The Harmonic Oscillator

Non-Homogeneous Linear Differential Equations

Consider the non-homogeneous differential equation:
$(y'' + p(x)y' + q(x)y = f(x))$, where $f(x) \neq 0$.

The general solution, $y(x)$, is given by:
$(y(x) = y_h(x) + y_p(x))$,
where:

- $y_h(x)$ is the general solution to the **complementary homogeneous equation** $(y'' + p(x)y' + q(x)y = 0)$.
- $y_p(x)$ is any **particular solution** to the non-homogeneous equation.

Methods for Finding $y_p(x)$:

1. **Undetermined Coefficients**:
   - If $f(x)$ is a polynomial, try y_p as a polynomial of the same degree.
   - For $f(x)$ involving an exponential, try $y_p = Ae^{rx}$.
   - For $f(x)$ with trigonometric functions like $\cos(rx)$ or $\sin(rx)$, try $y_p = A\cos(rx) + B\sin(rx)$.
2. **Modification**: If a trial term for $y_p(x)$ is already in $y_h(x)$, multiply by x to avoid redundancy.
3. **Variation of Parameters**:
   - Given $y_h(x) = C_1 y_1(x) + C_2 y_2(x)$, try $y_p(x) = u(x)y_1(x) + v(x)y_2(x)$.
   - Functions $u(x)$ and $v(x)$ are determined by:

$$u(x) = -\int \frac{y_2 f(x)}{W(y_1, y_2)} dx, \quad v(x) = \int \frac{y_1 f(x)}{W(y_1, y_2)} dx,$$

   where $W(y_1, y_2)$ is the **Wronskian**.

Example 17: Solution to $y'' + 4y = 24e^{2x}$

1. **Characteristic equation** for the complementary equation is $(r^2 + 4 = 0)$ with roots $r = \pm 2i$.
2. Solution for the complementary equation:
   $(y_h = C_1 \cos(2x) + C_2 \sin(2x))$.
3. **Particular Solution** guess: $y_p = Ae^{2x}$.
4. Substitute into the equation and solve for A, yielding $A = 1/4$.

General Solution:
$(y = C_1 \cos(2x) + C_2 \sin(2x) + \frac{1}{4}e^{2x})$.

Simple Harmonic Motion

Simple harmonic motion occurs when acceleration is proportional to displacement:
$(x'' + \omega^2 x = 0)$.

- The **angular frequency** ω determines the oscillation rate.

Solution:

1. The characteristic equation $(r^2 + \omega^2 = 0)$ has roots $\pm i\omega$.
2. General solution: $(x(t) = A\cos(\omega t) + B\sin(\omega t))$.

Alternatively:
$(x(t) = R\cos(\omega t - \phi))$, where $R = \sqrt{A^2 + B^2}$ and ϕ is the phase angle.

Damped Oscillations

For a **damped oscillator** with resistance proportional to velocity:
$(x'' + 2\gamma x' + \omega^2 x = 0)$.

- **Overdamped**: $(\gamma^2 > \omega^2)$ – solution decays without oscillating.
- **Critically damped**: $(\gamma^2 = \omega^2)$ – fastest decay to zero without oscillating.
- **Underdamped**: $(\gamma^2 < \omega^2)$ – oscillatory decay.

Example 23: Simple Harmonic Oscillator

For $(x'' + \omega^2 x = 0)$:

1. Characteristic equation $(r^2 + \omega^2 = 0)$ with roots $r = \pm i\omega$.
2. Solution:
   $(x(t) = A\cos(\omega t) + B\sin(\omega t))$ or $(x(t) = R\cos(\omega t - \phi))$.

Driven Harmonic Oscillator with Resonance

When an external force drives the oscillator:
$(x'' + \omega^2 x = F_0 \cos(\omega t))$.

If the driving frequency matches the natural frequency (ω), **resonance** occurs, leading to:

$$x(t) = R\cos(\omega t - \phi) + \frac{F_0}{2\omega} t \sin(\omega t).$$

Result: Amplitude increases over time due to resonance.
