 2k$

$0 \le t \le 2\pi$

$r'(t) = -2\sin t\, i + 2\cos t\, j + 0k$

$|r'(t)| = \sqrt{(-2\sin t)^2 + (2\cos t)^2 + 0^2}$

$= \sqrt{2^2[(\sin t)^2 + (\cos t)^2]}$

$= 2$

$\int_C (2x^2 + 2y^2 + z + y)\, ds$

$= \int_0^{2\pi} (8 + 2 + 2\sin t)\, |r'(t)|\, dt$

$= \int_0^{2\pi} 2(5 + \sin t)(2)\, dt$

$$= 4\int_0^{2\pi} (5 + \sin t)\,dt$$

$$= 4\left[5t - \cos t\right]_0^{2\pi}$$

$$= 4\left[(10\pi - \cos(2\pi)) - (0 - \cos 0)\right]$$

$$= 4\left[10\pi - 1 + 1\right] = 40\pi$$

Q 6

$$r(u,v) = u\,i + 2v^2 j + (u^2+v)k$$

$$r_u = i + 0j + 2u\,k$$

$$r_v = 0i + 4v\,j + k$$

$$r_u \times r_v = \begin{vmatrix} i & j & k \\ 1 & 0 & 2u \\ 0 & 4v & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 2u \\ 4v & 1 \end{vmatrix} i - \begin{vmatrix} 1 & 2u \\ 0 & 1 \end{vmatrix} j + \begin{vmatrix} 1 & 0 \\ 0 & 4v \end{vmatrix} k$$

$$= \left(0-(2u)(4v)\right) i - \left((1)(1)-0\right) j + \left((1)(4v)-0\right) k$$

$$= -8uv\,i - j + 4v\,k$$

At $(x,y,z) = (2,2,3)$, from

$r(u,v) = u\,i + 2v^2 j + (u^2+v)k$ get

$u = 2$, $2v^2 = 2$, $3 = u^2+v$

Hence $u = 2$, $v = \pm 1$. Now check $3 = u^2+v$

when $u = 2$, $v = 1$, $u^2+v = 4+1 = 5 \neq 3$
reject

when $u = 2$, $v = -1$, $u^2+v = 4-1 = 3$ o.k.

$\therefore\ u = 2,\ v = -1$ when $(x, y, z) = (2, 2, 3)$

$$r_u \times r_v = -8uv\,i - j + 4vk$$
$$= (-8)(2)(-1)\,i - j + 4(-1)k$$
$$= 16i - j - 4k \quad \text{at } (x, y, z) = (2, 2, 3)$$

which is a normal vector to the tangent plane to the surface S at $(x, y, z) = (2, 2, 3)$.

Hence the equation of the tangent plane is

$$16x - y - 4z = D$$

Now find the value of D. The poin $(2, 2, 3)$ is on the plane, so

$$16(2) - 2 - 4(3) = D$$
$$32 - 2 - 12 = D$$
$$D = 18$$

$\therefore$ The tangent plane is

$$16x - y - 4z = 18$$

$\therefore\ A = 16,\quad C = -4,\quad D = 18$

Q 7

(a) On the closed curve $x^2+4y^2=4$

$$\oint_C P(x,y)dx + Q(x,y)dy$$

$$= \oint_C \frac{-y}{x^2+4y^2}dx + \frac{x}{x^2+4y^2}dy$$

$$= \frac{1}{4}\oint_C -y\,dx + x\,dy$$

Apply Green's theorem to the above integral

get $$\frac{1}{4}\iint_D \left(\frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y}\right)dxdy$$

$$= \frac{1}{4}\iint_D (1-(-1))\,dxdy$$

$$= \frac{2}{4}\iint_D dx\,dy$$

$= \frac{1}{2}$ area of D where D is the region bounded by $x^2+4y^2=4$

$$= \frac{1}{2}(2\pi)$$

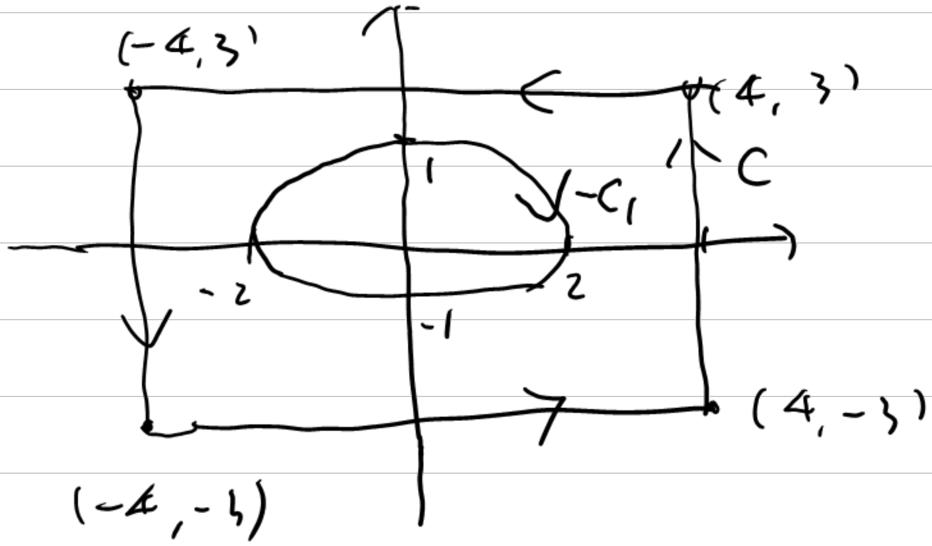
$$= \pi$$

Ans: π

(b) Let C_1 : the closed curve $x^2+4y^2=4$

Let $C_2 = C - C_1$ = the closed curve

consists of C and $-C_1$.



Apply Green's theorem to

$$\int_{C-C_1} P(x,y)\,dx + Q(x,y)\,dy$$

get the above integral equal to zero

$$\therefore \int_C P(x,y)\,dx + Q(x,y)\,dy = \int_{C_1} P(x,y)\,dx + Q(x,y)\,dy$$

$= \pi$ from (a)

Ans: π

Q 8

$$F(x,y,z) = \left(\frac{1}{y} - \frac{2y}{x^3}\right) i + \left(\frac{1}{x^2} - \frac{x}{y^2}\right) j + 2z^2 k$$

is conservative.

Assume $F = \nabla f$

$\therefore f_x = \frac{1}{y} - \frac{2y}{x^3}$ --- (1), $f_y = \frac{1}{x^2} - \frac{x}{y^2}$ -- (2), $f_z = 2z^2$ --- (3)

shall find f.

$$f(x,y,z) = \int f_x \, dx = \int \left(\frac{1}{y} - \frac{2y}{x^3}\right) dx$$

$$= \frac{x}{y} + \frac{y}{x^2} + C(y,z) \quad \text{-- (4)}$$

diff f above w.r.t y get

$$f_y = -\frac{x}{y^2} + \frac{1}{x^2} + C_y(y,z)$$

compare with (2) $f_y = \frac{1}{x^2} - \frac{x}{y^2}$ get

$C_y(y,z) = 0$. Hence $C(y,z) = \int C_y(y,z)\,dy = D(z)$

$\therefore$ from (4)

$$f(x,y,z) = \frac{x}{y} + \frac{y}{x^2} + D(z)$$

diff above w.r.t. z get

$f_z = 0 + 0 + D'(z)$ compare with (3)

get $D'(z) = 2z^2$

$$\therefore D(z) = \int D'(z)\,dz = 2\int z^2\,dz$$

$$= \frac{2}{3} z^3 + E$$

choose $E=0$, $D(z) = \frac{2}{3} z^3$

$$\therefore f(x,y,z) = \frac{x}{y} + \frac{y}{x^2} + \frac{2}{3} z^3$$

Use the fundamental theorem of line integrals

get

$$\int_C F\cdot dr = \int_C \nabla f\cdot dr$$

$$= f(\text{end pt}) - f(\text{starting pt})$$

where $C:\ r(t) = t\,i + t^2 j + (t-1)k,\ 1 \le t \le 2$

$r(\text{starting pt}) = r(1) = i + j + 0k$

$r(\text{end pt}) = r(2) = 2i + 4j + k$

$\therefore \int_C F\cdot dr = f(\text{end pt}) - f(\text{starting pt})$

where $f(x,y,z) = \frac{x}{y} + \frac{y}{x^2} + \frac{2}{3}z^2$

$$\therefore \int_C F\cdot dr = \left(\frac{2}{4} + \frac{4}{4} + \frac{2}{3}(1)\right) - \left(\frac{1}{1} + \frac{1}{1} + \frac{2}{3}(0)\right)$$

$$= \left(\frac{1}{2} + \frac{2}{3}\right) - 1 = \frac{7}{6} - 1 = \frac{1}{6}$$

Ans: $\frac{1}{6}$

Q9

(a) $$\lim_{n\to\infty} \frac{(n+1)^{n+1}}{n^n} \frac{1}{(n+1)^2}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n \frac{n+1}{(n+1)^2}$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n \frac{1}{(n+1)}$$

$$= e \cdot 0 = 0$$

(b) $$g(x) = (1+x^2)\cos(x^3)$$

$$= (1+x^2)\left(1 - \frac{(x^3)^2}{2!} + \frac{(x^3)^4}{4!} - \frac{(x^3)^6}{6!} + \cdots\right)$$

$$= 1 - \frac{(x^3)^2}{2!} + \frac{(x^3)^4}{4!} - \frac{(x^3)^6}{6!} + \cdots + x^2 - \frac{(x^3)^2}{2!}x^2 + \frac{(x^3)^4}{4!}x^2 + \cdots$$

coeff of $(x^3)^4 x^2$ = coeff. of x^{14}

is $\frac{1}{4!}$

$$\therefore g^{(14)}(0) = \frac{14!}{4!}$$

$$\left(\frac{g^{(14)}(0)}{14!} = \frac{1}{4!}\right)$$

Q10

(a) let $U_n = \frac{1}{3^n+(-2)^n} \frac{(5x+1)^{2n+1}}{n}$

$\therefore U_{n+1} = \frac{1}{3^{n+1}+(-2)^{n+1}} \frac{(5x+1)^{2(n+1)+1}}{n+1}$

$$\left|\frac{U_{n+1}}{U_n}\right| = \frac{3^n+(-2)^n}{3^{n+1}+(-2)^{n+1}} \frac{n}{n+1} |5x+1|^2$$

$$\lim_{n\to\infty} \left|\frac{U_{n+1}}{U_n}\right| = \left(\frac{1}{3}\right)(1)|5x+1|^2$$

Let $\lim_{n\to\infty} \left|\frac{U_{n+1}}{U_n}\right| < 1$ get

$$\frac{1}{3}|5x+1|^2 < 1$$

$$|5x+1|^2 < 3$$

$$|5x+1| < \sqrt{3}$$

$$5\left|5x+\frac{1}{5}\right| < \sqrt{3}$$

$$\left|x+\frac{1}{5}\right| < \frac{\sqrt{3}}{5}$$

Ans: the radius of convergence is $\frac{\sqrt{3}}{5}$.

(b)

$$f(x) = \sum_{n=0}^{\infty} \frac{4n}{n!} x^{4n} + \sum_{n=0}^{\infty} \frac{1}{n!} x^{4n}$$

$$= \sum_{n=1}^{\infty} \frac{4}{(n-1)!} (x^4)^n + \sum_{n=0}^{\infty} \frac{1}{n!} (x^4)^n$$

$$= 4 \sum_{n=1}^{\infty} \frac{1}{(n-1)!} (x^4)^{n-1} x^4 + \sum_{n=0}^{\infty} \frac{1}{n!} (x^4)^n$$

$$= 4x^4 \sum_{n=1}^{\infty} \frac{1}{(n-1)!} (x^4)^{n-1} + \sum_{n=0}^{\infty} \frac{1}{n!} (x^4)^n$$

$$= 4x^4 \sum_{n=0}^{\infty} \frac{1}{n!} (x^4)^n + \sum_{n=0}^{\infty} \frac{1}{n!} (x^4)^n$$

$$= 4x^4 e^{x^4} + e^{x^4}$$

$$f((\ln 2)^{\frac{1}{4}}) = 4(\ln 2) e^{\ln 2} + e^{\ln 2}$$

$$= 4(\ln 2)(2) + 2$$

$$= 2(4 \ln 2 + 1)$$