University of Massachusetts, Amherst, CICS
Instructor: Yair Zick
COMPSCI 590T: Algorithmic Fairness and Strategic Behavior

Assignment 5: Cooperative Games and Nash Bargaining

Issued: Week 6 — Due: two weeks after the issue date in Gradescope

Important Instructions:

- Please upload your solution to Gradescope by 23:59 on the due date.
- You may work with a partner on your homework; however, if you choose to do so, **you must** write down the name of your partner on the assignment.
- Provide a mathematical proof/computation steps to all questions, unless explicitly told not to. Solutions with no proof/explanations (even correct ones!) will be awarded no points.
- You may look up combinatorial inequalities and mathematical formulas online, but please do not look up the solutions. Many problems can be found in research papers/online; if you happen upon the solution before figuring it out yourself, mention this in the submission. **Unreferenced copies of online material will be considered plagiarism.**

1. (30 points) Given a space of possible outcomes $\mathcal{O}$, let $u_i(o)$ be the welfare of player i under the outcome o. We discussed two types of solutions (or objectives) in class:

   1. Utilitarian: $\max_{o\in\mathcal{O}} \sum_{i=1}^n u_i(o)$ maximizes the total utility of all agents.
   2. Nash: $\max_{o\in\mathcal{O}} \prod_{i=1}^n u_i(o)$ maximizes the product of agent utilities.

   For each of the following bargaining problems, find a socially optimal and a Nash bargaining solution, i.e. the solution that maximizes the product of players' utilities.

   (a) (10 points) $S = \left\{(x,y) \in \mathbb{R}_+^2 \mid \frac{x^2}{2} + y^2 \le 1\right\}$, with $u_1(x) = x$ and $u_2(y) = y$.

   (b) (10 points) $S = \left\{(x,y) \in \mathbb{R}_+^2 \mid x + y = 1\right\}$, where $u_1(x) = x^a$ and $u_2(y) = y$; here, $a > 1$ is a constant.

   (c) (10 points) $S = \left\{(x,y) \in \mathbb{R}_+^2 \mid x + y = 2\right\}$, where $u_1(x) = \log x$ and $u_2(y) = \log y$.

2. (30 points) Let us consider network flow games, discussed in class. We are given a directed graph $G = \langle V, E\rangle$ where $E = \{e_1, \dots, e_n\}$. There are two vertices $s, t \in V$ (the source and the target, respectively). The player set $N = \{1, \dots, n\}$ is such that each edge e_i is controlled by player i, and has a positive integer weight w_i. We abuse notation and refer to $S \subseteq N$ as the set of edges $\{e_i\}_{i\in S}$. The value of a coalition $S \subseteq N$ is the maximum flow that can pass from s to t, using only the edges controlled by S.

   (a) (10 points) Show that network flow games are superadditive.

   (b) (20 points) Show that the core of a network flow game is not empty.

   Hint: Given an optimal s-t flow f^* through G, let $C \subseteq N$ be a minimum cut. Argue that paying each player $i \in C$ the amount that flows through their edge is a core payoff.

3. (40 points) Recall that a coalition structure CS is a partition of players into disjoint sets, such that total welfare is maximized. In weighted voting games (and in simple games in general), an optimal coalition structure is one where players are partitioned into as many winning coalitions as possible. For example, if player weights are $\langle w_1, w_2, w_3, w_4\rangle = \langle 1, 1, 2, 3\rangle$ and the quota is 3, then the optimal coalition structure would form three coalitions: one with player 4 ($w_4 = 3$) alone, and one more with one player with weight 1 and the player with weight 2, say $CS^* = \{1,3\}; \{2\}; \{4\}$. Given a coalition structure CS, we let CS_+ be the set of winning coalitions. Given a cooperative game G, let $OPT(G)$ be the value of an optimal coalition structure for G.

(a) (10 points) Given a weighted voting game $\langle \vec{w}; q \rangle$, we say that a coalition structure CS is *lean* if every winning coalition in CS is minimally winning, and there is at most one losing coalition; that is, for every $S \in CS_+$, and every player $i \in S$, $S \setminus \{i\}$ is losing. Prove that there exists an optimal coalition structure CS^* that is lean.

(b) (10 points) Show that if CS^* is a lean, optimal coalition structure for a WVG, then $w(S) < 2q$ for every $S \in CS^*_+$ such that $|S| \geq 2$.

Hint: What happens if a winning coalition S of size ≥ 2 has $w(S) \geq 2q$?

(c) (10 points) Let $B = \{i \in N : w_i \geq q\}$ be the set of players whose weight is more than q. Let CS^*_+ be the set of winning coalitions in CS^* whose size is more than 2. Conclude that $\sum_{i \in N \setminus B}^{n} w_i < 2q|CS^*_+| + q$.

(d) (10 points) We have seen that the core of a cooperative game may be empty. This does not mean that we cannot have stable payoff divisions, just that we need to pay more than $OPT(G)$ in order to stabilize the game. Consider the following payoff division:

$$x_i^* = \begin{cases} \frac{w_i}{q} & \text{if } i \notin B \\ 1 & \text{if } i \in B \end{cases}$$

Show that $\vec{x} = (x_1^*, \dots, x_n^*)$ is a stable payoff division. Show that $\sum_{i=1}^{n} x_i^* \leq 2OPT(G) + 1$. In other words, we need to pay at most twice as much as we have in order to stabilize any weighted voting game.

Hint: Remember that $w_1, \dots, w_n$ and q are integers, and use (c)

4. (bonus points) The Nash bargaining solution maximizes the product of player utilities, and uniquely satisfies a set of desirable properties. However, independence of irrelevant alternatives seems a bit odd: players can leverage one another with bad offers in order to get a better outcome for themselves (see example in [1]). Let us introduce an alternative axiom, introduced by Kalai and Smorodinsky [1]. First, given a convex and compact set $S \subseteq \mathbb{R}^2$, let $b_1(S) = \max_{(x,y) \in S} x$ and $b_2(S) = \max_{(x,y) \in S} y$. In other words, $b_1(S)$ is the best alternative that player 1 can hope for, and $b_2(S)$ is the best that player 2 can hope for. Let $g_S(x)$ be the most that player 2 can receive when player 1 receives x.

Definition 1 (Monotonicity)**.** Let $(\vec{d}, S_1)$ and $(\vec{d}, S_2)$ be two bargaining problems such that $b_1(S_1) = b_1(S_2)$ and $g_{S_1}(x) \leq g_{S_2}(x)$ for all x. A solution f satisfies monotonicity if player 2 is paid weakly more under $(\vec{d}, S_2)$ than under $(\vec{d}, S_1)$.

Intuitively, the monotonicity axiom says that if under S_2 player 2 has better options than under S_1, and player 1's options are the same, then player 2 should receive more under S_2 than under S_1.

Kalai and Smorodinsky suggest the following solution: given a bargaining problem $(\vec{d}, S)$, draw a line connecting the disagreement point $\vec{d}$ and the point $(b_1(S), b_2(S))$. Select the point where this line intersects with the Pareto frontier. Prove Kalai and Smorodinsky's theorem: the K-S solution is the only one satisfying monotonicity, symmetry, Pareto efficiency and invariance under equivalent representations.

The Core and the Shapley Value:
Recall that a cooperative game is a tuple $\mathcal{G} = \langle N, v \rangle$ where $N = \{1, \dots, n\}$ is a set of players, and $v : 2^N \to \mathbb{R}_+$ is called the characteristic function of the game. The core of a cooperative game is the set

$$Core(\mathcal{G}) = \left\{ \vec{x} \in \mathbb{R}^n \mid \forall S \subseteq N : \sum_{i \in S} x_i \geq v(S) \right\}.$$

The Shapley value of a cooperative game is defined as follows: given a permutation $\sigma : N \to N$, let $P_i(\sigma) = \{j \in N : \sigma(j) < \sigma(i)\}$; the marginal contribution of i to σ is then simply $m_i(\sigma) = m_i(P_i(\sigma)) = v(P_i(\sigma) \cup \{i\}) - v(P_i(\sigma))$. The Shapley value is

$$\varphi_i(\mathcal{G}) = \mathbb{E}_{\sigma \sim U(\Pi(N))}[m_i(\sigma)] = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m_i(\sigma).$$

Classes of Coalitional Games:
A *simple game* is a game where every coalition's value is either 0 or 1. A game is *monotone* if for every $S \subseteq T$, $v(S) \leq v(T)$.
A *threshold task game* (TTG) is defined as follows. Each player $i \in N$ has a weight $w_i \in \mathbb{Z}_+$. We are given a list of tasks $\mathcal{T} = t_1, \dots, t_m$; each task t_j has a quota $q_j \in \mathbb{Z}_+$ and a value $v_j \in Z_+$. The value of a coalition $S \subseteq N$ is given by

$$v(S) = \max_{t_j \in \mathcal{T}} \{v_j : \sum_{i \in S} w_i \geq q_j\}.$$

In other words, the value of a coalition is the value of the best task that it can complete with its resources.
A *weighted voting game* (WVG) is a threshold task game with a single task whose threshold is q and whose value is 1. So a coalition is winning if its total weight is more than q and is losing otherwise.
An *induced subgraph game* (ISG) is defined as follows. Each player is a node in a weighted graph $G = \langle N, E \rangle$, with $w : E \to \mathbb{Z}_+$ being the weight function. Given a coalition $S \subseteq N$, let $E|_S = \{e \in E : e = \{i, j\} \wedge i, j \in S\}$ be the edges induced by the nodes in S, and let

$$v(S) = \sum_{e \in E|_S} w(e).$$

In other words, the value of a coalition is the total weight of the edges in its induced subgraph.

References

[1] Ehud Kalai and Meir Smorodinsky. Other solutions to nash's bargaining problem. *Econometrica*, 43 (3):513–518, 1975.