Global

Brian Zhang

DCW

§1 HMMT 2013

The answer is $2013\left(1-\left(\frac{2012}{2013}\right)^{2013}\right)$. This is the sum of the probability any given element is in the subset over all 2013 elements.

§2 Russia 1996

Let each committee member be in a_i committees. The key observation is that

$$\frac{\sum \binom{a_i}{2}}{\binom{16000}{2}} \geq \frac{1600 \cdot \binom{800}{2}}{\binom{16000}{2}}.$$

Since the RHS is greater than 3, there must exist some committee with having at least four common members.

§3 USAMO 1999/1

Note that the conditions imply that every checkered square is adjacent to another checkered square. We can construct the squares one after another such that the conditions are true, so each checkered square will "cover" 3 squares out of the total n^2 squares. However, the first and last square will each cover 4 squares, hence at least $(n^2-2)/3$ checkers must be used.

§4 BAMO 2017/4

Pick some "minimal" point X within $\mathcal{P}$ such that $\min(h_1, h_2, \ldots, h_n)$ is maximized, where $h_1, h_2, \ldots, h_n$ are the heights from X to each of the sides.

Then, the key claim is that setting $h = \min(h_1, h_2, \ldots, h_n)$ gives us the desired value of h. If we assume there is some uncovered region, then we pick some point X' with heights $h'_1, h'_2, \ldots h'_n$ to each of the sides within this uncovered region. This then implies that

$$\min(h'_1, h'_2, \ldots, h'_n) > \min(h_1, h_2, \ldots, h_n),$$

which is a contradiction.

Thus, we get

$$\sum s_i \cdot h \leq 2 \cdot [\mathcal{P}] = \sum s_i \cdot h_i,$$

as desired.

§5 JBMO 2007/3

The key observation is that

$$\binom{13}{3} - 2 \cdot \binom{13}{2} = 130.$$

More generally, the number of scalene triangles among n points with no three collinear is $\binom{n}{3} - 2 \cdot \binom{n}{2}$. In total, there are $\binom{n}{3}$ triangles. We subtract $2 \cdot \binom{n}{2}$ possible isosceles triangles, as each segment between two points can only be the base of 2 isosceles triangles, because if there were more then the vertices of the triangles would lie on the perpendicular bisector of the segment. Thus the number of scalene triangles is 13, as desired.