Linear Differential Equations

First-Order Linear Equations

A First Order Ordinary Differential Equations (ODE) has the form:

$$a(x)y' + b(x)y = c(x), \quad a(x) \neq 0$$

Solution Method - Integrating Factor:

1. Rewrite the equation in standard form:

$$y' + p(x)y = q(x)$$

2. Compute the **integrating factor**:

$$\mu(x) = e^{\int p(x)\,dx}$$

3. Multiply both sides by $\mu(x)$:

$$\mu(x)(y' + p(x)y) = \mu(x)q(x)$$

4. Integrate both sides to solve for (y).

Bernoulli Differential Equation

A **Bernoulli differential equation** is of the form:

$$y' + p(x)y = q(x)y^n$$

To solve:

1. Substitute $(v = y^{1-n})$, then rewrite the equation as a first-order linear differential equation in (v).

Example 9: General Solution of $(xy' - y = x^3)$

1. Rewrite in standard form:

$$y' - \frac{1}{x}y = x^2$$

2. Integrate using the factor $(\mu(x) = e^{-\ln x} = \frac{1}{x})$.
3. Solution:

$$\frac{y}{x} = \frac{x^2}{2} + C$$

Example 11: Modeling Free Fall with Drag

Using **Newton's Second Law** to model an object falling with drag:

$$mv' = mg - kv$$

where (g) is gravity and (k) is the drag coefficient.

1. Rewrite as $(v' + \frac{k}{m}v = g)$.
2. Use integrating factor $(e^{kt/m})$ to solve.
3. Solution:

$$v(t) = \frac{mg}{k}\left(1 - e^{-\frac{k}{m}t}\right)$$

4. To find displacement $(s(t))$, integrate $(v(t))$.

Higher-Order Linear Differential Equations

A **higher-order linear differential equation** with constant coefficients has the form:

$$a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = g(x)$$

- **Homogeneous case**: When $(g(x) = 0)$
- **Non-homogeneous case**: When $(g(x) \neq 0)$

Solution Approach

1. **Characteristic Equation**: Substitute $(y = e^{rx})$ to find roots.
2. **Distinct Real Roots**: Solution terms like (e^{rx}).
3. **Repeated Roots**: Include (xe^{rx}) for multiplicity.
4. **Complex Roots**: Use $(e^{\alpha x}\cos(\beta x))$ and $(e^{\alpha x}\sin(\beta x))$.

Example 13: Solve $(4y'' + 12y' + 9y = 0)$

1. Characteristic equation:

$$4r^2 + 12r + 9 = 0 \Rightarrow (2r+3)^2 = 0$$

2. Solution:

$$y(x) = C_1 e^{-\frac{3}{2}x} + C_2 x e^{-\frac{3}{2}x}$$

Superposition Principle

For a homogeneous linear differential equation, if $(y_1(x))$ and $(y_2(x))$ are solutions, then the general solution is:

$$y(x) = C_1 y_1(x) + C_2 y_2(x)$$