ℓ^p-norms and optimal windows

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1 Introduction

We consider the following problem:

$$\max_{g,\|g\|_2=1} \|\mathcal{V}_g(f)\|_p^p = \max_{g,\|g\|_2=1} \sum_{j,k} |\langle M_k T_j g, f\rangle|^p$$

with $p \geq 2$.

2 Functional derivative and solution

2.1 At infinity

We have the following result:

$$\lim_{p\to\infty} \left(\sum_{j,k} |\langle M_k T_j g, f\rangle|^p \right)^{1/p} = \max_{j,k} |\langle M_k T_j g, f\rangle|$$

The maximun of the scalar product is attained when $M_k T_j g = f$. The optimal window is equal to the signal (up to a translation and modulation).

2.2 For $p > 2$

Introduce $A = |\langle M_k T_j g, f\rangle|^2$, its derivative with respect to $\overline{g}$ is:

$$A'(g) = \langle M_k T_j g, f\rangle T_i M_k f.$$

One can write

$$\frac{\partial}{\partial \overline{g}} \sum_{j,k} |\langle M_k T_j g, f\rangle|^p = \sum_{j,k} \frac{\partial}{\partial \overline{g}} A^{p/2}(g) = \frac{p}{2} \sum_{j,k} A(g)^{\frac{p}{2}-1} A'(g)$$

The optimal g is given by the solution of

$$\frac{p}{2} \sum_{j,k} |\langle M_k T_j g, f\rangle|^{\frac{p}{2}-1} \langle M_k T_j g, f\rangle T_i M_k f = \lambda g$$

with λ the smallest eigenvalue of the Gabor multiplier with mask $A(g)^{\frac{p}{2}-1}$. For p close to 2, for our signals the optimal windows look like chirped Gaussians. Is there a simple way to explain that? It is not always true since for example we managed to obtain a square-shaped window for f being a square in the TF plane in our EUSIPCO paper (by solving the above eigenvalue problem).

2.3 Near two

Introduce $\gamma = 2 - p$. One can write

$$A^{\gamma} = \exp(p \ln A) = 1 + \gamma \ln A + \gamma^2 \ln^2 A + \cdots \tag{1}$$

For p tends to two the L^p-norm becomes:

$$\begin{aligned}\sum_{j,k} |\langle M_k T_j g, f\rangle|^p &= \sum_{j,k} |\langle M_k T_j g, f\rangle|^2 |\langle M_k T_j g, f\rangle|^{\gamma} \\ &= 1 + \frac{\gamma}{2} \sum_{j,k} |\langle M_k T_j g, f\rangle|^2 \ln |\langle M_k T_j g, f\rangle|^2 + \mathcal{O}(\gamma^2)\end{aligned}$$

The derivative with respect to $\overline{g}$ gives:

$$A'(g) = \langle M_k T_j g, f\rangle T_i M_k f,$$

and

$$(A \ln A)' = A'(1 + \ln A).$$

Notice that if $\{T_i M_k f\}$ is a tight frame,

$$\sum A' = g.$$

So that the extremum of the ℓ^p-norm is the solution of (case of tight frame):

$$-\sum_{j,k} \ln |\langle M_k T_j g, f\rangle|^2 \langle g, T_j M_k f\rangle T_i M_k f = g + \lambda g.$$

One has to diagonalize the gabor multiplier with mask

$$M = -\ln |\langle M_k T_j g, f\rangle|^2.$$

3 Optimization in the subclass of chirped Gaussians

Since the optimization problem with a small value for p leads to a chirped Gaussian-like atom, it may be more convenient to directly search for a solution in the subclass of chirped Gaussians. Let us introduce the subclass $\mathcal{G}$ of chirped Gaussian in $\mathbb{C}^N$. A function in this space is of the form $M_k T_j \phi_{\sigma,s}$ where

$$\phi_{\sigma,s}(t) = \sqrt[4]{\frac{2}{N\sigma}} e^{-\pi \frac{t^2}{N\sigma} + i\pi s \frac{t^2}{N}}, \tag{2}$$

where σ is positive and s is a real number.

We want to maximize the following quantity over σ and s:

$$\|\mathcal{V}_{\phi_{\sigma,s}} f\|^p,$$

where $p > 2$.

3.1 Gradient step

The gradient is given by

$$\frac{\partial \|\mathcal{V}_{\phi_{\sigma,s}} f\|^p}{\partial \sigma} = p \operatorname{Re}[\langle G_{f,M} \phi_{\sigma,s}, \frac{\partial \phi_{\sigma,s}}{\partial \sigma}\rangle], \tag{3}$$

$$\frac{\partial \|\mathcal{V}_{\phi_{\sigma,s}} f\|^p}{\partial s} = p \operatorname{Re}[\langle G_{f,M} \phi_{\sigma,s}, \frac{\partial \phi_{\sigma,s}}{\partial s}\rangle], \tag{4}$$

where $G_{f,M}$ is the Gabor multiplier with window f and mask $M = |\langle M_k T_j f, \phi_{\sigma,s}\rangle|^{p-2}$. One can see it by writing for $p > 2$ even, any window g and function f:

$$\sum_{j,k} |\langle M_k T_j g, f\rangle|^p = \sum_{j,k} \langle M_k T_j g, f\rangle^{p/2} \langle f, M_k T_j g\rangle^{p/2}$$

and

$$\frac{\partial}{\partial s} \langle M_k T_j \phi_{\sigma,s}, f\rangle^{p/2} = \frac{p}{2} \langle \frac{\partial}{\partial s} \phi_{\sigma,s}, T_j M_k f\rangle \langle \phi_{\sigma,s}, T_j M_k f\rangle^{p/2-1}.$$

At each step of the gradient method, one compute:

$$\sigma_{n+1} = \sigma_n + \alpha p \ \mathrm{Re}[\langle G_{f,M^{(n)}} \phi^{(n)}_{\sigma,s}, \frac{\partial \phi^{(n)}_{\sigma,s}}{\partial \sigma}\rangle] \tag{5}$$

$$s_{n+1} = s_n + \alpha p \ \mathrm{Re}[\langle G_{f,M^{(n)}} \phi^{(n)}_{\sigma,s}, \frac{\partial \phi^{(n)}_{\sigma,s}}{\partial s}\rangle] \tag{6}$$

and $\phi^{(n+1)}_{\sigma,s} = \phi_{\sigma_{n+1}, s_{n+1}}$.

The derivatives of the function $\phi_{\sigma,s}$ with respect to σ and s read:

$$\frac{\partial \phi_{\sigma,s}}{\partial \sigma}(t) = \left(\frac{\pi t^2}{\sigma^2 N} - \frac{1}{4\sigma}\right) \phi_{\sigma,s}(t)$$

and

$$\frac{\partial \phi_{\sigma,s}}{\partial s}(t) = \frac{i\pi}{N} t^2 \phi_{\sigma,s}(t).$$

3.2 Hessian and Newton's method

In order to calculate the Hessian matrix we would need the second derivatives with respect to the parameters:

$$\frac{\partial^2}{\partial s^2} \phi_{\sigma,s}(t) = -\left(\frac{\pi t^2}{N}\right)^2 \phi_{\sigma,s}(t) \tag{7}$$

$$\frac{\partial^2}{\partial s \partial \sigma} \phi_{\sigma,s}(t) = i \frac{\pi t^2}{N} \left(\frac{\pi t^2}{\sigma^2 N} - \frac{1}{4\sigma}\right) \phi_{\sigma,s}(t) \tag{8}$$

$$\frac{\partial^2}{\partial \sigma^2} \phi_{\sigma,s}(t) = \left[\frac{1}{4\sigma^2} - \frac{2\pi t^2}{N\sigma^3} + \left(\frac{\pi t^2}{\sigma^2 N} - \frac{1}{4\sigma}\right)^2\right] \phi_{\sigma,s}(t) \tag{9}$$

However, the derivative of the whole cost function leads to an operation which is not a Gabor multiplier...

3.3 Lattice parameters

In this section we present a way to provide the most appropriate lattice to our optimal window. For example if the window is a round shaped Gaussian, the best lattice is the quincux lattice as it packs the atoms in an optimal manner.

Our optimal windows are ellipsis-shaped atoms in the TF plane. They are characterized in the TF plane by their excentricity $e = L/l$ where l and L are the small and large diameter respectively. In addition, the ellipsis has an orientation given by the direction parallel to the large diameter. The difference from the horizontal direction is given by a shear parameter s. We recall the expression given in (2) of the optimal function:

$$\phi_{\sigma,s}(t) = \sqrt[4]{\frac{2}{N\sigma}} e^{-\pi \frac{t^2}{N\sigma} + i\pi s \frac{t^2}{N}}. \tag{10}$$

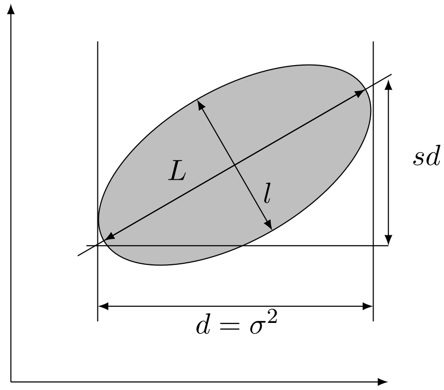

Figure 1: Atoms parameters.

This function is normalized such that $\|\phi_{\sigma,s}\|_2 = 1$. It has been chosen such that it is compatible with the LTFAT functions "pgauss" and "pchirp". The spreading d of the function along the time direction is given by $d = \sigma^2$ (variance of the Gaussian). σ has the same effect as the parameter "tfr" of "pgauss". The shear of the ellipsis is given by s as in the function "pchirp". A chirp with slope s revolves s times around the time-frequency plane in frequency. Here s may not be an integer. To link the excentricity and the variance, we know that a Gaussian with a variance d in time has a variance $1/d$ in frequency. We can deduce the relation with the excentricity of the ellipsis:

$$e = \sigma^2(1 + s^2). \tag{11}$$

3.4 Spread limitation

Must be revised, we can control the spreading by controlling the value of d. We don't really need to multiply by a weight function

Some optimal window are too much spread in time and prevent a precise localization of the TF component in time. We introduced an additional constraint in the optimization procedure in order to control this spreading. We introduce a weight function w of the form

$$w(t) = e^{-t^2/a^2}, \qquad a > 0.$$

The parameter a fixes the time spreading of the optimal window. In the gradient ascent loop, the computation is modified at each step by adding:

$$\psi_{\sigma,s}^{(n)} = w\phi_{\sigma,s}^{(n)}.$$

So that:

$$s_{n+1} = s_n + \nabla s,$$

where

$$\nabla s = p\ \langle G_{f,M^{(n)}}\psi_{\sigma,s}^{(n)}, \frac{\partial \psi_{\sigma,s}^{(n)}}{\partial s}\rangle,$$

with

$$M^{(n)} = |\langle M_k T_j f, \psi_{\sigma,s}^{(n)}\rangle|^{p-2}.$$

It is a sort of projection but instead of setting some coefficients to zero, we weight them with w. It is not a projection since $w\phi \neq w(w\phi)$.