MA1301 Introductory Mathematics
Semester 1: AY 2017/2018

Q1(a)(i).

$$\int \frac{1}{x(x+2)}\,dx = \int \frac{1}{2x} - \frac{1}{2(x+2)}\,dx = \frac{1}{2}\ln x - \frac{1}{2}\ln(x+2) + C$$

Q1(a)(ii).

$$\begin{aligned}
\int \frac{1}{e^x+2}\,dx &= \int \frac{1}{u+2}\left(\frac{1}{u}\,du\right) && u = e^x \Rightarrow \frac{du}{dx} = e^x \\
&= \int \frac{1}{u(u+2)}\,du \\
&= \frac{1}{2}\ln u - \frac{1}{2}\ln(u+2) + C && \text{(by part (i))} \\
&= \frac{1}{2}\ln e^x - \frac{1}{2}\ln(e^x+2) + C
\end{aligned}$$

Q1(b)(i).

$$\begin{aligned}
\frac{dy}{dx} &= 4 + \frac{k}{x^2} \\
0 &= 4 + \frac{k}{\left(\frac{1}{2}\right)^2} \Rightarrow k = -1
\end{aligned}$$

Q1(b)(ii).

$$\begin{aligned}
&y = \int 4 - \frac{1}{x^2}\,dx = 4x + \frac{1}{x} + C \\
&4 = 4\left(\frac{1}{2}\right) + \frac{1}{\frac{1}{2}} + C \Rightarrow C = 0 \\
&y = 4x + \frac{1}{x}
\end{aligned}$$

Q2(a).

$$
\begin{aligned}
y = v + 2x \Rightarrow \frac{dy}{dx} &= \frac{dv}{dx} + 2 \\
2 + \frac{1}{(2x-y)^2} &= \frac{dv}{dx} + 2 \\
\frac{1}{v^2} &= \frac{dv}{dx} \\
v^2 \frac{dv}{dx} &= 1 \\
\int v^2\, dv &= \int dx \\
\frac{1}{3} v^3 &= x + C \\
\frac{1}{3}(y-2x)^3 &= x + C \\
\frac{1}{3}(0-2(0))^3 &= 0 + C \Rightarrow C = 0 \\
\therefore \frac{1}{3}(y-2x)^3 = x \Rightarrow 3x = (y-2x)^3 \quad & \cancel{\therefore y = 5x}
\end{aligned}
$$

(This is the final answer you may give)

Q2(b).

$$
\begin{aligned}
(2y-1)\frac{dy}{dx} - 2e^y &= 0 \\
(2y-1)e^{-y}\frac{dy}{dx} &= 2 \\
\int (2y-1)e^{-y}\, dy &= \int 2\, dx \\
\int 2y - 1 d(-e^{-y}) &= 2x \\
(2y-1)(-e^{-y}) - \int -e^{-y} d(2y-1) &= 2x \\
(1-2y)e^{-y} + 2\int e^{-y} dy &= 2x \\
(1-2y)e^{-y} - 2e^{-y} &= 2x + C \\
(1-2(0))e^0 - 2e^0 &= 2(2) + C \Rightarrow C = -5 \\
\therefore (1-2y)e^{-y} - 2e^{-y} &= 2x - 5
\end{aligned}
$$

Q3(a)(i).

$$\sqrt{2}|z| = |z||1+i| = |z(1+i)| = \sqrt{32} \Rightarrow |z| = 4$$
$$\frac{\pi}{4} - \arg z = \arg(1-i) - \arg z = \arg\left(\frac{1-i}{z}\right) = \frac{\pi}{6} \Rightarrow \arg z = -\frac{\pi}{12}$$
$$z = 4\left(\cos\left(-\frac{\pi}{12}\right) + i\sin\left(-\frac{\pi}{12}\right)\right)$$

Q3(a)(ii).

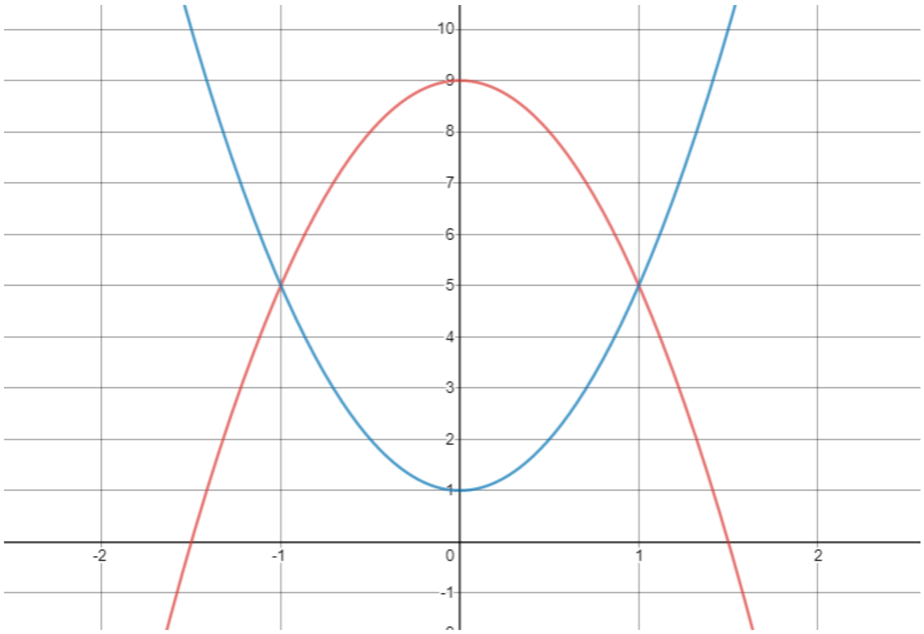
$$\begin{aligned} z^N \in \mathbb{R} \Rightarrow \sin\left(-\frac{\pi N}{12}\right) &= 0 \\ -\frac{\pi N}{12} &= 2k\pi \quad , k \in \mathbb{Z} \\ N &= -24k \\ \min_{N>0} N &= 24 \end{aligned}$$

Q3(b).

$$3z + w = 15 + 6i \tag{1}$$
$$6z + w = \frac{150}{1-7i} \tag{2}$$

$$\begin{aligned} 2\times(1)-(2): \quad w &= 2(15+6i) - \frac{150}{1-7i} \\ &= 30 + 12i - \frac{150(1+7i)}{1+49} \\ &= 30 + 12i - 3(1+7i) \\ &= 27 - 9i \\ \text{Sub } w = 27-9i \text{ into } (1): \quad 3z + (27-9i) &= 15 + 6i \\ z &= -4 + 5i \end{aligned}$$

Q4(i).



Q4(ii).

$$\begin{aligned}
\text{Area of } R &= 2\int_0^1 (9-4x^2)-(4x^2+1)\,dx \\
&= 2\int_0^1 -8x^2+8\,dx \\
&= 16\left[-\frac{1}{3}x^3+x\right]_0^1 \\
&= 16\left[-\frac{1}{3}+1\right] \\
&= \frac{32}{3}
\end{aligned}$$

Q4(iii).

$$\begin{aligned}\text{Volume of } R &= 2\pi\int_0^1 (9-4x^2)^2-(4x^2+1)^2\,dx\\ &= 2\pi\int_0^1 81-72x^2+16x^4-16x^4-8x^2-1\,dx\\ &= 2\pi\int_0^1 80-80x^2\,dx\\ &= 160\pi\left[x-\frac{1}{3}x^3\right]_0^1\\ &= 160\pi\left[1-\frac{1}{3}\right]\\ &= \frac{320}{3}\pi\end{aligned}$$

Q4(iv).

$$\begin{aligned}\text{Volume of } R &= \pi\left(\int_1^5 \frac{y-1}{4}\,dy+\int_5^9\frac{9-y}{4}\,dy\right)\\ &= \pi\left(\left[\frac{1}{8}(y-1)^2\right]_1^5+\left[-\frac{1}{8}(9-y)^2\right]_5^9\right)\\ &= \frac{\pi}{8}\left(16-(-16)\right)\\ &= 4\pi\end{aligned}$$

Q5(i).

Please note the notation $(a, b, c)^T$ is same as $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$\begin{aligned}(1+2\lambda, 1+\lambda, 1+\lambda)^T &= (4+\mu, 1, 10+3\mu)^T\\ 1+2\lambda &= 4+\mu\\ 1+\lambda &= 1\\ 1+\lambda &= 10+3\mu\\ \text{Solving, } \lambda &= 0, \mu=-3\\ \therefore L_1 &\text{ intersects } L_2\\ \text{point on intersection } &= (1.1.1)\end{aligned}$$

same as

$$\begin{pmatrix} 1+2\lambda \\ 1+\lambda \\ 1+\lambda \end{pmatrix} = \begin{pmatrix} 4+\mu \\ 1 \\ 10+3\mu \end{pmatrix}$$

Q5(ii).

$$\theta = \cos^{-1}\left|\frac{(2,1,1)^T\cdot(1,0,3)^T}{|(2,1,1)^T||(1,0,3)^T|}\right| = \cos^{-1}\frac{5}{\sqrt{6}\sqrt{10}} = 0.869$$

Q5(iii).

$$
\begin{aligned}
(1+2\lambda, 1+\lambda, 1+\lambda)^T &= (3,3,7)^T \\
1+2\lambda &= 3 \\
1+\lambda &= 3 \\
1+\lambda &= 7 \\
\text{Solving, no } \cancel{\text{slution}}\ \text{solution found} \\
\therefore \text{ point } A \text{ does not lie on } L_1
\end{aligned}
$$

Let point F be the foot of perpendicular from A to L_1.

$$
\begin{aligned}
\overrightarrow{AF} &= (1+2\lambda, 1+\lambda, 1+\lambda)^T - (3,3,7)^T = (2\lambda-2, \lambda-2, \lambda-6)^T \\
\overrightarrow{AF} \perp \mathbf{d}_1 &\Rightarrow (2\lambda-2, \lambda-2, \lambda-6)^T \cdot (2,1,1)^T = 0 \\
4\lambda - 4 + \lambda - 2 + \lambda - 6 &= 0 \\
\lambda &= 2 \\
F &= (5,3,3)
\end{aligned}
$$

Q6.

$$
\begin{aligned}
288\pi = V = \pi r^2 h \Rightarrow h = \frac{288}{r^2} \\
C = 60(2\pi r h) + 40(2(\pi r^2)) = 120\pi r\left(\frac{288}{r^2}\right) + 80\pi r^2 = \frac{34560\pi}{r} + 80\pi r^2 \\
C' = -\frac{34560\pi}{r^2} + 160\pi r = 0 \\
r = 6 \\
C'' = \frac{69120\pi}{r^3} + 160\pi > 0 \Rightarrow \min \text{ at } r = 6 \\
\min C = 8640\pi
\end{aligned}
$$