**NATIONAL UNIVERSITY OF SINGAPORE**

SEMESTER 1, 2018/2019

**MA1301 Introductory Mathematics** **Exam Solution**

**1.** (a) $\dfrac{dy}{dx} = 2\sin(x^2+x)\cos(x^2+x)(2x+1)$.

(b) Differentiate $x^2 + xy - y^3 = 7$ with respect to $x$ to get

$$2x + y + x\frac{dy}{dx} - 3y^2\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{2x+y}{3y^2-x}.$$

Let $(x, y) = (3,2)$ to get $\left.\dfrac{dy}{dx}\right|_{(3,2)} = \dfrac{8}{9}$. So the tangent line at $(3,2)$ is

$$y = \frac{8}{9}(x-3)+2 = \frac{8}{9}x - \frac{2}{3}.$$

(c) $\dfrac{dx}{dt} = t$ and $\dfrac{dy}{dt} = 1$; so $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{1}{t}$ and

$$\frac{d^2y}{dx^2} = \frac{1}{dx/dt}\cdot\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{1}{t}\cdot\frac{-1}{t^2} = -\frac{1}{t^3}.$$

(d) Let the length of the shadow be $L(t)$ at time $t$. Then

$$\frac{L(t)+10}{L(t)} = \frac{30-16t^2}{6} \Rightarrow L(t) = \frac{15}{6-4t^2}.$$

Then

$$\frac{dL}{dt} = \frac{30t}{(3-2t^2)^2} \quad \text{and} \quad \left.\frac{dL}{dt}\right|_{t=1} = 30\,\text{ft/s}.$$

(b) $f'(x) = 9x^8 - 2$. So for $x \approx 1$,

$$f(x) \approx f(1) + f'(1)(x-1) = f(1) + 7(x-1).$$

So $f(1.05) - f(1.00) \approx 7(1.05 - 1.00) = 0.35$.

(c) Note that $f'(x) = 0 \Leftrightarrow x = -1$ or $x = 2$ or $x = 7/2$.

| | $x < -1$ | $-1 < x < 2$ | $2 < x < 7/2$ | $x > 7/2$ |
|---|---|---|---|---|
| $f'(x)$ | $-$ | $+$ | $+$ | $-$ |
| $f(x)$ | $\searrow$ | $\nearrow$ | $\nearrow$ | $\searrow$ |

So $f$ has a local minimum at $x = -1$, a saddle point at $x = 2$, and a local maximum at $x = 7/2$.

(d) It is given that $300 = x \cdot 3x \cdot y$. So $y = 100/x^2$.

The total surface is $S = (3x^2 + 8xy) + (3x^2 + 8kxy) = 6x^2 + \dfrac{800(1+k)}{x}$.

$$\frac{dS}{dx} = 12x - \frac{800(1+k)}{x^2} = 0 \Rightarrow x = \sqrt[3]{\frac{200(1+k)}{3}}.$$

If $x = y$, then $x = \sqrt[3]{100}$. So $\dfrac{200(1+k)}{3} = 100 \Rightarrow k = \dfrac{1}{2}$.

**3.** (a) Let $u = x + x^2$. Then $\dfrac{du}{dx} = 1 + 2x$. So

$$\int (1+2x)(x+x^2)^{-1/2}\,dx = \int u^{-1/2}\,du = 2\sqrt{u} + C = 2\sqrt{x + x^2} + C.$$

(b) Let $u = x^9$. Then $\dfrac{du}{dx} = 9x^8$. So

$$\int x^8 \ln(x^9)\,dx = \frac{1}{9}\int \ln u\,du = \frac{1}{9}(u\ln u - u + C) = \frac{1}{9}x^9(\ln(x^9) - 1) + C.$$

(d) $x + 2y = 4 \Rightarrow 4 - 2y$. Then $A = \displaystyle\int_{-2}^{1} [(4-2y) - 2y^2]\,dy = \int_{-2}^{1} (4 - 2y - 2y^2)\,dy$.

$x = 2y^2 \Rightarrow y = \pm\sqrt{x/2}$. So

$$\begin{aligned} A &= \int_0^2 \left[\sqrt{x/2} - (-\sqrt{x/2})\right] dx + \int_2^8 \left[(4-x)/2 - (-\sqrt{x/2})\right] dx \\ &= \int_0^2 \sqrt{2x}\,dx + \int_2^8 \left(2 - x/2 + \sqrt{x/2}\right) dx. \end{aligned}$$

(e) The circle is $(x-3)^2 + y^2 = 2^2$; so $x = 3 \pm \sqrt{4 - y^2}$. So

$$V = \int_{-2}^{2} \pi \left[(3 + \sqrt{4-y^2})^2 - (3 - \sqrt{4-y^2})^2\right] dy = \pi \int_{-2}^{2} 12\sqrt{4-y^2}\,dy.$$

**4.** (a) $\dfrac{dy}{dx} = e^{8x} \cdot \dfrac{e^{y^2}}{y}$. So $\displaystyle\int e^{8x}\,dx = \int ye^{-y^2}\,dy$. It gives

$$\frac{1}{8}e^{8x} = -\frac{1}{2}e^{-y^2} + C.$$

(b) $\overrightarrow{AB} \times \overrightarrow{AC} = (2\boldsymbol{i} - \boldsymbol{k}) \times (3\boldsymbol{j} - \boldsymbol{k}) = 3\boldsymbol{i} + 2\boldsymbol{j} + 6\boldsymbol{k}$.

The plane through $A, B, C$ has normal vector $\boldsymbol{n} = 3\boldsymbol{i} + 2\boldsymbol{j} + 6\boldsymbol{k}$. So it is of the form

$$\boldsymbol{r} \bullet n = \boldsymbol{a} \bullet \boldsymbol{n} = 3x + 2y + 6z = 6 \Rightarrow \frac{x}{2} + \frac{y}{3} + \frac{z}{1} = 1.$$

(c) Let $(\boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k}) + \lambda(2\boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k}) = -2\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k} + \mu(\boldsymbol{i} + 5\boldsymbol{j} + \alpha\boldsymbol{k})$. Then

$$1 + 2\lambda = -2 + \mu, \quad 2 - 4\lambda = 1 + 5\mu, \quad -3 + \lambda = 3 + \alpha\mu.$$

Solve $1+2\lambda = -2+\mu$ and $2-4\lambda = 1+5\mu$ to get $\lambda = -1$ and $\mu = 1$. Then the last equation becomes $-4 = 3 + \alpha$; so $\alpha = -7$.

(d) The normal vector of $\Pi_1$ is

$$\boldsymbol{n} = (6\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k}) \times (4\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}) = -4\boldsymbol{i} - 4\boldsymbol{j} + 6\boldsymbol{k}.$$

So its equation is

$$-4x - 4y + 6z = 0 \Rightarrow 2x + 2y - 3z = 0.$$