University of Massachusetts, Amherst, CICS
Instructor: Yair Zick
COMPSCI 590T: Algorithmic Fairness and Strategic Behavior

Assignment 7: Fair Allocation of Indivisible Goods

Issued: Week 8 Due: two weeks after the issue date in Gradescope

Important Instructions:

- Please upload your solution to Gradescope by 23:59 on the due date.
- You may work with a partner on your homework; however, if you choose to do so, **you must** write down the name of your partner on the assignment.
- Provide a mathematical proof/computation steps to all questions, unless explicitly told not to. Solutions with no proof/explanations (even correct ones!) will be awarded no points.
- You may look up combinatorial inequalities and mathematical formulas online, but please do not look up the solutions. Many problems can be found in research papers/online; if you happen upon the solution before figuring it out yourself, mention this in the submission. **Unreferenced copies of online material will be considered plagiarism.**

1. (30 points) Let the Nash welfare of an allocation be given by

$$NW(A) = \prod_{i \in N} v_i(A_i).$$

We say that an allocation $A^* = A_1^*, \dots, A_n^*$ maximizes the Nash welfare if it maximizes the NW objective

$$\max_{A \in \mathcal{A}(N,G)} NW(A) \tag{1}$$

We assume that there exists at least one allocation A for which $NW(A) > 0$; therefore, the value of (1) is always positive.[1].

For this question, we assume that players have *additive* valuations, i.e. for all $i \in N$ and all S, $v_i(S) = \sum_{g \in S} v_i(\{g\})$. You may write $v_{i,g}$ to denote the value that player i assigns to the good g.

(a) (10 points) Show that an allocation A^* that maximizes the Nash welfare is Pareto efficient: that is, for any other allocation A', it is either the case that there exists some player $i \in N$ for which $v_i(A'_i) < v_i(A^*_i)$, or the two allocations offer the same utility to all players: $v_i(A^*_i) = v_i(A'_i)$ for all $i \in N$.

(b) (20 points) Show that an allocation A^* that maximizes the Nash welfare is envy-free up to one good: that is, for any two players $i, j \in N$, either the bundle of j is empty, or there exists some item $g \in A^*_j$ such that $v_i(A^*_i) \geq v_i(A_j \setminus \{g\})$.

Hint: Assume a contradiction, and consider a good g^* in the bundle of j that i values the most, and j values the least; use the fact that valuations are additive. Try and construct a new allocation A' with a higher Nash welfare than A^*, and look at the ratio $\frac{NW(A')}{NW(A^*)}$. If this ratio is strictly greater than 1 then you have arrived at a contradiction.

[1] If all allocations in $\mathcal{A}(N,G)$ are such that $NW(A) = 0$ (i.e. there always exists some player for which $v_i(A_i) = 0$), then we look at allocations that maximize the number of players with positive utility, and choose one that maximizes (1) out of these. This is not a case you need to consider though.

Hint: You may use the following inequality: let $A = \{a_1, \dots, a_m\}; B = \{b_1, \dots, b_m\}$ be two sets of non-negative values such that there is at least one strictly positive value in A, and one strictly positive value in B. Let a^*, b^* minimize the value $\min_{a\in A, b\in B} \frac{a}{b}$; then

$$\frac{a^*}{b^*} \leq \frac{\sum_{a\in A} a}{\sum_{b\in B} b}.$$

2. (15 points) Consider the 'decycling' algorithm described in class for computing EF-1 allocations, which we denote $\mathcal{A}$. Let $A = A_1, \dots, A_n$ be the allocation outputted by $\mathcal{A}$. Let

$$E_i(A) = \{j \in N \mid v_j(A_j) < v_j(A_i)\}$$

be the set of players who envy player i. We know that for every player $j \in E_i(A)$, there is some item g in A_i such that $v_j(A_j) \geq v_j(A_i \setminus \{g\})$; note that under the definition of EF-1, the choice of g may depend on the identity of player j. Argue that there exists some item g^* such that for **every player** $j \in E_i(A)$, removing g^* from A_i will cause j to not envy i.

Hint: Think about how the algorithm allocates items, and when can players start envying other players.

3. (15 points) Show that if there exists at least one MMS allocation, then there exists a Pareto-optimal MMS allocation.

4. (40 points) Following input from COMPSCI590T students, CICS has decided to implement a fair course allocation mechanism. They are wondering whether a simple Round-Robin (RR) mechanism would work. The RR mechanism is very simple (which is a very good thing - complex mechanisms are often rejected by their users!): we have $N = \{1, \dots, n\}$ students and m courses with capacities given by $C = \{c_1, \dots, c_m\}$ (c_j is a positive integer denoting the number of seats in course j). Students receive a fixed number of points k, which they can freely distribute amongst the courses. More formally, each student i has a non-negative (possibly zero) value v_{ij} for successfully enrolling to course j, such that $\sum_{j=1}^{m} v_{ij} = k$. In other words, we are implicitly assuming that students have additive valuations over modules[2]. The RR mechanism then determines some order of the students; with no loss of generality it picks the order $1, \dots, n$[3]. It assigns students to their favorite available courses in the order $1, \dots, n$; the pseudocode for the RR algorithm is given in Algorithm 1. Note that if a course j^* is assigned to student i, the Assign procedure sets the value of j^* to be 0 for i: students assign no value to being assigned twice to the same class.

Example 1. We have three students who have 10 points to assign to three courses; course capacities are written next to the module code.

	590T (1)	589 (1)	514 (2)
Alice	4	3	3
Bob	2	5	3
Claire	1	8	1

We set the order to be $A \to B \to C$. The algorithm proceeds as follows:

(i) Alice gets 590T

(ii) Bob gets 589

(iii) Claire gets 514

(iv) Alice gets 514 as well.

(v) Since all courses are at capacity, we terminate.

[2] Is this a realistic assumption?

[3] Order could be by first-come-first-serve, by student priority (e.g. students who are more senior need priority for courses they need to graduate) or some other metric, it does not matter for our analysis

Algorithm 1 The Round Robin (RR) Algorithm

Input: Players $N = \{1, \ldots, n\}$; courses $C = \{c_1, \ldots, c_m\}$; a valuation $v_{i1}, \ldots, v_{im}$ for each player $i \in N$, such that $\sum_{j=1}^{m} v_{ij} = k$; course capacities $c_1, \ldots, c_m$
for $i = 1, \ldots, n$ **do**
  $A_i \leftarrow \emptyset$ ▷ All players start with no courses assigned.
end for
while $\exists i \in N, j \in 1, \ldots, m : c_j > 0$ and $v_{ij} > 0$ **do**
  for $i = 1, \ldots, n$ **do**
    Assign $(C, \vec{v}_i, i, A_i)$
  end for
end while
return A

procedure Assign$(C, \vec{v}_i, i, A_i)$
  $j^* \leftarrow \arg\max_{j \in C : c_j > 0} v_{ij}$ ▷ j^* is i's most preferred available course
  if $v_{ij^*} > 0$ **then**
    $v_{ij^*} \leftarrow 0$ ▷ i has no value for j^* once they're assigned to it
    $A_i \leftarrow A_i \cup \{j^*\}$
    $c_{j^*} \leftarrow c_{j^*} - 1$
  end if
end procedure

Let us analyze the properties of the RR algorithm.

(a) (20 points) Show that the RR algorithm outputs an EF-1 allocation. For ease of notation, you may refer to player i's j-th favorite course as $v_{i(j)}$.

(b) (20 points) Show that the RR algorithm is not guaranteed to output a PO allocation, even for the case of two players.

5. (Bonus points) While the fair allocation problems do not allow us to use "actual" money to balance the market, there is nothing stopping us from using "funny money" in order to do so, i.e. giving each player a number of monetary credits that they can use in order to bid for items, but has no value outside of our mechanism[4]. Consider an instance of the fair allocation problem for players with additive valuations. For each player $i \in N$, we assign a *budget* $b_i > 0$, such that $\sum_{i=1}^{n} b_i = 1$. For this question, we require that all budgets are different, say that $b_1 < b_2 < \cdots < b_n$. In this setting, our solution is an allocation of items to players, as well as a price $p(g)$ for every item g.

Definition 1 (Fisher Market Equilibrium)**.** An allocation A and price vector $\vec{p}$ are a Fisher market equilibrium if for every player i and every bundle $S \subseteq G$, $v_i(A_i) \geq v_i(S)$ whenever $\sum_{g \in S} p(g) \leq b_i$, and $\sum_{g \in A_i} p(g) \leq b_i$. In other words, player i can afford the bundle A_i that they received, and they like it more than any other bundle S that they can afford.

Definition 2 (ℓ-out-of-d MMS share)**.** Given two integers $0 < \ell \leq d$ and a player i, the ℓ-out-of-d MMS share of player i (or (ℓ, d)-MMS_i for short) is computed by allowing player i partition the set of goods into d bundles, and then pick out the least valued ℓ bundles.

The maximin share we defined is simply the 1-out-of-n MMS share: the player divides the set of goods into n distinct bundles and then picks out the worst one; we now allow the player to divide the goods into d bundles and pick out the worst ℓ out of them.

Example 2 (adapted from [1])**.** Consider a setting with three items, A, B, C whose values for player 1 are $2, 5$ and 8 respectively. Their 2-out-of-3 MMS share is $2 + 5 = 7$ — their utility for items A and B — whereas their MMS share is their value for the worst item (item A), which is 2.

Prove the following theorem (from [1]).

[4]Some might argue that all money is "funny money", especially those dedicated to investing in cryptocurrencies, but this discussion is beyond the scope of this class.

Theorem 1. *Let $\langle A, \vec{p} \rangle$ be a Fisher market equilibrium for players with additive valuations and budgets $b_1 < \cdots < b_n$ such that $\sum_{i=1}^{n} b_i = 1$; then for every player $i \in N$ and every $0 < \ell \leq d$ such that $\frac{\ell}{d} \leq b_i$, player i is receiving their ℓ-out-of-d MMS share.*

Fair Allocation of Indivisible Goods

We are given a set of players $N = \{1, \ldots, n\}$ with additive valuations over a set of goods $G = \{g_1, \ldots, g_m\}$; that is, player i has a fixed non-negative value for each item g_j, given by $v_{ij} \in \mathbb{Z}_+$, and their valuation for a bundle of items $S \subseteq G$ is $v_i(S) = \sum_{g_j \in S} v_{ij}$. Let $\mathcal{A}(N, G)$ be the set of all feasible allocations of G to N — i.e. the set of all possible partitions of G into n disjoint bundles. An allocation A is:

1. Socially optimal if it maximizes $\sum_{i=1}^{n} v_i(A_i)$
2. Pareto optimal (PO) if for every allocation $A' \in \mathcal{A}(N, G)$, if there is some player $i \in N$ where $v_i(A_i) < v_i(A'_i)$, then there exists some player j for which $v_j(A_j) > v_j(A'_j)$.
3. Envy-free (EF) if for every two players $i, j \in N$, $v_i(A_i) \geq v_i(A_j)$.
4. Envy-free up to one good (EF-1) if for every two players $i, j \in N$ where $A_j \neq \emptyset$, there is some item g in A_j such that $v_i(A_i) \geq v_i(A_j \setminus \{g\})$.
5. Satisfies the maximin share guarantee if each player receives at least her maximin share. Recall that the maximin share of player i, denoted MMS_i is

$$MMS_i = \max_{A=(A_1,\ldots,A_n)\in\mathcal{A}(N,G)} \min_{A_j \in A} v_i(A_j)$$

References

[1] Moshe Babaioff, Noam Nisan, and Inbal Talgam-Cohen. Fair allocation through competitive equilibrium from generic incomes. In *Proceedings of the 2nd ACM Conference on Fairness, Accountability and Transparency (FAT*)*, page 180, 2019.