Solutions Guide to Differential Geometry

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Abstract

Solutions to the textbook "Elementary Differential Geometry", Revised Second Edition by Barrett O'Neill.

Contents

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1 Calculus on Euclidean Space

1.1 Euclidean Space

1. (a) $fg^2 = (x^2y)(y\sin(z))^2 = x^2y^3\sin^2(z)$
   (b) $\frac{\partial f}{\partial x}g + \frac{\partial g}{\partial y}f = (2xy)(y\sin(z)) + \sin(z)(x^2y) = (2xy^2 + x^2y)\sin(z)$
   (c) $fg = (x^2y)(y\sin(z)) = x^2y^2\sin(z) \implies \frac{\partial(fg)}{\partial z} = x^2y^2\cos(z) \implies \frac{\partial^2(fg)}{\partial y\partial z} = 2x^2y\cos(z)$
   (d) $\frac{\partial}{\partial y}\sin(f) = \cos(f)\frac{\partial f}{\partial y} = \cos(x^2y)\cdot 2xy$
2. (a) $1^2 \cdot 1 - 1^2 \cdot 1 = 0$
   (b) $3^2 \cdot (-1) - (-1)^2 \cdot \frac{1}{2} = -9.5$
   (c) $a^2 \cdot 1 - 1^2 \cdot (1-a) = a^2 + a - 1$
   (d) $t^2 \cdot t^2 - (t^2)^2 \cdot t^3 = t^4 - t^7$
3. (a) $\sin(xy) + x\cos(xy)y - y\sin(xz)z = \sin(xy) + xy\cos(xy) - yz\sin(xz)$
   (b) $\frac{\partial f}{\partial x} = \cos(g)\frac{\partial g}{\partial x} = \cos(e^h)e^h\frac{\partial h}{\partial x} = \cos(e^{x^2+y^2+z^2})e^{x^2+y^2+z^2}2x$
4. With a slight abuse of notation[1], the chain rule in this case gives us $\frac{\partial f}{\partial x} = \frac{\partial h}{\partial x}\frac{\partial g_1}{\partial x} + \frac{\partial h}{\partial y}\frac{\partial g_2}{\partial x} + \frac{\partial h}{\partial z}\frac{\partial g_3}{\partial x}$. For the given h, we can write this as $\frac{\partial f}{\partial x} = 2x\frac{\partial g_1}{\partial x} - z\frac{\partial g_2}{\partial x} - y\frac{\partial g_3}{\partial x}$. We need only compute the the partial derivatives with respect to x of each coordinate function.
   (a) $\frac{\partial f}{\partial x} = 2x \cdot 1 - z \cdot 0 - y \cdot 1 = 2x$
   (b) $\frac{\partial f}{\partial x} = 2x \cdot 0 - ze^{x+y} - ye^x = -ze^{x+y} - ye^x$
   (c) $\frac{\partial f}{\partial x} = 2x \cdot 1 - z \cdot (-1) - y \cdot 1 = 2x + z - y$

1.2 Tangent Vectors

1. (a) First, $\mathbf{v}_p = -2U_1(\mathbf{p}) + U_2(\mathbf{p}) - U_3(\mathbf{p})$ and $\mathbf{w}_p = U_2(\mathbf{p}) + 3U_3(\mathbf{p})$. Therefore, $3\mathbf{v}_p - 2\mathbf{w}_p = -6U_1(\mathbf{p}) + U_2(\mathbf{p}) - 9U_3(\mathbf{p})$.
   (b) Here is a code listing

```
import numpy as np

from pure_math import diff_geo

if __name__ == "__main__":
        p = np.array([1,1,0])
        v = np.array([-2, 1, -1])
        w = np.array([0, 1, 3])
        diff_geo.plot_arrows(p, [v, w, -2*v, v+w])
```

   And here is a plot in Figure 1.
2. $W - xV = 2x^2U_2 - U_3 - x(xU_1 + yU_2) = -x^2U_1 + (2x^2 - xy)U_2 - U_3$. At $\mathbf{p} = (-1, 0, 2)$ we have $(W - xV)(\mathbf{p}) = -U_1(\mathbf{p}) + 2U_2(\mathbf{p}) - U_3(\mathbf{p})$.
3. (a) $V = \frac{2}{7}z^2U_1 + 0U_2 - \frac{1}{7}xyU_3$
   (b) $V = xU_1 + (z-x)U_2 + 0U_3$
   (c) $V = xU_1 + 2yU_2 + xy^2U_3$
   (d) First we have to compute the difference between the points to construct the vector field, then make the standard conversion. In particular, $V(\mathbf{p}) = (1 + p_1, p_2p_3, p_2) - (p_1, p_2, p_3) = (1, p_2(p_3 - 1), p_3 - p_2)$. Thus, $V = 1U_1 + y(z-1)U_2 + (z-y)U_3$.

[1] The abuse is that when we say for example $\frac{\partial f}{\partial x}$ we really mean "partial derivative of f with respect to the first component." (And similarly for $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$).

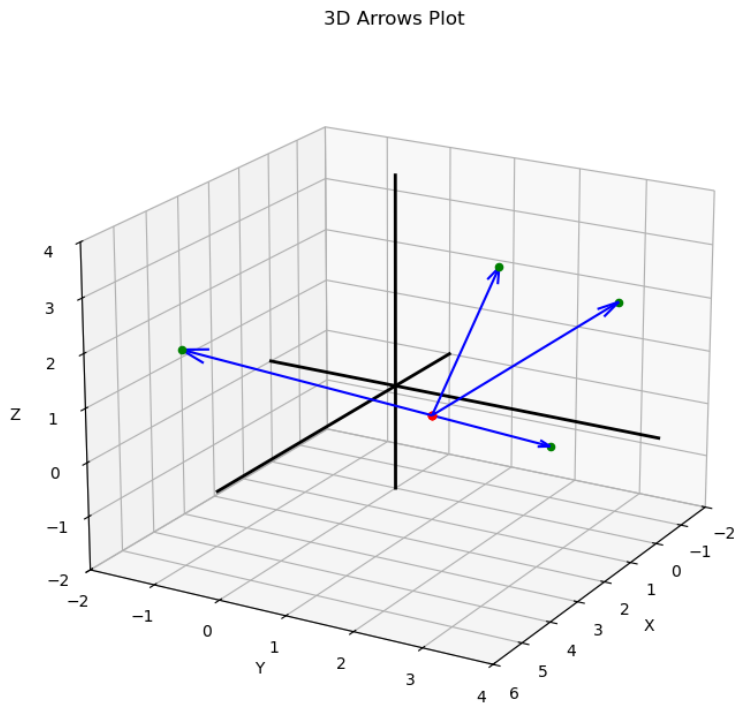

Figure 1: Plot for problem 1, part b in section 1.2.

(e) As above we have $V(\mathbf{p}) = (0,1,1) - (p_1, p_2, p_3) = (-p_1, -p_2, -p_3)$. Thus, $V = -xU_1 - yU_2 - zU_3$.

4. We do not care about any components of $fV + gW$ outside the U_1 term. That is equal to $f(x,y,z)y^2 + g(x,y,z)x^2$. Technically, without further restrictions we could just say $f = g = 0$. But that would be trivial and I think the question meant to emphasize that the U_1 term is 0, but the others are not. In particular, the U_2 term would only come from the fV part of $fV + gW$ and the U_3 term would only come from the gW part of $fV + gW$. It is clear these are non-zero if f and g are non-zero functions. In particular, $f(x,y,z) = x^2$ and $g(x,y,z) = -y^2$ will suffice to make $f(x,y,z)y^2 + g(x,y,z)x^2$ identically 0 without making f and g themselves 0.

5. (a) Suppose $c_1V_1(\mathbf{p}) + c_2V_2(\mathbf{p}) + c_3V_3(\mathbf{p}) = 0$. Writing this in standard form, we have $(c_1 + c_3x)U_1(\mathbf{p}) + (c_2)U_2(\mathbf{p}) + (-c_1x + c_3)U_3(\mathbf{p}) = 0$. For this to occur, each coefficient must be identically 0[2]. That is, $c_1 + c_3x = 0$ for all x so that $c_1 = c_3 = 0$ from the U_1 constraint. The U_2 immediately yields $c_2 = 0$. For the U_3 constraint we also require that $c_1 = c_3 = 0$. Thus, V_1, V_2, V_3 are linearly independent.

   If this "identically 0" business is confusing to you, then recall from the problem we must have 0 at *every* point $\mathbf{p}$ so take $\mathbf{p} = (1,0,0)$. Then the vector field at that point is $(c_1 + c_3)U_1(\mathbf{p}) + c_2U_2 + (c_3 - c_1)U_3(\mathbf{p})$. For this to be zero we need to solve $c_1 + c_3 = 0$, $c_2 = 0$, and $c_3 - c_1 = 0$. The solution is clearly $c_1 = c_2 = c_3 = 0$.

   (b) We need to invert the definitions. Observe that $xV_1 - V_3 = -(x^2+1)U_3$ and $xV_3 + V_1 = (x^2+1)U_1$. For all $x \in \mathbb{R}$, we know $x^2 + 1 > 0$. Therefore, $U_3 = \frac{-1}{x^2+1}(xV_1 - V_3)$ and $U_1 = \frac{1}{x^2+1}(V_1 + xV_3)$. Trivially we observe $U_2 = V_2$.

[2] I.e., the zero function.

Thus, $xU_1 + yU_2 + zU_3 = \left(\frac{x-zx}{x^2+1}\right) V_1 + yV_2 \left(\frac{x^2+z}{x^2+1}\right) V_3.$