Solution for Homework 4

1. **Solution:**

First of all, let's determine constant c. There has to be

$$1 = \int_{\mathbb{R}} f(x)dx = \int_0^2 cx^4 dx = c\frac{x^5}{5}\Big|_0^2 = c \cdot \frac{32}{5}$$

which implies that $c = \frac{5}{32}$.

(a)

The mean value is

$$\begin{aligned} EX &= \int_{\mathbb{R}} xf(x)dx = \int_0^2 x \cdot \frac{5}{32}x^4 dx = \frac{5}{32}\int_0^2 x^5 dx \\ &= \frac{5}{32}\frac{x^6}{6}\Big|_0^2 = \frac{5}{32} \cdot \frac{64}{6} = \frac{5}{3} \end{aligned}$$

(b)

In order to determine the variance, let's find the second moment. We have that

$$\begin{aligned} E(X^2) &= \int_{\mathbb{R}} x^2 f(x)dx = \int_0^2 x^2 \cdot \frac{5}{32}x^4 dx = \frac{5}{32}\int_0^2 x^6 dx \\ &= \frac{5}{32}\frac{x^7}{7}\Big|_0^2 = \frac{5}{32} \cdot \frac{128}{7} = \frac{20}{7} \end{aligned}$$

Hence, the variance is

$$Var(X) = E(X^2) - E(X)^2 = \frac{20}{7} - \frac{25}{9} = \frac{5}{63}$$

2. **Solution:**

Given Trains to destination A come at intervals of 15 minutes, starting 7a.m. And trains to destination B come at intervals of 15 minutes, starting 7:05a.m.
To find:(a) Proportion of time if a passenger arrives uniformly between 7a.m and 8a.m heads to destination A, given he takes the first train on arrival.
Solution: For the passenger taking the first train on arrival and heading to destination A, he/she must arrive after the train to B leaves and before the next train to A departs, that is he must arrive in one of the following time intervals:

$$(7.05 - 7.15), (7.20 - 7.30), (7.35 - 7.45), (7.50 - 8)$$

Therefore, total feasible time= 40minutes.
Thus, proportion of time he heads to destination A is $\dfrac{40}{60} = \dfrac{2}{3}$

To find: (b)Proportion of time if a passenger arrives uniformly between 7:10a.m and 8:10a.m heads to destination A, given he takes the first train on arrival.
Solution: For the passenger taking the first train on arrival and heading to destination A, he/she must arrive after the train to B leaves and before the next train to A departs, that is he must arrive in one of the following time intervals:

$$(7.10 - 7.15), (7.20 - 7.30), (7.35 - 7.45), (7.50 - 8)$$

Therefore, total feasible time= 35minutes.
Thus, proportion of time he heads to destination A is $\dfrac{35}{60} = \dfrac{7}{12}$

3. **Solution:**

Given that X is uniformly distributed over $(0, 1)$, i.e. $X \sim U(0, 1)$
To find the density function of $Y = e^X$, we first find the distribution function of Y.

$$F_Y(y) = P[Y \leq y] = P[e^X \leq y] = P[X \leq ln(y)] = \int_0^{ln(y)} dx = ln(y) \qquad 1 < y < e$$

Thus, the density function is given as

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d(ln(y))}{dy} = \frac{1}{y} \qquad 1 < y < e$$

Thus,

$$f_Y(y) = \begin{cases} \dfrac{1}{y} & 1 < y < e \\ 0 & o.w \end{cases}$$

4. **Solution:** The range of Y is $(0,1)$. So the PDF of Y outside $(0,1)$ is 0. For $t \in (0,1)$, we have

$$P(Y \leq t) = P(e^{-\lambda X} \leq t) = P(X \geq -\log(t)/\lambda) = \int_{-\log(t)/\lambda}^{\infty} \lambda e^{-\lambda x} dx = e^{\log(t)} = t.$$

Thus, by taking the differentiation, the PDF of Y in $(0,1)$ is the constant 1.

5. **Solution:**

Let T denote the time(in hours) required to repair a machine. Given that $T \sim Exp(\lambda = 1/2)$. Thus, we have

$$f_T(t) = \begin{cases} \dfrac{e^{-t/2}}{2} & t > 0 \\ 0 & o.w \end{cases}$$

Now,
(a) Probability that repair time exceeds two hours is

$$\begin{aligned} P[T > 2] &= \int_2^{\infty} \frac{e^{-t/2}}{2} dt \\ &= \frac{1}{2}\left[\frac{e^{-t/2}}{-1/2}\right]_2^{\infty} \\ &= -\left[e^{-t/2}\right]_2^{\infty} \\ &= -[0 - e^{-1}] \\ &= e^{-1} \end{aligned}$$

(b) Probability that repair time will exceed 10 hours given that it has exceeded 9 hours is given as $P[T > 10|T > 9]$. Here, we make use of the memoryless property of the exponential distribution, which states that for $t < s$

$$P[T > s|T > t] = P[T > s - t]$$

Thus, we have

$$\begin{aligned} P[T > 10|T > 9] &= P[T > 10 - 9] \\ &= P[T > 1] \\ &= \int_1^{\infty} \frac{e^{-t/2}}{2} dt \\ &= \frac{1}{2}\left[\frac{e^{-t/2}}{-1/2}\right]_1^{\infty} \\ &= -\left[e^{-t/2}\right]_1^{\infty} \\ &= -[0 - e^{-1/2}] \\ &= e^{-1/2} \end{aligned}$$

6. **Solution:**

Let X be the number of years a radio functions. We do not know the age of Jones's second-hand radio, but, given that it is still functioning, we can assume its age is the expected value of X, given by $E[X] = \frac{1}{\lambda} = \frac{1}{\frac{1}{8}} = 8$. Hence, the conditional probability that Jones's radio functions for **another** 8 years (as specified in the prompt) would be given by

$$P\{X \geq 16 | X \geq 8\} = \frac{P\{X \geq 16; X \geq 8\}}{P\{X \geq 8\}}$$

Of course, $P\{X \geq 16; X \geq 8\} = P\{X \geq 16\}$. Thus:

$$P\{X \geq 16 | X \geq 8\} = \frac{P\{X \geq 16\}}{P\{X \geq 8\}} = \frac{e^{-\frac{1}{8} \cdot 16}}{e^{-\frac{1}{8} \cdot 8}} = e^{-1}$$

7. **Solution:**

Given Height (in inches) of a 25 year man is a normal random variable with mean $\mu = 71$ and variance $\sigma^2 = 6.25$
To find: (a) What percentage of men are over 6 feet 2 inches (b) What percentage of men in the 6 footer club are over 6 feet 5 inches
Solution (a) To calculate the percentage of men, we first calculate the probability
P[Height of a 25 year old man is over 6 feet 2inches]=$P[X > 74in]$

$$\begin{aligned} P[X > 74] &= P\left[\frac{X - \mu}{\sigma} > \frac{74 - 71}{2.5}\right] \\ &= P[Z > 1.2] \\ &= 1 - P[Z \leq 1.2] \\ &= 1 - \Phi(1.2) \\ &= 1 - 0.8849 \\ &= 0.1151 \end{aligned}$$

Thus, percentage of 25 year old men that are above 6 feet 2 inches is 11.5%

(b)P[Height of 25 year old man is above 6 feet 5 inches given that he is above 6 feet]=P[X¿ 6ft 5in — X¿ 6ft]

$$\begin{aligned}
P[X > 6 \text{ ft } 5\text{in}|X > 6\text{ft}] &= P[X > 77|X > 72] \\
&= \frac{P\left[X > 77 \cap X > 72\right]}{P\left[X > 72\right]} \\
&= \frac{P\left[X > 77\right]}{P\left[X > 72\right]} \\
&= \frac{P\left[\dfrac{X-\mu}{\sigma} > \dfrac{77-71}{2.5}\right]}{P\left[\dfrac{X-\mu}{\sigma} > \dfrac{72-71}{2.5}\right]} \\
&= \frac{P[Z > 2.4]}{P[Z > 0.4]} \\
&= \frac{1 - P[Z \leq 2.4]}{1 - P[Z \leq 0.4]} \\
&= \frac{1 - 0.9918}{1 - 0.6554} \\
&= \frac{0.0082}{0.3446} \\
&= 0.024
\end{aligned}$$

Thus, Percentage of 25 year old men in the 6 footer club that are above 6 feet 5 inches are 2.4%

8. **Solution:**

Define random variable X that marks the life time of a randomly selected automobile tyre. We are given that $X \sim \mathcal{N}(34000, 4000^2)$.

(a)

We are required to find

$$P(X > 40000) = 1 - P(X \leq 40000) = 1 - P\left(\frac{X - 34000}{4000} \leq \frac{40000 - 34000}{4000}\right)$$
$$= 1 - \Phi(1.5) = 1 - 0.93319 = 0.06681$$

(b)

We are required to find

$$P(30000 \leq X \leq 35000) = P\left(\frac{30000 - 34000}{4000} \leq \frac{X - 34000}{4000} \leq \frac{35000 - 34000}{4000}\right)$$
$$= P\left(-1 \leq \frac{X - 34000}{4000} \leq 0.25\right) = \Phi(0.25) - \Phi(-1)$$
$$= 0.59871 - 0.15866 = 0.44$$

(c)

We are required to find

$$P(X \geq 40000 | X \geq 30000) = \frac{P(X \geq 40000, X \geq 30000)}{P(X \geq 30000)} = \frac{P(X \geq 40000)}{P(X \geq 30000)}$$

The expression above is equal to

$$\frac{P(X \geq 40000)}{P(X \geq 30000)} = \frac{1 - P\left(\frac{X-34000}{4000} \leq \frac{40000-34000}{4000}\right)}{1 - P\left(\frac{X-34000}{4000} \leq \frac{30000-34000}{4000}\right)} = \frac{1 - \Phi(1.5)}{1 - \Phi(-1)}$$
$$= \frac{1 - 0.93319}{1 - 0.15866} = 0.0794$$

9. **Solution:**

- The parameter of X is $\lambda = 1/1.5 = 2/3$. The probability is $P(X \geq 2) = e^{-4/3}$.
- By the memoryless property of exponential distribution, the probability is $P(X \geq 1) = e^{-2/3}$.