On recent results in nuclear dimension

Yao Ming Brian Chan

Supervisor: Stuart White

September 15, 2024

University of Oxford

A thesis submitted for the partial fulfilment of the Transfer of Status requirement for the DPhil Mathematics degree

Contents

Chapter 1

Introduction

The theory of C*-algebras is regarded as a non-commutative analogue of topology. Embodying this claim is the Gelfand-Naimark theorem ([Mur90, Theorem 2.1.10]) — an abelian C*-algebra is isomorphic to the C*-algebra of continuous functions on its spectrum which vanish at infinity, with the spectrum being a locally compact Hausdorff space. From this, it is unsurprising that many of the fundamental tools used to study topology can be adapted to investigate the non-commutative world of C*-algebras. Some examples include operator K-theory as a tool for the classification of certain C*-algebras ([RLL00] and [Mur90, Section 7.2]), the multiplier algebra of a C*-algebra as a non-commutative analogue of the Stone-Čech compactification ([HR00, Exercise 1.9.9]) and real rank as a non-commutative version of Lebesgue covering dimension ([BP91]).

Introduced by Winter and Zacharias in the paper [WZ10], *nuclear dimension* is one of many non-commutative generalisations of the covering dimension of a topological space. Similarly to how Lebesgue covering dimension is able to classify certain topological spaces (for instance, see [Pea75, Proposition 1.3] and [Pea75, Section 3.2]), nuclear dimension has enjoyed success in the classification of particular classes of C*-algebras, such as AF-algebras ([KW04, Example 4.1], [WZ10, Remarks 2.2 (iii)]) and simple unital C*-algebras ([CETWW21]).

Historically, nuclear dimension is the culmination of previous non-commutative covering dimensions, with the most recent of these being *decomposition rank*, first defined in [KW04]. The key property which distinguishes nuclear dimension from its predecessors is that finite nuclear dimension is preserved under extensions of C*-algebras ([WZ10, Proposition 2.9]). Consequently, finite nuclear dimension covers a far greater range of C*-algebras than its predecessors. The archetypal example of this is the Toeplitz algebra $\mathcal{T}$ which is an extension of $C(S^1)$ by compact operators K. By [WZ10, Proposition 2.9], the nuclear dimension of $\mathcal{T}$ is at most 2 whereas its decomposition rank is infinite.

Motivated by the successful classification of the nuclear dimension of simple unital C*-algebras in [CETWW21] (which have no non-trivial ideals), the next step is to compute the nuclear dimension of C*-algebras which arise from extensions (and hence contain non-trivial ideals). Provided that we know the nuclear dimensions of the ideal and quotient beforehand, one immediately obtains an upper bound for the nuclear dimension of an extension from [WZ10, Proposition 2.9]. In general, the main difficulty lies with winnowing these upper bounds to precise values.

The catalyst behind recent research into the nuclear dimension of extensions is the paper [BW19]. In it, Brake and Winter prove that the nuclear dimension of the Toeplitz algebra is in fact 1. Their argument consists of an application of Lin's theorem on almost normal matrices ([Lin97]) which eventually allows a c.c.p order zero map (see [WZ09]) from the quotient $C(S^1)$ to be extended to bounded Borel functions $B(S^1)$. The extra room afforded by this allows the sum of this map with another c.c.p order zero map to remain c.c.p order zero. Note that in general, the sum of two c.c.p order zero maps is not order zero. Subsequently, the number of c.c.p order zero maps in the nuclear dimension approximation for $\mathcal{T}$ is reduced from three to two, yielding a nuclear dimension of 1.

In this document, we will define nuclear dimension and study its basic properties. Then we will examine recent results on the nuclear dimension of extensions, placing particular emphasis on the fact that their proofs conform to a particular structure. Finally, we will end by discussing nuclear dimension in the context of graph C*-algebras of finite graphs, which are all constructed by repeated extensions of C*-algebras with finite nuclear dimension ([FS23]).

Chapter 2

Nuclear dimension

2.1 Nuclearity of abelian C*-algebras

The notion of nuclear dimension originates from two different concepts. Not only is nuclear dimension a non-commutative analogue of covering dimension, it is also a refined version of the completely positive approximation property (CPAP) satisfied by nuclear C*-algebras. With the goal of understanding where nuclear dimension comes from, we begin with the definition of a nuclear C*-algebra from [BO08, Section 2.3].

Definition 2.1.1. Let A be a C*-algebra. We say that A is **nuclear** if A satisfies the completely positive approximation property: If $\epsilon > 0$ and $\mathcal{F}$ is a finite subset of A then there exist a finite-dimensional C*-algebra G and c.c.p (contractive completely positive) maps $\varphi : A \to G$ and $\psi : G \to A$ such that if $a \in \mathcal{F}$ then

$$\|(\psi \circ \varphi)(a) - a\| < \epsilon.$$

Schematically, this is illustrated by the approximating commuting diagram below.

$$\begin{tikzcd} A \arrow[rr, dashed, "id_A"] \arrow[dr, "\varphi"'] & & A \\ & G \arrow[ur, "\psi"'] & \end{tikzcd} \tag{2.1}$$

By definition, finite-dimensional C*-algebras are nuclear. To this chapter, the most relevant example of a nuclear C*-algebra is an abelian C*-algebra.

Theorem 2.1.1. *Let A be an abelian C*-algebra. Then, A is a nuclear C*-algebra.*

Proof. Assume that A is an abelian C*-algebra. Since a C*-algebra is nuclear if and only if its unitization is nuclear ([BO08, Exercise 2.3.5]), it suffices to prove that A is nuclear in the case where A is unital.

To this end, assume that A is a unital abelian C*-algebra. Without loss of generality, we may assume that $A = C(X)$ where X is a compact Hausdorff topological space. Assume that $F \subsetneq A$ is a finite set and $\epsilon > 0$. Take an open cover $\{U_1, \ldots, U_n\}$ of X such that if $f \in F$, $i \in \{1, 2, \ldots, n\}$ and $x, y \in U_i$ then

$$|f(x) - f(y)| < \epsilon.$$

If $i \in \{1, \dots, n\}$ then let $y_i \in U_i$ and $\{\sigma_1, \dots, \sigma_n\}$ be a partition of unity with respect to the open cover $\{U_1, \dots, U_n\}$. First define φ by

$$\begin{array}{rccc} \varphi : & A & \to & \mathbb{C}^n \\ & f & \mapsto & (f(y_1), \dots, f(y_n)). \end{array}$$

It is straightforward to check that φ is a unital *-homomorphism and thus a u.c.p map (which is contractive by [Pau02, Corollary 2.9]). Next define the map

$$\begin{array}{rccc} \psi : & \mathbb{C}^n & \to & A \\ & (d_1, \dots, d_n) & \mapsto & \sum_{i=1}^n d_i \sigma_i. \end{array}$$

To see that ψ is positive, assume that $(r_1, \dots, r_n) \in (\mathbb{R}_{\geq 0})^n$. We know that if $i \in \{1, 2, \dots, n\}$ then $\sigma_i \in C(X, [0,1])$ and has support contained in U_i. So if $x \in X$ then there exist $k_1, \dots, k_j \in \{1, 2, \dots, n\}$ such that $x \in U_{k_1} \cap \dots \cap U_{k_j}$ and consequently,

$$\psi(r_1, \dots, r_n)(x) = \sum_{i=1}^n r_i \sigma_i(x) = \sum_{\ell=1}^j r_{k_\ell} \sigma_{k_\ell}(x) \in \mathbb{R}_{\geq 0}.$$

This shows that ψ is a positive map. Moreover, the range of ψ is A which is by assumption an abelian C*-algebra.

To show: (a) ψ is completely positive.

(a) Assume that $k \in \mathbb{Z}_{>0}$. If $T \in M_k(\mathbb{C})$ and $\sigma \in A = C(X)$ then define the element $T \otimes \sigma$ of $M_k(C(X)) \cong C(X, M_k(\mathbb{C}))$ by

$$\begin{array}{rccc} T \otimes \sigma : & X & \to & M_k(\mathbb{C}) \\ & x & \mapsto & \sigma(x) T. \end{array}$$

By identifying $M_k(\mathbb{C}^n)$ as the direct sum of n copies of $M_k(\mathbb{C})$, we find that the map ψ_k is defined explicitly by

$$\begin{array}{rccc} \psi_k : & M_k(\mathbb{C}^n) \cong \bigoplus_{i=1}^n M_k(\mathbb{C}) & \to & M_k(A) \\ & T_1 \oplus \dots \oplus T_n & \mapsto & \sum_{i=1}^n T_i \otimes \sigma_i. \end{array}$$

If $T_1, \dots, T_n \in M_k(\mathbb{C})$ are positive matrices and $i \in \{1, 2, \dots, n\}$ then $T_i \otimes \sigma_i$ are positive functions (because $\operatorname{im} \sigma_i \subseteq [0,1]$) and consequently, ψ_k is a positive map. So ψ is completely positive. This proves part (a) of the proof.

To see that ψ is contractive, note that if $x \in X$ then

$$\psi(1, 1, \dots, 1)(x) = \sum_{i=1}^n \sigma_i(x) = 1.$$

So ψ is a u.c.p map and hence contractive. Now if $f \in F$ and $x \in X$ then there exist $k_1, \ldots, k_j \in \{1, 2, \ldots, n\}$ such that $x \in U_{k_1} \cap \cdots \cap U_{k_j}$ and

$$\begin{aligned}
\|(\psi \circ \varphi)(f)(x) - f(x)\| &= \Big\| \sum_{i=1}^{n} f(y_i)\sigma_i(x) - (\sum_{i=1}^{n} \sigma_i(x))f(x) \Big\| \\
&= \Big\| \sum_{i=1}^{n} \big((f(x) - f(y_i))\sigma_i(x)\big) \Big\| \\
&\leq \Big\| \sum_{\ell=1}^{j} \big((f(x) - f(y_{k_\ell}))\sigma_{k_\ell}(x)\big) \Big\| \\
&\quad + \Big\| \sum_{i \notin \{k_1, \ldots, k_j\}} \big((f(x) - f(y_i))\sigma_i(x)\big) \Big\| \\
&< \Big\| \sum_{\ell=1}^{j} \epsilon \sigma_{k_\ell}(x) \Big\| + 0 = \epsilon.
\end{aligned}$$

Therefore if $f \in F$ then $\|(\psi \circ \varphi)(f) - f\| < \epsilon$ and subsequently A is a nuclear C*-algebra. □

The proof of Theorem 2.1.1 utilises an open cover for the compact Hausdorff space X and the upwards (in the sense of Diagram (2.1)) c.c.p map ψ was constructed from a partition of unity subordinate to the open cover. This leads to the following question: How does the extra structure on the open cover of X imposed by the covering dimension of X affect the proof of Theorem 2.1.1? To answer this question, we recall the definition of covering dimension as stated in [Cas21, Section 2.1].

Definition 2.1.2. Let X be a topological space and $\mathcal{U}$ be a finite open cover of X. Let $n \in \mathbb{Z}_{>0}$. The **order** of $\mathcal{U}$ is at most n if the following statement is satisfied: If $x \in X$ then x is contained in at most $n+1$ sets in $\mathcal{U}$.

The **covering dimension** of X is at most n, denoted by $\dim X \leq n$, if the following statement is satisfied: If $\mathcal{U}$ is a finite open cover of X then $\mathcal{U}$ has an open refinement of order at most n. The covering dimension of X, denoted by $\dim X$, is the quantity

$$\min\{n \in \mathbb{Z}_{\geq 0} \mid \dim X \leq n\}.$$

If no such $n \in \mathbb{Z}_{\geq 0}$ then X is said to have infinite covering dimension ($\dim X = \infty$).

A basic example of covering dimension is that if $n \in \mathbb{Z}_{>0}$ then $\dim \mathbb{R}^n = n$. As mentioned in [Cas21], this is non-trivial to prove.

In the situation of Theorem 2.1.1, we are dealing with a compact Hausdorff space X which is normal. The covering dimension of a normal topological space has a nice characterisation, due to Ostrand in [Ost65].

Theorem 2.1.2. *Let X be a normal topological space and $n \in \mathbb{Z}_{\geq 0}$. Then $\dim X \leq n$ if and only if the following statement is satisfied: If $\mathcal{V}$ is a finite open cover of X*

then there exists an open finite refinement $\mathcal{U}$ of $\mathcal{V}$ which can be decomposed into $n+1$ pairwise disjoint families

$$\mathcal{U} = \mathcal{U}_0 \sqcup \mathcal{U}_1 \sqcup \cdots \sqcup \mathcal{U}_n$$

such that if $j \in \{0, 1, \dots, n\}$ and $U, V \in \mathcal{U}_j$ then $U \cap V = \emptyset$.

A useful way of thinking about Theorem 2.1.2 is that if $\mathcal{V}$ is a finite open cover of X then we can find a finite open refinement $\mathcal{U}$ and colour the sets in $\mathcal{U}$ with $n+1$ colours in such a way that any two sets of the same colour are disjoint. Theorem 2.1.2 will be used to further analyse the proof of Theorem 2.1.1.

2.2 Completely positive maps of order zero

As we will see in the next section, completely positive maps of order zero are central to the definition of nuclear dimension. This notion was first introduced by Winter and Zacharias in [WZ09].

Definition 2.2.1. Let A and B be C*-algebras and $\varphi : A \to B$ be a c.p map. We say that φ is **order zero** if the following statement is satisfied: If $a, b \in A$ and $ab = 0$ then $\varphi(a)\varphi(b) = 0$.

In [WZ09], Winter and Zacharias prove the following fundamental structure theorem for c.p maps of order zero.

Theorem 2.2.1. *Let A and B be C*-algebras and $\varphi : A \to B$ be a c.p map of order zero. Define $C = C^*(\varphi(A))$ to be the C*-subalgebra of B generated by the image $\varphi(A)$. Let $\mathcal{M}(C)$ denote the multiplier algebra of C. Then there exist a positive element $h \in \mathcal{M}(C) \cap C'$ and a *-homomorphism $\pi_\varphi : A \to \mathcal{M}(C) \cap \{h\}'$ such that if $a \in A$ then*

$$\varphi(a) = \pi_\varphi(a)h = h\pi_\varphi(a) \qquad \text{and} \qquad \|h\| = \|\varphi\|.$$

Here are some consequences of Theorem 2.2.1 that we will use later. If $a \in A$ then by Theorem 2.2.1,

$$\varphi(a^*) = \pi_\varphi(a^*)h = \pi_\varphi(a)^*h = (h\pi_\varphi(a))^* = \varphi(a)^*. \tag{2.2}$$

Theorem 2.2.2. *Let A and B be C*-algebras and $\varphi : A \to B$ be a c.p map of order zero. Then $\varphi = \pi_\varphi h = h\pi_\varphi$ as in Theorem 2.2.1. If $a, b, c \in A$ then then*

$$\varphi(ab)\varphi(c) = \varphi(a)\varphi(bc)$$

Proof. Assume that A and B are C*-algebras. Assume that $\varphi : A \to B$ is a c.p map of order zero. By Theorem 2.2.1, if $a, b, c \in A$ then

$$\begin{aligned}
\varphi(ab)\varphi(c) &= \pi_\varphi(ab)h\pi_\varphi(c)h \\
&= \pi_\varphi(a)\pi_\varphi(b)h\pi_\varphi(c)h \\
&= \pi_\varphi(a)h\pi_\varphi(b)\pi_\varphi(c)h \\
&= \varphi(a)\varphi(bc).
\end{aligned}$$

This completes the proof. $\square$

The next consequence of Theorem 2.2.1 tells us that there are no non-trivial c.p order zero maps from a matrix algebra to an abelian C*-algebra.

Theorem 2.2.3. *Let $k \in \mathbb{Z}_{>0}$, A be an abelian C*-algebra and $\varphi : M_k(\mathbb{C}) \to A$ be a c.p map of order zero. If $k \in \mathbb{Z}_{>1}$ then $\varphi = 0$.*

Proof. Assume that $k \in \mathbb{Z}_{>0}$, A is an abelian C*-algebra and $\varphi : M_k(\mathbb{C}) \to A$ is a c.p map of order zero. Assume that $k \in \mathbb{Z}_{>1}$. Let $\{e_{i,j}\}_{i,j=1}^k$ be the matrix unit for $M_k(\mathbb{C})$. Assume that $i, j \in \{1, 2, \ldots, k\}$ are distinct (this uses the fact that $k > 1$). We claim that $\varphi(e_{i,j}) = 0$. Let $1_k \in M_k(\mathbb{C})$ denote the identity matrix. By Theorem 2.2.2, we have

$$\begin{aligned} \varphi(e_{i,i})\varphi(1_k) &= \varphi(e_{i,j}e_{j,i})\varphi(1_k) \\ &= \varphi(e_{i,j})\varphi(e_{j,i}) \\ &= \varphi(e_{j,i})\varphi(e_{i,j}) \qquad (A \text{ is abelian}) \\ &= \varphi(e_{j,i}e_{i,j})\varphi(1_k) \\ &= \varphi(e_{j,j})\varphi(1_k). \end{aligned}$$

Suppose for the sake of contradiction that $\varphi(1_k) \neq 0$. Then $\varphi(e_{i,i}) = \varphi(e_{j,j})$. Since φ is of order zero and $i \neq j$ then $\varphi(e_{i,i})\varphi(e_{j,j}) = 0$. So if $i \in \{1, 2, \ldots, k\}$ then $\varphi(e_{i,i}) = 0$ and

$$\varphi(1_k) = \sum_{i=1}^{k} \varphi(e_{i,i}) = 0 \tag{2.3}$$

which contradicts the assumption that $\varphi(1_k) \neq 0$. Therefore $\varphi(1_k) = 0$ and

$$\|\varphi(e_{j,i})\|^2 = \|\varphi(e_{i,j})\varphi(e_{j,i})\| = \|\varphi(e_{i,i})\varphi(1_k)\| = 0.$$

By equation (2.2), $\varphi(e_{j,i}) = 0 = \varphi(e_{i,j})$.

Now to see that $\varphi(e_{i,i}) = 0$, observe that the matrix

$$1_k - e_{i,i} = \sum_{j=1,\, j\neq i}^{k} e_{j,j}$$

is positive. Since φ is a positive map then $\varphi(e_{i,i}) \leq \varphi(1_k) = 0$ by equation (2.3). Therefore $\varphi(e_{i,i}) = 0$. Hence, φ is equal to zero on the matrix unit of $M_k(\mathbb{C})$ and so, $\varphi = 0$ as required. $\square$

A particularly important consequence of Theorem 2.2.1 is [WZ10, Corollary 3.1], which gives a bijective correspondence between c.c.p order zero maps $\psi : A \to B$ and *-homomorphisms $\Psi : C_0((0,1]) \otimes A \to B$.

Theorem 2.2.4. *Let A and B be C*-algebras. Then there is a bijective correspondence:*

$$\begin{Bmatrix} \textit{c.c.p order zero maps} \\ \psi : A \to B \end{Bmatrix} \longleftrightarrow \begin{Bmatrix} \textit{*-homomorphisms} \\ \Psi : C_0((0,1]) \otimes A \to B \end{Bmatrix}$$

$$\begin{array}{ccc} \varphi & \mapsto & \big(id_{(0,1]} \otimes a \mapsto \varphi(a)\big) \\ \big(a \mapsto \Theta(id_{(0,1]} \otimes a)\big) & \mapsfrom & \Theta \end{array}$$

2.3 The definition of nuclear dimension

As in the proof of Theorem 2.1.1, let A be a unital abelian C*-algebra. Without loss of generality, we may assume that $A = C(X)$ where X is a compact Hausdorff topological space. Assume that the covering dimension $\dim X = n$, F is a finite subset of $C(X)$ and $\epsilon > 0$. Since X is compact then there exists a finite open cover $\mathcal{V}$ of X such that if $f \in F$, $V \in \mathcal{V}$ and $x, y \in V$ then

$$|f(x) - f(y)| < \epsilon.$$

Since $\dim X = n$ then by Theorem 2.1.2, we obtain a finite open refinement $\mathcal{U}$ of $\mathcal{V}$ and a pairwise disjoint decomposition

$$\mathcal{U} = \mathcal{U}_0 \sqcup \mathcal{U}_1 \sqcup \cdots \sqcup \mathcal{U}_n \tag{2.4}$$

such that if $j \in \{0, 1, \ldots, n\}$ and $U, V \in \mathcal{U}_j$ then $U \cap V = \emptyset$. If $W \in \mathcal{U}$ then select $x_W \in \mathcal{U}$ and let $\{\sigma_U\}_{U \in \mathcal{U}}$ be a partition of unity subordinate to $\mathcal{U}$. Following the proof of Theorem 2.1.1, we obtain c.c.p maps

$$\begin{array}{rccc} \varphi : & A & \to & \mathbb{C}^{|\mathcal{U}|} \\ & f & \mapsto & f\big((x_U)_{U \in \mathcal{U}}\big) \end{array} \tag{2.5}$$

and

$$\begin{array}{rccc} \psi : & \mathbb{C}^{|\mathcal{U}|} & \to & A \\ & (d_U)_{U \in \mathcal{U}} & \mapsto & \sum_{U \in \mathcal{U}} d_U \sigma_U \end{array} \tag{2.6}$$

such that if $g \in F$ then $\|(\psi \circ \varphi)(g) - g\| < \epsilon$. We will now use the specific properties of the finite open cover $\mathcal{U}$. If $j \in \{0, 1, \ldots, n\}$ then define the c.p map

$$\begin{array}{rccc} \psi_j : & \mathbb{C}^{|\mathcal{U}|} & \to & A \\ & (d_U)_{U \in \mathcal{U}} & \mapsto & \sum_{U \in \mathcal{U}_j} d_U \sigma_U . \end{array} \tag{2.7}$$

The map ψ_j is obtained from ψ by restricting to the family $\mathcal{U}_j$. First observe that by the decomposition of $\mathcal{U}$ in equation (2.4),

$$\psi = \sum_{j=0}^{n} \psi_j .$$

If $(d_U)_{U \in \mathcal{U}} \in \mathbb{C}^{|\mathcal{U}|}$ then define $(d_{U,j})_{U \in \mathcal{U}}$ by

$$d_{U,j} = \begin{cases} d_U, & \text{if } U \in \mathcal{U}_j, \\ 0, & \text{otherwise.} \end{cases}$$

We have

$$\begin{aligned}
\|\psi_j((d_U)_{U\in\mathcal{U}})\| &= \sup_{x\in X}\Big|\sum_{U\in\mathcal{U}_j} d_U\sigma_U(x)\Big| \\
&= \sup_{x\in X}\Big|\sum_{U\in\mathcal{U}_j} d_U\sigma_U(x) + \sum_{U\notin\mathcal{U}_j} 0\sigma_U(x)\Big| \\
&= \sup_{x\in X}\Big|\sum_{U\in\mathcal{U}} d_{U,j}\sigma_U(x)\Big| = \|\psi((d_{U,j})_{U\in\mathcal{U}})\| \\
&\leq |(d_{U,j})_{U\in\mathcal{U}}| \leq |(d_U)_{U\in\mathcal{U}}|.
\end{aligned}$$

We conclude that if $j \in \{0, 1, \ldots, n\}$ then ψ_j is a c.c.p map. We now claim that ψ_j is order zero. Indeed, if $(d_U)_{U\in\mathcal{U}}$ and $(e_U)_{U\in\mathcal{U}}$ satisfy

$$(d_U)_{U\in\mathcal{U}}(e_U)_{U\in\mathcal{U}} = (0, 0, \ldots, 0)$$

then $d_U e_U = 0$ and

$$\psi_j((d_U))\psi_j((e_U)) = \sum_{U,V\in\mathcal{U}_j} d_U e_V \sigma_U \sigma_V = \sum_{U\in\mathcal{U}_j} d_U e_U \sigma_U = 0.$$

In the last equality, we used the fact that if $U, V \in \mathcal{U}_j$ are distinct then $U \cap V = \emptyset$.

In summary, from a unital abelian C*-algebra $A = C(X)$, we found that

1. If $F \subsetneq A$ is finite and $\epsilon > 0$ then there exist $r \in \mathbb{Z}_{>0}$ and c.c.p maps $\varphi : A \to \mathbb{C}^r$ and $\psi : \mathbb{C}^r \to A$ such that if $a \in A$ then $\|(\psi \circ \varphi)(a) - a\| < \epsilon$.
2. There is a decomposition $\mathbb{C}^r = \bigoplus_{i=0}^n \mathbb{C}^{r_i}$ into $n+1$ finite-dimensional subalgebras such that if $i \in \{0, 1, \ldots, n\}$ then the restriction $\psi_i = \psi|_{\mathbb{C}^{r_i}}$ is a c.c.p map of order zero.

The above observation form the blueprint for nuclear dimension which we formally define below, first for *-homomorphisms and then for C*-algebras.

Definition 2.3.1 (Nuclear dimension of a *-homomorphism)**.** Let A and B be C*-algebras and $n \in \mathbb{Z}_{>0}$. Let $\alpha : A \to B$ be a *-homomorphism. We say that α has **nuclear dimension** at most n, denoted by $\dim_{\text{nuc}}(\alpha) \leq n$, if the following statement is satisfied: If $F \subsetneq A$ is finite and $\epsilon > 0$ then there exist finite dimensional C*-algebras $F_0, F_1, \ldots, F_n$ and maps $\varphi : A \to \bigoplus_{j=0}^n F_j$ and $\psi : \bigoplus_{j=0}^n F_j \to B$ such that

1. φ is a c.c.p map,
2. If $a \in F$ then $\|(\psi \circ \varphi)(a) - \alpha(a)\| < \epsilon$,
3. If $j \in \{0, 1, \ldots, n\}$ then the restriction $\psi_j = \psi|_{F_j}$ is a c.c.p map of order zero.

This statement is represented diagrammatically by the approximately commuting diagram

$$\begin{tikzcd}
A \arrow[rr, dashed, "\alpha"] \arrow[dr, "\varphi"'] & & B \\
& \bigoplus_{j=0}^{n} F_j \arrow[ur, "\psi=\bigoplus_{j=0}^{n}\psi_j"'] &
\end{tikzcd}$$

We say that the nuclear dimension of α is n ($\dim_{\text{nuc}}(\alpha) = n$) if

$$n = \min\{m \in \mathbb{Z}_{>0} \mid \dim_{\text{nuc}}(\alpha) \leq m\}.$$

If no such n exists then α is said to have infinite nuclear dimension ($\dim_{\text{nuc}}(\alpha) = \infty$).

Definition 2.3.2 (Nuclear dimension of a C*-algebra)**.** Let A be a C*-algebra and $n \in \mathbb{Z}_{>0}$. The **nuclear dimension** of A, denoted by $\dim_{\text{nuc}} A$, is defined as

$$\dim_{\text{nuc}} A = \dim_{\text{nuc}}(id_A)$$

where id_A is the identity map on A. By Definition 2.3.1, the rough meaning of the RHS is depicted by the approximately commuting diagram below.

$$\begin{tikzcd}
A \arrow[rr, dashed, "id_A"] \arrow[dr, "\varphi"'] & & A \\
& \bigoplus_{j=0}^{n} F_j \arrow[ur, "\psi=\bigoplus_{j=0}^{n}\psi_j"'] &
\end{tikzcd}$$

Remark 2.3.1. Notice that in Definition 2.3.2, we did not ask for the c.p map ψ to be contractive. From Definition 2.3.1, we know a priori that if $\dim_{\text{nuc}} A \leq n$ then

$$\|\psi\| \leq \sum_{j=0}^{n} \|\psi_j\| = n + 1.$$

Insisting that ψ is contractive leads to the very similar notion of the *decomposition rank* of a C*-algebra from [KW04, Definition 3.1]. The decomposition rank of A will be denoted by dr A.

Remark 2.3.2. Assume that X is a compact Hausdorff space with covering dimension $\dim X = n \in \mathbb{Z}_{\geq 0}$. Using what we have done previously, if $j \in \{0, 1, \dots, n\}$ then let $F_j = \mathbb{C}^{|\mathcal{U}_j|}$ so that

$$\mathbb{C}^{|\mathcal{U}|} = \bigoplus_{j=0}^{n} F_j.$$

The map $\varphi : C(X) \to \bigoplus_{j=0}^{n} F_j$ from equation (2.5) is a c.c.p map and in tandem with the map $\psi : \bigoplus_{j=0}^{n} F_j \to C(X)$ from equation (2.6), satisfies $\|(\psi \circ \varphi)(g) - g\| < \epsilon$ for $g \in F$.

Furthermore if $j \in \{0, 1, \dots, n\}$ then the restriction $\psi|_{F_j} = \psi_j$, where ψ_j is defined in equation (2.7). Moreover, we showed previously that if $j \in \{0, 1, \dots, n\}$ then ψ_j is a c.c.p map of order zero. By Definition 2.3.2, we deduce that

$$\dim_{\text{nuc}} C(X) \le n = \dim X. \tag{2.8}$$

2.4 Examples and properties of nuclear dimension

In Remark 2.3.2, we demonstrated that if X is a compact Hausdorff space then the nuclear dimension of $C(X)$ is bounded above by the covering dimension of X. In Theorem 2.4.1, we will show that the reverse inequality also holds and that consequently, nuclear dimension lives up to its name as a non-commutative variant of covering dimension.

Theorem 2.4.1. *Let X be a compact Hausdorff space. Then*

$$\dim_{nuc} C(X) = \dim X.$$

Proof. Assume that X is a compact Hausdorff space. By Remark 2.3.2, it suffices to show that

$$\dim X \le \dim_{\text{nuc}} C(X).$$

If $\dim_{\text{nuc}} C(X) = \infty$ then there is nothing to show. So let $n = \dim_{\text{nuc}} C(X) \in \mathbb{Z}_{\ge 0}$. Let $\mathcal{U} = \{U_1, U_2, \ldots, U_m\}$ be a finite open cover for X. Let $\{h_a\}_{a=1}^m$ be a partition of unity subordinate to $\mathcal{U}$. Define

$$\mathcal{F} = \{h_1, \ldots, h_m, 1\} \subsetneq C(X).$$

The element $1 \in C(X)$ is the constant function mapping $x \in X$ to 1. By Definition 2.3.2, there exist finite-dimensional C*-algebras $F_0, F_1, \ldots, F_n$ and completely positive maps

$$\varphi : C(X) \to \bigoplus_{j=0}^{n} F_j \qquad \text{and} \qquad \psi : \bigoplus_{j=0}^{n} F_j \to C(X)$$

such that if $f \in \mathcal{F}$ then

$$\sup_{x \in X} |(\psi \circ \varphi)(f)(x) - f(x)| = \|(\psi \circ \varphi)(f) - f\| < \frac{1}{8m(n+1)^2}. \tag{2.9}$$

Moreover, φ is a c.c.p map and the restrictions $\psi_j = \psi|_{F_j}$ are c.c.p maps of order zero. If $j \in \{0, 1, \ldots, n\}$ then F_j is finite-dimensional and hence is a direct sum of matrix algebras. The map $\psi_j : F_j \to C(X)$ restricts to a c.c.p order zero map on each of these matrix algebras. Without loss of generality, we may assume that these restrictions are all non-zero. By the contrapositive of Theorem 2.2.3, these matrix algebras are isomorphic to $\mathbb{C}$ and consequently, there exists $k(j) \in \mathbb{Z}_{>0}$ such that $F_j \cong \mathbb{C}^{k(j)}$.

If $j \in \{0, 1, \ldots, n\}$ then let $\{e_i^{(j)}\}_{i=1}^{k(j)}$ denote the standard basis for $F_j \cong \mathbb{C}^{k(j)}$. Define the open set

$$W_{j,i} = \psi_j(\varphi(1)e_i^{(j)})^{-1}\Big(\big(\frac{1}{4(n+1)^2}, \infty\big)\Big) = \Big\{x \in X \;\Big|\; \psi_j(\varphi(1)e_i^{(j)})(x) > \frac{1}{4(n+1)^2}\Big\}.$$

If $i, i' \in \{1, 2, \ldots, k(j)\}$ are distinct then

$$W_{j,i} \cap W_{j,i'} = \emptyset$$

because ψ_j is an order zero map. Now if $j \in \{1, 2, \ldots, n\}$ then we define

$$W_0 = \big\{W_{0,i} \mid i \in \{1, 2, \ldots, k(0)\}\big\},$$

$$W_j = \big\{W_{j,i} \mid i \in \{1, 2, \ldots, k(j)\}\big\} \Big\backslash \bigcup_{\ell=0}^{j-1} W_\ell$$

and

$$W = W_0 \sqcup W_1 \sqcup \cdots \sqcup W_n.$$

We claim that W is an open cover for X. Suppose for the sake of contradiction that there exists $x \in X$ such that if $j \in \{0, 1, \ldots, n\}$ and $i \in \{1, 2, \ldots, k(i)\}$ then $x \notin W_{j,i}$. This means that

$$\psi_j(\varphi(1)e_i^{(j)})(x) \leq \frac{1}{4(n+1)^2}. \tag{2.10}$$

Summing over both i and j, we have

$$(\psi \circ \varphi)(1)(x) = \sum_{j=0}^{n} \sum_{i=1}^{k(j)} \psi_j(\varphi(1)e_i^{(j)})(x).$$

Since the maps $\psi_0, \ldots, \psi_n$ are all order zero maps then the sum on the RHS contains at most $n+1$ strictly positive summands. By equation (2.10), we find that

$$(\psi \circ \varphi)(1)(x) \leq \frac{1}{4(n+1)} \leq \frac{1}{4}. \tag{2.11}$$

However by equation (2.9)

$$|(\psi \circ \varphi)(1)(x) - 1(x)| = |(\psi \circ \varphi)(1)(x) - 1| < \frac{1}{8m(n+1)^2}$$

and

$$(\psi \circ \varphi)(1)(x) > 1 - \frac{1}{8m(n+1)^2} > 1 - \frac{1}{8} = \frac{7}{8}.$$

This contradicts equation (2.11). Therefore W is an open cover for X with order at most n.

We now claim that W is a refinement of $\mathcal{U}$.

To show: (a) If $j \in \{0,1,\dots,n\}$ and $i \in \{1,2,\dots,k(j)\}$ then there exists $b \in \{1,2,\dots,m\}$ such that $W_{j,i} \subseteq U_b$.

(a) Assume that $y \in W_{j,i}$. Since $1 = \sum_{a=1}^{m} h_a$ then

$$\sum_{a=1}^{m} \psi_j(\varphi(h_a)e_i^{(j)})(x) = \psi_j(\varphi(1)e_i^{(j)})(x) > \frac{1}{4(n+1)^2}.$$

So there exists $b \in \{1,2,\dots,m\}$ such that

$$\psi_j(\varphi(h_b)e_i^{(j)})(x) \geq \frac{1}{4m(n+1)^2}.$$

Now sum over all $i \in \{1,2,\dots,k(j)\}$. Since ψ_j is an order zero map, we find that

$$(\psi_j \circ \varphi)(h_b)(x) \geq \frac{1}{4m(n+1)^2} \qquad \text{and} \qquad (\psi \circ \varphi)(h_b)(x) \geq \frac{1}{4m(n+1)^2}.$$

By equation (2.9), we also have

$$|(\psi \circ \varphi)(h_b)(x) - h_b(x)| < \frac{1}{8m(n+1)^2}.$$

Therefore

$$\frac{1}{4m(n+1)^2} \leq (\psi \circ \varphi)(h_b)(x) < h_b(x) + \frac{1}{8m(n+1)^2}.$$

Consequently

$$0 < \frac{1}{8m(n+1)^2} < h_b(x) \qquad \text{and} \qquad x \in supp(h_b) \subseteq U_b.$$

We deduce that there exists $b \in \{1,2,\dots,m\}$ such that $W_{j,i} \subseteq U_b$.

By part (a), we conclude that W is a finite open cover of X with order at most n which refines the original cover $\mathcal{U}$. Hence

$$\dim X \leq n = \dim_{\text{nuc}} C(X)$$

as required. □

Next, we will list a few of the properties satisfied by nuclear dimension from [WZ10]. First, nuclear dimension behaves well with respect to basic constructions of C*-algebras.

Proposition 2.4.2. *[WZ10, Proposition 2.3] Let A, B, C, D and E be C*-algebras. Suppose that $C = \lim_i C_i$ is an inductive limit of C*-algebras and D is a quotient of E. Then*

1. $\dim_{nuc}(A \oplus B) = \max(\dim_{nuc} A, \dim_{nuc} B)$,

2. $\dim_{nuc}(A \otimes B) = (\dim_{nuc} A + 1)(\dim_{nuc} B + 1) - 1$,

3. $\dim_{nuc} C \leq \liminf \dim_{nuc} C_i$,

4. $\dim_{nuc} D \leq \dim_{nuc} E$.

The next property elucidates the relationship between nuclear dimension and nuclearity.

Theorem 2.4.3. *Let A be a C*-algebra and $n \in \mathbb{Z}_{\geq 0}$. If $\dim_{nuc} A \leq n$ then A is a nuclear C*-algebra.*

By the definition of nuclear dimension (Definition 2.3.2), Theorem 2.4.3 amounts to modifying the upwards c.p map in the nuclear dimension approximation for a C*-algebra A with finite nuclear dimension, so that it becomes contractive. In the unital case, the main idea is that the approximation factors through a finite-dimensional C*-algebra F, which is a direct sum of matrix algebras ([Mur90, Theorem 6.3.8]). Consequently, the image of the unit 1_A in F can be diagonalised. By projecting to an eigenspace whose eigenvalues are "large enough", the original c.p maps can then be modified so that they factor through a hereditary C*-subalgebra of F, they still approximate the identity map id_A and the upwards map is contractive.

Theorem 2.4.4. *[WZ10, Proposition 2.5] Let A be a C*-algebra and B be a hereditary C*-subalgebra of A. Then $\dim_{nuc} B \leq \dim_{nuc} A$.*

Observe that by Theorem 2.4.4, if I is an ideal of A then $\dim_{\mathrm{nuc}} I \leq \dim_{\mathrm{nuc}} A$. Finally, we will end the section with known examples of C*-algebras for which the precise value of nuclear dimension is known.

Theorem 2.4.5. *Let A be a C*-algebra. The following are equivalent:*

1. *A is an AF algebra,*

2. $\dim_{nuc} A = 0$,

3. $dr\ A = 0$.

As stated in the introduction, Theorem 2.4.5 is due to [KW04, Example 4.1] and [WZ10, Remarks 2.2 (iii)].

Example 2.4.1. In [CETWW21], it was proved that if A is a simple unital C*-algebra then $\dim_{\mathrm{nuc}} A$ is $0, 1$ or ∞. In particular, by Theorem 2.4.5, $\dim_{\mathrm{nuc}} A = 0$ if and only if A is an AF-algebra.

Example 2.4.2. In [WZ10, Theorem 7.4], it was shown that if $n \in \mathbb{Z}_{>1}$ then the nuclear dimension of the Cuntz algebra $\mathcal{O}_n$ is 1 and $\dim_{\mathrm{nuc}} \mathcal{O}_\infty \leq 2$. Recall that a *Kirchberg algebra* is a simple, separable, nuclear and purely infinite C*-algebra. The well-known Kirchberg-Phillips theorem states that Kirchberg algebras satisfying the UCT (universal coefficient theorem) are completely classified by their K-theory ([Phi00]).

One consequence of the Kirchberg-Phillips theorem is that every Kirchberg algebra satisfying the UCT is an inductive limit of C*-algebras assuming the form

$$\Big(\bigoplus_{i=1}^{r}(M_{k_i}(\mathbb{C})\otimes\mathcal{O}_{n_i})\Big)\otimes C(S^1).$$

Hence by Proposition 2.4.2, Winter and Zacharias concluded that the nuclear dimension of a Kirchberg algebra satisfying the UCT is at most five. This was improved in [BBSTWW19, Corollary 9.9], where it was proved that if A is a Kirchberg algebra then $\dim_{\text{nuc}} A = 1$.

Example 2.4.3. If $\theta \in \mathbb{R}$ then the rotation algebra A_θ is the universal C*-algebra generated by unitaries u, v satisfying

$$uv = e^{2\pi i\theta}vu.$$

By [WZ10, Example 6.1], the nuclear dimension of the rotation algebra A_θ is

$$\dim_{\text{nuc}} A_\theta = \begin{cases} 1, & \text{if } \theta \notin \mathbb{Q}, \\ 2, & \text{if } \theta \in \mathbb{Q}. \end{cases}$$

In the case where θ is irrational, A_θ is an inductive limit whose building blocks are finite direct sums of matrix algebras over $C(S^1)$ ([EE93, Theorem 4]). By the first and second properties in Proposition 2.4.2, these building blocks have nuclear dimension 1. Hence by the third property in Proposition 2.4.2, the fact that A_θ is not an AF-algebra ([Rie93, Introduction]) and Theorem 2.4.5, $\dim_{\text{nuc}} A_\theta = 1$.

2.5 Nuclear dimension and extensions of C*-algebras

As mentioned in the introduction, the key property which separates nuclear dimension from its predecessors such as decomposition rank is that finite nuclear dimension is preserved under extensions of C*-algebras. This result is due to Winter and Zacharias in [WZ10, Proposition 2.9].

Theorem 2.5.1. *Let*

$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0$$

be an extension of C-algebras. Then*

$$\max(\dim_{nuc} I, \dim_{nuc} B) \leq \dim_{nuc} A \leq \dim_{nuc} I + \dim_{nuc} B + 1. \tag{2.12}$$

Remark 2.5.1. In Theorem 2.5.1, note that $\dim_{\text{nuc}} A + 1$ is the least number of "colours" (more precisely, c.c.p order zero maps) needed for a nuclear dimension approximation of A. So the number of colours needed for a nuclear dimension approximation of the extension A is at most the total number of colours required to approximate both the ideal I and the quotient B. To see why this is the case, first observe that by [BO08, Theorem 1.2.1], there exists a quasicentral approximate unit $\{e_\lambda\}_{\lambda\in\Lambda}$ in I. If $x \in A$ then by [BO08, Proposition 1.2.2],

$$\left\| x - \left(e_\lambda^{\frac{1}{2}} x e_\lambda^{\frac{1}{2}} + (1_{\tilde{A}} - e_\lambda)^{\frac{1}{2}} x (1_{\tilde{A}} - e_\lambda)^{\frac{1}{2}} \right) \right\| \to 0.$$

Hence the key idea behind the upper bound in equation (2.12) is that any element $x \in A$ can be approximated in norm as a sum of two terms where the first term $e_\lambda^{\frac{1}{2}} x e_\lambda^{\frac{1}{2}} \in I$ and the second term $(1_{\tilde{A}} - e_\lambda)^{\frac{1}{2}} x (1_{\tilde{A}} - e_\lambda)^{\frac{1}{2}}$ is equal to x in the quotient B.

In one special case, the upper bound in equation (2.12) can be further refined.

Definition 2.5.1. Let

$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0$$

be an extension of C*-algebras. We say that this extension is **quasidiagonal** if there exists a quasicentral approximate unit of projections $\{p_\lambda\}_{\lambda\in\Lambda}$ for A contained in I.

If the extension in Theorem 2.5.1 is quasidiagonal then a colour from the ideal can be merged with a colour from the quotient because their ranges as c.c.p order zero maps become orthogonal to each other. Thus, we obtain

Corollary 2.5.2. *Let*

$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0$$

be a quasidiagonal extension of C-algebras. Then*

$$\dim_{nuc} A = \max(\dim_{nuc} I, \dim_{nuc} B).$$

We end this section with the following example which plays a major role in the remainder of the document.

Example 2.5.2. The archetypal example demonstrating the difference between nuclear dimension and decomposition rank regarding extensions of C*-algebras is the Toeplitz algebra $\mathcal{T}$, which fits into the following extension:

$$0 \longrightarrow K \longrightarrow \mathcal{T} \longrightarrow C(S^1) \longrightarrow 0$$

By Theorem 2.4.5, $\dim_{\text{nuc}} K = \text{dr}\ K = 0$ and by Theorem 2.4.1, $\dim_{\text{nuc}} C(S^1) = \text{dr}\ C(S^1) = 1$. Hence by Theorem 2.5.1, $\dim_{\text{nuc}} \mathcal{T} \leq 2$. Contrarily, $\mathcal{T}$ is generated by a proper isometry and is by definition, not stably finite. Since $\mathcal{T}$ is not stably finite then it is not quasidiagonal (by [BO08, Proposition 7.1.15]) and dr $\mathcal{T} = \infty$ (by [KW04, Proposition 5.1]).

Chapter 3

Recent results on the nuclear dimension of extensions

3.1 Brake and Winter's work on the Toeplitz algebra

Equation (2.12) yields an upper bound for the nuclear dimension of an extension of C*-algebras, provided that the associated ideal and quotient both have finite nuclear dimension. A natural question is whether the upper bound in equation (2.12) is optimal. In [BW19], this question was decisively answered in the negative by Brake and Winter, who demonstrated that the nuclear dimension of the Toeplitz algebra is equal to 1 and not the naïve upper bound of 2 in Example 2.5.2.

To illustrate the general principle behind Brake and Winter's proof, we first recall that the Toeplitz algebra fits into the extension

$$0 \longrightarrow K \longrightarrow \mathcal{T} \longrightarrow C(S^1) \longrightarrow 0.$$

Brake and Winter's argument utilises a specific type of quasicentral approximate unit.

Definition 3.1.1. Let A be a C*-algebra and $J \subseteq A$ be a separable ideal. Let $\{q_j\}_{j\in\mathbb{Z}_{>0}}$ be a quasicentral approximate unit in J. We say that $\{q_j\}$ is **idempotent** if the following statement is satisfied: If $i \in \mathbb{Z}_{>0}$ then $q_{i+1}q_i = q_i$.

By [BO08, Lemma 7.3.1], separable ideals always admit an idempotent quasicentral approximate unit. So let $\{h_n\}_{n\in\mathbb{Z}_{>0}}$ be an idempotent quasicentral approximate unit in K. By [BO08, Proposition 1.2.2], if $x, y \in \mathcal{T}$ and $x - y \in K$ then

$$x \approx h_n^{\frac{1}{2}} x h_n^{\frac{1}{2}} + (h_{n+1} - h_n)^{\frac{1}{2}} y (h_{n+1} - h_n)^{\frac{1}{2}} + (1 - h_{n+1})^{\frac{1}{2}} y (1 - h_{n+1})^{\frac{1}{2}} \tag{3.1}$$

where the approximation is in norm. Since $\{h_n\}_{n\in\mathbb{Z}_{>0}} \subsetneq K$ then the first term $h_n^{\frac{1}{2}} x h_n^{\frac{1}{2}}$ in equation (3.1) is in K and thus, contributes one colour (red) to a nuclear dimension approximation for $\mathcal{T}$ (because $\dim_{\mathrm{nuc}} K = 0$). The third term

$(1-h_{n+1})^{\frac{1}{2}}y(1-h_{n+1})^{\frac{1}{2}}$ in equation (3.1) is equal to x in the quotient $C(S^1)$ and since $\dim_{\text{nuc}} C(S^1) = 1$, it contributes two colours (blue and purple) to the nuclear dimension approximation for $\mathcal{T}$.

The three coloured nuclear dimension approximation for $\mathcal{T}$ in equation (3.1) is represented by the following diagram:

$$\begin{tikzcd}
x \in \mathcal{T} \arrow[rr, "\approx id_{\mathcal{T}}"] \arrow[dr, "\phi"'] & & \alpha + \beta + (1-h_{n+1})^{\frac{1}{2}}y(1-h_{n+1})^{\frac{1}{2}} \\
& F^{(0)} \oplus F^{(1)} \oplus F^{(2)} \arrow[ur, "(\psi_0,\psi_1,\psi_2)"'] &
\end{tikzcd}$$

Since $\{h_n\}$ is an idempotent quasicentral approximate unit then $h_n(1-h_{n+1}) = (1-h_{n+1})h_n = 0$ and subsequently, the first and third terms in equation (3.1) are orthogonal. This means that the red colour can be merged with one of the two colours from the third term:

$$\begin{tikzcd}
x \in \mathcal{T} \arrow[rr, "\approx id_{\mathcal{T}}"] \arrow[dr, "\phi"'] & & \alpha + \beta + (1-h_{n+1})^{\frac{1}{2}}y(1-h_{n+1})^{\frac{1}{2}} \\
& (F^{(0)} \oplus F^{(1)}) \oplus F^{(2)} \arrow[ur, "((\psi_0,\psi_1),\psi_2)"'] &
\end{tikzcd}$$

In [BW19], Brake and Winter show that with a slight perturbation, the purple colour can be reused to colour in the middle term, yielding a two coloured nuclear dimension approximation for $\mathcal{T}$:

$$\begin{tikzcd}
x \in \mathcal{T} \arrow[rr, "\approx id_{\mathcal{T}}"] \arrow[dr, "\phi"'] & & \alpha + \beta + (1-h_{n+1})^{\frac{1}{2}}y(1-h_{n+1})^{\frac{1}{2}} \\
& (F^{(0)} \oplus F^{(1)}) \oplus F^{(2)} \arrow[ur, "((\psi_0,\psi_1),\tilde{\psi_2})"'] &
\end{tikzcd} \tag{3.2}$$

As a result, $\dim_{\text{nuc}} \mathcal{T} = 1$.

Now that we have illustrated the general idea behind Brake and Winter's argument, we will now supplement further details. Firstly by the Choi-Effros theorem ([BO08, Theorem C.3]), there exists a c.c.p map $\mu : C(S^1) \to \mathcal{T}$ such that if $\pi : \mathcal{T} \to C(S^1)$ is the quotient *-homomorphism then $\pi \circ \mu = id_{C(S^1)}$. Secondly we may by density assume each $h_n \in K$ is a finite-rank operator. Now if $n \in \mathbb{Z}_{>0}$ then define the C*-subalgebras

$$A_n = h_n \mathcal{T} h_n, \quad B_n = (h_{n+1} - h_n)\mathcal{T}(h_{n+1} - h_n) \quad \text{and} \quad C_n = \overline{(1-h_{n+1})\mathcal{T}(1-h_{n+1})}.$$

Define the c.c.p maps

$$\begin{array}{rrcl} \alpha: & \mathcal{T} & \to & \prod_n A_n \\ & x & \mapsto & \left(h_n^{\frac{1}{2}} x h_n^{\frac{1}{2}}\right)_{n\in\mathbb{Z}_{>0}} \end{array} \tag{3.3}$$

$$\begin{array}{rrcl} \beta: & C(S^1) & \to & \prod_n B_n \\ & f & \mapsto & \left((h_{n+1}-h_n)^{\frac{1}{2}}\mu(f)(h_{n+1}-h_n)^{\frac{1}{2}}\right)_{n\in\mathbb{Z}_{>0}} \end{array} \tag{3.4}$$

$$\begin{array}{rrcl} \gamma: & C(S^1) & \to & \prod_n C_n \\ & f & \mapsto & \left((1-h_{n+1})^{\frac{1}{2}}\mu(f)(1-h_{n+1})^{\frac{1}{2}}\right)_{n\in\mathbb{Z}_{>0}} \end{array} \tag{3.5}$$

By equation (3.1), if $x \in \mathcal{T}$ then $x - (\mu \circ \pi)(x) \in K$ and

$$x \approx \alpha(x) + \beta(\pi(x)) + \gamma(\pi(x)). \tag{3.6}$$

In line with our analogy of c.c.p order zero maps as colours, we would ideally like β to be a c.c.p order zero map so that when we perturb it later, the perturbation remains a c.c.p order zero map. The issue is that β is not order zero. We do know that if $q_B : \prod_n B_n \to \prod_n B_n / \bigoplus_n B_n$ is the quotient map then the composite $q_B \circ \beta$ is a c.c.p order zero map from $C(S^1)$.

Remarkably, Brake and Winter showed that the composite $q_B \circ \beta$ can be lifted to a c.c.p order zero map from $C(S^1)$ to $\prod_n B_n$. The key innovation in [BW19] was a clever use of Lin's theorem on almost normal matrices ([Lin97]).

Theorem 3.1.1 (Lin). *If $\eta \in \mathbb{R}_{>0}$ then there exists $\delta \in \mathbb{R}_{>0}$ such that the following statement is satisfied: If $n \in \mathbb{Z}_{>0}$, $x \in M_n(\mathbb{C})$ and $\|xx^* - x^*x\| < \delta$ then there exists $y \in M_n(\mathbb{C})$ such that $yy^* = y^*y$ and $\|x - y\| < \eta$.*

Theorem 3.1.1 says that regardless of the size of the square matrix, if we are given an approximately normal matrix then we can always find an arbitrarily close normal matrix nearby. Before we state Brake and Winter's application of Theorem 3.1.1, recall the notion of an ultraproduct.

Definition 3.1.2. Let $\{A_n\}_{n\in\mathbb{Z}_{>0}}$ be a sequence of C*-algebras and $\prod_n A_n$ denote the product C*-algebra. Let

$$\bigoplus_n A_n = \left\{ (a_n)_n \in \prod_n A_n \;\middle|\; \lim_{n\to\infty} \|a_n\| = 0 \right\}.$$

Then $\bigoplus_n A_n$ is a closed ideal of $\prod_n A_n$. The **ultraproduct** of $\{A_n\}_n$ is the quotient C*-algebra $\prod_n A_n / \bigoplus_n A_n$.

Brake and Winter use Theorem 3.1.1 to prove a lifting theorem for c.c.p order zero maps from $C(S^1)$ to an ultraproduct of finite-dimensional C*-algebras.

Lemma 3.1.2. *Let $\{F_n\}_{n\in\mathbb{Z}_{>0}}$ be a sequence of finite-dimensional C*-algebras. Let $\beta : C(S^1) \to \prod_n F_n / \bigoplus_n F_n$ be a c.c.p order zero map. Then there exists a c.c.p order zero map*

$$\overline{\beta} : C(S^1) \to \prod_n F_n$$

*which lifts β. That is, if $\pi : \prod_n F_n \to \prod_n F_n / \bigoplus_n F_n$ is the quotient *-homomorphism then there exists a c.c.p order zero map $\overline{\beta}$ such that $\pi \circ \overline{\beta} = \beta$.*

Now since our idempotent quasicentral approximate unit $\{h_n\}_n$ consists of finite-rank operators then each B_n is a finite-dimensional C*-subalgebra of $\mathcal{T}$. Subsequently by Lemma 3.1.2, there exists a c.c.p order zero map $\overline{\beta} : C(S^1) \to \prod_n B_n$ which lifts $q_B \circ \beta$.

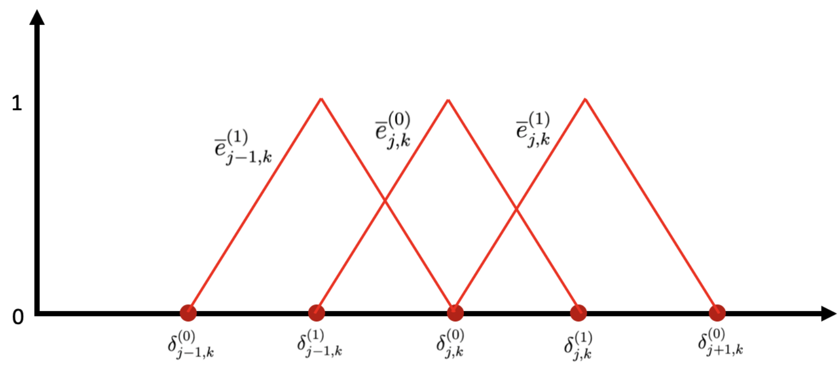
The second major step in Brake and Winter's argument was to show that, up to a small perturbation, the c.c.p order zero map $\overline{\beta}$ factors through a finite-dimensional C*-algebra and extends from $C(S^1)$ to bounded Borel functions $B(S^1)$. This factorisation uses a nuclear dimension approximation for $C(S^1)$ from Theorem 2.1.1. To be specific, if $k \in \mathbb{Z}_{>0}$, $j \in \{1, 2, \ldots, k\}$ and $\ell \in \{0, 1\}$ then define

$$\delta_{j,k}^{(\ell)} = e^{2\pi i \frac{2j+\ell}{2k}} \in S^1.$$

The downwards map in Brake and Winter's nuclear dimension approximation for $C(S^1)$ is $\psi_k = \psi_k^{(0)} \oplus \psi_k^{(1)} : C(S^1) \to \mathbb{C}^k \oplus \mathbb{C}^k$ where $\psi_k^{(\ell)}$ is given by

$$\begin{array}{rccc} \psi_k^{(\ell)} : & C(S^1) & \to & \mathbb{C}^k \\ & f & \mapsto & \big(f(\delta_{j,k}^{(\ell)})\big)_{j=1}^k . \end{array} \tag{3.7}$$

To define the upwards map, we need a partition of unity for S^1. In [BW19], this is given by the family of sawtooth functions $\{\overline{e}_{j,k}^{(\ell)}\}_{j\in\{1,2,\ldots,k\},\, \ell\in\{0,1\}}$ defined by the following diagram:



The upwards map is $\varphi_k = \varphi_k^{(0)} + \varphi_k^{(1)} : \mathbb{C}^k \oplus \mathbb{C}^k \to C(S^1)$ where $\varphi_k^{(\ell)}$ is given by

$$\begin{array}{rccc} \varphi_k^{(\ell)} : & \mathbb{C}^k & \to & C(S^1) \\ & (a_1, \ldots, a_k) & \mapsto & \sum_{j=1}^k a_j \overline{e}_{j,k}^{(\ell)} . \end{array} \tag{3.8}$$

With the above nuclear dimension approximation for $C(S^1)$, we can now state Brake and Winter's extension step as the following lemma.

Lemma 3.1.3. *There exist c.c.p order zero maps*

$$\Big\{\rho_k : \mathbb{C}^k \to \prod\nolimits_n B_n\Big\}_{k\in\mathbb{Z}_{>0}}$$

which satisfy the following two properties:

1. *If $f \in C(S^1)$ then*
$$\lim_{k\to\infty} \|(\rho_k \circ \psi_k^{(0)})(f) - \overline{\beta}(f)\| = 0.$$

2. *If $k \in \mathbb{Z}_{>0}$ then the c.c.p order zero map $\rho_k \circ \psi_k^{(0)}$ extends to a c.c.p order zero map on $B(S^1)$.*

The overall effect of Lemma 3.1.2 and Lemma 3.1.3 on the approximation in equation (3.6) is that the c.c.p map β in the middle term was converted into an order zero map $\overline{\beta}$ and then extended to $B(S^1)$, all at the small cost of a norm approximation. By a subsequent computation, Brake and Winter demonstrated that the "extra room" provided by extending to $B(S^1)$ allows the middle term in equation (3.6) to be coloured using one of the colours from the quotient term, thus implementing the crucial step outlined in Diagram (3.2).

3.2 Building on [BW19] — The general proof strategy

The result and method of proof in [BW19] sparked interest in the determination of the nuclear dimension of particular extensions of C*-algebras. Let

$$0 \longrightarrow J \longrightarrow A \longrightarrow B \longrightarrow 0$$

be an extension of C*-algebras where J is separable. The table below summarises the main results of papers building on the work done in [BW19].

J	A	B	Bound on $\dim_{\mathrm{nuc}} A$	Exact value	Reference
K	—	$C(X)$	$\dim X + 1$	$\dim X$	[GT22]
K	$\mathcal{T}_n$	$\mathcal{O}_n$	2	1	[CT20]
Stable AF	Unital ext.	Kirchberg	2	1	[CT20]
Stable AF	Full ext.	$\mathcal{O}_\infty$-stable	2	1	[Evi22]
Kirchberg	Essential ext.	$C(S^1)$	3	1	[ENSW]

[GT22, Theorem A] directly generalises Brake and Winter's result by replacing S^1 with an arbitrary compact Hausdorff space X. In [CT20], it was proved that if $n \in \mathbb{Z}_{>1}$ then the Cuntz-Toeplitz algebra $\mathcal{T}_n$, the universal C*-algebra generated by isometries $s_1, \ldots, s_n$ with pairwise orthogonal images, has nuclear dimension 1. Observe that [CT20] is also a generalisation of [BW19] because the Toeplitz algebra can be thought of as the Cuntz-Toeplitz algebra $\mathcal{T}_1$, whose accompanying quotient is $C(S^1)$ rather than a Cuntz algebra.

More generally, it was proved in [CT20, Theorem 1] that a unital extension of a Kirchberg algebra by a stable AF-algebra has nuclear dimension 1. By [KP00, Theorem 3.15], a Kirchberg algebra is $\mathcal{O}_\infty$-stable. So, if one replaces the Kirchberg algebra quotient in [CT20, Theorem 1] with a separable, nuclear and $\mathcal{O}_\infty$-stable C*-algebra, does the resulting extension still have nuclear dimension one? This was answered in the affirmative by Evington, who showed in [Evi22, Theorem 1] that full extensions of a separable nuclear and $\mathcal{O}_\infty$-stable C*-algebra by a stable AF-algebra have nuclear dimension 1. To be clear, we define the notion of a full extension below as stated in [KN06, Definition 1 (ii)-(iii)].

Definition 3.2.1. Let

$$0 \longrightarrow J \longrightarrow A \longrightarrow B \longrightarrow 0$$

be an extension of C*-algebras. Let $\mathcal{M}(J)$ be the multiplier algebra of J and $\phi : B \to \mathcal{M}(J)/J$ be a *-homomorphism. We say that ϕ is **norm-full** if the following statement is satisfied: If $b \in B$ is non-zero and positive then the ideal generated by $\phi(b)$ is $\mathcal{M}(J)/J$.

We say that the extension above is **full** if its associated Busby invariant $\beta : B \to \mathcal{M}(J)/J$ is norm-full.

Note that if the quotient B is simple then unital extensions with quotient B are full extensions. Hence, [Evi22, Theorem 1] generalises [CT20, Theorem 1].

The proofs of the main theorems in [CT20], [GT22] and [Evi22] share the same overarching strategy as the proof of [BW19, Theorem 1].

Step 1: Decompose the constant sequence embedding $A \hookrightarrow \prod_n A/\bigoplus_n A$ by using an idempotent quasicentral approximate unit from the ideal J.

In [BW19, Equation 1] and [GT22, Equation 4.6], this was achieved in an essentially identical manner by a straightforward application of [BO08, Proposition 1.2.2] (see equation (3.6)). However, [CT20, Equation 18] and [Evi22, Equation 12] use a slightly different decomposition. In [CT20] and [Evi22], the ideal B is $\mathcal{O}_\infty$-stable. So $B \cong B \otimes \mathcal{O}_\infty \otimes \mathcal{O}_\infty \otimes \dots$ and one can find an approximately central sequence of positive contractions $\{k_i\}_{i \in \mathbb{Z}_{>0}}$ in B such that if $i \in \mathbb{Z}_{>0}$ then the spectrum $\sigma(k_i) = [0,1]$. The family $\{k_i\}_i$ is then used to decompose the constant sequence embedding into a sum of three terms, two of which depend on $\{k_i\}_i$. The reason why $\{k_i\}_i$ was introduced will be explained in the next step.

As stated in [CT20], the idea to use an approximately central sequence in B is inspired by a technique used to study $\mathcal{O}_\infty$-stable C*-algebras. Notably, the technique was previously used in the proofs that Kirchberg algebras and $\mathcal{O}_\infty$-stable C*-algebras have nuclear dimension 1 (see [BBSTWW19, Section 9] and [BGSW22] respectively).

Step 2: Use results from classification to facilitate the reuse of one colour from the quotient B.

In [BW19] and [GT22], the decomposition of the constant sequence embedding $\iota : A \hookrightarrow \prod_n A/\bigoplus_n A$ is roughly

$$\iota = \text{Ideal term} + \text{Middle term} + \text{Quotient term}. \tag{3.9}$$

All three terms depend on an idempotent quasicentral approximate unit in the ideal J. As detailed in the previous section, Lin's theorem on almost normal matrices (see Theorem 3.1.1) was used to prove Lemma 3.1.2 in [BW19]. With Lemma 3.1.2, Brake and Winter were able to extend the middle term in equation (3.9) from $C(S^1)$ to $B(S^1)$. Consequently, they showed that they could recycle one of the available colours from the quotient term to colour this c.c.p order zero map (again see Diagram (3.2)).

In [GT22], the classification result Gardner and Tikuisis depended on was Lin's uniqueness theorem for approximately multiplicative maps from abelian C*-algebras to matrix algebras ([Lin17]). Analogously to the proof of Lemma 3.1.2, Lin's uniqueness theorem was used to prove [GT22, Corollary 2.7], which permits c.c.p order zero maps from $C(X)$ into an ultraproduct of a sequence of finite-dimensional C*-algebras to be extended to bounded Borel functions $B(X)$. Unlike Lemma 3.1.2, [GT22, Corollary 2.7] only applies to c.c.p order zero maps which satisfy a faithfulness condition. This condition is one of two assumptions in Lin's uniqueness theorem; the other one is a K-theoretic condition which is automatically satisfied by contractibility of the cone $(0, 1] \times X$. In [GT22, Corollary 3.2], Gardner and Tikuisis find a suitable idempotent quasicentral approximate unit needed for the middle term in equation (3.9) to satisfy the faithfulness condition so that they can extend it to $B(X)$ and then carry out a similar reuse of a colour to [BW19].

The extension steps done in [BW19] and [GT22] both rely on the fact that there are non-zero c.c.p order zero maps from the quotient B to a finite-dimensional C*-algebra. In [CT20] and [Evi22] however, the quotient B is simple and purely infinite. Since there are no non-trivial c.c.p order zero maps from B to a finite-dimensional C*-algebra, one cannot use the same extension argument as in [BW19] or [GT22]. Instead, [CT20] and [Evi22] take full advantage of the $\mathcal{O}_\infty$-stability of their quotient.

The decomposition of the constant sequence embedding used in [CT20] and [Evi22] also assumes the form in equation (3.9). However, the last two terms depend on both the idempotent quasicentral approximate unit $\{h_i\}_i$ in the ideal J and the approximately central sequence $\{k_i\}_i$ of positive contractions in the quotient B. To illustrate the general form of the argument used in both [CT20] and [Evi22], we will restrict our attention to the paper [CT20].

The first important result is [CT20, Proposition 5], which yields c.c.p maps from B with a specific factorisation through a matrix algebra. A key ingredient to the construction is Voiculescu's quasidiagonality of the cone $C_0(0, 1] \otimes B$ (see [BO08, Section 7.3]). We state [CT20, Proposition 5] as the lemma below.

Lemma 3.2.1. *Let B a fixed unital Kirchberg algebra. If $i \in \mathbb{Z}_{>0}$ then let J_i be a stable C*-algebra and*

$$\left\{0 \longrightarrow J_i \longrightarrow A_i \xrightarrow{\pi_i} B \longrightarrow 0\right\}_{i \in \mathbb{Z}_{>0}}$$

be a sequence of unital extensions. Then there exist a sequence of c.c.p maps $\{\phi_i :$

$B \to A_i\}_i$ *which factor through matrix algebras*

$$
\begin{tikzcd}
B \arrow[rr, "\phi_i"] \arrow[dr, "\eta_i"'] & & A_i \\
& M_{n_i}(\mathbb{C}) \arrow[ur, "\xi_i"'] &
\end{tikzcd}
\tag{3.10}
$$

Moreover, the factorisations in equation (3.10) *satisfy the following properties:*

1. *If $i \in \mathbb{Z}_{>0}$ then η_i is a c.c.p map.*
2. *If $i \in \mathbb{Z}_{>0}$ then ξ_i is a c.c.p order zero map.*
3. *The induced map $\phi : B \to \prod_i A_i / \bigoplus_i A_i$ is a c.c.p order zero map.*
4. *The *-homomorphism $\Phi : C_0(0,1] \otimes B \to \prod_i A_i / \bigoplus_i A_i$ induced by ϕ via Theorem 2.2.4 is full.*

There are two key features about Lemma 3.2.1. Firstly, the factorisations in equation (3.10) can be thought of as one-coloured models for the c.c.p maps $\{\phi_i\}_i$ because there is only one upwards c.c.p order zero map (one colour) in each factorisation. Secondly the ϕ_i, together with Theorem 2.2.4, combine to produce a full *-homomorphism from the cone $C_0(0,1] \otimes B$ to the ultraproduct $\prod_i A_i / \bigoplus_i A_i$.

How is the construction in Lemma 3.2.1 related to the decomposition in equation (3.9)? Herein lies the power of Gabe's classification theorem for full *-homomorphisms from cones over Kirchberg algebras to ultraproducts ([Gab24]). The middle and quotient terms in equation (3.9) are c.c.p order zero maps which by Theorem 2.2.4, induce respective *-homomorphisms $\Theta_m, \Theta_q : C_0(0,1] \otimes B \to \prod_i A_i / \bigoplus_i A_i$. The approximately central sequence $\{k_i\}_i$ in B we defined earlier ensures that Θ_m and Θ_q are both full *-homomorphisms.

Now if $i \in \mathbb{Z}_{>0}$ then define the hereditary C*-subalgebras

$$A_i = \overline{(1 - h_{i+1})A(1 - h_{i+1})} \qquad \text{and} \qquad J_i = \overline{(1 - h_{i+1})J(1 - h_{i+1})}.$$

Use Lemma 3.2.1 to construct sequences of c.c.p maps

$$\{\phi_i^{(m)} : B \to A\}_{i \in \mathbb{Z}_{>0}} \qquad \text{and} \qquad \{\phi_i^{(q)} : B \to A_i\}_{i \in \mathbb{Z}_{>0}}$$

corresponding to the constant sequence of extensions

$$\{0 \to J \to A \to B \to 0\}_i$$

and the sequence of extensions

$$\{0 \to J_i \to A_i \to B \to 0\}_i$$

respectively. As in the statement of Lemma 3.2.1, we obtain full *-homomorphisms Φ_m and Φ_q respectively. By Gabe's classification theorem (specifically [Gab24, Theorem B]), there exist unitary elements $u_m \in \prod_i M_2(A)/\bigoplus_i M_2(A)$ and $u_q \in \prod_i M_2(A_i)/\bigoplus_i M_2(A_i)$ such that if $x \in C_0(0,1] \otimes B$ then

$$u_m(\Phi_m(x) \oplus 0)u_m^* = \Theta_m(x) \oplus 0 \qquad \text{and} \qquad u_q(\Phi_q(x) \oplus 0)u_q^* = \Theta_q(x) \oplus 0.$$

In particular, by embedding $\prod_i A_i/\bigoplus_i A_i$ and $\prod_i A/\bigoplus_i A$ into $\prod_i M_2(A_i)/\bigoplus_i M_2(A_i)$ and $\prod_i M_2(A)/\bigoplus_i M_2(A)$, the middle and quotient terms in equation (3.9) are seen to be unitarily equivalent to $\phi^{(m)} \oplus 0$ and $\phi^{(q)} \oplus 0$ respectively.

The overall effect of applying Lemma 3.2.1 and Gabe's classification theorem is that we obtain one-coloured factorisations for the middle and quotient terms in equation (3.9), up to unitary equivalence and a matrix amplication. More simply, if α is the inclusion

$$\begin{aligned} \alpha : \ A &\to M_2(A) \\ a &\mapsto \begin{pmatrix} a & \\ & 0 \end{pmatrix} \end{aligned}$$

and $\tilde{\alpha} : \prod_i A/\bigoplus_i A \to \prod_i M_2(A)/\bigoplus_i M_2(A)$ is the induced map on ultraproducts then

$$\tilde{\alpha} \circ \iota = \tilde{\alpha} \circ \text{Ideal term} + \tilde{\alpha} \circ \text{Middle term} + \tilde{\alpha} \circ \text{Quotient term}. \tag{3.11}$$

As in [BW19], the ideal and quotient terms in equation (3.9) are orthogonal. Hence, the first and third terms in equation (3.11) are still orthogonal to each other and the singular colour from the third term can be reused to colour the ideal term.

$$\tilde{\alpha} \circ \iota = \tilde{\alpha} \circ \text{Ideal term} + \tilde{\alpha} \circ \text{Middle term} + \tilde{\alpha} \circ \text{Quotient term}. \tag{3.12}$$

The proof of [Evi22, Theorem 1] follows the same argument outlined above. In order to construct the zero-dimensional (or one-coloured) models required for comparison with the middle and quotient terms in equation (3.9), Evington utilised [BGSW22, Lemma 3.5] in lieu of Lemma 3.2.1. The specific version of Gabe's classification theorem Evington applied to carry out the comparison and obtain unitary equivalence is [Gab24, Theorem F].

Step 3: Build the appropriate nuclear dimension approximations for the identity map id_A or a closely related map.

After executing step 2 in [BW19] and [GT22], we arrive at the two-coloured decomposition

$$\iota \approx \text{Ideal term} + \text{Middle term} + \text{Quotient term}.$$

To be clear, the above approximation is in norm. From this, [BW19] and [GT22] construct a two-coloured nuclear dimension approximation for id_A by first lifting the c.c.p order zero middle and quotient terms to $\prod_n A$ using Loring's result in [Lor97, Theorem 10.2.1]. The effect of this is that ι approximately factors in norm as the composite

$$A \xrightarrow{I+M+Q} \prod_n A \xrightarrow{\quad q \quad} \prod_n A / \bigoplus_n A \tag{3.13}$$

where q is simply the quotient map. The definition of the quotient norm on the ultraproduct $\prod_n A/\bigoplus_n A$ then allows for individual elements of A to be approximated because $\iota : A \hookrightarrow \prod_n A/\bigoplus_n A$ is the constant sequence embedding. Hence, [BW19] and [GT22] obtain two-coloured nuclear dimension approximations for id_A, proving that $\dim_{\mathrm{nuc}} A = 1$.

In [CT20] and [Evi22], applying the second step results in equation (3.12); a two-coloured approximation for the composite $\tilde{\alpha} \circ \iota : A \to \prod_i M_2(A)/\bigoplus_i M_2(A)$ up to unitary equivalence. By [Ror02, Lemma 6.2.4], the unitary elements $u_m \in \prod_i M_2(A)/\bigoplus_i M_2(A)$ and $u_q \in \prod_i M_2(A_i)/\bigoplus_i M_2(A_i)$ implementing this equivalence can be lifted to $\prod_i M_2(A)$ and $\prod_i M_2(A_i)$ respectively. Consequently, a two-coloured nuclear dimension approximation for $\alpha = id_A \oplus 0$ was formed and $\dim_{\mathrm{nuc}}(\alpha) = 1$ (Definition 2.3.1). Since A is a hereditary C*-subalgebra of $M_2(A)$ then by [BGSW22, Proposition 1.6], the identity map id_A is the corestriction of α to A and

$$\dim_{\mathrm{nuc}} A = \dim_{\mathrm{nuc}} id_A = \dim_{\mathrm{nuc}}(\alpha) = 1.$$

Chapter 4

Graph C*-algebras

4.1 Definition and examples of graph C*-algebras

Graph C*-algebras are a class of C*-algebras which are defined with respect to a directed graph. The definition of a graph C*-algebra we will give in this section will follow the conventions in [Rae05]. The vertices of a directed graph $G = (V, E)$ will be denoted by V or E^0, whereas its edges will be denoted by E or E^1. If $e \in E$ is an edge then let $s(e), r(e) \in V$ be the source and range of e respectively.

Definition 4.1.1. Let $G = (V, E)$ be a directed graph. We say that G is **row-finite** if the following statement is satisfied: If $v \in V$ then the set $\{e \in E \mid r(e) = v\}$ is finite.

Definition 4.1.2. Let $G = (V, E)$ be a row-finite directed graph and A be a C*-algebra. A **Cuntz-Krieger G-family** is a set

$$\{S_e, P_v \mid e \in E,\ v \in V\} \subsetneq A$$

such that $\{S_e \mid e \in E\}$ is a set of partial isometries, $\{P_v \mid v \in V\}$ is a set of mutually orthogonal projections and the Cuntz-Krieger relations are satisfied:

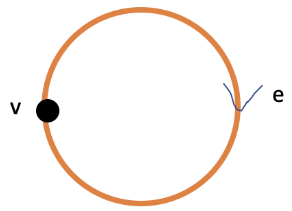
1. If $e \in E$ then $P_{s(e)} = S_e^* S_e$.
2. If $v \in V$ then $P_v = \sum_{e \in E,\ r(e)=v} S_e S_e^*$.

In the above definition, row-finiteness ensures that the second Cuntz-Krieger relation is well-defined.

Definition 4.1.3. Let $G = (V, E)$ be a row-finite directed graph. The **graph C*-algebra** associated to G, denoted by $C^*(G)$, is the universal C*-algebra generated by a Cuntz-Krieger G-family $\{S_e, P_v \mid e \in E,\ v \in V\}$. That is, $C^*(G)$ satisfies the following universal property (see [Rae05, Proposition 1.21]): If A is a C*-algebra and $\{T_e, Q_v \mid e \in E,\ v \in V\}$ is a Cuntz-Krieger G-family in A then there exists a *-homomorphism $\pi_{T,Q} : C^*(G) \to A$ such that if $v \in V$ and $e \in E$ then $\pi_{T,Q}(S_e) = T_e$ and $\pi_{T,Q}(P_v) = Q_v$.

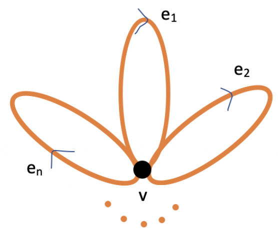
In [Rae05, Chapter 1], Raeburn explicitly constructs the graph C*-algebra for an arbitrary row-finite directed graph and demonstrates that it has the universal property outlined in the above definition. As we will see in the following examples, graph C*-algebras include some well-known examples of C*-algebras.

Example 4.1.4. Let G be the graph with a single vertex v and a loop e based at v.



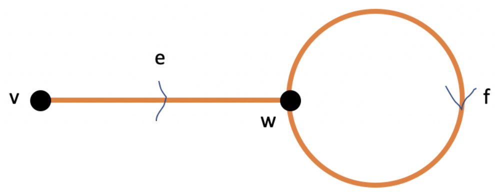
By the Cuntz-Krieger relations, $S_e^* S_e = P_v = S_e S_e^*$. Since P_v is a projection then P_v is the unit of $C^*(G)$. Hence by definition, $C^*(G)$ is the universal C*-algebra generated by the unitary S_e. This is the abelian C*-algebra $C(S^1)$.

Example 4.1.5. Let $n \in \mathbb{Z}_{>0}$ and G be the graph with a single vertex v and n loops $e_1, e_2, \ldots, e_n$ based at v.



By the Cuntz-Krieger relations, if $i \in \mathbb{Z}_{>0}$ then $S_{e_i}^* S_{e_i} = P_v$ and $P_v = \sum_{i=1}^n S_{e_i} S_{e_i}^*$. In this case, the graph C*-algebra $C^*(G)$ is the Cuntz algebra $\mathcal{O}_n$.

Example 4.1.6. Let G be the following graph:



By the Cuntz-Krieger relations, $P_v = S_e^* S_e$, $P_w = S_f^* S_f$ and $P_w = S_e S_e^* + S_f S_f^*$. One can verify by direct computation that $P_v + P_w$ is the unit for $C^*(G)$,

$$(S_e + S_f)(S_e^* + S_f^*) = P_w \qquad \text{and} \qquad (S_e + S_f)^*(S_e + S_f) = P_v + P_w. \tag{4.1}$$

In particular, the latter computation tells us that $S_e + S_f$ is an isometry in $C^*(G)$. In fact, $S_e + S_f$ is a proper isometry because

$$(P_v + P_w) - (S_e + S_f)(S_e^* + S_f^*) = P_v \neq 0.$$

Now observe that we can recover P_v, P_w, S_e and S_f from S_e+S_f. Indeed, in addition to equation (4.1)

$$(S_e + S_f)P_w = S_f S_f^* S_f = S_f \qquad \text{and} \qquad (S_e + S_f)P_v = S_e S_e^* S_e = S_e.$$

Therefore the graph C*-algebra $C^*(G)$ is generated by the proper isometry $S_e + S_f$. By Coburn's theorem ([Mur90, Theorem 3.5.18]), $C^*(G)$ is isomorphic to the Toeplitz algebra $\mathcal{T}$.

4.2 The structure of graph C*-algebras from a finite graph

In this section, we will connect graph C*-algebras to the overarching themes of nuclear dimension and extensions of C*-algebras. The first step towards this goal is to understand the connection between graph C*-algebras associated to a finite graph and extensions.

To do this, we follow [Rae05, Chapter 4] and begin by examining the ideal structure of graph C*-algebras from finite graphs. In [Rae05, Proposition 2.1], Raeburn constructed a specific action of the circle $\mathbb{T}$ on an arbitrary graph C*-algebra $C^*(G)$ from a row-finite directed graph G.

Theorem 4.2.1. *Let $G = (V, E)$ be a row-finite directed graph. Then there exists an action $\gamma : \mathbb{T} \to Aut(C^*(G))$ such that if $e \in E^1$, $v \in E^0$ and $z \in \mathbb{T}$ then*

$$\gamma_z(S_e) = zS_e \qquad \textit{and} \qquad \gamma_z(P_v) = P_v.$$

Definition 4.2.1. Let $G = (V, E)$ be a row-finite directed graph and I be an ideal of $C^*(G)$. The action from Theorem 4.2.1 is called the **gauge action** of $\mathbb{T}$ on $C^*(G)$.

The ideal I is called **gauge-invariant** if the following statement is satisfied: If $z \in \mathbb{T}$ then $\gamma_z(I) \subseteq I$.

The reason we are interested in the gauge-invariant ideals of a graph C*-algebra is because of [BPRS00, Theorem 4.1] — we can identify the gauge-invariant ideals by just examining the graph itself. Before we state [BPRS00, Theorem 4.1], we will first make some preliminary definitions. If $G = (V, E)$ is a directed graph then let E^* denote the set of paths in G. A cycle in G is a path which starts and ends at the same vertex.

Definition 4.2.2. Let $G = (V, E)$ be a row-finite directed graph and $H \subseteq V$. We say that H is **hereditary** if the following statement is satisfied: If $\mu \in E^*$ and

$r(\mu) \in H$ then the source vertex $s(\mu) \in H$.

We say that H is **saturated** if the following statement is satisfied: If $v \in E^0$ satisfies

1. $r^{-1}(v) = \{e \in E^1 \mid r(e) = v\} \neq \emptyset$,
2. $\{s(e) \mid e \in E^1 \text{ and } r(e) = v\} \subseteq H$,

then $v \in H$.

Hereditary means that if there is a path in the graph such that its range is in the hereditary subset H of vertices then the source vertex inherits membership in H from the range vertex. Saturated means that if a vertex v in the graph is not a source and the sources of all edges with range v are all in the saturated set S of vertices then v is in fact in S.

We now state [BPRS00, Theorem 4.1] below.

Theorem 4.2.2. *Let $G = (V, E)$ be a row-finite directed graph. If $H \subseteq E^0$ then define I_H to be the ideal in $C^*(G)$ generated by $\{P_v \mid v \in H\}$.*

1. *There is a bijective correspondence*

$$\begin{array}{ccc} \left\{\begin{matrix} \text{Hereditary saturated subsets} \\ \text{of } V = E^0 \end{matrix}\right\} & \longleftrightarrow & \left\{\begin{matrix} \text{Gauge-invariant ideals} \\ \text{of } C^*(G) \end{matrix}\right\} \\ \\ H & \mapsto & I_H \\ \\ \{v \in E^0 \mid P_v \in I\} & \leftarrow\!\shortmid & I. \end{array}$$

2. *Let H be a hereditary saturated subset of E^0. Let $Q = (U, F)$ be the row-finite directed graph defined by*

$$U = F^0 = E^0 \backslash H \qquad \text{and} \qquad F^1 = \{e \in E^1 \mid s(e), r(e) \notin H\}.$$

Then we have the isomorphism

$$C^*(Q) \cong C^*(G)/I_H. \tag{4.2}$$

3. *Let L be a hereditary subset of E^0. Let $R = (W, D)$ be the row-finite directed graph defined by*

$$W = D^0 = L \qquad \text{and} \qquad D^1 = \{e \in E^1 \mid s(e), r(e) \in L\}.$$

Then we have the isomorphism

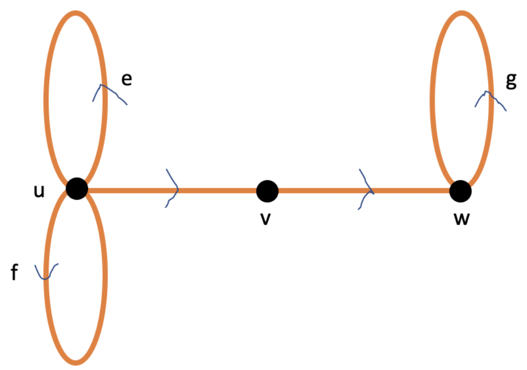
$$C^*(R) = C^*(\{S_e, P_v \mid v \in L, e \in D^1\}). \tag{4.3}$$

Moreover this subalgebra of $C^*(G)$ is a full corner in the ideal I_L.

A natural question stemming from Theorem 4.2.2 is: What conditions are required on the graph G so that the bijective correspondence in Theorem 4.2.2 recovers all the ideals of $C^*(G)$? The precise condition required for this to work was first introduced in [KPRR97] and is stated below.

Definition 4.2.3. Let $G = (V, E)$ be a row-finite directed graph. We say that G satisfies **Condition (K)** if the following statement is satisfied: If $v \in E^0$ then either there is no cycle based at v or there are at least two distinct cycles based at v.

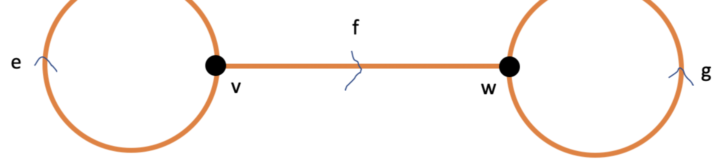
Example 4.2.4. The following example is from [Rae05, Examples 4.6]. Consider the following graph:



Condition (K) is satisfied at vertex u because there are two distinct cycles based at u, given by the length one paths e and f. Condition (K) is satisfied at v because there are no cycles based at v. However, Condition (K) is not satisfied at vertex w because there is only one cycle based at w. Note that if we add another loop at w then the resulting graph satisfies Condition (K).

As stated in [Rae05, Theorem 4.9], if we have a row-finite directed graph $G = (V, E)$ which satisfies Condition (K) then the bijective correspondence in Theorem 4.2.2 is between hereditary saturated subsets of V and ideals of $C^*(G)$. As we will see in the following example, Theorem 4.2.2 provides the important link between graph C*-algebras of finite graphs and extensions of C*-algebras.

Example 4.2.5. Let G be the following graph:



The set $\{v\}$ is a hereditary saturated subset of $V = \{v, w\}$. Let I be the ideal generated by $\{P_v\}$. By Theorem 4.2.2, I is a gauge-invariant ideal of $C^*(G)$ and the quotient C*-algebra $C^*(G)/I$ is isomorphic to the graph C*-algebra associated to the graph given by the single loop at w. By Example 4.1.4, the quotient is simply $C(S^1)$. Theorem 4.2.2 also tells us that the C*-algebra generated by S_e, which is $C(S^1)$ again, is a full corner in the ideal I. In fact, by the proof of Theorem 4.2.2 given in [BPRS00], $C(S^1) \cong P_v I P_v$. Overall, we obtain the following extension of graph C*-algebras

$$0 \longrightarrow I \longrightarrow C^*(G) \longrightarrow C(S^1) \longrightarrow 0.$$

What else can we say about the ideal I? Since $C(S^1)$ is a full corner in I then by Brown's theorem (see [Bro77, Theorem 2.8]), I and $C(S^1)$ are stably isomorphic. In fact, more is true — I itself is isomorphic to $C(S^1) \otimes K$. Thus, we have exhibited the graph C*-algebra $C^*(G)$ as an extension of $C(S^1)$ by $C(S^1) \otimes K$.

As mentioned in the introduction of [FS23], Theorem 4.2.2 implies that graph C*-algebras associated to a finite graph are built out of repeated extensions involving the following building blocks:

1. AF-algebras,
2. UCT Kirchberg algebras,
3. C*-algebras isomorphic to $C(S^1) \otimes K$.

By Theorem 2.4.5, [BBSTWW19, Corollary 9.9] and Proposition 2.4.2, all of these building blocks have finite nuclear dimension. By Theorem 2.5.1, we conclude the following:

Theorem 4.2.3. *Let $G = (V, E)$ be a directed graph with finitely many vertices. Then* $\dim_{nuc} C^*(G) < \infty$.

4.3 Nuclear dimension of graph C*-algebras and future work

In the previous section, we found that by Theorem 4.2.2 and Theorem 4.2.3, graph C*-algebras associated to finite graphs can be thought of as extensions with finite nuclear dimension. This leads to two questions. Firstly, what do we know so far about the nuclear dimension of these graph C*-algebras? Secondly, can we use the techniques developed in [BW19], [GT22], [Evi22] and [CT20] to compute precise values for their nuclear dimensions?

One of the most recent papers addressing the first question is [FS23]. In it, Faurot and Schafhauser prove [FS23, Theorem A] which is given below:

Theorem 4.3.1. *Let $G = (V, E)$ be a directed graph with countably many vertices satisfying Condition (K). Then* $\dim_{nuc} C^*(G) \leq 2$.

Theorem 4.3.1 generalises [RST15, Theorem 5.1] by Ruiz, Sims and Tomforde. When compared to Theorem 4.3.1, [RST15, Theorem 5.1] has one extra restriction on the graph G.

Let us briefly comment on the proof of Theorem 4.3.1. If G is a finite graph satisfying Condition (K) and $I \subsetneq C^*(G)$ is the ideal generated by the set

$$\{P_v \mid v \in E^0,\ v \text{ is a source vertex}\}$$

then we have the extension

$$0 \longrightarrow I \longrightarrow C^*(G) \longrightarrow C^*(G)/I \longrightarrow 0. \tag{4.4}$$

In the proof of [FS23, Theorem A], Faurot and Schafhauser showed that the ideal I is an AF-algebra and that the quotient is $\mathcal{O}_\infty$-stable. Subsequently by Theorem 2.5.1,

$$\dim_{\text{nuc}} C^*(G) \leq \dim_{\text{nuc}} I + \dim_{\text{nuc}} C^*(G)/I + 1 = 0 + 1 + 1 = 2.$$

In the case where G has no source vertices, $C^*(G)$ is $\mathcal{O}_\infty$-stable by [FS23, Theorem 3.3]. So by [BGSW22], $\dim_{\text{nuc}} C^*(G) = 1$.

One of the key points to Theorem 4.3.1 is that a graph satisfying Condition (K) is equivalent to saying that as a C*-algebra generated by repeated extensions, $C^*(G)$ is not built out of any C*-algebras isomorphic to $C(S^1) \otimes K$; it is built recursively by extensions involving only UCT Kirchberg algebras and AF-algebras.

Faurot and Schafhauser also prove [FS23, Theorem B], which shows that under some extra assumptions, the upper bound of 2 from Theorem 4.3.1 can be reduced to 1.

Definition 4.3.1. Let $G = (V, E)$ be a graph, $v \in V$ and $\mu \in E^*$ be a cycle. We say that v **connects to** μ if there exists a path $\nu \in E^*$ such that its source is $s(\nu) = v$ and its range $r(\nu)$ is on the cycle μ.

Theorem 4.3.2. *Let $G = (V, E)$ be a finite graph satisfying Condition (K). Suppose that G also satisfies one of the following conditions:*

1. *If $v \in V$ then v connects to every cycle in G,*

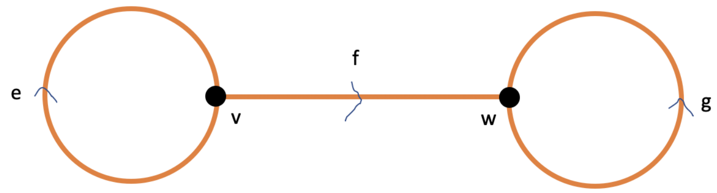
2. *If $v \in V$ then v connects to no cycles in G.*

Then $\dim_{nuc} C^*(G) \leq 1$. In particular, if G has no cycles then $\dim_{nuc} C^*(G) = 0$ and if G has a cycle then $\dim_{nuc} C^*(G) = 1$.

In the situation of Theorem 4.3.2, the extension in equation (4.4) was demonstrated to be full and the ideal I is stable. So by [Evi22, Theorem 1], $\dim_{\text{nuc}} C^*(G) \leq 1$, refining the previous upper bound of 2 from Theorem 4.3.1. To distinguish between the two cases in Theorem 4.3.2, note that by [KPR98, Corollary 2.3], if G has no cycles then $C^*(G)$ is finite-dimensional and hence has nuclear dimension 0.

After Faurot and Schafhauser's success in bounding the nuclear dimension of graph C*-algebras of a finite graph satisfying Condition (K), a natural and interesting direction of research would be to compute precise values for the nuclear dimension of graph C*-algebras from finite graphs which do not satisfy Condition (K). One of the simplest non-trivial examples of a graph C*-algebra fitting this description is given in Example 4.1.6 — the Toeplitz algebra $\mathcal{T}$. As emphasised previously, $\dim_{\mathrm{nuc}} \mathcal{T} = 1$ by [BW19]. This is an auspicious sign that the nuclear dimension of graph C*-algebras from finite graphs not satisfying Condition (K) are susceptible to the techniques developed in [BW19] and its successors.

The next simplest example of a finite graph not satisfying Condition (K) is given by Example 4.2.5.



Let $\mathcal{G}$ be the above graph. By Example 4.2.5, $C^*(\mathcal{G})$ fits into the following extension

$$0 \longrightarrow I \longrightarrow C^*(\mathcal{G}) \longrightarrow C(S^1) \longrightarrow 0.$$

where I is isomorphic to $C(S^1) \otimes K$. By Theorem 2.5.1, a crude upper bound for the nuclear dimension of $C^*(\mathcal{G})$ is

$$\dim_{\mathrm{nuc}} C^*(\mathcal{G}) \leq \dim_{\mathrm{nuc}}(C(S^1) \otimes K) + \dim_{\mathrm{nuc}} C(S^1) + 1 = 1 + 1 + 1 = 3.$$

Can we use the technique from [BW19] to reduce the above upper bound for $\dim_{\mathrm{nuc}} C^*(\mathcal{G})$ from 3 to 2? First of all, the following proposition details some useful facts about $C^*(\mathcal{G})$, all of which can be proved by direct computation.

Proposition 4.3.3. *Let $i, j, n \in \mathbb{Z}_{>0}$. Define*

$$a_{i1} = \begin{cases} P_v, & \text{if } i = 1, \\ S_f, & \text{if } i = 2, \\ S_g^{i-2} S_f, & \text{if } i > 2, \end{cases}$$

$a_{ij} = a_{i1} a_{j1}^*$, $e_n = \sum_{i=1}^n a_{ii}$ *and*

$$h_n = e_{2^n} + \sum_{i=1}^{2^n - 1} \left(1 - \frac{i}{2^n}\right) a_{ii}.$$

Then the following statements are satisfied:

1. *The family* $\{a_{ij}\}_{i,j\in\mathbb{Z}_{>0}}$ *is an infinite matrix unit in the ideal* I*,*

2. *The family* $\{e_i\}_{i\in\mathbb{Z}_{>0}}$ *is an approximate unit in* I*,*

3. *The family* $\{h_i\}_{i\in\mathbb{Z}_{>0}}$ *is an idempotent quasicentral approximate unit for* $C^*(\mathcal{G})$ *contained in* I*.*

Using the idempotent quasicentral approximate unit $\{h_n\}$ in Proposition 4.3.3, if $n \in \mathbb{Z}_{>0}$ then define the hereditary C*-subalgebras

$$A_n = \overline{h_n C^*(\mathcal{G}) h_n}, \qquad B_n = \overline{(h_{n+1} - h_n) C^*(\mathcal{G})(h_{n+1} - h_n)} \tag{4.5}$$

and

$$C_n = \overline{(1 - h_{n+1}) C^*(\mathcal{G})(1 - h_{n+1})}.$$

Let $\mu : C(S^1) \to C^*(\mathcal{G})$ be a c.c.p lifting of the identity map $id_{C(S^1)}$ which exists by the Choi-Effros theorem. Define the c.c.p maps

$$\begin{array}{rccc} \alpha: & C^*(\mathcal{G}) & \to & \prod_n A_n \\ & x & \mapsto & \big(h_n^{\frac{1}{2}} x h_n^{\frac{1}{2}}\big)_{n\in\mathbb{Z}_{>0}} \end{array}$$

$$\begin{array}{rccc} \beta: & C(S^1) & \to & \prod_n B_n \\ & f & \mapsto & \big((h_{n+1} - h_n)^{\frac{1}{2}} \mu(f) (h_{n+1} - h_n)^{\frac{1}{2}}\big)_{n\in\mathbb{Z}_{>0}} \end{array}$$

$$\begin{array}{rccc} \gamma: & C(S^1) & \to & \prod_n C_n \\ & f & \mapsto & \big((1 - h_{n+1})^{\frac{1}{2}} \mu(f) (1 - h_{n+1})^{\frac{1}{2}}\big)_{n\in\mathbb{Z}_{>0}}. \end{array}$$

Recall that for the Toeplitz algebra, Brake and Winter were able to use Lemma 3.1.2 to find a c.c.p order zero map which, after projecting to the ultraproduct $\prod_n B_n / \bigoplus_n B_n$, is equal to the map in equation (3.4). The fact that the idempotent quasicentral approximate unit for $\mathcal{T}$ in K consists of finite-rank operators is critical to this step because it ensures that the hereditary C*-subalgebras B_n are finite-dimensional so that Lemma 3.1.2 can be applied.

In this situation, the ideal in our extension is $C(S^1)\otimes K$ rather than K. So if $n \in \mathbb{Z}_{>0}$ then in general, B_n cannot be made a finite-dimensional C*-algebra. Nonetheless, we make the following observation.

Theorem 4.3.4. *If* $n \in \mathbb{Z}_{>0}$ *then let* B_n *be defined as in equation* (4.5) *and define*

$$F_n = span_{\mathbb{C}}\{a_{ij}\}_{i,j=2^n+1}^{2^{n+2}-1} \cong M_{3(2^n)-1}(\mathbb{C}) \cong \mathbb{C} id_{S^1} \otimes M_{3(2^n)-1}(\mathbb{C}).$$

If $x \in C^*(\mathcal{G})$ *and* $q_B : \prod_n B_n \to \prod_n B_n / \bigoplus_n B_n$ *is the quotient map then*

$$(q_B \circ \beta)(x) = q_B\big(\{(h_{n+1} - h_n)^{\frac{1}{2}} x (h_{n+1} - h_n)^{\frac{1}{2}}\}_{n\in\mathbb{Z}_{>0}}\big) \in \prod_n F_n / \bigoplus_n B_n.$$

Proof. Assume that if $n \in \mathbb{Z}_{>0}$ then B_n and F_n are the C*-algebras defined above. Assume that $x \in C^*(\mathcal{G})$ and q_B is the quotient map in the statement of the theorem. Note that by definition,

$$(h_{n+1} - h_n)^{\frac{1}{2}} x (h_{n+1} - h_n)^{\frac{1}{2}} \in B_n.$$

We also observe that F_n is a C*-subalgebra of B_n because if $i, j \in \{2^n + 1, \dots, 2^{n+2} - 1\}$ then

$$a_{ij} = (h_{n+1} - h_n)^{\frac{1}{2}} (\frac{1}{r_{ij}} a_{ij})(h_{n+1} - h_n)^{\frac{1}{2}} \in B_n$$

where $r_{ij} \in \mathbb{R}_{>0}$ is a constant depending on i and j. Now if $\mu = \mu_1 \mu_2 \dots \mu_p$ is a path in $C^*(\mathcal{G})$ then define $S_\mu = S_{\mu_p} \dots S_{\mu_2} S_{\mu_1}$. By [Rae05, Theorem 4.9],

$$C^*(\mathcal{G}) = \overline{span\{S_\mu S_\nu^* \mid \mu, \nu \in E^*\}}.$$

So it suffices to show that if $\mu, \nu \in E^*$ then

$$q_B\big(\{(h_{n+1} - h_n)^{\frac{1}{2}} S_\mu S_\nu^* (h_{n+1} - h_n)^{\frac{1}{2}}\}_n\big) \in \prod\nolimits_n F_n / \bigoplus\nolimits_n B_n.$$

There are two main cases to consider:

Case 1: Either μ or ν is a path with source v.

Without loss of generality, assume that μ is a path whose source vertex is v. Then S_μ as a product contains either S_e, S_f or P_v. These are elements of the ideal I. Hence $S_\mu S_\nu^* \in I$ and since $\{h_n\}_{n \in \mathbb{Z}_{>0}}$ is an approximate unit for I then

$$q_B\Big(\big\{(h_{n+1} - h_n)^{\frac{1}{2}} S_\mu S_\nu^* (h_{n+1} - h_n)^{\frac{1}{2}}\big\}_n\Big) = q_B(\{0\}_n) \in \prod\nolimits_n F_n / \bigoplus\nolimits_n B_n.$$

To be clear, $\{0\}_p$ is the zero sequence in $\prod_p F_p$.

Case 2: Both paths μ and ν have source w.

If $S_\mu = P_w$ then

$$(h_{n+1} - h_n)^{\frac{1}{2}} P_w = (h_{n+1} - h_n)^{\frac{1}{2}}.$$

and

$$q_B\Big(\big\{(h_{n+1} - h_n)^{\frac{1}{2}} P_w (h_{n+1} - h_n)^{\frac{1}{2}}\big\}_n\Big) = q_B\big(\big\{h_{n+1} - h_n\big\}_n\big) \in \prod\nolimits_n F_n / \bigoplus\nolimits_n B_n.$$

Next assume that $k \in \mathbb{Z}_{>0}$ and $S_\mu = S_g^k$. By the Cuntz-Krieger relations, if $m \in \mathbb{Z}_{>0}$ then

$$a_{mm} S_g^k = \begin{cases} 0, & \text{if } m \leq k, \\ a_{m,m-k}, & \text{if } m > k. \end{cases}$$

Hence,

$$
\begin{aligned}
&(h_{n+1}-h_n)^{\frac{1}{2}} S_g^k (h_{n+1}-h_n)^{\frac{1}{2}} \\
&= \Big(\sum_{i=1}^{2^n} (\frac{i}{2^n})^{\frac{1}{2}} a_{2^n+i,2^n+i-k} + \sum_{i=1}^{2^{n+1}-1} \Big(1-\frac{i}{2^{n+1}}\Big)^{\frac{1}{2}} a_{2^{n+1}+i,2^{n+1}+i-k} \Big) \\
&\cdot \Big(\sum_{i=1}^{2^n} (\frac{i}{2^n})^{\frac{1}{2}} a_{2^n+i,2^n+i} + \sum_{i=1}^{2^{n+1}-1} \Big(1-\frac{i}{2^{n+1}}\Big)^{\frac{1}{2}} a_{2^{n+1}+i,2^{n+1}+i} \Big).
\end{aligned}
$$

Note that in the above expression, if $j, l \in \{2^n+1, \ldots, 2^{n+2}-1\}$ and $l-k \leq 0$ then $a_{j,l-k} = 0$. We now have two cases to consider. If

$$\{2^n+1-k, \ldots, 2^{n+2}-1-k\} \cap \{2^n+1, \ldots, 2^{n+2}-1\} = \emptyset$$

then

$$(h_{n+1}-h_n)^{\frac{1}{2}} S_g^k (h_{n+1}-h_n)^{\frac{1}{2}} = 0 \in F_n.$$

If on the other hand

$$\{2^n+1-k, \ldots, 2^{n+2}-1-k\} \cap \{2^n+1, \ldots, 2^{n+2}-1\} \neq \emptyset$$

then $(h_{n+1}-h_n)^{\frac{1}{2}} S_g^k (h_{n+1}-h_n)^{\frac{1}{2}} \in F_n$ because it is a $\mathbb{C}$-linear combination of elements in $\{a_{ij}\}_{i=2^n+1}^{2^{n+2}-1}$. A similar argument demonstrates that if $k, \ell \in \mathbb{Z}_{>0}$ then

$$(h_{n+1}-h_n)^{\frac{1}{2}} S_g^k (S_g^*)^\ell (h_{n+1}-h_n)^{\frac{1}{2}} \in F_n.$$

We conclude that if $k, \ell \in \mathbb{Z}_{>0}$ then

$$q_B\Big(\big\{ (h_{n+1}-h_n)^{\frac{1}{2}} S_g^k (h_{n+1}-h_n)^{\frac{1}{2}} \big\}_n \Big) \in \prod\nolimits_n F_n / \bigoplus\nolimits_n B_n,$$

$$q_B\Big(\big\{ (h_{n+1}-h_n)^{\frac{1}{2}} (S_g^*)^\ell (h_{n+1}-h_n)^{\frac{1}{2}} \big\}_n \Big) \in \prod\nolimits_n F_n / \bigoplus\nolimits_n B_n$$

and

$$q_B\Big(\big\{ (h_{n+1}-h_n)^{\frac{1}{2}} S_g^k (S_g^*)^\ell (h_{n+1}-h_n)^{\frac{1}{2}} \big\}_n \Big) \in \prod\nolimits_n F_n / \bigoplus\nolimits_n B_n.$$

This completes the proof. □

By Theorem 4.3.4, the composite $q_B \circ \beta$ is a c.c.p order zero map from $C(S^1)$ to the ultraproduct $\prod_n F_n / \bigoplus_n B_n$ where $\{F_n\}_n$ is a family of finite-dimensional C*-algebras. Thus by Lemma 3.1.2, there exists a c.c.p order zero map $\tilde{\beta} : C(S^1) \to \prod_n F_n$ satisfying $q_B \circ \tilde{\beta} = q_B \circ \beta$.

Following Brake and Winter, the next step is to extend $\tilde{\beta}$ to $B(S^1)$ up to a norm approximation in exactly the same way as Lemma 3.1.3. Importantly, the proof of Lemma 3.1.3 in [BW19] does not rely on any specific features of $\mathcal{T}$. Subsequently, the rest of Brake and Winter's argument goes through smoothly and we find that

$$\dim_{\text{nuc}} C^*(\mathcal{G}) \leq 2. \tag{4.6}$$

Since $C^*(\mathcal{G})$ is not an AF-algebra then by Theorem 2.4.5, $\dim_{\text{nuc}} C^*(\mathcal{G}) \in \{1, 2\}$. Naturally we are led to ask whether the nuclear dimension of $C^*(\mathcal{G})$ is actually 1. This is certainly plausible because as mentioned in [FS23], it is currently unknown whether there exists a finite graph E satisfying $\dim_{\text{nuc}} C^*(E) > 1$.

In order to show that $\dim_{\text{nuc}} C^*(\mathcal{G}) = 1$, we need to construct a two-coloured nuclear dimension approximation for the identity map $id_{C^*(\mathcal{G})}$. The ideal and quotient, $C(S^1) \otimes K$ and $C(S^1)$ respectively, both have nuclear dimension 1 and contribute two colours each. This means that **we must reuse two colours** in order to construct our desired approximation. This is different from [BW19], [GT22], [CT20] and [Evi22], where only one colour from the quotient is reused.

Here is a proposed strategy for proving that $\dim_{\text{nuc}} C^*(\mathcal{G}) = 1$. If $n \in \mathbb{Z}_{>0}$ then define the hereditary C*-subalgebras

$$D_n = \overline{(h_{n+2} - h_{n+1}) C^*(\mathcal{G}) (h_{n+2} - h_{n+1})} \qquad \text{and} \qquad E_n = \overline{(1 - h_{n+2}) C^*(\mathcal{G}) (1 - h_{n+2})}.$$

Define the c.c.p maps

$$\begin{array}{rccc} \rho : & C(S^1) & \to & \prod_n D_n \\ & f & \mapsto & \left\{ (h_{n+2} - h_{n+1})^{\frac{1}{2}} \mu(f) (h_{n+2} - h_{n+1})^{\frac{1}{2}} \right\}_{n \in \mathbb{Z}_{>0}} \end{array}$$

and

$$\begin{array}{rccc} \sigma : & C(S^1) & \to & \prod_n E_n \\ & f & \mapsto & \left\{ (1 - h_{n+2})^{\frac{1}{2}} \mu(f) (1 - h_{n+2})^{\frac{1}{2}} \right\}_{n \in \mathbb{Z}_{>0}} \end{array}$$

If $\iota : C^*(\mathcal{G}) \hookrightarrow \prod_n C^*(\mathcal{G}) / \bigoplus_n C^*(\mathcal{G})$ is the constant sequence embedding then by applying [BO08, Proposition 1.2.2] twice,

$$\begin{aligned} \iota &= (\iota \circ \alpha) + (\iota \circ \beta \circ \pi) + (\iota \circ \rho \circ \pi) + (\iota \circ \sigma \circ \pi) \\ &= \text{Ideal term} + \text{First middle term} + \text{Second middle term} + \text{Quotient term}. \end{aligned}$$

By repeating the argument used to show $\dim_{\text{nuc}} C^*(\mathcal{G}) \leq 2$, we may use one of the colours in the quotient term to colour the second middle term.

$$\iota = \text{Ideal term} + \text{First middle term} + \text{Second middle term} + \text{Quotient term}.$$

The key step which needs to be implemented is to reuse one of the colours *from the ideal term* to colour in the first middle term.

$$\iota = \text{Ideal term} + \text{First middle term} + \text{Second middle term} + \text{Quotient term}. \tag{4.7}$$

Since $\{h_n\}_n$ from Proposition 4.3.3 is an idempotent quasicentral approximate unit then the ideal term and first middle term are both orthogonal to the quotient term. Also, the quotient term and the second middle term are both orthogonal to the ideal term. Hence, we may use blue to colour in the brown term and magenta to colour in the red term. This yields a two-coloured nuclear dimension approximation for $C^*(\mathcal{G})$.

$$\iota = \text{Ideal term} + \text{First middle term} + \text{Second middle term} + \text{Quotient term}.$$

So if we are able to reuse one of the colours from the ideal and obtain equation (4.7) then it follows that $\dim_{\text{nuc}} C^*(\mathcal{G}) = 1$. In currently unpublished work, Evington, Ng, Sims and White were able to reuse two colours, one from the quotient and one from the ideal, to prove [ENSW, Theorem A].

Theorem 4.3.5. *Let J be a stable Kirchberg algebra and X be a compact metric space. Let E be an essential extension of $C(X)$ by J. Then*

$$\dim_{nuc} E = \max\{1, \dim X\}.$$

Recall that $\dim X$ is the covering dimension of X (see Definition 2.1.2).

The reuse of two colours was necessary in the case where $\dim X = 1$ because the Kirchberg ideal J has nuclear dimension 1 ([BBSTWW19, Corollary 9.9]). The reuse of a second colour from the ideal was achieved in [ENSW] by a very similar argument to [CT20] and [Evi22], spearheaded by Gabe's classification theorem ([Gab24, Theorem B]). In the case of the graph C*-algebra $C^*(\mathcal{G})$ the ideal is $C(S^1) \otimes K$ which is not Kirchberg (since it is not simple c.f. [Tak79, Corollary 4.21]). Hence, the reuse of a colour from the ideal is likely to be facilitated by a different classification result to the ones seen so far from [BW19] to [ENSW].

Bibliography

[BBSTWW19] J. Bosa, N. P. Brown, Y. Sato, A. Tikuisis, S. White, W. Winter. *Covering dimension of C*-algebras and 2-coloured classification*, Mem. Amer. Math. Soc. **257**, 2019, ISBN: 978-1-4704-3470-0, MR3908669.

[BGSW22] J. Bosa, J. Gabe, A. Sims, S. White. *The nuclear dimension of $\mathcal{O}_\infty$-stable C*-algebras*, Adv. Math. **401**, 2022, 51 pages, MR4392219.

[BO08] NP. Brown, N. Ozawa. *C*-algebras and finite dimensional approximations*, American Mathematical Society, Providence RI, 2008, ISBN: 978-0-8218-4381-9, MR2391387.

[Bro77] L. G. Brown. *Stable isomorphism of hereditary subalgebras of C*-algebras*, Pacific J. Math. **71**, 1977, Pages 335-348, MR0454645.

[BP91] L. G. Brown, G. K. Pedersen. *C*-algebras of real rank zero*, J. Funct. Anal. **99**, 1991, Pages 131-149, MR1120918.

[BPRS00] T. Bates, D. Pask, I. Raeburn, W. Szymański. *The C*-algebras of row-finite graphs*, New York J. Math. **6**, 2000, Pages 307-324, MR1777234.

[BW19] L. Brake, W. Winter. *The Toeplitz algebra has nuclear dimension one*, Bull. Lond. Math. Soc. **51**, 2019, Pages 554-562, MR3964508.

[Cas21] J. Castillejos. *C*-algebras and their nuclear dimension*, Contemp. Math. **775**, American Mathematical Society, Providence, RI, 2021, Pages 41-63, ISBN: 978-1-4704-6536-0, MR4344306.

[CETWW21] J. Castillejos, S. Evington, A. Tikuisis, S. White, W. Winter. *Nuclear dimension of simple C*-algebras*, Invent. Math. **224**, 2021, Pages 245-290, MR4228503.

[CT20] P. Easo, E. Garijo, S. Kaubrys, D. Nkansah, M. Vrabec, D. Watt, C. Wilson, C. Bönicke, S. Evington, M. Forough, S. Girón Pacheco, N. Seaton, S. White, M. F. Whittaker, J. Zacharias. *The Cuntz-Toeplitz algebras have nuclear dimension one*, J. Funct. Anal. **279**, 2020, 14 pages, MR4116805.

[EE93] G. A. Elliott, D. E. Evans. *The structure of the irrational rotation C*-algebra*, Ann. of Math. **138**, 1993, Pages 477-501, MR1247990.

[ENSW] S. Evington, A. Ng, A. Sims, S. White. *Nuclear dimension of extensions of commutative C*-algebras by Kirchberg algebras*, Manuscript in preparation.

[Evi22] S. Evington. *Nuclear dimension of extensions of $\mathcal{O}_\infty$-stable algebras*, J. Operator Theory. **88**, 2022, Pages 171-187, MR4438603.

[FS23] G. Faurot, C. Schafhauser. *Nuclear dimension of graph C*-algebras with Condition (K)*, 2023, https://arxiv.org/abs/2310.15073.

[Gab24] J. Gabe. *Classification of $\mathcal{O}_\infty$-stable C*-algebras*, Mem. Amer. Math. Soc. **293**, 2024, 115 Pages, MR4684339.

[GT22] R. Gardner, A. Tikuisis. *The nuclear dimension of extensions of commutative C*-algebras by the compact operators*, 2022, https://arxiv.org/abs/2202.04695.

[HR00] N. Higson, J. Roe. *Analytic K-Homology*, Oxford Math. Monogr. Oxford University Press, Oxford, 2000, ISBN: 0-19-851176-0, MR1817560.

[KN06] D. Kucerovsky, P. W. Ng. *The corona factorization property and approximate unitary equivalence*, Houston J. Math. **32**, 2006, Pages 531-550, MR2219330.

[KP00] E. Kirchberg, N. C. Phillips. *Embedding of exact C*-algebras in the Cuntz algebra $\mathcal{O}_2$*, J. Reine Angew. Math. **525**, 2000, Pages 17-53, MR1780426.

[KPR98] A. Kumjian, D. Pask, I. Raeburn. *Cuntz-Krieger algebras of directed graphs*, Pacific. J. Math. **184**, 1998, Pages 161-174, MR1626528.

[KPRR97] A. Kumjian, D. Pask, I. Raeburn, J. Renault. *Graphs, groupoids, and Cuntz-Krieger algebras*, J. Funct. Anal. **144**, 1997, Pages 505-541, MR1432596.

[KW04] E. Kirchberg, W. Winter. *Covering dimension and quasidiagonality*, Internat. J. Math. **15**, 2004, Pages 63-85, MR2039212.

[Lin97] H. Lin. *Almost commuting selfadjoint matrices and applications*, Fields Inst. Commun. **13**, 1997, American Mathematical Society, Providence, RI, ISBN: 0-8218-0522-3, MR1424963.

[Lin17] H. Lin. *Homomorphisms from AH-algebras*, J. Topol. Anal. **9**, 2017, Pages 67-125, MR3594607.

[Lor97] T. A. Loring. *Lifting solutions to perturbing problems in C*-algebras*, Fields Institute Monographs, **8**, American Mathematical Society, Providence, RI, 1997, ISBN: 0-8218-0602-5, MR1420863.

[Mur90] G. J. Murphy. *C*-Algebras and Operator Theory*, Boston, Academic Press Inc, 1990, ISBN: 0-12-511360-9, MR1074574.

[Ost65] P. A. Ostrand. *Dimension of metric spaces and Hilbert's problem 13*, Bull. Amer. Math. Soc. **71**, 1965, Pages 619-622, MR0177391.

[Pau02] V. Paulsen. *Completely bounded maps and operator algebras*, Cambridge University Press, Cambridge, 2002, ISBN: 0-521-81669-6, MR1976867.

[Pea75] A. R. Pears. *Dimension theory of general spaces*, Cambridge University Press, Cambridge, 1975, ISBN: 0-521-20515-8, MR0394604.

[Phi00] N. C. Phillips. *A classification theorem for nuclear purely infinite simple C*-algebras*, Doc. Math. **5**, 2000, Pages 49-114, MR1745197.

[Rae05] I. Raeburn. *Graph algebras*, American Mathematical Society, Providence RI, 2005, ISBN: 0-8218-3660-9, MR2135030.

[Rie93] M. A. Rieffel. *C*-algebras associated with irrational rotations*, Pacific. J. Math. **93**, 1981, Pages 415-429, MR0623572.

[RLL00] M. Rørdam, F. Larson, N. Laustsen. *An introduction to K-theory for C*-algebras*, London Math. Soc. Stud. Texts, Cambridge University Press, Cambridge, 2000, ISBN: 0-521-78334-8, MR1783408.

[Ror02] M. Rørdam. *Classification of nuclear, simple C*-algebras*, Encyclopaedia Math. Sci. 126, Oper Alg Non-commut Geom, 7, Springer Verlag, Berlin, 2002, ISBN: 3-540-42305-X, MR1878882.

[RST15] E. Ruiz, A. Sims, M. Tomforde. *The nuclear dimension of graph C*-algebras*, Adv. Math. **272**, 2015, Page 96-123, MR3303230.

[Tak79] M. Takesaki. *Theory of operator algebras. I*, Springer-Verlag, New York-Heidelberg, 1979, ISBN: 0-387-90391-7, MR0548728.

[WZ09] W. Winter, J. Zacharias. *Completely positive maps of order zero*, Münster J. Math. **2**, 2009, Pages 311-324, MR2545617.

[WZ10] W. Winter, J. Zacharias. *The nuclear dimension of C*-algebras*, Adv. Math. **224**, 2010, Pages 461-498, MR2609012.