The R matrix of the six vertex model

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0.1 Finding the R matrix

The purpose of this document is to derive the form of the R matrix of the six vertex model directly from the RLL relations, which are depicted below graphically:

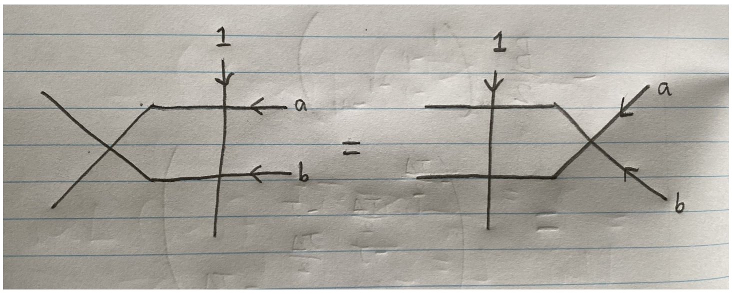


Figure 1: The RLL relations with weights a, b and 1.

The R matrices are depicted graphically by the cross, whereas the other intersections denote L matrices. The L matrices of the six vertex model are explicitly

$$L = \begin{pmatrix} a & & & \\ & b & c & \\ & c & b & \\ & & & a \end{pmatrix} \text{ and } L' = \begin{pmatrix} a' & & & \\ & b' & c' & \\ & c' & b' & \\ & & & a' \end{pmatrix}.$$

The goal is to find conditions on the Boltzmann weights a, b, c, a', b', c' such that there exists an invertible R-matrix which takes the form

$$R = \begin{pmatrix} a'' & & & \\ & b'' & c'' & \\ & c'' & b'' & \\ & & & a'' \end{pmatrix}$$

and satisfies the RLL relations. We follow the method described in [Zin21, Section 7.3]. Our first observation is that in the graphical depiction of the RLL relations, there are three incoming and outgoing edges, which can either be occupied or empty. Hence, there are $2^6 = 64$ equations which need to be satisfied simultaneously in order for the RLL relations to hold, depending on whether any given incoming/outgoing edge is occupied or empty.

While this might seem like a massive undertaking, there are quite a few observations which simplify the situation drastically. The first one is *line* *conservation* — the number of incoming occupied/empty edges must equal the number of outgoing occupied/empty edges. With this restriction, the number of relevant equations dwindles to

$$\binom{3}{0}^2 + \binom{3}{1}^2 + \binom{3}{2}^2 + \binom{3}{3}^2 = 20 \text{ equations.}$$

The binomial coefficient $\binom{3}{k}^2$ is the total number of configurations with k incoming edges and k outgoing edges. Obviously, $k \in \{0, 1, 2, 3\}$.

The next idea is *parity symmetry*. If we take a particular configuration and flip every occupied edge to an empty edge and vice versa, we obtain the same equation. This is because the six vertex model satisfies the ice rule, which renders each vertex "flip invariant". Hence, our 20 equations further reduce to

$$\binom{3}{0}^2 + \binom{3}{1}^2 = 10 \text{ equations.}$$

The final symmetry which we will apply is that if we rotate the LHS of the RLL relation by 180 degrees, we obtain the RHS of the RLL relation and vice versa. It turns out that out of the 10 remaining equations, four of them have 180° rotation invariant boundary conditions and are thus, automatically satisfied. They are

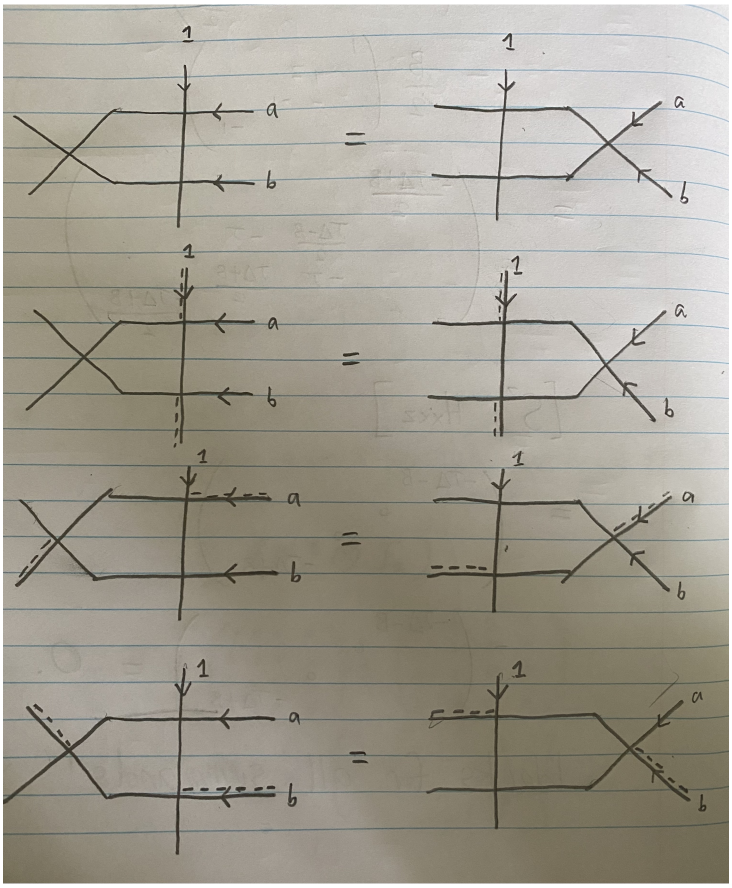
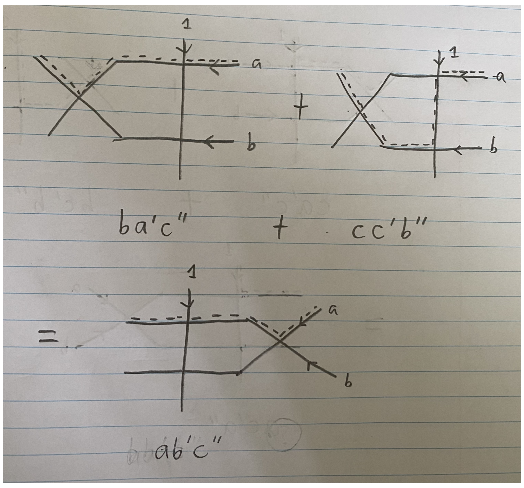
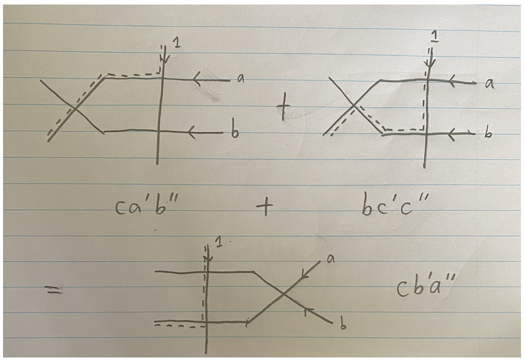
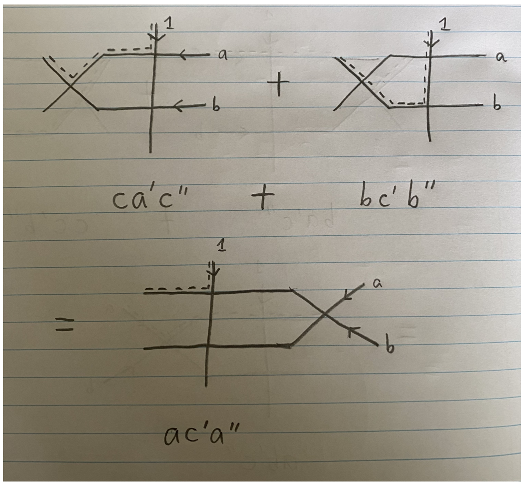


Figure 2: The equations which are invariant under a 180 degree rotation.

Finally, there are six equations left, which come in pairs of three. These are







We will restate the three equations below for convenience:

1. $ba'c'' + cc'b'' = ab'c''$
2. $ca'b'' + bc'c'' = cb'a''$
3. $ca'c'' + bc'b'' = ac'a''$

We want to find a solution to the linear system above. Express it as the matrix equation below:

$$\begin{pmatrix} 0 & cc' & ba' - ab' \\ -cb' & ca' & bc' \\ -ac' & bc' & ca' \end{pmatrix} \begin{pmatrix} a'' \\ b'' \\ c'' \end{pmatrix} = 0.$$

A non-trivial solution $(a'', b'', c'') \neq (0, 0, 0)$ exists if and only if the determinant

$$\begin{vmatrix} 0 & cc' & ba' - ab' \\ -cb' & ca' & bc' \\ -ac' & bc' & ca' \end{vmatrix} = 0. \tag{1}$$

After some tedious computations, we compute the LHS of (1) as

$$\begin{aligned} & -abcc'^3 + a'b'c^3c' + aa'^2bcc' - a'b^2b'cc' - a^2a'b'cc' + abb'^2cc' \\ &= aa'bb'cc'(a^2 + b^2 - c^2 - a'^2 - b'^2 + c'^2) \\ &= 2aa'bb'cc'(\Delta - \Delta') \end{aligned}$$

where Δ is

$$\Delta = \frac{a^2 + b^2 - c^2}{2ab}.$$

and Δ' admits the same expression, but with the primed variables. Thus, the expression $2aa'bb'cc'(\Delta - \Delta')$ is zero if and only if $\Delta = \Delta'$.

To unmask the identity of the R matrix, we will rewrite the matrix equation once again as

$$\begin{pmatrix} bc'' & -ac'' & b''c \\ b''c & -a''c & bc'' \\ cc'' & 0 & bb'' - aa'' \end{pmatrix} \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} = 0$$

where $a'', b'', c'' \neq 0$. By a similar argument to the above, we find that this has a non-trivial solution if and only if $\Delta = \Delta''$. Since $\Delta = \Delta' = \Delta''$, this reveals that the R matrix must have exactly the same form as the L matrices L and L'.

Before we proceed to the final step, we will parametrise the spectral parameters a, b, c. Assuming that $\Delta \neq \pm 1$, we can write

$$\Delta = \frac{q + q^{-1}}{2}$$

where $a(u) = qu - q^{-1}u^{-1}$, $b(u) = u - u^{-1}$ and $c(u) = q - q^{-1}$ for some spectral parameters q and u. The parameter q applies to all of the variables $a, b, c, a', b', c', a'', b''$ and c''. Let u, u' and u'' denote the other spectral parameters for the unprimed, primed and double primed variables respectively.

In order to determine the spectral parameter u'', we substitute the expressions in our parametrisation into the third equation $ca'c'' + bc'b'' = ac'a''$. The result is after a lot of simplification

$$q^2 u' - u'^{-1} - u' + q^{-2} u'^{-1} = q^2 u u'' - u u'' + q^{-2} u^{-1} u''^{-1} - u^{-1} u''^{-1}.$$

One can then check directly by inspection that $u'' = u'/u$ makes the equality true. Thus, we conclude that the R matrix of the six vertex model takes the same form as the L matrices L and L'. While L and L' have spectral parameters u and u' respectively, R must have spectral parameter $u'' = u'/u$ in order for the RLL relations to be satisfied.

The power of the RLL relation lies with the fact that it gives a direct proof of the fact that the transfer matrices (with periodic boundary conditions) of the six vertex model commute, regardless of the spectral parameters involved. Let $T(u)$ denote the periodic boundary transfer matrix with spectral parameter u. It is denoted graphically by

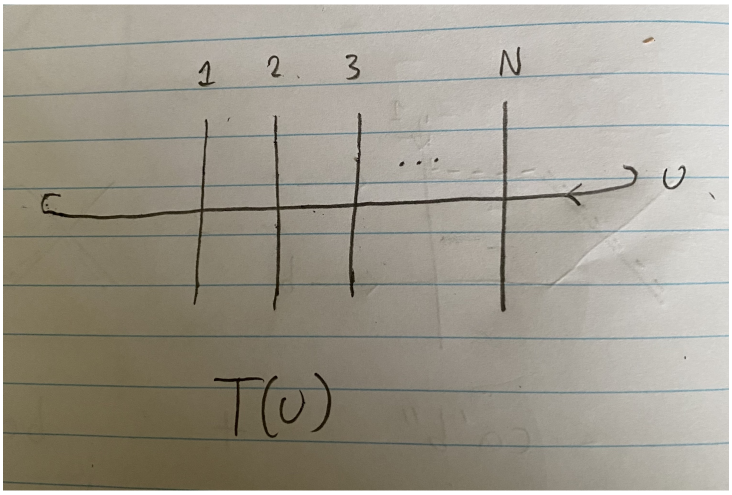


Figure 3: The transfer matrix $T(u)$ with periodic boundary conditions. Note that the vertical spaces all have spectral parameter 1.

The proof is best described by pictures. The general idea is to insert the matrix $R(u/v)R(u/v)^{-1}$, which is just the identity matrix, to the right of all the vertical spaces in the graphical representation of $T(u)T(v)$ and then apply the RLL relation repeatedly to move the R matrix so that it appears to the left of all the vertical spaces of the transfer matrices. Exploiting the periodic boundary conditions, we can move the R matrix back to the right.

However, this swaps the spectral parameters of the top and bottom rows. Cancelling $R(u/v)$ and $R(u/v)^{-1}$ then yields the desired result.

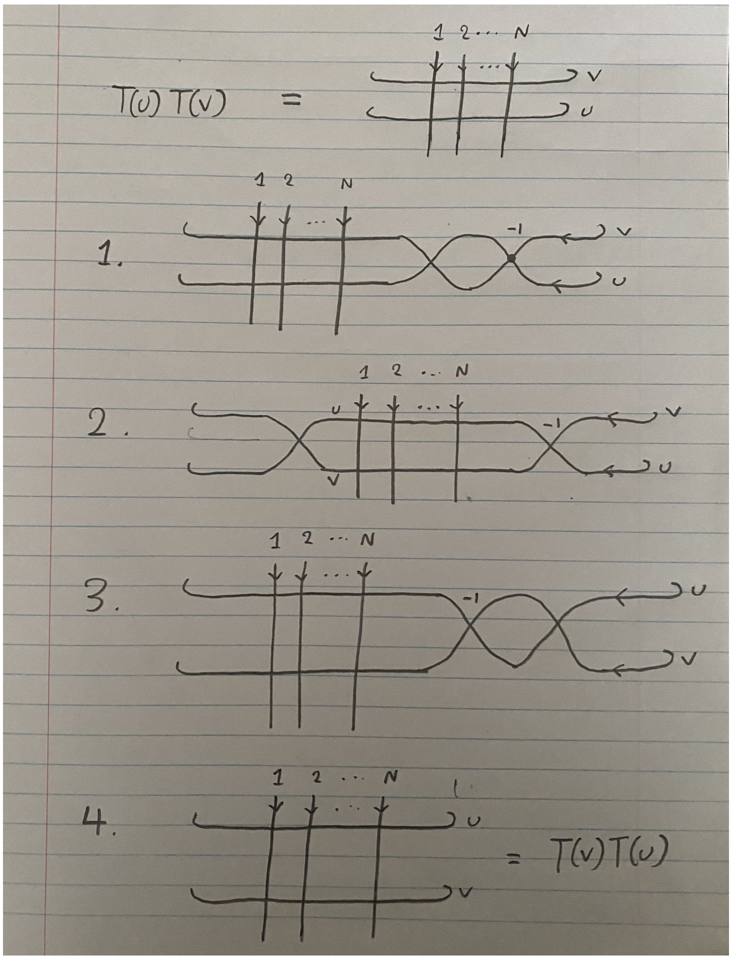


Figure 4: Proof that the transfer matrices commute

Similarly, the RTT relations (with monodromy matrices) and the Yang-Baxter equations can all be proved with an easy application of the RLL relation, not too dissimilar from the above application.

Bibliography

[Zin21] P. Zinn-Justin, *Exactly Solvable Models MAST90065, 2021*, October 20th 2021.