Art School

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§1 Shortlist 2019 G1

Let T' be the point on (ADE) such that $AT' \parallel BC$. Then, we have

$$\measuredangle TFD = \measuredangle TAD = \measuredangle DBF,$$

hence AT' is tangent to (BDF) as desired.

§2 IMO 2020/1

The 3 lines meet at the circumecenter O of $\triangle PAB$. Note that $ADPO$ is cyclic, as

$$\angle AOP = 2 \cdot \angle PBA = \angle PAD + \angle DPA.$$

Then, $OA = OP$ and $OB = OP$ as desired.

§3 APMO 2018/1

Claim 3.1 — $FM = FN$.

Proof. Note that

$$\measuredangle FMN = \measuredangle FKL = \measuredangle MBH = 90 - \measuredangle A = \measuredangle NCH = \measuredangle NLH = \measuredangle FNM,$$

as desired. This also implies that F lies on the radical axis of (BMH) and (CNH), or FH is tangent to the two circles. □

Claim 3.2 — J lies on FH.

Proof. Note that

$$\measuredangle FHM = \measuredangle MKH = \measuredangle NCH = \measuredangle FHN,$$

thus FH bisects $\angle MHN$ which gives us what we want. □

Claim 3.3 — $AMJN$ is cyclic.

Proof. Note that

$$\angle MJN = 90 + \frac{1}{2}\angle MHN = 90 + \angle MHJ = 90 + \angle MBH = 90 + 90 - \angle A = 180 - \angle A$$

as desired. □

Claim 3.4 — F is the circumcenter of $(AMJN)$.

Proof. It suffices to show that $FM = FJ$. Note that

$$\angle FJM = 180 - \angle MJH = 90 - \frac{1}{2}\angle MNH$$
$$\angle FMJ = \frac{1}{2}\angle HMN + \angle FMN = \frac{1}{2}\angle HMN + \frac{1}{2}\angle MHN$$

It is easy to see that these two are equal, so we are done. □

§4 USAMO 2021/1

Let (ABB_1A_2) and (CAA_1C_2) intersect at P. Then, the angle condition implies that $\angle BPC + \angle BC_1C = 180$, hence (BCC_1B_2) hits P. To finish, note that APB_2 is collinear, since $\angle AP_1C = \angle B_2PC = 90°$, thus B_1C_2, C_1A_2, A_1B_2 concur at the point P.