MA1512 Cheat Sheet

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Separable Equations

$M(x)dx = N(y)dy$

Homogeneous Functions

If $y' = f(x, y)$ and $f(tx, ty) = t^n f(x, y)$ for some n, then:

$$\frac{dz}{f(1,z) - z} = \frac{dx}{x}, \quad \text{where } z = \frac{y}{x}.$$

Linear Change of Variable

If $y' = f(ax + by + c)$, set $u = ax + by + c$ and solve.

Exact Equations

If $M(x, y)dx + N(x, y)dy = 0$ and $M_y = N_x$ (by the Mixed Derivatives Theorem), then let $f_x = M(x, y)$ and $f_y = N(x, y)$. Solve $f(x, y)$. Alternatively, use integrating factor $R(x) = e^{\int g(x)dx}$, where $g(x) = \frac{M_y - N_x}{N}$.

Linear First Order ODE

$y' + p(x)y = q(x)$
$u = e^{\int p(x)dx}$
$u(y' + py) = uq \implies (uy)' = uq$

Reduction of Order

If $f(x, y', y'') = 0$, set $y' = p$ and $y'' = p'$
If $f(x, y', y'') = 0$, set $y' = p$ and $y'' = pp'$

Bernoulli's Equation

If an equation has the form $y' + p(x)y = q(x)y^n$, divide by y^n and let $z = y^{1-n}$:

$$z' + (1 - n)p(x)z = (1 - n)q(x).$$

Homogeneous ODE

$$y'' + py' + qy = 0.$$

Characteristic equation: $r^2 + pr + q = 0$.
Case 1: Real distinct roots: $y = c_1e^{r_1x} + c_2e^{r_2x}$.
Case 2: Distinct Complex roots: If solution is a $\pm$ ib, then $y = e^{ax}(c_1 \cos(bx) + c_2 \sin(bx))$.
Case 3: Equal real roots: $y = c_1e^{r_1x} + c_2xe^{r_1x}$.

Method of Undetermined Coefficients

The solution to the equation

$$y'' + py' + qy = R(x)$$

is of the form

$$y = y_g + y_p$$

where y_g is the general solution found by letting $R(x)$ to be 0 and y_p is the particular solution that is to be determined
Case 1: $R(x) = P(x)e^{kx}$, Substitute $y_p = u(x)e^{kx}$
Case 2: $R(x)$ is a trigonometric function with angular frequency b. Substitute $y_p = u(x)e^{a+ib}$ and take the real or imaginary component of the resultant solution based on whether $R(x)$ has sin or cos. Alternatively, let $y_p = u(x)(A \sin bx + B \cos bx)$
Case 3: $R(x)$ is a polynomial. Substitute $y_p = A_0 + A_1x + ...A_nx^n$, $x(A_0 + A_1x + ... + A_nx^n)$, etc where the degree of the polynomial is the degree of $R(x)$.

Method of Variation of Parameters

$$y'' + p(x)y' + q(x)y = r(x)$$

Let $W(y_1, y_2) = y_1y_2' - y_1'y_2$. Then:

$$u = -\int \frac{y_2r(x)}{W(y_1, y_2)}dx, \quad v = \int \frac{y_1r(x)}{W(y_1, y_2)}dx.$$

Solution: $y_p = uy_1 + vy_2$.

Determining One Solution from Another

If y_1 is a solution to a homogeneous second-order differential equation, then $y_2 = vy_1$ where

$$v = \int \frac{1}{y_1^2} e^{-\int P dx} dx.$$

Superposition

If y_1 is the solution to the equation

$$y'' + p(x)y' + q(x)y = g(x)$$

and y_2 is the solution to the equation

$$y'' + p(x)y' + q(x)y = h(x)$$

then for all constants C_1 and C_1, the function $y = C_1y_1 + C_2y_2$ is a solution to the equation

$$y'' + p(x)y' + q(x)y = g(x) + h(x)$$

Laplace Transforms

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)dt, \ f(t) = \mathcal{L}^{-1}[F(s)]$$

$$1 \to \frac{1}{s}, \quad e^{at} \to \frac{1}{s-a}, \quad t^n \to \frac{n!}{s^{n+1}}, \quad \sqrt{t} \to \frac{\sqrt{\pi}}{s^{3/2}},$$

$$\cos(at) \to \frac{s}{s^2 + a^2}, \quad \sin(at) \to \frac{a}{s^2 + a^2},$$

$$\cosh(at) \to \frac{s}{s^2 - a^2}, \quad \sinh(at) \to \frac{a}{s^2 - a^2},$$

$$t\cos(at) \to \frac{s^2 - a^2}{(s^2 + a^2)^2}, \quad t\sin(at) \to \frac{2as}{s^2 + a^2},$$

$$\sin(at + b) \to \frac{s \sin b + a \cos b}{s^2 + a^2}, \quad \cos(at + b) \to \frac{s \cos b - a \sin b}{s^2 + a^2},$$

$$f(ct) \to \frac{1}{c}\mathcal{L}(f(s - c)), \quad u(t - c) \to \frac{e^{-cs}}{s}, \quad \delta(t - c) \to e^{-cs}$$

1. Given $f(t)$, $g(t)$ and $a, b \in \mathbb{R}$,
   $\mathcal{L}[af(t) + bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)]$.
2. If $\mathcal{L}[f(t)] = F(s)$, then $\mathcal{L}[t \cdot f(t)] = -\frac{d}{ds}F(s)$.
3. If $\mathcal{L}[f(t)] = F(s)$, then $\mathcal{L}[e^{at} \cdot f(t)] = F(s - a)$.
4. Given $y(t)$, $\mathcal{L}[y'(t)] = s\mathcal{L}[y] - y(0)$,
   Similarly, $\mathcal{L}[y''(t)] = s\mathcal{L}[y'] - y'(0) = s^2\mathcal{L}[y] - sy(0) - y'(0)$,
   $\mathcal{L}\left(\int_0^t y(\tau)\, d\tau\right) \to \frac{1}{s}\mathcal{L}(y)$.
5. Given $\mathcal{L}[f(t)] = F(s)$,
   $\mathcal{L}[f(t - c)\, u(t - c)] = e^{-sc}F(s)$.

Dirac Delta

$$\delta(t) \to 1, \quad \delta(t - a) \to e^{-as}, \quad \int_0^\infty \delta(t)\, dt = 1$$

Malthus Model

$$\frac{dN}{dt} = BN - DN = kN, \quad N(t) = \hat{N}e^{kt}.$$

Logistic Model

$$\frac{dN}{dt} = BN - DN = BN - (sN)N = BN - sN^2,$$

$$N_\infty = \frac{B}{s}, \quad N(t) = N_\infty \quad (\hat{N} = N_\infty)$$

$$N(t) = \frac{N_\infty}{1 + \left(\frac{N_\infty}{\hat{N}} - 1\right)e^{-Bt}}, \ (\hat{N} < N_\infty), \ y(t) = \frac{y_\infty}{1 + \left(\frac{y_\infty}{y_0} - 1\right)e^{-kt}},$$

$$N(t) = \frac{N_\infty}{1 - \left(1 - \frac{\hat{N}}{N_\infty}\right)e^{-Bt}}, \quad (\hat{N} > N_\infty).$$

Harvesting Model

$$\frac{dN}{dt} = (B - sN)N - E, \quad \frac{dy}{dt} = -\frac{k}{y_\infty}y^2 + ky - E$$

The quadratic curve has no solution when $E > \frac{B^2}{4s}$. This means the derivative will always be negative and population would dwindle to zero.
The quadratic curve has one solution when $E = \frac{B^2}{4s}$. This means there is one unstable equilibrium at $\frac{B}{2s}$.
In the last case, there is a stable and unstable equilibrium at two roots of the equation when the derivative is zero.

Wave Equations

$$c^2y_{xx} = y_{tt}, \quad y(t, 0) = 0, \quad y(t, \pi) = 0, \quad y(0, x) = f(x)$$

$$y_t(0, x) = 0, \quad y(t, x) = \frac{1}{2}[f(x + ct) + f(x - ct)]$$

Heat Equations

$$u_t = c^2u_{xx}, \quad u(L, t) = 0, \quad u(0, t) = 0$$

$$u_n(x, t) = e^{(-c^2\pi^2n^2t)/l^2}(\beta_n \sin \frac{\pi n}{l}x)$$

Boundary is from 0 to l, to determine n, β_n: use $u(x, 0) = f(x)$

Trigonometric Integration

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}, \quad \csc x = \frac{1}{\sin x}$$
$$\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}, \quad \sec x = \frac{1}{\cos x}$$
$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}, \quad \cot x = \frac{1}{\tan x}$$

Integration Rules

$$\int uv = u\int v\,dx - \int \frac{du}{dx}\int v\,dx\,dx, \quad \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v-uv'}{v^2}$$
$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}, \quad \int u\,dv = uv - \int v\,du$$
$$u = g(x), du = g'(x)\,dx, \quad \int f(g(x))g'(x)\,dx = \int f(u)\,du$$

Simple Harmonic Motion

$$k = mg/\Delta l, \quad \omega = \sqrt{k/m}, \quad y(t) = R\cos(\omega t - \delta)$$

Trigonometric Identities

$$1+\tan^2 u = \sec^2 u, \quad 1+\cot^2 u = \csc^2 u$$
$$\sin(-x) = -\sin x, \quad \cos(-x) = \cos x, \quad \tan(-x) = -\tan x$$
$$\sin\left(\frac{\pi}{2}-x\right) = \cos x, \quad \cos\left(\frac{\pi}{2}-x\right) = \sin x$$
$$\tan\left(\frac{\pi}{2}-x\right) = \cot x, \quad \cot\left(\frac{\pi}{2}-x\right) = \tan x$$
$$\sec\left(\frac{\pi}{2}-x\right) = \csc x, \quad \csc\left(\frac{\pi}{2}-x\right) = \sec x$$
$$\sin(x\pm y) = \sin x\cos y \pm \cos x\sin y$$
$$\cos(x\pm y) = \cos x\cos y \mp \sin x\sin y$$
$$\tan(x\pm y) = \frac{\tan x \pm \tan y}{1\mp\tan x\tan y}, \quad \cos^2 x + \sin^2 x = 1$$
$$\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$
$$\sin(2x) = 2\sin x\cos x, \quad \tan(2x) = \frac{2\tan x}{1-\tan^2 x}$$
$$\sin\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1-\cos x}{2}}, \quad \cos\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1+\cos x}{2}}$$
$$\tan\left(\frac{x}{2}\right) = \frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}$$
$$\sin^2 x = \frac{1-\cos(2x)}{2}, \quad \cos^2 x = \frac{1+\cos(2x)}{2}$$
$$\tan^2 x = \frac{1-\cos(2x)}{1+\cos(2x)}$$
$$\sin x\sin y = \frac{1}{2}\left[\cos(x-y) - \cos(x+y)\right]$$
$$\cos x\cos y = \frac{1}{2}\left[\cos(x-y) + \cos(x+y)\right]$$
$$\sin x\cos y = \frac{1}{2}\left[\sin(x+y) + \sin(x-y)\right]$$
$$\tan x\tan y = \frac{\tan x + \tan y}{\cot x + \cot y}, \quad \tan x\cot y = \frac{\tan x + \cot y}{\cot x + \tan y}$$
$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$
$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$
$$\tan x \pm \tan y = \frac{\sin(x\pm y)}{\cos x\cos y}$$
$$a\cos\theta \pm b\sin\theta = R\cos(\theta\mp\alpha)$$
$$a\sin\theta \pm b\cos\theta = R\sin(\theta\pm\alpha)$$
$$\alpha = \arctan\left(\frac{b}{a}\right), \quad R = \sqrt{a^2+b^2}$$

Hyperbolic Functions

$$\cosh t = \frac{e^t + e^{-t}}{2}, \quad \sinh t = \frac{e^t - e^{-t}}{2}, \quad \tanh t = \frac{\sinh t}{\cosh t}$$
$$\cosh^2 t - \sinh^2 t = 1, \quad (\sinh x)' = \cosh x$$
$$(\cosh x)' = \sinh x, \quad (\tanh x)' = \operatorname{sech}^2 x, \quad (\sinh^{-1} x)' = \frac{1}{\sqrt{1+x^2}}$$
$$(\cosh^{-1} x)' = \frac{1}{\sqrt{x^2-1}}, \quad (\tanh^{-1} x)' = \frac{1}{1-x^2}$$

Derivative Rules

$$\frac{d}{dx}(x^n) = nx^{n-1}, \quad \frac{d}{dx}(\ln x) = \frac{1}{x}, \quad \frac{d}{dx}(a^x) = a^x\ln a$$
$$\frac{d}{dx}(e^x) = e^x, \quad \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}, \quad v' = \frac{d}{dx}(y^{-1}) = -y^{-2}\cdot y'$$
$$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}, \quad \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$
$$\frac{d}{dx}(\sin x) = \cos x, \quad \frac{d}{dx}(\cos x) = -\sin x$$
$$\frac{d}{dx}(\tan x) = \sec^2 x, \quad \frac{d}{dx}(\cot x) = -\csc^2 x$$
$$\frac{d}{dx}(\sec x) = \sec x\tan x, \quad \frac{d}{dx}(\csc x) = -\csc x\cot x$$
$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, \quad \frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{x^2+1}, \quad \frac{d}{dx}(\sin^2 x) = 2\sin x\cos x = \sin(2x)$$
$$\frac{d}{dx}(\cos^2 x) = 2\cos x(-\sin x) = -\sin(2x)$$
$$\frac{d}{dx}(\tan^2 x) = 2\tan x\sec^2 x, \quad \frac{d}{dx}(\sin^3 x) = 3\sin^2 x\cdot\cos x$$
$$\frac{d}{dx}(\cot^2 x) = 2\cot x(-\csc^2 x) = -2\cot x\csc^2 x$$
$$\frac{d}{dx}(\sec^2 x) = 2\sec x\sec x\tan x = 2\sec^2 x\tan x$$
$$\frac{d}{dx}(\csc^2 x) = 2\csc x(-\csc x\cot x) = -2\csc^2 x\cot x$$
$$\frac{d}{dx}(\cos^4 x) = 4\cos^3 x\cdot(-\sin x) = -4\cos^3 x\sin x$$

Integral Formulas

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad \int 1\,dx = x + C, \quad \int x\ln x\,dx = \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$$
$$\int \sin(kx)\,dx = -\frac{\cos(kx)}{k} + C, \quad \int \cos(kx)\,dx = \frac{\sin(kx)}{k} + C$$
$$\int \sec^2(x)\,dx = \tan(x) + C, \quad \int \csc^2(x)\,dx = -\cot(x) + C$$
$$\int \sec(x)\tan(x)\,dx = \sec(x) + C, \quad \int \csc(x)\cot(x)\,dx = -\csc(x) + C$$
$$\int \frac{1}{x}\,dx = \ln|x| + C, \quad \int a^x\,dx = \frac{a^x}{\ln a} + C, \quad \int e^x\,dx = e^x + C$$
$$\int \sin^2(x)\,dx = \int \frac{1-\cos(2x)}{2}\,dx = -\frac{\sin(2x)-2x}{4} + C$$
$$\int \cos^2(x)\,dx = \int \frac{1+\cos(2x)}{2}\,dx = \frac{x - \frac{\sin(2x)}{2}}{2} + C$$

Standard Integrals

$$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{(n+1)a} + C, \quad \int \ln x\,dx = x\ln x - x$$
$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C, \quad \int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C$$
$$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + C,$$
$$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C,$$
$$\int \tan(ax+b)\,dx = \frac{1}{a}\ln|\sec(ax+b)| + C,$$
$$\int \sec(ax+b)\,dx = \frac{1}{a}\ln|\sec(ax+b) + \tan(ax+b)| + C,$$
$$\int \csc(ax+b)\,dx = -\frac{1}{a}\ln|\csc(ax+b) + \cot(ax+b)| + C,$$
$$\int \cot(ax+b)\,dx = -\frac{1}{a}\ln|\csc(ax+b)| + C,$$
$$\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + C,$$
$$\int \csc^2(ax+b)\,dx = -\frac{1}{a}\cot(ax+b) + C,$$
$$\int \sec(ax+b)\tan(ax+b)\,dx = \frac{1}{a}\sec(ax+b) + C,$$
$$\int \csc(ax+b)\cot(ax+b)\,dx = -\frac{1}{a}\csc(ax+b) + C$$