Wedge product matrices and applications

Brian Chan

University of Melbourne

yaoc6@student.unimelb.edu.au

September 8, 2022

Overview

1. Wedge product matrices
2. Quasideterminants
3. Eigenvector-eigenvalue identity
4. Smith normal form
5. Orbits of principal congruence subgroups

Wedge product matrices:

- R commutative ring, $A \in M_{n \times n}(R)$, $k \in \{1, 2, \ldots, n\}$

- $T_{\binom{n}{k}} = \{ S \subseteq \{1, 2, \ldots, n\} \mid |S| = k \}$

- In the free R-module $\underline{\Lambda^k(R^n)}$,

$$Ae_J = Ae_{j_1} \wedge \ldots \wedge Ae_{j_k} = \sum_{I \in T_{\binom{n}{k}}} (\Lambda^k(A))_{I,J} \, e_I$$

where $J = \{ j_1, \ldots, j_k \} \subseteq T_{\binom{n}{k}}$.

- The matrix $\Lambda^k(A) \in M_{\binom{n}{k} \times \binom{n}{k}}(R)$.

Properties of wedge product matrices:

- $\boxed{\Lambda^k(AB) = \Lambda^k(A)\,\Lambda^k(B)}$

- $(\Lambda^k(A))_{I,J} = \det(A_{I,J})$ where $I, J \in T_{\binom{n}{k}}$.

- $\Lambda^1(A) = A$ and $\Lambda^n(A) = (\det(A))$

- $\Lambda^k(I_n) = I_{\binom{n}{k}}$ where I_n is the $n \times n$ identity matrix.

- $\boxed{\det(\Lambda^k(A)) = \det(A)^{\binom{n-1}{k-1}}}$

Adjugate matrices:

— From <u>Laplace expansion</u>:

$$\det(A) = \sum_{L \in T_{\binom{n}{k}}} (\Lambda^k(A))_{L,H} \; S_{L,\{1,2,\ldots,n\}} \; S_{H,\{1,\ldots,n\}} \; (\Lambda^{n-k}(A))_{L^c,H^c}$$

$$\underbrace{\hspace{6cm}}_{\displaystyle (\Upsilon^{n-k}(A))_{H,L}}$$

— $\boxed{\Lambda^k(A) \, \Upsilon^{n-k}(A) = \Upsilon^{n-k}(A) \, \Lambda^k(A) = \det(A) \, I_{\binom{n}{k}}.}$

— Υ^{n-k} shares <u>similar properties</u> to Λ^k:

Quasideterminants:

- An analogue of the determinant for square matrices in a non-commutative ring.

- Molev: Quasideterminants used to factorise the quantum determinant

- In a commutative ring, quasideterminants are connected to elements of the adjugate matrix.

Gel'fand Retakh Molev definition:

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 3 & -4 \\ 7 & 6 & -3 \end{pmatrix}$$

$$|A|_{2,3} = \boxed{-4} - \begin{pmatrix} 0 & 3 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 7 & 6 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

$$= -\frac{37}{19}$$

$$\boxed{|A|_{j,\ell} = a_{j,\ell} - r_j^{\ell} \left(A^{j\ell} \right)^{-1} c_\ell^{j}}$$

– In a commutative ring,

$$\boxed{|A|_{j,\ell} \left(\Upsilon^{n-1}(A) \right)_{\ell,j} = \det(A).}$$

My general definition:

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 3 & -4 \\ 7 & 6 & -3 \end{pmatrix}$$

$$|A|_{\{1,2\},\{1,3\}} = \begin{pmatrix} 2 & 1 \\ 0 & -4 \end{pmatrix} - \begin{pmatrix} -1 \\ 3 \end{pmatrix} 6^{-1} \begin{pmatrix} 7 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{19}{6} & \frac{1}{2} \\ -\frac{7}{2} & -\frac{5}{2} \end{pmatrix}.$$

$$\boxed{|A|_{J,L} = A_{J,L} - A_{J,L^c} \left(A_{J^c,L^c} \right)^{-1} A_{J^c,L},}$$

$$J, L \in T_{\binom{n}{k}}$$

– In a commutative ring,

$$\boxed{\det \left(|A|_{J,L} \right) \left(\Upsilon^{n-k}(A) \right)_{L,J} = \det(A).}$$

Some properties of quasideterminants:

- $A \in M_{n \times n}(R)$, R commutative ring

$$\det(A) = \frac{1}{n} \begin{vmatrix} & A & \\ |A|_{j,1} & |A|_{j,2} \cdots |A|_{j,n} \\ & A & \end{vmatrix}, \quad j \in \{1, 2, \ldots, n\}$$

- If $J = \{j_1, \ldots, j_k\}$ and $L = \{l_1, \ldots, l_k\}$ are elements of $T_{\binom{n}{k}}$ then

$$\det(|A|_{J,L}) = \begin{vmatrix} |A|_{j_1, l_1}^{-1} & |A|_{j_1, l_2}^{-1} & \cdots & |A|_{j_1, l_k}^{-1} \\ \vdots & \vdots & \ddots & \vdots \\ |A|_{j_k, l_1}^{-1} & |A|_{j_k, l_2}^{-1} & \cdots & |A|_{j_k, l_k}^{-1} \end{vmatrix}^{-1}$$

⚠️

Bump, Hoffstein:

- Minimal parabolic Eisenstein series:

$$E_{v_1, v_2}(\tau) = \sum_{\gamma \in \Gamma_\infty(3) \backslash \Gamma(3)} \kappa(\gamma) \, I_{v_1, v_2}(\gamma \tau)$$

- Fourier coefficients of Eisenstein series computed by _particular_ representatives of $\Gamma_\infty(3) \backslash \Gamma(3)$.

$$\mathcal{O} = \mathbb{Z}[e^{2\pi i/3}]$$

$$\Gamma(3) = \{ A \in SL_3(\mathcal{O}) \mid A \equiv I_3 \mod 3\mathcal{O} \}$$

$$\Gamma_\infty(3) = \Gamma(3) \cap \underline{U}$$

Invariants of the coset $\Gamma_\infty(3) \cdot A$:

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \in \Gamma(3)$$

- $\underline{\Lambda^1 \text{ invariants}}$:

$$A_1 = g, \quad B_1 = h, \quad C_1 = i$$

- $\underline{\Lambda^2 \text{ invariants}}$:

$$A_2 = dh - eg, \quad B_2 = di - fg, \quad C_2 = ei - fh$$

$$\text{Inv}(A) = (A_1, B_1, C_1, A_2, B_2, C_2) \in \sigma^6$$

Invariant conditions:

$$(\text{I1}) \quad A_1 \equiv A_2 \equiv B_1 \equiv B_2 \equiv 0 \mod 3\sigma$$

$$(\text{I2}) \quad C_1 \equiv C_2 \equiv 1 \mod 3\sigma$$

$$(\text{I3}) \quad \gcd(A_1, B_1, C_1) = \gcd(A_2, B_2, C_2) = 1$$

$$(\text{I4}) \quad A_1 C_2 - B_1 B_2 + A_2 C_1 = 0$$

Theorem: If $A, B \in \Gamma(3)$ then

$$\Gamma_\infty(3) \cdot A = \Gamma_\infty(3) \cdot B \iff \text{Inv}(A) = \text{Inv}(B)$$

The connection to Bump and Hoffstein:

- For $A \in \Gamma(3)$, define

$$^\iota A = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} (A^{-1})^\top \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$$

- Invariants of A (BH): Bottom rows of A and $^\iota A$

$$\Lambda^2(A) = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix} (^\iota A) \begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix}$$

From invariants to a representative:

$\mathcal{O} = \mathbb{Z}[\omega]$, $\omega = e^{2\pi i/3}$

$(A_1, B_1, C_1, A_2, B_2, C_2) = (-3+6\omega, -3, -2-3\omega, -6+3\omega, 3-6\omega, 4+3\omega)$
satisfies the invariant conditions. (Either $A_1 \neq 0$ or $B_1 \neq 0$)

$$X = \begin{pmatrix} 1 & 0 & 2\omega \\ & 1 & 1+2\omega \\ & & 1 \end{pmatrix} \begin{pmatrix} -1-\omega & & \\ & 1+\omega & \\ & & 1+\omega \end{pmatrix} \begin{pmatrix} 2+2\omega & 1+6\omega & \\ 3+2\omega & 4+8\omega & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1+\omega & -1-\omega \\ & 3 & -3-\omega \end{pmatrix} \begin{pmatrix} 1+2\omega & \omega & \\ 2+3\omega & \omega & \\ & & 1 \end{pmatrix}$$

$X \in \Gamma(3)$

$\boxed{\text{Inv}(X) = (A_1, B_1, C_1, A_2, B_2, C_2)}$

Similar to
Bruhat decomposition

The End

Thank you for listening