University of Massachusetts, Amherst, CICS
Instructor: Yair Zick
COMPSCI 590T: Algorithmic Fairness and Strategic Behavior

Assignment 8: Mechanism Design and Rent Division

Issued: Week 12 Due: two weeks after the issue date in Gradescope

Important Instructions:

- Please upload your solution to Gradescope by 23:59 on the due date.
- You may work with a partner on your homework; however, if you choose to do so, **you must** write down the name of your partner on the assignment.
- Provide a mathematical proof/computation steps to all questions, unless explicitly told not to. Solutions with no proof/explanations (even correct ones!) will be awarded no points.
- You may look up combinatorial inequalities and mathematical formulas online, but please do not look up the solutions. Many problems can be found in research papers/online; if you happen upon the solution before figuring it out yourself, mention this in the submission. **Unreferenced copies of online material will be considered plagiarism.**

Rent Division

An instance of the rent division problem is given by valuations of n players for n rooms, where v_{ij} is the value player i has from room j. We require that $\sum_{j=1}^{n} v_{ij} = r$ for all i. An outcome $\langle \sigma, \vec{p} \rangle$ specifies the room assignment $\sigma : [n] \to [n]$, and the price p_j of room j, where $\sum_{j=1}^{n} = r$. We say that an outcome is envy-free (EF) if $v_{i\sigma(i)} - p_{\sigma(i)} \geq v_{ij} - p_j$ for all j.

1. (15 points) Show that if there is some player $i \in N$ such that $v_{i1} = v_{i2} = \cdots = v_{in}$ (i.e. player i values all rooms equally) then player i pays the least amount of rent in any envy-free outcome. Intuitively - players who are less picky are rewarded for it by paying less rent.
2. (15 points) Prove/disprove: if all players have the same valuation for all the rooms, then there is a unique EF price vector (the actual room allocation may not be unique).
3. (15 points) Prove that if a room j is least valued by all players, i.e. $v_{ij} \leq v_{ik}$ for all k, then the price of j is the lowest of all rooms under any envy-free outcome.

Mechanism Design

1. (25 points) Recall that in a multi-unit auction we are selling k items, with each player i having a value v_{ij} for item j. We say that the items are identical if $v_{ij} = v_{ij'}$ for any two items j, j' (think of bidding for tickets to a nightclub - we don't care which specific ticket we get, as long as we get one). Prove/disprove via counterexample the following claims.
   (a) (15 points) In a multi-unit VCG auction with identical items, it is possible that two successful bidders (i.e. two bidders who actually received an item) pay a different amount of money.
   (b) (10 points) In a multi-unit VCG auction with *different items* and buyers who receive a single good each, it is possible that two bidders who receive items they value the same will pay different prices.
2. (15 points) Consider a case where two bidders in a single item auction *cooperate*; that is, they both submit bids to the mechanism, but they want to maximize their joint utility. Can VCG still extract truthful valuations from the bidders?

3. (15 points) For each of the following auction formats, decide whether the auction is truthful; if it is, prove it. If not, provide a counterexample, i.e. an instance where at least one bidder has an incentive to misreport their true value for the item.

   (a) (7.5 points) The item goes to the highest bidder, and the winner pays the 3rd highest bid.

   (b) (7.5 points) The auctioneer has a reserve price $r \geq 0$. If all bidders submit bids that are less than or equal to r, the auctioneer keeps the item (so, no bidder gets it). Otherwise, the item goes to the highest bidder, and they pay the second highest bid, or the reserve price (whichever is higher). Formally, if the bids are $b_1 > b_2 > \cdots > b_n$, then nobody gets the item if $b_1 \leq r$; otherwise the highest bidder gets the item, and pays $\max\{b_2, r\}$.

4. (bonus points) We have seen in class that the 2nd price auction is incentive compatible, and that the 1st price auction is not. Suppose that the auctioneer runs a 1st price auction; in general, bidders have no good way of figuring out their best response to other bidders' actions, since they have no idea what their valuations are. What if they have an *idea* of what they are? More formally, suppose that each bidder samples their value for an item from the uniform distribution over $[0, 1]$. Prove that in this case, having player i bid $\frac{n-1}{n} v_i$ is a Nash equilibrium. Partial credit will be given for proving this claim for $n = 2$.

   You are actually finding a *Bayes-Nash* equilibrium, a concept similar to a Nash equilibrium, but not quite. In this context, you must prove that when player i has a true valuation of v_i, then their expected utility (over the random valuations of other players' types) is maximized when they bid $\frac{n-1}{n} v_i$, given that all the other players also bid a $\frac{n-1}{n}$ fraction of their true valuations.