Solutions Guide to Differential Geometry

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February 4, 2025

Abstract

Solutions to the textbook "Elementary Differential Geometry", Revised Second Edition by Barrett O'Neill.

Contents

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1 Calculus on Euclidean Space

1.1 Euclidean Space

1. (a) $fg^2 = (x^2y)(y\sin(z))^2 = x^2y^3\sin^2(z)$
   (b) $\frac{\partial f}{\partial x}g + \frac{\partial g}{\partial y}f = (2xy)(y\sin(z)) + \sin(z)(x^2y) = (2xy^2 + x^2y)\sin(z)$
   (c) $fg = (x^2y)(y\sin(z)) = x^2y^2\sin(z) \implies \frac{\partial(fg)}{\partial z} = x^2y^2\cos(z) \implies \frac{\partial^2(fg)}{\partial y\partial z} = 2x^2y\cos(z)$
   (d) $\frac{\partial}{\partial y}\sin(f) = \cos(f)\frac{\partial f}{\partial y} = \cos(x^2y)\cdot 2xy$
2. (a) $1^2\cdot 1 - 1^2\cdot 1 = 0$
   (b) $3^2\cdot(-1) - (-1)^2\cdot\frac{1}{2} = -9.5$
   (c) $a^2\cdot 1 - 1^2\cdot(1-a) = a^2 + a - 1$
   (d) $t^2\cdot t^2 - (t^2)^2\cdot t^3 = t^4 - t^7$
3. (a) $\sin(xy) + x\cos(xy)y - y\sin(xz)z = \sin(xy) + xy\cos(xy) - yz\sin(xz)$
   (b) $\frac{\partial f}{\partial x} = \cos(g)\frac{\partial g}{\partial x} = \cos(e^h)e^h\frac{\partial h}{\partial x} = \cos(e^{x^2+y^2+z^2})e^{x^2+y^2+z^2}2x$
4. With a slight abuse of notation[1], the chain rule in this case gives us $\frac{\partial f}{\partial x} = \frac{\partial h}{\partial x}\frac{\partial g_1}{\partial x} + \frac{\partial h}{\partial y}\frac{\partial g_2}{\partial x} + \frac{\partial h}{\partial z}\frac{\partial g_3}{\partial x}$. For the given h, we can write this as $\frac{\partial f}{\partial x} = 2x\frac{\partial g_1}{\partial x} - z\frac{\partial g_2}{\partial x} - y\frac{\partial g_3}{\partial x}$. We need only compute the the partial derivatives with respect to x of each coordinate function.
   (a) $\frac{\partial f}{\partial x} = 2x\cdot 1 - z\cdot 0 - y\cdot 1 = 2x$
   (b) $\frac{\partial f}{\partial x} = 2x\cdot 0 - ze^{x+y} - ye^x = -ze^{x+y} - ye^x$
   (c) $\frac{\partial f}{\partial x} = 2x\cdot 1 - z\cdot(-1) - y\cdot 1 = 2x + z - y$

1.2 Tangent Vectors

1. (a) First, $\mathbf{v}_p = -2U_1(\mathbf{p}) + U_2(\mathbf{p}) - U_3(\mathbf{p})$ and $\mathbf{w}_p = U_2(\mathbf{p}) + 3U_3(\mathbf{p})$. Therefore, $3\mathbf{v}_p - 2\mathbf{w}_p = -6U_1(\mathbf{p}) + U_2(\mathbf{p}) - 9U_3(\mathbf{p})$.
   (b) Here is a code listing

```
import numpy as np

from pure_math import diff_geo

if __name__ == "__main__":
        p = np.array([1,1,0])
        v = np.array([-2, 1, -1])
        w = np.array([0, 1, 3])
        diff_geo.plot_arrows(p, [v, w, -2*v, v+w])
```

   And here is a plot in Figure 1.
2. $W - xV = 2x^2U_2 - U_3 - x(xU_1 + yU_2) = -x^2U_1 + (2x^2 - xy)U_2 - U_3$. At $\mathbf{p} = (-1, 0, 2)$ we have $(W - xV)(\mathbf{p}) = -U_1(\mathbf{p}) + 2U_2(\mathbf{p}) - U_3(\mathbf{p})$.
3. (a) $V = \frac{2}{7}z^2U_1 + 0U_2 - \frac{1}{7}xyU_3$
   (b) $V = xU_1 + (z - x)U_2 + 0U_3$
   (c) $V = xU_1 + 2yU_2 + xy^2U_3$
   (d) First we have to compute the difference between the points to construct the vector field, then make the standard conversion. In particular, $V(\mathbf{p}) = (1 + p_1, p_2p_3, p_2) - (p_1, p_2, p_3) = (1, p_2(p_3 - 1), p_3 - p_2)$. Thus, $V = 1U_1 + y(z-1)U_2 + (z-y)U_3$.

[1] The abuse is that when we say for example $\frac{\partial f}{\partial x}$ we really mean "partial derivative of f with respect to the first component." (And similarly for $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$).

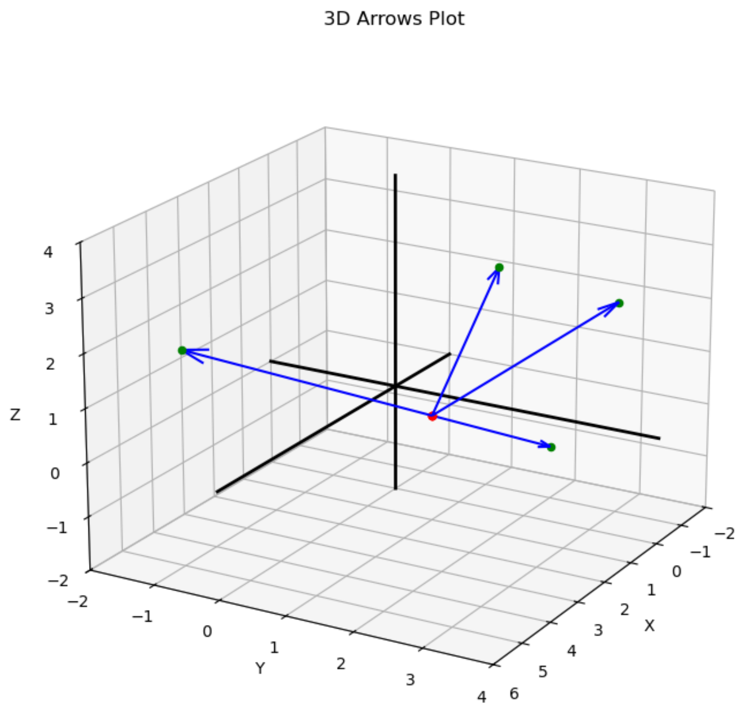

Figure 1: Plot for problem 1, part b in section 1.2.

(e) As above we have $V(\mathbf{p}) = (0,1,1) - (p_1, p_2, p_3) = (-p_1, -p_2, -p_3)$. Thus, $V = -xU_1 - yU_2 - zU_3$.

4. We do not care about any components of $fV + gW$ outside the U_1 term. That is equal to $f(x,y,z)y^2 + g(x,y,z)x^2$. Technically, without further restrictions we could just say $f = g = 0$. But that would be trivial and I think the question meant to emphasize that the U_1 term is 0, but the others are not. In particular, the U_2 term would only come from the fV part of $fV + gW$ and the U_3 term would only come from the gW part of $fV + gW$. It is clear these are non-zero if f and g are non-zero functions. In particular, $f(x,y,z) = x^2$ and $g(x,y,z) = -y^2$ will suffice to make $f(x,y,z)y^2 + g(x,y,z)x^2$ identically 0 without making f and g themselves 0.

5. (a) Suppose $c_1V_1(\mathbf{p}) + c_2V_2(\mathbf{p}) + c_3V_3(\mathbf{p}) = 0$. Writing this in standard form, we have $(c_1 + c_3x)U_1(\mathbf{p}) + (c_2)U_2(\mathbf{p}) + (-c_1x + c_3)U_3(\mathbf{p}) = 0$. For this to occur, each coefficient must be identically 0[2]. That is, $c_1 + c_3x = 0$ for all x so that $c_1 = c_3 = 0$ from the U_1 constraint. The U_2 immediately yields $c_2 = 0$. For the U_3 constraint we also require that $c_1 = c_3 = 0$. Thus, V_1, V_2, V_3 are linearly independent.

   If this "identically 0" business is confusing to you, then recall from the problem we must have 0 at *every* point $\mathbf{p}$ so take $\mathbf{p} = (1,0,0)$. Then the vector field at that point is $(c_1 + c_3)U_1(\mathbf{p}) + c_2U_2 + (c_3 - c_1)U_3(\mathbf{p})$. For this to be zero we need to solve $c_1 + c_3 = 0$, $c_2 = 0$, and $c_3 - c_1 = 0$. The solution is clearly $c_1 = c_2 = c_3 = 0$.

   (b) We need to invert the definitions. Observe that $xV_1 - V_3 = -(x^2+1)U_3$ and $xV_3 + V_1 = (x^2+1)U_1$. For all $x \in \mathbb{R}$, we know $x^2 + 1 > 0$. Therefore, $U_3 = \frac{-1}{x^2+1}(xV_1 - V_3)$ and $U_1 = \frac{1}{x^2+1}(V_1 + xV_3)$. Trivially we observe $U_2 = V_2$.

[2] I.e., the zero function.

Thus, $xU_1 + yU_2 + zU_3 = \left(\frac{x-zx}{x^2+1}\right) V_1 + yV_2 \left(\frac{x^2+z}{x^2+1}\right) V_3$.

1.3 Directional Derivatives

1. For each, we must compute $g(t) := f(\mathbf{p} + t\mathbf{v})$, compute the derivative with respect to t, and evaluate the resulting expression at $t = 0$. In this case $\mathbf{v} = (2, -1, 3)$ and $\mathbf{p} = (2, 0, -1)$ so that $\mathbf{p} + t\mathbf{v} = (2, 0, -1) + t(2, -1, 3) = (2 + 2t, -t, -1 + 3t)$. Thus, we set $x = 2 + 2t$, $y = -t$, and $z = -1 + 3t$.
   (a) $f(x, y, z) = y^2 z \implies g(t) = f(\mathbf{p} + t\mathbf{v}) = (-t)^2(-1 + 3t) = t^2(-1 + 3t) = 3t^3 - t^2$. Thus, $g'(t) = 9t^2 - 2t$ and $g'(0) = 0$.
   (b) $f(x, y, z) = x^7 \implies g(t) = f(\mathbf{p} + t\mathbf{v}) = (2 + 2t)^7$. Thus, $g'(t) = 7(2 + 2t)^6 \cdot 2 = 14(2 + 2t)^6$ and $g'(0) = 14 \cdot 2^6 = 896$.
   (c) $f(x, y, z) = e^x \cos(y) \implies g(t) = f(\mathbf{p} + t\mathbf{v}) = e^{2+2t}\cos(-t) = e^{2+2t}\cos(t)$. Thus, $g'(t) = 2e^{2+2t}\cos(t) - e^{2+2t}\sin(t)$ and $g'(0) = 2e^{2+2\cdot 0}\cos(0) - e^{2+2\cdot 0}\sin(0) = 2e^2$.
2. For each, we need to compute $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, and $\frac{\partial f}{\partial z}$. Then evaluate at $\mathbf{p} = (2, 0, -1)$ and compute the inner product with $\mathbf{v} = (2, -1, 3)$.
   (a) $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 2yz$, and $\frac{\partial f}{\partial z} = y^2$. Thus, $\frac{\partial f}{\partial x}(\mathbf{p}) = 0$, $\frac{\partial f}{\partial y}(\mathbf{p}) = 0$, and $\frac{\partial f}{\partial z}(\mathbf{p}) = 0$. We confirm that $2 \cdot 0 + (-1) \cdot 0 + 3 \cdot 0 = 0$.
   (b) $\frac{\partial f}{\partial x} = 7x^6$, $\frac{\partial f}{\partial y} = 0$, and $\frac{\partial f}{\partial z} = 0$. Thus, $\frac{\partial f}{\partial x}(\mathbf{p}) = 448$, $\frac{\partial f}{\partial y}(\mathbf{p}) = 0$, and $\frac{\partial f}{\partial z}(\mathbf{p}) = 0$. We confirm that $2 \cdot 448 + (-1) \cdot 0 + 3 \cdot 0 = 896$.
   (c) $\frac{\partial f}{\partial x} = e^x \cos(y)$, $\frac{\partial f}{\partial y} = -e^x \sin(y)$, and $\frac{\partial f}{\partial z} = 0$. Thus, $\frac{\partial f}{\partial x}(\mathbf{p}) = e^2$, $\frac{\partial f}{\partial y}(\mathbf{p}) = 0$, and $\frac{\partial f}{\partial z}(\mathbf{p}) = 0$. We confirm that $2 \cdot e^2 + (-1) \cdot 0 + 3 \cdot 0 = 2e^2$.
3. For reference, $V = y^2 U_1 - xU_3$, $f = xy$, and $g = z^3$.
   (a) $V[f] = y^2 \cdot y - x \cdot 0 = y^3$
   (b) $V[g] = y^2 \cdot 0 - x \cdot (3z^2) = -3xz^2$
   (c) $V[fg] = V[f] \cdot g + f \cdot V[g] = y^3 \cdot z^3 + xy \cdot (-3xz^2) = y^3 z^3 - 3x^2 yz^2$
   (d) $fV[g] - gV[f] = V[fg] - 2gV[f] = y^3 z^3 - 3x^2 yz^2 - 2 \cdot z^3 \cdot y^3 = -3x^2 yz^2 - z^3 \cdot y^3$
   (e) First, $f^2 = x^2 y^2$ and $g^2 = z^6$ so that $f^2 + g^2 = x^2 y^2 + z^6$. It follows that $V[f^2 + g^2] = y^2 \cdot (2xy^2) - x \cdot (6z^5) = 2xy^4 - 6xz^5$.
   (f) $V[V[f]] = y^2 \cdot 0 - x \cdot 0 = 0$
4. Per the hint, we act $V = \sum v_i U_i$ on x_j:
$$V[x_j] = \left(\sum v_i U_i\right)[x_j] = \sum v_i U_i[x_j].$$
   This is by part (3) of Corollary 3.4. The remarks at the top of page 14 tell us that $U_i[f] = \frac{\partial f}{\partial x_i}$ so that $U_i[x_j] = \frac{\partial x_j}{\partial x_i}$. This is 0 unless $i = j$, in which case it is 1. We conclude $V[x_j] = v_j$. This means that $V = \sum v_i U_i = \sum V[x_j] U_i$, as desired.
5. Let $Z := V - W$, then using part (3) of Corollary 3.4 in reverse, we conclude for all f, that $Z[f] = (V - W)[f] = 0$. Note that this means the vector field Z acting on f yields the zero *function*. If we let $f = x_j$, we have $Z[x_j] = 0$. Thus, for every $\mathbf{p}$, we have $Z(\mathbf{p})[x_j] = 0$. Denote the ith component of $Z(\mathbf{p})$ by z_i, then we have $0 = Z(\mathbf{p})[x_j] = \sum z_i \frac{\partial x_j}{\partial x_i} = z_j$. Thus, $z_j = 0$ for all j so that $Z(\mathbf{p}) = 0$ for all $\mathbf{p}$. That is, $Z = 0$ which implies $V - W = 0$ or $V = W$.