

# On a particular integral

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We record an interesting argument which proves the following result.

**Proposition 0.0.1.**

$$\int_0^1 \frac{\ln(x+1)}{x^2+1} dx = \frac{\pi}{8} \ln 2.$$

*Proof.* We begin with an integration by parts. This decomposes the LHS of the equation as

$$\begin{aligned} \int_0^1 \frac{\ln(x+1)}{x^2+1} dx &= [\ln(x+1) \arctan(x)]_0^1 - \int_0^1 \frac{\arctan(x)}{x+1} dx \\ &= \frac{\pi}{4} \ln(2) - \int_0^1 \frac{\arctan(x)}{x+1} dx \\ &= \frac{\pi}{4} \ln(2) - \int_1^2 \frac{\arctan(u-1)}{u} du. \end{aligned}$$

On the final line, we made the substitution  $x = u - 1$ . The problem reduces to finding the integral

$$\int_1^2 \frac{\arctan(u-1)}{u} du.$$

The leap here is to turn to the complex numbers for inspiration. Indeed, if  $z_1, z_2 \in \mathbb{C}$ , then  $\text{Arg}(z_1) + \text{Arg}(z_2) = \text{Arg}(z_1 z_2)$ , where  $\text{Arg}$  denote the principal argument. Set  $z_1 = x_1 + y_1 i$  and  $z_2 = x_2 + y_2 i$ . If  $x_1, x_2, y_1, y_2 \in \mathbb{Z}_{>0}$ , then

$$\arctan\left(\frac{y_1}{x_1}\right) + \arctan\left(\frac{y_2}{x_2}\right) = \arctan\left(\frac{x_1 y_2 + y_1 x_2}{x_1 x_2 - y_1 y_2}\right).$$

Now let  $y_1 = u - 1$ ,  $y_2 = x_2 = 1$  and  $x_1 = u$  for some  $u \in \mathbb{Z}_{>0}$ . Then,

$$\arctan\left(\frac{u-1}{u}\right) + \frac{\pi}{4} = \arctan(2u-1).$$

Replacing  $u$  with  $u/2$ , we obtain the equation

$$\arctan\left(\frac{u-2}{u}\right) + \frac{\pi}{4} = \arctan(u-1)$$

which is valid for  $u \in \mathbb{Z}_{>0}$ . Conveniently, this contains our region of integration.

In our integral, we now make the substitution  $u = 2/y$ . Then,  $du = -2/y^2 dy$  and our integral now becomes

$$\begin{aligned}
\int_2^1 \frac{\arctan(\frac{2}{y} - 1)}{\frac{2}{y}} \times -\frac{2}{y^2} dy &= \int_1^2 \arctan(\frac{2}{y} - 1) \times \frac{y}{2} \times \frac{2}{y^2} dy \\
&= \int_1^2 \frac{\arctan(\frac{2-y}{y})}{y} dy \\
&= - \int_1^2 \frac{\arctan(\frac{y-2}{y})}{y} dy
\end{aligned}$$

Hence, we have the following equation:

$$\int_1^2 \frac{\arctan(u-1)}{u} du = - \int_1^2 \frac{\arctan(\frac{u-2}{u})}{u} du.$$

Now, we set  $I = \int_1^2 \frac{\arctan(u-1)}{u} du$ . Then,

$$\begin{aligned}
2I &= \int_1^2 \frac{\arctan(u-1)}{u} du + \int_1^2 \frac{\arctan(u-1)}{u} du \\
&= \int_1^2 \frac{\arctan(u-1)}{u} du - \int_1^2 \frac{\arctan(\frac{u-2}{u})}{u} du \\
&= \int_1^2 \frac{\arctan(u-1) - \arctan(\frac{u-2}{u})}{u} du \\
&= \int_1^2 \frac{\pi}{4u} du \quad (\text{using the arctan result}) \\
&= \frac{\pi}{4} [\ln(u)]_1^2 \\
&= \frac{\pi}{4} \ln(2).
\end{aligned}$$

Hence,  $I = \frac{\pi}{8} \ln(2)$ . Our original integral now evaluates to

$$\int_0^1 \frac{\ln(x+1)}{x^2+1} dx = \frac{\pi}{4} \ln(2) - I = \frac{\pi}{8} \ln(2)$$

as required. □