

Art School

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DGX

§1 Shortlist 2019 G1

Let T' be the point on (ADE) such that $AT' \parallel BC$. Then, we have

$$\angle TFD = \angle TAD = \angle DBF,$$

hence AT' is tangent to (BDF) as desired.

§2 IMO 2020/1

The 3 lines meet at the circumcenter O of $\triangle PAB$. Note that $ADPO$ is cyclic, as

$$\angle AOP = 2 \cdot \angle PBA = \angle PAD + \angle DPA.$$

Then, $OA = OP$ and $OB = OP$ as desired.

§3 APMO 2018/1

Claim 3.1 — $FM = FN$.

Proof. Note that

$$\angle FMN = \angle FKL = \angle MBH = 90 - \angle A = \angle NCH = \angle NLH = \angle FNM,$$

as desired. This also implies that F lies on the radical axis of (BMH) and (CNH) , or FH is tangent to the two circles. \square

Claim 3.2 — J lies on FH .

Proof. Note that

$$\angle FHM = \angle MKH = \angle NCH = \angle FHN,$$

thus FH bisects $\angle MHN$ which gives us what we want. \square

Claim 3.3 — $AMJN$ is cyclic.

Proof. Note that

$$\angle MJN = 90 + \frac{1}{2}\angle MHN = 90 + \angle MHJ = 90 + \angle MBH = 90 + 90 - \angle A = 180 - \angle A$$

as desired. \square

Claim 3.4 — F is the circumcenter of $(AMJN)$.

Proof. It suffices to show that $FM = FJ$. Note that

$$\begin{aligned}\angle FJM &= 180 - \angle MJH = 90 - \frac{1}{2}\angle MNH \\ \angle FMJ &= \frac{1}{2}\angle HMN + \angle FMN = \frac{1}{2}\angle HMN + \frac{1}{2}\angle MHN\end{aligned}$$

It is easy to see that these two are equal, so we are done. \square

§4 USAMO 2021/1

Let (ABB_1A_2) and (CAA_1C_2) intersect at P . Then, the angle condition implies that $\angle BPC + \angle BC_1C = 180$, hence (BCC_1B_2) hits P . To finish, note that APB_2 is collinear, since $\angle AP_1C = \angle B_2PC = 90^\circ$, thus B_1C_2, C_1A_2, A_1B_2 concur at the point P .

§5 Shortlist 2013 G2

Let S be the midpoint of AT . Then, OS is the external angle bisector of $\angle MON$, which implies that $\triangle MON$ and $\triangle XOY$ are reflections about line OS . This means that $ANYT$ and $AMXT$ are isosceles trapezoids, which in turn implies that $MNYX$ is an isosceles trapezoid. Thus, K lies on OS , which implies that $KA = KT$, as desired.

§6 USAMO 2019/2

Let $(ABCD) = \omega$. Construct circles ω_A and ω_B centered at A and B with radii AD and BC , respectively. Note that they are orthogonal by the length condition. Let P' be the intersection of the radical axis of ω_A and ω_B with AB , $F = DP' \cap AC$, and $G = CP' \cap BD$. Note that when you invert wrt ω_A , line DP goes to ω , thus F lies on ω_B . Similarly, G lies on ω_A . Now, note that

$$\angle AP'D = \angle ADB = \angle ACB = \angle BP'C,$$

hence $P' = P$.

Note that since C, G, P are collinear, after inversion, this becomes circle $(AFGB)$. Thus,

$$\angle FGD = -\angle FGB = \angle FAB = \angle CDB,$$

which means that $FG \parallel CD$. Finally, Ceva's on $\triangle PCD$ gives us that PE bisects CE , as desired.

§7 Japan 2014/4

Let HD meet BC at T_1 , and IE meet BC at T_2 . Note that

$$\frac{BT_1}{BC} + \frac{CT_2}{BC} = \frac{BD}{BA} + \frac{CE}{CA} = \frac{BD}{BA} + 1 - \frac{AE}{CA} = 1,$$

hence $BT_1 + CT_2 = BC$, or $T = T_1 = T_2$.

Since

$$\angle ITC = \angle ABC = \angle IAC,$$

$(AICT)$ is cyclic. Thus, (HIT) is tangent to BC , as

$$\angle ITC = \angle IAC = \angle IHT.$$

Finally, let (HIT) hit (ABC) at points F' and G' , with F' closer to D . Note that by radax on $(ATCI)$, $(F'TG'I)$, $(AF'CG')$, we get that E lies on line $F'G'$. But we also can get that D lies on line $F'G'$, so line $F'G'$ is really just line FG , so we are done.

§8 IMO 2013/3

First, we prove this lemma that is true in general.

Lemma 8.1

(AB_1C_1) hits (ABC) at the midpoint of major arc BC .

Proof. Let L be the arc midpoint of major arc BC . It is known that I_B, L, A, I_C are collinear, by considering the 9 point circle of $I_AI_BI_C$, which is just (ABC) . Now, note that $BC_1 = CB_1$, as both are equal to the distance from A to the incircle touchpoints on AB and AC . Thus,

$$\triangle LBC_1 \cong \triangle LCB_1$$

by SAS congruence, hence L is the center of the spiral congruence taking $\triangle LBC_1$ to $\triangle LCB_1$. This means that L is the Miquel point of BCB_1C_1 , so L lies on (AB_1C_1) , as desired. Note that this also means $LB_1 = LC_1$, or L lies on the perpendicular bisector of B_1C_1 . \square

Returning to the original problem, we consider arc midpoints M and N of arcs CA and AB . Note that one of L, M, N must be the circumcenter of $(A_1B_1C_1)$, so WLOG we let it be L . We now claim that $\angle A = 90^\circ$.

Note that LM and LN are the perpendicular bisectors of sides A_1C_1 and A_1B_1 , respectively. Then, we have

$$\begin{aligned} \angle A &= \angle C_1LB_1 = \angle C_1LA_1 + \angle A_1LB_1 = 2(\angle MLA_1 + \angle NLA_1) = 2\angle MLN \\ &= 2(\angle A - \angle BAM - \angle CAN) \\ &= 2\left(\angle A - \left(90^\circ - \frac{\angle B}{2} - \angle C\right) - \left(90^\circ - \frac{\angle C}{2} - \angle B\right)\right) \\ &= 2\angle A - 360^\circ + 3(\angle B + \angle C) = 2\angle A - 360^\circ + 3(180^\circ - \angle A) \\ &= 90^\circ \end{aligned}$$

as desired.

§9 JMO 2020/2

The locus of R is arc AB of the circumcircle of $\triangle ABC$, and the arc $A'B'$ formed by taking a negative homothety at I with ratio -2 .

We fix $\triangle ABC$, and then let a point T vary on the incircle of $\triangle ABC$, where T is the tangency point of line PQ . If we let R_1 and R_2 be the intersections of TI with arc AB and arc $A'B'$ respectively, then the key observation is that R_1 and R_2 are the two possible locations of R for this choice of T . Obviously, over all choices of the point T ,

we can get all points on the locus by picking the point on the locus that satisfies the homothety.

First, we prove that $PR_1 = PA$ and $QR_1 = QB$. Since $\angle PIR_1 = \angle PIA$ and $IR_1 = IA$, then we have

$$\triangle PIR_1 \cong \triangle PIA$$

From this, it follows that

$$\angle PR_1A = \angle IR_1A + \angle PR_1I = \angle IAR_1 + \angle PAI = \angle DAR_1$$

hence $PR_1 = PA$. Similarly, $QR_1 = QB$ which is what we wanted.

Now, we show that $PR_2 = PA$ and $QR_2 = QB$. Note that under the negative homothety at I with ratio -2 , we have

$$\begin{aligned} T &\rightarrow R_1 \\ R_1 &\rightarrow R_2 \end{aligned}$$

Hence

$$R_2T = R_2I - IT = 2 \cdot R_1I - IT = R_1I + 2 \cdot IT - IT = R_1T$$

so $\triangle PTR_1 \cong \triangle PTR_2$, which gives us $PR_2 = PR_1 = PA$. Similarly, $QR_2 = QB$, so we are done.