On a particular integral

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We record an interesting argument which proves the following result.

Proposition 0.0.1.

$$\int_0^1 \frac{\ln(x+1)}{x^2+1}\,\mathrm{d}x = \frac{\pi}{8}\ln 2. \tag{1}$$

Proof. We begin with an integration by parts. This decomposes the LHS of equation (1) as

$$\begin{aligned}
\int_0^1 \frac{\ln(x+1)}{x^2+1}\,\mathrm{d}x &= [\ln(x+1)\arctan(x)]_0^1 - \int_0^1 \frac{\arctan(x)}{x+1}\,\mathrm{d}x \\
&= \frac{\pi}{4}\ln(2) - \int_0^1 \frac{\arctan(x)}{x+1}\,\mathrm{d}x \\
&= \frac{\pi}{4}\ln(2) - \int_1^2 \frac{\arctan(u-1)}{u}\,\mathrm{d}u.
\end{aligned}$$

On the final line, we made the substitution $x = u - 1$. The problem reduces to finding the integral

$$\int_1^2 \frac{\arctan(u-1)}{u}\,\mathrm{d}u. \tag{2}$$

The leap here is to turn to the complex numbers for inspiration. Indeed, if $z_1, z_2 \in \mathbb{C}$, then $\arg(z_1) + \arg(z_2) = \arg(z_1 z_2)$, where arg denote the principal argument. Set $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$. If $x_1, x_2, y_1, y_2 \in \mathbb{Z}_{>0}$ then $\arg(x_j + iy_j) = \arctan(y_j/x_j)$ for $j \in \{1, 2\}$ and

$$\arctan(\frac{y_1}{x_1}) + \arctan(\frac{y_2}{x_2}) = \arctan(\frac{x_1 y_2 + y_1 x_2}{x_1 x_2 - y_1 y_2}).$$

Now let $y_1 = u - 1$, $y_2 = x_2 = 1$ and $x_1 = u$ for some $u \in \mathbb{Z}_{>0}$. Then,

$$\arctan(\frac{u-1}{u}) + \frac{\pi}{4} = \arctan(2u-1).$$

Replacing u with $u/2$, we obtain the equation

$$\arctan(\frac{u-2}{u}) + \frac{\pi}{4} = \arctan(u-1). \tag{3}$$

which is valid for $u \in \mathbb{Z}_{>0}$. Conveniently, this contains our region of integration.

In equation (2), we now make the substitution $u = 2/y$. Then, $du = -2/y^2 dy$ and our integral now becomes

$$\begin{aligned}
\int_2^1 \frac{\arctan(\frac{2}{y} - 1)}{\frac{2}{y}} \times -\frac{2}{y^2}\, \mathrm{d}y &= \int_1^2 \arctan(\frac{2}{y} - 1) \times \frac{y}{2} \times \frac{2}{y^2}\, \mathrm{d}y \\
&= \int_1^2 \frac{\arctan(\frac{2-y}{y})}{y}\, \mathrm{d}y \\
&= -\int_1^2 \frac{\arctan(\frac{y-2}{y})}{y}\, \mathrm{d}y.
\end{aligned}$$

Hence, we have the following equation:

$$\int_1^2 \frac{\arctan(u-1)}{u}\, \mathrm{d}u = -\int_1^2 \frac{\arctan(\frac{u-2}{u})}{u}\, \mathrm{d}u.$$

Now, we set $I = \int_1^2 \frac{\arctan(u-1)}{u}\, \mathrm{d}u$. Then,

$$\begin{aligned}
2I &= \int_1^2 \frac{\arctan(u-1)}{u}\, \mathrm{d}u + \int_1^2 \frac{\arctan(u-1)}{u}\, \mathrm{d}u \\
&= \int_1^2 \frac{\arctan(u-1)}{u}\, \mathrm{d}u - \int_1^2 \frac{\arctan(\frac{u-2}{u})}{u}\, \mathrm{d}u \\
&= \int_1^2 \frac{\arctan(u-1) - \arctan(\frac{u-2}{u})}{u}\, \mathrm{d}u \\
&= \int_1^2 \frac{\pi}{4u}\, \mathrm{d}u \quad \text{(From equation (3))} \\
&= \frac{\pi}{4}[\ln(u)]_1^2 \\
&= \frac{\pi}{4}\ln(2).
\end{aligned}$$

Hence, $I = \frac{\pi}{8}\ln(2)$. Our original integral in equation (1) now evaluates to

$$\int_0^1 \frac{\ln(x+1)}{x^2+1}\, \mathrm{d}x = \frac{\pi}{4}\ln(2) - I = \frac{\pi}{8}\ln(2)$$

as required.

$\square$