Art School

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§1 Shortlist 2019 G1

Let T' be the point on (ADE) such that $AT' \parallel BC$. Then, we have

$$\measuredangle TFD = \measuredangle TAD = \measuredangle DBF,$$

hence AT' is tangent to (BDF) as desired.

§2 IMO 2020/1

The 3 lines meet at the circumecenter O of $\triangle PAB$. Note that $ADPO$ is cyclic, as

$$\angle AOP = 2 \cdot \angle PBA = \angle PAD + \angle DPA.$$

Then, $OA = OP$ and $OB = OP$ as desired.

§3 APMO 2018/1

Claim 3.1 — $FM = FN$.

Proof. Note that

$$\measuredangle FMN = \measuredangle FKL = \measuredangle MBH = 90 - \measuredangle A = \measuredangle NCH = \measuredangle NLH = \measuredangle FNM,$$

as desired. This also implies that F lies on the radical axis of (BMH) and (CNH), or FH is tangent to the two circles. $\square$

Claim 3.2 — J lies on FH.

Proof. Note that

$$\measuredangle FHM = \measuredangle MKH = \measuredangle NCH = \measuredangle FHN,$$

thus FH bisects $\angle MHN$ which gives us what we want. $\square$

Claim 3.3 — $AMJN$ is cyclic.

Proof. Note that

$$\angle MJN = 90 + \frac{1}{2}\angle MHN = 90 + \angle MHJ = 90 + \angle MBH = 90 + 90 - \angle A = 180 - \angle A$$

as desired. $\square$

Claim 3.4 — F is the circumcenter of $(AMJN)$.

Proof. It suffices to show that $FM = FJ$. Note that

$$\angle FJM = 180 - \angle MJH = 90 - \frac{1}{2}\angle MNH$$
$$\angle FMJ = \frac{1}{2}\angle HMN + \angle FMN = \frac{1}{2}\angle HMN + \frac{1}{2}\angle MHN$$

It is easy to see that these two are equal, so we are done. □

§4 USAMO 2021/1

Let (ABB_1A_2) and (CAA_1C_2) intersect at P. Then, the angle condition implies that $\angle BPC + \angle BC_1C = 180$, hence (BCC_1B_2) hits P. To finish, note that APB_2 is collinear, since $\angle AP_1C = \angle B_2PC = 90°$, thus B_1C_2, C_1A_2, A_1B_2 concur at the point P.

§5 Shortlist 2013 G2

Let S be the midpoint of AT. Then, OS is the external angle bisector of $\angle MON$, which implies that $\triangle MON$ and $\triangle XOY$ are reflections about line OS. This menas that $ANYT$ and $AMXT$ are isosceles trapezoids, which in turn implies that $MNYX$ is an isosceles trapezoid. Thus, K lies on OS, which implies that $KA = KT$, as desired.

§6 USAMO 2019/2

Let $(ABCD) = \omega$. Construct circles ω_A and ω_B centered at A and B with radii AD and BC, respectively. Note that they are orthogonal by the length condition. Let P' be the intersection of the radical axis of ω_A and ω_B with AB, $F = DP' \cap AC$, and $G = CP' \cap BD$. Note that when you invert wrt ω_A, line DP goes to ω, thus F lies on ω_B. Similarly, G lies on ω_A. Now, note that

$$\measuredangle AP'D = \measuredangle ADB = \measuredangle ACB = \measuredangle BP'C,$$

hence $P' = P$.

Note that since C, G, P are collinear, after inversion, this becomes circle $(AFGB)$. Thus,

$$\measuredangle FGD = -\measuredangle FGB = \measuredangle FAB = \measuredangle CDB,$$

which means that $FG \parallel CD$. Finally, Ceva's on $\triangle PCD$ gives us that PE bisects CE, as desired.

§7 Japan 2014/4

Let HD meet BC at T_1, and IE meet BC at T_2. Note that

$$\frac{BT_1}{BC} + \frac{CT_2}{BC} = \frac{BD}{BA} + \frac{CE}{CA} = \frac{BD}{BA} + 1 - \frac{AE}{CA} = 1,$$

hence $BT_1 + CT_2 = BC$, or $T = T_1 = T_2$.

Since

$$\measuredangle ITC = \measuredangle ABC = \measuredangle IAC,$$

$(AICT)$ is cyclic. Thus, (HIT) is tangent to BC, as

$$\measuredangle ITC = \measuredangle IAC = \measuredangle IHT.$$

Finally, let (HIT) hit (ABC) at points F' and G', with F' closer to D. Note that by radax on $(ATCI), (F'TG'I), (AF'CG')$, we get that E lies on line $F'G'$. But we also can get that D lies on line $F'G'$, so line $F'G'$ is really just line FG, so we are done.

§8 IMO 2013/3

First, we prove this lemma that is true in general.

Lemma 8.1
(AB_1C_1) hits (ABC) at the midpoint of major arc BC.

Proof. Let L be the arc midpoint of major arc BC. It is known that I_B, L, A, I_C are collinear, by considering the 9 point circle of $I_AI_BI_C$, which is just (ABC). Now, note that $BC_1 = CB_1$, as both are equal to the distance from A to the incircle touchpoints on AB and AC. Thus,

$$\triangle LBC_1 \cong \triangle LCB_1$$

by SAS congruence, hence L is the center of the spiral congruence taking $\triangle LBC_1$ to $\triangle LCB_1$. This means that L is the Miquel point of BCB_1C_1, so L lies on (AB_1C_1), as desired. Note that this also means $LB_1 = LC_1$, or L lies on the perpendicular bisector of B_1C_1. □

Returning to the original problem, we consider arc midpoints M and N of arcs CA and AB. Note that one of L, M, N must be the circumcenter of $(A_1B_1C_1)$, so WLOG we let it be L. We now claim that $\angle A = 90°$.

Note that LM and LN are the perpendicular bisectors of sides A_1C_1 and A_1B_1, respectively. Then, we have

$$\begin{aligned}
\angle A = \angle C_1LB_1 &= \angle C_1LA_1 + \angle A_1LB_1 = 2(\angle MLA_1 + \angle NLA_1) = 2\angle MLN \\
&= 2(\angle A - \angle BAM - \angle CAN) \\
&= 2\left(\angle A - \left(90^\circ - \frac{\angle B}{2} - \angle C\right) - \left(90^\circ - \frac{\angle C}{2} - \angle B\right)\right) \\
&= 2\angle A - 360^\circ + 3(\angle B + \angle C) = 2\angle A - 360^\circ + 3(180^\circ - \angle A) \\
&= 90^\circ
\end{aligned}$$

as desired.

§9 January TST 2013/2

bruh wtf are the point names for this
Let the foot from C to AB be D. Note that C lies on the perpendicular bisector of LN, so $CL = CN$. Similarly, $CM = CK$. Also, $NKLM$ is cyclic, since $NH \cdot LH = CH \cdot DH = KH \cdot MH$, which implies that C is the circumcenter of $NKLM$. Let this circle be ω.

Now, note that BK is tangent to ω, as $\angle BKC = 90°$. Pascals on $KKMLLN$ tells us that LM and KN meet on line AB. Finally, by Brocards, D is the Miquel point of $NKLM$, hence $MN \cap KL$ lies on AB, as desired.

§10 JMO 2020/2

The locus of R is arc AB of the circumcircle of $\triangle ABC$, and the arc $A'B'$ formed by taking a negative homothety at I with ratio -2.

We fix $\triangle ABC$, and then let a point T vary on the incircle of $\triangle ABC$, where T is the tangency point of line PQ. If we let R_1 and R_2 be the intersections of TI with arc AB and arc $A'B'$ respectively, then the key observation is that R_1 and R_2 are the two possible locations of R for this choice of T. Obviously, over all choices of the point T, we can get all points on the locus by picking the point on the locus that satisfies the homothety.

First, we prove that $PR_1 = PA$ and $QR_1 = QB$. Since $\angle PIR_1 = \angle PIA$ and $IR_1 = IA$, then we have

$$\triangle PIR_1 \cong \triangle PIA$$

From this, it follows that

$$\angle PR_1A = \angle IR_1A + \angle PR_1I = \angle IAR_1 + \angle PAI = \angle DAR_1$$

hence $PR_1 = PA$. Similarly, $QR_1 = QB$ which is what we wanted.

Now, we show that $PR_2 = PA$ and $QR_2 = QB$. Note that under the negative homothety at I with ratio -2, we have

$$\begin{aligned} T &\to R_1 \\ R_1 &\to R_2 \end{aligned}$$

Hence

$$R_2T = R_2I - IT = 2 \cdot R_1I - IT = R_1I + 2 \cdot IT - IT = R_1T$$

so $\triangle PTR_1 \cong \triangle PTR_2$, which gives us $PR_2 = PR_1 = PA$. Similarly, $QR_2 = QB$, so we are done.