Differential Equation

Introduction to Differential Equations

A **differential equation** involves an unknown function and its Derivate.

In the case of an **ordinary differential equation** (ODE), the equation involves:

- An independent variable (denoted t or x)
- A function (denoted $y(t)$ or $y(x)$)
- One or more of its derivatives

Key Points:

1. **Order** of a differential equation: The highest derivative in the equation determines the order.
2. **General Solution**: A family of solutions involving arbitrary constants.
3. **Particular Solution**: A specific solution obtained by applying initial conditions.
4. **Separable Differential Equation**: A first-order equation that can be separated into the form $f(x)dx = g(y)dy$ for integration.

Radioactive Decay Example

A classic example of a first-order differential equation is modeling **radioactive decay**, where the rate of decay is proportional to the substance present.

1. Let $G(t)$ be the amount of radioactive substance present at time t.
2. The equation is:

$$\frac{dG}{dt} = -kG$$

   where $k > 0$ is the decay constant.
3. **Solving the equation**:

$$\frac{1}{G} \cdot \frac{dG}{dt} = -k$$

   Integrating both sides:

$$\ln|G| = -kt + C$$

   Exponentiating:

$$G(t) = e^C \cdot e^{-kt}$$

   The general solution is:

$$G(t) = A \cdot e^{-kt}, \quad A = e^C$$

   A represents the initial amount of the substance.
4. **Determining the Decay Constant**:
   If the half-life of the substance is known (denoted t_{1/2}), we use:

$$G(t_{1/2}) = \frac{G_0}{2} = G_0 \cdot e^{-kt_{1/2}} \quad \Rightarrow \quad k = \frac{\ln 2}{t_{1/2}}$$

Another Example

Example: Find the solution x(t) for the equation $\frac{dx}{dt} = -kx$

$$\begin{aligned}
x(t) &= 0 \\
\frac{dx}{dt} &= -kx \\
\frac{1}{x} \cdot \frac{dx}{dt} &= -k \\
\int \frac{1}{x}\, dx &= \int -k\, dt \\
\ln|x| &= -kt + C_1 \\
x(t) &= e^{-kt+C_1} \\
x(t) &= e^{C_1} \cdot e^{-kt} \\
x(t) &= A \cdot e^{-kt} \quad \text{where } A = e^{C_1},\ A \in \mathbb{R}
\end{aligned}$$

This is the **General Solution**, where a **Particular Solution** would need us to determine the value of A.

Separation of Variables

Consider the equation:

$$\frac{dy}{dx} = e^x(1 + y^2)$$

This is separable. By rearranging and integrating both sides:

$$\frac{1}{1 + y^2}\, dy = e^x\, dx$$

Integrating both sides:

$$\tan^{-1}(y) = e^x + C$$

Thus, the solution is:

$$y = \tan(e^x + C)$$

Direction Fields and Equilibrium Points

Direction Field:

A **direction field** or **slope field** visualizes how the solution to a differential equation behaves at various points. It is constructed by plotting short line segments whose slopes are determined by the differential equation.

Equilibrium Solution:

An **equilibrium solution** is a constant solution where the derivative is zero. Stability of equilibrium points:

- **Stable**: Solutions approach the equilibrium point as t \to \infty.
- **Unstable**: Solutions move away from the equilibrium point.

Example: Solving a Nonlinear Equation

Consider the equation:

$$\frac{dy}{dx} = \frac{1 - 2y - 4x}{1 + y + 2x}$$

Let $u = y + 2x$, and substituting simplifies the equation to:

$$\frac{du}{dx} = \frac{1 - 2u}{1 + u}$$

This can now be solved by separation of variables.