The Laplace Transform

Basic Properties of the Laplace Transform

The **Laplace transform** of a function $f(t)$ is defined as:
$(L[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}\,dt)$.

The **inverse Laplace transform** of $F(s)$ is:
$(f(t) = L^{-1}[F(s)])$.

Theorems and Properties:

1. **Linearity**:
   $(L[af(t) + bg(t)] = aL[f(t)] + bL[g(t)])$.
2. **First Shifting Theorem**:
   If $F(s) = L[f(t)]$, then $(L[e^{at}f(t)] = F(s-a))$.
3. **Differentiation**:
   For $f(t)$ with derivatives, $(L[f'(t)] = sL[f(t)] - f(0))$ and
   $(L[f^{(n)}(t)] = s^nL[f(t)] - s^{n-1}f(0) - \cdots - f^{(n-1)}(0))$.
4. **Multiplication by t^n**:
   $(L[t^nf(t)] = (-1)^nF^{(n)}(s))$.

Examples

1. **Laplace Transform of $f(t) = t$**:

$$L[t] = \int_0^\infty te^{-st}\,dt = \frac{1}{s^2}, \quad s > 0$$

2. **Inverse Laplace of $F(s) = \frac{1}{s^2+2s-3}$**:
   Using partial fraction decomposition:

$$F(s) = -\frac{1}{4(s+3)} + \frac{1}{4(s-1)}$$

   Taking inverse transforms:

$$L^{-1}[F(s)] = -\frac{1}{4}e^{-3t} + \frac{1}{4}e^{t}$$

3. **Applying First Shifting Theorem**:

$$L[e^{at}\cos(\omega t)] = \frac{s-a}{(s-a)^2+\omega^2}$$

Initial Value Problems

Example: Solve $y' + 2y = 0$, with $y(0) = 2$.

1. Take the Laplace transform:

$$sL[y] - y(0) + 2L[y] = 0$$

2. Solve for $L[y]$:

$$L[y] = \frac{2}{s+2}$$

3. Inverse Laplace to find $y(t)$:

$$y(t) = 2e^{-2t}$$

Step Functions and the Unit Impulse

1. **Unit Step Function**:

$$u(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$$

   $(L[u(t-a)] = \frac{e^{-as}}{s})$.
2. **Dirac Delta (Impulse) Function**:
   The **Dirac delta** or **unit impulse** $\delta(t-a)$ has the transform:

$$L[\delta(t-a)] = e^{-as}$$

3. **Second Shifting Theorem**:
   If $F(s) = L[f(t)]$, then:

$$L[f(t-a)u(t-a)] = e^{-as}F(s)$$

Example Problems

1. **Transform of a Piecewise Function**:
   For a function defined as:

$$f(t) = \begin{cases} 2, & 0 \le t < 1 \\ t^2/2, & 1 \le t < \frac{c}{2} \\ \cos(t), & t \ge \frac{c}{2} \end{cases}$$

Rewrite using the unit step function:

$$f(t) = 2(1 - u(t-1)) + \frac{t^2}{2}(u(t-1) - u(t - \frac{c}{2})) + \cos(t)u(t - \frac{c}{2})$$

Then, take the Laplace transform term-by-term.

2. **Initial Value Problem with Unit Impulse**:
   Solve $y'' + 3y' + 2y = \delta(t-1)$ with $y(0) = y'(0) = 0$:
   - Transform both sides:

   $$(s^2 + 3s + 2)L[y] = e^{-s}$$

   - Solve for $L[y]$ and find $y(t)$ using inverse transforms, yielding:

   $$y(t) = \begin{cases} 0, & t < 1 \\ e^{-(t-1)} - e^{-2(t-1)}, & t \ge 1 \end{cases}$$