2223 S1 MA1511 Solutions

Q1 $f(x,y,z) = (z+x^3)\sin(xy+z) + ye^{zx^2}$

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \left[\frac{\partial}{\partial x}(z+x^3)\right]\sin(xy+z) + (z+x^3)\frac{\partial}{\partial x}\sin(xy+z) + y\frac{\partial}{\partial x}e^{zx^2} \\
&= (3x^2)\sin(xy+z) + (z+x^3)\cos(xy+z)\frac{\partial}{\partial x}(xy+z) + ye^{zx^2}\frac{\partial}{\partial x}(zx^2) \\
&= (3x^2)\sin(xy+z) + (z+x^3)\cos(xy+z)y + ye^{zx^2}(2zx)
\end{aligned}$$

Q2 Let $f(x,y,z) = x^2+4y^2+16z^2$ and $g(x,y,z) = xyz-1$

Local extreme ocurs at (x,y,z) if there exists λ such that

$$\begin{pmatrix} f_x(x,y,z) \\ f_y(x,y,z) \\ f_z(x,y,z) \end{pmatrix} = \lambda \begin{pmatrix} g_x(x,y,z) \\ g_y(x,y,z) \\ g_z(x,y,z) \end{pmatrix}$$

$f_x = 2x,\ f_y = 8y,\ f_z = 32z$

$g_x = yz,\ g_y = xz,\ g_z = xy$

$$f_x = \lambda g_x \Longrightarrow 2x = \lambda yz \tag{1}$$
$$f_y = \lambda g_y \Longrightarrow 8y = \lambda xz \tag{2}$$
$$f_z = \lambda g_z \Longrightarrow 32z = \lambda xy \tag{3}$$

From $xzy = 1$, we know $x \neq 0,\ y \neq 0,\ z \neq 0$.

From (1),

$$\lambda = \frac{2x}{yz} \tag{4}$$

From (2),

$$\lambda = \frac{8y}{xz} \tag{5}$$

From (3),

$$\lambda = \frac{32z}{xy} \tag{6}$$

From (4), (5), $\dfrac{2x}{yz} = \dfrac{8y}{xz} \qquad \therefore \quad x^2 = 4y^2$

From (5), (6), $\dfrac{8y}{xz} = \dfrac{32z}{xy} \qquad \therefore \quad y^2 = 4z^2$

$\therefore \quad z^2 = \frac{1}{4}y^2 = \frac{1}{4}\frac{1}{4}x^2$

$xyz = 1 \Longrightarrow \pm(x)(\frac{1}{2}x)(\frac{1}{4}x) = 1 \Longrightarrow x^3 = \pm(2)(4) \Longrightarrow x = \pm 2$

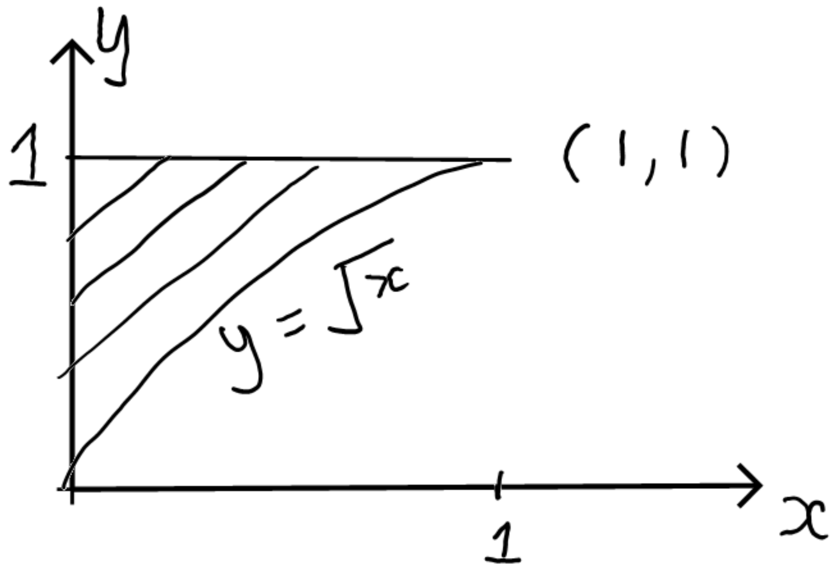
$x^2 = 4y^2 \Longrightarrow y = (\frac{1}{2})(\pm 2) = \pm 1$

$y^2 = 4z^2 \Longrightarrow z = (\frac{1}{2})(\pm 1) = \pm\frac{1}{2}$

local extreme at $x = \pm 2$, $y = \pm 1$, $z = \pm\frac{1}{2}$ is $x^2 + 4y^2 + 16z^2 = 4 + 4(1) + 16(\frac{1}{4}) = 8 + 4 = 12$

Ans : 12

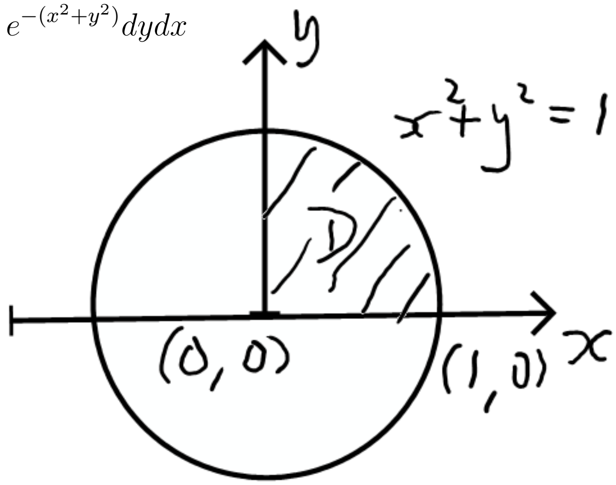
Q3



$$
\begin{aligned}
\int_0^1 \left[\int_{\sqrt{x}}^1 \sqrt{y^3+1}dy\right] dx &= \int_0^1 \left[\int_0^{y^2} \sqrt{y^3+1}dx\right] dy \\
&= \int_0^1 \left(\sqrt{y^3+1}\right) \left[\int_0^{y^2} dx\right] dy \\
&= \int_0^1 \left(\sqrt{y^3+1}\right) \left(y^2 - 0\right) dy \\
&= \int_0^1 y^2\sqrt{y^3+1}dy \\
&= \frac{1}{3}\int_0^1 \sqrt{y^3+1}d(y^3+1) \qquad \text{Let } u = y^3+1 \\
&= \frac{1}{3}\left[\frac{(y^3+1)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_0^1 \\
&= \frac{1}{3}\left[\frac{2^{\frac{3}{2}}}{\frac{3}{2}} - \frac{1}{\frac{3}{2}}\right] \\
&= \frac{1}{3}\frac{2}{3}\left[2^{\frac{3}{2}} - 1\right] \\
&= \frac{2}{9}\left[\sqrt{8} - 1\right] \\
&= \frac{2}{9}\left[2\sqrt{2} - 1\right]
\end{aligned}
$$

Ans : $\frac{2}{9}\left[\sqrt{8} - 1\right] = \frac{2}{9}\left[2\sqrt{2} - 1\right]$

Q4 $\int_0^1 \int_0^{\sqrt{1-x^2}} e^{-(x^2+y^2)} dydx$



Let $x = r\cos\theta$, $y = r\sin\theta$

$$\begin{aligned}
\int_0^1 \int_0^{\sqrt{1-x^2}} e^{-(x^2+y^2)} dydx &= \int_0^{\frac{\pi}{2}} \int_0^1 e^{-r^2} r dr d\theta \\
&= \int_0^{\frac{\pi}{2}} \left[(-\frac{1}{2}) \int_0^1 e^{-r^2} d(-r^2) \right] d\theta \quad \text{Let } u = -r^2 \\
&= \int_0^{\frac{\pi}{2}} (-\frac{1}{2}) \left[e^{-r^2} \right]_0^1 d\theta \\
&= \int_0^{\frac{\pi}{2}} (-\frac{1}{2}) \left[e^{-1} - 1 \right] d\theta \\
&= \frac{1}{2}(1 - \frac{1}{e}) \int_0^{\frac{\pi}{2}} d\theta \\
&= \frac{1}{2}(1 - \frac{1}{e}) \frac{\pi}{2} \\
&= \frac{\pi}{4}(1 - \frac{1}{e})
\end{aligned}$$

Ans : $\frac{\pi}{4}(1 - \frac{1}{e})$

Q5 A parametric equation of the curve of intersection is $r(t) = 2\cos t\ i + 2\sin t\ j + 2\sin t\ k$

$$\begin{aligned}
\int_C \frac{z}{\sqrt{2x^2+y^2}} ds &= \int_0^{2\pi} \frac{2\sin t}{\sqrt{2(4)\cos^2 t + 4\sin^2 t}} \sqrt{(-2\sin t)^2 + (2\cos t)^2 + (2\cos t)^2} dt \\
&= \int_0^{2\pi} \frac{2\sin t}{\sqrt{4 + 4\cos^2 t}} \sqrt{4 + 4\cos^2 t} dt \\
&= \int_0^{2\pi} 2\sin t dt = 2\left[-\cos t \right]_0^{2\pi} = -2\left[\cos t \right]_0^{2\pi} = -2[\cos 2\pi - \cos 0] = 0
\end{aligned}$$

Ans : 0

Q6

$$\frac{\partial f}{\partial x} = e^x + 2xy \tag{1}$$

$$\frac{\partial f}{\partial y} = (x^2 + \cos y) \tag{2}$$

From (1), $f(x,y) = \displaystyle\int \frac{\partial f}{\partial x} dx = \int (e^x + 2xy)dx = e^x + x^2 y + g(y)$

$\therefore \quad \frac{\partial f}{\partial y} = 0 + x^2 + g'(y)$

On the other hand $\frac{\partial f}{\partial y} = x^2 + \cos y$

$\therefore \quad 0 + x^2 + g'(y) = x^2 + \cos y \Longrightarrow g'(y) = \cos y$

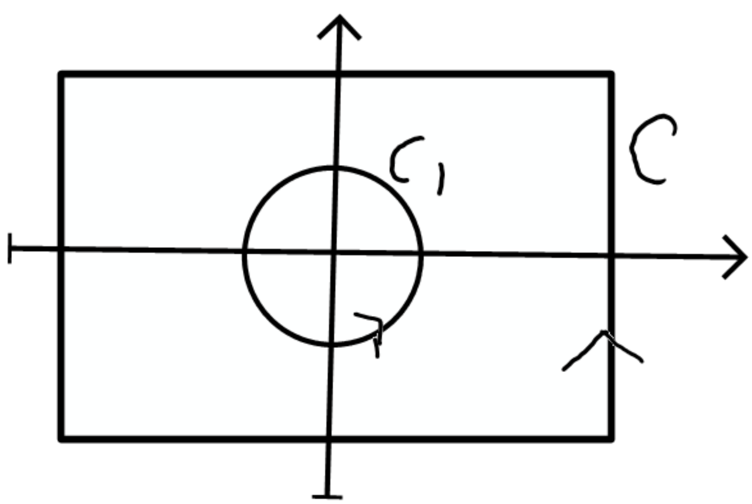
$g(y) = \int g'(y)dy = \int \cos y dy = \sin y + c$

$\therefore \quad f(x,y) = e^x + x^2 y + \sin y + c$

may choose $c = 0$

Ans : $f(x,y) = e^x + x^2 y + \sin y$

Q7



Let D be the region bounded by C and C_1. ∂D = boundary of $D = C - C_1$ taken in positive orientation.

$$P(x,y) = \frac{-y}{x^2+y^2} \qquad Q(x,y) = \frac{x}{x^2+y^2}.$$

P and Q have continuous partial derivatives in D. Note : $(0,0) \notin D$.

We can apply Green's Theorem to

$$\int_{\partial D} \frac{x}{x^2+y^2} dy + \frac{(-y)}{x^2+y^2} dx = \iint_D \left[\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2} \right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2} \right) \right] dxdy$$

$$\frac{\partial}{\partial y} \frac{(-y)}{x^2+y^2} = \frac{\left(\frac{\partial(-y)}{\partial y}\right)(x^2+y^2) - (-y)\frac{\partial}{\partial y}(x^2+y^2)}{(x^2+y^2)^2} = \frac{(-1)(x^2+y^2) - (-y)(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{x^2+y^2}$$

$$\frac{\partial}{\partial x}\frac{x}{x^2+y^2} = \frac{\left(\frac{\partial x}{\partial x}\right)(x^2+y^2) - x\frac{\partial}{\partial x}(x^2+y^2)}{(x^2+y^2)^2} = \frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{x^2+y^2}$$

$$\therefore \quad \int_{\partial D} \frac{x}{x^2+y^2}dy - \frac{y}{x^2+y^2}dx = 0$$

$$\therefore \quad \int_{C-C_1} \frac{x}{x^2+y^2}dy - \frac{y}{x^2+y^2}dx = 0$$

$$\int_C \frac{x}{x^2+y^2}dy - \frac{y}{x^2+y^2}dx = \int_{C_1} \frac{x}{x^2+y^2}dy - \frac{y}{x^2+y^2}dx = 2\pi$$

Ans : 2π

Q8 Parametric equation of the vertical line segment from $(3,4,5)$ to $(3,4,0)$.

$$\begin{aligned} r(t) &= \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}(1-t) + \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} t \quad ,0 \le t \le 1 \\ &= \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} + \begin{pmatrix} 3t-3t \\ 4t-4t \\ -5t \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -5t \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ 4 \\ 5-5t \end{pmatrix} \end{aligned}$$

$x(t) = 3,\ y(t) = 4,\ z(t) = 5-5t$

$$\int_0^1 4dx + (5-5t)dy + 3dz = \int_0^1 [4(0) + (5-5t)(0) + 3(-5)]dt = \int_0^1 (-15)dt = -15$$

Ans : -15

Q9 (a) use formula $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$.

$$\lim_{n\to\infty}\left(1+\frac{2}{n+1}\right)^{n+1}\left(\frac{2n+1}{n+2}\right) = \lim_{n\to\infty}\left(1+\frac{2}{n+1}\right)^{n+1}\lim_{n\to\infty}\left(\frac{2n+1}{n+2}\right) = (e^2)(2)$$

Ans : $2e^2$

(b)

$$\begin{aligned}
g(x) &= \ln\left[\left(\frac{1+2x}{1-2x}\right)^2\right] \quad , \qquad\qquad -1 < 2x < 1 \\
&= 2[\ln(1+2x) - \ln(1-2x)] \\
&= 2\left[\left((2x) - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} - \cdots\right) - \left((-2x) - \frac{(-2x)^2}{2} + \frac{(-2x)^3}{3} - \cdots\right)\right] \\
&= 2\left[2\left((2x) + \frac{(2x)^3}{3} + \frac{(2x)^5}{5} + \cdots\right)\right] \\
&= 4(2x) + \frac{4}{3}8x^3 + \cdots
\end{aligned}$$

$$\frac{g^3(0)}{3!} = \frac{(4)(8)}{3}$$

$$g^3(0) = \frac{(4)(8)}{3}(3)(2) = 64$$

Ans : 64

Q10 (a)

$$\frac{\frac{6(n+1)}{2^{n+1}+4^{n+1}}(5x+1)^{2(n+1)-1}}{\frac{6n}{2^n+4^n}(5x+1)^{2n-1}} = \frac{n+1}{n}\frac{2^n+4^n}{2^{n+1}+4^{n+1}}(5x+1)^2 \quad \to \quad (1)(\frac{1}{4})(5x+1)^2 \quad \text{as } n \to \infty$$

Let $(\frac{1}{4})(5x+1)^2 < 1$.

$$(\frac{1}{4})5^2\left(x+\frac{1}{5}\right)^2 < 1 \implies \left(x+\frac{1}{5}\right)^2 < \frac{4}{5^2} \implies -\frac{2}{5} < x + \frac{1}{5} < \frac{2}{5}$$

$\therefore \quad$ Radius of convergence $= \dfrac{2}{5}$

Ans : $\dfrac{2}{5}$

(b)

$$f(x) = \sum_{n=2}^{\infty} \frac{1}{(n-1)!}x^{2n} = \left(x^2\right)^2 + \frac{1}{2!}\left(x^2\right)^3 + \frac{1}{3!}\left(x^2\right)^4 + \cdots$$

Recall $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$$\begin{aligned}
f(x) &= x^2\left[x^2 + \frac{1}{2!}\left(x^2\right)^2 + \frac{1}{3!}\left(x^2\right)^3 + \cdots\right] \\
&= x^2\left[\left(1 + x^2 + \frac{1}{2!}\left(x^2\right)^2 + \frac{1}{3!}\left(x^2\right)^3 + \cdots\right) - 1\right] \\
&= x^2\left[e^{x^2} - 1\right] \\
f(2) &= 4\left[e^4 - 1\right]
\end{aligned}$$

Ans : $4\left(e^4 - 1\right)$