Solutions Guide to Differential Geometry

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Abstract

Solutions to the textbook "Elementary Differential Geometry", Revised Second Edition by Barrett O'Neill.

Contents

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1 Calculus on Euclidean Space

1.1 Euclidean Space

1. (a) $fg^2 = (x^2y)(y\sin(z))^2 = x^2y^3\sin^2(z)$
   (b) $\frac{\partial f}{\partial x}g + \frac{\partial g}{\partial y}f = (2xy)(y\sin(z)) + \sin(z)(x^2y) = (2xy^2 + x^2y)\sin(z)$
   (c) $fg = (x^2y)(y\sin(z)) = x^2y^2\sin(z) \implies \frac{\partial(fg)}{\partial z} = x^2y^2\cos(z) \implies \frac{\partial^2(fg)}{\partial y\partial z} = 2x^2y\cos(z)$
   (d) $\frac{\partial}{\partial y}\sin(f) = \cos(f)\frac{\partial f}{\partial y} = \cos(x^2y)\cdot 2xy$
2. (a) $1^2\cdot 1 - 1^2\cdot 1 = 0$
   (b) $3^2\cdot(-1) - (-1)^2\cdot\frac{1}{2} = -9.5$
   (c) $a^2\cdot 1 - 1^2\cdot(1-a) = a^2 + a - 1$
   (d) $t^2\cdot t^2 - (t^2)^2\cdot t^3 = t^4 - t^7$
3. (a) $\sin(xy) + x\cos(xy)y - y\sin(xz)z = \sin(xy) + xy\cos(xy) - yz\sin(xz)$
   (b) $\frac{\partial f}{\partial x} = \cos(g)\frac{\partial g}{\partial x} = \cos(e^h)e^h\frac{\partial h}{\partial x} = \cos(e^{x^2+y^2+z^2})e^{x^2+y^2+z^2}2x$
4. With a slight abuse of notation[1], the chain rule in this case gives us $\frac{\partial f}{\partial x} = \frac{\partial h}{\partial x}\frac{\partial g_1}{\partial x} + \frac{\partial h}{\partial y}\frac{\partial g_2}{\partial x} + \frac{\partial h}{\partial z}\frac{\partial g_3}{\partial x}$. For the given h, we can write this as $\frac{\partial f}{\partial x} = 2x\frac{\partial g_1}{\partial x} - z\frac{\partial g_2}{\partial x} - y\frac{\partial g_3}{\partial x}$. We need only compute the the partial derivatives with respect to x of each coordinate function.
   (a) $\frac{\partial f}{\partial x} = 2x\cdot 1 - z\cdot 0 - y\cdot 1 = 2x$
   (b) $\frac{\partial f}{\partial x} = 2x\cdot 0 - ze^{x+y} - ye^x = -ze^{x+y} - ye^x$
   (c) $\frac{\partial f}{\partial x} = 2x\cdot 1 - z\cdot(-1) - y\cdot 1 = 2x + z - y$

[1] The abuse is that when we say for example $\frac{\partial f}{\partial x}$ we really mean "partial derivative of f with respect to the first component." (And similarly for $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$).