Yoneda Lemma

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0.1 Statement and proof of the Yoneda lemma

We begin by setting the scene for the Yoneda lemma.

Definition 0.1.1. Let $\mathscr{C}$ be a category. We say that $\mathscr{C}$ is **locally small** if for any pair of objects $X, Y \in \mathscr{C}$, the class of morphisms $Hom_{\mathscr{C}}(X, Y)$ is a set.

Definition 0.1.2. Let $\mathscr{C}$ and $\mathscr{D}$ be categories. The **functor category** $\mathcal{F}(\mathscr{C}, \mathscr{D})$ is the category whose objects are functors $F : \mathscr{C} \to \mathscr{D}$ and morphism are natural transformations between functors.

The isomorphisms in the functor category $\mathcal{F}(\mathscr{C}, \mathscr{D})$ are the natural isomorphisms between functors from $\mathscr{C}$ to $\mathscr{D}$. For two functors $F : \mathscr{C} \to \mathscr{D}$ and $F' : \mathscr{C} \to \mathscr{D}$, we write $Nat(F, F')$ to denote the set of natural transformations from F to F'.

If $\mathscr{C}$ is a locally small category and $X \in \mathscr{C}$ is an object then we have the functor

$$\begin{array}{rccc} Hom_{\mathscr{C}}(-, X) : & \mathscr{C}^{op} & \to & \mathbf{Set} \\ & Y & \mapsto & Hom_{\mathscr{C}}(Y, X) \\ & f : Y \to Y' & \mapsto & Hom_{\mathscr{C}}(f, X) \end{array}$$

In turn, the function $Hom_{\mathscr{C}}(f, X)$ of sets is defined by

$$\begin{array}{rccc} Hom_{\mathscr{C}}(f, X) : & Hom_{\mathscr{C}}(Y', X) & \to & Hom_{\mathscr{C}}(Y, X) \\ & g & \mapsto & g \circ f. \end{array}$$

This functor is important in the definition of the Yoneda functor.

Definition 0.1.3. Let $\mathscr{C}$ be a locally small category. The **Yoneda functor** is defined by

$$\begin{array}{rccc} Y : & \mathscr{C} & \to & \mathcal{F}(\mathscr{C}^{op}, \mathbf{Set}) \\ & X & \mapsto & Y(X) = Hom_{\mathscr{C}}(-, X) \\ & f : X \to X' & \mapsto & Y(f) \end{array} \tag{1}$$

If $f : X \to X'$ is a morphism in $\mathscr{C}$ then $Y(f)$ is a natural transformation defined by the family of maps

$$\{Y(f)_A : Hom_{\mathscr{C}}(A, X) \to Hom_{\mathscr{C}}(A, X') \mid A \in \mathscr{C}\}$$

where we have for each object $A \in \mathscr{C}$ the morphism of sets

$$\begin{array}{rccc} Y(f)_A : & Hom_{\mathscr{C}}(A, X) & \to & Hom_{\mathscr{C}}(A, X') \\ & g & \mapsto & f \circ g. \end{array}$$

In order for the proof of the Yoneda lemma to go through smoothly, we will first prove the following lemma:

Lemma 0.1.1. *Let $\mathscr{C}$ be a locally small category and $F : \mathscr{C}^{op} \to$ **Set** be a functor. Let $C \in \mathscr{C}$ be an object. Define the map*

$$\begin{array}{rccc} \Phi_{C,F} : & Nat(Y(C), F) & \to & F(C) \\ & \alpha & \mapsto & \alpha_C(id_C) \end{array}$$

Explicitly, Y is the functor from equation (1), α_C is a morphism of sets from $Y(C)(C) = Hom_{\mathscr{C}}(C, C)$ to $F(C)$ and id_C is the identity map on the object C. Then, $\Phi_{C,F}$ is a bijection, which satisfies the following two properties:

1. If $f : C \to C'$ is a morphism in $\mathscr{C}$ then the following square in ***Set*** *commutes:*

$$\begin{tikzcd} Nat(Y(C), F) \arrow[r, "\Phi_{C,F}"] & F(C) \\ Nat(Y(C'), F) \arrow[r, "\Phi_{C',F}"] \arrow[u, "(-)\circ Y(f)"] & F(C') \arrow[u, "F(f)"'] \end{tikzcd}$$

2. If $\beta : F \to F'$ is a natural transformation then the following diagram in ***Set*** *commutes:*

$$\begin{tikzcd} Nat(Y(C), F) \arrow[r, "\Phi_{C,F}"] \arrow[d, "\beta\circ(-)"'] & F(C) \arrow[d, "\beta_C"] \\ Nat(Y(C), F') \arrow[r, "\Phi_{C,F'}"] & F'(C) \end{tikzcd}$$

Proof. Assume that $\mathscr{C}$ is a locally small category and $C \in \mathscr{C}$ is an object. Assume that $F : \mathscr{C}^{op} \to$ **Set** is a functor.

To show: (a) The map $\Phi_{C,F}$ is surjective.

(b) The map $\Phi_{C,F}$ is injective.

(c) The first property in the statement of the lemma is satisfied.

(d) The second property in the statement of the lemma is satisfied.

(a) Assume that $X \in F(C)$ and D is an object in $\mathscr{C}^{op}$. Define the map $N(X)_D$ by

$$\begin{array}{rrcl} N(X)_D: & Y(C)(D) = Hom_{\mathscr{C}}(D,C) & \rightarrow & F(D) \\ & g & \mapsto & F(g)(X) \end{array}$$

Recall that F is a contravariant functor by assumption so that $F(g)$ is a morphism in **Set** from $F(C)$ to $F(D)$.

To show: (aa) $N(X) \in Nat(Y(C), F)$.

(aa) We will show that if $h : D \to D'$ is a morphism in $\mathscr{C}^{op}$ then the following diagram in **Set** commutes:

$$\begin{tikzcd} Y(C)(D') \arrow[r, "Y(C)(h)"] \arrow[d, "N(X)_{D'}"'] & Y(C)(D) \arrow[d, "N(X)_D"] \\ F(D') \arrow[r, "F(h)"] & F(D) \end{tikzcd}$$

Assume that $\xi \in Y(C)(D') = Hom_{\mathscr{C}}(D', C)$. We compute directly that

$$\begin{aligned} (N(X)_D \circ Y(C)(h))(\xi) &= (N(X)_D \circ Hom_{\mathscr{C}}(h, C))(\xi) \\ &= N(X)_D(\xi \circ h) \\ &= F(\xi \circ h)(X) \\ &= (F(h) \circ F(\xi))(X) \\ &= (F(h) \circ N(X)_{D'})(\xi). \end{aligned}$$

Hence, the above diagram in **Set** commutes and $N(X) \in Nat(Y(C), F)$.

(a) We claim that $\Phi_{C,F}(N(X)) = X$. Using the definitions of $\Phi_{C,F}$ and $N(X)$, we find that

$$\begin{aligned} \Phi_{C,F}(N(X)) &= N(X)_C(id_C) \\ &= F(id_C)(X) = id_{F(C)}(X) = X. \end{aligned}$$

Therefore, the map $\Phi_{C,F}$ is surjective.

(b) Assume that $\alpha, \beta \in Nat(Y(C), F)$ such that $\Phi_{C,F}(\alpha) = \Phi_{C,F}(\beta)$. Assume that $f \in Hom_{\mathscr{C}}(D, C)$ for some object $D \in \mathscr{C}$. By naturality of α, the following diagram in **Set** commutes:

$$\begin{tikzcd}
Y(C)(C) \arrow[r, "Y(C)(f)"] \arrow[d, "\alpha_C"'] & Y(C)(D) \arrow[d, "\alpha_D"] \\
F(C) \arrow[r, "F(f)"] & F(D)
\end{tikzcd}$$

We then have

$$\begin{aligned}
(F(f) \circ \Phi_{C,F})(\alpha) &= F(f)(\alpha_C(id_C)) \\
&= (\alpha_D \circ Y(C)(f))(id_C) \\
&= \alpha_D(Hom_{\mathscr{C}}(f, C)(id_C)) \\
&= \alpha_D(id_C \circ f) = \alpha_D(f).
\end{aligned}$$

Since $\Phi_{C,F}(\alpha) = \Phi_{C,F}(\beta)$ by assumption, $\alpha_C(id_C) = \beta_C(id_C)$. But, β is also a natural transformation between the functors $Y(C)$ and F. So, the following diagram in **Set** commutes:

$$\begin{tikzcd}
Y(C)(C) \arrow[r, "Y(C)(f)"] \arrow[d, "\beta_C"'] & Y(C)(D) \arrow[d, "\beta_D"] \\
F(C) \arrow[r, "F(f)"] & F(D)
\end{tikzcd}$$

If $f \in Hom_{\mathscr{C}}(D, C) = Y(C)(D)$ then

$$\begin{aligned}
\alpha_D(f) &= \alpha_D(id_C \circ f) \\
&= \alpha_D(Hom_{\mathscr{C}}(f, C)(id_C)) \\
&= (\alpha_D \circ Y(C)(f))(id_C) \\
&= F(f)(\alpha_C(id_C)) \\
&= F(f)(\beta_C(id_C)) \quad (\text{since } \alpha_C(id_C) = \beta_C(id_C)) \\
&= (\beta_D \circ Y(C)(f))(id_C) \\
&= \beta_D(Hom_{\mathscr{C}}(f, C)(id_C)) = \beta_D(f).
\end{aligned}$$

Therefore, $\alpha_D = \beta_D$. Since the object $D \in \mathscr{C}$ was arbitrary, we deduce that $\alpha = \beta$ as natural transformations from $Y(C)$ to F. Therefore, $\Phi_{C,F}$ is injective.

Combining parts (a) and (b), we deduce that $\Phi_{C,F}$ is indeed a bijective map. Its inverse is given explicitly by

$$\begin{array}{rccc} \Phi_{C,F}^{-1}: & F(C) & \to & Nat(Y(C),F) \\ & X & \mapsto & N(X) \end{array}$$

where $N(X)$ is the natural transformation in parts (a) and (aa). Recall that it is defined by

$$\begin{array}{rccc} N(X)_D: & Y(C)(D) = Hom_{\mathscr{C}}(D,C) & \to & F(D) \\ & g & \mapsto & F(g)(X) \end{array}$$

(c) Now assume that $f : C \to C'$ is a morphism in $\mathscr{C}$. We want to show that the following square in **Set** commutes:

$$\begin{tikzcd} Nat(Y(C),F) \arrow[r, "\Phi_{C,F}"] & F(C) \\ Nat(Y(C'),F) \arrow[u, "(-)\circ Y(f)"] \arrow[r, "\Phi_{C',F}"] & F(C') \arrow[u, "F(f)"'] \end{tikzcd}$$

Assume that $\alpha \in Nat(Y(C'),F)$. We compute directly that

$$\begin{aligned} (\Phi_{C,F} \circ (-) \circ Y(f))(\alpha) &= \Phi_{C,F}(\alpha \circ Y(f)) \\ &= (\alpha \circ Y(f))_C(id_C) \\ &= (\alpha_C \circ Y(f)_C)(id_C) \\ &= \alpha_C(Y(f)_C(id_C)) \\ &= \alpha_C(f \circ id_C) = \alpha_C(f) \end{aligned}$$

and

$$\begin{aligned} (F(f) \circ \Phi_{C',F})(\alpha) &= F(f)(\alpha_{C'}(id_{C'}) \\ &= (F(f) \circ \alpha_{C'})(id_{C'}) \\ &= (\alpha_C \circ Y(C')(f))(id_{C'}) \quad \text{(Naturality of } \alpha) \\ &= \alpha_C(Hom_{\mathscr{C}}(f,C')(id_{C'})) \\ &= \alpha_C(f). \end{aligned}$$

So, the above diagram in **Set** is commutative.

(d) Assume that $\beta \in Nat(F,F')$. We want to show that the following diagram in **Set** commutes:

$$\begin{array}{ccc} Nat(Y(C),F) & \xrightarrow{\Phi_{C,F}} & F(C) \\ {\scriptstyle \beta\circ(-)}\downarrow & & \downarrow{\scriptstyle \beta_C} \\ Nat(Y(C),F') & \xrightarrow{\Phi_{C,F'}} & F'(C) \end{array}$$

Assume that $\chi \in Nat(Y(C), F)$. We compute directly that

$$\begin{aligned}(\beta_C \circ \Phi_{C,F})(\chi) &= \beta_C(\chi_C(id_C)) \\ &= (\beta_C \circ \chi_C)(id_C) \\ &= (\beta \circ \chi)_C(id_C) \\ &= \Phi_{C,F'}(\beta \circ \chi) \\ &= (\Phi_{C,F'} \circ \beta \circ (-))(\chi)\end{aligned}$$

Therefore, the above diagram in **Set** commutes. This completes the proof. □

In Lemma 0.1.1, the property proved in part (c) tells us that $\Phi_{C,F}$ is natural in the object $C \in \mathscr{C}$. Correspondingly, the property proved in part (d) tells us that $\Phi_{C,F}$ is natural with respect to the functor $F : \mathscr{C}^{op} \to \mathbf{Set}$.

Now, we can state and prove the Yoneda lemma. We state it as a theorem due to it being one of the most important results in category theory.

Theorem 0.1.2 (Yoneda lemma)**.** *Let $\mathscr{C}$ be a locally small category. The Yoneda functor (see equation* (1)*), which is the functor*

$$\begin{array}{cccc} Y: & \mathscr{C} & \to & \mathcal{F}(\mathscr{C}^{op}, \boldsymbol{Set}) \\ & X & \mapsto & Y(X) = Hom_{\mathscr{C}}(-,X) \\ & f : X \to X' & \mapsto & Y(f) \end{array}$$

is a fully faithful functor.

Proof. Assume that $\mathscr{C}$ is a locally small category and that the Yoneda embedding Y is the functor defined as above. Let X, X' be objects in $\mathscr{C}$. Then, the functor Y induces a mapping

$$Y_{X,X'} : Hom_{\mathscr{C}}(X, X') \to Hom_{\mathcal{F}(\mathscr{C}^{op},\mathbf{Set})}(Y(X), Y(X'))$$

Note that $Hom_{\mathcal{F}(\mathscr{C}^{op},\mathbf{Set})}(Y(X), Y(X')) = Nat(Y(X), Y(X'))$ and $Y_{X,X'}(f) = Y(f)$.

To show: (a) $Y_{X,X'}$ is bijective.

(a) By Lemma 0.1.1, it suffices to show that $Y_{X,X'}$ is the inverse to the bijection $\Phi_{X,Y(X')} : Nat(Y(X), Y(X')) \to Y(X')(X)$. Assume that $f \in Hom_{\mathscr{C}}(X, X')$. Then,

$$\begin{aligned}(\Phi_{X,Y(X')} \circ Y_{X,X'})(f) &= \Phi_{X,Y(X')}(Y(f)) \\ &= Y(f)_X(id_X) \\ &= f \circ id_X = f.\end{aligned}$$

Hence, $Y_{X,X'}$ is a bijection as required.

Part (a) shows that the Yoneda functor is a fully faithful functor as required. $\square$