Introduction to Partial Differential Equations (PDEs)

Definition

- A **Partial Differential Equation (PDE)** involves partial derivatives of an unknown function dependent on two or more variables.

Key Concepts

1. **Order**:
   - Determined by the highest-order derivative in the equation.
   - Example: $u_{xx} + u_{yy} = 0$ is a second-order PDE.
2. **Linearity**:
   - A PDE is linear if the unknown function and its derivatives appear to the first degree.
   - Example: $u_{xx} + u_{yy} = 0$ is linear; $u_{xx}u_{yy} = 0$ is nonlinear.
3. **Homogeneity**:
   - A PDE is homogeneous if every term contains the unknown function or its derivatives.
   - Example: $u_{xx} + u_{yy} = 0$ (homogeneous), $u_{xx} + u_{yy} = f(x, y)$ (nonhomogeneous).

Examples of PDEs

1. **Heat Equation**: $u_t = c^2 u_{xx}$
   - Describes the dispersion of heat in a one-dimensional rod.
2. **Wave Equation**: $u_{tt} = c^2 u_{xx}$
   - Models vibrations of strings or sound waves.
3. **Laplace Equation**: $u_{xx} + u_{yy} = 0$
   - Governs potential fields in electrostatics or fluid flow.

Methods of Solving PDEs

Separation of Variables

1. **Assume** a solution: $u(x, t) = X(x)T(t)$.
2. Substitute into the PDE and separate variables:
   - $\frac{T'(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda$ (constant).
3. Solve the resulting ODEs for $X(x)$ and $T(t)$.
4. Combine solutions to find $u(x, t)$.

Example: Heat Equation

- Solve $u_t = c^2 u_{xx}$, $u(0, t) = u(L, t) = 0$, $u(x, 0) = f(x)$:
  1. Separate variables: $u(x, t) = X(x)T(t)$.
  2. $X'' + \lambda X = 0$, $T' + c^2 \lambda T = 0$.
  3. Solutions: $X(x) = \sin\left(\frac{n\pi x}{L}\right)$, $T(t) = e^{-n^2\pi^2c^2t/L^2}$.
  4. Combine: $u(x, t) = \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2c^2t/L^2} \sin\left(\frac{n\pi x}{L}\right)$.

The Superposition Principle

- If $u_1(x, t)$ and $u_2(x, t)$ solve a homogeneous linear PDE, then:$$
  u(x, t) = c_1u_1(x, t) + c_2u_2(x, t), \quad c_1, c_2 \in \mathbb{R}

$$is also a solution.$$

Example: Laplace Equation

- PDE: $u_{xx} + u_{yy} = 0$.
- Solutions: $u_1(x, y) = x^2 - y^2$, $u_2(x, y) = x \cos y$.
- General solution: $u(x, y) = c_1(x^2 - y^2) + c_2 x \cos y$.

Applications of PDEs

1. **Heat Transfer**:
   - **PDE**: $u_t = c^2 u_{xx}$.
   - **Boundary/Initial Conditions**: $u(0, t) = u(L, t) = 0$, $u(x, 0) = f(x)$.
2. **Wave Propagation**:
   - **PDE**: $u_{tt} = c^2 u_{xx}$.
   - Models vibrations and oscillations.
3. **Fluid Flow**:
   - **PDE**: $u_{xx} + u_{yy} = 0$.
   - Governs steady-state flows and electric potentials.