Notes on C*-algebras

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Contents

0.1 Purpose

These notes are for the study of the basic theory of C*-algebras. The content is typically encountered in a first graduate course in C*-algebras. The main reference for the first chapter of these notes is [Put19]. The main reference for the second chapter is [Mur90]. The main reference for the third chapter is [BO08].

Chapter 1

An introduction to C*-algebras from [Put19]

1.1 Definition and examples

In this section, we will build up to the definition of a C*-algebra and then give some useful examples.

Definition 1.1.1. Let A be an associative algebra over $\mathbb{C}$ (or $\mathbb{R}$). We say that A is a **Banach algebra** if A is also a Banach space. That is, A is equipped with a norm $\|-\|$ which makes A complete — every Cauchy sequence in A converges with respect to the norm $\|-\|$. Additionally, if $x, y \in A$, the norm must satisfy

$$\|xy\| \leq \|x\|\|y\|. \tag{1.1}$$

Equation (1.1) ensures that the norm respects the algebraic structure of A, by rendering multiplication continuous. Let $m_x : A \to A$ be the operator which sends $y \in A$ to xy. If $\epsilon \in \mathbb{R}_{>0}$, choose $\delta = \epsilon/\|x\|$ and suppose that $\|y_1 - y_2\| < \delta$. Then,

$$\begin{aligned}\|m_x(y_1) - m_x(y_2)\| &= \|xy_1 - xy_2\| \\ &\leq \|x\|\|y_1 - y_2\| \\ &< \epsilon.\end{aligned}$$

Hence, m_x is a continuous operator on A for all $x \in A$.

Definition 1.1.2. Let A be a Banach algebra over $\mathbb{C}$. We say that A is a **Banach *-algebra** if A is equipped with a map $^* : A \to A$ which satisfies for all $x, y \in A$ and $\lambda \in \mathbb{C}$,

1. $(x^*)^* = x$ (Involution)

2. $(x + y)^* = x^* + y^*$

3. $(\lambda x)^* = \overline{\lambda} x^*$

4. $(xy)^* = y^* x^*$ (Anti-multiplicative)

The middle two properties of the map $^* : A \to A$ means that * is anti-linear (or conjugate linear). If $a \in A$ then the element a^* is called the **adjoint** of a.

Definition 1.1.3. Let A be a Banach *-algebra over $\mathbb{C}$. We say that A is a **C*-algebra** if for all $x \in A$, $\|x^*x\| = \|x\|^2$.

The first theorem we will state gives us the primary example of a C*-algebra.

Theorem 1.1.1. *Let H be a Hilbert space over $\mathbb{C}$ and $B(H)$ denote the Banach space of bounded linear operators $\phi : H \to H$. Then, $B(H)$ is a C*-algebra.*

Proof. Assume that H is a Hilbert space over $\mathbb{C}$ and that $B(H)$ is the Banach space of bounded linear operators from H to H. Let $\|-\|_H$ denote the norm on H and $\langle -.- \rangle$ denote the inner product on H. Define the map $^* : H \to B(H)$ by

$$\begin{array}{cccc} ^* : & B(H) & \to & B(H) \\ & h & \to & h^*. \end{array}$$

where h^* is the adjoint of h, which satisfies for all $\xi, \eta \in H$,

$$\langle h^*(\xi), \eta \rangle = \langle \xi, h(\eta) \rangle \tag{1.2}$$

To show: (a) $B(H)$ is a Banach algebra.

(b) $B(H)$ is a Banach *-algebra.

(c) $B(H)$ is a C*-algebra.

(a) Observe that $B(H)$ is an associative algebra over $\mathbb{C}$, where scalar multiplication and addition are defined as usual and multiplication is given by composition of linear operators, which we will denote by $\circ$. We also know that $B(H)$ is a Banach space when equipped with the operator norm

$$\|h\| = \sup_{\|x\|_H=1} \|h(x)\|_H$$

To show: (aa) If $f, g \in B(H)$, then $\|f \circ g\| \leq \|f\|\|g\|$.

(aa) Assume that $f, g \in B(H)$. Then, from the definition of the operator norm, we have

$$\begin{aligned}
\|f \circ g\| &= \sup_{\|x\|_H=1} \|f(g(x))\|_H \\
&\leq \sup_{\|x\|_H=1} \|f\|\|g(x)\|_H \\
&= \|f\|\|g\|.
\end{aligned}$$

Therefore, $B(H)$ is a Banach algebra.

(b) To show: (ba) The map * is an involution.

(bb) The map * is anti-linear.

(bc) The map * is anti-multiplicative.

(ba) Assume that $h \in B(H)$. By equation (1.2), $h^{**} \in B(H)$ must satisfy for all $\xi, \eta \in H$,

$$\langle h^{**}(\xi), \eta \rangle = \langle \xi, h^*(\eta) \rangle = \langle h(\xi), \eta \rangle.$$

Therefore, $h^{**}(\xi) = h(\xi)$ for all $\xi \in H$. So, $h^{**} = h$, revealing that $^* : B(H) \to B(H)$ is an involution.

(bb) Assume that $g, h \in B(H)$. Then, for all $\xi, \eta \in H$, we have

$$\begin{aligned}
\langle (g+h)^*(\xi), \eta \rangle &= \langle \xi, (g+h)(\eta) \rangle \\
&= \langle \xi, g(\eta) \rangle + \langle \xi, h(\eta) \rangle \\
&= \langle (g^* + h^*)(\xi), \eta \rangle.
\end{aligned}$$

So, $(g+h)^* = g^* + h^*$. Now assume that $\lambda \in \mathbb{C}$. Then,

$$\begin{aligned}
\langle (\lambda h)^*(\xi), \eta\rangle &= \langle \xi, (\lambda h)(\eta)\rangle \\
&= \overline{\lambda}\langle \xi, h(\eta)\rangle \\
&= \overline{\lambda}\langle h^*(\xi), \eta\rangle \\
&= \langle (\overline{\lambda} h^*)(\xi), \eta\rangle.
\end{aligned}$$

So, $(\lambda h)^* = \overline{\lambda} h^*$. This demonstrates that * is anti-linear.

(bc) We compute directly that for all $\xi, \eta \in H$,

$$\begin{aligned}
\langle (g \circ h)^*(\xi), \eta\rangle &= \langle \xi, g(h(\eta))\rangle \\
&= \langle g^*(\xi), h(\eta)\rangle \\
&= \langle (h^* \circ g^*)(\xi), \eta\rangle.
\end{aligned}$$

Therefore, $(g \circ h)^* = h^* \circ g^*$. Hence, the map * is anti-linear. So, $B(H)$ is a Banach *-algebra.

(c) To show: (ca) For all $h \in B(H)$, $\|h^* \circ h\| = \|h\|^2$.

(ca) Assume that $h \in B(H)$ and that $\|h\| > 0$ (the statement holds when $h = 0$). We have already shown that $\|h^* \circ h\| \leq \|h^*\|\|h\|$.

To show: (caa) $\|h^*\| = \|h\|$.

(caa) Observe that

$$\begin{aligned}
\|h^* \circ h\| &= \sup_{\|\xi\|_H = 1} \|h^*(h(\xi))\|_H \\
&= \sup_{\|\xi\|_H = 1} \sup_{\|\eta\|_H = 1} |\langle h^*(h(\xi)), \eta\rangle| \\
&\geq \sup_{\|\xi\|_H = 1} |\langle h^*(h(\xi)), \xi\rangle| \\
&= \sup_{\|\xi\|_H = 1} \|h(\xi)\|_H^2 \\
&= \|h\|^2.
\end{aligned}$$

Therefore, $\|h\|^2 \leq \|h^*\|\|h\|$ and $\|h\| \leq \|h^*\|$. To establish the reverse inequality, we can interchange the roles of h and h^* in the above calculation so that

$$\begin{aligned}\|h \circ h^*\| &= \sup_{\|\xi\|_H=1} \|h(h^*(\xi))\|_H \\ &= \sup_{\|\xi\|_H=1} \sup_{\|\eta\|_H=1} |\langle h(h^*(\xi)), \eta\rangle| \\ &\geq \sup_{\|\xi\|_H=1} |\langle h(h^*(\xi)), \xi\rangle| \\ &= \sup_{\|\xi\|_H=1} \|h^*(\xi)\|_H^2 \\ &= \|h^*\|^2.\end{aligned}$$

So, $\|h^*\|^2 \leq \|h^*\|\|h\|$ and $\|h^*\| \leq \|h\|$. In tandem with $\|h\| \leq \|h^*\|$, we deduce that $\|h^*\| = \|h\|$.

(ca) Recall from part (caa) that $\|h\|^2 \leq \|h^* \circ h\|$ and from the beginning of part (ca) that $\|h^* \circ h\| \leq \|h^*\|\|h\|$. Since $\|h^*\| = \|h\|$, $\|h^* \circ h\| \leq \|h\|^2$ and consequently, $\|h\|^2 = \|h^* \circ h\|$ as required. □

Example 1.1.1. Here, we will give another important example of a C*-algebra. Let X be a compact, Hausdorff space and $Cts(X, \mathbb{C})$ denote the space of continuous functions from X to $\mathbb{C}$. Then, $Cts(X, \mathbb{C})$ is a C*-algebra with scalar multiplication, addition and multiplication defined pointwise on $\mathbb{C}$. The norm on $Cts(X, \mathbb{C})$ is

$$\|f\| = \sup_{x \in X} |f(x)|.$$

and the map $^* : Cts(X, \mathbb{C}) \to Cts(X, \mathbb{C})$ is defined by the equation

$$f^*(x) = \overline{f(x)}.$$

Example 1.1.2. The complex numbers $\mathbb{C}$ with addition, multiplication and complex conjugation is an example of a C*-algebra.

Example 1.1.3. As a special case of Theorem 1.1.1, if we set $H = \mathbb{C}^n$ for $n \in \mathbb{Z}_{>0}$, we find that the $\mathbb{C}$-algebra of $n \times n$ matrices $M_{n\times n}(\mathbb{C})$ is a C*-algebra.

Next, we define some specific types of C*-algebras.

Definition 1.1.4. Let A be a C*-algebra. We say that A is **unital** if as an associative algebra, A has a multiplicative unit which is usually denoted by 1_A.

Definition 1.1.5. Let A be a C*-algebra. We say that A is **commutative** if as an associative algebra, A is commutative. That is, if $a, b \in A$ then $ab = ba$.

One of the defining properties of a C*-algebra is that the involution map $^* : A \to A$ is isometric (distance preserving).

Theorem 1.1.2. *Let A be a C*-algebra. If $a \in A$ then $\|a\| = \|a^*\|$.*

Proof. Assume that A is a C*-algebra. Assume that $a \in A$. Then,

$$\|a\|^2 = \|a^*a\| \leq \|a^*\|\|a\|$$

and

$$\|a^*\|^2 = \|aa^*\| \leq \|a\|\|a^*\|.$$

The first equation shows that $\|a\| \leq \|a^*\|$ and the second shows that $\|a^*\| \leq \|a\|$. Hence, $\|a\| = \|a^*\|$. □

Next, we define some more terminology related to C*-algebras.

Definition 1.1.6. Let A be a C*-algebra.

1. We say that $a \in A$ is **self-adjoint** if $a^* = a$.
2. We say that $a \in A$ is **normal** if $a^*a = aa^*$.
3. We say that $a \in A$ is a **projection** if a is self-adjoint and $a^2 = a$.
4. We say that $a \in A$ is a **partial isometry** if a^*a is a projection.
5. We say that $a \in A$ is **positive** if there exists an element $b \in A$ such that $a = b^*b$. We write $a \geq 0$ to mean that a is positive.

Now let A be a unital C*-algebra.

1. We say that $a \in A$ is **unitary** if $a^*a = aa^* = 1_A$. That is a is invertible and $a^{-1} = a^*$.
2. We say that $a \in A$ is an **isometry** if $a^*a = 1_A$.

These definitions are consistent with those found in [Sol18], which focuses on the C*-algebra of bounded linear operators on a Hilbert space.

From a finite number of C*-algebras, we can construct a C*-algebra from them, which is called the direct sum.

Theorem 1.1.3 (Finite direct sum of C*-algebras). *Let $n \in \mathbb{Z}_{>0}$ and A_i be C*-algebras for $i \in \{1, 2, \dots, n\}$. Define the* ***direct sum*** *of the family $\{A_i\}_{i=1}^n$ as the associative algebra over $\mathbb{C}$*

$$\bigoplus_{i=1}^n A_i = \{(a_1, \dots, a_n) \mid a_i \in A_i \text{ for } i \in \{1, 2, \dots, n\}\}.$$

The algebraic operations of multiplication, scalar multiplication, addition and involution on $\bigoplus_{i=1}^n A_i$ are defined coordinate-wise. The norm on $\bigoplus_{i=1}^n A_i$ is given by

$$\|(a_1, \dots, a_n)\| = \max_{i \in \{1,2,\dots,n\}} \|a_i\|.$$

Then, $\bigoplus_{i=1}^n A_i$ is a C-algebra.*

Proof. Assume that $n \in \mathbb{Z}_{>0}$ and A_i are C*-algebras for $i \in \{1, 2, \dots, n\}$. Assume that the direct sum $A = \bigoplus_{i=1}^n A_i$ is defined as in the statement of the theorem.

To see that $\bigoplus_{i=1}^n A_i$ is a Banach algebra, first note that A is the direct sum of Banach spaces and is hence, a Banach space. Next, assume that $(a_1, \dots, a_n), (b_1, \dots, b_n) \in A$. We compute directly that

$$\begin{aligned}
\|(a_1, \dots, a_n)(b_1, \dots, b_n)\| &= \|(a_1 b_1, \dots, a_n b_n)\| \\
&= \max_{i \in \{1,2,\dots,n\}} \|a_i b_i\| \\
&\leq \max_{i \in \{1,2,\dots,n\}} \|a_i\| \|b_i\| \\
&\leq \big(\max_{i \in \{1,2,\dots,n\}} \|a_i\|\big)\big(\max_{j \in \{1,2,\dots,n\}} \|b_j\|\big) \\
&= \|(a_1, \dots, a_n)\| \|(b_1, \dots, b_n)\|.
\end{aligned}$$

Hence, A is a Banach algebra.

By definition of the involution map $* : A \to A$ pointwise, it is straightforward to check that A is a Banach *-algebra.

Finally, to see that A is a C*-algebra, we compute for $(a_1, \dots, a_n) \in A$ that

$$\begin{aligned}\|(a_1,\dots,a_n)^*(a_1,\dots,a_n)\| &= \|(a_1^*,\dots,a_n^*)(a_1,\dots,a_n)\| \\ &= \|(a_1^*a_1,\dots,a_n^*a_n)\| \\ &= \max_{i\in\{1,2,\dots,n\}} \|a_i^*a_i\| \\ &= \max_{i\in\{1,2,\dots,n\}} \|a_i\|^2 \\ &= \|(a_1,\dots,a_n)\|^2.\end{aligned}$$

Therefore, A is a C*-algebra. □

Now, if we have a countable family of C*-algebras $\{A_i\}_{i=1}^{\infty}$ then the definition of the direct sum in Theorem 1.1.3 does not carry over to this situation. In this case, the direct sum is defined by

$$\bigoplus_{n=1}^{\infty} A_n = \{(a_1,a_2,\dots) \mid a_i \in A_i \text{ for } i \in \{1,2,\dots,n\}, \lim_{n\to\infty} \|a_n\| = 0\}.$$

Example 1.1.4. This example encapsulates [Put19, Exercise 1.2.1]. Consider the $\mathbb{C}$-algebra $\mathbb{C}^2 \cong \mathbb{C} \oplus \mathbb{C}$ with the operations of multiplication, addition and complex conjugation defined pointwise. We claim that $\mathbb{C}^2$ with the norm

$$\|(z_1,z_2)\| = |z_1| + |z_2|$$

is not a C*-algebra.

Consider the element $(1,1) \in \mathbb{C}^2$. Then,

$$\|(1,1)^*(1,1)\| = \|(1,1)\| = 1+1 = 2$$

and

$$\|(1,1)\|^2 = (1+1)^2 = 4.$$

So, $\|(1,1)\|^2 \neq \|(1,1)^*(1,1)\|$ and hence, $\mathbb{C}^2$ is not a C*-algebra with the above norm.

In fact, the only norm which makes $\mathbb{C}^2$ a C*-algebra is the one given in Theorem 1.1.3. In order to prove this, we will follow the outline given in [Put19, Exercise 1.2.1].

Suppose that $\|-\|$ is a norm on $\mathbb{C}^2$, which makes $\mathbb{C}^2$ into a C*-algebra.

To show: (a) $\|(1,0)\| = \|(1,1)\| = \|(0,1)\| = 1$.

(a) By the defining property of C*-algebras, we find that

$$\|(1,0)\| = \|(1,0)^*(1,0)\| = \|(1,0)\|^2.$$

Similarly, $\|(1,1)\| = \|(1,1)\|^2$ and $\|(0,1)\| = \|(0,1)\|^2$. Therefore, $\|(1,0)\| = \|(1,1)\| = \|(0,1)\| = 1$.

Now let $(z_1, z_2) \in \mathbb{C}^2$. Write $z_1 = |z_1|e^{i\theta_1}$ and $z_2 = |z_2|e^{i\theta_2}$, where $\theta_1, \theta_2 \in [0, 2\pi)$. Then, the element $(e^{-i\theta_1}, e^{-i\theta_2}) \in \mathbb{C}^2$ satisfies

$$(e^{-i\theta_1}, e^{-i\theta_2})^*(e^{-i\theta_1}, e^{-i\theta_2}) = (1,1) = (e^{-i\theta_1}, e^{-i\theta_2})(e^{-i\theta_1}, e^{-i\theta_2})^*.$$

So, $(e^{-i\theta_1}, e^{-i\theta_2})$ is a unitary element of $\mathbb{C}^2$ satisfying $(e^{-i\theta_1}, e^{-i\theta_2})(z_1, z_2) = (|z_1|, |z_2|)$

To show: (b) $\|(z_1, z_2)\| = \|(|z_1|, |z_2|)\|$.

(b) By the defining property of a C*-algebra, we have

$$\begin{aligned}
\|(z_1, z_2)\|^2 &= \|(z_1, z_2)^*(z_1, z_2)\| \\
&= \|(z_1, z_2)^*(e^{-i\theta_1}, e^{-i\theta_2})^*(e^{-i\theta_1}, e^{-i\theta_2})(z_1, z_2)\| \\
&= \|(|z_1|, |z_2|)^*(|z_1|, |z_2|)\| \\
&= \|(|z_1|, |z_2|)\|^2.
\end{aligned}$$

So, $\|(z_1, z_2)\| = \|(|z_1|, |z_2|)\|$.

Next, assume that $(a_1, a_2), (b_1, b_2) \in \mathbb{C}^2$ such that $\|(a_1, a_2)\| = \|(b_1, b_2)\| = 1$. Assume that $t \in [0, 1]$.

To show: (c) $\|t(a_1, a_2) + (1-t)(b_1, b_2)\| \leq 1$.

(c) We compute directly that

$$
\begin{aligned}
\|t(a_1,a_2)+(1-t)(b_1,b_2)\|^2 &= \|(ta_1+(1-t)b_1, ta_2+(1-t)b_2)\|^2 \\
&= \|(ta_1+(1-t)b_1, ta_2+(1-t)b_2)^* \\
&\quad (ta_1+(1-t)b_1, ta_2+(1-t)b_2)\| \\
&= \|(|ta_1+(1-t)b_1|^2, |ta_2+(1-t)b_2|^2)\| \\
&\le \|\big((t|a_1|+(1-t)|b_1|)^2, (t|a_2|+(1-t)|b_2|)^2\big)\| \\
&\le \|(t^2|a_1|^2, t^2|a_2|^2)\| + \|(t(1-t)|a_1||b_1|, t(1-t)|a_2||b_2|)\| \\
&\quad +\|((1-t)^2|b_1|^2, (1-t)^2|b_2|^2)\| \\
&= t^2\|(|a_1|^2, |a_2|^2)\| + t(1-t)\|(|a_1||b_1|, |a_2||b_2|)\| \\
&\quad +(1-t)^2\|(|b_1|^2, |b_2|^2)\| \\
&= t^2\|(a_1,a_2)^*(a_1,a_2)\| + t(1-t)\|(|a_1b_1|, |a_2b_2|)\| \\
&\quad +(1-t)^2\|(b_1,b_2)^*(b_1,b_2)\| \\
&\le t^2\|(a_1,a_2)\|^2 + t(1-t)\|(|a_1|,|a_2|)\|\|(|b_1|,|b_2|)\| \\
&\quad +(1-t)^2\|(b_1,b_2)\|^2 \\
&\le t^2 + t(1-t) + (1-t)^2 \quad \text{(by part (b))} \\
&= t + (1-t)^2 = 1 - t + t^2 \le 1.
\end{aligned}
$$

This proves part (c). Now assume that $\alpha \in \mathbb{C}$ such that $|\alpha| \le 1$.

To show: (d) $\|(1,\alpha)\| = 1$ and $\|(\alpha,1)\| = 1$.

(d) We compute directly that

$$
\begin{aligned}
\|(1,\alpha)\|^2 &= \|(1,\alpha)^*(1,\alpha)\| \\
&= \|(1,|\alpha|^2)\| \\
&= \||\alpha|^2(1,1) + (1-|\alpha|^2)(1,0)\| \\
&\le 1 \quad \text{(by parts (a) and (c))}.
\end{aligned}
$$

The inequality $\|(\alpha,1)\| \le 1$ follows from a similar computation.

Next, observe that

$$
\begin{aligned}
1 &= \|(1,0)\| \quad \text{(by part (a))} \\
&= \|(1,0)(1,\alpha)\| \\
&\le \|(1,0)\|\|(1,\alpha)\| = \|(1,\alpha)\|.
\end{aligned}
$$

Therefore, $1 = \|(1, \alpha)\|$. Similarly, $1 = \|(\alpha, 1)\|$.

Now we put all of these observations together to obtain the required conclusion. Assume that $(\alpha, \beta) \in \mathbb{C}^2$ such that $0 < |\alpha| \leq |\beta|$. The norm of this element is by part (d)

$$\|(\alpha, \beta)\| = |\beta| \|(\frac{\alpha}{\beta}, 1)\| = |\beta|$$

Similarly, if $0 < |\beta| \leq |\alpha|$ then $\|(\alpha, \beta)\| = |\alpha|$. Furthermore, by part (a)

$$\|(\alpha, 0)\| = |\alpha| \|(1, 0)\| = |\alpha|$$

and $\|(0, \alpha)\| = |\alpha|$. Finally, $\|(0, 0)\| = 0$ by definition of the norm. Hence,

$$\|(\alpha, \beta)\| = \max(|\alpha|, |\beta|).$$

Consequently, the above norm is the only norm which makes $\mathbb{C}^2$ into a C*-algebra.

As usual with mathematical structures, we can define the notion of a map between C*-algebras.

Definition 1.1.7. Let A and B be C*-algebras with involution maps $*_A$ and $*_B$ respectively. A ***-homomorphism** is a function $\phi : A \to B$ such that

1. If $a_1, a_2 \in A$ then $\phi(a_1 + a_2) = \phi(a_1) + \phi(a_2)$.
2. If $\lambda \in \mathbb{C}$ and $a \in A$ then $\phi(\lambda a) = \lambda \phi(a)$.
3. If $a_1, a_2 \in A$ then $\phi(a_1 a_2) = \phi(a_1)\phi(a_2)$.
4. If $a_1 \in A$ then $\phi(a_1^{*_A}) = \phi(a_1)^{*_B}$.

If ϕ is bijective then we say that ϕ is a ***-isomorphism**.

If A and B are unital C*-algebras and $\phi : A \to B$ is a *-homomorphism such that $\phi(1_A) = 1_B$ then ϕ is called a **unital *-homomorphism**.

C*-algebras, together with the class of *-homomorphisms, form the category of C*-algebras.

1.2 The spectrum

The concept of the spectrum is used widely in linear algebra and for bounded operators on a Hilbert space. In this section, we will define the spectrum for elements in a C*-algebra. First, we prove the following important connection between the algebraic and topological structures on a C*-algebra.

Theorem 1.2.1. *Let A be a unital Banach *-algebra. If $a \in A$ and $\|a - 1_A\| < 1$ then a is invertible. Moreover, the set of invertible elements of A is open.*

Proof. Assume that A is a unital Banach *-algebra. First assume that $a \in A$ and $\|a - 1_A\| < 1$.

To show: (a) a is invertible.

(a) Consider the following infinite sum in A:

$$\sum_{n=0}^{\infty}(1_A - a)^n.$$

To see that this converges in A, we must show that its norm is finite. We compute directly that

$$\|\sum_{n=0}^{\infty}(1_A - a)^n\| \leq \sum_{n=0}^{\infty}\|1_A - a\|^n < \infty$$

because $\|1_A - a\| < 1$. Hence, $\sum_{n=0}^{\infty}(1_A - a)^n \in A$. Let $b = \sum_{n=0}^{\infty}(1_A - a)^n$. To see that $b = a^{-1}$, we compute directly that

$$\begin{aligned}
ab &= a\sum_{n=0}^{\infty}(1_A - a)^n \\
&= a\Big(\lim_{N\to\infty}\sum_{n=0}^{N}(1_A - a)^n\Big) \\
&= \lim_{N\to\infty} a\Big(\sum_{n=0}^{N}(1_A - a)^n\Big) \\
&= \lim_{N\to\infty}(1_A - (1_A - a))\Big(\sum_{n=0}^{N}(1_A - a)^n\Big) \\
&= \lim_{N\to\infty}\big(1_A - (1_A - a)^{N+1}\big) = 1_A.
\end{aligned}$$

A similar computation demonstrates that $ba = 1_A$. Therefore, $b = a^{-1}$ and a is invertible.

Next, assume that $a \in A$ is invertible. Select $x \in A$ such that $\|x\| < \frac{1}{2\|a^{-1}\|}$.

To show: (b) $a + x$ is invertible.

(b) Observe that

$$\begin{aligned}\|a^{-1}x\| &\leq \|a^{-1}\|\|x\| \\ &< \|a^{-1}\|\frac{1}{2\|a^{-1}\|} \\ &= \frac{1}{2} < 1.\end{aligned}$$

By the first part of the theorem, we deduce that $1_A + a^{-1}x$ is an invertible element of A. Hence,

$$a + x = a(1_A + a^{-1}x)$$

is an invertible element of A because it is the product of two invertible elements.

Let $I \subseteq A$ denote the set of invertible elements of A. By part (b), we find that the open ball

$$B(a, \frac{1}{2\|a^{-1}\|}) \subseteq I.$$

Hence, I is an open subset of A. □

Now we arrive at the familiar notion of the spectrum. Perhaps the easiest example of the spectrum arises from linear algebra. If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation then the spectrum of T is given by computing the roots of the characteristic polynomial of T.

Definition 1.2.1. Let A be a unital associative algebra. Let $a \in A$. The **spectrum** of a, denoted by $\sigma(a)$, is the set

$$\sigma(a) = \{\lambda \in \mathbb{C} \mid \lambda 1_A - a \text{ is not invertible}\}.$$

The **resolvent set** of a, denoted by $\rho(a)$, is the set $\rho(a) = \mathbb{C} - \sigma(a)$.

The **spectral radius** of a, denoted by $r(a)$, is the quantity

$$r(a) = \sup_{\lambda \in \sigma(a)} |\lambda|.$$

Note that the spectral radius of a is defined provided that $\sigma(a)$ is non-empty. Also, we allow the possibility that $\sigma(a) = \infty$.

Let us return to the world of linear algebra for a brief moment and work with the C*-algebra $M_{n\times n}(\mathbb{C})$. If $n \in \mathbb{Z}_{>0}$ and $a \in M_{n\times n}(\mathbb{C})$ then the following are equivalent:

1. a is invertible
2. The linear transformation $a : \mathbb{C}^n \to \mathbb{C}^n$ is injective.
3. The linear transformation $a : \mathbb{C}^n \to \mathbb{C}^n$ is surjective.
4. The determinant $\det(a) \neq 0$.

In particular, the characteristic polynomial of a is computed by using the determinant. In this very special case, the spectrum of a is indeed quite straightforward to compute. However, if H is an infinite dimensional Hilbert space then the above result completely falls apart for the C*-algebra $B(H)$ — the space of bounded linear operators on H. Additionally, a notion of the determinant does not exist for an arbitrary element of $B(H)$. So, one cannot expect to compute spectrums easily for the elements of $B(H)$.

Here are some fundamental results about the spectrum. First, we will prove that the spectrum is always a non-empty, compact subset of $\mathbb{C}$. We will follow [Sol18, Chapter 1].

Theorem 1.2.2. *Let A be a unital Banach *-algebra and $h \in A$. Then,*

$$\{\lambda \in \mathbb{C} \mid |\lambda| > \|h\|\} \subseteq \rho(h).$$

Proof. Assume that A is a unital Banach *-algebra and $h \in A$. Assume that $\lambda \in \mathbb{C}$ such that $|\lambda| > \|h\|$. We claim that the sum

$$\frac{1}{\lambda} \sum_{n=0}^{\infty} \lambda^{-n} h^n \tag{1.3}$$

converges in A and that it is equal to $(\lambda 1_A - h)^{-1}$. To see that the sum in equation (1.3) converges, it suffices to show that its norm is finite. But,

$$
\begin{aligned}
\|\frac{1}{\lambda}\sum_{n=0}^{\infty}\lambda^{-n}h^n\| &= \frac{1}{|\lambda|}\|\sum_{n=0}^{\infty}\lambda^{-n}h^n\| \\
&\leq \frac{1}{|\lambda|}\sum_{n=0}^{\infty}|\lambda|^{-n}\|h\|^n \\
&= \frac{C}{|\lambda|}
\end{aligned}
$$

where $C = \sum_{n=0}^{\infty}|\lambda|^{-n}\|h\|^n \in \mathbb{R}_{>0}$. The sum $\sum_{n=0}^{\infty}|\lambda|^{-n}\|h\|^n$ is a convergent geometric series because $|\lambda| > \|h\|$.

Hence, the sum in equation (1.3) converges in A and is consequently, a well-defined element in A. For all $m \in \mathbb{Z}_{>0}$, let

$$S_m = \frac{1}{\lambda}\sum_{n=0}^{m}\lambda^{-n}h^n.$$

Then,

$$
\begin{aligned}
S_m(\lambda 1_A - h) &= \frac{1}{\lambda}(1_A + \lambda^{-1}h + \lambda^{-2}h^2 + \cdots + \lambda^{-m}h^m)(\lambda 1_A - h) \\
&= \frac{1}{\lambda}((\lambda 1_A - h) + (h - \lambda^{-1}h^2) + \cdots + (\lambda^{-m+1}h^m - \lambda^{-m}h^{m+1})) \\
&= \frac{1}{\lambda}(\lambda 1_A - \lambda^{-m}h^{m+1}) \\
&= 1_A - \lambda^{-m-1}h^{m+1}.
\end{aligned}
$$

A similar calculation gives $(\lambda 1_A - h)S_m = 1_A - \lambda^{-m-1}h^{m+1}$. Now take the limit as $m \to \infty$. Observe that in A,

$$\lim_{m\to\infty}\lambda^{-m-1}h^{m+1} = 0$$

because

$$\lim_{m\to\infty}(\frac{\|h\|}{|\lambda|})^{m+1} = 0.$$

So, in the limit as $m \to \infty$,

$$(\lambda 1_A - h)\Big(\frac{1}{\lambda}\sum_{n=0}^{\infty}\lambda^{-n}h^n\Big) = \Big(\frac{1}{\lambda}\sum_{n=0}^{\infty}\lambda^{-n}h^n\Big)(\lambda 1_A - h) = 1_A.$$

Therefore,

$$\left(\frac{1}{\lambda}\sum_{n=0}^{\infty}\lambda^{-n}h^{n}\right) = (\lambda 1_A - h)^{-1}$$

and subsequently, $\lambda \in \rho(h)$ and

$$\{\lambda \in \mathbb{C} \mid |\lambda| > \|h\|\} \subseteq \rho(h).$$

□

Rewriting the conclusion of Theorem 1.2.2 in terms of the spectrum, we find that

$$\sigma(h) \subseteq \{\lambda \in \mathbb{C} \mid |\lambda| \leq \|h\|\} \tag{1.4}$$

If $h \in A$ then $\sigma(h)$ is a bounded set. We now want to show that $\sigma(h)$ is a closed set.

Theorem 1.2.3. *Let A be a unital Banach *-algebra, $h \in B(H)$ and $\lambda_0 \in \rho(h)$. Suppose that $\lambda \in \mathbb{C}$ such that*

$$|\lambda - \lambda_0| < \frac{1}{\|(\lambda_0 1_A - h)^{-1}\|}.$$

Then,

$$(\lambda 1_A - h)^{-1} = \sum_{n=0}^{\infty}(\lambda_0 - \lambda)^n(\lambda_0 1_A - x)^{-n-1}.$$

and consequently, $\lambda \in \rho(h)$.

Proof. Assume that A is a unital Banach *-algebra, $h \in A$ and $\lambda_0 \in \rho(h)$. Assume that $\lambda \in \mathbb{C}$ such that

$$|\lambda - \lambda_0| < \frac{1}{\|(\lambda_0 1_A - h)^{-1}\|}.$$

To see that the sum

$$\sum_{n=0}^{\infty}(\lambda_0 - \lambda)^n(\lambda_0 1_A - x)^{-n-1}$$

converges in A, we must show that its norm is finite. We have

$$\|\sum_{n=0}^{\infty}(\lambda_0-\lambda)^n(\lambda_0 1_A-h)^{-n-1}\| \leq \sum_{n=0}^{\infty}|\lambda_0-\lambda|^n\|(\lambda_0 1_A-h)^{-1}\|^{n+1}$$
$$= \frac{1}{|\lambda_0-\lambda|}\sum_{n=0}^{\infty}|\lambda_0-\lambda|^{n+1}\|(\lambda_0 1_A-h)^{-1}\|^{n+1}$$
$$= \frac{D}{|\lambda_0-\lambda|}$$

for some $D \in \mathbb{R}_{>0}$. Hence, the sum $\sum_{n=0}^{\infty}(\lambda_0-\lambda)^n(\lambda_0 1_A-h)^{-n-1}$ converges in A and is a well-defined element in A. If $m \in \mathbb{Z}_{>0}$ then let

$$T_m = \sum_{n=0}^{m}(\lambda_0-\lambda)^n(\lambda_0 1_A-h)^{-n-1}.$$

First, observe that

$$\lambda 1_A - h = (\lambda_0 1_A - h)(1_A + (\lambda-\lambda_0)(\lambda_0 1_A-h)^{-1})$$

Then, a direct calculation yields

$$\begin{aligned}
T_m(\lambda 1_A-h) &= \Big(\sum_{n=0}^{m}(\lambda_0-\lambda)^n(\lambda_0 1_A-h)^{-n-1}\Big)((\lambda_0 1_A-h)(1_A+(\lambda-\lambda_0)(\lambda_0 1_A-h)^{-1}))\\
&= \Big(\sum_{n=0}^{m}(\lambda_0-\lambda)^n(\lambda_0 1_A-h)^{-n}\Big)(1_A+(\lambda-\lambda_0)(\lambda_0 1_A-h)^{-1})\\
&= \sum_{n=0}^{m}(\lambda_0-\lambda)^n(\lambda_0 1_A-h)^{-n} - \sum_{n=0}^{m}(\lambda_0-\lambda)^{n+1}(\lambda_0 1_A-h)^{-n-1}\\
&= 1_A - (\lambda_0-\lambda)^{m+1}(\lambda_0 1_A-h)^{-m-1}.
\end{aligned}$$

Note that by a similar computation,

$$(\lambda 1_A-h)T_m = 1_A - (\lambda_0-\lambda)^{m+1}(\lambda_0 1_A-h)^{-m-1}$$

as well. Taking the limit as $m \to \infty$, we find that
$(\lambda_0-\lambda)^{m+1}(\lambda_0 1_A-h)^{-m-1} \to 0$ in $B(H)$ because its norm tends to 0 as $m \to \infty$. Hence,

$$(\lambda 1_A-h)\Big(\sum_{n=0}^{\infty}(\lambda_0-\lambda)^n(\lambda_0 1_A-h)^{-n-1}\Big) = \Big(\sum_{n=0}^{\infty}(\lambda_0-\lambda)^n(\lambda_0 1_A-h)^{-n-1}\Big)(\lambda 1_A-h) = 1_A$$

and

$$\Big(\sum_{n=0}^{\infty}(\lambda_0-\lambda)^n(\lambda_0 1_A-h)^{-n-1}\Big)=(\lambda 1_A-h)^{-1}$$

which demonstrates that $\lambda\in\rho(h)$. $\square$

Theorem 1.2.3 tells us that if $h\in A$ then $\rho(h)$ is an open subset of $\mathbb{C}$. In tandem with equation (1.4), the spectrum $\sigma(h)$ is a closed and bounded subset of $\mathbb{C}$ and is thus, compact. Let us summarise this important finding as a theorem.

Theorem 1.2.4. *Let A be a unital Banach *-algebra and $a\in A$. Then, the spectrum $\sigma(a)$ is a compact subset of $\mathbb{C}$.*

Now, we will show that the spectrum is a non-empty subset of $\mathbb{C}$. This is exhibited by the following method of computing the spectral radius.

Theorem 1.2.5. *Let A be a unital Banach *-algebra and $h\in A$. Then, $\sigma(h)\neq\emptyset$, the sequence $\{\|h^n\|^{1/n}\}_{n\in\mathbb{Z}_{>0}}$ converges in $\mathbb{R}$ and*

$$\lim_{n\to\infty}\|h^n\|^{\frac{1}{n}}=r(h)=\sup_{\lambda\in\sigma(h)}|\lambda|.$$

Proof. Assume that A is a unital Banach *-algebra and $h\in A$. Define

$$\alpha(h)=\inf_{n\in\mathbb{Z}_{>0}}\|h^n\|^{\frac{1}{n}}.$$

We will show that the sequence $\{\|h^n\|^{1/n}\}_{n\in\mathbb{Z}_{>0}}$ converges to $\alpha(h)$. Assume that $\epsilon\in\mathbb{R}_{>0}$. From the definition of infimum, there exists an index $n_\epsilon\in\mathbb{Z}_{>0}$ such that

$$\|h^{n_\epsilon}\|^{\frac{1}{n_\epsilon}}\le\alpha(h)+\epsilon.$$

Take any $n\in\mathbb{Z}_{>0}$ and use the Euclidean algorithm to write $n=qn_\epsilon+r$, where $q\in\mathbb{Z}_{>0}$ and $r\in\{0,1,\dots,n_\epsilon-1\}$. Then,

$$\begin{aligned}\|h^n\|&=\|h^{qn_\epsilon+r}\|\\&\le\|h^{n_\epsilon}\|^q\|h\|^r\\&\le(\alpha(h)+\epsilon)^{qn_\epsilon}\|h\|^r\\&=(\alpha(h)+\epsilon)^{n-r}\|h\|^r.\end{aligned}$$

Taking the n^{th} root of both sides, we obtain the inequality

$$\|h^n\|^{\frac{1}{n}} \leq (\alpha(h)+\epsilon)^{1-\frac{r}{n}}\|h\|^{\frac{r}{n}}.$$

Consequently,

$$\alpha(h) \leq \liminf_{n\to\infty}\|h^n\|^{\frac{1}{n}} \leq \limsup_{n\to\infty}\|h^n\|^{\frac{1}{n}} \leq \alpha(h)+\epsilon.$$

This demonstrates that the sequence $\{\|h^n\|^{\frac{1}{n}}\}$ converges in $\mathbb{R}$ to $\alpha(h)$. The next step is to show that

$$\alpha(h) = r(h) = \sup_{\lambda\in\sigma(h)}|\lambda|. \tag{1.5}$$

To show: (a) $\alpha(h) \geq \sup_{\lambda\in\sigma(h)}|\lambda|$.

(b) $\alpha(h) \leq \sup_{\lambda\in\sigma(h)}|\lambda|$.

(a) Suppose for the sake of contradiction that $\alpha(h) < |\lambda|$ for some $\lambda \in \sigma(h)$. By the root test, the series

$$\sum_{n=0}^{\infty}\frac{\|h^n\|}{|\lambda|^n}$$

in $\mathbb{R}$ is convergent. Therefore, the sum

$$\sum_{n=0}^{\infty}\lambda^{-n}h^n$$

converges in A and is a well-defined element of A. By using similar arguments to Theorem 1.2.2 and Theorem 1.2.3, we deduce that

$$\sum_{n=0}^{\infty}\lambda^{-n}h^n = (1_A - \frac{h}{\lambda})^{-1}.$$

So, $\lambda 1_A - h$ is invertible and $\lambda \in \rho(h)$. But this contradicts the fact that $\lambda \in \sigma(h)$. Therefore, $\alpha(h) \geq \sup_{\lambda\in\sigma(h)}|\lambda|$.

(b) We will divide this into two cases. First, we note that if $x, y \in A$ then

$$\begin{aligned}\alpha(xy) &= \lim_{n\to\infty}\|(xy)^n\|^{\frac{1}{n}}\\ &= \lim_{n\to\infty}\|x^ny^n\|^{\frac{1}{n}}\\ &\leq \lim_{n\to\infty}\|x^n\|^{\frac{1}{n}}\|y^n\|^{\frac{1}{n}}\\ &= \lim_{n\to\infty}\|x^n\|^{\frac{1}{n}}\lim_{n\to\infty}\|y^n\|^{\frac{1}{n}}\\ &= \alpha(x)\alpha(y).\end{aligned}$$

Case 1: $\alpha(h)=0$.

If $\alpha(h)=0$ then h is not invertible because otherwise,

$$1=\alpha(1_A)=\alpha(hh^{-1})\leq\alpha(h)\alpha(h^{-1})=0.$$

So, $0\in\sigma(h)$ and since $\alpha(h)\geq\sup_{\lambda\in\sigma(h)}|\lambda|$ from part (a),

$$\alpha(h)=\sup_{\lambda\in\sigma(h)}|\lambda|=0.$$

Case 2: $\alpha(h)>0$.

Assume that $\alpha(h)>0$ and $\alpha(h)>\sup_{\lambda\in\sigma(h)}|\lambda|$. Since the spectrum $\sigma(h)$ is a compact subset of $\mathbb{C}$, there exists $r\in(0,\alpha(h))$ such that

$$\sigma(h)\subseteq\{\lambda\in\mathbb{C}\mid|\lambda|\leq r\}.$$

Let $D=\{\lambda\in\mathbb{C}\mid|\lambda|>r\}$. Then, $D\subseteq\rho(h)$. Let φ be a continuous linear functional on A and define the map

$$\begin{array}{rccc}\psi: & D & \to & \mathbb{C}\\ & \lambda & \mapsto & \varphi((\lambda 1_A-h)^{-1})\end{array}$$

The map ψ is holomorphic due to the series expansion in Theorem 1.2.2. In particular, if $|\lambda|>\alpha(h)$ then we have the series expansion

$$\varphi((\lambda 1_A-h)^{-1})=\sum_{n=0}^{\infty}\lambda^{-n-1}\varphi(h^n).$$

The series $\sum_{n=0}^{\infty}\lambda^{-n-1}h^n$ converges in A because

$$\|\sum_{n=0}^{\infty}\lambda^{-n-1}h^n\|\leq\sum_{n=0}^{\infty}|\lambda|^{-n-1}\|h^n\|$$

and by applying the root test on $\sum_{n=0}^{\infty} \lambda^{-n}\|h^n\|$, we find that

$$\lim_{n\to\infty} \frac{\|h^n\|^{\frac{1}{n}}}{|\lambda|} = \frac{\alpha(h)}{|\lambda|} < 1.$$

Moreover, $\sum_{n=0}^{\infty} \lambda^{-n-1}\varphi(h^n) \in \mathbb{C}$ vanishes as $\lambda \to \infty$. To see why this is the case, replace λ by $\lambda\mu$ and take the limit as $|\mu| \to \infty$. We obtain for $|\mu| > 1$

$$\begin{aligned} |\sum_{n=0}^{\infty} (\lambda\mu)^{-n-1}\varphi(h^n)| &\leq |\sum_{n=0}^{\infty} (\lambda\mu)^{-n-1}\|\varphi\|\|h^n\|| \\ &\leq \frac{\|\varphi\|}{|\mu|} \sum_{n=0}^{\infty} \frac{\|h^n\|}{|\lambda|^{n+1}} |\mu|^{-n} \\ &\leq \frac{\|\varphi\|}{|\mu|} \sum_{n=0}^{\infty} \frac{\|h^n\|}{|\lambda|^{n+1}} \\ &\to 0 \end{aligned}$$

as $|\mu| \to \infty$. Consequently, the function $f : \mathbb{C} \to \mathbb{C}$ defined by

$$f(\mu) = \begin{cases} 0, \text{ if } \mu = 0, \\ \varphi((\frac{1}{\mu} 1_A - h)^{-1}), \text{ if } 0 < |\mu| < \frac{1}{r}. \end{cases}$$

is a holomorphic function on the set

$$B(0, 1/r) = \{\mu \in \mathbb{C} \mid |\mu| < \frac{1}{r}\}.$$

The Taylor expansion of f in the disk $B(0, 1/r)$ is

$$f(\mu) = \sum_{n=0}^{\infty} \mu^{n+1}\varphi(h^n).$$

Furthermore, if $\mu \in B(0, 1/r)$, then

$$\lim_{n\to\infty} \mu^{n+1}\varphi(h^n) = 0.$$

Now, we take $\lambda_0 \in \mathbb{C}$ such that $r < |\lambda_0| < \alpha(h)$. Then, $\frac{1}{\lambda_0} \in B(0, 1/r)$ and

$$\lim_{n\to\infty} \lambda_0^{-n-1}\varphi(h^n) = 0.$$

Let A^* denote the dual space of A and define for all $n \in \mathbb{Z}_{>0}$

$$\begin{array}{rccc} \rho_n : & A^* & \to & \mathbb{C} \\ & \varphi & \mapsto & \lambda_0^{-n-1}\varphi(h^n) \end{array}$$

The family $\{\rho_n\}_{n\in\mathbb{Z}_{>0}}$ is a family of continuous linear functionals on A^*. By the *uniform boundedness principle*, there exists a constant $M \in \mathbb{R}_{>0}$ such that

$$\begin{aligned}\sup_{n\in\mathbb{Z}_{>0}} |\lambda_0|^{-n-1}|\varphi(h^n)| &\leq \|\varphi\| \sup_{n\in\mathbb{Z}_{>0}} |\lambda_0|^{-n-1}\|h^n\| \\ &\leq M.\end{aligned}$$

Letting $N = M/\|\varphi\|$, we have $\sup_{n\in\mathbb{Z}_{>0}}|\lambda_0|^{-n-1}\|h^n\| \leq N$. Hence, if $n \in \mathbb{Z}_{>0}$ then

$$\|h^n\| \leq N|\lambda_0|^{n+1}$$

and from this inequality, we have

$$\begin{aligned}\alpha(h) &= \lim_{n\to\infty} \|h^n\|^{\frac{1}{n}} \\ &\leq \lim_{n\to\infty} N^{\frac{1}{n}}|\lambda_0|^{1+\frac{1}{n}} \\ &= |\lambda_0| < \alpha(h).\end{aligned}$$

This contradicts the assumption that $\alpha(h) > \sup_{\lambda\in\sigma(h)}|\lambda|$. Therefore, $\alpha(h) \leq \sup_{\lambda\in\sigma(h)}|\lambda|$.

Combining parts (a) and (b), we deduce that

$$\lim_{n\to\infty} \|h^n\|^{\frac{1}{n}} = \alpha(h) = r(h) = \sup_{\lambda\in\sigma(h)} |\lambda|.$$

□

Theorem 1.2.5 and Theorem 1.2.4 have particularly important consequences for C*-algebras, which we explore below.

Theorem 1.2.6. *Let A be a unital C*-algebra. Let $a \in A$ be normal. Then, the spectral radius $r(a) = \|a\|$.*

Proof. Assume that A is a unital C*-algebra and $a \in A$ is normal. If $n \in \mathbb{Z}_{>0}$ then a^n is also normal and

$$\|a^{2^n}\| = \|a\|^{2^n}.$$

To see why this is the case, we will proceed by induction. For the base case, if $n = 1$ then we compute directly that

$$
\begin{aligned}
\|a^2\|^2 &= \|(a^2)^*a^2\| \\
&= \|a^*(a^*a)a\| \\
&= \|a^*(aa^*)a\| \\
&= \|(a^*a)^*a^*a\| = \|a^*a\|^2 \\
&= \|a\|^4.
\end{aligned}
$$

So, $\|a^2\| = \|a\|^2$. This proves the base case. For the inductive hypothesis, assume that $\|a^{2^k}\| = \|a\|^{2^k}$ for some $k \in \mathbb{Z}_{>0}$. Then,

$$
\begin{aligned}
\|a^{2^{k+1}}\|^2 &= \|(a^{2^k})^2\|^2 \\
&= \|((a^{2^k})^2)^*(a^{2^k})^2\| \\
&= \|(a^*)^{2^k}(a^*)^{2^k}a^{2^k}a^{2^k}\| \\
&= \|(a^*)^{2^k}a^{2^k}(a^*)^{2^k}a^{2^k}\| \\
&= \|((a^*a)^{2^k})^*(a^*a)^{2^k}\| \quad \text{(since } a \text{ is normal)} \\
&= \|(a^*a)^{2^k}\|^2 = \|(a^{2^k})^*a^{2^k}\|^2 \\
&= \|a^{2^k}\|^4 = \|a\|^{2^{k+2}}.
\end{aligned}
$$

Hence, $\|a^{2^{k+1}}\| = \|a\|^{2^{k+1}}$, which completes the induction.

Now consider the sequence $\{\|a^n\|^{\frac{1}{n}}\}_{n\in\mathbb{Z}_{>0}}$ in $\mathbb{R}$. We know from Theorem 1.2.5 that this sequence converges to $r(a)$. However, the subsequence $\{\|a^{2^k}\|^{2^{-k}}\}_{k\in\mathbb{Z}_{>0}}$ is a constant sequence which converges to $\|a\|$. Therefore,

$$
r(a) = \lim_{n\to\infty}\|a^n\|^{\frac{1}{n}} = \lim_{k\to\infty}\|a^{2^k}\|^{2^{-k}} = \|a\|.
$$

□

It is remarked in [Put19] that Theorem 1.2.6 seems restrictive because it only applies to normal elements of a unital C*-algebra. However, we observe that Theorem 1.2.6 can be used to yield information about the norm of an *arbitrary* element of a unital C*-algebra. If A is a unital C*-algebra and $a \in A$ then $a^*a \in A$ is self-adjoint and by Theorem 1.2.6,

$$
\|a\| = \|a^*a\|^{\frac{1}{2}} = r(a^*a)^{\frac{1}{2}}.
$$

The next theorems demonstrate why the above observation is useful.

Theorem 1.2.7. *Let A and B be unital C*-algebras and $\phi : A \to B$ be a unital *-homomorphism. If $a \in A$ then $\|\phi(a)\| \leq \|a\|$. In particular, $\|\phi\| \leq 1$ and ϕ is called* ***contractive****.*

Proof. Assume that A and B are unital C*-algebras and $\rho : A \to B$ is a unital *-homomorphism.

First, assume that $b \in A$ is self-adjoint.

To show: (a) $\rho(b) \subseteq \rho(\phi(b))$.

(a) Assume that $\lambda \in \rho(b)$ so that $\lambda 1_A - b \in A$ is invertible. Then, there exists $(\lambda 1_A - b)^{-1} \in A$ such that

$$(\lambda 1_A - b)^{-1}(\lambda 1_A - b) = (\lambda 1_A - b)(\lambda 1_A - b)^{-1} = 1_A.$$

Applying the unital *-homomorphism ϕ to the above equation, we find that $\phi(\lambda 1_A - b) = \lambda 1_B - \phi(b)$ is invertible in B. So, $\lambda \in \rho(\phi(b))$ and $\rho(b) \subseteq \rho(\phi(b))$.

By part (a), $\sigma(\phi(b)) \subseteq \sigma(b)$. Since b and $\phi(b)$ are both self-adjoint, we have by Theorem 1.2.6

$$\|\phi(b)\| = \sup_{\lambda \in \sigma(\phi(b))} |\lambda| \leq \sup_{\lambda \in \sigma(b)} |\lambda| = \|b\|.$$

Now let $a \in A$ be an arbitrary element. We compute directly that

$$\begin{aligned} \|\phi(a)\| &= \|\phi(a^* a)\|^{\frac{1}{2}} \\ &\leq \|a^* a\|^{\frac{1}{2}} = \|a\|. \end{aligned}$$

Hence, $\phi : A \to B$ is contractive. □

Another surprising result is that the norm of a unital C*-algebra is unique.

Theorem 1.2.8. *Let A be a unital C*-algebra. Then, the norm on A is unique. That is, if a unital Banach *-algebra possesses a norm which turns it into a unital C*-algebra then this is the only such norm.*

Proof. Assume that A is a unital C*-algebra. By Theorem 1.2.6,

$$\|a\| = \|a^* a\|^{\frac{1}{2}} = (\sup\{|\lambda| \mid \lambda 1_A - a^* a \text{ is not invertible}\})^{\frac{1}{2}}.$$

The RHS of the above equation only depends on the algebraic structure of A. Hence, it uniquely determines the norm on A. □

To solidify the concepts introduced in this section, we will now work through some examples.

Example 1.2.1. We work in the C*-algebra $M_{2\times 2}(\mathbb{C})$. Let $t \in \mathbb{R}_{\geq 0}$ and

$$a = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

We will compute the spectrum, spectral radius and norm of a. The characteristic polynomial of a is

$$\det(\lambda I_2 - a) = (\lambda - 1)^2.$$

Here, I_2 is the 2×2 identity matrix. Hence, the spectrum $\sigma(a) = \{1\}$. By definition, the spectral radius of a is

$$r(a) = \sup_{\lambda \in \sigma(a)} |\lambda| = 1.$$

Computing the norm of a is more involved. Note that by a quick computation, $a^*a \neq aa^*$. Hence, Theorem 1.2.6 does not apply. Instead, we use the characterisation of the norm in Theorem 1.2.8.

We compute directly that

$$a^*a = \begin{pmatrix} 1 & t \\ t & t^2+1 \end{pmatrix}.$$

By computing the roots of the characteristic polynomial of a^*a, we find that

$$\sigma(a^*a) = \left\{ \frac{t^2 + 2 \pm t\sqrt{t^2+4}}{2} \right\}.$$

So,

$$r(a^*a) = \frac{t^2 + 2 + t\sqrt{t^2+4}}{2} = 1 + \frac{t(t + \sqrt{t^2+4})}{2}$$

and

$$\|a\| = \sqrt{r(a^*a)} = \sqrt{1 + \frac{t(t + \sqrt{t^2+4})}{2}}.$$

Example 1.2.2. Consider the $\mathbb{C}$-algebra of polynomials $\mathbb{C}[x]$. If $p(x) \in \mathbb{C}[x]$ is a non-constant polynomial then its degree is greater than zero and its spectrum is

$$\sigma(p(x)) = \mathbb{C}.$$

We claim that there is no norm on $\mathbb{C}[x]$ which makes it a C*-algebra. Suppose for the sake of contradiction that there exists a norm on $\mathbb{C}[x]$ which makes $\mathbb{C}[x]$ a C*-algebra. By Theorem 1.2.8, if $p(x) \in \mathbb{C}[x]$ is a non-constant polynomial then

$$\|p(x)\| = r(\overline{p(x)}p(x))^{\frac{1}{2}} = \infty.$$

This contradicts the fact that in a unital Banach *-algebra (or a C*-algebra), the spectral radius is bounded ($r(p(x)) \leq \|p(x)\|$) by Theorem 1.2.5. Hence, $\mathbb{C}[x]$ cannot be a C*-algebra.

Example 1.2.3. Now consider the $\mathbb{C}$-algebra of rational functions $\mathbb{C}(x)$ — the field of fractions of the polynomial ring $\mathbb{C}[x]$. If $p(x) \in \mathbb{C}(x)$ is non-constant then

$$\sigma(p(x)) = \emptyset.$$

Suppose for the sake of contradiction that there exists a norm on $\mathbb{C}(x)$ which makes $\mathbb{C}(x)$ a C*-algebra. If $p(x) \in \mathbb{C}(x)$ is non-constant then the spectrum $\sigma(p(x)) = \emptyset$, which contradicts the fact that the spectrum must be non-empty (see Theorem 1.2.5). Therefore, $\mathbb{C}(x)$ is also not a C*-algebra.

1.3 Commutative unital C*-algebras

The prototypical example of a commutative unital C*-algebra is $Cts(X, \mathbb{C})$ where X is a compact Hausdorff topological space. In this section, we will demonstrate that in a sense, this is the *only* example — every commutative unital C*-algebra is isomorphic to $Cts(X, \mathbb{C})$ for some compact Hausdorff space X. For the sake of clarity, let us define the term isomorphic in this context.

Definition 1.3.1. Let A and B be C*-algebras. We say that A and B are **isomorphic** if there exists an isometric *-isomorphism $\phi : A \to B$.

Actually, we can get away with omitting the term "isometric" in the above definition. We will eventually prove Theorem 1.6.4, which states that any injective *-homomorphism is isometric.

Definition 1.3.2. Let A be a $\mathbb{C}$-algebra. Define $\mathcal{M}(A)$ to be the set of non-zero $\mathbb{C}$-algebra homomorphisms from A to $\mathbb{C}$.

As explained in [Put19], the above notation originates from the fact that a $\mathbb{C}$-algebra homomorphism is a multiplicative linear map. We begin by proving some facts about the set $\mathcal{M}(A)$.

Theorem 1.3.1. *Let A be a commutative unital C*-algebra.*

1. *If $a \in A$ and $\phi \in \mathcal{M}(A)$ then $\phi(a) \in \sigma(a)$.*
2. *If $\phi \in \mathcal{M}(A)$ then $\|\phi\| = 1$.*
3. *If $a \in A$ and $\phi \in \mathcal{M}(A)$ then $\phi(a^*) = \overline{\phi(a)}$.*

Proof. Assume that A is a commutative unital C*-algebra. First assume that $a \in A$ and $\phi \in \mathcal{M}(A)$. Since ϕ is a non-zero $\mathbb{C}$-algebra homomorphism, the kernel $\ker \phi$ is a proper ideal of A.

We claim that if $a \in \ker \phi$ then a is not invertible. We prove the contrapositive statement. Assume that $a \in A$ is invertible so that there exists a^{-1} such that $aa^{-1} = a^{-1}a = 1_A$. By applying ϕ, we find that

$$\phi(a)\phi(a^{-1}) = \phi(1_A) = 1$$

where the last equality follows from the fact that ϕ is a $\mathbb{C}$-algebra homomorphism and thus, preserves multiplicative units. Therefore, $\phi(a) \neq 0$ and $a \notin \ker \phi$.

Now note that $\phi(a)1_A - a \in \ker \phi$ and by the previous claim, $\phi(a)1_A - a$ is not invertible in A. Consequently, $\phi(a) \in \sigma(a)$.

Next, we will show that $\|\phi\| = 1$.

To show: (a) $\|\phi\| \leq 1$.

(b) $\|\phi\| \geq 1$.

(a) Using the fact that $r(a) \leq \|a\|$, we compute directly that

$$\begin{aligned}
\|\phi\| &= \sup_{\|a\|=1} |\phi(a)| \\
&\leq \sup_{\|a\|=1} r(a) \quad (\text{since } \phi(a) \in \sigma(a)) \\
&\leq \sup_{\|a\|=1} \|a\| = 1.
\end{aligned}$$

(b) Consider the element $\phi(1_A) \in \mathbb{C}$. Since ϕ preserves multiplicative units, $\phi(1_A) = 1$. Now, $\|1\|^2 = \|\bar{1}1\| = \|1\|$. So, $\|1\| = 1$ and

$$\|\phi\| = \sup_{\|a\|=1} |\phi(a)| \geq |\phi(1_A)| = 1.$$

By combining parts (a) and (b), we deduce that $\|\phi\| = 1$.

For the final part of the proof, take $a \in A$ and write $a = b + ic$ where

$$b = \frac{1}{2}(a + a^*) \qquad \text{and} \qquad c = \frac{i(a^* - a)}{2}.$$

Note that b and c are both self-adjoint elements of A. So, it suffices to prove that if $b = b^*$ then $\phi(b) \in \mathbb{R}$. To this end, assume that $b \in A$ is self-adjoint.

For $t \in \mathbb{R}$, define

$$u_t = e^{itb} = \sum_{n=0}^{\infty} \frac{(itb)^n}{n!}$$

The norm of u_t is finite because

$$\begin{aligned}
\|u_t\| &= \|\sum_{n=0}^{\infty} \frac{(itb)^n}{n!}\| \\
&\leq \sum_{n=0}^{\infty} \frac{|t|^n \|b\|^n}{n!} \\
&= e^{|t|\|b\|} < \infty.
\end{aligned}$$

So, u_t is a well-defined element of A. Furthermore, one can verify that $u_{-t} = u_t^*$ and $u_t u_{-t} = 1_A$. So,

$$\|u_t\| = \|u_t u_t^*\|^{\frac{1}{2}} = \|u_t u_{-t}\| = \|1_A\| = 1.$$

Since $\|\phi\| = 1$ (from the second part of the theorem), if $t \in \mathbb{R}$ then

$$1 = \|\phi\| \geq |\phi(u_t)| = |e^{it\phi(b)}|.$$

Thus, $\phi(b) \in \mathbb{R}$ which completes the proof because if $a = b + ic \in A$ then

$$\phi(a^*) = \phi(b - ic) = \phi(b) - i\phi(c) = \overline{\phi(b) + i\phi(c)} = \overline{\phi(a)}.$$

$\square$

At this point, one might suspect that the set $\mathcal{M}(A)$ plays an important role in constructing the isomorphism alluded to in the beginning of this section. To elucidate this point, we need to better understand $\mathcal{M}(A)$ as a topological space, rather than just a set. Theorem 1.3.1 provides the first step towards this goal. In particular, the second statement in Theorem 1.3.1 reveals that

$$\mathcal{M}(A) \subseteq \{\psi \in A^* \mid \|\psi\| \leq 1\}.$$

Now recall that Banach-Alaoglu theorem, which states that the closed unit ball

$$\{\psi \in A^* \mid \|\psi\| \leq 1\} \subseteq A^*$$

is compact with respect to the weak-* topology on the dual space A^*. This suggests that we consider $\mathcal{M}(A)$ with the weak-* topology.

Here is a refresher on the weak-* topology.

Definition 1.3.3. Let X be a Banach space and X^* be its dual space. The **weak-* topology** on X^* is the weakest topology on X^* such that the evaluation maps

$$\begin{array}{rccc} ev_x : & X^* & \to & \mathbb{C} \\ & \varphi & \mapsto & \varphi(x) \end{array}$$

are continuous for $x \in X$.

Recall that a net $\{\phi_n\}_{n\in I}$ in the dual space X^* converges to $\phi \in X^*$ in the weak-* topology if and only if for $x \in X$, the net $\{\phi_n(x)\}_{n\in I}$ converges to $\phi(x)$ in $\mathbb{C}$.

Theorem 1.3.2. *Let A be a commutative unital C^*-algebra. The set $\mathcal{M}(A)$ is a compact subset of the closed unit ball*

$$\{\psi \in A^* \mid \|\psi\| \leq 1\} \subseteq A^*$$

with respect to the weak-* topology on A^*.

Proof. Assume that A is a commutative unital C*-algebra. As mentioned previously, Theorem 1.3.1 yields the inclusion

$$\mathcal{M}(A) \subseteq \{\psi \in A^* \mid \|\psi\| \leq 1\}.$$

By the Banach-Alaoglu theorem, the closed unit ball is compact with respect to the weak-* topology on A^*.

To show: (a) $\mathcal{M}(A)$ is a closed subset of the closed unit ball of A^*.

(a) Assume that $\phi \in \overline{\mathcal{M}(A)}$. Then, there exists a net $\{\phi_n\}_{n\in I}$ which converges to ϕ with respect to the weak-* topology. This means that if $x \in X$ then the net $\{\phi_n(x)\}_{n\in I}$ in $\mathbb{C}$ converges to $\phi(x)$.

By definition, ϕ is a linear functional on A. To see that ϕ is non-zero, we compute the norm of ϕ as

$$\begin{aligned}\|\phi\| &= \sup_{\|x\|=1} |\phi(x)| \\ &= \sup_{\|x\|=1} \lim_{n\in I} |\phi_n(x)| \\ &= \lim_{n\in I} \sup_{\|x\|=1} |\phi_n(x)| \\ &= \lim_{n\in I} \|\phi_n\| = 1.\end{aligned}$$

Finally, to see that ϕ is multiplicative, we have for $a, b \in A$ that

$$\begin{aligned}\phi(ab) &= \lim_{n\in I} \phi_n(ab) \\ &= \lim_{n\in I} \phi_n(a)\phi_n(b) \\ &= \phi(a)\phi(b).\end{aligned}$$

Therefore, $\phi \in \mathcal{M}(A)$ and $\overline{\mathcal{M}(A)} = \mathcal{M}(A)$. So, $\mathcal{M}(A)$ is a closed subset of the closed unit ball of A^* with respect to the weak-* topology.

Since $\mathcal{M}(A)$ is a closed subset of a compact set, it must be compact with respect to the weak-* topology as required. □

To see that $\mathcal{M}(A)$ is Hausdorff, note that A^* with the weak-* topology is Hausdorff. Indeed, if $\phi, \psi \in A^*$ such that $\phi \neq \psi$ then there exists $x \in A$ such that

$$ev_x(\phi) = \phi(x) \neq \psi(x) = ev_x(\psi).$$

Hence, $\mathcal{M}(A)$ is a subspace of a Hausdorff space and is thus, Hausdorff. Therefore, $\mathcal{M}(A)$ is a compact Hausdorff topological space. Since we want to show that every commutative unital C*-algebra is of the form $Cts(X, \mathbb{C})$ for some compact Hausdorff space X, we will show that A as a commutative unital C*-algebra is isomorphic to $Cts(\mathcal{M}(A), \mathbb{C})$.

Theorem 1.3.3. *Let A be a commutative unital C*-algebra and $a \in A$. Then, the evaluation map*

$$\begin{array}{rccc} ev_a : & \mathcal{M}(A) & \to & \mathbb{C} \\ & \phi & \mapsto & \phi(a) \end{array}$$

is a continuous map from $\mathcal{M}(A)$ onto the spectrum $\sigma(a)$.

Proof. Assume that A is a commutative unital C*-algebra. Assume that $a \in A$. By definition of the weak-* topology on A^*, the evaluation map $ev_a : A^* \to \mathbb{C}$ is continuous. Furthermore, recall from Theorem 1.3.1 that if $\phi \in \mathcal{M}(A)$ then $ev_a(\phi) = \phi(a) \in \sigma(a)$.

To show: (a) If $\lambda \in \sigma(a)$ then there exists $\psi \in A^*$ such that $ev_a(\psi) = \lambda$.

(a) Assume that $\lambda \in \sigma(a)$. Define the set S by

$$S = \{\text{Ideals } I \subsetneq A \mid \lambda 1_A - a \in I\}.$$

Then, S is a poset when equipped with the partial order of inclusion. It is non-empty because $\lambda 1_A - a$ is an element of the ideal $(\lambda 1_A - a)$ generated by $\lambda 1_A - a$.

Now let $S' \subseteq S$ be a totally ordered subset of S. Define

$$J = \sum_{I \in S'} I.$$

Then, J is an ideal of A because it is the sum of ideals of A. Moreover if $I \in S'$ then $I \subseteq J$. Hence, J is an upper bound for the totally ordered set S'.

By Zorn's lemma, there exists a maximal element $I_{max} \in S$. This means that $\lambda 1_A - a$ is contained in the maximal proper ideal $I_{max} \subseteq A$. We now claim that I_{max} is closed.

To show: (aa) I_{max} is closed.

(aa) The closure $\overline{I_{max}}$ is an ideal of A such that $I_{max} \subseteq \overline{I_{max}}$. We claim that $\overline{I_{max}}$ is a proper ideal of A. To see why, it suffices to show that the unit $1_A \notin \overline{I_{max}}$.

Suppose for the sake of contradiction that $1_A \in \overline{I_{max}}$. Then, $\overline{I_{max}} = A$. Let V denote the set of invertible elements of A. Then, $V \subseteq \overline{I_{max}} = A$. Since

I_{max} is a proper ideal of A, it cannot contain any invertible elements of A (as otherwise, $1_A \in I_{max}$). Hence, $V \cap I_{max} = \emptyset$.

Now since V is an open subset of A, there exists $\epsilon \in \mathbb{R}_{>0}$ such that the open ball

$$B(1_A, \epsilon) = \{a \in V \mid \|1_A - a\| < \epsilon\} \subseteq V.$$

By assumption, $1_A \in \overline{I_{max}}$. So, $B(1_A, \epsilon) \cap I_{max} \neq \emptyset$. However, this means that $V \cap I_{max} \neq \emptyset$, which contradicts the fact that $V \cap I_{max} = \emptyset$. Therefore, $1_A \notin \overline{I_{max}}$ and $\overline{I_{max}}$ is a proper ideal of A.

In particular, $\overline{I_{max}} \in S$ and $I_{max} \subseteq \overline{I_{max}}$. Since I_{max} is maximal in S, $I_{max} = \overline{I_{max}}$ and consequently, I_{max} is a closed ideal of A.

(a) The idea behind part (aa) is that since I_{max} is a closed maximal ideal of A, the quotient A/I_{max} is both a Banach space and a field. Hence, A/I_{max} is a Banach *-algebra with quotient norm

$$\|a + I_{max}\| = \inf_{j \in I_{max}} \|a + j\|.$$

To show: (ab) As Banach *-algebras, $A/I_{max} \cong \mathbb{C}$.

(ab) Let $b + I_{max} \in A/I_{max}$. Since A/I_{max} is a Banach *-algebra, then by Theorem 1.2.5 the spectrum $\sigma(b + I_{max}) \neq \emptyset$. So, there exists $\mu \in \mathbb{C}$ such that $\mu 1_A - b - I_{max}$ is not invertible. Since A/I_{max} is a field, the only non-invertible element of A/I_{max} is $0 + I_{max}$. Therefore,

$$\mu 1_A - b - I_{max} = 0 + I_{max}$$

and $b + I_{max} = \mu 1_A + I_{max} = \mu(1_A + I_{max})$. This means that every element of A/I_{max} is a scalar multiple of the unit $1_A + I_{max} \in A/I_{max}$. Thus, we have the isomorphism of Banach *-algebras, $A/I_{max} \cong \mathbb{C}$.

(a) Following on from part (ab), consider the quotient map

$$\begin{array}{rccc} \pi : & A & \to & A/I_{max} \\ & b & \mapsto & b + I_{max} = \mu(1_A + I_{max}). \end{array}$$

This is a continuous map. Next, let $\delta : A/I_{max} \to \mathbb{C}$ be the isomorphism given by

$$\delta(b + I_{max}) = \delta(\mu(1_A + I_{max})) = \mu.$$

Then, the composite $\delta \circ \pi \in A^*$ and since $\lambda 1_A - a \in I_{max}$ and $\lambda \in \sigma(a)$ by assumption, then $\lambda 1_A + I_{max} = a + I_{max}$ and

$$(\delta \circ \pi)(a) = \delta(a + I_{max}) = \delta(\lambda 1_A + I_{max}) = \delta(\lambda(1_A + I_{max})) = \lambda.$$

So, $ev_a(\delta \circ \pi) = \lambda$ which demonstrates that the image of ev_a is the spectrum $\sigma(a)$ as required. $\square$

Of course, we do not expect the evaluation map in Theorem 1.3.3 to be injective. Under a particular hypothesis, the evaluation map becomes injective, as we will see in the next theorem.

Theorem 1.3.4. *Let A be a commutative unital C*-algebra. Assume that there exists $a \in A$ such that A is generated as a C*-algebra by the set $\{1_A, a, a^*\}$. Then, the evaluation map*

$$\begin{array}{rccc} ev_a: & \mathcal{M}(A) & \to & \mathbb{C} \\ & \phi & \mapsto & \phi(a) \end{array}$$

is a homeomorphism from $\mathcal{M}(A)$ to the spectrum $\sigma(a)$.

Proof. Assume that A is a commutative unital C*-algebra. Assume that there exists $a \in A$ such that A is generated as a C*-algebra by the set $\{1_A, a, a^*\}$.

By Theorem 1.3.3, the evaluation map ev_a is continuous and surjective onto the spectrum $\sigma(a)$.

To show: (a) ev_a is injective.

(a) Assume that $\phi, \psi \in \mathcal{M}(A)$ such that $\phi(a) = \psi(a)$. By Theorem 1.3.1, $\phi(a^*) = \overline{\phi(a)} = \overline{\psi(a)} = \psi(a^*)$. Since ϕ, ψ are non-zero $\mathbb{C}$-algebra homomorphisms, $\phi(1_A) = 1 = \psi(1_A)$. So, ϕ and ψ agree on the generating set $\{1_A, a, a^*\}$ for A. Hence, $\phi = \psi$ in $\mathcal{M}(A)$ and ev_a is injective. $\square$

Now, we are ready to prove our main result.

Theorem 1.3.5. *Let A be a commutative unital C*-algebra. Define the map*

$$\begin{array}{rccc} \Lambda: & A & \to & Cts(\mathcal{M}(A), \mathbb{C}) \\ & a & \mapsto & ev_a \end{array}$$

*where ev_a is the evaluation map in Theorem 1.3.3. Then, Λ is an isometric *-isomorphism from A to $Cts(\mathcal{M}(A), \mathbb{C})$.*

Proof. Assume that A is a commutative unital C*-algebra. Assume that Λ is the map defined as above. First observe that Λ is well-defined because if $a \in A$ then $\Lambda(a) = ev_a$ is a continuous function by definition of the weak-* topology on $\mathcal{M}(A)$.

It is straightforward to verify that Λ is a $\mathbb{C}$-algebra homomorphism. If $a_1, a_2 \in A$, $\mu \in \mathbb{C}$ and $\phi \in \mathcal{M}(A)$ then

$$\begin{aligned}\Lambda(a_1 + a_2)(\phi) &= ev_{a_1+a_2}(\phi) \\ &= \phi(a_1 + a_2) \\ &= \phi(a_1) + \phi(a_2) \\ &= ev_{a_1}(\phi) + ev_{a_2}(\phi) = \Lambda(a_1)(\phi) + \Lambda(a_2)(\phi),\end{aligned}$$

$$\begin{aligned}\Lambda(\mu a_1)(\phi) &= ev_{\mu a_1}(\phi) \\ &= \phi(\mu a_1) \\ &= \mu\phi(a_1) \\ &= \mu ev_{a_1}(\phi) = \mu\Lambda(a_1)(\phi)\end{aligned}$$

and

$$\begin{aligned}\Lambda(a_1 a_2)(\phi) &= ev_{a_1 a_2}(\phi) \\ &= \phi(a_1 a_2) \\ &= \phi(a_1)\phi(a_2) \\ &= ev_{a_1}(\phi)ev_{a_2}(\phi) = \Lambda(a_1)(\phi)\Lambda(a_2)(\phi).\end{aligned}$$

To see that Λ is a *-homomorphism, we have by Theorem 1.3.1

$$\begin{aligned}\Lambda(a_1^*)(\phi) &= ev_{a_1^*}(\phi) \\ &= \phi(a_1^*) = \overline{\phi(a_1)} \\ &= \overline{ev_{a_1}(\phi)} \\ &= \overline{\Lambda(a_1)(\phi)} = \Lambda(a_1)^*(\phi).\end{aligned}$$

Next, we will show that Λ is isometric.

To show: (a) If $a \in A$ then $\|\Lambda(a)\| = \|a\|$.

(a) Assume that $a \in A$. First assume that a is self-adjoint. Using Theorem 1.2.6, we compute directly that

$$\begin{aligned}\|a\| &= r(a)\\ &= \sup_{\lambda\in\sigma(a)} |\lambda|\\ &= \sup_{\phi\in\mathcal{M}(A)} |\phi(a)| \quad \text{(by Theorem 1.3.3)}\\ &= \|ev_a\| = \|\Lambda(a)\|.\end{aligned}$$

Now assume that $a \in A$ is an arbitrary element. Then,

$$\|a\| = \|a^*a\|^{\frac{1}{2}} = \|\Lambda(a^*a)\|^{\frac{1}{2}} = \|\Lambda(a)^*\Lambda(a)\|^{\frac{1}{2}} = \|\Lambda(a)\|.$$

Finally, we show that the image of Λ is $Cts(\mathcal{M}(A), \mathbb{C})$.

To show: (b) im $\Lambda = Cts(\mathcal{M}(A), \mathbb{C})$.

(b) The idea here is that since $\mathcal{M}(A)$ with the weak-* topology is a compact Hausdorff topological space, we could potentially use the Stone-Weierstrass theorem on $Cts(\mathcal{M}(A), \mathbb{C})$.

Here is how the theorem applies. The image of Λ is a unital *-subalgebra of $Cts(\mathcal{M}(A), \mathbb{C})$. To see that im Λ separates points, let $\phi, \psi \in \mathcal{M}(A)$ such that $\phi \neq \psi$. Then, there exists $a \in A$ such that $\phi(a) \neq \psi(a)$. This means that $\Lambda(a)(\phi) \neq \Lambda(a)(\psi)$. So, the image im Λ separates points.

Now we can apply the Stone-Weierstrass theorem to deduce that im Λ is dense in $Cts(\mathcal{M}(A), \mathbb{C})$. Since Λ is an isometry and A is complete, we find that im Λ is closed. Hence,

$$\text{im } \Lambda = \overline{\text{im } \Lambda} = Cts(\mathcal{M}(A), \mathbb{C}).$$

This completes the proof. $\square$

Next, we will consider how Theorem 1.3.5 applies to a unital, not necessarily commutative C*-algebra B. Let $a \in B$ and A be the C*-subalgebra of B with generating set $\{1_B, a, a^*\}$. Observe that A is commutative whenever a and a^* commute with each other — that is, when a is normal. As a more formal statement, $a \in B$ is normal if and only if there exists a commutative unital C*-subalgebra $A \subseteq B$ such that $a \in A$.

If we apply Theorem 1.3.5 to A, we obtain the C*-algebra isomorphism $A \cong Cts(\mathcal{M}(A), \mathbb{C})$. Moreover, Theorem 1.3.4 yields a homeomorphism from $\mathcal{M}(A)$ with the weak-* topology to the spectrum $\sigma(a)$. However, by construction of A as a C*-subalgebra of B, the spectrum $\sigma(a)$ here is ill-defined. The problem is that if $c \in A$ then c could have an inverse in B, but not in A. That is, the invertibility of c depends on whether we consider c an element of A or an element of B.

We will prove below that in this special case, the above observation is not an issue — invertibility in A is equivalent to invertibility in B. Since the spectrum of c depends on whether we think of it in A or B, we introduce the notation

$$\sigma_A(c) = \{\lambda \in \mathbb{C} \mid \lambda 1_B - c \text{ is not invertible in } A\}.$$

and

$$\sigma_B(c) = \{\lambda \in \mathbb{C} \mid \lambda 1_B - c \text{ is not invertible in } B\}.$$

Note that in our case, $\sigma_B(c) \subseteq \sigma_A(c)$.

Theorem 1.3.6. *Let B be a unital C*-algebra and A be a commutative C*-subalgebra of B such that $1_B \in A$. Let $a \in A$. Then, a has an inverse in A if and only if it has an inverse in B. Consequently, $\sigma_A(a) = \sigma_B(a)$.*

Proof. Assume that B is a unital C*-algebra and A is a commutative C*-subalgebra of B such that $1_B \in A$. Assume that $a \in A$. We already know that $\sigma_B(a) \subseteq \sigma_A(a)$. This means that if a is invertible in A then it is invertible in B.

To show: (a) If a is invertible in B then it is invertible in A.

(a) Assume that a has an inverse in B. Its adjoint a^* must have inverse given by $(a^*)^{-1} = (a^{-1})^*$. Therefore, the self-adjoint element a^*a is invertible. By Theorem 1.3.3 and the first statement of Theorem 1.3.1, we have

$$\sigma_A(a^*a) = \{\phi(a^*a) \mid \phi \in \mathcal{M}(A)\}.$$

By the third statement of Theorem 1.3.1, we have

$$\phi(a^*a) = \phi(a^*)\phi(a) = \overline{\phi(a)}\phi(a) = |\phi(a)|^2 \in \mathbb{R}_{\geq 0}.$$

The second statement of Theorem 1.3.1 tells us that if $\phi \in \mathcal{M}(A)$ then $\|\phi\| = 1$ and

$$\frac{|\phi(a^*a)|}{\|a^*a\|} \leq \sup_{x\neq 0} \frac{|\phi(x)|}{\|x\|} = \|\phi\| = 1.$$

So, $|\phi(a^*a)| = \phi(a^*a) \leq \|a^*a\|$. By combining the previous two observations, we find that $0 \leq \phi(a^*a) \leq \|a^*a\|$ and

$$\sigma_B(a^*a) \subseteq \sigma_A(a^*a) \subseteq [0, \|a^*a\|].$$

Now a^*a is invertible in B. So, $0 \notin \sigma_B(a^*a)$. Thus, there exists $\delta \in \mathbb{R}_{>0}$ such that

$$\sigma_B(a^*a) \subseteq [\delta, \|a^*a\|].$$

Now consider the spectrum $\sigma_B(\|a^*a\|1_B - a^*a)$. We want to show that it is contained in a particular interval of $\mathbb{R}_{\geq 0}$. The key is to notice that if $\lambda \in \mathbb{C}$ then

$$\lambda 1_B - (\|a^*a\|1_B - a^*a) = -\big((\|a^*a\| - \lambda)1_B - a^*a\big).$$

Therefore,

$$\sigma_B(\|a^*a\|1_B - a^*a) \subseteq \{\|a^*a\| - \lambda \mid \lambda \in \sigma_B(a^*a)\} \subseteq [0, \|a^*a\| - \delta].$$

Now the element $\|a^*a\|1_B - a^*a$ is self-adjoint. By Theorem 1.2.6,

$$\big\|\|a^*a\|1_B - a^*a\big\| = r(\|a^*a\|1_B - a^*a) \leq \|a^*a\| - \delta < \|a^*a\|.$$

By the second part of Theorem 1.3.1, if $\phi \in \mathcal{M}(A)$ then

$$|\phi(\|a^*a\|1_B - a^*a)| \leq \big\|\|a^*a\|1_B - a^*a\big\| < \|a^*a\|.$$

But,

$$\phi(\|a^*a\|1_B - a^*a) = \|a^*a\| - |\phi(a)|^2.$$

Subsequently, $\phi(a) \neq 0$ for $\phi \in \mathcal{M}(A)$. By Theorem 1.3.3 and Theorem 1.3.1, we deduce that $0 \notin \sigma_A(a)$. Therefore, a has an inverse in A.

So, $a \in A$ has an inverse in B if and only if it has an inverse in A. The equality of spectra $\sigma_A(a) = \sigma_B(a)$ follows directly. □

Now we return to the situation where B is a unital C*-algebra and A is a commutative unital C*-subalgebra of B, generated by the set $\{1_B, a, a^*\}$ for some normal element $a \in B$. The next definition we make exploits the fact that we have an inverse to the isometric *-isomorphism in Theorem 1.3.5.

Definition 1.3.4. Let B be a unital C*-algebra and $a \in B$ be normal. Let A be the commutative unital C*-subalgebra of B generated by the set $\{1_B, a\}$. By Theorem 1.3.4, we have a homeomorphism $\mathcal{M}(A) \cong \sigma(a)$. Hence, we can identify $Cts(\mathcal{M}(A), \mathbb{C})$ with the space of continuous functions $Cts(\sigma(a), \mathbb{C})$.

Let $\Lambda : A \to Cts(\sigma(a), \mathbb{C})$ denote the isometric *-isomorphism in Theorem 1.3.5 and $f \in Cts(\sigma(a), \mathbb{C})$. Since Λ is surjective, we let $f(a)$ be the unique element of A such that $\Lambda(f(a)) = f$. That is, if $\phi \in \mathcal{M}(A)$ then $\phi(a) \in \sigma(a)$ and

$$\phi(f(a)) = ev_{f(a)}(\phi) = \Lambda(f(a))(\phi) = f(\phi(a)).$$

Using the above definition, we can restate Theorem 1.3.5 as follows:

Theorem 1.3.7 (Continuous functional calculus)**.** *Let B be a unital C*-algebra and $a \in B$ be normal. The map*

$$\begin{array}{rccc} \Lambda^{-1}: & Cts(\sigma(a), \mathbb{C}) \cong Cts(\mathcal{M}(A), \mathbb{C}) & \to & B \\ & f & \mapsto & f(a) \end{array}$$

*is an isometric *-isomorphism from $Cts(\sigma(a), \mathbb{C})$ to the C*-subalgebra of B generated by the set $\{1, a\}$. Moreover, if $f(z) = \sum_{k,l} a_{k,l} z^k \overline{z}^l$ is any polynomial in the variables z and $\overline{z}$ then*

$$\Lambda^{-1}(f(z)) = f(a) = \sum_{k,l} a_{k,l} a^k (a^*)^l.$$

Theorem 1.3.7 is the well-known continuous functional calculus on normal elements of a unital C*-algebra. In the reference [Sol18], the continuous functional calculus is done for the unital C*-algebra of bounded linear operators on a Hilbert space. The continuous functional calculus is first proved for self-adjoint operators and then after establishing the notion of a *joint spectrum*, it is extended to normal operators.

Here are some important consequences of Theorem 1.3.7.

Theorem 1.3.8. *Let B be a unital C*-algebra and $a \in B$ be normal. Then, a is self-adjoint if and only if $\sigma(a) \subseteq \mathbb{R}$.*

Proof. Assume that B is a unital C*-algebra and $a \in B$ is normal. Let $id_{\sigma(a)}$ denote the identity function on the spectrum $\sigma(a) \subseteq \mathbb{C}$. Then, $id_{\sigma(a)} \in Cts(\sigma(a), \mathbb{C})$ and by the continuous functional calculus in Theorem 1.3.7,

$$\Lambda^{-1}(id_{\sigma(a)}) = a.$$

So, a is self-adjoint if and only if $a = a^*$ if and only if $id_{\sigma(a)} = \overline{id_{\sigma(a)}}$ if and only if $\sigma(a) \subseteq \mathbb{R}$. □

Theorem 1.3.9. *Let A and B be unital C*-algebras. Let $\phi : A \to B$ be a unital *-homomorphism. If $a \in A$ is normal then $\sigma(\phi(a)) \subseteq \sigma(a)$. Moreover, if $f \in Cts(\sigma(a), \mathbb{C})$ then $f(\phi(a)) = \phi(f(a))$.*

Proof. Assume that A and B are unital C*-algebras. Assume that $\phi : A \to B$ is a unital *-homomorphism.

To show: (a) If $a \in A$ is normal then $\sigma(\phi(a)) \subseteq \sigma(a)$.

(b) If $f \in Cts(\sigma(a), \mathbb{C})$ then $f(\phi(a)) = \phi(f(a))$.

(a) Assume that $a \in A$ is normal. If $\lambda \in \rho(a)$ then $\lambda 1_A - a$ is invertible and $\phi(\lambda 1_A - a) = \lambda 1_B - \phi(a)$ is invertible in B. Hence, $\rho(a) \subseteq \rho(\phi(a))$ and $\sigma(\phi(a)) \subseteq \sigma(a)$.

(b) Assume that $f \in Cts(\sigma(a), \mathbb{C})$. Let C be the C*-subalgebra of A generated by the set $\{1_A, a\}$ and D be the C*-subalgebra of B generated by the set $\{1_B, \phi(a)\}$. Then, the restriction $\phi|_C : C \to D$ is a unital *-homomorphism.

Using the notation in Theorem 1.3.7, let $\Lambda_C : C \to Cts(\sigma(a), \mathbb{C})$ and $\Lambda_D : D \to Cts(\sigma(\phi(a)), \mathbb{C})$ denote the continuous functional calculi on C and D respectively. To see that $\phi(f(a)) = f(\phi(a))$, assume that $\psi \in \mathcal{M}(D)$. Then $\psi \circ \phi \in \mathcal{M}(C)$ and

$$\begin{aligned}
\Lambda_D(\phi(f(a)))(\psi) &= ev_{\phi(f(a))}(\psi) = \psi(\phi(f(a))) \\
&= (\psi \circ \phi)(f(a)) \\
&= ev_{f(a)}(\psi \circ \phi) = \Lambda_C(f(a))(\psi \circ \phi) \\
&= f((\psi \circ \phi)(a)) \\
&= f(\psi(\phi(a))) = \Lambda_D(f(\phi(a)))(\psi).
\end{aligned}$$

Since Λ_D is injective and $\psi \in \mathcal{M}(D)$ was arbitrary, we must have $\phi(f(a)) = f(\phi(a))$. □

Next, we would like to extend Theorem 1.2.7, which states that a unital *-homomorphism between C*-algebras is necessarily contractive. We will prove that if the unital *-homomorphism is also injective then it is necessarily an isometry.

Theorem 1.3.10. *Let A and B be unital C*-algebras and $\phi : A \to B$ be an injective, unital *-homomorphism. If $a \in A$ is normal then $\sigma(a) = \sigma(\phi(a))$.*

Proof. Assume that A and B are unital C*-algebras. Assume that $\phi : A \to B$ is an injective unital *-homomorphism. Assume that $a \in A$ is normal. We already have the inclusion $\sigma(\phi(a)) \subseteq \sigma(a)$.

Suppose for the sake of contradiction that $\sigma(\phi(a)) \subsetneq \sigma(a)$. Then, there exists a continuous function $f \in Cts(\sigma(a), \mathbb{C})$ such that $f \neq 0$, but if $\lambda \in \sigma(\phi(a))$ then $f(\lambda) = 0$.

Now by Theorem 1.3.9, $f(\phi(a)) = \phi(f(a))$. By the continuous functional calculus in Theorem 1.3.7 and the fact that the restriction $f|_{\sigma(\phi(a))} = 0$, $f(\phi(a)) = 0$. This contradicts the assumption that ϕ is injective. Therefore, $\sigma(\phi(a)) = \sigma(a)$. $\square$

Theorem 1.3.11. *Let A and B be unital C*-algebras. Let $\phi : A \to B$ be an injective *-homomorphism. Then, ϕ is an isometry.*

Proof. Assume that A and B are unital C*-algebras. Assume that $\phi : A \to B$ is an injective *-homomorphism.

To show: (a) If $a \in A$ is self-adjoint then $\|\phi(a)\| = \|a\|$.

(a) Assume that $a \in A$ is self-adjoint. By Theorem 1.2.6 and Theorem 1.3.10, we have

$$\|a\| = r(a) = \sup_{\lambda \in \sigma(a)} |\lambda| = \sup_{\lambda \in \sigma(\phi(a))} |\lambda| = r(\phi(a)) = \|\phi(a)\|.$$

Now assume that $a \in A$ is an arbitrary element. Then,

$$\|a\| = \|a^*a\|^{\frac{1}{2}} = \|\phi(a^*a)\|^{\frac{1}{2}} = \|\phi(a)^*\phi(a)\|^{\frac{1}{2}} = \|\phi(a)\|.$$

Hence, ϕ is an isometry. $\square$

Now recall Theorem 1.3.6, which states that if B is a unital C*-algebra and A is a commutative C*-subalgebra containing the unit then an element $a \in A$ is invertible in A if and only if it is invertible in B. We will now extend this result slightly, by removing the commutativity assumption.

Theorem 1.3.12 (Spectral permanence). *Let B be a unital C*-algebra and A be a C*-subalgebra of B such that $1_B \in A$. If $a \in A$ then a is invertible in A if and only if a is invertible in B. That is, the spectrums $\sigma_A(a) = \sigma_B(a)$.*

Proof. Assume that B is a unital C*-algebra and A is a C*-subalgebra of B. Assume that $1_B \in A$. Assume that $a \in A$. If a is invertible in A then a is invertible in B since $A \subseteq B$.

To show: (a) If a has inverse $a^{-1} \in B$ then $a^{-1} \in A$.

(a) First assume that $a \in A$ is normal and has inverse $a^{-1} \in B$. The inclusion map $\iota : A \hookrightarrow B$ is an injective unital *-homomorphism. By Theorem 1.3.10, we find that $\sigma_A(a) = \sigma_B(a)$. In particular, $0 \notin \sigma_A(a)$ if and only if $0 \notin \sigma_B(a)$. So, a is invertible in A if and only if a is invertible in B.

Now assume that $a \in A$ is arbitrary. If a has an inverse $a^{-1} \in B$ then a^* also has an inverse in B. So, the normal element $a^*a \in A$ must have an inverse in B. But by the previous case, the inverse $(a^*a)^{-1} \in A$. Consequently,

$$a^{-1} = (a^*a)^{-1}(a^*a)a^{-1} = (a^*a)^{-1}a^* \in A.$$

Hence, $a^{-1} \in A$ which completes the proof. □

A particularly useful consequence of the continuous functional calculus is the spectral mapping theorem, which yields a straightforward way of computing the spectrum of elements obtained by Theorem 1.3.7. We first need the following result.

Theorem 1.3.13. *Let X be a compact Hausdorff space and $f \in Cts(X, \mathbb{C})$. Then, the spectrum $\sigma(f) = im\ f$.*

Proof. Assume that X is a compact Hausdorff space and $f \in Cts(X, \mathbb{C})$. Let $\mathbb{1} \in Cts(X, \mathbb{C})$ be the function which sends $x \in X$ to 1. Then, $\mathbb{1}$ is the multiplicative unit in the C*-algebra $Cts(X, \mathbb{C})$.

First, assume that $\lambda \in \sigma(f)$. Then, $\lambda\mathbb{1} - f$ is not invertible and consequently, there exists $x \in X$ such that

$$\lambda - f(x) = (\lambda\mathbb{1} - f)(x) = 0.$$

So, $f(x) = \lambda$ and $\lambda \in \text{im } f$. This shows that $\sigma(f) \subseteq \text{im } f$.

Conversely, assume that $\lambda \in \text{im } f$. Then, there exists $y \in X$ such that $\lambda = f(y)$. So,

$$(\lambda\mathbb{1} - f)(y) = \lambda - f(y) = 0.$$

This means that the function $\frac{1}{(\lambda\mathbb{1}-f)}$ is not defined and hence, $\lambda\mathbb{1} - f$ is not invertible. So, $\lambda \in \sigma(f)$ and $\text{im } f \subseteq \sigma(f)$. We conclude that $\sigma(f) = \text{im } f$. □

Now we will state and prove the spectral mapping theorem.

Theorem 1.3.14 (Spectral mapping theorem)**.** *Let B be a unital C*-algebra and $a \in B$ be normal. If $f \in Cts(\sigma(a), \mathbb{C})$ then*

$$\sigma(f(a)) = f(\sigma(a)) = \{f(\lambda) \mid \lambda \in \sigma(a)\}.$$

Proof. Assume that B is a unital C*-algebra and $a \in B$ is normal. Assume that $f \in Cts(\sigma(a), \mathbb{C})$.

The point is that the spectrum $\sigma(a)$ is a compact Hausdorff space with the subspace topology induced from the usual topology on $\mathbb{C}$. Applying the known result to $f \in Cts(\sigma(a), \mathbb{C})$, we find that the spectrum of f is the image $f(\sigma(a))$.

Now let A be the C*-subalgebra of B generated by the set $\{1_B, a\}$. By the continuous functional calculus in Theorem 1.3.7, the map $f \mapsto f(a)$ is an isometric *-isomorphism from $Cts(\sigma(a), \mathbb{C})$ to A. So, $\sigma(f) = \sigma_A(f(a))$. Finally by Theorem 1.3.12, $\sigma_A(f(a)) = \sigma_B(f(a))$. Therefore,

$$\sigma_B(f(a)) = \sigma_A(f(a)) = \sigma(f) = f(\sigma(a))$$

where the last equality follows from Theorem 1.3.13. □

1.4 Positive elements

Let A be a C*-algebra. Recall that an element $a \in A$ is positive if there exists $b \in A$ such that $a = b^*b$. Note that by definition, any positive element is self-adjoint. In the last section, we proved Theorem 1.3.8, which states that a normal element $c \in A$ is self-adjoint if and only if $\sigma(c) \subseteq \mathbb{R}$. In this section, we will prove a similar characterisation of positive elements of a C*-algebra.

Theorem 1.4.1. *Let A be a unital C*-algebra and $a \in A$ be self-adjoint. For $x \in \mathbb{R}$, let $f(x) = \max\{x, 0\}$ and $g(x) = \max\{-x, 0\}$. Then, $f(a)$ and $g(a)$ are positive elements of A which satisfy*

1. $a = f(a) - g(a)$,
2. $af(a) = f(a)^2$,
3. $ag(a) = -g(a)^2$,
4. $f(a)g(a) = 0$.

Moreover, if $\sigma(a) \subseteq [0, \infty)$ then a is positive.

Proof. Assume that A is a unital C*-algebra and $a \in A$ is self-adjoint. Assume that f and g are the functions defined as above. Then, f and g restrict to continuous real-valued functions on the spectrum $\sigma(a)$, which is contained in $\mathbb{R}$ by Theorem 1.3.8. By definition of f and g, it is straightforward to check that if $x \in \sigma(a)$ then

1. $x = f(x) - g(x)$,
2. $xf(x) = f(x)^2$,
3. $xg(x) = -g(x)^2$,
4. $f(x)g(x) = 0$.

By the continuous functional calculus in Theorem 1.3.7, we obtain the required equations in A.

Next, observe that if $x \in \sigma(a)$ then $f(x) \geq 0$. Hence, the square root $\sqrt{f}$ is a continuous real-valued function which satisfies $(\sqrt{f})^2 = f$. So,

$$f(a) = (\sqrt{f}(a))^2 = (\sqrt{f}(a))^* \sqrt{f}(a).$$

Therefore, $f(a) \in A$ is a positive element. Similarly, $g(a) \in A$ is also a positive element. Finally, if $\sigma(a) \subseteq [0, \infty)$ then $g = 0$ and $a = f(a)$. Since $f(a) \in A$ is positive, a must be positive in this case. $\square$

Keeping the decomposition of Theorem 1.4.1 in mind, we prove the following characterisation of a spectrum contained in $[0, \infty)$.

Theorem 1.4.2. *Let A be a unital C*-algebra and $a \in A$ be self-adjoint. Then, the following are equivalent:*

1. $\sigma(a) \subseteq [0, \infty)$
2. *If* $t \geq \|a\|$ *then* $\|t1_A - a\| \leq t$
3. *There exists* $t \geq \|a\|$ *such that* $\|t1_A - a\| \leq t$.

Proof. Assume that A is a unital C*-algebra and $a \in A$ is self-adjoint. It is obvious that the second statement implies the third.

To show: (a) $\sigma(a) \subseteq [0, \infty)$ if and only if for $t \geq \|a\|$, $\|t1_A - a\| \leq t$.

(a) Since a is self-adjoint, $\sigma(a) \subseteq \mathbb{R}$ and its spectral radius $r(a) = \|a\|$. Therefore, $\sigma(a) \subseteq [-\|a\|, \|a\|]$.

Next, assume that $t \geq \|a\|$ and define the function

$$\begin{array}{rccc} f_t : & [-\|a\|, \|a\|] & \to & \mathbb{R} \\ & x & \mapsto & t - x \end{array}$$

Then, f_t is positive and monotonically decreasing. So, the norm of the restriction

$$\|f_t|_{\sigma(a)}\| = \sup_{x \in \sigma(a)} |f_t(x)| = \sup_{x \in \sigma(a)} |t - x| = t - \inf_{x \in \sigma(a)} x = f_t(\inf_{x \in \sigma(a)} x).$$

By the continuous functional calculus in Theorem 1.3.7,

$$\|t1_A - a\| = \|f_t|_{\sigma(a)}\| = f_t(\inf_{x \in \sigma(a)} x).$$

Finally, observe that $f_t(x) \leq t$ if and only if $x \geq 0$. Hence, $\sigma(a) \subseteq \mathbb{R}_{<0}$ if and only if $\|t1_A - a\| > t$. By taking the contrapositive statement, we find that $\sigma(a) \subseteq \mathbb{R}_{\geq 0}$ if and only if $\|t1_A - a\| \leq t$ as required. $\square$

One application of Theorem 1.4.2 is the fact that spectrums contained in $\mathbb{R}_{\geq 0}$ "add" in a sense made precise in the next theorem.

Theorem 1.4.3. *Let A be a unital C*-algebra and $a, b \in A$ be self-adjoint. Assume that $\sigma(a), \sigma(b) \subseteq [0, \infty)$. Then, $\sigma(a + b) \subseteq [0, \infty)$.*

Proof. Assume that A is a unital C*-algebra and $a, b \in A$ are self-adjoint. Assume that $\sigma(a), \sigma(b) \subseteq [0, \infty)$. Set $t = \|a\| + \|b\|$. Then,

$$\begin{aligned} \|t1_A - a - b\| &= \|(\|a\|1_A - a) + (\|b\|1_A - b)\| \\ &\leq \|\|a\|1_A - a\| + \|\|b\|1_A - b\| \\ &\leq \|a\| + \|b\| = t \quad \text{(by Theorem 1.4.2)}. \end{aligned}$$

By another application of Theorem 1.4.2, we deduce that $\sigma(a+b) \subseteq [0, \infty)$ as required. $\square$

Before we proceed to our promised characterisation of positive elements, we need yet another theorem about the spectrum.

Theorem 1.4.4. *Let A be a unital C*-algebra A and $g, h \in A$. Then, $\sigma(gh) - \{0\} = \sigma(hg) - \{0\}$.*

Proof. Assume that A is a unital C*-algebra and $g, h \in A$. It suffices to show that if $\lambda \in \mathbb{C} - \{0\}$ such that $\lambda 1_A - gh$ is invertible then $\lambda 1_A - hg$ is also invertible.

To this end, assume that $\lambda \in \mathbb{C} - \{0\}$ such that $\lambda 1_A - gh$ is invertible. Define

$$x = \frac{1}{\lambda} 1_A + \frac{1}{\lambda} h(\lambda 1_A - gh)^{-1} g.$$

We compute directly that

$$\begin{aligned}
x(\lambda 1_A - hg) &= \big(\frac{1}{\lambda} 1_A + \frac{1}{\lambda} h(\lambda 1_A - gh)^{-1} g\big)(\lambda 1_A - hg) \\
&= 1_A - \frac{1}{\lambda} hg + h(\lambda 1_A - gh)^{-1} g - \frac{1}{\lambda} h(\lambda 1_A - gh)^{-1} ghg \\
&= 1_A + h\big((\lambda 1_A - gh)^{-1} - \frac{1}{\lambda} 1_A - \frac{1}{\lambda}(\lambda 1_A - gh)^{-1} gh\big) g \\
&= 1_A + h\big((\lambda 1_A - gh)^{-1} - \frac{1}{\lambda}(\lambda 1_A - gh)^{-1}(\lambda 1_A - gh) \\
&\quad - \frac{1}{\lambda}(\lambda 1_A - gh)^{-1} gh\big) g \\
&= 1_A
\end{aligned}$$

and

$$\begin{aligned}
(\lambda 1_A - hg)x &= (\lambda 1_A - hg)\big(\frac{1}{\lambda} 1_A + \frac{1}{\lambda} h(\lambda 1_A - gh)^{-1} g\big) \\
&= 1_A + h(\lambda 1_A - gh)^{-1} g - \frac{1}{\lambda} hg - \frac{1}{\lambda} hgh(\lambda 1_A - gh)^{-1} g \\
&= 1_A + h\big((\lambda 1_A - gh)^{-1} - \frac{1}{\lambda}(\lambda 1_A - gh)(\lambda 1_A - gh)^{-1} \\
&\quad - \frac{1}{\lambda} gh(\lambda 1_A - gh)^{-1}\big) g \\
&= 1_A.
\end{aligned}$$

Hence, $\lambda 1_A - hg$ is also invertible and $\sigma(gh) - \{0\} = \sigma(hg) - \{0\}$. $\square$

Now we are ready to prove our characterisation of positive elements.

Theorem 1.4.5. *Let A be a unital C*-algebra and $a \in A$ be self-adjoint. Then, a is positive if and only if $\sigma(a) \subseteq [0, \infty)$.*

Proof. Assume that A is a unital C*-algebra and $a \in A$ is self-adjoint. We already know that if $\sigma(a) \subseteq [0, \infty)$ then a is positive by Theorem 1.4.1.

Now assume that $a \in A$ is positive. Then, there exists $b \in A$ such that $a = b^*b$. Recall that in Theorem 1.4.1, we defined positive elements $f(a), g(a) \in A$. The idea is to consider the element $c = bg(a)$. We compute directly that

$$c^*c = g(a)^*b^*bg(a) = g(a)b^*bg(a) = g(a)ag(a) = g(a)(-g(a))g(a) = -g(a)^3.$$

Now write $c = d + ie$, where $d, e \in A$ are self-adjoint. Another computation yields

$$cc^* + c^*c = 2d^2 + 2e^2.$$

So, $cc^* = 2d^2 + 2e^2 - c^*c = 2d^2 + 2e^2 + g(a)^3$. Since the real-valued functions $2x^2$ and $g(x)$ are positive, we can use the spectral mapping theorem (see Theorem 1.3.14) to deduce that

$$\sigma(2d^2), \sigma(2e^2), \sigma(g(a)^3) \subseteq [0, \infty).$$

Consequently by Theorem 1.4.3,

$$\sigma(cc^*) = \sigma(2d^2 + 2e^2 + g(a)^3) \subseteq [0, \infty).$$

On the other hand, by Theorem 1.3.14

$$\sigma(c^*c) \subseteq (-\infty, 0].$$

Next, recall from Theorem 1.4.4 that

$$\sigma(c^*c) - \{0\} = \sigma(cc^*) - \{0\}.$$

Since $\sigma(c^*c) \subseteq (-\infty, 0]$ and $\sigma(cc^*) \subseteq [0, \infty)$, then $\sigma(c^*c) = \sigma(cc^*) = \{0\}$. Therefore, $c^*c = 0 = -g(a)^3$ and consequently, $g(a) = 0$. This means that the restricted function $g|_{\sigma(a)}$ is the zero function. Recalling from Theorem 1.4.1 that $g = \max\{-x, 0\}$, we conclude that $\sigma(a) \subseteq [0, \infty)$ as required. $\square$

A useful consequence of Theorem 1.4.5 is the connection between normal elements of a unital C*-algebra and the continuous functional calculus.

Theorem 1.4.6. *Let A be a unital C*-algebra and $a \in A$ be normal. If $f \in Cts(\sigma(a), \mathbb{R})$ then $f(a) \in A$ is self-adjoint. Moreover, if $f \in Cts(\sigma(a), \mathbb{R})$ is positive then $f(a) \in A$ is positive.*

Proof. Assume that A is a unital C*-algebra and $a \in A$ is normal. Assume that $f \in Cts(\sigma(a), \mathbb{R})$. By the spectral mapping theorem (see Theorem 1.3.14),

$$\sigma(f(a)) = f(\sigma(a)) = \{f(\lambda) \mid \lambda \in \sigma(a)\} \subseteq \mathbb{R}.$$

So, $f(a) \in A$ is self-adjoint by Theorem 1.3.8. If $f \in Cts(\sigma(a), \mathbb{R})$ is positive then $\sigma(f(a)) \subseteq \mathbb{R}_{\geq 0}$ and by Theorem 1.4.5, $f(a) \in A$ is normal as required. □

The *polar decomposition* gives a method of decomposing an arbitrary element of a unital C*-algebra into the product of a positive element and a unitary element. This decomposition is analogous to computing the exponential form of a complex number. Here, we will consider the polar decomposition of invertible elements, in line with [Put19, Exercise 1.6.1]. We begin with a preliminary result.

Theorem 1.4.7. *Let A be a unital C*-algebra and $a \in A$. Let f be the function defined explicitly by*

$$\begin{array}{rccc} f: & \mathbb{R}_{\geq 0} & \to & \mathbb{R} \\ & x & \mapsto & x^{\frac{1}{2}}. \end{array}$$

Define $|a| = f(a^*a)$. If a is invertible then $|a|$ is invertible.

Proof. Assume that A is a unital C*-algebra and $a \in A$. Assume that f is the square root function defined as above. Define the function $g : \mathbb{C} \to \mathbb{C}$ by $g(z) = \overline{z}z = |z|^2$. By the continuous functional calculus in Theorem 1.3.7, $g(a) = a^*a$. Now by the spectral mapping theorem in Theorem 1.3.14,

$$\sigma(a^*a) = \sigma(g(a)) = g(\sigma(a)) = \{|\lambda|^2 \mid \lambda \in \sigma(a)\}.$$

By considering the restriction of f to $\sigma(a^*a) \subseteq \mathbb{R}_{\geq 0}$, we find that by another application of the spectral mapping theorem,

$$\sigma(f(a^*a)) = f(\sigma(a^*a)) = \{|\lambda| \mid \lambda \in \sigma(a)\}.$$

Hence, $0 \in \sigma(a)$ if and only if $0 \in \sigma(f(a^*a))$. Thus, if a is invertible then $|a|$ is also invertible. □

The element $|a| \in A$ constructed in Theorem 1.4.7 is commonly referred to as the **modulus/absolute value** of a. The next theorem yields the second half of the polar decomposition.

Theorem 1.4.8. *Let A be a unital C*-algebra and $a \in A$. Let $|a| \in A$ be the element constructed in Theorem 1.4.7. If a is invertible then the element*

$$u = a|a|^{-1} \in A$$

is unitary.

Proof. Assume that A is a unital C*-algebra and $a \in A$. Assume that $|a| \in A$ is the element defined in Theorem 1.4.7. Assume that a is invertible. By Theorem 1.4.7, $|a| \in A$ is invertible. Hence, we can define the element $u = a|a|^{-1}$.

To show: (a) $u^*u = uu^* = 1_A$.

(a) Recall from Theorem 1.4.7 that

$$\sigma(|a|) = \{|\lambda| \mid \lambda \in \sigma(a)\}.$$

Since $\sigma(|a|) \subseteq \mathbb{R}_{\geq 0}$, we deduce by Theorem 1.4.5 that $|a|$ is positive and hence, self-adjoint. This means that

$$|a|^{-1} = (|a|^*)^{-1} = (|a|^{-1})^*.$$

So,

$$u^*u = (|a|^{-1})^*a^*a|a|^{-1} = |a|^{-1}(a^*a)|a|^{-1} = |a|^{-1}|a| = 1_A$$

and

$$uu^*a = a|a|^{-1}(|a|^{-1})^*a^*a = a|a|^{-1}(|a|^{-1}a^*a) = a|a|^{-1}|a| = a.$$

By multiplying both sides on the right by a^{-1}, we deduce that $uu^* = 1_A$.

By part (a), u is unitary. □

Theorem 1.4.8 establishes that if $a \in A$ is invertible then $a = u|a|$. This is referred to as the **polar decomposition** of a. The next theorem provides a criterion for the elements u and $|a|$ to commute.

Theorem 1.4.9. *Let A be a unital C*-algebra and $a \in A$ be invertible. Let u and $|a|$ be the elements defined in Theorem 1.4.8 and Theorem 1.4.7 respectively. Then, u and $|a|$ commute if and only if a is normal.*

Proof. Assume that A is a unital C*-algebra and $a \in A$ is invertible.

To show: (a) If a is normal then the elements u and $|a|$ commute.

(b) If u and $|a|$ commute then a is normal.

(a) Assume that $a \in A$ is normal. By the continuous functional calculus in Theorem 1.3.7, the elements a and $|a|$ must commute. Using this, we now have

$$\begin{aligned}|a|u &= |a|a|a|^{-1} \\ &= a|a||a|^{-1} \\ &= a = u|a|.\end{aligned}$$

Therefore, the elements u and $|a|$ commute.

(b) Assume that the elements u and $|a|$ commute. To see that a is normal, we compute directly that

$$\begin{aligned}a^*a &= (u|a|)^*u|a| \\ &= |a|u^*u|a| \quad (|a| \text{ is self-adjoint}) \\ &= |a|1_A|a| = |a|^2 \quad (\text{Theorem 1.4.8})\end{aligned}$$

and

$$\begin{aligned}aa^* &= u|a|(u|a|)^* \\ &= u|a||a|u^* \quad (|a| \text{ is self-adjoint}) \\ &= u|a|^2u^* \\ &= |a|^2uu^* \\ &= |a|^2 1_A = |a|^2.\end{aligned}$$

Therefore, $a^*a = aa^*$ and a is normal. □

1.5 Finite dimensional C*-algebras

Finite dimensional C*-algebras have a specific structure. This section is dedicated to proving the following theorem.

Theorem 1.5.1. *Let A be a unital, finite dimensional C*-algebra. Then, there exists positive integers $k, N_1, \ldots, N_k \in \mathbb{Z}_{>0}$ such that as C*-algebras,*

$$A \cong \bigoplus_{i=1}^{k} M_{N_i \times N_i}(\mathbb{C})$$

Moreover, k is unique and the integers $N_1, \ldots, N_k$ are unique up to permutation.

It is remarked in [Put19, Section 1.7] that Theorem 1.5.1 is valid without the C*-algebra being unital. In fact, it is a consequence of Theorem 1.5.1 that every finite dimensional C*-algebra is unital. This result will be proved later.

The proof of Theorem 1.5.1 revolves around the existence of projections in a finite dimensional C*-algebra. In general, C*-algebras may or may not have non-trivial projections. The reference [Put19, Section 1.7] gives the example of $Cts(X, \mathbb{C})$, where X is a compact Hausdorff space. It turns out that in this case, $Cts(X, \mathbb{C})$ has no non-trivial projections if and only if X is connected.

Let us start off with some notation pertaining to bounded operators on a Hilbert space.

Definition 1.5.1. Let H be a Hilbert space and $\xi, \eta \in H$. Define the map $|\xi\rangle\langle\eta|$ by

$$\begin{array}{rccc} |\xi\rangle\langle\eta| : & H & \to & H \\ & \zeta & \mapsto & \langle\zeta, \eta\rangle\xi \end{array}$$

This is the well-known bra-ket notation from quantum mechanics. In the next theorem, we prove various properties about the map $|\xi\rangle\langle\eta|$, which we will make use of later.

Theorem 1.5.2. *Let H be a Hilbert space, $\eta, \xi, \zeta, \omega \in H$ and $a \in B(H)$. Then,*

1. $\| |\xi\rangle\langle\eta| \| = \|\xi\| \|\eta\|$.
2. *If* $\eta \neq 0$ *then* $|\xi\rangle\langle\eta| H = span\{\xi\}$.
3. $(|\xi\rangle\langle\eta|)^* = |\eta\rangle\langle\xi|$.
4. $(|\xi\rangle\langle\eta|)(|\zeta\rangle\langle\omega|) = \langle\zeta, \eta\rangle(|\xi\rangle\langle\omega|)$.

5. $a|\xi\rangle\langle\eta| = |a\xi\rangle\langle\eta|$

6. $|\xi\rangle\langle\eta|a = |\xi\rangle\langle a^*\eta|$.

Proof. Assume that H is a Hilbert space, $\eta, \xi, \zeta, \omega \in H$ and $a \in B(H)$.

1. We compute directly that

$$\||\xi\rangle\langle\eta|\| = \sup_{\|\alpha\|=1} |\langle\alpha, \eta\rangle|\|\xi\| \leq \|\eta\|\|\xi\|.$$

We also have

$$\|\eta\|\|\xi\| = |\langle\frac{\eta}{\|\eta\|}, \eta\rangle|\|\xi\| \leq \sup_{\|\alpha\|=1} |\langle\alpha, \eta\rangle|\|\xi\| = \||\xi\rangle\langle\eta|\|.$$

Therefore, $\||\xi\rangle\langle\eta|\| = \|\eta\|\|\xi\|$.

2. Assume that $\eta \neq 0$. By the Hahn-Banach extension theorem, there exists a functional $\Gamma \in H^*$ such that $\Gamma(\eta) \neq 0$. By the Riesz representation theorem, there exists $\gamma \in H$ such that $\Gamma(\eta) = \langle\gamma, \eta\rangle \neq 0$. So, $|\xi\rangle\langle\eta|\gamma = \langle\gamma, \eta\rangle\xi$. Now if $a \in \mathbb{C}$ then

$$a\xi = \frac{a}{\langle\gamma, \eta\rangle}\langle\gamma, \eta\rangle\xi = \frac{a}{\langle\gamma, \eta\rangle}|\xi\rangle\langle\eta|\gamma = |\xi\rangle\langle\eta|(\frac{a}{\langle\gamma, \eta\rangle}\gamma).$$

Hence, $span\{\xi\} \subseteq |\xi\rangle\langle\eta|H$ and subsequently, $|\xi\rangle\langle\eta|H = span\{\xi\}$.

3. If $\alpha, \beta \in H$ then

$$\begin{aligned}
\langle(|\xi\rangle\langle\eta|)^*(\alpha), \beta\rangle &= \langle\alpha, |\xi\rangle\langle\eta|(\beta)\rangle \\
&= \langle\alpha, \langle\beta, \eta\rangle\xi\rangle \\
&= \langle\alpha, \xi\rangle\langle\eta, \beta\rangle \\
&= \langle\langle\alpha, \xi\rangle\eta, \beta\rangle \\
&= \langle|\eta\rangle\langle\xi|(\alpha), \beta\rangle.
\end{aligned}$$

So, $(|\xi\rangle\langle\eta|)^* = |\eta\rangle\langle\xi|$.

4. If $\alpha \in H$ then

$$\begin{aligned}(|\xi\rangle\langle\eta|)(|\zeta\rangle\langle\omega|)\alpha &= |\xi\rangle\langle\eta|(\langle\alpha,\omega\rangle\zeta) \\ &= \langle\alpha,\omega\rangle\langle\zeta,\eta\rangle\xi \\ &= \langle\zeta,\eta\rangle|\xi\rangle\langle\omega|\alpha.\end{aligned}$$

   Hence, $(|\xi\rangle\langle\eta|)(|\zeta\rangle\langle\omega|) = \langle\zeta,\eta\rangle|\xi\rangle\langle\omega|$.

5. If $\alpha \in H$ then

$$a|\xi\rangle\langle\eta|\alpha = a(\langle\alpha,\eta\rangle\xi) = \langle\alpha,\eta\rangle a\xi = |a\xi\rangle\langle\eta|\alpha.$$

6. If $\alpha \in H$ then

$$|\xi\rangle\langle\eta|a\alpha = \langle a\alpha,\eta\rangle\xi = \langle\alpha,a^*\eta\rangle\xi = |\xi\rangle\langle a^*\eta|\alpha.$$

□

The first prominent step towards our classification of finite dimensional C*-algebras is to prove that every normal element is a linear combination of projections.

Theorem 1.5.3. *Let A be a finite dimensional unital C*-algebra. If $a \in A$ is normal then a has finite spectrum and a is a linear combination of projections.*

Proof. Assume that A is a finite dimensional unital C*-algebra. Assume that $a \in A$ is normal.

To show: (a) The spectrum of a is finite.

(b) a can be written as a linear combination of projections.

(a) By the continuous functional calculus in Theorem 1.3.7, the C*-algebra $Cts(\sigma(a),\mathbb{C})$ is isomorphic to a C*-subalgebra of A, which is finite dimensional because A itself is finite dimensional. Consequently, $\sigma(a)$ must be finite.

(b) Let $\lambda \in \sigma(a)$ and define the function p_λ by

$$\begin{aligned} p_\lambda: \quad \sigma(a) &\to \quad \mathbb{C} \\ \mu &\mapsto \begin{cases} 0, & \text{if } \mu \neq \lambda, \\ 1, & \text{if } \mu = \lambda. \end{cases} \end{aligned}$$

Then, $p_\lambda = \overline{p_\lambda}$ and $p_\lambda^2 = p_\lambda$ where we recall that multiplication in the C*-algebra $Cts(\sigma(a), \mathbb{C})$ is defined pointwise. By the continuous functional calculus in Theorem 1.3.7, $p_\lambda(a) \in A$ is self-adjoint and idempotent. Hence, it is a projection.

Notice that if $z \in \sigma(a)$ then

$$\sum_{\lambda \in \sigma(a)} \lambda p_\lambda(z) = z.$$

By another application of the continuous functional calculus, we have

$$\sum_{\lambda \in \sigma(a)} \lambda p_\lambda(a) = a.$$

Hence, a is a linear combination of projections as required. □

A consequence of the proof of Theorem 1.5.3 is that projections exist in finite dimensional unital C*-algebras. Studying this structure is key to the proof of our main result.

Definition 1.5.2. Let A be a C*-algebra. If $p, q \in A$ are projections then we define the relation $p \geq q$ if and only if $pq = q$.

Theorem 1.5.4. *Let A be a C*-algebra. The relation $\leq$ on the projections of A is a partial order.*

Proof. Assume that A is a C*-algebra. Reflexivity of the relation $\leq$ follows from the fact that a projection $p \in A$ is idempotent.

Next, assume that $p, q \in A$ are projections such that $p \leq q$ and $q \leq p$. Then, $qp = p$ and $pq = q$. Since p and q are self-adjoint, $pq = p = q$.

Finally to see that $\leq$ is transitive, assume that $p, q, r \in A$ are projections such that $p \leq q$ and $q \leq r$. Then, $qp = p$ and $rq = q$. So, $rp = rqp = qp = p$. Hence, $\leq$ defines a partial order on the set of projections in A. □

Under some conditions, there exist minimal non-zero projections with regards to the partial order $\leq$. This is established in the next theorem.

Theorem 1.5.5. *Let A be a C*-algebra and $p, q \in A$ be projections. Then,*

1. *$p \geq q$ if and only if $qAq \subseteq pAp$. In particular, if $pAp = qAq$ then $p = q$.*
2. *If pAp has finite dimension greater than 1 then there exists a projection $q \neq 0$ such that $p \geq q$.*
3. *If A is unital and finite dimensional then there exists non-zero projections which are minimal with respect to the order $\leq$.*

Proof. Assume that A is a C*-algebra and $p, q \in A$ are projections.

To show: (a) $p \geq q$ if and only if $qAq \subseteq pAp$.

(a) First assume that $p \geq q$. Then, $qAq = pqAqp \subseteq pAp$. Conversely, assume that $qAq \subseteq pAp$. Then, $q = qqq \in qAq = pqAqp \subseteq pAp$. So, there exists $a \in A$ such that $q = pap$. Thus, $pq = ppap = pap = q$ and subsequently, $q \leq p$.

Next, assume that $qAq = pAp$. By part (a), $q \leq p$ and $p \leq q$. Since $\leq$ is a partial order, $p = q$.

Before we proceed to proving the second statement, we first observe that pAp is a Banach *-algebra. Moreover, if A is finite dimensional then A is closed and pAp is a C*-subalgebra of A. It is also unital, with $p = ppp$ acting as a unit for the C*-subalgebra pAp.

To show: (b) If $a \in pAp$ is self-adjoint and $\sigma(a) = \{\lambda\}$ for some $\lambda \in \mathbb{C}$ then a is a scalar multiple of p.

(b) Assume that $a \in pAp$ is self-adjoint. Assume that there exists $\lambda \in \mathbb{C}$ such that $\sigma(a) = \{\lambda\}$. If $z \in \sigma(a)$ then the functions $f(z) = z$ and $g(z) = \lambda$ are equal. By the continuous functional calculus in Theorem 1.3.7,

$$a = f(a) = g(a) = \lambda p.$$

Hence, a is a scalar multiple of p.

Now suppose that pAp has dimension strictly greater than 1. By the contrapositive statement of part (b), there exists a self-adjoint element $b \in A$ such that $\sigma(b)$ contains at least two points. Let f be a surjective function from $\sigma(b)$ to $\{0,1\}$. Since $\{0,1\}$ is equipped with the discrete topology, f must be continuous.

By the continuous functional calculus, the element $f(b) = q$ is a non-zero projection. Since $q \in pAp$, we must have $pq = q$. So, $p \geq q$ as required.

Finally, assume that A is unital and finite dimensional. By Theorem 1.5.3, there exists a non-zero projection $p \in A$. The dimension of the C*-subalgebra pAp is at least one. If it is strictly greater than 1 then by part (b) of the proof, there exists a non-zero projection $q \in A$ such that $p \geq q$. By part (a), this holds if and only if $qAq \subseteq pAp$. Since $q \neq p$, qAq is strictly contained in pAp and $\dim qAq < \dim pAp$.

By iterating this procedure, we eventually obtain a non-zero projection $q' \in A$ such that $\dim q'Aq' = 1$. Hence, q' qualifies as a minimal projection with respect to the partial order $\leq$. $\square$

One consequence of Theorem 1.5.5 is that if $p \in A$ is a non-zero minimal projection then $pAp = \mathbb{C}p$. Next, we want to now when a set of non-zero minimal projections are linearly independent.

Theorem 1.5.6. *Let A be a unital finite dimensional C*-algebra and $p_1, \dots, p_K$ be non-zero minimal projections in A. Suppose that if $i, j \in \{1, 2, \dots, K\}$ such that $i \neq j$ then $p_i A p_j = 0$. Then, $p_1, \dots, p_K$ are linearly independent.*

Proof. Assume that A is a unital finite dimensional C*-algebra and $p_1, \dots, p_K$ are non-zero minimal projections in A. Assume that $p_i A p_j = 0$ for $i \neq j$. Fix $i \in \{1, 2, \dots, K\}$. Suppose for the sake of contradiction that

$$p_i = \sum_{j \neq i} \alpha_j p_j$$

for some $\alpha_j \in \mathbb{C}$. By Theorem 1.5.5, we have

$$\mathbb{C}p_i = p_i A p_i = p_i A \big(\sum_{j \neq i} \alpha_j p_j \big) \subseteq \sum_{j \neq i} p_i A p_j = 0.$$

This is a contradiction. Therefore, the minimal projections $p_1, \dots, p_K$ are linearly independent. $\square$

Theorem 1.5.6 tells us that a linearly independent set of non-zero minimal projections cannot contain more than $\dim A$ elements. Hence, we can select a maximal independent set of non-zero minimal projections and it will be finite.

Now suppose that A is a unital finite dimensional C*-algebra and $\{p_1, \ldots, p_K\}$ is a maximal set of independent minimal non-zero projections in A. Fix $i \in \{1, 2, \ldots, K\}$ and let $a, b \in Ap_i$. Then, $ap_i = a$ and $bp_i = b$. Observe that

$$b^*a = (bp_i)^*ap_i = p_i b^* a p_i \in p_i A p_i = \mathbb{C}p_i$$

as a consequence of the second statement in Theorem 1.5.5. Let $\langle a, b\rangle \in \mathbb{C}$ denote the scalar such that $b^*a = \langle a, b\rangle p_i$. We claim that $\langle -, -\rangle : Ap_i \times Ap_i \to \mathbb{C}$ defines an inner product on Ap_i.

First assume that $a, a', b, b' \in Ap_i$. Then,

$$\begin{aligned}\langle a + a', b\rangle p_i &= b^*(a + a') \\ &= b^*a + b^*a' \\ &= (\langle a, b\rangle + \langle a', b\rangle)p_i\end{aligned}$$

and

$$\begin{aligned}\langle a, b + b'\rangle p_i &= (b^* + (b')^*)a \\ &= b^*a + (b')^*a \\ &= (\langle a, b\rangle + \langle a, b'\rangle)p_i.\end{aligned}$$

If $\lambda \in \mathbb{C}$ then

$$\langle \lambda a, b\rangle p_i = b^*(\lambda a) = \lambda b^* a = \lambda\langle a, b\rangle p_i$$

and

$$\langle a, \lambda b\rangle p_i = (\lambda b)^* a = \overline{\lambda} b^* a = \overline{\lambda}\langle a, b\rangle p_i.$$

Hence, the map $\langle -, -\rangle$ is $\mathbb{C}$-bilinear. Also, if $a \in A$ then $\langle a, a\rangle = 0$ if and only if $a^*a = 0$ if and only if $a = 0$. Hence, $\langle -, -\rangle$ defines an inner product on Ap_i.

Since A is finite dimensional, Ap_i is also finite dimensional and thus, a closed subspace of A. We conclude that if $i \in \{1, 2, \ldots, K\}$ then Ap_i is a finite dimensional Hilbert space.

Next, we claim that if $i \in \{1, 2, \ldots, K\}$ then the subspace Ap_iA is a unital C*-subalgebra of A. Taking advantage of the fact that Ap_i is a Hilbert space, we can let B_i be an orthonormal basis for Ap_i. If $a \in Ap_i$ then

$$\sum_{b \in B_i} bb^*a = \sum_{b \in B_i} b\langle a, b\rangle p_i = \sum_{b \in B_i} \langle a, b\rangle bp_i = \sum_{b \in B_i} \langle a, b\rangle b = a.$$

In light of this computation, let $q_i = \sum_{b \in B_i} bb^* = \sum_{b \in B_i} bp_ib^*$. Then, $q_i \in Ap_iA$ is self-adjoint and from the previous computation, if $a \in Ap_i$ then $q_ia = a$.

We claim that q_i is the unit of Ap_iA. To this end, assume that $a, a' \in A$. We compute directly that

$$q_i(ap_ia') = (q_iap_i)a' = ap_ia'$$

and

$$(ap_ia')q_i = \big((q_i(a')^*p_i)a^*\big)^* = ((a')^*p_ia^*)^* = ap_ia'.$$

Therefore, Ap_iA is a C*-subalgebra of A with unit q_i.

Now if $k, \ell \in \{1, 2, \ldots, K\}$ with $k \neq \ell$ then $p_kAp_\ell = 0$. So, $(Ap_kA)(Ap_\ell A) = 0$. In particular, $q_kAp_\ell = 0$ and $q_kAp_\ell A = 0$, which means that the projections q_k are pairwise orthogonal (since $q_kq_\ell = 0$).

Now define

$$q = \sum_{k=1}^{K} q_k.$$

Then, q is a projection because each q_k is self-adjoint and $q^2 = q$ since the projections q_k are pairwise orthogonal. It is also a central projection. We claim that $q = 1_A$ in A.

Suppose for the sake of contradiction that $q \neq 1_A$. Then, there exists $b \in A$ such that $bq \neq b$. Hence, $b - bq$ is a non-zero element of the set

$$q^\perp = \{a \in A \mid qa = 0\}.$$

It is straightforward to verify that $q^\perp$ is a C*-subalgebra of A. Hence, $q^\perp$ must be finite dimensional. Since $q^\perp$ contains a non-zero element, then it must contain a non-zero minimal projection by the third part of Theorem 1.5.5. We denote this projection by p. Thus, if $k \in \{1, 2, \ldots, K\}$ then

$$pq_k \leq pq = 0.$$

If $k \in \{1, 2, \ldots, K\}$ then $pq_k = 0$. Therefore,

$$pAp_k \subseteq pAp_kA = pq_kAp_kA = 0$$

where the second last equality follows from the fact that q_k is the unit for the C*-subalgebra Ap_kA. The point here is that the projections $p, p_1, \ldots, p_K$ form an independent set of non-zero minimal projections by Theorem 1.5.6. However, this contradicts the maximality of the set $\{p_1, \ldots, p_K\}$. Therefore, $q = 1_A$. The primary consequence of this claim is that

$$A = \big(\sum_{k=1}^{K} q_k\big)A = \bigoplus_{k=1}^{K} q_kA = \bigoplus_{k=1}^{K} Ap_kA.$$

because $q_k \in Ap_kA$. Finally, recall that $B(Ap_i)$ is the C*-algebra of bounded linear operators on the Hilbert space Ap_i. For $i \in \{1, 2, \ldots, K\}$, define the map

$$\begin{array}{rccc} \pi_i : & A & \to & B(Ap_i) \\ & a & \mapsto & (b \mapsto ab). \end{array}$$

The fact that π_i is a *-homomorphism follows from routine computations. Observe that if $k \neq i$ then the restriction $\pi_i|_{Ap_kA} = 0$ because $(Ap_kA)Ap_i = A(p_kAp_i) = 0$.

Now we claim that π_i is an isomorphism from Ap_iA to $B(Ap_i)$. First, we show that π_i is injective. The key idea here is that the span of the orthonormal basis B_i is Ap_i by definition. Hence, $p_iA = (Ap_i)^*$ is the span of the adjoints of elements of B_i. Hence, $Ap_iA = (Ap_i)(Ap_i)^*$ is contained in the span of elements of the form bc^*, where $b, c \in B_i$. For $b, c, b_0, c_0 \in B_i$, let $\alpha_{b,c} \in \mathbb{C}$. Then,

$$\begin{aligned} \langle \pi_i\big(\sum_{b,c\in B_i} \alpha_{b,c} bc^*\big)c_0, b_0\rangle p_i &= \sum_{b,c\in B_i} \alpha_{b,c} \langle \pi_i\big(bc^*\big)c_0, b_0\rangle p_i \\ &= \sum_{b,c\in B_i} \alpha_{b,c} \langle bc^*c_0, b_0\rangle p_i \\ &= \sum_{b,c\in B_i} \alpha_{b,c} b_0^*bc^*c_0 = \alpha_{b_0,c_0} p_i. \end{aligned}$$

The last equality follows from the assumption that B_i is an orthonormal basis for the Hilbert space Ap_i. To see that the restriction $\pi_i|_{Ap_iA}$ is injective, assume that $\pi_k(\sum_{b,c\in B_i} \alpha_{b,c}bc^*) = 0$. By the previous computation, each coefficient $\alpha_{b,c} = 0$ for $b, c \in Ap_i$. Therefore, $\sum_{b,c\in B_i} \alpha_{b,c}bc^* = 0$ in Ap_iA and consequently, $\pi_i|_{Ap_iA}$ is injective.

To see that π_i is surjective, assume that $a, b \in Ap_i$ so that $ab^* \in Ap_iA$. If $c \in Ap_i$ then

$$\begin{aligned}\pi_i(ab^*)c &= a(b^*c)\\ &= a(\langle c, b\rangle p_i)\\ &= \langle c, b\rangle ap_i = \langle c, b\rangle a\\ &= |a\rangle\langle b|c.\end{aligned}$$

The above computation reveals that the rank one operator $|a\rangle\langle b|$ is in the image of π_i. So, π_i is surjective because the span of operators of the form $|a\rangle\langle b|$ is all of $B(Ap_i)$. Hence, $\pi_i|_{Ap_iA}$ is a *-isomorphism.

Let us summarise our findings pertaining to the structure of finite dimensional C*-algebras. This is [Put19, Theorem 1.7.8]. Recall that A is a finite dimensional C*-algebra and $p_1, \ldots, p_K$ is a maximal set of independent minimal non-zero projections in A.

1. If $k \in \{1, 2, \ldots, K\}$ then Ap_k is a finite dimensional Hilbert space with inner product given by $\langle a, b\rangle p_k = b^*a$ for $a, b \in Ap_k$.
2. If $k \in \{1, 2, \ldots, K\}$ then Ap_kA is a unital C*-subalgebra of A, with unit $q_k = \sum_{b\in B_k} bb^*$.
3. $\bigoplus_{k=1}^{K} Ap_kA = A$
4. If $k \in \{1, 2, \ldots, K\}$ then $\pi_k : A \to B(Ap_k)$ is a *-homomorphism. Moreover, if $i \neq k$ then $\pi_k|_{Ap_iA} = 0$ and $\pi_k|_{Ap_kA}$ is a *-isomorphism.

The next theorem describes the centres of the C*-algebra $M_{n\times n}(\mathbb{C})$.

Theorem 1.5.7. *Let $n \in \mathbb{Z}_{>0}$. The centre of $M_{n\times n}(\mathbb{C})$ is*

$$\{aI_n \mid a \in \mathbb{C}\}$$

where $I_n \in M_{n\times n}(\mathbb{C})$ is the $n \times n$ identity matrix. Furthermore, if $n_1, \ldots, n_K \in \mathbb{Z}_{>0}$ then the centre of the direct sum $\bigoplus_{k=1}^{K} M_{n_k\times n_k}(\mathbb{C})$ is isomorphic to $\mathbb{C}^K$ and is spanned by the identity elements of the direct summands.

Proof. Assume that $n \in \mathbb{Z}_{>0}$. The case where $n = 1$ is trivial. So, assume that $n \geq 2$. Obviously, we can identify $M_{n\times n}(\mathbb{C})$ with the space of bounded linear transformations $B(\mathbb{C}^n)$. Assume that $a : \mathbb{C}^n \to \mathbb{C}^n$ is in the centre of $B(\mathbb{C}^n)$ and $\xi \in \mathbb{C}^n$ is non-zero. Note that

$$|a\xi\rangle\langle\xi| = a|\xi\rangle\langle\xi| = |\xi\rangle\langle\xi|a = |\xi\rangle\langle a^*\xi|.$$

Subsequently,

$$a\xi\langle\xi,\xi\rangle = \langle\xi, a^*\xi\rangle\xi.$$

We conclude that if $\xi \in \mathbb{C}^n$ is non-zero then there exists a scalar $r \in \mathbb{C}$ such that $a\xi = r\xi$. We want to show that the scalar r is independent of the choice of non-zero $\xi \in \mathbb{C}^n$.

Let $\eta \in \mathbb{C}^n$ such that the set $\{\xi, \eta\}$ is linearly independent. Then, there exists $r, s, t \in \mathbb{C}$ such that $a(\xi + \eta) = r(\xi + \eta)$, $a\xi = s\xi$ and $a\eta = t\eta$. So, $r\xi + r\eta = s\xi + t\eta$ and by linear independence, $r = s = t$.

Therefore, if $\xi \in \mathbb{C}^n$ is non-zero then $a\xi$ is a scalar multiple of ξ and this scalar is independent of ξ. Consequently, a is a multiple of the identity transformation $id_{\mathbb{C}^n} \in B(\mathbb{C}^n)$ as required.

The second statement follows from the first. □

Now we can finally put together a proof of Theorem 1.5.1.

Proof of Theorem 1.5.1. Assume that A is a unital, finite dimensional C*-algebra. By Theorem 1.5.6, we can select a maximal independent set of non-zero minimal projections, which has finite cardinality. Let $\{p_1, \ldots, p_K\}$ be such a set. We know that

$$A = \bigoplus_{k=1}^{K} Ap_kA.$$

Moreover, if $k \in \{1, 2, \ldots, K\}$ then the map

$$\begin{array}{rccc} \pi_k : & A & \to & B(Ap_k) \\ & a & \mapsto & (b \mapsto ab) \end{array}$$

is a *-isomorphism from the C*-subalgebra Ap_kA to $B(Ap_k)$. So, the direct sum

$$\bigoplus_{k=1}^{K} \pi_k : \; A \cong \bigoplus_{k=1}^{K} Ap_kA \longrightarrow \bigoplus_{k=1}^{K} B(Ap_k) \cong \bigoplus_{k=1}^{K} M_{n_k \times n_k}(\mathbb{C})$$

is a *-isomorphism. Here, $n_1, \ldots, n_K \in \mathbb{Z}_{>0}$ are finite because each Hilbert space Ap_k is finite dimensional for $k \in \{1, 2, \ldots, K\}$.

To see that $K, n_1, \ldots, n_K \in \mathbb{Z}_{>0}$ is unique, we first note that from Theorem 1.5.7, K is the dimension of the centre of $\bigoplus_{k=1}^{K} M_{n_k \times n_k}(\mathbb{C}) \cong A$. Hence, K must be unique. Next, recall that if $k, \ell \in \{1, 2, \ldots, K\}$ with $k \neq \ell$ then Ap_kA has unit q_k which satisfies $q_k Ap_\ell A = 0$. By the *-isomorphism and Theorem 1.5.7, we find that n_k is the square root of the dimension of $q_kA = Ap_kA$. This completes the proof. □

We finish this section by stating [Put19, Exercise 1.7.1] as a theorem about partial isometries.

Theorem 1.5.8. *Let A be a C*-algebra and $e \in A$ be a partial isometry so that e^*e is a projection. Then, $ee^*e = e$ and ee^* is a projection.*

Proof. Assume that A is a C*-algebra. Assume that $e \in A$ is a partial isometry so that e^*e is a projection.

To see that $ee^*e = e$, consider the expression $(ee^*e - e)^*(ee^*e - e)$. By computing this element, we find that

$$\begin{aligned}(ee^*e - e)^*(ee^*e - e) &= (e^*ee^* - e^*)(ee^*e - e) \\ &= (e^*ee^*e)e^*e - e^*ee^*e - e^*ee^*e - e^*e \\ &= e^*e - e^*e - e^*e + e^*e = 0.\end{aligned}$$

By taking the norms of both sides, we find that

$$\|ee^*e - e\|^2 = \|(ee^*e - e)^*(ee^*e - e)\| = 0.$$

Therefore, $ee^*e = e$.

Next, to see that ee^* is also a projection (or e^* is a partial isometry), we simply note that $ee^*ee^* = (ee^*e)e^* = ee^*$. Since ee^* is also self-adjoint, we deduce that ee^* is a projection as required. □

1.6 Non-unital C*-algebras

The main idea we want to focus on in this section is that from a C*-algebra which is not necessarily unital, one can always construct a unique unital C*-algebra from it. This is stated more precisely in the following theorem.

Theorem 1.6.1. *Let A be a C*-algebra. Then, there exists a unique unital C*-algebra $\tilde{A}$ such that A is contained in $\tilde{A}$ as a closed two-sided ideal and the quotient $\tilde{A}/A \cong \mathbb{C}$.*

Proof. Assume that A is a C*-algebra. As a *vector space*, we define $\tilde{A} = \mathbb{C} \oplus A$. The next step is to endow $\tilde{A}$ with the structure necessary for it to be a C*-algebra — multiplication, involution and a suitable norm.

Multiplication: We define multiplication on $\tilde{A}$ by

$$(\lambda, a)(\mu, b) = (\lambda\mu, \lambda b + \mu a + ab).$$

To be clear, $\lambda, \mu \in \mathbb{C}$ and $a, b \in A$.

Involution: We define an involution map on $\tilde{A}$ by

$$(\lambda, a)^* = (\overline{\lambda}, a^*).$$

Again, $\lambda \in \mathbb{C}$ and $a \in A$. It is easy to check that multiplication and involution defined as above satisfies the required properties.

Before we move on to the problem of defining a norm on $\tilde{A}$, we first make the following observations:

1. It is straightforward to check that $(1, 0) \in \tilde{A}$ is the multiplicative unit for $\tilde{A}$.

2. Define the map

$$\begin{array}{rccc} \iota : & A & \to & \tilde{A} \\ & a & \mapsto & (0, a). \end{array}$$

   Then, ι is an injective *-homomorphism. Moreover, its image

$$\text{im } \iota = \{(0, a) \mid a \in A\}$$

   is a two-sided ideal in $\tilde{A}$. This shows that A is contained in $\tilde{A}$ as a closed two-sided ideal. Moreover by definition, the quotient $\tilde{A}/A$ is isomorphic to $\mathbb{C}$. For instance if $(\lambda, a) \in \tilde{A}$ then $[(\lambda, a)] = [(\lambda, 0)]$ in $\tilde{A}/A$.

Norm: Before we proceed, let us highlight a useful remark from [Put19]. One initial guess for a norm on $\tilde{A}$ is

$$\|(\lambda, a)\|_1 = |\lambda| + \|a\|.$$

With this norm, $\tilde{A}$ becomes a Banach *-algebra whose involution map is isometric. However, this norm does not satisfy the C*-algebra condition. If $(\lambda, a) \in \tilde{A}$ then

$$\begin{aligned}
\|(\lambda, a)\|_1^2 &= (|\lambda| + \|a\|)^2 \\
&= |\lambda|^2 + 2|\lambda|\|a\| + \|a\|^2 \\
&= |\lambda|^2 + 2|\lambda|\|a\| + \|a^*a\| \\
&= |\lambda|^2 + |\lambda|\|a\| + |\lambda|\|a^*\| + \|a^*a\|
\end{aligned}$$

and

$$\begin{aligned}
\|(\lambda, a)^*(\lambda, a)\|_1 &= \|(\overline{\lambda}, a^*)(\lambda, a)\|_1 \\
&= \|(|\lambda|^2, \overline{\lambda}a + \lambda a^* + a^*a)\|_1 \\
&= |\lambda|^2 + \|\overline{\lambda}a + \lambda a^* + a^*a\|.
\end{aligned}$$

One can see that both expression are not equal in general. By the triangle inequality, we only have $\|(\lambda, a)^*(\lambda, a)\|_1 \leq \|(\lambda, a)\|_1^2$ at best.

How do we proceed from here? The key tangential observation is that since A is a Banach space, we can study the related Banach space $B(A)$ of bounded linear operators on A. Define the map

$$\begin{array}{rccc}
\pi: & A & \to & B(A) \\
 & a & \mapsto & (b \mapsto ab)
\end{array}$$

By definition of the operator norm, we have $\|\pi(a)\| \leq \|a\|$. For the reverse inequality, we note that

$$\|\pi(a)a^*\| = \|aa^*\| = \|a\|^2 = \|a\|\|a^*\|.$$

So, $\|a\|\|a^*\| \leq \|\pi(a)\|\|a^*\|$ and consequently, $\|a\| = \|\pi(a)\|$. In this way, we see that the norm of A is simply the operator norm on $B(A)$ acting on A

itself.

This observation suggests that we construct the required norm for $\tilde{A}$ by thinking of $\tilde{A}$ as acting on A. If $(\lambda, a) \in \tilde{A}$ then we define

$$\|(\lambda, a)\| = \sup\{|\lambda|, \|(\lambda, a)(0, b)\|, \|(0, b)(\lambda, a)\| \mid b \in A, \|b\| \leq 1\}. \tag{1.6}$$

A quick computation shows that $(\lambda, a)(0, b) = (0, \lambda b + ab)$ and $(0, b)(\lambda, a) = (0, \lambda b + ba)$. So, the norm appearing on the RHS of equation (1.6) is in fact, the norm on A. Here we implicitly use our embedding of A in $\tilde{A}$ via ι.

First we will deal with the existence of the expression in equation (1.6). By its definition and the triangle inequality, we have

$$\begin{aligned}
\|(\lambda, a)\| &= \sup\{|\lambda|, \|(\lambda, a)(0, b)\|, \|(0, b)(\lambda, a)\| \mid b \in A, \|b\| \leq 1\} \\
&= \sup\{|\lambda|, \|\lambda b + ab\|, \|\lambda b + ba\| \mid b \in A, \|b\| \leq 1\} \\
&\leq \sup_{b \in A, \|b\| \leq 1} (|\lambda|\|b\| + \|a\|\|b\|) \\
&= |\lambda| + \|a\| = \|(\lambda, a)\|_1.
\end{aligned}$$

Therefore, the supremum in equation (1.6) does indeed exist.

To show: (a) The map $\|-\|$ in equation (1.6) is a norm.

(a) Assume that $\mu \in \mathbb{C}$ and $(\lambda, a) \in \tilde{A}$. Then,

$$\begin{aligned}
\|\mu(\lambda, a)\| &= \|(\mu\lambda, \mu a)\| \\
&= \sup\{|\mu\lambda|, \|(\mu\lambda, \mu a)(0, b)\|, \|(0, b)(\mu\lambda, \mu a)\| \mid b \in A, \|b\| \leq 1\} \\
&= \sup\{|\mu\lambda|, \|\mu\lambda b + \mu ab\|, \|\mu\lambda b + \mu ba\| \mid b \in A, \|b\| \leq 1\} \\
&= |\mu|\|(\lambda, a)\|.
\end{aligned}$$

Next, assume that $(\sigma, c) \in \tilde{A}$. Then,

$$\begin{aligned}
\|(\lambda, a) + (\sigma, c)\| &= \|(\lambda + \sigma, a + c)\| \\
&= \sup\{|\lambda + \sigma|, \|(\lambda + \sigma)b + (a + c)b\|, \\
&\quad \|(\lambda + \sigma)b + b(a + c)\| \mid b \in A, \|b\| \leq 1\} \\
&\leq \sup\{|\lambda| + |\sigma|, \|\lambda b + ab\| + \|\sigma b + cb\|, \\
&\quad \|\lambda b + ba\| + \|\sigma b + bc\| \mid b \in A, \|b\| \leq 1\} \\
&\leq \|(\lambda, a)\| + \|(\sigma, c)\|.
\end{aligned}$$

By equation (1.6), if $(\lambda, a) = (0, 0)$ then $\|(\lambda, a)\| = 0$. For the converse statement, assume that $\|(\lambda, a)\| = 0$. Then,

$$\sup\{|\lambda|, \|\lambda b + ab\|, \|\lambda b + ba\| \mid b \in A, \|b\| \leq 1\} = 0.$$

Firstly, we have $|\lambda| = 0$. Now if $b = a^*/\|a\|$ then

$$\|\lambda b + ab\| = \|\lambda b + ba\| = \frac{\|aa^*\|}{\|a\|} = \|a\| = 0.$$

Hence, $(\lambda, a) = (0, 0)$ as required. So, the map in equation (1.6) is indeed a norm.

To show: (b) $\|(0, a)\| = \|a\|$.

(c) $\|(\lambda, a)(\sigma, c)\| \leq \|(\lambda, a)\|\|(\sigma, c)\|$.

(b) First observe that

$$\begin{aligned}
\|(0, a)\| &= \sup\{\|ab\|, \|ba\| \mid b \in A, \|b\| \leq 1\} \\
&\leq \sup_{b \in A, \|b\| \leq 1} \|a\|\|b\| = \|a\|.
\end{aligned}$$

Since $\|(0, 0)\| = 0$ by equation (1.6), we may assume for the reverse inequality that $a \neq 0$. Then,

$$\begin{aligned}
\|(0, a)\| &= \sup\{\|ab\|, \|ba\| \mid b \in A, \|b\| \leq 1\} \\
&\geq \frac{\|a^*a\|}{\|a\|} = \|a\|.
\end{aligned}$$

In the above computation, we set $b = a^*/\|a\|$. Hence, $\|(0, a)\| = \|a\|$.

A notable consequence of this above computation is that if $a \in A$ then

$$\|a\| = \frac{\|a^*a\|}{\|a\|} = \|a\frac{a^*}{\|a\|}\|. \tag{1.7}$$

(c) Observe that if $(\lambda, a), (\sigma, c) \in \tilde{A}$ then

$$\begin{aligned}
\|(\lambda, a)(\sigma, c)\| &= \|(\lambda\sigma, \lambda c + \sigma a + ac)\| \\
&= \sup\{|\lambda\sigma|, \|\lambda\sigma b + \lambda cb + \sigma ab + acb\|, \\
&\quad \|\lambda\sigma b + \lambda bc + \sigma ba + bac\| \mid b \in A, \|b\| \leq 1\} \\
&\leq \sup\{|\lambda|, \|\lambda b + ab\|, \|\lambda b + ba\| \mid b \in A, \|b\| \leq 1\} \\
&\quad \sup\{|\sigma|, \|\sigma b + cb\|, \|\sigma b + bc\| \mid b \in A, \|b\| \leq 1\} \\
&= \|(\lambda, a)\|\|(\sigma, c)\|.
\end{aligned}$$

Next, we will show that $\tilde{A}$ is complete with respect to the norm in equation (1.6). Let $\{(\lambda_n, a_n)\}_{n\in\mathbb{Z}_{>0}}$ be a Cauchy sequence in $\tilde{A}$. By equation (1.6), the sequence $\{\lambda_n\}_{n\in\mathbb{Z}_{>0}}$ in $\mathbb{C}$ is Cauchy and hence, converges to some $\lambda \in \mathbb{C}$.

We claim that $\{a_n\}_{n\in\mathbb{Z}_{>0}}$ is a Cauchy sequence in A. To this end, if $m, n \in \mathbb{Z}_{>0}$ then

$$\begin{aligned}
\|a_m - a_n\| &= \|(a_m - a_n)^*(a_m - a_n)\|^{\frac{1}{2}} \\
&\leq \|a_m - a_n\|^{\frac{1}{2}}\|(a_m - a_n)^*\|^{\frac{1}{2}} \\
&= \begin{cases} 0, \text{ if } a_m - a_n = 0, \\ \|a_m - a_n\|^{\frac{1}{2}}\|(a_m - a_n)\frac{(a_m-a_n)^*}{\|a_m-a_n\|}\|^{\frac{1}{2}}, \text{ if } a_m - a_n \neq 0. \end{cases} \\
&\leq \|a_m - a_n\|^{\frac{1}{2}} \sup_{b\in A, \|b\|\leq 1} \|(a_m - a_n)b\|^{\frac{1}{2}} \\
&\leq \|a_m - a_n\|^{\frac{1}{2}}\|(0, a_m - a_n)\|^{\frac{1}{2}}.
\end{aligned}$$

In the third line, we used equation (1.7). Consequently,

$$\begin{aligned}
\|a_m - a_n\| &\leq \|(0, a_m - a_n)\| \\
&\leq \|(\lambda_m, a_m) - (\lambda_n, a_n)\| + \|(\lambda_m - \lambda_n, 0)\| \\
&\leq \|(\lambda_m, a_m) - (\lambda_n, a_n)\| + |\lambda_m - \lambda_n| \to 0
\end{aligned}$$

as $m, n \to \infty$. To be clear, the second inequality follows from the triangle inequality and the last inequality follows straight from the definition of the norm on $\tilde{A}$. Therefore, $\{a_n\}_{n\in\mathbb{Z}_{>0}}$ is a Cauchy sequence in A. Hence, it must converge to some $a \in A$.

Now we claim that the sequence $\{(\lambda_n, a_n)\}_{n\in\mathbb{Z}_{>0}}$ converges to $(\lambda, a) \in A$. Assume that $\epsilon \in \mathbb{R}_{>0}$. Then, there exists $N_1, N_2 \in \mathbb{Z}_{>0}$ such that if $m > N_1$ and $n > N_2$ then

$$|\lambda_n - \lambda| < \frac{\epsilon}{2} \qquad \text{and} \qquad \|a_n - a\| < \frac{\epsilon}{2}.$$

So, if $m > \max\{N_1, N_2\}$ then

$$\begin{aligned}
\|(\lambda_m, a_m) - (\lambda, a)\| &= \|(\lambda_m - \lambda, a_m - a)\| \\
&\leq \|(\lambda_m - \lambda, a_m - a)\|_1 \\
&= |\lambda_m - \lambda| + \|a_m - a\| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Hence, the sequence $\{(\lambda_n, a_n)\}_{n\in\mathbb{Z}_{>0}}$ in $\tilde{A}$ converges to (λ, a). This demonstrates that $\tilde{A}$ is a unital Banach *-algebra.

Now we will show that $\tilde{A}$ is a unital C*-algebra. Again, assume that $(\lambda, a) \in \tilde{A}$. Then,

$$\begin{aligned}
\|(\lambda, a)^*\| &= \|(\overline{\lambda}, a^*)\| \\
&= \sup\{|\overline{\lambda}|, \|\overline{\lambda}b + a^*b\|, \|\overline{\lambda}b + ba^*\| \mid b \in A, \|b\| \leq 1\} \\
&= \sup\{|\overline{\lambda}|, \|\overline{\lambda}b^* + a^*b^*\|, \|\overline{\lambda}b^* + b^*a^*\| \mid b \in A, \|b\| \leq 1\} \\
&= \sup\{|\lambda|, \|\lambda b + ba\|, \|\lambda b + ab\| \mid b \in A, \|b\| \leq 1\} \\
&= \|(\lambda, a)\|.
\end{aligned}$$

Since the involution * in $\tilde{A}$ is isometric with respect to the norm in equation (1.6), we have

$$\|(\lambda, a)^*(\lambda, a)\| \leq \|(\lambda, a)^*\|\|(\lambda, a)\| = \|(\lambda, a)\|^2.$$

To obtain the reverse inequality, note that if $b \in A$ and $\|b\| \leq 1$ then

$$\begin{aligned}\|(\lambda,a)^*(\lambda,a)(0,b)\| &\geq \|(0,b)^*\|\|(\lambda,a)^*(\lambda,a)(0,b)\| \\ &\geq \|(0,b)^*(\lambda,a)^*(\lambda,a)(0,b)\| \\ &= \|(\lambda b+ab)^*(\lambda b+ab)\| \quad \text{(by part (b))} \\ &= \|\lambda b+ab\|^2 \quad (A \text{ is a C*-algebra}) \\ &= \|(\lambda,a)(0,b)\|^2 \quad \text{(by part (b))}.\end{aligned}$$

By replacing (λ, a) with $(\overline{\lambda}, a^*)$ and b with b^*, we deduce that $\|(\lambda,a)(\lambda,a)^*(0,b)^*\| \geq \|(\lambda,a)^*(0,b)^*\|^2$. Using the fact that the involution on $\tilde{A}$ is isometric, we find that

$$\|(0,b)(\lambda,a)(\lambda,a)^*\| \geq \|(0,b)(\lambda,a)\|^2.$$

Finally, we have

$$\begin{aligned}\|(\lambda,a)^*(\lambda,a)\| &= \sup\{|\lambda|^2, \|(\lambda,a)^*(\lambda,a)(0,b)\|, \|(0,b)(\lambda,a)^*(\lambda,a)\| \\ &\qquad \mid b \in A, \|b\| \leq 1\} \\ &\geq \sup\{|\lambda|^2, \|(\lambda,a)(0,b)\|^2, \|(0,b)(\lambda,a)\|^2 \mid b \in A, \|b\| \leq 1\} \\ &= \|(\lambda,a)\|^2.\end{aligned}$$

Therefore, $\tilde{A}$ is a unital C*-algebra as required.

Uniqueness: To see that $\tilde{A}$ is unique, assume that B is another unital C*-algebra which contains A as a closed two-sided ideal and the quotient $B/A \cong \mathbb{C}$. Let $1_B \in B$ be the unit of B. We define

$$\begin{array}{rccc}\rho: & \tilde{A} & \to & B \\ & (\lambda,a) & \mapsto & \lambda 1_B + a.\end{array}$$

It is straightforward to check that ρ is a unital *-homomorphism. Now let $q: B \to B/A$ be the quotient map. Since A is a closed two-sided ideal of B which is not all of B, then $1_B \notin A$ and subsequently, $q(1_B) \neq 0$.

To show: (d) ρ is injective.

(e) ρ is surjective.

(d) Assume that $(\lambda, a) \in \ker \rho$ so that $\rho(\lambda, a) = \lambda 1_B + a = 0$. Then, $\lambda 1_B = -a$ and by applying the quotient map q, we find that $\lambda q(1_B) = 0$ in $B/A \cong \mathbb{C}$. Since $q(1_B) \neq 0$, $\lambda = 0$ in $\mathbb{C}$ and $a = 0$ in A. Therefore, ρ is injective.

(e) Assume that $b \in B$. Note that the composite $q \circ \rho$ sends $(\lambda, a) \in \tilde{A}$ to $\lambda q(1_B) \in B/A \cong \mathbb{C}$. Hence, $q \circ \rho$ is surjective because if $\mu \in \mathbb{C}$ then

$$(q \circ \rho)(\frac{\mu}{q(1_B)}, 0) = \mu.$$

So, there exists $(\sigma, c) \in \tilde{A}$ such that

$$(q \circ \rho)(\sigma, c) = q(\sigma 1_B + c) = q(b).$$

This means that there exists $a' \in A \subseteq B$ such that $\sigma 1_B + c - a' = b$. Thus, $\rho(\sigma, c - a') = b$ and ρ is surjective.

By part (d) of the proof, ρ is an injective *-homomorphism. By Theorem 1.3.11, ρ is an isometry. In tandem with part (e), we conclude that ρ is a *-isomorphism from $\tilde{A}$ to B. Hence, $\tilde{A}$ is unique which completes the proof. $\square$

The C*-algebra $\tilde{A}$ is referred to as the **unitization** of A. The next theorem tells us how $\tilde{A}$ as constructed in Theorem 1.6.1 is connected to A when A is a unital C*-algebra itself.

Theorem 1.6.2. *Let A be a unital C*-algebra. Let $\tilde{A}$ be the unital C*-algebra constructed in Theorem 1.6.1. Then, $\tilde{A} \cong \mathbb{C} \oplus A$ as C*-algebras.*

Proof. Assume that A is a unital C*-algebra. Assume that $\tilde{A}$ is the unital C*-algebra constructed in Theorem 1.6.1. Recall from Theorem 1.1.3 that $\mathbb{C} \oplus A$ is a C*-algebra with scalar multiplication, multiplication, addition and involution defined pointwise. It has norm given by

$$\|(\lambda, a)\|_D = \max\{|\lambda|, \|a\|\}.$$

In $\tilde{A}$, scalar multiplication, involution and addition are the same as $\mathbb{C} \oplus A$. However, multiplication is given by

$$(\lambda, a)(\mu, b) = (\lambda\mu, \lambda b + \mu a + ab).$$

The norm on $\tilde{A}$ is given explicitly by

$$\|(\lambda, a)\| = \sup\{|\lambda|, \|\lambda b + ab\|, \|\lambda b + ba\| \mid b \in A, \|b\| \leq 1\}.$$

The key is to observe how the norms $\|-\|_D$ and $\|-\|$ are related. In particular, since A is unital,

$$\|(\lambda, a - \lambda 1_A)\| = \sup\{|\lambda|, \|ab\|, \|ba\| \mid b \in A, \|b\| \leq 1\}.$$

By the triangle inequality, we obtain

$$\|(\lambda, a - \lambda 1_A)\| \leq \max\{|\lambda|, \|a\|\} = \|(\lambda, a)\|_D.$$

But, we also have

$$\begin{aligned}
\|(\lambda, a)\|_D &= \max\{|\lambda|, \|a\|\} \\
&= \max\{|\lambda|, \|a1_A\|, \|1_A a\|\} \\
&\leq \sup\{|\lambda|, \|ab\|, \|ba\| \mid b \in A, \|b\| \leq 1\} \\
&= \|(\lambda, a - \lambda 1_A)\|.
\end{aligned}$$

Therefore, if $\lambda \in \mathbb{C}$ and $a \in A$ then $\|(\lambda, a)\|_D = \|(\lambda, a - \lambda 1_A)\|$. This suggests that we define the map

$$\begin{array}{rccc}
\varphi : & \mathbb{C} \oplus A & \to & \tilde{A} \\
& (\lambda, a) & \mapsto & (\lambda, a - \lambda 1_A)
\end{array}$$

We claim that φ is an isometric *-isomorphism. By our previous computation, φ is an isometry. Next, observe that if $(\lambda, a), (\mu, b) \in \mathbb{C} \oplus A$ and $\sigma \in \mathbb{C}$ then

$$\begin{aligned}
\varphi((\lambda, a) + (\mu, b)) &= \varphi(\lambda + \mu, a + b) \\
&= (\lambda + \mu, a + b - \lambda 1_A - \mu 1_A) \\
&= (\lambda, a - \lambda 1_A) + (\mu, b - \mu 1_A) \\
&= \varphi(\lambda, a) + \varphi(\mu, b),
\end{aligned}$$

$$\varphi((\lambda, a)^*) = \varphi(\overline{\lambda}, a^*) = (\overline{\lambda}, a^* - \overline{\lambda} 1_A) = \varphi(\lambda, a)^*,$$

$$\begin{aligned}
\varphi(\sigma(\lambda, a)) &= \varphi(\sigma\lambda, \sigma a) \\
&= (\sigma\lambda, \sigma a - \sigma\lambda 1_A) \\
&= \sigma(\lambda, a - \lambda 1_A) = \sigma\varphi(\lambda, a),
\end{aligned}$$

and

$$\begin{aligned}
\varphi((\lambda, a)(\mu, b)) &= \varphi(\lambda\mu, ab) \\
&= (\lambda\mu, ab - \lambda\mu 1_A) \\
&= (\lambda\mu, (\lambda b - \lambda\mu 1_A) + (\mu a - \lambda\mu 1_A) + (ab - \mu a - \lambda b + \lambda\mu 1_A)) \\
&= (\lambda, a - \lambda 1_A)(\mu, b - \mu 1_A) \\
&= \varphi(\lambda, a)\varphi(\mu, b).
\end{aligned}$$

Finally, to see that φ is bijective, observe that φ has an inverse given by

$$\begin{array}{rccc}
\varphi^{-1}: & \tilde{A} & \to & \mathbb{C} \oplus A \\
& (\lambda, a) & \mapsto & (\lambda, a + \lambda 1_A)
\end{array}$$

One can check by direct calculation that φ^{-1} is also a *-homomorphism. Consequently, φ is an isometric *-isomorphism and $\tilde{A} \cong \mathbb{C} \oplus A$ as C*-algebras. □

The unital C*-algebra $\tilde{A}$ constructed in Theorem 1.6.1 satisfies the following *universal property.* This is discussed in the reference [Ter20].

Theorem 1.6.3. *Let A be a C*-algebra. Then, the unital C*-algebra $\tilde{A}$ in Theorem 1.6.1 satisfies the following universal property: If B is a unital C*-algebra and $f : A \to B$ is a *-homomorphism then there exists a unique unital *-homomorphism $\tilde{f} : \tilde{A} \to B$ such that the following diagram commutes:*

$$\begin{tikzcd}
A \arrow[r, hook, "\iota"] \arrow[dr, "f"'] & \tilde{A} \arrow[d, dashed, "\tilde{f}"] \\
& B
\end{tikzcd}$$

Here, $\iota : A \to \tilde{A}$ denotes the inclusion map $a \mapsto (0, a)$.

Proof. Assume that A is a C*-algebra and that $\tilde{A}$ is the unital C*-algebra constructed in Theorem 1.6.1. Assume that B is a unital C*-algebra and that $f : A \to B$ is a *-homomorphism. Define the map $\tilde{f}$ by

$$\begin{array}{rccc}
\tilde{f}: & \tilde{A} & \to & B \\
& (\lambda, a) & \mapsto & \lambda 1_B + f(a)
\end{array}$$

A barrage of brief computations shows that $\tilde{f}$ is a unital *-homomorphism such that $\tilde{f} \circ \iota = f$, where $\iota : A \to \tilde{A}$ is the inclusion map $a \mapsto (0, a)$.

To see that $\tilde{f}$ is unique, assume that $g : \tilde{A} \to B$ is another unital *-homomorphism such that $g \circ \iota = f$. If $a \in A$ then

$$g(0, a) = (g \circ \iota)(a) = f(a) = \tilde{f}(0, a).$$

Since g is a unital *-homomorphism and $(1, 0) \in \tilde{A}$ is the unit of $\tilde{A}$, we have $g(1, 0) = 1_B$. Subsequently, if $\lambda \in \mathbb{C} - \{0\}$ and $a \in A$ then

$$\begin{aligned}
g(\lambda, a) &= g(\lambda(1, \frac{1}{\lambda}a)) \\
&= \lambda g(1, \frac{1}{\lambda}a) \\
&= \lambda g((1, 0) + (0, \frac{1}{\lambda}a)) \\
&= \lambda(g(1, 0) + g(0, \frac{1}{\lambda}a)) \\
&= \lambda(1_B + f(\frac{1}{\lambda}a)) \\
&= \lambda 1_B + f(a) = \tilde{f}(\lambda, a).
\end{aligned}$$

Therefore, $g = \tilde{f}$ which completes the proof. □

We will use the universal property in Theorem 1.6.3 to extend Theorem 1.3.11.

Theorem 1.6.4. *Let A and B be C*-algebras and $\phi : A \to B$ be an injective *-homomorphism. Then, ϕ is an isometry.*

Proof. Assume that A and B are C*-algebras and $\phi : A \to B$ is an injective *-homomorphism. By the universal property of unitization in Theorem 1.6.3, there exists a unique unital *-homomorphism $\tilde{\phi} : \tilde{A} \to \tilde{B}$ such that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, hook, "\iota_A"] \arrow[dr, "\iota_B \circ \phi"'] & \tilde{A} \arrow[d, dashed, "\tilde{\phi}"] \\
& \tilde{B}
\end{tikzcd}$$

Here, ι_A and ι_B are the inclusion $A \hookrightarrow \tilde{A}$ and $B \hookrightarrow \tilde{B}$.

We claim that $\tilde{\phi}$ is injective. Explicitly, it is given by

$$\begin{array}{cccc}
\tilde{\phi} : & \tilde{A} & \to & \tilde{B} \\
& (\lambda, a) & \mapsto & \lambda(1, 0) + (0, \phi(a)) = (\lambda, \phi(a)).
\end{array}$$

Assume that $(\mu, c) \in \ker \tilde{\phi}$. Then, $\tilde{\phi}(\mu, c) = (\mu, \phi(c)) = (0, 0)$. Then, $\mu = 0$ and $c = 0$ because ϕ is injective. So, $(\mu, c) = (0, 0)$ and $\tilde{\phi}$ is injective.

By Theorem 1.3.11, $\tilde{\phi}$ is an isometry. If $a \in A$ then

$$\|\phi(a)\| = \|(0, \phi(a))\| = \|\tilde{\phi}(0, a)\| = \|(0, a)\| = \|a\|.$$

Hence, ϕ is an isometry as required. $\square$

Returning to the exposition in [Put19], we are now interested in extending Theorem 1.3.5 and the continuous functional calculus in Theorem 1.3.7 to the case of non-unital C*-algebras. The most direct way to do this is to remove the assumption that the C*-algebra is unital in both theorems. Before we proceed, we remind ourselves of continuous functions which *vanish at infinity.* First recall that a topological space X is **locally compact** if for $x \in X$, there exists an open set U and a compact set K such that $x \in U \subseteq K$.

Definition 1.6.1. Let X be a locally compact Hausdorff space. We say that $f \in Cts(X, \mathbb{C})$ **vanishes at infinity** if for $\epsilon \in \mathbb{R}_{>0}$, there exists a compact subset $K \subseteq X$ such that if $x \in X \backslash K$ then $|f(x)| < \epsilon$. Equivalently, if $\epsilon \in \mathbb{R}_{>0}$ then the set

$$\{x \in X \mid |f(x)| \geq \epsilon\}$$

is compact. Furthermore, we define

$$Cts_0(X, \mathbb{C}) = \{f \in Cts(X, \mathbb{C}) \mid f \text{ vanishes at infinity}\}.$$

Theorem 1.6.5. *Let X be a locally compact Hausdorff space. Then,*

1. *$Cts_0(X, \mathbb{C})$ is a subalgebra of the algebra*

$$Cts_b(X, \mathbb{C}) = \{f \in Cts(X, \mathbb{C}) \mid f \textit{ is bounded}\}.$$

2. *$Cts_0(X, \mathbb{C})$ is a commutative C*-algebra with norm given by*

$$\|f\| = \sup_{x \in X} |f(x)|$$

for $f \in Cts_0(X, \mathbb{C})$.

Proof. Assume that X is a locally compact Hausdorff space. Let $Cts_b(X,\mathbb{C})$ be the algebra of bounded functions from X to $\mathbb{C}$.

To show: (a) $Cts_0(X,\mathbb{C}) \subseteq Cts_b(X,\mathbb{C})$.

(b) $Cts_0(X,\mathbb{C})$ is closed under scalar multiplication.

(c) $Cts_0(X,\mathbb{C})$ is closed under addition.

(d) $Cts_0(X,\mathbb{C})$ is closed under multiplication.

(a) Assume that $f \in Cts_0(X,\mathbb{C})$. If $\epsilon \in \mathbb{R}_{>0}$ then the set

$$S_\epsilon = \{x \in X \mid |f(x)| \geq \epsilon\}$$

is a compact subset of X. Fix $\epsilon \in \mathbb{R}_{>0}$. Since S_ϵ is compact and f is continuous then the image $f(S_\epsilon)$ is a compact subset of $\mathbb{C}$. By the Heine-Borel theorem, $f(S_\epsilon)$ must be closed and bounded. So, there exists $r \in \mathbb{R}_{>0}$ such that

$$f(S_\epsilon) \subseteq B(0,r) = \{\lambda \in \mathbb{C} \mid |\lambda| < r\}.$$

So, if $x \in S_\epsilon$ then $|f(x)| < r$. On the other hand, if $x \notin S_\epsilon$ then $|f(x)| < \epsilon$. Therefore, if $x \in X$ then

$$|f(x)| < \max\{r, \epsilon\}.$$

So, $f \in Cts_b(X,\mathbb{C})$.

(b) Assume that $\lambda \in \mathbb{C} - \{0\}$ and $\epsilon \in \mathbb{R}_{>0}$. If $f \in Cts_0(X,\mathbb{C})$ then the set

$$\{x \in X \mid |(\lambda f)(x)| \geq \epsilon\} = \{x \in X \mid |f(x)| \geq \frac{\epsilon}{|\lambda|}\}$$

is a compact subset of X. So, $\lambda f \in Cts_0(X,\mathbb{C})$. If $\lambda = 0$ then by definition of $Cts_0(X,\mathbb{C})$, the zero function $0 = \lambda f \in Cts_0(X,\mathbb{C})$.

(c) Assume that $f, g \in Cts_0(X,\mathbb{C})$ and $\epsilon \in \mathbb{R}_{>0}$. Define the sets

$$K_f = \{x \in X \mid |f(x)| \geq \frac{\epsilon}{2}\} \qquad \text{and} \qquad K_g = \{x \in X \mid |g(x)| \geq \frac{\epsilon}{2}\}.$$

Also define

$$K = \{x \in X \mid |(f+g)(x)| \geq \epsilon\}.$$

Then, $K_f^c \cap K_g^c \subseteq K^c$ and by taking complements, $K \subseteq K_f \cup K_g$. Since $f, g \in Cts_0(X, \mathbb{C})$ then $K_f \cup K_g$ is a compact subset of X. To see that K is compact, it suffices to show that K is closed.

Assume that $\{k_n\}_{n \in \mathbb{Z}_{>0}}$ is a sequence in K which converges to some $k \in X$. We compute directly that

$$\begin{aligned} |(f+g)(k)| &= |f(k) + g(k)| \\ &= \lim_{n \to \infty} |f(k_n) + g(k_n)| \\ &= \lim_{n \to \infty} |(f+g)(k_n)| \geq \epsilon. \end{aligned}$$

Let us justify the inequality in the above working. Suppose for the sake of contradiction that $|(f+g)(k)| < \epsilon$. Since $\{(f+g)(k_n)\}_{n \in \mathbb{Z}_{>0}}$ converges to $(f+g)(k)$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $m > N$ and $t \in (0, \epsilon)$ then

$$|(f+g)(k_m)| - |(f+g)(k)| < t - |(f+g)(k)|.$$

So, $|(f+g)(k_m)| < t < \epsilon$ which contradicts the assumption that $k_m \in K$. Thus, $|(f+g)(k)| \geq \epsilon$ and $k \in K$. This means that K is a closed subset of the compact set $K_f \cup K_g$ and subsequently, K itself is compact. We conclude that $f + g \in Cts_0(X, \mathbb{C})$.

(d) Assume that $f, g \in Cts_0(X, \mathbb{C})$ and $\epsilon \in \mathbb{R}_{>0}$. Define the sets

$$L_f = \{x \in X \mid |f(x)| \geq \epsilon^{\frac{1}{2}}\}, \qquad L_g = \{x \in X \mid |g(x)| \geq \epsilon^{\frac{1}{2}}\}.$$

and

$$L = \{x \in X \mid |(fg)(x)| \geq \epsilon\}.$$

Then, $L_f^c \cap L_g^c \subseteq L^c$ and $L \subseteq L_f \cup L_g$. Since $f, g \in Cts_0(X, \mathbb{C})$ then $L_f \cup L_g$ is compact. Arguing in the same manner as part (c), we deduce that L is closed and hence, compact. This uses the fact that multiplication in $\mathbb{C}$ is continuous. So, $fg \in Cts_0(X, \mathbb{C})$.

By parts (a), (b), (c) and (d), we find that $Cts_0(X, \mathbb{C})$ is indeed a subalgebra of $Cts_b(X, \mathbb{C})$.

Now $Cts_0(X, \mathbb{C})$ is commutative because multiplication is defined pointwise and multiplication in $\mathbb{C}$ is commutative. To see that $Cts_0(X, \mathbb{C})$ is closed under the involution inherited from $Cts(X, \mathbb{C})$, assume that $f \in Cts_0(X, \mathbb{C})$. If $\epsilon \in \mathbb{R}_{>0}$ then

$$\{x \in X \mid |\overline{f}(x)| \geq \epsilon\} = \{x \in X \mid |\overline{f(x)}| \geq \epsilon\} = \{x \in X \mid |f(x)| \geq \epsilon\}$$

is a compact subset of X. So, $\overline{f} \in Cts_0(X, \mathbb{C})$. The norm on $Cts_0(X, \mathbb{C})$

$$\begin{array}{cccc} \|-\|: & Cts_0(X, \mathbb{C}) & \to & \mathbb{R}_{\geq 0} \\ & f & \mapsto & \sup_{x \in X} |f(x)| \end{array}$$

which is inherited from the norm on $Cts(X, \mathbb{C})$ satisfies the C*-algebra property. Therefore, to see that $Cts_0(X, \mathbb{C})$ is a commutative C*-algebra, it suffices to show that $Cts_0(X, \mathbb{C})$ is a closed subspace of $Cts(X, \mathbb{C})$.

Assume that $\{g_n\}_{n \in \mathbb{Z}_{>0}}$ is a sequence in $Cts_0(X, \mathbb{C})$ which converges to a function $g \in Cts(X, \mathbb{C})$. If $\epsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $n > N$ then

$$\|g_n - g\| = \sup_{x \in X} |g_n(x) - g(x)| < \frac{\epsilon}{2}.$$

Fix $m > N$ and define the sets

$$P = \{x \in X \mid |g_m(x)| \geq \frac{\epsilon}{2}\}, \qquad Q = \{x \in X \mid |g_m(x) - g(x)| \geq \frac{\epsilon}{2}\}$$

and

$$R = \{x \in X \mid |g(x)| \geq \epsilon\}.$$

Then, $P^c \cap Q^c \subseteq R^c$ and $R \subseteq P \cup Q$. But, by our construction of the index $N \in \mathbb{Z}_{>0}$, $Q = \emptyset$ and $R \subseteq P$. Since $g_m \in Cts_0(X, \mathbb{C})$ then P is a compact subset of X. By adapting the argument made in part (c), we find that R is a closed subset of P and hence, compact. So, $g \in Cts_0(X, \mathbb{C})$ and $Cts_0(X, \mathbb{C})$ is a closed subspace of $Cts(X, \mathbb{C})$. Consequently, $Cts_0(X, \mathbb{C})$ is a commutative C*-algebra as required. □

There is a simple criterion to determine whether $Cts_0(X, \mathbb{C})$ is unital.

Theorem 1.6.6. *Let X be a locally compact Hausdorff space. Then, $Cts_0(X, \mathbb{C})$ is a unital C*-algebra if and only if X is compact.*

Proof. Assume that X is a locally compact Hausdorff space.

To show: (a) If $Cts_0(X, \mathbb{C})$ is a unital C*-algebra then X is compact.

(b) If X is compact then $Cts_0(X, \mathbb{C})$ is a unital C*-algebra.

(a) Assume that the C*-algebra $Cts_0(X, \mathbb{C})$ is unital. Let $1_X : X \to \mathbb{C}$ be the function defined by $1_X(x) = 1$ for $x \in X$. Then, 1_X is the multiplicative unit for $Cts_0(X, \mathbb{C})$.

Since $1_X \in Cts_0(X, \mathbb{C})$ by assumption, there exists a compact subset $K \subseteq X$ such that if $x \in X - K$ then $|1_X(x)| < \frac{1}{2}$. However, $1_X(x) = 1$ for arbitrary $x \in X$. Hence, $X - K = \emptyset$ and consequently, $X = K$ must be compact.

(b) Assume that X is compact. To see that $Cts_0(X, \mathbb{C})$ is unital, it suffices to show that $1_X \in Cts_0(X, \mathbb{C})$. Assume that $\epsilon \in \mathbb{R}_{>0}$. If $\epsilon > 1$ then $|1_X(x)| < \epsilon$ for $x \in X = X - \emptyset$. If $\epsilon \leq 1$ then $|1_X(x)| < \epsilon$ for $x \in \emptyset = X - X$. Since X and $\emptyset$ are compact sets, we deduce that $1_X \in Cts_0(X, \mathbb{C})$. So, $Cts_0(X, \mathbb{C})$ is a unital C*-algebra. □

If X happens to be a compact Hausdorff space then a particular C*-subalgebra of $Cts(X, \mathbb{C})$ can be related to functions which vanish at infinity in the following manner:

Theorem 1.6.7. *Let X be a compact Hausdorff space and $x_0 \in X$. Then, we have the isomorphism of C*-algebras*

$$\{f \in Cts(X, \mathbb{C}) \mid f(x_0) = 0\} \cong Cts_0(X - \{x_0\}, \mathbb{C}).$$

Proof. Assume that X is a compact Hausdorff space. Define the map

$$\begin{array}{rccc} r: & \{f \in Cts(X, \mathbb{C}) \mid f(x_0) = 0\} & \to & Cts_0(X - \{x_0\}, \mathbb{C}) \\ & \phi & \mapsto & \phi|_{X - \{x_0\}} \end{array}$$

The map r is restriction to $X - \{x_0\}$. To see that r is well-defined, assume that $\varphi \in Cts(X, \mathbb{C})$ and $\varphi(x_0) = 0$. Firstly, $X - \{x_0\}$ is locally compact because it is an open subset of the compact space X. Now assume that $\epsilon \in \mathbb{R}_{>0}$. Since φ is continuous, there exists an open neighbourhood U_0 of X such that $x_0 \in U_0$ and if $u \in U_0$ then $|f(u)| < \epsilon$.

Now the set $X - U_0$ is a closed subset of X, which is compact. Hence, $X - U_0$ is also compact. Observe that

$$\{x \in X \mid |f(x)| \geq \epsilon\} \subseteq X - U_0.$$

Since the set $\{x \in X \mid |f(x)| \geq \epsilon\}$ is closed, it must be compact by the above inclusion. Therefore, $f|_{X-\{x_0\}} \in Cts_0(X - \{x_0\}, \mathbb{C})$ and consequently, the map r is well-defined.

The fact that r is a *-homomorphism follows from its definition. It is also easy to see that r is injective. To see that r is surjective, assume that $g \in Cts_0(X - \{x_0\}, \mathbb{C})$. Define $\tilde{g} : X \to \mathbb{C}$ by $\tilde{g}(x) = g(x)$ if $x \neq x_0$ and $\tilde{g}(x_0) = 0$. Then, $\tilde{g} \in Cts(X, \mathbb{C})$ (see [Mur20]) and $r(\tilde{g}) = g$ by construction. So, r is surjective.

We conclude that r is a *-isomorphism and consequently, we obtain the isomorphism of C*-algebras

$$\{f \in Cts(X, \mathbb{C}) \mid f(x_0) = 0\} \cong Cts_0(X - \{x_0\}, \mathbb{C}).$$

□

Now we will prove our extension of Theorem 1.3.5.

Theorem 1.6.8. *Let A be a commutative C*-algebra and $\tilde{A}$ be the unital C*-algebra constructed in Theorem 1.6.1. Let $\pi : \tilde{A} \to \tilde{A}/A \cong \mathbb{C}$ be the canonical quotient map. Then, $\pi \in \mathcal{M}(\tilde{A})$. Moreover, the restriction of the *-isomorphism in Theorem 1.3.5*

$$\begin{array}{rccc} \Lambda : & \tilde{A} & \to & Cts(\mathcal{M}(\tilde{A}), \mathbb{C}) \\ & a & \mapsto & ev_a \end{array}$$

*to A is an isometric *-isomorphism from A to $Cts_0(\mathcal{M}(\tilde{A}) - \{\pi\}, \mathbb{C})$.*

Proof. Assume that A is a commutative C*-algebra. Since $(1, 0)$ is the multiplicative unit of $\tilde{A}$, $\pi((1, 0)) = 1 \neq 0$. So, π is non-zero and $\pi \in \mathcal{M}(\tilde{A})$.

Assume that Λ is the *-isomorphism defined as above on $\tilde{A}$. If $a \in A$ then

$$\Lambda(a)(\pi) = ev_a(\pi) = \pi(a) = 0.$$

Hence, the image of A under Λ is contained in the C*-subalgebra

$$\{f \in Cts(\mathcal{M}(\tilde{A}), \mathbb{C}) \mid f(\pi) = 0\} \cong Cts_0(\mathcal{M}(\tilde{A}) - \{\pi\}, \mathbb{C})$$

by Theorem 1.6.7. The proof that the restriction $\Lambda|_A$ is a *-homomorphism and an isometry follows what was done in Theorem 1.3.5. To see that it is injective, assume that $a_1, a_2 \in A$ and that $\Lambda(a_1) = \Lambda(a_2)$. Then, $ev_{a_1} = ev_{a_2}$ and if $f \in Cts(\mathcal{M}(\tilde{A}), \mathbb{C})$ then $f(a_1) = f(a_2)$. Hence, $a_1 = a_2$ and $\Lambda|_A$ is injective.

Finally to see that $\Lambda|_A$ is surjective, note that A has codimension 1 in $\tilde{A}$ since $\tilde{A}/A \cong \mathbb{C}$. Also, $Cts_0(\mathcal{M}(\tilde{A}) - \{\pi\}, \mathbb{C})$ has codimension 1 in $Cts(\mathcal{M}(\tilde{A}), \mathbb{C})$ by Theorem 1.6.7. Therefore, the restriction $\Lambda|_A$ is an isometric *-isomorphism from A to $Cts_0(\mathcal{M}(\tilde{A}) - \{\pi\}, \mathbb{C})$. □

Observe that we have the following important consequence of Theorem 1.6.8.

Theorem 1.6.9. *Let A be a commutative C*-algebra. Then, there exists a locally compact Hausdorff space X such that $A \cong Cts_0(X, \mathbb{C})$ as C*-algebras.*

Now we proceed to a generalisation of the continuous functional calculus in Theorem 1.3.7. As stated in [Put19, Page 39], there is a subtlety here which needs to be addressed. If B is a possibly non-unital C*-algebra and $a \in B$ is normal then we can consider the unital C*-algebra $\tilde{B}$ and then apply Theorem 1.3.7 to $\tilde{B}$. However, the image of the *-isomorphism in Theorem 1.3.7 is the C*-algebra generated by a and the multiplicative unit of $\tilde{B}$, which lies outside of B (since B is not assumed to be unital).

The way this issue is circumvented is to strengthen the conclusion of Theorem 1.3.7 by showing if $f \in Cts(\sigma(a), \mathbb{C})$ satisfies $f(0) = 0$ then $f(a) \in B$ lies in the C*-subalgebra generated by a. In fact, this is useful even if B is unital. Hence, our extension of Theorem 1.3.7 will be stated for unital C*-algebras.

Theorem 1.6.10. *Let B be a unital C*-algebra and $a \in B$ be normal. The *-isomorphism in Theorem 1.3.7 restricts to an isometric *-isomorphism from $Cts_0(\sigma(a) - \{0\}, \mathbb{C})$ to the C*-subalgebra of B generated by a.*

Proof. Assume that B is a unital C*-algebra and $a \in B$ be normal. We have two different cases to consider.

Case 1: $0 \notin \sigma(a)$.

First assume that $0 \notin \sigma(a)$. Then, a is invertible and

$$Cts_0(\sigma(a) - \{0\}, \mathbb{C}) = Cts(\sigma(a), \mathbb{C})$$

because $\sigma(a) - \{0\} = \sigma(a)$ is a compact Hausdorff space. Hence, we need to show that the C*-algebra generated by a and its unit is the same as the C*-algebra generated by a alone.

To this end, let f be a continuous function on $\sigma(a) \cup \{0\}$ satisfying $f(0) = 0$ and $f(x) = 1$ for $x \in \sigma(a)$. Such a continuous function exists by Urysohn's lemma. Assume that $\epsilon \in \mathbb{R}_{>0}$. By Weierstrass's approximation theorem, there exists a polynomial $p(z, \overline{z})$ such that

$$|p(z, \overline{z}) - f(z)| < \frac{\epsilon}{2}.$$

Now $p(0, 0)$ yields the constant term of the polynomial $p(z, \overline{z})$. So, $p(z, \overline{z}) - p(0, 0)$ is a polynomial with no constant term which satisfies

$$|p(z, \overline{z}) - p(0, 0) - f(z)| = |p(z, \overline{z}) - p(0, 0) - f(z) - f(0)| < \epsilon.$$

By applying the map Λ^{-1} in Theorem 1.3.7, we deduce that $f(a) = 1_B$ and

$$\|p(a, a^*) - 1_B\| < \epsilon.$$

By Theorem 1.3.7, $p(a, a^*)$ is an element of the C*-subalgebra generated by a. Therefore, 1_B is also in the C*-subalgebra of B generated by a. This demonstrates that the C*-subalgebra generated by the set $\{1_B, a\}$ is exactly the C*-subalgebra generated by a.

By Theorem 1.6.7, Λ^{-1} maps $Cts_0(\sigma(a) - \{0\}, \mathbb{C})$ to the C*-subalgebra generated by a. Moreover, it is an isometric *-isomorphism by Theorem 1.3.7.

Case 2: $0 \in \sigma(a)$.

Assume that $0 \in \sigma(a)$. Let $f \in Cts_0(\sigma(a) - \{0\}, \mathbb{C})$. By the isomorphism in Theorem 1.6.7, we can think of f as a continuous function on $\sigma(a)$ satisfying $f(0) = 0$. Again, assume that $\epsilon \in \mathbb{R}_{>0}$. By Weierstrass's approximation theorem, there exists a polynomial $p(z, \overline{z})$ such that $|p(z, \overline{z}) - f(z)| < \epsilon/2$. By repeating the same argument as in Case 1, we find that $f(a)$ is approximated to within ϵ by $p(a, a^*)$ which is an element of the C*-subalgebra generated by only a. Hence, $f(a)$ is in fact, an element of the C*-subalgebra generated by a.

As in Case 1, we conclude that the restriction of Λ^{-1} to $Cts_0(\sigma(a) - \{0\}, \mathbb{C})$ is an isometric *-isomorphism from $Cts_0(\sigma(a) - \{0\}, \mathbb{C})$ to the C*-subalgebra generated by only a. $\square$

1.7 Ideals and Quotients

Let A be a C*-algebra and I be a subspace of A. What conditions on I are required for the quotient A/I to be a C*-algebra itself? Algebraically, if I is a two-sided ideal then A/I is a $\mathbb{C}$-algebra. Topologically, if I is a closed as a subset of A then A/I is a Banach space. This suggests that we focus on closed two-sided ideals I of A in order to answer our question.

We begin by formalising and proving the topological statement about the quotient A/I. First, we will recall a useful criterion for proving whether a normed vector space is a Banach space.

Definition 1.7.1. Let V be a normed vector space. A sequence $\{x_n\}_{n\in\mathbb{Z}_{>0}}$ is called **absolutely summable** if the quantity

$$\sum_{n=1}^{\infty}\|x_n\| < \infty.$$

The sequence $\{x_n\}_{n\in\mathbb{Z}_{>0}}$ is called **summable** if the sequence of partial sums $\{\sum_{n=1}^{N} x_n\}_{N\in\mathbb{Z}_{>0}}$ converges to some $x \in V$.

Theorem 1.7.1. *Let V be a normed vector space. Then, V is a Banach space if and only if every absolutely summable sequence is summable.*

Proof. Assume that V is a normed vector space.

To show: (a) If V is a Banach space then every absolutely summable sequence is summable.

(b) If every absolutely summable sequence is summable then V is a Banach space.

(a) Assume that V is a Banach space so that V is complete with respect to its norm. Assume that $\{x_n\}_{n\in\mathbb{Z}_{>0}}$ is an absolutely summable sequence in V. To see that the sequence of partial sums $\{\sum_{n=1}^{N} x_n\}_{N\in\mathbb{Z}_{>0}}$ converges, it suffices to show that it is a Cauchy sequence.

Now the sequence $\{\sum_{n=1}^{N}\|x_n\|\}_{N\in\mathbb{Z}_{>0}}$ converges in $\mathbb{R}$ to $\sum_{n=1}^{\infty}\|x_n\| < \infty$. So, it is a Cauchy sequence. Assume that $\epsilon \in \mathbb{R}_{>0}$. Then, there exists $N \in \mathbb{Z}_{>0}$ such that if $N_1, N_2 > N$ with $N_1 < N_2$ then

$$\Big|\sum_{n=1}^{N_1}\|x_n\| - \sum_{n=1}^{N_2}\|x_n\|\Big| = \sum_{n=N_1+1}^{N_2}\|x_n\| < \epsilon.$$

Therefore,

$$\|\sum_{n=1}^{N_1} x_n - \sum_{n=1}^{N_2} x_n\| = \|\sum_{n=N_1+1}^{N_2} x_n\| \leq \sum_{n=N_1+1}^{N_2} \|x_n\| < \epsilon.$$

So, the sequence $\{\sum_{n=1}^{N} x_n\}_{N\in\mathbb{Z}_{>0}}$ in V is Cauchy and hence, converges. Thus, the sequence $\{x_n\}_{n\in\mathbb{Z}_{>0}}$ is summable.

(b) Assume that in V, every absolutely summable sequence is summable. Assume that $\{x_n\}_{n\in\mathbb{Z}_{>0}}$ is a Cauchy sequence in V. To see that $\{x_n\}_{n\in\mathbb{Z}_{>0}}$ converges, it suffices to construct a convergent subsequence $\{x_{n_k}\}$.

Assume that $k \in \mathbb{Z}_{>0}$. Then, there exists $N_k \in \mathbb{Z}_{>0}$ such that if $m, n > N_k$ then

$$\|x_m - x_n\| < 2^{-k}.$$

Now define $n_k = \sum_{i=1}^{k} N_i$. If $i, j \in \mathbb{Z}_{>0}$ and $i < j$ then $n_i < n_j$. Moreover, if $k \in \mathbb{Z}_{>0}$ then by construction,

$$\|x_{n_k} - x_{n_{k+1}}\| < 2^{-k}.$$

We claim that the subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges. To this end, consider the sequence $\{x_{n_k} - x_{n_{k+1}}\}_{k\in\mathbb{Z}_{>0}}$ in V. This sequence is absolutely summable because

$$\sum_{k=1}^{\infty} \|x_{n_k} - x_{n_{k+1}}\| < \sum_{k=1}^{\infty} 2^{-k} = 1 < \infty.$$

By our assumption, we conclude that $\{x_{n_k} - x_{n_{k+1}}\}_{k\in\mathbb{Z}_{>0}}$ is summable, which means that the sequence $\{\sum_{k=1}^{L} x_{n_k} - x_{n_{k+1}}\}_{L\in\mathbb{Z}_{>0}}$ in V. But,

$$\sum_{k=1}^{L} (x_{n_k} - x_{n_{k+1}}) = x_{n_1} - x_{n_{L+1}}.$$

Consequently, the sequence $\{x_{n_k}\}$ converges and so does $\{x_n\}_{n\in\mathbb{Z}_{>0}}$. So, V is complete and hence, a Banach space. $\square$

Theorem 1.7.2. *Let A be a Banach space and I be a closed subspace of A. Then, the quotient space A/I is a Banach space equipped with the norm*

$$\|a + I\| = \inf_{b\in I} \|a + b\|.$$

Proof. Assume that A is a Banach space and I is a closed subspace of A. For $a \in A$, we define

$$\|a + I\| = \inf_{b \in I} \|a + b\|.$$

To show: (a) The map $\|-\| : A/I \to \mathbb{R}_{\geq 0}$ is well-defined.

(a) Assume that $a_1 + I = a_2 + I$ in A/I. Then, $a_1 - a_2 \in I$ and

$$\begin{aligned}
\|a_1 + I\| &= \inf_{b \in I} \|a_1 + b\| \\
&= \inf_{b \in I} \|a_2 + (a_1 - a_2 + b)\| \\
&= \inf_{b \in I} \|a_2 + b\| = \|a_2 + I\|.
\end{aligned}$$

Hence, the map $\|-\| : A/I \to \mathbb{R}_{\geq 0}$ is well-defined.

Next, we show that $\|-\|$ defines a norm on A/I. If $\alpha \in \mathbb{C}$ then

$$\begin{aligned}
\|\alpha(a + I)\| &= \|\alpha a + I\| \\
&= \inf_{b \in I} \|\alpha a + b\| \\
&= \inf_{b \in I} \|\alpha a + \alpha b\| \\
&= |\alpha| \inf_{b \in I} \|a + b\| = |\alpha| \|a + I\|.
\end{aligned}$$

By definition of the map $\|-\|$, if $a + I \in A/I$ then $\|a + I\| \geq 0$. Observe that $\|a + I\| = 0$ if and only if

$$\inf_{b \in I} \|a + b\| = 0.$$

This means that there exists a sequence $\{b_n\}_{n \in \mathbb{Z}_{>0}}$ in I such that $\lim_{n \to \infty} \|a - b_n\| = 0$. Since I is a closed subspace of A, $a \in I$ and $a + I = 0$ in A/I.

Finally, if $a_1 + I, a_2 + I \in A/I$ then

$$\begin{aligned}
\|(a_1+I)+(a_2+I)\| &= \|(a_1+a_2)+I\| \\
&= \inf_{b\in I}\|a_1+a_2+b\| \\
&= \inf_{b\in I}\|a_1+b+a_2+b\| \\
&\le \inf_{b\in I}\|a_1+b\| + \inf_{b\in I}\|a_2+b\| \\
&= \|a_1+I\| + \|a_2+I\|.
\end{aligned}$$

Hence, $\|-\|$ defines a norm on A/I.

To show: (b) A/I is a Banach space.

(b) We will use Theorem 1.7.1. Suppose that $\{a_n+I\}_{n\in\mathbb{Z}_{>0}}$ is an absolutely summable sequence in A/I. Then, the quantity

$$\sum_{n=1}^{\infty}\|a_n+I\| = \sum_{n=1}^{\infty}\inf_{b\in I}\|a_n+b\| < \infty.$$

If $n\in\mathbb{Z}_{>0}$ then we can select $b_n\in I$ such that $\|a_n+b_n\| < 2\inf_{b\in I}\|a_n+b\|$. We claim that the sequence $\{a_n+b_n\}_{n\in\mathbb{Z}_{>0}}$ in A is absolutely summable.

To see why this is the case, we compute directly that

$$\sum_{n=1}^{\infty}\|a_n+b_n\| \le 2\sum_{n=1}^{\infty}\|a_n+b\| < \infty.$$

Since A is complete, the absolutely summable sequence $\{a_n+b_n\}$ is in turn summable by Theorem 1.7.1. Hence, the sequence $\{\sum_{n=1}^{N}(a_n+b_n)\}_{N\in\mathbb{Z}_{>0}}$ converges to some $c\in A$. Consequently,

$$\begin{aligned}
\|\Big(\sum_{n=1}^{N}(a_n+I)\Big)-(c+I)\| &\le \|\Big(\sum_{n=1}^{N}a_n-c\Big)+\sum_{n=1}^{N}b_n\| \\
&= \|\Big(\sum_{n=1}^{N}(a_n+b_n)\Big)-c\| \\
&\to 0
\end{aligned}$$

as $N\to\infty$. Therefore, the sequence $\{a_n+I\}_{n\in\mathbb{Z}_{>0}}$ is summable. By Theorem 1.7.1, we deduce that A/I is complete and thus, a Banach space. $\square$

A natural application of Theorem 1.7.2 is to the case where A is a Banach algebra — a Banach *-algebra without an involution operation. If A is a Banach algebra and I is a closed, two sided ideal then the quotient A/I is simultaneously a $\mathbb{C}$-algebra and a Banach space. It is a Banach algebra itself because if $a+I, b+I \in A/I$ then

$$\begin{aligned}
\|(a+I)(b+I)\| &= \|ab+I\| \\
&= \inf_{j \in I} \|ab+j\| \\
&= \inf_{i,j \in I} \|ab+(ai+bj+ij)\| \\
&\leq \big(\inf_{i \in I} \|a+i\|\big)\big(\inf_{j \in I} \|b+j\|\big) \\
&= \|a+I\|\|b+I\|.
\end{aligned}$$

Now let A be a C*-algebra. In order for the quotient A/I to be a C*-algebra, there are two sticking points which must be addressed.

1. What is the involution operation on A/I? The obvious definition of an involution on A/I would be $(a+I)^* = a^* + I$, but in order for this map to be well-defined, the closed two-sided ideal I must be closed under involution.

2. Does the norm on A/I satisfy the C*-algebra condition?

Note that in order to answer the second question, we must have an answer to the first question. We will prove shortly that any closed two-sided ideal I of a C*-algebra is closed under involution. The reference [Sol18, Section A.5.2] proves this result for the special case of a unital C*-algebra. Our proof uses the following technical result.

Theorem 1.7.3. *Let A be a C*-algebra and $a \in A$. If $\epsilon \in \mathbb{R}_{>0}$ then there exists $f \in Cts([0,\infty), \mathbb{C})$ such that $e = f(a^*a) \in A$, is positive, $\|e\| \leq 1$ and $\|a - ae\| < \epsilon$.*

Proof. Assume that A is a C*-algebra and $a \in A$. First, recall that by Theorem 1.4.5, the spectrum $\sigma(a^*a) \subseteq [0,\infty)$ since a^*a is positive. Assume that $\epsilon \in \mathbb{R}_{>0}$. Define the function f by

$$\begin{array}{rccc}
f: & [0,\infty) & \to & \mathbb{C} \\
& x & \mapsto & \frac{x}{x+\epsilon} = 1 - \frac{\epsilon}{x+\epsilon}.
\end{array}$$

Note that $f(0) = 0$. By Theorem 1.6.10 and Theorem 1.6.7, the element $e = f(a^*a)$ is contained in the C*-algebra generated by a^*a and hence, in A.

Now observe that if $t \in [0, \infty)$ then $f(t) \in (0, 1)$. By the continuous functional calculus in Theorem 1.6.10,

$$\|e\| = \sup_{t \in [0,\infty)} |f(t)| \leq 1.$$

By the spectral mapping theorem in Theorem 1.3.14 and Theorem 1.4.5, $e = f(a^*a)$ is a positive element of A.

To see that $\|a - ae\| < \epsilon$, we turn to the unitization $\tilde{A}$ of A. Recalling that $(1, 0) \in \tilde{A}$ is the multiplicative unit of $\tilde{A}$, we compute directly that

$$\begin{aligned}
\|a - ae\|^2 &= \|(0, a - ae)\|^2 \\
&= \|(0, a)\big((1, 0) - (0, e)\big)\|^2 \\
&= \|(0, a)\big((1, 0) - (0, f(a^*a))\big)\|^2 \\
&= \|\big((1, 0) - (0, f(a^*a))\big)(0, a^*)(0, a)\big((1, 0) - (0, f(a^*a))\big)\| \\
&= \|\big((1, 0) - (0, f(a^*a))\big)(0, a^*a)\big((1, 0) - (0, f(a^*a))\big)\| \\
&= \|g(a^*a)\| \leq \|g\|_\infty
\end{aligned}$$

where $g(t) = t(1 - f(t))^2$. The function $g(t)$ obtains it maximum at $t = \epsilon$ and $g(\epsilon) = \epsilon/4$. Therefore, $\|a - ae\| < \frac{\sqrt{\epsilon}}{2}$ and we are done. □

The main theorem of this section uses Theorem 1.7.3.

Theorem 1.7.4. *Let A be a C*-algebra and I be a (topologically) closed, two-sided ideal. Then, I is closed under the involution on A and A/I with the quotient norm in Theorem 1.7.2 is a C*-algebra.*

Proof. Assume that A is a C*-algebra and I is a closed, two-sided ideal.

To show: (a) If $a \in I$ then $a^* \in I$.

(a) Assume that $a \in I$ and $\epsilon \in \mathbb{R}_{>0}$. By Theorem 1.7.3, there exists a sequence $\{e_i\}_{i \in \mathbb{Z}_{\geq 0}}$ in A such that

$$\|a - ae_i\| < \frac{\epsilon}{2^i}.$$

Each e_i is in the C*-algebra generated by a^*a. Since $a \in I$, then $a^*a \in I$ and the C*-algebra generated by a^*a is contained in I. Hence, if $i \in \mathbb{Z}_{\geq 0}$ then $e_i \in I$ and $\{ae_i\}_{i \in \mathbb{Z}_{\geq 0}}$ is a sequence in I which converges (in the norm topology on A) to a.

Now

$$\|e_i a^* - a^*\| = \|(a - ae_i)^*\| = \|a - ae_i\| < \frac{\epsilon}{2^i}.$$

Since $e_i \in I$ and I is an ideal, then $\{e_i a^*\}_{i \in \mathbb{Z}_{\geq 0}}$ is a sequence in I which converges (in the norm topology on A) to a^*. Since I is closed topologically, we deduce that $a^* \in I$.

To show: (b) If $a \in A$ then $\|(a + I)(a^* + I)\| = \|a^* a + I\| = \|a + I\|^2$.

(b) Again, assume that $a \in A$. By part (a), the involution operation on A/I given by $(a + I)^* = a^* + I$ is well-defined. Observe that

$$\begin{aligned}
\|a^* + I\| &= \inf_{b \in I} \|a^* + b\| \\
&= \inf_{b \in I} \|a^* + b^*\| \\
&= \inf_{b \in I} \|(a + b)^*\| = \inf_{b \in I} \|a + b\| \\
&= \|a + I\|.
\end{aligned}$$

This means that

$$\|a^* a + I\| \leq \|a + I\| \|a^* + I\| = \|a + I\|^2.$$

To see that the reverse inequality holds, we claim that

$$\|a + I\| = \inf\{\|a - ae\| \mid e \in I, e \geq 0, \|e\| \leq 1\}.$$

First, note that if $e \in I$ is positive and $\|e\| < 1$ then $\sigma((0, e)) \subseteq [0, 1]$. Here, we used Theorem 1.6.1 to consider the spectrum of an element of A in the unitization $\tilde{A}$. By the continuous functional calculus in Theorem 1.3.7 applied to the unitization $\tilde{A}$ and the spectral mapping Theorem 1.3.14, we find that the spectrum $\sigma((1, 0) - (0, e)) \subseteq [0, 1]$ and $\|(1, 0) - (0, e)\| = \|(1, -e)\| \leq 1$. So,

$$\|a + I\| = \inf_{b \in I} \|a + b\| \leq \inf\{\|a - ae\| \mid e \in I, e \geq 0, \|e\| \leq 1\}.$$

For the reverse inequality, let $b \in I$ and $\epsilon \in \mathbb{R}_{>0}$. By Theorem 1.7.3, there exists $e \in I$ such that e is positive, $\|e\| \leq 1$ and $\|b - be\| < \epsilon$. By the reverse triangle inequality, we have

$$\begin{aligned}
\|a+b\| &= \|(0,a+b)\| \\
&\geq \|(0,a+b)\|\|(1,0)-(0,e)\| \\
&\geq \|(0,a)\big((1,0)-(0,e)\big)\| - \|(0,b)\big((1,0)-(0,e)\big)\| \\
&= \|(0,a)\big((1,0)-(0,e)\big)\| - \|b-be\| \\
&\geq \|a-ae\| - \epsilon.
\end{aligned}$$

Since $\epsilon \in \mathbb{R}_{>0}$ was arbitrary, we deduce that

$$\inf_{b\in I}\|a+b\| \geq \inf\{\|a-ae\| \mid e\in I, e\geq 0, \|e\|\leq 1\}.$$

Combined with the previous inequality, we obtain equality. With part (b), we now compute for $e \in I$ positive with $\|e\| < 1$ that

$$\begin{aligned}
\|a-ae\|^2 &= \|(0,a)\big((1,0)-(0,e)\big)\|^2 \\
&= \|\big((1,0)-(0,e)\big)(0,a^*a)\big((1,0)-(0,e)\big)\| \\
&\leq \|\big((1,0)-(0,e)\big)\|\|(0,a^*a)\big((1,0)-(0,e)\big)\| \\
&\leq \|(0,a^*a)\big((1,0)-(0,e)\big)\| = \|a^*a - a^*ae\|.
\end{aligned}$$

Taking the infimum over all such e, we find that

$$\begin{aligned}
\|a+I\|^2 &= \inf_{b\in I}\|a+b\|^2 \\
&= \inf\{\|a-ae\|^2 \mid e\in I, e\geq 0, \|e\|\leq 1\} \\
&\leq \inf\{\|a^*a-a^*ae\| \mid e\in I, e\geq 0, \|e\|\leq 1\} \\
&= \inf_{b\in I}\|a^*a+b\| = \|a^*a+I\|.
\end{aligned}$$

Since we also have $\|a^*a+I\| \leq \|a+I\|^2$, then $\|a^*a+I\| = \|a+I\|^2$. Thus, A/I is a C*-algebra as required. $\square$

A direct consequence of Theorem 1.7.4 is the following theorem regarding the projection map $A \to A/I$.

Theorem 1.7.5. *Let A be a C*-algebra and I be a closed two-sided ideal with $I \neq A$. Then, the projection map*

$$\begin{array}{rccc}
\pi: & A & \to & A/I \\
 & a & \mapsto & a+I
\end{array}$$

*is a *-homomorphism with norm 1.*

Proof. Assume that A is a C*-algebra and I is a closed two-sided ideal of A. The fact that the projection map $\pi : A \to A/I$ is a *-homomorphism follows from direct computation.

To see that $\|\pi\| = 1$, first observe that if $a \in A$ then

$$\|\pi(a)\| = \|a + I\| = \inf_{j \in I} \|a + j\| \leq \|a + 0\| = \|a\|.$$

So, $\|\pi\| \leq 1$. Alternatively, we could use Theorem 1.2.7.

To see that $\|\pi\| \geq 1$, let $b \in A - I$. Then, $\pi(b) = b + I \neq 0$ in A/I. If $x \in I$ then

$$\|b + I\| = \|(b + x) + I\| = \|\pi(b + x)\| \leq \|\pi\|\|b + x\|.$$

By taking the infimum over all $x \in I$, we deduce that

$$\|b + I\| \leq \|\pi\| \inf_{x \in I} \|b + x\| = \|\pi\|\|b + I\|.$$

So, $\|\pi\| \geq 1$. Hence, $\|\pi\| = 1$. $\square$

It is remarked in [Put19, Page 42] that the projection map being a *-homomorphism is often useful. With Theorem 1.7.4, we will prove some more useful facts about *-homomorphisms. The first of these is [Put19, Exercise 1.9.3].

Theorem 1.7.6. *Let A and B be C*-algebras and $\phi : A \to B$ be a *-homomorphism. Then, the image $\phi(A)$ is closed and is a C*-subalgebra of B.*

Proof. Assume that A and B are C*-algebras and $\phi : A \to B$ is a *-homomorphism.

To show: (a) The image $\phi(A)$ is closed.

(a) Consider the kernel $\ker \phi$, which is a closed two-sided ideal. By the universal property of the quotient, there exists a unique *-homomorphism ϕ' such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\pi"] \arrow[dr, "\phi"'] & A/\ker\phi \arrow[d, dashed, "\tilde{\phi}"] \\ & B \end{tikzcd}$$

Observe that $\tilde{\phi}$ is injective. If $a+\ker\phi \in \ker\tilde{\phi}$ then $\tilde{\phi}(a+\ker\phi) = \phi(a) = 0$. Then, $a \in \ker\phi$ and $a+\ker\phi = \ker\phi$ in $A/\ker\phi$. So, $\tilde{\phi}$ is injective.

By Theorem 1.6.4, $\tilde{\phi}$ must be an isometry. To see that the image $\tilde{\phi}(A/\ker\phi)$ is closed, assume that $\{b_i\}_{i\in\mathbb{Z}_{>0}}$ is a sequence in $\tilde{\phi}(A/\ker\phi)$ which converges to some $b \in B$. If $i \in \mathbb{Z}_{>0}$ then there exists $a_i + \ker\phi \in A/\ker\phi$ such that $\tilde{\phi}(a_i+\ker\phi) = b_i$. Since $\{b_i\}_{i\in\mathbb{Z}_{>0}}$ converges, it must be Cauchy.

Assume that $\epsilon \in \mathbb{R}_{>0}$. Then, there exists $N \in \mathbb{Z}_{>0}$ such that if $m, n > N$ then

$$\|b_m - b_n\| < \epsilon.$$

Using the fact that $\tilde{\phi}$ is an isometry, we find that

$$\|a_m+\ker\phi-(a_n+\ker\phi)\| = \|\tilde{\phi}(a_m+\ker\phi)-\tilde{\phi}(a_n+\ker\phi)\| = \|b_m-b_n\| < \epsilon.$$

Hence, $\{a_i+\ker\phi\}$ is a Cauchy sequence in the C*-algebra $A/\ker\phi$. So, it must converge to some $a+\ker\phi \in A/\ker\phi$.

Now we claim that $\tilde{\phi}(a+\ker\phi) = b$. We argue that for $m \in \mathbb{Z}_{>0}$ large enough,

$$\begin{aligned}
\|\tilde{\phi}(a+\ker\phi) - b\| &\leq \|\tilde{\phi}(a+\ker\phi) - \tilde{\phi}(a_m+\ker\phi)\| + \|\tilde{\phi}(a_m+\ker\phi) - b\| \\
&= \|(a+\ker\phi) - (a_m+\ker\phi)\| + \|b_m - b\| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Since $\epsilon \in \mathbb{R}_{>0}$ was arbitrary, $\|\tilde{\phi}(a+\ker\phi) - b\| = 0$ and $\tilde{\phi}(a+\ker\phi) = b$. Therefore, $b \in \tilde{\phi}(A/\ker\phi)$ and the image $\tilde{\phi}(A/\ker\phi)$ is closed.

Subsequently, we deduce that $\phi(A) = \tilde{\phi}(A/\ker\phi)$ is closed. Furthermore, the image $\phi(A)$ is closed under scalar multiplication, multiplication, addition and involution. So, $\phi(A)$ is a C*-subalgebra of B as required. $\square$

1.8 Trace on a C*-algebra

By definition, a C*-algebra is a Banach space. By the Hahn-Banach extension theorem, a C*-algebra has plenty of linear functionals. It is often useful to consider linear functionals with extra properties. For instance, a $\mathbb{C}$-algebra homomorphism from a C*-algebra A to $\mathbb{C}$ is simply a linear functional on A which is multiplicative. These types of functionals featured prominently in our analysis of commutative unital C*-algebras.

However, as remarked in [Put19, Section 1.10], most C*-algebras do not have $\mathbb{C}$-algebra homomorphisms to $\mathbb{C}$. In order to deal with these C*-algebras, we need to consider maps which are somewhere in between a linear functional and a $\mathbb{C}$-algebra homomorphism. This gives rise to the notion of a *trace*.

Definition 1.8.1. Let A be a unital C*-algebra and $\phi : A \to \mathbb{C}$ be a linear functional. We say that ϕ is **positive** if for $a \in A$, $\phi(a^*a) \geq 0$.

A **trace** on A is a positive linear functional $\tau : A \to \mathbb{C}$ such that $\tau(1_A) = 1$ and if $a, b \in A$ then

$$\tau(ab) = \tau(ba).$$

The last property is referred to as the **trace property**. The trace is called **faithful** if $\tau(a^*a) = 0$ implies that $a = 0$.

Any $\mathbb{C}$-algebra homomorphism satisfies the trace property. If A is a commutative unital C*-algebra then every positive linear functional $\phi : A \to \mathbb{C}$ with $\phi(1_A) = 1$ is a trace.

In order to illustrate the concept of a trace, we will construct a trace on the C*-algebra of bounded linear operators on a finite-dimensional Hilbert space. Recall that if H is a Hilbert space then the C*-algebra of bounded linear operators on H is denoted by $B(H)$.

Theorem 1.8.1. *Let H be a Hilbert space with finite dimension $n \in \mathbb{Z}_{>0}$. Let $\{\xi_1, \ldots, \xi_n\}$ be an orthonormal basis for H. On the C*-algebra $B(H)$, define the map*

$$\begin{array}{rccc} \tau : & B(H) & \to & \mathbb{C} \\ & a & \mapsto & \frac{1}{n}\sum_{i=1}^{n}\langle a\xi_i, \xi_i\rangle. \end{array}$$

Then, τ is a faithful trace on $B(H)$, which is unique.

Before we delve into the proof, we note that by identifying $B(H)$ with the C*-algebra of matrices $M_{n\times n}(\mathbb{C})$, the map τ is expressed for $A = (a_{ij}) \in M_{n\times n}(\mathbb{C})$ as

$$\tau(A) = \frac{1}{n}\sum_{i=1}^{n} a_{ii} = \frac{1}{n}Tr(A).$$

Proof. Assume that H is a Hilbert space with finite dimension $n \in \mathbb{Z}_{>0}$. Assume that $\{\xi_1, \ldots, \xi_n\}$ is an orthonormal basis for H. Assume that τ is defined as above. By linearity of the inner product, we deduce that τ is a linear functional on $B(H)$.

To show: (a) τ is positive and if $id_H \in B(H)$ is the identity operator then $\tau(id_H) = 1$.

(b) If $a, b \in B(H)$ then $\tau(ab) = \tau(ba)$.

(c) τ is unique.

(d) τ is a faithful trace.

(a) Assume that $a \in B(H)$. We compute directly that

$$\begin{aligned}
\tau(a^*a) &= \frac{1}{n}\sum_{i=1}^{n}\langle a^*a\xi_i, \xi_i\rangle \\
&= \frac{1}{n}\sum_{i=1}^{n}\langle a\xi_i, a\xi_i\rangle \\
&= \frac{1}{n}\sum_{i=1}^{n}\|a\xi_i\|^2 \geq 0.
\end{aligned}$$

So, τ is a positive linear functional. Assume that $id_H \in B(H)$ is the identity operator. By the above computation,

$$\tau(id_H) = \tau(id_H^* id_H) = \frac{1}{n}\sum_{i=1}^{n}\|\xi_i\|^2 = 1.$$

(b) Since the linear span of rank one operators on H is $B(H)$, it suffices to prove the trace property for rank one operators a and b. We first observe that if $\xi, \eta \in H$ then

$$
\begin{aligned}
\tau(|\xi\rangle\langle\eta|) &= \frac{1}{n}\sum_{i=1}^{n}\langle|\xi\rangle\langle\eta|)\xi_i, \xi_i\rangle \\
&= \frac{1}{n}\sum_{i=1}^{n}\langle\langle\xi_i, \eta\rangle\xi, \xi_i\rangle \\
&= \frac{1}{n}\sum_{i=1}^{n}\langle\xi_i, \eta\rangle\langle\xi, \xi_i\rangle \\
&= \frac{1}{n}\langle\sum_{i=1}^{n}\langle\xi, \xi_i\rangle\xi_i, \eta\rangle \\
&= \frac{1}{n}\langle\xi, \eta\rangle.
\end{aligned}
$$

Now let $a = |\sigma_1\rangle\langle\eta_1|$ and $b = |\sigma_2\rangle\langle\eta_2|$, where $\sigma_1, \sigma_2, \eta_1, \eta_2 \in H$. Then,

$$
ab = \langle\sigma_2, \eta_1\rangle|\sigma_1\rangle\langle\eta_2| \qquad \text{and} \qquad ba = \langle\sigma_1, \eta_2\rangle|\sigma_2\rangle\langle\eta_1|.
$$

We compute directly that

$$
\tau(ab) = \frac{1}{n}\langle\sigma_2, \eta_1\rangle\langle\sigma_1, \eta_2\rangle = \tau(ba).
$$

Since the linear span of rank one operators on H is all of $B(H)$, τ satisfies the trace property and is therefore, a trace.

(c) Assume that φ is another trace on $B(H)$. Let $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$. We will first show that φ and τ agree on the rank one operator $|\xi_i\rangle\langle\xi_j|$.

Let $a = |\xi_i\rangle\langle\xi_i|$ and $b = |\xi_i\rangle\langle\xi_j|$. Then, $ab = |\xi_i\rangle\langle\xi_j| = b$ and $ba = 0$. We compute directly that

$$
\phi(b) = \phi(ab) = \phi(ba) = 0 = \frac{1}{n}\langle\xi_i, \xi_j\rangle = \tau(b).
$$

Now assume that $i, j \in \{1, 2, \ldots, n\}$, which may or may not be distinct. Observe that $b^*b = (|\xi_j\rangle\langle\xi_i|)(|\xi_i\rangle\langle\xi_j|) = |\xi_j\rangle\langle\xi_j|$ and $bb^* = |\xi_i\rangle\langle\xi_i|$. Since ϕ is a trace, we must have

$$
\phi(|\xi_j\rangle\langle\xi_j|) = \phi(b^*b) = \phi(bb^*) = \phi(|\xi_i\rangle\langle\xi_i|).
$$

Now observe that

$$1 = \phi(id_H) = \phi(\sum_{i=1}^{n} |\xi_i\rangle\langle\xi_i|) = n\phi(|\xi_1\rangle\langle\xi_1|).$$

So,

$$\phi(|\xi_i\rangle\langle\xi_i|) = \phi(|\xi_1\rangle\langle\xi_1|) = \frac{1}{n} = \tau(|\xi_i\rangle\langle\xi_i|).$$

Hence, we have shown that τ and ϕ agree on the set $\{|\xi_i\rangle\langle\xi_j| \mid i, j \in \{1, 2, \ldots, n\}\}$, which is a spanning set for $B(H)$. Hence, τ and ϕ agree on all of $B(H)$, rendering the trace τ unique.

(d) Assume that $a \in B(H)$ and $\tau(a^*a) = 0$. Then,

$$\frac{1}{n}\sum_{i=1}^{n} \|a\xi_i\|^2 = 0$$

and subsequently, if $i \in \{1, 2, \ldots, n\}$ then $\|a\xi_i\| = 0$. So, $a = 0$ and τ is a faithful trace on $B(H)$. $\square$

Here are a few consequences of Theorem 1.8.1.

Theorem 1.8.2. *Let H be a finite-dimensional Hilbert space with dimension $\dim H \in \mathbb{Z}_{>0}$. Let τ be the unique trace on $B(H)$ constructed in Theorem 1.8.1. If p is a projection then $\dim pH = \tau(p) \dim H$.*

Proof. Assume that H is a finite dimensional Hilbert space. Assume that τ is the trace on $B(H)$ constructed in Theorem 1.8.1. Assume that p is a projection. We choose an orthonormal basis $\{\xi_1, \ldots, \xi_n\}$ for H in such a way that $\{\xi_1, \ldots, \xi_k\}$ is a basis for pH.

We compute directly that

$$\begin{aligned} \tau(p) \dim H &= n \cdot \frac{1}{n} \sum_{i=1}^{n} \langle p\xi_i, \xi_i\rangle \\ &= \sum_{i=1}^{n} \langle p\xi_i, \xi_i\rangle = \sum_{i=1}^{k} \langle \xi_i, \xi_i\rangle \\ &= k = \dim pH. \end{aligned}$$

$\square$

Theorem 1.8.2 states that if the trace is applied to projections then it recovers the geometric notion of the dimension of the range. Theorem 1.8.1 also tells us that if $n \in \mathbb{Z}_{>1}$ then there is no non-zero *-homomorphism from $M_{n\times n}(\mathbb{C})$ to $\mathbb{C}$.

Theorem 1.8.3. *Let $n \in \mathbb{Z}_{>1}$. Then, there does not exist a non-zero *-homomorphism from $M_{n\times n}(\mathbb{C})$ to $\mathbb{C}$.*

Proof. Assume that $n \in \mathbb{Z}_{>1}$. We identify $M_{n\times n}(\mathbb{C})$ with $B(H)$, where H is a Hilbert space with finite dimension n. Let $\{\xi_1, \dots, \xi_n\}$ be an orthonormal basis for H. Suppose for the sake of contradiction that $\alpha : B(H) \to \mathbb{C}$ is a non-zero *-homomorphism. Then, α is a trace.

By uniqueness in Theorem 1.8.1, $\alpha = \tau$ where τ is the trace constructed in Theorem 1.8.1. However, if $p \in B(H)$ is the projection operator onto the span of $\{\xi_1, \dots, \xi_k\}$ (with $k < n$) then $\tau(p) = \tau(p^2) = \tau(p)^2$. But, by the computation in Theorem 1.8.2

$$\tau(p) = \frac{k}{n} \neq \frac{k^2}{n^2} = \tau(p)^2.$$

This yields the desired contradiction. So, there is no non-zero *-homomorphism from $M_{n\times n}(\mathbb{C})$ to $\mathbb{C}$. □

1.9 Representations of C*-algebras

As mentioned at the beginning of this document, the prototypical example of a C*-algebra is the space of bounded linear operators $B(H)$ on some Hilbert space H. This particular C*-algebra was studied extensively in [Sol18]. Keeping this example in mind, we would like to find the ways an arbitrary C*-algebra can act as bounded linear operators on H. This gives rise to representations of C*-algebras.

Definition 1.9.1. Let A be a *-algebra (a $\mathbb{C}$-algebra with an involution map). A **representation** of A is a pair (π, H), where H is a Hilbert space and $\pi : A \to B(H)$ is a *-homomorphism.

We also say that π is a representation of A on H. In this section, we will focus on the basic definitions and properties regarding representations of *-algebras. We will begin with the necessary definitions.

Definition 1.9.2. Let A be a *-algebra. We say that two representations of A, (π_1, H_1) and (π_2, H_2) are **unitarily equivalent** if there exists a unitary operator $u : H_1 \to H_2$ such that if $a \in A$ then

$$u\pi_1(a) = \pi_2(a)u.$$

If the representations (π_1, H_1) and (π_2, H_2) are unitarily equivalent then we write $(\pi_1, H_1) \sim_u (\pi_2, H_2)$ or $\pi_1 \sim_u \pi_2$.

As remarked in [Put19, Page 46], unitarily equivalent representations are considered to be the same. Before we press on with the definitions, we will formalise this remark and prove that unitary equivalence is an equivalence relation.

Theorem 1.9.1. *Let A be a *-algebra. The relation of unitary equivalence between two representations of A, denoted by $\sim_u$, is an equivalence relation.*

Proof. Assume that A is a *-algebra. For reflexivity, if (π_1, H_1) is a representation of A then $(\pi_1, H_1) \sim_u (\pi_1, H_1)$ because if $a \in A$ and $id_{H_1} \in B(H_1)$ is the identity operator then

$$id_{H_1}\pi_1(a) = \pi_1(a)id_{H_1}.$$

For symmetry, assume that $(\pi_1, H_1) \sim_u (\pi_2, H_2)$. Then, there exists a unitary operator $u : H_1 \to H_2$ such that if $a \in A$ then $u\pi_1(a) = \pi_2(a)u$. Its inverse $u^{-1} : H_2 \to H_1$ is also unitary,

$$\pi_1(a) = u^{-1}\pi_2(a)u \qquad \text{and} \qquad \pi_1(a)u^{-1} = u^{-1}\pi_2(a).$$

So, $(\pi_2, H_2) \sim_u (\pi_1, H_1)$.

For transitivity, assume that $(\pi_1, H_1) \sim_u (\pi_2, H_2)$ and $(\pi_2, H_2) \sim_u (\pi_3, H_3)$. Then, there exists unitary operators $s : H_1 \to H_2$ and $t : H_2 \to H_3$ such that if $a \in A$ then $s\pi_1(a) = \pi_2(a)s$ and $t\pi_2(a) = \pi_3(a)t$. The composite $ts : H_1 \to H_3$ is a unitary operator which satisfies

$$ts\pi_1(a) = t\pi_2(a)s = \pi_3(a)ts.$$

Hence, $(\pi_1, H_1) \sim_u (\pi_3, H_3)$ and $\sim_u$ is therefore an equivalence relation. □

A major operation one can perform on representations of a fixed *-algebra is to take the direct sum of representations.

Definition 1.9.3. Let A be a *-algebra and $\{(\pi_i, H_i)\}_{i\in I}$ be a family of representations of A. The **direct sum** of the representations $\{(\pi_i, H_i)\}_{i\in I}$ is given by the pair

$$(\bigoplus_{i\in I} \pi_i, \bigoplus_{i\in I} H_i)$$

where

$$\bigoplus_{i\in I} H_i = \{(\xi_i)_{i\in I} \mid \sum_{i\in I} \|\xi_i\|^2 < \infty\}$$

and $\bigoplus_{i\in I} \pi_i$ is the *-homomorphism

$$\begin{array}{rccc} \bigoplus_{i\in I} \pi_i : & A & \to & B(\bigoplus_{i\in I} H_i) \\ & a & \mapsto & \big((\xi_i)_{i\in I} \mapsto (\pi_i(a)\xi_i)_{i\in I}\big) \end{array}$$

For a refresher on the direct sum of Hilbert spaces, a good reference is [Con90, Chapter I, §6]. The next few definitions mirror foundational definitions in representation theory.

Definition 1.9.4. Let A be a *-algebra and (π, H) be a representation of A. A subspace $N \subseteq H$ is called **invariant** if for $a \in A$, $\pi(a)N \subset N$.

Definition 1.9.5. Let A be a *-algebra and (π, H) be a representation of A. We say that the representation (π, H) is **non-degenerate** if the following statement is satisfied: If $\xi \in H$, $a \in A$ and $\pi(a)\xi = 0$ then $\xi = 0$.

Otherwise, the representation is called **degenerate**.

Definition 1.9.6. Let A be a *-algebra and (π, H) be a representation of A. Let $\xi \in H$. We say that ξ is **cyclic** if the linear subspace $\pi(A)\xi$ is dense in H. We say that the representation (π, H) is **cyclic** if there exists a cyclic vector $\xi \in H$.

Observe that if A is *-algebra and (π, H) is a cyclic representation of A then (π, H) must be non-degenerate. Assume that $\xi \in H$ is the cyclic vector, $\eta \in H$, $a \in A$ and $\pi(a)\eta = 0$. Using the inner product on H, we have

$$0 = \langle \pi(a)\eta, \xi\rangle = \langle \eta, \pi(a)^*\xi\rangle = \langle \eta, \pi(a^*)\xi\rangle.$$

So, $\eta \in (\pi(A)\xi)^\perp$ and since $\pi(A)\xi$ is dense in H, $(\pi(A)\xi)^\perp = \{0\}$. Therefore, $\eta = 0$ and the representation (π, H) is cyclic as required.

Definition 1.9.7. Let A be a *-algebra and (π, H) be a representation of A. The representation (π, H) is called **irreducible** if the only closed invariant subspaces of H are the zero subspace 0 and H. Otherwise, the representation is called **reducible**.

When dealing with representations of *-algebras, we are generally interested in invariant subspaces which are closed topologically. The following theorem tells us when a closed subspace is invariant.

Theorem 1.9.2. *Let A be a *-algebra and (π, H) be a representation of A. Let $N \subseteq H$ be a closed subspace. Then, N is invariant if and only if the orthogonal complement $N^{\perp}$ is invariant.*

Proof. Assume that A is a *-algebra and (π, H) be a representation of A. Assume that $N \subseteq H$ is a closed subspace.

To show: (a) If N is invariant then $N^{\perp}$ is invariant.

(b) If $N^{\perp}$ is invariant then N is invariant.

(a) Assume that the closed subspace N is invariant. This means that if $a \in A$ then $\pi(a)N \subset N$. To see that $N^{\perp}$ is invariant, assume that $a \in A$ and $\eta \in N^{\perp}$. If $\xi \in H$ then

$$\langle \pi(a)\eta, \xi \rangle = \langle \eta, \pi(a)^{*}\xi \rangle = \langle \eta, \pi(a^{*})\xi \rangle = 0$$

because $\pi(a^{*})\xi \in N$ and $\eta \in N^{\perp}$. Hence, $\pi(a)\eta \in N^{\perp}$ and the subspace $N^{\perp}$ is invariant.

(b) Assume that $N^{\perp}$ is invariant. If we apply part (a), we find that $(N^{\perp})^{\perp} = N$ is invariant because N is closed. □

In the scenario of Theorem 1.9.2, we can define two further representations of A, by restricting the operators to either N or $N^{\perp}$. More precisely in the case of N, the Hilbert space N together with the *-homomorphism

$$\begin{array}{rccc} \pi|_N : & A & \to & B(N) \\ & a & \mapsto & \pi(a)|_N \end{array}$$

is a representation of A. Unsurprisingly, the following two representations of A are unitarily equivalent

$$(\pi, H) \sim_u (\pi|_N, N) \oplus (\pi|_{N^{\perp}}, N^{\perp}).$$

The following result is a direct consequence of the definitions introduced.

Theorem 1.9.3. *Let A be a unital *-algebra with multiplicative unit 1_A and (π, H) be a representation of A. The representation (π, H) is non-degenerate if and only if $\pi(1_A) = id_H$ where id_H is the identity operator on H.*

Proof. Assume that A is a unital *-algebra and (π, H) is a representation of A.

To show: (a) If (π, H) is non-degenerate then $\pi(1_A) = id_H$.

(b) If $\pi(1_A) = id_H$ then (π, H) is non-degenerate.

(a) Assume that the representation (π, H) is non-degenerate. Assume that $\xi \in H - \{0\}$ is non-zero. Since the representation (π, H) is non-degenerate, there exists $a \in A$ such that $\pi(a)\xi \neq 0$. But,

$$0 = \pi(a)\xi - \pi(a)\xi = \pi(1_A)\pi(a)\xi - id_H(\pi(a)\xi).$$

Since $\pi(a)\xi \neq 0$, we deduce that $\pi(1_A) = id_H$ as required.

Since the representation (π, H) is non-degenerate, then $\pi(a) = 0$.

(b) Assume that $\pi(1_A) = id_H$. Assume that $\xi \in H$ satisfies $\pi(a)\xi = 0$ for $a \in A$. Then,

$$\|\xi\|^2 = \langle \xi, \xi \rangle = \langle \pi(1_A)\xi, \pi(1_A)\xi \rangle = 0.$$

So, $\xi = 0$ and the representation (π, H) is non-degenerate. $\square$

Here is a useful characterisation of non-degenerate representations.

Theorem 1.9.4. *Let A be a *-algebra and (π, H) be a representation of A. Then, (π, H) is non-degenerate if and only if $\overline{\pi(A)H} = H$.*

Proof. Assume that A is a *-algebra and (π, H) is a representation of A.

To show: (a) If (π, H) is non-degenerate then $\overline{\pi(A)H} = H$.

(b) If $\overline{\pi(A)H} = H$ then (π, H) is non-degenerate.

(a) Assume that (π, H) is non-degenerate. Since $H = (\pi(A)H)^\perp \oplus \overline{\pi(A)H}$, it suffices to show that $(\pi(A)H)^\perp = \{0\}$. To this end, assume that $\eta \in (\pi(A)H)^\perp$. If $a \in A$ and $\xi \in H$ then

$$\langle \pi(a^*)\xi, \eta \rangle = \langle \xi, \pi(a)\eta \rangle = 0.$$

So, $\pi(a)\eta = 0$ and since (π, H) is non-degenerate, $\eta = 0$. Hence, $(\pi(A)H)^\perp = \{0\}$ and $\overline{\pi(A)H} = H$.

(b) Assume that $\overline{\pi(A)H} = H$. Assume that $\xi \in H$ satisfies $\pi(a)\xi = 0$ for $a \in A$. Since $\overline{\pi(A)H} = H$ and $H = (\pi(A)H)^\perp \oplus \overline{\pi(A)H}$, $(\pi(A)H)^\perp = \{0\}$. If $\pi(b)\eta \in \pi(A)H$ then

$$\langle \pi(b)\eta, \xi\rangle = \langle \eta, \pi(b^*)\xi\rangle = 0.$$

So, $\xi \in (\pi(A)H)^\perp = \{0\}$ and $\xi = 0$. Hence, (π, H) is a non-degenerate representation of A. $\square$

When dealing with representations of *-algebras, we can focus on non-degenerate representations, as evidenced by the following theorem:

Theorem 1.9.5. *Let A be a *-algebra and (π, H) be a representation of A. Then, (π, H) is the direct sum of a non-degenerate representation and the zero representation.*

Proof. Assume that A is a *-algebra and (π, H) is a representation of A. Consider the closure $\overline{\pi(A)H}$, where $\pi(A)H$ is the subspace

$$\pi(A)H = \{\pi(a)\xi \mid a \in A, \xi \in H\}.$$

Then, H can be written as the direct sum

$$H = (\pi(A)H)^\perp \oplus \overline{\pi(A)H}.$$

We claim that $(\pi|_{(\pi(A)H)^\perp}, (\pi(A)H)^\perp)$ is the zero representation. If $a \in A$, $\xi \in (\pi(A)H)^\perp$ and $\eta \in H$ then

$$\langle \pi(a)\xi, \eta\rangle = \langle \xi, \pi(a^*)\eta\rangle = 0.$$

We deduce that if $a \in A$ and $\xi \in (\pi(A)H)^\perp$ then

$$\pi(a)\xi = \pi(a)|_{(\pi(A)H)^\perp}\xi = \pi|_{(\pi(A)H)^\perp}(a)\xi = 0.$$

Thus, the representation $(\pi|_{(\pi(A)H)^\perp}, (\pi(A)H)^\perp)$ is the zero representation.

To see that the representation $(\pi|_{\overline{\pi(A)H}}, \overline{\pi(A)H})$ is non-degenerate, it suffices to show that

$$\overline{\pi|_{\overline{\pi(A)H}}(A)\overline{\pi(A)H}} = \overline{\pi(A)H}$$

by Theorem 1.9.4. Hence, it suffices to show that

$$\pi(A)\overline{\pi(A)H} = \pi(A)H.$$

We already have the inclusion $\pi(A)\overline{\pi(A)H} \subseteq \pi(A)H$. For the reverse inclusion, assume that $\pi(a)\xi \in \pi(A)H$. Write $\xi = \xi_1 + \xi_2$, where $\xi_1 \in \overline{\pi(A)H}$ and $\xi_2 \in (\pi(A)H)^\perp$. Then, $\pi(a)\xi = \pi(a)\xi_1 \in \pi(A)\overline{\pi(A)H}$ because $\pi(a)\xi_2 = 0$ as found previously. Therefore, $\pi(A)H \subseteq \pi(A)\overline{\pi(A)H}$

and $\pi(A)H = \pi(A)\overline{\pi(A)H}$. Consequently by Theorem 1.9.4, the representation $(\pi|_{\overline{\pi(A)H}}, \overline{\pi(A)H})$ is non-degenerate.

Finally, we observe that as representations of A,

$$(\pi, H) \sim_u (\pi|_{(\pi(A)H)^\perp}, (\pi(A)H)^\perp) \oplus (\pi|_{\overline{\pi(A)H}}, \overline{\pi(A)H}).$$

$\square$

Next, we link irreducible representations to cyclic representations.

Theorem 1.9.6. *Let A be a *-algebra and (π, H) be a non-degenerate representation of A. Then, (π, H) is irreducible if and only if every non-zero vector is cyclic.*

Proof. Assume that A is a *-algebra and (π, H) is a non-degenerate representation of A.

To show: (a) If (π, H) is irreducible then every non-zero vector is cyclic.

(b) If every non-zero vector of H is cycilc then (π, H) is irreducible.

(a) Assume that (π, H) is an irreducible representation. Assume that $\xi \in H - \{0\}$ and let

$$\pi(A)\xi = \{\pi(a)\xi \mid a \in A\}.$$

Then, $\pi(A)\xi$ is an invariant subspace of H and its closure $\overline{\pi(A)\xi}$ is a closed invariant subspace of H. Since (π, H) is irreducible, either $\overline{\pi(A)\xi} = 0$ or $\overline{\pi(A)\xi} = H$. Suppose for the sake of contradiction that $\overline{\pi(A)\xi} = 0$. Then, $\pi(A)\xi = 0$ and the representation (π, H) is degenerate. This contradicts our original assumption that (π, H) is non-degenerate. So, $\overline{\pi(A)\xi} = H$ and ξ is cyclic.

(b) We will prove the contrapositive statement. Assume that the representation (π, H) is reducible. Then, there exists a closed invariant subspace $N \subsetneq H$. Select a non-zero $\eta \in N$. If $a \in A$ then $\pi(a)\eta \in N$ since N is invariant. Since N is closed and $N \neq H$, the subspace $\pi(A)\eta$ cannot be dense in H. So, $\eta \in H - \{0\}$ is not a cyclic vector. $\square$

We have another, more useful criterion, for a representation to be irreducible.

Theorem 1.9.7. *Let A be a *-algebra and (π, H) be a non-degenerate representation. Then, (π, H) is irreducible if and only if the only positive operators which commute with its image are scalars.*

Proof. Assume that A is a *-algebra and (π, H) is a non-degenerate representation.

To show: (a) If the only positive operators which commute with operators in the image $\pi(A)$ are scalars then (π, H) is irreducible.

(b) If (π, H) is an irreducible representation then the only positive operators which commute with operators in the image $\pi(A)$ are scalars.

(a) We will prove the contrapositive statement. Assume that the representation (π, H) is reducible. Then, there exists a non-trivial closed invariant subspace $N \subseteq H$. Now let $p \in B(H)$ be the projection operator onto N. Since p is a projection, it satisfies $p = p^* = p^2$ and is positive. Moreover, p is not a scalar operator because both N and $N^\perp$ are non-zero subspaces of H.

Now assume that $a \in A$. We will show that the operator $\pi(a)$ commutes with p. There are two different cases to consider:

Case 1: $\xi \in N$.

Assume that $\xi \in N$. Since N is invariant, then $\pi(a)\xi \in N$ and

$$(p\pi(a))(\xi) = p(\pi(a)\xi) = \pi(a)\xi = (\pi(a)p)\xi.$$

Case 2: $\xi \in N^\perp$.

Assume that $\xi \in N^\perp$. By Theorem 1.9.2, $N^\perp$ is also an invariant subspace of H and $\pi(a)\xi \in N^\perp$. So,

$$(p\pi(a))(\xi) = p(\pi(a)\xi) = 0 = \pi(a)(0) = \pi(a)(p\xi) = (\pi(a)p)(\xi).$$

By combining both cases, we deduce that as operators on H, p is a non-scalar, positive operator which commutes with $\pi(a)$ for $a \in A$.

(b) Again, we proceed by proving the contrapositive statement. Assume that h is a positive non-scalar operator on H which commutes with every element of $\pi(A)$. We claim that the spectrum $\sigma(h)$ contains at least two points.

Suppose for the sake of contradiction that the spectrum $\sigma(h)$ contains a single point. By the continuous functional calculus in Theorem 1.3.7, h must be a scalar operator. This contradicts our assumption that h is a non-scalar operator. So, $\sigma(h)$ must contain at least two points.

By Urysohn's lemma applied to $\sigma(h)$ (which is a normal topological space with the subspace topology inherited from $\mathbb{C}$), we can construct two non-zero functions $f, g \in Cts(\sigma(h), \mathbb{C})$ such that if $x \in \sigma(h)$ then $f(x)g(x) = 0$. By using the continuous functional calculus in Theorem 1.3.7, the operator $f(h) \in B(H)$ is non-zero because the function f is non-zero on $\sigma(h)$.

Now let $N = \overline{\text{im } f(h)}$. Then, N is a non-zero subspace of H. However by the same reasoning as before, $g(h) \in B(H)$ is a non-zero operator which is zero on im $f(h)$ and hence, on N. This means that N is a proper subspace of H.

Next, we claim that $f(h)$ commutes with the operators in $\pi(A)$.

To show: (ba) If $a \in A$ then $\pi(a)f(h) = f(h)\pi(a)$.

(ba) Assume that $a \in A$ and $\epsilon \in \mathbb{R}_{>0}$. By the Weierstrass theorem, there exists a polynomial function $p \in Cts(\sigma(h), \mathbb{C})$ such that $\|p - f\|_\infty < \epsilon$. By Theorem 1.3.7, $\|p(h) - f(h)\| < \epsilon$.

Since $\pi(a)$ commutes with h by assumption, $\pi(a)$ must also commute with $p(h)$. By the standard ϵ argument, we have

$$\begin{aligned}
\|\pi(a)f(h) - f(h)\pi(a)\| &= \|\pi(a)f(h) - \pi(a)p(h) + p(h)\pi(a) - f(h)\pi(a)\| \\
&\leq \|\pi(a)f(h) - \pi(a)p(h)\| + \|f(h)\pi(a) - p(h)\pi(a)\| \\
&\leq 2\|\pi(a)\|\|f(h) - p(h)\| \\
&< 2\|\pi(a)\|\epsilon.
\end{aligned}$$

Since $\epsilon \in \mathbb{R}_{>0}$ was arbitrary, we deduce that $f(h)$ commutes with the operators in $\pi(A)$.

Our final claim is that if $a \in A$ then N is invariant under $\pi(a)$. Since $N = \overline{\text{im } f(h)}$, then it suffices to show that the image im $f(h)$ is invariant.

If $\xi \in H$ and $a \in A$ then by part (ba),

$$\pi(a)(f(h)\xi) = (\pi(a)f(h))\xi = (f(h)\pi(a))\xi \in \text{im } h.$$

We conclude that N is a non-trivial closed, invariant subspace of H. So, (π, H) is a reducible representation of A as required. □

1.10 Representations of matrix C*-algebras

Complementary to the theory developed in the previous section, we will this time investigate representations of the matrix C*-algebra $M_{n\times n}(\mathbb{C})$ where $n \in \mathbb{Z}_{>0}$.

We first need to prove various properties about $M_{n\times n}(\mathbb{C})$. The next few results originate from [Put19, Exercise 1.9.2].

Theorem 1.10.1. *Let $n \in \mathbb{Z}_{>1}$ and $A = M_{n\times n}(\mathbb{C})$. Let $I \subseteq A$ be a right ideal. Then, the set*

$$I\mathbb{C}^n = \{a\xi \mid a \in I, \xi \in \mathbb{C}^n\}$$

is a subspace of $\mathbb{C}^n$.

Proof. Assume that $n \in \mathbb{Z}_{>1}$ and $A = M_{n\times n}(\mathbb{C})$. Assume that $I \subseteq A$ is a right ideal. Assume that $I\mathbb{C}^n$ is defined as above. To see that $I\mathbb{C}^n$ is closed under scalar multiplication, assume that $a \in I$, $\xi \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$. Then,

$$\lambda a\xi = a(\lambda\xi) \in I\mathbb{C}^n.$$

Checking that $I\mathbb{C}^n$ is closed under addition is more tricky. Assume that $a, b \in I$ and $\xi, \eta \in \mathbb{C}^n$. If at least one of ξ or η is the zero vector in $\mathbb{C}^n$ then $a\xi + b\eta \in I\mathbb{C}^n$ by inspection. So, assume that both $\xi, \eta \neq 0$. Let $\widehat{\xi} = \frac{\xi}{\|\xi\|}$ and $\widehat{\eta} = \frac{\eta}{\|\eta\|}$. Then

$$a\xi + b\eta = a\|\xi\|\widehat{\xi} + b\|\eta\|\widehat{\eta}.$$

The key here is to realise that

$$|\widehat{\xi}\rangle\langle\widehat{\eta}|\widehat{\eta} = \langle\widehat{\eta}, \widehat{\eta}\rangle\widehat{\xi} = \widehat{\xi}.$$

Therefore, $a\xi + b\eta$ further simplifies to

$$a\|\xi\|\widehat{\xi} + b\|\eta\|\widehat{\eta} = a\|\xi\||\widehat{\xi}\rangle\langle\widehat{\eta}|\widehat{\eta} + b\|\eta\|\widehat{\eta} = \big(a\|\xi\||\widehat{\xi}\rangle\langle\widehat{\eta}| + b\|\eta\|\big)\widehat{\eta}.$$

For $\lambda \in \mathbb{C}$, define $D_\lambda = diag[\lambda, \ldots, \lambda] \in M_{n\times n}(\mathbb{C})$. Then,

$$a\xi + b\eta = \big(aD_{\|\xi\|}|\widehat{\xi}\rangle\langle\widehat{\eta}| + bD_{\|\eta\|}\big)\widehat{\eta}.$$

Since $a, b \in I$, $aD_{\|\xi\|}|\widehat{\xi}\rangle\langle\widehat{\eta}| + bD_{\|\eta\|} \in I$ (because I is a right ideal) and $a\xi + b\eta \in I\mathbb{C}^n$ as a result.

Hence, $I\mathbb{C}^n$ is a vector subspace of $\mathbb{C}^n$. $\square$

There is a correspondence between right ideals of $M_{n\times n}(\mathbb{C})$ and vector subspaces of $\mathbb{C}^n$. This is illustrated by the following theorem.

Theorem 1.10.2. *Let $n \in \mathbb{Z}_{>1}$ and $A = M_{n\times n}(\mathbb{C})$. Then, there is a bijection of sets*

$$\begin{array}{ccc} \Big\{\textit{Right ideals of } A\Big\} & \longleftrightarrow & \Big\{\textit{Vector subspaces of } \mathbb{C}^n\Big\} \\ I & \mapsto & I\mathbb{C}^n \end{array}$$

where

$$I\mathbb{C}^n = \{a\xi \mid a \in I, \xi \in \mathbb{C}^n\}.$$

Proof. Assume that $n \in \mathbb{Z}_{>1}$ and $A = M_{n\times n}(\mathbb{C})$. Let Θ denote the set function $I \mapsto I\mathbb{C}^n$, where $I\mathbb{C}^n$ is defined as above. By Theorem 1.10.1, $I\mathbb{C}^n$ is a vector subspace of $\mathbb{C}^n$.

Firstly, to see that Θ is injective, assume that $I \in \ker\Theta$ so that $I\mathbb{C}^n = \{0\}$. Then, $I = 0$ as a right ideal in A. Therefore, Θ is injective.

To see that Θ is surjective, assume that V is a vector subspace of $\mathbb{C}^n$. Let $\{v_1, v_2, \ldots, v_k\}$ be a basis for V, where $k \in \{1, 2, \ldots, n\}$. For $i \in \{1, 2, \ldots, k\}$, write $v_i = (v_{i,1}, v_{i,2}, \ldots, v_{i,k})$. Let $V_i \in M_{n\times n}(\mathbb{C})$ be the matrix whose i^{th} column is v_i^T. The remaining entries of V_i are zeros. Let $e_i \in \mathbb{C}^n$ be the vector with a 1 in the i^{th} position and zeros elsewhere. If $i \in \{1, 2, \ldots, k\}$ then $V_i e_i = v_i$.

If we let $(V_1, V_2, \ldots, V_k)$ be the right ideal generated by the matrices $V_1, V_2, \ldots, V_k$ then $\Theta((V_1, V_2, \ldots, V_k)) = V$. So, Θ is surjective and consequently, bijective. $\square$

The next result requires a definition.

Definition 1.10.1. Let A be a C*-algebra. We say that A is **simple** if the only two closed two-sided ideals of A are the zero ideal 0 and A itself.

We will now prove that the matrix C*-algebra $M_{n\times n}(\mathbb{C})$ is a simple C*-algebra.

Theorem 1.10.3. *Let $n \in \mathbb{Z}_{>1}$ and $A = M_{n\times n}(\mathbb{C})$. Then, A is simple.*

Proof. Assume that $n \in \mathbb{Z}_{>1}$ and $A = M_{n\times n}(\mathbb{C})$. Suppose for the sake of contradiction that I is a non-trivial ideal of A. By using the bijection Θ in Theorem 1.10.2, we obtain a vector subspace $I\mathbb{C}^n$ of $\mathbb{C}^n$.

Assume that $a \in I$ and $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n$ so that $a\xi = (\xi_1', \xi_2', \ldots, \xi_n') \in I\mathbb{C}^n - \{0\}$. Let ξ_k' be the first non-zero element of the vector $a\xi$. For $i \in \{1, 2, \ldots, n\}$, let $W_i \in M_{n\times n}(\mathbb{C})$ be the matrix whose ik entry is $\frac{1}{\xi_k'}$ and whose remaining entries are zeros. If $i \in \{1, 2, \ldots, n\}$ then

$$W_i a\xi = e_i$$

where $e_i \in \mathbb{C}^n$ is the vector with a 1 in the i^{th} position and zeros elsewhere. Crucially, I is both a left and right sided ideal. So, $W_i a \in I$ and subsequently, $e_i = W_i a\xi \in I\mathbb{C}^n$ for $i \in \{1, 2, \ldots, n\}$. So, $I\mathbb{C}^n = \mathbb{C}^n$, which means that $\Theta(I) = \Theta(A)$. Here, Θ is the bijection in Theorem 1.10.2.

Since Θ is injective, $I = A$. However, this contradicts the assumption that I is a non-trivial ideal of A. Therefore, A does not have any non-trivial two-sided ideals. So, A is a simple C*-algebra. □

Let us assume that $n \in \mathbb{Z}_{\geq 1}$ and $A = M_{n\times n}(\mathbb{C})$. We have the representation $(\rho, \mathbb{C}^n)$ of A, where ρ is the *-homomorphism given by matrix multiplication:

$$\begin{array}{rccc} \rho: & A & \to & B(\mathbb{C}^n) = M_{n\times n}(\mathbb{C}) \\ & a & \mapsto & (\xi \mapsto a\xi). \end{array}$$

We will return to this representation later. Let (π, H) be a non-degenerate representation of A. If $i, j \in \{1, 2, \ldots, n\}$ then let $e_{i,j} \in A$ be the matrix with a 1 in the ij position and zeros elsewhere.

We highlight the following result about simple C*-algebras.

Theorem 1.10.4. *Let A be a simple unital C*-algebra and B be a unital C*-algebra. Let $\phi : A \to B$ be a unital *-homomorphism. Then, ϕ is injective.*

Proof. Assume that A is a simple unital C*-algebra and B is a unital C*-algebra. Assume that $\phi : A \to B$ is a unital *-homomorphism. Then,

$\phi(1_A) = 1_B$. In particular, the *-homomorphism ϕ is non-zero.

The kernel $\ker \phi$ is a closed two-sided ideal of A. Since A is simple, then either $\ker \phi = \{0\}$ or $\ker \phi = H$. If $\ker \phi = H$ then ϕ is the zero map, which contradicts the fact that $\phi(1_A) = 1_B$. Hence, $\ker \phi = \{0\}$ and ϕ is injective. □

Since (π, H) is a non-degenerate representation of $A = M_{n\times n}(\mathbb{C})$, then by Theorem 1.9.3 π is a unital *-homomorphism. Utilising Theorem 1.10.4, we deduce that π is injective. In particular, $\pi(e_{1,1}) \neq 0$.

Next, let $\xi \in \pi(e_{1,1})H$ be a unit vector. Then, there exists $\eta \in H$ such that $\xi = \pi(e_{1,1})\eta$. Note that if $i \in \{1, 2, \ldots, n\}$ then $e_{i,1}e_{1,1} = e_{i,1}$. We claim that the set

$$\{\xi, \pi(e_{2,1})\xi, \ldots, \pi(e_{n,1})\xi\} \subset H$$

is an orthonormal set. Firstly, if $i \in \{2, 3, \ldots, n\}$ then

$$\begin{aligned}
\|\pi(e_{i,1})\xi\|^2 &= \langle \pi(e_{i,1})\xi, \pi(e_{i,1})\xi \rangle \\
&= \langle \xi, \pi(e_{i,1})^*\pi(e_{i,1})\xi \rangle \\
&= \langle \xi, \pi(e_{1,i})\pi(e_{i,1})\xi \rangle \\
&= \langle \xi, \pi(e_{1,1})\xi \rangle \\
&= \langle \xi, \xi \rangle = \|\xi\|^2 = 1.
\end{aligned}$$

By a similar computation, if $i, j \in \{1, 2, \ldots, n\}$ are distinct then

$$\begin{aligned}
\langle \pi(e_{i,1})\xi, \pi(e_{j,1})\xi \rangle &= \langle \xi, \pi(e_{i,1})^*\pi(e_{j,1})\xi \rangle \\
&= \langle \xi, \pi(e_{1.i})\pi(e_{j,1})\xi \rangle \\
&= \langle \xi, 0\xi \rangle = 0.
\end{aligned}$$

Hence, $\{\xi, \pi(e_{2,1})\xi, \ldots, \pi(e_{n,1})\xi\}$ is an orthonormal set in H. We would like to find the span of this set. We claim that

$$\pi(A)\xi = span\{\xi, \pi(e_{2,1})\xi, \ldots, \pi(e_{n,1})\xi\}.$$

We certainly have the inclusion $span\{\xi, \pi(e_{2,1})\xi, \ldots, \pi(e_{n,1})\xi\} \subseteq \pi(A)\xi$ because $\pi(e_{1,1})\xi = \xi$. Now assume that

$$\sum_{i=1}^{n} \lambda_i \pi(e_{1,1})\xi \in span\{\xi, \pi(e_{2,1})\xi, \ldots, \pi(e_{n,1})\xi\}$$

where $\lambda_i \in \mathbb{C}$ for $i \in \{1, 2, \ldots, n\}$. Then,

$$\sum_{i=1}^{n} \lambda_i \pi(e_{1,1})\xi = \pi(\sum_{i=1}^{n} \lambda_i e_{1,1})\xi \in \pi(A)\xi.$$

Therefore,

$$\pi(A)\xi = span\{\xi, \pi(e_{2,1})\xi, \ldots, \pi(e_{n,1})\xi\} \tag{1.8}$$

as required. Note that $\pi(A)\xi$ is a finite dimensional subspace of the Hilbert space H and is thus, closed. We now have enough to state our first main result.

Theorem 1.10.5. *Let $n \in \mathbb{Z}_{\geq 1}$ and $A = M_{n\times n}(\mathbb{C})$. Let (π, H) be a non-degenerate representation of A. Let $\xi \in \pi(e_{1,1})H$ be a unit vector. Then, the restriction $(\pi|_{\pi(A)\xi}, \pi(A)\xi)$ is unitarily equivalent to $(\rho, \mathbb{C}^n)$ where we recall that ρ is the *-homomorphism*

$$\begin{array}{rccc} \rho: & A & \to & B(\mathbb{C}^n) = M_{n\times n}(\mathbb{C}) \\ & a & \mapsto & (\xi \mapsto a\xi). \end{array}$$

Proof. Assume that $A = M_{n\times n}(\mathbb{C})$ and (π, H) is a non-degenerate representation of A. Assume that $\xi \in \pi(e_{1,1})H$ is a unit vector.

To see that the representations $(\pi|_{\pi(A)\xi}, \pi(A)\xi)$ and $(\rho, \mathbb{C}^n)$ are unitarily equivalent, it suffices to show that there exists a unitary operator $u : \pi(A)\xi \to \mathbb{C}^n$ such that if $a \in A$ then

$$u\pi|_{\pi(A)\xi}(a) = \rho(a)u.$$

Recall from equation (1.8) that $\pi(A)\xi$ is the $\mathbb{C}$-span of the set $\{\xi, \pi(e_{2,1})\xi, \ldots, \pi(e_{n,1})\xi\}$. Define the map u by

$$\begin{array}{rccc} u: & \pi(A)\xi & \to & \mathbb{C}^n \\ & \pi(e_{i,1})\xi & \mapsto & E_i = (0, \ldots, 0, 1, 0, \ldots, 0) \end{array}$$

where $i \in \{1, 2, \ldots, n\}$ and E_i is the n-tuple with a 1 in the i^{th} position and zeros elsewhere. To see that u is unitary, observe that if $i, j \in \{1, 2, \ldots, n\}$ then

$$\delta_{ij} = \langle E_i, E_j\rangle = \langle u(\pi(e_{i,1})\xi), u(\pi(e_{j,1})\xi)\rangle = \langle \pi(e_{i,1})\xi, \pi(e_{j,1})\xi\rangle.$$

By linearity of the inner product, we find that if $\xi_1, \xi_2 \in \pi(A)\xi$ then $\langle u\xi_1, u\xi_2\rangle = \langle \xi_1, \xi_2\rangle$. Hence, u is a unitary map.

Now assume that $\eta \in \pi(A)\xi$. Then, there exists $\gamma_1, \ldots, \gamma_n \in \mathbb{C}$ such that

$$\eta = \sum_{i=1}^{n} \gamma_i \pi(e_{i,1})\xi.$$

We compute directly that

$$\begin{aligned}
(\rho(a)u)(\eta) &= (\rho(a)u)(\sum_{i=1}^{n} \gamma_i \pi(e_{i,1})\xi) \\
&= \rho(a)\big(\sum_{i=1}^{n} \gamma_i E_i\big) \\
&= a\big(\sum_{i=1}^{n} \gamma_i E_i\big) = \sum_{i=1}^{n} \gamma_i a(E_i)
\end{aligned}$$

and

$$\begin{aligned}
(u\pi(a))(\eta) &= (u\pi(a))(\sum_{i=1}^{n} \gamma_i \pi(e_{i,1})\xi) \\
&= u\big(\sum_{i=1}^{n} \gamma_i \pi(a e_{i,1})\xi\big) \\
&= u\big(\sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_i a_{j,i} \pi(e_{i,1})\xi\big) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_i a_{j,i} E_i = \sum_{i=1}^{n} \gamma_i a(E_i).
\end{aligned}$$

Hence, if $a \in A$ then $u\pi|_{\pi(A)\xi}(a) = \rho(a)u$. So, the representations $(\pi|_{\pi(A)\xi}, \pi(A)\xi)$ and $(\rho, \mathbb{C}^n)$ are unitarily equivalent as required. $\square$

Theorem 1.10.5 tells us that our non-degenerate representation of A is unitarily equivalent to the "matrix multiplication representation" of A, provided that we restrict our representation appropriately. This suggests that we can build the non-degenerate representation (π, H) by using a direct sum of matrix multiplication representations. It turns out that this is indeed the case and the clue here is that we took $\xi \in \pi(e_{1,1})H$ to be an arbitrary unit vector. This in turn, suggests that we consider an orthonormal basis for the Hilbert space $\pi(e_{1,1})H$.

Let $\mathcal{B}$ be an orthonormal basis for $\pi(e_{1,1})H$. We claim that if $\eta, \xi \in \mathcal{B}$ are distinct then $\pi(A)\xi$ and $\pi(A)\eta$ are orthogonal subspaces to each other. Recall that

$$\pi(A)\xi = span\{\xi, \pi(e_{2,1})\xi, \ldots, \pi(e_{n,1})\xi\}.$$

A similar statement holds for $\pi(A)\eta$. If $i, j \in \{1, 2, \ldots, n\}$ then

$$\begin{aligned}
\langle \pi(e_{i,1})\xi, \pi(e_{j,1})\eta \rangle &= \langle \xi, \pi(e_{i,1})^* \pi(e_{j,1})\eta \rangle \\
&= \langle \xi, \pi(e_{1,i} e_{j,1})\eta \rangle \\
&= \begin{cases} 0, \text{ if } i \neq j, \\ \langle \xi, \pi(e_{1,1})\eta \rangle, \text{ if } i = j. \end{cases} \\
&= \begin{cases} 0, \text{ if } i \neq j, \\ \langle \xi, \eta \rangle = 0, \text{ if } i = j. \end{cases}
\end{aligned}$$

Therefore, $\pi(A)\xi$ and $\pi(A)\eta$ are orthogonal subspaces to each other.

Next, we can decompose H as the direct sum $H = \pi(A)\xi \oplus (\pi(A)\xi)^\perp$. Since $\pi(A)\eta \subseteq (\pi(A)\xi)^\perp$, then $\pi(A)\eta$ is a closed subspace of $(\pi(A)\xi)^\perp$ because it is finite dimensional. So, H decomposes further as

$$H = \pi(A)\xi \oplus \pi(A)\eta \oplus \big((\pi(A)\xi)^\perp \cap (\pi(A)\eta)^\perp\big).$$

Since the subspaces $\pi(A)\eta$ are all mutually orthogonal for $\eta \in \mathcal{B}$, we can repeat the above argument to obtain

$$H = \Big(\bigoplus_{\eta \in \mathcal{B}} \pi(A)\eta\Big) \oplus \bigcap_{\eta \in \mathcal{B}} (\pi(A)\eta)^\perp.$$

Now if $\delta \in \bigcap_{\eta \in \mathcal{B}} (\pi(A)\eta)^\perp$ then $\langle \delta, \eta \rangle = 0$ for $\eta \in \mathcal{B}$. Since $\mathcal{B}$ is an orthonormal basis, then $\delta = 0$. So,

$$H = \bigoplus_{\eta \in \mathcal{B}} \pi(A)\eta$$

as required. With this, we can now state the main theorem of this section.

Theorem 1.10.6. *Let $n \in \mathbb{Z}_{\geq 1}$ and $A = M_{n \times n}(\mathbb{C})$. Let (π, H) be a non-degenerate representation of A. Let $\mathcal{B}$ be an orthonormal basis for $\pi(e_{1,1})H$. Then, (π, H) is unitarily equivalent to the direct sum*

$$\bigoplus_{\eta \in \mathcal{B}} (\rho, \mathbb{C}^n)$$

*where ρ is the *-homomorphism*

$$\begin{array}{rccc} \rho : & A & \to & B(\mathbb{C}^n) = M_{n\times n}(\mathbb{C}) \\ & a & \mapsto & (\xi \mapsto a\xi). \end{array}$$

Proof. Assume that $n \in \mathbb{Z}_{\geq 1}$ and $A = M_{n\times n}(\mathbb{C})$. Assume that (π, H) is a non-degenerate representation of A and $\mathcal{B}$ is an orthonormal basis for $\pi(e_{1,1})H$. By Theorem 1.10.5, if $\xi \in \mathcal{B}$ then

$$(\pi|_{\pi(A)\xi}, \pi(A)\xi) \sim_u (\rho, \mathbb{C}^n).$$

By taking direct sums over all elements of $\mathcal{B}$, we find that

$$(\bigoplus_{\xi \in \mathcal{B}} \pi|_{\pi(A)\xi}, \bigoplus_{\xi \in \mathcal{B}} \pi(A)\xi) \sim_u \bigoplus_{\xi \in \mathcal{B}} (\rho, \mathbb{C}^n).$$

But, we know that $H = \bigoplus_{\eta \in \mathcal{B}} \pi(A)\eta$. Therefore,

$$(\pi, H) \sim_u \bigoplus_{\xi \in \mathcal{B}} (\rho, \mathbb{C}^n).$$

□

Thus, Theorem 1.10.6 demonstrates that any non-degenerate representation of $M_{n\times n}(\mathbb{C})$ can be built directly from the matrix multiplication representation $(\rho, \mathbb{C}^n)$. In this sense, the representation $(\rho, \mathbb{C}^n)$ is the only representation of $M_{n\times n}(\mathbb{C})$ one needs to know in order to understand arbitrary non-degenerate representations of $M_{n\times n}(\mathbb{C})$.

In fact, $(\rho, \mathbb{C}^n)$ is irreducible, as proven below.

Theorem 1.10.7. *Let $n \in \mathbb{Z}_{\geq 1}$ and $A = M_{n\times n}(\mathbb{C})$. Then, the matrix multiplication representation $(\rho, \mathbb{C}^n)$ is irreducible.*

Proof. Assume that $n \in \mathbb{Z}_{\geq 1}$ and $A = M_{n\times n}(\mathbb{C})$. Since $(\rho, \mathbb{C}^n)$ is a non-degenerate representation of A, it suffices to prove that every non-zero vector of $\mathbb{C}^n$ is cyclic by Theorem 1.9.6.

Assume that $\eta = (\eta_1, \ldots, \eta_n) \in \mathbb{C}^n - \{0\}$. If $i, j \in \{1, 2, \ldots, n\}$ then let $e_{i,j} \in A$ denote the matrix with a 1 in the ij position and zeros elsewhere.

Let η_k be the first non-zero element of the n-tuple η. If $i \in \{1, 2, \ldots, n\}$ then

$$\rho(\frac{1}{\eta_k}e_{i,k})\eta = \frac{1}{\eta_k}e_{i,k}\eta = E_i$$

where $E_i \in \mathbb{C}^n$ is the n-tuple with a 1 in the i^{th} position and zeros elsewhere. Therefore,

$$\mathbb{C}^n = span\{E_1, E_2, \ldots, E_n\} \subseteq \rho(A)\xi.$$

So, $\rho(A)\xi = \mathbb{C}^n$ and ξ is a cyclic vector as required. By Theorem 1.9.6, the representation $(\rho, \mathbb{C}^n)$ is irreducible. $\square$

1.11 The Gelfand-Naimark-Segal construction

In the last few sections, we defined representations of C*-algebras, investigated a few of their properties and in the case of the last section, delved into a particular example of a representation. Here, we are interested in constructing a representation of an arbitrary C*-algebra.

To motivate this sections, let us consider the parallel situation for groups. Groups naturally occur as symmetries and one often looks for ways that abstract groups act as symmetries. The simplest way this occurs is to consider a group G acting on itself via left multiplication. The main result stemming from this is Cayley's theorem (see [DF04, Section 4.2]).

As remarked in [Put19, Section 1.12], we will apply a similar line of thinking to producing representations of a C*-algebra. The multiplication operation on a C*-algebra allows one to think of its elements as linear transformations acting on the C*-algebra itself. The problem here is that C*-algebras are not generally Hilbert spaces. The GNS construction (Gelfand-Naimark-Segal) produces the desired inner product on the C*-algebra by using the linear functionals on the C*-algebra.

The key property for the functionals we consider in the GNS construction is positivity, which was introduced in the context of traces. We briefly recall that a linear functional ϕ on a C*-algebra A is positive if for $a \in A$, $\phi(a^*a) \geq 0$.

Definition 1.11.1. Let A be a unital C*-algebra and ϕ be a linear functional on A. We say that ϕ is a **state** if ϕ is positive and $\phi(1_A) = 1$.

We have the following characterisation of states.

Theorem 1.11.1. *Let A be a unital C*-algebra and ϕ be a linear functional on A. If $\phi(1_A) = \|\phi\| = 1$ then ϕ is a state.*

Proof. Assume that A is a unital C*-algebra and ϕ is a linear functional on A. Assume that $\phi(1_A) = \|\phi\| = 1$.

To show: (a) ϕ is positive.

(a) We will first show that if $a \in A$ is self-adjoint then $\phi(a) \in \mathbb{R}$. So, assume that $a \in A$ is self-adjoint. Suppose for the sake of contradiction that $Im(\phi(a)) \neq 0$. Without loss of generality, we may assume that $Im(\phi(a)) > 0$. Select $r \in \mathbb{R}_{>0}$ satisfying the inequality

$$0 < r < Im(\phi(a)) \leq \|a\|.$$

Then, define the function f by

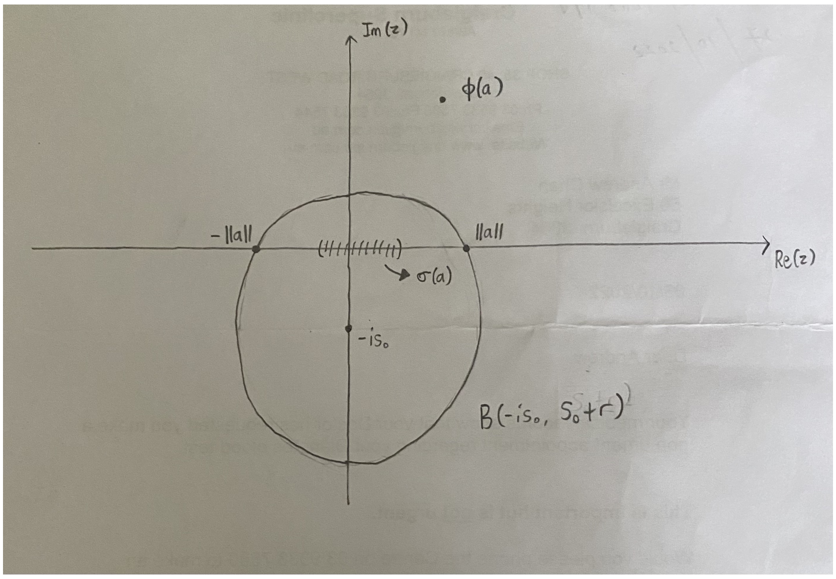
$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ s &\mapsto \sqrt{\|a\|^2 + s^2} - s. \end{aligned}$$

Observe that $f(0) > r$. Observe that the limit

$$\begin{aligned} \lim_{s\to\infty} f(s) &= \lim_{s\to\infty} (\sqrt{\|a\|^2 + s^2} - s) \\ &= \lim_{s\to\infty} \frac{\|a\|^2 + s^2 - s^2}{\sqrt{\|a\|^2 + s^2} + s} \\ &= \lim_{s\to\infty} \frac{\|a\|^2}{\sqrt{\|a\|^2 + s^2} + s} \\ &= 0. \end{aligned}$$

Hence, there exists $s_0 \in \mathbb{R}_{>0}$ such that $f(s_0) = r$.

The argument we make here is geometric in nature. Here is a diagram of the situation.



Let $B(-is_0, s_0 + r)$ be the open ball centred at $-is_0$ with radius $s_0 + r$. Note that $s_0 + r = s_0 + f(s_0) = \sqrt{\|a\|^2 + s_0^2}$. So, $[-\|a\|, \|a\|] \subseteq B(-is_0, s_0 + r)$. Since a is self-adjoint, then the spectrum $\sigma(a) \subseteq [-\|a\|, \|a\|]$. So, $\sigma(a) \subseteq B(-is_0, s_0 + r)$.

From the diagram, we also see that $\phi(a) \notin B(-is_0, s_0 + r)$ because by construction,

$$r = (s_0 + r) - s_0 < Im(\phi(a))$$

With this information, we now compute

$$\begin{aligned}
|\phi(a - is_0 1_A)| &= |\phi(a) - is_0\phi(1_A)| \\
&= |\phi(a) - is_0| \quad \text{(since } \phi(1_A) = 1 \text{ by assumption)} \\
&> r + s_0 = \sqrt{\|a\|^2 + s_0^2}.
\end{aligned}$$

Note that by the diagram, the spectrum $\sigma(a)$ is contained in $B(-is_0, s_0 + r)$ and consequently,

$$\|a - is_0 1_A\| \leq \sqrt{\|a\|^2 + s_0^2}.$$

So,

$$\phi\left(\frac{a - is_0 1_A}{\sqrt{\|a\|^2 + s_0^2}}\right) > 1$$

which contradicts the assumption that $\|\phi\| = 1$. So, $Im(\phi(a)) = 0$ and $\phi(a) \in \mathbb{R}$.

Now let $a \in A$ be positive. By the previous part, $\phi(a) \in \mathbb{R}$. It suffices to show that $\phi(a) \geq 0$. By Theorem 1.4.2, $\|\|a\|1_A - a\| \leq \|a\|$. Since $\phi(1_A) = 1 = \|\phi\|$, we have

$$\begin{aligned}
|\phi(\|a\|1_A - a)| &= |\|a\| - \phi(a)| \\
&= \|a\||1 - \phi(\frac{a}{\|a\|})| \\
&\leq \|a\|.
\end{aligned}$$

So, $\phi(a) \geq 0$ as required. □

Next, we prove some more properties of positive linear functionals.

Theorem 1.11.2. *Let A be a unital C*-algebra and ϕ be a positive linear functional.*

1. *If $a, b \in A$ then $|\phi(b^*a)|^2 \leq \phi(a^*a)\phi(b^*b)$.*
2. *If $a \in A$ then $\phi(a^*) = \overline{\phi(a)}$.*
3. *$\phi(1_A) = \|\phi\|$*
4. *If $a, b \in A$ then $\phi(b^*a^*ab) \leq \|a\|^2\phi(b^*b)$.*

Proof. Assume that A is a unital C*-algebra and ϕ is a positive linear functional. Assume that $a, b \in A$ and $\lambda \in \mathbb{C}$. Then,

$$0 \leq \phi((\lambda a + b)^*(\lambda a + b)) = |\lambda|^2\phi(a^*a) + \overline{\lambda}\phi(a^*b) + \lambda\phi(b^*a) + \phi(b^*b). \quad (1.9)$$

Since $|\lambda|^2\phi(a^*a), \phi(b^*b) \in \mathbb{R}_{\geq 0}$, the sum

$$\overline{\lambda}\phi(a^*b) + \lambda\phi(b^*a) \in \mathbb{R}.$$

Now set $\lambda = i$ and $b = 1_A$. Then, $i\phi(a) - i\phi(a^*) \in \mathbb{R}$. On the other hand, if we set $\lambda = 1$ and $b = 1_A$ then $\phi(a^*) + \phi(a) \in \mathbb{R}$. Now let $\phi(a) = \alpha + \beta i$ and $\phi(a^*) = \gamma + \delta i$. By the two equations, we find that $\alpha - \gamma = 0$ and $\beta + \delta = 0$. So,

$$\phi(a^*) = \gamma + \delta i = \alpha - \beta i = \overline{\phi(a)}.$$

To prove the first statement, note that by equation (1.9),

$$\begin{aligned}
-\overline{\lambda}\phi(a^*b) - \lambda\phi(b^*a) &= -\overline{\lambda\phi(b^*a)} - \lambda\phi(b^*a) \\
&= -2Re(\lambda\phi(b^*a)) \\
&\leq |\lambda|^2\phi(a^*a) + \phi(b^*b).
\end{aligned}$$

There are two cases to consider.

Case 1: $\phi(a^*a) = 0$.

Assume that $\phi(a^*a) = 0$. Then, $-Re(\lambda\phi(b^*a)) \leq \phi(b^*b)$. Since this holds for arbitrary $\lambda \in \mathbb{C}$, then $\phi(b^*a) = 0$. So, the inequality $|\phi(b^*a)|^2 \leq \phi(a^*a)\phi(b^*b)$ holds in this case.

Case 2: $\phi(a^*a) \neq 0$.

Assume that $\phi(a^*a) \neq 0$. Select $z \in \mathbb{C}$ such that $|z| = 1$ and $z\phi(b^*a) = |\phi(b^*a)|$. If $\lambda = -z\phi(a^*a)^{-\frac{1}{2}}\phi(b^*b)^{\frac{1}{2}}$ then

$$\begin{aligned}
-2Re(\lambda\phi(b^*a)) &= 2Re(z\phi(a^*a)^{-\frac{1}{2}}\phi(b^*b)^{\frac{1}{2}}\phi(b^*a)) \\
&= 2\phi(a^*a)^{-\frac{1}{2}}\phi(b^*b)^{\frac{1}{2}}|\phi(b^*a)| \\
&\leq |\lambda|^2\phi(a^*a) + \phi(b^*b) \\
&= \phi(b^*b) + \phi(b^*b) = 2\phi(b^*b).
\end{aligned}$$

By the above calculation,

$$2\phi(a^*a)^{-\frac{1}{2}}\phi(b^*b)^{-\frac{1}{2}}|\phi(b^*a)| \leq 2\phi(b^*b).$$

So, $|\phi(b^*a)| \leq \phi(a^*a)^{\frac{1}{2}}\phi(b^*b)^{\frac{1}{2}}$. Squaring both sides, we obtain the first statement of the theorem.

Next, we show that $\phi(1_A) = \|\phi\|$. Since $\|1_A\| = 1$, we have by definition of the operator norm $|\phi(1_A)| \leq \|\phi\|$.

Now let $b = 1_A$ in the inequality $|\phi(b^*a)|^2 \leq \phi(a^*a)\phi(b^*b)$. Then, $|\phi(a)|^2 \leq \phi(a^*a)\phi(1_A)$. So,

$$\begin{aligned}
|\phi(a)| &\leq \phi(a^*a)^{\frac{1}{2}}\phi(1_A)^{\frac{1}{2}} \\
&\leq \|\phi\|^{\frac{1}{2}}\|a^*a\|^{\frac{1}{2}}\phi(1_A)^{\frac{1}{2}} \\
&= \|\phi\|^{\frac{1}{2}}\|a\|\phi(1_A)^{\frac{1}{2}}.
\end{aligned}$$

Taking the supremum over all $a \in A$ satisfying $\|a\| = 1$, we deduce that $\|\phi\|^{\frac{1}{2}} \leq \phi(1_A)^{\frac{1}{2}}$. So, $\phi(1_A) = \|\phi\|$ as required.

To the prove the fourth and final statement, define the map $\psi : A \to \mathbb{C}$ by $\psi(c) = \phi(b^*cb)$. Then, ψ is a positive linear functional and by the third part, $\|\psi\| = \psi(1) = \phi(b^*b)$. Therefore,

$$\phi(b^*a^*ab) = \psi(a^*a) \leq \|\psi\|\|a^*a\| = \phi(b^*b)\|a\|^2.$$

□

Now we begin the GNS construction. Let A be a unital C*-algebra and ϕ be a state on A. Define

$$N_\phi = \{a \in A \mid \phi(a^*a) = 0\}.$$

To see that N_ϕ is a left ideal in A, assume that $a, b \in N_\phi$. Then,

$$\begin{aligned}
\phi((a+b)^*(a+b)) &= \phi(a^*a + a^*b + b^*a + b^*b) \\
&= \phi(a^*a) + \phi(a^*b + b^*a) + \phi(b^*b) \\
&= \phi(a^*b + b^*a).
\end{aligned}$$

By the first statement in Theorem 1.11.2,

$$|\phi(a^*b + b^*a)| \leq |\phi(a^*b)| + |\phi(b^*a)| \leq 2\phi(a^*a)^{\frac{1}{2}}\phi(b^*b)^{\frac{1}{2}} = 0.$$

Hence, $\phi((a+b)^*(a+b)) = 0$ and $a + b \in N_\phi$.

Now assume that $c \in A$. Then,

$$\begin{aligned}
\phi((ca)^*ca) &= \phi(a^*c^*ca) \\
&\leq \|c\|^2\phi(a^*a) = 0.
\end{aligned}$$

The inequality follows from the fourth statement in Theorem 1.11.2. Hence, $ca \in N_\phi$ and N_ϕ is a left ideal of A.

To see that N_ϕ is closed, suppose that $\{a_n\}_{n\in I}$ is a net in N_ϕ which converges to some $a \in A$. Since ϕ is continuous,

$$\phi(a^*a) = \phi(\lim_{n\in I} a_n^*a_n) = \lim_{n\in I} \phi(a_n^*a_n) = 0.$$

So, $a \in N_\phi$ and N_ϕ is therefore a closed left ideal of A.

Next, we take the quotient A/N_ϕ. The next task is to define an inner product on A/N_ϕ. Define the map

$$\begin{array}{rccc} \langle -,- \rangle : & A/N_\phi \times A/N_\phi & \to & \mathbb{C} \\ & (a+N_\phi, b+N_\phi) & \mapsto & \phi(b^*a). \end{array}$$

First, we have to show that the above map is well-defined. Assume that $a_1 + N_\phi = a_2 + N_\phi$. Then, $\phi((a_1-a_2)^*(a_1-a_2)) = 0$ and by the first statement of Theorem 1.11.2,

$$|\phi(b^*(a_1-a_2))|^2 \leq \phi(b^*b)\phi((a_1-a_2)^*(a_1-a_2)) = 0$$

for $b \in A$. Hence, if $b \in A$ then $\phi(b^*(a_1-a_2)) = 0$ and consequently,

$$\begin{aligned} \langle a_1 + N_\phi, b + N_\phi \rangle &= \phi(b^*a_1) \\ &= \phi(b^*(a_1-a_2)) + \phi(b^*a_2) \\ &= \phi(b^*a_2) = \langle a_2 + N_\phi, b + N_\phi \rangle. \end{aligned}$$

By a similar argument, the map $\langle -,- \rangle$ is well-defined in the second argument. So, the map $\langle -,- \rangle$ is well-defined overall. It is straightforward to check that $\langle -,- \rangle$ is linear in the first argument and antilinear in the second argument.

To see that the map $\langle -,- \rangle$ is a non-degenerate inner product, assume that $\langle a + N_\phi, a + N_\phi \rangle = 0$. This holds if and only if $\phi(a^*a) = 0$ if and only if $a \in N_\phi$. So, $a + N_\phi = N_\phi$ in A/N_ϕ. Thus, $\langle a + N_\phi, a + N_\phi \rangle = 0$ if and only if $a + N_\phi = N_\phi$.

Hence, $\langle -,- \rangle$ defines an inner product on A/N_ϕ. Now define $\mathcal{H}_\phi$ to be the completion of the inner product space A/N_ϕ. Then, $\mathcal{H}_\phi$ is a Hilbert space.

Next, we need to define a *-homomorphism on $\mathcal{H}_\phi$. To do this, we will first define the *-homomorphism on A/N_ϕ and then use the universal property of the completion to extend it to a *-homomorphism on $\mathcal{H}_\phi$.

Let $b + N_\phi \in A/N_\phi$. Define the map π_ϕ on A by

$$\pi_\phi(a)(b + N_\phi) = ab + N_\phi$$

First, we show that if $a \in A$ then $\pi_\phi(a)$ is a well-defined and bounded operator. First, assume that $b_1 + N_\phi = b_2 + N_\phi$ in A/N_ϕ. Then, $b_1 - b_2 \in N_\phi$. Since N_ϕ is a left ideal of A, $ab_1 - ab_2 \in N_\phi$ and $ab_1 + N_\phi = ab_2 + N_\phi$. So, $\pi_\phi(a)(b_1 + N_\phi) = \pi_\phi(a)(b_2 + N_\phi)$ and $\pi_\phi(a)$ is well-defined.

To see that $\pi_\phi(a)$ is bounded, we compute directly that

$$\begin{aligned}
\|\pi_\phi(a)\|^2 &= \sup_{\|b+N_\phi\|=1} \|ab + N_\phi\|^2 \\
&= \sup_{\|b+N_\phi\|=1} \phi(b^*a^*ab) \\
&\leq \sup_{\|b+N_\phi\|=1} \|a\|^2\phi(b^*b) \quad \text{(by Theorem 1.11.2)} \\
&= \sup_{\|b+N_\phi\|=1} \|a\|^2\|b + N_\phi\|^2 = \|a\|^2.
\end{aligned}$$

Hence, $\pi_\phi(a)$ is a bounded operator on A/N_ϕ.

Next, we show that π_ϕ is a *-homomorphism. The fact that π_ϕ is linear and multiplicative follows from direct computations. To see that π_ϕ preserves adjoints, assume that $a, b, c \in A$. Then,

$$\begin{aligned}
\langle \pi_\phi(a^*)(b + N_\phi), c + N_\phi \rangle &= \langle a^*b + N_\phi, c + N_\phi \rangle \\
&= \phi(c^*a^*b) = \phi((ac)^*b) \\
&= \langle b + N_\phi, ac + N_\phi \rangle \\
&= \langle b + N_\phi, \pi_\phi(a)(c + N_\phi) \rangle \\
&= \langle \pi_\phi(a)^*(b + N_\phi), c + N_\phi \rangle.
\end{aligned}$$

Since $b, c \in A$ was arbitrary, we deduce that $\pi_\phi(a)^* = \pi_\phi(a^*)$. So, π_ϕ is a *-homomorphism. If $a \in A$ then by the universal property of the completion, $\pi_\phi(a)$ extends to a bounded operator on $\mathcal{H}_\phi$ (the completion of A/N_ϕ). Therefore, π_ϕ defines a representation of A on $\mathcal{H}_\phi$.

Finally, define the vector

$$u_\phi = 1_A + N_\phi \in A/N_\phi \subseteq \mathcal{H}_\phi.$$

We claim that u_ϕ is a cyclic vector for the representation $(\pi_\phi, \mathcal{H}_\phi)$ with norm 1.

Assume that $b + N_\phi \in A/N_\phi$. Then, $b + N_\phi = \pi_\phi(b)(u_\phi)$ and $A/N_\phi \subseteq \pi_\phi(A)u_\phi$. Since A/N_ϕ is dense in $\mathcal{H}_\phi$, then $\pi_\phi(A)u_\phi$ is dense in $\mathcal{H}_\phi$ and so, u_ϕ is a cyclic vector.

Since ϕ is a state, $\phi(1_A) = 1$ and

$$\|u_\phi\| = \|1_A + N_\phi\| = \phi(1_A) = 1.$$

This completes the GNS construction. We summarise it below with the following definition.

Definition 1.11.2. Let A be a unital C*-algebra and ϕ be a state on A. The triple $(\pi_\phi, \mathcal{H}_\phi, u_\phi)$ is called the **GNS representation** of ϕ. This is a representation of A, as shown previously.

In summary, the GNS construction takes a state ϕ on a unital C*-algebra A and produces a representation $(\pi_\phi, \mathcal{H}_\phi)$ and a unit cyclic vector u_ϕ. The next theorem shows that this process can be reversed — from a representation with a unit cyclic vector, we can construct a state on A.

Theorem 1.11.3. *Let A be a unital C*-algebra and (π, H) be a representation of A with unit cyclic vector ξ. Then, the map*

$$\begin{array}{rccc} \phi: & A & \to & \mathbb{C} \\ & a & \mapsto & \langle \pi(a)\xi, \xi \rangle \end{array}$$

is a state on A. Moreover, the GNS representation of ϕ is unitarily equivalent to (π, H).

Proof. Assume that A is a unital C*-algebra. Assume that (π, H) is a representation of A and that $\xi \in H$ is a unit cyclic vector of this representation. Assume that ϕ is the linear functional defines as above.

To show: (a) ϕ is a state.

(a) First observe that ϕ is a positive linear functional because if $a \in A$ then

$$\phi(a^*a) = \langle \pi(a^*a)\xi, \xi \rangle = \langle \pi(a)\xi, \pi(a)\xi \rangle = \|\pi(a)\xi\|^2 \geq 0.$$

Now, if $a \in A$ then

$$\begin{aligned} \phi(a) &= \langle \pi(a)\xi, \xi \rangle \\ &= \langle \pi(1_A a)\xi, \xi \rangle = \langle \pi(1_A)\pi(a)\xi, \xi \rangle \\ &= \langle \pi(a)\xi, \pi(1_A)\xi \rangle. \end{aligned}$$

This means that if $a \in A$ then

$$0 = \langle \pi(a)\xi, \pi(1_A)\xi - \xi \rangle.$$

Since $\xi \in H$ is cyclic, the subspace $\pi(A)\xi$ is dense in H. So, $(\pi(A)\xi)^\perp = \{0\}$ and $\pi(1_A)\xi = \xi$. Therefore,

$$\phi(1_A) = \langle \pi(1_A)\xi, \xi \rangle = \|\xi\|^2 = 1.$$

Therefore, ϕ is a state.

Next, we show that (π, H) is unitarily equivalent to the GNS representation $(\pi_\phi, H_\phi, \xi_\phi)$ of ϕ. Define the map $u : \pi(A)\xi \to A/N_\phi$ by

$$\begin{array}{rrcl} u: & \pi(A)\xi & \to & A/N_\phi \\ & \pi(a)\xi & \mapsto & a + N_\phi. \end{array}$$

We will use $\langle -, - \rangle_\phi$ to represent the inner product on H_ϕ constructed in the GNS representation. To see that u is isometric, we compute directly that

$$\begin{aligned} \langle \pi(a)\xi, \pi(a)\xi \rangle &= \langle \pi(a^*a)\xi, \xi \rangle \\ &= \phi(a^*a) = \langle a + N_\phi, a + N_\phi \rangle_\phi \\ &= \langle u(\pi(a)\xi), u(\pi(a)\xi) \rangle_\phi. \end{aligned}$$

We also have $u(\xi) = u(\pi(1_A)\xi) = 1_A + N_\phi = \xi_\phi$. By the universal property of the Hilbert space H (which is complete), u extends to a unitary operator $\tilde{u} : H \to H_\phi$.

To see that $\tilde{u}\pi(a)\tilde{u}^* = \pi_\phi(a)$ for $a \in A$, assume that $h \in H_\phi$ so that there exists a net $\{h_n + N_\phi\}_{n \in I}$ which converges to h. Then,

$$\begin{aligned} \pi_\phi(a)(h) &= \pi_\phi(a)(\lim_{n \in I}(h_n + N_\phi)) \\ &= \lim_{n \in I} \pi_\phi(a)(h_n + N_\phi) \\ &= \lim_{n \in I}(ah_n + N_\phi) = ah \end{aligned}$$

and

$$\begin{aligned}
\tilde{u}\pi(a)\tilde{u}^*(h) &= \tilde{u}\pi(a)\tilde{u}^*(\lim_{n\in I}(h_n + N_\phi)) \\
&= \lim_{n\in I}\left(u\pi(a)u^*(h_n + N_\phi)\right) \\
&= \lim_{n\in I} u\pi(a)(\pi(h_n)\xi) \\
&= \lim_{n\in I}(ah_n + N_\phi) = ah.
\end{aligned}$$

Hence, the representations (π, H) and $(\pi_\phi, H_\phi, \xi_\phi)$ are unitarily equivalent. $\square$

The GNS construction, in tandem with Theorem 1.11.3, tells us that if A is a unital C*-algebra then there is a bijection of sets

$$\begin{array}{ccc}
\{\text{States on } A\} & \longleftrightarrow & \left\{\begin{matrix}\text{Representations of } A \\ \text{with unit cyclic vector}\end{matrix}\right\} \\
\phi & \mapsto & (\pi_\phi, H_\phi, \xi_\phi) \\
(a \mapsto \langle \pi(a)\xi, \xi\rangle) & \leftarrow\!\shortmid & (\pi, H, \xi)
\end{array}$$

The next question we will answer is this: where do the irreducible representations of A map to under the above bijection? As remarked in [Put19], irreducible representations of A map to the "extreme points" in the set of states. Theses states are often called *pure states.*

Theorem 1.11.4. *Let A be a unital C*-algebra and ϕ be a state on A. The GNS representation $(\pi_\phi, H_\phi, \xi_\phi)$ is irreducible if and only if ϕ is not a non-trivial convex combination of two other states — if there exists states ϕ_0, ϕ_1 and $t \in (0, 1)$ such that $\phi = t\phi_0 + (1-t)\phi_1$ then $\phi = \phi_0 = \phi_1$.*

Proof. Assume that A is a unital C*-algebra and ϕ is a state on A.

To show: (a) If ϕ is not a non-trivial convex combination of two other states then the GNS representation of ϕ is irreducible.

(b) If the GNS representation of ϕ is irreducible then ϕ is not a non-trivial convex combination of two other states.

(a) We will prove the contrapositive of this statement. Assume that the GNS representation $(\pi_\phi, H_\phi, \xi_\phi)$ is reducible. Then, there exists a non-trivial closed invariant subspace $\mathcal{N} \subseteq H_\phi$.

Since $H_\phi = \mathcal{N} \oplus \mathcal{N}^\perp$, we can decompose $\xi_\phi = \xi_0 + \xi_1$, where $\xi_0 \in \mathcal{N}$ and $\xi_1 \in \mathcal{N}^\perp$. We claim that $\xi_0, \xi_1 \neq 0$.

Suppose for the sake of contradiction that $\xi_1 = 0$. Then, $\xi_\phi = \xi_0 \in \mathcal{N}$. Consequently, $\pi(A)\xi_\phi \subseteq \pi(A)\mathcal{N} \subseteq \mathcal{N}$. However, since ξ_ϕ is a cyclic vector, the subspace $\pi(A)\xi_\phi$ is dense in H_ϕ. Therefore, $\mathcal{N} = \overline{\mathcal{N}} = H_\phi$, which contradicts the assumption that $\mathcal{N}$ is a non-trivial closed, invariant subspace of H_ϕ. So, $\xi_1 \neq 0$. The argument that $\xi_0 \neq 0$ is similar and uses Theorem 1.9.2.

If $a \in A$ and $i \in \{0, 1\}$ then define

$$\phi_i(a) = \|\xi_i\|^{-2}\langle \pi_\phi(a)\xi_i, \xi_i\rangle.$$

We claim that ϕ_i is a state. If $a \in A$ then

$$\phi_i(a^*a) = \|\xi_i\|^{-2}\|\pi_\phi(a)\xi_i\|^2 \geq 0.$$

So, ϕ_i is a positive linear functional. Now since $\mathcal{N}$ and $\mathcal{N}^\perp$ are both closed invariant subspaces of H_ϕ,

$$\begin{aligned}
\phi_i(1_A) &= \|\xi_i\|^{-2}\langle \pi_\phi(1_A)\xi_i, \xi_i\rangle \\
&= \|\xi_i\|^{-2}\langle \pi_\phi(1_A)\xi_i, \xi_i + \xi_{1-i}\rangle \\
&= \|\xi_i\|^{-2}\langle \pi_\phi(1_A)\xi_i, \xi_\phi\rangle \\
&= \|\xi_i\|^{-2}\langle \xi_i, \pi_\phi(1_A)\xi_\phi\rangle \\
&= \|\xi_i\|^{-2}\langle \xi_i, \xi_\phi\rangle \\
&= \|\xi_i\|^{-2}\|\xi_i\|^2 = 1.
\end{aligned}$$

Hence, ϕ_0 and ϕ_1 are both states.

Next, we claim that $\phi = \|\xi_0\|^2\phi_0 + \|\xi_1\|^2\phi_1$. We compute directly that if $a \in A$ then

$$\begin{aligned}
\phi(a) &= \langle \pi_\phi(a)\xi_\phi, \xi_\phi\rangle \\
&= \langle \pi_\phi(a)(\xi_0 + \xi_1), \xi_0 + \xi_1\rangle \\
&= \langle \pi_\phi(a)\xi_0, \xi_0\rangle + \langle \pi_\phi(a)\xi_0, \xi_1\rangle + \langle \pi_\phi(a)\xi_1, \xi_0\rangle + \langle \pi_\phi(a)\xi_1, \xi_1\rangle \\
&= \langle \pi_\phi(a)\xi_0, \xi_0\rangle + \langle \pi_\phi(a)\xi_1, \xi_1\rangle \\
&= \|\xi_0\|^2\phi_0(a) + \|\xi_1\|^2\phi_1(a).
\end{aligned}$$

Finally, we claim that $\phi_0 \neq \phi_1$. Let $C = \min\{\|\xi_0\|, \|\xi_1\|\} > 0$. Since the vector ξ_ϕ is cyclic, there exists $a \in A$ such that

$$\|\pi_\phi(a)\xi_\phi - \xi_0\| < 2^{-1}C.$$

Now, $\pi_\phi(a)\xi_\phi = \pi_\phi(a)\xi_0 + \pi_\phi(a)\xi_1$, where $\pi_\phi(a)\xi_0 \in \mathcal{N}$ and $\pi_\phi(a)\xi_1 \in \mathcal{N}^\perp$. Then,

$$\begin{aligned}
\|\pi_\phi(a)\xi_1\|^2 &= \langle \pi_\phi(a)\xi_1, \pi_\phi(a)\xi_1 \rangle \\
&= \langle \pi_\phi(a)\xi_\phi, \pi_\phi(a)\xi_1 \rangle \\
&= \langle \pi_\phi(a)\xi_\phi - \xi_0, \pi_\phi(a)\xi_1 \rangle \\
&\leq \|\pi_\phi(a)\xi_0 - \xi_0\|\|\pi_\phi(a)\xi_1\|.
\end{aligned}$$

and

$$\begin{aligned}
\|\pi_\phi(a)\xi_0 - \xi_0\|^2 &= \langle \pi_\phi(a)\xi_0 - \xi_0, \pi_\phi(a)\xi_0 - \xi_0 \rangle \\
&= \langle \pi_\phi(a)\xi_0 + \pi_\phi(a)\xi_1 - \xi_0, \pi_\phi(a)\xi_0 - \xi_0 \rangle \\
&= \langle \pi_\phi(a)\xi_\phi - \xi_0, \pi_\phi(a)\xi_0 - \xi_0 \rangle \\
&\leq \|\pi_\phi(a)\xi_\phi - \xi_0\|\|\pi_\phi(a)\xi_0 - \xi_0\|.
\end{aligned}$$

Hence, we have the inequalities

$$\|\pi_\phi(a)\xi_1\| \leq 2^{-1}C \qquad \text{and} \qquad \|\pi_\phi(a)\xi_0 - \xi_0\| \leq 2^{-1}C.$$

From both inequalities we obtain the upper bounds

$$\begin{aligned}
|\phi_0(a) - 1| &= |\|\xi_0\|^{-2}\langle \pi_\phi(a)\xi_0, \xi_0 \rangle - 1| \\
&= |\|\xi_0\|^{-2}\langle \pi_\phi(a)\xi_0, \xi_0 \rangle - \|\xi_0\|^{-2}\|\xi_0\|^2| \\
&= \|\xi_0\|^{-2}|\langle \pi_\phi(a)\xi_0 - \xi_0, \xi_0 \rangle| \\
&\leq \|\xi_0\|^{-1}\|\pi_\phi(a)\xi_0 - \xi_0\| \\
&< \|\xi_0\|^{-1}2^{-1}C \leq \frac{1}{2}
\end{aligned}$$

and

$$\begin{aligned}
|\phi_1(a)| &= |\|\xi_1\|^{-2}\langle \pi_\phi(a)\xi_1, \xi_1 \rangle| \\
&\leq \|\xi_1\|^{-1}\|\pi_\phi(a)\xi_1\| \\
&< \|\xi_1\|^{-1}2^{-1}C \leq \frac{1}{2}.
\end{aligned}$$

From these upper bounds, we deduce that $\phi_0(a) \neq \phi_1(a)$. Hence, the state ϕ is a non-trivial convex combination of the states ϕ_0 and ϕ_1.

(b) We will prove the contrapositive of this statement. Suppose that there exists states ϕ_0, ϕ_1 and $t \in (0,1)$ such that $\phi = t\phi_0 + (1-t)\phi_1$. Define the bilinear form

$$\begin{array}{cccc} B: & A/N_\phi \times A/N_\phi & \to & \mathbb{C} \\ & (a + N_\phi, b + N_\phi) & \mapsto & t\phi_0(b^*a). \end{array}$$

Observe that $B(a + N_\phi, b + N_\phi)$ is bounded above as follows:

$$\begin{aligned} |B(a + N_\phi, b + N_\phi)|^2 &= t^2|\phi_0(b^*a)|^2 \\ &\leq |t\phi_0(a^*a)||t\phi_0(b^*b)| \quad \text{(by Theorem 1.11.2)} \\ &\leq |\phi_0(a^*a)||\phi_0(b^*b)| \\ &= \|a + N_\phi\|^2\|b + N_\phi\|^2. \end{aligned}$$

By a similar argument to Theorem 1.11.3, we find that the bilinear form B is well-defined on A/N_ϕ. Since it is also bounded (and hence, continuous), B extends to a bounded bilinear form $\tilde{B} : H_\phi \times H_\phi \to \mathbb{C}$.

Since H_ϕ is a Hilbert space, there exists a unique positive operator $h \in B(H_\phi)$ such that if $a, b \in A$ then

$$t\phi_0(b^*a) = B(a + N_\phi, b + N_\phi) = \langle h(a + N_\phi), b + N_\phi \rangle.$$

By Theorem 1.9.7, it suffices to show that h commutes with $\pi_\phi(a)$ for $a \in A$ and that h is not a scalar multiple of the identity operator id_{H_ϕ}.

To show: (ba) h is not a scalar multiple of id_{H_ϕ}.

(bb) If $a \in A$ then $\pi_\phi(a)h = h\pi_\phi(a)$.

(ba) Suppose for the sake of contradiction that $h = \lambda id_{H_\phi}$ for some $\lambda \in \mathbb{C}$. Then,

$$t\phi_0(b^*a) = \lambda\langle a + N_\phi, b + N_\phi \rangle.$$

If $b = 1_A$ then $t\phi_0(a) = \lambda\langle a + N_\phi, 1_A + N_\phi \rangle = \lambda\phi(a)$. Thus, ϕ is a multiple of ϕ_0. However, ϕ and ϕ_0 are both states. So, $\phi(1_A) = \phi_0(1_A) = 1$. Therefore, $\phi = \phi_0$. Since $\phi = t\phi_0 + (1-t)\phi_1$, then $\phi_1 = \phi$. This contradicts the assumption that $\phi_0 \neq \phi_1$. Hence, h is not a scalar multiple of id_{H_ϕ}.

(bb) Assume that $a, b, c \in A$. Then,

$$\begin{aligned}
\langle \pi_\phi(a)h(b+N_\phi), c+N_\phi\rangle &= \langle h(b+N_\phi), \pi_\phi(a)^*(c+N_\phi)\rangle \\
&= \langle h(b+N_\phi), a^*c+N_\phi\rangle \\
&= t\phi_0((a^*c)^*b) = t\phi_0(c^*ab) \\
&= \langle h(ab+N_\phi), c+N_\phi\rangle \\
&= \langle h\pi_\phi(a)(b+N_\phi), c+N_\phi\rangle
\end{aligned}$$

Since $b, c \in A$ were arbitrary, we deduce that if $a \in A$ then $h\pi_\phi(a) = \pi_\phi(a)h$.

(b) By combining parts (ba) and (bb), we find that by Theorem 1.9.7, the GNS representation $(\pi_\phi, H_\phi, \xi_\phi)$ is reducible as required. $\square$

One of the most important applications of the GNS construction is the following strong theorem:

Theorem 1.11.5. *Let A be a C*-algebra. Then, there exists a Hilbert space H and a C*-subalgebra $B \subset B(H)$ such that $A \cong B$ as C*-algebras.*

Put simply, every unital C*-algebra is isomorphic to a C*-algebra of operators, reinforcing the notion that the space of bounded linear operators $B(H)$ is, in this sense, the prototypical example of a C*-algebra.

In order to prove Theorem 1.11.5, we require the following preliminary result.

Theorem 1.11.6. *Let A be a C*-algebra and $a \in A$ be self-adjoint. Then, there exists an irreducible representation π of A such that $\|\pi(a)\| = \|a\|$.*

Proof. We will first prove Theorem 1.11.6 for unital C*-algebras. Assume that A is a unital C*-algebra and $a \in A$ is self-adjoint. The idea is to use the GNS construction to construct the required representation of A.

However, we first need a state on A to do this. Let B be the C*-subalgebra of A generated by the set $\{1_A, a\}$. Then, B is commutative and unital. Recall that $\mathcal{M}(B)$ is the set of non-zero $\mathbb{C}$-algebra homomorphisms from B to $\mathbb{C}$. By Theorem 1.3.4, $\mathcal{M}(B)$ is homeomorphic to the spectrum $\sigma(a)$, which is a compact subset of $\mathbb{R}$.

Since a is self-adjoint, its spectral radius $r(a)$ is equal to $\|a\|$ by Theorem 1.2.6. Now choose $\phi_0 \in \mathcal{M}(B)$ such that

$$|\phi_0(a)| = \sup_{x \in \sigma(a)} |x| = \|a\|.$$

Note that $\phi_0(1_A) = 1$ by definition of $\mathcal{M}(B)$. By the Hahn-Banach extension theorem, we obtain a linear functional ϕ on A which extends ϕ_0 and has $\|\phi\| = \|\phi_0\|$. Hence,

$$\phi(1_A) = \phi_0(1_A) = 1 = \|\phi_0\| = \|\phi\|.$$

By Theorem 1.11.1, ϕ is a state on A. Note that by construction of ϕ, $|\phi(a)| = |\phi_0(a)| = \|a\|$.

Now let $(\pi_\phi, H_\phi, \xi_\phi)$ be the GNS representation of ϕ. Since ξ_ϕ is a unit vector, we have the inequality

$$\|\pi_\phi(a)\| = \|\pi_\phi(a)\|\|\xi_\phi\|^2 \geq |\langle \pi_\phi(a)\xi_\phi, \xi_\phi\rangle| = |\phi(a)| = \|a\|.$$

Recall from the GNS construction that if $a, b \in A$ then π_ϕ is defined by

$$\pi_\phi(a)(b + N_\phi) = ab + N_\phi.$$

In particular, $\pi_\phi(1_A) = id_{H_\phi}$ (the identity operator on H_ϕ). So, π_ϕ is a unital *-homomorphism and by Theorem 1.2.7, π_ϕ is a contraction which means that $\|a\| \leq \|\pi_\phi(a)\|$. Therefore, $\|a\| = \|\pi_\phi(a)\|$.

It remains to show that the representation $(\pi_\phi, H_\phi, \xi_\phi)$. The idea is to use Theorem 1.11.4. Let S be the set of states ϕ on A satisfying $|\phi(a)| = |\phi_0(a)| = \|a\|$. Let $\mathcal{B}$ be the closed unit ball in the dual space A^*. Then, $S \subseteq \mathcal{B}$.

To show: (a) S is a convex set.

(b) S is closed with respect to the weak *-topology on A^*.

(a) Assume that $\alpha, \beta \in S$ and $t \in [0, 1]$. Then, the convex combination $t\alpha + (1-t)\beta$ is a positive linear functional on A which satisfies

$$t\alpha(1_A) + (1-t)\beta(1_A) = t + 1 - t = 1.$$

Hence, $t\alpha + (1-t)\beta$ is a state on A. Moreover,

$$\|a\| = t\|a\| + (1-t)\|a\| = t|\alpha(a)| + (1-t)|\beta(a)|.$$

Hence, $t\alpha + (1-t)\beta \in S$ and S is a convex set.

(b) Assume that $\{\alpha_n\}_{n\in I}$ is a net in S which converges to some $\alpha \in A^*$ with respect to the weak *-topology on A^*. We want to show that $\alpha \in S$.

We find that

$$|\alpha(a)| = |\lim_{n\in I} \alpha_n(a)| = \lim_{n\in I} \|a\| = \|a\|.$$

To see that α is a state on A, note that

$$\alpha(1_A) = \lim_{n\in I} \alpha_n(1_A) = \lim_{n\in I} 1 = 1.$$

If $d \in A$ then

$$\alpha(d^*d) = \lim_{n\in I} \alpha_n(d^*d) \geq 0.$$

So, α is a state on A satisfying $|\alpha(a)| = \|a\|$. Therefore, $\alpha \in S$ and S is closed with respect to the weak *-topology.

Since the closed unit ball $\mathcal{B}$ is compact with respect to the weak *-topology on A (Banach-Alaoglu), S is also compact by part (b). Note also that S is non-empty because by construction, $\phi \in S$. Due to parts (a) and (b), we can apply the Krein-Milman theorem (see [Zim90, Section 2.3]) to deduce that S is the closure of the convex hull of the extreme points of S.

In particular, S has extreme points. So, we can choose $\phi \in S$ so that ϕ is one of the extreme points of S. By the definition of S, it is straightforward to show from this that ϕ is extreme among the set of states — it cannot be written as a convex combination of states. By Theorem 1.11.4, we deduce that $(\pi_\phi, H_\phi, \xi_\phi)$ is an irreducible representation of A.

For the general case, assume that A is a C*-algebra and $a \in A$ is self-adjoint. Then, there exists an irreducible representation $(\pi_\phi, H_\phi, \xi_\phi)$ of the unitization $\tilde{A}$ such that $\|\pi_\phi(a)\| = \|a\|$. Since A is a closed two-sided ideal of $\tilde{A}$, the restriction $(\pi_\phi|_A, H_\phi)$ defines a representation of A with $\|\pi_\phi(a)\| = \|a\|$.

Note that the GNS representation $(\pi_\phi, H_\phi, \xi_\phi)$ of $\tilde{A}$ is non-degenerate because it is a cyclic representation. By Theorem 1.9.6, every non-zero vector of $\tilde{A}$ is cyclic. In particular, every non-zero vector of $A \subseteq \tilde{A}$ is cyclic. Since the restricted representation $(\pi_\phi|_A, H_\phi)$ is also non-degenerate, then by Theorem 1.9.6, it is an irreducible representation as required. $\square$

Now we prove Theorem 1.11.5.

Proof of Theorem 1.11.5. Assume that A is a C*-algebra. Let

$$B = \{a \in A \,|\, \|a\| \leq 1\}$$

denote the closed unit ball in A. By Theorem 1.11.6, if $a \in B$ then there exists an irreducible representation (π_a, H_a) such that $\|\pi_a(a^*a)\| = \|a^*a\|$. Since $B(H_a)$ and A are both C*-algebras,

$$\|\pi_a(a)\| = \|\pi_a(a^*a)\|^{\frac{1}{2}} = \|a^*a\|^{\frac{1}{2}} = \|a\|.$$

Taking the direct sum over all $a \in B$, we obtain the *-homomorphism

$$\begin{array}{rccc} \bigoplus_{a\in B} \pi_a : & A & \to & \bigoplus_{a\in B} B(H_a) \\ & d & \mapsto & (\pi_a(d))_{a\in B} \end{array}$$

where on the RHS, we have a direct sum of C*-algebras. If $d \in A$ then

$$\begin{aligned} \|\big(\bigoplus_{a\in B} \pi_a\big)(d)\| &= \|(\pi_a(d))_{a\in B}\| \\ &= \sup_{a\in B} \|\pi_a(d)\| \\ &= \sup_{a\in B} \Big(\|\pi_a(\frac{d}{\|d\|})\|\|d\|\Big) \\ &\geq \|\pi_{\frac{d}{\|d\|}}(\frac{d}{\|d\|})\|\|d\| = \|d\|. \end{aligned}$$

Next, we will show that $\bigoplus_{a\in B} \pi_a$ is injective. Suppose for the sake of contradiction that there exists $k \in A - \{0\}$ such that $\big(\bigoplus_{a\in B} \pi_a\big)(k) = (0)_{a\in B}$. For clarity, $(0)_{a\in B} \in \bigoplus_{a\in B} B(H_a)$ is a sequence of zero operators. By the above inequality, we have

$$0 = \|\big(\bigoplus_{a\in B} \pi_a\big)(k)\| \geq \|k\|.$$

So, $k = 0$. This contradicts the assumption that k is non-zero. So, $\ker \bigoplus_{a\in B} \pi_a = \{0\}$ and $\bigoplus_{a\in B} \pi_a$ is an injective *-homomorphism.

By Theorem 1.6.4, $\bigoplus_{a\in B}\pi_a$ is an isometric *-homomorphism and an isometric *-isomorphism onto its image $\mathcal{I}$. By Theorem 1.7.6, $\mathcal{I}$ is a C*-subalgebra of $\bigoplus_{a\in B}B(H_a)\cong B(\bigoplus_{a\in B}H_a)$. This proves Theorem 1.11.5. $\square$

Chapter 2

Topics from [Mur90]

2.1 The Gelfand representation for abelian Banach algebras

In this section, we follow [Mur90, Section 1.3]. The goal of this section is to make precise the *Gelfand representation*, a way to represent an abelian Banach algebra as an algebra of continuous functions on an appropriate LCH (locally compact Hausdorff) space.

Note that we already accomplished this for commutative C*-algebras, as seen from Theorem 1.3.5 and Theorem 1.6.8. Thus, we will recycle results from section 1.3 if their proofs also work for abelian Banach algebras.

We will commence by proving a few preliminary results.

Definition 2.1.1. Let A be a $\mathbb{C}$-algebra. An ideal $I \subseteq A$ is called **modular** if there exists an element $u \in A$ such that if $a \in A$ then

$$a - au \in I \quad \text{and} \quad a - ua \in I.$$

Theorem 2.1.1. *Let A be a Banach algebra and I be a modular ideal. If I is a proper ideal then its closure $\overline{I}$ is also a proper ideal. If I is a maximal ideal then it is closed (topologically).*

Proof. Assume that A is a Banach algebra and I is a modular ideal. Then, there exists $u \in A$ such that if $a \in A$ then $a - au \in I$ and $a - ua \in I$. First, assume that I s a proper ideal of A.

To show: (a) If $b \in I$ then $\|u - b\| \geq 1$.

(a) Suppose for the sake of contradiction that there exists $b \in I$ such that $\|u - b\| < 1$. Consider the unitization $\tilde{A}$ of A (which is a Banach algebra by the construction in Theorem 1.6.1) and let $v = 1_{\tilde{A}} - u + b$.

We claim that $v \in \tilde{A}$ is invertible. Consider the sum

$$\sum_{n=0}^{\infty} \|(u-b)^n\|.$$

Note that $(u-b)^0 = 1_{\tilde{A}}$. This is a convergent sum in $\mathbb{R}$ because $\|u - b\| < 1$. By Theorem 1.7.1, the sequence $\{(u-b)^n\}_{n=0}^{\infty}$ in $\tilde{A}$ is summable. Thus, the sequence $\{\sum_{n=0}^{N}(u-b)^n\}_{N=1}^{\infty}$ converges to an element of $\tilde{A}$. We call this element $\sum_{n=0}^{\infty}(u-b)^n$.

To see that $\sum_{n=0}^{\infty}(u-b)^n$ is the inverse of v, we compute directly that

$$\begin{aligned}
v\Big(\sum_{n=0}^{\infty}(u-b)^n\Big) &= \lim_{N\to\infty} v\Big(\sum_{n=0}^{N}(u-b)^n\Big) \\
&= \lim_{N\to\infty} (1_{\tilde{A}} - (u-b))\Big(\sum_{n=0}^{N}(u-b)^n\Big) \\
&= \lim_{N\to\infty} (1_{\tilde{A}} - (u-b)^{N+1}) = 1_{\tilde{A}}.
\end{aligned}$$

Similarly, $\big(\sum_{n=0}^{\infty}(u-b)^n\big)v = 1_{\tilde{A}}$. Hence, $v^{-1} = \big(\sum_{n=0}^{\infty}(u-b)^n\big)$ and v is invertible in $\tilde{A}$.

By Theorem 1.6.1, A is a closed two-sided ideal of $\tilde{A}$. So, $Av \subseteq A$. Since v is invertible in $\tilde{A}$, $A = Av^{-1}v \subseteq Av$. Thus, $A = Av$. Now if $a \in A$ then

$$av = (a - au) + ab \in I.$$

Thus, $A = Av \subseteq I$. However, this contradicts the assumption that I is a proper ideal of A. We conclude that if $b \in I$ then $\|u - b\| \geq 1$.

Now define

$$B(u, \frac{1}{2}) = \{a \in A \mid \|u - a\| < \frac{1}{2}\}.$$

By part (a), $B(u, \frac{1}{2}) \cap I = \emptyset$. Hence, $u \in A$ but $u \notin \overline{I}$. Therefore, $\tilde{I}$ is a proper ideal of A.

Finally, if I is a maximal ideal of A then $\tilde{I}$ is a proper ideal of A containing I. So, $I = \overline{I}$ and I is closed. $\square$

We briefly remark that if L is a left ideal of a Banach algebra A then it is called modular if there exists $u \in A$ such that if $a \in A$ then $a - au \in L$. Theorem 2.1.1 carries over for modular left ideals.

Theorem 2.1.2. *Let A be a commutative Banach algebra and $I \subseteq A$ be a modular maximal ideal of A. Then, A/I is a field.*

Proof. Assume that A is a commutative Banach algebra. Assume that I is a modular maximal ideal of A. Let $u \in A$ be such that if $a \in A$ then $a - au \in I$ and $a - ua \in I$. Then, I is closed by Theorem 2.1.1. Thus, we can form the quotient Banach algebra A/I. It is abelian and unital, with multiplicative unit $u + I \in A/I$.

Let $\pi : A \to A/I$ be the canonical projection map. Then, π is a $\mathbb{C}$-algebra homomorphism. So, if J is an ideal of A/I then by the correspondence principle, the preimage $\pi^{-1}(J)$ is an ideal of A containing I. Since I is a maximal ideal, then either $\pi^{-1}(J) = I$ or $\pi^{-1}(J) = A$. Therefore, either $J = A/I$ or $J = 0$.

We conclude that A/I and 0 are the only ideals of A/I. So, A/I is a field. $\square$

Now recall from section 1.3 that if A is a $\mathbb{C}$-algebra then $\mathcal{M}(A)$ is the set of non-zero $\mathbb{C}$-algebra homomorphisms from A to $\mathbb{C}$. In [Mur90], the elements of $\mathcal{M}(A)$ are referred to as *characters*.

By Theorem 1.3.1, if A is a commutative unital C*-algebra then every element of $\mathcal{M}(A)$ has norm 1. A close examination of the proof of Theorem 1.3.1 shows that this conclusion also extends to the case where A is a commutative unital Banach algebra.

A relevant fact about $\mathcal{M}(A)$ which we have not proved yet is a bijection from $\mathcal{M}(A)$ to the set of maximal ideals of A.

Theorem 2.1.3. *Let A be a unital abelian Banach algebra. Then, there is a bijection of sets*

$$\begin{array}{rccc} K: & \mathcal{M}(A) & \to & \{\textit{Maximal ideals of } A\} \\ & \tau & \mapsto & \ker \tau \end{array}$$

Moreover, $\mathcal{M}(A) \neq \emptyset$.

Proof. Assume that A is a unital abelian Banach algebra. Assume that K is the function of sets defined as above. First, we will show that K is well-defined.

Assume that $\tau \in \mathcal{M}(A)$ so that $K(A) = \ker \tau$. Then, $\ker \tau$ is a closed ideal of A. Since $\tau \in \mathcal{M}(A)$, τ is not the zero map. So, $\ker \tau$ is a proper ideal of A. Now if $a \in A$ and 1_A is the unit of A then

$$\tau(a - \tau(a)1_A) = \tau(a) - \tau(a)\tau(1_A) = \tau(a) - \tau(a) = 0.$$

We conclude that if $a \in A$ then $a - \tau(a)1_A \in \ker \tau$. This means that $\ker \tau + \mathbb{C}1_A = A$ as ideals. To see that $\ker \tau$ is a maximal ideal of A, suppose for the sake of contradiction that I is a proper ideal of A which contains $\ker \tau$. We claim that $I \subseteq \ker \tau$. Assume that $\alpha \in I$. Since $\ker \tau + \mathbb{C}1_A = A$, there exists $\lambda \in \mathbb{C}$ and $\kappa \in \ker \tau$ such that

$$\alpha = \kappa + \lambda 1_A.$$

Now $\lambda 1_A = \alpha - \kappa \in I$. Hence, $1_A \in I$ and $I = A$. This contradicts the assumption that I is proper. Hence, $\ker \tau$ is a maximal ideal and the map K is well-defined.

To show: (a) K is injective.

(b) K is surjective.

(a) Assume that $\tau_1, \tau_2 \in \mathcal{M}(A)$ such that $K(\tau_1) = K(\tau_2)$. Then, $\ker \tau_1 = \ker \tau_2$. Note that if $a \in A$ then $a - \tau_2(a)1_A \in \ker \tau_2$ and

$$\tau_1(a - \tau_2(a)1_A) = \tau_1(a) - \tau_2(a) = 0.$$

So, $\tau_1 = \tau_2$ and K is injective.

(b) Assume that I is a maximal ideal of A. Since A is unital, I is also a modular ideal. By Theorem 2.1.1, I is a closed (two-sided) ideal. This means that we can form the quotient Banach algebra A/I. By Theorem 2.1.2, A/I is a field with unit $1_A + I$.

In particular, A/I is a unital Banach algebra where every non-zero element is invertible. Now we claim that $A/I = \mathbb{C}(1_A + I)$. Suppose for the sake of contradiction that $A/I \neq \mathbb{C}(1_A + I)$. Then, there exists $a + I \in A/I$ such that $a + I \notin \mathbb{C}(1_A + I)$. In particular, $a + I$ is non-zero. If $\lambda \in \mathbb{C}$ then

$(\lambda 1_A - a) + I$ is a non-zero element of A/I and is thus invertible. Consequently, the spectrum $\sigma(a + I) = \emptyset$ which contradicts the fact that $\sigma(a + I)$ must be non-empty (because $a + I$ is non-zero and A/I is a unital Banach algebra).

Hence, $A/I = \mathbb{C}(1_A + I)$ and $A = I \oplus \mathbb{C}1_A$. Now define the map

$$\begin{array}{rccc} \phi: & A = I \oplus \mathbb{C}1_A & \to & \mathbb{C} \\ & a + \lambda 1_A & \mapsto & \lambda. \end{array}$$

Then, ϕ is a non-zero $\mathbb{C}$-algebra homomorphism such that $K(\phi) = \ker \phi = I$. So, K is surjective.

By parts (a) and (b) of the proof, we find that K is a bijection. Note that since A is unital, it has maximal ideals. By the bijection K, $\mathcal{M}(A) \neq \emptyset$. $\square$

Now we will use Theorem 2.1.3 to generalise part 1 of Theorem 1.3.1.

Theorem 2.1.4. *Let A be an abelian Banach algebra and $a \in A$. If A is unital then*

$$\sigma(a) = \{\tau(a) \mid \tau \in \mathcal{M}(A)\}.$$

Furthermore, if A is non-unital then

$$\sigma(a) = \{\tau(a) \mid \tau \in \mathcal{M}(A)\} \cup \{0\}.$$

Proof. Assume that A is an abelian Banach algebra and $a \in A$. First, assume that A is unital with multiplicative unit 1_A. By the proof of part 1 of Theorem 1.3.1, we have the inclusion

$$\{\tau(a) \mid \tau \in \mathcal{M}(A)\} \subseteq \sigma(a).$$

For the reverse inclusion, assume that $\lambda \in \sigma(a)$. Then, $\lambda 1_A - a$ is not invertible in A and the ideal $I = (\lambda 1_A - a)A$ is a proper ideal of A. Hence, I must be contained in a maximal ideal I_{max}. By the bijection established in Theorem 2.1.3, $I_{max} = \ker \phi$ for some $\phi \in \mathcal{M}(A)$.

Now observe that $\phi(\lambda 1_A - a) = 0$. We find that $\phi(a) = \lambda$. Hence, $\sigma(a) \subseteq \{\tau(a) \mid \tau \in \mathcal{M}(A)\}$ and subsequently,

$$\sigma(a) = \{\tau(a) \mid \tau \in \mathcal{M}(A)\}.$$

For the more general case, assume that A is non-unital. Let $\sigma_{\tilde{A}}(a)$ denote the spectrum of $a \in \tilde{A}$, where $\tilde{A}$ is the unitization of A. By the previous case, we have

$$\sigma_{\tilde{A}}(a) = \{\tau(a) \mid \tau \in \mathcal{M}(\tilde{A})\}.$$

By the universal property of unitization in Theorem 1.6.3, we have

$$\mathcal{M}(\tilde{A}) = \{\tilde{\tau} \mid \tau \in \mathcal{M}(A)\} \cup \{\tau_\infty\}$$

where $\tilde{\tau}$ is the unique homomorphism extending τ from the universal property and $\tau_\infty((\lambda, b)) = \lambda$ for $\lambda \in \mathbb{C}$ and $b \in A$. Since $a \in A$, we find that

$$\sigma(a) = \sigma_{\tilde{A}}(a) = \{\tau(a) \mid \tau \in \mathcal{M}(\tilde{A})\} = \{\beta(a) \mid \beta \in \mathcal{M}(A)\} \cup \{0\}$$

because $\tau_\infty((0, a)) = 0$ as required. □

By Theorem 2.1.4, if A is an abelian Banach algebra then $\mathcal{M}(A)$ is contained in the closed unit ball of the dual space A^*. This is analogous to Theorem 1.3.2. The space $\mathcal{M}(A)$ with the weak *-topology is called the *character space* of A or the *spectrum* of A.

The proof of Theorem 1.3.2 readily extends to the case where A is an abelian *unital* Banach algebra. Here is what happens in the non-unital case.

Theorem 2.1.5. *Let A be an abelian Banach algebra. Then, $\mathcal{M}(A)$ is a locally compact Hausdorff space when equipped with the weak *-topology from A^*.*

Proof. Assume that A is an abelian Banach algebra. If $\tau \in \mathcal{M}(A)$ then $\|\tau\| \leq 1$ by the proof of Theorem 1.3.1. In this case, $\mathcal{M}(A)$ is still contained in the closed unit ball of A^*.

By the same argument in Theorem 1.3.2, $\mathcal{M}(A) \cup \{0\}$ is a compact subset of A^* with respect to the weak *-topology. By the one point compactification, $\mathcal{M}(A)$ is a locally compact Hausdorff space as required. □

If A is an abelian Banach algebra and $a \in A$ then we can again define the evaluation map

$$\begin{array}{rccc} ev_a : & \mathcal{M}(A) & \to & \mathbb{C} \\ & \tau & \mapsto & \tau(a) \end{array}$$

By definition of the weak *-topology on $\mathcal{M}(A)$, the evaluation map ev_a is continuous. We claim that $ev_a \in Cts_0(\mathcal{M}(A), \mathbb{C})$. That is, the evaluation map on a vanishes at infinity.

Assume that $\epsilon \in \mathbb{R}_{>0}$. We want to show that the set

$$\mathcal{M}(A)_\epsilon = \{\tau \in \mathcal{M}(A) \mid |\tau(a)| \geq \epsilon\}$$

is compact with respect to the weak *-topology on $\mathcal{M}(A)$ and A^*. By the Banach-Alaoglu theorem, it suffices to show that $\mathcal{M}(A)_\epsilon$ is closed. To this end, assume that $\{\tau_n\}_{n\in I}$ is a net in $\mathcal{M}(A)_\epsilon$ which converges to $\tau \in \mathcal{M}(A)$ in the weak *-topology. Then, $\tau_n(a) \to \tau(a)$ in $\mathbb{C}$ in the limit over $n \in I$. To see that $|\tau(a)| \geq \epsilon$, note that if $m \in \mathbb{Z}_{>0}$ then there exists $N_m \in I$ such that if $n > N_m$ in I then

$$|\tau_n(a) - \tau(a)| < \frac{1}{2^m}.$$

So,

$$|\tau(a)| \geq \big||\tau(a) - \tau_n(a)| - |-\tau_n(a)|\big| > \epsilon - \frac{1}{2^m}.$$

Since $m \in \mathbb{Z}_{>0}$ was arbitrary, we deduce that $|\tau(a)| \geq \epsilon$. Hence, $\mathcal{M}(A)_\epsilon$ is closed with respect to the weak *-topology and the evaluation map $ev_a \in Cts_0(\mathcal{M}(A), \mathbb{C})$.

The evaluation map ev_a is called the **Gelfand transform** of a. Now we are able to extend Theorem 1.3.5 and prove the *Gelfand representation*.

Theorem 2.1.6 (Gelfand representation)**.** *Let A be an abelian Banach algebra and assume that $\mathcal{M}(A)$ is non-empty. Define the map*

$$\begin{array}{rccc} \Lambda: & A & \to & Cts_0(\mathcal{M}(A), \mathbb{C}) \\ & a & \mapsto & ev_a. \end{array}$$

Then, Λ is a norm-decreasing $\mathbb{C}$-algebra homomorphism (and hence, a homomorphism of Banach algebras) with $r(a) = \|ev_a\|_\infty$. Furthermore, if A is unital then $ev_a(\mathcal{M}(A)) = \sigma(a)$ and if A is non-unital then $\sigma(a) = ev_a(\mathcal{M}(A)) \cup \{0\}$.

Proof. Assume that A is an abelian Banach algebra and that $\mathcal{M}(A)$ is non-empty. Assume that Λ is the map defined as above. By Theorem 2.1.4, we have

$$ev_a(\mathcal{M}(A)) = \sigma(a)$$

if A is unital and $ev_a(\mathcal{M}(A)) \cup \{0\} = \sigma(a)$ if A is non-unital. Recalling the definition of the spectral radius, we have for $a \in A$,

$$\|ev_a\|_\infty = \sup_{\|\tau\|=1} |\tau(a)| = \sup_{\lambda \in \sigma(a)} |\lambda| = r(a).$$

This means that

$$\|\Lambda(a)\| = \|ev_a\|_\infty = r(a) \leq \|a\|$$

and consequently, $\|\Lambda\| \leq 1$. The fact that Λ is a $\mathbb{C}$-algebra homomorphism follows from direct computation. □

The Gelfand representation Λ in Theorem 2.1.6 allows us to define the notion of a radical.

Definition 2.1.2. Let A be an abelian Banach algebra. Let Λ be the Gelfand representation in Theorem 2.1.6. The **radical** of A, denoted by $Rad(A)$, is defined as

$$Rad(A) = \ker \Lambda = \{a \in A \mid \Lambda(a) = ev_a = 0\}.$$

The abelian Banach algebra A is said to be **semisimple** if the radical $Rad(A) = \ker \Lambda = \{0\}$.

One consequence of the Gelfand representation in Theorem 2.1.6 is that we can prove further properties about the spectral radius.

Theorem 2.1.7. *Let A be a Banach algebra. Let $a, b \in A$ satisfy $ab = ba$. Then, $r(a+b) \leq r(a) + r(b)$ and $r(ab) \leq r(a)r(b)$.*

Proof. Assume that A is a Banach algebra. Assume that $a, b \in A$ such that $ab = ba$. Without loss of generality, we may assume that A is abelian and unital (if not, pass to the unitization $\tilde{A}$ and consider the commutative Banach subalgebra generated by the set $\{1_{\tilde{A}}, a, b\}$).

By the Gelfand representation in Theorem 2.1.6, we have

$$r(a+b) = \|ev_{a+b}\|_\infty = \|ev_a + ev_b\|_\infty \leq \|ev_a\|_\infty + \|ev_b\|_\infty = r(a) + r(b).$$

and

$$r(ab) = \|ev_{ab}\|_\infty = \|ev_a ev_b\|_\infty \leq \|ev_a\|_\infty \|ev_b\|_\infty = r(a)r(b).$$

□

It is mentioned in [Mur90] that proofs of Theorem 2.1.6 which do not use the Gelfand representation are much more complicated that the proof we gave. In fact, in [Sol18, Section 1.2], the inequality $r(ab) \leq r(a)r(b)$ is mentioned as one of the properties of spectral radius, but is not proved.

Returning to Theorem 2.1.7, we note that if $a, b \in A$ do not commute then the inequalities do not necessarily hold. As an example, let $A = M_{2\times 2}(\mathbb{C})$,

$$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad b = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then, $r(a) = r(b) = 0$. But, $r(a+b) = 1$ and $r(ab) = 1$. Therefore, Theorem 2.1.7 does not hold in this general case.

Now we will state and prove the appropriate generalisation of Theorem 1.3.4.

Theorem 2.1.8. *Let A be a unital Banach algebra generated by the set $\{1_A, a\}$ where $a \in A$. Then, A is abelian and the evaluation map $ev_a \in Cts_0(\mathcal{M}(A), \mathbb{C})$ is a homeomorphism from $\mathcal{M}(A)$ to the spectrum $\sigma(a)$.*

Proof. Assume that A is a unital Banach algebra generated by the set $\{1_A, a\}$ for some $a \in A$. Since 1_A and a commute with each other, A must be abelian. By Theorem 2.1.4, the evaluation map ev_a is a bijection from the compact space $\mathcal{M}(A)$ to the compact Hausdorff space $\sigma(a) \subseteq \mathbb{C}$. Hence, ev_a is a homeomorphism. $\square$

We end this section with a particular application of the material, taken from [Mur90, Example 1.3.1].

Example 2.1.1. Let $\ell^1(\mathbb{Z})$ denote the set

$$\ell^1(\mathbb{Z}) = \{f : \mathbb{Z} \to \mathbb{C} \mid \sum_{n=-\infty}^{\infty} |f(n)| < \infty\}.$$

We know that $\ell^1(\mathbb{Z})$ is a Banach space, with scalar multiplication and addition defined pointwise. The norm of $\ell^1(\mathbb{Z})$ is given by

$$\|f\|_1 = \sum_{n=-\infty}^{\infty} |f(n)|.$$

If $f, g \in \ell^1(\mathbb{Z})$ and $m \in \mathbb{Z}$ then we define the *convolution* of f and g by

$$(f * g)(m) = \sum_{n=-\infty}^{\infty} f(m-n)g(n).$$

We claim that $f * g \in \ell^1(\mathbb{Z})$. A quick computation reveals that

$$\begin{aligned}
\|f * g\|_1 &= \sum_{m=-\infty}^{\infty} |(f * g)(m)| \\
&= \sum_{m=-\infty}^{\infty} |\sum_{n=-\infty}^{\infty} f(m-n)g(n)| \\
&\leq \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} |f(m-n)g(n)| \\
&= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} |f(m-n)g(n)| \\
&= \sum_{n=-\infty}^{\infty} |g(n)| \sum_{m=-\infty}^{\infty} |f(m-n)| = \|f\|_1 \|g\|_1 < \infty.
\end{aligned}$$

Hence, $f * g \in \ell^1(\mathbb{Z})$.

Next, we claim that $\ell^1(\mathbb{Z})$ is an abelian unital Banach algebra with multiplication defined by convolution. The established bound $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$ shows that multiplication is continuous in $\ell^1(\mathbb{Z})$. Let $\chi_{\{0\}}$ denote the characteristic function on the set $\{0\}$. Then, $\chi_{\{0\}} \in \ell^1(\mathbb{Z})$ and

$$\begin{aligned}
(f * \chi_{\{0\}})(m) &= \sum_{n=-\infty}^{\infty} f(m-n)\chi_{\{0\}}(n) \\
&= f(m-0) = f(m).
\end{aligned}$$

Similarly, if $m \in \mathbb{Z}$ then $(\chi_{\{0\}} * f)(m) = f(m)$. So, $\chi_{\{0\}}$ is the multiplicative unit for $\ell^1(\mathbb{Z})$.

Finally, to see that $\ell^1(\mathbb{Z})$ is abelian, we compute directly that if $f, g \in \ell^1(\mathbb{Z})$ and $m \in \mathbb{Z}$ then

$$
\begin{aligned}
(f * g)(m) &= \sum_{n=-\infty}^{\infty} f(m-n)g(n) \\
&= \sum_{m-n=-\infty}^{\infty} f(m-(m-n))g(m-n) \\
&= \sum_{n=-\infty}^{\infty} f(m-n)g(n) = (g * f)(m).
\end{aligned}
$$

So, $\ell^1(\mathbb{Z})$ is an abelian unital Banach algebra. Hence, we can look at its Gelfand representation in Theorem 2.1.6. First, observe that if $f \in \ell^1(\mathbb{Z})$ then

$$
f = \sum_{n=-\infty}^{\infty} f(n)(\chi_{\{1\}})^n.
$$

where $(\chi_{\{1\}})^n$ is the convolution of $\chi_{\{1\}}$ with itself n times for n positive. In fact, one can show that if $n \in \mathbb{Z}$ then $(\chi_{\{1\}})^n = \chi_{\{n\}}$. Let $U = \{z \in \mathbb{C} \mid |z| = 1\}$ and $z \in U$. We define τ_z by

$$
\begin{array}{rccc}
\tau_z : & \ell^1(\mathbb{Z}) & \to & \mathbb{C} \\
& f & \mapsto & \sum_{n=-\infty}^{\infty} f(n)z^n.
\end{array}
$$

Note that $\tau_z \in \mathcal{M}(\ell^1(\mathbb{Z}))$. In particular, if $f, g \in \ell^1(\mathbb{Z})$ and $z \in U$ then

$$
\begin{aligned}
\tau_z(f * g) &= \sum_{n=-\infty}^{\infty} (f * g)(n)z^n \\
&= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} f(n-m)g(m)z^n \\
&= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} f(n)g(m)z^{m+n} \\
&= \big(\sum_{n=-\infty}^{\infty} f(n)z^n \big) \big(\sum_{m=-\infty}^{\infty} g(m)z^m \big) \\
&= \tau_z(f)\tau_z(g)
\end{aligned}
$$

Thus, we have a map given by

$$
\begin{array}{rccc}
\Gamma : & U & \to & \mathcal{M}(\ell^1(\mathbb{Z})) \\
& z & \mapsto & \tau_z.
\end{array}
$$

Next, we claim that Γ is a bijection. To see that Γ is injective, assume that $z_1, z_2 \in U$ such that $\tau_{z_1} = \tau_{z_2}$. If $f \in \ell^1(\mathbb{Z})$ then

$$\sum_{n=-\infty}^{\infty} f(n) z_1^n = \sum_{n=-\infty}^{\infty} f(n) z_1^n.$$

In particular, if $f = \chi_{\{1\}}$ then $z_1 = z_2$. So, Γ is injective.

To see that Γ is surjective, assume that $\phi \in \mathcal{M}(\ell^1(\mathbb{Z}))$. First, observe that if $f = \sum_{n=-\infty}^{\infty} f(n)(\chi_{\{1\}})^n \in \ell^1(\mathbb{Z})$ then

$$\phi(f) = \phi(\sum_{n=-\infty}^{\infty} f(n)(\chi_{\{1\}})^n) = \sum_{n=-\infty}^{\infty} f(n)\phi(\chi_{\{1\}}^n) = \sum_{n=-\infty}^{\infty} f(n)\phi(\chi_{\{1\}})^n.$$

Since $\phi \in \mathcal{M}(\ell^1(\mathbb{Z}))$, then $\|\phi\| = 1$ and $|\phi(\chi_{\{1\}})| \leq 1$. We also have

$$1 = |\phi(\chi_{\{0\}})| = |\phi(\chi_{\{1\}})\phi(\chi_{\{1\}})^{-1}| = |\phi(\chi_{\{1\}})||\phi(\chi_{\{-1\}})|.$$

Since $|\phi(\chi_{\{-1\}})| \leq 1$, then $|\phi(\chi_{\{1\}})| \geq 1$ and consequently, $|\phi(\chi_{\{1\}})| = 1$. So, $\phi(\chi_{\{1\}}) \in U$ and

$$\phi = \tau_{\phi(\chi_{\{1\}})} = \Gamma(\phi(\chi_{\{1\}})).$$

Thus, Γ is surjective and hence, a bijection.

Let us now take a step further and prove that Γ is in fact, a homeomorphism. Since U is a compact subset of $\mathbb{C}$ and $\mathcal{M}(\ell^1(\mathbb{Z}))$ is a Hausdorff space, it suffices to show that Γ is continuous.

To this end, it suffices to show that if $f \in \ell^1(\mathbb{Z})$ then the composite $ev_f \circ \Gamma : U \to \mathbb{C}$ is continuous. This is straightforward to see because if $z \in U$ then $(ev_f \circ \Gamma)(z) = \tau_z(f)$ is the uniform limit of the sequence of continuous functions of z

$$\Big\{ \sum_{|n| \leq N} f(n) z^n \Big\}_{N \in \mathbb{Z}_{>0}}.$$

In turn, this holds because the sum $\sum_{n=-\infty}^{\infty} |f(n)z^n| = \|f\|_1 < \infty$. So, Γ is a homeomorphism and U can be identified with $\mathcal{M}(\ell^1(\mathbb{Z}))$ as topological spaces.

If $f \in \ell^1(\mathbb{Z})$ then the Gelfand transform of f is the evaluaton map $ev_f : \mathcal{M}(\ell^1(\mathbb{Z})) \to \mathbb{C}$. By the homeomorphism Γ, the Gelfand transform of f is a continuous function $\widehat{f} : U \to \mathbb{C}$ such that

$$\widehat{f}(z) = (ev_f \circ \Gamma)(z) = \tau_z(f) = \sum_{n=-\infty}^{\infty} f(n) z^n.$$

We recognise that $\widehat{f}$ is similar to a Fourier transform. Indeed, the coefficients $f(n)$ are given by

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} \widehat{f}(e^{it}) e^{-imt}\, dt &= \frac{1}{2\pi}\int_0^{2\pi} \sum_{n=-\infty}^{\infty} f(n) e^{int} e^{-imt}\, dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} \sum_{n=-\infty}^{\infty} f(n) e^{i(n-m)t}\, dt \\
&= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_0^{2\pi} f(n) e^{i(n-m)t}\, dt \\
&= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} 2\pi f(n) \delta_{m,n} = f(m).
\end{aligned}$$

We conclude that the set of Gelfand transforms of $\ell^1(\mathbb{Z})$ is the set of functions $h \in Cts(U, \mathbb{C})$ whose Fourier series is absolutely convergent.

2.2 More on positive elements of unital C*-algebras

In this section, we will prove various results about positive elements of unital C*-algebras. The relevant sections we will follow are [Mur90, Section 2.2] and [Sol18, Section 3.1]. First, we begin with [Mur90, Remark 2.2.1].

Example 2.2.1. Let $A = Cts_0(X, \mathbb{C})$, where X is a LCH space. Let A_{sa} be the set of real-valued functions in A. We can turn A_{sa} into a poset given by the relation $f \leq g$ if and only if $f(x) \leq g(x)$ for $x \in X$.

In this case, the function $f \in A$ is positive if and only if there exists $g \in A$ such that $f = \overline{g}g$. Also, f has a unique positive square root, which is the function $x \mapsto \sqrt{f(x)}$. In this section, we want to generalise this situation to an arbitrary C*-algebra.

First, we will prove the existence of a unique square root of a positive element. Usually, the proof is done using the continuous functional calculus. However, the proof we will give relies on the Gelfand representation of a commutative C*-algebra in Theorem 1.6.8.

Definition 2.2.1. Let A be a C*-algebra. We define A^+ to be the set of positive elements of A.

Theorem 2.2.1. *Let A be a C*-algebra and $a \in A^+$. Then, there exists a unique element $b \in A^+$ such that $b^2 = a$.*

Proof. Assume that A is a C*-algebra and $a \in A^+$. Define B to be the C*-subalgebra generated by the set $\{a\}$. Since a is positive, it is self-adjoint. So, B is a commutative C*-algebra.

The idea is to apply the Gelfand representation in Theorem 1.6.8 to B. As a C*-algebra, B is isomorphic to $Cts_0(X, \mathbb{C})$ for some locally compact Hausdorff space X. Let Λ denote the isometric *-isomorphism from B to $Cts_0(X, \mathbb{C})$. Since a is positive, $\Lambda(a)$ is a positive function in $Cts_0(X, \mathbb{C})$.

By the previous remark, $\Lambda(a)$ has a unique square root, which we denote by $\Lambda(b) \in Cts_0(X, \mathbb{C})$. By applying the inverse map, we obtain a positive element $b \in A$ such that

$$b^2 = \Lambda^{-1}(\Lambda(b))^2 = \Lambda^{-1}(\Lambda(b)^2) = \Lambda^{-1}(\Lambda(a)) = a.$$

Now we will address the uniqueness of b. Assume that there exists $c \in A^+$ such that $c^2 = a$. Then, c must commute with a. Since $b \in B$ and B is the C*-subalgebra generated by $\{a\}$, c must also commute with b, as b is the limit of polynomials in a.

Now let C be the C*-subalgebra of A generated by the set $\{b, c\}$. Then, C is a commutative C*-algebra. By another application of Theorem 1.6.8, let $\Lambda_C : C \to Cts_0(Y, \mathbb{C})$ be the isometric *-isomorphism defining the Gelfand representation of C. Here, Y is a LCH space.

Observe that $\Lambda_C(a) = \Lambda_C(b)^2 = \Lambda_C(c)^2$. So, $\Lambda_C(b)$ and $\Lambda_C(c)$ are positive square roots of $\Lambda_C(a) \in Cts_0(Y, \mathbb{C})$ because b and c are positive. By the previous remark again, we deduce that by uniqueness, $\Lambda_C(b) = \Lambda_C(c)$ and so, $b = c$. This proves uniqueness. $\square$

The square root of a positive element a is denoted by $a^{\frac{1}{2}}$.

Definition 2.2.2. Let A be a unital C*-algebra and $a, b \in A$. We say that $a \leq b$ if the element $b - a \in A^+$. That is, the element $b - a$ is positive.

With the relation in Definition 2.2.2, we will prove that the pair $(A, \leq)$ is a poset. Let us first prove some more properties about the relation in Definition 2.2.2.

Theorem 2.2.2. *Let A be a unital C*-algebra. Define*

$$-A^+ = \{-x \mid x \in A^+\}.$$

1. *If $x \in A^+$ and $\lambda \in \mathbb{R}_{\geq 0}$ then $\lambda x \in A^+$.*
2. *If $x, y \in A^+$ then $x + y \in A^+$.*
3. *$A^+ \cap (-A^+) = \{0\}$*
4. *If $x \in A^+$ and $y \in A$ then $y^* x y \in A^+$.*

Proof. Assume that A is a unital C*-algebra.

(1) Assume that $x \in A^+$ and $\lambda \in \mathbb{R}_{\geq 0}$. Then,

$$\begin{aligned} \sigma(\lambda x) &= \{\alpha \in \mathbb{C} \mid \alpha 1_A - \lambda x \text{ is not invertible}\} \\ &= \{\lambda\beta \in \mathbb{C} \mid \beta 1_A - x \text{ is not invertible}\} \subseteq \mathbb{R}_{\geq 0}. \end{aligned}$$

Hence, $\lambda x \in A^+$.

(2) This follows from Theorem 1.4.3 and Theorem 1.4.5.

(3) Assume that $x \in A^+$ and $x \in -A^+$. Then, the spectrum $\sigma(x) \subseteq [0, \infty)$. By the spectral mapping theorem in Theorem 1.3.14, $\sigma(-x) \subseteq (-\infty, 0]$. But since $x \in -A^+$, $-x \in A^+$ and $\sigma(-x) \subseteq [0, \infty)$. Consequently, $\sigma(-x) = \{0\}$. This means that the spectral radius $r(-x) = 0$ and by Theorem 1.2.6, $\|-x\| = r(-x) = 0$. So, $x = 0$ and we conclude that $A^+ \cap (-A^+) = \{0\}$.

(4) Assume that $x \in A^+$ and $y \in A$. By Theorem 1.4.2, there exists $t \geq \|x\|$ such that $\|t1_A - x\| \leq t$. Now consider $t\|y\|^2 \in \mathbb{R}_{\geq 0}$. Then, $t\|y\|^2 \geq \|y^* x y\|$ and

$$\big\|t\|y\|^2 1_A - y^* x y\big\| \leq t\|y\|^2 + \|y^* x y\| \leq t\|y\|^2 + t\|y\|^2 = 2t\|y\|^2.$$

By Theorem 1.4.2, we deduce that $\sigma(y^* x y) \subseteq [0, \infty)$ and consequently, $y^* x y \in A^+$ as required. $\square$

Now we are ready to prove that $(A, \leq)$ is a poset.

Theorem 2.2.3. *Let A be a unital C*-algebra and $\leq$ be the relation defined in Definition 2.2.2. Then, the pair $(A, \leq)$ is a poset.*

Proof. Assume that A is a unital C*-algebra. Assume that $\leq$ is the relation defined in Definition 2.2.2.

To show: (a) If $x \in A$ then $x \leq x$.

(b) If $x, y, z \in A$, $x \leq y$ and $y \leq z$ then $x \leq z$.

(c) If $x, y \in A$, $x \leq y$ and $y \leq x$ then $x = y$.

(a) Assume that $x \in A$. Then, $x - x = 0 \in A^+$. So, $x \leq x$.

(b) Assume that $x, y, z \in A$, $x \leq y$ and $y \leq z$. Then, $y - x, z - y \in A^+$ and since A^+ is closed under addition,

$$z - x = (z - y) + (y - x) \in A^+.$$

So, $x \leq z$.

(c) Assume that $x, y \in A$, $x \leq y$ and $y \leq x$. Then, $y - x, x - y \in A^+$. This means that $y - x \in A^+ \cap (-A^+)$. But, $A^+ \cap (-A^+) = \{0\}$ by Theorem 2.2.2. Hence, $y - x = 0$ and $x = y$.

Consequently, the pair $(A, \leq)$ is a poset. $\square$

We will prove some more properties about positive elements in a unital C*-algebra. The next few results originate from [Mur90, Theorem 2.2.5].

Theorem 2.2.4. *Let A be a unital C*-algebra. Let $a, b, c \in A$. If $a \leq b$ then $c^*ac \leq c^*bc$.*

Proof. Assume that A is a unital C*-algebra. Assume that $a, b, c \in A$ and $a \leq b$. Then, $b - a \in A^+$ and by part 4 of Theorem 2.2.2,

$$c^*bc - c^*ac = c^*(b - a)c \in A^+.$$

Therefore, $c^*ac \leq c^*bc$. $\square$

Theorem 2.2.5. *Let A be a unital C*-algebra. Let $a, b \in A$ such that $0 \leq a \leq b$. Then, $\|a\| \leq \|b\|$.*

Proof. Assume that A is a unital C*-algebra. Assume that $a, b \in A$ and $0 \leq a \leq b$. First, we claim that $b \leq \|b\|1_A$. Let C_b be the C*-subalgebra of A generated by the set $\{1_A, b\}$. Then, C_b is a commutative C*-algebra and thus, we can consider its Gelfand representation from Theorem 1.6.8.

Let $\Lambda_b : C_b \to Cts_0(X_b, \mathbb{C})$ be the isometric *-isomorphism from C_b to $Cts_0(X_b, \mathbb{C})$, where X_b is a locally compact Hausdorff space. Since Λ_b is isometric, then

$$\sup_{x \in X} |\Lambda_b(b)(x)| = \|\Lambda_b(b)\| = \|b\|.$$

Now let $\mathbb{1}$ be the multiplicative unit of $Cts_0(X_b, \mathbb{C})$ (the constant function 1). Then, the function $\|b\|\mathbb{1} - \Lambda_b(b)$ is positive and by applying the inverse map Λ_b^{-1}, we deduce that the element $\|b\|1_A - b \in C_b$ is positive. So, $\|b\|1_A \geq b$.

Since $0 \leq a \leq b$ by assumption, then $a \leq \|b\|1_A$. Next, let C_a be the C*-subalgebra of A generated by the set $\{1_A, a\}$. Again, C_a is a commutative C*-algebra and thus, we let $\Lambda_a : C_a \to Cts_0(X_a, \mathbb{C})$ denote the Gelfand representation of C_a. Then, the function $\|b\|\mathbb{1} - \Lambda_a(a)$ is positive and if $x \in X_a$ then

$$0 \leq \Lambda_a(a)x \leq \|b\|x.$$

By taking the absolute value and then the supremum over all $x \in X_a$, we find that $\|a\| = \|\Lambda_a(a)\| \leq \big\| \|b\|\mathbb{1} \big\|$. Hence, $(\|b\| - \|a\|)\mathbb{1}$ is a positive function and consequently, $\|b\| \geq \|a\|$. $\square$

Theorem 2.2.6. *Let A be a unital C*-algebra and $a, b \in A$ be positive invertible elements of A. If $a \leq b$ then $0 \leq b^{-1} \leq a^{-1}$.*

Proof. Assume that A is a unital C*-algebra. Assume that $a, b \in A$ are positive invertible elements of A.

To show: (a) If $c \geq 1_A$ then c is invertible.

(a) Assume that $c \in A$ such that $c \geq 1_A$. Let C_c denote the C*-subalgebra generated by the set $\{1_A, c\}$ and $\Lambda_c : C_c \to Cts_0(X_c, \mathbb{C})$ be the Gelfand representation of C_c. Here, X_c is a locally compact Hausdorff space.

Since $c \geq 1_A$, then the function $\Lambda_c(c) - \mathbb{1}$ is positive. Here, $\mathbb{1}$ is the multiplicative unit of $Cts_0(X_c, \mathbb{C})$. Since $0 \leq c$, the function $\Lambda_c(c)$ is real-valued and positive. If $x \in X_c$ then $\Lambda_c(c)(x) \geq 1$. Thus, we can define the inverse function $\Lambda_c(c)^{-1}$ by

$$\begin{array}{rccc} \Lambda_c(c)^{-1}: & X_c & \to & \mathbb{C} \\ & x & \mapsto & \frac{1}{\Lambda_c(c)(x)}. \end{array}$$

Then, $\Lambda_c(c)^{-1} \in Cts_0(X_c, \mathbb{C})$ and $\Lambda_c(c)\Lambda_c(c)^{-1} = \mathbb{1}$. By applying the inverse map Λ_c^{-1}, we find that c is an invertible element of A, with inverse given by

$$c^{-1} = \Lambda_c^{-1}(\Lambda_c(c)^{-1}).$$

Note that by construction, we also have $c^{-1} \leq 1_A$.

Now assume that $a \leq b$. Since a is invertible, then

$$1_A = a^{-\frac{1}{2}} a a^{-\frac{1}{2}} \leq a^{-\frac{1}{2}} b a^{-\frac{1}{2}}.$$

By part (a), we deduce that

$$a^{\frac{1}{2}} b^{-1} a^{\frac{1}{2}} \leq 1_A.$$

Consequently, $b^{-1} \leq a^{-\frac{1}{2}} a^{-\frac{1}{2}} = a^{-1}$ as required. □

2.3 Approximate units

Theorem 1.7.3 can be strengthened, giving rise to the powerful notion of an **approximate unit**. It is powerful because every C*-algebra has an approximate unit.

Definition 2.3.1. Let A be a C*-algebra. An **approximate unit** is an increasing net $\{u_\lambda\}_{\lambda \in L}$ of positive elements in the closed unit ball of A (L is a upwards directed set) such that if $a \in A$ then

$$a = \lim_\lambda a u_\lambda = \lim_\lambda u_\lambda a.$$

Let A be a C*-algebra and A^+ be the poset of positive elements in A (see Theorem 2.2.3). Let A_1^+ denote the set of positive elements in A with norm less than 1. Then, A_1^+ is a poset, inheriting its relation from A^+. We will prove that the set A_1^+ is upwards directed.

Theorem 2.3.1. *Let A be a C*-algebra and A_1^+ be the poset of positive elements of A with norm less than 1. Let $a, b \in A_1^+$. Then, there exists $c \in A_1^+$ such that $a \leq c$ and $b \leq c$.*

Proof. Assume that A is a C*-algebra and A_1^+ is the poset of positive elements of A with norm less than 1. If $a \in A^+$ then $1_{\tilde{A}} + a$ is invertible in the unitization $\tilde{A}$ (since $\sigma_A(a) = \sigma_{\tilde{A}}(a)$ and $r(a) \leq \|a\| \leq 1$) and $a(1_{\tilde{A}} + a)^{-1} = 1_{\tilde{A}} - (1_{\tilde{A}} + a)^{-1}$.

To show: (a) If $a, b \in A^+$ and $a \leq b$ then $a(1_{\tilde{A}} + a)^{-1} \leq b(1_{\tilde{A}} + b)^{-1}$.

(a) Assume that $a, b \in A^+$ and $a \leq b$. Then, $1_{\tilde{A}} + a \leq 1_{\tilde{A}} + b$ in the unitization $\tilde{A}$. By Theorem 2.2.6, $(1_{\tilde{A}} + b)^{-1} \leq (1_{\tilde{A}} + a)^{-1}$. Therefore,

$$a(1_{\tilde{A}} + a)^{-1} = 1_{\tilde{A}} - (1_{\tilde{A}} + a)^{-1} \leq 1_{\tilde{A}} - (1_{\tilde{A}} + b)^{-1} = b(1_{\tilde{A}} + b)^{-1}$$

as required.

Next, we claim that if $a \in A^+$ then $a(1_{\tilde{A}} + a)^{-1} \in A_1^+$. Let C be the C*-subalgebra of $\tilde{A}$ generated by the set $\{1_{\tilde{A}}, a\}$ and $\Lambda : C \to Cts_0(X, \mathbb{C})$ denote the Gelfand representation of C, where X is a locally compact Hausdorff space. Since Λ is isometric, we have

$$\|a(1_{\tilde{A}} + a)^{-1}\| = \|\frac{\Lambda(a)}{\mathbb{1} + \Lambda(a)}\|_\infty < 1.$$

where $\mathbb{1} \in Cts_0(X, \mathbb{C})$ is the unit (the map $x \mapsto 1$). Obviously, $a(1_{\tilde{A}} + a)^{-1}$ is still positive.

Now assume that $a, b \in A_1^+$. Define $a' = a(1_{\tilde{A}} - a)^{-1}$ and $b' = b(1_{\tilde{A}} - b)^{-1}$. By the spectral mapping theorem in Theorem 1.3.14 and the fact that if $x \in [0, 1)$ then

$$\frac{x}{1 - x} \geq 0,$$

we deduce that the elements a' and b' are both positive. Furthermore,

$$a'(1_{\tilde{A}} + a')^{-1} = a(1_{\tilde{A}} - a)^{-1}(1_{\tilde{A}} + a(1_{\tilde{A}} - a)^{-1})^{-1} = a$$

and similarly $b'(1_{\tilde{A}} + b')^{-1} = b$. Now define $c = (a' + b')(1_{\tilde{A}} + a' + b')^{-1}$. Then, $a' \leq a' + b'$ and $b' \leq a' + b'$. By the proof of part (a), $c \in A_1^+$ and thus,

$$a = a'(1_{\tilde{A}} + a')^{-1} \leq (a' + b')(1_{\tilde{A}} + a' + b')^{-1} = c.$$

Similarly, $b \leq c$. Therefore, the poset A_1^+ is upwards directed. $\square$

Now we will proceed to prove the existence of an approximate unit.

Theorem 2.3.2. *Let A be a C*-algebra. Let $\Lambda = A_1^+$ be the set of positive elements of A with norm less than 1, which is an upwards directed poset by Theorem 2.3.1. If $\lambda \in \Lambda$ then define $u_\lambda = \lambda$. Then, $\{u_\lambda\}_{\lambda\in\Lambda}$ is an approximate unit for A (sometimes called the canonical approximate unit).*

Proof. Assume that A is a C*-algebra and that Λ is defined as above. By Theorem 2.3.1, $\{u_\lambda\}_{\lambda\in\Lambda}$ is an increasing net of positive elements contained in the closed unit ball of A. Hence, it suffices to show that if $a \in A$ then $a = \lim_\lambda u_\lambda a$. Since Λ linearly spans A, we may assume that $a \in \Lambda$.

Assume that $\epsilon \in \mathbb{R}_{>0}$ and $C^*(a)$ is the C*-subalgebra of the unitization $\tilde{A}$ generated by the set $\{1_{\tilde{A}}, a\}$. Using Theorem 1.6.8, let $\varphi : C^*(a) \to Cts_0(X, \mathbb{C})$ denote the Gelfand representation of $C^*(a)$, where X is a compact Hausdorff space. If $f = \varphi(a)$ then

$$K = \{\omega \in X \mid |f(\omega)| \geq \epsilon\} \subseteq X$$

is compact. By Urysohn's lemma, we can construct a continuous function $g : X \to [0, 1]$ with compact support such that if $\omega \in K$ then $g(\omega) = 1$. Let $\delta \in \mathbb{R}_{>0}$ such that $\delta < 1$ and $1 - \delta < \epsilon$. Then,

$$\|f - \delta g f\|_\infty \leq \|f\|_\infty \|\mathbb{1} - \delta g\|_\infty = \|a\| \|\mathbb{1} - \delta g\|_\infty < \epsilon.$$

The last inequality follows from the assumption that $a \in \Lambda = A_1^+$. Now define $\lambda_0 = \varphi^{-1}(\delta g)$. Then, $\lambda_0 \in \Lambda$ and by the above inequality,

$$\|a - u_{\lambda_0} a\| = \|a - \lambda_0 a\| = \|\varphi(a - \lambda_0 a)\|_\infty = \|f - \delta g f\|_\infty < \epsilon.$$

To see that $\lim_\lambda u_\lambda a = a$, let $\lambda \in \Lambda$ such that $\lambda \geq \lambda_0$. Then, $1_{\tilde{A}} - u_\lambda \leq 1_{\tilde{A}} - u_{\lambda_0}$ and by Theorem 2.2.4,

$$a(1_{\tilde{A}} - u_\lambda)a = a^*(1_{\tilde{A}} - u_\lambda)a \leq a^*(1_{\tilde{A}} - u_{\lambda_0})a = a(1_{\tilde{A}} - u_{\lambda_0})a.$$

Therefore,

$$\begin{aligned}
\|a - u_\lambda a\|^2 &= \|(1_{\tilde{A}} - u_\lambda)^{\frac{1}{2}}(1_{\tilde{A}} - u_\lambda)^{\frac{1}{2}} a\|^2 \\
&\leq \|(1_{\tilde{A}} - u_\lambda)^{\frac{1}{2}}\|^2 \|(1_{\tilde{A}} - u_\lambda)^{\frac{1}{2}} a\|^2 \\
&= \|1_{\tilde{A}} - u_\lambda\| \|(1_{\tilde{A}} - u_\lambda)^{\frac{1}{2}} a\|^2 \\
&\leq \|1_{\tilde{A}}\| \|(1_{\tilde{A}} - u_\lambda)^{\frac{1}{2}} a\|^2 \quad \text{(Theorem 2.2.5)} \\
&= \|(1_{\tilde{A}} - u_\lambda)^{\frac{1}{2}} a\|^2 \\
&= \|a(1_{\tilde{A}} - u_\lambda)a\| \\
&\leq \|a(1_{\tilde{A}} - u_{\lambda_0})a\| \quad \text{(Theorem 2.2.5)} \\
&\leq \|a\| \|a - u_{\lambda_0} a\| < \epsilon.
\end{aligned}$$

So, $\lim_\lambda u_\lambda a = a$. The equality $\lim_\lambda a u_\lambda = a$ follows from an analogous argument. □

The rest of this section is dedicated to proving useful properties about approximate units.

Theorem 2.3.3. *Let A be a C*-algebra and I be a closed left ideal of A. Then, there exists an increasing net $\{u_\lambda\}_{\lambda \in L}$ of positive elements contained in the closed unit ball of I such that if $a \in I$ then $a = \lim_\lambda a u_\lambda$.*

Proof. Assume that A is a C*-algebra and I is a closed left ideal of A. Define $B = I \cap I^*$. Then, B is a C*-algebra and thus, we can use Theorem 2.3.2 to construct an approximate unit $\{u_\lambda\}_{\lambda \in L}$. If $a \in I$ then $a^* a \in B$. So,

$$\lim_\lambda a^* a (1_{\tilde{B}} - u_\lambda) = \lim_\lambda (a^* a - a^* a u_\lambda) = 0$$

and subsequently

$$\begin{aligned}
\lim_\lambda \|a - a u_\lambda\|^2 &= \lim_\lambda \|a(1_{\tilde{B}} - u_\lambda)\|^2 \\
&= \lim_\lambda \|(1_{\tilde{B}} - u_\lambda) a^* a (1_{\tilde{B}} - u_\lambda)\| \\
&\leq \lim_\lambda \|(1_{\tilde{B}} - u_\lambda)\| \|a^* a (1_{\tilde{B}} - u_\lambda)\| \\
&\leq \lim_\lambda \|a^* a (1_{\tilde{B}} - u_\lambda)\| = 0.
\end{aligned}$$

So, $a = \lim_\lambda a u_\lambda$. □

Theorem 2.3.4. *Let A be a C*-algebra and (φ, H) be a non-degenerate representation of A. Let $\{u_\lambda\}_{\lambda \in \Lambda}$ be an approximate unit for A. Then, $\{\varphi(u_\lambda)\}_{\lambda \in \Lambda}$ is an approximate unit for the image $\varphi(A)$ which converges strongly to the identity operator id_H.*

For a refresher on the strong operator topology on $B(H)$, consult the beginning of section 2.4.

Proof. Assume that A is a C*-algebra and (φ, H) is a non-degenerate representation of A. Assume that $\{u_\lambda\}_{\lambda\in\Lambda}$ is an approximate unit for A. Since φ is continuous, the net $\{\varphi(u_\lambda)\}_{\lambda\in\Lambda}$ is an approximate unit for the C*-algebra $\varphi(A)$.

If $b \in \varphi(A)$ then

$$\lim_\lambda b\varphi(u_\lambda) = \lim_\lambda \varphi(u_\lambda)b = b.$$

Since (φ, H) is non-degenerate, we can use Theorem 1.9.4 to show that $\overline{\varphi(A)H} = H$. If $\xi \in H$ then

$$\lim_\lambda (b\varphi(u_\lambda))\xi = \lim_\lambda (\varphi(u_\lambda)b)\xi = b\xi.$$

Since this holds for arbitrary $b\xi \in \varphi(A)H$, we deduce that if $\psi \in H$ then

$$\lim_\lambda \varphi(u_\lambda)\psi = \psi.$$

Therefore, the net $\{\varphi(u_\lambda)\}_{\lambda\in\Lambda}$ converges strongly to the identity operator id_H. □

Theorem 2.3.5. *Let A be a C*-algebra and τ be a bounded linear functional on A. The following are equivalent:*

1. *τ is positive.*
2. *If $\{u_\lambda\}_{\lambda\in L}$ is an approximate unit for A then $\|\tau\| = \lim_\lambda \tau(u_\lambda)$.*
3. *There exists an approximate unit $\{u_\lambda\}_{\lambda\in L}$ for A satisfying $\|\tau\| = \lim_\lambda \tau(u_\lambda)$.*

Proof. Assume that A is a C*-algebra and τ is a bounded linear functional on A. Without loss of generality, assume that $\|\tau\| = 1$.

First assume that τ is positive and $\{u_\lambda\}_{\lambda\in L}$ is an approximate unit for A. By the construction in Theorem 2.3.2, u_λ is positive (and self-adjoint) for $\lambda \in L$ and the sequence $\{\tau(u_\lambda)\}_{\lambda\in L}$ is increasing in $\mathbb{R}$. Hence, it must converge to its supremum. Since $\|\tau\| = 1$ then $\sup_{\lambda\in L} \tau(u_\lambda) \le 1$. Consequently, $\lim_\lambda \tau(u_\lambda) \le 1$.

Now assume that $a \in A$ and $\|a\| \leq 1$. Then,

$$|\tau(u_\lambda a)|^2 \leq \tau(u_\lambda^2)\tau(a^*a) = \tau(u_\lambda)\tau(a^*a) \leq \tau(u_\lambda)\|\tau\|\|a^*a\| \leq \lim_\lambda \tau(u_\lambda).$$

By taking the limit over $\lambda \in L$, we deduce that $|\tau(a)|^2 \leq \lim_\lambda \tau(u_\lambda)$. Taking the supremum over all $a \in A$ with $\|a\| \leq 1$, we find that $1 \leq \lim_\lambda \tau(u_\lambda)$. Therefore, $1 = \lim_\lambda \tau(u_\lambda)$.

It is obvious that the second statement implies the third. Finally, assume that there exists an approximate unit $\{u_\lambda\}_{\lambda \in L}$ such that $1 = \lim_\lambda \tau(u_\lambda)$. First, assume that $a \in A$ is self-adjoint and $\|a\| \leq 1$.

To show: (a) $\tau(a) \in \mathbb{R}$.

(a) Assume that $\tau(a) = \alpha + i\beta$, where $\alpha, \beta \in \mathbb{R}$. We may assume further that $\beta \leq 0$. If $n \in \mathbb{Z}_{>0}$ then

$$\begin{aligned} |\tau(a - inu_\lambda)|^2 &\leq \|\tau\|^2 \|a - inu_\lambda\|^2 \\ &= \|a - inu_\lambda\|^2 \\ &= \|(a + inu_\lambda)(a - inu_\lambda)\| \\ &= \|a^2 + n^2 u_\lambda^2 - in(au_\lambda - u_\lambda a)\| \\ &\leq 1 + n^2 + n\|au_\lambda - u_\lambda a\|. \end{aligned}$$

Taking the limit over $\lambda \in L$, we find that by our assumption

$$\begin{aligned} |\tau(a) - in|^2 &= \lim_\lambda |\tau(a - inu_\lambda)|^2 \\ &\leq \lim_\lambda (1 + n^2 + n\|au_\lambda - u_\lambda a\|) \\ &= 1 + n^2. \end{aligned}$$

So, $|\alpha + i(\beta - n)|^2 \leq 1 + n^2$. By expanding the LHS of the inequality and then rearranging, we find that

$$-2\beta n \leq 1 - \beta^2 - \alpha^2$$

Since $\beta \leq 0$ and the above inequality holds for $n \in \mathbb{Z}_{>0}$, then $\beta = 0$. Hence, $\tau(a) \in \mathbb{R}$.

Finally, assume that a is positive and $\|a\| \leq 1$. Then, $u_\lambda - a$ is self-adjoint and $\|u_\lambda - a\| \leq \|u_\lambda\| \leq 1$ by Theorem 2.2.5. By part (a), $\tau(u_\lambda - a) \in \mathbb{R}$ and subsequently, $\tau(u_\lambda - a) \leq \|\tau\|\|u_\lambda - a\| \leq 1$. Taking the limit over $\lambda \in L$, we find that

$$1 \geq \lim_\lambda \tau(u_\lambda - a) = \lim_\lambda(\tau(u_\lambda) - \tau(a)) = 1 - \tau(a).$$

Hence, $\tau(a) \geq 0$ and τ must be positive. This completes the proof. $\square$

An intriguing consequence of Theorem 2.3.5 is the following result.

Theorem 2.3.6. *Let A be a C*-algebra and τ, τ' be positive linear functionals on A. Then, $\|\tau + \tau'\| = \|\tau\| + \|\tau'\|$.*

Proof. Assume that A is a C*-algebra. Assume that τ and τ' are positive linear functionals on A. Let $\{u_\lambda\}_{\lambda\in\Lambda}$ be an approximate unit for A. By Theorem 2.3.5, $\|\tau\| = \lim_\lambda \tau(u_\lambda)$ and $\|\tau'\| = \lim_\lambda \tau'(u_\lambda)$. So,

$$\|\tau + \tau'\| = \lim_\lambda(\tau(u_\lambda) + \tau'(u_\lambda)) = \|\tau\| + \|\tau'\|.$$

$\square$

Theorem 2.3.7. *Let A be a unital C*-algebra and $\tau : A \to \mathbb{C}$ be a bounded linear functional. Then, τ is positive if and only if $\tau(1_A) = \|\tau\|$.*

Proof. Assume that A is a unital C*-algebra. Assume that $\tau : A \to \mathbb{C}$ is a bounded linear functional. Let $\{u_\lambda\}_{\lambda\in L}$ be an approximate unit for A. By Theorem 2.3.5, τ is positive if and only if

$$\|\tau\| = \lim_\lambda \tau(u_\lambda) = \lim_\lambda \tau(1_A u_\lambda) = \tau(1_A).$$

$\square$

Our next use of approximate units is to generalise certain results contained in Theorem 1.11.2.

Theorem 2.3.8. *Let A be a C*-algebra and $\tau : A \to \mathbb{C}$ be a positive linear functional.*

1. *If $a \in A$ then $\tau(a^*a) = 0$ if and only for $b \in A$, $\tau(ba) = 0$*
2. *If $a, b \in A$ then $\tau(b^*a^*ab) \leq \|a^*a\|\tau(b^*b)$.*
3. *If $a \in A$ then $\tau(a^*) = \overline{\tau(a)}$.*

4. If $a \in A$ then $|\tau(a)|^2 \leq \|\tau\|\tau(a^*a)$.

Proof. Assume that A is a C*-algebra and $\tau : A \to \mathbb{C}$ is a positive linear functional. The map

$$\begin{array}{rccc} \langle -, - \rangle : & A \times A & \to & \mathbb{C} \\ & (a, b) & \mapsto & \tau(b^*a) \end{array}$$

defines a sesquilinear form on A and thus, satisfies the Cauchy-Schwarz inequality. If $a \in A$ satisfies $\tau(a^*a) = 0$ then this holds if and only if for $b \in A$,

$$|\tau(ba)|^2 \leq \tau(a^*a)\tau(bb^*) = 0$$

if and only if $\tau(ba) = 0$.

For the second statement, assume that $a, b \in A$. Observe that if $\tau(b^*b) = 0$ then by the first part, $\tau(b^*a^*ab) = 0$ and the inequality is trivially satisfied. So, assume that $\tau(b^*b) > 0$. Define the function

$$\begin{array}{rccc} \rho : & A & \to & \mathbb{C} \\ & c & \mapsto & \frac{\tau(b^*cb)}{\tau(b^*b)}. \end{array}$$

It is easy to verify that ρ is a positive linear functional. Now let $\{u_\lambda\}_{\lambda \in L}$ be an approximate unit for A. By Theorem 2.3.5,

$$\|\rho\| = \lim_\lambda \rho(u_\lambda) = \lim_\lambda \frac{\tau(b^*u_\lambda b)}{\tau(b^*b)} = \lim_\lambda \frac{\tau(b^*b)}{\tau(b^*b)} = 1.$$

Hence,

$$\frac{\tau(b^*a^*ab)}{\tau(b^*b)} = \rho(a^*a) \leq \|a^*a\|$$

which gives the desired inequality. For the third statement, assume that $a \in A$. Then,

$$\tau(a^*) = \lim_\lambda \tau(a^*u_\lambda) = \lim_\lambda \overline{\tau(u_\lambda a)} = \overline{\tau(a)}.$$

Note that the second last equality in the above equation follows from the fact that $\langle -, - \rangle$ is a sesquilinear form on A. For the final statement, we argue directly that

$$
\begin{aligned}
|\tau(a)|^2 &= \lim_\lambda |\tau(u_\lambda a)|^2 \\
&\leq \lim_\lambda |\tau(u_\lambda^2)||\tau(a^*a)| \qquad \text{(Cauchy-Schwarz inequality)} \\
&\leq \|\tau\|\tau(a^*a).
\end{aligned}
$$

□

2.4 An introduction to von Neumann algebras

Recall Theorem 1.6.9, which states that if A is a commutative C*-algebra then there exists a locally compact Hausdorff space X such that $A \cong Cts_0(X, \mathbb{C})$ as C*-algebras. In this sense, the theory of C*-algebras can be thought of as a theory of "non-commutative topology".

Von Neumann algebras are a specific class of C*-algebras which happen to be the setting for "non-commutative measure theory". The reason for this is because abelian von Neumann algebras are, up to isomorphism, of the form $L^\infty(X, \mu)$, where (X, μ) is a measure space. This section serves as an introduction to the rich theory of von Neumann algebras, with [Mur90, Chapter 4] serving as the main reference. The main result we cover is the well-known *double commutant theorem*.

Let H be a Hilbert space. There are a wide variety of topologies on the space of bounded linear operators $B(H)$. The most important one for this section is the *strong operator topology*, which we will introduce in a more general context.

Definition 2.4.1. Let V and W be Banach spaces and $B(V, W)$ be the Banach space of bounded linear operators $T : V \to W$. The **strong operator topology** on $B(V, W)$ is the $\mathcal{F}$-weak topology, where $\mathcal{F}$ is the set

$$\mathcal{F} = \{ev_v : B(V, W) \to W \mid v \in V\}$$

and $ev_v(T) = T(v)$. That is, the strong operator topology on $B(V, W)$ is the weakest topology on $B(V, W)$ which makes the evaluation maps in $\mathcal{F}$ continuous.

Note that the ordinary norm topology on $B(V, W)$ also makes the evaluation maps in $\mathcal{F}$ continuous. Hence, the strong operator topology is weaker than the norm topology. Here is how convergence is expressed in the strong operator topology.

Theorem 2.4.1. *Let V and W be Banach spaces and $B(V,W)$ denote the Banach space of bounded linear operators $T : V \to W$. Then, a sequence $\{A_n\}_{n\in\mathbb{Z}_{>0}}$ in $B(V,W)$ converges to an operator A in the strong operator topology if and only if for $v \in V$, $A_n v \to Av$ in W.*

Proof. Assume that V and W are Banach spaces and $B(V,W)$ is the Banach space of bounded linear operators.

To show: (a) If a sequence $A_n \to A$ in the strong operator topology on $B(V,W)$ then for $v \in V$, $A_n v \to Av$ in W.

(b) If for $v \in V$ $A_n v \to Av$ in W then $A_n \to A$ in the strong operator topology on $B(V,W)$.

(a) Assume that $\{A_n\}_{n\in\mathbb{Z}_{>0}}$ is a sequence in $B(V,W)$ which converges to $A \in B(V,W)$ in the strong operator topology. Assume that $\epsilon \in \mathbb{R}_{>0}$. Then, there exists $N \in \mathbb{Z}_{>0}$ such that if $n > N$ then

$$A_n - A \in L_{v,0,\epsilon} = \{T \in B(V,W) \mid \|Tv\| < \epsilon\}.$$

So, $\|(A_n - A)v\| < \epsilon$ and consequently, $A_n v \to Av$ in W.

(b) Assume that if $v \in V$ then $A_n v \to Av$ in W. Assume that $\epsilon \in \mathbb{R}_{>0}$. Then, there exists $N \in \mathbb{Z}_{>0}$ such that if $n > N$ then $\|A_n v - Av\| < \epsilon$. This means that

$$A_n - A \in L_{v,0,\epsilon} = \{T \in B(V,W) \mid \|Tv\| < \epsilon\}.$$

Since $L_{v,0,\epsilon}$ is a basic open set in the space $B(V,W)$ with the strong operator topology, we deduce that $A_n \to A$ in the strong operator topology. □

Using Theorem 2.4.1, we will provide an example illustrating the difference between the strong operator topology and the norm topology.

Example 2.4.1. Let $V = W = \ell^2(\mathbb{C})$. For $n \in \mathbb{Z}_{>0}$, define

$$\begin{array}{cccc} L_n : & \ell^2(\mathbb{C}) & \to & \ell^2(\mathbb{C}) \\ & (x_1, x_2, \dots) & \mapsto & (x_{n+1}, x_{n+2}, \dots) \end{array}$$

We will show that the sequence L_n converges to zero in the strong operator topology, but does not converge in the norm topology.

If $x = (x_1, x_2, \dots) \in \ell^2(\mathbb{C})$ then

$$\|L_n x\|^2 = \sum_{i=n+1}^{\infty} |x_i|^2 \to 0$$

as $n \to \infty$. Hence, $L_n x \to 0x$ and consequently, $L_n \to 0$ in the strong operator topology.

To see that L_n does not converge to zero in the norm topology, observe that

$$\begin{aligned} \|L_n\| &= \sup_{\|x\|=1} \|L_n x\| \\ &\geq \|L_n(0, 0, \dots, 1, 0, \dots)\| \\ &= \|(1, 0, 0, \dots)\| = 1. \end{aligned}$$

Since $\|L_n\| \geq 1$ for $n \in \mathbb{Z}_{>0}$, L_n can never converge to zero in the norm topology.

If H is a Hilbert space then $B(H)$ is a topological vector space. So, the operations of addition and scalar multiplication are continuous with respect to the strong operator topology on $B(H)$. However, multiplication and involution are not in general strongly continuous.

Example 2.4.2. Let H be an infinite-dimensional Hilbert space with orthonormal basis $\{e_n\}_{n=1}^{\infty}$. If $n \in \mathbb{Z}_{>0}$ then let $u_n = |e_1\rangle\langle e_n|$. If $x \in H$ then

$$u_n(x) = \langle x, e_n \rangle e_1.$$

and

$$\lim_{n\to\infty} \|u_n(x)\| = \lim_{n\to\infty} |\langle x, e_n \rangle| = 0$$

because if we write $x = \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i$ then all but finitely many of the coefficients $\langle x, e_i \rangle$ are zero.

Now, $u_n^* = |e_n\rangle\langle e_1|$ and if $x \in X$ then

$$\lim_{n\to\infty} \|u_n^*(x)\| = \lim_{n\to\infty} |\langle x, e_1 \rangle|.$$

This clearly does not converge in the strong operator topology on $B(H)$ because if $x = e_1$ then $\lim_{n\to\infty}\|u_n^*(x)\| = 1$. Hence, the involution operation on $B(H)$ is not strongly continuous.

So far, we have mentioned that the strong operator topology on $B(H)$ behaves badly in some aspects. Next, we will discuss a useful feature of the strong operator topology. Let $B(H)_{sa}$ denote the set of self-adjoint operators on H. Then, $(B(H)_{sa}, \leq)$ is a poset, where $x \leq y$ if and only if $y - x \geq 0$ or alternatively, if and only if $y - x$ is a positive operator. This is elaborated on further in [Sol18, Section 3.1].

We will prove that if we have a net $\{u_i\}_{i\in I}$ of self-adjoint operators which is increasing (with respect to the partial order $\leq$) and bounded above then it must strongly converge.

Theorem 2.4.2. *Let H be a Hilbert space over $\mathbb{C}$ and $\{x_i\}_{i\in I}$ be a net of self-adjoint operators such that if $i \geq j$, then $x_i \geq x_j$. Assume that there exists $C \in \mathbb{R}_{>0}$ such that if $i \in I$ then $\|x_i\| \leq C$.*

Then, there exists a self-adjoint operator $x \in B(H)$ such that if $i \in I$ then $x \geq x_i$ and if $y \in B(H)$ is an operator satisfying $y \geq x_i$ for $i \in I$ then $y \geq x$. Moreover, $\{x_i\}_{i\in I}$ converges to x in the strong topology.

Proof. Assume that $\{x_i\}_{i\in I}$ is a net of self-adjoint operators satisfying the properties above. If $\xi \in H$ then the net $\{\langle \xi, x_i(\xi)\rangle\}_{i\in I}$ in $\mathbb{R}$ is bounded and non-decreasing. Hence, it must converge to its supremum:

$$\lim_{i\to\infty}\langle \xi, x_i(\xi)\rangle = \sup_{i\in I}\langle \xi, x_i(\xi)\rangle.$$

Now observe that by the polarization identity, the net $\{\langle \xi, x_i(\eta)\rangle\}_{i\in I}$ in $\mathbb{C}$ must also converge. To see why this is the case, write

$$\langle \xi, x_j(\eta)\rangle = \frac{1}{4}\sum_{k=0}^{3} i^k \langle \xi + i^k\eta, x_j(\xi + i^k\eta)\rangle$$

for $j \in I$ and then take the limit of both sides as $j \to \infty$. Since the RHS converges in the limit, the LHS must converge as well.

Now let $F(\xi, \eta) = \lim_{i\to\infty}\langle \xi, x_i(\eta)\rangle$. This is a sesquilinear form which is bounded because

$$|F(\xi, \eta)| = |\lim_{i\to\infty}\langle \xi, x_i(\eta)\rangle| = \lim_{i\to\infty}|\langle \xi, x_i(\eta)\rangle| \leq C\|\xi\|\|\eta\|.$$

For $\eta \in H$, define the map

$$\begin{array}{rccc} \phi_\xi: & H & \to & \mathbb{C} \\ & \eta & \mapsto & \overline{F(\xi,\eta)} \end{array}$$

This is a continuous/bounded linear functional. By the Riesz representation theorem, there exists unique $\tau \in H$ such that

$$\overline{F(\xi,\eta)} = \lim_i \langle \xi, x_i(\eta)\rangle = \langle \xi, \tau\rangle.$$

Consequently, there exists a unique operator $x \in B(H)$ such that

$$F(\xi,\eta) = \langle \xi, x(\eta)\rangle = \lim_i \langle \xi, x_i(\eta)\rangle$$

for $\xi, \eta \in H$. Notice that $F(\xi,\xi) \in \mathbb{R}$ since it is the limit of a sequence in $\mathbb{R}$. Hence, x must be a self-adjoint operator.

To see that $x \geq x_i$ for $i \in I$, we compute directly that

$$\langle \xi, x(\xi)\rangle = \lim_j \langle \xi, x_j(\xi)\rangle = \sup_{j\in I} \langle \xi, x_j(\xi)\rangle \geq \langle \xi, x_i(\xi)\rangle.$$

Now assume that $y \in B(H)$ satisfies $y \geq x_i$ for $i \in I$. If $\xi \in H$ then

$$\langle \xi, y(\xi)\rangle \geq \sup_{i\in I} \langle \xi, x_i(\xi)\rangle = \langle \xi, x(\xi)\rangle.$$

So, $y \geq x$. Finally, to see that $\{x_i\}$ converges to x in the strong topology, we find that if $\xi \in H$ then

$$\begin{aligned}
\|x(\xi) - x_i(\xi)\|^2 &= \|(x - x_i)\xi\|^2 \\
&= \|(x - x_i)^{\frac{1}{2}}(x - x_i)^{\frac{1}{2}}\xi\|^2 \qquad \text{(since } x \geq x_i\text{)} \\
&\leq \|(x - x_i)^{\frac{1}{2}}\|^2 \|(x - x_i)^{\frac{1}{2}}\xi\|^2 \\
&= \|x - x_i\| \|(x - x_i)^{\frac{1}{2}}\xi\|^2 \qquad \text{(since } x - x_i \text{ is self-adjoint)} \\
&\leq (\|x\| + \|x_i\|)\|(x - x_i)^{\frac{1}{2}}\xi\|^2 \\
&\leq 2C\|(x - x_i)^{\frac{1}{2}}\xi\|^2 \\
&= 2C\langle (x - x_i)^{\frac{1}{2}}\xi, (x - x_i)^{\frac{1}{2}}\xi\rangle \\
&= 2C\langle \xi, (x - x_i)(\xi)\rangle \to 0
\end{aligned}$$

in the limit over $i \in I$. □

Note that by multiplying by -1, Theorem 2.4.2 also tells us that a non-increasing sequence of self-adjoint operators which is bounded below must be strongly convergent.

Now suppose that H is a Hilbert space and that $\{p_i\}_{i\in I}$ is a net of projections in $B(H)$, which strongly converges to an operator p. Firstly, p is self-adjoint by similar reasoning to Theorem 2.4.2. To see that p is idempotent, we compute directly that if $\xi, \eta \in H$ then

$$\begin{aligned} \langle p\xi, \eta\rangle &= \langle \lim_{i\to\infty} p_i\xi, \eta\rangle \\ &= \lim_{i\to\infty} \langle p_i\xi, \eta\rangle \\ &= \lim_{i\to\infty} \langle p_i^2\xi, \eta\rangle \\ &= \lim_{i\to\infty} \langle p_i\xi, p_i\eta\rangle \\ &= \langle p\xi, p\eta\rangle = \langle p^2\xi, \eta\rangle. \end{aligned}$$

Therefore, $p = p^2$ and p is a projection.

Recall that if V is a normed vector space then a sequence $\{x_n\}_{n\in\mathbb{Z}_{>0}}$ is summable if the sequence of partial sums $\{\sum_{n=1}^{N} x_n\}_{N\in\mathbb{Z}_{>0}}$ converges to some $x \in V$. We will extend this definition to locally convex spaces.

Definition 2.4.2. Let X be a locally convex space and $\{x_i\}_{i\in I}$ be a net in X. The net $\{x_i\}_{i\in I}$ is **summable** to a point $x \in X$ if the net $\{\sum_{i\in F} x_i\}_{F\subseteq I,\, F \text{ finite}}$ converges to x. We write $x = \sum_{i\in I} x_i$. Note that F runs over all non-empty finite subsets of I.

Here is a result regarding nets of pairwise orthogonal projections.

Theorem 2.4.3. *Let H be a Hilbert space and $\{p_i\}_{i\in I}$ be a net of projections which are pairwise orthogonal. That is, if $i, j \in I$ are distinct then $p_ip_j = 0$. Then, $\{p_i\}_{i\in I}$ is summable to a projection p with respect to the strong operator topology on $B(H)$. Moreover, p satisfies for $\xi \in H$*

$$\|p\xi\| = (\sum_{i\in I} \|p_i\xi\|^2)^{\frac{1}{2}}$$

and if $p = id_H$ then the map

$$\begin{array}{ccc} H & \to & \bigoplus_{i\in I} p_i(H) \\ \xi & \mapsto & (p_i\xi)_{i\in I} \end{array}$$

is a unitary operator.

Proof. Assume that H is a Hilbert space and $\{p_i\}_{i\in I}$ is a net of projections which are pairwise orthogonal. If F is a non-empty finite subset of I then $\sum_{i\in F} p_i$ is itself a projection because the projections $\{p_i\}_{i\in I}$ are pairwise orthogonal.

So, the net $\{\sum_{i\in F} p_i\}_{F\subseteq I,\, F \text{ finite}}$ is increasing and bounded above. By Theorem 2.4.2, it must strongly converge to some projection p. Alternatively, the net $\{p_i\}_{i\in I}$ is strongly summable. Moreover, if $\xi \in H$ then

$$\|p\xi\|^2 = \lim_F \|\sum_{i\in F} p_i\xi\|^2 = \lim_F \sum_{i\in F}\|p_i\xi\|^2 = \sum_{i\in I}\|p_i\xi\|^2.$$

The second equality follows from the fact that the projections $\{p_i\}_{i\in I}$ are pairwise orthogonal.

Now consider the case where $p = id_H$ (p is the identity operator on H). Define the map

$$\Delta : \begin{array}{ccc} \bigoplus_{i\in I} p_i(H) & \to & H \\ (p_i\xi)_{i\in I} & \mapsto & \sum_{i\in I} p_i\xi = \xi. \end{array}$$

The map Δ is bijective. To see that Δ is a unitary operator, we compute directly that if $\xi, \eta \in H$ then

$$\langle (p_i\xi)_{i\in I}, \Delta^*\eta\rangle = \langle \Delta\big((p_i\xi)_{i\in I}\big), \eta\rangle = \langle \xi, \eta\rangle$$

and

$$\langle (p_i\xi)_{i\in I}, \Delta^{-1}\eta\rangle = \langle (p_i\xi)_{i\in I}, (p_i\eta)_{i\in I}\rangle = \sum_{i\in I}\langle p_i\xi, p_i\eta\rangle = \sum_{i,j\in I}\langle p_i\xi, p_j\eta\rangle = \langle \xi, \eta\rangle.$$

So, $\Delta^* = \Delta^{-1}$ and consequently, Δ is a unitary operator as required. □

Here is the one of the most important definitions in this section.

Definition 2.4.3. Let A be a $\mathbb{C}$-algebra and C be a subset of A. The **commutant** of C, denoted by C', is defined as the set

$$C' = \{a \in A \mid \text{If } c \in C \text{ then } ac = ca\}.$$

Let us prove the following useful properties satisfied by the commutant.

Theorem 2.4.4. *Let A be a $\mathbb{C}$-algebra and C be a subset of A.*

1. *The commutant C' is a subalgebra of A.*

2. $C \subseteq C''$.

3. $C' = C'''$.

4. *If A is a normed algebra then the commutant C' is closed.*

5. *If A is a *-algebra and C is closed under involution then C' is a *-subalgebra of A.*

Proof. Assume that A is a $\mathbb{C}$-algebra and C is a subset of A. Assume that $a, b \in C'$ and $\lambda \in \mathbb{C}$. If $c \in C$ then $(\lambda a)c = a(\lambda c)$, $(a+b)c = c(a+b)$ and $(ab)c = acb = c(ab)$. Hence, C' is a subalgebra of A.

If $c \in C$ then c commutes with C' by assumption. Hence, $C \subseteq C''$. In particular, this means that $C' \subseteq C'''$. On the other hand, if $k \in C'''$ then k commutes with the double commutant C''. Since $C \subseteq C''$, k must commute with C. Hence, $k \in C'$ and $C''' \subseteq C'$. Subsequently, we have $C''' = C'$.

Now assume that A is a normed algebra. To see that C' is closed, assume that $\{c'_n\}_{n \in \mathbb{Z}_{>0}}$ is a sequence in C' which converges to some $c' \in A$. If $c \in C$ then

$$\begin{aligned}
\|cc' - c'c\| &= \|cc' - cc'_n + c'_n c - c'c\| \\
&\leq \|cc' - cc'_n\| + \|c'_n c - c'c\| \\
&\leq \|c\|\|c' - c'_n\| + \|c'_n - c'\|\|c\| \\
&< 2\|c\|(\frac{\epsilon}{2\|c\|}) = \epsilon.
\end{aligned}$$

Since $\epsilon \in \mathbb{R}_{>0}$ was arbitrary, we deduce that $cc' = c'c$ and $c' \in C'$. Hence, C' is closed.

Finally, assume that A is a *-algebra and that C is closed under involution. We know from the first part that A is a $\mathbb{C}$-subalgebra of A. If $c' \in C'$ and $c \in C$ then the adjoint $c^* \in C$ and

$$(c')^* c = (c')^*(c^*)^* = (c^* c')^* = (c' c^*)^* = c(c')^*.$$

Hence, C' is a *-subalgebra of A as required. $\square$

As we will see shortly, the double commutant theorem is a consequence of the following theorem.

Theorem 2.4.5. *Let H be a Hilbert space and A a *-subalgebra of $B(H)$. Assume that $id_H \in A$. Then, A is strongly dense in its double commutant A''. That is, A is dense in A'' with respect to the strong operator topology on $B(H)$.*

Proof. Assume that H is a Hilbert space and A is a *-subalgebra of $B(H)$. Assume that $id_H \in A$. If $\xi \in H$ then define

$$K = \overline{\{v(\xi) \mid v \in A\}}.$$

Then, K is a closed vector subspace of H. Let $p \in B(H)$ denote the projection onto K.

To show: (a) $p \in A'$.

(a) Assume that $\eta \in H$. Then, $p\eta \in K$. Hence, there exists a net $\{v_n\xi\}_{n\in I}$ in K which converges to $p\eta$. Now, if $w \in A$ and $n \in I$ then $wv_n\xi \in K$. Since w is a bounded operator, the net $\{wv_n\xi\}_{n\in I}$ converges to $wp\eta \in K$.

In particular, $wp\eta = pwp\eta$ and since $\eta \in H$ was arbitrary, we deduce that if $w \in A$ then $wp = pwp$.

Now if $w \in A$ and $\xi, \eta \in H$ then

$$\begin{aligned}
\langle wp\eta, \xi\rangle &= \langle pwp\eta, \xi\rangle \\
&= \langle \eta, pw^*p\xi\rangle \\
&= \langle \eta, w^*p\xi\rangle \quad \text{(since } A \text{ is a *-algebra)} \\
&= \langle pw\eta, \xi\rangle.
\end{aligned}$$

So, $p \in A'$.

Assume that $u \in A''$. Since $p \in A'$ by part (a), $pu = up$. Recall that by assumption, the identity operator $id_H \in A$. Hence, $\xi \in \{v\xi \mid v \in A\} \subseteq K$ and

$$u\xi = up\xi = pu\xi \in K.$$

Assume that $\epsilon \in \mathbb{R}_{>0}$. Since $u\xi \in K$, then there exists $v \in A$ such that $\|u\xi - v\xi\| < \epsilon$. This means that

$$v \in \{w \in B(H) \mid \|w\xi - u\xi\| < \epsilon\}.$$

We recognise that the set $\{w \in B(H) \mid \|w\xi - u\xi\| < \epsilon\}$ is a basic open set in the strong operator topology on $B(H)$. Consequently, every open set of $B(H)$ containing $u \in A''$ contains an element of A. Hence, A'' is the strong closure of A. $\square$

Now we introduce the notion of a von Neumann algebra.

Definition 2.4.4. Let H be a Hilbert space. A **von Neumann algebra** on H is a strongly closed *-subalgebra of $B(H)$.

Notice that if A is a von Neumann algebra then A is strongly closed and hence, closed with respect to the norm topology. Hence, A is also a C*-algebra.

One example of a von Neumann algebra is $B(H)$ for a Hilbert space H. We will give another example from [Put19, Example 1.13.3].

Example 2.4.3. Let (X, μ) be a measure space. The space of essentially bounded functions $L^\infty(X, \mu)$ acts on the Hilbert space $L^2(X, \mu)$ as multiplication operators (see [Sol18, Section 4.1]). It turns out that $L^\infty(X, \mu)$ is a commutative von Neumann algebra.

A useful result which gives us some more examples of von Neumann algebras is that the commutant of a *-algebra on a Hilbert space is a von Neumann algebra.

Theorem 2.4.6. *Let H be a Hilbert space and A be a *-subalgebra of $B(H)$. Then, the commutant A' is a von Neumann algebra.*

Proof. Assume that H is a Hilbert space and A is a *-subalgebra of $B(H)$. By Theorem 2.4.4, the commutant A' is a *-subalgebra of $B(H)$.

To show: (a) A' is strongly closed.

(a) Assume that $\{a'_n\}_{n \in I}$ is a net in A' which strongly converges to some $a' \in B(H)$. To see that $a' \in A'$, assume that $a \in A$ and $\epsilon \in \mathbb{R}_{>0}$. Then, there exists $N \in I$ such that if $n > N$ in I and $\xi \in H$ then

$$\|a'_n(\xi) - a'(\xi)\| < \frac{\epsilon}{2\|a\|}.$$

By taking the supremum over all $\xi \in H$ with $\|\xi\| = 1$, we obtain the inequality $\|a'_n - a'\| < \frac{\epsilon}{2\|a\|}$. Hence,

$$\begin{aligned}\|a'a - aa'\| &= \|a'a - a'_n a + aa'_n - aa'\| \\ &\leq 2\|a\|\|a' - a'_n\| \\ &< 2\|a\|(\frac{\epsilon}{2\|a\|}) = \epsilon.\end{aligned}$$

Since $\epsilon \in \mathbb{R}_{>0}$ was arbitrary, we deduce that $a'a = aa'$ and $a' \in A'$ as required.

Since A' is strongly closed by part (a), we deduce that A' is a von Neumann algebra. □

Now we will state and prove the important double commutant theorem, attributed to von Neumann himself.

Theorem 2.4.7 (Double commutant theorem)**.** *Let H be a Hilbert space and A be a *-subalgebra of $B(H)$. Assume that the identity operator $id_H \in A$. Then, A is a von Neumann algebra on H if and only if $A = A''$.*

Proof. Assume that H is a Hilbert space and A is a *-subalgebra of $B(H)$. Assume that the identity operator $id_H \in A$. If A is a von Neumann algebra then it is strongly closed. By Theorem 2.4.5, we have $A = \overline{A} = A''$.

On the other hand, if $A = A''$ then by Theorem 2.4.5, $\overline{A} = A'' = A$. So, A is strongly closed and is therefore, a von Neumann algebra. □

Example 2.4.4. If H is a Hilbert space then $B(H)' = \mathbb{C}id_H$. To see why, note that $\mathbb{C}id'_H = B(H)$. Now $\mathbb{C}id_H$ is a von Neumann algebra containing the identity id_H. By the double commutant theorem, $\mathbb{C}id''_H = \mathbb{C}id_H$. So, $\mathbb{C}id_H = \mathbb{C}id''_H = B(H)'$.

In particular, $B(H)'' = \mathbb{C}id'_H = B(H)$. By the double commutant theorem, we deduce that $B(H)$ is itself a von Neumann algebra.

As we will see later, the important elements of a von Neumann algebra are its projections. First, we will prove some results related to projections.

Theorem 2.4.8. *Let H be a Hilbert space and K be a closed vector subspace of H. Let $p : H \to K$ be the projection of H onto K. Then, the map*

$$\begin{array}{rccc} P: & pB(H)p & \to & B(K) \\ & u & \mapsto & u_p = u|_K \end{array}$$

*is a *-isomorphism.*

Proof. Assume that H is a Hilbert space and K is a closed vector subspace of H. Assume that P is the map defined as above. It is straightforward to check that P is a *-homomorphism.

To see that P is injective, assume that $u, v \in pB(H)p$ such that $P(u) = P(v)$. Then, there exists $u', v' \in B(H)$ such that $u = pu'p$ and $v = pv'p$. Since $u|_K = v|_K$ then

$$u = (pu'p)p = up = vp = (pv'p)p = v.$$

So, P is injective. Now assume that $w \in B(K)$. Then,

$$P(pwp) = pwp|_K = pw = w.$$

So, P is surjective. Thus, P is a *-isomorphism. □

If H is a Hilbert space, A is a *-subalgebra of $B(H)$ and $p \in A'$ is a projection then we define

$$A_p = \{u_p \mid u \in A\}.$$

Theorem 2.4.9. *Let H be a Hilbert space and A be a *-subalgebra of $B(H)$. Let $p \in A'$ be a projection. Then, pAp is a *-subalgebra of $B(H)$, A_p is a *-subalgebra on $B(pH)$ and the map*

$$\begin{array}{rccc} P: & pAp & \to & A_p \\ & u & \mapsto & u_p \end{array}$$

*is a *-isomorphism. Moreover, if $p \in A''$ then $(A')_p = (A_p)'$.*

Proof. Assume that H is a Hilbert space and A is a *-subalgebra of $B(H)$. Assume that $p \in A'$ is a projection.

To show: (a) pAp is a *-subalgebra of $B(H)$.

(b) A_p is a *-subalgebra of $B(pH)$.

(c) P is a *-isomorphism.

(d) If $p \in A''$ then $(A')_p = (A_p)'$.

(a) This follows from direct computations. We note that pAp is closed under multiplication because $p^2 = p$ and that pAp is closed under involution because $p^* = p$.

(b) First assume that $u \in A$. Recall that

$$A_p = \{u_p \mid u \in A\}.$$

By definition, $u_p = u|_{pH}$ and if $\xi \in H$ then

$$u_p(p\xi) = u(p\xi) = p(u\xi) \in pH$$

where the last equality follows from the fact that $p \in A'$. Hence, $u_p \in B(pH)$ and $A_p \subseteq B(pH)$. From here, the fact that A_p is a *-subalgebra of $B(pH)$ follows from direct computation.

(c) We omit the computation necessary to show that P is a *-homomorphism. To see that P is injective, assume that $pap, pbp \in pAp$ such that $P(pap) = P(pbp)$. Then, $pap|_{pH} = pbp|_{pH}$ and

$$pap = (pap)p = pap|_{pH} = pbp|_{pH} = (pbp)p = pbp.$$

So, P is injective. To see that P is surjective, assume that $u \in A_p$. Then, there exists $v \in A$ such that $u = v_p = v|_{pH}$. We compute directly that

$$P(pvp) = pvp|_{pH} = ppv|_{pH} = v|_{pH} = v_p = u.$$

Hence, P is surjective. We conclude that P is a *-isomorphism.

(d) Assume that $p \in A''$. Assume that $w \in (A')_p$. Then, there exists $v \in A'$ such that $w = v_p = v|_{pH}$. To see that $w \in (A_p)'$, assume that $u_p \in A_p$. If $\xi \in H$ then

$$wu_p(p\xi) = v_p up(p\xi) = vpupp(\xi) = pupv(p\xi) = pu_p v_p(p\xi) = u_p w(p\xi).$$

For clarity, we used the fact that $u_p \in B(pH)$ in the last equality. Therefore, $w \in (A_p)'$ and $(A')_p \subseteq (A_p)'$.

Conversely, assume that $z \in (A_p)'$. If $u_p \in A_p$ then $zu_p = u_p z$. We want to show that $z \in (A')_p$. So, write $z = pvp$ for some $v \in B(H)$. Then,

$$zu_p = pvppup = pvpup = pupvp = u_p z.$$

So, as operators in $A_p \subseteq B(pH)$, $pvpu = pupv$. Therefore, $vu = uv$ and consequently, $v \in A'$. Hence, $z \in (A')_p$ which completes the proof. $\square$

In [Mur90, Page 118], it is mentioned that sometimes, a von Neumann algebra on a Hilbert space H is defined to be a *-subalgebra A of $B(H)$ such that $A'' = A$. This ensures that $id_H \in A$ and that A is unital. We will see in the next result, that von Neumann algebras are indeed unital, even if we stick to our original definition.

Theorem 2.4.10. *Let A be a non-zero von Neumann algebra. Then, A is unital.*

Proof. Let H be a Hilbert space and $A \subseteq B(H)$ be a von Neumann algebra. Let $\{u_\lambda\}_{\lambda\in\Lambda}$ be an approximate unit for A (see Theorem 2.3.2). By Theorem 2.4.2, the net $\{u_\lambda\}_{\lambda\in\Lambda}$ converges in the strong operator topology to a self-adjoint element p. Since A is a von Neumann algebra, it is strongly closed and so $p \in A$.

We claim that $p \in A$ is the unit of A. If $\xi \in H$ and $u \in A$ then

$$pu\xi = \lim_{\lambda} u_\lambda u\xi = u\xi.$$

So, $pu = u$. By a similar argument, $up = u$ (use Theorem 2.3.2). Therefore, p is a unit for A and A is unital. $\square$

Let H be a Hilbert space and A be a von Neumann algebra on H. By Theorem 2.4.10, A has a unit which we denote by p. Since p is a unit, it is a projection and $p \in A'$. By Theorem 2.4.9, the map

$$\begin{array}{rcl} A = pAp & \to & A_p \\ u & \mapsto & u_p \end{array} \tag{2.1}$$

is a *-isomorphism. So, $A_p \subseteq B(pH)$ is a von Neumann algebra on pH. Furthermore, $id_{pH} = p_p \in A_p$. By Theorem 2.4.7, $(A_p)'' = A_p$. This construction is frequently used to reduce to the case where a von Neumann algebra is equal to its double commutant.

Definition 2.4.5. Let H be a Hilbert space and $u \in B(H)$. Define the **range projection** of u to be the projection of H onto $\overline{uH}$. The range projection of u is denoted by $[u]$.

Recall that if H is a Hilbert space and $x \in B(H)$ then $x = u|x|$, where u is a partial isometry and $|x| = (x^*x)^{\frac{1}{2}}$. This is called the **polar decomposition** of x. See [Sol18, Section 3.4] and [Mur90, Theorem 2.3.4] for the details on polar decomposition.

Theorem 2.4.11. *Let H be a Hilbert space and $A \subseteq B(H)$ be a von Neumann algebra. Then,*

$$\{[u] \mid u \in A\} \subseteq A.$$

That is, A contains the range projections of all of its elements.

Proof. Assume that H is a Hilbert space and $A \subseteq B(H)$ is a von Neumann algebra. Let $u \in A$. In order to show that $[u] \in A$ where $[u]$ is the range projection of u, we want to reduce to the case where u is positive.

To show: (a) $[u] = [(uu^*)^{\frac{1}{2}}]$.

(a) In order to show that $[u] = [(uu^*)^{\frac{1}{2}}]$, it suffices to show that $\overline{uH}^{\perp} = \overline{(uu^*)^{\frac{1}{2}}H}^{\perp}$. We compute directly that

$$\overline{uH}^{\perp} = \ker u^* = \ker(uu^*)^{\frac{1}{2}} = \overline{(uu^*)^{\frac{1}{2}}H}^{\perp}.$$

The middle equality follows from the polar decomposition of u^*.

By part (a), we may assume that $u \in A$ is positive. By scaling, we can also assume that $\|u\| \leq 1$ so that $u \leq id_H$ (see the proof of Theorem 2.2.5). If $n \in \mathbb{Z}_{>0}$ then let $u_n = u^{\frac{1}{2^n}}$. By our assumptions, $\{u_n\}_{n\in\mathbb{Z}_{>0}}$ is an increasing sequence of positive elements contained in the closed unit ball of A. To see why it is increasing, we have $u \leq id_H$ and by applying Theorem 2.2.4,

$$u^2 = u^{\frac{1}{2}}uu^{\frac{1}{2}} \leq u^{\frac{1}{2}}id_H u^{\frac{1}{2}} = u.$$

If $i \in \mathbb{Z}_{>1}$ then $\|u_i\| \leq 1$ since $\|u\| \leq 1$. Hence, $u_i \leq id_H$ and we can replace u with u_i in the above computation. This yields $u_{i-1} \leq u_i$.

By Theorem 2.4.2, the sequence $\{u_n\}_{n\in\mathbb{Z}_{>0}}$ strongly converges to a positive operator, which we denote by p. Since A is strongly closed, $p \in A$. Now if $x \in H$ then

$$\begin{aligned}\|(p^2 - u_n^2)(x)\| &\leq \|(p^2 - u_n p)(x)\| + \|(u_n p - u_n^2)(x)\| \\ &\leq \|(p - u_n)(p(x))\| + \|(p - u_n)(x)\|.\end{aligned}$$

We conclude that the net $\{u_n^2\}_{n\in\mathbb{Z}_{>0}}$ strongly converges to p^2. But, if $n \in \mathbb{Z}_{>1}$ then $u_n^2 = u_{n-1}$. Therefore, $p^2 = p$.

To show: (b) $\overline{uH} = pH$.

(b) Assume that $\xi \in H$. Then, $p\xi = \lim_{n\to\infty} u_n\xi$ and by the continuous functional calculus in Theorem 1.6.10, the sequence $\{u_n\}_{n\in\mathbb{Z}_{>0}}$ is contained in the C*-subalgebra generated by u. Therefore, $\lim_{n\to\infty} u_n\xi \in \overline{uH}$ and $pH \subseteq \overline{uH}$.

Conversely, assume that $\mu \in \overline{uH}$ so that there exists a sequence $\{u\xi_i\}_{i\in\mathbb{Z}_{>0}}$ which converges to μ. If $n \in \mathbb{Z}_{>0}$ then define

$$\begin{array}{rccc} f_n: & \sigma(u) \subseteq \mathbb{R}_{>0} & \to & \mathbb{R} \\ & t & \mapsto & t^{1+2^{-n}}. \end{array}$$

Then, $\{f_n\}_{n\in\mathbb{Z}_{>0}}$ is an increasing sequence which converges pointwise to the identity map $id_{\sigma(u)}$. By Dini's theorem, the convergence must be uniform. Applying the continuous functional calculus, we find that

$$u = \lim_{n\to\infty} u^{1+2^{-n}} = \lim_{n\to\infty} uu_n = up.$$

Similarly, $u = pu$. Subsequently, we have

$$\mu = \lim_{i\to\infty} u\xi_i = \lim_{i\to\infty} pu\xi_i = p(\lim_{i\to\infty} u\xi_i) = p\mu.$$

We conclude that $\overline{uH} \subseteq pH$. So, $\overline{uH} = pH$.

To summarise, we have constructed a positive operator $p \in A$ such that $p^2 = p$ and $pH = \overline{uH}$ by part (b). Hence, $p = [u] \in A$ and consequently, A contains the range projections of all of its elements. □

The next result tells us how von Neumann algebras interact with the polar decomposition.

Theorem 2.4.12. *Let H be a Hilbert space and $A \subseteq B(H)$ be a von Neumann algebra. Let $v = u|v| \in A$ be the polar decomposition of v. Then, $u \in A$.*

Proof. Assume that H is a Hilbert space and $A \subseteq B(H)$ be a von Neumann algebra. Assume that $v \in A$ such that $v = u|v|$ is its polar decomposition. Let $w \in A'$ be a unitary operator and $\tilde{w} = w^*uw$. Then,

$$\tilde{w}^*\tilde{w} = w^*u^*ww^*uw = w^*u^*uw.$$

So, $\tilde{w}^*\tilde{w}$ is a projection and hence, $\tilde{w}$ is a partial isometry. Furthermore,

$$\tilde{w}|v| = w^*uw|v| = w^*u|v|w = w^*vw = v.$$

Here, we used the fact that $w \in A'$ and $v, |v| \in A$. To see that $\ker \tilde{w} = \ker v$, recall from the construction of polar decomposition that u^*u is a projection from H onto the subspace $\overline{|v|H}$. Notably, u^*u is the range projection of $|v| \in A$. By Theorem 2.4.11, $u^*u \in A$ and

$$\tilde{w}^*\tilde{w} = w^*u^*uw = u^*uw^*w = u^*u.$$

Therefore, the partial isometries $\tilde{w}$ and u have the same initial subspace — $\overline{|v|H}$. By the characterisation of partial isometries on $B(H)$ in [Sol18, Proposition 3.12], $\tilde{w}$ and u are both isometries on $\overline{|v|H}$ and equal to zero on $\overline{|v|H}^{\perp} = \ker|v|$. Therefore,

$$\ker \tilde{w} = \ker u = \ker v.$$

So, $\tilde{w}$ is a partial isometry satisfying $v = \tilde{w}|v|$ and $\ker \tilde{w} = \ker v$. By uniqueness of the polar decomposition as in [Mur90, Theorem 2.3.4], $w^*uw = \tilde{w} = u$. In particular, $w^*uw = w^*wu$ and by multiplying on the left by w, we deduce that $uw = wu$.

Now, the commutant A' is the linear span of its unitary elements (by Theorem 2.11.4). So, u commutes with A' and $u \in A'' = (A + \mathbb{C}id_H)''$. By Theorem 2.4.5, there exists nets $\{u_\lambda\}_{\lambda \in \Lambda}$ and $\{\alpha_\lambda\}_{\lambda \in \Lambda}$ in A and $\mathbb{C}$ respectively such that if $\xi \in H$ then

$$\lim_{\lambda}(u_\lambda \xi + \alpha_\lambda \xi) = u\xi.$$

Now let $p = [|v|]$. By Theorem 2.4.11, $p \in A$. We know that by the polar decomposition, the image

$$(id_H - p)H = \overline{|v|H}^{\perp} = \ker|v| = \ker v = \ker u.$$

So, $u(id_H - p) = 0$ and consequently, $u = up$. This means that u is the strong limit of the net $\{u_\lambda p + \alpha_\lambda p\}_{\lambda \in \Lambda}$ which lies entirely in A. Since A is strongly closed then $u \in A$ as required. $\square$

We will now prove the important fact that a von Neumann algebra is the closed linear span of its projections.

Theorem 2.4.13. *Let H be a Hilbert space and $A \subseteq B(H)$ be a von Neumann algebra.*

1. *A is the closed linear span of its projections.*

2. *If $id_H \in A$, $u \in A$ is normal, S is a Borel subset of $\sigma(u)$ and E is the spectral resolution of the identity for u then $E(S) \in A$.*

3. *If $id_H \in A$ and $v \in B(H)$ then $v \in A$ if and only if v commutes with all the projections of A'.*

Proof. Assume that H is a Hilbert space and $A \subseteq B(H)$ is a von Neumann algebra. We will prove the second statement first.

Assume that $id_H \in A$, $u \in A$ is normal and E is the spectral resolution of the identity for u. Recall from [Mur90, Theorem 2.5.6] that E is the unique spectral measure such that if $f \in Cts(\sigma(u), \mathbb{C})$ and
$\Lambda_C : Cts(\sigma(u), \mathbb{C}) \to B(H)$ is the continuous functional calculus then

$$\Lambda_C(f) = \int f \, dE.$$

If $v \in A'$ then $uv = vu$ and $vu^* = u^*v$. Let $Bor(\sigma(u), \mathbb{C})$ denote the space of Borel functions from $\sigma(u)$ to $\mathbb{C}$. Since v commutes with u, u^* and id_H, we can apply the Stone-Weierstrass theorem to deduce that if
$f \in Bor(\sigma(u), \mathbb{C})$ then $vf(u) = f(u)v$. In particular, recall the construction of the spectral measure E — if $S \subseteq \sigma(u)$ is Borel then $E(S) = \chi_S(u)$. So, $vE(S) = E(S)v$. By the double commutant theorem in Theorem 2.4.7, we must have $E(S) \in A'' = A$ as required.

We will now prove the first statement. Without loss of generality, we may assume that $id_H \in A$ (see the remark at equation (2.1)). We know that $Bor(\sigma(u), \mathbb{C})$ is the closed linear span of its characteristic functions. That is,

$$Bor(\sigma(u), \mathbb{C}) = \overline{span_{\mathbb{C}}\{\chi_S \mid S \text{ is a Borel subset of } \sigma(u)\}}.$$

In particular, the identity map $id_{\sigma(u)} \in Bor(\sigma(u), \mathbb{C})$ is the norm limit of a sequence $\{f_n\}_{n \in \mathbb{Z}_{>0}}$ contained in the span of the characteristic functions on Borel subsets of $\sigma(u)$. By applying the Borel functional calculus (see [Sol18, Theorem 7.7]), we deduce that $u \in A$ is the norm limit of the sequence $\{f_n(u)\}_{n \in \mathbb{Z}_{>0}}$. Since every element of A can be written as the sum of two self-adjoint elements in A, every element of A can be written as a norm limit of a sequence of elements spanned by projections in A. So, A is the closed linear span of its projections.

Finally, assume that $id_H \in A$ and $v \in B(H)$. By the first statement, A' is a von Neumann algebra and thus, the closed linear span of its projections. Hence, $v \in A$ if and only if v commutes with all the projections of A'. $\square$

Let us investigate some consequences of the important Theorem 2.4.13.

Theorem 2.4.14. *Let A be a non-zero C*-algebra acting on a Hilbert space H. Then, the following are equivalent:*

1. *A acts irreducibly on H*
2. *$A' = \mathbb{C}id_H$.*
3. *A is strongly dense in $B(H)$*

Proof. Assume that A is a non-zero C*-algebra acting on a Hilbert space H. We will first prove that the first statement is equivalent to the second statement. Assume that A acts irreducibly on H. Let $p \in B(H)$ be a projection.

To show: (a) $p \in A'$ if and only if the closed subspace pH is invariant for A.

(a) Assume that $p \in A'$ and $a \in A$. Then, $apH = paH \subseteq pH$. Therefore, pH is invariant for A. Conversely, assume that pH is invariant for A. If $a \in A$ then $apH \subseteq pH$ and $ap = pap$. Note that $a \in B(pH)$. Subsequently, we can use Theorem 2.4.8 to show that there exists $b \in B(H)$ such that $a = pbp$. Therefore, $pap = ap = (pbp)p = pbp = a$ and

$$pa = p(pap) = p(ap) = ap.$$

So, $p \in A'$. This proves part (a).

Since A acts irreducibly on H then there are no non-trivial closed subspaces of H which are invariant for A. By part (a), A' only consists of the trivial projections; the identity map id_H and the zero map. Now A' is a von Neumann algebra. By Theorem 2.4.13, A' is the closed linear span of its projections. So, $A' = \mathbb{C}id_H$.

Conversely, assume that A acts reducibly on H (the representation of A on H is reducible). Then, there exists a non-trivial closed subspace $K \subsetneq H$ which is invariant for A. Let p_K be the projection operator from H onto K. By part (a), p_K is a non-trivial projection in A'. Therefore, $A' \neq \mathbb{C}id_H$.

This proves that the first and second statements are equivalent.

Now we prove that the second and third statements are equivalent. First, assume that $A' = \mathbb{C}id_H$. Then, $(A + \mathbb{C}id_H)' = \mathbb{C}id_H$ and $(A + \mathbb{C}id_H)'' = B(H)$. By Theorem 2.4.5, $A + \mathbb{C}id_H$ is strongly dense in $B(H)$. Since $A' = \mathbb{C}id_H$ then A acts irreducibly on H. By Theorem 2.3.4, if $\{u_\lambda\}_{\lambda\in\Lambda}$ is an approximate unit for A then $\{u_\lambda\}_{\lambda\in\Lambda}$ converges strongly to the identity id_H. Therefore,

$$\overline{A}^{SOT} = \overline{A + \mathbb{C}id_H}^{SOT} = B(H)$$

where $\overline{A}^{SOT}$ is the closure of A with respect to the strong operator topology. We conclude that A is strongly dense in $B(H)$.

Conversely, assume that A is strongly dense in $B(H)$. Then, $A' = B(H)' = \mathbb{C}id_H$. This proves the equivalence of the second and third statements, completing the proof. $\square$

As demonstrated in the proof of Theorem 2.4.13, von Neumann algebras have a large supply of projections. In the next result, we will give a construction which reinforces this important idea.

Definition 2.4.6. Let A be a C*-algebra and $B \subseteq A$ be a C*-subalgebra of A. We say that B is **hereditary** if the following statement is satisfied: If $a \in A^+$, $b \in B^+$ and $a \leq b$ then $a \in B$.

Theorem 2.4.15. *Let H be a Hilbert space and $B \subseteq B(H)$ be a von Neumann algebra. Let $A \subseteq B$ be a hereditary C*-subalgebra of A. Let $a \in A$ be a positive element such that $\|a\| \leq 1$. Then, there exists a sequence of projections $\{p_n\}_{n\in\mathbb{Z}_{>0}}$ in A such that*

$$a = \sum_{n=1}^{\infty} \frac{p_n}{2^n}.$$

Proof. First, assume that H is a Hilbert space and $A \subseteq B(H)$ is a von Neumann algebra. Without loss of generality, we may assume that $id_H \in A$ (again, see the remark punctuated by equation (2.1)). Assume that $a \in A$ is a positive element with norm $\|a\| \leq 1$. We will use induction to construct a sequence of projections $\{p_n\}_{n\in\mathbb{Z}_{>0}}$ such that if $n \in \mathbb{Z}_{>0}$ then

$$0 \leq a - \sum_{j=1}^{n} \frac{p_j}{2^j} \leq \frac{1}{2^n} id_H.$$

The idea here is to use the Borel functional calculus. Let $\chi : \sigma(a) \to \mathbb{C}$ be defined by

$$\chi(t) = \begin{cases} 1, & \text{if } t \geq 1/2, \\ 0, & \text{if } t < 1/2. \end{cases}$$

Since χ is a sum of characteristic functions then $\chi \in Bor(\sigma(a), \mathbb{C})$. Let $\iota : \sigma(a) \hookrightarrow \mathbb{C}$ denote the inclusion map. Then,

$$0 \leq \iota - \frac{1}{2}\chi \leq \frac{1}{2}$$

because $\sigma(a) \subseteq [0,1]$. Applying the Borel functional calculus, we find that

$$0 \leq a - \frac{1}{2}\chi(a) \leq \frac{1}{2} id_H.$$

Let $p_1 = \chi$. Then, p_1 is a projection and by the second statement in Theorem 2.4.13, $p_1 \in A$.

Now we argue inductively. Suppose that we have projections $p_1, p_2, \ldots, p_n \in A$ such that

$$0 \leq a - \sum_{j=1}^{n} \frac{p_j}{2^j} \leq \frac{1}{2^n} id_H.$$

Let $b = a - \sum_{j=1}^{n} \frac{p_j}{2^j}$. By the inductive hypothesis, b is a positive element of A and $\sigma(b) \subseteq [0, 2^{-n}]$. Define $\chi_{n+1} : \sigma(b) \to \mathbb{C}$ by

$$\chi_{n+1}(t) = \begin{cases} 1, & \text{if } t \in [2^{-n-1}, 2^{-n}], \\ 0, & \text{otherwise.} \end{cases}$$

Arguing in the same manner, we note that $0 \leq \iota - \frac{\chi_{n+1}}{2^{n+1}} \leq \frac{1}{2^{n+1}}$ and by the Borel functional calculus on b,

$$0 \leq b - \frac{\chi_{n+1}(b)}{2^{n+1}} \leq \frac{1}{2^{n+1}} id_H.$$

By setting $p_{n+1} = \chi_{n+1}(b)$, we find that p_{n+1} is a projection and an element of A (we argue in the same way as for p_1). Therefore, we have projections $p_1, \ldots, p_{n+1}$ in A such that

$$0 \leq a - \sum_{i=1}^{n+1} \frac{p_i}{2^i} \leq \frac{1}{2^{n+1}} id_H.$$

This completes the induction. By taking the limit as $n \to \infty$, we obtain a sequence of projections $\{p_n\}_{n \in \mathbb{Z}_{>0}}$ in A such that

$$0 \leq a - \sum_{j=1}^{\infty} \frac{p_j}{2_j} \leq 0 \qquad \text{and} \qquad a = \sum_{j=1}^{\infty} \frac{p_j}{2_j}.$$

This proves the statement in this particular case. For the general case, assume that $B \subseteq B(H)$ is a von Neumann algebra and $A \subseteq B$ is a hereditary C*-subalgebra of B. Assume that $a \in A$ is a positive element such that $\|a\| \leq 1$. By the previous case, there exists a sequence of projections $\{p_n\}_{n \in \mathbb{Z}_{>0}}$ in B such that $a = \sum_{i=1}^{\infty} \frac{p_i}{2^i}$. Now observe that if $n \in \mathbb{Z}_{>0}$ then

$$a = \sum_{i=1}^{\infty} \frac{p_i}{2^i} \geq \frac{p_n}{2^n}.$$

Since A is a hereditary C*-subalgebra of B, we find that $p_n \in A$. This completes the proof. $\square$

A consequence of Theorem 2.4.15 is the following result.

Theorem 2.4.16. *Let H be a Hilbert space. Let $B \subseteq B(H)$ be a von Neumann algebra and $A \subseteq B$ be a hereditary C*-subalgebra of B. Then, A is the closed linear span of its projections.*

Proof. Assume that H is a Hilbert space, $B \subseteq B(H)$ is a von Neumann algebra and A is a hereditary C*-subalgebra of B. We know that A is the linear span of its positive elements. By Theorem 2.4.15, A^+ is contained in the closed linear span of the projections of A. This means that A is contained in the closed linear span of the projections of A. The reverse inclusion also holds. So, A is the closed linear span of its projections. $\square$

We will end this section with some results about the ideal $B_0(H) \subseteq B(H)$ of bounded compact operators on a Hilbert space H. First, we make an important definition.

Definition 2.4.7. Let A be a C*-algebra and $p, q \in A$ be projections. We say that p and q are **(Murray-von Neumann) equivalent** if there exists $u \in A$ such that $p = u^*u$ and $q = uu^*$. We write $p \sim q$ to mean that p and q are equivalent.

It is easy to check that Murray-von Neumann equivalence is an equivalence relation. It is remarked in [Mur90, Page 122] that Murray-von Neumann equivalence is of fundamental importance in the classification of von

Neumann algebras and in the K-theory of C*-algebras. For now, we only focus on one particular application of the equivalence.

Example 2.4.5. Let H be a separable Hilbert space and $p, q \in B(H)$ be infinite-rank projections. We claim that $p \sim q$. To see why this is the case, let $\{e_n\}_{n\in\mathbb{Z}_{>0}}$ and $\{f_n\}_{n\in\mathbb{Z}_{>0}}$ be orthonormal bases for pH and qH respectively. Define the map $u : pH \to qH$ by

$$\begin{array}{rccc} u: & pH & \to & qH \\ & e_n & \mapsto & f_n. \end{array}$$

Note that u is a unitary map. Recalling that $pH \oplus (id_H - p)H = H$, we define

$$\begin{array}{rccc} v: & H & \to & H \\ & \xi & \mapsto & \begin{cases} u\xi, & \text{if } \xi \in pH, \\ 0, & \text{if } \xi \in (id_H - p)H. \end{cases} \end{array}$$

Then, $v \in B(H)$. If $\xi, \mu \in H$ then

$$\begin{aligned} \langle v^*v\xi, \mu\rangle &= \langle v\xi, v\mu\rangle \\ &= \langle v(p\xi + (id_H - p)\xi), v(p\mu + (id_H - p)\mu)\rangle \\ &= \langle up\xi, up\mu\rangle = \langle p\xi, p\mu\rangle = \langle p\xi, \mu\rangle. \end{aligned}$$

Analogously,

$$\begin{aligned} \langle vv^*\xi, \mu\rangle &= \langle v^*\xi, v^*\mu\rangle \\ &= \langle v^*(q\xi + (id_H - q)\xi), v^*(q\mu + (id_H - q)\mu)\rangle \\ &= \langle u^*q\xi, u^*q\mu\rangle = \langle q\xi, q\mu\rangle = \langle q\xi, \mu\rangle. \end{aligned}$$

So, $p = v^*v$, $q = vv^*$ and $p \sim q$.

Theorem 2.4.17. *Let H be a separable infinite-dimensional Hilbert space. Then, the space of bounded compact operators $B_0(H)$ is the unique non-trivial closed ideal of $B(H)$.*

Proof. Assume that H is a separable infinite-dimensional Hilbert space. Let $I \subseteq B(H)$ be a non-zero closed ideal of $B(H)$. Let $F(H)$ be the ideal of finite-rank operators on H. Then, $\overline{F(H)} = B_0(H)$, where closure is taken with respect to the norm topology. We also have $F(H) \subseteq I$ by [Mur90, Theorem 2.4.7]. Since I is closed then $B_0(H) \subseteq I$.

Now assume that $I \not\subseteq B_0(H)$. By Theorem 2.4.16, $B_0(H)$ cannot contain all the projections of I. We can apply Theorem 2.4.16 in the first place because $B(H)$ is a von Neumann algebra and the closed ideal I is a hereditary C*-subalgebra of $B(H)$ (see [Mur90, Corollary 3.2.3]). So, there exists an infinite-rank projection $p \in I$.

If q is another infinite-rank projection on H then, by the previous example, there exists an element $u \in B(H)$ such that $p = u^*u$ and $q = uu^*$. But, $q = q^2 = upu^* \in I$. Therefore, any projection is an element of I, whether it is finite-rank or infinite-rank. We conclude that $I = B(H)$. Hence, the only three closed ideals of $B(H)$ are 0, $B_0(H)$ and $B(H)$ and the only non-trivial closed ideal of $B(H)$ is $B_0(H)$. □

Definition 2.4.8. Let H be a Hilbert space. The **Calkin algebra** is the quotient C*-algebra $B(H)/B_0(H)$.

Theorem 2.4.18. *Let H be a separable infinite-dimensional Hilbert space. Then, the Calkin algebra $B(H)/B_0(H)$ is a simple C*-algebra.*

Proof. Assume that H is a separable infinite-dimensional Hilbert space. Let $\pi : B(H) \to B(H)/B_0(H)$ denote the canonical quotient map onto the Calkin algebra.

Now let I be a closed ideal of $B(H)/B_0(H)$. Then, the preimage $\pi^{-1}(I)$ is a closed ideal of $B(H)$. By Theorem 2.4.17, $\pi^{-1}(I)$ is equal to one of $0, B_0(H), B(H)$. If $\pi^{-1}(I) = 0$ then $I = 0$. If $\pi^{-1}(I) = B(H)$ then $I = B(H)/B_0(H)$. Finally, if $\pi^{-1}(I) = B_0(H)$ then $I = 0$. We conclude that the Calkin algebra is a simple C*-algebra. □

We will conclude this section with an application of Theorem 2.4.15 to compact operators. First, we need the following result.

Theorem 2.4.19. *Let H be a Hilbert space and $u, v \in B(H)$ such that $uu^* \leq vv^*$. Then, there exists $w \in B(H)$ such that $u = vw$.*

Proof. Assume that H is a Hilbert space and $u, v \in B(H)$ such that $uu^* \leq vv^*$. If $\xi \in H$ then

$$\|u^*\xi\|^2 = \langle uu^*\xi, \xi\rangle \leq \langle vv^*\xi, \xi\rangle = \|v^*\xi\|^2.$$

Now define the linear map w_0 by

$$\begin{array}{rccc} w_0 : & v^*H & \to & H \\ & v^*\xi & \mapsto & u^*\xi \end{array}$$

We claim that w_0 is well-defined. Assume that $\xi_1, \xi_2 \in H$ such that $v^*\xi_1 = v^*\xi_2$. Then, $v^*(\xi_1 - \xi_2) = 0$ and since $\|u^*(\xi_1 - \xi_2)\| \leq \|v^*(\xi_1 - \xi_2)\|$

$$w_0(v^*\xi_1) = u^*\xi_1 = u^*\xi_2 = w_0(v^*\xi_2).$$

So, w_0 is a well-defined, norm-decreasing linear map on v^*H. We can extend w_0 to a bounded linear map $w_1 \in B(H)$. If $w = w_1^*$ and $\xi, \mu \in H$ then

$$\begin{aligned} \langle vw\xi, \mu\rangle &= \langle vw_1^*\xi, \mu\rangle \\ &= \langle \xi, w_1 v^*\mu\rangle = \langle \xi, w_0 v^*\mu\rangle \\ &= \langle \xi, u^*\mu\rangle = \langle u\xi, \mu\rangle. \end{aligned}$$

So, $u = vw$ as required. □

Theorem 2.4.20. *Let H be a Hilbert space and $u \in B(H)$. Then, u is a compact operator if and only if the image uH contains no infinite-dimensional closed vector subspace of H.*

Proof. Assume that H is a Hilbert space and $u \in B(H)$. First, assume that u is a compact operator. Let K be a closed vector subspace of uH and p_K be the projection operator of H onto K. Define the linear map

$$\begin{array}{rccc} v : & (\ker pu)^{\perp} & \to & K \\ & x & \mapsto & pu(x). \end{array}$$

To show: (a) v is compact.

(b) v is bijective.

(a) v is compact because $u \in B_0(H)$ and $B_0(H)$ is an ideal of $B(H)$.

(b) To see that v is injective, assume that $\xi_1, \xi_2 \in (\ker pu)^{\perp}$ such that $pu(\xi_1) = pu(\xi_2)$. Then, $\xi_1 - \xi_2 \in \ker pu$ and

$$\langle \xi_2, \xi_1 - \xi_2\rangle = \langle \xi_1, \xi_1 - \xi_2\rangle = 0$$

Therefore, $\|\xi_1 - \xi_2\|^2 = 0$ and $\xi_1 = \xi_2$. Hence, v is injective.

To see that v is surjective, assume that $\kappa \in K$. Since K is a closed vector subspace of uH then there exists $\delta \in H$ such that $u\delta = \kappa$. Because $\kappa \in K$, we also have $pu\delta = p\kappa = \kappa$. Using the fact that $H = \ker pu \oplus (\ker pu)^{\perp}$, write $\delta = \delta_1 + \delta_2$, where $\delta_1 \in \ker pu$ and $\delta_2 \in (\ker pu)^{\perp}$. Thus,

$$v(\delta_2) = pu(\delta_2) = pu(\delta_1 + \delta_2) = pu(\delta) = \kappa.$$

Thus, v is surjective and subsequently bijective.

By part (b), we can apply the inverse mapping theorem to deduce that the operator v^{-1} is bounded. By part (a), v is compact and thus, the identity $id_K = vv^{-1}$ is a compact operator. We deduce that K is finite dimensional.

Conversely, assume that the image uH does not contain an infinite-dimensional closed vector subspace of H. We will first justify why we can assume that u is positive. Let $u^* = w|u^*|$ be the polar decomposition of u^*.

To show: (c) $uH = |u^*|H$.

(d) u is compact if and only if $|u^*|$ is compact.

(c) By the polar decomposition,

$$(|u^*|H)^{\perp} = \ker|u^*| = \ker u^* = (uH)^{\perp}.$$

Therefore, $|u^*|H = uH$.

(d) Since $u^* = w|u^*|$ then $u = |u^*|w^*$. Hence, if $|u^*|$ is compact then u is compact. By the polar decomposition, we also have $|u^*| = w^*u^*$. Taking adjoints, we have $|u^*| = uw$. This shows that if u is compact then $|u^*|$ is compact.

Therefore, by replacing u with $|u^*|$, we may assume without loss of generality that $u \in B(H)$ is positive. By rescaling, we may assume that $\|u\| \leq 1$ so that $0 \leq u \leq id_H$. So, $0 \leq u^2 \leq id_H$ and by Theorem 2.4.15, there exists a sequence of projections $\{p_n\}_{n \in \mathbb{Z}_{>0}}$ such that

$$u^2 = \sum_{n=1}^{\infty} \frac{p_n}{2^n}.$$

If $n \in \mathbb{Z}_{>0}$ then

$$u^2 = uu^* \geq \frac{p_n}{2^n} = \frac{p_n}{(2^n)^{\frac{1}{2}}} \left(\frac{p_n}{(2^n)^{\frac{1}{2}}} \right)^*$$

By Theorem 2.4.19, there exists $w \in B(H)$ such that

$$\frac{p_n}{(2^n)^{\frac{1}{2}}} = uw \qquad \text{and} \qquad p_n = u((2^n)^{\frac{1}{2}}w).$$

Let $w_n = (2^n)^{\frac{1}{2}}w$. Then, w_n satisfies $p_n = uw_n$. So, p_nH is a closed vector subspace of uH. Since uH cannot contain an infinite-dimensional closed vector subspace of H, p_nH must be finite-dimensional. Consequently, if $n \in \mathbb{Z}_{>0}$ then p_n is a finite-rank projection and hence compact. Therefore, $u^2 = \sum_{n=1}^{\infty} \frac{p_n}{2^n}$ is compact and u must be compact as required. □

2.5 The weak and ultraweak topologies

In this section, we will first define the weak operator topology and the ultraweak topology on $B(H)$ where H is a Hilbert space. We will investigate some results linking the weak operator topology to von Neumann algebras.

The definition of the ultraweak topology requires the isomorphism $B_1(H)^* \cong B(H)$. To reiterate, $B_1(H)$ is the Banach algebra of trace-class operators on H. One can think of the dualities $B_0(H)^* \cong B_1(H)$ and $B_1(H)^* \cong B(H)$ as the non-commutative analogues of the well-known sequence space dualities $c_0^* \cong \ell^1$ and $(\ell^1)^* \cong \ell^\infty$.

We summarise the dualities $B_0(H)^* \cong B_1(H)$ and $B_1(H)^* \cong B(H)$ below, drawing from the references [Mur90, Section 4.2] and [Sol18, Section 6.2].

Definition 2.5.1. Let H be a Hilbert space and $\{\xi_j\}_{j\in J}$ be an orthonormal basis for H. Let $t \in B(H)_+$. The **trace** of t, denoted by $Tr(t)$, is defined by

$$Tr(t) = \sum_{j\in J} \langle t\xi_j, \xi_j\rangle \in [0, \infty].$$

The space of **trace-class operators**, denoted by $B_1(H)$, is defined by

$$B_1(H) = \{x \in B(H) \mid Tr(|x|) < \infty\}.$$

Here, if $x \in B(H)$ then $|x| = (x^*x)^{\frac{1}{2}}$.

It is a fact that the trace is independent of the choice of orthonormal basis for H. The space $B_1(H)$ is a Banach algebra with the **trace norm**, defined below as

$$\begin{array}{rccc} \|-\|_1 : & B_1(H) & \to & \mathbb{R}_{\geq 0} \\ & x & \mapsto & Tr(|x|). \end{array}$$

The fact that $B_1(H)$ is complete with respect to the trace norm is actually a consequence of the isomorphism $B_0(H)^* \cong B_1(H)$, which we state below.

Theorem 2.5.1. *Let H be a Hilbert space and $x \in B_1(H)$. Define the map*

$$\begin{array}{rccc} \varphi_x : & B_0(H) & \to & \mathbb{C} \\ & y & \mapsto & Tr(xy) \end{array}$$

Then, $\varphi_x \in B_0(H)^*$ and the map $x \mapsto \varphi_x$ is a bijective isometry from $B_1(H)$ with the norm $\|-\|_1$ to $B_0(H)^*$ with the operator norm.

We also have the isomorphism $B_1(H)^* \cong B(H)$.

Theorem 2.5.2. *Let H be a Hilbert space. If $y \in B(H)$ then define*

$$\begin{array}{rccc} \psi_y : & B_1(H) & \to & \mathbb{C} \\ & x & \mapsto & Tr(yx). \end{array}$$

Then, $\psi_y \in B_1(H)^*$ and the map $y \mapsto \psi_y$ is a bijective isometry from $B(H)$ to $B_1(H)^*$.

We will make use of Theorem 2.5.2 to define the ultraweak topology on $B(H)$ later. First, we will introduce the weak operator topology on $B(H)$. In a similar vein to the strong operator topology introduced in section 2.4, we will first define the weak operator topology on the space of bounded linear operators $B(V, W)$, where V and W are Banach spaces.

Definition 2.5.2. Let V and W be Banach spaces and $B(V, W)$ be the Banach space of bounded linear operators $T : V \to W$. The **weak operator topology** on $B(V, W)$ is the $\mathcal{G}$-weak topology, where

$$\mathcal{G} = \{\lambda \circ ev_v : B(V, W) \to \mathbb{C} \mid v \in V, \lambda \in W^*\}.$$

That is, the weak operator topology on $B(V, W)$ is the weakest topology on $B(V, W)$ which makes the maps in $\mathcal{G}$ continuous.

We now characterise convergence in the weak operator topology.

Theorem 2.5.3. *Let V and W be Banach spaces and $B(V, W)$ denote the Banach space of bounded linear operators $T : V \to W$. Then, a sequence $\{A_n\}_{n \in \mathbb{Z}_{>0}}$ in $B(V, W)$ converges to an operator A in the weak operator topology if and only if for $v \in V$ and $\lambda \in W^*$, $\lambda(A_n v) \to \lambda(Av)$ in $\mathbb{C}$.*

Proof. Assume that V and W are Banach spaces and $B(V,W)$ is the Banach space of bounded linear operators $T : V \to W$.

To show: (a) If a sequence $A_n \to A$ in the weak operator topology on $B(V,W)$ then if $v \in V$ and $\lambda \in W^*$ then $\lambda(A_n v) \to \lambda(Av)$ in $\mathbb{C}$.

(b) If $v \in V$ and $\lambda \in W^*$, $\lambda(A_n v) \to \lambda(Av)$ in $\mathbb{C}$ then $A_n \to A$ in the weak operator topology on $B(V,W)$.

(a) Assume that $A_n \to A$ in the weak operator topology. Assume that $\epsilon \in \mathbb{R}_{>0}$. Then, there exists $N \in \mathbb{Z}_{>0}$ such that if $n > N$ then

$$A_n - A \in L_{v,\lambda,0,\epsilon} = \{T \in B(V,W) \mid |\lambda(Tv)| < \epsilon\}$$

where $\lambda \in W^*$ and $v \in V$. So,

$$|\lambda((A_n - A)v)| = |\lambda(A_n v) - \lambda(Av)| < \epsilon.$$

Hence, $\lambda(A_n v) \to \lambda(Av)$ in $\mathbb{C}$.

(b) Assume that if $v \in V$ and $\lambda \in W^*$ then $\lambda(A_n v) \to \lambda(Av)$ in $\mathbb{C}$. Assume that $\epsilon \in \mathbb{R}_{>0}$. Then, there exists $N \in \mathbb{Z}_{>0}$ such that if $n > N$ then

$$|\lambda(A_n v) - \lambda(Av)| < \epsilon.$$

This means that

$$A_n - A \in L_{v,\lambda,0,\epsilon} = \{T \in B(V,W) \mid |\lambda(Tv)| < \epsilon\}.$$

Since $L_{v,\lambda,0,\epsilon}$ is a basic open subset of $B(V,W)$ with the weak operator topology, we deduce that $A_n \to A$ in the weak operator topology as required. $\square$

Example 2.5.1. Let $V = W = \ell^2(\mathbb{C})$. For $n \in \mathbb{Z}_{>0}$, define

$$\begin{array}{cccc} R_n : & \ell^2(\mathbb{C}) & \to & \ell^2(\mathbb{C}) \\ & (x_1, x_2, \dots) & \mapsto & (0, 0, \dots, 0, x_1, x_2) \end{array}$$

In the above definition, the x_i is in the $(n+i)^{th}$ position for $i \in \mathbb{Z}_{>0}$.

To see that $R_n \to 0$ in the weak operator topology, assume that $x = (x_1, x_2, \dots) \in \ell^2(\mathbb{C})$ and $\lambda \in \ell^2(\mathbb{C})^*$. By the Riesz representation theorem, $\lambda = \langle (\lambda_1, \lambda_2, \dots), - \rangle$, where $(\lambda_1, \lambda_2, \dots) \in \ell^2(\mathbb{C})$. So,

$$\begin{aligned}\lambda(R_n(x_1, x_2, \dots)) &= \lambda(0, 0, \dots, x_1, x_2, \dots) \\ &= \langle(\lambda_1, \lambda_2, \dots), (0, 0, \dots, x_1, x_2, \dots)\rangle \\ &= \sum_{i=1}^{\infty} \overline{\lambda_{n+i}} x_i \\ &\leq \|(\overline{\lambda_{n+1}}, \overline{\lambda_{n+2}}, \dots)\| \|x\| \\ &= \|L_n(\overline{\lambda_1}, \overline{\lambda_2}, \dots)\| \|x\| \to 0\end{aligned}$$

as $n \to \infty$. We used the shift operator L_n in Example 2.4.1. In the second last inequality, we used the Cauchy-Schwarz inequality. Therefore, $R_n \to 0$ in the weak operator topology.

To see that R_n does not converge to zero in the norm topology, we compute directly that

$$\begin{aligned}\|R_n\| &= \sup_{\|x\|=1} \|R_n x\| \\ &\geq \|R_n(1, 0, 0, \dots)\| \\ &= \|(0, \dots, 0, 1, 0, 0, \dots)\| = 1.\end{aligned}$$

So, R_n does not converge to zero in the norm topology.

Finally, to see that R_n does not converge to zero in the strong operator topology, we observe that if $x \in \ell^2(\mathbb{C})$ and $n \in \mathbb{Z}_{>0}$ then $\|R_n x\| = \|x\|$.

Now, we specialise to the weak operator topology on $B(H)$, where H is a Hilbert space. The definition of the weak operator topology on $B(H)$ implicitly uses the Riesz representation theorem.

Definition 2.5.3. Let H be a Hilbert space. The **weak operator topology** on $B(H)$ is the $\mathcal{G}$-weak topology, where

$$\mathcal{G} = \{u \mapsto \langle u(x), y\rangle \mid x, y \in H\}.$$

For clarity, $u \in B(H)$.

By Theorem 2.5.3, a net $\{x_n\}_{n \in I}$ in $B(H)$ converges to $x \in B(H)$ in the weak operator topology if and only if for $\xi, \mu \in H$

$$\langle x_n(\xi), \mu\rangle \to \langle x(\xi), \mu\rangle$$

in $\mathbb{C}$.

Now we will define the ultraweak topology on $B(H)$. Let X and Y be Banach spaces and Y^* be the dual space of Y. Let $B(X, Y^*)$ denote the Banach space of bounded linear transformations from X to Y^*. Our goal is to define a weak-* topology on $B(X, Y^*)$. This will be accomplished by proving that $B(X, Y^*)$ is isometrically isomorphic to the dual space Z^* for some Banach space Z.

We first construct the Banach space Z. If $x \in X$ and $y \in Y$ then define the map

$$\begin{array}{rccc} ev_{x,y}: & B(X, Y^*) & \to & \mathbb{C} \\ & L & \mapsto & L(x)(y). \end{array}$$

This is an element of the dual space $B(X, Y^*)^*$ because firstly, $ev_{x,y}$ is a linear functional and secondly,

$$\|ev_{x,y}\| = \sup_{\|L\|=1} |L(x)(y)| \leq \|x\|\|y\|.$$

In fact, we claim that $\|ev_{x,y}\| \geq \|x\|\|y\|$. If $f \in X^*$ and $g \in Y^*$ then define the linear map

$$\begin{array}{rccc} L_{f,g}: & X & \to & Y^* \\ & x & \mapsto & f(x)g. \end{array}$$

To see that $L_{f,g}$ is bounded, we compute directly that

$$\|L_{f,g}\| = \sup_{\|x\|=1} \sup_{\|y\|=1} |f(x)g(y)| \leq \|f\|\|g\|.$$

Now recall that by the Hahn-Banach extension theorem, there exists a bounded linear functional $\psi_x \in X^*$ such that $\psi_x(x) = \|x\|$ and $\|\psi_x\| = 1$. Similarly, there exists $\psi_y \in Y^*$ such that $\psi_y(y) = \|y\|$ and $\|\psi_y\| = 1$. So, $L_{\psi_x,\psi_y} \in B(X, Y^*)$ satisfies

$$\|L_{\psi_x,\psi_y}\| \leq \|\psi_x\|\|\psi_y\| = 1.$$

Subsequently, we have

$$\begin{aligned} \|ev_{x,y}\| &= \sup_{\|L\|\leq 1} |ev_{x,y}(L)| \\ &\geq |ev_{x,y}(L_{\psi_x,\psi_y})| = |L_{\psi_x,\psi_y}(x)(y)| \\ &= |\psi_x(x)\psi_y(y)| = \|x\|\|y\|. \end{aligned}$$

Therefore, if $x \in X$ and $y \in Y$ then $\|ev_{x,y}\| = \|x\|\|y\|$. It is straightforward to check that if $x_1, x_2 \in X$, $y_1, y_2 \in Y$ and $\lambda \in \mathbb{C}$ then

$$\begin{aligned} ev_{x_1+x_2,y_1} &= ev_{x_1,y_1} + ev_{x_2,y_1}, \\ ev_{x_1,y_1+y_2} &= ev_{x_1,y_1} + ev_{x_1,y_2}, \\ ev_{\lambda x_1,y_1} &= \lambda ev_{x_1,y_1}, \\ ev_{x_1,\lambda y_1} &= \lambda ev_{x_1,y_1}. \end{aligned}$$

This suggests that we define the Banach space Z by

$$Z = \overline{span_{\mathbb{C}}\{ev_{x,y} \mid x \in X, y \in Y\}}.$$

Now define the map

$$\begin{array}{rccc} ev: & B(X,Y^*) & \to & Z^* \\ & L & \mapsto & (ev_{x,y} \mapsto ev_{x,y}(L)) \end{array} \tag{2.2}$$

We claim that the map ev in equation (2.2) is an isometric isomorphism. It is straightforward to show that ev is a linear map. To see that ev is isometric, we compute directly that

$$\|ev(L)\| = \sup_{\|ev_{x,y}\| \le 1} |ev(L)(ev_{x,y})| = \sup_{\|ev_{x,y}\| \le 1} |ev_{x,y}(L)| \le \|L\|.$$

On the other hand,

$$\begin{aligned} \|L\| &= \sup_{\|x\|=1} \sup_{\|y\|=1} |L(x)(y)| \\ &= \sup_{\|x\|=1} \sup_{\|y\|=1} |ev_{x,y}(L)| \le \sup_{\|x\|\|y\|=1} |ev_{x,y}(L)| \\ &= \sup_{\|ev_{x,y}\|=1} |ev(L)(ev_{x,y})| = \|ev(L)\|. \end{aligned}$$

Therefore, if $L \in B(X,Y^*)$ then $\|ev(L)\| = \|L\|$ and consequently, the linear map ev in equation (2.2) is isometric.

To see that ev is injective, assume that $L_1, L_2 \in B(X,Y^*)$ such that $ev(L_1) = ev(L_2)$. If $x \in X$ and $y \in Y$ then

$$L_1(x)(y) = ev(L_1)(ev_{x,y}) = ev(L_2)(ev_{x,y}) = L_2(x)(y).$$

So, $L_1 = L_2$ and consequently, ev is injective. To see that ev is surjective, assume that $\phi \in Z^*$. If $x \in X$ then define the map

$$\begin{array}{rccc} \phi_x: & Y & \to & \mathbb{C} \\ & y & \mapsto & \phi(ev_{x,y}). \end{array}$$

Once again, it is easy to check that ϕ_x is a linear functional. It is also bounded because

$$\|\phi_x\| = \sup_{\|y\|=1} |\phi_x(y)| = \sup_{\|y\|=1} |\phi(ev_{x,y})| \leq \|\phi\| \|x\|.$$

Now define the map

$$\begin{array}{cccc} L_\phi : & X & \to & Y^* \\ & x & \mapsto & \phi_x \end{array}$$

To see that L_ϕ is a linear operator, assume that $x_1, x_2 \in X$ and $\lambda \in \mathbb{C}$. If $y \in Y$ then

$$\begin{aligned} L_\phi(x_1 + x_2)(y) &= \phi_{x_1+x_2}(y) \\ &= \phi(ev_{x_1+x_2,y}) = \phi(ev_{x_1,y}) + \phi(ev_{x_2,y}) \\ &= L_\phi(x_1)(y) + L_\phi(x_2)(y) \end{aligned}$$

and

$$\begin{aligned} L_\phi(\lambda x_1)(y) &= \phi_{\lambda x_1}(y) \\ &= \phi(ev_{\lambda x_1,y}) = \lambda \phi(ev_{x_1,y}) \\ &= \lambda L_\phi(x_1)(y). \end{aligned}$$

So, $L_\phi(x_1 + x_2) = L_\phi(x_1) + L_\phi(x_2)$ and $L_\phi(\lambda x_1) = \lambda L_\phi(x_1)$. This demonstrates that L_ϕ is a linear operator. It is also bounded because

$$\|L_\phi\| = \sup_{\|x\|\leq 1} \|\phi_x\| \leq \|\phi\|.$$

Finally, we observe that if $x \in X$ and $y \in Y$

$$ev(L_\phi)(ev_{x,y}) = ev_{x,y}(L_\phi) = L_\phi(x)(y) = \phi_x(y) = \phi(ev_{x,y}).$$

So, $ev(L_\phi) = \phi$ and hence, ev is surjective. We conclude that the map ev from equation (2.2) is an isometric isomorphism.

Using the isomorphism in equation (2.2), we will now endow $B(X, Y^*)$ with a particular topology. The dual space Z^* can be endowed with the weak-* topology, which is the weakest topology such that the subset of maps

$$\{ev_z : \varphi \mapsto \varphi(z) \mid \varphi \in Z^*, z \in Z\}$$

are all continuous. The weak-* topology is generated by the sub-basis

$$\{ev_z^{-1}(B(x,\epsilon)) \mid z \in Z,\ x \in \mathbb{C},\ \epsilon \in \mathbb{R}_{>0}\}.$$

where $B(x,\epsilon) \subseteq \mathbb{C}$ is the open ball centred at $x \in \mathbb{C}$ with radius $\epsilon \in \mathbb{R}_{>0}$.

Definition 2.5.4. Let X and Y be Banach spaces. Let ev denote the isometric *-isomorphism in equation (2.2). The **ultraweak topology** on $B(X,Y^*)$ is the weak-* topology with sub-basis

$$\{ev^{-1}(ev_z^{-1}(B(x,\epsilon))) \mid z \in Z,\ x \in \mathbb{C},\ \epsilon \in \mathbb{R}_{>0}\}.$$

We remark that in [Pau02], the ultraweak topology is called the **BW (bounded weak) topology**. The name "bounded weak" is a result of the following theorem.

Theorem 2.5.4. *Let X and Y be Banach spaces. Let $\{L_\lambda\}_{\lambda\in\Lambda}$ be a bounded net in $B(X,Y^*)$. Then, L_λ converges to $L \in B(X,Y^*)$ in the ultraweak topology if and only if for $x \in X$, $L_\lambda(x)$ converges weakly to $L(x)$.*

Proof. Assume that X and Y are Banach spaces. First, assume that the bounded net $\{L_\lambda\}_{\lambda\in\Lambda}$ converges to L in the ultraweak topology on $B(X,Y^*)$. Assume that $\epsilon \in \mathbb{R}_{>0}$. Then, there exists $\lambda' \in \Lambda$ such that if $x \in X$ and $y \in Y$ then

$$L_{\lambda'} - L \in ev^{-1}(ev^{-1}_{ev_{x,y}}(B(0,\epsilon))).$$

So, $ev(L_{\lambda'} - L) \in ev^{-1}_{ev_{x,y}}(B(0,\epsilon))$ and

$$|ev_{ev_{x,y}}(ev(L_{\lambda'} - L))| = |ev(L_{\lambda'} - L)(ev_{x,y})| = |L_{\lambda'}(x)(y) - L(x)(y)| < \epsilon.$$

Since $y \in Y$ was arbitrary, we find that if $x \in X$ then $L_\lambda(x)$ converges weakly to $L(x)$ on Y^*.

Conversely, assume that if $x \in X$ then $\{L_\lambda(x)\}_{\lambda\in\Lambda}$ converges weakly to $L(x)$ in Y^*. If $x \in X$ and $y \in Y$ then $L_\lambda(x)(y)$ converges to $L(x)(y)$. Equivalently, $ev_{ev_{x,y}}(ev(L_\lambda))$ converges to $ev_{ev_{x,y}}(ev(L))$ and subsequently,

$$ev_z(ev(L_\lambda)) \to ev_z(ev(L))$$

where $z \in span_{\mathbb{C}}\{ev_{x,y} \mid x \in X, y \in Y\}$. Now since $\{L_\lambda\}_{\lambda\in\Lambda}$ is a bounded net, we deduce that the convergence $ev_z(ev(L_\lambda)) \to ev_z(ev(L))$ holds for arbitrary $z \in Z$ and therefore, L_λ converges in the ultraweak topology on $B(X,Y^*)$ to L. This completes the proof. □

As explained in [Pau02, Chapter 7], the Banach space Z such that $Z^* \cong B(H)$ can be identified either with the space of *ultraweakly continuous linear functionals* in equation (2.2) or the space of trace-class operators in Theorem 2.5.2. The general definition of the ultraweak topology given in Definition 2.5.4 focuses on the duality with ultraweakly continuous linear functionals established in equation (2.2). By contrast, the definition of the ultraweak topology on $B(H)$ will focus on the duality with trace-class operators. Note that the general definition of the ultraweak topology will recur in section 3.3.

Definition 2.5.5. Let H be a Hilbert space. The **ultraweak topology** on $B(H)$ is the $\mathcal{G}$-weak topology where

$$\mathcal{G} = \{u \mapsto \psi_u(x) = Tr(ux) \mid x \in B_1(H)\}.$$

Here, $\psi_y \in B_1(H)^*$ is the functional defined in Theorem 2.5.2.

As with the strong operator and weak operator topologies, we will now describe convergence in the ultraweak topology.

Theorem 2.5.5. *Let H be a Hilbert space. Then, a net $\{A_n\}_{n\in I}$ in $B(H)$ converges to an operator $A \in B(H)$ in the ultraweak topology if and only if for $x \in B_1(H)$, $Tr(A_n x) \to Tr(Ax)$ in $\mathbb{C}$.*

Proof. Assume that H is a Hilbert space. Assume that $\{A_n\}_{n\in I}$ is a net in $B(H)$.

To show: (a) If $A_n \to A$ in the ultraweak topology on $B(H)$ then if $x \in B_1(H)$ then $Tr(A_n x) \to Tr(Ax)$ in $\mathbb{C}$.

(b) If for $x \in B_1(H)$, $Tr(A_n x) \to Tr(Ax)$ in $\mathbb{C}$ then $A_n \to A$ in the ultraweak topology on $B(H)$.

(a) Assume that $A_n \to A$ in the ultraweak topology on $B(H)$. Assume that $\epsilon \in \mathbb{R}_{>0}$. Then, there exists $N \in I$ such that if $n > N$ and $x \in B_1(H)$ then

$$A_n - A \in L_{x,0,\epsilon} = \{y \in B(H) \mid |Tr(yx)| < \epsilon\}.$$

This means that if $x \in B_1(H)$ then

$$|Tr(A_n x) - Tr(Ax)| = |Tr((A_n - A)x)| < \epsilon.$$

So, $Tr(A_n x) \to Tr(Ax)$ in $\mathbb{C}$.

(b) Assume that if $x \in B_1(H)$ then $Tr(A_n x) \to Tr(Ax)$. Then, there exists $N \in I$ such that if $n > N$ then

$$|Tr(A_n x) - Tr(Ax)| = |Tr((A_n - A)x)| < \epsilon.$$

Equivalently,

$$A_n - A \in L_{x,0,\epsilon} = \{y \in B(H) \mid |Tr(yx)| < \epsilon\}.$$

Since $L_{x,0,\epsilon}$ is a basic open set of $B(H)$ with the ultraweak topology, we deduce that $A_n \to A$ in the ultraweak topology as required. $\square$

Example 2.5.2. Let H be a Hilbert space. We will give an example which demonstrates that the weak operator topology is weaker than the ultraweak topology on $B(H)$. Let $\{u_n\}_{n\in I}$ be a net which converges ultraweakly to an operator $u \in B(H)$. Recall that $B_1(H)$ is the closure of the linear span of rank-one operators in $B(H)$. That is,

$$B_1(H) = \overline{span_{\mathbb{C}}\{|\xi\rangle\langle\mu| \mid \xi, \mu \in H\}}.$$

By Theorem 2.5.5, if $\xi, \mu \in H$ then

$$Tr(u_n|\xi\rangle\langle\mu|) \to Tr(u|\xi\rangle\langle\mu|)$$

in $\mathbb{C}$. If $\{\xi_j\}_{j\in J}$ is an orthonormal basis for H then

$$\begin{aligned}
Tr(u|\xi\rangle\langle\mu|) &= \sum_{j\in J} \langle u|\xi\rangle\langle\mu|\xi_j, \xi_j\rangle \\
&= \sum_{j\in J} \langle u\langle\xi_j, \mu\rangle\xi, \xi_j\rangle = \sum_{j\in J} \langle\xi_j, \mu\rangle\langle u\xi, \xi_j\rangle \\
&= \langle u\xi, \sum_{j\in J} \langle\mu, \xi_j\rangle\xi_j\rangle = \langle u\xi, \mu\rangle.
\end{aligned}$$

Hence, if $\xi, \mu \in H$ then $\langle u_n\xi, \mu\rangle$ converges to $\langle u\xi, \mu\rangle$. By Theorem 2.5.3, this means that if $u_n \to u$ in the ultraweak topology then $u_n \to u$ in the weak operator topology.

By its definition in Definition 2.5.5, the ultraweak topology on $B(H)$ is the weak-* topology on $B(H) \cong B_1(H)^*$. Hence, the Banach-Alaoglu theorem tells us that the closed unit ball of $B(H)$ is ultraweakly compact. We include this result in the next theorem.

Theorem 2.5.6. *Let H be a Hilbert space. Then, the closed unit ball of $B(H)$ is compact with respect to the ultraweak topology on $B(H)$. Moreover, the weak operator and ultraweak topologies on the closed unit ball of $B(H)$ coincide and therefore, the closed unit ball of $B(H)$ is weakly compact.*

Proof. Assume that H is a Hilbert space. Let

$$B_1 = \{x \in B(H) \mid \|x\| \leq 1\} \subseteq B(H)$$

be the closed unit ball of $B(H)$. By the Banach-Alaoglu theorem, B_1 is compact with respect to the ultraweak topology on $B(H)$. In what follows, we will abbreviate the ultraweak topology as UWT and the weak operator topology as WOT.

Consider the identity map

$$\begin{array}{rccc} id: & (B_1, UWT) & \to & (B_1, WOT) \\ & x & \mapsto & x. \end{array}$$

Since continuity in the ultraweak topology implies continuity in the weak operator topology, id is a continuous bijection. Furthermore, B_1 is ultraweakly compact and B_1 is Hausdorff in the weak operator topology. Therefore, id is a homeomorphism which demonstrates that the ultraweak topology and the weak operator topology coincide on B_1 and that B_1 is compact in the weak operator topology. □

In light of Theorem 2.5.6, one might wonder whether the closed unit ball $B_1 \subseteq B(H)$ is compact in the strong operator topology. We will show that this is false. Suppose for the sake of contradiction that B_1 is strongly compact. Then, the identity map

$$\begin{array}{rccc} id: & (B_1, SOT) & \to & (B_1, WOT) \\ & x & \mapsto & x \end{array}$$

is a continuous bijection from a compact space to a Hausdorff space. Hence, id is a homeomorphism and the strong operator and weak operator topologies coincide on B_1. However, the involution operation is weakly continuous, whereas by Example 2.4.2, involution is not strongly continuous. This contradicts the finding that the strong and weak operator topologies are the same on B_1. Hence, B_1 is not strongly compact.

Theorem 2.5.7. *Let H be a Hilbert space, A be a *-subalgebra of $B(H)$ such that $id_H \in A$ and C be a subset of $B(H)$. Let $\overline{C}^{WOT}$ denote the weak closure of C. Then,*

1. $\overline{C}^{WOT} \subseteq C''$

2. $\overline{A}^{WOT} = A''$

3. *A is a von Neumann algebra if and only if A is weakly closed.*

Proof. Assume that H is a Hilbert space, A is a *-subalgebra of $B(H)$ such that $id_H \in A$ and C is a subset of $B(H)$.

To show: (a) $\overline{C}^{WOT} \subseteq C''$.

(a) Assume that $u \in \overline{C}^{WOT}$. Then, there exists a net $\{u_\lambda\}_{\lambda \in \Lambda}$ in C such that if $\xi, \mu \in H$ then $\langle u_\lambda \xi, \mu \rangle$ converges to $\langle u\xi, \mu \rangle$ in $\mathbb{C}$. Assume that $w \in C'$. If $\xi, \mu \in H$ then

$$\begin{aligned} \langle uw\xi, \mu \rangle &= \lim_\lambda \langle u_\lambda w\xi, \mu \rangle \\ &= \lim_\lambda \langle wu_\lambda \xi, \mu \rangle \\ &= \lim_\lambda \langle u_\lambda \xi, w^* \mu \rangle = \langle u\xi, w^*\mu \rangle \\ &= \langle wu\xi, \mu \rangle. \end{aligned}$$

So, $uw = wu$ and $u \in C''$. Hence, $\overline{C}^{WOT} \subseteq C''$.

We know from part (a) that $\overline{A}^{WOT} \subseteq A''$. By Theorem 2.4.5, A is strongly dense in A''. Since convergence in the strong operator topology implies convergence in the weak operator topology then

$$\overline{A}^{WOT} \subseteq A'' = \overline{A}^{SOT} \subseteq \overline{A}^{WOT}$$

and we deduce that A'' is the weak closure of A.

Finally by part (a) and the double commutant theorem in Theorem 2.4.7, A is a von Neumann algebra if and only if $A = A''$. This holds if and only if $\overline{A}^{WOT} \subseteq A'' = A$ and subsequently, if and only if $A = \overline{A}^{WOT}$. $\square$

We are interested in slightly extending the last statement of Theorem 2.5.7, by removing the assumption that $id_H \in A$. Along the way, we will encounter particular situations where the strong and weak operator topologies agree.

Theorem 2.5.8. *Let H be a Hilbert space and $\tau \in B(H)^*$. Then, the following are equivalent:*

1. *τ is weakly continuous.*

2. *τ is strongly continuous.*

3. *There exists $x_1, \dots, x_n, y_1, \dots, y_n \in H$ such that if $u \in B(H)$ then*

$$\tau(u) = \sum_{j=1}^{n} \langle ux_j, y_j \rangle.$$

Proof. Assume that H is a Hilbert space and $\tau \in B(H)^*$. We will first prove that the third statement implies the first, Assume that there exists $x_1, \dots, x_n, y_1, \dots, y_n \in H$ such that if $u \in B(H)$ then

$$\tau(u) = \sum_{j=1}^{n} \langle ux_j, y_j \rangle.$$

To see that τ is weakly continuous, note that by definition of the weak operator topology, if $j \in \{1, 2, \dots, n\}$ then the map $u \mapsto \langle ux_j, y_j \rangle$ is weakly continuous. Since τ is a sum of weakly continuous linear functionals then τ is weakly continuous.

Now we prove that the first statement implies the second. Assume that $\tau : B(H) \to \mathbb{C}$ is weakly continuous. Let $U \subseteq \mathbb{C}$ be an open set (with respect to the usual Euclidean topology on $\mathbb{C}$). Since τ is weakly continuous then the preimage $\tau^{-1}(U)$ is weakly open in $B(H)$. This means that if $x \in \tau^{-1}(U)$ then there exists a basic weakly open set $L_{\xi,\mu,x,\epsilon} \subseteq \tau^{-1}(U)$ such that $\xi, \mu \in H$, $\epsilon \in \mathbb{R}_{>0}$ and

$$x \in L_{\xi,\mu,x,\epsilon} = \{y \in B(H) \mid |\langle (y - x)\xi, \mu \rangle| < \epsilon\}.$$

Now let $S_{\xi,x,\epsilon}$ be the basic strongly open set

$$S_{\xi,x,\epsilon/\|\mu\|} = \Big\{y \in B(H) \mid \|(y - x)\xi\| < \frac{\epsilon}{\|\mu\|}\Big\}.$$

We have the inclusions

$$x \in S_{\xi,x,\epsilon/\|\mu\|} \subseteq L_{\xi,\mu,x,\epsilon} \subseteq \tau^{-1}(U).$$

Thus, the preimage $\tau^{-1}(U)$ is strongly open and τ is strongly continuous.

Finally, assume that τ is strongly continuous. Recalling the definition of continuity for linear maps between locally convex spaces (see [RS80, Theorem V.2]), there exists $M \in \mathbb{R}_{>0}$ and $x_1, \ldots, x_n \in H$ such that if $u \in B(H)$ then

$$|\tau(u)| \leq M \max_{j \in \{1,2,\ldots,n\}} \|ux_j\|.$$

By scaling appropriately, we may assume that $M = 1$. Then,

$$|\tau(u)| \leq \max_{j \in \{1,2,\ldots,n\}} \|ux_j\| \leq \Big(\sum_{j=1}^{n} \|ux_j\|^2 \Big)^{\frac{1}{2}}.$$

Now let $H^{(n)} = \bigoplus_{i=1}^{n} H$ and

$$K_0 = \{(ux_1, \ldots, ux_n) \mid u \in B(H)\} \subseteq H^{(n)}.$$

Then, K_0 is a vector subspace of $H^{(n)}$. Let K be the norm closure of K_0 and define the function

$$\begin{array}{cccc} \sigma : & K_0 & \to & \mathbb{C} \\ & (ux_1, \ldots, ux_n) & \mapsto & \tau(u). \end{array}$$

Then, σ is linear and bounded. In fact,

$$\begin{aligned} |\sigma(ux_1, \ldots, ux_n)| &= |\tau(u)| \\ &\leq \Big(\sum_{j=1}^{n} \|ux_j\|^2 \Big)^{\frac{1}{2}} \\ &= \|(ux_1, \ldots, ux_n)\|_{H^{(n)}}. \end{aligned}$$

So, $\|\sigma\| \leq 1$ and thus, we can extend σ to a norm-decreasing linear functional on K, which we will denote by σ again. Since K is a closed vector subspace of $H^{(n)}$ then K is a Hilbert space and by the Riesz representation theorem, there exists $(y_1, \ldots, y_n) \in K$ such that

$$\tau(u) = \sigma(ux_1, \ldots, ux_n) = \langle (ux_1, \ldots, ux_n), (y_1, \ldots, y_n) \rangle_{H^{(n)}} = \sum_{j=1}^{n} \langle ux_j, y_j \rangle.$$

This completes the proof. □

Here is a particular situation where the strong and weak operator topologies on $B(H)$ agree.

Theorem 2.5.9. *Let H be a Hilbert space and $C \subseteq B(H)$ be a convex subset of $B(H)$. Then, C is strongly closed if and only if it is weakly closed.*

Proof. Assume that H is a Hilbert space. Assume that C is a convex subset of $B(H)$.

To show: (a) If C is weakly closed then C is strongly closed.

(b) If C is strongly closed then C is weakly closed.

(a) Assume that C is weakly closed. Then, $B(H) - C$ is weakly open. Arguing in the same manner as in Theorem 2.5.8, we find that $B(H) - C$ is strongly open, as the weak operator topology is weaker than the strong operator topology. So, C is strongly closed.

(b) Assume that C is strongly closed. Let $u \in \overline{C}^{WOT}$. We want to show that $u \in C$. Since u is an element of the weak closure of C then there exists a net of operators $\{u_\lambda\}_{\lambda \in \Lambda}$ which converges weakly to u. So, if $\tau \in B(H)^*$ is a weakly continuous linear functional then $\tau(u) = \lim_\lambda \tau(u_\lambda)$.

By Theorem 2.5.8, the weakly continuous linear functionals are precisely the strongly continuous linear functionals. Since C is convex, we can use [Mur90, Corollary A.8] to show that $u \in \overline{C}^{SOT} = C$. So, C is weakly closed. □

A major consequence of Theorem 2.5.9 is the following theorem.

Theorem 2.5.10. *Let H be a Hilbert space and $A \subseteq B(H)$ be a *-subalgebra of $B(H)$. Then, A is a von Neumann algebra if and only if A is weakly closed.*

Proof. Assume that H is a Hilbert space and $A \subseteq B(H)$ is a *-subalgebra of $B(H)$. By Theorem 2.5.9, A is weakly closed if and only if A is strongly closed because A is convex. Hence, A is weakly closed if and only if A is a von Neumann algebra. □

Another consequence of Theorem 2.5.9 we will frequently make use of is the following:

Theorem 2.5.11. *Let H be a Hilbert space and $C \subseteq B(H)$ be a convex subset of $B(H)$. Then, $\overline{C}^{SOT} = \overline{C}^{WOT}$.*

Proof. Assume that H is a Hilbert space and $C \subseteq B(H)$ is a convex subset of $B(H)$. It is straightforward to check that $\overline{C}^{SOT}$ and $\overline{C}^{WOT}$ are both convex sets, by using Theorem 2.4.1 and Theorem 2.5.3. By Theorem 2.5.9,

$\overline{C}^{SOT}$ is a weakly closed set containing C. Therefore, $\overline{C}^{WOT} \subseteq \overline{C}^{SOT}$.

By Theorem 2.5.9 again, $\overline{C}^{WOT}$ is a strongly closed set containing C. Therefore, $\overline{C}^{SOT} \subseteq \overline{C}^{WOT}$ and $\overline{C}^{SOT} = \overline{C}^{WOT}$. $\square$

Another important property satisfied by von Neumann algebras is that any von Neumann algebra is the dual space of a Banach space. Note that this does not hold for arbitrary C*-algebras. In fact, it was proved by Sakai that if a C*-algebra is the dual space of a Banach space then it is isomorphic to a von Neumann algebra.

Theorem 2.5.12. *Let H be a Hilbert space and $A \subseteq B(H)$ be a von Neumann algebra. Then, there exists a Banach space X and an isometric isomorphism $\theta : A \to X^*$.*

Proof. Assume that H is a Hilbert space and that $A \subseteq B(H)$ is a von Neumann algebra. We will begin by constructing the Banach space X.

Define

$$A^{\perp} = \{v \in B_1(H) \mid \text{If } u \in A \text{ then } Tr(uv) = 0\}.$$

Then, $A^{\perp}$ is a vector subspace of $B_1(H)$. Furthermore, it is closed with respect to the trace norm on $B_1(H)$. So, let $X = B_1(H)/A^{\perp}$. Then, X is a Banach space, equipped with its quotient norm

$$\begin{array}{rccc} \|-\|_X : & X = B_1(H)/A^{\perp} & \to & \mathbb{R}_{\geq 0} \\ & \varphi + A^{\perp} & \mapsto & \inf_{b \in A^{\perp}} \|\varphi + b\|_1. \end{array}$$

If $u \in A$ then define

$$\begin{array}{rccc} \theta(u) : & X & \to & \mathbb{C} \\ & v + A^{\perp} & \mapsto & Tr(uv). \end{array}$$

By linearity of the trace, $\theta(u)$ is a linear map. To see that $\theta(u)$ is bounded, we compute directly that

$$
\begin{aligned}
\|\theta(u)\| &= \sup_{\|v+A^\perp\|_X=1} |Tr(uv)| \\
&= \sup_{\inf_{b\in A^\perp}\|v+b\|_1=1} |Tr(uv)+Tr(ub_0)| \\
&= \sup_{\inf_{b\in A^\perp}\|v+b\|_1=1} |Tr(u(v+b_0))| \\
&\leq \sup_{\inf_{b\in A^\perp}\|v+b\|_1=1} \|u\||Tr(|v+b_0|)| \\
&= \sup_{\inf_{b\in A^\perp}\|v+b\|_1=1} \|u\|\|v+b_0\|_1 = \|u\|.
\end{aligned}
$$

In the above computation, we used the fact that if $v \in B_1(H)$ then there exists $b_0 \in A^\perp$ such that $\|v+b_0\|_1 = \inf_{b\in A^\perp}\|v+b\|_1$. Therefore, if $u \in A$ then $\theta(u)$ is a well-defined bounded linear functional on X.

Now let $\theta : A \to X^*$ denote the map $u \mapsto \theta(u)$. By linearity of the trace, θ is a linear map.

To show: (a) θ is injective.

(b) θ is surjective.

(c) θ is isometric.

(a) Assume that $u \in A$ such that $\theta(u) = 0$ in X^*. Let $\psi : B(H) \to B_1(H)^*$ be the isometric isomorphism in Theorem 2.5.2. If $v \in B_1(H)$ then

$$\theta(u)(v + A^\perp) = Tr(uv) = \psi_u(v) = 0.$$

So, $\psi(u) = \psi_u = 0$ in $B_1(H)^*$. Since ψ is injective then $u = 0$. Thus, θ is injective.

(b) Assume that $\tau \in X^*$. Let $\pi : B_1(H) \to X$ denote the canonical quotient map. Then, $\tau \circ \pi \in B_1(H)^*$ and by surjectivity of ψ, there exists $u \in B(H)$ such that $\psi(u) = \tau \circ \pi$. To see that $u \in A$, first observe that if $w \in A^\perp$ then

$$Tr(uw) = \psi_u(w) = \psi(u)(w) = (\tau \circ \pi)(w) = \tau(0) = 0.$$

Suppose for the sake of contradiction that $u \notin A$. Then, $u \in B(H) - A$. By [Mur90, Corollary A.9], there exists a strongly continuous linear functional $\phi : X \to \mathbb{C}$ such that if $a \in A$ then $\phi(a) = 0$ and $\phi(u) = 1$.

By Theorem 2.5.8, there exists $x_1, \ldots, x_n, y_1, \ldots, y_n \in H$ such that if $v \in B(H)$ then

$$\phi(v) = \sum_{j=1}^{n} \langle vx_j, y_j \rangle.$$

Recall that if $j \in \{1, 2, \ldots, n\}$ then the rank-one operator $|x_j\rangle\langle y_j|$ is a trace-class operator. Let $v = \sum_{i=1}^{n} |x_i\rangle\langle y_j| \in B_1(H)$. To see that $v \in A^{\perp}$, we compute directly that if $a \in A$ then

$$Tr(av) = \sum_{i=1}^{n} Tr(a|x_i\rangle\langle y_i|) = \sum_{i=1}^{n} \langle ax_i, y_i \rangle = \psi(a) = 0.$$

The last equality follows from the construction of the linear functional ϕ. Therefore, $v \in A^{\perp}$ and it follows that $Tr(uv) = 0$. But,

$$Tr(uv) = \sum_{i=1}^{n} Tr(u|x_i\rangle\langle y_i|) = \sum_{i=1}^{n} \langle ux_i, y_i \rangle = \phi(u) = 1$$

which contradicts the finding that $Tr(uv) = 0$. We conclude that $u \in A$. Now if $w + A^{\perp} \in X$ then

$$\theta(u)(w + A^{\perp}) = Tr(uw) = \psi(u)(w) = (\tau \circ \pi)(w) = \tau(w + A^{\perp}).$$

Since $w + A^{\perp} \in X$ was arbitrary then $\theta(u) = \tau$. Hence, θ is surjective.

(c) We have already found that if $u \in A$ then $\|\theta(u)\| \leq \|u\|$. To prove the reverse inequality, assume that $\epsilon \in \mathbb{R}_{>0}$. Then, there exists $v \in B_1(H)$ such that

$$\|v\|_1 \leq 1 \qquad \text{and} \qquad |\psi(u)(v)| > \|\psi(u)\| - \epsilon.$$

So,

$$\|\theta(u)\| \geq |\theta(u)(\pi(v))| = |Tr(uv)| = |\psi(u)(v)| > \|\psi(u)\| - \epsilon.$$

By Theorem 2.5.2, $\|\theta(u)\| > \|u\| - \epsilon$. Since $\epsilon \in \mathbb{R}_{>0}$ was arbitrary then $\|\theta(u)\| \geq \|u\|$ and consequently, if $u \in A$ then $\|\theta(u)\| = \|u\|$.

By parts (a), (b) and (c), θ is an isometric isomorphism as required. $\square$

By using the isomorphism $\theta : A \to X^*$ in Theorem 2.5.12, we can define a weak-* topology on a von Neumann algebra A. Explicitly, it is the $\mathcal{F}$-weak topology given by the set

$$\mathcal{F} = \{ev_x \circ \theta : A \to \mathbb{C} \mid x \in X\}.$$

where if $\tau \in X^*$ then $ev_x(\tau) = \tau(x)$. If $a \in A$ and $v + A^\perp \in X$ then

$$(ev_{v+A^\perp} \circ \theta)(a) = ev_{v+A^\perp}(\theta(a)) = Tr(av).$$

By the construction of X in Theorem 2.5.12, we find that

$$\mathcal{F} = \{a \mapsto Tr(av) \mid v \in B_1(H)\}.$$

We conclude that weak-* topology on A induced by the isomorphism in Theorem 2.5.12 is exactly the ultraweak topology on A, inherited from the ultraweak topology on $B(H)$. We will use this to give a characterisation of thw ultraweakly continuous functionals on the von Neumann algebra A. First, we require the following result, which is [Mur90, Theorem A.2].

Theorem 2.5.13. *Let X be a normed vector space. Then, a linear functional $\theta : X^* \to \mathbb{C}$ is weak-* continuous if and only if there exists $x \in X$ such that $\theta = ev_x$.*

Proof. Assume that X is a normed vector space. By definition of the weak-* topology on X^*, if $x \in X$ then the evaluation map $ev_x : X^* \to \mathbb{C}$ is weak-* continuous.

Conversely, assume that $\theta : X^* \to \mathbb{C}$ is weak-* continuous. Thinking of X^* as a locally convex space, this means that there exists $M \in \mathbb{R}_{>0}$ and $x_1, x_2, \ldots, x_n \in X$ such that if $\tau \in X^*$ then

$$|\theta(\tau)| \leq M \max_{i \in \{1,2,\ldots,n\}} |\tau(x_i)| = M \max_{i \in \{1,2,\ldots,n\}} |ev_{x_i}(\tau)|.$$

Therefore, $\theta = 0$ on the subspace $\bigcap_{i=1}^n \ker ev_{x_i}$. Now define the vector subspace

$$V = \{(y_1, \ldots, y_n) \in \mathbb{C}^n \mid \text{There exists } \tau \in X^* \text{ such that } \tau(x_i) = y_i\}$$

and the linear map

$$\begin{array}{cccc} g : & V & \to & \mathbb{C} \\ & (\tau(x_1), \ldots, \tau(x_n)) & \mapsto & \theta(\tau) \end{array}$$

To see that g is well-defined, assume that $\tau, \tau' \in X^*$ such that if $i \in \{1, 2, \ldots, n\}$ then $\tau(x_i) = \tau'(x_i)$. Then, $\tau - \tau' \in \bigcap_{i=1}^n \ker ev_{x_i}$ and since $\ker \theta \subseteq \bigcap_{i=1}^n \ker ev_{x_i}$ then $\theta(\tau - \tau') = 0$. So, g is a well-defined linear map.

Now take a basis for V and extend it to a basis for $\mathbb{C}^n$. By doing this, we can extend the linear map g to a linear map $\tilde{g} : \mathbb{C}^n \to \mathbb{C}$. Let $\{e_1, \dots, e_n\}$ be the standard basis for $\mathbb{C}^n$. If $\tau \in X^*$ then

$$\theta(\tau) = \tilde{g}\big((\tau(x_1), \dots, \tau(x_n))\big) = \tilde{g}(\sum_{i=1}^{n} ev_{x_i}(\tau)e_i) = \sum_{i=1}^{n} \tilde{g}(e_i)ev_{x_i}(\tau).$$

So, there exists $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ such that

$$\theta = \lambda_1 ev_{x_1} + \cdots + \lambda_n ev_{x_n}.$$

Therefore, if $x = \lambda_1 x_1 + \cdots + \lambda_n x_n$ then $\theta = ev_x$ as required. $\square$

Theorem 2.5.14. *Let H be a Hilbert space and $A \subseteq B(H)$ be a von Neumann algebra. Let $\tau : A \to \mathbb{C}$ be a linear functional. Then, τ is ultraweakly continuous if and only if there exists $u \in B_1(H)$ such that if $v \in A$ then $\tau(v) = Tr(uv)$.*

Proof. Assume that H is a Hilbert space and that $A \subseteq B(H)$ is a von Neumann algebra. Assume that $\tau : A \to \mathbb{C}$ is a linear functional. Then, τ is ultraweakly continuous if and only if τ is weak-* continuous.

Let $\theta : A \to X^*$ be the isomorphism constructed in Theorem 2.5.12. Then, τ is weak-* continuous if and only if $\tau \circ \theta^{-1}$ is weak-* continuous as a functional on X^*. By Theorem 2.5.13, this holds if and only if there exists $u \in B_1(H)$ such that

$$ev_{u+A^\perp} = \tau \circ \theta^{-1}.$$

Thus, $ev_{u+A^\perp} \circ \theta = \tau$ and if $v \in A$ then

$$\tau(v) = ev_{u+A^\perp}(\theta(v)) = \theta(v)(u + A^\perp) = Tr(uv).$$

So, we conclude that $\tau \in A^*$ is ultraweakly continuous if and only if there exists $u \in B_1(H)$ such that if $v \in A$ then $\tau(v) = Tr(uv)$. $\square$

2.6 The Kaplansky Density Theorem

It is remarked in [Ped18, Section 2.3] that the Kaplansky Density theorem is “Kaplansky’s great gift to mankind”. Following [Mur90, Section 4.3], we will first prove a few results pertaining to the strong operator topology on $B(H)$ for a Hilbert space H. Recall from Example 2.4.2 that the involution map on $B(H)$ is not strongly continuous. We will see in the next theorem that involution becomes strongly continuous when restricted to a particular subset of operators.

Theorem 2.6.1. *Let H be a Hilbert space. Let $N \subseteq B(H)$ be the set of normal operators in $B(H)$. Then, the involution map on N*

$$\begin{array}{rccc} {}^*: & N & \to & N \\ & u & \mapsto & u^* \end{array}$$

is strongly continuous.

Proof. Assume that H is a Hilbert space and that $N \subseteq B(H)$ is the set of normal operators in $B(H)$. Assume that $u, v \in N$. If $\xi \in H$ then

$$\begin{aligned} \|(u^* - v^*)\xi\|^2 &= \langle (u^* - v^*)\xi, (u^* - v^*)\xi \rangle \\ &= \langle u^*\xi, u^*\xi \rangle + \langle v^*\xi, v^*\xi \rangle - \langle u^*\xi, v^*\xi \rangle - \langle v^*\xi, u^*\xi \rangle \\ &= \|u\xi\|^2 + \|v\xi\|^2 - \langle vu^*\xi, \xi \rangle - \langle uv^*\xi, \xi \rangle \\ &= \|u\xi\|^2 - \|v\xi\|^2 + \langle vv^*\xi, \xi \rangle + \langle vv^*\xi, \xi \rangle - \langle vu^*\xi, \xi \rangle - \langle uv^*\xi, \xi \rangle \\ &= \|u\xi\|^2 - \|v\xi\|^2 + \langle v(v^* - u^*)\xi, \xi \rangle + \langle (v - u)v^*\xi, \xi \rangle \\ &\le \|u\xi\|^2 - \|v\xi\|^2 + 2\|(v - u)v^*\xi\|\|\xi\|. \end{aligned}$$

To be clear, in the third equality we used the fact that u and v are normal. Now let $\{u_\lambda\}_{\lambda \in \Lambda}$ be a net in N which strongly converges to u. By the above inequality, we have

$$\begin{aligned} \lim_\lambda \|(u^* - u_\lambda^*)\xi\|^2 &\le \lim_\lambda \left(\|u\xi\|^2 - \|u_\lambda \xi\|^2 + 2\|(u_\lambda - u)u_\lambda^*\xi\|\|\xi\| \right) \\ &= \lim_\lambda \left(\|u\xi\|^2 - \|u_\lambda \xi\|^2 \right) + \lim_\lambda \left(2\|(u_\lambda - u)u_\lambda^*\xi\|\|\xi\| \right) \\ &= 0. \end{aligned}$$

Therefore, the net $\{u_\lambda^*\}_{\lambda \in \Lambda}$ strongly converges to u^*. So, the involution map when restricted to the set of normal operators N is strongly continuous. $\square$

Similarly to Theorem 2.6.1, there is also a particular situation where multiplication is strongly continuous.

Theorem 2.6.2. *Let H be a Hilbert space and S be a bounded subset of $B(H)$. Then, the multiplication map*

$$\begin{array}{rccc} \cdot: & S \times B(H) & \to & B(H) \\ & (u, v) & \mapsto & uv \end{array}$$

is strongly continuous.

Proof. Assume that H is a Hilbert space and S is a bounded subset of $B(H)$. Assume that $u \in S$ and $v \in B(H)$. Let $\{u_\lambda\}_{\lambda\in\Lambda}$ and $\{v_\lambda\}_{\lambda\in\Lambda}$ be nets which converge strongly to u and v respectively. If $\xi \in H$ then

$$\begin{aligned}
\lim_\lambda \|uv\xi - u_\lambda v_\lambda \xi\| &= \lim_\lambda \|uv\xi - uv_\lambda\xi + uv_\lambda\xi - u_\lambda v_\lambda\xi\| \\
&\leq \lim_\lambda \big(\|u\|\|v\xi - v_\lambda\xi\| + \|(u - u_\lambda)v_\lambda\xi\|\big) \\
&= 0.
\end{aligned}$$

Therefore, $\{u_\lambda v_\lambda\}_{\lambda\in\Lambda}$ strongly converges to uv. So, the multiplication map from $S \times B(H)$ to $B(H)$ is strongly continuous. $\square$

For the next major result required for the proof of the Kaplansky Density theorem, we need the following definition.

Definition 2.6.1. Let $f \in Cts(\mathbb{R}, \mathbb{C})$. We say that f is **strongly continuous** if the following statement is satisfied: If H is a Hilbert space and $\{u_\lambda\}_{\lambda\in\Lambda}$ is a net of self-adjoint operators in $B(H)$ which converge strongly to a self-adjoint operator $u \in B(H)$ then $\{f(u_\lambda)\}_{\lambda\in\Lambda}$ converges strongly to $f(u)$.

To be clear, the above definition of a strongly continuous function uses the continuous functional calculus from Theorem 1.3.7. The next theorem gives us a variety of examples of strongly continuous functions.

Theorem 2.6.3. *Let $f \in Cts(\mathbb{R}, \mathbb{C})$. If f is continuous and bounded then f is strongly continuous.*

Proof. Recall from Definition 1.6.1 that $Cts_0(\mathbb{R}, \mathbb{C})$ is the C*-algebra of continuous functions $f : \mathbb{R} \to \mathbb{C}$ which vanish at infinity. Let $A \subseteq Cts(\mathbb{R}, \mathbb{C})$ be the set of strongly continuous functions. With the pointwise defined operations, A is a vector subspace of $Cts(\mathbb{R}, \mathbb{C})$.

We claim that if $f, g \in A$ and f is bounded then $fg \in A$.

To show: (a) If $f, g \in A$ and f is bounded then $fg \in A$.

(a) Assume that $f, g \in A$ and that f is bounded. Assume that H is a Hilbert space and that $\{u_\lambda\}_{\lambda\in\Lambda}$ is a net of self-adjoint operators in $B(H)$ which converge strongly to a self-adjoint operator $u \in B(H)$. Since f and g are strongly continuous then the nets $\{f(u_\lambda)\}_{\lambda\in\Lambda}$ and $\{g(u_\lambda)\}_{\lambda\in\Lambda}$ converge strongly to $f(u)$ and $g(u)$ respectively.

By Theorem 2.6.2, $\{fg(u_\lambda)\}_{\lambda\in\Lambda}$ must converge strongly to $fg(u)$ because f is bounded. So, $fg \in A$.

Now let $A_0 = A \cap Cts_0(\mathbb{R}, \mathbb{C})$. We claim that A_0 is a closed *-subalgebra of $Cts_0(\mathbb{R}, \mathbb{C})$.

To show: (b) A_0 is a closed subalgebra of $Cts_0(\mathbb{R}, \mathbb{C})$.

(c) A_0 is self-adjoint (closed under the involution map).

(b) Since A and $Cts_0(\mathbb{R}, \mathbb{C})$ are both vector spaces, A_0 is closed under scalar multiplication and addition. To see that A_0 is closed under multiplication, assume that $f, g \in A_0$. Since $Cts_0(\mathbb{R}, \mathbb{C})$ is closed under multiplication then $fg \in Cts_0(\mathbb{R}, \mathbb{C})$. By Theorem 1.6.5, f and g are both bounded functions and by part (a) of the proof $fg \in A$. So, $fg \in A_0$.

To see that A_0 is closed, let $\{f_\mu\}_{\mu\in M}$ be a net in A_0 which converges to f. By Theorem 1.6.5, $f \in Cts_0(\mathbb{R}, \mathbb{C})$. Assume that H is a Hilbert space and $\{u_\lambda\}_{\lambda\in\Lambda}$ is a net of self-adjoint operators in $B(H)$ which converge strongly to a self-adjoint operator $u \in B(H)$. If $\mu \in M$ then the net $\{f_\mu(u_\lambda)\}_{\lambda\in\Lambda}$ converges strongly to $f_\mu(u)$.

Observe that if $\xi \in H$ then

$$\begin{aligned}\|f(u_\lambda)\xi - f(u)\xi\| &\leq \|(f(u_\lambda) - f_\mu(u_\lambda))\xi\| + \|(f_\mu(u_\lambda) - f_\mu(u))\xi\| + \|(f_\mu(u) - f(u))\xi\| \\ &\to 0\end{aligned}$$

in the limit over $\mu \in M$. The first and last terms vanish in this limit because the continuous functional calculus is an isometric isomorphism (see Theorem 1.3.7). Therefore, $\{f(u_\lambda)\}_{\lambda\in\Lambda}$ converges strongly to $f(u)$. So, $f \in A$ and hence, $f \in A_0$. This shows that A_0 is a closed subalgebra of $Cts_0(\mathbb{R}, \mathbb{C})$.

(c) Assume that $f \in A_0$, H is a Hilbert space and $\{u_\lambda\}_{\lambda\in\Lambda}$ is a net of self-adjoint operators in $B(H)$ which converge strongly to a self-adjoint operator $u \in B(H)$. Since $f\overline{f} = \overline{f}f$ then $f(u_\lambda)f(u_\lambda)^* = f(u_\lambda)^*f(u_\lambda)$. By Theorem 2.6.1, the net $\{f(u_\lambda)^*\}_{\lambda\in\Lambda}$ strongly converges to $f(u)^*$. So, $\overline{f} \in A_0$ and consequently, A_0 is closed under the involution map.

The idea behind the proof is to exploit parts (b) and (c) by using the Stone-Weierstrass theorem. First, we must show that the conditions of the

Stone-Weierstrass theorem hold.

To show: (d) If $x \in \mathbb{R}$ then there exists $f \in A_0$ such that $f(x) \neq 0$.

(e) If $x, y \in \mathbb{R}$ and $x \neq y$ then there exists $g \in A_0$ such that $g(x) \neq g(y)$ (A_0 separates points).

(d) Recall the isomorphism of C*-algebras given in Theorem 1.6.7. Define the functions

$$\begin{array}{rccc} f: & \mathbb{R} & \to & \mathbb{C} \\ & x & \mapsto & \frac{1}{1+x^2} \end{array}$$

and

$$\begin{array}{rccc} g: & \mathbb{R} & \to & \mathbb{C} \\ & x & \mapsto & \frac{x}{1+x^2}, \end{array}$$

Then, $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0$. Now define the function $\tilde{f} : \mathbb{R} \cup \{\infty\} \to \mathbb{C}$ by $\tilde{f}(x) = f(x)$ if $x \neq \infty$ and $\tilde{f}(\infty) = 0$. The function $\tilde{g} : \mathbb{R} \cup \{\infty\} \to \mathbb{C}$ is defined similarly. The point is that $\tilde{f}, \tilde{g} \in Cts(\mathbb{R} \cup \{\infty\}, \mathbb{C})$ and by the isomorphism in Theorem 1.6.7, $f, g \in Cts_0(\mathbb{R}, \mathbb{C})$.

Now we will show that $f, g \in A_0$. First we will show that $g \in A_0$. Assume that H is a Hilbert space and $u, v \in B(H)$ are self-adjoint. Using the continuous functional calculus, we compute directly that

$$\begin{aligned} g(u) - g(v) &= u(1+u^2)^{-1} - v(1+v^2)^{-1} \\ &= \big(u(1+u^2)^{-1}(1+v^2) - v\big)(1+v^2)^{-1} \\ &= (1+u^2)^{-1}\big(u(1+v^2) - (1+u^2)v\big)(1+v^2)^{-1} \\ &= (1+u^2)^{-1}\big((u-v) + u(v-u)v\big)(1+v^2)^{-1}. \end{aligned}$$

Thus, if $\xi \in H$ then

$$\begin{aligned} \|g(u)\xi - g(v)\xi\| &= \|(1+u^2)^{-1}\big((u-v) + u(v-u)v\big)(1+v^2)^{-1}\xi\| \\ &\leq \|(1+u^2)^{-1}(u-v)(1+v^2)^{-1}\xi\| + \|(1+u^2)^{-1}u(v-u)v(1+v^2)^{-1}\xi\| \\ &\leq \|(1+u^2)^{-1}\|\|(u-v)(1+v^2)^{-1}\xi\| + \|(1+u^2)^{-1}u\|\|(v-u)v(1+v^2)^{-1}\xi\| \\ &= \|f(u)\|\|(u-v)(1+v^2)^{-1}\xi\| + \|g(u)\|\|(v-u)v(1+v^2)^{-1}\xi\| \\ &\leq \|(u-v)(1+v^2)^{-1}\xi\| + \|(v-u)v(1+v^2)^{-1}\xi\|. \end{aligned}$$

In the last inequality, we used the fact that $\|f(u)\| = \|f\|_\infty \leq 1$ and $\|g(u)\| = \|g\|_\infty \leq 1$. So, if $\{u_\lambda\}_{\lambda\in\Lambda}$ is a net of self-adjoint operators which converges strongly to the self-adjoint operator $u \in B(H)$ then by the above inequality, $\{g(u_\lambda)\}_{\lambda\in\Lambda}$ converges strongly to $g(u)$. This shows that $g \in A$ and hence, $g \in A_0$.

To see that $f \in A_0$, first note that the identity map $x \in A$. Since $g(x)$ is bounded then $xg(x) \in A$. So, $f(x) = 1 - xg(x) \in A$ and hence, $f(x) \in A_0$.

Now observe that if $x \in \mathbb{R} - \{0\}$ then $g(x) \neq 0$ and $f(0) \neq 0$. Thus, the first condition of the Stone-Weierstrass theorem is satisfied (A_0 vanishes nowhere).

(e) Assume that $x, y \in \mathbb{R}$ and that $x \neq y$. If $x^2 \neq y^2$ then $f(x) \neq f(y)$. If $x^2 = y^2$ then $g(x) \neq g(y)$ and consequently, the set $\{f(x), g(x)\} \subseteq A_0$ separates points.

By parts (d) and (e), we can finally invoke the Stone-Weierstrass theorem to deduce that $A_0 = Cts_0(\mathbb{R}, \mathbb{C})$.

Now assume that $h \in Cts(\mathbb{R}, \mathbb{C})$ and that h is bounded. Then, $hf, hg \in Cts_0(\mathbb{R}, \mathbb{C}) = A_0$. In particular, $hf, hg \in A$. So, $xhg \in A$ and

$$h(x)f(x) + xh(x)g(x) = \frac{h(x)}{1+x^2} + \frac{h(x)x^2}{1+x^2} = h(x) \in A.$$

Thus, h is strongly continuous. $\square$

Before we state and prove the Kaplansky Density theorem, we recall Theorem 2.5.9 and Theorem 2.5.10. In particular, Theorem 2.5.10 tells us that if H is a Hilbert space and $A \subseteq B(H)$ is a *-subalgebra of $B(H)$ then the weak closure $\overline{A}^{WOT}$ is a von Neumann algebra.

Theorem 2.6.4. *Let H be a Hilbert space and $A \subseteq B(H)$ be a C^*-subalgebra of $B(H)$. Let $B = \overline{A}^{SOT}$ be the strong closure of A. Let A_{sa} denote the set of self-adjoint elements in A.*

1. *A_{sa} is strongly dense in B_{sa}.*

2. *The closed unit ball of A_{sa} is strongly dense in the closed unit ball of B_{sa}.*

3. *The closed unit ball of A is strongly dense in the closed unit ball of B*

4. *If $id_H \in A$ then the set of unitary elements of A are strongly dense in the set of unitary elements of B.*

Proof. Assume that H is a Hilbert space and $A \subseteq B(H)$ is a C*-subalgebra of $B(H)$. Assume that $B = \overline{A}^{SOT}$ is the strong closure of A.

First, assume that $u \in B_{sa}$. Then, $u \in B = \overline{A}^{SOT}$ and there exists a net $\{u_\lambda\}_{\lambda \in \Lambda}$ in A which strongly converges to u. The involution map on $B(H)$ is weakly continuous. So, the net $\{u_\lambda^*\}_{\lambda \in \Lambda}$ weakly converges to u^*. Thus, the net $\{\frac{1}{2}(u_\lambda + u_\lambda^*)\}_{\lambda \in \Lambda}$ weakly converges to

$$\frac{1}{2}(u + u^*) = u.$$

Note that the net $\{\frac{1}{2}(u_\lambda + u_\lambda^*)\}_{\lambda \in \Lambda}$ is entirely contained in A_{sa}. Therefore, $\overline{A_{sa}}^{WOT} = B_{sa}$. But since A_{sa} is convex, we can use Theorem 2.5.11 to deduce that

$$\overline{A_{sa}}^{SOT} = \overline{A_{sa}}^{WOT} = B_{sa}.$$

Next, assume that u is an element of the closed unit ball of B_{sa}. By the first part, there exists a net $\{u_\lambda\}_{\lambda \in \Lambda}$ in A_{sa} which strongly converges to u. Define the function f by

$$\begin{array}{rccc} f: & \mathbb{R} & \to & \mathbb{C} \\ & x & \mapsto & \begin{cases} x, & \text{if } x \in [-1,1], \\ \frac{1}{x}, & \text{otherwise.} \end{cases} \end{array}$$

Then, $f \in Cts_0(\mathbb{R}, \mathbb{C})$. By the proof Theorem 2.6.3, the function f is strongly continuous. Consequently, the net $\{f(u_\lambda)\}_{\lambda \in \Lambda}$ strongly converges to $f(u)$. Since $\|u\| \leq 1$ and u is self-adjoint then $\sigma(u) \subseteq [-1,1]$ and $f(u) = u$. Furthermore, observe that the net $\{f(u_\lambda)\}_{\lambda \in \Lambda}$ is in the closed unit ball of A_{sa} because f is real-valued and $\|f\|_\infty \leq 1$.

Hence, the net $\{f(u_\lambda)\}_{\lambda \in \Lambda}$ in the closed unit ball of A_{sa} strongly converges to u in the closed unit ball of B_{sa}. This demonstrates that the closed unit ball of A_{sa} is strongly dense in the closed unit ball of B_{sa}.

Next, assume that v is an element of the closed unit ball of B. The idea here is to consider the C*-subalgebra $M_{2\times 2}(A) \subseteq M_{2\times 2}(B(H)) = B(H^{(2)})$, where $H^{(2)} = H \oplus H$. Since A is strongly dense in B then $M_{2\times 2}(A)$ is strongly dense in the von Neumann algebra $M_{2\times 2}(B)$.

Let

$$w = \begin{pmatrix} 0 & v \\ v^* & 0 \end{pmatrix} \in M_{2\times 2}(B) = \overline{M_{2\times 2}(A)}^{SOT}.$$

Then, w is self-adjoint. We also observe that

$$\|w\| = \sup_{\|(\xi,\mu)\|_{H^{(2)}}=1} \|w(\xi,\mu)\|_{H^{(2)}} = \sup_{\max\{\|\xi\|,\|\mu\|\}=1} \max\{\|v\mu\|, \|v^*\xi\|\} \leq 1.$$

By the second part of the proof, there exists a net $\{w_\lambda\}_{\lambda\in\Lambda}$ in the closed unit ball of $M_{2\times 2}(A)_{sa}$ which strongly converges to w. Therefore, the net $\{(w_\lambda)_{1,2}\}_{\lambda\in\Lambda}$ is contained in the closed unit ball of A and strongly converges to $w_{1,2} = v$. Hence, the closed unit ball of A is strongly dense in the closed unit ball of B.

For the final part of the proof, assume that the identity operator $id_H \in A$. Let $U(A)$ and $U(B)$ the set of unitary elements in A and B respectively. If $u \in U(B)$ then there exists a self-adjoint element $v \in B$ such that $u = e^{iv}$ (note that $v \in B$ due to part 2 of Theorem 2.4.13).

By the first part of this theorem, there exists a net $\{v_\lambda\}_{\lambda\in\Lambda}$ in A_{sa} which strongly converges to v, which is an element of B_{sa}. Now, the function

$$\begin{array}{rccc} h: & \mathbb{R} & \to & \mathbb{C} \\ & t & \mapsto & e^{it} \end{array}$$

is continuous and bounded. By Theorem 2.6.3, h is strongly continuous and thus, the net $\{h(v_\lambda)\}_{\lambda\in\Lambda}$ is contained in $U(A)$ and strongly converges to $h(v) = u$. Therefore, the set $U(A)$ is strongly dense in $U(B)$ as required. $\square$

One application of the Kaplansky Density theorem is to prove the following analogue of Theorem 1.7.6 which applies to von Neumann algebras rather than C*-algebras.

Theorem 2.6.5. *Let H_1 and H_2 be Hilbert spaces. Let $A \subseteq B(H_1)$ be a von Neumann algebra and $\varphi : A \to B(H_2)$ be a weakly continuous *-homomorphism. Then, the image $\varphi(A) \subseteq B(H_2)$ is a von Neumann algebra.*

Proof. Assume that H_1 and H_2 are Hilbert spaces and that $A \subseteq B(H_1)$ is a von Neumann algebra. Assume that $\varphi : A \to B(H_2)$ is a weakly continuous *-homomorphism. By the remark involving equation (2.1), we may assume that $id_{H_1} \in A$.

Let $v \in \varphi(A)$ such that $\|v\| < 1$. Then, there exists $\alpha \in (0, 1)$ such that $\|v\| < \alpha < 1$. There also exists $u \in A$ such that $v = \varphi(u)$. Let $u = w|u|$ be the polar decomposition of u. By Theorem 2.4.12, $w \in A$.

Now let E be the spectral resolution of the identity for $|u|$ and

$$G = \{\lambda \in \sigma(|u|) \mid \lambda \geq \alpha\}.$$

By the second statement in Theorem 2.4.13, $E(G)$ is a projection in A. By definition of the set G, we have

$$\alpha E(G) \leq |u|E(G) \qquad \text{and} \qquad |u|(1_A - E(G)) \leq \alpha(1_A - E(G)).$$

Applying φ, we find that $0 \leq \alpha\varphi(E(G)) \leq \varphi(|u|)\varphi(E(G))$ and

$$\begin{aligned} 0 &\leq \alpha\|\varphi(E(G))\| \\ &\leq \|\varphi(|u|)\|\|\varphi(E(G))\| \\ &\leq \|\varphi(|u|)\| = \|\varphi(w^*w|u|)\| \\ &\leq \|\varphi(w^*)\|\|\varphi(u)\| \leq \|\varphi(u)\| \\ &\leq \|v\| < \alpha. \end{aligned}$$

We conclude that $\|\varphi(E(G))\| < 1$. Now since $\varphi(E(G))$ is a projection in $B(H_2)$, we deduce that $\varphi(E(G)) = 0$. Consequently, $v = \varphi(u(1_A - E(G)))$ and

$$\|u(1_A - E(G))\| \leq \||u|(1_A - E(G))\| \leq \alpha\|1_A - E(G)\| \leq \alpha < 1.$$

The second inequality follows from Theorem 2.2.5. The third inequality follows from the fact that $1_A - E(G)$ is a projection in $A \subseteq B(H_1)$.

Now set

$$R = \{u_1 \in A \mid \|u_1\| < 1\}.$$

By our previous argument, we conclude that the image

$$\varphi(R) = \{v \in \varphi(A) \mid \|v\| < 1\}.$$

By Theorem 2.5.6, the closed unit ball of A is weakly compact because it is a weakly closed subset of the closed unit ball of $B(H_1)$ which is compact.

Let S be the closed unit ball of A. Since φ is weakly continuous then $\varphi(S)$ is weakly compact in $B(H_2)$.

It is straightforward to check that $\overline{R}^{WOT} = S$. We now claim that $\varphi(S)$ is actually the closed unit ball of $\varphi(A)$. To see why this is the case, first note that $\varphi(S)$ is contained in the closed unit ball of $\varphi(A)$ by definition of S. Now assume that w is an element of the closed unit ball of $\varphi(A)$. Let $\{\epsilon_n\}_{n\in\mathbb{Z}_{>0}}$ be a sequence contained in the interval $(0, 1)$ which converges to 1. There exists $b \in A$ such that $\varphi(b) = w$. If $n \in \mathbb{Z}_{>0}$ then

$$\epsilon_n w = \varphi(\epsilon_n b) \in \varphi(A) \qquad \text{and} \qquad \|\epsilon_n w\| < 1.$$

So, $\epsilon_n w \in \{v \in \varphi(A) \mid \|v\| < 1\} = \varphi(R)$. If $n \in \mathbb{Z}_{>0}$ then there exists $u_n \in R$ such that $\varphi(u_n) = \epsilon_n w$. Since $R \subseteq S$ then the sequence $\{\epsilon_n w\}_{n\in\mathbb{Z}_{>0}}$ is a sequence in $\varphi(S)$ which converges in the norm topology to w. By Theorem 1.7.6, $\varphi(S)$ is closed in the norm topology on $B(H_2)$ and subsequently, $w \in \varphi(S)$. Hence, $\varphi(S)$ is the closed unit ball of $\varphi(A)$.

Finally, we show that $\varphi(A)$ is a von Neumann algebra. By Theorem 2.5.7, it suffices to show that $\varphi(A)$ is weakly closed. Obviously, $\varphi(A) \subseteq \overline{\varphi(A)}^{WOT}$. Conversely, let $p \in \overline{\varphi(A)}^{WOT} - \{0\}$. Then, $p/\|p\|$ is in the closed unit ball of $\overline{\varphi(A)}^{WOT}$. Let $(\overline{\varphi(A)}^{WOT})_1$ denote the closed unit ball of $\overline{\varphi(A)}^{WOT}$. Since φ is a *-homomorphism then $\varphi(A)$ is a C*-subalgebra of $B(H_2)$. By the Kaplansky Density theorem, we have

$$\begin{aligned}
\frac{p}{\|p\|} &\in (\overline{\varphi(A)}^{WOT})_1 \\
&= (\overline{\varphi(A)}^{SOT})_1 \quad \text{(by Theorem 2.5.11 because } \varphi(A) \text{ is convex)} \\
&= \overline{(\varphi(A))_1}^{SOT} \quad \text{(by Theorem 2.6.4 with } \varphi(A) \text{ and } \overline{\varphi(A)}^{SOT}) \\
&= \overline{(\varphi(A))_1}^{WOT} \quad \text{(by Theorem 2.5.11 because } (\varphi(A))_1 \text{ is convex)} \\
&= \overline{\varphi(S)}^{WOT} = \varphi(S).
\end{aligned}$$

The last line follows from the fact that $\varphi(S)$ is weakly compact. Since $\varphi(S)$ is the closed unit ball of $\varphi(A)$, $p \in \varphi(A)$. Therefore, $\varphi(A) = \overline{\varphi(A)}^{WOT}$ and $\varphi(A)$ is a von Neumann algebra. This completes the proof. $\square$

2.7 Enveloping C*-algebras

In the next few sections, we follow the exposition in [Mur90, Chapter 6]. We are particularly interested in the construction of various C*-algebras. In this section, we begin with enveloping C*-algebras.

Definition 2.7.1. Let A be a *-algebra. A **C*-seminorm** on A is a seminorm $p : A \to \mathbb{R}_{\geq 0}$ such that if $a, b \in A$ then

$$p(ab) \leq p(a)p(b), \qquad p(a^*) = p(a) \qquad \text{and} \qquad p(a^*a) = p(a)^2.$$

If in addition p is a norm then p is called a **C*-norm**.

Example 2.7.1. Assume that A is a *-algebra and B is a C*-algebra. Let $\varphi : A \to B$ be a *-homomorphism. Define the map

$$\begin{array}{rccc} p: & A & \to & \mathbb{R}_{\geq 0} \\ & a & \mapsto & \|\varphi(a)\|. \end{array}$$

We claim that p defines a C*-seminorm on A. Since $\|-\|$ is a norm on B, it is straightforward to check that if $\lambda \in \mathbb{C}$ and $a, b \in A$ then

$$p(\lambda a) = |\lambda| p(a) \qquad \text{and} \qquad p(a+b) \leq p(a) + p(b).$$

Now since B is a C*-algebra and φ is a *-homomorphism by assumption, then

$$p(ab) \leq p(a)p(b), \qquad p(a^*a) = p(a)^2 \qquad \text{and} \qquad p(a^*) = p(a).$$

So, p is a C*-seminorm on A.

Now assume that φ is injective. To see that p is a C*-norm, it suffices to show that if $p(a) = 0$ then $a = 0$. If $p(a) = 0$ then $\|\varphi(a)\| = 0$. So, $\varphi(a) = 0$ in B and since φ is injective, $a = 0$ in A. Therefore, if φ is injective then p is a C*-norm.

The concept of a C*-seminorm plays an important role in the construction of the enveloping C*-algebra. The first step to this construction is given by the following theorem.

Theorem 2.7.1. *Let A be a *-algebra and p be a C*-seminorm on A. Let $N = p^{-1}(\{0\})$. Then, N is a self-adjoint ideal of A and we can consider the quotient *-algebra A/N. The map*

$$\begin{array}{rccc} \|-\|: & A/N & \to & \mathbb{R}_{\geq 0} \\ & a+N & \mapsto & p(a) \end{array}$$

defines a C-norm on A/N.*

Proof. Assume that A is a *-algebra and p is a C*-seminorm on A. Assume that N is the preimage

$$p^{-1}(\{0\}) = \{a \in A \mid p(a) = 0\}.$$

To show: (a) $p^{-1}(\{0\})$ is a self-adjoint ideal.

(a) Assume that $a, b \in N$. Then, $p(a+b) \leq p(a) + p(b) = 0$. Hence, $a + b \in N$. Now assume that $c \in A$. Since p is a C*-seminorm, $p(ca) \leq p(c)p(a) = 0$. Hence, $ca \in N$. Finally, we also have $p(a^*) = p(a) = 0$. So, $a^* \in N$ and consequently, N is a self-adjoint ideal of A.

Now consider the quotient *-algebra A/N and assume that the map $\|-\| : A/N \to \mathbb{R}_{\geq 0}$ is defined as above.

To show: (b) $\|-\|$ is a C*-norm.

(b) Observe that $\|-\|$ is a C*-seminorm because p is. To see that $\|-\|$ is a C*-norm, it suffices to check that $\|a + N\| = 0$ if and only if $a + N = 0 + N$. Firstly, if $a + N = 0 + N$ then $a \in N$ and $\|a + N\| = p(a) = 0$. Conversely, if $\|a + N\| = 0$ then $p(a) = 0$ and $a \in N$. Hence, $\|-\|$ is a C*-norm on A/N. $\square$

In the scenario of Theorem 2.7.1, let B denote the Banach space completion of the normed vector space $(A/N, \|-\|)$. We claim that B is in fact, a C*-algebra.

We need to define multiplication and involution on B. Let $b \in B$. Then, there exists a sequence $\{a_n + N\}_{n \in \mathbb{Z}_{>0}}$ in A/N such that $a_n \to b$ as $n \to \infty$. We know that if $n \in \mathbb{Z}_{>0}$ then $\|a_n + N\| = \|a_n^* + N\|$. So, the sequence $\{a_n^* + N\}_{n \in \mathbb{Z}_{>0}}$ is Cauchy and hence, also converges. We define the adjoint of b to be the limit

$$b^* = \lim_{n \to \infty} (a_n^* + N).$$

Now let $b' \in B$ so that there exists a sequence $\{a_n' + N\}_{n \in \mathbb{Z}_{>0}}$ which converges to b'. The product bb' is defined by

$$bb' = \big(\lim_{n \to \infty} (a_n + N)\big)\big(\lim_{n \to \infty} (a_n' + N)\big) = \lim_{m \to \infty} \lim_{n \to \infty} (a_m a_n' + N).$$

Observe that

$$\|bb'\| = \lim_{m\to\infty}\lim_{n\to\infty}\|a_m a'_n + N\| \leq \lim_{m\to\infty}\|a_m + N\| \lim_{n\to\infty}\|a'_n + N\| = \|b\|\|b'\|.$$

We also have

$$\|b^*\| = \lim_{n\to\infty}\|a_n^* + N\| = \lim_{n\to\infty}\|a_n + N\| = \|b\|$$

and

$$\|b\|^2 = \lim_{n\to\infty}\|a_n + N\|^2 = \lim_{n\to\infty}\|a_n^* a_n + N\| = \|b^* b\|.$$

Thus, B is a C*-algebra.

Definition 2.7.2. Let A be a *-algebra and p be a C*-seminorm on A. Let $N = p^{-1}(\{0\})$ and $\|-\| : A/N \to \mathbb{R}_{\geq 0}$ be the C*-norm given by $\|a + N\| = p(a)$. The C*-algebra B constructed as above is called the **enveloping C*-algebra** of the pair (A, p).

The reason for the name "enveloping" is because if we define the map

$$\begin{array}{rccc} \iota : & A & \to & B \\ & a & \mapsto & a + N \end{array}$$

from the *-algebra A to its enveloping C*-algebra B then the image $\iota(A) = A/N$ is a dense *-subalgebra of B. The map ι is sometimes called the *canonical map* from A to B.

If p is a C*-norm to begin with then the enveloping C*-algebra B is referred to as the **C*-completion of** A. In this case, A is a dense *-subalgebra of B.

2.8 Direct limit of C*-algebras

We will use the construction of the enveloping C*-algebra to define the direct limit of a *sequence of C*-algebras*. We first begin with the definition of a direct sequence of C*-algebras.

Definition 2.8.1. Let $\{A_n\}_{n\in\mathbb{Z}_{>0}}$ be a sequence of C*-algebras and

$$\{\varphi_n : A_n \to A_{n+1}\}_{n\in\mathbb{Z}_{>0}}$$

be a sequence of *-homomorphisms. The sequence of pairs $\{(A_n, \varphi_n)\}_{n\in\mathbb{Z}_{>0}}$ is called a **direct sequence of C*-algebras**.

If $\{(A_n, \varphi_n)\}_{n\in\mathbb{Z}_{>0}}$ is a direct sequence of C*-algebras then the infinite product

$$\prod_{k=1}^{\infty} A_k = \{(a_1, a_2, \dots) \mid \text{If } i \in \mathbb{Z}_{>0} \text{ then } a_i \in A_i\}.$$

is a *-algebra with the operations of scalar multiplication, addition and multiplication defined pointwise. Now define

$$\prod_{k=1}^{\infty}{}' A_k = \left\{ (a_k)_{k\in\mathbb{Z}_{>0}} \in \prod_{k=1}^{\infty} A_k \;\middle|\; \begin{matrix} \text{There exists } N \in \mathbb{Z}_{>0} \text{ such that} \\ a_{\ell+1} = \varphi_\ell(a_\ell) \text{ for } \ell \geq N \end{matrix} \right\}.$$

Then, $\prod_{k=1}^{\infty}{}' A_k$ is a *-subalgebra of $\prod_{k=1}^{\infty} A_k$. Notice that we have stuck to talking about *-algebras for now. At the moment, it is impractical to give $\prod_{k=1}^{\infty} A_k$ a norm, since there are issues with convergence. As we will see shortly, this is remedied when we work with $\prod_{k=1}^{\infty}{}' A_k$.

Recall that if $i \in \mathbb{Z}_{>0}$ then φ_i is a *-homomorphism and is thus, contractive. This means that if $(a_k)_{k\in\mathbb{Z}_{>0}} \in \prod_{k=1}^{\infty}{}' A_k$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $i \geq N$ then $a_{i+1} = \varphi_i(a_i)$ and consequently, $\|a_{i+1}\| \leq \|a_i\|$ for $i \geq N$. Hence, the sequence $\{\|a_k\|\}_{k\in\mathbb{Z}_{>0}}$ in $\mathbb{R}$ is decreasing and bounded below. So, it must converge. Next, we define the map p by

$$\begin{array}{rccc} p: & \prod_{k=1}^{\infty}{}' A_k & \to & \mathbb{R}_{\geq 0} \\ & (a_k)_{k\in\mathbb{Z}_{>0}} & \mapsto & \lim_{k\to\infty} \|a_k\| \end{array} \tag{2.3}$$

Observe that p defines a C*-seminorm on $\prod_{k=1}^{\infty}{}' A_k$. The fact that p is a seminorm follows from the properties of a norm. The fact that p is a C*-seminorm follows from the fact that each A_i is a C*-algebra.

Now we proceed to defining the direct limit of the direct sequence $\{(A_n, \varphi_n)\}_{n\in\mathbb{Z}_{>0}}$.

Definition 2.8.2. Let $\{(A_n, \varphi_n)\}_{n\in\mathbb{Z}_{>0}}$ be a direct sequence of C*-algebras. The **direct limit** of the sequence $\{(A_n, \varphi_n)\}_{n\in\mathbb{Z}_{>0}}$ is the enveloping C*-algebra of the pair $(\prod_{k=1}^{\infty}{}' A_k, p)$, where

$$\prod_{k=1}^{\infty}{}' A_k = \left\{ (a_k)_{k\in\mathbb{Z}_{>0}} \in \prod_{k=1}^{\infty} A_k \;\middle|\; \begin{matrix} \text{There exists } N \in \mathbb{Z}_{>0} \text{ such that} \\ a_{\ell+1} = \varphi_\ell(a_\ell) \text{ for } \ell \geq N \end{matrix} \right\}.$$

and $p : \prod_{k=1}^{\infty}{}' A_k \to \mathbb{R}_{\geq 0}$ is defined by $p((a_k)_{k\in\mathbb{Z}_{>0}}) = \lim_{k\to\infty} \|a_k\|$. The direct limit is usually denoted by $\varinjlim A_k$.

Before we proceed, we will give a brief explicit description of the direct limit of the direct sequence $\{(A_n, \varphi_n)\}_{n\in\mathbb{Z}_{>0}}$ of C*-algebras. The map in equation (2.3) is a C*-seminorm. By Theorem 2.7.1, we form the quotient *-algebra

$$(\prod_{k=1}^{\infty}{}' A_k)/p^{-1}(\{0\}) = \left\{ (a_k)_{k\in\mathbb{Z}_{>0}} + p^{-1}(\{0\}) \;\middle|\; (a_k)_{k\in\mathbb{Z}_{>0}} \in \prod_{k=1}^{\infty}{}' A_k \right\}$$

which has C*-norm

$$\|(a_k)_{k\in\mathbb{Z}_{>0}} + p^{-1}(\{0\})\| = p\big((a_k)_{k\in\mathbb{Z}_{>0}}\big) = \lim_{k\to\infty} \|a_k\|.$$

The direct limit $\varinjlim A_k$ is then the Banach space completion of $(\prod_{k=1}^{\infty}{}' A_k)/p^{-1}(\{0\})$ with respect to the C*-norm $\|-\|$.

Again, we work with the direct sequence $\{(A_n, \varphi_n)\}_{n=1}^{\infty}$ of C*-algebras. Now assume that $n, m \in \mathbb{Z}_{>0}$ with $n \leq m$. We define $\varphi_{n,n} = id_{A_n}$. The *-homomorphism $\varphi_{n,m} : A_n \to A_m$ is defined as the composite

$$\varphi_{n,m} = \varphi_{m-1} \circ \varphi_{m-2} \circ \cdots \circ \varphi_n.$$

If $b \in A_n$ then we define the element $\widehat{\varphi}^n(b) \in \prod_{k=1}^{\infty}{}' A_k$ to be the sequence

$$\widehat{\varphi}^n(b) = (0, \ldots, 0, b, \varphi_n(b), \varphi_{n,n+2}(b), \varphi_{n,n+3}(b), \ldots).$$

Note that in the above definition, the first $n-1$ entries of the sequence are zeros. Now let $\iota : \prod_{k=1}^{\infty}{}' A_k \to \varinjlim A_k$ be the canonical map. If $n \in \mathbb{Z}_{>0}$ then define the map

$$\begin{array}{rccc} \varphi^n : & A_n & \to & \varinjlim A_k \\ & a & \mapsto & \iota(\widehat{\varphi}^n(a)) \end{array}$$

Since ι and $\widehat{\varphi}^n$ are both *-homomorphisms, then φ^n is a *-homomorphism for $n \in \mathbb{Z}_{>0}$. We also observe that if $b \in A_n$ then

$$\begin{aligned} \varphi^{n+1}(\varphi_n(b)) &= \iota(\widehat{\varphi}^{n+1}(\varphi_n(b))) \\ &= \iota((0, \ldots, 0, \varphi_n(b), \varphi_{n,n+2}(b), \varphi_{n,n+3}(b), \ldots)) \\ &= \iota((0, \ldots, 0, b, \varphi_n(b), \varphi_{n,n+2}(b), \varphi_{n,n+3}(b), \ldots)) \\ &= \iota(\widehat{\varphi}^n(b)) = \varphi^n(b). \end{aligned}$$

Let us justify the third equality. The difference of the two sequences is

$$\begin{gathered} (0, \ldots, 0, \varphi_n(b), \varphi_{n,n+2}(b), \ldots) - (0, \ldots, 0, b, \varphi_n(b), \varphi_{n,n+2}(b), \ldots) \\ = (0, \ldots, 0, -b, 0, 0, \ldots). \end{gathered}$$

Note that the $-b$ appears in the n^{th} position. The norm of its equivalence class in $\varinjlim A_n$ is 0. Hence, $\varphi^{n+1}(\varphi_n(b)) = \varphi^n(b)$ in $\varinjlim A_n$. The map φ^n is called the **natural map** from A_n to $\varinjlim A_n$.

Theorem 2.8.1. *Let $\{(A_n, \varphi_n)\}_{n\in\mathbb{Z}_{>0}}$ be a direct sequence of C*-algebras and $A = \varinjlim A_n$ be its direct limit. If $n \in \mathbb{Z}_{>0}$ then let $\varphi^n : A_n \to A$ denote the natural map from A_n to A.*

1. *The sequence of C*-algebras $\{\varphi^n(A_n)\}_{n\in\mathbb{Z}_{>0}}$ is increasing.*
2. *The union $\bigcup_{n\in\mathbb{Z}_{>0}} \varphi^n(A_n)$ is a dense *-subalgebra of A.*
3. *If $a \in A_n$ then $\|\varphi^n(a)\| = \lim_{k\to\infty}\|\varphi_{n,n+k}(a)\|$.*

Proof. Assume that $n \in \mathbb{Z}_{>0}$. We already showed previously that $\varphi^{n+1} \circ \varphi_n = \varphi^n$. So,

$$\varphi^n(A_n) = (\varphi^{n+1})(\varphi_n(A_n)) \subseteq \varphi^{n+1}(A_{n+1}).$$

Thus, the sequence $\{\varphi^n(A_n)\}_{n\in\mathbb{Z}_{>0}}$ is increasing with respect to inclusion. Also note that $\varphi^n(A_n)$ is a C*-algebra by Theorem 1.7.6.

To see that the union $\bigcup_{n\in\mathbb{Z}_{>0}} \varphi^n(A_n)$ is a dense *-subalgebra of A, assume that $(a_1, a_2, \dots) \in A$. Consider the sequence

$$\{\varphi^n(a_n)\}_{n\in\mathbb{Z}_{>0}}$$

in the union $\bigcup_{n\in\mathbb{Z}_{>0}} \varphi^n(A_n)$. Note that if $n \in \mathbb{Z}_{>0}$ then

$$\varphi^n(a_n) = \iota(\widehat{\varphi}^n(a_n)) = (a_1, \dots, a_n, \varphi_n(a_n), \varphi_{n,n+2}(a_n), \dots)$$

in A (by considering equivalence classes). Therefore,

$$\lim_{n\to\infty} \|(a_1, a_2, \dots) - \varphi^n(a_n)\| = 0$$

and the sequence $\{\varphi^n(a_n)\}_{n\in\mathbb{Z}_{>0}}$ converges to $(a_1, a_2, \dots)$. Thus, the union $\bigcup_{n\in\mathbb{Z}_{>0}} \varphi^n(A_n)$ is a dense C*-subalgebra of A.

Finally by the definition of the natural map φ^n,

$$\|\varphi^n(a)\| = \|\iota((0, \dots, 0, a, \varphi_n(a), \varphi_{n,n+2}(a), \dots))\| = \lim_{k\to\infty}\|\varphi_{n,n+k}(a)\|.$$

□

The direct limit of a direct sequence of C*-algebras satisfies the following universal property given in [Mur90, Theorem 6.1.2].

Theorem 2.8.2. *Let $\{(A_n, \varphi_n)\}_{n\in\mathbb{Z}_{>0}}$ be a direct sequence of C*-algebras and $A = \varinjlim A_n$ denote its direct limit. For $n \in \mathbb{Z}_{>0}$, let $\varphi^n : A_n \to A$ be the natural map.*

1. *Assume that $\epsilon \in \mathbb{R}_{>0}$, $a \in A_n$ and $b \in A_m$ satisfy $\varphi^n(a) = \varphi^m(b)$. Then, there exists $k \in \mathbb{Z}_{>0}$ such that $k \geq \max(m, n)$ and $\|\varphi_{n,k}(a) - \varphi_{m,k}(b)\| < \epsilon$.*
2. *If B is a C*-algebra and there exists a *-homomorphism $\psi^n : A_n \to B$ such that the following diagram commutes for each $n \in \mathbb{Z}_{>0}$*

$$\begin{tikzcd} A_n \arrow[r, "\varphi_n"] \arrow[dr, "\psi^n"'] & A_{n+1} \arrow[d, "\psi^{n+1}"] \\ & B \end{tikzcd}$$

*then there exists a unique *-homomorphism $\psi : A \to B$ such that if $n \in \mathbb{Z}_{>0}$ then the following diagram commutes:*

$$\begin{tikzcd} A_n \arrow[r, "\varphi^n"] \arrow[dr, "\psi^n"'] & A \arrow[d, dashed, "\psi"] \\ & B \end{tikzcd}$$

Proof. Assume that $\{(A_n, \varphi_n)\}_{n\in\mathbb{Z}_{>0}}$ is a direct sequence of C*-algebras and $A = \varinjlim A_n$ is its direct limit.

Assume that $\epsilon \in \mathbb{R}_{>0}$, $a \in A_n$, $b \in A_m$ and $\varphi^n(a) = \varphi^m(b)$. By Theorem 2.8.1,

$$\|\varphi^n(a)\| = \|\varphi^m(b)\| = \lim_{k\to\infty} \|\varphi_{n,n+k}(a)\|.$$

Thus, there exists $k \geq \max(m, n)$ such that

$$\|\varphi_{n,k}(a) - \varphi_{m,k}(b)\| < \epsilon.$$

Now assume that B is a C*-algebra and $\{\psi^n : A_n \to B\}_{n\in\mathbb{Z}_{>0}}$ is a sequence of *-homomorphisms satisfying $\psi^n = \psi^{n+1} \circ \varphi_n$. The idea is to first define our required map on the C*-subalgebra $\bigcup_{n\in\mathbb{Z}_{>0}} \varphi^n(A_n)$. Again, assume that

$a \in A_n$ and $b \in A_m$ satisfy $\varphi^n(a) = \varphi^m(b)$.

To show: (a) $\psi^n(a) = \psi^m(a)$.

(a) By the first part of this theorem, there exists $k \geq \max(m, n)$ such that

$$\|\varphi_{n,k}(a) - \varphi_{m,k}(b)\| < \epsilon.$$

So,

$$\begin{aligned} \|\psi^n(a) - \psi^m(b)\| &= \|\psi^k(\varphi_{n,k}(a)) - \psi^k(\varphi_{m,k}(b))\| \\ &\leq \|\psi^k\| \|\varphi_{n,k}(a) - \varphi_{m,k}(b)\| \\ &< \epsilon. \end{aligned}$$

Since $\epsilon \in \mathbb{R}_{>0}$ was arbitrary, we find that $\psi^n(a) = \psi^n(b)$ as required.

Now we define the map ψ by

$$\begin{array}{cccc} \psi: & \bigcup_{n \in \mathbb{Z}_{>0}} \varphi^n(A_n) & \to & B \\ & \varphi^n(a) & \mapsto & \psi^n(a). \end{array}$$

Part (a) of the proof demonstrates that ψ is well-defined. To see that ψ is a *-homomorphism, assume that $a \in A_n$, $b \in A_m$ and $\lambda \in \mathbb{C}$. Assume without loss of generality that $n \geq m$. Then,

$$\psi(\varphi^n(a)^*) = \psi(\varphi^n(a^*)) = \psi^n(a^*) = \psi^n(a)^* = \psi(\varphi^n(a))^*,$$

$$\psi(\lambda\varphi^n(a)) = \psi(\varphi^n(\lambda a)) = \psi^n(\lambda a) = \lambda\psi^n(a) = \lambda\psi(\varphi^n(a)),$$

$$\begin{aligned} \psi(\varphi^n(a) + \varphi^m(b)) &= \psi(\varphi^n(a) + \varphi^n(\varphi_{m,n}(b))) \\ &= \psi(\varphi^n(a + \varphi_{m,n}(b))) \\ &= \psi^n(a + \varphi_{m,n}(b)) \\ &= \psi^n(a) + \psi^n(\varphi_{m,n}(b)) \\ &= \psi(\varphi^n(a)) + \psi^m(b) = \psi(\varphi^n(a)) + \psi(\varphi^m(b)) \end{aligned}$$

and

$$\begin{aligned}
\psi(\varphi^n(a)\varphi^m(b)) &= \psi(\varphi^n(a)\varphi^n(\varphi_{m,n}(b))) \\
&= \psi(\varphi^n(a\varphi_{m,n}(b))) \\
&= \psi^n(a\varphi_{m,n}(b)) \\
&= \psi^n(a)\psi^n(\varphi_{m,n}(b)) \\
&= \psi(\varphi^n(a))\psi^m(b) = \psi(\varphi^n(a))\psi(\varphi^m(b)).
\end{aligned}$$

So, ψ is a *-homomorphism. By Theorem 2.8.1, $\bigcup_{n\in\mathbb{Z}_{>0}} \varphi^n(A_n)$ is dense in the direct limit A. Therefore, ψ extends to a *-homomorphism from A to B which satisfies $\psi \circ \varphi^n = \psi^n$ for $n \in \mathbb{Z}_{>0}$.

To see that ψ is unique, suppose that $\phi : A \to B$ is another *-homomorphism such that if $n \in \mathbb{Z}_{>0}$ then $\phi \circ \varphi^n = \psi^n$. Then, $\phi = \psi$ on $\bigcup_{n\in\mathbb{Z}_{>0}} \varphi^n(A_n)$, which is dense in A. Therefore, $\phi = \psi$. □

We finish this section with a consequence of Theorem 2.8.2.

Theorem 2.8.3. *Let A be a C*-algebra and let $\{A_n\}_{n\in\mathbb{Z}_{>0}}$ be an increasing sequence of C*-subalgebras of A. Assume that*

$$\overline{\bigcup_{n\in\mathbb{Z}_{>0}} A_n} = A.$$

Let $\iota_n : A_n \to A_{n+1}$ denote the inclusion map. Then, $A \cong \varinjlim A_n$ as C-algebras, where $\varinjlim A_n$ is the direct limit of the direct sequence $\{(A_n, \iota_n)\}_{n\in\mathbb{Z}_{>0}}$.*

Proof. Assume that A is a C*-algebra and $\{A_n\}_{n\in\mathbb{Z}_{>0}}$ is an increasing sequence of C*-subalgebras of A. Assume that

$$\overline{\bigcup_{n\in\mathbb{Z}_{>0}} A_n} = A.$$

Assume that $\varinjlim A_n$ and ι_n are defined as above. Let $i^n : A_n \to A$ denote the inclusion map. If $n \in \mathbb{Z}_{>0}$ then $i^n = i^{n+1} \circ \iota_n$. By the universal property of the direct limit in Theorem 2.8.2, there exists a unique *-homomorphism $\phi : \varinjlim A_n \to A$ such that the following diagram commutes for $n \in \mathbb{Z}_{>0}$:

$$\begin{tikzcd}
A_n \arrow[r, "\iota^n"] \arrow[dr, "i^n"'] & \varinjlim A_n \arrow[d, dashed, "\phi"] \\
& A
\end{tikzcd}$$

Here, $\iota^n : A_n \to \varinjlim A_n$ is the natural map. To see that ϕ is surjective, assume that $a \in A$. Since $\bigcup_{n\in\mathbb{Z}_{>0}} A_n$ is dense in A, there exists a sequence $\{a_n\}_{n\in\mathbb{Z}_{>0}}$ such that $a_n \in A_n$ and $\lim_{n\to\infty}\|a_n - a\| = 0$. Since the sequence $\{\|a_n\|\}_{n\in\mathbb{Z}_{>0}}$ in $\mathbb{R}$ converges, the sequence $\{\|\iota^n(a_n)\|\}_{n\in\mathbb{Z}_{>0}}$ must also converge. Hence, the sequence $\{\iota^n(a_n)\}_{n\in\mathbb{Z}_{>0}}$ in $\varinjlim A_n$ converges to some $\tilde{a} \in \varinjlim A_n$. Thus,

$$\phi(\tilde{a}) = \phi(\lim_{n\to\infty} \iota^n(a_n)) = \lim_{n\to\infty} \phi(\iota^n(a_n)) = \lim_{n\to\infty} i^n(a_n) = i^n(a) = a.$$

Therefore, ϕ is surjective. To see that ϕ is injective, assume that $k \in \varinjlim A_n$ satisfies $\phi(k) = 0$. Let $k = (k_1, k_2, \dots)$ and assume that $\epsilon \in \mathbb{R}_{>0}$. By construction of the direct limit $\varinjlim A_n$, there exists

$$k' \in ({\prod_{i=1}^{\infty}}' A_i)/p^{-1}(\{0\})$$

such that $\|k - k'\| < \epsilon$. Recall that p is the C*-seminorm in equation (2.3). By Definition 2.8.2, if $k' = (k'_1, k'_2, \dots)$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $n, m \geq N$ then $k'_m = k'_n$.

Now consider $k'_N \in A_N$. In A we have

$$k'_N = i^N(k'_N) = \phi(\iota^N(k'_N)) = \phi(k) = 0.$$

Hence, $k' = 0$ in the direct limit $\varinjlim A_n$ and since $\epsilon \in \mathbb{R}_{>0}$ was arbitrary, $k = 0$. So, ϕ is injective and subsequently, a *-isomorphism from $\varinjlim A_n$ to A. $\square$

2.9 The multiplier algebra

In section 1.6, we studied the unitization of a C*-algebra. To summarise, unitization is a method of constructing a unital C*-algebra from an arbitrary C*-algebra. The original C*-algebra is then an ideal of the unitization. In this section, we will briefly study another construction of a unital C*-algebra — the **multiplier algebra** of a C*-algebra.

The multiplier algebra of a C*-algebra satisfies a universal property, akin to the unitization in Theorem 1.6.3. Roughly speaking, the universal property satisfied by the multiplier algebra states that the multiplier algebra of a C*-algebra A is the largest unital C*-algebra which contains A.

There are multiple ways to construct the multiplier algebra. We will follow [Mur90, Section 2.1] and construct the multiplier algebra through the use of *double centralisers*.

Definition 2.9.1. Let A be a C*-algebra. A **double centraliser** is a pair (L, R) of bounded linear maps on A such that if $a, b \in A$ then

$$L(ab) = L(a)b, \qquad R(ab) = aR(b) \qquad \text{and} \qquad R(a)b = aL(b).$$

The set of double centralisers of A is denoted by $M(A)$.

Here is a basic example of a double centraliser.

Example 2.9.1. Let A be a C*-algebra and $c \in A$. Define the linear maps L_c and R_c on A by $L_c(a) = ca$ and $R_c(a) = ac$. To see that L_c and R_c are bounded, observe that $\|L_c\| \leq \|c\|$ and $\|R_c\| \leq \|c\|$. We also have

$$\|c\| = \|c\frac{c^*}{\|c\|}\| \leq \sup_{\|a\|=1} \|L_c(a)\| = \|L_c\|.$$

Similarly, $\|R_c\| = \|c\|$. By direct computation, the pair (L_c, R_c) is a double centraliser of A. So, $(L_c, R_c) \in M(A)$.

Our goal is to show that the set $M(A)$ becomes a C*-algebra, when equipped with the necessary operations and norm. We will first work on defining a viable norm on $M(A)$.

Theorem 2.9.1. *Let A be a C*-algebra and (L, R) be a double centraliser on A. Then, $\|L\| = \|R\|$.*

Proof. Assume that A is a C*-algebra. Assume that (L, R) is a double centraliser on A. If $b \in A$ then

$$\begin{aligned}
\|L(b)\| &= \sup_{\|a\|=1} \|aL(b)\| \\
&= \sup_{\|a\|=1} \|R(a)b\| \\
&\leq \sup_{\|a\|=1} \|R\|\|a\|\|b\| \\
&= \|R\|\|b\|.
\end{aligned}$$

Taking the supremum over all $b \in A$ with $\|b\| = 1$ yields the inequality $\|L\| \leq \|R\|$. Similarly,

$$\begin{aligned}\|R(b)\| &= \sup_{\|a\|=1} \|R(b)a\| \\ &= \sup_{\|a\|=1} \|aL(b)\| \\ &\leq \sup_{\|a\|=1} \|a\|\|L\|\|b\| \\ &= \|L\|\|b\|.\end{aligned}$$

Taking the supremum over all $b \in A$ with $\|b\| = 1$ yields the inequality $\|R\| \leq \|L\|$. So, $\|L\| = \|R\|$. □

In light of Theorem 2.9.1, if $(L, R) \in M(A)$ then we define the norm of (L, R) as

$$\|(L, R)\| = \|L\| = \|R\|.$$

Now we proceed to defining the operations on the set $M(A)$.

Addition: If $(L_1, R_1), (L_2, R_2) \in M(A)$ then define

$$(L_1, R_1) + (L_2, R_2) = (L_1 + L_2, R_1 + R_2).$$

If $a, b \in A$ then

$$(L_1 + L_2)(ab) = L_1(ab) + L_2(ab) = L_1(a)b + L_2(a)b = (L_1 + L_2)(a)b,$$

$$(R_1 + R_2)(ab) = R_1(ab) + R_2(ab) = aR_1(b) + aR_2(b) = a(R_1 + R_2)(b)$$

and

$$(R_1 + R_2)(a)b = R_1(a)b + R_2(a)b = aL_1(b) + aL_2(b) = a(L_1 + L_2)(b).$$

So, $(L_1, R_1) + (L_2, R_2) \in M(A)$.

Scalar multiplication: If $\lambda \in \mathbb{C}$ then we define

$$\lambda(L_1, R_1) = (\lambda L_1, \lambda R_1).$$

If $a, b \in A$ then

$$(\lambda L_1)(ab) = \lambda L_1(ab) = \lambda L_1(a)b = (\lambda L_1)(a)b,$$

$$(\lambda R_1)(ab) = \lambda R_1(ab) = \lambda a R_1(b) = a(\lambda R_1)(b)$$

and

$$(\lambda R_1)(a)b = \lambda R_1(a)b = \lambda a L_1(b) = a(\lambda L_1)(b).$$

So, $\lambda(L_1, R_1) \in M(A)$.

Multiplication: We define

$$(L_1, R_1) \cdot (L_2, R_2) = (L_1 \circ L_2, R_2 \circ R_1).$$

If $a, b \in A$ then

$$(L_1 \circ L_2)(ab) = L_1(L_2(ab)) = L_1(L_2(a)b) = (L_1 \circ L_2)(a)b,$$

$$(R_2 \circ R_1)(ab) = R_2(R_1(ab)) = R_2(aR_1(b)) = a(R_2 \circ R_1)(b),$$

and

$$(R_2 \circ R_1)(a)b = R_2(R_1(a))b = R_1(a)L_2(b) = a(L_1 \circ L_2)(b).$$

Hence, $(L_1 \circ L_2, R_2 \circ R_1) \in M(A)$.

Involution: Firstly, if $L : A \to A$ is a bounded linear operator then we define

$$\begin{array}{rccc} L^* : & A & \to & A \\ & a & \mapsto & (L(a^*))^* \end{array}$$

Then, L^* is also a bounded linear map on A which satisfies for $a \in A$

$$L^{**}(a) = (L^*(a^*))^* = (L(a))^{**} = L(a)$$

and

$$\begin{aligned} (L_1 \circ L_2)^*(a) &= ((L_1 \circ L_2)(a^*))^* \\ &= L_1(L_2(a^*))^* \\ &= L_1^*(L_2(a^*)^*) \\ &= (L_1^* \circ L_2^*)(a). \end{aligned}$$

In fact, it is easy to check that the map $L \mapsto L^*$ is isometric. If $(L, R) \in M(A)$ then we define

$$(L,R)^* = (R^*, L^*).$$

If $a, b \in A$ then

$$R^*(ab) = (R(b^*a^*))^* = (b^*R(a^*))^* = R^*(a)b,$$

$$L^*(ab) = (L(b^*a^*))^* = (L(b^*)a^*)^* = aL^*(b)$$

and

$$L^*(a)b = (L(a^*))^*b = (b^*L(a^*))^* = (R(b^*)a^*)^* = aR^*(b).$$

Therefore, $(R^*, L^*) \in M(A)$. The computations which check that the map $(L,R) \mapsto (L,R)^*$ satisfies the properties of an involution is suppressed here.

Theorem 2.9.2. *Let A be a C*-algebra. Let $M(A)$ denote the set of double centralisers on A. With the operations, norm and involution defined as above, $M(A)$ is a C*-algebra.*

Proof. Assume that A is a C*-algebra. Assume that $M(A)$ is the set of double centralisers on A. Let $B(A)$ denote the space of bounded linear operators on A. Since A is complete, $B(A)$ is a Banach space when equipped with the operator norm.

To show: (a) $M(A)$ is a closed vector subspace of $B(A) \oplus B(A)$.

(b) $M(A)$ is a Banach *-algebra.

(c) If $T = (L,R) \in M(A)$ then $\|T^*T\| = \|T\|^2$.

(a) The direct sum $B(A) \oplus B(A)$ is a Banach space when equipped with the norm

$$\|(S,T)\| = \max(\|S\|, \|T\|).$$

We already know that $M(A)$ is a vector subspace of $B(A) \oplus B(A)$. To see that $M(A)$ is closed, assume that $(L,R) \in \overline{M(A)}$ so that there exists a sequence of double centralisers $\{(L_n, R_n)\}_{n \in \mathbb{Z}_{>0}}$ which converges to (L,R). Then, $L_n \to L$ and $R_n \to R$ as $n \to \infty$. If $a, b \in A$ then

$$L(ab) = \lim_{n\to\infty} L_n(ab) = \lim_{n\to\infty} L_n(a)b = L(a)b,$$

$$R(ab) = \lim_{n\to\infty} R_n(ab) = \lim_{n\to\infty} aR_n(b) = aR(b),$$

and

$$R(a)b = \lim_{n\to\infty} R_n(a)b = \lim_{n\to\infty} aL_n(b) = aL(b).$$

Therefore, $(L, R) \in M(A)$ and $M(A)$ is a closed vector subspace of $B(A) \oplus B(A)$. This means that $M(A)$ is itself a Banach space.

(b) To see that $M(A)$ is a Banach *-algebra, assume that $(L_1, R_1), (L_2, R_2) \in M(A)$. Then, $(L_1, R_1) \cdot (L_2, R_2) = (L_1 \circ L_2, R_2 \circ R_1)$ and

$$\|(L_1 \circ L_2, R_2 \circ R_1)\| = \|L_1 \circ L_2\| \le \|L_1\|\|L_2\| = \|(L_1, R_1)\|\|(L_2, R_2)\|.$$

Hence, $M(A)$ is a Banach *-algebra.

(c) Assume that $T = (L, R) \in M(A)$. If $a \in A$ satisfies $\|a\| = 1$ then

$$\|L(a)\|^2 = \|L(a)^*L(a)\| = \|L^*(a^*)L(a)\| = \|a^*R^*(L(a))\| \le \|R^*L\| = \|T^*T\|.$$

Taking the supremum over all such a, we find that $\|T\|^2 \le \|T^*T\|$. We also have $\|T^*T\| \le \|T^*\|\|T\| = \|T\|^2$. Therefore, $\|T^*T\| = \|T\|^2$ and consequently, $M(A)$ is a C*-algebra as required. $\square$

If A is a C*-algebra then the C*-algebra $M(A)$ in Theorem 2.9.2 is called the **multiplier algebra**. It is unital because if $id_A : A \to A$ is the identity map on A then (id_A, id_A) is the multiplicative unit of $M(A)$.

Now define the map

$$\begin{array}{rccc} \iota : & A & \to & M(A) \\ & a & \mapsto & (L_a, R_a) \end{array}$$

where the maps L_a and R_a are defined by $L_a(b) = ab$ and $R_a(b) = ba$. By the previous example, we find that ι is a isometric *-homomorphism. So, we are able to identify A as a C*-subalgebra of $M(A)$. Additionally, we can also identify A as an ideal of $M(A)$.

Before we state the universal property of the multiplier algebra, we require the following definition.

Definition 2.9.2. Let A be a C*-algebra and I be a closed ideal of A. We say that I is **essential** in A if the following statement is satisfied: If $a \in A$ and $aI = 0$ then $a = 0$.

We claim that if A is a C*-algebra then A is an essential ideal in its multiplier algebra $M(A)$. Assume that $(L, R) \in M(A)$ satisfies $(L, R)A = 0$. Let $(L_a, R_a) \in A$ for $a \in A$. Then,

$$(L, R)(L_a, R_a) = (L \circ L_a, R_a \circ R) = (0, 0).$$

Here, 0 denotes the zero map on A. If $b \in A$ then $(L \circ L_a)(b) = L(ab) = 0$ and $(R_a \circ R)(b) = R(b)a = 0$. Since a was arbitrary, $R(b) = 0$. But, we also have $R(b)a = bL(a) = 0$. So, $L(a) = 0$ and consequently, $(L, R) = (0, 0)$. Therefore, A is an essential ideal of $M(A)$.

Theorem 2.9.3. *Let A be a C*-algebra and I be a closed ideal in A. Let $\iota_I : I \to M(I)$ denote the inclusion map. Then, there exists a unique *-homomorphism $\varphi : A \to M(I)$ extending ι_I. Furthermore, if I is an essential ideal of A then φ is injective.*

Proof. Assume that A is a C*-algebra. Assume that I is a closed ideal of A. Assume that ι_I is the inclusion of I into its multiplier algebra $M(I)$. Define the map φ by

$$\begin{array}{rccc} \varphi : & A & \to & M(I) \\ & a & \mapsto & (L_a|_I, R_a|_I) \end{array}$$

Here, $L_a|_I$ is the restriction of the left multiplication map $L_a : A \to A$ to the closed ideal I. Similarly, $R_a|_I$ is the restriction of the right multiplication map $R_a : A \to A$ to I. It is easy to check that φ is a *-homomorphism which extends ι_I.

To prove uniqueness, assume that there exists another *-homomorphism $\psi : A \to M(I)$ which extends ι_I. If $b \in I$ and $a \in A$ then

$$\varphi(a)b = \varphi(ab) = ab = \psi(ab) = \psi(a)b.$$

This shows that if $a \in A$ then $(\varphi(a) - \psi(a))I = 0$. Since I is an essential ideal of $M(I)$, then $\varphi(a) = \psi(a)$. So, $\varphi = \psi$.

Finally, assume that I is an essential ideal of A. Assume that $a \in \ker \varphi$. Then, $L_a|_I$ is the zero map on I. This means that $aI = 0$ and since I is essential in A, $a = 0$. Hence, φ is injective. $\square$

Every closed ideal of a C*-algebra is itself a C*-algebra. Hence, Theorem 2.9.3 states that $M(I)$ is the largest unital C*-algebra containing I as an essential ideal.

Theorem 2.9.4. *Let A be a C*-algebra. If A is unital then $M(A) = A$.*

Proof. Assume that A is a unital C*-algebra. We know that $M(A)$ contains A as an essential ideal. To see that $M(A) \subseteq A$, notice that if $a \in A$ then

$$(1_A - 1_{M(A)})a = a - 1_{M(A)}a = 0.$$

So, $(1_A - 1_{M(A)})A = 0$ and since A is an essential ideal of $M(A)$, $1_A - 1_{M(A)}$. Hence, $1_A = 1_{M(A)}$ and $M(A) \subseteq A$. Consequently, $M(A) = A$. □

2.10 Uniformly hyperfinite algebras

In this section, we follow [Mur90, Section 6.2]. We will first set up the relevant theory before defining uniformly hyperfinite algebras. Our first task is to characterise finite-dimensional simple C*-algebras.

Definition 2.10.1. Let A be a C*-algebra. We say that A is **liminal** if for every non-zero irreducible representation (π, H), the image $\pi(A) = B_0(H)$, where $B_0(H) \subseteq B(H)$ is the space of compact operators on H.

We recall that if H is a Hilbert space and $x : H \to H$ is a bounded operator on H then x is **compact** if there exists a net of $\{x_n\}_{n \in I}$ of finite dimensional (or finite rank) operators such that

$$\lim_{n \in I} \|x_n - x\| = 0.$$

Theorem 2.10.1. *Let A be a C*-algebra. If A is finite dimensional then A is liminal.*

Proof. Assume that A is a finite dimensional C*-algebra. Let (π, H) be a non-zero, non-degenerate irreducible representation of A. If $\xi \in H - \{0\}$ then ξ is a cyclic vector (see Theorem 1.9.6) and

$$\overline{\pi(A)\xi} = H.$$

Since A is finite dimensional, the subspace $\pi(A)\xi$ is also finite dimensional and hence, closed. So, $\pi(A)\xi = H$ and H must be finite dimensional. Hence, $\pi(A) \subseteq B_0(H) = B(H)$ and by [Mur90, Theorem 2.4.9], $B_0(H) \subseteq \pi(A)$. So, $\pi(A) = B_0(H)$ and A is liminal. □

[Mur90, Theorem 2.4.9] states that if (π, H) is an irreducible representation of a C*-algebra A and $\pi(A) \cap B_0(H) \neq \emptyset$ then $B_0(H) \subseteq \pi(A)$. Now we will use Theorem 2.10.1 to characterise finite dimensional simple C*-algebras.

Theorem 2.10.2. *A non-zero finite dimensional C*-algebra is simple if and only if there exists $n \in \mathbb{Z}_{>0}$ such that A is isomorphic to the matrix algebra $M_{n\times n}(\mathbb{C})$.*

Proof. We already know that if $n \in \mathbb{Z}_{>0}$ then the matrix C*-algebra $M_{n\times n}(\mathbb{C})$ is simple by Theorem 1.10.3. So, assume that A is a non-zero finite dimensional simple C*-algebra.

By Theorem 2.10.1, A is a liminal C*-algebra. That is, if (π, H) is a non-zero irreducible representation of A then $\pi(A) = B_0(H)$ and subsequently, H is finite dimensional (because A is finite dimensional). Moreover, the kernel $\ker \pi$ is a closed ideal of A. Since A is simple, $\ker \pi = 0$.

Thus, if $n = \dim H$ then A is isomorphic as a C*-algebra to $K(H) = B(H)$ (as H is finite dimensional). In turn, $B(H)$ is isomorphic to $M_{n\times n}(\mathbb{C})$ as required. □

Recall the structure theorem for unital finite dimensional C*-algebras in Theorem 1.5.1. We can now remove the assumption that A is unital.

Theorem 2.10.3. *Let A be a non-zero finite dimensional C*-algebra. Then, there exist $k, n_1, \ldots, n_k \in \mathbb{Z}_{>0}$ such that we have the isomorphism of C*-algebras*

$$A \cong \bigoplus_{i=1}^{k} M_{n_i\times n_i}(\mathbb{C}).$$

Proof. Assume that A is a non-zero finite dimensional C*-algebra. The proof divides into two cases:

Case 1: A is simple.

If A is simple then by Theorem 2.10.2, A is isomorphic to a matrix algebra.

Case 2: A is not simple.

Assume that A is not a simple C*-algebra. We will prove the statement by induction on the dimension of A. For the base case, assume that $\dim A = 1$.

Then, $A \cong \mathbb{C} \cong M_{1\times 1}(\mathbb{C})$. This proves the base case.

For the inductive hypothesis, assume that there exists $m \in \mathbb{Z}_{>0}$ such that if $i \in \{1, 2, \ldots, m\}$ then the statement of the theorem holds for C*-algebras with dimension i. Assume that $\dim A = m+1$. Since A is not simple then there exists a non-zero proper closed ideal of A which we denote by I. Since I is finite dimensional, we may assume that I has minimum dimension.

By doing this, I must have no non-trivial ideals. So, I is a non-zero finite dimensional simple C*-algebra and by Theorem 2.10.2, there exists $n_1 \in \mathbb{Z}_{>0}$ such that $I \cong M_{n_1\times n_1}(\mathbb{C})$ as C*-algebras. In particular, this means that I has a unit. Let p be the unit of I. By arguing as in Theorem 3.4.11, we deduce that $I = Ap$ and p commutes with the elements of A.

Now let $1_{\tilde{A}}$ be the unit of the unitization $\tilde{A}$ of A. Define the map

$$\begin{array}{rccl} \phi: & A & \to & Ap \oplus A(1_{\tilde{A}} - p) \\ & a & \mapsto & (ap, a(1_{\tilde{A}} - p)). \end{array}$$

It is straightforward to check that ϕ is a *-isomorphism. Now observe that the C*-subalgebra $A(1_{\tilde{A}} - p)$ of A has dimension less than or equal to m. By the inductive hypothesis, there exist $k, n_2, \ldots, n_k \in \mathbb{Z}_{>0}$ such that

$$A(1_{\tilde{A}} - p) \cong \bigoplus_{i=2}^{k} M_{n_i\times n_i}(\mathbb{C}).$$

By using the *-isomorphism ϕ, we obtain the isomorphism of C*-algebras

$$A \cong Ap \oplus A(1_{\tilde{A}} - p) \cong \bigoplus_{i=1}^{k} M_{n_i\times n_i}(\mathbb{C}).$$

This completes the induction. $\square$

Here is another useful characterisation of simple C*-algebras.

Theorem 2.10.4. *Let A be a C*-algebra. If any surjective *-homomorphism $\pi : A \to B$ onto a non-zero C*-algebra B is also injective then A is a simple C*-algebra.*

Proof. Assume that A is a C*-algebra. Suppose that any surjective *-homomorphism from A to a non-zero C*-algebra B is injective. Let I be a closed ideal of A and suppose for the sake of contradiction that I is a proper ideal of A. Then, the projection map

$$\begin{array}{rccc} \pi: & A & \to & A/I \\ & a & \mapsto & a+I \end{array}$$

is a surjective *-homomorphism onto A/I which is a non-zero C*-algebra (non-zero because I is proper). By our assumption, π must also be injective. However, this means that $A = I$ which contradicts the assumption that I is a proper ideal of A.

So, I is not a proper ideal and is hence, either 0 or A. This means that A is a simple C*-algebra as required. □

Next, we require a few technical results pertaining to projections.

Theorem 2.10.5. *Let A be a unital C*-algebra and $p, q \in A$ be projections satisfying $\|q-p\| < 1$. Then, there exists a unitary element $u \in A$ such that $q = upu^*$ and $\|1_A - u\| \leq \sqrt{2}\|q-p\|$.*

Proof. Assume that A is a unital C*-algebra and $p, q \in A$ are projections satisfying $\|p-q\| < 1$. Consider the element $v = 1_A - p - q + 2qp$.

To show; (a) v is invertible.

(a) The idea here is to consider the elements v^*v and vv^*. By direct computation, $v^* = 1_A - p - q + 2pq$ and

$$v^*v = vv^* = 1_A - (q-p)^2.$$

This means that v is normal. Since $\|q-p\| < 1$, then $\|(q-p)^2\| = \|q-p\|^2 < 1$ (by C*-algebra condition). Since the spectrum $\sigma((q-p)^2) \subseteq [0, \|(q-p)^2\|]$, $1 \notin \sigma((q-p)^2)$. Therefore, $v^*v = 1_A - (q-p)^2$ is invertible. Since v is normal, then by the spectral mapping theorem in Theorem 1.3.14,

$$\sigma(v^*v) = \{|\lambda|^2 \mid \lambda \in \sigma(v)\}.$$

Since $0 \notin \sigma(v^*v)$, $0 \notin \sigma(v)$. Therefore, v is invertible.

Recall the polar decomposition of v from Theorem 1.4.7, Theorem 1.4.8 and Theorem 1.4.9. In particular, $v = u|v|$, where $|v|$ is invertible (by Theorem 1.4.7) and $u = v|v|^{-1}$ is unitary (by Theorem 1.4.8).

To show: (b) $q = upu^*$.

(c) $\|1_A - u\| \leq \sqrt{2}\|q-p\|$.

(b) First, we observe that

$$vp = (1_A - p - q + 2qp)p = p - p - qp + 2qp = qp$$

and

$$qv = q(1_A - p - q + 2qp) = q - qp - q + 2qp = qp.$$

This means that $v^*vp = (v^*q)v = pqv = pv^*v$. Since p commutes with v^*v, it must also commute with $|v| = (v^*v)^{\frac{1}{2}}$ and $|v|^{-1}$. Now, we have

$$up = v|v|^{-1}p = vp|v|^{-1} = qv|v|^{-1} = qu$$

and consequently, $q = upu^*$.

(c) Let $Re(v)$ denote the real part of the operator v. Then,

$$Re(v) = \frac{1}{2}(v + v^*) = 1_A - p - q + qp + pq = 1_A - (q-p)^2 = |v|^2$$

and

$$Re(u) = \frac{1}{2}(u + u^*) = \frac{1}{2}(v + v^*)|v|^{-1} = |v|.$$

Hence, we have the upper bound

$$\begin{aligned}
\|1_a - u\|^2 &= \|(1_A - u^*)(1_A - u)\| \\
&= \|21_A - u - u^*\| \\
&= 2\|1_A - Re(u)\| \\
&= 2\|1_A - |v|\| \leq 2\|1_A - |v|^2\|.
\end{aligned}$$

The inequality follows from the fact that if $t \in [0,1]$ then $1 - t \leq 1 - t^2$ (see Theorem 1.3.7). Now since $1_A - |v|^2 = 1_A - v^*v = (q-p)^2$, we have

$$\|1_A - u\|^2 \leq 2\|1_A - |v|^2\| = 2\|q-p\|^2.$$

By taking square roots, we are done. $\square$

Theorem 2.10.6. *Let A be a C^*-algebra and $a \in A$ be self-adjoint. Suppose that $\|a - a^2\| < \frac{1}{4}$. Then, there exists a projection $p \in A$ such that $\|a - p\| < \frac{1}{2}$.*

Proof. Assume that A is a C*-algebra and $a \in A$ is self-adjoint. By Theorem 1.6.8, we may assume that A is abelian and that $A = Cts_0(X, \mathbb{C})$ where X is a locally compact Hausdorff space.

Consider the absolute value of the function a. Notice that $\frac{1}{2} \notin \text{im } |a|$. Otherwise, there exists $x \in X$ such that $|a(x)| = \frac{1}{2}$. Since a is self-adjoint, it is a real-valued function. If $a(x) = \frac{1}{2}$ then

$$|a(x) - a^2(x)| = |\frac{1}{2} - \frac{1}{4}| = \frac{1}{4}$$

which contradicts the assumption that $\|a - a^2\| < \frac{1}{4}$. If $a(x) = -\frac{1}{2}$ then

$$|a(x) - a^2(x)| = |-\frac{1}{2} - \frac{1}{4}| = \frac{3}{4}$$

which again contradicts the assumption that $\|a - a^2\| < \frac{1}{4}$. Hence, $\frac{1}{2} \notin \text{im } |a|$.

Now let $S = |a|^{-1}((\frac{1}{2}, \infty))$. Then, S is an open subset of X. It is also compact because

$$S = \{x \in X \mid |a(x)| \geq \frac{1}{2}\}$$

and X is a LCH space. Now consider the characteristic function χ_S. Since χ_S is self-adjoint and idempotent, it is a projection in A. If $x \notin S$ then

$$|a(x) - \chi_S(x)| < \frac{1}{2} + 0 = \frac{1}{2}.$$

If $x \in S$ then first observe that by the reverse triangle inequality,

$$\frac{1}{4} > \|a - a^2\|_\infty \geq |\|a\|_\infty - \|a^2\|_\infty| = |\|a\|_\infty - \|a\|_\infty^2|.$$

Consequently, $\|a\|_\infty \in [0, \frac{1}{2}(1 + \sqrt{2})] - \{\frac{1}{2}\}$. If $\|a\|_\infty \in [0, \frac{1}{2})$ then $S = \emptyset$ and the result is satisfied trivially. On the other hand, if $\|a\|_\infty \in (\frac{1}{2}, \frac{1}{2}(1 + \sqrt{2})]$ and $x \in S$ then

$$|a(x) - \chi_S(x)| = |1 - a(x)| < |\frac{1}{2}(1 + \sqrt{2}) - 1| < \frac{1}{2}.$$

Therefore, $\|a - \chi_S\|_\infty < \frac{1}{2}$ as required. $\square$

In a previous section, we defined the concept of a trace on a unital C*-algebra. The notion of a trace generalises to an arbitrary C*-algebra. In [Mur90], positive linear functionals which are traces are called **tracial**. We already know one example of a tracial positive linear functional — from Theorem 1.8.1. We will give another example from [Mur90, Remark 6.2.2]. Before we do this, here is how the notion of a state generalises to an arbitrary C*-algebra.

Definition 2.10.2. Let A be a C*-algebra. A **state** on A is a positive linear functional $\phi : A \to \mathbb{C}$ such that $\|\phi\| = 1$.

Example 2.10.1. Let H be an infinite dimensional Hilbert space and $B_0(H)$ denote the space of bounded compact operators on H. We will show that there does not exist a tracial state on $B_0(H)$.

Suppose for the sake of contradiction that $\tau : B_0(H) \to \mathbb{C}$ is a tracial state on $B_0(H)$. The idea is to show that all rank one projections on H are unitarily equivalent. Let $p, q \in B_0(H)$ be rank one projections. Then, there exists unit vectors $\xi, \psi \in H$ such that

$$p = |\xi\rangle\langle\xi| \qquad \text{and} \qquad q = |\psi\rangle\langle\psi|.$$

Since ξ and ψ are both unit vectors, there exists a unitary operator $u \in B(H)$ such that $u\xi = \psi$. Therefore,

$$q = |\psi\rangle\langle\psi| = |u\xi\rangle\langle u\xi| = u|\xi\rangle\langle\xi|u^* = upu^*.$$

The second last equality follows from Theorem 1.5.2. Hence, p and q are unitarily equivalent. So,

$$\tau(q) = \tau(upu^*) = \tau(u^*(up)) = \tau(p).$$

Every rank one projection takes the same value under τ. Let $t \in \mathbb{C}$ be this value. Since τ is a positive linear functional, $t \in \mathbb{R}_{\geq 0}$. Now let $n \in \mathbb{Z}_{>0}$ and $\{e_i\}_{i \in \mathbb{Z}_{>0}}$ be an orthonormal basis for H. Define

$$p_n = \sum_{i=1}^{n} |e_i\rangle\langle e_i|.$$

Then, p_n is a sum of n rank one projections and

$$\tau(p_n) = \sum_{i=1}^{n} \tau(|e_i\rangle\langle e_i|) = nt.$$

However, since $\{e_i\}_{i \in \mathbb{Z}_{>0}}$ is an orthonormal basis, p_n is also a projection. So, $\tau(p_n) \leq \|\tau\|\|p_n\| \leq 1$. This means that $nt \leq 1$ and $n \leq 1/t$ for arbitrary $n \in \mathbb{Z}_{>0}$. This gives the required contradiction. Hence, $B_0(H)$ does not admit a tracial state.

Recall that if A is a C*-algebra and $\phi : A \to \mathbb{C}$ is a positive linear functional then

$$N_\phi = \{a \in A \mid \phi(a^*a) = 0\}$$

is a left ideal of A. This was derived as part of the GNS construction. If ϕ is tracial then

$$\begin{aligned} N_\phi &= \{a \in A \mid \phi(a^*a) = 0\} \\ &= \{a \in A \mid \phi(aa^*) = 0\} \\ &= \{a^* \mid a \in N_\phi\} = N_\phi^*. \end{aligned}$$

So, N_ϕ is a closed ideal of A.

Theorem 2.10.7. *Let A be a simple C*-algebra and $\tau : A \to \mathbb{C}$ be a non-zero tracial positive linear functional. Then, τ is faithful.*

Proof. Assume that A is a simple C*-algebra. Assume that τ is a non-zero tracial positive linear functional on A. By the previous discussion,

$$N_\tau = \{a \in A \mid \tau(a^*a) = 0\}$$

is a closed ideal of A. Since A is simple, N_τ is either the zero ideal or A. Since τ is non-zero, $N_\tau \neq A$. Therefore, $N_\tau = 0$ and τ is faithful. $\square$

We will now list a particular application of Theorem 2.10.7 for the purpose of motivating the definition of a uniformly hyperfinite algebra.

Example 2.10.2. Let $(A_n)_{n\in\mathbb{Z}_{>0}}$ be an increasing sequence of C*-subalgebras of a C*-algebra A such that

$$A = \overline{\bigcup_{n=1}^{\infty} A_n}.$$

Suppose further that A is unital with multiplicative unit 1_A and that if $n \in \mathbb{Z}_{>0}$ then $1_A \in A_n$. We claim that if there exists a unique tracial state τ_n on A_n then there exists a unique tracial state τ on A.

First, note that if $m, n \in \mathbb{Z}_{>0}$ and $m \leq n$ then the restriction $\tau_n|_{A_m} = \tau_m$ by uniqueness of the tracial state τ_m on A_m. Keeping this in mind, we define the map

$$\begin{array}{rccc} \tau : & \bigcup_{n\in\mathbb{Z}_{>0}} A_n & \to & \mathbb{C} \\ & a \in A_m & \mapsto & \tau_m(a). \end{array}$$

By the most recent remark, τ is a well-defined map. It is straightforward to check that τ is a linear map. Since each τ_n is a state, then

$$|\tau(a)| = |\tau_m(a)| \leq \|\tau_m\|\|a\| = \tau_m(1_A)\|a\| = \|a\|.$$

The second last equality follows from Theorem 1.11.2. Hence, τ is norm decreasing.

Since A is the closure (with respect to the norm topology) of $\bigcup_{n\in\mathbb{Z}_{>0}} A_n$, we can extend τ to a bounded linear functional on all of A. To see that τ is positive, assume that $a \in A$. Then, there exists a sequence $\{a_n\}_{n\in\mathbb{Z}_{>0}}$ in $\bigcup_n A_n$ such that $a_n \in A_n$ and $\lim_{n\to\infty}\|a_n - a\| = 0$. So,

$$\tau(a^*a) = \tau(\lim_{n\to\infty} a_n^*a_n) = \lim_{n\to\infty} \tau(a_n^*a_n) = \lim_{n\to\infty} \tau_n(a_n^*a_n) \geq 0$$

and τ is a positive linear functional on A. To see that τ is a state, note that

$$\tau(1_A) = \tau_n(1_A) = 1$$

since $1_A \in A_n$ for $n \in \mathbb{Z}_{>0}$. To see that τ is tracial, assume that $a, b \in A$. Then,

$$\tau(ab) = \lim_{n\to\infty} \tau(a_nb_n) = \lim_{n\to\infty} \tau_n(a_nb_n) = \lim_{n\to\infty} \tau(b_na_n) = \tau(ba).$$

So, τ defines a tracial state on A. Finally, the fact that τ is unique follows from the fact that τ_m is the unique tracial state on A_m for $m \in \mathbb{Z}_{>0}$.

Definition 2.10.3. A **uniformly hyperfinite algebra** or **UHF algebra** is a unital C*-algebra A which has an increasing sequence $\{A_n\}_{n\in\mathbb{Z}_{>0}}$ of finite dimensional simple C*-subalgebras such that $1_A \in A_m$ for $m \in \mathbb{Z}_{>0}$ and the union $\bigcup_{n=1}^{\infty} A_n$ is dense in A.

Let us unpack the definition of a UHF algebra with what we know so far. Let A be a UHF algebra with increasing sequence $\{A_n\}_{n=1}^{\infty}$. By definition, if $n \in \mathbb{Z}_{>0}$ then A_n is finite dimensional and simple. By Theorem 2.10.2, A_n is isomorphic as a C*-algebra to the matrix algebra $M_{k\times k}(\mathbb{C})$ for some $k \in \mathbb{Z}_{>0}$. By Theorem 1.8.1, each A_n admits a unique tracial state. Hence, A also admits a unique tracial state.

We claim that A is a simple C*-algebra. To see why this is the case, we have to prove a few results first. Fortunately, most of the work has already been done.

Theorem 2.10.8. *Let $\mathcal{S}$ be a non-empty set of simple C*-subalgebras of a C*-algebra A. Suppose that the set $\mathcal{S}$ is upwards-directed — that is, if $B, C \in \mathcal{S}$ then there exists $D \in \mathcal{S}$ such that $B \subseteq D$ and $C \subseteq D$. Suppose also that the union $\bigcup_{T \in \mathcal{S}} T$ is dense in A. Then, A is a simple C*-algebra.*

Proof. Assume that A is a C*-algebra and $\mathcal{S}$ is a non-empty set of simple C*-subalgebras of A. Assume that $\mathcal{S}$ is upwards-directed and that the union $\bigcup_{T \in \mathcal{S}} T$ is dense in A.

We will use Theorem 2.10.4 to prove that A is simple. Assume that B is a non-zero C*-algebra and $\pi : A \to B$ is a surjective *-homomorphism. If $C \in \mathcal{S}$ then the restriction $\pi|_C$ is either the zero map or a surjective *-homomorphism on its non-zero image. In the latter case, $\pi|_C$ is injective because C is simple. Hence, $\pi|_C$ is an isometry.

By assumption, the map π is not the zero map on $\bigcup_{T \in \mathcal{S}} T$. Since $\mathcal{S}$ is upwards-directed, π cannot be the zero map on any non-zero $C \in \mathcal{S}$. Otherwise, if $\pi|_C = 0$ then the restriction $\pi|_D = 0$ where $D \in \mathcal{S}$ such that $D \subseteq C$. Using the fact that $\mathcal{S}$ is upwards-directed, there exists $E \in \mathcal{S}$ such that $C \subseteq E$. Since E is a simple C*-algebra, the closed ideal $\ker \pi|_E$ is either 0 or E. However, $C \subseteq \ker \pi|_E$ and C is non-zero. So, $\ker \pi|_E = E$ and $\pi|_E = 0$. By iterating this argument, we find that π is the zero map on the union $\bigcup_{T \in \mathcal{S}} T$.

Therefore, π is an isometry on $\bigcup_{T \in \mathcal{S}} T$. Since the union $\bigcup_{T \in \mathcal{S}} T$ is dense in A, π must also be an isometry on A by continuity. Hence, π is injective on A and by Theorem 2.10.4, A is a simple C*-algebra. □

Theorem 2.10.9 is an application of Theorem 2.10.8.

Theorem 2.10.9. *Let $\{(A_n, \varphi_n)\}_{n \in \mathbb{Z}_{>0}}$ be a direct sequence of simple C*-algebras. Then, the direct limit $\varinjlim A_n$ is also a simple C*-algebra.*

Proof. Assume that $\{(A_n, \varphi_n)\}_{n \in \mathbb{Z}_{>0}}$ is a direct sequence of simple C*-algebras. If $n \in \mathbb{Z}_{>0}$ then let $\varphi^n : A_n \to \varinjlim A_n$ denote the natural map. Define

$$\mathcal{S} = \{\varphi^n(A_n) \mid n \in \mathbb{Z}_{>0}\}.$$

Recall from Theorem 2.8.1 that $\mathcal{S}$ is an upwards-directed set whose union $\bigcup_{n=1}^{\infty} \varphi^n(A_n)$ is dense in $\varinjlim A_n$. By Theorem 2.10.8, we deduce that the direct limit $\varinjlim A_n$ is a simple C*-algebra. □

Theorem 2.10.10. *Let A be a UHF algebra. Then, A is a simple C*-algebra.*

Proof. Assume that A is a UHF algebra. By Theorem 2.8.3, A is the direct limit of the direct sequence of simple C*-algebras $\{(A_n, \iota_n)\}_{n\in\mathbb{Z}_{>0}}$ where $\iota_n : A_n \hookrightarrow A_{n+1}$ is the inclusion map. By Theorem 2.10.9, we find that A is a simple C*-algebra as required. $\square$

In [Gli59], Glimm proved that there are uncountably many UHF algebras which are not isomorphic to each other as C*-algebras. We will now prove this fact. Let $n, d \in \mathbb{Z}_{>0}$. Define the unital *-homomorphism

$$\begin{array}{rccc} \varphi : & M_{n\times n}(\mathbb{C}) & \to & M_{dn\times dn}(\mathbb{C}) \\ & a & \mapsto & \begin{pmatrix} a & & & \\ & a & & \\ & & \ddots & \\ & & & a \end{pmatrix}. \end{array}$$

We call φ the canonical map from $M_{n\times n}(\mathbb{C})$ to $M_{dn\times dn}(\mathbb{C})$. Let $\mathcal{S}$ denote the set of all functions $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$. If $s \in \mathcal{S}$ then define the function

$$\begin{array}{rccc} s! : & \mathbb{Z}_{>0} & \to & \mathbb{Z}_{>0} \\ & n & \mapsto & s(1)s(2)\dots s(n). \end{array}$$

If $n \in \mathbb{Z}_{>0}$ then let $\varphi_n : M_{s!(n)\times s!(n)}(\mathbb{C}) \to M_{s!(n+1)\times s!(n+1)}(\mathbb{C})$ denote the canonical map. Define M_s to be the direct limit of the sequence

$$\{(M_{s!(n)\times s!(n)}(\mathbb{C}), \varphi_n)\}_{n\in\mathbb{Z}_{>0}}.$$

By construction, M_s is the direct limit of a sequence of finite dimensional simple algebras. Hence, it is a UHF algebra. Now let $\mathcal{P} \subseteq \mathbb{Z}_{>0}$ denote the set of prime numbers. If $s \in \mathcal{S}$ then we define

$$\begin{array}{rccc} \epsilon_s : & \mathcal{P} & \to & \mathbb{Z}_{>0} \cup \{\infty\} \\ & p & \mapsto & \sup\{m \in \mathbb{Z}_{>0} \mid \ p^m \text{ divides some } s!(n)\ \}. \end{array}$$

The point of ϵ_s is that it can tell us when M_s and M_t are isomorphic where $s, t \in \mathcal{S}$.

Theorem 2.10.11. *Let $s, s' \in \mathcal{S}$ and assume that $M_s \cong M_{s'}$ as C*-algebras. Then, $\epsilon_s = \epsilon_{s'}$.*

Proof. Assume that $s, s' \in \mathcal{S}$ and that M_s is isomorphic to $M_{s'}$ as C*-algebras. Let $\pi : M_s \to M_{s'}$ be a *-isomorphism. Let τ and τ' be the unique tracial states of M_s and $M_{s'}$ respectively. If $n \in \mathbb{Z}_{>0}$ then let

$\varphi^n : M_{s!(n)\times s!(n)}(\mathbb{C}) \to M_s$ and $\psi^n : M_{s'!(n)\times s'!(n)}(\mathbb{C}) \to M_{s'}$ be the natural maps.

First, observe that the composite $\tau' \circ \pi$ is a tracial state on M_s. By uniqueness, we have $\tau' \circ \pi = \tau$.

To show: (a) $\epsilon_s \leq \epsilon_{s'}$.

(a) In order to prove this statement, we will show that if $n \in \mathbb{Z}_{>0}$ then there exists $m \in \mathbb{Z}_{>0}$ such that $s!(n)$ divides $s'!(m)$. This is enough because if $p \in \mathbb{Z}_{>0}$ is prime and $k \in \mathbb{Z}_{>0}$ such that p^k divides $s!(n)$ then from the statement we want to prove, p^k must also divide $s'!(m)$ and subsequently, $\epsilon_s(p) \leq \epsilon_s(p')$.

So, assume that $n \in \mathbb{Z}_{>0}$ and q is a rank one projection in $M_{s!(n)\times s!(n)}(\mathbb{C})$. Consider the composite $\tau \circ \varphi^n : M_{s!(n)\times s!(n)}(\mathbb{C}) \to \mathbb{C}$. It is a tracial state on $M_{s!(n)\times s!(n)}(\mathbb{C})$. By uniqueness, the composite $\tau \circ \varphi^n$ is the tracial state given in Theorem 1.8.1 and

$$\tau(\varphi^n(q)) = \frac{1}{s!(n)}.$$

Since π and φ^n are both *-homomorphisms, $\pi(\varphi^n(q)) \in M_{s'}$ is a projection. Now we use the fact that the *-subalgebra

$$\bigcup_{k\in\mathbb{Z}_{>0}} \psi^k(M_{s'!(k)\times s'!(k)}(\mathbb{C}))$$

is dense in $M_{s'}$ to obtain a positive integer $m \in \mathbb{Z}_{>0}$ and a self-adjoint element $a \in M_{s'!(m)\times s'!(m)}(\mathbb{C})$ such that

$$\|\pi(\varphi^n(q)) - \psi^m(a)\| < \frac{1}{8} \qquad \text{and} \qquad \|\pi(\varphi^n(q)) - \psi^m(a^2)\| < \frac{1}{8}.$$

So,

$$\begin{aligned}
\|a - a^2\| &= \|\psi^m(a) - \psi^m(a^2)\| \\
&\leq \|\psi^m(a) - \pi(\varphi^n(q))\| + \|\pi(\varphi^n(q)) - \psi^m(a^2)\| \\
&< \frac{1}{8} + \frac{1}{8} = \frac{1}{4}.
\end{aligned}$$

By Theorem 2.10.6, there exists a projection r in $M_{s'!(m)\times s'!(m)}(\mathbb{C})$ such that $\|a - r\| < \frac{1}{2}$. So, $\psi^m(r)$ is a projection in $M_{s'}$ satisfying

$$\begin{aligned}\|\pi(\varphi^n(q)) - \psi^m(r)\| &\le \|\pi(\varphi^n(q)) - a\| + \|a - r\| \\ &< \frac{1}{8} + \frac{1}{2} < 1.\end{aligned}$$

By Theorem 2.10.5, the projections $\pi(\varphi^n(q))$ and $\psi^m(r)$ in $M_{s'}$ are unitarily equivalent. Since τ' is the unique tracial state on $M_{s'}$,

$$\tau'(\psi^m(r)) = \tau'(\pi(\varphi^n(q))) = \tau(\varphi^n(q)) = \frac{1}{s!(n)}.$$

But, $\tau' \circ \psi^m$ is a tracial state on $M_{s'!(m)\times s'!(m)}(\mathbb{C})$, which must be unique. Since r is a projection in $M_{s'!(m)\times s'!(m)}(\mathbb{C})$, there exists $d \in \mathbb{Z}_{>0}$ such that

$$\tau'(\psi^m(r)) = \frac{d}{s'!(m)}.$$

Hence, $\frac{d}{s'!(m)} = \frac{1}{s!(n)}$ and $ds!(n) = s'!(m)$ as required. This proves part (a).

The inequality $\epsilon_{s'} \le \epsilon_s$ follows by a symmetric argument to that used in part (a). Hence, $\epsilon_s = \epsilon_{s'}$. $\square$

Theorem 2.10.12. *There exists an uncountable number of UHF algebras which are not isomorphic to each other.*

Proof. If $n \in \mathbb{Z}_{>0}$ then let p_n denote the n^{th} prime number. If $s \in \mathcal{S}$ then we define

$$\begin{array}{rccl}\overline{s}: & \mathbb{Z}_{>0} & \to & \mathbb{Z}_{>0} \\ & n & \mapsto & p_n^{s(n)}.\end{array}$$

Then, $\epsilon_{\overline{s}}(p_n) = s(n)$. Now let $s' \in \mathcal{S}$. If $\epsilon_{\overline{s}} = \epsilon_{\overline{s'}}$ then

$$p_n^{s(n)} = \epsilon_{\overline{s}}(p_n) = \epsilon_{\overline{s'}}(p_n) = p_n^{s'(n)}$$

and $s = s'$.

Now consider the family of UHF algebras $\{M_{\overline{s}}\}_{s\in\mathcal{S}}$. This is an uncountable family because $\mathcal{S}$ itself is uncountable (it is isomorphic as sets to $\mathbb{R}$). If $s, t \in \mathcal{S}$ are distinct then $\epsilon_{\overline{s}} \ne \epsilon_{\overline{t}}$ and by Theorem 2.10.11, $M_{\overline{s}}$ is not isomorphic to $M_{\overline{t}}$. $\square$

We will now investigate an application of UHF algebras to the theory of von Neumann algebras.

Definition 2.10.4. Let H be a Hilbert space. A **factor** on H is a von Neumann algebra A on H such that $A \cap A' = \mathbb{C}id_H$ (recall that A' is the commutant of A).

A basic example of a factor is once again $B(H)$. Our application will give an example of an infinite dimensional factor which is not isomorphic to $B(H)$ for any Hilbert space H.

Definition 2.10.5. Let H be a Hilbert space and A be a von Neumann algebra on H. We say that A is **hyperfinite** if there exists a weakly dense C*-subalgebra W of A such that W is a UHF algebra and whose unit is id_H.

Example 2.10.3. We claim that if H is a separable Hilbert space then $B(H)$ is hyperfinite. This is easy to see if H is finite dimensional. To prove this for the infinite dimensional case, let A be an infinite dimensional UHF algebra and (φ, H) be a non-zero irreducible representation of A.

Since A is a UHF algebra, it is simple by Theorem 2.10.10. By Theorem 2.10.4, the *-homomorphism $\varphi : A \to B(H)$ defines a *-isomorphism from A to its image $\varphi(A)$. Hence, $\varphi(A)$ is a UHF algebra.

Now let $x \in H - \{0\}$. By Theorem 1.9.6, x is a cyclic vector for the representation (φ, H). So,

$$H = \overline{\varphi(A)x}.$$

Now since A is infinite dimensional, H must also be infinite dimensional. Also, since A is separable, H must also be a separable Hilbert space. Since (φ, H) is irreducible, then by Theorem 1.9.7, $\varphi(A)' = \mathbb{C}id_H$ and the double commutant $\varphi(A)'' = B(H)$. Therefore, $\varphi(A)$ is a UHF subalgebra of $B(H)$ which contains the identity map id_H and is weakly dense in $B(H)$ (see [Mur90, Theorem 4.2.5]). So, $B(H)$ is a hyperfinite algebra.

Now we will give our example of a factor in the following theorem.

Theorem 2.10.13. *Let A be a UHF algebra with unique tracial state τ. Let $(\varphi_\tau, H_\tau, \xi_\tau)$ be the GNS representation of A. Then, the von Neumann algebra $\varphi_\tau(A)''$ is a hyperfinite factor which admits a faithful tracial state.*

Proof. Assume that A is a UHF algebra with unique tracial state τ. Assume that $(\varphi_\tau, H_\tau, \xi_\tau)$ is the GNS representation of A. Let $B = \varphi_\tau(A)$.

To show: (a) B is hyperfinite.

(a) Since A is a UHF algebra, it is simple by Theorem 2.10.10. By Theorem 2.10.4, φ_τ defines a *-isomorphism from A to B. Therefore, B is a UHF algebra. By [Mur90, Theorem 4.2.5], B is weakly dense in its double commutant B''.

Since the GNS representation $(\varphi_\tau, H_\tau, \xi_\tau)$ is cyclic, it is non-degenerate. Since A is unital, then by Theorem 1.9.3, $\varphi_\tau(1_A) = id_{H_\tau}$. Hence, $id_{H_\tau} \in B \subseteq B''$ and consequently, B'' is a hyperfinite algebra.

Now we will construct a faithful tracial state on B''. Assume that $u, u' \in B$. Then, there exists $a, a' \in A$ such that $\varphi_\tau(a) = u$ and $\varphi_\tau(a') = u'$. So,

$$\begin{aligned}
\langle uu'(\xi_\tau), \xi_\tau\rangle &= \langle \varphi_\tau(aa')(\xi_\tau), \xi_\tau\rangle \\
&= \langle \varphi_\tau(aa')(1_A + N_\tau), 1_A + N_\tau\rangle \\
&= \langle aa' + N_\tau, 1_A + N_\tau\rangle \\
&= \tau(1_A aa') = \tau(aa') = \tau(a'a) \\
&= \langle a'a + N_\tau, 1_A + N_\tau\rangle \\
&= \langle \varphi_\tau(a'a)(1_A + N_\tau), 1_A + N_\tau\rangle \\
&= \langle u'u(\xi_\tau), \xi_\tau\rangle.
\end{aligned}$$

Since B is weakly dense in B'', the above identity also holds for $u, u' \in B''$. Now define

$$\begin{array}{rccc}
\omega : & B'' & \to & \mathbb{C} \\
 & u & \mapsto & \langle u(\xi_\tau), \xi_\tau\rangle.
\end{array}$$

Then, ω is a tracial state on B''.

To show: (b) ω is faithful.

(b) Assume that $u \in B''$ satisfies $\omega(u^*u) = 0$. Then,

$$\omega(u^*u) = \langle u^*u(\xi_\tau), \xi_\tau\rangle = \langle u(\xi_\tau), u(\xi_\tau)\rangle = \|u(\xi_\tau)\|^2 = 0.$$

Hence, $u(\xi_\tau) = 0$. We claim that this implies that $u = 0$. If $v \in B$ then

$$\|uv(\xi_\tau)\|^2 = \langle v^*u^*uv(\xi_\tau), \xi_\tau\rangle = \langle vv^*u^*u(\xi_\tau), \xi_\tau\rangle$$

where the last equality follows from the tracial condition on ω. Since $u(\xi_\tau) = 0$, then $\|uv(\xi_\tau)\|^2 = 0$. From this, we find that $uB\xi_\tau = u\varphi_\tau(A)\xi_\tau = 0$. Since ξ_τ is a cyclic vector, $\overline{\varphi_\tau(A)\xi_\tau} = H$ and consequently, $u = 0$. Therefore, ω is a faithful tracial state on B''.

To show: (c) B'' is a factor.

(c) By Theorem 2.4.6, B'' is a von Neumann algebra. Let $p \in B' \cap B''$ be a projection. Define

$$\begin{array}{rccc} \omega' : & B'' & \to & \mathbb{C} \\ & u & \mapsto & \omega(pu) \end{array}$$

Then, ω' defines a weakly continuous trace on B''. If ω' is restricted to the UHF algebra B, which has a unique tracial state we denote by ω_B, then there exists $t \in \mathbb{C}$ such that if $v \in B$ then

$$\omega'(v) = t\omega(v).$$

Since B is weakly dense in B'' and ω and ω' are both weakly continuous, we deduce that $\omega' = t\omega$ as functionals on B''. Hence,

$$\omega'(id_{H_\tau}) = \langle p(\xi_\tau), \xi_\tau \rangle = \langle t\xi_\tau, \xi_\tau \rangle = t\omega(id_{H_\tau}).$$

and $\omega(p) = t$. Now consider $\omega'(id_{H_\tau} - p)$. We have

$$\omega'(id_{H_\tau} - p) = \omega(p - p^2) = \omega(p - p) = 0.$$

However,

$$\omega'(id_{H_\tau} - p) = t\omega(id_{H_\tau} - p) = \omega(p)\omega(id_{H_\tau} - p).$$

We conclude that either $\omega(p) = 0$ or $\omega(id_{H_\tau} - p) = 0$. Since the tracial state ω is faithful, we deduce that either $p = 0$ or $p = id_{H_\tau}$. Consequently, the only projections in the von Neumann algebra $B' \cap B''$ are trivial.

Since a von Neumann algebra is the closed linear span of its projections, we deduce that $B' \cap B'' = \mathbb{C}id_{H_\tau}$. So, B'' is a factor as required. $\square$

Now suppose that in the statement of Theorem 2.10.13, we let A be an infinite dimensional UHF factor. Then, $\varphi_\tau(A)''$ is a hyperfinite factor which is not *-isomorphic to $B(H)$ for any Hilbert space H. This is because if H is infinite dimensional then $B(H)$ does not admit a faithful tracial state.

We can weaken the definition of a UHF algebra slightly in order to obtain AF-algebras (approximately finite).

Definition 2.10.6. Let A be a C*-algebra. We say that A is an **AF-algebra** if A contains an increasing sequence $\{A_n\}_{n=1}^{\infty}$ of finite dimensional C*-subalgebras such that the union $\bigcup_{n=1}^{\infty} A_n$ is dense in A.

Example 2.10.4. Suppose that A is a direct limit of a direct sequence $\{(A_n, \varphi_n)\}_{n=1}^{\infty}$ of C*-algebras where the A_n are finite dimensional. Then, A is an AF-algebra by definition of the direct limit (see Theorem 2.8.1).

By Theorem 2.8.3, any AF-algebra is isomorphic as a C*-algebra to a direct limit of finite dimensional C*-algebras.

A useful property about AF-algebras is that they are stable under taking closed ideals and quotients.

Theorem 2.10.14. *Let A be an AF-algebra and I be a closed ideal of A. Then, I and A/I are AF-algebras.*

Proof. Assume that A is an AF-algebra. Assume that I is a closed ideal of A.

To show: (a) A/I is an AF-algebra.

(b) I is an AF-algebra.

(a) Let $\pi : A \to A/I$ denote the projection map. Since A is a UHF algebra, it contains a increasing sequence $\{A_n\}_{n\in\mathbb{Z}_{>0}}$ of finite dimensional C*-subalgebras such that $\bigcup_{n=1}^{\infty} A_n$ is dense in A. By Theorem 1.7.6, the sequence $\{\pi(A_n)\}_{n\in\mathbb{Z}_{>0}}$ is an increasing sequence of finite dimensional C*-subalgebras of A/I. Moreover,

$$\overline{\bigcup_{n\in\mathbb{Z}_{>0}} \pi(A_n)} = \pi(\overline{\bigcup_{n\in\mathbb{Z}_{>0}} A_n}) = \pi(A) = A/I.$$

Therefore, A/I is an AF-algebra.

(b) If $n \in \mathbb{Z}_{>0}$ then define $I_n = I \cap A_n$. Then, the sequence $\{I_n\}_{n\in\mathbb{Z}_{>0}}$ is an increasing sequence of finite dimensional C*-subalgebras. We need to show that

$$\overline{\bigcup_{n\in\mathbb{Z}_{>0}} I_n} = I.$$

Let $J = \overline{\bigcup_{n\in\mathbb{Z}_{>0}} I_n}$. Define

$$\begin{array}{rccc} \varphi: & A/J & \to & A/I \\ & a+J & \mapsto & a+I \end{array}$$

Then, φ is a well-defined *-homomorphism.

To show: (ba) φ is isometric.

(ba) By part (a), it suffices to prove that φ is isometric on each C*-subalgebra in the increasing sequence

$$\{(A_n + J)/J\}_{n\in\mathbb{Z}_{>0}}$$

because the increasing union $\bigcup_n (A_n + J)/J$ is a dense *-subalgebra of A/J. If $n \in \mathbb{Z}_{>0}$ then let $\psi : (A_n + J)/J \to A_n/(A_n \cap J)$ and $\theta : (A_n + I)/I \to A_n/(A_n \cap I)$ be *-isomorphisms (the second isomorphism theorem). Let $\iota_n : (A_n + I)/I \hookrightarrow A/I$ denote the inclusion.

Observe that $I_n = A_n \cap I = A_n \cap J$ and that the restriction of φ to $(A_n + J)/J$ is simply the composite $\iota_n \circ \theta^{-1} \circ \psi$. Notably, the restriction of φ is a composite of isometries. Hence, φ is an isometry on $(A_n + J)/J$ and consequently, an isometry on A/J.

(b) Since φ is isometric, we conclude that $J = I$. Hence, I is an AF-algebra as required. □

A useful consequence of Theorem 2.10.14 is the following theorem:

Theorem 2.10.15. *Let A be a C*-algebra. Then, A is an AF-algebra if and only if its unitization $\tilde{A}$ is an AF-algebra.*

2.11 Tensor products of C*-algebras

We begin by recalling the tensor product for the category of $\mathbb{C}$-vector spaces. If H and K are $\mathbb{C}$-vector spaces then their **algebraic tensor product** is the $\mathbb{C}$-vector space

$$H \otimes K = \mathrm{span}\{x \otimes y \mid x \in H, y \in K\}.$$

The primary use of tensor products is to turn bilinear (or multilinear) maps into linear maps via its universal property.

Theorem 2.11.1. *Let H and K be $\mathbb{C}$-vector spaces. Let ϕ be the $\mathbb{C}$-bilinear map*

$$\begin{array}{rccc} \phi : & H \times K & \to & H \otimes K \\ & (h,k) & \mapsto & h \otimes k. \end{array}$$

Let V be another $\mathbb{C}$-vector space and $f : H \times K \to V$ be a $\mathbb{C}$-bilinear map. Then, there exists a unique $\mathbb{C}$-linear map $\overline{f} : H \otimes K \to V$ such that the following diagram commutes:

$$\begin{tikzcd} H \times K \arrow[r, "\phi"] \arrow[dr, "f"'] & H \otimes K \arrow[d, dashed, "\overline{f}"] \\ & V \end{tikzcd}$$

Here is a quick application of Theorem 2.11.1.

Example 2.11.1. Let H and K be $\mathbb{C}$-vector spaces. Let $\tau : H \to \mathbb{C}$ and $\rho : K \to \mathbb{C}$ be linear functionals. Define the map

$$\begin{array}{rccc} \tau \times \rho : & H \times K & \to & \mathbb{C} \\ & (h,k) & \mapsto & \tau(h)\rho(k) \end{array}$$

The map $\tau \times \rho$ is $\mathbb{C}$-bilinear. By the universal property in Theorem 2.11.1, there exists a unique linear functional $\tau \otimes \rho : H \otimes K \to \mathbb{C}$ such that the following diagram commutes:

$$\begin{tikzcd} H \times K \arrow[r, "\phi"] \arrow[dr, "\tau\times\rho"'] & H \otimes K \arrow[d, dashed, "\tau\otimes\rho"] \\ & \mathbb{C} \end{tikzcd}$$

In particular, $\tau \otimes \rho$ is defined by $(\tau \otimes \rho)(h \otimes k) = \tau(h)\rho(k)$.

Now suppose that $\sum_{j=1}^{n} x_j \otimes y_j = 0$ where $x_j \in H$ and $y_j \in K$. We claim that if the set $\{y_1, y_2, \dots, y_n\}$ is linearly independent in K then $x_1 = \cdots = x_n = 0$.

If $j \in \{1, 2, \dots, n\}$ then let ρ_j be the linear functional defined by $\rho_j(y_i) = \delta_{i,j}$ where $\delta_{i,j}$ is the Kronecker delta. If $\tau : H \to \mathbb{C}$ is a linear functional then

$$\begin{aligned}
0 &= (\tau \otimes \rho_j)(\sum_{i=1}^{n} x_i \otimes y_i) \\
&= \sum_{i=1}^{n} \tau(x_i)\rho_j(y_i) \\
&= \sum_{i=1}^{n} \tau(x_i)\delta_{i,j} = \tau(x_j).
\end{aligned}$$

So, $\tau(x_j) = 0$ for arbitrary $\tau \in H^*$. Therefore, if $j \in \{1, 2, \ldots, n\}$ then $x_j = 0$. Analogously, if the set $\{x_1, \ldots, x_n\}$ is linearly independent in H then $y_1 = \cdots = y_n = 0$.

Now if H and K are normed vector space then there are multiple different norms one can equip the tensor product $H \otimes K$ with. As we will see shortly, this makes the business of constructing tensor products of C*-algebras quite complicated. First, we will see that the tensor product of two Hilbert spaces is a relatively simple affair.

Theorem 2.11.2. *Let H and K be Hilbert spaces. Then, there exists a unique inner product $\langle -, - \rangle$ on the tensor product $H \otimes K$, defined by*

$$\begin{array}{rccc}
\langle -, - \rangle : & (H \otimes K) \times (H \otimes K) & \to & \mathbb{C} \\
& \langle x \otimes y, x' \otimes y' \rangle & \mapsto & \langle x, x' \rangle \langle y, y' \rangle.
\end{array}$$

Proof. Assume that H and K are Hilbert spaces. First observe that if $\tau : H \to \mathbb{C}$ and $\phi : K \to \mathbb{C}$ are conjugate-linear maps then there exists a unique conjugate-linear map

$$\begin{array}{rccc}
\tau \otimes \phi : & H \otimes K & \to & \mathbb{C} \\
& (h, k) & \mapsto & \tau(h)\phi(k).
\end{array}$$

Here is how the map $\tau \otimes \phi$ is constructed. The composites $\overline{\tau}$ and $\overline{\phi}$ are linear maps. By Theorem 2.11.1, we obtain a unique linear functional $\overline{\tau} \otimes \overline{\phi}$ and then set $\tau \otimes \phi = \overline{\overline{\tau} \otimes \overline{\phi}}$.

Now assume that $x \in H$ and $y \in K$. Let $\phi_x : H \to \mathbb{C}$ be the conjugate-linear functional defined by $h \mapsto \langle x, h \rangle$. Similarly, let $\phi_y : K \to \mathbb{C}$ be the conjugate-linear functional defined by $k \mapsto \langle y, k \rangle$. Then, there exists a unique conjugate-linear functional $\phi_x \otimes \phi_y : H \otimes K \to \mathbb{C}$ defined by

$$(\phi_x \otimes \phi_y)(h \otimes k) = \phi_x(h)\phi_y(k) = \langle x, h \rangle \langle y, k \rangle.$$

Next, let X be the vector space of conjugate-linear functionals on $H \otimes K$ and define

$$\begin{array}{rccc} M': & H \times K & \to & X \\ & (h,k) & \mapsto & \phi_h \otimes \phi_k \end{array}$$

Then, M' is a $\mathbb{C}$-bilinear map. By Theorem 2.11.1, there exists a unique linear map $M : H \otimes K \to X$ such that $M(h \otimes k) = \phi_h \otimes \phi_k$.

Now, we define a sesquilinear form on $H \otimes K$ by

$$\begin{array}{rccc} \langle -, - \rangle : & (H \otimes K) \times (H \otimes K) & \to & \mathbb{C} \\ & \langle x \otimes y, x' \otimes y' \rangle & \mapsto & M(x \otimes y)(x' \otimes y'). \end{array}$$

Since the maps M and $\phi_x \otimes \phi_y$ are all unique, the map $\langle -, - \rangle$ must also be unique. It is straightforward to check that $\langle -, - \rangle$ defines a sesquilinear form on $H \otimes K$.

To see that $\langle -, - \rangle$ is an inner product on $H \otimes K$, it suffices to show that if $z \in H \otimes K$ and $\langle z, z \rangle = 0$ then $z = 0$. So, assume that $z \in H \otimes K$ and $\langle z, z \rangle = 0$. Then, $z = \sum_{j=1}^{n} (x_j \otimes y_j)$. Now let $\{e_i\}_{i \in \mathbb{Z}_{>0}}$ be an orthonormal basis for K. Then, there exists $x'_1, \ldots, x'_n$ such that

$$z = \sum_{j=1}^{n} (x'_j \otimes e_j).$$

Now we compute directly that

$$\begin{aligned} \langle z, z \rangle &= \langle \sum_{j=1}^{n} (x'_j \otimes e_j), \sum_{j=1}^{n} (x'_j \otimes e_j) \rangle \\ &= \sum_{i,j=1}^{n} \langle x'_i \otimes e_i, x'_j \otimes e_j \rangle \\ &= \sum_{i,j=1}^{n} \langle x'_i, x'_j \rangle \langle e_i, e_j \rangle \\ &= \sum_{i=1}^{n} \|x'_i\|^2 = 0. \end{aligned}$$

Hence, if $i \in \{1, 2, \ldots, n\}$ then $\|x'_i\| = 0$ and $x'_i = 0$. Consequently, $z = 0$ and $\langle -, - \rangle$ is the unique inner product on $H \otimes K$ as required. $\square$

In the scenario of Theorem 2.11.2, we regard $H \otimes K$ as a pre-Hilbert space, equipped with its unique inner product.

Definition 2.11.1. Let H and K be Hilbert spaces and $H \otimes K$ be the pre-Hilbert space with the unique inner product in Theorem 2.11.2. The completion of $H \otimes K$, denoted by $H \hat{\otimes} K$, is called the **Hilbert space tensor product** of H and K.

Here are some easy consequences of Theorem 2.11.2. If $x \in H$ and $y \in K$ then

$$\|x \otimes y\| = \|x\|\|y\|.$$

Theorem 2.11.3. *Let H and K be Hilbert spaces. Let E_1 and E_2 be orthonormal bases for H and K respectively. Then,*

$$E_1 \otimes E_2 = \{x \otimes y \mid x \in E_1, y \in E_2\}$$

is an orthonormal basis for $H \hat{\otimes} K$.

If H' and K' are closed vector subspaces of H and K respectively then the inclusion map $\iota : H' \otimes K' \to H \hat{\otimes} K$ is isometric where $H' \otimes K'$ inherits its inner product from $H \otimes K$. Hence, we can consider $H' \hat{\otimes} K'$ as a closed vector subspace of $H \hat{\otimes} K$.

Before we move onto the next result regarding the Hilbert space tensor product, we recall the following result about bounded linear operators on a Hilbert space.

Theorem 2.11.4. *Let H be a Hilbert space and $x \in B(H)$. Then, x is a $\mathbb{C}$-linear combination of four unitary operators.*

Theorem 2.11.5. *Let H and K be Hilbert space. Let $u \in B(H)$ and $v \in B(K)$. Then, there exists a unique operator $u \hat{\otimes} v \in B(H \hat{\otimes} K)$ such that if $x \in H$ and $y \in K$ then*

$$(u \hat{\otimes} v) = u(x) \otimes v(y).$$

We also have $\|u \hat{\otimes} v\| = \|u\|\|v\|$.

Proof. Assume that H and K are Hilbert space. Assume that $u \in B(H)$ and $v \in B(K)$. Then, the map $(u, v) \mapsto u \otimes v$ is $\mathbb{C}$-bliinear. By Theorem 2.11.1, there exists a unique $\mathbb{C}$-linear map $u \otimes v : H \otimes K \to H \otimes K$, defined by $(u \otimes v)(x \otimes y) = u(x) \otimes v(y)$.

We would like to extend $u \otimes v$ to $H \hat{\otimes} K$ by using the universal property of completion. In order to do this, we first need to show that $u \otimes v$ is bounded. By Theorem 2.11.4, we may assume without loss of generality that u and v are unitary operators. If $z \in H \otimes K$ then we write $z = \sum_{j=1}^{n}(x_j \otimes k_j)$ where $\{k_1, \ldots, k_n\}$ is an orthogonal subset of K. We have

$$\begin{aligned}
\|(u \otimes v)(z)\|^2 &= \|(u \otimes v)(\sum_{j=1}^{n}(x_j \otimes k_j))\|^2 \\
&= \|\sum_{j=1}^{n}(u(x_j) \otimes v(k_j))\|^2 = \sum_{j=1}^{n}\|u(x_j) \otimes v(k_j)\|^2 \\
&= \sum_{j=1}^{n}\|u(x_j)\|^2\|v(k_j)\|^2 = \sum_{j=1}^{n}\|x_j\|^2\|k_j\|^2 \\
&= \sum_{j=1}^{n}\|x_j \otimes k_j\|^2 = \|z\|^2.
\end{aligned}$$

The third equality follows from the fact that the set $\{v(k_1), v(k_2), \ldots, v(k_n)\}$ is orthogonal. The third last equality follows from the fact that u and v are isometries (because they were assumed to be unitary). Finally, the last equality follows from the fact that if $i, j \in \{1, 2, \ldots, n\}$ are distinct then $x_i \otimes k_i$ is orthogonal to $x_j \otimes k_j$.

We deduce that $\|u \otimes v\| = 1$. Since the linear map $u \otimes v$ is bounded for $u \in B(H)$ and $v \in B(K)$ we can apply the universal property of completion in order to extend $u \otimes v$ to a bounded linear map $u \hat{\otimes} v$ on the Hilbert space $H \hat{\otimes} K$.

To show: (a) If $u \in B(H)$ and $v \in B(K)$ then $\|u \hat{\otimes} v\| = \|u\|\|v\|$.

(a) Define the maps

$$\begin{array}{rccc}
\iota_H : & B(H) & \to & B(H \hat{\otimes} K) \\
& u & \mapsto & u \hat{\otimes} id_K
\end{array}$$

and

$$\begin{array}{rccc}
\iota_K : & B(K) & \to & B(H \hat{\otimes} K) \\
& v & \mapsto & id_H \hat{\otimes} v
\end{array}$$

Then, ι_H and ι_K are injective $\mathbb{C}$-linear maps. We claim that ι_H and ι_K are in fact, *-homomorphisms. If $x \in H$, $y \in K$ and $u_1, u_2 \in B(H)$ then

$$\begin{aligned}
\iota_H(u_1u_2)(x \otimes y) &= (u_1u_2\hat{\otimes} id_K)(x \otimes y) \\
&= u_1(u_2(x))\hat{\otimes} y \\
&= (u_1\hat{\otimes} id_K)(u_2(x) \otimes y) \\
&= \iota_H(u_1)\iota_H(u_2)(x \otimes y).
\end{aligned}$$

Hence, $\iota_H(u_1u_2) = \iota_H(u_1)\iota_H(u_2)$ on $H \otimes K$. Since $H \otimes K$ is dense in $H\hat{\otimes}K$, $\iota_H(u_1u_2) = \iota_H(u_1)\iota_H(u_2)$ on $H\hat{\otimes}K$. Similarly, if $v_1, v_2 \in B(K)$ then $\iota_K(v_1v_2) = \iota_K(v_1)\iota_K(v_2)$. To show that ι_H preserves adjoints, first assume that $x_1, x_2 \in H$, $y_1, y_2 \in K$, $u \in B(H)$ and $v \in B(K)$. Then,

$$\begin{aligned}
\langle (u\hat{\otimes}v)^*(x_1 \otimes y_1), x_2 \otimes y_2\rangle &= \langle x_1 \otimes y_1, (u\hat{\otimes}v)(x_2 \otimes y_2)\rangle \\
&= \langle x_1 \otimes y_1, u(x_2) \otimes v(y_2)\rangle \\
&= \langle x_1, u(x_2)\rangle\langle y_1, v(y_2)\rangle \\
&= \langle u^*(x_1), x_2\rangle\langle v^*(y_1), y_2\rangle \\
&= \langle (u^*\hat{\otimes}v^*)(x_1 \otimes y_1), x_2 \otimes y_2\rangle.
\end{aligned}$$

So, $(u\hat{\otimes}v)^* = u^*\hat{\otimes}v^*$ on $H \otimes K$ and subsequently, on $H\hat{\otimes}K$. Consequently,

$$\iota_H(u^*) = u^*\hat{\otimes} id_K = (u\hat{\otimes} id_K)^* = \iota_H(u)^*.$$

Of course, we also have $\iota_K(v^*) = \iota_K(v)^*$. Therefore, ι_H and ι_K are injective *-homomorphisms and are thus, isometries by Theorem 1.6.4. Hence,

$$\|u\hat{\otimes}v\| = \|(u\hat{\otimes} id_K)(id_H\hat{\otimes}v)\| \leq \|\iota_H(u)\|\|\iota_K(v)\| = \|u\|\|v\|.$$

To prove the reverse inequality, assume that $\epsilon \in \mathbb{R}_{>0}$. Assume without loss of generality that $u, v \neq 0$. Then, there exists unit vectors $x \in H$ and $y \in K$ such that

$$\|u(x)\| > \|u\| - \epsilon > 0 \qquad \text{and} \qquad \|v(y)\| > \|v\| - \epsilon > 0.$$

This means that

$$\|(u\hat{\otimes}v)(x \otimes y)\| = \|u(x)\|\|v(y)\| > (\|u\| - \epsilon)(\|v\| - \epsilon).$$

Since $\epsilon \in \mathbb{R}_{>0}$ was arbitrary, we obtain the desired inequality. So, $\|u\hat{\otimes}v\| = \|u\|\|v\|$. □

So far, we understand how to construct Hilbert spaces by using the tensor product (in the category of $\mathbb{C}$-vector spaces) and then completing to a Hilbert space. In our quest to define the tensor product of C*-algebras, we now have to define a suitable notion of multiplication and involution.

We will first address multiplication. Let A and B be $\mathbb{C}$-algebras. Since A and B are $\mathbb{C}$-vector spaces, we can construct the tensor product $A \otimes B$. We will construct multiplication on $A \otimes B$ so that $A \otimes B$ is itself a $\mathbb{C}$-algebra. If $a \in A$ then let L_a denote left multiplication by a and let X be the vector space of linear maps on $A \otimes B$. Then, if $a \in A$ and $b \in B$ then $L_a \otimes L_b \in X$ and the map $(a, b) \mapsto L_a \otimes L_b$ is $\mathbb{C}$-bilinear.

By Theorem 2.11.1, there exists a unique linear map M, defined explicitly as

$$\begin{array}{rccc} M: & A \otimes B & \to & X \\ & a \otimes b & \mapsto & L_a \otimes L_b. \end{array}$$

Now consider the map

$$\begin{array}{rccc} \cdot: & (A \otimes B)^2 & \to & A \otimes B \\ & (c, d) & \mapsto & M(c)(d). \end{array} \tag{2.4}$$

This map is $\mathbb{C}$-bilinear and since M is unique, it defines the unique multiplication on $A \otimes B$. Explicitly, if $a, a' \in A$ and $b, b' \in B$ then

$$(a \otimes b) \cdot (a' \otimes b') = M(a \otimes b)(a' \otimes b') = aa' \otimes bb'.$$

Definition 2.11.2. Let A and B be $\mathbb{C}$-algebras. The tensor product of $\mathbb{C}$-vector spaces $A \otimes B$, equipped with the bilinear multiplication in equation (2.4), is a $\mathbb{C}$-algebra called the **algebra tensor product** of A and B.

Next, we address involution. Let A and B be *-algebras so that $A \otimes B$ is a $\mathbb{C}$-algebra. We will define involution on $A \otimes B$ so that $A \otimes B$ becomes a *-algebra. The obvious way to do this is to define

$$(a \otimes b)^* = a^* \otimes b^*$$

for $a \in A$ and $b \in B$. It is straightforward to check that this satisfies all the properties demanded by an involution. However, we have to show that involution is well-defined. We do this by showing that if $\sum_{j=1}^{n}(a_j \otimes b_j) = 0$ then $\sum_{j=1}^{n}(a_j^* \otimes b_j^*) = 0$.

To this end, assume that $\sum_{j=1}^{n}(a_j \otimes b_j) = 0$. Let $\{c_1, \ldots, c_m\}$ be a linearly independent set in B with the same linear span as the set $\{b_1, \ldots, b_n\}$. If $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, m\}$ then there exists scalars $\lambda_{i,j} \in \mathbb{C}$ such that

$$b_i = \sum_{j=1}^{m} \lambda_{i,j} c_j.$$

Therefore,

$$\begin{aligned}
0 &= \sum_{i=1}^{n}(a_i \otimes b_i) \\
&= \sum_{i=1}^{n}(a_i \otimes \sum_{j=1}^{m} \lambda_{i,j} c_j) \\
&= \sum_{i=1}^{n}(\sum_{j=1}^{m} \lambda_{i,j} a_i \otimes c_j)
\end{aligned}$$

So, $\sum_{j=1}^{m} \lambda_{i,j} a_i = 0$ in A since the set $\{c_1, c_2, \ldots, c_m\}$ is linearly independent. Consequently, $\sum_{j=1}^{m} \overline{\lambda_{i,j}} a_i^* = 0$ and

$$\begin{aligned}
\sum_{i=1}^{n}(a_i^* \otimes b_i^*) &= \sum_{i=1}^{n}(a_i^* \otimes \sum_{j=1}^{m} \overline{\lambda_{i,j}} c_j^*) \\
&= \sum_{i=1}^{n}(\sum_{j=1}^{m} \overline{\lambda_{i,j}} a_i^* \otimes c_j^*) \\
&= \sum_{i=1}^{n}(0 \otimes c_j^*) = 0.
\end{aligned}$$

Definition 2.11.3. Let A and B be *-algebras. The algebra tensor product $A \otimes B$, equipped with the involution above, is a *-algebra called the ***-algebra tensor product** of A and B.

In a similar manner to before, if A' and B' are *-subalgebras of A and B respectively then we can regard $A' \otimes B'$ as a *-subalgebra of $A \otimes B$.

Here is how we can form *-homomorphisms on *-algebra tensor products.

Theorem 2.11.6. *Let A, B, C, D be *-algebras. Let $\phi : A \to B$ and $\psi : C \to D$ be *-homomorphisms. Then, the map*

$$\begin{array}{rccc} \phi \otimes \psi : & A \otimes C & \to & B \otimes D \\ & a \otimes c & \mapsto & \phi(a) \otimes \psi(c) \end{array}$$

*is a *-homomorphism.*

Proof. Assume that A, B, C and D are *-algebras. Assume that $\phi : A \to B$ and $\psi : C \to D$ are *-homomorphisms. By Theorem 2.11.1, $\phi \otimes \psi$ is a linear map. Assume that $a, a' \in A$ and $c, c' \in C$. Then,

$$\begin{aligned} (\phi \otimes \psi)((a \otimes c)(a' \otimes c')) &= (\phi \otimes \psi)(aa' \otimes cc') \\ &= \phi(aa') \otimes \psi(cc') \\ &= \phi(a)\phi(a') \otimes \psi(c)\psi(c') \\ &= (\phi(a) \otimes \psi(c))(\phi(a') \otimes \psi(c')) \\ &= (\phi \otimes \psi)(a \otimes c)(\phi \otimes \psi)(a' \otimes c'). \end{aligned}$$

We also have

$$(\phi \otimes \psi)((a \otimes c)^*) = (\phi \otimes \psi)(a^* \otimes c^*) = \phi(a)^* \otimes \psi(c)^* = (\phi \otimes \psi)(a \otimes c)^*.$$

Therefore, $\phi \otimes \psi$ is a *-homomorphism from $A \otimes C$ to $B \otimes D$. □

At this point, we have set up enough machinery to discuss tensor products of C*-algebras. Previously, we mentioned that a tensor product of C*-algebras could have more than one possible norm. The next result will be used to demonstrate this claim.

Theorem 2.11.7. *Let A and B be C*-algebras. Let (φ, H) and (ψ, K) be representations of A and B respectively. Then, there exists a unique *-homomorphism $\pi : A \otimes B \to B(H \hat{\otimes} K)$ such that if $a \in A$ and $b \in B$ then*

$$\pi(a \otimes b) = \varphi(a) \hat{\otimes} \psi(b).$$

Furthermore, if φ and ψ are injective then π is injective.

Proof. Assume that A and B are C*-algebras with representations (φ, H) and (ψ, K) respectively. Define the maps

$$\begin{array}{rccc} \varphi' : & A & \to & B(H \hat{\otimes} K) \\ & a & \mapsto & \varphi(a) \hat{\otimes} id_K \end{array}$$

and

$$\begin{array}{rccc} \psi' : & B & \to & B(H\hat{\otimes}K) \\ & b & \mapsto & id_H\hat{\otimes}\psi(b). \end{array}$$

By Theorem 2.11.6, φ' and ψ' are *-homomorphisms. Furthermore, the images $\varphi'(A)$ and $\psi'(B)$ commute. Now define the map

$$\begin{array}{rccc} \alpha : & A\times B & \to & B(H\hat{\otimes}K) \\ & (a,b) & \mapsto & \varphi'(a)\psi'(b). \end{array}$$

Then, α is a bilinear map. By the universal property of the tensor product in Theorem 2.11.1, there exists a unique linear map $\pi : A\otimes B \to B(H\hat{\otimes}K)$ defined by

$$\pi(a\otimes b) = \alpha(a,b) = \varphi'(a)\psi'(b) = \varphi(a)\hat{\otimes}\psi(b).$$

Since φ' and ψ' are both *-homomorphisms, we deduce that π is also a *-homomorphism as required.

Now, assume that φ and ψ are both injective. Assume that $z \in \ker\pi$. Then, $z = \sum_{j=1}^{n}(a_j\otimes b_j)$, where $a_j \in A$ and $b_j \in B$. We may also assume that $\{b_1, b_2, \ldots, b_n\}$ is linearly independent. Since ψ is injective, then the set $\{\psi(b_1), \psi(b_2), \ldots, \psi(b_n)\}$ is linearly independent.

We have

$$\pi(z) = \sum_{j=1}^{n}(\varphi(a_j)\hat{\otimes}\psi(b_j)) = 0.$$

To show: (a) If $j \in \{1, 2, \ldots, n\}$ then $\varphi(a_j) = 0$.

(a) Assume that $h\hat{\otimes}k \in H\hat{\otimes}K$. Then, we can construct an orthonormal set $\{e_1, e_2, \ldots, e_m\}$ such that

$$\mathbb{C}\varphi(a_1)(h\hat{\otimes}k) + \cdots + \mathbb{C}\varphi(a_n)(h\hat{\otimes}k) \subseteq \mathbb{C}e_1 + \cdots + \mathbb{C}e_m.$$

Then, there exists $\lambda_{i,j} \in \mathbb{C}$ such that

$$\varphi(a_i)(h\hat{\otimes}k) = \sum_{j=1}^{m}\lambda_{i,j}e_j.$$

Now, if $h'\hat{\otimes}k' \in H\hat{\otimes}K$ then

$$\begin{aligned}
0 &= \sum_{i=1}^{n}(\varphi(a_i)(h\hat{\otimes}k)\hat{\otimes}\psi(b_i)(h'\hat{\otimes}k')) \\
&= \sum_{i=1}^{n}(\sum_{j=1}^{m}\lambda_{i,j}e_m\hat{\otimes}\psi(b_i)(h'\hat{\otimes}k')) \\
&= (\sum_{j=1}^{m}e_m\hat{\otimes}\sum_{i=1}^{n}\lambda_{i,j}\psi(b_i)(h'\hat{\otimes}k')).
\end{aligned}$$

Therefore, $\sum_{i=1}^{n}\lambda_{i,j}\psi(b_i)(h'\hat{\otimes}k') = 0$ and since the element $h'\hat{\otimes}k' \in H\hat{\otimes}K$ is arbitrary, then $\sum_{i=1}^{n}\lambda_{i,j}\psi(b_i) = 0$. By linear independence of $\{\psi(b_1), \dots, \psi(b_n)\}$, if $i \in \{1, 2, \dots, n\}$ and $j \in \{1, 2, \dots, m\}$ then $\lambda_{i,j} = 0$. Thus, if $i \in \{1, 2, \dots, n\}$ then $\varphi(a_i) = 0$ as required.

By part (a) and the fact that φ is injective, we find that $a_1 = \cdots = a_n = 0$. Hence, $z = \sum_{j=1}^{n}(a_j \otimes b_j) = 0$ and π is injective as required. □

The *-homomorphism π in Theorem 2.11.7 is usually denoted as $\varphi\hat{\otimes}\psi$.

Definition 2.11.4. Let A be a C*-algebra and $S(A)$ denote the set of states of A. If $\phi \in S(A)$ then let $(\pi_\phi, H_\phi, \xi_\phi)$ be the GNS representation associated to A. Then, the direct sum of all GNS representations

$$(\bigoplus_{\phi\in S(A)}\pi_\phi, \bigoplus_{\phi\in S(A)}H_\phi)$$

is called the **universal representation** of A. Note that the universal representation is faithful — the *-homomorphism $\bigoplus_{\phi\in S(A)}\pi_\phi$ is injective.

Now let A and B be C*-algebras with universal representations (φ, H) and (ψ, K) respectively. By Theorem 2.11.7, there exists a unique injective *-homomorphism $\pi : A \otimes B \to B(H\hat{\otimes}K)$ such that $\pi(a \otimes b) = \varphi(a)\hat{\otimes}\psi(b)$. Define the function

$$\begin{array}{rccc}
\|-\|_* : & A \otimes B & \to & \mathbb{R}_{\geq 0} \\
& c & \mapsto & \|\pi(c)\|.
\end{array}$$

Since the norm on $B(H\hat{\otimes}K)$ satisfies the property of a C*-algebra, $\|-\|_*$ defines a C*-norm on $A \otimes B$.

Definition 2.11.5. Let A and B be C*-algebras with universal representations (φ, H) and (ψ, K) respectively. Let $\pi : A \otimes B \to B(H\hat{\otimes}K)$ denote the unique injective *-homomorphism constructed in Theorem 2.11.7. Then, the C*-norm $\|-\|_*$ on $A \otimes B$ is called the **spatial C*-norm**.

One of the basic properties of the spatial C*-norm is that if $a \in A$ and $b \in B$ then

$$\|a \otimes b\|_* = \|\pi(a \otimes b)\| = \|\varphi(a)\hat{\otimes}\psi(b)\| = \|\varphi(a)\|\|\psi(b)\| = \|a\|\|b\|$$

where in the last equality, we used Theorem 1.6.4.

Definition 2.11.6. Let A and B be C*-algebras. The C*-completion of $A \otimes B$ with respect to the spatial norm $\|-\|_*$ is called the **spatial tensor product** of A and B and is denoted by $A \otimes_* B$.

We will now demonstrates that there can be more than one C*-norm on $A \otimes B$.

Definition 2.11.7. Let A and B be C*-algebras and γ be a C*-norm on $A \otimes B$. The C*-completion of $A \otimes B$ with respect to γ will be denoted by $A \otimes_\gamma B$.

Theorem 2.11.8. *Let A and B be C*-algebras and γ be a C*-norm on $A \otimes B$. If $a' \in A$ and $b' \in B$ then define*

$$\begin{array}{rccc} \iota_{b'}: & A & \to & A \otimes_\gamma B \\ & a & \mapsto & a \otimes b' \end{array}$$

and

$$\begin{array}{rccc} \iota_{a'}: & B & \to & A \otimes_\gamma B \\ & b & \mapsto & a' \otimes b \end{array}$$

Then, $\iota_{a'}$ and $\iota_{b'}$ are both continuous maps.

Proof. Assume that A and B are C*-algebras. Assume that γ is a C*-norm on $A \otimes B$. Assume that $a' \in A$, $b' \in B$ and that the maps $\iota_{a'}$ and $\iota_{b'}$ are defined as above. Since A, B and $A \otimes_\gamma B$ are Banach spaces, the *closed graph theorem* applies.

Hence, to show that $\iota_{a'}$ and $\iota_{b'}$ are continuous, it suffices to show that their graphs are closed.

To show: (a) If $\{a_n\}_{n\in\mathbb{Z}_{>0}}$ is a sequence converging to 0 in A and the sequence $\{\iota_{b'}(a_n)\}_{n\in\mathbb{Z}_{>0}}$ converges to $c \in A \otimes_\gamma B$ then $c = 0$.

(a) Assume that $\{a_n\}_{n\in\mathbb{Z}_{>0}}$ is a sequence which converges to 0 in A. Assume that $\{\iota_{b'}(a_n)\}_{n\in\mathbb{Z}_{>0}}$ converges to $c \in A \otimes_\gamma B$. Note that we can assume this without loss of generality because every Banach space is a topological vector space and addition is itself continuous in a topological vector space.

Furthermore, we can assume that if $n \in \mathbb{Z}_{>0}$ then a_n and b' are positive (if not, we can simply replace a_n with $a_n^* a_n$ and b' by $(b')^* b'$). Hence, the sequence $\{\iota_{b'}(a_n)\}_{n\in\mathbb{Z}_{>0}}$ converges to $c \in A \otimes_\gamma B$ which is positive.

Now let τ be a positive linear functional on $A \otimes_\gamma B$. The composite $\rho = \tau \circ \iota_{b'}$ is a positive linear functional on A and hence, continuous. So,

$$\tau(c) = \tau(\lim_{n\to\infty} \iota_{b'}(a_n)) = \lim_{n\to\infty} (\tau \circ \iota_{b'})(a_n) = \lim_{n\to\infty} \rho(a_n) = \rho(0) = 0.$$

So, $\tau(c) = 0$ for any positive functional τ on $A \otimes_\gamma B$. Therefore, $c = 0$ and consequently, the graph of $\iota_{b'}$ is closed.

A similar argument to part (a) establishes that the graph of $\iota_{a'}$ is closed. By the closed graph theorem, $\iota_{a'}$ and $\iota_{b'}$ are both continuous. □

The next result is integral to the construction of our next C*-norm.

Theorem 2.11.9. *Let A and B be non-zero C*-algebras and γ be a C*-norm on $A \otimes B$. Let (π, H) be a non-degenerate representation of $A \otimes_\gamma B$. Then, there exists unique *-homomorphisms $\varphi : A \to B(H)$ and $\psi : B \to B(H)$ such that if $a \in A$ and $b \in B$ then*

$$\pi(a \otimes b) = \varphi(a)\psi(b) = \psi(b)\varphi(a).$$

Moreover, the representations (φ, H) and (ψ, H) are both non-degenerate.

Proof. Assume that A and B are non-zero C*-algebras and that $A \otimes_\gamma B$ is defined as above. Assume that (π, H) is a non-degenerate representation of $A \otimes_\gamma B$.

Let $H_0 = \pi(A \otimes B)H$. If $z \in H_0$ and $i \in \{1, 2, \ldots, n\}$ then there exists $a_i \in A$, $b_i \in B$ and $x_i \in H$ such that

$$z = \sum_{i=1}^{n} \pi(a_i \otimes b_i)(x_i).$$

We want to define a well-defined map on H_0. We will deal with whether this mysterious map is well-defined first before defining it. Suppose that $z \in H_0$ has two expressions of the above form:

$$z = \sum_{i=1}^{n} \pi(a_i \otimes b_i)(x_i) = \sum_{j=1}^{m} \pi(c_j \otimes d_j)(y_j).$$

By using Theorem 1.7.3, let $\{v_\mu\}_{\mu \in M}$ be an approximate unit for B. If $a \in A$ then

$$\pi(a \otimes v_\mu)(z) = \sum_{i=1}^{n} \pi(a \otimes v_\mu)\pi(a_i \otimes b_i)(x_i) = \sum_{i=1}^{n} \pi(aa_i \otimes v_\mu b_i)(x_i).$$

But, we also have

$$\pi(a \otimes v_\mu)(z) = \sum_{j=1}^{m} \pi(ac_j \otimes v_\mu d_j)(y_j).$$

By taking the limit with respect to the variable μ, we find that

$$\begin{aligned} \lim_\mu \pi(a \otimes v_\mu)(z) &= \lim_\mu \sum_{i=1}^{n} \pi(aa_i \otimes v_\mu b_i)(x_i) \\ &= \sum_{i=1}^{n} \pi(\lim_\mu (aa_i \otimes v_\mu b_i))(x_i) \\ &= \sum_{i=1}^{n} \pi(aa_i \otimes b_i)(x_i) \end{aligned}$$

where the last equality follows from the fact that tensoring with aa_i is continuous (see Theorem 2.11.8). Therefore,

$$\sum_{i=1}^{n} \pi(aa_i \otimes b_i)(x_i) = \sum_{j=1}^{m} \pi(ac_j \otimes d_j)(y_j).$$

Therefore, if $a \in A$ then the map

$$\begin{array}{cccc} \varphi(a) : & H_0 & \to & H_0 \\ & z = \sum_{i=1}^{n} \pi(a_i \otimes b_i)(x_i) & \mapsto & \sum_{i=1}^{n} \pi(aa_i \otimes b_i)(x_i) \end{array}$$

is well-defined. Since $\varphi(a)(z) = \lim_\mu \pi(a \otimes v_\mu)(z)$, we deduce that $\varphi(a)$ is a linear map on H_0. To see that $\varphi(a)$ is bounded, we first use Theorem 2.11.8 to show that there exists $M \in \mathbb{Z}_{>0}$ depending on a such that

$$\|\pi(a \otimes b)\| \leq M\|b\|.$$

Hence,

$$
\begin{aligned}
\|\varphi(a)(z)\| &= \|\lim_{\mu} \pi(a \otimes v_\mu)(z)\| \\
&\leq \lim_{\mu} \|\pi(a \otimes v_\mu)\| \|z\| \\
&\leq \lim_{\mu} M \|v_\mu\| \|z\| \\
&\leq M \|z\|.
\end{aligned}
$$

So, $\|\varphi(a)\| \leq M$. This means that $\varphi(a)$ is a bounded linear map on H_0. Since the representation (π, H) of $A \otimes_\gamma B$ is non-degenerate, H_0 is dense in H by Theorem 1.9.4. Therefore, we can extend $\varphi(a)$ to a bounded linear map on H, which we also denote by $\varphi(a)$ as an abuse of notation.

By arguing in nearly the same fashion as before, if $b \in B$ then the map

$$
\begin{array}{cccc}
\psi(b) : & H_0 & \to & H_0 \\
& z = \sum_{i=1}^n \pi(a_i \otimes b_i)(x_i) & \mapsto & \sum_{i=1}^n \pi(a_i \otimes bb_i)(x_i)
\end{array}
$$

is a well-defined bounded linear operator on H_0, which extends to a bounded linear operator on H, which we also denote by $\psi(b)$.

Now consider the maps $\varphi : A \to B(H)$ and $\psi : B \to B(H)$, given by $a \mapsto \varphi(a)$ and $b \mapsto \psi(b)$.

To show: (a) φ is a *-homomorphism.

(b) $\pi(a \otimes b) = \varphi(a)\psi(b) = \psi(b)\varphi(a)$.

(a) Since H_0 is dense in H is suffices to check that φ is a *-homomorphism by using elements of H_0. Let $a, a' \in A$. If $z = \sum_{i=1}^n \pi(a_i \otimes b_i)(x_i) \in H_0$ then

$$
\begin{aligned}
\varphi(a + a')(z) &= \lim_{\mu} \pi((a + a') \otimes v_\mu)(z) \\
&= \lim_{\mu} \Big(\pi(a \otimes v_\mu)(z) + \pi(a' \otimes v_\mu)(z)\Big) \\
&= \varphi(a)(z) + \varphi(a')(z).
\end{aligned}
$$

If $\alpha \in \mathbb{C}$ then

$$
\varphi(\alpha a)(z) = \lim_{\mu} \pi(\alpha a \otimes v_\mu)(z) = \alpha \lim_{\mu} \pi(a \otimes v_\mu)(z) = \alpha\varphi(a)(z).
$$

We also have

$$\begin{aligned}
\varphi(a^*)(z) &= \lim_\mu \pi(a^* \otimes v_\mu)(z) \\
&= \lim_\mu \pi((a \otimes v_\mu^*)^*)(z) \\
&= \lim_\mu \pi(a \otimes v_\mu^*)^*(z) \\
&= \lim_\mu \pi(a \otimes v_\mu)^*(z) \quad \text{(by Theorem 2.3.2)} \\
&= \varphi(a)(z)^*.
\end{aligned}$$

and

$$\begin{aligned}
\varphi(aa')(z) &= \lim_\mu \pi((aa') \otimes v_\mu)(z) \\
&= \lim_\mu \lim_{\mu'} \pi((aa') \otimes v_\mu v_{\mu'})(z) \\
&= \big(\lim_\mu \pi(a \otimes v_\mu)(z)\big)\big(\lim_{\mu'} \pi(a' \otimes v_{\mu'})(z)\big) \\
&= \varphi(a)(z)\varphi(a')(z).
\end{aligned}$$

Since $z \in H_0$ was arbitrary, φ is a *-homomorphism. Note that by similar computations, ψ is also a *-homomorphism.

(b) Let $\{u_\lambda\}_{\lambda \in L}$ be an approximate unit for A. If $z \in H_0$ then

$$\begin{aligned}
\varphi(a)\psi(b)(z) &= \varphi(a)(\lim_\lambda \pi(u_\lambda \otimes b)(z)) \\
&= \lim_\lambda \lim_\mu \pi(a \otimes v_\mu)\pi(u_\lambda \otimes b)(z) \\
&= \lim_\lambda \lim_\mu \pi(au_\lambda \otimes v_\mu b)(z) \\
&= \lim_\lambda \pi(au_\lambda \otimes b)(z) \\
&= \pi(a \otimes b)(z).
\end{aligned}$$

By a similar computation, we also have $\psi(b)\varphi(a)(z) = \pi(a \otimes b)(z)$.

Now, we will address the uniqueness of φ and ψ. Assume that there exists *-homomorphisms $\varphi' : A \to B(H)$ and $\psi' : B \to B(H)$ such that if $a \in A$ and $b \in B$ then

$$\pi(a \otimes b) = \varphi'(a)\psi'(b) = \psi'(b)\varphi'(a).$$

The idea is to use the approximate units $\{u_\lambda\}_{\lambda \in L}$ and $\{v_\mu\}_{\mu \in M}$. First, we need to show that the representations (φ', H) and (ψ', H) are non-degenerate.

To this end, assume that $z \in H$ satisfies $\varphi'(a)(z) = 0$. By assumption, if $b \in B$ then

$$\pi(a \otimes b)(z) = \psi'(b)\varphi'(a)(z) = 0.$$

This holds for arbitrary $a \in A$ and $b \in B$. So, $\pi(A \otimes B)z = 0$ and since the representation (π, H) is non-degenerate, $z = 0$. Therefore, (φ', H) is a non-degenerate representation of A. By a similar argument, the representations $(\psi', H), (\psi, H)$ and (φ, H) are all non-degenerate.

Since the representations (ψ', H) and (φ', H) are non-degenerate, the nets $\{\psi'(v_\mu)\}_{\mu \in M}$ and $\{\varphi'(u_\lambda)\}_{\lambda \in L}$ both strongly converge to the identity operator id_H by Theorem 2.3.4. Now, we have

$$\lim_\mu \pi(a \otimes v_\mu) = \lim_\mu \varphi'(a)\psi'(v_\mu) = \varphi'(a)$$

But, $\lim_\mu \pi(a \otimes v_\mu) = \varphi(a)$. So, $\varphi'(a) = \varphi(a)$ and $\varphi' = \varphi$. An analogous argument reveals that $\psi' = \psi$. Hence, ψ and φ are unique. $\square$

The maps φ and ψ in Theorem 2.11.9 will be referred to as π_A and π_B respectively.

Theorem 2.11.10. *Let A and B be C*-algebras. Let γ be a C*-seminorm on $A \otimes B$. If $a \in A$ and $b \in B$ then*

$$\gamma(a \otimes b) \leq \|a\|\|b\|.$$

Proof. Assume that A and B are C*-algebras. Assume that γ is a C*-seminorm on $A \otimes B$. Let $\delta = \max(\gamma, \|-\|_*)$. Then, δ is a C*-norm on $A \otimes B$ because the spatial norm $\|-\|_*$ is itself a C*-norm on $A \otimes B$. Hence, we can form the C*-algebra $A \otimes_\delta B$.

Now let (π, H) denote the universal representation of $A \otimes_\delta B$. Then, (π, H) is faithful and non-degenerate. By Theorem 2.11.9, if $a \in A$ and $b \in B$ then $\pi(a \otimes b) = \pi_A(a)\pi_B(b)$. Since π is isometric, we have

$$\delta(a \otimes b) = \|\pi(a \otimes b)\| \leq \|\pi_A(a)\|\|\pi_B(b)\| \leq \|a\|\|b\|.$$

Therefore, $\gamma(a \otimes b) \leq \|a\|\|b\|$. $\square$

Now we will begin the construction of our next important C*-norm. Let A and B be C*-algebras. Let Γ be the set of all C*-norms on $A \otimes B$. If $c \in A \otimes B$ then define

$$\|c\|_{\max} = \sup_{\gamma \in \Gamma} \gamma(c).$$

By Theorem 2.11.10, if $c = \sum_{j=1}^{n}(a_j \otimes b_j)$ and $\gamma \in \Gamma$ then

$$\gamma(c) \leq \sum_{j=1}^{n} \gamma(a_j \otimes b_j) \leq \sum_{j=1}^{n} \|a_j\|\|b_j\|.$$

Hence, $\|c\|_{\max} < \infty$. Furthermore, it is a C*-norm on $A \otimes B$.

Definition 2.11.8. Let A and B be C*-algebras. The norm $\|-\|_{\max}$ defined above is called the **maximal C*-norm** on $A \otimes B$. The C*-completion of the tensor product $A \otimes B$ with respect to the maximal C*-norm $\|-\|_{\max}$ is called the **maximal tensor product** of A and B and is denoted by $A \otimes_{\max} B$.

The name "maximal tensor product" is not just for show. In fact, the maximal tensor product satisfies the following universal property:

Theorem 2.11.11. *Let A, B and C be C*-algebras. Let $\varphi : A \to C$ and $\psi : B \to C$ be *-homomorphisms such that the image $\varphi(A)$ commutes with $\psi(B)$. Then, there exists a unique *-homomorphism $\pi : A \otimes_{\max} B \to C$ such that if $a \in A$ and $b \in B$ then*

$$\pi(a \otimes b) = \varphi(a)\psi(b).$$

Proof. Assume that A, B and C are C*-algebras. Assume that $\varphi : A \to C$ and $\psi : B \to C$ are *-homomorphisms such that $\varphi(A)$ commutes with $\psi(B)$.

By Theorem 2.11.1, there exists a unique *-homomorphism $\pi : A \otimes B \to C$ such that if $a \in A$ and $b \in B$ then

$$\pi(a \otimes b) = \varphi(a)\psi(b).$$

Now consider the map

$$\begin{array}{rccc} \gamma : & A \otimes B & \to & \mathbb{R}_{\geq 0} \\ & c & \mapsto & \|\pi(c)\|. \end{array}$$

Then, γ is a C*-seminorm on $A \otimes B$ (seminorm because π may not be injective). By Theorem 2.11.10, $\gamma(c) \leq \delta(c) \leq \|c\|_{\max}$ because δ is a

C*-norm on $A \otimes B$. Therefore, $\|\pi(c)\| \leq \|c\|_{\max}$ and π is a norm-decreasing *-homomorphism on $A \otimes B$.

Consequently, we can extend π to a norm-decreasing *-homomorphism on the maximal tensor product $A \otimes_{\max} B$. $\square$

Now, we arrive at the definition of a class of C*-algebras which has been the subject of much research.

Definition 2.11.9. Let A be a C*-algebra. We say that A is **nuclear** if for all C*-algebras B, the tensor product $A \otimes B$ has exactly one C*-norm.

Here is an important remark we will use shortly. Let A be a *-algebra equipped with a complete C*-norm $\|-\|$. We claim that this is the only C*-norm on A. Let γ be another C*-norm on A and B be the completion of A with respect to γ. Define the inclusion map

$$\begin{array}{rccc} \iota : & (A, \|-\|) & \to & (B, \gamma) \\ & a & \mapsto & a. \end{array}$$

Then, ι is an injective *-homomorphism and thus, isometric. Hence, if $a \in A$ then $\gamma(a) = \|a\|$ and as a result, $\gamma = \|-\|$.

Example 2.11.2. Let $n \in \mathbb{Z}_{\geq 1}$. We claim that the C*-algebra $M_{n\times n}(\mathbb{C})$ is nuclear. Assume that A is a C*-algebra. To see that the tensor product $M_{n\times n}(\mathbb{C}) \otimes A$ has exactly one C*-norm, it suffices by the previous remark to show that $M_{n\times n}(\mathbb{C}) \otimes A$ admits a complete C*-norm.

Define the map

$$\begin{array}{rccc} \pi' : & M_{n\times n}(\mathbb{C}) \times A & \to & M_{n\times n}(A) \\ & ((\lambda_{i,j}), a) & \mapsto & (\lambda_{i,j} a). \end{array}$$

It is straightforward to verify that π' is a bilinear map. By the universal property in Theorem 2.11.1, there exists a unique linear map $\pi : M_{n\times n}(\mathbb{C}) \otimes A \to M_{n\times n}(A)$ such that

$$\pi((\lambda_{i,j}) \otimes a) = (\lambda_{i,j} a).$$

Next, we show that π is a *-homomorphism. Assume that $(\lambda_{i,j}) \in M_{n\times n}(\mathbb{C})$ and $a \in A$. Then,

$$\begin{aligned} \pi(((\lambda_{i,j}) \otimes a)^*) &= \pi((\overline{\lambda_{j,i}}) \otimes a^*) \\ &= (\overline{\lambda_{j,i}} a^*) \\ &= (\lambda_{i,j} a)^* = \pi((\lambda_{i,j}) \otimes a)^* \end{aligned}$$

and π is also multiplicative by direct computation. Therefore, π is a *-homomorphism. We now claim that π is actually a *-isomorphism.

To show: (a) π is surjective.

(b) π is injective.

(a) Assume that $X = (x_{ij}) \in M_{n\times n}(A)$. If $i, j \in \{1, 2, \ldots, n\}$ then let $e_{i,j} \in M_{n\times n}(\mathbb{C})$ be the matrix unit with a 1 in the ij position and zeros elsewhere. Then, $\pi(e_{i,j} \otimes x_{i,j})$ is the matrix with $x_{i,j} \in A$ in the ij position and zeros elsewhere. By linearity of π, we have

$$\pi(\sum_{i=1}^{n}\sum_{j=1}^{n}(e_{i,j} \otimes x_{i,j})) = X.$$

So, π is surjective.

(b) Assume that $(\lambda_{i,j}) \otimes a \in \ker \pi$ so that

$$\pi((\lambda_{i,j}) \otimes a) = (\lambda_{i,j}a) = 0.$$

Then either $a = 0$ or if $i, j \in \{1, 2, \ldots, n\}$ then $\lambda_{i,j} = 0$ and the matrix $(\lambda_{i,j})$ is the zero matrix. In either case, $(\lambda_{i,j}) \otimes a = 0$ and π must be injective.

By parts (a) and (b), we deduce that π is a *-isomorphism. Now, we define the map $\|-\|$ by

$$\begin{array}{rccc} \|-\|: & M_{n\times n}(\mathbb{C}) \otimes A & \to & \mathbb{C} \\ & (\lambda_{i,j}) \otimes a & \mapsto & \|\pi((\lambda_{i,j}) \otimes a)\| \end{array}$$

Since $M_{n\times n}(A)$ is a C*-algebra with its norm (see [Mur90, Theorem 3.4.2]), $\|-\|$ defines a C*-norm on the tensor product $M_{n\times n}(\mathbb{C}) \otimes A$. It is complete because π is a *-isomorphism and is thus, isometric. By the preceding remark, $\|-\|$ is the only C*-norm on $M_{n\times n}(\mathbb{C}) \otimes A$. Since A was arbitrary, we deduce that $M_{n\times n}(\mathbb{C})$ is a nuclear C*-algebra.

Recalling the structure of finite-dimensional C*-algebras from Theorem 2.10.3, we suspect that in light of above example, every finite-dimensional C*-algebra must be nuclear. We prove this in Theorem 2.11.12 below.

Theorem 2.11.12. *Let A be a finite-dimensional C*-algebra. Then, A is a nuclear C*-algebra.*

Proof. Assume that A is a finite-dimensional C*-algebra. By Theorem 2.10.3, there exists $k \in \mathbb{Z}_{>0}$ and $n_1, \ldots, n_k \in \mathbb{Z}_{>0}$ such that

$$A \cong \bigoplus_{i=1}^{k} M_{n_i \times n_i}(\mathbb{C}).$$

as C*-algebras. Let B be a C*-algebra and define the map

$$\begin{array}{cccc} \phi : & A \times B & \to & \bigoplus_{i=1}^{k}(M_{n_i \times n_i}(\mathbb{C}) \otimes B) \\ & (a,b) = ((a_1, \ldots, a_k), b) & \mapsto & (a_1 \otimes b, \ldots, a_k \otimes b) \end{array}$$

where if $i \in \{1, 2, \ldots, k\}$ then $a_i \in M_{n_i \times n_i}(\mathbb{C})$. It is easy to verify that ϕ is a bilinear map. By Theorem 2.11.1, there exists a unique linear map ψ, defined explicitly by

$$\begin{array}{cccc} \psi : & A \otimes B & \to & \bigoplus_{i=1}^{k}(M_{n_i \times n_i}(\mathbb{C}) \otimes B) \\ & a \otimes b = (a_1, \ldots, a_k) \otimes b & \mapsto & (a_1 \otimes b, \ldots, a_k \otimes b). \end{array}$$

By direct computation, ψ is in fact, a *-homomorphism. To see that ψ is injective, assume that $(a_1, \ldots, a_k) \otimes b \in \ker \psi$ so that $\psi((a_1, \ldots, a_k) \otimes b) = 0$. Then, either $b = 0$ or if $i \in \{1, \ldots, k\}$ then $a_i = 0$. In either case, $(a_1, \ldots, a_k) \otimes b = 0$. So, ψ is injective.

To see that ψ is surjective, assume that

$$(a_1 \otimes b_1, \ldots, a_k \otimes b_k) \in \bigoplus_{i=1}^{k}(M_{n_i \times n_i}(\mathbb{C}) \otimes B).$$

If $i \in \{1, 2, \ldots, k\}$ then let $e_i \in \bigoplus_{i=1}^{k} M_{n_i \times n_i}(\mathbb{C}) \cong A$ be the tuple with the identity matrix in the i^{th} position and zero matrices elsewhere. Since ψ is linear, we have

$$\psi(\sum_{i=1}^{k}(a_i e_i \otimes b_i)) = (a_1 \otimes b_1, \ldots, a_k \otimes b_k).$$

Hence, ψ is a *-isomorphism. Now define the map

$$\begin{array}{cccc} \|-\| : & A \otimes B & \to & \mathbb{R}_{\geq 0} \\ & a \otimes b = (a_1, \ldots, a_k) \otimes b & \mapsto & \|\psi((a_1, \ldots, a_k) \otimes b)\|. \end{array}$$

Since the direct sum $\bigoplus_{i=1}^{k}(M_{n_i \times n_i}(\mathbb{C}) \otimes B)$ is a C*-algebra, then $\|-\|$ must be a C*-norm on $A \otimes B$. It is also complete because ψ is a *-isomorphism and thus, isometric. Hence, $\|-\|$ is the only C*-norm on $A \otimes B$ and hence, A is a nuclear C*-algebra. $\square$

Now, we will show that there are many examples of C*-algebras with the next few results.

Theorem 2.11.13. *Let A be a C*-algebra and $\mathcal{S}$ be a non-empty, upwards directed set of C*-subalgebras of A. Suppose that the union $\bigcup_{T\in\mathcal{S}} T$ is dense in A and that every element of $\mathcal{S}$ is nuclear. Then, A is a nuclear C*-algebra.*

Proof. Assume that A is a C*-algebra and $\mathcal{S}$ is defined as above. Assume that every C*-subalgebra of A in $\mathcal{S}$ is nuclear.

Assume that B is a C*-algebra and that β and γ are C*-norms on $A \otimes B$. We will show that $\beta = \gamma$. Let $C = (\bigcup_{T\in\mathcal{S}} T) \otimes B$, where we regard $T \otimes B$ as a *-subalgebra of $A \otimes B$ for $T \in \mathcal{S}$. Then, C is a C*-subalgebra of $A \otimes B$ and is dense in $A \otimes_\beta B$ and $A \otimes_\gamma B$ (see Theorem 2.11.8).

We assumed that if $T \in \mathcal{S}$ then T is nuclear. Since $T \otimes B$ only has one C*-norm, $\beta = \gamma$ on $T \otimes B$. Since $T \in \mathcal{S}$ was arbitrary, $\beta = \gamma$ on C. Hence, the identity map $id_C : (C, \beta) \to (C, \gamma)$ is bounded and by density of C, extends to a *-homomorphism $\pi : A \otimes_\beta B \to A \otimes_\gamma B$.

Now assume that $a \in A$ and $b \in B$. Then, there exists a sequence $\{a_n\}_{n\in\mathbb{Z}_{>0}}$ in $\bigcup_{T\in\mathcal{S}} T$ such that $\lim_{n\to\infty} a_n = a$. By Theorem 2.11.8,

$$\lim_{n\to\infty} (a_n \otimes b) = a \otimes b.$$

Therefore,

$$\pi(a \otimes b) = \lim_{n\to\infty} \pi(a_n \otimes b) = \lim_{n\to\infty} (a_n \otimes b) = a \otimes b$$

where the convergence in the last equality is with respect to the C*-norm γ. We conclude that π is the identity map on $A \otimes B$. Consequently, if $c \in A \otimes B$ then

$$\gamma(c) = (\gamma \circ \pi)(c) = \beta(c).$$

So, $\beta = \gamma$ on $A \otimes B$ and thus, A is a nuclear C*-algebra. □

As a consequence of Theorem 2.11.13, if H is a Hilbert space then the C*-algebra of compact operators $B_0(H)$ is nuclear. See [Mur90, Example 6.3.2].

Theorem 2.11.14. *Let A be an AF-algebra. Then, A is a nuclear C*-algebra.*

Proof. Assume that A is an AF-algebra. Then, there exists an increasing sequence of finite-dimensional C*-algebras $\{B_n\}_{n\in\mathbb{Z}_{>0}}$ such that $\bigcup_{i=1}^{n} B_n$ is dense in A.

The set $\{B_n \mid n \in \mathbb{Z}_{>0}\}$ of C*-subalgebras of A is upwards directed and each element of the set is a nuclear C*-algebra by Theorem 2.11.12. By Theorem 2.11.13, we find that A is a nuclear C*-algebra as required. $\square$

In particular, since UHF algebras are AF-algebras by definition, UHF algebras are also nuclear C*-algebras. By Theorem 2.10.12, we technically have uncountably many examples of nuclear C*-algebras.

2.12 More on irreducible representations and pure states

Let A and B be C*-algebras and $\|-\|_{\max}$ denote the maximal C*-norm on the tensor product $A \otimes B$. By construction, $\|-\|_{\max}$ is the largest of all possible C*-norms. From this, one might wonder if there is a *smallest* possible C*-norm on $A \otimes B$. It turns out that the spatial C*-norm $\|-\|_*$ is the least C*-norm on $A \otimes B$, but the proof of this is highly technical and makes use of results about irreducible representations of C*-algebras and pure states. With the interest of following [Mur90, Section 6.4] in the next section, this section is dedicated to stating and proving the preliminary results we need in the following section.

We will follow [Mur90, Section 5.1]. In these notes, the GNS construction was previously done for a **unital C*-algebra**. We note here that the construction generalises in a straightforward fashion to an arbitrary C*-algebra (see [Mur90, Section 3.4]). However, we will have to do something different to recover the unit cyclic vector associated to the GNS representation of an arbitrary C*-algebra.

Theorem 2.12.1. *Let A be a C*-algebra and $\tau : A \to \mathbb{C}$ be a state on A. Let (φ_τ, H_τ) denote the GNS representation of A. Then, there exists a unique unit cyclic vector $\xi_\tau \in A$ such that if $a \in A$ then*

$$\tau(a) = \langle a + N_\tau, \xi_\tau\rangle \qquad \textit{and} \qquad \varphi_\tau(a)\xi_\tau = a + N_\tau.$$

Proof. Assume that A is a C*-algebra and that τ is a state on A. Assume that (φ_τ, H_τ) is the GNS representation of A. Define the map

$$\begin{array}{rccc} \rho_0 : & A/N_\tau & \to & \mathbb{C} \\ & a + N_\tau & \mapsto & \tau(a). \end{array}$$

Obviously, ρ_0 is linear. To see that ρ_0 is well-defined, assume that $a + N_\tau = b + N_\tau$. Then, $\tau((a-b)^*(a-b)) = 0$. If $c \in A$ then by the Cauchy-Schwarz inequality,

$$\begin{aligned} |\tau(c^*(a-b))|^2 &= |\langle a - b + N_\tau, c + N_\tau \rangle|^2 \\ &\leq \|a - b + N_\tau\|^2 \|c + N_\tau\|^2 \\ &= \tau((a-b)^*(a-b)) \|c + N_\tau\|^2 = 0. \end{aligned}$$

Therefore, $\langle a - b + N_\tau, c + N_\tau \rangle = \tau(c^*(a-b)) = 0$ and subsequently, $a - b + N_\tau = N_\tau$ and $a + N_\tau = b + N_\tau$ as required. So, ρ_0 is well-defined.

To see that ρ_0 is bounded, recall that since τ is a state, $\|\tau\| = 1$. Let $\pi : A \to A/N_\tau$ denote the projection *-homomorphism. Then, π is bounded (recall that $\|\pi\| = 1$) and surjective. By the open mapping theorem, it is an open map. So, there exists $r \in \mathbb{R}_{>0}$ such that

$$B_{A/N_\tau}(0,1) \subseteq \pi(B_A(0,r))$$

where $B_{A/N_\tau}(0,1)$ is the open ball centred at $0 = N_\tau$ with radius 1 and $B_A(0,r)$ is the open ball centred at $0 \in A$ with radius r. By increasing $r \in \mathbb{R}_{>0}$ as necessary, we may assume that

$$\overline{B}_{A/N_\tau}(0,1) \subseteq \pi(B_A(0,r))$$

where $\overline{B}_{A/N_\tau}(0,1)$ is the closed unit ball of A/N_τ. Now assume that $a + N_\tau \in \overline{B}_{A/N_\tau}(0,1)$. Then, $\|a\| < r$ and

$$|\rho_0(a + N_\tau)| = |\tau(a)| \leq \|\tau\| \|a\| < r.$$

By taking the supremum over all element of the closed ball $\overline{B}_{A/N_\tau}(0,1)$, we deduce that $\|\rho_0\| < r$. Hence, ρ_0 is bounded.

Recall from the GNS construction that H_τ is the completion of A/N_τ. Since ρ_0 is a bounded linear functional on A/N_τ, we can extend it to a bounded linear functional on H_τ, which we will call ρ. Since H_τ is a Hilbert space, we can apply the Riesz representation theorem to find a unique element $\xi_\tau \in H_\tau$ such that if $y \in H_\tau$ then

$$\rho(y) = \langle y, \xi_\tau \rangle.$$

Hence, if $a \in A$ then

$$\tau(a) = \rho_0(a + N_\tau) = \rho(a + N_\tau) = \langle a + N_\tau, \xi_\tau \rangle.$$

If in addition $b \in A$ then

$$\begin{aligned}\langle b + N_\tau, \varphi_\tau(a)\xi_\tau \rangle &= \langle \varphi_\tau(a^*)(b + N_\tau), \xi_\tau \rangle \\ &= \langle a^*b + N_\tau, \xi_\tau \rangle \\ &= \tau(a^*b) = \langle b + N_\tau, a + N_\tau \rangle.\end{aligned}$$

Since this hold for arbitrary $b + N_\tau \in A/N_\tau$, it must also hold in H_τ. Therefore, $\varphi_\tau(a)\xi_\tau = a + N_\tau$. Note from this that the image $\varphi_\tau(A)\xi_\tau = A/N_\tau$ which is dense in H_τ. This means that ξ_τ is a cyclic vector for the GNS representation of A.

Finally, to see that ξ_τ is a unit vector, let $\{u_\lambda\}_{\lambda \in L}$ be an approximate unit for A. Then, the family $\{\varphi_\tau(u_\lambda)\}_{\lambda \in L}$ is an approximate unit for $\varphi_\tau(A)$ and thus, it must converge strongly to the identity operator id_{H_τ} by Theorem 2.3.4. So,

$$\begin{aligned}\|\xi_\tau\|^2 &= \langle \xi_\tau, \xi_\tau \rangle \\ &= \lim_\lambda \langle \varphi_\tau(u_\lambda)(\xi_\tau), \xi_\tau \rangle \\ &= \lim_\lambda \tau(u_\lambda) \\ &= \|\tau\| = 1.\end{aligned}$$

Note that the second last equality follows from Theorem 2.3.5. This completes the proof. □

For the next result, we make a quick definition. If ρ and τ are positive linear functionals on a C*-algebra A then we say that $\rho \leq \tau$ if $\tau - \rho$ is itself a positive linear functional on A.

Theorem 2.12.2. *Let A be a C*-algebra. Let $\tau : A \to \mathbb{C}$ be a state and $\rho : A \to \mathbb{C}$ be a positive linear functional. Let $(\varphi_\tau, H_\tau, \xi_\tau)$ be the GNS representation of A associated to the state τ. Then, there exists a unique operator $v \in \varphi_\tau(A)'$ such that $0 \leq v \leq id_{H_\tau}$ and if $a \in A$ then*

$$\rho(a) = \langle \varphi_\tau(a)v\xi_\tau, \xi_\tau \rangle.$$

Proof. Assume that A is a C*-algebra. Assume that τ is a state on A and ρ is a positive linear functional on A. Assume that $(\varphi_\tau, H_\tau, \xi_\tau)$ is the GNS representation of A associated to the state τ.

First, define the map σ by

$$\begin{array}{rccc} \sigma: & A/N_\tau \times A/N_\tau & \to & \mathbb{C} \\ & (a+N_\tau, b+N_\tau) & \mapsto & \langle a+N_\tau, b+N_\tau\rangle = \rho(b^*a). \end{array}$$

First, we claim that σ is well-defined. Assume that $a_1, a_2 \in A$ satisfy $a_1 + N_\tau = a_2 + N_\tau$. Arguing in a similar fashion to the GNS construction and using Theorem 2.3.8, we deduce that if $b \in A$ then $\tau(b^*(a_1 - a_2)) = 0$. Subsequently,

$$\sigma(a_1 - a_2 + N_\tau, b + N_\tau) = \rho(b^*(a_1 - a_2)) \leq \tau(b^*(a_1 - a_2)) = 0.$$

Therefore, $\sigma(a_1 + N_\tau, b + N_\tau) = \sigma(a_2 + N_\tau, b + N_\tau)$. By a similar argument for the second argument, we deduce that σ is well-defined. We also see that σ is a sesquilinear form because ρ is linear.

Next, to see that σ is bounded, we use the Cauchy-Schwarz inequality to deduce that

$$\sigma(a + N_\tau, b + N_\tau)^2| \leq \|a + N_\tau\|^2 \|b + N_\tau\|^2 = \rho(a^*a)\rho(b^*b) \leq \tau(a^*a)\tau(b^*b).$$

By taking the supremum over all $a, b \in A$ such that $\|a + N_\tau\| = \|b + N_\tau\| = 1$, we find that $\|\sigma\| \leq 1$ because τ is a state and hence, satisfies $\|\tau\| = 1$. Since σ is a bounded sesquilinear form on A/N_τ and H_τ is the completion of A/N_τ, we can extend σ to a sesquilinear form on H_τ which has norm less than or equal to 1. We will abuse notation and call the sesquilinear form σ.

Assume that $\psi \in H_\tau$. By applying the Riesz representation theorem to each of the linear functionals $\xi \mapsto \sigma(\xi, \psi)$, we obtain an operator $v \in B(H_\tau)$ such that if $\xi \in H_\tau$ then

$$\sigma(\xi, \psi) = \langle v\xi, \psi\rangle.$$

To be clear, the inner product on the RHS is the inner product from H_τ. Observe that if $\xi, \psi \in H_\tau$ then

$$\begin{aligned}
\|v\|^2 &= \sup_{\|\xi\|=1} \|v\xi\|^2 \\
&= \sup_{\|\xi\|=1} \langle v\xi, v\xi\rangle \\
&\leq \sup_{\|\xi\|=1} \sigma(\xi, v\xi) \\
&\leq \sup_{\|\xi\|=1} \|\sigma\|\|\xi\|\|v\xi\| \\
&\leq \sup_{\|\xi\|=1} \|v\|\|\xi\|^2 = \|v\|.
\end{aligned}$$

Therefore, $\|v\| \leq 1$ and consequently, $v \leq id_{H_\tau}$. Next, to see that v is positive, we will show that if $a \in A$ then $\langle v(a + N_\tau), a + N_\tau\rangle \geq 0$. So, assume that $a \in A$. Then,

$$\begin{aligned}
\langle v(a + N_\tau), a + N_\tau\rangle &= \sigma(a + N_\tau, a + N_\tau) \\
&= \rho(a^*a) \geq 0.
\end{aligned}$$

Therefore, v is positive. Now, we will show that $v \in \varphi_\tau(A)'$. Assume that $a, b, c \in A$. Then,

$$\begin{aligned}
\langle \varphi_\tau(a)v(b + N_\tau), c + N_\tau\rangle &= \langle v(b + N_\tau), \varphi_\tau(a^*)(c + N_\tau)\rangle \\
&= \langle v(b + N_\tau), a^*c + N_\tau\rangle \\
&= \sigma(b + N_\tau, a^*c + N_\tau) = \rho(c^*ab) \\
&= \sigma(ab + N_\tau, c + N_\tau) \\
&= \langle v(ab + N_\tau), c + N_\tau\rangle \\
&= \langle v\varphi_\tau(a)(b + N_\tau), c + N_\tau\rangle.
\end{aligned}$$

Since this holds for arbitrary $b, c \in A$, we deduce that if $a \in A$ then $\varphi_\tau(a)v = v\varphi_\tau(a)$ because A/N_τ is dense in H_τ. Therefore, $v \in \varphi_\tau(A)'$. Now let $\{u_\lambda\}_{\lambda\in L}$ be an approximate unit for A. If $b \in B$ then

$$\begin{aligned}
\rho(u_\lambda b) &= \sigma(b + N_\tau, u_\lambda + N_\tau) \\
&= \sigma(b + N_\tau, \varphi_\tau(u_\lambda)\xi_\tau) \\
&= \langle v(b + N_\tau), \varphi_\tau(u_\lambda)\xi_\tau\rangle \\
&= \langle \varphi_\tau(u_\lambda)v(b + N_\tau), \xi_\tau\rangle \\
&= \langle v\varphi_\tau(u_\lambda b)\xi_\tau, \xi_\tau\rangle.
\end{aligned}$$

Taking the limit over $\lambda \in L$, we obtain $\rho(b) = \langle v\varphi_\tau(b)\xi_\tau, \xi_\tau\rangle$. Finally, to see that v is unique, suppose that there exists $w \in \varphi_\tau(A)'$ such that if $a \in A$ then

$$\rho(a) = \langle \varphi_\tau(a)w\xi_\tau, \xi_\tau\rangle = \langle \varphi_\tau(a)v\xi_\tau, \xi_\tau\rangle.$$

If $a, b \in A$ then

$$\begin{aligned}\langle w(b + N_\tau), a + N_\tau\rangle &= \langle w(b + N_\tau), \varphi_\tau(a)\xi_\tau\rangle \\ &= \langle \varphi_\tau(a^*)w(b + N_\tau), \xi_\tau\rangle \\ &= \langle \varphi_\tau(a^*)v(b + N_\tau), \xi_\tau\rangle \\ &= \langle v(b + N_\tau), a + N_\tau\rangle.\end{aligned}$$

Since A/N_τ is dense in H_τ, we deduce that $w = v$. □

It is remarked in [Mur90, Page 142] that when faced with an arbitrary representation of a C*-algebra, we can always reduce to the case of a non-degenerate representation by Theorem 1.9.5.

The next main result we will prove is that a non-degenerate representation can be written as a direct sum of cyclic representations. We require the following preliminary definition and result.

Definition 2.12.1. Let H be a Hilbert space and A be a *-subalgebra of $B(H)$. We say that A is **non-degenerate** on H if $\overline{AH} = H$.

Theorem 2.12.3. *Let H be a Hilbert space and A be a non-degenerate *-subalgebra of $B(H)$. If $\xi \in H$ then $\xi \in \overline{A\xi}$.*

Proof. Assume that H is a Hilbert space and A is a non-degenerate *-subalgebra of $B(H)$. Assume that $\xi \in H$ and let P be the projection operator onto the closed subspace $\overline{A\xi}$. If $T \in A$ then $PTP = TP$ and by taking adjoints, we find that $PT^* = PT^*P$. Since A is self-adjoint, if $T \in A$ then

$$PT = TP = PTP.$$

We conclude that $P \in A'$. Now, if $\xi' = P\xi$ and $\xi'' = (id_H - P)\xi$ then $\xi = \xi' + \xi''$. However,

$$T\xi' = TP\xi = PT\xi = T\xi.$$

Therefore, if $T \in A$ then $T\xi'' = 0$. Since A is non-degenerate, $\xi'' = 0$ and $\xi = \xi' \in \overline{A\xi}$. □

Theorem 2.12.4. *Let A be a C*-algebra and (φ, H) be a non-degenerate representation of A. Then, (φ, H) can be written as a direct sum of cyclic representations of A.*

Proof. Assume that A is a C*-algebra and (φ, H) is a non-degenerate representation of A. If $x \in H$ then define $H_x = \overline{\varphi(A)x}$. Let Λ be the set whose elements are non-empty subsets S of $H - \{0\}$ such that if $x, y \in S$ then the closed subspaces H_x and H_y are pairwise orthogonal.

Our strategy is to apply Zorn's lemma to Λ which is a poset when equipped with the relation of inclusion. To see that Λ is non-empty, assume that $x \in H - \{0\}$. Then, $H = H_x \oplus H_x^{\perp}$. Let $y \in H_x^{\perp}$ be non-zero. Then, H_y is orthogonal to H_x, $\{x, y\} \in \Lambda$ and hence, Λ is non-empty.

Now assume that $\mathcal{S}$ is a totally ordered subset of Λ. Consider the set

$$\bigcup_{S \in \mathcal{S}} S.$$

If $V \in \mathcal{S}$ then $V \subseteq \bigcup_{S \in \mathcal{S}} S$. Furthermore, it is easy to check that $\bigcup_{S \in \mathcal{S}} S \in \Lambda$. Hence, $\mathcal{S}$ has an upper bound.

By Zorn's lemma, Λ has a maximal element — we will denote this set by $\mathcal{M}$. We will show that $H = \bigoplus_{x \in \mathcal{M}} H_x$.

To show: (a) $(\bigcup_{x \in \mathcal{M}} H_x)^{\perp} = \{0\}$.

(a) Suppose for the sake of contradiction that there exists $y \in (\bigcup_{x \in \mathcal{M}} H_x)^{\perp}$ such that $y \neq 0$. If $a, b \in A$ and $x \in \mathcal{M}$ then

$$\langle \varphi(b^* a)x, y \rangle = \langle \varphi(a)x, \varphi(b)y \rangle = 0.$$

We conclude that if $x \in \mathcal{M}$ then H_x is orthogonal to H_y. Since (φ, H) is a non-degenerate representation, then by Theorem 2.12.3, $y \in \overline{\varphi(A)y} = H_y$. So, $\mathcal{M} \cup \{y\} \in \Lambda$ which contradicts the maximality of $\mathcal{M}$. Therefore, $y = 0$ and $(\bigcup_{x \in \mathcal{M}} H_x)^{\perp} = \{0\}$.

Hence, $H = \bigoplus_{x \in \mathcal{M}} H_x$ and the restricted representation $(\varphi|_{H_x}, H_x)$ of A is cyclic with cyclic vector x. Therefore, (φ, H) is the direct sum of representations (φ_x, H_x) for $x \in \mathcal{M}$. $\square$

Our next result concerns unitary equivalence.

Theorem 2.12.5. *Let A be a C*-algebra. Let (φ_1, H_1) and (φ_2, H_2) be representations of A with cyclic vectors x_1 and x_2 respectively. Then, there exists a unitary map $u : H_1 \to H_2$ such that $x_2 = u(x_1)$ and $\varphi_2(a) = u\varphi_1(a)u^*$ for $a \in A$ if and only if*

$$\langle \varphi_1(a)(x_1), x_1\rangle = \langle \varphi_2(a)(x_2), x_2\rangle$$

for $a \in A$.

Proof. Assume that A is a C*-algebra. Assume that (φ_1, H_1) and (φ_2, H_2) are representations of A with cyclic vectors x_1 and x_2 respectively. First suppose that there exists a unitary map $u : H_1 \to H_2$ such that $x_2 = u(x_1)$ and if $a \in A$ then $\varphi_2(a) = u\varphi_1(a)u^*$.

If $a \in A$ then

$$\begin{aligned}\langle \varphi_2(a)(x_2), x_2\rangle &= \langle u\varphi_1(a)u^*(x_2), x_2\rangle \\ &= \langle \varphi_1(a)u^*(x_2), u^*(x_2)\rangle \\ &= \langle \varphi_1(a)u^{-1}(x_2), u^{-1}(x_2)\rangle \\ &= \langle \varphi_1(a)(x_1), x_1\rangle.\end{aligned}$$

Conversely, assume that if $a \in A$ then $\langle \varphi_1(a)(x_1), x_1\rangle = \langle \varphi_2(a)(x_2), x_2\rangle$. Define the map u_0 by

$$\begin{array}{cccc} u_0 : & \varphi_1(A)x_1 & \to & H_2 \\ & \varphi_1(a)x_1 & \mapsto & \varphi_2(a)x_2 \end{array}$$

To see that u_0 is well-defined, suppose that $a, b \in A$ satisfy $\varphi_1(a)x_1 = \varphi_1(b)x_1$. If $c \in A$ then

$$\begin{aligned}\langle u_0(\varphi_1(a)(x_1)), \varphi_2(c)(x_2)\rangle &= \langle \varphi_2(a)(x_2), \varphi_2(c)(x_2)\rangle \\ &= \langle \varphi_1(c^*a)(x_1), x_1\rangle \\ &= \langle \varphi_1(c^*b)(x_1), x_1\rangle \\ &= \langle \varphi_2(c^*b)(x_2), x_2\rangle \\ &= \langle u_0(\varphi_1(b)(x_1)), \varphi_2(c)(x_2)\rangle.\end{aligned}$$

Since x_2 is a cyclic vector for (φ_2, H_2), then $\overline{\varphi_2(A)x_2} = H_2$ and consequently, $u_0(\varphi_1(a)(x_1)) = u_0(\varphi_1(b)(x_1))$. So, u_0 is well-defined.

To see that u_0 is an isometry, observe that if $a \in A$ then

$$\begin{aligned}\langle u_0(\varphi_1(a)(x_1)), u_0(\varphi_1(a)(x_1))\rangle &= \langle \varphi_2(a)(x_2), \varphi_2(a)(x_2)\rangle \\ &= \langle \varphi_2(a^*a)(x_2), x_2\rangle \\ &= \langle \varphi_1(a^*a)(x_1), x_1\rangle \\ &= \langle \varphi_1(a)(x_1), \varphi_1(a)(x_1)\rangle.\end{aligned}$$

Hence, u_0 is an isometry from $\varphi_1(A)x_1$ to H_2. Since $\varphi_1(A)x_1$ is dense in H_1, we can extend u_0 to an isometric linear map $u : H_1 \to H_2$. Notice that the image im $u = \overline{\varphi_2(A)x_2} = H_2$. SInce u is surjective, u must be unitary.

To see that $u(x_1) = x_2$, first observe that if $a, b \in A$ then

$$u\varphi_1(a)\varphi_1(b)x_1 = \varphi_2(ab)x_2 = \varphi_2 u(\varphi_1(b)(x_1)).$$

So, if $a \in A$ then $u\varphi_1(a) = \varphi_2(a)u$. In particular,

$$\varphi_2(a)u(x_1) = u\varphi_1(a)(x_1) = \varphi_2(a)(x_2).$$

Therefore, if $a \in A$ then $\varphi_2(a)(u(x_1) - x_2) = 0$. Since the representation (φ_2, H_2) is non-degenerate, we deduce that $u(x_1) = x_2$ as required. $\square$

We have already encountered pure states in the context of GNS representations of unital C*-algebras. Let us define pure states for arbitrary C*-algebras.

Definition 2.12.2. Let A be a C*-algebra and τ be a state on A. We say that τ is **pure** if it has the following property: If ρ is a positive linear functional on A such that $\rho \leq \tau$ then there exists $t \in [0, 1]$ such that $\rho = t\tau$.

We denote the set of pure states on A by $PS(A)$.

Here is a characterisation of pure states.

Theorem 2.12.6. *Let A be a C*-algebra and τ be a state on A. Then*

1. *τ is pure if and only if its associated GNS representation $(\varphi_\tau, H_\tau, \xi_\tau)$ of A is irreducible.*
2. *If A is commutative then $PS(A) = \mathcal{M}(A)$.*

Proof. Assume that A is a C*-algebra and τ is a state on A. Let $(\varphi_\tau, H_\tau, \xi_\tau)$ be the GNS representation associated to τ. We will first prove that τ is pure if and only if $(\varphi_\tau, H_\tau, \xi_\tau)$ is irreducible.

Assume that τ is a pure state and $v \in \varphi_\tau(A)'$ such that $0 \leq v \leq id_{H_\tau}$.

To show: (a) $v \in \mathbb{C}id_{H_\tau}$.

(a) Similarly to Theorem 1.11.4, define the map

$$\begin{array}{cccc} \rho : & A & \to & \mathbb{C} \\ & a & \mapsto & \langle \varphi_\tau(a)v(\xi_\tau), \xi_\tau \rangle \end{array}$$

Then, ρ is a linear functional. To see that ρ is positive, note that if $a \in A$ then

$$\begin{aligned} \rho(a^*a) &= \langle \varphi_\tau(a^*a)v(\xi_\tau), \xi_\tau \rangle \\ &= \langle \varphi_\tau(a)v(\xi_\tau), \varphi_\tau(a)(\xi_\tau) \rangle \\ &= \langle v\varphi_\tau(a)(\xi_\tau), \varphi_\tau(a)(\xi_\tau) \rangle \geq 0 \end{aligned}$$

where the inequality follows from the assumption that v is a positive operator. To see that $\rho \leq \tau$, we have by Theorem 2.12.1,

$$\begin{aligned} \rho(a) &= \langle \varphi_\tau(a)v(\xi_\tau), \xi_\tau \rangle \\ &= \langle v\varphi_\tau(a)(\xi_\tau), \xi_\tau \rangle \\ &= \langle v(a + N_\tau), \xi_\tau \rangle \\ &\leq \|v\| \langle a + N_\tau, \xi_\tau \rangle \\ &\leq \langle a + N_\tau, \xi_\tau \rangle = \tau(a). \end{aligned}$$

Hence, ρ is a positive linear functional satisfying $\rho \leq \tau$. Since τ is pure, there exists $t \in [0,1]$ such that $\rho = t\tau$. We claim that $v = t \cdot id_{H_\tau}$. If $a, b \in A$ then

$$\begin{aligned} \langle v(a + N_\tau), b + N_\tau \rangle &= \langle v\varphi_\tau(a)(\xi_\tau), \varphi_\tau(b)(\xi_\tau) \rangle \\ &= \langle v\varphi_\tau(b^*a)(\xi_\tau), \xi_\tau \rangle \\ &= \rho(b^*a) = t\tau(b^*a) \\ &= \langle t(b^*a + N_\tau), \xi_\tau \rangle \\ &= \langle t(a + N_\tau), b + N_\tau \rangle. \end{aligned}$$

Since A/N_τ is dense in H_τ, we conclude that $v = t \cdot id_{H_\tau}$. So, $v \in \mathbb{C}id_{H_\tau}$.

From part (a), we conclude that $\varphi_\tau(A)' = \mathbb{C}id_{H_\tau}$. By Theorem 1.9.7, we deduce that the GNS representation $(\varphi_\tau, H_\tau, \xi_\tau)$ is irreducible.

Now assume that the GNS representation $(\varphi_\tau, H_\tau, \xi_\tau)$ is irreducible. Let $\rho : A \to \mathbb{C}$ be a positive linear functional such that $\rho \leq \tau$. By Theorem 2.12.2, there exists a unique operator $v \in \varphi_\tau(A)'$ such that $0 \leq v \leq id_{H_\tau}$ and if $a \in A$ then

$$\rho(a) = \langle \varphi_\tau(a) v \xi_\tau, \xi_\tau \rangle.$$

By Theorem 1.9.7, $\varphi_\tau(A)' = \mathbb{C}id_{H_\tau}$. Hence, there exists $t \in [0, 1]$ such that $v = t id_{H_\tau}$. So, $\rho = t\tau$ and τ is a pure state.

Next, assume that A is commutative. Assume that τ is a pure state on A. We will show that $\tau \in \mathcal{M}(A)$. Certainly, τ is a linear functional. It is non-zero because $\|\tau\| = 1$ by definition of a state. By Theorem 2.3.8, if $a \in A$ then $\tau(a^*) = \overline{\tau(a)}$.

To see that τ is multiplicative, we know from part (a) of the proof that $\varphi_\tau(A)' = \mathbb{C}id_{H_\tau}$. Since A is commutative, $\varphi_\tau(A) \subseteq \varphi_\tau(A)'$. So, $\varphi_\tau(A)$ consists of scalar operators and $B(H_\tau) \subseteq \varphi_\tau(A)'$. Consequently, $B(H_\tau) = \varphi_\tau(A)' = \mathbb{C}id_{H_\tau}$ and if $a, b \in A$ then

$$\begin{aligned}
\tau(ab) &= \langle \varphi_\tau(ab)(\xi_\tau), \xi_\tau \rangle \\
&= \langle \varphi_\tau(b)(\xi_\tau), \xi_\tau \rangle \\
&= \varphi_\tau(a) \langle \varphi_\tau(b)(\xi_\tau), \xi_\tau \rangle \\
&= \varphi_\tau(a) \langle \xi_\tau, \xi_\tau \rangle \langle \varphi_\tau(b)(\xi_\tau), \xi_\tau \rangle \\
&= \langle \varphi_\tau(a)(\xi_\tau), \xi_\tau \rangle \langle \varphi_\tau(b)(\xi_\tau), \xi_\tau \rangle = \tau(a)\tau(b).
\end{aligned}$$

So, τ is multiplicative and $\tau \in \mathcal{M}(A)$. We conclude that $PS(A) \subseteq \mathcal{M}(A)$.

Finally, assume that $\tau \in \mathcal{M}(A)$. Let ρ be a positive linear functional on A such that $\rho \leq \tau$. If $\tau(a) = 0$ then $\tau(a^*a) = 0$ and $\rho(a^*a) = 0$. Since ρ is positive, we have

$$|\rho(a)| \leq \rho(a^*a)^{\frac{1}{2}} = 0.$$

So, $\rho(a) = 0$ and $\ker \tau \subseteq \ker \rho$. Hence, there exists $t \in \mathbb{R}$ such that $\rho = t\tau$. Now pick $a \in A$ such that $\tau(a) = 1$. Then, $\tau(a^*a) = 1$ and

$$0 \leq \rho(a^*a) = t\tau(a^*a) = t \leq \tau(a^*a) = 1.$$

So, $t \in [0,1]$ and $\tau \in PS(A)$. So, $\mathcal{M}(A) \subseteq PS(A)$ and consequently, $PS(A) = \mathcal{M}(A)$. $\square$

Recall that the GNS construction takes a state τ on a C*-algebra A and produces a representation of A. In the next result, we go in the opposite direction. We begin with a representation of A which has a unit cyclic vector and then produce a state on A. This is similar to what was done in Theorem 1.11.4.

Theorem 2.12.7. *Let A be a C*-algebra and (φ, H) be a representation of A. Let x be a unit cyclic vector for (φ, H). Then, the function*

$$\begin{array}{rccc} \tau : & A & \to & \mathbb{C} \\ & a & \mapsto & \langle \varphi(a)(x), x\rangle \end{array}$$

is a state of A and (φ, H) is unitarily equivalent to the GNS representation $(\varphi_\tau, H_\tau, \xi_\tau)$. Moreover, if (φ, H) is an irreducible representation then τ is pure.

Proof. Assume that A is a C*-algebra and (φ, H) is a cyclic representation of A, with unit cyclic vector x. Assume that τ is defined as above. It is straightforward to check that τ is a positive linear functional.

To see that $\|\tau\| = 1$, assume that $\{u_\lambda\}_{\lambda \in L}$ is an approximate unit for A. Then,

$$\|\tau\| = \lim_\lambda \tau(u_\lambda) = \lim_\lambda \langle \varphi(u_\lambda)(x), x\rangle = \langle x, x\rangle = 1$$

because the net $\{\varphi(u_\lambda)(x)\}_{\lambda \in L}$ strongly converges to id_H. Therefore, τ is a state on A.

Now let $(\varphi_\tau, H_\tau, \xi_\tau)$ be the GNS representation associated to A. To see that $(\varphi_\tau, H_\tau, \xi_\tau)$ is unitarily equivalent to (φ, H), we observe that if $a \in A$ then

$$\langle \varphi_\tau(a)(\xi_\tau), \xi_\tau\rangle = \tau(a) = \langle \varphi(a)(x), x\rangle.$$

By Theorem 2.12.5, $(\varphi_\tau, H_\tau, \xi_\tau)$ is unitarily equivalent to (φ, H). Finally, if (φ, H) is an irreducible representation then $(\varphi_\tau, H_\tau, \xi_\tau)$ is also irreducible and by Theorem 2.12.6, τ is a pure state as required. $\square$

Recall that in the specific case where A is unital, the pure states of A are the extreme points of the set of states of A, which is a convex set (see Theorem 1.11.4).

Theorem 2.12.8. *Let A be a C*-algebra. Define*

$$S = \{\phi \in A^* \mid \phi \text{ is positive and } \|\phi\| \leq 1\}.$$

Then, S is a convex and weak- compact set. Moreover, the extreme points of S are the zero functional 0 and the pure states of A.*

Proof. Assume that A is a C*-algebra. Assume that S is the set of positive linear and norm-decreasing functionals on A, defined as above.

To show: (a) S is a convex set.

(b) S is a weak-* compact set.

(a) Assume that $\phi, \psi \in S$ and $t \in [0, 1]$. Then, $t\phi + (1-t)\psi$ is a positive linear functional with norm

$$\|t\phi + (1-t)\psi\| \leq t\|\phi\| + (1-t)\|\psi\| \leq 1.$$

So, $t\phi + (1-t)\psi \in S$ and S is a convex set.

(b) Note that S is a subset of the closed unit ball of A^*, which is a weak-* compact set by the Banach-Alaoglu theorem. Hence, it suffices to show that S is weak-* closed. Assume that $\{\phi_n\}_{n\in I}$ is a sequence in S which converges to some $\phi \in A^*$ with respect to the weak-* topology. To see that $\phi \in S$, first note that ϕ is contained in the closed unit ball of A^* so that $\|\phi\| \leq 1$.

To see that ϕ is positive, observe that if $a \in A$ then

$$\phi(a^*a) = \lim_n \phi_n(a^*a) \geq 0$$

because each ϕ_n is positive. Therefore, $\phi \in S$ and we conclude that S is a weak-* closed set.

By parts (a) and (b), we deduce that S is a convex and weak-* compact subset of A^*.

To show: (c) 0 is an extreme point of S.

(d) If $\tau \in PS(A)$ then τ is an extreme point of S.

(c) Assume that there exists $\alpha, \beta \in S$ and $t \in (0, 1)$ such that $0 = t\alpha + (1-t)\beta$. If $a \in A$ then

$$0 \geq -t\alpha(a^*a) = (1-t)\beta(a^*a) \geq 0.$$

Therefore, $\alpha = \beta = 0$ and 0 is an extreme point of S.

(d) Assume that $\tau \in PS(A)$. Assume that $\tau = t\gamma + (1-t)\delta$, where $t \in (0,1)$ and $\gamma, \delta \in S$. Then, $t\gamma \leq \tau$ and since τ is a pure state, there exists $t' \in [0,1]$ such that $t\gamma = t'\tau$. Since γ and δ are positive linear functionals, we have

$$1 = \|\tau\| = t\|\gamma\| + (1-t)\|\delta\|$$

by Theorem 2.3.6. Therefore, $\|\gamma\| = \|\delta\| = 1$ and

$$t = t\|\gamma\| = t'\|\tau\| = t'.$$

Therefore, $\gamma = \tau$ and $(1-t)\delta = \tau - t\gamma = (1-t)\tau$. So, $\delta = \tau$ as well and consequently, τ is an extreme point of S.

Finally, assume that ρ is a non-zero extreme point of S. We will demonstrate that ρ is a pure state. First, observe that

$$\rho = \|\rho\|(\frac{\rho}{\|\rho\|}) + (1 - \|\rho\|)0$$

and $0, \rho/\|\rho\| \in S$. Since ρ is a non-zero extreme point of S, we conclude that $\|\rho\| = 1$. Now assume that τ is a non-zero positive linear functional on A such that $\tau < \rho$. Then, $\|\tau\| \in (0,1)$ and

$$\begin{aligned}
\|\tau\|(\frac{\tau}{\|\tau\|}) + (1-\|\tau\|)(\frac{\rho-\tau}{\|\rho-\tau\|}) &= \tau + (1-\|\tau\|)(\frac{\rho-\tau}{\|\rho-\tau\|}) \\
&= \tau + (1-\|\tau\|)(\frac{\rho-\tau}{\|\rho\|-\|\tau\|}) \\
&= \tau + (1-\|\tau\|)(\frac{\rho-\tau}{1-\|\tau\|}) \\
&= \rho.
\end{aligned}$$

Since $\rho > \tau$, then $(\rho - \tau)/\|\rho - \tau\| \in S$. Consequently, $\rho = \tau/\|\tau\|$ and thus, ρ is a pure state on A as required. $\square$

One particular consequence of Theorem 2.12.8 is that by the Krein-Milman theorem, the set S in Theorem 2.12.8 is the closed convex hull of its extreme points, which in this case are the zero functional and the pure states on A.

We end this section by noting the following generalisation of Theorem 1.11.6, which is [Mur90, Theorem 5.1.12].

Theorem 2.12.9. *Let A be a C*-algebra and $a \in A$ be arbitrary. Then, there exists an irreducible representation (φ, H) of A such that $\|a\| = \|\varphi(a)\|$.*

Proof. Assume that A is a C*-algebra and $a \in A$. Then, a^*a is self-adjoint and by Theorem 1.11.6, there exists an irreducible representation (φ, H) of A such that $\|a^*a\| = \|\varphi(a^*a)\|$. Since A and $B(H)$ are C*-algebras,

$$\|a\| = \|a^*a\|^{\frac{1}{2}} = \|\varphi(a^*a)\|^{\frac{1}{2}} = \|\varphi(a)\|.$$

$\square$

2.13 The spatial C*-norm is minimal

Now we will embark on the long proof that the spatial C*-norm on a tensor product of C*-algebras $A \otimes B$ is the smallest C*-norm. Along the way, we will prove a significant number of useful results we will depend on in this section. First, there is a canonical approximate unit on the tensor product $A \otimes B$.

Theorem 2.13.1. *Let A and B be C*-algebras with approximate units $\{u_\lambda\}_{\lambda \in L}$ and $\{v_\mu\}_{\mu \in M}$ respectively. Let γ be a C*-norm on $A \otimes B$. Then, $A \otimes_\gamma B$ admits an approximate unit of the form $\{u_\delta \otimes v_\delta\}_{\delta \in D}$.*

Proof. Assume that A and B are C*-algebras. Assume that $\{u_\lambda\}_{\lambda \in L}$ and $\{v_\mu\}_{\mu \in M}$ are approximate units for A and B respectively. Assume that γ is a C*-norm on $A \otimes B$.

We first impose a partial order on $L \times M$. If $(\lambda, \mu), (\lambda', \mu') \in L \times M$ then $(\lambda, \mu) \leq (\lambda', \mu')$ if and only if $\lambda \leq \lambda'$ and $\mu \leq \mu'$. Then, $L \times M$ is reflexive, transitive and upwards-directed. In particular, it is upwards-directed because L and M are both upwards-directed.

If $(\lambda, \mu) \in L \times M$ then set $u'_{(\lambda,\mu)} = u_\lambda$ and $v'_{(\lambda,\mu)} = v_\mu$. We claim that $\{u'_{(\lambda,\mu)} \otimes v'_{(\lambda,\mu)}\}_{(\lambda,\mu) \in L \times M}$ is an approximate unit for $A \otimes B$. Assume that $z \in A \otimes B$. If $i \in \{1, 2, \ldots, n\}$ then there exists $a_i \in A$ and $b_i \in B$ such that

$$z = \sum_{i=1}^{n} (a_i \otimes b_i).$$

We compute directly that

$$\begin{aligned}
\gamma\big(z(u'_{(\lambda,\mu)} \otimes v'_{(\lambda,\mu)}) - z\big) &= \gamma\big(\sum_{i=1}^{n}(a_i u'_{(\lambda,\mu)} \otimes b_i v'_{(\lambda,\mu)}) - \sum_{i=1}^{n}(a_i \otimes b_i)\big) \\
&= \gamma\big(\sum_{i=1}^{n}(a_i u_\lambda \otimes b_i v_\mu) - \sum_{i=1}^{n}(a_i \otimes b_i)\big) \\
&\to \gamma\big(\sum_{i=1}^{n}(a_i \otimes b_i) - \sum_{i=1}^{n}(a_i \otimes b_i)\big) = 0
\end{aligned}$$

where the limit is taken over the variables λ and μ. Note that in the limit above, we implicitly used Theorem 2.11.8. Therefore, $\{u'_{(\lambda,\mu)} \otimes v'_{(\lambda,\mu)}\}_{(\lambda,\mu)\in L\times M}$ is an approximate unit for $A \otimes B$. □

The next results concerns direct sums of representations.

Theorem 2.13.2. *Let A and B be C*-algebras. Let $(\varphi_\lambda, H_\lambda)_{\lambda\in L}$ and $(\psi_\mu, K_\mu)_{\mu\in M}$ be families of representations of A and B respectively. Let $(\varphi, H) = \bigoplus_{\lambda\in L}(\varphi_\lambda, H_\lambda)$ and $(\psi, K) = \bigoplus_{\mu\in M}(\psi_\mu, K_\mu)$. If $c \in A \otimes B$ then*

$$\|(\varphi\hat{\otimes}\psi)(c)\| = \sup_{\lambda\in L,\mu\in M}\|(\varphi_\lambda\hat{\otimes}\psi_\mu)(c)\|.$$

Proof. Assume that A and B are C*-algebras. Define

$$\begin{array}{cccc}
\tilde{u}: & H \times K & \to & \bigoplus_{\lambda\in L,\mu\in M}(H_\lambda\hat{\otimes}K_\mu) \\
& \big((x_\lambda)_{\lambda\in L}, (y_\mu)_{\mu\in M}\big) & \mapsto & (x_\lambda \otimes y_\mu)_{\lambda,\mu}
\end{array}$$

Then, $\tilde{u}$ is $\mathbb{C}$-bilinear and by Theorem 2.11.1, there exists a unique linear map, defined by

$$\begin{array}{cccc}
u': & H \otimes K & \to & \bigoplus_{\lambda\in L,\mu\in M}(H_\lambda\hat{\otimes}K_\mu) \\
& (x_\lambda)_{\lambda\in L} \otimes (y_\mu)_{\mu\in M} & \mapsto & (x_\lambda \otimes y_\mu)_{\lambda,\mu}
\end{array}$$

It is clear by definition that u' is surjective. To see that u' is bounded, observe that

$$\|u'((x_\lambda)_{\lambda\in L} \otimes (y_\mu)_{\mu\in M})\| = \|(x_\lambda \otimes y_\mu)_{\lambda,\mu}\| = \|(x_\lambda)_{\lambda\in L}\|\|(y_\mu)_{\mu\in M}\|.$$

Thus, u' is a unitary operator and since $H \otimes K$ is dense in $H\hat{\otimes}K$, we can extend u' to a unitary operator u defined on $H\hat{\otimes}K$.

Next, observe that if $a \otimes b \in A \otimes B$ and $(x_\lambda)_{\lambda\in L} \otimes (y_\mu)_{\mu\in M} \in H \otimes K$ then

$$
\begin{aligned}
&u^*\big(\bigoplus_{\lambda\in L,\mu\in M}(\varphi_\lambda\hat{\otimes}\psi_\mu)(a\otimes b)\big)u((x_\lambda)_{\lambda\in L}\otimes(y_\mu)_{\mu\in M})\\
&=u^*\big(\bigoplus_{\lambda\in L,\mu\in M}(\varphi_\lambda\hat{\otimes}\psi_\mu)(a\otimes b)\big)\big((x_\lambda\otimes y_\mu)_{\lambda,\mu}\big)\\
&=u^*\big(\bigoplus_{\lambda\in L,\mu\in M}(\varphi_\lambda(a)\otimes\psi_\mu(b))\big)\big((x_\lambda\otimes y_\mu)_{\lambda,\mu}\big)\\
&=u^*\big(\bigoplus_{\lambda\in L,\mu\in M}\varphi_\lambda(a)(x_\lambda)\otimes\psi_\mu(b)(y_\mu)\big)\\
&=\varphi(a)((x_\lambda)_{\lambda\in L})\otimes\psi(b)((y_\mu)_{\mu\in M})\\
&=(\varphi\hat{\otimes}\psi)(a\otimes b)((x_\lambda)_{\lambda\in L}\otimes(y_\mu)_{\mu\in M}).
\end{aligned}
$$

Since u is unitary, we conclude that if $c\in A\otimes B$ then

$$
\|(\varphi\hat{\otimes}\psi)(c)\|=\|\bigoplus_{\lambda\in L,\mu\in M}(\varphi_\lambda\hat{\otimes}\psi_\mu)(c)\|=\sup_{\lambda\in L,\mu\in M}\|(\varphi_\lambda\hat{\otimes}\psi_\mu)(c)\|.
$$

□

We can apply Theorem 2.13.2 to the spatial C*-norm to obtain the following result.

Theorem 2.13.3. *Let A and B be non-zero C*-algebras. Let $c\in A\otimes B$. Then,*

$$
\|c\|_*=\sup_{\tau\in S(A),\rho\in S(B)}\|(\varphi_\tau\hat{\otimes}\varphi_\rho)(c)\|.
$$

Proof. Assume that A and B are non-zero C*-algebras and $c\in A\otimes B$. Let (φ,H) and (ψ,H) be the universal representations of A and B respectively. By definition of the spatial C*-norm,

$$
\|c\|_*=\|(\varphi\hat{\otimes}\psi)(c)\|.
$$

By the definition of the universal representation and Theorem 2.13.2, we have

$$
\|c\|_*=\|(\varphi\hat{\otimes}\psi)(c)\|=\sup_{\tau\in S(A),\rho\in S(B)}\|(\varphi_\tau\hat{\otimes}\varphi_\rho)(c)\|.
$$

□

Theorem 2.13.4. *Let A and B be C*-algebras. Let τ and ρ be states on A and B respectively. Then, $\tau\otimes\rho$ is continuous on $A\otimes B$ with respect to the spatial C*-norm.*

Proof. Assume that A and B are C*-algebras. Assume that $\tau \in S(A)$, $\rho \in S(B)$ and $a \otimes b \in A \otimes B$. Let $(\varphi_\tau, H_\tau, \xi_\tau)$ and $(\varphi_\rho, H_\rho, \xi_\rho)$ be the GNS representations associated to τ and ρ respectively. Then,

$$\begin{aligned}
\langle(\varphi_\tau \hat{\otimes} \varphi_\rho)(a \otimes b)(\xi_\tau \otimes \xi_\rho), \xi_\tau \otimes \xi_\rho\rangle &= \langle(\varphi_\tau(a) \otimes \varphi_\rho(b))(\xi_\tau \otimes \xi_\rho), \xi_\tau \otimes \xi_\rho\rangle \\
&= \langle\varphi_\tau(a)\xi_\tau \otimes \varphi_\rho(b)(\xi_\rho), \xi_\tau \otimes \xi_\rho\rangle \\
&= \langle\varphi_\tau(a)\xi_\tau, \xi_\tau\rangle\langle\varphi_\rho(b)\xi_\rho, \xi_\rho\rangle \\
&= \tau(a)\rho(b) = (\tau \otimes \rho)(a \otimes b)
\end{aligned}$$

Since $a \otimes B \in A \otimes B$ was arbitrary, we conclude that if $c \in A \otimes B$ then

$$(\tau \otimes \rho)(c) = \langle(\varphi_\tau \hat{\otimes} \varphi_\rho)(c)(\xi_\tau \otimes \xi_\rho), \xi_\tau \otimes \xi_\rho\rangle.$$

By Theorem 2.13.3, we first have

$$\|(\varphi_\tau \hat{\otimes} \varphi_\rho)(c)\| \leq \sup_{\tau \in S(A), \rho \in S(B)} \|(\varphi_\tau \hat{\otimes} \varphi_\rho)(c)\| = \|c\|_*.$$

Therefore,

$$\begin{aligned}
|(\tau \otimes \rho)(c)| &\leq \|(\varphi_\tau \hat{\otimes} \varphi_\rho)(c)\| \|\xi_\tau \otimes \xi_\rho\|^2 \\
&= \|(\varphi_\tau \hat{\otimes} \varphi_\rho)(c)\| \\
&\leq \|c\|_*.
\end{aligned}$$

So, $\tau \otimes \rho$ is continuous with respect to the spatial C*-norm on $A \otimes B$. $\square$

Before we move onto the next result, we state the following remark in [Mur90, Remark 6.4.2]. Let A and B be C*-algebras. Let (φ, H) and (φ', H') be unitarily equivalent representations of A and (ψ, K) and (ψ', K') be unitarily equivalent representations of B. Then, there exists a unitary map $u : H \hat{\otimes} K \to H' \hat{\otimes} K'$ such that if $c \in A \otimes B$ then

$$(\varphi' \hat{\otimes} \psi')(c) = u(\varphi \hat{\otimes} \psi)(c)u^*.$$

Theorem 2.13.5. *Let A and B be C*-algebras. Let (φ, H) and (ψ, K) be representations of A and B respectively. Let $c \in A \otimes B$. Then,*

$$\|(\varphi \hat{\otimes} \psi)(c)\| \leq \|c\|_*.$$

Proof. Assume that A and B are C*-algebras. Assume that (φ, H) and (ψ, K) are representations of A and B respectively. We would like to assume that the representations (φ, H) and (ψ, K) are non-degenerate.

To this end, let $H' = \overline{\varphi(A)H}$ and $K' = \overline{\psi(B)K}$. Then,

$$H' \hat{\otimes} K' = \overline{(\varphi \hat{\otimes} \psi)(A \otimes B)(H \hat{\otimes} K)}$$

and if $a, a' \in A$, $b, b' \in B$, $h \in H$ and $k \in K$ then

$$\begin{aligned}(\varphi \hat{\otimes} \psi)(a \otimes b)(\varphi(a')(h) \otimes \psi(b')(k)) &= (\varphi(a) \otimes \psi(b))(\varphi(a')(h) \otimes \psi(b')(k)) \\ &= \varphi(aa')(h) \otimes \psi(bb')(k) \\ &= (\varphi|_{H'} \hat{\otimes} \psi|_{K'})(a \otimes b)(\varphi(a')(h) \otimes \psi(b')(k)).\end{aligned}$$

Since $a \otimes b \in A \otimes B$ and $\varphi(a')(h) \otimes \psi(b')(k) \in H' \hat{\otimes} K'$ were arbitrary, we deduce that if $c \in A \otimes B$ then

$$(\varphi \hat{\otimes} \psi)(c)|_{H' \hat{\otimes} K'} = (\varphi|_{H'} \hat{\otimes} \psi|_{K'})(c).$$

By Theorem 1.9.5,

$$\|(\varphi \hat{\otimes} \psi)(c)\| = \|(\varphi \hat{\otimes} \psi)(c)|_{H' \hat{\otimes} K'}\| = \|(\varphi|_{H'} \hat{\otimes} \psi|_{K'})(c)\|$$

Hence, we may assume without loss of generality that the representations (φ, H) and (ψ, K) are non-degenerate. By Theorem 2.12.4, we can write (φ, H) and (ψ, K) as the direct sum of cyclic representations. Next, recall from the proof of Theorem 2.12.6 that each non-zero cyclic representation of A or B is unitarily equivalent to a GNS representation associated to some state on A or B.

By replacing the cyclic representations in the direct sums comprising (φ, H) and (ψ, K) with unitarily equivalent representations if necessary (this invokes the previous remark), we may assume that

$$(\varphi, H) = \bigoplus_{\lambda \in \Lambda} (\varphi_{\tau_\lambda}, H_{\tau_\lambda}, \xi_{\tau_\lambda})$$

for some indexing set Λ and states $\tau_\lambda \in S(A)$ for $\lambda \in \Lambda$. Similarly,

$$(\psi, K) = \bigoplus_{\mu \in M} (\varphi_{\rho_\mu}, H_{\rho_\mu}, \xi_{\rho_\mu}).$$

Now if $c \in A \otimes B$ then by Theorem 2.13.2,

$$\|(\varphi\hat{\otimes}\psi)(c)\| = \sup_{\lambda\in\Lambda,\mu\in M}\|(\varphi_{\tau_\lambda}\hat{\otimes}\varphi_{\rho_\mu})(c)\|.$$

By Theorem 2.13.3, we have

$$\|(\varphi\hat{\otimes}\psi)(c)\| \leq \sup_{\tau\in S(A),\rho\in S(B)}\|(\varphi_\tau\hat{\otimes}\varphi_\rho)(c)\| = \|c\|_*.$$

□

Next, we need [Mur90, Remark 6.4.3]. Let $p \in M_{n\times n}(\mathbb{C})$ be a rank-one projection. Then, write $p = x \otimes x$ where $x \in \mathbb{C}^n$. Let $\{e_1, \ldots, e_n\}$ be the canonical orthonormal basis of $\mathbb{C}^n$. Then, there exists $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ such that

$$x = \sum_{i=1}^{n} \lambda_i e_i.$$

Now $e_i \otimes e_j \in M_{n\times n}(\mathbb{C})$ is the matrix with a 1 in the i, j entry and zeros elsewhere. So,

$$p = \sum_{i,j=1}^{n} \lambda_i\overline{\lambda_j}(e_i \otimes e_j) = (\lambda_i\overline{\lambda_j})_{i,j}.$$

The next result we need concerns the tensor product of positive linear functionals.

Theorem 2.13.6. *Let A and B be C*-algebras. Let $\tau : A \to \mathbb{C}$ and $\rho : B \to \mathbb{C}$ be positive linear functionals. Then, the linear functional $\tau \otimes \rho$ on $A \otimes B$ is positive.*

Proof. Assume that A and B are C*-algebras. Assume that τ and ρ are positive linear functionals on A and B respectively. Assume that $c = \sum_{j=1}^{n}(a_j \otimes b_j) \in A \otimes B$. We compute directly that

$$(\tau \otimes \rho)(c^*c) = (\tau \otimes \rho)(\sum_{i,j=1}^{n}(a_i^*a_j \otimes b_i^*b_j) = \sum_{i,j=1}^{n}\tau(a_i^*a_j)\rho(b_i^*b_j).$$

Now define the matrix $u = (\rho(b_i^*b_j))_{i,j} \in M_{n\times n}(\mathbb{C})$. If $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ then

$$\sum_{i,j=1}^{n}\rho(b_i^*b_j)\overline{\lambda_i}\lambda_j = \rho\Big((\sum_{i=1}^{n}\lambda_i b_i)^*(\sum_{i=1}^{n}\lambda_i b_i)\Big) \geq 0$$

because ρ is positive. We conclude that u is a positive element of $M_{n\times n}(\mathbb{C})$. So, it can be diagonalised and written as

$$u = \sum_{j=1}^{n} t_j p_j$$

where $t_1, \ldots, t_n \in \mathbb{R}_{>0}$ and $p_1, \ldots, p_n$ are rank-one projections in $M_{n\times n}(\mathbb{C})$.

To see that $(\tau \otimes \rho)(c^*c) \geq 0$, it suffices to show that if $p = (p_{i,j})$ is a rank-one projection in $M_{n\times n}(\mathbb{C})$ then $\sum_{i,j=1}^{n} \tau(a_i^* a_j) p_{i,j} \geq 0$. By [Mur90, Remark 6.4.3], if $i, j \in \{1, 2, \ldots, n\}$ then $p_{i,j} = \overline{\lambda_i}\lambda_j$ for some $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$. So,

$$\begin{aligned}
\sum_{i,j=1}^{n} \tau(a_i^* a_j) p_{i,j} &= \sum_{i,j=1}^{n} \tau(a_i^* a_j) \overline{\lambda_i}\lambda_j \\
&= \tau\Big((\sum_{i=1}^{n} \lambda_i a_i)^* (\sum_{j=1}^{n} \lambda_j a_j) \Big) \geq 0
\end{aligned}$$

because τ is positive. Therefore,

$$(\tau \otimes \rho)(c^*c) = \sum_{i,j=1}^{n} \tau(a_i^* a_j)\rho(b_i^* b_j) \geq 0$$

and $\tau \otimes \rho$ defines a positive linear functional on $A \otimes B$. □

The point of Theorem 2.13.6 is that in the next result, we want to use two states τ and ρ on C*-algebras A and B to obtain a unique state on $A \otimes_\gamma B$.

Theorem 2.13.7. *Let A and B be C*-algebras and γ be a C*-norm on $A \otimes B$. Let $\tau \in S(A)$ and $\rho \in S(B)$. If $\tau \otimes \rho$ is continuous with respect to γ then $\tau \otimes \rho$ extends uniquely to a state ω on $A \otimes_\gamma B$.*

Proof. Assume that A and B are C*-algebras. Assume that γ is a C*-norm on $A \otimes B$, $\tau \in S(A)$ and $\rho \in S(B)$. Assume that the linear functional $\tau \otimes \rho$ is continuous with respect to γ. Since $A \otimes B$ is dense in $A \otimes_\gamma B$, we can extend $\tau \otimes \rho$ to a unique continuous linear functional ω on $A \otimes_\gamma B$.

To show: (a) ω is positive.

(b) $\|\omega\| = 1$.

(a) Assume that $c \in A \otimes_\gamma B$. Then, there exists a sequence $\{c_n\}_{n\in\mathbb{Z}_{>0}}$ in $A \otimes B$ such that $\lim_{n\to\infty} \gamma(c - c_n) = 0$. So,

$$\omega(c^*c) = \lim_{n\to\infty} \omega(c_n^* c_n) = \lim_{n\to\infty} (\tau \otimes \rho)(c_n^* c_n) \geq 0$$

where the last inequality follows from Theorem 2.13.6. Hence, ω is positive.

(b) By Theorem 2.13.1, let $\{u_\lambda \otimes v_\lambda\}_{\lambda\in L}$ be an approximate unit for $A \otimes B$, where $\{u_\lambda\}_{\lambda\in L}$ and $\{v_\lambda\}_{\lambda\in L}$ are approximate units for A and B respectively. By Theorem 2.3.5, we have

$$\begin{aligned}\|\omega\| &= \lim_\lambda \omega(u_\lambda \otimes v_\lambda)\\ &= \lim_\lambda (\tau \otimes \rho)(u_\lambda \otimes v_\lambda)\\ &= \lim_\lambda \tau(u_\lambda)\rho(v_\lambda)\\ &= \|\tau\|\|\rho\| = 1.\end{aligned}$$

By parts (a) and (b), ω is the unique state on $A \otimes_\gamma B$ which extends the continuous linear functional $\tau \otimes \rho$ on $A \otimes B$. □

In Theorem 2.13.7, we denote the state ω by $\tau \otimes_\gamma \rho$.

Theorem 2.13.8. *Let A and B be C^*-algebras. Let γ be a C^*-norm on $A \otimes B$. Let $\tau \in S(A)$ and $\rho \in S(B)$ such that the linear functional $\tau \otimes \rho$ is continuous with respect to γ. Then, there exists a unitary map $u : H_\tau \hat{\otimes} H_\rho \to H_{\tau\otimes_\gamma\rho}$ such that if $c \in A \otimes B$ then*

$$\varphi_{\tau\otimes_\gamma\rho}(c) = u(\varphi_\tau \hat{\otimes} \varphi_\rho)(c)u^*.$$

Proof. Assume that A and B are C*-algebras. Let $\omega = \tau \otimes_\gamma \rho$ and $\pi = \varphi_\tau \hat{\otimes} \varphi_\rho$. Let ξ_τ and ξ_ρ be the unit cyclic vectors corresponding to the GNS representations of τ and ρ respectively. Let $y = \xi_\tau \otimes \xi_\rho \in H_\tau \hat{\otimes} H_\rho$.

Let $(\varphi_\omega, H_\omega, \xi_\omega)$ be the GNS representation associated to the state ω constructed as an extension of $\tau \otimes \rho$ in Theorem 2.13.7. If $a \otimes b \in A \otimes B$ then

$$\begin{aligned}\langle \varphi_\omega(c)(\xi_\omega), \xi_\omega\rangle &= \omega(a \otimes b)\\ &= (\tau \otimes \rho)(a \otimes b) = \tau(a)\rho(b)\\ &= \langle \varphi_\tau(a)(\xi_\tau), \xi_\tau\rangle\langle \varphi_\rho(b)(\xi_\rho), \xi_\rho\rangle\\ &= \langle \varphi_\tau(a)(\xi_\tau) \otimes \varphi_\rho(b)(\xi_\rho), \xi_\tau \otimes \xi_\rho\rangle\\ &= \langle (\varphi_\tau \hat{\otimes} \varphi_\rho)(a \otimes b)(\xi_\tau \otimes \xi_\rho), \xi_\tau \otimes \xi_\rho\rangle\\ &= \langle \pi(a \otimes b)(y), y\rangle.\end{aligned}$$

Now let $H_0 = \varphi_\omega(A \otimes B)\xi_\omega$ and $K_0 = \pi(A \otimes B)(y)$. Define the map

$$\begin{array}{rccc} u_0 : & K_0 & \to & H_0 \\ & \pi(c)(y) & \mapsto & \varphi_\omega(c)(\xi_\omega). \end{array}$$

To see that u_0 is well-defined, assume that $c_1, c_2 \in A \otimes B$ such that $\pi(c_1)(y) = \pi(c_2)(y)$. Then,

$$\langle \varphi_\omega(c_1)(\xi_\omega), \xi_\omega \rangle = \langle \pi(c_1)(y), y \rangle = \langle \pi(c_2)(y), y \rangle = \langle \varphi_\omega(c_2)(\xi_\omega), \xi_\omega \rangle$$

and since ξ_ω is cyclic, $\varphi_\omega(c_1)(\xi_\omega) = \varphi_\omega(c_2)(\xi_\omega)$. To see that u_0 is isometric, we compute directly that if $c \in A \otimes B$ then

$$\begin{aligned} \|u_0(\pi(c)(y))\|^2 &= \|\varphi_\omega(c)(\xi_\omega)\|^2 \\ &= \langle \varphi_\omega(c)(\xi_\omega), \varphi_\omega(c)(\xi_\omega) \rangle \\ &= \langle \varphi_\omega(c^*c)(\xi_\omega), \xi_\omega \rangle \\ &= \langle \pi(c^*c)(y), y \rangle = \|\pi(c)(y)\|^2. \end{aligned}$$

So, u_0 is a well-defined isometric linear map and since K_0 is dense in $H_\tau \hat{\otimes} H_\rho$ and H_0 is dense in H_ω, we can extend u_0 uniquely to a unitary map $u : H_\tau \hat{\otimes} H_\rho \to H_\omega$.

Finally, if $c, d \in A \otimes B$ and $\varphi_\omega(d)(\xi_\omega) \in H_0$ then

$$\begin{aligned} u\pi(c)u^*(\varphi_\omega(d)(\xi_\omega)) &= u\pi(c)\big(\pi(d)(y)\big) \\ &= u(\pi(cd)(y)) = \varphi_\omega(cd)(\xi_\omega) \\ &= \varphi_\omega(c)\big(\varphi_\omega(d)(\xi_\omega)\big). \end{aligned}$$

Since $u\pi(c)u^*$ and $\varphi_\omega(c)$ agree on H_0 and H_0 is dense in H_ω, $u\pi(c)u^* = \varphi_\omega(c)$ on H_ω. □

Next, we will use Theorem 2.13.7 and Theorem 2.13.8 to give an alternative characterisation of the spatial C*-norm.

Theorem 2.13.9. *Let A and B be non-zero C*-algebras. Suppose that $c \in A \otimes B$. Then,*

$$\|c\|_*^2 = \sup_{\substack{\tau \in S(A) \\ \rho \in S(B)}} \sup_{\substack{d \in A \otimes B \\ (\tau \otimes \rho)(d^*d) > 0}} \frac{(\tau \otimes \rho)(d^*c^*cd)}{(\tau \otimes \rho)(d^*d)}.$$

Proof. Assume that A and B are non-zero C*-algebras. Let ω be a state on the spatial tensor product $A \otimes_* B$. If $(\varphi_\omega, H_\omega, \xi_\omega)$ is the GNS representation of $A \otimes_* B$ associated to ω and $c, d \in A \otimes B$ then

$$\begin{aligned}
\sup_{\|d+N_\omega\|\neq 0} \frac{\|\varphi_\omega(c)(d+N_\omega)\|^2}{\|d+N_\omega\|^2} &= \sup_{\|d+N_\omega\|\neq 0} \frac{\|cd+N_\omega\|^2}{\|d+N_\omega\|^2} \\
&= \sup_{\|d+N_\omega\|\neq 0} \frac{\omega(d^*c^*cd)}{\omega(d^*d)} \\
&= \sup_{\omega(d^*d)>0} \frac{\omega(d^*c^*cd)}{\omega(d^*d)}.
\end{aligned}$$

Since $\varphi_\omega(A \otimes B)\xi_\omega$ is dense in H_ω and $d + N_\omega = \varphi_\omega(d)\xi_\omega$, we can use Theorem 2.12.1 to deduce that

$$\|\varphi_\omega(c)\|^2 = \sup_{\omega(d^*d)>0} \frac{\omega(d^*c^*cd)}{\omega(d^*d)}.$$

By Theorem 2.13.3, we have

$$\|c\|_* = \sup_{\substack{\tau\in S(A)\\ \rho\in S(B)}} \|(\varphi_\tau \hat{\otimes} \varphi_\rho)(c)\|.$$

By Theorem 2.13.4, if $\tau \in S(A)$ and $\rho \in S(B)$ then the tensor product $\tau \otimes \rho : A \otimes B \to \mathbb{C}$ is continuous with respect to the spatial C*-norm $\|-\|_*$ on $A \otimes B$. Furthermore, by Theorem 2.13.7, we can construct the state $\tau \otimes_{\|-\|_*} \rho$ as the unique extension of $\tau \otimes \rho$ to $A \otimes_* B$. Hence,

$$\begin{aligned}
\|c\|_*^2 &= \sup_{\tau\in S(A),\, \rho\in S(B)} \|(\varphi_\tau \hat{\otimes} \varphi_\rho)(c)\|^2 \\
&= \sup_{\tau\in S(A),\, \rho\in S(B)} \|\varphi_{\tau \otimes_{\|-\|_*} \rho}(c)\|^2 \quad \text{(by Theorem 2.13.8)} \\
&= \sup_{\substack{\tau\in S(A)\\ \rho\in S(B)}} \sup_{\substack{d\in A\otimes B\\ (\tau\otimes\rho)(d^*d)>0}} \frac{(\tau\otimes\rho)(d^*c^*cd)}{(\tau\otimes\rho)(d^*d)}.
\end{aligned}$$

This completes the proof. □

Next, we will use the characterisation of the spatial C*-norm in Theorem 2.13.9 to prove a result involving tensor products of C*-algebras and unitizations.

Theorem 2.13.10. *Let A and B be C*-algebras. Let $\tilde{B}$ be the unitization of B. Then, the restriction of the spatial C*-norm on $A \otimes \tilde{B}$ to $A \otimes B$ is the spatial C*-norm on $A \otimes B$.*

Proof. Assume that A and B are C*-algebras. Assume that $\tilde{B}$ is the unitization of B. Let γ be the restriction of the spatial C*-norm on $A \otimes \tilde{B}$ to $A \otimes B$. By Theorem 2.13.9, if $c \in A \otimes B$ then

$$\gamma(c)^2 = \sup_{\substack{\tau \in S(A) \\ \rho \in S(\tilde{B})}} \sup_{\substack{d \in A \otimes \tilde{B} \\ (\tau \otimes \rho)(d^* d) > 0}} \frac{(\tau \otimes \rho)(d^* c^* c d)}{(\tau \otimes \rho)(d^* d)}$$

and

$$\|c\|_*^2 = \sup_{\substack{\tau \in S(A) \\ \rho \in S(B)}} \sup_{\substack{d \in A \otimes B \\ (\tau \otimes \rho)(d^* d) > 0}} \frac{(\tau \otimes \rho)(d^* c^* c d)}{(\tau \otimes \rho)(d^* d)}$$

where $\|-\|_*$ is the spatial C*-norm on $A \otimes B$. Now observe that if $\rho \in S(B)$ then there exists a unique extension $\tilde{\rho}$ of ρ such that $\tilde{\rho} \in S(\tilde{B})$. Thus, if $c \in A \otimes B$ then $\gamma(c) \geq \|c\|_*$.

To see that $\gamma(c) \leq \|c\|_*$, let (φ, H) and (ψ, K) denote the universal representations of A and $\tilde{B}$ respectively. Let $\psi|_B$ denote the restriction of the *-homomorphism ψ to B. By the definition of the spatial norm on $A \otimes B$ and Theorem 2.11.7,

$$\gamma(c) = \|(\varphi \hat{\otimes} \psi)(c)\| = \|(\varphi \hat{\otimes} \psi|_B)(c)\| \leq \|c\|_*.$$

Thus, if $c \in A \otimes B$ then $\gamma(c) \leq \|c\|_*$. So, $\gamma = \|-\|_*$ on $A \otimes B$. □

The next result tells us that we can extend a C*-norm on $A \otimes B$ to $A \otimes \tilde{B}$.

Theorem 2.13.11. *Let A and B be C*-algebras with B non-unital. Let γ be a C*-norm on $A \otimes B$. Then, there exists a C*-norm on $A \otimes \tilde{B}$ which extends γ.*

Proof. Assume that A and B are C*-algebras, with B non-unital. Assume that γ is a C*-norm on $A \otimes B$. Let (π, H) be a faithful non-degenerate representation of the C*-algebra $A \otimes_\gamma B$. By Theorem 2.11.9, there exists unique *-homomorphisms $\pi_A : A \to B(H)$ and $\pi_B : B \to B(H)$ such that if $a \in A$ and $b \in B$ then

$$\pi(a \otimes b) = \pi_A(a)\pi_B(b) = \pi_B(b)\pi_A(a).$$

Since π is injective, π_A and π_B are both injective. Since π_B is an injective *-homomorphism. By the universal property of unitization in Theorem 1.6.3, there exists a unique unital *-homomorphism $\pi'_B : \tilde{B} \to B(H)$ such that if $\iota : B \hookrightarrow \tilde{B}$ denotes the inclusion map then

$$\pi'_B \circ \iota = \pi_B.$$

To show: (a) π'_B is injective.

(a) Assume that $(\lambda, b) \in \ker \pi'_B$ so that $\pi'_B((\lambda, b)) = 0$. Then,

$$\pi'_B((\lambda, b)) = \lambda \pi'_B((1, 0)) + \pi'_B(0, b) = \lambda id_{B(H)} + \pi_B(b) = 0.$$

Suppose for the sake of contradiction that $\lambda \neq 0$. Then, $\pi_B(-\lambda^{-1}b) = id_{B(H)}$. Since π_B is injective, we deduce that $-\lambda^{-1}b \in B$ is a unit for B. However, this contradicts the assumption that B is non-unital. Therefore, $\lambda = 0$ and consequently, $b = 0$. So, $(\lambda, b) = (0, 0)$ and π'_B is injective.

Since the images $\pi_A(A)$ and $\pi_B(B)$ commute, the images $\pi_A(A)$ and $\pi'_B(\tilde{B})$ must also commute. By invoking the universal property in Theorem 2.11.1, there exists a unique *-homomorphism $\pi' : A \otimes \tilde{B} \to B(H)$ such that $\pi'|_{A \otimes B} = \pi$.

By injectivity of π, if $c \in A \otimes_\gamma B$ then $\|\pi(c)\| = \gamma(c)$ because π is isometric by Theorem 1.6.4. At this point, it suffices to show that π' is injective, as in this case, the map $c \mapsto \|\pi'(c)\|$ becomes a C*-norm on $A \otimes \tilde{B}$ extending γ.

To show: (b) π' is injective.

(b) Assume that $d \in \ker \pi'$. If $c \in A \otimes B$ then $dc \in A \otimes B$ and $\pi(dc) = 0$. Since π is injective, $dc = 0$. Now let $\theta = \pi_A \hat{\otimes} \pi'_B$. Then,

$$\theta(d)\theta(c) = \theta(dc) = (\pi_A \hat{\otimes} \pi'_B)(dc) = 0$$

where the last equality follows from Theorem 2.11.7 and the fact that π_A and π'_B are both injective. Since this holds for arbitrary $c \in A \otimes B$, then $\theta(d) = 0$ on the subspace

$$K_0 = \theta(A \otimes B)(H \hat{\otimes} H).$$

Since (π, H) is a non-degenerate representation of $A \otimes_\gamma B$, then (π_A, H) and (π_B, H) are both non-degenerate representations of A and B respectively by Theorem 2.11.9. By Theorem 1.9.4,

$$\overline{\pi_A(A)H} = H \qquad \text{and} \qquad \overline{\pi_B(B)H} = H.$$

By definition of θ,

$$\theta(A \otimes B)(H \hat{\otimes} H) = (\pi_A \hat{\otimes} \pi'_B)(A \otimes B)(H \hat{\otimes} H) = \pi_A(A)(H) \hat{\otimes} \pi'_B(B)(H).$$

Hence, K_0 is dense in $H \hat{\otimes} H$ and $\theta(d) = 0$ on $H \hat{\otimes} H$. Since θ is injective by Theorem 2.11.7, $d = 0$ and π' is injective. This completes the proof. □

Theorem 2.13.12. *Let A and B be C*-algebras. Let $u \in \tilde{A}$ and $v \in \tilde{B}$ be unitary elements. If γ is a C*-norm on $A \otimes B$ then the unique *-isomorphism*

$$\begin{array}{rccc} \pi: & A \otimes B & \to & A \otimes B \\ & a \otimes b & \mapsto & uau^* \otimes vbv^* \end{array}$$

*is an isometry. Note that in the definition of π, we regard $A \otimes B$ as a *-subalgebra of $\tilde{A} \otimes \tilde{B}$.*

Proof. Assume that A and B are C*-algebras and that π is the *-isomorphism on $A \otimes B$ defined as above. Note that the inverse of π is the map $a \otimes b \mapsto u^*au \otimes v^*bv$. By symmetry, it suffices to show that π is norm-decreasing.

To show: (a) If γ is a C*-norm on $A \otimes B$ and $c \in A \otimes B$ then $\gamma(\pi(c)) \leq \gamma(c)$.

(a) Assume that γ is a C*-norm on $A \otimes B$. By Theorem 2.13.1, let $\{u_\lambda \otimes v_\lambda\}_{\lambda \in L}$ be an approximate unit for $A \otimes_\gamma B$, where $\{u_\lambda\}_{\lambda \in L}$ and $\{v_\lambda\}_{\lambda \in L}$ are approximate units for A and B respectively.

If $\lambda \in L$ then let $w_\lambda = u_\lambda \otimes v_\lambda$. Let $w = u \otimes v$ so that if $c \in A \otimes B$ then $\pi(c) = wcw^*$. If $a \in A$ and $b \in B$ then

$$uau^* = \lim_\lambda uu_\lambda au_\lambda u^* \qquad \text{and} \qquad vbv^* = \lim_\lambda vv_\lambda bv_\lambda v^*.$$

So,

$$\begin{aligned} w(a \otimes b)w^* &= uau^* \otimes vbv^* \\ &= \lim_\lambda (uu_\lambda au_\lambda u^* \otimes vv_\lambda bv_\lambda v^*) \\ &= \lim_\lambda ww_\lambda (a \otimes b) w_\lambda w^* \end{aligned}$$

in $A \otimes_\gamma B$. Therefore,

$$\pi(c) = wcw^* = \lim_\lambda ww_\lambda cw_\lambda w^* = \lim_\lambda \pi(w_\lambda cw_\lambda).$$

Finally, we argue that

$$\begin{aligned}
\gamma(\pi(c)) &= \gamma(\lim_\lambda \pi(w_\lambda cw_\lambda)) \\
&= \lim_\lambda \gamma(ww_\lambda cw_\lambda w^*) \\
&\leq \sup_{\lambda \in L} \gamma(ww_\lambda)\gamma(c)\gamma(w_\lambda w^*) \\
&\leq \sup_{\lambda \in L} \|uu_\lambda\|\|vv_\lambda\|\gamma(c)\|u_\lambda u^*\|\|v_\lambda v^*\| \\
&\leq \gamma(c).
\end{aligned}$$

This completes the proof. □

Within the onslaught of these results, we now present a definition that is the key to proving that abelian C*-algebras are nuclear and that the spatial C*-norm is minimal.

Definition 2.13.1. Let A and B be C*-algebras and γ be a C*-norm on $A \otimes B$. Let $PS(A)$ denote the topological space of pure states on A, where $PS(A)$ is endowed with the weak-* topology from A^*. Define

$$S_\gamma = \left\{ (\tau, \rho) \in PS(A) \times PS(B) \;\middle|\; \begin{matrix} \tau \otimes \rho \text{ is continuous} \\ \text{on } A \otimes B \text{ with respect to } \gamma \end{matrix} \right\}. \tag{2.5}$$

The next few results are dedicated to proving properties about S_γ.

Theorem 2.13.13. *Let A and B be C*-algebras. Let γ be a C*-norm on $A \otimes B$. Let $PS(A)$ and $PS(B)$ denote the topological spaces of pure states on A and B respectively, where the topology is the weak-* topology induced from A^* and B^*. Then, S_γ is closed in $PS(A) \times PS(B)$.*

Moreover, if $u \in \tilde{A}$ and $v \in \tilde{B}$ are unitary elements and $(\tau, \rho) \in S_\gamma$ then $(\tau^u, \rho^v) \in S_\gamma$ where $\tau^u(a) = \tau(uau^*)$ and $\rho^v(b) = \rho(vbv^*)$ (recall that A is a closed two-sided ideal of $\tilde{A}$ and so, $uau^* \in A$. Similarly, $vbv^* \in B$).

Proof. Assume that A and B are C*-algebras. Assume that γ is a C*-norm on $A \otimes B$. Assume that $u \in \tilde{A}$ and $v \in \tilde{B}$ are unitary elements. If π is the *-isomorphism from Theorem 2.13.12 then

$$\tau^u \otimes \rho^v = (\tau \otimes \rho)\pi.$$

By Theorem 2.13.12, π is continuous with respect to γ. Since $\tau \otimes \rho \in S_\gamma$, it is continuous with respect to γ. So, $\tau^u \otimes \rho^v \in S_\gamma$.

To see that S_γ is closed, assume that $\{(\tau_n, \rho_n)\}_{n\in\mathbb{Z}_{>0}}$ is a sequence in S_γ which converges to some $(\tau, \rho) \in PS(A) \times PS(B)$ with respect to the topology on $PS(A) \times PS(B)$. If $n \in \mathbb{Z}_{>0}$ and $c \in A \otimes B$ then

$$|(\tau_n \otimes \rho_n)(c)| \leq \|\tau_n \otimes \rho_n\|\gamma(c) = \|\tau_n\|\|\rho_n\|\gamma(c) = \gamma(c).$$

Now let $d = \sum_{i=1}^n (a_i \otimes b_i) \in A \otimes B$ with $\|d\| \leq 1$. Then,

$$\begin{aligned}(\tau \otimes \rho)(d) &= (\tau \otimes \rho)(\sum_{i=1}^n (a_i \otimes b_i)) \\ &= \sum_{i=1}^n \tau(a_i)\rho(b_i) \\ &= \lim_{m\to\infty} \sum_{i=1}^n \tau_m(a_i)\rho_m(b_i) \\ &= \lim_{m\to\infty} (\tau_m \otimes \rho_m)(d).\end{aligned}$$

So, if $d \in A \otimes B$ and $\|d\| = 1$ then

$$|(\tau \otimes \rho)(d)| = \lim_{m\to\infty} |(\tau_m \otimes \rho_m)(d)| \leq \lim_{m\to\infty} \gamma(d) = \gamma(d).$$

Therefore, $\tau \otimes \rho$ is continuous with respect to γ and subsequently, $\tau \otimes \rho \in S_\gamma$. So, S_γ is a closed subset of $PS(A) \otimes PS(B)$. $\square$

In the next result, we will construct a particular decomposition of a state on $A \otimes_\gamma B$.

Theorem 2.13.14. *Let A and B be C^*-algebras. Let γ be a C^*-norm on $A \otimes B$ and ω be a state on $A \otimes_\gamma B$. Let $(\pi, H_\omega, \xi_\omega)$ be the GNS representation of $A \otimes_\gamma B$ associated to ω. Using Theorem 2.11.9, let $\pi_A : A \to B(H_\omega)$ and $\pi_B : B \to B(H_\omega)$ be the unique *-homomorphism such that if $a \in A$ and $b \in B$ then*

$$\pi(a \otimes b) = \pi_A(a)\pi_B(b) = \pi_B(b)\pi_A(a).$$

Define the states ω_A and ω_B on A and B respectively by

$$\omega_A(a) = \langle \pi_A(a)(\xi_\omega), \xi_\omega\rangle \qquad \textit{and} \qquad \omega_B(b) = \langle \pi_B(b)(\xi_\omega), \xi_\omega\rangle.$$

If $(\tau, \rho) \in S_\gamma$ *and* $\omega = \tau \otimes_\gamma \rho$ *then* $\tau = \omega_A$ *and* $\rho = \omega_B$.

Proof. Assume that A and B are C*-algebras. Assume that γ is a C*-norm on $A \otimes B$. Assume that ω is a state on $A \otimes_\gamma B$. Assume that $\pi, \pi_A, \pi_B, \omega_A$ and ω_B are defined as above. Then, ω_A and ω_B are states by Theorem 2.12.7.

Let $\{u_\lambda\}_{\lambda \in L}$ be an approximate unit for A. Since (π_A, H_ω) is non-degenerate by Theorem 2.11.9, then $\xi_\omega = \lim_\lambda \pi_A(u_\lambda)(\xi_\omega)$ by Theorem 2.3.4. Hence, if $b \in B$ then

$$\begin{aligned}
\omega_B(b) &= \langle \pi_B(b)(\xi_\omega), \xi_\omega\rangle \\
&= \lim_\lambda \langle \pi_B(b)\pi_A(u_\lambda)(\xi_\omega), \xi_\omega\rangle \\
&= \lim_\lambda \langle \pi(u_\lambda \otimes b)(\xi_\omega), \xi_\omega\rangle \\
&= \lim_\lambda \omega(u_\lambda \otimes b).
\end{aligned}$$

By a similar argument if $\{v_\mu\}_{\mu \in M}$ is an approximate unit for B and $a \in A$ then

$$\omega_A(a) = \lim_\mu \omega(a \otimes v_\mu).$$

Assume that $(\tau, \rho) \in S_\gamma$ and $\omega = \tau \otimes_\gamma \rho$. Then,

$$\omega_A(a) = \lim_\mu \omega(a \otimes v_\mu) = \lim_\mu \tau(a)\rho(v_\mu) = \lim_\mu \tau(a)\|\rho\| = \lim_\mu \tau(a) = \tau(a)$$

by Theorem 2.3.5. By a similar argument, $\omega_B = \rho$. □

Theorem 2.13.15. *Let A and B be C*-algebras. Suppose that either one of A or B is commutative. Let γ be a C*-norm on $A \otimes B$ and $(\tau, \rho) \in S_\gamma$. Then, $\tau \otimes_\gamma \rho$ (see Theorem 2.13.7) is a pure state of $A \otimes_\gamma B$.*

Proof. Assume that A and B are C*-algebras and that γ is a C*-norm on $A \otimes B$. Without loss of generality, we may assume that A is abelian. Assume that $(\tau, \rho) \in S_\gamma$ and define $\omega = \tau \otimes_\gamma \rho$. By the construction in Theorem 2.13.7, ω defines a state on $A \otimes_\gamma B$.

Let (π, H, ξ) denote the GNS representation of $A \otimes_\gamma B$ associated to ω. Let $\pi_A : A \to B(H)$ and $\pi_B : B \to B(H)$ denote the unique *-homomorphisms (see Theorem 2.11.9) satisfying for $a \in A$ and $b \in B$

$$\pi(a \otimes b) = \pi_A(a)\pi_B(b) = \pi_B(b)\pi_A(a).$$

Define $K = \overline{\pi_A(A)\xi}$. Then, K is a closed vector space of H, which is invariant for $\pi_A(A)$. Define the map

$$\begin{array}{rccc} \psi : & A & \to & B(K) \\ & a & \mapsto & \pi_A(a)|_K. \end{array}$$

Since π_A is a *-homomorphism, then ψ is also a *-homomorphism. By definition of K, ξ is a unit cyclic vector for the representation (ψ, K). Now observe that

$$\langle \psi(a)(\xi), \xi \rangle = \langle \pi_A(a)(\xi), \xi \rangle = \tau(a)$$

by Theorem 2.13.14. Subsequently by Theorem 2.12.5, the representation (ψ, K) of A is unitarily equivalent to the GNS representation $(\varphi_\tau, H_\tau, \xi_\tau)$. Since τ is a pure state by assumption, the GNS representation $(\varphi_\tau, H_\tau, \xi_\tau)$ is irreducible. Therefore, (ψ, K) is an irreducible representation of A.

By Theorem 1.9.7, the commutant $\psi(A)' = \mathbb{C} id_K$. Invoking the assumption that A is abelian, we find that $\psi(A) \subseteq \psi(A)'$. Thus, if $a \in A$ then there exists a scalar $\lambda \in \mathbb{C}$ such that $\psi(a) = \lambda id_K$. In particular, we can deduce the identity of this scalar by noticing that

$$\tau(a) = \langle \psi(a)(\xi), \xi \rangle = \langle \lambda\xi, \xi \rangle = \lambda.$$

Therefore, if $a \in A$ then $\psi(a) = \tau(a) id_K$.

To show: (a) If $a \in A$ and $b \in B$ then $\pi_A(a)\pi_B(b) = \tau(a)\pi_B(b)$.

(a) Assume that $a \in A$ and $b \in B$. It suffices to prove the identity $\pi_A(a)\pi_B(b) = \tau(a)\pi_B(b)$ on the subspace

$$\pi_A(A)\pi_B(B)(\xi) = \pi(A \otimes B)(\xi)$$

because ξ is a cyclic vector for the GNS representation (π, H, ξ) of $A \otimes_\gamma B$. Assume that $a' \in A$ and $b' \in B$. Then,

$$\begin{aligned}
\pi_A(a)\pi_B(b)\big(\pi_A(a')\pi_B(b')\xi\big) &= \pi_A(a)\pi_A(a')\pi_B(b)\pi_B(b')(\xi) \\
&= \pi_A(a')\pi_A(a)\pi_B(b)\pi_B(b')(\xi) \\
&= \pi_A(a')\pi_B(b)\pi_B(b')\pi_A(a)(\xi) \\
&= \pi_A(a')\pi_B(b)\pi_B(b')\pi_A(a)|_K(\xi) \\
&= \tau(a)\pi_A(a')\pi_B(b)\pi_B(b')(\xi) \\
&= \tau(a)\pi_B(b)\big(\pi_A(a')\pi_B(b')(\xi)\big).
\end{aligned}$$

So, $\pi_A(a)\pi_B(b) = \tau(a)\pi_B(b)$.

From part (a), we obtain the equality

$$\pi(A \otimes B) = \pi_A(A)\pi_B(B) = \pi_B(B).$$

Therefore, ξ is a unit cyclic vector for the non-degenerate representation (π_B, H). By Theorem 2.13.14, if $b \in B$ then

$$\rho(b) = \langle \pi_B(b)(\xi), \xi \rangle.$$

Arguing in a similar manner to before, we deduce that the representations (π_B, ξ) and $(\varphi_\rho, H_\rho, \xi_\rho)$ are unitarily equivalent. Since ρ is a pure state on B by assumption, the GNS representation $(\varphi_\rho, H_\rho, \xi_\rho)$ is irreducible. Therefore, (π_B, ξ) is an irreducible representation of B. By Theorem 1.9.7,

$$\pi(A \otimes B)' = \pi_B(B)' = \mathbb{C} id_H$$

and consequently, (π, H, ξ) is an irreducible representation of $A \otimes_\gamma B$. By Theorem 2.12.7, the state ω on $A \otimes_\gamma B$ is pure. □

Theorem 2.13.16. *Let A and B be C*-algebras. Let γ be a C*-norm on $A \otimes B$ and ω be a pure state on $A \otimes_\gamma B$ such that the state ω_A on A is pure (see Theorem 2.13.14). Then, $(\omega_A, \omega_B) \in S_\gamma$ and $\omega = \omega_A \otimes_\gamma \omega_B$.*

Theorem 2.13.16 can be thought of as a partial converse to Theorem 2.13.14.

Proof. Assume that A and B are C*-algebras. Assume that γ is a C*-norm on $A \otimes B$. Assume that ω is a pure state on $A \otimes_\gamma B$ such that the state ω_A from Theorem 2.13.14 on A is also pure.

To simplify the notation we will use, let $(\pi, H, \xi) = (\varphi_\omega, H_\omega, \xi_\omega)$ be the GNS representation of $A \otimes_\gamma B$ associated to ω. Let $\tau = \omega_A$ and $\rho = \omega_B$. Define the subspace $K = \overline{\pi_A(A)\xi}$. Let ψ be the *-homomorphism

$$\begin{array}{rccc} \psi : & A & \to & B(K) \\ & a & \mapsto & \pi_A(a)|_K. \end{array}$$

By a similar argument to that of Theorem 2.13.15, the vector ξ is a cyclic vector for the representation (ψ, K). Now if $a \in A$ then

$$\langle \psi(a)(\xi), \xi \rangle = \langle \pi_A(a)|_K(\xi), \xi \rangle = \tau(a).$$

Therefore, the representations (ψ, K) and $(\varphi_\tau, H_\tau, \xi_\tau)$ are unitarily equivalent. Since τ is pure by assumption, the GNS representation $(\varphi_\tau, H_\tau, \xi_\tau)$ is irreducible by Theorem 2.12.6. Now let $p : H \to K$ denote the projection onto the closed subspace K of H. Since K is an invariant subspace for $\pi_A(A)$, if $a \in A$ and $\mu \in H$ then

$$p\pi_A(a)(\mu) = \pi_A(a)(\mu) = \pi_A(a)p(\mu).$$

So, $p \in \pi_A(A)'$. Now let q be a projection in $p\pi_A(A)'p$. Then, the image $q(H)$ is a closed vector space of K. Furthermore, it is an invariant subspace for (ψ, K). Since (ψ, K) is an irreducible representation of A, either $q(H) = 0$ or $q(H) = K$. This means that either $q = 0$ or $q = p$.

We conclude that the von Neumann algebra $p\pi_A(A)'p$ contains only scalar projections (scalars of the projection p). Recalling the fact that a von Neumann algebra is the closed linear span of its projections, we deduce that $p\pi_A(A)'p = \mathbb{C}p$.

By Theorem 2.11.9, $\pi_B(B) \subseteq \pi_A(A)'$. If $b \in B$ then there exists a scalar $\lambda \in \mathbb{C}$ such that $p\pi_B(b)p = \lambda p$. Using Theorem 2.13.14, we find that

$$\begin{aligned} \rho(b) &= \langle \pi_B(b)(\xi), \xi \rangle \\ &= \langle \pi_B(b)p(\xi), p(\xi) \rangle \\ &= \langle p\pi_B(b)p(\xi), \xi \rangle \\ &= \langle \lambda\xi, \xi \rangle = \lambda. \end{aligned}$$

Therefore, $p\pi_B(b)p = \rho(b)p$. Now if $a \in A$ then

$$\begin{aligned}
\omega(a \otimes b) &= \langle \pi(a \otimes b)(\xi), \xi \rangle \\
&= \langle \pi_A(a)\pi_B(b)(\xi), \xi \rangle \\
&= \langle \pi_A(a)\pi_B(b)p(\xi), p(\xi) \rangle \\
&= \langle p\pi_A(a)\pi_B(b)p(\xi), \xi \rangle \\
&= \langle \pi_A(a)(\rho(b)\xi), \xi \rangle \\
&= \rho(b)\tau(a).
\end{aligned}$$

So, ω extends the functional $\tau \otimes \rho$ to a linear functional on $A \otimes_\gamma B$. By Theorem 2.13.7, we deduce that $\omega = \tau \otimes_\gamma \rho$.

Finally, we show that $(\tau, \rho) = (\omega_A, \omega_B) \in S_\gamma$. First, we use Theorem 2.13.8 to obtain a unitary map $u : H_\tau \hat{\otimes} H_\rho \to H$ such that if $c \in A \otimes B$ then

$$\pi(c) = u(\varphi_\tau \hat{\otimes} \varphi_\rho)(c)u^*.$$

Suppose for the sake of contradiction that ρ is not pure. Then, the associated GNS representation $(\varphi_\rho, H_\rho, \xi_\rho)$ of B is not irreducible and hence, there exists a non-trivial closed vector subspace L of H_ρ, which is invariant for $\varphi_\rho(B)$. Define $L' = H_\tau \hat{\otimes} L$. Then, L' is a non-trivial closed vector subspace of $H_\tau \hat{\otimes} H_\rho$, which is invariant for $(\varphi_\tau \hat{\otimes} \varphi_\rho)(A \otimes B)$.

Now define $L'' = u(L')$. Then, L'' is a non-trivial closed vector subspace of H, which is invariant for $\pi(A \otimes_\gamma B)$. To see why this is the case, let $c \in A \otimes B$. Then,

$$\pi(c)L'' = u(\varphi_\tau \hat{\otimes} \varphi_\rho)(c)u^*u(L') = u(\varphi_\tau \hat{\otimes} \varphi_\rho)(c)(L') \subseteq L''.$$

Hence, L'' is invariant for $\pi(A \otimes B)$ and hence, also for $\pi(A \otimes_\gamma B)$. This contradicts the assumption that ω is a pure state on $A \otimes_\gamma B$. So, ρ is pure and $(\tau, \rho) \in S_\gamma$ as required. □

Now we have reached one of the two pinnacles of this section.

Theorem 2.13.17. *Let A be an abelian C*-algebra. Then, A is a nuclear C*-algebra.*

Proof. Assume that A is an abelian C*-algebra and B is an arbitrary C*-algebra. Let γ be a C*-norm on $A \otimes B$. Let $\omega \in PS(A \otimes_\gamma B)$ and let (π, H, ξ) be the GNS representation of $A \otimes_\gamma B$ associated to ω. Let $\tau = \omega_A$ and $\rho = \omega_B$ where ω_A and ω_B are defined as in Theorem 2.13.14.

Now let π_A and π_B be the unique *-homomorphisms constructed in Theorem 2.11.9. Since ω is a pure state, the GNS representation (π, H, ξ) is irreducible. Since A is abelian, we deduce that

$$\pi_A(A) \subseteq \pi(A \otimes_\gamma B)' = \mathbb{C}id_H.$$

Therefore, if $a \in A$ then there exists $\lambda \in \mathbb{C}$ such that $\pi_A(a) = \lambda id_H$. Arguing in a similar fashion to Theorem 2.13.15, we find that if $a \in A$ then $\pi_A(a) = \tau(a) id_H$. Since π_A is a *-homomorphism, τ is a multiplicative state on A. By Theorem 2.12.6, we deduce that τ is a pure state on A.

By Theorem 2.13.16, we deduce that ρ is a pure state of B, $(\tau, \rho) \in S_\gamma$ and $\omega = \tau \otimes_\gamma \rho$. Next, Theorem 2.13.12 tells us that if $c \in A \otimes B$ then

$$\|\pi(c)\| = \|(\varphi_\tau \hat{\otimes} \varphi_\rho)(c)\|.$$

We can relate this to the C*-norm γ by observing that if $c \in A \otimes B$ then

$$\gamma(c) = \sup_{\omega \in PS(A \otimes_\gamma B)} \|\varphi_\omega(c)\|.$$

This is due to Theorem 2.12.9 and Theorem 2.12.7. By Theorem 2.13.8,

$$\gamma(c) = \sup_{\omega \in PS(A \otimes_\gamma B)} \|\varphi_\omega(c)\| = \sup_{(\tau,\rho) \in S_\gamma} \|(\varphi_\tau \hat{\otimes} \varphi_\rho)(c)\|.$$

The key idea behind this proof is that if we show that
$S_\gamma = PS(A) \times PS(B)$ then

$$\gamma(c) = \sup_{\substack{\tau \in PS(A) \\ \rho \in PS(B)}} \|(\varphi_\tau \hat{\otimes} \varphi_\rho)(c)\|$$

and since the RHS of the above equation is independent of γ, we can conclude that $A \otimes B$ has a unique C*-norm.

To show: (a) $S_\gamma = PS(A) \times PS(B)$.

(a) Suppose for the sake of contradiction that $S_\gamma \neq PS(A) \times PS(B)$. By Theorem 2.13.13, S_γ is a closed subset of $PS(A) \times PS(B)$. By the product topology on $PS(A) \times PS(B)$, there exists a pair of weak-* open sets $U \subseteq PS(A)$ and $V \subseteq PS(B)$ such that $S_\gamma \cap (U \times V) = \emptyset$.

For the next part, we want to assume that U and V are unitarily equivalent. In order to justify this, let $u \in \tilde{A}$ and $v \in \tilde{B}$ be unitary elements, where $\tilde{A}$ and $\tilde{B}$ are the unitizations of A and B respectively. Define

$$U^u = \{\tau^u \mid \tau \in U\} \qquad \text{and} \qquad V^v = \{\rho^v \mid \rho \in V\}$$

where we adopt the notation from Theorem 2.13.13. Define $U' = \bigcup_u U^u$ and $V' = \bigcup_v V^v$. Then, U' and V' are unitarily invariant sets. That is, if $\tau \in U'$ and $u' \in \tilde{A}$ is unitary then $\tau^{u'} \in U'$. Similarly, if $\rho \in V'$ and $v' \in \tilde{B}$ is unitary then $\rho^{v'} \in V'$.

We claim that U' and V' are weak-* open. To see why this is the case, it suffices to show that if $u \in \tilde{A}$ and $v \in \tilde{B}$ are unitary then U^u and V^v are weak-* open in $PS(A)$ and $PS(B)$ respectively.

To this end, assume that $u \in \tilde{A}$ is unitary. Let $\{\tau_n\}_{n\in\mathbb{Z}_{>0}}$ be a sequence in $(U^u)^c$ which weakly converges to $\tau \in PS(A)$. This means that if $a \in A$ then $\lim_{n\to\infty} \tau_n(a) = \tau(a)$. In particular,

$$\lim_{n\to\infty} (\tau_n)^{u^*}(a) = \lim_{n\to\infty} \tau_n(u^* a u) = \tau(u^* a u) = \tau^{u^*}(a).$$

Since $\tau_n = ((\tau_n)^{u^*})^u$ and $\tau_n \notin U^u$ for $n \in \mathbb{Z}_{>0}$, then $(\tau_n)^{u^*} \notin U$. By assumption, U is a weak-* open subset of $PS(A)$. Hence its complement U^c is weak-* closed and consequently, $\tau^{u^*} \notin U$. Therefore, $\tau = ((\tau)^{u^*})^u \notin U^u$. We conclude that $(U^u)^c$ is weak-* closed and subsequently, that U^u is weak-* open in $PS(A)$. By an analogous argument, V^v is weak-* open in $PS(B)$.

Therefore, U' and V' are unitarily invariant and weak-* open subsets of $PS(A)$ and $PS(B)$ respectively. Moreover, $S_\gamma \cap (U' \times V') = \emptyset$ by the contrapositive of Theorem 2.13.13. Hence, if U and V are not initially unitarily invariant, we can always replace them by U' and V' respectively.

Returning to the problem at hand, we may assume without loss of generality that U and V are unitarily invariant. Then, the complements $S_A = PS(A)\backslash U$ and $S_B = PS(B)\backslash V$ are weak-* closed and unitarily invariant sets in $PS(A)$ and $PS(B)$ respectively. By assumption, $S_A \neq PS(A)$ and $S_B \neq PS(B)$. Hence, the orthogonal complements $S_A^\perp$ and $S_B^\perp$ are non-zero closed ideals (of A and B respectively) and hence, contain non-zero positive elements. Let $a \in S_A^\perp$ and $b \in S_B^\perp$ be such positive elements.

If $(\tau, \rho) \in S_\gamma$ then because $S_\gamma \cap (U \times V) = \emptyset$, either $\tau \notin U$ or $\rho \notin V$ because U and V are unitarily invariant. In either case,

$$(\tau \otimes_\gamma \rho)(a \otimes b) = \tau(a)\rho(b) = 0$$

because $a \in S_A^\perp$ and $b \in S_B^\perp$. Now by Theorem 2.12.9, there exists an irreducible representation (φ, H) of $A \otimes_\gamma B$ such that $\gamma(a \otimes b) = \|\varphi(a \otimes b)\|$. By the proof of Theorem 1.11.6, the representation (φ, H) is the GNS representation associated to some state ω on $A \otimes_\gamma B$. By Theorem 2.12.7, ω is a pure state on $A \otimes_\gamma B$ such that

$$\omega(a \otimes b) = \|\varphi(a \otimes b)\| = \gamma(a \otimes b).$$

By the first part of the proof of this theorem, we deduce that there exists pure states $\tau \in PS(A)$ and $\rho \in PS(B)$ such that $(\tau, \rho) \in S_\gamma$ and $\omega = \tau \otimes_\gamma \rho$. By the previous finding and our construction of a and b, we deduce that

$$\omega(a \otimes b) = (\tau \otimes_\gamma \rho)(a \otimes b) = 0 = \gamma(a \otimes b).$$

So, $a \otimes b = 0$ and either $a = 0$ or $b = 0$. In either case, this contradicts the assumption that $a \in S_A^\perp$ and $b \in S_B^\perp$ are non-zero.

Part (a) shows that $S_\gamma = PS(A) \times PS(B)$ and therefore,

$$\gamma(c) = \sup_{\substack{\tau \in PS(A) \\ \rho \in PS(B)}} \|(\varphi_\tau \hat{\otimes} \varphi_\rho)(c)\|$$

is the only C*-norm on $A \otimes B$. Therefore, A is a nuclear C*-algebra. $\square$

Before we move on, we first recall the notion of a partition of unity . Let X be a compact Hausdorff space, $n \in \mathbb{Z}_{>0}$ and $U_1, \ldots, U_n$ be open subsets of X such that $X = U_1 \cup U_2 \cup \cdots \cup U_n$. Then, there exists continuous functions $h_1, \ldots, h_n \in Cts(X, [0,1])$ such that if $i \in \{1, 2, \ldots, n\}$ then

$$supp(h_i) = \overline{\{x \in X \mid h_i(x) \neq 0\}} \subseteq U_i.$$

and $\sum_{i=1}^n h_i = \mathbb{1}$ where $\mathbb{1}$ is the constant function which sends $x \in X$ to 1.

Definition 2.13.2. Let Ω be a LCH space and X be a Banach space. Define $Cts_0(\Omega, X)$ to be the Banach space of all continuous functions $g : \Omega \to X$ such that the continuous map $\omega \mapsto \|g(\omega)\|$ vanishes at infinity. The operations on $Cts_0(\Omega, X)$ are defined pointwise and its norm is the supremum norm.

If $f \in Cts_0(\Omega, \mathbb{C})$ and $x \in X$ then we define $fx \in Cts_0(\Omega, X)$ by $(fx)(\omega) = f(\omega)x$.

We remark here that if Ω is a LCH space and X is a C*-algebra then $Cts_0(\Omega, X)$ is also a C*-algebra, with multiplication and involution defined pointwise.

Theorem 2.13.18. *Let Ω be a LCH space and X be a Banach space. Then, $Cts_0(\Omega, X)$ is the closed linear span of the set*

$$\{fx \mid f \in Cts_0(\Omega, \mathbb{C}),\ x \in X\}.$$

Proof. Assume that Ω is a LCH space and that X is a Banach space. Let $\tilde{\Omega}$ be the one-point compactification of Ω. The point at infinity of $\tilde{\Omega}$ is denoted by ∞.

Assume that $g \in Cts_0(\Omega, X)$. Define the function $\tilde{g} : \tilde{\Omega} \to X$ by

$$\tilde{g}(\omega) = \begin{cases} g(\omega), & \text{if } \omega \in \Omega, \\ 0, & \text{if } \omega = \infty. \end{cases}$$

Since g is continuous and vanishes at infinity, the function $\tilde{g}$ is continuous. Now assume that $\epsilon \in \mathbb{R}_{>0}$. Since $\tilde{\Omega}$ is a compact Hausdorff space, the image $\tilde{g}(\tilde{\Omega})$ is compact and hence, totally bounded. So, it is precompact and thus, there exists $x_1, \ldots, x_n \in \tilde{g}(\tilde{\Omega})$ such that if $j \in \{1, 2, \ldots, n\}$ and

$$U_j = \{\omega \in \tilde{\Omega} \mid \|\tilde{g}(\omega) - x_j\| < \epsilon\}$$

then $\tilde{\Omega} = U_1 \cup \cdots \cup U_n$. We also note that $U_1, \ldots, U_n$ are all open subsets of $\tilde{\Omega}$. This means that we can construct a partition of unity. So, there exists $h_1, \ldots, h_n \in Cts(\tilde{\Omega}, [0, 1])$ such that if $j \in \{1, 2, \ldots, n\}$ then the support $supp(h_j) \subseteq U_j$ and $\sum_{i=1}^{n} h_i = \mathbb{1}$.

If $\omega \in \tilde{\Omega}$ then there exists distinct $j_1, \ldots, j_k \in \{1, 2, \ldots, n\}$ such that if $i \in \{1, 2, \ldots, k\}$ then $\omega \in U_{j_i}$. So,

$$
\begin{aligned}
\|\tilde{g}(\omega) - \sum_{i=1}^{n} h_i(\omega)x_i\| &= \|\tilde{g}(\omega)\sum_{i=1}^{n} h_i(\omega) - \sum_{i=1}^{n} h_i(\omega)x_i\| \\
&= \|\sum_{i=1}^{n} h_i(\omega)(\tilde{g}(\omega) - x_i)\| \\
&\leq \sum_{i=1}^{n} h_i(\omega)\|\tilde{g}(\omega) - x_i\| \\
&= \sum_{i\in\{j_1,\ldots,j_k\}} h_i(\omega)\|\tilde{g}(\omega) - x_i\| + \sum_{i\notin\{j_1,\ldots,j_k\}} h_i(\omega)\|\tilde{g}(\omega) - x_i\| \\
&< \sum_{i\in\{j_1,\ldots,j_k\}} h_i(\omega)\epsilon + \sum_{i\notin\{j_1,\ldots,j_k\}} h_i(\omega)\|\tilde{g}(\omega) - x_i\| \\
&= \sum_{i\in\{j_1,\ldots,j_k\}} h_i(\omega)\epsilon \leq \sum_{i=1}^{n} h_i(\omega)\epsilon = \epsilon.
\end{aligned}
$$

The second last inequality follows from the fact that if $i \notin \{j_1, \ldots, j_k\}$ then $\omega \notin U_i$ and $h_i(\omega) = 0$ because ω lies outside the support of h_i.

In particular, if $\omega = \infty$ then

$$
\|\tilde{g}(\infty) - \sum_{i=1}^{n} h_i(\omega)x_i\| = \|\sum_{i=1}^{n} h_i(\omega)x_i\| = 0.
$$

Hence, if $\epsilon \in \mathbb{R}_{>0}$ then $\|\sum_{i=1}^{n} h_i(\omega)x_i\| \leq \epsilon$. If $i \in \{1, 2, \ldots, n\}$ then define $f_i = h_i|_\Omega$. Then, $f_i \in Cts_0(\Omega, \mathbb{C})$ and if $\omega \in \Omega$ then

$$
\begin{aligned}
\|g(\omega) - \sum_{i=1}^{n} f_i(\omega)x_i\| &= \|\tilde{g}(\omega) - \sum_{i=1}^{n} h_i(\omega)x_i\| + \|\tilde{g}(\infty) - \sum_{i=1}^{n} h_i(\infty)x_i\| \\
&\leq \epsilon + \epsilon = 2\epsilon.
\end{aligned}
$$

By taking the supremum over all $\omega \in \Omega$, we find that $g = \sum_{i=1}^{n} f_i x_i$. $\square$

We need one more result before we can prove that the spatial C*-norm on a tensor product of C*-algebras is the smallest one. This requires the following construction. Let Ω be a locally compact Hausdorff space and A be a C*-algebra. Define the map

$$
\begin{array}{rccc}
B: & Cts_0(\Omega, \mathbb{C}) \times A & \to & Cts_0(\Omega, A) \\
& (f, a) & \mapsto & fa.
\end{array}
$$

Then, B is a $\mathbb{C}$-bilinear map. By the universal property of the tensor product in Theorem 2.11.1, there exists a unique linear map $\pi : Cts_0(\Omega, \mathbb{C}) \otimes A \to Cts_0(\Omega, A)$ such that if $f \in Cts_0(\Omega, \mathbb{C})$ and $a \in A$ then

$$\pi(f \otimes a) = fa. \tag{2.6}$$

The map in equation (2.6) is called the **canonical map** from $Cts_0(\Omega, \mathbb{C}) \otimes A$ to $Cts_0(\Omega, A)$.

Theorem 2.13.19. *Let Ω be a LCH space and A be a C^*-algebra. Let π denote the canonical map in equation* (2.6)*. Then, π extends uniquely to a *-isomorphism from $Cts_0(\Omega, \mathbb{C}) \otimes_* A$ to $Cts_0(\Omega, X)$, where $\otimes_*$ denotes the spatial tensor product.*

Proof. Assume that Ω is a locally compact Hausdorff space and that A is a C*-algebra. Assume that π is the canonical map defined in equation (2.6). To see that π is a *-homomorphism in this case, assume that $f, g \in Cts_0(\Omega, \mathbb{C})$, $a, b \in A$ and $\omega \in \Omega$. Then,

$$\begin{aligned}
\pi((f \otimes a) \cdot (g \otimes b))(\omega) &= \pi(fg \otimes ab)(\omega) \\
&= (fg)ab(\omega) \\
&= f(\omega)g(\omega)ab = f(\omega)ag(\omega)b \\
&= (fa)(\omega)(gb)(\omega) \\
&= \pi(f \otimes a)(\omega) \cdot \pi(g \otimes b)(\omega)
\end{aligned}$$

and

$$\begin{aligned}
\pi((f \otimes a)^*)(\omega) &= \pi(\overline{f} \otimes a^*)(\omega) \\
&= (\overline{f}a^*)(\omega) \\
&= \overline{f}(\omega)a^* = \overline{f(\omega)}a^* \\
&= (f(\omega)a)^* = ((fa)(\omega))^* \\
&= (\pi(f \otimes a)(\omega))^*.
\end{aligned}$$

So, the canonical map π is a *-homomorphism. Next, we claim that π is injective.

To show: (a) π is an injective *-homomorphism.

(a) Assume that $c \in \ker \pi$. Then, we can write $c = \sum_{i=1}^{n}(f_i \otimes a_i)$, where $f_i \in Cts_0(\Omega, \mathbb{C})$ and the $a_i \in A$ are linearly independent. We compute directly that

$$\pi(c) = \pi(\sum_{i=1}^{n}(f_i \otimes a_i)) = \sum_{i=1}^{n}(f_i a_i) = 0.$$

If $\omega \in \Omega$ then $\sum_{i=1}^{n} f_i(\omega)a_i = 0$. Since the set $\{a_1, \ldots, a_n\}$ is linearly independent, we deduce that $f_1 = \cdots = f_n = 0$. Therefore, $c = 0$ and π is injective.

Now define the map $\|-\|$ by

$$\begin{array}{rccc} \pi: & Cts_0(\Omega, \mathbb{C}) \otimes A & \to & \mathbb{R}_{\geq 0} \\ & c & \mapsto & \|\pi(c)\|. \end{array}$$

Since π is an injective *-homomorphism, the map $\|-\|$ defines a C*-norm on $Cts_0(\Omega, \mathbb{C}) \otimes A$. However, $Cts_0(\Omega, \mathbb{C})$ is an abelian C*-algebra and is thus, nuclear by Theorem 2.13.17. Hence, if $c \in Cts_0(\Omega, \mathbb{C}) \otimes A$ then

$$\|c\| = \|\pi(c)\| = \|c\|_*.$$

This means that we can extend π uniquely to an isometric *-homomorphism π' on the spatial tensor product $Cts_0(\Omega, \mathbb{C}) \otimes_* A$. Finally, to see that π' is surjective, we observe that

$$\{fa \mid f \in Cts_0(\Omega, \mathbb{C}), a \in A\} \subseteq \text{im } \pi'$$

By Theorem 2.13.18, we deduce that π' is surjective and consequently, a *-isomorphism from $Cts_0(\Omega, \mathbb{C}) \otimes_* A$ to $Cts_0(\Omega, X)$. □

Now let A and B be *-algebras and $\theta : A \otimes B \to B \otimes A$ be the unique linear map defined by $\theta(a \otimes b) = b \otimes a$. Then, θ is a *-isomorphism. The key observation we make here is that if $A \otimes B$ admits a unique C*-norm then $B \otimes A$ also admits a unique C*-norm. This observation is used in the proof of the second main theorem of the section.

Theorem 2.13.20. *Let A and B be C*-algebras. Then, the spatial C*-norm $\|-\|_*$ is the smallest C*-norm on the tensor product $A \otimes B$.*

Proof. Assume that A and B are C*-algebras. Let γ be a C*-norm on $A \otimes B$. If B is non-unital then by Theorem 2.13.11, we can extend γ to a C*-norm on $A \otimes \tilde{B}$. By Theorem 2.13.10, the spatial C*-norm on $A \otimes \tilde{B}$ restricts to the spatial C*-norm on $A \otimes B$. Therefore, it suffices to prove the theorem in the case where B is unital. Hence, assume that B is a unital C*-algebra. Recall the definition of S_γ from equation (2.5).

To show: (a) $S_\gamma = PS(A) \times PS(B)$.

(a) Suppose for the sake of contradiction that $S_\gamma \neq PS(A) \times PS(B)$. By the proof of Theorem 2.13.17, there exists weak-* closed unitarily invariant subsets $S_A \subsetneq PS(A)$ and $S_B \subsetneq PS(B)$ such that

$$S_\gamma \subseteq (S_A \times PS(B)) \cup (PS(A) \times S_B).$$

Furthermore, $S_A^\perp$ and $S_B^\perp$ contain non-zero positive elements a_0 and b_0 respectively. Now if $(\tau, \rho) \in S_\gamma$ then because

$$S_\gamma \cap \big((PS(A) \backslash S_A) \times (PS(B) \backslash S_B)\big) = \emptyset$$

either $\tau \in S_A$ or $\rho \in S_B$. In either case,

$$(\tau \otimes_\gamma \rho)(a_0 \otimes b_0) = \tau(a_0)\rho(b_0) = 0.$$

Now let C be the C*-subalgebra generated by the set $\{1_B, b_0\}$. Then, C is abelian and hence nuclear by Theorem 2.13.17. Since $A \otimes C$ has a unique C*-norm, $\gamma = \|-\|_*$ on $A \otimes C$. Therefore, the spatial tensor product $A \otimes_* C$ can be considered a C*-subalgebra of $A \otimes_\gamma B$.

Now let $\tau \in PS(A)$ and $\rho \in PS(C)$ such that $\tau(a_0) = \|a_0\| > 0$ and $\rho(b_0) = \|b_0\| > 0$ by Theorem 1.11.6. Now by Theorem 2.13.4 and Theorem 2.13.15, $\tau \otimes \rho$ is continuous with respect to the spatial C*-norm and consequently, extends to a pure state ω' on $A \otimes_* C$. Now by [Mur90, Theorem 5.1.13], ω' extends to a pure state ω on $A \otimes_\gamma B$.

Now let ω_A and ω_B be the states on A and B respectively, defined in Theorem 2.13.14. If $a \in A$ and $\{v_\mu\}_{\mu \in M}$ is an approximate unit then

$$\omega_A(a) = \lim_\mu \omega(a \otimes v_\mu) = \omega(a \otimes 1_B) = (\tau \otimes \rho)(a \otimes 1_B) = \tau(a)\rho(1_B) = \tau(a).$$

Consequently, $\tau = \omega_A$ is a pure state on A. By Theorem 2.13.16, $(\omega_A, \omega_B) \in S_\gamma$ and $\omega = \omega_A \otimes_\gamma \omega_B$. Therefore,

$$\omega(a_0 \otimes b_0) = (\tau \otimes \rho)(a_0 \otimes b_0) = \tau(a_0)\rho(b_0) = \|a_0\|\|b_0\| > 0.$$

However, since $(\omega_A, \omega_B) \in S_\gamma$, then

$$\omega(a_0 \otimes b_0) = \omega_A(a_0)\omega_B(b_0) = 0.$$

This contradicts the fact that $\omega(a_0 \otimes b_0) > 0$. Therefore, $S_\gamma = PS(A) \times PS(B)$.

Next, observe that by Theorem 2.12.8 and the Krein-Milman theorem, the states of a C*-algebra are weak-* limits of nets of convex combinations of the extreme points — the zero functional and the pure states. Now let τ and ρ be positive linear functionals on A and B respectively, which are convex combinations of the zero functional and pure states. Then, there exists scalars

$$t_1, \ldots, t_n, s_1, \ldots, s_m \in \mathbb{R}_{\geq 0}$$

and functionals

$$\tau_1, \ldots, \tau_n \in \{0\} \cup PS(A) \qquad \text{and} \qquad \rho_1, \ldots, \rho_m \in \{0\} \cup PS(B)$$

such that

$$\tau = \sum_{i=1}^{n} t_i \tau_i \qquad \text{and} \qquad \rho = \sum_{j=1}^{m} s_j \rho_j.$$

Now we have

$$\tau \otimes \rho = \sum_{i=1}^{n} \sum_{j=1}^{m} t_i s_j (\tau_i \otimes \rho_j).$$

By part (a), $S_\gamma = PS(A) \times PS(B)$. So, if $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, m\}$ then $\tau_i \otimes \rho_j$ is continuous with respect to γ. Therefore, $\tau \otimes \rho$ is also continuous with respect to γ.

Now suppose that τ and ρ are arbitrary states of A and B respectively. Then, there exists nets $\{\tau_\lambda\}_{\lambda \in L}$ and $\{\rho_\mu\}_{\mu \in M}$ of positive linear functionals on A and B respectively such that $\{\tau_\lambda\}$ weakly converges to τ and $\{\rho_\mu\}$ weakly converges to ρ. Moreover, if $\lambda \in L$ and $\mu \in M$ then $\|\tau_\lambda\|, \|\rho_\mu\| \leq 1$ and $\tau_\lambda \otimes \rho_\mu$ is continuous with respect to γ.

By Theorem 2.13.7, $\tau_\lambda \otimes \rho_\mu$ uniquely extends to a state on $A \otimes_\gamma B$, which is a positive linear functional of norm $\|\tau_\lambda\|\|\rho_\mu\|$. So, if $c \in A \otimes B$ then

$$|(\tau_\lambda \otimes \rho_\mu)(c)| \leq \|\tau_\lambda\|\|\rho_\mu\|\gamma(c) \leq \gamma(c).$$

Now if $c \in A \otimes B$ then $(\tau \otimes \rho)(c) = \lim_{\lambda,\mu}(\tau_\lambda \otimes \rho_\mu)(c)$. So, $|(\tau \otimes \rho)(c)| \le \gamma(c)$ and hence, $\tau \otimes \rho$ is continuous with respect to γ.

Now let D be the unitization of $A \otimes_\gamma B$, $\tau \in S(A)$ and $\rho \in S(B)$. Let ω be the unique state on D extending $\tau \otimes_\gamma \rho$. This uses the previous observation and the universal property of the unitization. Now if $d \in D$ then the linear functional

$$\begin{array}{rccc} \omega^d : & D & \to & \mathbb{C} \\ & c & \mapsto & \omega(d^* c d) \end{array}$$

is positive. If $c \in D$ and 1_D is the multiplicative unit of D then $\gamma(c^*c)1_D - c^*c \ge 0$ by Theorem 2.2.5. Therefore,

$$\omega^d(\gamma(c^*c)1_D - c^*c) = \gamma(c^*c)\omega^d(1_D) - \omega^d(c^*c) \ge 0.$$

Now if $\omega(d^*d) = \omega^d(1_D) > 0$ then

$$\gamma(c)^2 = \gamma(c^*c) = \frac{\omega^d(c^*c)}{\omega^d(1_D)} = \frac{\omega(d^*c^*cd)}{\omega(d^*d)} \ge 0.$$

By Theorem 2.13.9, we deduce that if $c \in A \otimes B$ then

$$\begin{aligned} \|c\|_*^2 &= \sup_{\substack{\tau \in S(A) \\ \rho \in S(B)}} \sup_{\substack{d \in A \otimes B \\ (\tau \otimes \rho)(d^*d) > 0}} \frac{(\tau \otimes \rho)(d^*c^*cd)}{(\tau \otimes \rho)(d^*d)} \\ &\le \sup_{\substack{\tau \in S(A) \\ \rho \in S(B)}} \sup_{\substack{d \in D \\ \omega(d^*d) > 0}} \frac{\omega(d^*c^*cd)}{\omega(d^*d)} \\ &= \sup_{\substack{\tau \in S(A) \\ \rho \in S(B)}} \sup_{\substack{d \in D \\ \omega(d^*d) > 0}} \gamma(c)^2 = \gamma(c)^2. \end{aligned}$$

Therefore, the spatial C*-norm $\|-\|_*$ is the smallest norm on $A \otimes B$ as required. $\square$

We finish this long section with two corollaries of Theorem 2.13.20.

Theorem 2.13.21. *Let A and B be C*-algebras. Let γ be a C*-norm on $A \otimes B$. If $a \in A$ and $b \in B$ then $\gamma(a \otimes b) = \|a\|\|b\|$.*

Proof. Assume that A and B are C*-algebras. Assume that γ is a C*-norm on $A \otimes B$. If $a \in A$ and $b \in B$ then

$$\|a\|\|b\| = \|a \otimes b\|_* \le \gamma(a \otimes b) \le \|a\|\|b\|.$$

Recall that the first equality follows from the original definition of the spatial C*-norm. The final inequality follows from Theorem 2.11.10. □

Theorem 2.13.22. *Let A and B be C*-algebras. Let (φ, H) and (ψ, K) be faithful representations of A and B respectively. If $c \in A \otimes B$ then*

$$\|(\varphi \hat{\otimes} \psi)(c)\| = \|c\|_*.$$

Proof. Assume that A and B are. C*-algebras. Assume that (φ, H) and (ψ, K) are faithful representations of A and B respectively. Define

$$\begin{array}{rccc} \gamma : & A \otimes B & \to & \mathbb{R}_{\geq 0} \\ & c & \mapsto & \|(\varphi \hat{\otimes} \psi)(c)\|. \end{array}$$

By Theorem 2.11.7, $\varphi \hat{\otimes} \psi$ is an injective *-homomorphism. By Theorem 1.6.4, γ is a C*-norm on $A \otimes B$. By Theorem 2.13.20, if $c \in A \otimes B$ then $\gamma(c) \geq \|c\|_*$.

But by Theorem 2.13.3,

$$\gamma(c) = \|(\varphi \hat{\otimes} \psi)(c)\| \leq \sup_{\substack{\tau \in S(A) \\ \rho \in S(B)}} \|(\varphi_\tau \hat{\otimes} \varphi_\rho)(c)\| = \|c\|_*.$$

Hence, $\gamma = \|-\|_*$. □

2.14 Short exact sequences of C*-algebras

In this section, we follow [Mur90, Section 6.5]. The notion of a short exact sequence of C*-algebras is exactly the notion of a short exact sequence of objects in an abelian category.

Definition 2.14.1. Suppose that we have the following sequence in the category of C*-algebras:

$$0 \longrightarrow J \xrightarrow{j} A \xrightarrow{\pi} B \longrightarrow 0$$

We say that the above sequence is a **short exact sequence** if j is an injective *-homomorphism, π is a surjective *-homomorphism and $\operatorname{im} j = \ker \pi$.

We also refer to A as an **extension** of B by J.

We will explore how short exact sequences of C*-algebras interact with nuclearity. We begin with the following result, which is reminiscent of Theorem 2.11.7.

Theorem 2.14.1. *Let A, B, A', B' be C*-algebras and $\varphi : A \to A'$ and $\psi : B \to B'$ be *-homomorphisms. Then, there exists a unique *-homomorphism $\pi : A \otimes_* B \to A' \otimes_* B'$ such that if $a \in A$ and $b \in B$ then*

$$\pi(a \otimes b) = \varphi(a) \otimes \psi(b)$$

Moreover, if φ and ψ are injective then π is also injective.

Proof. Assume that A, B, A' and B' are C*-algebras. Assume that $\varphi : A \to A'$ and $\psi : B \to B'$ are *-homomorphisms. Let (φ', H') and (ψ', K') be faithful representations of A' and B' respectively. By Theorem 2.11.7, we obtain the unique *-homomorphism $\varphi' \hat{\otimes} \psi'$, defined by

$$\begin{array}{rccc} \varphi' \hat{\otimes} \psi' : & A' \otimes B' & \to & B(H' \hat{\otimes} K') \\ & a \otimes b & \mapsto & \varphi'(a) \hat{\otimes} \psi'(b) \end{array}$$

By Theorem 2.13.22, $\varphi' \hat{\otimes} \psi'$ is an isometry on $A' \otimes B'$ with respect to the spatial C*-norm. Using Theorem 2.11.6, form the *-homomorphism $\pi' = \varphi \otimes \psi$ and define

$$\phi = (\varphi' \hat{\otimes} \psi') \circ \pi'.$$

Then $\phi = \varphi' \varphi \mathbin{\hat{\otimes}} \psi' \psi$. By Theorem 2.13.5, if $c \in A \otimes B$ then

$$\|\phi(c)\| \leq \|c\|_*.$$

Since $\varphi' \hat{\otimes} \psi'$ is an isometry then

$$\|\pi'(c)\|_* = \|(\varphi' \hat{\otimes} \psi')(\pi'(c))\| = \|\phi(c)\| \leq \|c\|_*.$$

Hence, we can extend π' to a *-homomorphism $\pi : A \otimes_* B \to A' \otimes_* B'$ satisfying

$$\pi(a \otimes b) = \pi'(a \otimes b) = \varphi(a) \otimes \psi(b).$$

Now assume that φ and ψ are injective *-homomorphisms. Then, the composites $\varphi' \circ \varphi$ and $\psi' \circ \psi$ are both injective *-homomorphisms. By Theorem 2.13.22, if $c \in A \otimes B$ then

$$\|\phi(c)\| = \|(\varphi' \varphi \otimes \psi' \psi)(c)\| = \|c\|_*.$$

Hence, $\|\pi'(c)\|_* = \|c\|_*$, which renders the *-homomorphism π an isometry. Consequently, if $x \in A \otimes_* B$ such that $\pi(x) = 0$ then $x = 0$ and thus, π is injective as required. □

The *-homomorphism π constructed in Theorem 2.14.1 will be denoted by the notation $\varphi \otimes_* \psi$. Let us highlight a special case of Theorem 2.14.1 before we proceed. Suppose that A' and B' are C*-algebras and $A \subseteq A'$ and $B \subseteq B'$ are C*-subalgebras. Let $\iota_A : A \hookrightarrow A'$ and $\iota_B : B \hookrightarrow B'$ be the inclusion *-homomorphisms. By Theorem 2.14.1, we can construct the *-homomorphism $\iota_A \otimes_* \iota_B : A \otimes_* B \to A' \otimes_* B'$. Since ι_A and ι_B are injective then $\iota_A \otimes_* \iota_B$ is also an injective *-homomorphism. In this manner, we may regard $A \otimes_* B$ as a C*-subalgebra of $A' \otimes_* B'$.

The main theorem of this short section demonstrates a very nice property nuclear C*-algebras have with regards to short exact sequences.

Theorem 2.14.2. *Let D be a C*-algebra and*

$$0 \longrightarrow J \xrightarrow{j} A \xrightarrow{\pi} B \longrightarrow 0$$

be a short exact sequence in the category of C-algebras. Assume that the tensor product $B \otimes D$ has a unique C*-norm (note that this is the case if B or D are nuclear). Then, the sequence*

$$0 \longrightarrow J \otimes_* D \xrightarrow{j \otimes_* id_D} A \otimes_* D \xrightarrow{\pi \otimes_* id_D} B \otimes_* D \longrightarrow 0$$

is a short exact sequence of C-algebras.*

Proof. Assume that D is a C*-algebra such that the tensor product $B \otimes D$ has a unique C*-norm.

To show: (a) The *-homomorphism $j \otimes_* id_D$ is injective.

(b) The *-homomorphism $\pi \otimes_* id_D$ is surjective.

(c) $\mathrm{im}\, j = \ker \pi$.

(a) Since the *-homomorphisms $j : J \to A$ and $id_D : D \to D$ are injective then $j \otimes_* id_D$ is also an injective *-homomorphism by Theorem 2.14.1.

(b) Observe that the image of $\pi \otimes_* id_D$ contains the subspace

$$(\pi \otimes_* id_D)(A \otimes D) = \pi(A) \otimes D = B \otimes D.$$

Hence, im $\pi \otimes_* id_D = B \otimes D$ and $\pi \otimes_* id_D$ is a surjective *-homomorphism.

(c) Let $Q = (A \otimes_* D)/\mathrm{im}(j \otimes_* id_D)$ and $\phi : A \otimes_* D \to Q$ be the projection *-homomorphism. Observe that

$$(\pi \otimes_* id_D)(\mathrm{im}(j \otimes_* id_D)) = 0$$

because $\ker \pi = \mathrm{im}\ j$. By the universal property of the quotient, there exists a unique *-homomorphism $\pi' : Q \to B \otimes_* D$ such that the following diagram commutes:

$$\begin{tikzcd} A \otimes_* D \arrow[r, "\phi"] \arrow[dr, "\pi \otimes_* id_D"'] & Q \arrow[d, dashed, "\pi'"] \\ & B \otimes_* D \end{tikzcd}$$

To see that $\mathrm{im}(j \otimes_* id_D) = \ker(\pi \otimes_* id_D)$, we will show that π' is a *-isomorphism.

Since $\pi \otimes_* id_D$ is a surjective *-homomorphism then π' is also a surjective *-homomorphism. To see that π' is injective, we will construct a left inverse for π'.

Define the map

$$\begin{array}{rccc} \psi' : & B \times D & \to & Q \\ & (\pi(a), d) & \mapsto & (a \otimes d) + \mathrm{im}(j \otimes_* id_D) \end{array}$$

It is straightforward to check that ψ' is a bilinear map. By the universal property of the tensor product in Theorem 2.11.1, we obtain a linear map $\psi : B \otimes D \to Q$, defined by

$$\psi(\pi(a) \otimes d) = (a \otimes d) + \mathrm{im}(j \otimes_* id_D).$$

The fact that ψ is a *-homomorphism follows from direct computations. Now define the map

$$\begin{array}{rccc} \|-\|' : & B \otimes D & \to & \mathbb{R}_{\geq 0} \\ & c & \mapsto & \max(\|\psi(c)\|, \|c\|_*) \end{array}$$

This is a C*-norm on $B \otimes D$. By assumption, $B \otimes D$ has a unique C*-norm. Hence if $c \in B \otimes D$ then $\|c\|' = \|c\|_*$ and $\|\psi(c)\| \leq \|c\|_*$. This means that we can extend ψ to all of $B \otimes_* D$. This unique extension of ψ will once again be denoted by ψ.

To see that ψ is a left inverse for π', we compute directly that if $a \in A$ and $d \in D$ then

$$\begin{aligned}\psi(\pi'((a \otimes d) + \mathrm{im}(j \otimes_* id_D))) &= \psi(\pi'(\phi(a \otimes d))) \\ &= \psi(\pi(a) \otimes d) \\ &= (a \otimes d) + \mathrm{im}(j \otimes_* id_D).\end{aligned}$$

Therefore, π' is injective and thus, a *-isomorphism. Since $\pi \otimes_* id_D = \pi' \circ \phi$ then

$$\ker(\pi \otimes_* id_D) = \ker \phi = \mathrm{im}(j \otimes_* id_D).$$

By combining parts (a), (b) and (c) of the proof, we find that the sequence

$$0 \longrightarrow J \otimes_* D \xrightarrow{j \otimes_* id_D} A \otimes_* D \xrightarrow{\pi \otimes_* id_D} B \otimes_* D \longrightarrow 0$$

is a short exact sequence of C*-algebras. □

A major consequence of Theorem 2.14.2 is that any extension of a nuclear C*-algebra by another nuclear C*-algebra is again nuclear.

Theorem 2.14.3. *Suppose that we have the short exact sequence of C*-algebras*

$$0 \longrightarrow J \xrightarrow{j} A \xrightarrow{\pi} B \longrightarrow 0.$$

If B and J are nuclear C-algebras then A is also a nuclear C*-algebra.*

Proof. Assume that the following sequence

$$0 \longrightarrow J \xrightarrow{j} A \xrightarrow{\pi} B \longrightarrow 0.$$

in the category of C*-algebras is a short exact sequence. Assume that J and B are nuclear C*-algebras. Since B is nuclear then by Theorem 2.14.2, the sequence of C*-algebras

$$0 \longrightarrow J \otimes_* D \xrightarrow{j \otimes_* id_D} A \otimes_* D \xrightarrow{\pi \otimes_* id_D} B \otimes_* D \longrightarrow 0$$

is short exact. Recall that the maximal C*-norm $\|-\|_{max}$ is the largest C*-norm on the tensor product $A \otimes D$, whereas the spatial C*-norm $\|-\|_*$ is the smallest C*-norm by Theorem 2.13.20. Hence, to show that A is a nuclear C*-algebra, it suffices to show that $\|-\|_{max} = \|-\|_*$ on $A \otimes D$.

Let $id_{A\otimes D}$ denote the identity map on $A \otimes D$. Since $\|-\|_* \leq \|-\|_{max}$ then $id_{A\otimes D}$ extends uniquely to a *-homomorphism $i : A \otimes_{max} D \to A \otimes_* D$. By Theorem 1.6.4, it suffices to show that i is injective.

To show: (a) i is an injective *-homomorphism.

(a) We begin by constructing various *-homomorphisms. First, we use the universal property in Theorem 2.11.1 to construct the unique *-homomorphism

$$\begin{array}{rccc} \kappa : & J \otimes D & \to & A \otimes_{max} D \\ & a \otimes d & \mapsto & j(a) \otimes d \end{array}$$

Analogously to Theorem 2.14.2, the map

$$\begin{array}{rccc} \|-\|' : & J \otimes D & \to & \mathbb{R}_{\geq 0} \\ & c & \mapsto & \max(\|\kappa(c)\|_{max}, \|c\|_*) \end{array}$$

defines a C*-norm on $J \otimes D$. Since J is a nuclear C*-algebra then $\|-\|' = \|-\|_*$ and if $c \in J \otimes D$ then $\|\kappa(c)\|_{max} \leq \|c\|_*$. Thus, we can extend κ to a *-homomorphism from $J \otimes_* D$ to $A \otimes_{\max} D$. We denote this extension by κ again. We note that if $\ell \in J$ and $d \in D$ then

$$(i \circ \kappa)(\ell \otimes d) = i(j(\ell) \otimes d) = j(\ell) \otimes d = (j \otimes_* id_D)(\ell \otimes d).$$

Next, we again use Theorem 2.11.1 to construct the unique *-homomorphism

$$\begin{array}{rccc} \pi' : & A \otimes D & \to & B \otimes_* D \\ & a \otimes d & \mapsto & \pi(a) \otimes d. \end{array}$$

Observe that if $e \in A \otimes D$ then $\|\pi'(e)\|_* \leq \|e\|_{max}$. This is because the map $e \mapsto \max(\|\pi'(e)\|_*, \|e\|_{max})$ is a C*-norm on $A \otimes D$. Therefore, we can extend π' to a *-homomorphism from $A \otimes_{max} D$ to $B \otimes_* D$. Abusing notation, we denote the extension by π'.

Next, let Q be the quotient C*-algebra

$$Q = (A \otimes_{max} D)/\text{im } \kappa$$

and $\psi : A \otimes_{max} D \to Q$ be the associated projection map. By the nuclearity of B and the construction in Theorem 2.14.2, there exists a unique *-homomorphism defined by

$$\begin{array}{rccc} \theta : & B \otimes_* D & \to & Q \\ & \pi(a) \otimes d & \mapsto & (a \otimes d) + \mathrm{im}\ \kappa. \end{array}$$

So far, we have the commutative diagram of C*-algebras

$$\begin{tikzcd}
J \otimes_* D \arrow[r, "j \otimes_* id_D"] \arrow[dr, "\kappa"'] & A \otimes_* D \arrow[r, "\pi \otimes_* id_D"] & B \otimes_* D \arrow[d, "\theta"] \\
& A \otimes_{max} D \arrow[u, "i"] \arrow[ur, "\pi'"] \arrow[r, "\psi"] & Q
\end{tikzcd}$$

Now we perform a diagram chase. Assume that $c \in \ker i$. Then, $\pi'(c) = 0$ and $\psi(c) = (\theta \circ \pi')(c) = 0$. This means that $c \in \mathrm{im}\ \kappa$ and there exists $h \in J \otimes_* D$ such that $\kappa(h) = c$. So,

$$(j \otimes_* id_D)(h) = (i \circ \kappa)(h) = i(c) = 0.$$

Since $j \otimes_* id_D$ is injective then $h = 0$ and $c = \kappa(0) = 0$. Therefore, i is injective.

By part (a) and Theorem 1.6.4, i is isometric and hence, the maximal and spatial C*-norms on $A \otimes D$ are equal. Therefore, $A \otimes D$ has only one C*-norm and A is a nuclear C*-algebra as a result. $\square$

2.15 An introduction to the K-theory of C*-algebras

The idea behind the K-theory of C*-algebras is to adapt various methods and techniques from homological algebra to the category of C*-algebras. As stated in [Mur90, Chapter 7], the K-theory of C*-algebras was crucial to solving a few longstanding open problems in C*-algebra theory.

The basic idea behind the K-theory of C*-algebras is to distinguish C*-algebras by defining for a C*-algebra A, two abelian groups $K_0(A)$ and $K_1(A)$. These two groups can then be used to distinguish one C*-algebra from another. The next few sections are based on [Mur90, Chapter 7].

We begin by recalling some facts about block matrices. Let A be a *-algebra, $r = (r_1, \ldots, r_m) \in \mathbb{Z}^m_{>0}$ and $c = (c_1, \ldots, c_n) \in \mathbb{Z}^n_{>0}$. If

$i \in \{1, 2, \ldots, m\}$ and $j \in \{1, 2, \ldots, n\}$ then let $A_{ij} \in M_{r_i \times c_j}(A)$. Then, the $r \times c$ block matrix

$$a = \begin{pmatrix} A_{11} & A_{12} & \ldots & A_{1n} \\ A_{21} & A_{22} & \ldots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \ldots & A_{mn} \end{pmatrix} \tag{2.7}$$

is regarded as a $(r_1 + \cdots + r_m) \times (c_1 + \cdots + c_n)$ matrix whose elements are in A. Its adjoint is the $c \times r$ block matrix

$$a^* = \begin{pmatrix} A_{11}^* & A_{21} & \ldots & A_{n1}^* \\ A_{12}^* & A_{22}^* & \ldots & A_{n2}^* \\ \vdots & \vdots & \ddots & \vdots \\ A_{1m}^* & A_{2m}^* & \ldots & A_{mn}^* \end{pmatrix}$$

Moreover, if $d = (d_1, \ldots, d_p) \in \mathbb{Z}_{>0}^p$ and b is a $c \times d$ block matrix given by

$$b = \begin{pmatrix} B_{11} & B_{12} & \ldots & B_{1p} \\ B_{21} & B_{22} & \ldots & B_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ B_{n1} & B_{n2} & \ldots & B_{np} \end{pmatrix}$$

then the product ab is the $r \times d$ block matrix

$$ab = \begin{pmatrix} C_{11} & C_{12} & \ldots & C_{1p} \\ C_{21} & C_{22} & \ldots & C_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ C_{m1} & C_{m2} & \ldots & C_{mp} \end{pmatrix}$$

where if $i \in \{1, 2, \ldots, m\}$ and $j \in \{1, 2, \ldots, p\}$ then

$$C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}.$$

Succinctly, multiplying block matrices is exactly the same as multiplying regular matrices.

Definition 2.15.1. Let A be a *-algebra and a be the block matrix in equation (2.7). If $m = n$ and $A_{ij} = 0$ whenever $i, j \in \{1, 2, \ldots, n\}$ are distinct then we denote the matrix a by $A_{11} \oplus A_{22} \oplus \cdots \oplus A_{nn}$.

The $n \times n$ zero matrix with entries in A will be denoted by 0_n and if A is unital then the $n \times n$ identity matrix will be denoted by I_n. For clarity, the diagonal entries of I_n are all 1_A and the off-diagonal entries are all zeros.

Definition 2.15.2. Let A be a *-algebra. Define the set

$$P[A] = \bigcup_{n=1}^{\infty} \{p \in M_{n\times n}(A) \mid p \text{ is a projection}\}.$$

If $p, q \in P[A]$ then we say that p and q are **equivalent** if there exists $c \in \mathbb{Z}_{>0}$ and a matrix $u \in M_{n\times c}(A)$ such that $p = u^*u$ and $q = uu^*$. As usual, equivalence in $P[A]$ will be denoted by $p \sim q$.

Of course, we must show that the above definition of equivalence is an equivalence relation.

Theorem 2.15.1. *Let A be a *-algebra. Then, the equivalence $\sim$ on the set $P[A]$ is an equivalence relation.*

Proof. Assume that A is a *-algebra. Assume that $p, q, r \in P[A]$. Then, $p \sim p$ because $p = p^2 = p^*p = pp^*$. Next, assume that $p \sim q$. Then, there exists $c \in \mathbb{Z}_{>0}$ and $u \in M_{n\times c}(A)$ such that $p = u^*u$ and $q = uu^*$. So, $q = (u^*)^*u^*$ and $p = u^*(u^*)^*$ and $q \sim p$.

Finally, assume that $p \sim q$ and $q \sim r$. Then, there exists $d \in \mathbb{Z}_{>0}$ and $v \in M_{c\times d}(A)$ such that $q = v^*v$ and $r = vv^*$. Now observe that

$$(vu)^*vu = u^*(v^*v)u = u^*(uu^*)u = p^2 = p$$

and

$$vu(vu)^* = v(uu^*)v^* = v(v^*v)v^* = r^2 = r.$$

Therefore, $p \sim r$ and $\sim$ defines an equivalence relation on $P[A]$. $\square$

We note that if $p, q \in P[A]$ happen to have the same dimension then $p \sim q$ if and only if p and q are Murray-von Neumann equivalent projections (see Definition 2.4.7).

Here is an important observation we will use later. Suppose that A is a C*-algebra and $p, q \in P[A]$ satisfy $p \sim q$. Then, there exists a rectangular matrix u such that $p = u^*u$ and $q = uu^*$. By replacing u with qup, we find that

$$\begin{aligned} (qup)^*qup &= pu^*qqup = pu^*qup = p^4 = p, \\ qup(qup)^* &= quppu^*q = qupu^*q = q^4 = q. \end{aligned} \tag{2.8}$$

Moreover,

$$(qup)(qup)^*(qup) = qupp = qup.$$

This means that if $p, q \in P[A]$ satisfy $p = u^*u$ and $q = uu^*$ then we can assume without loss of generality that $u = uu^*u$ by replacing u with qup if necessary.

Next, we prove some more properties pertaining to the equivalence relation of $P[A]$.

Theorem 2.15.2. *Let A be a *-algebra and $p, q, p', q' \in P[A]$.*

1. *If $p \sim p'$ and $q \sim q'$ then $p \oplus q \sim p' \oplus q'$.*
2. *$p \oplus q \sim q \oplus p$.*
3. *If $p, q \in M_{n\times n}(A)$ and $pq = 0$ then $p + q \sim p \oplus q$.*

Proof. Assume that A is a *-algebra and $p, q, p', q' \in P[A]$. First, assume that $p \sim p'$ and $q \sim q'$ so that there exists rectangular matrices u, v with entries in A such that $p = u^*u$, $p' = uu^*$, $q = v^*v$ and $q' = vv^*$. Define $w = u \oplus v$. Then, $p \oplus q = w^*w$ and $p' \oplus q' = ww^*$. So, $p \oplus q \sim p' \oplus q$.

To see that $p \oplus q \sim q \oplus p$, define

$$x = \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix}.$$

We compute directly that

$$x^*x = \begin{pmatrix} 0 & p^* \\ q^* & 0 \end{pmatrix} \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix} = \begin{pmatrix} p^*p & 0 \\ 0 & q^*q \end{pmatrix} = p \oplus q$$

and

$$xx^* = \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix} \begin{pmatrix} 0 & p^* \\ q^* & 0 \end{pmatrix} = \begin{pmatrix} q^*q & 0 \\ 0 & p^*p \end{pmatrix} = q \oplus p.$$

Therefore, $p \oplus q \sim q \oplus p$.

Finally, assume that $p, q \in M_{n\times n}(A)$ and $pq = 0$. Let $0_{n,m}$ be the $n \times m$ matrix whose entries are all zero. Define

$$y = \begin{pmatrix} p & 0_{n,m} \end{pmatrix} \in M_{n\times(n+m)}(A).$$

Then, $y^*y = p \oplus 0_m$ and $yy^* = p$. So, $p \sim p \oplus 0_m$. Now define

$$z = \begin{pmatrix} p & q \\ 0_n & 0_n \end{pmatrix}.$$

We have

$$z^*z = \begin{pmatrix} p^* & 0_n \\ q^* & 0_n \end{pmatrix} \begin{pmatrix} p & q \\ 0_n & 0_n \end{pmatrix} = \begin{pmatrix} p^2 & pq \\ qp & q^2 \end{pmatrix} = p \oplus q$$

because $pq = qp = 0$ and

$$zz^* = \begin{pmatrix} p & q \\ 0_n & 0_n \end{pmatrix} \begin{pmatrix} p^* & 0_n \\ q^* & 0_n \end{pmatrix} = \begin{pmatrix} p^2 + q^2 & 0_n \\ 0_n & 0_n \end{pmatrix} = (p + q) \oplus 0_n.$$

Therefore, $p \oplus q \sim (p + q) \oplus 0_n \sim p + q$ as required. $\square$

Now we will begin constructing the group $K_0(A)$ associated to a unital *-algebra A. We make a few definitions below.

Definition 2.15.3. Let A be a unital *-algebra and $p, q \in P[A]$. We say that p and q are **stably equivalent** if there exists $n \in \mathbb{Z}_{>0}$ such that $I_n \oplus p \sim I_n \oplus q$. Stable equivalence will be denoted by $\approx$.

Theorem 2.15.3. *Let A be a unital *-algebra. Then, the notion of stable equivalence on the set $P[A]$ defines an equivalence relation.*

Proof. Assume that A is a unital *-algebra. The fact that stable equivalence $\approx$ is symmetric and reflexive follows from the fact that $\sim$ is an equivalence relation on $P[A]$.

To see that $\approx$ is transitive, assume that $p, q, r \in P[A]$, $p \approx q$ and $q \approx r$. Then, there exists $m, n \in \mathbb{Z}_{>0}$ such that $I_m \oplus p \sim I_m \oplus q$ and $I_n \oplus q \sim I_n \oplus r$. Using Theorem 2.15.2, we find that

$$\begin{aligned} I_{n+m} \oplus p &= I_n \oplus (I_m \oplus p) \\ &\sim I_n \oplus (I_m \oplus q) \\ &= I_m \oplus (I_n \oplus q) \\ &\sim I_m \oplus (I_n \oplus r) = I_{n+m} \oplus r. \end{aligned}$$

So, $p \approx r$ and $\approx$ is transitive. Therefore, stable equivalence on $P[A]$ is an equivalence relation. $\square$

We will prove one more property about stable equivalence before moving on with our construction.

Theorem 2.15.4. *Let A be a unital *-algebra and $p, q, p', q' \in P[A]$. If $p \approx p'$ and $q \approx q'$ then $p \oplus q \approx p' \oplus q'$.*

Proof. Assume that A is a unital *-algebra. Assume that $p, q, p', q' \in P[A]$ satisfy $p \approx p'$ and $q \approx q'$. Then, there exist $m, n \in \mathbb{Z}_{>0}$ such that $I_m \oplus p \sim I_m \oplus p'$ and $I_n \oplus q \sim I_n \oplus q'$. Using Theorem 2.15.2, we have

$$\begin{aligned} I_{m+n} \oplus (p \oplus q) &= I_n \oplus (I_m \oplus p) \oplus q \\ &\sim (q \oplus I_n) \oplus (I_m \oplus p) \\ &\sim (I_n \oplus q) \oplus (I_m \oplus p) \\ &\sim (I_n \oplus q') \oplus (I_m \oplus p') \\ &\sim I_{m+n} \oplus (p' \oplus q'). \end{aligned}$$

Therefore, $p \oplus q \approx p' \oplus q'$. $\square$

Definition 2.15.4. Let A be a unital *-algebra. Define the set $K_0(A)^+$ as the quotient

$$K_0(A)^+ = P[A]/\approx$$

That is, the elements of $K_0(A)^+$ are the stable equivalence classes of $P[A]$. The elements of $K_0(A)^+$ will be denoted by the notation $[p]$ (or $[p]_A$ if we need to be clear).

The idea here is that we can turn $K_0(A)^+$ into a *cancellative* commutative monoid. Let us explain what this means.

Definition 2.15.5. Let M be a monoid. We say that M is **left cancellative** if the following statement is satisfied: if $g, h, k \in M$ and $gh = gk$ then $h = k$.

We say that M is **right cancellative** if the following statement is satisfied: if $g, h, k \in M$ and $hg = kg$ then $h = k$.

We say that M is **cancellative** if it is both left cancellative and right cancellative.

Theorem 2.15.5. *Let A be a unital *-algebra. Then, the set $K_0(A)^+$ is a cancellative commutative monoid whose binary operation is defined by*

$$\begin{array}{rccc} + : & K_0(A)^+ \times K_0(A)^+ & \to & K_0(A)^+ \\ & ([p],[q]) & \mapsto & [p \oplus q] \end{array}$$

and whose identity element is $[0]$.

Proof. Assume that A is a unital *-algebra and that the binary operation $+$ on the set $K_0(A)^+$ is defined as above. By Theorem 2.15.4, the binary operation $+$ is well-defined.

Assume that $p, q, r \in P[A]$. Since stable equivalence is an equivalence relation, we have

$$[p] + [q] = [p \oplus q] = [q \oplus p] = [q] + [p]$$

and

$$([p] + [q]) + [r] = [p \oplus q] + [r] = [(p \oplus q) \oplus r] = [p \oplus (q \oplus r)] = [p] + ([q] + [r]).$$

Hence, $+$ is associative and commutative. We also note that if $n \in \mathbb{Z}_{>0}$ then

$$[p] + [0_n] = [p \oplus 0_n] = [p]$$

by the proof of the third property in Theorem 2.15.2. So, $[0]$ is the identity element of $K_0(A)^+$ (the zero matrix can be of any dimension) and hence, $K_0(A)^+$ is a commutative monoid.

To see that $K_0(A)^+$ is cancellative, it suffices to show that $K_0(A)^+$ is left cancellative because $K_0(A)^+$ is commutative. So, assume that $[p] + [q] = [p] + [r]$ in $K_0(A)^+$. Then, $p \oplus q \approx p \oplus r$ and there exists $m \in \mathbb{Z}_{>0}$ such that

$$I_m \oplus p \oplus q \sim I_m \oplus p \oplus r.$$

Now suppose that $p \in M_{n \times n}(A)$. By the above relation, we have $p \oplus q \oplus I_m \sim p \oplus r \oplus I_m$ and

$$(I_n - p) \oplus p \oplus q \oplus I_m \sim (I_n - p) \oplus p \oplus r \oplus I_m.$$

By the third property in Theorem 2.15.2, $(I_n - p) \oplus p \sim I_n$. Thus, $q \oplus I_{m+n} \sim r \oplus I_{m+n}$ and $[q] = [r]$ in $K_0(A)^+$. Therefore, $K_0(A)^+$ is a cancellative commutative monoid. $\square$

In order to obtain the abelian group $K_0(A)$ from $K_0(A)^+$, we perform a well-known construction attributed to Grothendieck.

Theorem 2.15.6. *Let N be a cancellative commutative monoid. Define the equivalence relation $\sim$ on $N \times N$ by setting $(x, y) \sim (z, w)$ if $x + w = y + z$. Let $[x, y]$ be the equivalence class of $(x, y) \in N \times N$ under this equivalence relation and $G(N)$ be the set of all such equivalence classes. Then, $G(N)$ is an abelian group with binary operation*

$$\begin{array}{cccc} + : & G(N) \times G(N) & \to & G(N) \\ & ([x, y], [z, w]) & \mapsto & [x + z, y + w]. \end{array}$$

Proof. Assume that N is a cancellative commutative monoid and that the identity element is 0. Assume that $G(N)$ is the set of equivalence classes defined as above. Assume that the binary operation $+$ is defined as above.

To be complete, we will first show that $\sim$ is an equivalence relation on $N \times N$. Since N is abelian then $\sim$ is reflexive. Next, assume that $(x, y), (z, w) \in N \times N$ such that $(x, y) \sim (z, w)$. Then, $x + w = y + z$ and $(z, w) \sim (x, y)$ because N is abelian. Finally, assume that $(s, t) \in N \times N$, $(x, y) \sim (z, w)$ and $(z, w) \sim (s, t)$. Then,

$$\begin{aligned} z + (x + t) &= x + (z + t) = x + (w + s) \\ &= (x + w) + s = (y + z) + s \\ &= z + (y + s). \end{aligned}$$

Since N is cancellative then $x + t = y + s$ and $(x, y) \sim (s, t)$. Therefore, $\sim$ defines an equivalence relation on $N \times N$.

To show: (a) $+$ is well-defined.

(b) $+$ is commutative.

(c) $+$ is associative.

(a) To see that $+$ is well-defined, assume that $[x_1, y_1] = [x_2, y_2]$ and $[z_1, w_1] = [z_2, w_2]$ in $G(N)$. Then, $x_1 + y_2 = y_1 + x_2$ and $z_1 + w_2 = w_1 + z_2$. Using this, we find that

$$\begin{aligned}(x_1 + z_1) + (y_2 + w_2) &= (x_1 + y_2) + (z_1 + w_2)\\ &= (y_1 + x_2) + (w_1 + z_2)\\ &= (y_1 + w_1) + (x_2 + z_2).\end{aligned}$$

Hence, $[x_1, y_1] + [z_1, w_1] = [x_2, y_2] + [z_2, w_2]$ and consequently, the binary operation $+$ is well-defined.

(b) If $[x, y], [z, w] \in G(N)$ then

$$[x, y] + [z, w] = [x + z, y + w] = [z + x, w + y] = [z, w] + [x, y].$$

The second equality follows from the fact that N is abelian. So, $+$ is commutative.

(c) If $[s, t] \in G(N)$ then

$$\begin{aligned}([x, y] + [z, w]) + [s, t] &= [x + z, y + w] + [s, t]\\ &= [(x + z) + s, (y + w) + t] = [x + (z + s), y + (w + t)]\\ &= [x, y] + ([z, w] + [s, t]).\end{aligned}$$

So, the binary operation $+$ is associative.

It is easy to see that $[0, 0] \in G(N)$ is the identity element of $+$. Now assume that $[x, y] \in G(N)$. Then,

$$[x, y] + [y, x] = [x + y, x + y] = [0, 0].$$

Hence, $[y, x]$ is the additive inverse of $[x, y]$ and consequently, $(G(N), +)$ is an abelian group. $\square$

Definition 2.15.6. Let N be a cancellative commutative monoid. The group $G(N)$ constructed in Theorem 2.15.6 is called the **Grothendieck group** of N. It is sometimes called the **enveloping group** of N.

Next, we will prove the universal property satisfied by the Grothendieck group.

Theorem 2.15.7. *Let N be a cancellative commutative monoid. Then, the map*

$$\begin{array}{rccc} \varphi: & N & \to & G(N) \\ & x & \mapsto & [x,0] \end{array}$$

is an injective monoid homomorphism. Moreover, if G is an abelian group and $\psi : N \to G$ is a monoid homomorphism then there exists a unique group homomorphism $\tilde{\psi} : G(N) \to G$ such that the following diagram commutes:

$$\begin{tikzcd} N \arrow[r, hook, "\varphi"] \arrow[dr, "\psi"'] & G(N) \arrow[d, dashed, "\tilde{\psi}"] \\ & G \end{tikzcd}$$

Proof. Assume that N is a cancellative commutative monoid. Assume that $0 \in N$ is the identity element and that $x, y \in N$. Then,

$$\varphi(x+y) = [x+y, 0] = [x,0] + [y,0] = \varphi(x) + \varphi(y)$$

and $\varphi(0) = [0,0]$. Thus, φ is a monoid homomorphism. To see that φ is injective, assume that $\varphi(x) = \varphi(y)$. Then, $[x,0] = [y,0]$ and $x = y$. So, φ is injective.

Now assume that G is an abelian group and $\psi : N \to G$ is a monoid homomorphism. Define the map $\tilde{\psi}$ by

$$\begin{array}{rccc} \tilde{\psi}: & G(N) & \to & G \\ & [x,y] & \mapsto & \psi(x) - \psi(y). \end{array}$$

To show: (a) $\tilde{\psi}$ is well-defined.

(b) $\tilde{\psi}$ is a group homomorphism.

(c) $\tilde{\psi}$ is the unique group homomorphism such that $\psi = \tilde{\psi} \circ \varphi$.

(a) Assume that $(x,y), (z,w) \in N \times N$ such that $[x,y] = [z,w]$. Then, $x + w = y + z$, $\psi(x) + \psi(w) = \psi(y) + \psi(z)$ and

$$\tilde{\psi}([x,y]) = \psi(x) - \psi(y) = \psi(z) - \psi(w) = \tilde{\psi}([z,w]).$$

Hence, ψ is well-defined.

(b) To see that $\tilde{\psi}$ is a group homomorphism, assume that $[x,y], [s,t] \in G(N)$. Then,

$$\tilde{\psi}([x+s,y+t]) = \psi(x+s-y-t) = \psi(x-y)+\psi(s-t) = \tilde{\psi}([x,y])+\tilde{\psi}([s,t])$$

and

$$\tilde{\psi}([0,0]) = \psi(0) - \psi(0) = 0.$$

So, $\tilde{\psi}$ is a group homomorphism.

(c) If $x \in N$ then

$$\tilde{\psi}(\varphi(x)) = \tilde{\psi}([x,0]) = \psi(x) - \psi(0) = \psi(x).$$

For uniqueness, assume that there exists another group homomorphism $\rho : G(N) \to G$ such that $\rho \circ \varphi = \psi$. If $x, y \in N$ then $\rho([x,0]) = \psi(x)$ and

$$\rho([x,y]) = \rho([x,0])+\rho([0,y]) = \rho([x,0])-\rho([y,0]) = \psi(x)-\psi(y) = \tilde{\psi}([x,y]).$$

Therefore, $\tilde{\psi} = \rho$ and $\tilde{\psi}$ must be unique. This completes the proof. □

We briefly remark that we can extend the construction of the Grothendieck group to non-cancellative commutative monoids. This requires the equivalence relation in the definition of the Grothendieck group to be tweaked slightly.

Definition 2.15.7. Let A be a unital *-algebra. We define the abelian group $K_0(A)$ to be the Grothendieck group of the cancellative commutative monoid $K_0(A)^+$. That is,

$$K_0(A) = G(K_0(A)^+).$$

Next, we will show that if A is a unital *-algebra then the constructed group $K_0(A)$ can be interpreted as a functor.

Theorem 2.15.8. *Let* ***Ab*** *denote the category of abelian groups and* ***U*-Alg*** *denote the category of unital *-algebras. Then, the map*

$$\begin{array}{rccc} K_0 : & \textbf{\textit{U*-Alg}} & \to & \textbf{\textit{Ab}} \\ & A & \mapsto & K_0(A) \end{array}$$

is a covariant functor.

Proof. Assume that the map K_0 is defined as above, where **U*-Alg** is the category of unital *-algebras and **Ab** is the category of abelian groups. By construction, K_0 is well-defined on the objects of **U*-Alg**.

We need to understand what K_0 does to the morphisms of **U*-Alg**, which are the unital *-homomorphisms. Assume that $\varphi : A \to B$ is a unital *-homomorphism between unital *-algebras A and B. If $m, n \in \mathbb{Z}_{>0}$ then define the map

$$\begin{array}{rccc} \varphi' : & M_{m\times n}(A) & \to & M_{m\times n}(B) \\ & (a_{ij}) & \mapsto & (\varphi(a_{ij})) \end{array}$$

If $a = (a_{ij}) \in M_{m\times n}(A)$ and $b = (b_{ij}) \in M_{n\times p}(A)$ then it is straightforward to check that $\varphi'(ab) = \varphi'(a)\varphi'(b)$. and $\varphi'(a^*) = \varphi'(a)^*$.

To show: (a) If $p, q \in P[A]$ and $p \sim q$ then $\varphi'(p) \sim \varphi'(q)$.

(b) If $p, q \in P[A]$ and $p \approx q$ then $\varphi'(p) \approx \varphi'(q)$.

(a) Assume that $p, q \in P[A]$. Assume that $p \sim q$. Then, there exists a rectangular matrix u with entries in A such that $p = u^*u$ and $q = uu^*$. So,

$$\varphi'(p) = \varphi'(u^*u) = \varphi'(u)^*\varphi'(u)$$

and similarly, $\varphi'(q) = \varphi'(u)\varphi'(u)^*$. Thus, $\varphi'(p) \sim \varphi'(q)$.

(b) Now assume that $p \approx q$. Then, there exists $r \in \mathbb{Z}_{>0}$ such that $I_{r,A} \oplus p \sim I_{r,A} \oplus q$ where $I_{r,A} \in M_{r\times r}(A)$ is the $r \times r$ identity matrix with entries in A. Since φ is a unital *-homomorphism then $\varphi'(I_{r,A}) = I_{r,B}$. By part (a), we find that

$$I_{r,B} \oplus \varphi'(p) = \varphi'(I_{r,A}) \oplus \varphi'(p) = \varphi'(I_{r,A} \oplus p) \sim \varphi'(I_{r,A} \oplus q) = I_{r,B} \oplus \varphi'(q).$$

Therefore, $\varphi'(p) \approx \varphi'(q)$.

Now we define the map

$$\begin{array}{rccc} \varphi_* : & K_0(A)^+ & \to & K_0(B)^+ \\ & [p] & \mapsto & [\varphi'(p)] \end{array}$$

By part (b), φ_* is well-defined. Furthermore, if $[p], [q] \in K_0(A)^+$ then

$$\varphi_*([p] + [q]) = \varphi_*([p \oplus q]) = [\varphi'(p \oplus q)] = [\varphi'(p) \oplus \varphi'(q)] = \varphi_*([p]) + \varphi_*([q]).$$

We also have $\varphi_*([0]) = [\varphi'(0)] = [0]$ because φ is a *-homomorphism. So, the map φ_* is a monoid homomorphism.

Now let $\iota_B : K_0(B)^+ \to K_0(B)$ be defined by $\iota_B([t]) = [t, 0]$. By Theorem 2.15.7, ι_B is an injective monoid homomorphism. The composite $\iota \circ \varphi_*$ from $K_0(A)^+$ to $K_0(B)$ is a monoid homomorphism. By using the universal property of the Grothendieck group in Theorem 2.15.7, there exists a unique group homomorphism $K_0(\varphi)$ which makes the following diagram commute:

$$\begin{tikzcd} K_0(A)^+ \arrow[r, hook, "\iota_A"] \arrow[dr, "\iota_B \circ \varphi_*"'] & K_0(A) \arrow[d, dashed, "K_0(\varphi)"] \\ & K_0(B) \end{tikzcd}$$

To be clear, ι_A is defined in a similar manner to ι_B. We observe that if $id_A : A \to A$ is the identity map on the unital *-algebra A then $(id_A)_* = id_{K_0(A)^+}$ and the group homomorphism $K_0(id_A)$ is defined by (see Theorem 2.15.7)

$$\begin{array}{rccc} K_0(id_A) : & K_0(A) & \to & K_0(A) \\ & [p, q] & \mapsto & \iota_A([p]) - \iota_A([q]) = [p, q]. \end{array}$$

So, $K_0(id_A) = id_{K_0(A)}$.

To show: (c) If $\varphi : A \to B$ and $\psi : B \to C$ are unital *-homomorphisms then $K_0(\psi \circ \varphi) = K_0(\psi) \circ K_0(\varphi)$.

(c) Assume that $\varphi : A \to B$ and $\psi : B \to C$ are unital *-homomorphisms. By construction, the group homomorphism $K_0(\psi \circ \varphi)$ makes the following diagram commute:

$$\begin{tikzcd} K_0(A)^+ \arrow[r, hook, "\iota_A"] \arrow[dr, "\iota_C \circ (\psi \circ \varphi)_*"'] & K_0(A) \arrow[d, dashed, "K_0(\psi \circ \varphi)"] \\ & K_0(C) \end{tikzcd}$$

Note that $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$ and

$$\begin{aligned} K_0(\psi) \circ (K_0(\varphi) \circ \iota_A) &= K_0(\psi) \circ \iota_B \circ \varphi_* \\ &= \iota_C \circ \psi_* \circ \varphi_* = \iota_C \circ (\psi \circ \varphi)_*. \end{aligned}$$

By uniqueness, we deduce that $K_0(\psi \circ \varphi) = K_0(\psi) \circ K_0(\varphi)$. Consequently, K_0 is a functor from **U*-Alg** to **Ab**. □

We end this section with a few examples regarding the functor K_0 in Theorem 2.15.8.

Example 2.15.1. Let H be a separable infinite-dimensional Hilbert space. By Example 2.4.5, if $p, q \in B(H)$ are infinite-rank projections then $p \sim q$. Note that $\sim$ is the Murrary-von Neumann equivalence in Definition 2.4.7.

If $n \in \mathbb{Z}_{>0}$ then we also have a *-isomorphism $\psi : M_{n\times n}(B(H)) \to B(H^{(n)})$ given by equation (3.1). Note that $H^{(n)} = \bigoplus_{i=1}^{n} H$. Now assume that $p \in P[B(H)]$. Then, there exists $m \in \mathbb{Z}_{>0}$ such that p is a projection in $M_{m\times m}(B(H))$.

The projection $I_1 \in M_{1\times 1}(B(H))$ corresponds to the identity map id_H under the *-isomorphism ψ. Since H is infinite-dimensional then id_H is an infinite-rank projection. Therefore, $I_1 \oplus p \sim I_1$ as both are infinite-rank projections. Consequently, $[p] = [0]$ in $P[B(H)]$ and we deduce that $K_0(B(H)) = 0$.

Example 2.15.2. In this example, we consider the unital *-algebra $\mathbb{C}$. Let $p, q \in P[\mathbb{C}]$. We claim that $p \sim q$ if and only if p and q have the same rank.

Assume that p and q have the same rank $r \in \mathbb{Z}_{>0}$. Assume that $p \in M_{a\times a}(\mathbb{C})$ and $q \in M_{b\times b}(\mathbb{C})$. Let $m_p(x), m_q(x) \in \mathbb{C}[x]$ be the minimal polynomials of p and q respectively. Since $p^2 = p$ and $q^2 = q$ then $m_p(x)$ and $m_q(x)$ must divide $x^2 - x = x(x-1)$. Hence, there exist matrices $P \in GL_a(\mathbb{C})$ and $Q \in GL_b(\mathbb{C})$ such that

$$p = P\ (I_r \oplus 0_{a-r})\ P^{-1} \qquad \text{and} \qquad q = Q\ (I_r \oplus 0_{b-r})\ Q^{-1}.$$

Without loss of generality, assume that $a \leq b$. Then,

$$p \oplus 0_{b-a} = (P \oplus I_{b-a})(I_r \oplus 0_{b-r})(P^{-1} \oplus I_{b-a})$$

We know from the proof of Theorem 2.15.2 that $p \sim p \oplus 0_{b-a}$. Now define

$$u = qQ(P \oplus I_{b-a})^*(p \oplus 0_{b-a}).$$

Then, we have

$$\begin{aligned}
u^*u &= (qQ(P \oplus I_{b-a})^*(p \oplus 0_{b-a}))^* qQ(P \oplus I_{b-a})^*(p \oplus 0_{b-a}) \\
&= (p \oplus 0_{b-a})(P \oplus I_{b-a})Q^*qqQ(P \oplus I_{b-a})^*(p \oplus 0_{b-a}) \\
&= (p \oplus 0_{b-a})(P \oplus I_{b-a})(Q^*qQ)(P \oplus I_{b-a})^*(p \oplus 0_{b-a}) \\
&= (p \oplus 0_{b-a})(P \oplus I_{b-a})(I_r \oplus 0_{b-r})(P \oplus I_{b-a})^*(p \oplus 0_{b-a}) \\
&= (p \oplus 0_{b-a})^3 = (p \oplus 0_{b-a})
\end{aligned}$$

and by a similar computation, $uu^* = q$. Therefore, $p \sim p \oplus 0_{b-a} \sim q$.

Conversely, assume that $p \sim q$. Then, there exists $v \in M_{b\times a}(\mathbb{C})$ such that $p = v^*v$ and $q = vv^*$. So,

$$rank(p) = rank(v^*v) = rank(v) = rank(v^*) = rank(vv^*) = rank(q).$$

Observe that if $p, q \in P[\mathbb{C}]$ then $rank(p \oplus q) = rank(p) + rank(q)$. Now define the map

$$\begin{array}{rccc} R' : & K_0(\mathbb{C})^+ & \to & \mathbb{Z} \\ & [p] & \mapsto & rank(p) \end{array}$$

By our previous observations, R' is a well-defined monoid homomorphism. Using the universal property in Theorem 2.15.7, there exists a unique group homomorphism $R : K_0(\mathbb{C}) \to \mathbb{Z}$ such that the following diagram commutes:

$$\begin{tikzcd} K_0(\mathbb{C})^+ \arrow[r, hook] \arrow[dr, "R'"'] & K_0(\mathbb{C}) \arrow[d, dashed, "R"] \\ & \mathbb{Z} \end{tikzcd}$$

The group homomorphism R is given explicitly by

$$\begin{array}{rccc} R : & K_0(\mathbb{C}) & \to & \mathbb{Z} \\ & [p, q] & \mapsto & rank(p) - rank(q). \end{array}$$

We claim that R is an isomorphism. To see that R is surjective, assume that $k \in \mathbb{Z}$. Select $k_0, k_1 \in \mathbb{Z}_{\geq 0}$ such that $k = k_0 - k_1$. Then, $R([I_{k_0}, I_{k_1}]) = k_0 - k_1 = k$. To be clear, we define I_0 to be the zero matrix.

To see that R is injective, assume that $[p, q] \in \ker R$ so that $R([p, q]) = 0$. Then, $rank(p) = rank(q)$ and subsequently, $p \sim q$. In the monoid $K_0(\mathbb{C})^+$, we have

$$[p] + [0] = [p \oplus 0] = [p] = [q] = [q \oplus 0] = [q] + [0]$$

and $[p, q] = [0, 0]$ in $K_0(\mathbb{C})$. Thus, R is injective and we conclude that R is a group isomorphism from $K_0(\mathbb{C})$ to $\mathbb{Z}$.

2.16 K-theory of AF-algebras and Elliott's theorem

Let A be a unital AF-algebra. In this section, we will see that the group $K_0(A)$ has a naturally defined partial ordering, together with a "basepoint" at $[I_1]$. This extra structure on $K_0(A)$ is integral to Elliott's theorem, which states that $K_0(A)$ with this partial ordering is an "isomorphism invariant" of A.

The concept required to obtain the partial ordering on the K_0-group of a unital AF-algebra is that of *stable finiteness*.

Definition 2.16.1. Let A be a unital C*-algebra. We say that A is **stably finite** if the following statement is satisfied: if $n \in \mathbb{Z}_{>0}$, $u \in M_{n\times n}(A)$ and $u^*u = I_n$ then $uu^* = I_n$. Here, $I_n \in M_{n\times n}(A)$ is the identity matrix — the multiplicative unit of $M_{n\times n}(A)$.

The idea is that unital AF-algebras are stably finite. To understand the proof of this statement, we will use the fact that if A is a C*-algebra and $n \in \mathbb{Z}_{>0}$ then $M_{n\times n}(A)$ is also a C*-algebra. This is dealt with later in Theorem 3.1.3.

Theorem 2.16.1. *Let A be an AF-algebra and $n \in \mathbb{Z}_{>0}$. Then, $M_{n\times n}(A)$ is an AF-algebra.*

Proof. Assume that A is an AF-algebra and that $n \in \mathbb{Z}_{>0}$. Then, there exists an increasing sequence of finite dimensional C*-subalgebras $\{A_m\}_{m\in\mathbb{Z}_{>0}}$ of A such that

$$\overline{\bigcup_{m=1}^{\infty} A_m} = A.$$

If $n \in \mathbb{Z}_{>0}$ then by Theorem 3.1.3, $M_{n\times n}(A)$ is a C*-algebra and $\{M_{n\times n}(A_m)\}_{m\in\mathbb{Z}_{>0}}$ is an increasing sequence of finite dimensional C*-subalgebras contained in $M_{n\times n}(A)$.

We claim that $\overline{\bigcup_{m=1}^{\infty} M_{n\times n}(A_m)} = M_{n\times n}(A)$. Assume that $a = (a_{ij}) \in M_{n\times n}(A)$. If $i, j \in \{1, 2, \ldots, n\}$ then there exists a sequence $\{a_{ij}^{(m)}\}_{m\in\mathbb{Z}_{>0}}$ such that $a_{ij}^{(m)} \in A_m$ and

$$\lim_{m\to\infty} \|a_{ij}^{(m)} - a_{ij}\| = 0.$$

A consequence of Theorem 3.1.3 and Theorem 3.1.2 is the inequality

$$\|a_{ij}\| \le \|a\| \le \sum_{k,l=1}^{n} \|a_{kl}\|.$$

If $m \in \mathbb{Z}_{>0}$ then define $a_m = (a_{ij}^{(m)})$. Then, $a_m \in M_{n\times n}(A_m)$ and

$$\|a_m - a\| \le \sum_{k,l=1}^{n} \|a_{ij}^{(m)} - a_{ij}\| \to 0$$

as $m \to \infty$. Consequently, the sequence $\{a_m\}_{m\in\mathbb{Z}_{>0}}$ converges to $a \in M_{n\times n}(A)$ and $\overline{\bigcup_{m=1}^{\infty} M_{n\times n}(A_m)} = M_{n\times n}(A)$. So, $M_{n\times n}(A)$ is an AF-algebra. □

Theorem 2.16.2. *Let A be a unital AF-algebra. Then, A is stably finite.*

Proof. First, observe that if $A = M_{n\times n}(\mathbb{C})$ and $u \in A$ has a left inverse (there exists $v \in A$ such that $vu = I_n$ where I_n is the identity matrix in A) then $\dim\ker u = 0$ and by the rank-nullity theorem, $uv = I_n$.

Now if A is finite-dimensional then by Theorem 2.10.3, there exist positive integers $k, n_1, \ldots, n_k$ such that

$$A \cong \bigoplus_{i=1}^{k} M_{n_i\times n_i}(\mathbb{C}).$$

By the previous observation, we deduce that if $a \in A$ has a left inverse then a is invertible.

Now assume that A is a unital AF-algebra. By Theorem 2.16.1, the matrix algebra $M_{n\times n}(A)$ is a unital AF-algebra. We need to show that if $n \in \mathbb{Z}_{>0}$, $u \in M_{n\times n}(A)$ and $u^*u = I_n$ then $uu^* = I_n$.

Since $M_{n\times n}(A)$ is a unital AF-algebra then it suffices to show the statement holds for $n = 1$ by the proof of Theorem 2.16.1. To this end, assume that $u \in A$ satisfies $u^*u = 1_A$. Since A is an AF-algebra then there exists a sequence $\{A_n\}_{n\in\mathbb{Z}_{>0}}$ of finite dimensional C*-subalgebras of A such that

$$\overline{\bigcup_{n=1}^{\infty} A_n} = A.$$

So, there exists a sequence $\{u_n\}_{n\in\mathbb{Z}_{>0}}$ such that $u_n \in A_n$ and the sequence converges to u. Therefore,

$$1_A = u^*u = \lim_{n\to\infty} u_n^* u_n.$$

By passing to an appropriate subsequence of the sequence $\{u_n^*u_n\}_{n\in\mathbb{Z}_{>0}}$, we may assume that $\|1_A - u_n^*u_n\| < 1$. So, $1 \notin \sigma(1_A - u_n^*u_n)$ and $u_n^*u_n$ is invertible. In particular, u_n is left invertible and since A_n is finite dimensional, u_n is invertible.

Now we have

$$u = \lim_{n\to\infty} u_n = \lim_{n\to\infty} (u_n^*)^{-1} u_n^* u_n.$$

To show: (a) $\lim_{n\to\infty}(u_n^*)^{-1} = u$.

(a) If $n \in \mathbb{Z}_{>0}$ then

$$\begin{aligned}\|u - (u_n^*)^{-1}\| &\le \|u - (u_n^*)^{-1}u_n^*u_n\| + \|(u_n^*)^{-1}u_n^*u_n - (u_n^*)^{-1}\| \\ &\le \|u - (u_n^*)^{-1}u_n^*u_n\| + \|(u_n^*)^{-1}\|\|1_A - u_n^*u_n\|\end{aligned}$$

We claim that the quantity $\|(u_n^*)^{-1}\|$ is bounded. We compute directly that

$$\begin{aligned}\|(u_n^*)^{-1}\| &= \|u_n u_n^{-1}(u_n^*)^{-1}\| \\ &\le \|u_n\|\|(u_n^*u_n)^{-1}\| \\ &= \|u_n\| \sup_{\lambda\in\sigma(u_n^*u_n)} \frac{1}{|\lambda|} < \infty.\end{aligned}$$

The equality $\|(u_n^*u_n)^{-1}\| = \sup_{\lambda\in\sigma(u_n^*u_n)} \frac{1}{|\lambda|}$ is a consequence of the spectral mapping theorem (see Theorem 1.3.14). The final inequality follows from the fact that $\sigma(u_n^*u_n)$ is a bounded closed subset of $\mathbb{R} - \{0\}$. Therefore,

$$\|u - (u_n^*)^{-1}\| \le \|u - (u_n^*)^{-1}u_n^*u_n\| + \|(u_n^*)^{-1}\|\|1_A - u_n^*u_n\| \to 0$$

and $u = \lim_{n\to\infty}(u_n^*)^{-1}$.

By part (a), we find that $uu^* = \lim_{n\to\infty}(u_n^*)^{-1}u_n^* = 1_A$. This completes the proof. $\square$

Now, we are ready to discuss how stable finiteness endows the K_0-group with additional structure. We make the following preliminary definitions.

Definition 2.16.2. A **partially ordered group** is a pair $(G, \leq)$ consisting of an abelian group G and a partial order $\leq$ on G satisfying the following properties:

1. If $G^+ = \{x \in G \mid x \geq 0\}$ then $G = G^+ - G^+$
2. If $x, y, z \in G$ and $x \leq y$ then $x + z \leq y + z$.

Definition 2.16.3. Let G be an abelian group and N be a subset of G. We say that N is a **cone** on G if

$$N + N = \{n_1 + n_2 \mid n_1, n_2 \in N\} \subseteq N,$$
$$G = \{n_1 - n_2 \mid n_1, n_2 \in N\} = N - N$$

and $N \cap (-N) = \{0\}$.

Observe that we can construct partially ordered groups in the manner outlined by the theorem below.

Theorem 2.16.3. *Let G be an abelian group and $N \subseteq G$ be a cone on G. Define a partial order $\leq_N$ on G by setting $x \leq_N y$ if $y - x \in N$. Then, the pair $(G, \leq_N)$ is a partially ordered group with $G^+ = N$.*

Proof. Assume that G, N and $\leq_N$ are defined as above.

To show: (a) $G^+ = N$.

(b) $G = G^+ - G^+$.

(c) If $x, y, z \in G$ and $x \leq_N y$ then $x + z \leq_N y + z$.

(a) We have

$$G^+ = \{x \in G \mid x \geq_N 0\} = \{x \in G \mid x - 0 \in N\} = N.$$

(b) Using part (a) and the fact that N is a cone, we have $G = N - N = G^+ - G^+$.

(c) Assume that $x, y, z \in G$ and $x \leq_N y$. Then,

$$(y + z) - (x + z) = y - x \in N.$$

So, $x + z \leq_N y + z$. $\square$

The partial order $\leq_N$ defined in Theorem 2.16.3 is called the partial order induced by N. In the next theorem, we will prove that $K_0(A)$ can be constructed as a partially ordered group by using Theorem 2.16.3.

Theorem 2.16.4. *Let A be a stably finite unital C*-algebra. Let $\leq_{K_0(A)^+}$ be the partial order on the abelian group $K_0(A)$ induced by $K_0(A)^+$. Then, the pair $(K_0(A), \leq_{K_0(A)^+})$ is a partially ordered group.*

Proof. Assume that A is a stably finite unital C*-algebra. Assume that $\leq_{K_0(A)^+}$ be the partial order on $K_0(A)$ induced by $K_0(A)^+$. By Theorem 2.16.3, it suffices to show that $K_0(A)^+$ defines a cone on $K_0(A)$.

To show: (a) $K_0(A)^+$ defines a cone on $K_0(A)$.

(a) We identify $K_0(A)^+$ as a subset of $K_0(A)$ by using the injective monoid homomorphism

$$\begin{array}{rccc} \varphi : & K_0(A)^+ & \to & K_0(A) \\ & [p] & \mapsto & [p,0]. \end{array}$$

If $p, q \in P[A]$ then $[p,0] + [q,0] = [p+q,0] \in K_0(A)^+$. So,

$$K_0(A)^+ + K_0(A)^+ \subseteq K_0(A)^+.$$

Next, we have the inclusion $K_0(A)^+ - K_0(A)^+ \subseteq K_0(A)$. To obtain the reverse inclusion, assume that $p, q \in P[A]$. Then,

$$[p,q] = [p,0] + [0,q] = [p,0] - [q,0] \in K_0(A)^+ - K_0(A)^+.$$

Now assume that $[q,0] \in K_0(A)^+ \cap (-K_0(A)^+)$. Then, there exists $p \in P[A]$ such that $[q,0] = -[p,0] = [0,p]$. This means that in $K_0(A)^+$,

$$[p+q] = [p \oplus q] = [0].$$

So, there exists $r \in \mathbb{R}_{>0}$ such that $I_r \oplus (p \oplus q) \sim I_r \oplus 0 \sim I_r$. If $s \in \mathbb{Z}_{>0}$ then

$$I_s \oplus (I_r \oplus (p \oplus q)) \sim I_s \oplus I_r = I_{s+r}.$$

This means that there exists $u \in M_{s+r}(A)$ such that $I_{s+r} = u^*u$ and

$$uu^* = I_s \oplus (I_r \oplus (p \oplus q)) = I_{s+r} \oplus (p \oplus q).$$

Since A is stably finite then $uu^* = I_{s+r}$. Subsequently, $p \oplus q = 0$ and the projections $p, q \in P[A]$ are zero projections. So, $[p,q] = [0,0]$ and

$K_0(A)^+ \cap (-K_0(A)^+) = 0.$

We deduce that $K_0(A)^+$ is a cone on $K_0(A)$ and so, $(K_0(A), \leq_{K_0(A)^+})$ is a partially ordered group. $\square$

We give another example of a partially ordered group below.

Example 2.16.1. Let H be a Hilbert space. Let $B(H)_+$ be the set of positive operators on H. Then, $B(H)_+$ defines a cone on $B(H)$ (see Theorem 2.2.2), which is an abelian group with addition as its binary operation.

The partial order $\leq_{B(H)_+}$ induced by $B(H)_+$ is defined by setting $x \leq_{B(H)_+} y$ if $y - x \in B(H)_+$. This is just the relation given by Definition 2.2.2. By Theorem 2.16.3, the pair $(B(H), \leq_{B(H)_+})$ is a partially ordered group.

Here is a structure theorem regarding direct sums of projections.

Theorem 2.16.5. *Let A be a C*-algebra and $p_1, \ldots, p_n \in P[A]$. Let q be a projection in A such that $q \sim p_1 \oplus \cdots \oplus p_n$. Then there exists pairwise orthogonal projections $q_1, \ldots, q_n \in A$ such that if $i \in \{1, 2, \ldots, n\}$ then $q_i \sim p_i$ and $q = q_1 + \cdots + q_n$.*

Proof. Assume that A is a C*-algebra. Assume that $p_1, p_2, \ldots, p_n$ are projections in $P[A]$. Assume that q is a projection in A such that $q \sim p_1 \oplus \cdots \oplus p_n$. Assume that if $i \in \{1, 2, \ldots, n\}$ then $p_i \in M_{m_i \times m_i}(A)$. Let $m = \sum_{i=1}^n m_i$.

There exists a rectangular matrix $u \in M_{m\times 1}(A)$ such that

$$q = u^*u \qquad \text{and} \qquad p_1 \oplus \cdots \oplus p_n = uu^*.$$

Write the matrix u as

$$u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$$

where if $i \in \{1, 2, \ldots, n\}$ then $u_i \in M_{m_i \times 1}(A)$. Then,

$$q = \sum_{i=1}^n u_i^* u_i$$

and

$$p_1 \oplus \cdots \oplus p_n = uu^* = \begin{pmatrix} u_1u_1^* & u_1u_2^* & \dots & u_1u_n^* \\ u_2u_1^* & u_2u_2^* & \dots & u_2u_n^* \\ \vdots & \vdots & \ddots & \vdots \\ u_nu_1^* & u_nu_2^* & \dots & u_nu_n^* \end{pmatrix}$$

If $i \in \{1, 2, \dots, n\}$ then define $q_i = u_i^* u_i \in M_{1\times 1}(A) \cong A$. Observe that $p_i = u_i u_i^*$ so that $q_i \sim p_i$. Moreover, if $i, j \in \{1, 2, \dots, n\}$ are distinct then

$$q_i q_j = u_i^* u_i u_j^* u_j = u_i^*(0)u_j = 0.$$

Finally, $q = \sum_{i=1}^n u_i^* u_i = \sum_{i=1}^n q_i$. This completes the proof. □

So far, we have defined partially ordered groups and constructed a few examples. Next, we will define maps/morphisms between partially ordered groups.

Definition 2.16.4. Let G and H be partially ordered groups. Let $\varphi : G \to H$ be a group homomorphism. We say that φ is **positive** if the image $\varphi(G^+) \subseteq H^+$.

Now let $\psi : G \to H$ be a positive group homomorphism. We say that ψ is an **order isomorphism** if ψ is a group isomorphism and ψ^{-1} is positive. We say that the partially ordered groups G and H are **order isomorphic** if there exists an order isomorphism from G to H.

We will now provide a specific example of an order isomorphism. Recall that the matrix algebra $M_{n\times n}(\mathbb{C})$ has a linear basis given by its matrix units:

$$\{e_{ij} \mid i, j \in \{1, 2, \dots, n\}\}.$$

Recall that if $i, j \in \{1, 2, \dots, n\}$ then $e_{ij} \in M_{n\times n}(\mathbb{C})$ is the matrix with a 1 in the ij position and zeros elsewhere. We also know that $M_{n\times n}(\mathbb{C}) = B(\mathbb{C}^n)$.

Now let $m \in \mathbb{Z}_{>0}$. Then,

$$M_{m\times m}(M_{n\times n}(\mathbb{C})) = M_{m\times m}(B(\mathbb{C}^n)) = B(\mathbb{C}^{mn}).$$

So, if $p \in P[M_{n\times n}(\mathbb{C})]$ then it can diagonalised so that it is unitarily equivalent to a diagonal matrix with ones and zeros on the diagonal. So, there exists $k \in \mathbb{Z}_{>0}$ such that in $K_0(M_{n\times n}(\mathbb{C}))^+$,

$$[p] = \sum_{i=1}^{k} [e_{11}] = [\bigoplus_{i=1}^{k} e_{11}] = k[e_{11}].$$

Now let $n_1, n_2, \ldots, n_k \in \mathbb{Z}_{>0}$ and

$$A = \bigoplus_{i=1}^{k} M_{n_i \times n_i}(\mathbb{C}).$$

If $\ell \in \{1, 2, \ldots, k\}$ then let $\{e_{ij}^{\ell} \mid i, j \in \{1, 2, \ldots, n\}\}$ be the canonical basis for the matrix algebra $M_{n_\ell \times n_\ell}(\mathbb{C})$. Then,

$$\{e_{ij}^{\ell} \mid i, j \in \{1, 2, \ldots, n\}, \ell \in \{1, 2, \ldots, k\}\}$$

defines a basis for A.

If $k \in \mathbb{Z}_{>0}$ then $\mathbb{Z}^k$ is a partially ordered group when equipped with the partial order induced by the cone $(\mathbb{Z}_{>0})^k$. This is last ingredient required to construct our order isomorphism.

Theorem 2.16.6. *Let $n_1, \ldots, n_k \in \mathbb{Z}_{>0}$ and*

$$A = \bigoplus_{i=1}^{k} M_{n_i \times n_i}(\mathbb{C}).$$

Define the map

$$\begin{array}{cccc} \tau: & \mathbb{Z}^k & \to & K_0(A) \\ & (m_1, \ldots, m_k) & \mapsto & \sum_{i=1}^{k} m_i [e_{11}^{i}, 0]. \end{array}$$

Then, τ is an order isomorphism, where $\mathbb{Z}^k$ is equipped with the partial order induced by $(\mathbb{Z}_{>0})^k$ and $K_0(A)$ is equipped with the partial order induced by $K_0(A)^+$ (see Theorem 2.16.4).

Proof. Assume that $A = \bigoplus_{i=1}^{k} M_{n_i \times n_i}(\mathbb{C})$. Assume that τ is the map defined as above. First observe that τ is well-defined because if $i \in \{1, 2, \ldots, k\}$ then by the aforementioned reasoning, $m_i[e_{11}^i, 0] = m_i[e_{11}^i]$ is equal to $[p_i]$ in $K_0(M_{n_i \times n_i}(\mathbb{C}))^+$ where $p_i \in P[M_{n_i \times n_i}(\mathbb{C})]$.

It is straightforward to check that τ is a group homomorphism. To see that τ is positive, assume that $(j_1, \ldots, j_k) \in (\mathbb{Z}_{>0})^k$. Then,

$$\tau(j_1, \ldots, j_k) = \sum_{i=1}^{k} j_i [e_{11}^i, 0] = [\sum_{i=1}^{k} j_i e_{11}^i, 0] \in K_0(A)^+.$$

Therefore, τ is a positive group homomorphism. To see that τ is surjective, assume that $[p, q] \in K_0(A)$. By writing $[p, q]$ as $[p, 0] - [q, 0]$, it suffices to show that there exists $(f_1, \ldots, f_k) \in \mathbb{Z}^k$ such that $\tau(f_1, \ldots, f_k) = [p, 0]$.

Since p is a projection in $P[A]$ then $p = p_1 + p_2 + \cdots + p_k$, where if $i \in \{1, 2, \ldots, k\}$ then $p_i \in P[M_{n_i \times n_i}(\mathbb{C})]$. In turn, we have

$$[p_i] = \alpha_i [e_{11}^i] = [\bigoplus_{a=1}^{\alpha_i} e_{11}^i]$$

in $K_0(M_{n_i \times n_i}(\mathbb{C}))^+$ where $\alpha_i \in \mathbb{Z}_{>0}$. Subsequently, the set

$$\{[e_{11}^{\ell}, 0] \mid \ell \in \{1, 2, \ldots, k\}\}$$

generates $K_0(A)$ and τ is surjective.

To see that τ is injective, assume that $(m_1, \ldots, m_k) \in \mathbb{Z}^k$ such that

$$\tau(m_1, \ldots, m_k) = \sum_{i=1}^{k} m_i [e_{11}^i, 0] = 0$$

If $i \in \{1, 2, \ldots, k\}$ then $m_i[e_{11}^i, 0] = 0$ and

$$\bigoplus_{j=1}^{|m_i|} e_{11}^i = 0.$$

Therefore, $m_i = 0$ and $(m_1, \ldots, m_k) = (0, \ldots, 0)$. So, τ is injective. Since the inverse τ^{-1} is positive by direct computation then τ is an order isomorphism. $\square$

In proving Theorem 2.16.6, we learnt how to compute the K_0 group of a finite-dimensional C*-algebra.

Theorem 2.16.7. *Let A be a non-zero finite-dimensional C*-algebra. Then, there exists $k \in \mathbb{Z}_{>0}$ such that $K_0(A) \cong \mathbb{Z}^k$ as groups. Moreover, if $\{x_1, \ldots, x_k\}$ is a basis for $K_0(A)$ then*

$$K_0(A)^+ = \mathbb{Z}_{>0} x_1 + \cdots + \mathbb{Z}_{>0} x_k.$$

Proof. Assume that A is a non-zero finite-dimensional C*-algebra. By Theorem 2.10.3, there exists $k, n_1, \ldots, n_k \in \mathbb{Z}_{>0}$ such that as C*-algebras,

$$A \cong \bigoplus_{i=1}^{k} M_{n_i \times n_i}(\mathbb{C}).$$

By the proof of Theorem 2.16.6, $K_0(A) \cong \mathbb{Z}^k$ and if $\{x_1, \dots, x_k\}$ is a basis for the free abelian group $K_0(A)$ then

$$K_0(A)^+ = \mathbb{Z}_{>0}x_1 + \cdots + \mathbb{Z}_{>0}x_k$$

as required. $\square$

Definition 2.16.5. Let A and B be unital C*-algebras. Let $\tau : K_0(A) \to K_0(B)$ be a group homomorphism. We say that τ is **unital** if $\tau([I_1, 0]) = [I_1, 0]$, where I_1 is the 1×1 identity matrix.

If A is a C*-algebra and $u \in A$ is unitary then we define the *-homomorphism

$$\begin{array}{rccc} Ad\, u : & A & \to & A \\ & a & \mapsto & uau^*. \end{array} \tag{2.9}$$

Before we proceed, let us observe a situation where stable equivalence and Murray-von Neumann equivalence of projections are equivalent.

Theorem 2.16.8. *Let A be a finite dimensional C*-algebra and $p, q \in P[A]$. Then, $p \approx q$ if and only if $p \sim q$.*

Proof. Assume that A is a finite dimensional C*-algebra. By Theorem 2.10.3, it suffices to prove the statement of the theorem for $A = M_{n\times n}(\mathbb{C})$ where $n \in \mathbb{Z}_{>0}$.

So, assume that $p, q \in P[M_{n\times n}(\mathbb{C})]$ and assume that $p \sim q$. Assume that $p \in M_{a\times a}(M_{n\times n}(\mathbb{C})) \cong M_{an\times an}(\mathbb{C})$ and $q \in M_{b\times b}(M_{n\times n}(\mathbb{C})) \cong M_{bn\times bn}(\mathbb{C})$. Then, there exists a rectangular matrix $u \in M_{bn\times an}(\mathbb{C})$ such that

$$p = u^*u \qquad \text{and} \qquad q = uu^*.$$

Let v be the block matrix

$$v = \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix} \in M_{(bn+1)\times(an+1)}(\mathbb{C}).$$

Then, $v^*v = I_1 \oplus p$ and $vv^* = I_1 \oplus q$. So, $p \approx q$.

Conversely, assume that $p \approx q$. Then, there exists $r \in \mathbb{Z}_{>0}$ such that $I_r \oplus p \sim I_r \oplus q$. So, there exists a rectangular matrix $w \in M_{(bn+r)\times(an+r)}(\mathbb{C})$

such that $w^*w = I_r \oplus p$ and $ww^* = I_r \oplus q$.

Since $Tr(w^*w) = Tr(ww^*)$ then $Tr(I_r \oplus p) = Tr(I_r \oplus q)$ and

$$Tr(p) = Tr(I_r \oplus p) - r = Tr(I_r \oplus q) - r = Tr(q).$$

By diagonalising p and q, there exist unitary matrices $u \in M_{an\times an}(\mathbb{C})$ and $v \in M_{b\times b}(\mathbb{C})$ such that

$$p = u(I_\alpha \oplus 0_{an-\alpha})u^* \qquad \text{and} \qquad q = v(I_\alpha \oplus 0_{bn-\alpha})v^*.$$

Here, $\alpha = rank(p) = rank(q) \in \mathbb{Z}_{>0}$. We conclude that p and q are projection matrices with the same rank. By the result proven in Example 2.15.2, $p \sim q$ as required. □

The next theorem tells us that the functor K_0 in Theorem 2.15.8 behaves nicely with respect to finite dimensional C*-algebras.

Theorem 2.16.9. *Let A and B be non-zero finite dimensional C*-algebras.*

1. *If $\tau : K_0(A) \to K_0(B)$ is a unital positive group homomorphism then there exists a unital *-homomorphism $\varphi : A \to B$ such that $K_0(\varphi) = \tau$.*

2. *If $\varphi, \psi : A \to B$ are unital *-homomorphisms then $K_0(\varphi) = K_0(\psi)$ if and only if there exists a unitary element $u \in B$ such that $\psi = (Ad\ u) \circ \varphi$.*

Proof. Assume that A and B are non-zero finite dimensional C*-algebras.

Firstly, assume that $\tau : K_0(A) \to K_0(B)$ is a unital positive group homomorphism. By Theorem 2.10.3, we may assume that there exist $k, n_1, \ldots, n_k \in \mathbb{Z}_{>0}$ such that

$$A = \bigoplus_{i=1}^{k} M_{n_i\times n_i}(\mathbb{C}).$$

Before we proceed, let us set up some notation. Let

$$\big\{e^\ell_{ij} \mid i, j \in \{1, 2, \ldots, n\}, \ell \in \{1, 2, \ldots, k\}\big\}$$

denote the canonical basis elements of A. If $\ell \in \{1, 2, \ldots, k\}$ then let e_ℓ be the unit of $M_{n_\ell\times n_\ell}(\mathbb{C})$. Let 1_A and 1_B be the units of A and B.

Since τ is a positive group morphism then there exists $p_\ell \in P[B]$ such $\tau([e_\ell, 0]) = [p_\ell, 0]$. So,

$$[p_1 \oplus \cdots \oplus p_k, 0] = \tau(\sum_{\ell=1}^{k} [e_\ell, 0]) = \tau([1_A, 0]) = [1_B, 0]$$

because τ is unital. Therefore, $[p_1 \oplus \cdots \oplus p_k] = [1_B]$ in $K_0(B)^+$ and by Theorem 2.16.8, $p_1 \oplus \cdots \oplus p_k \sim 1_B$. By Theorem 2.16.5, there exist pairwise orthogonal projections $q_1, q_2, \ldots, q_k \in B$ such that if $\ell \in \{1, 2, \ldots, k\}$ then $q_\ell \sim p_\ell$ and

$$1_B = q_1 + q_2 + \cdots + q_k.$$

We note that $\tau([e_\ell, 0]) = [p_\ell, 0] = [q_\ell, 0]$.

Now we can repeat the above argument for the elements $[e^\ell_{11}, 0] \in K_0(A)$ where $\ell \in \{1, 2, \ldots, k\}$. There exists a projection $p^\ell_{11} \in P[B]$ such that $\tau([e^\ell_{11}, 0]) = [p^\ell_{11}, 0]$. Therefore,

$$n_\ell [p^\ell_{11}, 0] = \tau(n_\ell [e^\ell_{11}, 0]) = \tau([e_\ell, 0]) = [q_\ell, 0].$$

Theorem 2.16.8 tells us that if $\ell \in \{1, 2, \ldots, k\}$ then $\bigoplus_{i=1}^{n_\ell} p^\ell_{11} \sim q_\ell$. Applying Theorem 2.16.5, we deduce that there exist pairwise orthogonal projections $q^\ell_{11}, \ldots, q^\ell_{n_\ell, n_\ell}$ in B such that

$$q^\ell_{11} + q^\ell_{22} + \cdots + q^\ell_{n_\ell, n_\ell} = q_\ell$$

and if $j \in \{1, 2, \ldots, n_\ell\}$ then $q^\ell_{jj} \sim p^\ell_{11}$. So, there exist $u^\ell_j \in B$ such that

$$q^\ell_{jj} = u^\ell_j (u^\ell_j)^* \qquad \text{and} \qquad q^\ell_{11} = (u^\ell_j)^* u^\ell_j.$$

If $i, j \in \{1, 2, \ldots, n_\ell\}$ then define $q^\ell_{ij} = u^\ell_i (u^\ell_j)^*$. If $i = j$ then we recover the projection $q^\ell_{ii} \in B$ from this definition. We compute directly that

$$(q^\ell_{ij})^* = u^\ell_j (u^\ell_i)^* = q^\ell_{ji}$$

and if $m, n \in \{1, 2, \ldots, n_\ell\}$ then

$$q^\ell_{ij} q^\ell_{mn} = u^\ell_i (u^\ell_j)^* u^\ell_m (u^\ell_n)^* = \delta_{jm} q^\ell_{in}$$

where δ_{jm} is the Kronecker delta. Hence, define the map

$$\begin{array}{rccc} \varphi : & A & \to & B \\ & e^\ell_{ij} & \mapsto & q^\ell_{ij}. \end{array}$$

Using the previous computations above, we find that φ is a *-homomorphism. Furthermore, φ is unital because

$$\varphi(1_A) = \varphi(\sum_{\ell=1}^{k}\sum_{i=1}^{n_\ell} e_{ii}^{\ell}) = \sum_{\ell=1}^{k}(\sum_{i=1}^{n_\ell} q_{ii}^{\ell}) = \sum_{\ell=1}^{k} q_\ell = 1_B.$$

Recall from the proof of Theorem 2.16.6 that the set

$$\{[e_{11}^{\ell}, 0] \mid \ell \in \{1, 2, \ldots, k\}\}$$

generates $K_0[A]$. So,

$$K_0(\varphi)([e_{11}^{\ell}, 0]) = [\varphi(e_{11}^{\ell}), 0] - [\varphi(0), 0] = [q_{11}^{\ell}, 0] = [p_{11}^{\ell}, 0] = \tau([e_{11}^{\ell}, 0]).$$

Hence, $K_0(\varphi) = \tau$ as required.

Next, assume that $\psi, \varphi : A \to B$ are unital *-homomorphisms. Assume that there exists a unitary element $u \in B$ such that $\psi = (Ad\ u) \circ \varphi$. To be clear, $Ad\ u$ is the map in equation (2.9). Observe that if $a_1, a_2 \in P[A]$ then

$$\begin{aligned} K_0(Ad\ u)([a_1, a_2]) &= [(Ad\ u)(a_1), 0] - [(Ad\ u)(a_2), 0] \\ &= [ua_1u^*, 0] - [ua_2u^*, 0] \\ &= [a_1, 0] - [a_2, 0] = [a_1, 0] + [0, a_2] \\ &= [a_1, a_2]. \end{aligned}$$

So, $K_0(Ad\ u) = id_{K_0(B)}$ and consequently,

$$K_0(\psi) = K_0(Ad\ u) \circ K_0(\varphi) = K_0(\varphi).$$

Conversely, assume that $K_0(\varphi) = K_0(\psi)$. If $i, j \in \{1, 2, \ldots, n_\ell\}$ and $\ell \in \{1, 2, \ldots, k\}$ then define

$$r_{ij}^{\ell} = \varphi(e_{ij}^{\ell}) \qquad \text{and} \qquad s_{ij}^{\ell} = \psi(e_{ij}^{\ell}).$$

Since $K_0(\psi) = K_0(\varphi)$ then

$$[r_{ij}^{\ell}, 0] = K_0(\varphi)([e_{ij}^{\ell}, 0]) = K_0(\psi)([e_{ij}^{\ell}, 0]) = [s_{ij}^{\ell}, 0].$$

In particular, $[r_{ij}^{\ell}] = [s_{ij}^{\ell}]$ in $K_0(B)^+$. By Theorem 2.16.8, $r_{ij}^{\ell} \sim s_{ij}^{\ell}$. So, there exist $u_\ell \in B$ such that

$$r_{11}^{\ell} = u_\ell^* u_\ell \qquad \text{and} \qquad s_{11}^{\ell} = u_\ell u_\ell^*.$$

Now define

$$u = \sum_{\ell=1}^{k} \sum_{i=1}^{n_\ell} r_{i1}^\ell u_\ell^* s_{1i}^\ell.$$

We claim that $u \in B$ is unitary. We compute directly that

$$\begin{aligned}
u^*u &= \Big(\sum_{\ell=1}^{k} \sum_{i=1}^{n_\ell} r_{i1}^\ell u_\ell^* s_{1i}^\ell \Big)^* \Big(\sum_{m=1}^{k} \sum_{j=1}^{n_m} r_{j1}^m u_m^* s_{1j}^m \Big) \\
&= \sum_{\ell=1}^{k} \sum_{i=1}^{n_\ell} \sum_{m=1}^{k} \sum_{j=1}^{n_m} s_{i1}^\ell u_\ell r_{1i}^\ell r_{j1}^m u_m^* s_{1j}^m = \sum_{\ell=1}^{k} \sum_{i=1}^{n_\ell} s_{i1}^\ell u_\ell r_{1i}^\ell r_{i1}^\ell u_\ell^* s_{1i}^\ell \\
&= \sum_{\ell=1}^{k} \sum_{i=1}^{n_\ell} s_{i1}^\ell u_\ell r_{11}^\ell u_\ell^* s_{1i}^\ell = \sum_{\ell=1}^{k} \sum_{i=1}^{n_\ell} s_{i1}^\ell u_\ell u_\ell^* u_\ell u_\ell^* s_{1i}^\ell \\
&= \sum_{\ell=1}^{k} \sum_{i=1}^{n_\ell} s_{i1}^\ell s_{11}^\ell s_{1i}^\ell = \sum_{\ell=1}^{k} \sum_{i=1}^{n_\ell} s_{ii}^\ell = \sum_{\ell=1}^{k} \sum_{i=1}^{n_\ell} \psi(e_{ii}^\ell) \\
&= \psi(1_A) = 1_B
\end{aligned}$$

and

$$\begin{aligned}
uu^* &= \Big(\sum_{\ell=1}^{k} \sum_{i=1}^{n_\ell} r_{i1}^\ell u_\ell^* s_{1i}^\ell \Big) \Big(\sum_{m=1}^{k} \sum_{j=1}^{n_m} r_{j1}^m u_m^* s_{1j}^m \Big)^* \\
&= \sum_{\ell=1}^{k} \sum_{i=1}^{n_\ell} \sum_{m=1}^{k} \sum_{j=1}^{n_m} r_{i1}^\ell u_\ell^* s_{1i}^\ell s_{j1}^m u_m r_{1j}^m = \sum_{\ell=1}^{k} \sum_{i=1}^{n_\ell} r_{i1}^\ell u_\ell^* s_{1i}^\ell s_{i1}^\ell u_\ell r_{1i}^\ell \\
&= \sum_{\ell=1}^{k} \sum_{i=1}^{n_\ell} r_{i1}^\ell u_\ell^* s_{11}^\ell u_\ell r_{1i}^\ell = \sum_{\ell=1}^{k} \sum_{i=1}^{n_\ell} r_{i1}^\ell u_\ell^* u_\ell u_\ell^* u_\ell r_{1i}^\ell \\
&= \sum_{\ell=1}^{k} \sum_{i=1}^{n_\ell} r_{i1}^\ell r_{11}^\ell r_{1i}^\ell = \sum_{\ell=1}^{k} \sum_{i=1}^{n_\ell} r_{ii}^\ell = \sum_{\ell=1}^{k} \sum_{i=1}^{n_\ell} \varphi(e_{ii}^\ell) \\
&= \varphi(1_A) = 1_B
\end{aligned}$$

Hence, u is a unitary element of B. We also note that if $m \in \{1, 2, \ldots, k\}$ and $x, y \in \{1, 2, \ldots, n_m\}$ then

$$\begin{aligned}
us_{xy}^m u^{-1} &= \Big(\sum_{\ell=1}^{k}\sum_{i=1}^{n_\ell} r_{i1}^\ell u_\ell^* s_{1i}^\ell\Big) s_{xy}^m \Big(\sum_{\ell=1}^{k}\sum_{i=1}^{n_\ell} r_{i1}^\ell u_\ell^* s_{1i}^\ell\Big)^* \\
&= r_{x1}^m u_m^* s_{1y}^m \Big(\sum_{\ell=1}^{k}\sum_{i=1}^{n_\ell} s_{i1}^\ell u_\ell r_{1i}^\ell\Big) \\
&= r_{x1}^m u_m^* s_{11}^m u_m r_{1y}^m = r_{x1}^m r_{11}^m r_{1y}^m = r_{xy}^m.
\end{aligned}$$

Consequently, if $\ell \in \{1, 2, \ldots, k\}$ and $i, j \in \{1, 2, \ldots, n_\ell\}$ then

$$(Ad\ u^* \circ \varphi)(e_{ij}^\ell) = (Ad\ u^*)(r_{ij}^\ell) = u^* r_{ij}^\ell u = s_{ij}^\ell = \psi(e_{ij}^\ell).$$

Since the set $\{e_{ij}^\ell\}$ forms a basis for $A = \bigoplus_{i=1}^k M_{n_i \times n_i}(\mathbb{C})$ then
$\psi = (Ad\ u^*) \circ \varphi$. This completes the proof. $\square$

We require a few more results before we state and prove Elliott's theorem. The next result gives a specific condition required for Murray-von Neumann equivalence.

Theorem 2.16.10. *Let A be a C*-algebra and $p, q \in A$ be projections. Assume that there exists $u \in A$ such that*

$$\|p - u^*u\| < 1, \qquad \|q - uu^*\| < 1 \qquad \textit{and} \qquad u = qup.$$

Then, $p \sim q$.

Proof. Assume that A is a C*-algebra. Assume that $p, q \in A$ are projections and that there exists $u \in A$ such that the above three conditions are satisfied.

Recall that p is the unit for the C*-subalgebra pAp. Since $u = qup$, $u^*u = pu^*up \in pAp$. Since $\|p - u^*u\| < 1$ then u^*u is invertible in the C*-subalgebra pAp. Similarly, uu^* is an invertible element of the C*-subalgebra qAq.

By considering the polar decomposition of u, we have $u^*u = |u|^2$. Let z be the inverse of $|u|$ in pAp and define $w = uz$. Then, z is self-adjoint because it is the inverse of a self-adjoint element and

$$w^*w = z^*u^*uz = z^*|u|^2 z = z|u|p = p^2 = p.$$

Moreover,

$$uu^*ww^* = uu^*uzz^*u^* = u(u^*u)z^2u^* = u|u|^2z^2u^* = uu^*.$$

Now let r be the inverse of uu^* in the C*-subalgebra qAq. Then, $ww^* = ruu^* = q$. Therefore, $p \sim q$. $\square$

Theorem 2.16.11. *Let A be unital C*-algebra. Let $\{A_n\}_{n\in\mathbb{Z}_{>0}}$ be an increasing sequence of C*-subalgebras of A such that if $m \in \mathbb{Z}_{>0}$ then $1_A \in A_m$. Assume that*

$$\overline{\bigcup_{n=1}^{\infty} A_n} = A.$$

1. *If $p \in P[A]$ then there exists $k \in \mathbb{Z}_{>0}$ and $q \in P[A_k]$ such that $[p]_A = [q]_A$ in $K_0(A)^+$.*
2. *If $\ell \in \mathbb{Z}_{>0}$, $p, q \in P[A_\ell]$ and $[p]_A = [q]_A$ then there exists $m \in \mathbb{Z}_{>\ell}$ such that $[p]_{A_m} = [q]_{A_m}$ in $K_0(A_m)^+$.*

Proof. Assume that A and $\{A_n\}_{n\in\mathbb{Z}_{>0}}$ are defined as above. Note that if $\ell \in \mathbb{Z}_{>0}$ then $\{M_{\ell\times\ell}(A_n)\}_{n\in\mathbb{Z}_{>0}}$ is an increasing sequence of C*-subalgebras of $M_{\ell\times\ell}(A)$ such that

$$\overline{\bigcup_{n=1}^{\infty} M_{\ell\times\ell}(A_n)} = M_{\ell\times\ell}(A).$$

Also, each C*-subalgebra $M_{\ell\times\ell}(A_n)$ contains the unit of $M_{\ell\times\ell}(A)$. Therefore, it suffices to prove the first statement in the case where p is a projection in A. Note that it also suffices to prove the second statement in the case where p and q are projections in A_ℓ for some $\ell \in \mathbb{Z}_{>0}$.

So assume that p is a projection in A. Since $\bigcup_{n=1}^{\infty} A_n$ is dense in A then there exists a sequence $\{u_n\}_{n\in\mathbb{Z}_{>0}}$ with $u_n \in A_n$ converging to p. By replacing each u_n with $Re(u_n) = \frac{1}{2}(u_n + u_n^*)$ if necessary, we may assume that each u_n is self-adjoint.

We would like to use Theorem 2.16.10. Since $\{u_n^2\}_{n\in\mathbb{Z}_{>0}}$ converges to p then there exists $k \in \mathbb{Z}_{>0}$ such that

$$\|p - u_k\| < \frac{1}{2} \qquad \text{and} \qquad \|u_k - u_k^2\| < \frac{1}{4}.$$

By Theorem 2.10.6, there exists a projection q in the C*-subalgebra generated by $\{u_k\}$ such that $\|q - u_k\| < \frac{1}{2}$. Note that by construction of q, there exists $j \in \mathbb{Z}_{>0}$ such that $q \in A_j$.

By the triangle inequality, $\|p - q\| < 1$. So, by Theorem 2.10.5, there exists a unitary element $u \in A$ such that $q = upu^*$. By Theorem 2.16.10, we deduce that $q \sim p$ as required.

Next, assume that $\ell \in \mathbb{Z}_{>0}$ and p, q are projections in A_ℓ such that $[p]_A = [q]_A$ in $K_0(A)^+$. Then, there exists $u \in A$ such that $p = u^*u$, $q = uu^*$ and $u = qup$ (see equation (2.8)). Let $\{u_n\}_{n \in \mathbb{Z}_{>0}}$ be a sequence in $\bigcup_{n=1}^{\infty} A_n$ converging to u. Without loss of generality, we may assume that $u_m = qu_m p$ by replacing u_m by $qu_m p$ if necessary.

Since $\{u_n\}$ converges to u then there exists $N \in \mathbb{Z}_{>0}$ such that if $n \geq N$ then

$$\|p - u_n^* u_n\| < 1 \qquad \text{and} \qquad \|q - u_n u_n^*\| < 1.$$

Choose $n' \in \mathbb{Z}_{\geq N}$ large enough so that there exists $m \in \mathbb{Z}_{\geq \ell}$ such that $u_{n'} \in A_m$. By Theorem 2.16.10, we deduce that $[p]_{A_m} = [q]_{A_m}$ in the monoid $K_0(A_m)^+$ as required. $\square$

The next result plays an important role in the proof of Elliott's theorem.

Theorem 2.16.12. *Let A, B and C be unital stably finite C*-algebras. Assume that A is finite dimensional. Let $\tau : K_0(A) \to K_0(C)$ and $\rho : K_0(B) \to K_0(C)$ be positive group homomorphisms such that $\tau(K_0(A)^+) \subseteq \rho(K_0(B)^+)$. Then, there exists a unique positive group homomorphism $\tau' : K_0(A) \to K_0(B)$ such that the following diagram (in the category of partially ordered groups) commutes:*

$$\begin{tikzcd} K_0(A) \arrow[r, dashed, "\tau'"] \arrow[dr, "\tau"'] & K_0(B) \arrow[d, "\rho"] \\ & K_0(C) \end{tikzcd}$$

Proof. Assume that A, B and C are unital stably finite C*-algebras. Assume that A is finite dimensional and that $\tau : K_0(A) \to K_0(C)$ and $\rho : K_0(B) \to K_0(C)$ are positive group homomorphisms. Assume that

$$\tau(K_0(A)^+) \subseteq \rho(K_0(B)^+).$$

By Theorem 2.16.7, there exists $k \in \mathbb{Z}_{>0}$ such that $K_0(A) \cong \mathbb{Z}^k$ as partially ordered groups. Moreover, there exists a basis $\{x_1, \dots, x_k\}$ of $K_0(A)$ such that

$$K_0(A)^+ = \mathbb{Z}_{>0}x_1 + \mathbb{Z}_{>0}x_2 + \cdots + \mathbb{Z}_{>0}x_k.$$

Recall that $\tau(K_0(A)^+) \subseteq \rho(K_0(B)^+)$. So, if $i \in \{1, 2, \dots, k\}$ then there exists $y_i \in K_0(B)^+$ such that $\rho(y_i) = \tau(x_i)$. Subsequently, we define τ to be the unique map

$$\begin{array}{cccc} \tau' : & K_0(A) & \to & K_0(B) \\ & \sum_{i=1}^k \alpha_i x_i & \mapsto & \sum_{i=1}^k \alpha_i y_i \end{array}$$

Here, we consider $K_0(B)^+$ as a subset of $K_0(B)$. By construction of τ', if $i \in \{1, 2, \dots, k\}$ then

$$(\rho \circ \tau')(x_i) = \rho(y_i) = \tau(x_i).$$

Since $\{x_1, \dots, x_k\}$ is a basis for $K_0(A)$ then $\rho \circ \tau' = \tau$. Furthermore, τ' is a group homomorphism. To see that τ' is positive, assume that $\beta_1, \dots, \beta_k \in \mathbb{Z}_{>0}$. Then,

$$\tau'(\sum_{i=1}^k \beta_i x_i) = \sum_{i=1}^k \beta_i y_i \in K_0(B)^+$$

because $y_i \in K_0(B)^+$. So, τ' is a positive group homomorphism. □

Now we will state and prove Elliott's theorem on unital AF-algebras.

Theorem 2.16.13 (Elliott's theorem)**.** *Let A and B be unital AF-algebras and $\tau : K_0(A) \to K_0(B)$ be a unital order isomorphism. Then, there exists a *-isomorphism $\varphi : A \to B$ such that $K_0(\varphi) = \tau$.*

Proof. Assume that A and B are unital AF-algebras. Assume that $\tau : K_0(A) \to K_0(B)$ is a unital order isomorphism so that $K_0(A)$ and $K_0(B)$ are isomorphic as partially ordered groups (see Theorem 2.16.4).

Since A and B are unital AF-algebras then there exist increasing sequences $\{A_n\}_{n \in \mathbb{Z}_{>0}}$ and $\{B_n\}_{n \in \mathbb{Z}_{>0}}$ of finite dimensional C*-subalgebras of A and B respectively, such that

$$\overline{\bigcup_{n=1}^{\infty} A_n} = A \qquad \text{and} \qquad \overline{\bigcup_{n=1}^{\infty} B_n} = B.$$

We may assume that if $n \in \mathbb{Z}_{>0}$ then $1_A \in A_n$ and $1_B \in B_n$. We also let $\varphi^n : A_n \hookrightarrow A$ and $\psi^n : B_n \to B$ denote the inclusion *-homomorphisms. Let ρ be the inverse of the unital order isomorphism τ. Then,
$\rho : K_0(B) \to K_0(A)$ is also a unital order isomorphism.

Consider the increasing sequence

$$\{K_0(\varphi^n)(K_0(A_n)^+)\}_{n\in\mathbb{Z}_{>0}}.$$

We claim that $K_0(A)^+$ is the union of the above increasing sequence.

To show: (a) $K_0(A)^+ = \bigcup_{n=1}^{\infty} K_0(\varphi^n)(K_0(A_n)^+)$.

(a) Since $K_0(\varphi^n)$ is a positive group homomorphism from $K_0(A_n)$ to $K_0(A)$ then $\bigcup_{n=1}^{\infty} K_0(\varphi^n)(K_0(A_n)^+) \subseteq K_0(A)^+$. For the reverse inclusion, assume that $p \in P[A]$ so that the stable equivalence class $[p]_A \in K_0(A)^+$. By Theorem 2.16.11, there exists $k \in \mathbb{Z}_{>0}$ and $q \in P[A_k]$ such that $[p]_A = [q]_A$. Now note that $[q, 0]_{A_k} \in K_0(A_k)$ satisfies

$$K_0(\varphi^k)([q, 0]_{A_k}) = [q, 0]_A.$$

Consequently, $[p]_A = [q]_A \in K_0(\varphi^k)(K_0(A_k)^+)$ and

$$K_0(A)^+ \subseteq \bigcup_{n=1}^{\infty} K_0(\varphi^n)(K_0(A_n)^+).$$

This proves part (a).

By part (a), we also have

$$K_0(B)^+ = \bigcup_{n=1}^{\infty} K_0(\psi^n)(K_0(B_n)^+).$$

Subsequently, we deduce that

$$K_0(A) = \bigcup_{n=1}^{\infty} K_0(\varphi^n)(K_0(A_n)) \qquad \text{and} \qquad K_0(B) = \bigcup_{n=1}^{\infty} K_0(\psi^n)(K_0(B_n)).$$

Let $n_1 = 1$. By Theorem 2.16.7, there exists a basis $\{x_1, \ldots, x_k\}$ for the free abelian group $K_0(A_{n_1})$ such that

$$K_0(A_{n_1})^+ = \mathbb{Z}_{>0}x_1 + \mathbb{Z}_{>0}x_2 + \cdots + \mathbb{Z}_{>0}x_k.$$

Applying the composite $\tau \circ K_0(\varphi^{n_1}) : K_0(A_{n_1}) \to K_0(B)$, we find that

$$(\tau K_0(\varphi^{n_1}))(K_0(A_{n_1})^+) = \mathbb{Z}_{>0}(\tau K_0(\varphi^{n_1}))(x_1) + \cdots + \mathbb{Z}_{>0}(\tau K_0(\varphi^{n_1}))(x_k).$$

We know that $K_0(B)^+$ is the increasing union of the sequence of sets $\{K_0(\psi^m)(K_0(B_m)^+)\}_{m\in\mathbb{Z}_{>0}}$. Therefore, if $i \in \{1, 2, \dots, k\}$ then there exists $m \in \mathbb{Z}_{>n_1}$ such that

$$(\tau K_0(\varphi^{n_1}))(x_i) \in K_0(\psi^m)(K_0(B_m)^+).$$

Consequently, $(\tau K_0(\varphi^{n_1}))(K_0(A_{n_1})^+) \subseteq K_0(\psi^m)(K_0(B_m)^+)$. By Theorem 2.16.12, there exists a positive group homomorphism $\tilde{\tau} : K_0(A_{n_1}) \to K_0(B_m)$ such that the following diagram commutes:

$$\begin{tikzcd}
K_0(A_{n_1}) \arrow[r, "\tilde{\tau}"] \arrow[d, "K_0(\varphi^{n_1})"'] & K_0(B_m) \arrow[d, "K_0(\psi^m)"] \\
K_0(A) \arrow[r, "\tau"'] & K_0(B)
\end{tikzcd}$$

Now consider the element $[1_A, 0]_{A_{n_1}} \in K_0(A_{n_1})$. Let $[e, 0]_{B_m} = \tilde{\tau}([1_A, 0]_{A_{n_1}})$. By commutativity of the above diagram,

$$[e, 0]_B = K_0(\psi^m)([e, 0]_{B_m}) = (\tau \circ K_0(\varphi^{n_1}))([1_A, 0]_{A_{n_1}}) = [1_B, 0]_B.$$

So, $e, 1_B$ are elements of B_m satisfying $[e]_B = [1_B]_B$ in $K_0(B)^+$. By the second part of Theorem 2.16.11, there exists $m_1 \in \mathbb{Z}_{>m}$ such that in $K_0(B_{m_1})^+$,

$$[e]_{B_{m_1}} = [1_B]_{B_{m_1}}.$$

Let $\psi^{m,m_1} : B_m \hookrightarrow B_{m_1}$ be the inclusion map and define

$$\tau^1 = K_0(\psi^{m,m_1}) \circ \tilde{\tau} : K_0(A_{n_1}) \to K_0(B_{m_1}).$$

Then, τ^1 is a unital positive group homomorphism which makes the following diagram commute:

$$\begin{tikzcd}
K_0(A_{n_1}) \arrow[r, "\tau^1"] \arrow[d, "K_0(\varphi^{n_1})"'] & K_0(B_{m_1}) \arrow[d, "K_0(\psi^{m_1})"] \\
K_0(A) \arrow[r, "\tau"'] & K_0(B)
\end{tikzcd}$$

Let us repeat the argument just used, except for the commutative square above. Let $[f,0]_A = \rho([e,0]_B)$. By the commutativity of the above diagram,

$$[f,0]_A = (\rho \circ K_0(\psi^{m_1}))([e,0]_{m_1}) = K_0(\varphi^{n_1})([1_A,0]_{A_{n_1}}) = [1_A,0]_A.$$

Therefore, $[f]_A = [1_A]_A$ in $K_0(A)^+$. By the second part of Theorem 2.16.11, there exists $n \in \mathbb{Z}_{>1}$ such that $[f]_{A_n} = [1_A]_{A_n}$ in $K_0(A_n)^+$. Now let $\varphi^{n_1,n} : A_{n_1} \hookrightarrow A_n$ be the inclusion map and define

$$\tilde{\rho} = K_0(\varphi^{n_1,n}) \circ \rho \circ K_0(\psi^{m_1}) : K_0(B_{m_1}) \to K_0(A_n)$$

Then, $\tilde{\rho}$ is a unital positive group homomorphism which makes the following diagram commute:

$$\begin{tikzcd}
K_0(B_{m_1}) \arrow[r, "K_0(\psi^{m_1})"] \arrow[d, "\tilde{\rho}"'] & K_0(B) \arrow[d, "\rho"] \\
K_0(A_n) \arrow[r, "K_0(\varphi^n)"'] & K_0(A)
\end{tikzcd}$$

Next, consider the basis $\{x_1, \ldots, x_k\}$ of A_{n_1}. If $j \in \{1, 2, \ldots, k\}$ then we write $x_j = [p_j, 0]_{A_{n_1}}$ where $p_j \in P[A_{n_1}]$. We also write

$$(\tilde{\rho} \circ \tau^1)(x_j) = [q_j, 0]_{A_n}$$

where $q_j \in P[A_n]$. We compute directly that

$$\begin{aligned}
[q_j, 0]_A &= (K_0(\varphi^n) \circ \tilde{\rho} \circ \tau^1)(x_j) \\
&= (\rho \circ K_0(\psi^{m_1}) \circ \tau^1)(x_j) \\
&= (\rho \circ \tau \circ K_0(\varphi^{n_1}))([p_j, 0]_{A_{n_1}}) \\
&= [p_j, 0]_A.
\end{aligned}$$

So, $[p_j]_A = [q_j]_A$ in $K_0(A)^+$. By the second part of Theorem 2.16.11, there exists $n_2 \in \mathbb{Z}_{>n}$ such that if $j \in \{1, 2, \ldots, k\}$ then $p_j, q_j \in P[A_{n_2}]$ and

$$[p_j]_{A_{n_2}} = [q_j]_{A_{n_2}}.$$

Again, let $\varphi^{n,n_2} : A_n \hookrightarrow A_{n_2}$ be the inclusion map and define the unital positive group homomorphism

$$\rho^1 = K_0(\varphi^{n,n_2}) \circ \tilde{\rho} : K_0(B_{m_1}) \to K_0(A_{n_2}).$$

Then, the following square commutes:

$$\begin{tikzcd}
K_0(B_{m_1}) \arrow[r, "K_0(\psi^{m_1})"] \arrow[d, "\rho^1"'] & K_0(B) \arrow[d, "\rho"] \\
K_0(A_{n_2}) \arrow[r, "K_0(\varphi^{n_2})"'] & K_0(A)
\end{tikzcd}$$

Now we claim that the composite $\rho^1 \circ \tau^1 = K_0(\varphi^{n_1,n_2})$.

To show: (b) $\rho^1 \circ \tau^1 = K_0(\varphi^{n_1,n_2})$

(b) We compute directly that if $j \in \{1, 2, \ldots, k\}$ then

$$\begin{aligned}
(\rho^1 \circ \tau^1)(x_j) &= (K_0(\varphi^{n,n_2}) \circ \tilde{\rho} \circ \tau^1)(x_j) \\
&= K_0(\varphi^{n,n_2})([q_j, 0]_{A_n}) \\
&= [q_j, 0]_{A_{n_2}} = [p_j, 0]_{A_{n_2}} \\
&= K_0(\varphi^{n_1,n_2})([p_j, 0]_{A_{n_1}}) = K_0(\varphi^{n_1,n_2})(x_j).
\end{aligned}$$

So, $\rho^1 \circ \tau^1 = K_0(\varphi^{n_1,n_2})$.

The idea here is that we can continue the above fiendish construction. Inductively, we obtain two sequences of positive integers $\{n_i\}_{i\in\mathbb{Z}_{>0}}$ and $\{m_i\}_{i\in\mathbb{Z}_{>0}}$ such that

$$n_1 < m_1 < n_2 < m_2 < \ldots$$

We also obtain positive group homomorphisms $\tau^k : K_0(A_{n_k}) \to K_0(B_{m_k})$ and $\rho^k : K_0(B_{m_k}) \to K_0(A_{n_{k+1}})$ such that the following diagrams below commute:

$$\begin{tikzcd}
K_0(A_{n_k}) \arrow[r, "K_0(\varphi^{n_k})"] \arrow[d, "\tau^k"'] & K_0(A) \arrow[d, "\tau"] \\
K_0(B_{m_k}) \arrow[r, "K_0(\psi^{m_k})"'] & K_0(B)
\end{tikzcd} \tag{2.10}$$

$$\begin{tikzcd}
K_0(B_{m_k}) \arrow[r, "K_0(\psi^{m_k})"] \arrow[d, "\rho^k"'] & K_0(B) \arrow[d, "\rho"] \\
K_0(A_{n_{k+1}}) \arrow[r, "K_0(\varphi^{n_{k+1}})"'] & K_0(A)
\end{tikzcd} \tag{2.11}$$

Moreover, the maps ρ^k and τ^k also satisfy

$$\rho^k \circ \tau^k = K_0(\varphi^{n_k,n_{k+1}}) \qquad \text{and} \qquad \tau^{k+1} \circ \rho^k = K_0(\psi^{m_k,m_{k+1}}). \tag{2.12}$$

Note that since τ^1 is unital, an induction argument using the above composition relations demonstrates that if $k \in \mathbb{Z}_{>0}$ then the group homomorphisms τ^k and ρ^k are unital.

The next step is to exploit Theorem 2.16.9. We find that if $k \in \mathbb{Z}_{>0}$ then there exist unital *-homomorphisms $\alpha^k : A_{n_k} \to B_{m_k}$ and $\beta_k : B_{m_k} \to A_{n_{k+1}}$ such that

$$K_0(\alpha^k) = \tau^k \qquad \text{and} \qquad K_0(\beta^k) = \rho^k.$$

Moreover, by equation (2.12),

$$\beta^k \circ \alpha^k = \varphi^{n_k,n_{k+1}} \qquad \text{and} \qquad \alpha^{k+1} \circ \beta^k = \psi^{m_k,m_{k+1}}.$$

Now observe that if $a \in A_{n_k}$ then

$$\alpha^k(a) = (\psi^{m_k,m_{k+1}} \circ \alpha^k)(a) = \alpha^{k+1}\beta^k\alpha^k(a) = \alpha^{k+1}\varphi^{n_k,n_{k+1}}(a) = \alpha^{k+1}(a).$$

Now define the map

$$\begin{array}{rccc} \varphi : & \bigcup_{k=1}^{\infty} A_{n_k} & \to & B \\ & a \in A_{n_k} & \mapsto & \alpha^k(a). \end{array}$$

We know that if $a \in A_{n_k}$ then $\alpha^k(a) = \alpha^{k+1}(a)$. Thus, the map φ is well-defined. Recall that $\alpha^k : A_{n_k} \to B_{m_k}$ are *-homomorphism and are norm-decreasing. This means that φ is also a (norm-decreasing) *-homomorphism. So, we can extend φ from the dense *-subalgebra $\bigcup_{k=1}^{\infty} A_{n_k}$ to all of A. We denote this extension by φ again.

Similarly, define the map

$$\begin{array}{rccc} \psi : & \bigcup_{k=1}^{\infty} B_{m_k} & \to & A \\ & b \in B_{m_k} & \mapsto & \beta^k(b). \end{array}$$

The map ψ is well-defined because if $b \in B_{m_k}$ then

$$\beta^k(b) = (\varphi^{n_{k+1},n_{k+2}} \circ \beta^k)(b) = \beta^{k+1}\alpha^{k+1}\beta^k(b) = \beta^{k+1}\psi^{m_k,m_{k+1}}(b) = \beta^{k+1}(b).$$

By employing a similar argument to the one used to construct φ, we extend ψ to a *-homomorphism $\psi : B \to A$.

Now, if $a \in A_{n_k}$ then

$$(\psi \circ \varphi)(a) = \psi(\alpha^k(a)) = \beta^k(\alpha^k(a)) = \varphi^{n_k, n_{k+1}}(a) = a$$

and if $b \in B_{m_k}$ then

$$(\varphi \circ \psi)(b) = \varphi(\beta^k(b)) = \alpha^{k+1}(\beta^k(b)) = \psi^{m_k, m_{k+1}}(b) = b.$$

Therefore, $\varphi \circ \psi = id_B$ and $\psi \circ \varphi = id_A$. This means that φ is a *-isomorphism.

To show: (c) $K_0(\varphi) = \tau$.

(c) Assume that $p \in P[A_{n_k}]$. Then,

$$\begin{aligned}
\tau([p, 0]_A) &= (\tau \circ K_0(\varphi^{n_k}))([p, 0]_{A_{n_k}}) \\
&= (K_0(\psi^{m_k}) \circ \tau^k)([p, 0]_{A_{n_k}}) \qquad \text{(by (2.10))} \\
&= (K_0(\psi^{m_k}) \circ K_0(\alpha^k))([p, 0]_{A_{n_k}}) \\
&= K_0(\psi^{m_k})([\alpha^k(p), 0]_{B_{m_k}}) \\
&= [\alpha^k(p), 0]_B = [\varphi(p), 0]_B \\
&= K_0(\varphi)([p, 0]_A).
\end{aligned}$$

So, $\tau = K_0(\varphi)$ on the sets $K_0(\varphi^{n_k})(K_0(A_{n_k})^+)$ and because

$$K_0(A)^+ = \bigcup_{k=1}^{\infty} K_0(\varphi^{n_k})(K_0(A_{n_k})^+)$$

and it generates $K_0(A)$, we deduce that $\tau = K_0(\varphi)$ as required. $\square$

A very straightforward consequence of Theorem 2.16.13 is stated below.

Theorem 2.16.14. *Let A and B be unital AF-algebras. Then, $A \cong B$ as C*-algebras if and only if $K_0(A) \cong K_0(B)$ as partially ordered groups. That is, there exists a *-isomorphism from A to B if and only if there exists a unital order isomorphism from $K_0(A)$ to $K_0(B)$.*

Chapter 3

Topics from [BO08]

3.1 Completely positive maps and Stinespring's theorem

As explained in [BO08, Section 1.5], completely positive maps form the foundations of C*-approximation theory. In fact, nuclear C*-algebras are defined using completely positive maps in [BO08, Definition 2.3.1]. This section is dedicated to outlining the basic theory behind completely positive maps, following the treatment in [BO08, Section 1.5].

In order to define completely positive maps, we must first define the objects they map from.

Definition 3.1.1. Let A be a unital C*-algebra. An **operator system** E is a self-adjoint subspace of A such that $1_A \in E$.

By definition, a unital C*-algebra is an operator system.

Theorem 3.1.1. *Let E be an operator system. Then, E is spanned by its positive elements. That is, E is spanned by the set*

$$\{e \in E \mid e \textit{ is positive}\}.$$

Proof. Assume that E is an operator system. Then, E is a self-adjoint subspace of a unital C*-algebra A such that $1_A \in E$. Assume that $e \in E$. Then, e can be written as

$$e = \frac{1}{2}(e + e^*) + i\Big(\frac{1}{2i}(e - e^*)\Big).$$

In particular, e is a linear combination of two self-adjoint elements of E. Hence, it suffices to prove that a self-adjoint element of E can be written as a linear combination of positive elements in E.

So, assume that $s \in E$ is self-adjoint. By Theorem 1.2.6, the spectral radius $r(s) = \|s\|$. If $t > \|s\|$ then $s + t1_A$ and $-s + t1_A$ are positive elements of E and

$$s = \frac{1}{2}(s + t1_A) - \frac{1}{2}(-s + t1_A).$$

Thus, any self-adjoint element in E is a linear combination of two positive elements in E. We conclude that any element in E is a linear combination of four positive elements in E. So, E is spanned by its positive elements. □

Before we define completely positive maps, we must first explain the structure of the matrix algebra $M_{n\times n}(A)$ as a C*-algebra, where $n \in \mathbb{Z}_{>0}$ and A is a C*-algebra. We already made use of this fact when we proved that the matrix algebra $M_{n\times n}(\mathbb{C})$ is a nuclear C*-algebra.

If $n \in \mathbb{Z}_{>0}$ and A is a *-algebra then $M_{n\times n}(A)$ is a *-algebra where scalar multiplication, addition and multiplication is defined analogously to $M_{n\times n}(\mathbb{C})$. Involution in $M_{n\times n}(A)$ is given by $(a_{ij})^* = (a_{ji}^*)$.

Definition 3.1.2. Let A and B be *-algebras and $\varphi : A \to B$ be a *-homomorphism. The **inflation** of φ, which is also denoted by φ, is the *-homomorphism

$$\begin{array}{rccc} \varphi : & M_{n\times n}(A) & \to & M_{n\times n}(B) \\ & (a_{ij}) & \mapsto & (\varphi(a_{ij})) \end{array}$$

Now if H is a Hilbert space and $n \in \mathbb{Z}_{>0}$ then we define the Hilbert space

$$H^{(n)} = \bigoplus_{i=1}^{n} H.$$

If $u \in M_{n\times n}(B(H))$ then define the map $\psi : M_{n\times n}(B(H)) \to B(H^{(n)})$, where $\psi(u)$ is given explicitly by

$$\begin{array}{rccc} \psi(u) : & H^{(n)} & \to & H^{(n)} \\ & (x_1, \ldots, x_n) & \mapsto & \left(\sum_{j=1}^n u_{1j}(x_j), \ldots, \sum_{j=1}^n u_{nj}(x_j) \right) \end{array} \tag{3.1}$$

It is straightforward to verify that ψ is a *-isomorphism. We call ψ the **canonical *-isomorphism** of $M_{n\times n}(B(H))$ onto $B(H^{(n)})$.

Definition 3.1.3. Let H be a Hilbert space and $n \in \mathbb{Z}_{>0}$. Let $v \in B(H^{(n)})$ and $\psi : M_{n\times n}(B(H)) \to B(H^{(n)})$ be the canonical *-isomorphism. Let $u \in M_{n\times n}(B(H))$ be such that $\psi(u) = v$. Then, u is called the **operator matrix** of v.

To define a norm on $M_{n\times n}(B(H))$ which makes it a C*-algebra, we set

$$\begin{array}{rccc} \|{-}\| : & M_{n\times n}(B(H)) & \to & \mathbb{R}_{\geq 0} \\ & u & \mapsto & \|\psi(u)\|. \end{array}$$

Theorem 3.1.2. *Let H be a Hilbert space and $n \in \mathbb{Z}_{>0}$. If $i, j \in \{1, 2, \ldots, n\}$ and $u \in M_{n\times n}(B(H))$ then*

$$\|u_{ij}\| \leq \|u\| \leq \sum_{k,l=1}^{n} \|u_{kl}\|.$$

Proof. Assume that H is a Hilbert space and $n \in \mathbb{Z}_{>0}$. Let ψ denote the canonical *-isomorphism from $M_{n\times n}(B(H))$ to $B(H^{(n)})$. If $i, j \in \{1, 2, \ldots, n\}$ and $u = (u_{ij}) \in M_{n\times n}(B(H))$ then

$$\begin{aligned} \|u\| &= \|\psi(u)\| \\ &= \sup_{\|(x_1,\ldots,x_n)\|=1} \|\psi(u)(x_1, \ldots, x_n)\| \\ &= \sup_{\|(x_1,\ldots,x_n)\|=1} \|\big(\sum_{j=1}^{n} u_{1j}(x_j), \ldots, \sum_{j=1}^{n} u_{nj}(x_j)\big)\| \\ &= \sup_{\|(x_1,\ldots,x_n)\|=1} \max_{k\in\{1,2,\ldots,n\}} \|\sum_{j=1}^{n} u_{kj}(x_j)\| \\ &\geq \sup_{\|(x_1,\ldots,x_n)\|=1} \|\sum_{j=1}^{n} u_{ij}(x_j)\| \geq \sup_{\|x_j\|=1} \|u_{ij}(x_j)\| = \|u_{ij}\|. \end{aligned}$$

We also compute directly that

$$
\begin{aligned}
\|u\| = \|\psi(u)\| &= \sup_{\|(x_1,\ldots,x_n)\|=1} \|\psi(u)(x_1,\ldots,x_n)\| \\
&= \sup_{\|(x_1,\ldots,x_n)\|=1} \|\big(\sum_{j=1}^{n} u_{1j}(x_j),\ldots,\sum_{j=1}^{n} u_{nj}(x_j)\big)\| \\
&= \sup_{\|(x_1,\ldots,x_n)\|=1} \max_{k\in\{1,2,\ldots,n\}} \|\sum_{j=1}^{n} u_{kj}(x_j)\| \\
&\le \sup_{\|(x_1,\ldots,x_n)\|=1} \max_{k\in\{1,2,\ldots,n\}} \sum_{l=1}^{n} \|u_{kl}(x_l)\| \\
&\le \sup_{\|(x_1,\ldots,x_n)\|=1} \sum_{k,l=1}^{n} \|u_{kl}(x_l)\| = \sum_{k,l=1}^{n} \|u_{kl}\|.
\end{aligned}
$$

□

Theorem 3.1.3. *Let A be a C*-algebra and $n \in \mathbb{Z}_{>0}$. Then, there exists a unique norm on $M_{n\times n}(A)$ such that $M_{n\times n}(A)$ is a C*-algebra.*

Proof. Assume that A is a C*-algebra and $n \in \mathbb{Z}_{>0}$. Let (φ, H) denote the universal representation of A, which is faithful. Since $\varphi : A \to B(H)$ is injective, its inflation $\varphi : M_{n\times n}(A) \to M_{n\times n}(B(H))$ is also injective. Now define the map

$$
\begin{array}{rccc}
\|-\| : & M_{n\times n}(A) & \to & \mathbb{R}_{\ge 0} \\
 & a & \mapsto & \|\varphi(a)\|.
\end{array}
$$

It is straightforward to verify that $\|-\|$ is a norm on $M_{n\times n}(A)$. If $a \in A$ then

$$\|a\|^2 = \|\varphi(a)\|^2 = \|\varphi(a)^*\varphi(a)\| = \|\varphi(a^*a)\| = \|a^*a\|.$$

To see that $M_{n\times n}(A)$ is complete with respect to its norm, let $\{a_m\}_{m\in\mathbb{Z}_{>0}}$ be a Cauchy sequence in $M_{n\times n}(A)$. If $\epsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $k, l > N$ then

$$\|a_k - a_l\| = \|\varphi(a_k) - \varphi(a_l)\| < \epsilon.$$

By Theorem 3.1.2, if $i, j \in \{1, 2, \ldots, n\}$ then

$$\|\varphi((a_k)_{ij}) - \varphi((a_l)_{ij})\| < \epsilon.$$

Therefore, the sequence $\{\varphi((a_k)_{ij})\}_{k\in\mathbb{Z}_{>0}}$ is a Cauchy sequence in $B(H)$ and hence converges to $b_{ij} \in B(H)$. By Theorem 1.7.6, there exists $a_{ij} \in A$ such that $\varphi(a_{ij}) = b_{ij}$. Now let $\Lambda = (a_{ij}) \in M_{n\times n}(A)$. We claim that the sequence $\{a_m\}_{m\in\mathbb{Z}_{>0}}$ converges to Λ. We compute directly that if $\epsilon \in \mathbb{R}_{>0}$ then

$$\begin{aligned}
\|a_k - \Lambda\| &= \|\varphi(a_k) - \varphi(\Lambda)\| \\
&= \|\varphi(a_k - \Lambda)\| \\
&\leq \sum_{l,m=1}^{n} \|\big(\varphi(a_k - \Lambda)\big)_{lm}\| \\
&= \sum_{l,m=1}^{n} \|\varphi((a_k)_{lm}) - \varphi(a_{lm})\| \\
&< \sum_{l,m=1}^{n} \frac{\epsilon}{n^2} = \epsilon.
\end{aligned}$$

We conclude that $M_{n\times n}(A)$ is complete with respect to the norm $\|{-}\|$ defined as above. Therefore, $\|{-}\|$ makes $M_{n\times n}(A)$ into a C*-algebra. By Theorem 1.2.8, it is the unique norm which does this. $\square$

Since injective *-homomorphisms are isometric, if A is a C*-algebra, $n \in \mathbb{Z}_{>0}$, $a = (a_{ij}) \in M_{n\times n}(A)$ and $i, j \in \{1, 2, \ldots, n\}$ then

$$\|a_{ij}\| \leq \|a\| \leq \sum_{k,l=1}^{n} \|a_{kl}\|.$$

This follows directly from Theorem 3.1.2.

Definition 3.1.4. Let A and B be C*-algebras. A linear map $\varphi : A \to B$ is **positive** if for a positive element $a \in A$, $\varphi(a)$ is a positive element of B.

Let E be an operator system and B be a C*-algebra. We say that the linear map $\varphi : E \to B$ is **completely positive** if for $n \in \mathbb{Z}_{>0}$ the map

$$\begin{array}{rccc}
\varphi_n : & M_{n\times n}(E) & \to & M_{n\times n}(B) \\
& (a_{ij}) & \mapsto & (\varphi(a_{ij}))
\end{array}$$

is positive. The set of completely positive maps from E to B is denoted by $CP(E, B)$.

If A is a unital C*-algebra and $E \subseteq A$ is an operator system then $M_{n\times n}(E)$ inherits positivity from $M_{n\times n}(A)$. We say that $\Lambda \in M_{n\times n}(E)$ is positive if and only if Λ is positive in $M_{n\times n}(A)$. Note that the definition of a completely positive map can be extended to cover maps between C*-algebras.

The term "completely positive" is commonly abbreviated as c.p, "unital completely positive" is abbreviated as u.c.p and "contractive completely positive" is abbreviated as c.c.p.

Example 3.1.1. Let A and B be C*-algebras and $\pi : A \to B$ be a *-homomorphism. Then, π is completely positive because if $n \in \mathbb{Z}_{>0}$ then its inflations $\pi : M_{n\times n}(A) \to M_{n\times n}(B)$ are *-homomorphisms themselves and hence, preserve positivity.

More generally, let H be a Hilbert space and $\varphi : A \to B(H)$ be a map of the form $\varphi(a) = V^*\pi(a)V$, where $\pi : A \to B(H')$ is a *-homomorphism and $V : H \to H'$ is an operator. We claim that φ is a completely positive map. Assume that $n \in \mathbb{Z}_{>0}$ and $\Lambda = (\lambda_{ij}) \in M_{n\times n}(A)$ is positive. Then, there exists $\Gamma = (\gamma_{ij}) \in M_{n\times n}(A)$ such that $\Lambda = \Gamma^*\Gamma$. If $i, j \in \{1, 2, \dots, n\}$ then

$$\big(\varphi_n(\Lambda)\big)_{ij} = V^*\pi(\lambda_{ij})V = V^*\pi(\sum_{k=1}^{n} \gamma_{ki}^*\gamma_{kj})V = \sum_{k=1}^{n} V^*\pi(\gamma_{ki}^*\gamma_{kj})V.$$

Then, $\varphi_n(\Lambda) = \Delta^*\Delta$ where $\Delta_{kj} = \pi(\gamma_{kj})V \in M_{n\times n}(B(H, H'))$. Making use of the isomorphism $\psi : M_{n\times n}(B(H)) \to B(H^{(n)})$, assume that $x = (x_1, \dots, x_n) \in H^{(n)}$. Then,

$$\langle(\psi \circ \varphi_n)(\Lambda)x, x\rangle = \langle\psi(\Delta^*\Delta)x, x\rangle = \|\psi(\Delta)x\|^2 \geq 0.$$

This means that $(\psi \circ \varphi_n)(\Lambda)$ is a positive element of $B(H^{(n)})$. By composing with the *-homomorphism ψ^{-1}, we deduce that $\varphi_n(\Lambda)$ is positive. Since $n \in \mathbb{Z}_{>0}$ was arbitrary, we deduce that φ is completely positive.

Example 3.1.2. Let A be a unital C*-algebra and $E \subseteq A$ be an operator system. Let $f : E \to \mathbb{C}$ be a positive linear functional. We claim that f is completely positive. Assume that $n \in \mathbb{Z}_{>0}$. The idea is to take advantage of the isomorphism $M_{n\times n}(\mathbb{C}) \cong B(\mathbb{C}^n)$ and the fact that $\mathbb{C}^n$ is a Hilbert space.

Assume that $A = (a_{ij}) \in M_{n\times n}(E)$ is positive. Assume that $x = (x_1, \dots, x_n) \in \mathbb{C}^n$. Then,

$$
\begin{aligned}
\langle f_n(A)x, x\rangle &= \left\langle \begin{pmatrix} f(a_{11}) & \dots & f(a_{1n}) \\ \vdots & \ddots & \vdots \\ f(a_{n1}) & \dots & f(a_{nn}) \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right\rangle \\
&= \left\langle \begin{pmatrix} \sum_{i=1}^n f(a_{1i})x_i \\ \vdots \\ \sum_{i=1}^n f(a_{ni})x_i \end{pmatrix}, \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right\rangle \\
&= \sum_{i,j=1}^n f(a_{ji})x_i\overline{x_j} = f(\sum_{i,j=1}^n a_{ji}x_i\overline{x_j}) = f(x^*Ax).
\end{aligned}
$$

Since A is positive, there exists $B = (b_{ij}) \in M_{n\times n}(E)$ such that $A = B^*B$. Then, we have

$$
\langle f_n(A)x, x\rangle = f(x^*B^*Bx) = f((Bx)^*Bx) = f(\|(Bx)^*\|^2) \geq 0.
$$

Hence, the map $f_n : M_{n\times n}(E) \to M_{n\times n}(\mathbb{C})$ is positive for arbitrary $n \in \mathbb{Z}_{>0}$. So, f is a c.p map.

Here is an example of a map which is not c.p.

Example 3.1.3. Let $n \in \mathbb{Z}_{>0}$ and $\varphi : M_{n\times n}(\mathbb{C}) \to M_{n\times n}(\mathbb{C})$ be the adjoint map $A \mapsto A^*$. We claim that φ is not completely positive. First, we observe that φ is positive. Assume that $A = B^*B \in M_{n\times n}(\mathbb{C})$ is a positive element. Then,

$$
\varphi(B^*B) = (B^*B)^* = B^*B = A.
$$

Hence, φ is positive. To see that φ is not completely positive, we will give a counterexample.

Let $n = 2$. Then, $M_{2\times 2}(M_{2\times 2}(\mathbb{C})) \cong M_{4\times 4}(\mathbb{C})$ as C*-algebras. Let

$$
A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \in M_{4\times 4}(\mathbb{C}).
$$

If $x = (x_1, x_2, x_3, x_4) \in \mathbb{C}^4$ then

$$
\langle Ax, x\rangle = (x_1 + x_4)(\overline{x_1} + \overline{x_4}) = |x_1 + x_4|^2 \geq 0.
$$

So, A is a positive element of $M_{4\times 4}(\mathbb{C})$. However,

$$\varphi_2(A) = \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^* & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^* \\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}^* & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}^* \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

If $x = (x_1, x_2, x_3, x_4) \in \mathbb{C}^4$ then

$$\langle \varphi_2(A)x, x\rangle = |x_1|^2 + x_2\overline{x_3} + x_3\overline{x_2} + |x_4|^2.$$

If $x = (0, 1, -1, 0)$ then $\langle \varphi_2(A)x, x\rangle = -2$. Therefore, $\varphi_2(A)$ is not a positive element of $M_{4\times 4}(\mathbb{C})$. So, φ is not a completely positive map.

Stinespring's dilation theorem characterises completely positive maps to $B(H)$, where H is a Hilbert space.

Theorem 3.1.4. *Let A be a unital C*-algebra, H be a Hilbert space and $\varphi : A \to B(H)$ be a c.p map. Then, there exists a representation $(\pi, \widehat{H})$ and an operator $V : H \to \widehat{H}$ such that if $a \in A$ then*

$$\varphi(a) = V^*\pi(a)V.$$

Moreover, $\|V^*V\| = \|\varphi(1_A)\|$.

Proof. Assume that A is a unital C*-algebra and H is a Hilbert space. Assume that $\varphi : A \to B(H)$ is a completely positive map. We begin by defining a sesquilinear form on the algebraic tensor product $A \otimes H$.

Define

$$\begin{array}{cccc} \langle -, - \rangle : & (A \otimes H)^2 & \to & \mathbb{C} \\ & (\sum_j b_j \otimes \eta_j, \sum_i a_i \otimes \xi_i) & \mapsto & \sum_{i,j} \langle \varphi(a_i^* b_j)\eta_j, \xi_i\rangle_H \end{array}$$

To be clear, $\langle -, - \rangle_H$ is the inner product on H.

To show: (a) $\langle -, - \rangle$ defines a positive semidefinite sesquilinear form.

(a) Assume that $\lambda \in \mathbb{C}$. Then,

$$\begin{aligned}
\langle \lambda \sum_j b_j \otimes \eta_j, \sum_i a_i \otimes \xi_i \rangle &= \langle \sum_j (\lambda b_j) \otimes \eta_j, \sum_i a_i \otimes \xi_i \rangle \\
&= \sum_{i,j} \langle \varphi(a_i^* \lambda b_j)\eta_j, \xi_i \rangle_H \\
&= \lambda \sum_{i,j} \langle \varphi(a_i^* b_j)\eta_j, \xi_i \rangle_H \\
&= \lambda \langle \sum_j b_j \otimes \eta_j, \sum_i a_i \otimes \xi_i \rangle
\end{aligned}$$

and

$$\begin{aligned}
\langle \sum_j b_j \otimes \eta_j, \lambda \sum_i a_i \otimes \xi_i \rangle &= \langle \sum_j b_j \otimes \eta_j, \sum_i (\lambda a_i) \otimes \xi_i \rangle \\
&= \sum_{i,j} \langle \varphi((\lambda a_i)^* b_j)\eta_j, \xi_i \rangle_H \\
&= \overline{\lambda} \sum_{i,j} \langle \varphi(a_i^* b_j)\eta_j, \xi_i \rangle_H \\
&= \overline{\lambda} \langle \sum_j b_j \otimes \eta_j, \sum_i a_i \otimes \xi_i \rangle.
\end{aligned}$$

Now assume that $b_1, b_2 \in A$ and $\eta_1, \eta_2 \in H$. Then,

$$\begin{aligned}
\langle (b_1 \otimes \eta_1) + (b_2 \otimes \eta_2), \sum_i a_i \otimes \xi_i \rangle &= \sum_i \sum_{j=1}^{2} \langle \varphi(a_i^* b_j)\eta_j, \xi_i \rangle_H \\
&= \sum_i \big(\langle \varphi(a_i^* b_1)\eta_1, \xi_i \rangle_H + \langle \varphi(a_i^* b_2)\eta_2, \xi_i \rangle_H \big) \\
&= \sum_i \langle \varphi(a_i^* b_1)\eta_1, \xi_i \rangle_H + \sum_i \langle \varphi(a_i^* b_2)\eta_2, \xi_i \rangle_H \\
&= \langle b_1 \otimes \eta_1, \sum_i a_i \otimes \xi_i \rangle + \langle b_2 \otimes \eta_2, \sum_i a_i \otimes \xi_i \rangle.
\end{aligned}$$

By a similar computation, we also have

$$\langle \sum_i a_i \otimes \xi_i, (b_1 \otimes \eta_1) + (b_2 \otimes \eta_2) \rangle = \langle \sum_i a_i \otimes \xi_i, b_1 \otimes \eta_1 \rangle + \langle \sum_i a_i \otimes \xi_i, b_2 \otimes \eta_2 \rangle.$$

Hence, $\langle -, - \rangle$ is a sesquilinear form. Next, assume that $\sum_{i=1}^{n} a_i \otimes \xi_i \in A \otimes H$. Then,

$$\begin{aligned}
\langle\sum_{i=1}^{n} a_i \otimes \xi_i, \sum_{i=1}^{n} a_i \otimes \xi_i\rangle &= \sum_{i,j=1}^{n} \langle \varphi(a_i^* a_j)\xi_j, \xi_i\rangle_H \\
&= \left\langle \varphi_n((a_i^* a_j)) \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}, \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} \right\rangle_{H^{(n)}} \\
&= \left\langle \varphi_n(\alpha^* \alpha) \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}, \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} \right\rangle_{H^{(n)}}
\end{aligned}$$

where

$$\alpha = \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \in M_{n\times n}(A).$$

In the above computation, we recall that the inner product on the direct sum of Hilbert spaces $H^{(n)}$ is given by

$$\langle (g_1, \dots, g_n), (h_1, \dots, h_n)\rangle_{H^{(n)}} = \sum_{i=1}^{n} \langle g_i, h_i\rangle_H.$$

Since φ is completely positive, φ_n is a positive map. Therefore,

$$\langle\sum_{i=1}^{n} a_i \otimes \xi_i, \sum_{i=1}^{n} a_i \otimes \xi_i\rangle = \left\langle \varphi_n(\alpha^* \alpha) \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}, \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} \right\rangle_{H^{(n)}} \geq 0.$$

Finally, assume that $a \otimes \xi = 0$ in $A \otimes H$. By definition of $\langle -, -\rangle$, we compute directly that $\langle a \otimes \xi, a \otimes \xi\rangle = 0$. Hence, we conclude that $\langle -, -\rangle$ is a positive semidefinite sesquilinear form on $A \otimes H$.

Now define

$$\mathcal{N} = \{u \in A \otimes H \mid \langle u, u\rangle = 0\}.$$

Since $\langle -, -\rangle$ is a sesquilinear form on $A \otimes H$, it must satisfy the Cauchy-Schwarz inequality. If $u, v \in A \otimes H$ then

$$|\langle u, v\rangle|^2 \leq \langle u, u\rangle\langle v, v\rangle.$$

Thus,

$$\mathcal{N} = \{u \in A \otimes H \mid \text{If } v \in A \otimes H \text{ then } \langle u, v\rangle = 0\}. \tag{3.2}$$

Using equation (3.2), it is straightforward to check that $\mathcal{N}$ is a vector subspace of $A \otimes H$. To see that $\mathcal{N}$ is closed, assume that $\{u_n\}_{n\in\mathbb{Z}_{>0}}$ is a sequence in $\mathcal{N}$ which converges to $u \in A \otimes H$. If $v \in A \otimes H$ then

$$\langle u, v\rangle = \langle \lim_{n\to\infty} u_n, v\rangle = \lim_{n\to\infty} \langle u_n, v\rangle = 0.$$

Hence, $u \in \mathcal{N}$ and $\mathcal{N}$ is a closed subspace of $A \otimes H$.

Now consider the quotient space $(A \otimes H)/\mathcal{N}$. Define the map $\langle -, -\rangle'$ by

$$\begin{array}{cccc} \langle -, -\rangle' : & ((A \otimes H)/\mathcal{N})^2 & \to & \mathbb{C} \\ & ([\sum_j b_j \otimes \eta_j], [\sum_i a_i \otimes \xi_i]) & \mapsto & \langle \sum_j b_j \otimes \eta_j, \sum_i a_i \otimes \xi_i\rangle. \end{array}$$

Here, $[\sum_j b_j \otimes \eta_j]$ refers to the equivalence class of $\sum_j b_j \otimes \eta_j \in A \otimes H$ in $(A \otimes H)/\mathcal{N}$. By construction, the pair $((A \otimes H)/\mathcal{N}, \langle -, -\rangle')$ is an inner product space. Analogously to the GNS construction, let $\widehat{H}$ be the completion of $(A \otimes H)/\mathcal{N}$ with respect to the inner product $\langle -, -\rangle'$. Then, $\widehat{H}$ is a Hilbert space by construction.

Following the notation in [BO08, Theorem 1.5.3], let $(\sum_i a_i \otimes \xi_i)^\wedge$ be the element in $\widehat{H}$ corresponding to $\sum_i a_i \otimes \xi_i \in H$. Define

$$\begin{array}{cccc} V: & H & \to & \widehat{H} \\ & x & \mapsto & (1_A \otimes x)^\wedge. \end{array}$$

Then, V is a linear operator. To see that V is bounded, we compute directly that

$$\begin{aligned} \|V\|^2 &= \sup_{\|x\|_H=1} \|V(x)\|^2 \\ &= \sup_{\|x\|_H=1} \|(1_A \otimes x)^\wedge\|^2 \\ &= \sup_{\|x\|_H=1} \langle [1_A \otimes x], [1_A \otimes x]\rangle' \\ &= \sup_{\|x\|_H=1} \langle \varphi(1_A^* 1_A)x, x\rangle_H \\ &\leq \sup_{\|x\|_H=1} \|\varphi(1_A)\|\|x\|^2 = \|\varphi(1_A)\|. \end{aligned}$$

In fact, we claim that the reverse inequality also holds. We have

$$
\begin{aligned}
\|\varphi(1_A)\|^2 &= \sup_{\|x\|_H=1} \|\varphi(1_A)(x)\|_H^2 \\
&= \sup_{\|x\|_H=1} \langle \varphi(1_A)(x), \varphi(1_A)(x)\rangle_H \\
&= \sup_{\|x\|_H=1} \langle [1_A \otimes x], [1_A \otimes \varphi(1_A)(x)]\rangle' \\
&= \sup_{\|x\|_H=1} \langle (1_A \otimes x)^\wedge, (1_A \otimes \varphi(1_A)(x))^\wedge\rangle \\
&\le \sup_{\|x\|_H=1} \|(1_A \otimes x)^\wedge\| \|(1_A \otimes \varphi(1_A)(x))^\wedge\| \\
&= \sup_{\|x\|_H=1} \|V(x)\| \|V(\varphi(1_A)(x))\| \\
&\le \sup_{\|x\|_H=1} \|V\|^2 \|x\|_H \|\varphi(1_A)(x)\|_H = \|V\|^2 \|\varphi(1_A)\|.
\end{aligned}
$$

So, $\|\varphi(1_A)\| \le \|V\|^2$ and consequently, $\|V^*V\| = \|V\|^2 = \|\varphi(1_A)\|$.

For the next step, we define the map

$$
\begin{array}{rccc}
\pi : & A & \to & B(\widehat{H}) \\
 & a & \mapsto & \Big((\sum_i b_i \otimes \eta_i)^\wedge \mapsto (\sum_i ab_i \otimes \eta_i)^\wedge \Big)
\end{array}
$$

To show: (b) π is a *-homomorphism.

(b) Since $(A \otimes H)/\mathcal{N}$ is dense in $\widehat{H}$, it suffices to check that π is a *-homomorphism on $(A \otimes H)/\mathcal{N}$. It is straightforward to check that π is a linear map. To see that π is a *-homomorphism, assume that $a \in A$ and

$$
(\sum_i b_i \otimes \eta_i)^\wedge, (\sum_j c_j \otimes \xi_j)^\wedge \in \widehat{H}.
$$

We compute directly that

$$
\begin{aligned}
\langle \pi(a)^*(\sum_i b_i \otimes \eta_i)^\wedge, (\sum_j c_j \otimes \xi_j)^\wedge \rangle &= \langle (\sum_i b_i \otimes \eta_i)^\wedge, \pi(a)(\sum_j c_j \otimes \xi_j)^\wedge \rangle \\
&= \langle (\sum_i b_i \otimes \eta_i)^\wedge, (\sum_j ac_j \otimes \xi_j)^\wedge \rangle \\
&= \sum_{i,j} \langle \varphi(c_j^* a^* b_i)\eta_i, \xi_j \rangle_H \\
&= \langle (\sum_i a^* b_i \otimes \eta_i)^\wedge, (\sum_j c_j \otimes \xi_j)^\wedge \rangle \\
&= \langle \pi(a^*)(\sum_i b_i \otimes \eta_i)^\wedge, (\sum_j c_j \otimes \xi_j)^\wedge \rangle
\end{aligned}
$$

and if $b \in A$ then

$$
\pi(ab)(\sum_j c_j \otimes \xi_j)^\wedge = (\sum_j abc_j \otimes \xi_j)^\wedge = \pi(a)(\sum_j bc_j \otimes \xi_j)^\wedge = \pi(a)\pi(b)(\sum_j c_j \otimes \xi_j)^\wedge.
$$

Therefore, π is a *-homomorphism.

By part (b), the pair $(\pi, \widehat{H})$ is a representation of A. Now assume that $x, y \in H$ and $a \in A$. Then,

$$
\begin{aligned}
\langle V^*\pi(a)V(x), y \rangle_H &= \langle \pi(a)V(x), V(y) \rangle_H \\
&= \langle \pi(a)(1_A \otimes x)^\wedge, (1_A \otimes y)^\wedge \rangle \\
&= \langle (a \otimes x)^\wedge, (1_A \otimes y)^\wedge \rangle \\
&= \langle \varphi(1_A^* a)x, y \rangle_H \\
&= \langle \varphi(a)x, y \rangle_H.
\end{aligned}
$$

Finally, we conclude that if $a \in A$ then $V^*\pi(a)V = \varphi(a)$. $\square$

In the statement of Theorem 3.1.4, we note that $\|\varphi(1_A)\| = \|\varphi\|$. This was stated in [BO08, Theorem 1.5.3]. However, this fact is not obvious.

Definition 3.1.5. Let A be a unital C*-algebra and H be a Hilbert space. Let $\varphi : A \to B(H)$ be a c.p map. The triplet $(\pi, \widehat{H}, V)$ in Theorem 3.1.4 is called a **Stinespring dilation** of φ.

If φ is a u.c.p map then

$$V^*V = V^*\pi(1_A)V = \varphi(1_A) = id_{B(H)}.$$

In this case, V is an isometry and $VV^* \in B(\widehat{H})$ is a projection operator. We call VV^* the **Stinespring projection**.

In general, there are many different Stinespring dilations. It is explained in [BO08] that we can always choose a *minimal* Stinespring dilation in the following manner.

Definition 3.1.6. Let A be a unital C*-algebra and H be a Hilbert space. Let $\varphi : A \to B(H)$ be a c.p map and $(\pi, \widehat{H}, V)$ be a Stinespring dilation of φ. The Stinespring dilation $(\pi, \widehat{H}, V)$ is called **minimal** if the subspace $\pi(A)VH$ is dense in $\widehat{H}$.

Note that in the proof of Theorem 3.1.4, the Stinespring dilation we constructed is minimal because $\pi(A)VH = (A \otimes H)/\mathcal{N}$ which is dense in $\widehat{H}$.

Theorem 3.1.5. *Let A be a unital C*-algebra and H be a Hilbert space. Let $\pi : A \to B(H)$ be a c.p map and $(\pi, \widehat{H}, V)$ be a minimal Stinespring dilation. Then, $(\pi, \widehat{H}, V)$ is unique up to unitary equivalence.*

Proof. Assume that $\varphi : A \to B(H)$ is a c.p map and $(\pi, \widehat{H}, V)$ be a minimal Stinespring dilation of φ. Suppose that (π', H', W) is another minimal Stinespring dilation of φ. If $a \in A$ then

$$\varphi(a) = W^*\pi'(a)W = V^*\pi(a)V.$$

Since the Stinespring dilations $(\pi, \widehat{H}, V)$ and (π', H', W) are both minimal, the subspaces $\pi(A)VH$ and $\pi'(A)WH$ are both dense in $\widehat{H}$ and H' respectively. Define the map

$$\begin{array}{rccc} u: & \pi(A)VH & \to & \pi'(A)WH \\ & \pi(a)V(x) & \mapsto & \pi'(a)W(x). \end{array}$$

To see that u is isometric, assume that $a \in A$ and $x \in X$. Then,

$$\begin{aligned} \langle u(\pi(a)V(x)), u(\pi(a)V(x))\rangle &= \langle \pi'(a)W(x), \pi'(a)W(x)\rangle \\ &= \langle W^*\pi'(a^*a)W(x), x\rangle \\ &= \langle \varphi(a^*a)(x), x\rangle \\ &= \langle V^*\pi(a^*a)V(x), x\rangle \\ &= \langle \pi(a)V(x), \pi(a)V(x)\rangle. \end{aligned}$$

By the universal property of completeness, we can extend u to a unitary operator $\tilde{u} : \widehat{H} \to H'$. Now observe that if $a \in A$ and $\pi(b)V(x) \in \pi(A)VH$ then

$$\tilde{u}\pi(a)\big(\pi(b)V(x)\big) = \pi'(ab)W(x) = \pi'(a)\big(\pi'(b)W(x)\big) = \pi'(a)\tilde{u}\big(\pi(b)V(x)\big).$$

Since $\pi(A)VH$ is dense in $\widehat{H}$, we deduce that $\tilde{u}\pi(a) = \pi'(a)\tilde{u}$ on $\widehat{H}$. Therefore, the Stinespring dilations $(\pi, \widehat{H}, V)$ and (π', H', W) are unitarily equivalent as required. □

As explained in [BO08, Remark 1.5.4], Stinespring's theorem also holds for non-unital C*-algebras. We also claim that Stinespring's theorem is a generalisation of the GNS construction . Let A be a unital C*-algebra and τ be a state on A. Since τ is a state, it is a positive linear functional on A and is thus, completely positive.

By identifying $\mathbb{C}$ with $B(\mathbb{C})$, let (π, H, V) be a minimal Stinespring dilation of τ. If $a \in A$ then $\tau(a) = V^*\pi(a)V$. Define $\xi = V(1) \in H$. If $\lambda \in \mathbb{C}$ then $V(\lambda) = \lambda\xi$ and by Theorem 3.1.4 and Theorem 1.11.2,

$$\|V\|^2 = \|V^*V\| = |\tau(1_A)| = \|\tau\| = 1.$$

So, $\|V\| = 1$ and

$$\|V\| = \sup_{|\lambda|=1} \|\lambda\xi\| = \|\xi\| = 1.$$

Hence, ξ is a unit vector in H. Moreover, if $a \in A$ then

$$\langle \pi(a)\xi, \xi\rangle = \langle \pi(a)V(1), V(1)\rangle = \langle V^*\pi(a)V(1), 1\rangle = \langle \tau(a)1, 1\rangle = \tau(a).$$

Finally, to see that ξ is a cyclic vector, observe that since (π, H, V) is a minimal Stinespring dilation of τ, the subspace $\pi(A)V\mathbb{C}$ is dense in H. Hence, $\overline{\pi(A)\xi} = \overline{\pi(A)V\mathbb{C}} = H$ and ξ is a unit cyclic vector for the representation (π, H). This connects Stinespring's theorem to the GNS construction.

The next result we prove is an analogue of Theorem 2.12.2, applied to a minimal Stinespring dilation.

Theorem 3.1.6. *Let A be a unital C*-algebra and H be a Hilbert space. Let $\varphi : A \to B(H)$ be a c.c.p map and $(\pi, \widehat{H}, V)$ be the minimal Stinespring dilation of φ. Then, there exists a *-homomorphism*

$$\rho : \varphi(A)' \to \pi(A)' \subseteq B(\widehat{H})$$

such that if $a \in A$ and $x \in \varphi(A)'$ then

$$\varphi(a)x = V^*\pi(a)\rho(x)V.$$

Proof. Assume that $\varphi : A \to B(H)$ is a c.c.p map and $(\pi, \widehat{H}, V)$ is a minimal Stinespring dilation of φ. Define the map $\rho : \varphi(A)' \to B(\widehat{H})$ by

$$\begin{array}{cccc} \rho(x) : & \pi(A)VH & \to & \pi(A)VH \\ & \sum_i \pi(a_i)V\xi_i & \mapsto & \sum_i \pi(a_i)Vx\xi_i \end{array}$$

Note that $\rho(x)$ is a linear operator.

To show: (a) If $x \in \varphi(A)'$ then $\rho(x)$ is well-defined and bounded.

(a) Assume that $x \in \varphi(A)'$. Assume that $\sum_i \pi(a_i)V\xi_i = \sum_j \pi(b_j)V\mu_j$ in $\pi(A)VH$. If $c \in A$ and $\lambda \in H$ then

$$\begin{aligned}
\langle \rho(x)\big(\sum_i \pi(a_i)V\xi_i\big), \pi(c)V\lambda\rangle &= \langle \sum_i \pi(a_i)Vx\xi_i, \pi(c)V\lambda\rangle \\
&= \langle \sum_i V^*\pi(c^*a_i)Vx\xi_i, \lambda\rangle \\
&= \langle \sum_i \varphi(c^*a_i)x\xi_i, \lambda\rangle = \langle \sum_i x\varphi(c^*a_i)\xi_i, \lambda\rangle \\
&= \langle \sum_i \pi(a_i)V\xi_i, \pi(c)Vx^*\lambda\rangle \\
&= \langle \sum_j \pi(b_j)V\mu_j, \pi(c)Vx^*\lambda\rangle \\
&= \langle \sum_j xV^*\pi(c^*b_j)V\mu_j, \lambda\rangle = \langle \sum_j \varphi(c^*b_j)x\mu_j, \lambda\rangle \\
&= \langle \sum_j \pi(b_j)x\mu_j, \pi(c)V\lambda\rangle \\
&= \langle \rho(x)\big(\sum_j \pi(b_j)V\mu_j\big), \pi(c)V\lambda\rangle.
\end{aligned}$$

Since $\pi(A)VH$ is dense in $\widehat{H}$, we conclude that

$$\rho(x)\big(\sum_i \pi(a_i)V\xi_i\big) = \rho(x)\big(\sum_j \pi(b_j)V\mu_j\big).$$

Therefore, $\rho(x)$ is well-defined. To see that $\rho(x)$ is bounded, let $\xi = [\xi_1, \dots, \xi_n]^T \in H^n$. Let $diag(x)$ be the $n \times n$ matrix whose diagonal elements are x and non-diagonal elements are zeros. Then,

$$\begin{aligned}
\|\rho(x) \sum_i \pi(a_i) V \xi_i\|^2 &= \langle \sum_i \pi(a_i) V x \xi_i, \sum_j \pi(a_j) V x \xi_j \rangle \\
&= \sum_{i,j} \langle x^* V^* \pi(a_j^* a_i) V x \xi_i, \xi_j \rangle \\
&= \sum_{i,j} \langle x^* \varphi(a_j^* a_i) x \xi_i, \xi_j \rangle \\
&= \langle diag(x)^* \varphi_n((a_j^* a_i)) diag(x) \xi, \xi \rangle_{H^n} \\
&= \langle diag(x)^* diag(x) \varphi_n((a_j^* a_i)) \xi, \xi \rangle_{H^n} \\
&\leq \|x\|^2 \langle \varphi_n((a_j^* a_i)) \xi, \xi \rangle_{H^n} \\
&= \|x\|^2 \sum_{i,j} \langle \varphi(a_j^* a_i) \xi_i, \xi_j \rangle \\
&= \|x\|^2 \|\sum_i \pi(a_i) V \xi_i\|^2.
\end{aligned}$$

Therefore, $\|\rho(x)\| \leq \|x\|$ and $\rho(x)$ is a bounded operator.

By part (a), if $x \in \varphi(A)'$ then $\rho(x)$ can be extended to a bounded linear operator on $\widehat{H}$ because $\overline{\pi(A)VH} = \widehat{H}$.

To show: (b) ρ is a *-homomorphism.

(c) $\rho(\varphi(A)') \subseteq \pi(A)'$.

(d) If $a \in A$ and $x \in \varphi(A)'$ then $\varphi(a)x = V^* \pi(a) \rho(x) V$.

(b) It is straightforward to check that ρ is a linear operator. Now assume that $x, y \in \varphi(A)'$ and $\sum_i \pi(a_i) V \xi_i \in \pi(A)VH$. Then,

$$\begin{aligned}
\rho(xy)\big(\sum_i \pi(a_i) V \xi_i\big) &= \sum_i \pi(a_i) V xy \xi_i \\
&= \rho(x)\big(\sum_i \pi(a_i) V y \xi_i\big) \\
&= \rho(x)\rho(y)\big(\sum_i \pi(a_i) V \xi_i\big).
\end{aligned}$$

We also have

$$\begin{aligned}
\langle \rho(x)^*\big(\sum_i \pi(a_i)V\xi_i\big), \sum_j \pi(b_j)V\mu_j\rangle &= \langle \sum_i \pi(a_i)V\xi_i, \rho(x)\big(\sum_j \pi(b_j)V\mu_j\big)\rangle \\
&= \langle \sum_i \pi(a_i)V\xi_i, \rho(x)\big(\sum_j \pi(b_j)V\mu_j\big)\rangle \\
&= \langle \sum_i \pi(a_i)V\xi_i, \sum_j \pi(b_j)Vx\mu_j\rangle \\
&= \sum_{i,j} \langle x^*V^*\pi(b_j^*a_i)V\xi_i, \mu_j\rangle \\
&= \sum_{i,j} \langle x^*\varphi(b_j^*a_i)\xi_i, \mu_j\rangle \\
&= \sum_{i,j} \langle \varphi(b_j^*a_i)x^*\xi_i, \mu_j\rangle \\
&= \langle \sum_i \pi(a_i)Vx^*\xi_i, \sum_j \pi(b_j)V\mu_j\rangle \\
&= \langle \rho(x^*)\big(\sum_i \pi(a_i)V\xi_i\big), \sum_j \pi(b_j)V\mu_j\rangle.
\end{aligned}$$

Since $\pi(A)VH$ is dense in $\widehat{H}$, we conclude that if $x, y \in \varphi(A)'$ then $\rho(xy) = \rho(x)\rho(y)$ and $\rho(x^*) = \rho(x)^*$. So, ρ is a *-homomorphism.

(c) Assume that $\pi(a)V\xi \in \pi(A)VH$, $b \in A$ and $x \in \varphi(A)'$. Then,

$$\begin{aligned}
(\rho(x)\pi(b))\big(\pi(a)V\xi\big) &= \rho(x)\big(\pi(ba)V\xi\big) \\
&= \pi(ba)Vx\xi = \pi(b)\pi(a)Vx\xi \\
&= (\pi(b)\rho(x))\big(\pi(a)V\xi\big).
\end{aligned}$$

Hence, $\pi(b)\rho(x) = \rho(x)\pi(b)$ in $B(\widehat{H})$ because $\pi(A)VH$ is dense in $\widehat{H}$. So, $\rho(\varphi(A)') \subseteq \pi(A)'$.

(d) Assume that $a \in A$ and $x \in \varphi(A)'$. Recall from the construction in Theorem 3.1.4 that $\pi : A \to B(\widehat{H})$ is a unital *-homomorphism. If $\xi \in H$ then

$$V^*\pi(a)\rho(x)V\xi = V^*\pi(a)\rho(x)\pi(1_A)V\xi = V^*\pi(a)\pi(1_A)Vx\xi = \varphi(a)x\xi.$$

This completes the proof. □

3.2 Multiplicative domains and conditional expectations

The definition of a multiplicative domain is motivated by the following properties of a c.c.p map.

Theorem 3.2.1. *Let A and B be C*-algebras and $\varphi : A \to B$ be a c.c.p map.*

1. *(Schwarz Inequality) If $a \in A$ then $\varphi(a)^*\varphi(a) \leq \varphi(a^*a)$.*
2. *(Bimodule property) If $a \in A$ such that $\varphi(a^*a) = \varphi(a)^*\varphi(a)$ and $\varphi(aa^*) = \varphi(a)\varphi(a)^*$ then if $b \in A$ then $\varphi(ba) = \varphi(b)\varphi(a)$ and $\varphi(ab) = \varphi(a)\varphi(b)$.*
3. *The subspace*

$$A_\varphi = \{a \in A \mid \varphi(a^*a) = \varphi(a)^*\varphi(a) \text{ and } \varphi(aa^*) = \varphi(a)\varphi(a)^*\}$$

is a C-subalgebra of A.*

Proof. Assume that A and B are C*-algebras. Assume that $\varphi : A \to B$ is a c.c.p map. Let (ψ, H) be a faithful representation of B. Since ψ is an isometric *-homomorphism by Theorem 1.6.4, the composite $\psi \circ \varphi : A \to B(H)$ is a c.c.p map. Let $(\pi, \widehat{H}, V)$ be a minimal Stinespring dilation of $\psi \circ \varphi$. If $a \in A$ then

$$\begin{aligned}\psi(\varphi(a^*a) - \varphi(a)^*\varphi(a)) &= (\psi \circ \varphi)(a^*a) - (\psi \circ \varphi)(a)^*(\psi \circ \varphi)(a)\\ &= V^*\pi(a^*a)V - V^*\pi(a^*)VV^*\pi(a)V\\ &= V^*\pi(a)^*(id_{\widehat{H}} - VV^*)\pi(a)V\end{aligned}$$

By Theorem 3.1.4, $\|VV^*\| = \|(\psi \circ \varphi)(1_A)\| \leq \|(\psi \circ \varphi)\| \leq \|\varphi\| \leq 1$. So, $\|V\| \leq 1$ and V is a contraction. By the proof of Theorem 2.2.5, $id_{\widehat{H}} - VV^*$ is a positive element of $B(\widehat{H})$. By Theorem 2.2.4, $\psi(\varphi(a^*a) - \varphi(a)^*\varphi(a))$ is positive. Since B is isomorphic to $\psi(B)$ as C*-algebras, we deduce that $\varphi(a^*a) - \varphi(a)^*\varphi(a) \geq 0$ in B as required.

Now assume that $a \in A$ satisfies $\varphi(a^*a) = \varphi(a)^*\varphi(a)$ and $\varphi(aa^*) = \varphi(a)\varphi(a)^*$. By the first part, this means that

$$\psi(\varphi(a^*a) - \varphi(a)^*\varphi(a)) = V^*\pi(a)^*(id_{\widehat{H}} - VV^*)\pi(a)V = 0.$$

In particular, $(id_{\widehat{H}} - VV^*)^{\frac{1}{2}}\pi(a)V = 0$. Therefore, if $b \in A$ then

$$\psi(\varphi(ab) - \varphi(a)\varphi(b)) = V^*\pi(b)(id_{\widehat{H}} - VV^*)\pi(a)V = 0.$$

Since ψ is injective, then $\varphi(ab) = \varphi(a)\varphi(b)$. By symmetry, $\varphi(ba) = \varphi(b)\varphi(a)$. This proves the bimodule property.

Before we proceed, we make the following observation. If $c \in A$ then by Theorem 3.1.4,

$$\psi(\varphi(c^*)) = V^*\pi(c^*)V = V^*\pi(c)^*V = (V^*\pi(c)V)^* = \psi(\varphi(c))^* = \psi(\varphi(c)^*).$$

Since ψ is injective, $\varphi(c^*) = \varphi(c)^*$.

Finally, assume that A_φ is the set defined in the statement of the theorem. Assume that $a, b \in A_\varphi$ and $\lambda \in \mathbb{C}$. By using the bimodule property of φ, we have

$$\varphi((\lambda a)^*\lambda a) = \varphi(|\lambda|^2 a^*a) = \lambda\overline{\lambda}\varphi(a)^*\varphi(a) = \varphi(\lambda a)^*\varphi(\lambda a),$$

$$\begin{aligned}\varphi((a+b)^*(a+b)) &= \varphi(a^*a) + \varphi(a^*b) + \varphi(b^*a) + \varphi(b^*b)\\ &= \varphi(a)^*\varphi(a) + \varphi(a)^*\varphi(b) + \varphi(b)^*\varphi(a) + \varphi(b)^*\varphi(b)\\ &= (\varphi(a)^* + \varphi(b)^*)(\varphi(a) + \varphi(b))\\ &= (\varphi(a+b))^*\varphi(a+b),\end{aligned}$$

$$\begin{aligned}\varphi((ab)^*ab) &= \varphi(b^*a^*ab)\\ &= \varphi(b^*a^*)\varphi(a)\varphi(b)\\ &= \varphi((ab)^*)\varphi(a)\varphi(b)\\ &= \varphi(ab)^*\varphi(ab)\end{aligned}$$

and

$$\varphi((a^*)^*a^*) = \varphi(a)\varphi(a)^* = \varphi(a^*)^*\varphi(a^*).$$

So, A_φ is closed under scalar multiplication, addition, multiplication and involution. In order to show that A_φ is a C*-subalgebra, it suffices to show that A_φ is closed with respect to the norm topology on A. Assume that $\{a_n\}_{n\in\mathbb{Z}_{>0}}$ is a sequence in A_φ which converges to some $a \in A$. Since φ is a c.c.p map, it is continuous. So,

$$\varphi(a^*a) = \lim_{n\to\infty} \varphi(a_n^* a_n) = \lim_{n\to\infty} \varphi(a_n)^*\varphi(a_n) = \varphi(a)^*\varphi(a).$$

Similarly, $\varphi(aa^*) = \varphi(a)\varphi(a)^*$. So, $a \in A_\varphi$ and A_φ is a closed subspace of A. Hence, A_φ is a C*-subalgebra of A. This completes the proof. $\square$

Definition 3.2.1. Let A and B be C*-algebras and $\varphi : A \to B$ be a c.c.p map. Let A_φ denote the C*-subalgebra of A defined in Theorem 3.2.1. Then, A_φ is called the **multiplicative domain** of φ.

By the bimodule property of c.c.p maps, A_φ is the largest C*-subalgebra where φ restricts to a *-homomorphism.

We now turn to an important class of c.c.p maps.

Definition 3.2.2. Let A and B be C*-algebras with $B \subsetneq A$. A **conditional expectation** from A onto B is a c.c.p map $E : A \to B$ satisfying the following two properties:

1. If $b \in B$ then $E(b) = b$.
2. If $x \in A$ and $b, b' \in B$ then $E(bxb') = bE(x)b'$.

The second property of a conditional expectation tells us that conditional expectations are B-bimodule homomorphisms. In the next theorem, we prove an important characterisation of conditional expectations. First, we make a definition.

Definition 3.2.3. Let A be a C*-algebra and (π, H) be the universal representation of A. The **enveloping von Neumann algebra** of A is the double commutant $\pi(A)''$.

The reason why the enveloping von Neumann algebra is useful is due to the following theorem (see [BO08, Theorem 1.4.1]):

Theorem 3.2.2. *Let A be a C*-algebra and E be the enveloping von Neumann algebra of A. Let A^{**} be the Banach space double dual of A. Then, E is isometrically isomorphic to A^{**}.*

Recall that A embeds into its Banach space double dual A^{**} and that $A = A^{**}$ if and only if A is a reflexive Banach space.

Theorem 3.2.3. *Let A and B be unital C*-algebras such that $B \subsetneq A$. Let $E : A \to B$ be a linear map such that if $b \in B$ then $E(b) = b$. Then, the following are equivalent:*

1. *E is a conditional expectation.*
2. *E is a c.c.p map.*
3. *E is contractive.*

Proof. Assume that A and B are unital C*-algebras with $B \subsetneq A$. Assume that $E : A \to B$ is a linear map such that if $b \in B$ then $E(b) = b$. It suffices to prove that if E is contractive then E is a conditional expectation.

To show: (a) If E is contractive then E is a conditional expectation.

(a) Assume that $E : A \to B$ is contractive. We can pass to the double dual map $E^{**} : A^{**} \to B^{**}$ and without loss of generality, assume that A and B are von Neumann algebras.

To show: (aa) E is a B-bimodule map.

(ab) E is a completely positive map.

(aa) Since B was assumed to be a von Neumann algebra, it is the closed linear span of its projections (in the norm topology). Thus, it suffices to check that E is a B-bimodule map on the projections of B. Let $p \in B$ be a projection and $p^\perp = 1_A - p$. By our assumption on E, if $x \in A$ then

$$E(pE(p^\perp x)) = pE(p^\perp x).$$

So, if $t \in \mathbb{R}$ then

$$\begin{aligned}
(1+t)^2\|pE(p^\perp x)\|^2 &= \|pE(p^\perp x) + tpE(p^\perp x)\|^2 \\
&= \|pE(p^\perp x) + tppE(p^\perp x)\|^2 \\
&= \|pE(p^\perp x) + pE(tpE(p^\perp x))\|^2 \\
&= \|pE(p^\perp x + tpE(p^\perp x))\|^2 \\
&\leq \|p^\perp x + tpE(p^\perp x\|^2 \\
&\leq \|p^\perp x\|^2 + t^2\|pE(p^\perp x)\|^2.
\end{aligned}$$

The first inequality follows from the fact that $\|p\| \leq 1$ and E is contractive. So, if $t \in \mathbb{R}$ then

$$(1+2t)\|pE(p^\perp x)\|^2 \leq \|p^\perp x\|^2$$

and subsequently, $pE(p^\perp x) = 0$. In particular, if $p = 1_B$ (the multiplicative unit of B) then $E(1_B^\perp x) = 1_B E(1_B^\perp x) = 0$.

Now observe that if $x \in A$ and $p \in B$ is a projection then

$$\begin{aligned} 0 &= (1_B - p)E((1_B - p)^\perp x) \\ &= (1_B - p)E((1_A - 1_B + p)x) \\ &= (1_B - p)\big(E((1_A - 1_B)x) + E(px)\big) \\ &= (1_B - p)\big(E(1_B^\perp x) + E(px)\big) \\ &= (1_B - p)E(px) \end{aligned}$$

and consequently,

$$E(px) = 1_B E(px) = pE(px) = pE(x - p^\perp x) = pE(x) - pE(p^\perp x) = pE(x).$$

By repeating the same argument with $xp^\perp$ rather than $p^\perp x$, we also find that $E(xp) = E(x)p$. Therefore, E is a B-bimodule map as required.

(ab) First we note that E is unital. If $b \in B$ then

$$bE(1_A) = E(b) = b$$

because E is a B-bimodule map by part (aa). Since E is unital and contractive, it must be positive. To see that E is a c.p map, let $(x_{ij}) \in M_{n\times n}(A)$ be a positive element. Let (π, H) be a representation of B with cyclic vector ξ (for instance, a GNS representation). If $b_1, b_2, \ldots, b_n \in B$ then

$$\begin{aligned} \sum_{i,j} \langle \pi(E(x_{ij}))\pi(b_j)\xi, \pi(b_i)\xi\rangle &= \langle \sum_{i,j} \pi(b_i^*)\pi(E(x_{ij}))\pi(b_j)\xi, \xi\rangle \\ &= \langle \pi(\sum_{i,j} b_i^* E(x_{ij})b_j)\xi, \xi\rangle \\ &= \langle \pi(E(\sum_{i,j} b_i^* x_{ij} b_j))\xi, \xi\rangle \geq 0 \end{aligned}$$

because $\sum_{i,j} b_i^* x_{ij} b_j$ is a positive element of A. So, the matrix $\big(\pi(E(x_{ij}))\big) \geq 0$ in $M_{n\times n}(\pi(B))$. Since π is an arbitrary cyclic representation of B, we deduce that $(E(x_{ij}))$ is a positive element of

$M_{n\times n}(B)$. So, E is a c.p map.

(a) By parts (aa) and (ab), we deduce that E is a conditional expectation as required. □

Theorem 3.2.3 is attributed to Tomiyama. We also have the following useful result from [BO08, Lemma 1.5.11].

Theorem 3.2.4. *Let M be a von Neumann algebra and $\tau : M \to \mathbb{C}$ be a faithful normal tracial state. Let $N \subseteq M$ be a von Neumann subalgebra such that $1_M \in N$. Then, there exists a unique conditional expectation $E : M \to N$ which is trace-preserving and normal.*

We omit the proof because it relies on material we have not covered in these notes.

3.3 Matrix algebras and Arveson's extension theorem

There are useful bijective correspondences involving completely positive maps to and from a matrix algebra. In this section, we will prove some of these correspondences and end with the proof of Arveson's extension theorem, which can be thought of as an analogue of the well-known Hahn-Banach theorem for c.c.p maps.

Theorem 3.3.1. *Let A be a C*-algebra and $\{e_{i,j}\}_{i,j\in\{1,2,\ldots,n\}}$ be the matrix units of $M_{n\times n}(\mathbb{C})$. Then, the following map is bijective:*

$$\begin{array}{rccc} \mathcal{C}: & CP(M_{n\times n}(\mathbb{C}), A) & \to & M_{n\times n}(A)_+ \\ & \varphi & \mapsto & \big(\varphi(e_{i,j})\big) \end{array}$$

Proof. Assume that A is a C*-agebra and that $\{e_{i,j}\}_{i,j\in\{1,2,\ldots,n\}}$ is the set of matrix units of $M_{n\times n}(\mathbb{C})$. Assume that the map $\mathcal{C}$ is defined as above.

To show: (a) $\mathcal{C}$ is well-defined.

(a) Assume that $\varphi \in CP(M_{n\times n}(\mathbb{C}), A)$. The matrix $(e_{i,j}) \in M_{n\times n}(M_{n\times n}(\mathbb{C}))$ is positive because

$$\big(e_{i,j}\big) = \begin{pmatrix} e_{1,1} & e_{1,2} & \ldots & e_{1,n} \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix}^* \begin{pmatrix} e_{1,1} & e_{1,2} & \ldots & e_{1,n} \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix}.$$

Since φ is completely positive, the matrix $(\varphi(e_{i,j}))$ is a positive element of $M_{n\times n}(A)$. So, $\mathcal{C}$ is well-defined.

To show: (b) $\mathcal{C}$ is injective.

(c) $\mathcal{C}$ is surjective.

(b) Assume that $\varphi, \psi \in CP(M_{n\times n}(\mathbb{C}), A)$ such that $\mathcal{C}(\varphi) = \mathcal{C}(\psi)$. If $i, j \in \{1, 2, \ldots, n\}$ then $\varphi(e_{i,j}) = \psi(e_{i,j})$. By linearity, we find that if $B \in M_{n\times n}(\mathbb{C})$ then $\varphi(B) = \psi(B)$. So, $\varphi = \psi$ and $\mathcal{C}$ is injective.

(c) Assume that $X = (x_{i,j}) \in M_{n\times n}(A)_+$. Define a linear map ψ by

$$\begin{array}{rccc} \psi: & M_{n\times n}(\mathbb{C}) & \to & A \\ & e_{i,j} & \mapsto & x_{i,j} \end{array}$$

Then, $(\psi(e_{i,j})) = X$. We will now show that ψ is completely positive. Since X is a positive element of $M_{n\times n}(A)$, let $X^{\frac{1}{2}} = (b_{i,j}) \in M_{n\times n}(A)$ denote the square root of X. Then,

$$x_{i,j} = \varphi(e_{i,j}) = \sum_{k=1}^{n} b_{k,i}^* b_{k,j}.$$

Now let (π, H) be a faithful representation of A. If $n \in \mathbb{Z}_{>0}$ then let $\{\xi_1, \ldots, \xi_n\}$ be the standard orthonormal basis for the Hilbert space $\mathbb{C}^n$. Define the operator V by

$$\begin{array}{rccc} V: & H & \to & \mathbb{C}^n \otimes \mathbb{C}^n \otimes H \\ & \xi & \mapsto & \sum_{j,k=1}^{n} \xi_j \otimes \xi_k \otimes \pi(b_{k,j})\xi \end{array}$$

If $T = (t_{i,j}) \in M_{n\times n}(\mathbb{C})$ and $\xi, \eta \in H$ then

$$\begin{aligned}
\langle V^*(T\otimes id_{\mathbb{C}^n}\otimes id_H)V\eta,\xi\rangle &= \langle (T\otimes id_{\mathbb{C}^n}\otimes id_H)V\eta, V\xi\rangle \\
&= \langle \sum_{j,k=1}^n T\xi_j\otimes\xi_k\otimes\pi(b_{k,j})\eta, \sum_{\ell,m=1}^n \xi_\ell\otimes\xi_m\otimes\pi(b_{m,\ell})\xi\rangle_{\mathbb{C}^n\otimes\mathbb{C}^n\otimes H} \\
&= \sum_{j,k,\ell,m=1}^n \langle T\xi_j,\xi_\ell\rangle\langle\xi_k,\xi_m\rangle\langle\pi(b_{k,j})\eta,\pi(b_{m,\ell})\xi\rangle \\
&= \sum_{j,k,\ell=1}^n \langle T\xi_j,\xi_\ell\rangle\langle\pi(b_{k,\ell}^*b_{k,j})\eta,\xi\rangle \\
&= \sum_{j,\ell=1}^n t_{\ell,j}\langle\sum_{k=1}^n \pi(b_{k,\ell}^*b_{k,j})\eta,\xi\rangle \\
&= \sum_{j,\ell=1}^n t_{\ell,j}\langle\pi(\psi(e_{\ell,j}))\eta,\xi\rangle = \langle\pi(\psi(\sum_{j,\ell=1}^n t_{\ell,j}e_{\ell,j}))\eta,\xi\rangle \\
&= \langle\pi(\psi(T))\eta,\xi\rangle.
\end{aligned}$$

Therefore, if $T\in M_{n\times n}(\mathbb{C})$ then

$$V^*(T\otimes id_{\mathbb{C}^n}\otimes id_H)V = \pi(\psi(T)).$$

So, $\pi\circ\psi$ is completely positive. Since $\pi: A\to B(H)$ is injective, we deduce that ψ is a completely positive map.

By parts (b) and (c), $\mathcal{C}$ is a bijection. $\square$

Example 3.3.1. Let A be a C*-algebra and $n\in\mathbb{Z}_{>0}$. Let $a_1,\ldots,a_n\in A$. If $i,j\in\{1,2,\ldots,n\}$ then define the linear map

$$\begin{array}{rccc}\varphi: & M_{n\times n}(\mathbb{C}) & \to & A \\ & e_{i,j} & \mapsto & a_ia_j^*.\end{array}$$

We compute directly that the matrix

$$\begin{aligned}
\big(\varphi(e_{i,j})\big) &= \begin{pmatrix} a_1a_1^* & a_1a_2^* & \ldots & a_1a_n^* \\ a_2a_1^* & a_2a_2^* & \ldots & a_2a_n^* \\ \vdots & \vdots & \ddots & \vdots \\ a_na_1^* & a_na_2^* & \ldots & a_na_n^* \end{pmatrix} \\
&= \begin{pmatrix} a_1 & 0 & \ldots & 0 \\ a_2 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_n & 0 & \ldots & 0 \end{pmatrix} \begin{pmatrix} a_1 & 0 & \ldots & 0 \\ a_2 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_n & 0 & \ldots & 0 \end{pmatrix}^* \geq 0.
\end{aligned}$$

By Theorem 3.3.1, we deduce that φ is a completely positive map.

We also have a bijective correspondence for completely positive maps from a C*-algebra to a matrix algebra.

Theorem 3.3.2. *Let A be a unital C*-algebra. Then, the following map is bijective:*

$$\begin{array}{cccc} \mathcal{D}: & CP(A, M_{n\times n}(\mathbb{C})) & \to & M_{n\times n}(A)^*_+ \\ & \varphi & \mapsto & \big(\, (a_{i,j}) \mapsto \sum_{i,j=1}^n (\varphi(a_{i,j}))_{i,j} \,\big) \end{array}$$

Proof. Assume that A is a unital C*-algebra and that the map $\mathcal{D}$ is defined as above.

To show: (a) The map $\mathcal{D}$ is well-defined.

(a) Assume that $\varphi : A \to M_{n\times n}(\mathbb{C})$ is a completely positive map and that $\widehat{\varphi} = \mathcal{D}(\varphi)$. It is straightforward to verify that $\widehat{\varphi}$ is a linear functional on $M_{n\times n}(A)$. To see that $\widehat{\varphi}$ is positive, let $\{\xi_1, \ldots, \xi_n\}$ be the standard orthonormal basis for $\mathbb{C}^n$. Let $\xi = [\xi_1, \ldots, \xi_n]^T \in (\mathbb{C}^n)^n$. If $(a_{i,j}) \in M_{n\times n}(A)$ and φ_n is the inflation of φ then

$$\begin{aligned} \langle \varphi_n((a_{i,j}))\xi, \xi\rangle &= \langle [\sum_{k=1}^n \varphi_n((a_{i,j}))_{1,k}\xi_k, \ldots, \sum_{k=1}^n \varphi_n((a_{i,j}))_{n,k}\xi_k]^T, \xi\rangle \\ &= \langle [\sum_{k=1}^n \varphi(a_{1,k})_{1,k}\xi_k, \ldots, \sum_{k=1}^n \varphi_n(a_{n,k})_{n,k}\xi_k]^T, \xi\rangle \\ &= \sum_{\ell,k=1}^n \varphi(a_{\ell,k})_{\ell,k} = \widehat{\varphi}((a_{i,j})). \end{aligned}$$

Since φ is a c.p map, the inflation φ_n is a positive map. So, if $(a_{i,j}) \in M_{n\times n}(A)$ then $\widehat{\varphi}((a_{i,j})) \geq 0$ and consequently, $\widehat{\varphi}$ is a positive linear functional. Hence, $\mathcal{D}$ is well-defined.

To show: (b) $\mathcal{D}$ is injective.

(c) $\mathcal{D}$ is surjective.

(b) Assume that $\varphi, \psi \in CP(A, M_{n\times n}(\mathbb{C}))$ such that $\widehat{\varphi} = \widehat{\psi}$. If $(a_{i,j}) \in M_{n\times n}(A)$ then by part (a),

$$\langle \varphi_n((a_{i,j}))\xi, \xi\rangle = \sum_{i,j=1}^{n} (\varphi(a_{i,j}))_{i,j} = \sum_{i,j=1}^{n} (\psi(a_{i,j}))_{i,j} = \langle \psi_n((a_{i,j}))\xi, \xi\rangle.$$

So, $\langle (\varphi_n - \psi_n)((a_{i,j}))\xi, \xi\rangle = 0$. If $n = 1$ and $a \in A$ then $\langle (\varphi - \psi)(a)1, 1\rangle = 0$ (the inner product is on $\mathbb{C}$) and consequently, $\varphi(a) = \psi(a)$. So, $\mathcal{D}$ is injective.

(c) Assume that $\delta : M_{n\times n}(A) \to \mathbb{C}$ is a positive linear functional. Since A is a unital C*-algebra, we let $\{e_{i,j}\}_{i,j\in\{1,2,\ldots,n\}}$ be the matrix units on $M_{n\times n}(A)$. Define the map $\phi_\delta : A \to M_{n\times n}(\mathbb{C})$ by

$$\big(\phi_\delta(a)\big)_{i,j} = \delta(a \cdot e_{i,j}).$$

We compute directly that if $(a_{i,j}) \in M_{n\times n}(A)$ then

$$\begin{aligned}
\mathcal{D}(\phi_\delta)((a_{i,j})) &= \sum_{i,j=1}^{n} (\phi_\delta(a_{i,j}))_{i,j} \\
&= \sum_{i,j=1}^{n} (\phi_\delta(a_{i,j}))_{i,j} \\
&= \sum_{i,j=1}^{n} \delta(a_{i,j} \cdot e_{i,j}) = \delta((a_{i,j})).
\end{aligned}$$

It remains to show that ϕ_δ is a c.p map. Let $(\pi_\delta, H_\delta, \xi_\delta)$ be the GNS representation associated to δ. Reusing the standard orthonormal basis $\{\xi_1, \ldots, \xi_n\}$ for $\mathbb{C}^n$ from part (a), define the operator V by

$$\begin{array}{rccc} V: & \mathbb{C}^n & \to & H \\ & \xi_i & \mapsto & \pi_\delta(e_{1,i})\xi_\delta \end{array}$$

Of course, V is extended to a linear map on all of $\mathbb{C}^n$. If $j, k \in \{1, 2, \ldots, n\}$ and $a \in A$ then

$$\begin{aligned}
\langle V^*\pi_\delta(diag[a, a, \ldots, a])V\xi_j, \xi_k\rangle &= \langle \pi_\delta(diag[a, a, \ldots, a])V\xi_j, V\xi_k\rangle \\
&= \langle \pi_\delta(diag[a, a, \ldots, a])\pi_\delta(e_{1,j})\xi_\delta, \pi_\delta(e_{1,k})\xi_\delta\rangle \\
&= \langle \pi_\delta(e_{k,1}diag[a, a, \ldots, a]e_{1,j})\xi_\delta, \xi_\delta\rangle \\
&= \langle \pi_\delta(a \cdot e_{k,j})\xi_\delta, \xi_\delta\rangle \\
&= \delta(a \cdot e_{k,j}) = \big(\phi_\delta(a)\big)_{k,j} \\
&= \langle \phi_\delta(a)\xi_j, \xi_k\rangle.
\end{aligned}$$

By linearity, we find that

$$\phi_\delta = V^* \pi_\delta(diag[a, \dots, a])V.$$

Therefore, ϕ_δ is a c.p map and $\mathcal{D}$ is surjective.

By parts (b) and (c), we find that $\mathcal{D}$ is a bijection as required. $\square$

We will now work towards Arveson's extension theorem by proving a few extension related results.

Theorem 3.3.3. *Let A be a unital C^*-algebra and $E \subseteq A$ be an operator system. Let $\psi : E \to \mathbb{C}$ be a positive linear functional. Then, $\|\psi\| = \psi(1_A)$.*

Proof. Assume that A is a unital C*-algebra and that $E \subseteq A$ is an operator system. Assume that $\psi : E \to \mathbb{C}$ is a positive linear functional and $\epsilon \in \mathbb{R}_{>0}$. Pick $x \in E$ such that $\|x\| \leq 1$ and

$$\|\psi\| - \epsilon < |\psi(x)|.$$

Multiplying by a complex scalar of norm 1, we may assume that $\psi(x) \in \mathbb{R}_{>0}$. Since ψ is positive, it is self-adjoint. So,

$$\psi(x) = \frac{1}{2}\psi(x + x^*).$$

Thus, we may assume that x is self-adjoint. By the proof of Theorem 2.2.5, $x \leq \|x\|1_A$ and consequently, $\psi(x) \leq \|x\|\psi(1_A)$. By taking the supremum over all $x \in E$ with $\|x\| \leq 1$, we find that $\|\psi\| \leq \psi(1_A)$ and consequently, $\psi(1_A) = \|\psi\|$. $\square$

A useful consequence of Theorem 3.3.3 is that if there exists an extension $\psi' : A \to \mathbb{C}$ such that $\psi'|_E = \psi$ and $\|\psi'\| = \|\psi\|$ then

$$\psi'(1_A) = \psi(1_A) = \|\psi\| = \|\psi'\|.$$

If $a \in A$ then

$$\psi'(a^*a) = \|a\|^2\psi'(\frac{a^*a}{\|a\|^2})$$

By Theorem 2.3.7, the norm-preserving extension ψ' is also a positive linear functional.

Theorem 3.3.4. *Let A be a unital C^*-algebra and $E \subseteq A$ be an operator system. Let $\varphi : E \to M_{n\times n}(\mathbb{C})$ be a c.p map. Then, φ extends to a c.p map $\varphi' : A \to M_{n\times n}(\mathbb{C})$.*

Proof. Assume that A is a unital C*-algebra and $E \subseteq A$ is an operator system. Assume that $\varphi : E \to M_{n\times n}(\mathbb{C})$ is a c.p map. By a similar argument to Theorem 3.3.2, we can define the positive linear functional $\widehat{\varphi} : M_{n\times n}(E) \to \mathbb{C}$.

By the Hahn-Banach extension theorem, we can extend $\widehat{\varphi}$ to a linear functional $\widehat{\varphi}' : M_{n\times n}(A) \to \mathbb{C}$ such that $\|\widehat{\varphi}'\| = \|\widehat{\varphi}\|$. By Theorem 3.3.3, $\widehat{\varphi}'$ is a positive linear functional.

Finally, by Theorem 3.3.2, we obtain a c.p map $\varphi' : A \to M_{n\times n}(\mathbb{C})$ which extends φ. □

The proof of Arveson's extension theorem relies on an application of the Banach-Alaoglu theorem, which we will make clear by following [Pau02, Chapter 7].

Recall the ultraweak topology as defined in Definition 2.5.4. We want to understand the ultraweak topology in the special case where $Y = B_1(H)$, where H is a Hilbert space and $B_1(H)$ is the Banach algebra of trace class operators on H. Recall that $B_1(H)^* \cong B(H)$ from Theorem 2.5.2. Hence, if X is a Banach space then we can endow $B(X, B(H))$ with the ultraweak topology.

Theorem 3.3.5. *Let X be a Banach space and H be a Hilbert space. Then, a bounded net $\{L_\lambda\}_{\lambda\in\Lambda}$ converges in the ultraweak topology to L if and only if for $\xi, \mu \in H$ and $x \in X$*

$$\langle L_\lambda(x)\xi, \mu\rangle \quad \textit{converges to} \quad \langle L(x)\xi, \mu\rangle.$$

Proof. Assume that X is a Banach space and H is a Hilbert space. Let $\{L_\lambda\}_{\lambda\in\Lambda}$ be a bounded net in $B(X, B(H))$. By Theorem 2.5.4, $\{L_\lambda\}_{\lambda\in\Lambda}$ converges in the ultraweak topology on $B(X, B(H))$ to L if and only if for $x \in X$, $L_\lambda(x)$ converges weakly to $L(x)$.

We recall the following bijective isometry from $B(H)$ to $B_1(H)^*$:

$$\begin{array}{rccc} \psi : & B(H) & \to & B_1(H)^* \\ & y & \mapsto & (x \mapsto Tr(yx)) \end{array}$$

Here, Tr refers to the trace on $B(H)_+$. Hence, $\{L_\lambda\}_{\lambda\in\Lambda}$ converges in the ultraweak topology to L if and only if for $x \in X$ and $T \in B_1(H)$, $\psi(L_\lambda(x))(T) = Tr(L_\lambda(x)T)$ converges to $\psi(L(x))(T) = Tr(L(x)T)$. Now recall that the Banach space of trace class operators $B_1(H)$ is the closure of the linear span of rank-one operators in $B(H)$. That is,

$$B_1(H) = \overline{span_{\mathbb{C}}\{|\xi\rangle\langle\mu| \mid \xi, \mu \in H\}}.$$

Since the net $\{L_\lambda\}_{\lambda\in\Lambda}$ is bounded, it is enough to consider the case where $T = |\xi\rangle\langle\mu|$ where $\xi, \mu \in H$. In this case, if $\{\psi_n\}_{n\in\mathbb{Z}_{>0}}$ is an orthonormal basis for H then

$$\begin{aligned} Tr(L_\lambda(x)|\xi\rangle\langle\mu|) &= \sum_{n=1}^{\infty}\langle L_\lambda(x)|\xi\rangle\langle\mu|\varphi_n, \varphi_n\rangle \\ &= \sum_{n=1}^{\infty}\langle L_\lambda(x)\langle\varphi_n, \mu\rangle\xi, \varphi_n\rangle \\ &= \langle L_\lambda(x)\xi, \sum_{n=1}^{\infty}\langle\mu, \varphi_n\rangle\varphi_n\rangle \\ &= \langle L_\lambda(x)\xi, \mu\rangle. \end{aligned}$$

Therefore, $\{L_\lambda\}$ converges in the ultraweak topology to L if and only if for $x \in X$ and $\xi, \mu \in H$, $\langle L_\lambda(x)\xi, \mu\rangle$ converges to $\langle L(x)\xi, \mu\rangle$. □

For the proof of Arveson's extension theorem, we also need the following result about the closed unit ball of $B(X, B(H))$.

Theorem 3.3.6. *Let X be a Banach space and H be a Hilbert space. Let*

$$\mathcal{B} = \{\varphi \in B(X, B(H)) \mid \|\varphi\| \leq 1\}.$$

Then, $\mathcal{B}$ is a compact subset of $B(X, B(H))$ with respect to the ultraweak topology on $B(X, B(H))$.

Proof. Assume that X is a Banach space and H is a Hilbert space. Assume that $\mathcal{B}$ is the unit ball defined as above. By the isometric isomorphism in equation (2.2),

$$ev(\mathcal{B}) = \{\psi \in Z^* \mid \|\psi\| \leq 1\}$$

where Z is the Banach space such that $ev : B(X, B(H)) \to Z^*$ is an isometric isomorphism of Banach spaces. By the Banach-Alaoglu theorem, $ev(\mathcal{B})$ is a compact subset of Z^* with respect to the weak-* topology on Z^*. By applying the continuous inverse ev^{-1}, we deduce that

$$\mathcal{B} = ev^{-1}(ev(\mathcal{B}))$$

is compact with respect to the ultraweak topology on $B(X, B(H))$, since it is the continuous image of a compact set. $\square$

Now we embark on the proof of the fundamental Arveson's extension theorem.

Theorem 3.3.7 (Arveson's extension theorem). *Let A be a unital C*-algebra and $E \subseteq A$ be an operator system. Let H be a Hilbert space. If $\varphi : E \to B(H)$ is a c.c.p map then there exists a c.c.p map $\tilde{\varphi} : A \to B(H)$ such that $\tilde{\varphi}|_E = \varphi$.*

Proof. Assume that A is a unital C*-algebra. Assume that $E \subseteq A$ is an operator system. Assume that H is a Hilbert space and $\varphi : E \to B(H)$ is a c.c.p map.

Let $\{P_i\}_{i \in I} \subseteq B(H)$ be an increasing net of finite-rank projections which converge to id_H in the strong operator topology. Define the map

$$\begin{array}{rccc} \varphi_i : & E & \to & P_i B(H) P_i \subsetneq B(H) \\ & e & \mapsto & P_i \varphi(e) P_i. \end{array}$$

Since P_i is a finite-rank projection, we can regard φ_i as a c.p map which takes values in a matrix algebra. By Theorem 3.3.4, we may assume that if $i \in I$ then φ_i is defined on all of A.

Now regard φ_i as taking values in $B(H)$. By Theorem 3.3.6, the unit ball in $B(A, B(H))$ is compact with respect to the ultraweak topology on $B(A, B(H))$. Since $\{\varphi_i\}_{i \in I}$ is a net in the unit ball of $B(A, B(H))$, it converges in the ultraweak topology to an accumulation point $\tilde{\varphi}$ in the unit ball of $B(A, B(H))$. By Theorem 3.3.5, if $a \in A$ and $\xi, \mu \in H$ then

$$\langle \varphi_i(a)\xi, \mu \rangle \quad \text{converges to} \quad \langle \tilde{\varphi}(a)\xi, \mu \rangle.$$

To see that $\tilde{\varphi}$ extends φ, assume that $e \in E$. If $\xi, \mu \in H$ then

$$\langle \varphi(e)\xi, \mu \rangle = \langle \lim_i P_i \varphi(e) P_i \xi, \mu \rangle = \lim_i \langle \varphi_i(e)\xi, \mu \rangle = \langle \tilde{\varphi}(e)\xi, \mu \rangle$$

The first equality follows from the fact that $\{P_i\}_{i \in I}$ is an increasing net of finite-rank projections which converge to id_H in the strong operator topology on $B(H)$. Hence, $\tilde{\varphi}|_E = \varphi$.

Finally, to see that $\tilde{\varphi}$ is a c.c.p map, first recall that $\tilde{\varphi}$ is an element of the unit ball of $B(A, B(H))$. So, $\|\tilde{\varphi}\| \leq 1$. Now assume that

$Y = (y_{i,j}) \in M_{n\times n}(A)$ so that $\tilde{\varphi}(Y) = (\tilde{\varphi}(y_{i,j})) \in M_{n\times n}(B(H))$.

Recall that we have the canonical *-isomorphism ψ from $M_{n\times n}(B(H))$ to $B(H^{(n)})$, where

$$\begin{array}{rccc} \psi(\tilde{\varphi}(Y)): & H^{(n)} & \to & H^{(n)} \\ & (h_1, \ldots, h_n) & \mapsto & \big(\sum_{j=1}^n \tilde{\varphi}(y_{1,j})(h_j), \ldots, \sum_{j=1}^n \tilde{\varphi}(y_{n,j})(h_j)\big) \end{array}$$

Using the inner product on the direct sum $H^{(n)}$, we compute directly that if $h = (h_1, \ldots, h_n) \in H^{(n)}$ then

$$\begin{aligned} \langle \psi(\tilde{\varphi}(Y))h, h\rangle_{H^{(n)}} &= \langle \big(\sum_{j=1}^n \tilde{\varphi}(y_{1,j})(h_j), \ldots, \sum_{j=1}^n \tilde{\varphi}(y_{n,j})(h_j)\big), h\rangle_{H^{(n)}} \\ &= \sum_{i,j=1}^n \langle \tilde{\varphi}(y_{i,j})(h_j), h_i\rangle = \sum_{i,j=1}^n \lim_k \langle \varphi_k(y_{i,j})(h_j), h_i\rangle \\ &= \lim_k \sum_{i,j=1}^n \langle \varphi_k(y_{i,j})(h_j), h_i\rangle \\ &= \lim_k \langle \big(\sum_{j=1}^n \varphi_k(y_{1,j})(h_j), \ldots, \sum_{j=1}^n \varphi_k(y_{n,j})(h_j)\big), h\rangle_{H^{(n)}} \\ &= \lim_k \langle \psi(\varphi_k(Y))h, h\rangle \geq 0 \end{aligned}$$

since $\psi(\varphi_k(Y))$ is a positive element of $B(H^{(n)})$. So, $\psi(\tilde{\varphi}(Y))$ is a positive element of $B(H^{(n)})$ and

$$\tilde{\varphi}(Y) = \psi^{-1}(\psi(\tilde{\varphi}(Y)))$$

is positive in $M_{n\times n}(B(H))$. Since $n \in \mathbb{Z}_{>0}$ was arbitrary, we deduce that $\tilde{\varphi}$ is a completely positive map. This completes the proof. $\square$

In the language of category theory, Arveson's extension theorem (Theorem 3.3.7) tells us that if H is a Hilbert space then $B(H)$ is an injective object in the category of operator systems with the morphisms being c.c.p maps. We finish this section with an application of Theorem 3.3.7.

Theorem 3.3.8. *Let M be a von Neumann algebra and (π, H) be a faithful representation of M. Then, M is injective in the category of operator systems with c.c.p maps as morphisms if and only if there exists a conditional expectation $E : B(H) \to M$.*

Proof. Assume that M is a von Neumann algebra and (π, H) is a faithful representation of M. Let $\mathcal{O}_{c.c.p}$ denote the category of operator systems whose morphisms are c.c.p maps. By definition of a von Neumann algebra, M is an object in the category $\mathcal{O}_{c.c.p}$.

To show: (a) If M is an injective object in $\mathcal{O}_{c.c.p}$ then there exists a conditional expectation $E : B(H) \to M$.

(b) If there exists a conditional expectation $E : B(H) \to M$ then M is an injective object in $\mathcal{O}_{c.c.p}$.

(a) Assume that M is an injective object in the category $\mathcal{O}_{c.c.p}$. We treat M as a closed subspace of $B(H)$ (it is isomorphic to $\pi(M)$, which is a closed subspace of $B(H)$ by Theorem 1.7.6). Since M is injective, the identity map $id_M : M \to M$ is a c.c.p map which extends to a c.c.p map $E : B(H) \to M$. By construction, if $m \in M$ then $E(m) = id_M(m) = m$. By Theorem 3.2.3, E is a conditional expectation as required.

(b) Assume that $E : B(H) \to M$ is a conditional expectation. Let A be a unital C*-algebra, $S \subseteq A$ be an operator system and $\varphi : S \to M$ be a c.c.p map. As in part (a), we can treat φ as a map from S to $B(H)$. By Arveson's extension theorem (Theorem 3.3.7), we can extend φ to a c.c.p map $\tilde{\varphi} : A \to B(H)$. If $s \in S$ then

$$(E \circ \tilde{\varphi})(s) = (E \circ \varphi)(s) = E(\varphi(s)) = \varphi(s).$$

The last equality follows from the assumption that E is a conditional expectation and $\varphi(s) \in M$. By Theorem 3.2.3, $E \circ \tilde{\varphi} : A \to M$ is a c.c.p map which extends φ. Therefore, M is an injective object in $\mathcal{O}_{c.c.p}$. □

As a consequence of the proof of Theorem 3.3.8, the injectivity of a von Neumann algebra M in the category $\mathcal{O}_{c.c.p}$ does not depend on the choice of faithful representation of M.

3.4 Quasicentral approximate units

So far, we have been working with approximate units of a C*-algebra or an ideal of a C*-algebra. By Theorem 2.3.3, every closed left ideal of a C*-algebra has an approximate unit. In this section, we will consider the following extension of the notion of an approximate unit:

Definition 3.4.1. Let A be a C*-algebra and I be a closed two-sided ideal of A. Let $\{u_\lambda\}_{\lambda\in\Lambda}$ be an approximate unit for I. We say that $\{u_\lambda\}$ is a **quasicentral approximate unit** for I if for $a \in A$,

$$\lim_{\lambda} \|u_\lambda a - a u_\lambda\| = 0.$$

There is a subtle difference between the definition of a quasicentral approximate unit for an ideal I and the approximate unit constructed in Theorem 2.3.3. In Theorem 2.3.3 the approximation property must hold for every element in the ideal. In Definition 3.4.1, the corresponding approximation property must hold for every element in the entire C*-algebra.

The goal of this section is to prove that every closed two-sided ideal of a C*-algebra admits a quasicentral approximate unit, as stated in [BO08, Theorem 1.2.1]. The main references we will consult for this section are [Arv77] and [Dav96]. The preliminary results we will prove in this section are from [Mur90].

It is worth noting that in both [Dav96] and [BO08], all ideals are assumed to be closed and two-sided. This was stated in [Dav96, Section I.5] and [BO08, Section 1.1] respectively. For clarity, we will still refer to them as closed two-sided ideals in this section.

We begin from [Mur90, Page 91]. Let X be a compact Hausdorff space and $Cts_{\mathbb{R}}(X, \mathbb{R})$ denote the *real* Banach space of real-valued continuous functions on X. As usual, the operations on $Cts_{\mathbb{R}}(X, \mathbb{R})$ are defined pointwise and the norm is the supremum norm.

Let $M_r(X)$ denote the set of real-valued measures on X. A consequence of the Riesz-Markov-Kakutani representation theorem states that the following map is an isometric isomorphism

$$\begin{array}{ccc} M_r(X) & \to & Cts_{\mathbb{R}}(X, \mathbb{R})^* \\ \mu & \mapsto & \Big(f \mapsto \int f \, \mathrm{d}\mu\Big). \end{array}$$

Note that if $\mu \in M_r(X)$ then the norm on μ is the *total variation* of μ, usually written as $\|\mu\| = |\mu|(X)$. Surjectivity of the above map arises from the Riesz-Markov-Kakutani representation theorem. See [Rud87, Theorem 6.19] for the complete statement and proof of this theorem. For a reference on the Riesz-Markov-Kakutani representation theorem, see [Rud87, Theorem 2.14].

Another result we will need from measure theory is the *Jordan decomposition theorem*, which states that if μ is a real-valued measure on X then there exists positive measures μ^+ and μ^- on X such that

$$\mu = \mu^+ - \mu^-$$

and $\|\mu\| = \|\mu^+\| + \|\mu^-\|$. Combining this with the Riesz-Markov-Kakutani representation theorem, we obtain the following result about linear functionals.

Theorem 3.4.1. *Let X be a compact Hausdorff space and $\tau : Cts_{\mathbb{R}}(X, \mathbb{R}) \to \mathbb{R}$ be a bounded real-linear functional. Then, there exists positive bounded real-linear functionals τ_+ and τ_- on $Cts_{\mathbb{R}}(X, \mathbb{R})$ such that*

$$\tau = \tau_+ - \tau_- \qquad \textit{and} \qquad \|\tau\| = \|\tau_+\| + \|\tau_-\|.$$

Our first goal is to prove a version of Theorem 3.4.1 for C*-algebras. So, let A be a C*-algebra and $\tau \in A^*$. We claim that

$$\|\tau\| = \sup_{\|a\| \leq 1} |Re(\tau(a))|. \tag{3.3}$$

First, note that if $a \in A$ and $\|a\| \leq 1$ then

$$|Re(\tau(a))| \leq |\tau(a)| \leq \|\tau\|.$$

On the other hand, there exists $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$ and $\lambda\tau(a) = \tau(\lambda a) \in \mathbb{R}$. So,

$$|\tau(a)| = |\tau(\lambda a)| = |Re(\tau(\lambda a))| \leq \sup_{\|a\| \leq 1} |Re(\tau(a))|.$$

Hence, the equality in equation (3.3) is proved. If $\tau \in A^*$ then we define the linear functional $\tau^* \in A^*$ by

$$\begin{array}{rccc} \tau^* : & A & \to & \mathbb{C} \\ & a & \mapsto & \overline{\tau(a^*)}. \end{array} \tag{3.4}$$

It is straightforward to verify that the map $\tau \mapsto \tau^*$ is conjugate-linear, $\tau^{**} = \tau$ and $\|\tau^*\| = \|\tau\|$.

Definition 3.4.2. Let A be a C*-algebra and $\tau : A \to \mathbb{C}$ be a bounded linear functional. We say that τ is **self-adjoint** if $\tau = \tau^*$ where τ^* is defined by equation (3.4).

If $\tau \in A^*$ then let

$$\tau_1 = \frac{1}{2}(\tau + \tau^*) \qquad \text{and} \qquad \tau_2 = \frac{1}{2i}(\tau - \tau^*).$$

Then, τ_1 and τ_2 are self-adjoint bounded linear functionals on A satisfying $\tau = \tau_1 + i\tau_2$.

Now observe that $\tau \in A^*$ is self-adjoint if and only if $\tau(A_{sa}) \subseteq \mathbb{R}$. Hence, the restriction

$$\begin{array}{rccc} \tau' = \tau|_{A_{sa}} : & A_{sa} & \to & \mathbb{R} \\ & a & \mapsto & \tau(a) \end{array}$$

defines a bounded real-linear functional on A_{sa}. Moreover, $\|\tau\| = \|\tau'\|$ because by equation (3.3), Theorem 2.3.8 and Theorem 3.4.1,

$$\begin{aligned}
\|\tau\| &= \sup_{\|a\|\leq 1} |Re(\tau(a))| \\
&= \sup_{\|a\|\leq 1} \left|\frac{1}{2}(\tau(a) + \overline{\tau(a)})\right| \\
&= \sup_{\|a\|\leq 1} \left|\frac{1}{2}(\tau(a) + \tau(a^*))\right| \quad \text{(by Theorem 2.3.8 and Theorem 3.4.1)} \\
&= \sup_{\|a\|\leq 1} \left|\tau(\frac{1}{2}(a + a^*))\right| \\
&\leq \sup_{\|b\|\leq 1,\, b\in A_{sa}} |\tau(b)| \\
&= \|\tau'\| \leq \|\tau\|.
\end{aligned}$$

If A is a C*-algebra then we define A^*_{sa} to be the set of self-adjoint functionals in A^*. We also define A^*_+ to be the set of positive functionals in A^*.

Now, we will define some temporary notation in preparation for the next theorem, analogously to [Mur90, Page 92]. If X is a real-linear Banach space then its dual space over the field $\mathbb{R}$ will be denoted by $X^\natural$. We observe that

1. If A is a C*-algebra then A_{sa} is a real-linear Banach space.

2. The space A^*_{sa} of self-adjoint functionals in A^* is a real-linear vector subspace of A^*.

3. The map given by

$$\begin{array}{rccc} r: & A^*_{sa} & \to & A^\natural_{sa} \\ & \tau & \mapsto & \tau' = \tau|_{A_{sa}} \end{array}$$

is an isometric real-linear isomorphism with inverse given by

$$\begin{array}{rccc} r^{-1}: & A^\natural_{sa} & \to & A^*_{sa} \\ & \alpha & \mapsto & \big(a \mapsto \alpha(\frac{1}{2}(a+a^*)) + i\alpha(\frac{1}{2i}(a-a^*))\big) \end{array}$$

The above statements are not too difficult to show.

Theorem 3.4.2. *Let A be a non-zero C*-algebra and $a \in A$ be normal. Then, there exists a state $\tau : A \to \mathbb{C}$ such that $|\tau(a)| = \|a\|$.*

Proof. Assume that A is a non-zero C*-algebra. Assume that $a \in A$ is normal. Let $\tilde{A}$ be the unitization of A and B the C*-algebra generated by the set $\{1_{\tilde{A}}, a\}$. By Theorem 1.2.6, $\|a\| \in \sigma(a)$. Since B is commutative and unital then we can use Theorem 1.3.4 to obtain $\tau_2 \in \mathcal{M}(B)$ such that

$$|\tau_2(a)| = \|a\|.$$

By the Hahn-Banach extension theorem, there exists a bounded linear functional $\tau_1 : \tilde{A} \to \mathbb{C}$ such that $\|\tau_1\| = 1$ and $\tau_1|_B = \tau_2$. Now observe that $\tau_1(1_{\tilde{A}}) = \tau_2(1_{\tilde{A}}) = 1$. By Theorem 2.3.7, τ_1 is a positive linear functional on A.

Now define $\tau = \tau_1|_A$. Then, τ is a positive linear functional on A satisfying $|\tau(a)| = \|a\|$. This means that

$$\|\tau\|\|a\| \geq |\tau(a)| = \|a\|$$

and so, $1 \leq \|\tau\|$. We also have

$$\|\tau\| = \sup_{\|b\|\leq 1,\, b \in A} |\tau(b)| \leq \sup_{\|b\|\leq 1,\, b\in\tilde{A}} |\tau_1(b)| = \|\tau_1\| = 1.$$

Therefore, $\|\tau\| = 1$ and so, τ is a state on A such that $|\tau(a)| = \|a\|$. □

We now prove our analogue of the Jordan decomposition in Theorem 3.4.1 for C*-algebras.

Theorem 3.4.3. *Let A be a C*-algebra and $\tau \in A^*_{sa}$ be a self-adjoint bounded linear functional on A. Then, there exist positive linear functionals $\tau_+, \tau_- \in A^*$ such that*

$$\tau = \tau_+ - \tau_- \qquad \textit{and} \qquad \|\tau\| = \|\tau_+\| + \|\tau_-\|.$$

Proof. Assume that A is a C*-algebra. Assume that $\tau \in A^*_{sa}$ is a self-adjoint bounded linear functional on A. Define the set

$$\Omega = \{\tau \in A^*_+ \mid \|\tau\| \leq 1\}.$$

Then, Ω is a weak-* closed subset of the closed unit ball of A^*, which is weak-* compact by the Banach-Alaoglu theorem. Therefore, Ω is a weak-* compact and Hausdorff set. Now define the evaluation map ev by

$$\begin{array}{rccc} ev: & A_{sa} & \to & Cts_{\mathbb{R}}(\Omega, \mathbb{R}) \\ & a & \mapsto & \big(\tau \mapsto \tau(a) = \tau'(a)\big) \end{array}$$

Then, ev is real-linear. It is also order-preserving because if $a \in A_+$ and $\tau \in \Omega$ then $ev(\tau)(a) = \tau(a) \geq 0$. Now we claim that ev is isometric. Firstly, if $a \in A_{sa}$ then

$$\|ev(a)\| = \sup_{\|\tau\| \leq 1} |ev(a)(\tau)| = \sup_{\|\tau\| \leq 1} |\tau(a)| \leq \|a\|.$$

For the reverse inequality, note that there exists a state $\phi : A \to \mathbb{C}$ such that $|\phi(a)| = \|a\|$ by Theorem 3.4.2. So,

$$\|a\| = |\phi(a)| = |ev(a)(\phi)| \leq \|ev(a)\|.$$

Therefore, ev is an isometry.

Now if $\tau \in A^*_{sa}$ then $\tau' \in A^\natural_{sa}$ by definition. By the Hahn-Banach extension theorem, there exists a real-linear functional $\rho \in Cts_{\mathbb{R}}(\Omega, \mathbb{R})^\natural$ such that $\rho \circ ev = \tau'$ and $\|\rho\| = \|\tau'\|$. By Theorem 3.4.1, there exists positive bounded real-linear functionals ρ_+ and ρ_- on $Cts_{\mathbb{R}}(\Omega, \mathbb{R})$ such that $\rho = \rho_+ - \rho_-$ and $\|\rho_+\| + \|\rho_-\| = \|\rho\|$.

Now we define

$$\tau'_+ = \rho_+ \circ ev \qquad \text{and} \qquad \tau'_- = \rho_- \circ ev.$$

Then, $\tau'_+, \tau'_- \in A^\natural_{sa}$. Now recall that the map

$$\begin{array}{rccc} r: & A^*_{sa} & \to & A^\natural_{sa} \\ & \tau & \mapsto & \tau' = \tau|_{A_{sa}} \end{array}$$

is an isometric real-linear isomorphism. Let $\tau_+ = r^{-1}(\tau'_+)$ and $\tau_- = r^{-1}(\tau'_-)$. Then,

$$\tau = r^{-1}(\tau') = r^{-1}(\tau'_+ - \tau'_-) = \tau_+ - \tau_-$$

and

$$\begin{aligned} \|\tau\| &= \|r^{-1}(\tau')\| \\ &= \|\tau'\| = \|\rho \circ ev\| \\ &= \|\rho\| = \|\rho_+\| + \|\rho_-\| \\ &\geq \|\tau'_+\| + \|\tau'_-\| \\ &= \|\tau_+\| + \|\tau_-\| \\ &\geq \|\tau_+ - \tau_-\| = \|\tau\|. \end{aligned}$$

So, $\|\tau\| = \|\tau_+\| + \|\tau_-\|$ as required. Moreover, τ_+ and τ_- are positive linear functionals because ρ_+ and ρ_- are positive and ev is order-preserving. $\square$

Every linear functional on a C*-algebra A can be written as a linear combination of two self-adjoint linear functionals on A. By Theorem 3.4.3, it follows that every linear functional on A is a linear combination of positive linear functionals. By scaling, we find that every linear functional on A can be written as a linear combinations of states on A. This has a particular consequence regarding the universal representation of A, which we will use in the proof of the existence of a quasicentral approximate unit.

Theorem 3.4.4. *Let A be a C*-algebra and*

$$(\pi_u, H_u) = \Big(\bigoplus_{\phi \in S(A)} \pi_\phi, \bigoplus_{\phi \in S(A)} H_\phi \Big)$$

be the universal representation of A. Let $\alpha : A \to \mathbb{C}$ be a bounded linear functional. Then, there exists $x, y \in H_u$ such that if $a \in A$ then

$$\alpha(a) = \langle \pi_u(a)x, y \rangle.$$

Proof. Assume that A is a C*-algebra and that (π_u, H_u) is the universal representation of A. Assume that $\alpha : A \to \mathbb{C}$ is a bounded linear functional on A. By Theorem 3.4.3 and the remark preceding the theorem, we can write α as a linear combination of states on A. So,

$$\alpha = \sum_{i=1}^{n} \lambda_i \phi_i$$

where $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $\phi_1, \ldots, \phi_n \in S(A)$. If $\psi \in S(A)$ then let $\xi_\psi \in H_\psi$ be the cyclic vector associated to the GNS representation of ψ. Now let $x, y \in H_u$ be defined by

$$x = \bigoplus_{\phi \in S(A)} \chi(\phi)\xi_\phi \qquad \text{and} \qquad y = \bigoplus_{\phi \in S(A)} \lambda(\phi)\xi_\phi$$

where if $i \in \{1, 2, \ldots, n\}$ then

$$\chi(\phi) = \begin{cases} \xi_{\phi_i}, & \text{if } \phi = \phi_i, \\ 0, & \text{otherwise.} \end{cases} \qquad \text{and} \qquad \lambda(\phi) = \begin{cases} \overline{\lambda_i}\xi_{\phi_i}, & \text{if } \phi = \phi_i, \\ 0, & \text{otherwise.} \end{cases}$$

If $a \in A$ then

$$\begin{aligned} \langle \pi_u(a)x, y\rangle &= \sum_{i=1}^{n} \langle \pi_{\phi_i}(a)\xi_{\phi_i}, \overline{\lambda_i}\xi_{\phi_i}\rangle \\ &= \sum_{i=1}^{n} \lambda_i \langle \pi_{\phi_i}(a)\xi_{\phi_i}, \overline{\lambda_i}\xi_{\phi_i}\rangle \\ &= \sum_{i=1}^{n} \lambda_i \phi_i = \alpha. \end{aligned}$$

□

As mentioned in the statement of [BO08, Theorem 1.2.1], the quasicentral approximate unit we will construct arises from the *convex hull* of an existing approximate unit. We will show that the convex hull of an approximate unit is, in its own right, an approximate unit.

Definition 3.4.3. Let X be a subset of a vector space V. The **convex hull** of X, denoted by $conv(X)$, is the set

$$conv(X) = \left\{ \sum_{i=1}^{n} \lambda_i x_i \;\middle|\; x_i \in X,\ \lambda_i \in [0,1],\ \sum_{i=1}^{n} \lambda_i = 1 \right\}.$$

Theorem 3.4.5. *Let A be a C*-algebra and I be a closed two-sided ideal of A. Let $\{u_\lambda\}_{\lambda\in\Lambda}$ be an approximate unit for I. Then, the convex hull of $\{u_\lambda\}_{\lambda\in\Lambda}$ is also an approximate unit for I.*

Proof. Assume that A is a C*-algebra and that I is a closed two-sided ideal of A. Assume that $\{u_\lambda\}_{\lambda\in\Lambda}$ is an approximate ideal for I. Define

$$\Lambda' = conv\big(\{u_\lambda \mid \lambda \in \Lambda\}\big).$$

To see that Λ' is upwards directed, assume that $\sum_{i=1}^{n} \mu_i u_{\lambda_i} \in \Lambda'$. Since Λ is upwards directed then there exists u_λ such that if $i \in \{1, 2, \dots, n\}$ then $u_{\lambda_i} \leq u_\lambda$. Hence,

$$\sum_{i=1}^{n} \mu_i u_{\lambda_i} \leq \sum_{i=1}^{n} \mu_i u_\lambda = u_\lambda$$

and consequently, Λ' is upwards directed with respect to the order on positive elements of I.

To see that Λ' is an approximate unit for I, assume that $j \in I$ and $u_\lambda \in \Lambda$. If $v \in \Lambda'$ satisfies $v \geq u_\lambda$ then

$$\begin{aligned}\|j - vj\|^2 &= \|j^*(1_{\tilde{A}} - v)^2 j\| \\ &\leq \|j^*(1_{\tilde{A}} - v)j\| \leq \|j^*(1_{\tilde{A}} - u_\lambda)j\| \\ &\leq \|j\|\|j - u_\lambda j\|\end{aligned}$$

Taking the limit over $\lambda \in \Lambda$, we deduce that $\lim_\lambda \|j - vj\| = 0$. By a similar computation, $\lim_\lambda \|j - jv\| = 0$. Therefore, the convex hull Λ' is an approximate unit for I. □

We also require a few results pertaining to the extension and restriction of representations. These results originate from [Dav96, Lemma I.9.14, Lemma I.9.15].

Theorem 3.4.6. *Let A be a C*-algebra and I be a closed two-sided ideal of A. Let (π, H) be a non-degenerate representation of I. Then, there exists a unique representation $\tilde{\pi} : A \to B(H)$ of A such that $\tilde{\pi}|_I = \pi$. Furthermore, $\tilde{\pi}$ is irreducible if and only if π is irreducible.*

Proof. Assume that A is a C*-algebra and that I is a closed two-sided ideal of A. Assume that (π, H) is a representation of I. First, define the map $\tilde{\pi}'$ by

$$\begin{array}{rccc}\tilde{\pi}' : & A & \to & B(\pi(I)H) \\ & a & \mapsto & \big(\pi(j)x \mapsto \pi(aj)x\big).\end{array}$$

The fact that $\tilde{\pi}'$ is linear follows from direct calculations.

To show: (a) If $a \in A$ then $\tilde{\pi}'(a)$ is well-defined.

(a) Assume that $a \in A$. Assume that $j_1, j_2 \in I$ and $x_1, x_2 \in H$ such that $\pi(j_1)x_1 = \pi(j_2)x_2$. Let $\{u_\lambda\}_{\lambda\in\Lambda}$ be an approximate unit for I (see Theorem 2.3.3). We compute directly that

$$\begin{aligned}\tilde{\pi}'(a)(\pi(j_1)x_1) &= \pi(aj_1)x_1\\ &= \lim_\lambda \pi(au_\lambda j_1)x_1\\ &= \lim_\lambda \pi(au_\lambda)\pi(j_1)x_1\\ &= \lim_\lambda \pi(au_\lambda)\pi(j_2)x_2\\ &= \tilde{\pi}'(a)(\pi(j_2)x_2).\end{aligned}$$

Therefore, $\tilde{\pi}'$ is well-defined.

We also have

$$\|\tilde{\pi}'(a)(\pi(j)x)\| \leq \lim_\lambda \|\pi(au_\lambda)\|\|\pi(j)x\| \leq \|a\|\|\pi(j)x\| < \infty.$$

Since (π, H) is a non-degenerate representation then $\pi(I)H$ is dense in H. Hence, we can extend $\tilde{\pi}'$ to a linear map $\tilde{\pi} : A \to B(H)$.

Now, we will show that $\tilde{\pi}$ is a *-homomorphism. Assume that $a_1, a_2 \in A$, $j \in I$ and $x \in H$. Then,

$$\begin{aligned}\tilde{\pi}(a_1a_2)(\pi(j)x) &= \tilde{\pi}'(a_1a_2)(\pi(j)x)\\ &= \pi(a_1a_2j)x\\ &= \tilde{\pi}(a_1)(\pi(a_2j)x)\\ &= \tilde{\pi}(a_1)\tilde{\pi}(a_2)(\pi(j)x)\end{aligned}$$

and if $j' \in I$ and $y \in H$ then

$$\begin{aligned}\langle\tilde{\pi}(a^*)(\pi(j)x), \pi(j')y\rangle &= \langle\pi(a^*j)x, \pi(j')y\rangle\\ &= \langle x, \pi(j^*aj')y\rangle\\ &= \langle\pi(j)x, \tilde{\pi}(a)(\pi(j')y)\rangle\\ &= \langle\tilde{\pi}(a)^*(\pi(j)x), \pi(j')y\rangle.\end{aligned}$$

Since $\pi(A)H$ is dense in H then we deduce that $\tilde{\pi}$ is a *-homomorphism. Next, we will show that $\tilde{\pi}$ is unique. Suppose that $\psi : A \to B(H)$ is another *-homomorphism such that $\psi|_I = \pi$. If $j, k \in I$ and $x \in H$ then

$$\psi(j)(\pi(k)x) = \pi(j)\pi(k)x = \pi(jk)x = \tilde{\pi}(j)(\pi(k)x).$$

So, $\psi = \tilde{\pi}$ and hence, $\tilde{\pi}$ is the unique *-homomorphism such that $\tilde{\pi}|_I = \pi$.

Now assume that $(\tilde{\pi}, H)$ is not an irreducible representation of A. Then, there exists a proper closed invariant subspace L for $\tilde{\pi}(A)$, which also qualifies as a proper closed invariant subspace for $\pi(I)$. So, (π, H) is not an irreducible representation of I.

Conversely, assume that π is not an irreducible representation of I. Then, there exists a proper closed invariant subspace M of $\pi(I)$. Using the fact that (π, H) is non-degenerate,

$$H = \overline{\pi(I)H} = \overline{\pi(I)(M \oplus M^{\perp})} \subsetneq \overline{\pi(I)M} \oplus \overline{\pi(I)M^{\perp}}.$$

Note that $\pi(I)M^{\perp} \subsetneq M^{\perp}$ by Theorem 1.9.2. So, $M = \overline{\pi(I)M}$ and

$$\tilde{\pi}(A)M = \overline{\tilde{\pi}(A)\pi(I)M} = \overline{\pi(I)M} = M.$$

Hence, $(\tilde{\pi}, H)$ is not an irreducible representation of A, which completes the proof. $\square$

Now we consider the restriction of a representation.

Theorem 3.4.7. *Let A be a C*-algebra and (π, H) be a representation of A. Let I be a closed two-sided of A and p be the projection onto the closed subspace $\overline{\pi(I)H}$. Then, p is an element of the centre of $\pi(A)''$. Moreover, if (π, H) is irreducible and $\pi(I) \neq 0$ then the restricted representation $(\pi|_I, H)$ is also irreducible.*

Proof. Assume that A is a C*-algebra and that (π, H) is a representation of A. Assume that I is a closed two-sided ideal of A and that p is the projection operator onto the closed subspace $\overline{\pi(I)H}$. Note that $\pi(A)(\pi(I)H) = \pi(I)H$. So, $\overline{\pi(I)H}$ is a proper closed invariant subspace for $\pi(A)$. and consequently, $p\pi(a) = \pi(a)p$ because $\pi(A)\overline{\pi(I)H} = \overline{\pi(I)H}$. This means that $p \in \pi(A)'$.

Now assume that $\alpha \in \pi(I)'$. We compute directly that if $j \in I$ and $\xi \in H$ then $\alpha\pi(j)\xi = \pi(j)(\alpha\xi) \in \pi(I)H$. This means that $\overline{\pi(I)H} = pH$ is a closed invariant subspace for $\pi(I)'$. Hence, $p \in \pi(I)''$ and

$$p \in \pi(I)'' \cap \pi(A)' \subsetneq \pi(A)'' \cap \pi(A)'.$$

We conclude that p is in the centre of $\pi(A)''$ as required.

Now assume that (π, H) is an irreducible representation of A. By Theorem 2.4.14, $\pi(A)' = \mathbb{C}id_H$. Since $p \in \pi(A)'$ and p is a projection operator, we deduce that $p = id_H$. Consequently, $\overline{\pi(I)H} = H$ and the restricted representation $(\pi|_I, H)$ of I is irreducible. □

The crux of the proof that quasicentral approximate units exist is detailed by the following theorem.

Theorem 3.4.8. *Let A be a C*-algebra and I be a closed two-sided ideal of A. Let $\{u_\lambda\}_{\lambda \in \Lambda}$ be an approximate unit for I and*

$$\mathcal{E} = conv\,\{u_\lambda \mid \lambda \in \Lambda\}\,.$$

If $a_1, \ldots, a_n \in A$ and $\lambda \in \Lambda$ then there exists $f \in \mathcal{E}$ such that $f \geq u_\lambda$ and if $i \in \{1, 2, \ldots, n\}$ then

$$\|a_i f - f a_i\| < \frac{1}{n}.$$

Proof. Assume that A is a C*-algebra and that I is a closed two-sided ideal of A. Assume that $\{u_\lambda\}_{\lambda \in \Lambda}$ is an approximate unit for I. Assume that $\mathcal{E}$ is the convex hull of $\{u_\lambda\}_{\lambda \in \Lambda}$, $a_1, \ldots, a_n \in A$ and $\lambda \in \Lambda$. By Theorem 3.4.5, $\mathcal{E}$ forms an approximate unit for I.

If $i \in \mathbb{Z}_{>0}$ then let

$$u_\lambda^{(i)} = \bigoplus_{j=1}^{i} u_\lambda \in B(A^{(i)}) \cong M_{i \times i}(A)$$

be the direct sum of i copies of u_λ. Then, the net $\{u_\lambda^{(n)}\}_{\lambda \in \Lambda}$ forms an approximate unit for $M_{n \times n}(I)$, which is an ideal of the C*-algebra $M_{n \times n}(A)$. Let

$$a = a_1 \oplus a_2 \oplus \cdots \oplus a_n \in M_{n \times n}(A)$$

be the operator (on the direct sum $A^{(n)}$) with diagonal entries $a_1, \ldots, a_n$. Define

$$\mathcal{F} = conv\{E_\mu \mid \mu \in \Lambda,\ \mu \geq \lambda\}$$

and

$$\mathcal{S} = \{af^{(n)} - f^{(n)}a \mid f \in \mathcal{F}\} \subseteq M_{n\times n}(A).$$

We note that if $f \in \mathcal{F}$ then $f^{(n)} \in M_{n\times n}(A)$ is the diagonal matrix whose diagonal entries are all f. The idea of the proof is to show that $0 \in \overline{\mathcal{S}}$.

To show: (a) $0 \in \overline{\mathcal{S}}$.

(a) Suppose for the sake of contradiction that $0 \notin \overline{\mathcal{S}}$. We first observe that $\mathcal{S}$ is a convex set. By the separation version of the Hahn-Banach theorem (see [Mur90, Theorem A.7]), there exists a linear functional $\phi : M_{n\times n}(A) \to \mathbb{C}$ such that if $\mu \in \Lambda$ and $\mu \geq \lambda$ then

$$Re\big(\phi(au_\mu^{(n)} - u_\mu^{(n)}a)\big) \geq 1.$$

Now let (π, H) be the universal representation of $M_{n\times n}(A)$. By Theorem 3.4.4, there exists vectors $x, y \in H$ such that if $z \in M_{n\times n}(A)$ then

$$\phi(z) = \langle \pi(z)x, y\rangle.$$

Now let $p \in B(H)$ be the projection operator onto the subspace $\overline{\pi(M_{n\times n}(I))H}$. We observe that if $\xi \in H$ then

$$\begin{aligned}
\|\pi(u_\lambda^{(n)})\xi - p\xi\|^2 &= \langle \pi(u_\lambda^{(n)})\xi - p\xi, \pi(u_\lambda^{(n)})\xi - p\xi\rangle \\
&= \langle p(\pi(u_\lambda^{(n)})\xi - \xi), p(\pi(u_\lambda^{(n)})\xi - \xi)\rangle \\
&= \langle \pi(u_\lambda^{(n)})\xi - \xi, p(\pi(u_\lambda^{(n)})\xi - \xi)\rangle \\
&= \|\pi(u_\lambda^{(n)})\xi\|^2 + \|p\xi\|^2 - \langle \pi(u_\lambda^{(n)})\xi, \xi\rangle - \langle \xi, \pi(u_\lambda^{(n)})\xi\rangle \\
&\leq 2\|\xi\|^2 - \langle \pi(u_\lambda^{(n)})\xi, \xi\rangle - \langle \xi, \pi(u_\lambda^{(n)})\xi\rangle \\
&\to 2\|\xi\|^2 - \|\xi\|^2 - \|\xi\|^2 \quad \text{(by Theorem 2.3.4)} \\
&= 0
\end{aligned}$$

The limit is taken over Λ. Hence, p is the strong limit of the net $\{\pi(u_\lambda^{(n)})\}$. By Theorem 3.4.7, p is in the centre of $\pi(I)''$. In particular, $p \in \pi(A)'$ and consequently,

$$\begin{aligned}
\lim_\lambda \phi(au_\lambda^{(n)} - u_\lambda^{(n)}a) &= \lim_\lambda \langle \pi(au_\lambda^{(n)} - u_\lambda^{(n)}a)x, y\rangle \\
&= \langle \pi(a)px - p\pi(a)x, y\rangle = 0.
\end{aligned}$$

However, this contradicts the fact that if $\mu \geq \lambda$ then

$$Re\big(\phi(au_\mu^{(n)} - u_\mu^{(n)}a)\big) \geq 1.$$

Therefore, $0 \in \overline{S}$.

Using part (a) of the proof, there exists a sequence $\{af_j^{(n)} - f_j^{(n)}a\}_{j \in \mathbb{Z}_{>0}}$ in $\mathcal{S}$ such that

$$\lim_{j \to \infty} \|af_j^{(n)} - f_j^{(n)}a\| = \lim_{j \to \infty} \max_{i \in \{1,\ldots,n\}} \|a_i f_j - f_j a_i\| = 0.$$

So, there exists $k \in \mathbb{Z}_{>0}$ such that if $i \in \{1, 2, \ldots, n\}$ then

$$\|a_i f_k - f_k a_i\| \leq \frac{1}{n}.$$

Moreover, $f_k \in \mathcal{F}$ which means that $f_k \geq u_\lambda$. $\square$

We can finally use Theorem 3.4.8 to prove the existence of quasicentral approximate units.

Theorem 3.4.9. *Let A be a C*-algebra and I be a closed two-sided ideal of A. Let $\{u_\lambda\}_{\lambda \in \Lambda}$ be an approximate unit for I and*

$$\mathcal{E} = conv\{u_\lambda \mid \lambda \in \Lambda\}.$$

Then, there exists a quasicentral approximate unit for I, arising from the convex hull $\mathcal{E}$.

Proof. Assume that A is a C*-algebra and I is a closed two-sided ideal of A. Assume that $\{u_\lambda\}_{\lambda \in \Lambda}$ is an approximate unit for I and that $\mathcal{E}$ is its convex hull. By Theorem 3.4.5, $\mathcal{E}$ is an approximate unit for I.

Let $\mathcal{F}$ be the set of all finite subsets of A. Then, $\mathcal{F}$ is a poset with the binary relation of inclusion. We also note that the product of sets $\mathcal{F} \times \Lambda$ is also a poset. That is, $(X, \lambda) \preceq (Y, \mu)$ in $\mathcal{F} \times \Lambda$ if and only if $X \subseteq Y$ and $\lambda \leq \mu$.

Now assume that $X \in \mathcal{F}$ and $\lambda \in \Lambda$. By Theorem 3.4.8, there exists $F_{X,\lambda} \in \mathcal{E}$ such that $F_{X,\lambda} \geq u_\lambda$ and if $x \in X$ then

$$\|xF_{X,\lambda} - F_{X,\lambda}x\| < \frac{1}{|X|}.$$

Here, $|X|$ is the cardinality of X. Now define the set

$$Q = \{F_{X,\lambda} \mid X \in \mathcal{F}, \lambda \in \Lambda\}.$$

Then, Q is a poset when we define $F_{X,\lambda} \preceq F_{Y,\mu}$ if and only if $(X, \lambda) \preceq (Y, \mu)$ (using the relation on $\mathcal{F} \times \Lambda$) and $F_{X,\lambda} \leq F_{Y,\mu}$ (using the positivity relation).

To see that Q is upwards-directed, assume that $X, Y \in \mathcal{F}$ and $\mu, \lambda \in \Lambda$. Since $F_{X,\lambda}, F_{Y,\mu} \in \mathcal{E}$ then we can write

$$F_{X,\lambda} = \sum_{i=1}^{\ell} s_i u_{\lambda_i} \qquad \text{and} \qquad F_{Y,\mu} = \sum_{i=1}^{n} t_i u_{\mu_i}.$$

Now let $Z = X \cup Y$ and $\nu \in \Lambda$ such that if $i \in \{1, 2, \ldots, \ell\}$ and $j \in \{1, 2, \ldots, n\}$ then $\nu \geq \lambda_i$ and $\nu \geq \mu_j$. Then, $u_\nu \geq F_{X,\lambda}$ and $u_\nu \geq F_{Y,\mu}$.

Now, using the fact that Λ is upwards-directed, we can select $\nu' \in \Lambda$ such that $\nu' \geq \nu$, $\nu' \geq \lambda$ and $\nu' \geq \mu$. By the construction in Theorem 3.4.8, we have

$$F_{Z,\nu'} \geq u_{\nu'} \geq u_\nu \geq F_{X,\lambda}$$

and similarly, $F_{Z,\nu'} \geq F_{Y,\mu}$. Using the relation on the poset $\mathcal{F} \times \Lambda$, we also have

$$(Z, \nu') \geq (X, \lambda) \qquad \text{and} \qquad (Z, \nu') \geq (Y, \mu).$$

Therefore, $F_{Z,\nu'} \succeq F_{X,\lambda}$ and $F_{Z,\nu'} \succeq F_{Y,\mu}$ and consequently, Q is an upwards-directed set.

To see that Q defines a quasicentral approximate unit for I, assume that $a \in A$ and $j \in I$. Since $Q \subseteq \mathcal{E}$ and $\mathcal{E}$ is an approximate unit for I then

$$\lim_{q \in Q} \|jq - j\| = \lim_{q \in Q} \|qj - j\| = 0.$$

Now observe that if $X \in \mathcal{F}$ and $a \in X$ then

$$\|aF_{X,\lambda} - F_{X,\lambda}a\| < \frac{1}{|X|} \to 0$$

when we take the limit over Q. This is because by the poset relation on Q, the finite set X containing a must increase in cardinality when the limit is taken over Q. Therefore, Q is a quasicentral approximate unit for I. $\square$

The result [BO08, Proposition 1.2.2] is a particular type of approximation which arises from a quasicentral approximate unit. We will prove this below.

Theorem 3.4.10. *Let A be a unital C^*-algebra and I be a closed two-sided ideal of A. Let $\{u_\lambda\}_{\lambda\in\Lambda}$ be a quasicentral approximate unit for I. If $a, b \in A$ and $a - b \in I$ then*

$$\lim_\lambda \left\| a - \Big((1_A - u_\lambda)^{\frac{1}{2}} b (1_A - u_\lambda)^{\frac{1}{2}} + u_\lambda^{\frac{1}{2}} a u_\lambda^{\frac{1}{2}} \Big) \right\| = 0$$

Proof. Assume that A is a unital C*-algebra and I is a closed two-sided ideal of A. Assume that $\{u_\lambda\}_{\lambda\in\Lambda}$ is a quasicentral approximate unit for I.

To show: (a) If $n \in \mathbb{Z}_{\geq 0}$ and $a \in A$ then $\lim_\lambda \|u_\lambda^n a - a u_\lambda^n\| = 0$.

(a) Assume that $n \in \mathbb{Z}_{\geq 0}$ and $a \in A$. Then,

$$\lim_\lambda \|u_\lambda^0 a - a u_\lambda^0\| = \|1_A a - a 1_A\| = 0$$

and $\lim_\lambda \|u_\lambda a - a u_\lambda\| = 0$ because $\{u_\lambda\}_{\lambda\in\Lambda}$ is a quasicentral approximate unit. This will be used as the base case.

For the inductive hypothesis, assume that there exists $k \in \mathbb{Z}_{>0}$ such that if $a \in A$ then $\lim_\lambda \|u_\lambda^k a - a u_\lambda^k\| = 0$. Observe that

$$\|u_\lambda^{k+1} a - a u_\lambda^{k+1}\| \leq \|u_\lambda^k (u_\lambda a) - (u_\lambda a) u_\lambda^k\| + \|u_\lambda (a u_\lambda^k) - (a u_\lambda^k) u_\lambda\|.$$

Taking the limit of $\lambda \in \Lambda$, the RHS of the above inequality vanishes due to the inductive hypothesis and the fact that $\{u_\lambda\}_{\lambda\in\Lambda}$ is a quasicentral approximate unit for I. So, $\lim_\lambda \|u_\lambda^{k+1} a - a u_\lambda^{k+1}\| = 0$ which proves the claim.

Now let $p(z) = \sum_{i=1}^n b_i z^i$ be a polynomial with $b_1, \ldots, b_n \in \mathbb{C}$. By the continuous functional calculus, if $x \in A$ then

$$\begin{aligned} \|p(u_\lambda)x - xp(u_\lambda)\| &= \|\big(\sum_{i=1}^n b_i u_\lambda^i\big)x - x\big(\sum_{i=1}^n b_i u_\lambda^i\big)\| \\ &\leq \sum_{i=1}^n |b_i| \|u_\lambda^i x - x u_\lambda^i\| \to 0 \end{aligned}$$

where we take the limit over $\lambda \in \Lambda$. By the Weierstrass approximation theorem, the elements of $Cts(\sigma(u_\lambda), \mathbb{C})$ can be approximated by polynomials. We conclude that if $x \in A$ then

$$\lim_\lambda \|u_\lambda^{\frac{1}{2}} x - x u_\lambda^{\frac{1}{2}}\| = 0 \qquad \text{and} \qquad \lim_\lambda \|(1_A - u_\lambda)^{\frac{1}{2}} x - x(1_A - u_\lambda)^{\frac{1}{2}}\| = 0.$$

To show: (b) If $x \in A$ then $\|x + I\| = \lim_\lambda \|(1_A - u_\lambda)x\|$.

(b) Assume that $x \in A$. Recall that the norm on the quotient C*-algebra A/I is given by

$$\|x + I\| = \inf_{j \in I} \|x + j\|.$$

Assume that $\epsilon \in \mathbb{R}_{>0}$. Then, there exists $k \in I$ such that

$$\|x + k\| \leq \|x + I\| + \frac{\epsilon}{2}.$$

Since $\{u_\lambda\}_{\lambda \in \Lambda}$ is an approximate unit for I, there exists $\mu \in \Lambda$ such that if $\lambda \geq \mu$ then

$$\|k - k u_\lambda\| < \frac{\epsilon}{2}.$$

Therefore, if $\lambda \in \Lambda$ and $\lambda \geq \mu$ then

$$\begin{aligned}
\|x - x u_\lambda\| &= \|x(1_A - u_\lambda)\| \\
&= \|(x + k - k)(1_A - u_\lambda)\| \\
&\leq \|(x + k)(1_A - u_\lambda)\| + \|k u_\lambda - k\| \\
&\leq \|x + k\| \|1_A - u_\lambda\| + \|k - k u_\lambda\| \\
&\leq \|x + k\| + \|k - k u_\lambda\| \\
&< \|x + I\| + \epsilon.
\end{aligned}$$

Note that $\|1_A - u_\lambda\| \leq 1$ because u_λ is positive with norm at most 1 (see Theorem 2.2.5). Hence, $\lim_\lambda \|x - x u_\lambda\| = \|x + I\|$. By replacing x with x^*, we also find that $\lim_\lambda \|x - u_\lambda x\| = \|x + I\|$. This proves part (b).

Now assume that $a, b \in A$ such that $a - b \in I$. Then,

$$
\begin{aligned}
&\|(1_A - u_\lambda)^{\frac{1}{2}}a(1_A - u_\lambda)^{\frac{1}{2}} - (1_A - u_\lambda)^{\frac{1}{2}}b(1_A - u_\lambda)^{\frac{1}{2}}\| \\
&= \|(1_A - u_\lambda)^{\frac{1}{2}}(a-b)(1_A - u_\lambda)^{\frac{1}{2}}\| \\
&\le \left\|\big((1_A - u_\lambda)^{\frac{1}{2}}(a-b)\big)(1_A - u_\lambda)^{\frac{1}{2}} - (1_A - u_\lambda)(a-b)\right\| + \|(1_A - u_\lambda)(a-b)\| \\
&= \left\|\big((1_A - u_\lambda)^{\frac{1}{2}}(a-b)\big)(1_A - u_\lambda)^{\frac{1}{2}} - (1_A - u_\lambda)(a-b)\right\| + \|(a-b)+I\| \\
&= \left\|(1_A - u_\lambda)^{\frac{1}{2}}(a-b)(1_A - u_\lambda)^{\frac{1}{2}} - (1_A - u_\lambda)(a-b)\right\| \to 0
\end{aligned}
$$

Note that the second last equality follows from part (b) and that the final equality follows from the assumption that $a - b \in I$.

Combining all the previous observations, we find that

$$
\begin{aligned}
&\left\|a - \left((1_A - u_\lambda)^{\frac{1}{2}}b(1_A - u_\lambda)^{\frac{1}{2}} + u_\lambda^{\frac{1}{2}}au_\lambda^{\frac{1}{2}}\right)\right\| \\
= &\left\|au_\lambda + a(1_A - u_\lambda) - \left((1_A - u_\lambda)^{\frac{1}{2}}b(1_A - u_\lambda)^{\frac{1}{2}} + u_\lambda^{\frac{1}{2}}au_\lambda^{\frac{1}{2}}\right)\right\| \\
\le &\ \|au_\lambda - u_\lambda^{\frac{1}{2}}au_\lambda^{\frac{1}{2}}\| + \|a(1_A - u_\lambda) - (1_A - u_\lambda)^{\frac{1}{2}}a(1_A - u_\lambda)^{\frac{1}{2}}\| \\
&+\|(1_A - u_\lambda)^{\frac{1}{2}}a(1_A - u_\lambda)^{\frac{1}{2}} - (1_A - u_\lambda)^{\frac{1}{2}}b(1_A - u_\lambda)^{\frac{1}{2}}\| \\
\to &\ 0
\end{aligned}
$$

in the limit of $\lambda \in \Lambda$. □

We finish this section with the following remark. The theories of C*-algebras and von Neumann algebras are parallel in the sense that results about von Neumann algebras are “exact”, whereas the analogous results about C*-algebras are “approximate”. For instance, von Neumann algebras are unital by Theorem 2.4.10. The analogous result for C*-algebras is the existence of an approximate unit in Theorem 2.3.2.

Another pair of parallel results is Theorem 1.7.6 for C*-algebras and Theorem 2.6.5 for von Neumann algebras. The analogous result for von Neumann algebras to the existence of a quasicentral approximate unit in Theorem 3.4.9 is given below.

Theorem 3.4.11. *Let H be a Hilbert space and $A \subseteq B(H)$ be a von Neumann algebra. Let $I \subseteq A$ be a weakly closed two-sided ideal of A (closed in the weak operator topology). Then, there exists a projection p in the centre of A such that $I = pM$.*

Proof. Assume that H is a Hilbert space and $A \subseteq B(H)$ is a von Neumann algebra. Assume that $I \subseteq A$ is a weakly closed two-sided ideal of A. Since I is a weakly closed *-subalgebra of $B(H)$ then by Theorem 2.5.10, I is itself a von Neumann algebra. By Theorem 2.4.10, I must have a unit, which we denote by p. Obviously, $p^2 = pp = p$ and if $b \in I$ then

$$bp^* = (pb^*)^* = b \qquad \text{and} \qquad p^*b = (b^*p)^* = b.$$

Hence, p is a projection in A.

To show: (a) p is in the centre of A.

(b) $I = pA$.

(a) Assume that $a \in A$ and that 1_A is the unit of A (see Theorem 2.4.10). Then, $pa, ap \in I$ and

$$pa = (pa)p = p(ap) = ap.$$

We conclude that p is in the centre of A.

(b) We know that $pA \subseteq I$. To see the reverse inclusion holds, assume that $b \in I$. Then, $b = pb \in pA$. So, $I = pA$. □

In the proof of Theorem 3.4.11, the fact that p is an element of the centre of the von Neumann algebra A mirrors the "approximate commutativity" exhibited by the quasicentral approximate unit in Theorem 3.4.9.

3.5 Nuclear and weakly nuclear maps

In [Mur90], Murphy defines nuclear C*-algebras by using tensor products (see Definition 2.11.9). Historically, this was how nuclear C*-algebras were defined. In the upcoming sections, we follow [BO08, Chapter 2] and approach the notion of nuclearity via c.c.p maps.

The first port of call is to define *nuclear maps* between C*-algebras.

Definition 3.5.1. Let A and B be C*-algebras. Let $\theta : A \to B$ be a linear map. We say that θ is **nuclear** if there exist c.c.p maps

$$\varphi_n : A \to M_{k(n)}(\mathbb{C}) \qquad \text{and} \qquad \psi_n : M_{k(n)}(\mathbb{C}) \to B$$

such that if $a \in A$ then $(\psi_n \circ \varphi_n)(a) \to \theta(a)$ in the norm topology on A. That is, if $a \in A$ then

$$\lim_n \|(\psi_n \circ \varphi_n)(a) - \theta(a)\| = 0.$$

Note that in the above definition, we have switched to using the standard notation $M_{k(n)}(\mathbb{C})$ for the matrix algebra rather than $M_{k(n)\times k(n)}(\mathbb{C})$. By definition, a nuclear map is a c.c.p map.

The definition of a nuclear map can be thought of as constructing an "approximately commutative triangle". If A and B are C*-algebras and $\theta : A \to B$ is a nuclear map then there exist c.c.p maps ψ_n, φ_n such that the following diagram "approximately commutes":

$$\begin{tikzcd}
A \arrow[rr, "\theta"] \arrow[dr, dashed, "\varphi_n"'] & & B \\
& M_{k(n)}(\mathbb{C}) \arrow[ur, dashed, "\psi_n"'] &
\end{tikzcd}$$

This type of diagram was used in [BO08, Definition 2.3.1] — the definition of nuclearity given by Brown and Ozawa.

It should be pointed out that in the definition of nuclear maps, it is not explicitly specified whether to work with nets or sequences. As stated in [BO08, Page 27], it does not really matter because nuclearity of maps is a *local property* by Theorem 3.5.8 which we will prove later. One should use nets in general and reserve the use of sequences for the separable setting.

It turns out that there is a counterpart to nuclear maps for von Neumann algebras.

Definition 3.5.2. Let M and N be von Neumann algebras. We say that a linear map $\varphi : M \to N$ is **normal** if the following statement is satisfied: If $\{x_i\}_{i\in I}$ is a norm bounded, monotonically increasing net of self-adjoint elements in M then

$$\varphi(\sup_{i\in I} x_i) = \sup_{i\in I} \varphi(x_i).$$

If M and N are von Neumann algebras then they have a predual by Theorem 2.5.12. We have already shown that the weak-* topologies induced on M and N are the ultraweak topologies. Thus, it is reasonable to conclude that the important linear maps between M and N are those which are ultraweakly continuous. Fortunately, this matches up with the definition of normal maps above because of the following result, which we state without proof.

Theorem 3.5.1. *Let M and N be von Neumann algebras and $\phi : M \to N$ be a positive linear map. Then, ϕ is normal if and only if ϕ is ultraweakly continuous.*

See [KR86, Theorem 7.1.2], [KR86, Proposition 7.4.5] and the comment in [Arg16] for the details on Theorem 3.5.1.

Definition 3.5.3. Let A be a C*-algebra and N be a von Neumann algebra. Let $\theta : A \to N$ be a linear map. Let N_* be the predual of N (the Banach space satisfying $(N_*)^* = N$. See Theorem 2.5.12). We say that θ is **weakly nuclear** if there exist c.c.p maps

$$\varphi_n : A \to M_{k(n)}(\mathbb{C}) \qquad \text{and} \qquad \psi_n : M_{k(n)}(\mathbb{C}) \to N$$

such that if $a \in A$ then $(\psi_n \circ \varphi_n)(a) \to \theta(a)$ in the ultraweak topology on N. That is, if $a \in A$ and $\eta \in N_*$ is a normal functional then

$$\eta((\psi_n \circ \varphi_n)(a)) \to \eta(\theta(a)).$$

The reason why we use normal linear functional in the above definition of weakly nuclear maps is due to *Sakai's predual theorem* which states that a von Neumann algebra M is the dual space of the Banach space of normal linear functionals on M. See [BO08, Theorem 1.3.5] and Theorem 2.5.12.

Before we proceed, we note the following result.

Theorem 3.5.2. *Let H be a Hilbert space and S be a (norm) bounded subset of $B(H)$. Then, the weak operator topology and ultraweak topology on S coincide.*

Proof. Assume that H is a Hilbert space and S is a norm bounded subset of $B(H)$. Then, there exists $r \in \mathbb{R}_{>0}$ such that

$$S \subseteq \{x \in B(H) \mid \|x\| \leq r\} = B_r.$$

Recall Theorem 2.5.6, which states that the weak operator topology and the ultraweak topology on $B(H)$ coincide on the closed unit ball of $B(H)$. Now define the maps

$$\begin{array}{rccc} \varphi_1: & (B_r, WOT) & \mapsto & (B_1, WOT) \\ & x & \mapsto & \frac{1}{r}x \end{array}$$

and

$$\begin{array}{rccc} \varphi_2: & (B_1, UWT) & \mapsto & (B_r, UWT) \\ & x & \mapsto & rx \end{array}$$

Then, φ_1 and φ_2 are homeomorphisms. Let $id : (B_1, UWT) \to (B_1, WOT)$ be the identity map on B_1, as defined in Theorem 2.5.6. Then, $\varphi_2 \circ id \circ \varphi_1$ is a homeomorphism and thus, the ultraweak and weak operator topologies coincide on B_r. Since $S \subseteq B_r$ then the ultraweak and weak operator topologies coincide on S. □

It is a fact that the nuclearity of certain linear maps depends on its range. A common occurrence is that a map $\theta : A \to B$ may not be nuclear, but becomes nuclear after embedding B into a larger C*-algebra. This phenomenon is readily seen in the context of von Neumann algebras — there exist von Neumann algebras $M \subseteq B(H)$ such that the identity map $id_M : M \to M$ is not nuclear. However, we have the following theorem:

Theorem 3.5.3. *Let H be a Hilbert space and $M \subseteq B(H)$ be a von Neumann algebra. Then, the inclusion map $\iota : M \hookrightarrow B(H)$ is a weakly nuclear map.*

Proof. Assume that H is a Hilbert space and $M \subseteq B(H)$ is a von Neumann algebra. Assume that $\iota : M \hookrightarrow B(H)$ is the inclusion map. Let $\{p_i\}_{i \in I}$ be a net of finite-rank projections which strongly converge to the identity map id_H.

If $n \in I$ then let $k(n) \in \mathbb{Z}_{>0}$ be the rank of the projection p_n. Define the linear maps

$$\begin{array}{rccc} \varphi_n: & M & \to & M_{k(n)}(\mathbb{C}) \cong p_n B(H) p_n \\ & a & \mapsto & p_n a p_n \end{array}$$

and

$$\begin{array}{rccc} \psi_n: & M_{k(n)}(\mathbb{C}) \cong p_n B(H) p_n & \to & B(H) \\ & A & \mapsto & A \end{array}$$

Observe that if $n \in I$ then φ_n and ψ_n are *-homomorphisms and thus, c.c.p maps.

To show: (a) If $a \in M$ then $(\psi_n \circ \varphi_n)(a) \to \iota(a)$ in the ultraweak topology on $B(H)$.

(a) Observe that if $a \in M$ and $n \in \mathbb{Z}_{>0}$ then

$$\|(\psi_n \circ \varphi_n)(a)\| = \|p_n a p_n\| \leq \|a\|.$$

By Theorem 3.5.2, it suffices to show that if $a \in M$ then $(\psi_n \circ \varphi_n)(a) \to \iota(a)$ in the weak operator topology on $B(H)$. Assume that $\xi, \eta \in H$ and $a \in M$. Then,

$$\begin{aligned}
|\langle p_n a\xi - p_n a p_n \xi, \eta\rangle| &= |\langle p_n a(id_H - p_n)\xi, \eta\rangle| \\
&\leq \|p_n a\| \|(id_H - p_n)\xi\| \|\eta\| \\
&= \|p_n a\| \|\xi - p_n \xi\| \|\eta\| \to 0
\end{aligned}$$

where the limit is taken over I because the net $\{p_i\}_{i\in I}$ converges strongly to the identity map id_H. Similarly, the quantities

$$|\langle (id_H - p_n)ap_n\xi, \eta\rangle| \qquad \text{and} \qquad |\langle (id_H - p_n)a(id_H - p_n)\xi, \eta\rangle|$$

converge to zero. Now, we have

$$\begin{aligned}
\left|\left\langle \big((\psi_n \circ \varphi_n)(a) - \iota(a)\big)\xi, \eta\right\rangle\right| &= \left|\left\langle \big(p_n a p_n - a\big)\xi, \eta\right\rangle\right| \\
&= |\langle p_n a p_n \xi - a\xi, \eta\rangle| \\
&\leq |\langle p_n a p_n \xi - a p_n \xi, \eta\rangle| + |\langle a p_n \xi - a\xi, \eta\rangle| \\
&\leq |\langle p_n a p_n \xi - a p_n \xi, \eta\rangle| + |\langle a p_n \xi - a\xi + p_n a\xi - p_n a p_n \xi, \eta\rangle| \\
&\quad + |\langle p_n a p_n \xi - p_n a \xi, \eta\rangle| \\
&= |\langle (id_H - p_n)ap_n\xi, \eta\rangle| + |\langle (id_H - p_n)a(id_H - p_n)\xi, \eta\rangle| \\
&\quad + |\langle p_n a(id_H - p_n)\xi, \eta\rangle| \\
&\to 0
\end{aligned}$$

in the limit over I. Thus, $(\psi_n \circ \varphi_n)(a) \to \iota(a)$ in the ultraweak topology and consequently, the inclusion ι is weakly nuclear. □

In the next few results, we will prove various properties of nuclear maps, which are the content of [BO08, Exercises 2.1.1-2.1.9]. According to [BO08, Chapter 2], the exercises tend to be used often without proof in the literature. The first result deals with restrictions of nuclear maps.

Theorem 3.5.4. *Let A and B be C*-algebras and $C \subseteq A$ be a C*-subalgebra of A. Let $\theta : A \to B$ be a nuclear map. Then, the restriction $\theta|_C : C \to B$ is also a nuclear map.*

Proof. Assume that A and B are C*-algebras. Assume that C is a C*-subalgebra of A. Assume that $\theta : A \to B$ is a nuclear map. Then there exist c.c.p maps

$$\varphi_n : A \to M_{k(n)}(\mathbb{C}) \qquad \text{and} \qquad \psi_n : M_{k(n)}(\mathbb{C}) \to B$$

such that if $a \in A$ then $(\psi_n \circ \varphi_n)(a) \to \theta(a)$ in the norm topology on B. Now the restriction

$$\varphi_n|_C : C \to M_{k(n)}(\mathbb{C})$$

is a c.c.p map. Furthermore, if $c \in C$ then

$$\|(\psi_n \circ \varphi_n|_C)(c) - \theta|_C(c)\| = \|(\psi_n \circ \varphi_n)(c) - \theta(c)| \to 0$$

where the limit is taken over n. Therefore, the restriction $\theta|_C : C \to B$ is a nuclear map. $\square$

Next, we will see how nuclear maps behave under composition.

Theorem 3.5.5. *Let A, B and C be C*-algebras. Let $\theta : A \to B$ and $\sigma : B \to C$ be c.c.p maps. If either θ or σ is nuclear then the composite $\sigma \circ \theta$ is also a nuclear map.*

Proof. Assume that A, B and C are C*-algebras. Assume that $\theta : A \to B$ and $\sigma : B \to C$ are c.c.p maps. Assume that θ is a nuclear map. Then there exist c.c.p maps

$$\varphi_n : A \to M_{k(n)}(\mathbb{C}) \qquad \text{and} \qquad \psi_n : M_{k(n)}(\mathbb{C}) \to B$$

such that if $a \in A$ then $(\psi_n \circ \varphi_n)(a)$ converges to $\theta(a)$ in the norm topology on B. Observe that the composites $\sigma \circ \psi_n$ are c.c.p maps from $M_{k(n)}(\mathbb{C})$ to C. Furthermore, if $a \in A$ then

$$\|(\sigma \circ \psi_n \circ \varphi_n)(a) - (\sigma \circ \theta)(a)\| \leq \|\sigma\|\|(\psi_n \circ \varphi_n)(a) - \theta(a)\| \to 0$$

in the limit over all n. Therefore, $\sigma \circ \theta$ is a nuclear map. The case where σ is a nuclear map is dealt with by a similar argument. $\square$

There are two special cases of Theorem 3.5.5 we wish to highlight.

Theorem 3.5.6. *Let A and B be C*-algebras and $\theta : A \to B$ be a c.c.p map. If the identity map $id_A : A \to A$ is a nuclear map then θ is a nuclear map.*

Proof. Assume that A and B are C*-algebras. Assume that $\theta : A \to B$ is a c.c.p map. Assume that the identity map $id_A : A \to A$ on A is nuclear. By Theorem 3.5.5, $\theta = \theta \circ id_A$ must be a nuclear map. $\square$

Theorem 3.5.7. *Let H be a Hilbert space and $A \subseteq B(H)$ be a concretely represented C*-algebra. Assume that the inclusion map $\iota : A \hookrightarrow B(H)$ is a nuclear map. If K is a Hilbert space and $\theta : A \to B(K)$ is a c.c.p map then θ is a nuclear map.*

Proof. Assume that H is a Hilbert space and $A \subseteq B(H)$ is a C*-algebra such that the inclusion map $\iota : A \hookrightarrow B(H)$ is a nuclear map.

Assume that K is a Hilbert space and $\theta : A \to B(K)$ is a c.c.p map. By Arveson's extension theorem (Theorem 3.3.7), there exists a c.c.p map $\tilde{\theta} : B(H) \to B(K)$ such that the restriction $\tilde{\theta}|_A = \theta$. Since ι is nuclear then by Theorem 3.5.5, the composite $\tilde{\theta} \circ \iota : A \to B(K)$ is a nuclear map. But, if $a \in A$ then

$$(\tilde{\theta} \circ \iota)(a) = \tilde{\theta}(a) = \theta(a).$$

Hence, θ is a nuclear map as required. $\square$

Now, we will state and prove alternative characterisations of nuclear maps.

Theorem 3.5.8. *Let A and B be C*-algebras and $\theta : A \to B$ be a c.c.p map. Then, θ is a nuclear map if and only if the following statement is satisfied: If $F \subseteq A$ is a finite subset of A and $\epsilon \in \mathbb{R}_{>0}$ then there exist $n \in \mathbb{Z}_{>0}$ and c.c.p maps*

$$\varphi : A \to M_n(\mathbb{C}) \qquad \textit{and} \qquad \psi : M_n(\mathbb{C}) \to B$$

such that if $a \in F$ then

$$\|\theta(a) - (\psi \circ \varphi)(a)\| < \epsilon.$$

Proof. Assume that A and B are C*-algebras. Assume that $\theta : A \to B$ is a c.c.p map.

First assume that $\theta : A \to B$ is a nuclear map. If $n \in I$ then there exist c.c.p maps

$$\varphi_n : A \to M_{k(n)}(\mathbb{C}) \qquad \text{and} \qquad \psi_n : M_{k(n)}(\mathbb{C}) \to B$$

such that if $a \in A$ then $\|(\psi_n \circ \varphi_n)(a) - \theta(a)\| \to 0$.

Now assume that F is a finite subset of A and $\epsilon \in \mathbb{R}_{>0}$. Since θ is nuclear, we may choose $N \in I$ such that if $n \geq N$ and $a \in A$ then

$$\|\theta(a) - (\psi_n \circ \varphi_n)(a)\| < \epsilon.$$

Fix $m \geq N$. Then, the positive integer $k(m)$ and the c.c.p maps φ_m and ψ_m satisfy

$$\|\theta(a) - (\psi_m \circ \varphi_m)(a)\| < \epsilon.$$

In particular, the above inequality holds for $a \in A$ and thus, for $a \in F$. So, the second statement must be satisfied.

Conversely, assume that the second statement holds. We want to show that $\theta : A \to B$ is a nuclear map. Let $\mathcal{S}$ be the set

$$\mathcal{S} = \{(F, n) \mid F \subseteq A \text{ is finite}, n \in \mathbb{R}_{>0}\}.$$

Then, $\mathcal{S}$ is a poset with relation $\prec$ defined by declaring that $(F, n) \prec (G, m)$ if and only if $F \subsetneq G$ and $n > m$. It is straightforward to see that the pair $(\mathcal{S}, \prec)$ is an upwards-directed set.

By the second statement, if $\alpha = (F, n) \in \mathcal{S}$ then there exists $n_\alpha \in \mathbb{Z}_{>0}$ and c.c.p maps

$$\varphi_\alpha : A \to M_{n_\alpha}(\mathbb{C}) \qquad \text{and} \qquad \psi_\alpha : M_{n_\alpha}(\mathbb{C}) \to B$$

such that if $a \in F$ then

$$\|(\psi_\alpha \circ \varphi_\alpha)(a) - \theta(a)\| < n.$$

We claim that the net $\{\psi_\alpha \circ \varphi_\alpha\}_{\alpha \in \mathcal{S}}$ consists of the c.c.p maps required for θ to be a nuclear map. Assume that $a \in A$ and $\epsilon \in \mathbb{R}_{>0}$. Then, there exists $\alpha = (F, \epsilon) \in \mathcal{S}$ such that $a \in F$ and

$$\|(\psi_\alpha \circ \varphi_\alpha)(a) - \theta(a)\| < \epsilon.$$

Now assume that $\beta = (G, \delta) \succ \alpha$. Then, $a \in G$, $\delta < \epsilon$ and

$$\|(\psi_\beta \circ \varphi_\beta)(a) - \theta(a)\| < \delta < \epsilon.$$

We deduce that if $a \in A$ then

$$\lim_{\alpha \in \mathcal{S}} \|(\psi_\alpha \circ \varphi_\alpha)(a) - \theta(a)\| = 0.$$

Consequently, θ is a nuclear map as required. $\square$

By adapting the proof given in Theorem 3.5.8, we will also show that weak nuclearity of maps is also a local property.

Theorem 3.5.9. *Let A be a C*-algebra, N be a von Neumann algebra and $\theta : A \to N$ be a c.c.p map. Then, θ is a weakly nuclear map if and only if the following statement is satisfied: If $F \subsetneq A$ is a finite subset of A, $\chi \subsetneq N_*$ is a finite subset of normal linear functionals and $\epsilon \in \mathbb{R}_{>0}$ then there exist $n \in \mathbb{Z}_{>0}$ and c.c.p maps*

$$\varphi : A \to M_n(\mathbb{C}) \qquad \textit{and} \qquad \psi : M_n(\mathbb{C}) \to B$$

such that if $a \in F$ and $\eta \in \chi$ then

$$|\eta(\theta(a)) - \eta((\psi \circ \varphi)(a))| < \epsilon.$$

Proof. Assume that A is a C*-algebra and that N is a von Neumann algebra. Assume that $\theta : A \to N$ is a c.c.p map.

First, assume that $\theta : A \to N$ is a weakly nuclear map. Assume that $F \subsetneq A$ is a finite set and that $\chi \subsetneq N_*$ is a finite set of normal linear functionals on N. Assume that $\epsilon \in \mathbb{R}_{>0}$. If $n \in I$ then there exist c.c.p maps

$$\varphi_n : A \to M_{k(n)}(\mathbb{C}) \qquad \text{and} \qquad \psi_n : M_{k(n)}(\mathbb{C}) \to N$$

such that if $a \in A$ and $\eta \in N_*$ is a normal linear functional then $|\eta((\psi_n \circ \varphi_n)(a)) - \eta(\theta(a))| \to 0$ in the limit over n. Now choose $m \in I$ such that

$$|\eta((\psi_n \circ \varphi_n)(a)) - \eta(\theta(a))| < \epsilon.$$

Since the inequality above holds for arbitrary $a \in A$ and for an arbitrary normal functional $\eta \in N_*$, it must hold for $a \in F$ and $\eta \in \chi$. So, the second statement is satisfied.

Conversely, assume that the second statement holds. We want to show that $\theta : A \to N$ is a nuclear map. Arguing as in Theorem 3.5.8, let $\mathcal{S}$ be the set

$$\mathcal{S} = \left\{ (F, \chi, n) \;\middle|\; \begin{array}{c} F \subsetneq A \text{ is finite, } n \in \mathbb{R}_{>0} \\ \chi \subsetneq N_* \text{ is a finite set} \end{array} \right\}.$$

Then, $\mathcal{S}$ is a poset when equipped with the relation $\prec$. If $(F, \chi, n), (G, \delta, m) \in \mathcal{S}$ then we define $(F, \chi, n) \prec (G, \delta, m)$ if and only if $F \subsetneq G$, $\chi \subsetneq \delta$ and $m < n$. Similarly to Theorem 3.5.8, it is straightforward to see that the pair $(\mathcal{S}, \prec)$ is an upwards-directed set.

By the second statement, if $\alpha = (F, \chi, n) \in \mathcal{S}$ then there exist $n_\alpha \in \mathbb{Z}_{>0}$ and c.c.p maps

$$\varphi_\alpha : A \to M_{n_\alpha}(\mathbb{C}) \qquad \text{and} \qquad \psi_\alpha : M_{n_\alpha}(\mathbb{C}) \to N$$

such that if $a \in F$ and $\eta \in \chi$ then

$$|\eta((\psi_\alpha \circ \varphi_\alpha)(a)) - \eta(\theta(a))| < \epsilon.$$

We claim that $\{\psi_\alpha \circ \varphi_\alpha\}_{\alpha \in \mathcal{S}}$ is then net of c.c.p maps required for $\theta : A \to N$ to be a weakly nuclear map. To this end, assume that $a \in A$, $\eta \in N_*$ is a normal linear functional and $\epsilon \in \mathbb{R}_{>0}$. Then, there exists $\alpha = (F, \chi, \epsilon) \in \mathcal{S}$ such that $a \in F$, $\eta \in \chi$ and

$$|\eta((\psi_\alpha \circ \varphi_\alpha)(a)) - \eta(\theta(a))| < \epsilon.$$

Now, if $\beta = (G, \delta, \tau) \succeq (F, \chi, \epsilon)$ then $a \in G$, $\eta \in \delta$ and

$$|\eta((\psi_\beta \circ \varphi_\beta)(a)) - \eta(\beta(a))| < \tau < \epsilon.$$

Thus, if $a \in A$ and $\eta \in N_*$ is a normal linear functional on N then

$$\lim_{\alpha \in \mathcal{S}} \|(\psi_\alpha \circ \varphi_\alpha)(a) - \theta(a)\| = 0.$$

So, θ is a weakly nuclear map which completes the proof. $\square$

Theorem 3.5.10. *Let A and B be C*-algebras and $\theta : A \to B$ be a c.c.p map. Then, θ is a nuclear map if and only if there exist finite dimensional C*-algebras C_n and c.c.p maps*

$$\varphi_n : A \to C_n \qquad \text{and} \qquad \psi_n : C_n \to B$$

such that if $a \in A$ then

$$\lim_n \|(\psi_n \circ \varphi_n)(a) - \theta(a)\| = 0.$$

Proof. Assume that A and B are C*-algebras and that $\theta : A \to B$ is a c.c.p map. If θ is a nuclear map then there exist c.c.p maps

$$\varphi_n : A \to M_{k(n)}(\mathbb{C}) \qquad \text{and} \qquad \psi_n : M_{k(n)}(\mathbb{C}) \to B$$

and finite dimensional C*-algebra $M_{k(n)}(\mathbb{C})$ such that if $a \in A$ then

$$\|(\psi_n \circ \varphi_n)(a) - \theta(a)\| \to 0$$

in the limit over n. So, the second statement must be satisfied.

Conversely, assume that there exist finite dimensional C*-algebras C_n and c.c.p maps

$$\varphi_n : A \to C_n \qquad \text{and} \qquad \psi_n : C_n \to B$$

such that if $a \in A$ then

$$\lim_n \|(\psi_n \circ \varphi_n)(a) - \theta(a)\| = 0.$$

By Theorem 2.10.3, there exist $k_m, n_{1,m}, \ldots, n_{k_m,m} \in \mathbb{Z}_{>0}$ such that

$$C_m \cong \bigoplus_{i=1}^{k_m} M_{n_{i,m}}(\mathbb{C}).$$

Let $n_m = \sum_{i=1}^{k_m} n_{i,m}$. We have the unital embedding

$$\begin{array}{rccc} \iota_m : & C_m \cong \bigoplus_{i=1}^{k_m} M_{n_{i,m}}(\mathbb{C}) & \to & M_{n_m}(\mathbb{C}) \\ & (a_1, a_2, \ldots, a_{k_m}) & \mapsto & a_1 \oplus a_2 \oplus \cdots \oplus a_{k_m}. \end{array}$$

By the isomorphism $C_m \cong \bigoplus_{i=1}^{k_m} M_{n_{i,m}}(\mathbb{C})$, $C_m \subseteq M_{n_m}(\mathbb{C})$ is a direct sum of matrix blocks. This means that there exist pairwise orthogonal projections $p_1, p_2, \ldots, p_{k_m}$ such that

$$C_m = \sum_{i=1}^{k_m} p_i M_{n_m}(\mathbb{C}) p_i.$$

This propels us to define the map

$$\begin{array}{rccc} E : & M_{n_m}(\mathbb{C}) & \to & C_m \\ & X & \mapsto & \sum_{i=1}^{k_m} p_i X p_i. \end{array}$$

We claim that E is a conditional expectation. Firstly, if $X \in C_m$ then there exists $Y \in M_{n_m}(\mathbb{C})$ such that $X = \sum_{i=1}^{k_m} p_i Y p_i$. So,

$$E(X) = \sum_{i=1}^{k_m} p_i (\sum_{j=1}^{k_m} p_j Y p_j) p_i = \sum_{i=1}^{k_m} p_i Y p_i = X$$

because if $i, j \in \{1, 2, \ldots, k_m\}$ are distinct then $p_i p_j = p_j p_i = 0$. It is obvious that E is a linear map.

To see that E is a contractive map, observe that if $X \in M_{n_m}(\mathbb{C})$ then

$$\|E(X)\| = \|\sum_{i=1}^{k_m} p_i X p_i\| = \max_{i \in \{1,\ldots,k_m\}} \|p_i X p_i\| \leq \|X\|.$$

In the second equality, we used the norm of the direct sum $\bigoplus_{i=1}^{k_m} M_{n_{i,m}}(\mathbb{C})$ which is isomorphic to C_m. By Theorem 3.2.3, E is a conditional expectation.

Now consider the composites

$$\iota_n \circ \varphi_n : A \to M_{n_n}(\mathbb{C}) \qquad \text{and} \qquad \psi_n \circ E : M_{n_n}(\mathbb{C}) \to B.$$

By Theorem 3.2.3, $\psi_n \circ E$ is a c.c.p map. Since the embedding $\iota_n : C_n \to M_{n_n}(\mathbb{C})$ is a *-homomorphism then it is a c.c.p map. Consequently, the composite $\iota_n \circ \varphi_n$ is also a c.c.p map. Finally, if $a \in A$ then

$$\begin{aligned}
\lim_n \|(\psi_n \circ E \circ \iota_n \circ \varphi_n)(a) - \theta(a)\| &= \lim_n \|(\psi_n \circ E)(\iota_n(\varphi_n(a))) - \theta(a)\| \\
&= \lim_n \|(\psi_n \circ \varphi_n)(a) - \theta(a)\| \\
&= 0
\end{aligned}$$

where the second line follows from the fact that E is a conditional expectation. Therefore, θ is a nuclear map. □

The next two results constitute examples of how the nuclearity of maps depends on the ranges of the maps.

Theorem 3.5.11. *Let A and B be C^*-algebras and $\theta : A \to B$ be a nuclear map. Let $C \subseteq B$ be a C^*-subalgebra of B satisfying the following two properties:*

1. *The image $\theta(A) \subseteq C$*
2. *There exists a conditional expectation $\Phi : B \to C$.*

Then, the map $\theta : A \to C$ is also a nuclear map.

Proof. Assume that A and B are C*-algebras and that $C \subseteq B$ is a C*-subalgebra of B. Assume that $\theta : A \to B$ is a nuclear map. Assume that $\theta(A) \subseteq C$ and that we have a conditional expectation $\Phi : B \to C$.

Since θ is a nuclear map then there exist c.c.p maps

$$\varphi_n : A \to M_{k(n)}(\mathbb{C}) \qquad \text{and} \qquad \psi_n : M_{k(n)}(\mathbb{C}) \to B$$

such that if $a \in A$ then $\|(\psi_n \circ \varphi_n)(a) - \theta(a)\| \to 0$ in the limit over n.

The composite $\Phi \circ \psi_n : M_{k(n)}(\mathbb{C}) \to C$ is a composite of c.c.p maps and thus, is a c.c.p map itself. To see that $\theta : A \to C$ is a nuclear map, let θ_B be the nuclear map $\theta : A \to B$ and θ_C be the c.c.p map $\theta : A \to C$. Since Φ is a conditional expectation then $\theta_C = \Phi \circ \theta_B$ because $\theta(A) \subseteq C$. So,

$$\begin{aligned} \|(\Phi \circ \psi_n \circ \varphi_n)(a) - \theta_C(a)\| &= \|(\Phi \circ \psi_n \circ \varphi_n)(a) - (\Phi \circ \theta_B)(a)\| \\ &\leq \|(\psi_n \circ \varphi_n)(a) - \theta_B(a)\| \to 0 \end{aligned}$$

in the limit over n. The inequality follows from the fact that Φ is contractive. Therefore, the map $\theta_C = \theta : A \to C$ is a nuclear map. $\square$

Theorem 3.5.12. *Let A and B be C*-algebras and $\theta : A \to B$ be a nuclear map. Let $C \subseteq B$ be a C*-subalgebra satisfying the following two properties:*

1. *The image $\theta(A) \subseteq C$.*
2. *There exist a net of c.c.p maps $\Phi_n : B \to C$ such that if $c \in C$ then $\|\Phi_n|_C(c) - id_C(c)\| = \|\Phi_n(c) - c\| \to 0$ in the limit over n.*

Then, $\theta : A \to C$ is also a nuclear map.

Proof. Assume that A and B are C*-algebras. Assume that $\theta : A \to B$ is a nuclear map. Assume that C is a C*-subalgebra of B. Assume that $\theta(A) \subseteq C$ and there exist a net of c.c.p maps $\Phi_n : B \to C$ such that if $c \in C$ then

$$\|\Phi_n|_C(c) - id_C(c)\| = \|\Phi_n(c) - c\| \to 0$$

in the limit over n.

Since θ is a nuclear map then there exist c.c.p maps

$$\varphi_n : A \to M_{k(n)}(\mathbb{C}) \qquad \text{and} \qquad \psi_n : M_{k(n)}(\mathbb{C}) \to B$$

such that if $a \in A$ then $\|(\psi_n \circ \varphi_n)(a) - \theta(a)\| \to 0$ in the limit over n. Now since Φ_n and ψ_n are c.c.p maps then the composites

$$\Phi_n \circ \psi_n : M_{k(n)}(\mathbb{C}) \to C$$

are also c.c.p maps. Let θ_C be the map $\theta : A \to C$ and θ_B be the map $\theta : A \to B$. If $a \in A$ then

$$\begin{aligned}
\|(\Phi_n \circ \psi_n \circ \varphi_n)(a) - \theta_C(a)\| &\leq \|(\Phi_n \circ \psi_n \circ \varphi_n)(a) - (\Phi_n \circ \theta_B)(a)\| \\
&\quad + \|(\Phi_n \circ \theta_B)(a) - \theta_C(a)\| \\
&\leq \|(\psi_n \circ \varphi_n)(a) - \theta_B(a)\| + \|(\Phi_n \circ \theta_B)(a) - \theta_C(a)\| \\
&= \|(\psi_n \circ \varphi_n)(a) - \theta_B(a)\| + \|(\Phi_n \circ \theta_B)(a) - \theta_B(a)\| \\
&\to 0
\end{aligned}$$

in the limit over n. The second inequality follows from the fact that Φ_n is contractive. The final equality follows from the fact that $\theta(A) \subseteq C$. Therefore, the map $\theta_C = \theta : A \to C$ is a nuclear map as required. $\square$

The final result of this section concerns nuclear maps from quotient C*-algebras. It requires the following definition.

Definition 3.5.4. Let A be a unital C*-algebra and

$$0 \longrightarrow B \longrightarrow A \xrightarrow{\pi} C \longrightarrow 0$$

be a short exact sequence of C*-algebras. We say that the short exact sequence is **locally split** if for each finite dimensional operator system $E \subsetneq C$, there exists a u.c.p (unital completely positive) map $\sigma : E \to A$ such that $\pi \circ \sigma = id_E$.

In order to state the next theorem properly, we need to make a construction. Let A and B be unital C*-algebras and $\theta : A \to B$ be a unital nuclear map. Suppose that there exists a closed two-sided ideal J such that $\theta|_J = 0$. Define the map

$$\begin{array}{rccc} \tilde{\theta}: & A/J & \to & B \\ & a + J & \mapsto & \theta(a) \end{array} \tag{3.5}$$

Since $\theta|_J = 0$ then $\tilde{\theta}$ is a well-defined linear map. Note that $\tilde{\theta}$ is unital because $\tilde{\theta}(1_A + J) = \theta(1_A) = 1_B$. To see that $\tilde{\theta}$ is completely positive, assume that $n \in \mathbb{Z}_{>0}$ and let $\tilde{\theta}_n : M_n(A/J) \to M_n(B)$ be the inflation of $\tilde{\theta}$.

Let $X \in M_n(A/J)$ be positive. Then, there exists $Y \in M_n(A/J)$ such that $X = Y^*Y$. Let $Y = (y_{ij} + J)$. Then,

$$\begin{aligned}\tilde{\theta}_n(X) = \tilde{\theta}_n(Y^*Y) &= \tilde{\theta}_n\big((\sum_{k=1}^{n} y_{ki}^* y_{kj} + J)\big)\\ &= \big(\theta(\sum_{k=1}^{n} y_{ki}^* y_{kj})\big)\end{aligned}$$

which is a positive matrix in $M_n(B)$. Thus, the linear map $\tilde{\theta}$ in equation (3.5) is a well-defined u.c.p map.

The last theorem we will prove in this section gives criteria for the map in equation (3.5) to be nuclear.

Theorem 3.5.13. *Let A and B be unital C*-algebras and $\theta : A \to B$ be a unital nuclear map. Suppose that there exists a closed two-sided ideal J such that $\theta|_J = 0$. Let $\pi : A \to A/J$ be the projection map. If the short exact sequence*

$$0 \longrightarrow J \hookrightarrow A \xrightarrow{\pi} A/J \longrightarrow 0$$

is locally split then the u.c.p map $\tilde{\theta} : A/J \to B$ in equation (3.5) is a nuclear map.

Proof. Assume that A and B are unital C*-algebras. Assume that $\theta : A \to B$ is a unital nuclear map and that there exists a closed two-sided ideal J such that $\theta|_J = 0$. Assume that the short exact sequence

$$0 \longrightarrow J \hookrightarrow A \xrightarrow{\pi} A/J \longrightarrow 0$$

is locally split. Note that since A is unital then A/J is also a unital C*-algebra with unit given by $1_A + J$.

We will use Theorem 3.5.8 to prove that $\tilde{\theta}$ is nuclear. Assume that $F \subsetneq A/J$ is a finite subset of A/J and $\epsilon \in \mathbb{R}_{>0}$. Let E_F be the finite dimensional C*-algebra generated by the finite set $\{1_A + J\} \cup F$. Note that E_F is a finite dimensional operator system. Since the short exact sequence above is locally split then there exists a u.c.p map $\sigma : E_F \to A$ such that $\pi \circ \sigma = id_{E_F}$.

We know that $\|\sigma\| = \|\sigma(1_A + J)\| = 1$ by [Pau02, Corollary 2.9]. We emphasise that we can apply [Pau02, Corollary 2.9] because E_F is a unital C*-algebra by construction. This means that σ is a c.c.p map. The image $\sigma(F)$ is a finite subset of A. Since θ is nuclear then there exist c.c.p maps

$$\varphi : A \to M_n(\mathbb{C}) \qquad \text{and} \qquad \psi : M_n(\mathbb{C}) \to B$$

such that if $a \in \sigma(F)$ then

$$\|(\psi \circ \varphi)(a) - \theta(a)\| < \epsilon.$$

The composite $\varphi \circ \sigma : E_F \to M_n(\mathbb{C})$ is a c.c.p map. By Arveson's extension theorem (see Theorem 3.3.7), there exists a c.c.p map $\rho : A/J \to M_n(\mathbb{C})$ such that $\rho|_{E_F} = \varphi \circ \sigma$. So if $f + J \in F$ then

$$\begin{aligned} \|(\psi \circ \rho)(f + J) - \tilde{\theta}(f + J)\| &= \|(\psi \circ \varphi \circ \sigma)(f + J) - \theta(f)\| \\ &\leq \|(\psi \circ \varphi)(\sigma(f + J)) - \theta(\sigma(f + J))\| \\ &\quad + \|\theta(\sigma(f + J)) - \theta(f)\|. \end{aligned}$$

Notice that

$$\pi(\sigma(f + J) - f) = (\pi \circ \sigma)(f + J) - (f + J) = 0 + J$$

because $\pi \circ \sigma = id_{E_F}$. This means that $\sigma(f + J) - f \in J$. Using the fact that $\theta|_J = 0$, we find that

$$\|(\psi \circ \rho)(f + J) - \tilde{\theta}(f + J)\| \leq \|(\psi \circ \varphi)(\sigma(f + J)) - \theta(\sigma(f + J))\| < \epsilon.$$

By Theorem 3.5.8, $\tilde{\theta}$ is a nuclear map. $\square$

3.6 Extensions of c.c.p maps

As stated in [BO08, Section 2.2], many arguments involving c.c.p maps are easier in the presence of unital C*-algebras. This sections contains results pertaining to extensions of completely positive maps which involve unital C*-algebras.

Recall from Theorem 3.2.2 (which is [BO08, Theorem 1.4.1]) that the enveloping von Neumann algebra of a C*-algebra is isometrically isomorphic (as a Banach space) to the double dual of the C*-algebra. We will make liberal use of this identification in the following extension theorem.

Theorem 3.6.1. *Let A and B be C*-algebras. Assume that A is not unital and that B is unital. Let $\varphi : A \to B$ be a c.c.p map. Then, there exists a linear map*

$$\begin{array}{rccc} \tilde{\varphi} : & \tilde{A} & \to & B \\ & a + \lambda 1_{\tilde{A}} & \mapsto & \varphi(a) + \lambda 1_B \end{array}$$

defined on the unitization $\tilde{A}$ such that $\tilde{\varphi}$ is a u.c.p map and $\tilde{\varphi}|_A = \varphi$.

Proof. Assume that A is a non-unital C*-algebra and that B is a unital C*-algebra. Assume that $\varphi : A \to B$ is a c.c.p map. It is easy to see that the map $\tilde{\varphi}$ defined as above is a linear map. The map $\tilde{\varphi}$ is unital because

$$\tilde{\varphi}(1_{\tilde{A}}) = \varphi(0) + 1_B = 1_B.$$

Also, if $a \in A$ then $\tilde{\varphi}(a) = \varphi(a) + 01_B = \varphi(a)$.

It remains to show that $\tilde{\varphi}$ is a completely positive map. The idea behind the proof is to consider the double adjoint map $\varphi^{**} : A^{**} \to B^{**}$.

To show: (a) The map φ^{**} is positive.

(a) Here, we identify the double duals A^{**} and B^{**} with the enveloping von Neumann algebras of A and B respectively. Assume that $\alpha \in A^{**}$ is positive. By Theorem 2.5.10, Theorem 2.5.11 and Theorem 2.4.5, the enveloping von Neumann algebra A^{**} is the weak closure of A (more accurately, the image of A under its universal representation, but this is isomorphic to A as a C*-algebra). Thus, there exists a net $\{a_i\}_{i\in I}$ of positive elements in A which converges to α in the weak-* topology on A^{**}.

Now assume that $\beta \in B^*$ is a positive linear functional. We compute directly that

$$\begin{aligned} \varphi^{**}(\alpha)(\beta) = \alpha(\varphi^*(\beta)) &= \lim_i a_i(\varphi^*(\beta)) \\ &= \lim_i \varphi^*(\beta)(a_i) \\ &= \lim_i \beta(\varphi(a_i)) \geq 0. \end{aligned}$$

In the first and last equalites, we used the definition of the adjoint map. The inequality follows from the fact that β and φ are positive maps. Thus, φ^{**} is a positive map.

Now we claim that φ^{**} is a completely positive map. If $n \in \mathbb{Z}_{>0}$ then let

$$(\varphi^{**})_n : M_n(A^{**}) \to M_n(B^{**})$$

be the inflation of φ^{**}. To see that $(\varphi^{**})_n$ is positive, we will use the fact that if C is an arbitrary C*-algebra and $n \in \mathbb{Z}_{>0}$ then $M_n(C^{**}) \cong (M_n(C))^{**}$. Under this isomorphism, the linear map $(\varphi^{**})_n$ becomes

$$(\varphi_n)^{**} : M_n(A)^{**} \to M_n(B)^{**}$$

which by part (a) is positive. Since $n \in \mathbb{Z}_{>0}$ was arbitrary then φ^{**} is completely positive.

Now, identify the unitization $\tilde{A}$ with $A + \mathbb{C}1_{A^{**}} \subsetneq A^{**}$ by sending $a + \lambda 1_{\tilde{A}} \in \tilde{A}$ to $a + \lambda 1_{A^{**}}$. Hence, we can consider φ^{**} as a map on $\tilde{A}$. Now we will show that $\tilde{\varphi}$ is completely positive.

To show: (b) $\tilde{\varphi}$ is a completely positive map.

(b) Assume that $n \in \mathbb{Z}_{>0}$ and $(a_{ij} + \lambda_{ij}1_{A^{**}}) \in M_n(\tilde{A})$ be a positive element. Then,

$$\tilde{\varphi}_n\big((a_{ij} + \lambda_{ij}1_{A^{**}})\big) = (\varphi(a_{ij}) + \lambda 1_B).$$

Since the map φ^{**} is completely positive then it suffices to show that

$$\tilde{\varphi}_n\big((a_{ij} + \lambda_{ij}1_{A^{**}})\big) \geq (\varphi^{**})_n\big((a_{ij} + \lambda_{ij}1_{A^{**}})\big).$$

We compute directly that

$$\begin{aligned}
&\tilde{\varphi}_n\big((a_{ij} + \lambda_{ij}1_{A^{**}})\big) - (\varphi^{**})_n\big((a_{ij} + \lambda_{ij}1_{A^{**}})\big) \\
&= (\varphi(a_{ij}) + \lambda_{ij}1_B) - (\varphi(a_{ij}) + \lambda_{ij}\varphi^{**}(1_{A^{**}})) \\
&= \big(\lambda_{ij}(1_B - \varphi^{**}(1_{A^{**}})\big) \\
&= (\lambda_{ij}1_B)\; diag\big((1_B - \varphi^{**}(1_{A^{**}})\big).
\end{aligned}$$

To be clear, $diag\big((1_B - \varphi^{**}(1_{A^{**}})\big) \in M_n(B^{**})$ is the diagonal matrix whose diagonal entries are all $1_B - \varphi^{**}(1_{A^{**}}) \in B^{**}$. Since $(a_{ij} + \lambda_{ij}1_{A^{**}}) \geq 0$ and $M_n(\mathbb{C})$ is a quotient of $M_n(\tilde{A})$ then $(\lambda_{ij}1_B) \geq 0$. Furthermore, the positive matrix $(\lambda_{ij}1_B)$ commutes with the scalar matrix $diag\big((1_B - \varphi^{**}(1_{A^{**}})\big)$.

To see that $diag\big((1_B - \varphi^{**}(1_{A^{**}})\big) \geq 0$, note that

$$0 \leq \varphi^{**}(1_{A^{**}}) = \varphi^{**}(1_{A^{**}})1_{B^{**}} \leq \|\varphi\| 1_{B^{**}} \leq 1_B$$

because φ is a c.c.p map. Therefore, the product of matrices

$$(\lambda_{ij} 1_B)\ diag\big((1_B - \varphi^{**}(1_{A^{**}})\big)$$

is a product of commuting positive operators, which is again positive. Therefore,

$$\tilde{\varphi}_n\big((a_{ij} + \lambda_{ij} 1_{A^{**}})\big) \geq (\varphi^{**})_n\big((a_{ij} + \lambda_{ij} 1_{A^{**}})\big)$$

and $\tilde{\varphi} : \tilde{A} \to B$ is a u.c.p map as required. □

We observe that in Theorem 3.6.1, the norm of $\tilde{\varphi}$ is generally larger than that of φ. In particular, we claim that $\|\tilde{\varphi}\| > \|\varphi\|$ if and only if $\|\varphi\| < 1$.

Assume that $\|\varphi\| < 1$. The map $\tilde{\varphi} : \tilde{A} \to B$ constructed in Theorem 3.6.1 is a u.c.p map between the unital C*-algebras $\tilde{A}$ and B. By [Pau02, Corollary 2.9],

$$\|\tilde{\varphi}\| = \|\tilde{\varphi}(1_{\tilde{A}})\| = \|1_B\| = 1.$$

Conversely, if $\|\tilde{\varphi}\| > \|\varphi\|$ then $\|\varphi\| < \|\tilde{\varphi}\| = 1$. We also note that if we insist that $\tilde{\varphi}(1_{\tilde{A}}) = \|\varphi\| 1_B$ then the proof of Theorem 3.6.1 still works and in addition, $\|\tilde{\varphi}\| = \|\varphi\|$. Usually, the fact that $\tilde{\varphi}$ is a unital map is more useful than forcing $\|\tilde{\varphi}\| = \|\varphi\|$.

Theorem 3.6.2. *Let A and B be unital C*-algebras and $\varphi : A \to B$ be a c.c.p map. Then, the map*

$$\begin{array}{rccc} \tilde{\varphi}: & A \oplus \mathbb{C} & \to & B \\ & a \oplus \lambda & \mapsto & \varphi(a) + \lambda(1_B - \varphi(1_A)) \end{array}$$

is a u.c.p map such that $\tilde{\varphi}|_A = \varphi$.

Proof. Assume that A and B are unital C*-algebras. Assume that $\varphi : A \to B$ is a c.c.p map. The map $\tilde{\varphi}$ is by direct computation a unital linear map satisfying $\tilde{\varphi}|_A = \varphi$.

It remains to show that $\tilde{\varphi}$ is completely positive. Using the fact that φ is a c.c.p map, we have

$$\varphi(1_A) = \varphi(1_A)1_B \leq \|\varphi\| 1_B \leq 1_B.$$

This means that the linear map

$$\begin{array}{rccc} \phi: & \mathbb{C} & \to & B \\ & \lambda & \mapsto & \lambda(1_B - \varphi(1_A)) \end{array}$$

is positive. Moreover, if $n \in \mathbb{Z}_{>0}$ and $(\lambda_{ij}) \in M_n(\mathbb{C})$ is positive then

$$\phi_n\big((\lambda_{ij})\big) = (\lambda_{ij}(1_B - \varphi(1_A)) = (\lambda_{ij}1_B)diag(1_B - \varphi(1_A)).$$

Arguing in a similar manner to Theorem 3.6.1, we find that $\phi_n\big((\lambda_{ij})\big)$ is a product of two commuting positive operators in $M_n(B)$, which is again positive. Therefore, ϕ is completely positive.

Now, $\tilde{\varphi} = \varphi + \phi$ by definition. Since $\tilde{\varphi}$ is the sum of two c.p maps, it must also be completely positive. Thus, $\tilde{\varphi}$ is a u.c.p map satisfying $\tilde{\varphi}|_A = \varphi$. □

Next, we will show that extending c.c.p maps with Theorem 3.6.1 preserves nuclearity.

Theorem 3.6.3. *Let A and B be C*-algebras and $\theta : A \to B$ be a nuclear map.*

1. *If A is not unital and B is a unital C*-algebra then the u.c.p extension of φ given in Theorem 3.6.1 is a nuclear map.*
2. *If A and B are both not unital then the unique unital extension $\overline{\theta} : \tilde{A} \to \tilde{B}$ is also a nuclear map.*

Proof. Assume that A and B are C*-algebras and that $\theta : A \to B$ is a nuclear map. Then, there exist c.c.p maps

$$\varphi_n : A \to M_{k(n)}(\mathbb{C}) \qquad \text{and} \qquad \psi_n : M_{k(n)}(\mathbb{C}) \to B$$

such that if $a \in A$ then

$$\lim_n \|(\psi_n \circ \varphi_n)(a) - \theta(a)\| = 0.$$

First assume that A is not unital and B is unital. Let $\tilde{\theta} : \tilde{A} \to B$ be the u.c.p extension of θ constructed in Theorem 3.6.1. By applying Theorem 3.6.1 to the c.c.p maps φ_n, we can extend them to u.c.p maps

$$\tilde{\varphi}_n : \tilde{A} \to M_{k(n)}(\mathbb{C}) \oplus \mathbb{C}$$

where we regard $M_{k(n)}(\mathbb{C})$ as a C*-subalgebra of $M_{k(n)}(\mathbb{C}) \oplus \mathbb{C}$. By [Pau02, Corollary 2.9], $\|\tilde{\varphi}_n\| = 1$ and so, $\tilde{\varphi}_n$ is a c.c.p map.

By Theorem 3.6.2, we can extend each ψ_n to u.c.p maps

$$\tilde{\psi}_n : M_{k(n)}(\mathbb{C}) \oplus \mathbb{C} \to B$$

By another application of [Pau02, Corollary 2.9], $\tilde{\psi}_n$ is a c.c.p map. We compute directly that if $a \in A$ and $\lambda \in \mathbb{C}$ then

$$\begin{aligned}
&\|(\tilde{\psi}_n \circ \tilde{\varphi}_n)(a + \lambda 1_{\tilde{A}}) - \tilde{\theta}(a + \lambda 1_{\tilde{A}})\| \\
&\leq \|(\tilde{\psi}_n \circ \tilde{\varphi}_n)(a) - \tilde{\theta}(a)\| + |\lambda| \|(\tilde{\psi}_n \circ \tilde{\varphi}_n)(1_{\tilde{A}}) - \tilde{\theta}(1_{\tilde{A}})\| \\
&= \|(\psi_n \circ \varphi_n)(a) - \theta(a)\| + |\lambda| \|1_B - 1_B\| \\
&= \|(\psi_n \circ \varphi_n)(a) - \theta(a)\| \to 0
\end{aligned}$$

in the limit over n. In the first equality above, we used the fact that the maps $\tilde{\psi}_n \circ \tilde{\varphi}_n$ and $\tilde{\theta}$ are unital. By Theorem 3.5.10, we find that $\tilde{\theta}$ is a nuclear map.

Next assume that both A and B are not unital. The unique unital extension $\overline{\theta} : \tilde{A} \to \tilde{B}$ is given by applying Theorem 3.6.1 to $\iota \circ \theta : A \to \tilde{B}$, where $\iota : B \hookrightarrow \tilde{B}$ is the inclusion map, which is a c.c.p map. By a similar argument to the proof of the first statement, we obtain unital c.c.p maps

$$\tilde{\varphi}_n : \tilde{A} \to M_{k(n)}(\mathbb{C}) \oplus \mathbb{C}.$$

By applying Theorem 3.6.2 to the composite $\iota \circ \psi_n : M_{k(n)}(\mathbb{C}) \to \tilde{B}$, we obtain unital c.c.p maps

$$\tilde{\psi}_n : M_{k(n)}(\mathbb{C}) \oplus \mathbb{C} \to \tilde{B}.$$

Hence, if $a \in A$ and $\lambda \in \mathbb{C}$ then

$$\begin{aligned}
&\|(\tilde{\psi}_n \circ \tilde{\varphi}_n)(a + \lambda 1_{\tilde{A}}) - \overline{\theta}(a + \lambda 1_{\tilde{A}})\| \\
&\leq \|(\tilde{\psi}_n \circ \tilde{\varphi}_n)(a) - \overline{\theta}(a)\| + |\lambda| \|(\tilde{\psi}_n \circ \tilde{\varphi}_n)(1_{\tilde{A}}) - \overline{\theta}(1_{\tilde{A}})\| \\
&= \|(\iota \circ \psi_n \circ \varphi_n)(a) - (\iota \circ \theta)(a)\| + |\lambda| \|1_{\tilde{B}} - 1_{\tilde{B}}\| \\
&\leq \|(\psi_n \circ \varphi_n)(a) - \theta(a)\| \to 0
\end{aligned}$$

By Theorem 3.5.10, we deduce that $\overline{\theta}$ is nuclear as required. □

Theorem 3.6.3 is commonly used to assume that C*-algebras are unital in arguments involving nuclear maps. The next theorem allows us to further reduce these arguments to the case of unital maps.

Theorem 3.6.4. *Let A be a unital C*-algebra and $\tilde{\varphi} : A \to M_n(\mathbb{C})$ be a completely positive map. Then, there exists a u.c.p map $\varphi : A \to M_n(\mathbb{C})$ such that if $a \in A$ then*

$$\tilde{\varphi}(a) = \tilde{\varphi}(1_A)^{\frac{1}{2}}\varphi(a)\tilde{\varphi}(1_A)^{\frac{1}{2}}.$$

Proof. Assume that A is a unital C*-algebra and that $\tilde{\varphi} : A \to M_n(\mathbb{C})$ is a completely positive map. There are two cases to consider:

Case 1: $\tilde{\varphi}(1_A) \in M_n(\mathbb{C})$ is invertible.

If the matrix $\tilde{\varphi}(1_A) \in M_n(\mathbb{C})$ is invertible then we define the map φ by

$$\begin{array}{rccc} \varphi : & A & \to & M_n(\mathbb{C}) \\ & a & \mapsto & \tilde{\varphi}(1_A)^{-\frac{1}{2}}\tilde{\varphi}(a)\tilde{\varphi}(1_A)^{-\frac{1}{2}}. \end{array}$$

Then, φ is a unital linear map such that if $a \in A$ then

$$\tilde{\varphi}(a) = \tilde{\varphi}(1_A)^{\frac{1}{2}}\varphi(a)\tilde{\varphi}(1_A)^{\frac{1}{2}}.$$

To see that φ is completely positive, assume that $m \in \mathbb{Z}_{>0}$ and $(a_{ij}) \in M_m(A)$. We compute directly that

$$\varphi_m((a_{ij})) = \left(\tilde{\varphi}(1_A)^{\frac{1}{2}}\tilde{\varphi}(a_{ij})\tilde{\varphi}(1_A)^{\frac{1}{2}}\right) = diag(\tilde{\varphi}(1_A)^{\frac{1}{2}})(\tilde{\varphi}(a_{ij}))diag(\tilde{\varphi}(1_A)^{\frac{1}{2}}).$$

Hence, $\varphi_m((a_{ij}))$ is positive by property 4 in Theorem 2.2.2. Therefore, φ is a completely positive map.

Case 2: $\tilde{\varphi}(1_A) \in M_n(\mathbb{C})$ is not invertible.

We will apply the reasoning in the first case to deal with this case. Let P be the projection operator onto $\ker \tilde{\varphi}(1_A) \subseteq M_n(\mathbb{C})$. Let $I_n \in M_n(\mathbb{C})$ be the unit and $P^{\perp} = I_n - P$ be the projection operator onto the orthogonal complement $(\ker \tilde{\varphi}(1_A))^{\perp}$.

To show: (a) If $a \in A$ then $\tilde{\varphi}(a) = P^{\perp}\tilde{\varphi}(a) = \tilde{\varphi}(a)P^{\perp}$.

(a) By Theorem 1.4.1, it suffices to prove the assertion in the case where $a \in A$ is positive. By the proof of Theorem 2.2.5, if $a \in A$ is positive then $0 \leq a \leq \|a\|1_A$. By linearity, we may assume that $\|a\| = 1$ so that $0 \leq a \leq 1_A$. Since $\tilde{\varphi}$ is positive then $0 \leq \tilde{\varphi}(a) \leq \tilde{\varphi}(1_A)$.

We will show that $\ker \tilde{\varphi}(1_A) \subseteq \ker \tilde{\varphi}(a)$. We identify $M_n(\mathbb{C})$ with $B(\mathbb{C}^n)$. Assume that $\xi \in \ker \tilde{\varphi}(1_A) \subsetneq \mathbb{C}^n$ so that $\tilde{\varphi}(1_A)\xi = 0$. By assumption, $\tilde{\varphi}(1_A - a)$ is a positive operator and consequently,

$$0 \leq \langle \tilde{\varphi}(1_A - a)\xi, \xi \rangle = -\langle \tilde{\varphi}(a)\xi, \xi \rangle.$$

Since $\tilde{\varphi}(a)$ is a positive operator then $\langle \tilde{\varphi}(a)\xi, \xi \rangle \geq 0$ and consequently, $\langle \tilde{\varphi}(a)\xi, \xi \rangle = 0$.

Now define the map

$$\begin{array}{rccc} F: & \mathbb{C}^n \times \mathbb{C}^n & \to & \mathbb{C} \\ & (\xi, \mu) & \mapsto & \langle \tilde{\varphi}(a)\xi, \mu \rangle. \end{array}$$

Then, F is a sesquilinear form. By the polarization identity, we find that if $\xi, \mu \in \ker \tilde{\varphi}(1_A)$ then

$$\begin{aligned} \langle \tilde{\varphi}(a)\xi, \mu \rangle &= F(\xi, \mu) \\ &= \frac{1}{4} \sum_{k=0}^{3} i^k F(\xi + i^k \mu, \xi + i^k \mu) \\ &= \frac{1}{4} \sum_{k=0}^{3} i^k \langle \tilde{\varphi}(a)(\xi + i^k \mu), \xi + i^k \mu \rangle = 0 \end{aligned}$$

because $\xi + i^k \mu \in \ker \tilde{\varphi}(1_A)$. Now if $\alpha \in \mathbb{C}^n$ and $\xi \in \ker \tilde{\varphi}(1_A)$ then

$$0 = \langle \tilde{\varphi}(a)\xi, P\alpha \rangle = \langle P\tilde{\varphi}(a)\xi, \alpha \rangle$$

since $\text{im } P = \ker \tilde{\varphi}(1_A)$. Since $\alpha \in \mathbb{C}^n$ was arbitrary, we deduce that if $\xi \in \ker \tilde{\varphi}(1_A)$ then $P\tilde{\varphi}(a)\xi = 0$ and subsequently, $P\tilde{\varphi}(a)P = 0$. In particular,

$$P^{\perp}\tilde{\varphi}(a)P = \tilde{\varphi}(a)P \qquad \text{and} \qquad P\tilde{\varphi}(a)P^{\perp} = P\tilde{\varphi}(a).$$

So if $\xi \in \ker \tilde{\varphi}(1_A)$ then $P^{\perp}\tilde{\varphi}(a)\xi = \tilde{\varphi}(a)\xi$ and

$$\tilde{\varphi}(a)\xi - \tilde{\varphi}(a)P^{\perp}\xi = \tilde{\varphi}(a)P\xi = \tilde{\varphi}(a)\xi \in \ker \tilde{\varphi}(1_A)$$

However, $\tilde{\varphi}(a)\xi \in (\ker \tilde{\varphi}(1_A))^{\perp}$ because $P^{\perp}\tilde{\varphi}(a)\xi = \tilde{\varphi}(a)\xi$. Therefore,

$$\tilde{\varphi}(a)\xi \in \ker \tilde{\varphi}(1_A) \cap (\ker \tilde{\varphi}(1_A))^{\perp} = \{0\}$$

and $\ker \tilde{\varphi}(1_A) \subseteq \ker \tilde{\varphi}(a)$ as required.

Using this fact, we now have

$$\tilde{\varphi}(a) = \tilde{\varphi}(a)P + \tilde{\varphi}(a)P^{\perp} = \tilde{\varphi}(a)P^{\perp}$$

and by taking adjoints, $\tilde{\varphi}(a) = P^{\perp}\tilde{\varphi}(a)$.

The idea is that since $\tilde{\varphi}(a) = \tilde{\varphi}(a)P^{\perp} = P^{\perp}\tilde{\varphi}(a)$ then we can think of $\tilde{\varphi}$ as a c.p map from A to $P^{\perp}M_n(\mathbb{C})P^{\perp}$. By applying the proof of the first case, we obtain a u.c.p map $\varphi_1 : A \to P^{\perp}M_n(\mathbb{C})P^{\perp}$ satisfying the statement of the theorem. Note that $\varphi_1(1_A) = P^{\perp}$.

Now let $\eta : A \to \mathbb{C}$ be a state on A and define the map

$$\begin{array}{rccc} \varphi : & A & \to & M_n(\mathbb{C}) \\ & a & \mapsto & \varphi_1(a) \oplus \eta(a)P. \end{array}$$

Since η is a state on the unital C*-algebra A then $\eta(1_A) = 1$ and consequently, φ is a unital linear map. It is straightforward to check that φ is completely positive and that if $a \in A$ then

$$\tilde{\varphi}(a) = P^{\perp}\varphi(a)P^{\perp} = \tilde{\varphi}(1_A)^{\frac{1}{2}}\varphi(a)\tilde{\varphi}(1_A)^{\frac{1}{2}}.$$

□

Now we will use Theorem 3.6.4 to prove a characterisation of unital nuclear maps.

Theorem 3.6.5. *Let A and B be unital C*-algebras and $\theta : A \to B$ be a unital nuclear map. Then, there exist u.c.p maps*

$$\varphi_n : A \to M_{k(n)}(\mathbb{C}) \qquad \text{and} \qquad \psi_n : M_{k(n)}(\mathbb{C}) \to B$$

such that if $a \in A$ then

$$\|(\psi_n \circ \varphi_n)(a) - \theta(a)\| \to 0$$

in the limit over n.

Proof. Assume that A and B are unital C*-algebras and that $\theta : A \to B$ is a unital nuclear map. Then, there exist c.c.p maps

$$\tilde{\varphi}_n : A \to M_{k(n)}(\mathbb{C}) \qquad \text{and} \qquad \tilde{\psi}_n : M_{k(n)}(\mathbb{C}) \to B$$

such that if $a \in A$ then $\lim_n \|(\psi_n \circ \varphi_n)(a) - \theta(a)\| = 0$.

By Theorem 3.6.4, there exist u.c.p maps $\varphi_n : A \to M_{k(n)}(\mathbb{C})$ such that if $a \in A$ then

$$\tilde{\varphi}_n(a) = \tilde{\varphi}(1_A)^{\frac{1}{2}}\varphi_n(a)\tilde{\varphi}(1_A)^{\frac{1}{2}}.$$

The right u.c.p maps to replace the c.c.p maps ψ_n are trickier to describe. First, we note that since θ is a nuclear map then

$$\|\theta(1_A) - (\tilde{\psi}_n \circ \tilde{\varphi}_n)(1_A)\| = \|1_B - (\tilde{\psi}_n \circ \tilde{\varphi}_n)(1_A)\| \to 0$$

in the limit over n. This means that if n is sufficiently large then the element

$$(\tilde{\varphi}_n \circ \tilde{\varphi}_n)(1_A)$$

is positive and invertible in B. So, we define

$$b_n = \tilde{\psi}_n(\tilde{\varphi}_n(1_A))^{-\frac{1}{2}} \in B \qquad \text{and} \qquad Y_n = \tilde{\varphi}_n(1_A)^{\frac{1}{2}} \in M_{k(n)}(\mathbb{C}).$$

By the continuous functional calculus on $(\tilde{\psi}_n \circ \tilde{\varphi}_n)(1_A)$, we have

$$\|1_B - b_n\| = \|1_B - \tilde{\psi}_n(\tilde{\varphi}_n(1_A))^{-\frac{1}{2}}\| \to 0$$

in the limit over n. Now define the map ψ_n by

$$\begin{array}{rccc} \psi_n : & M_{k(n)}(\mathbb{C}) & \to & B \\ & T & \mapsto & b_n\tilde{\psi}_n(Y_nTY_n)b_n. \end{array}$$

Then, ψ_n is a u.c.p map (see the fourth statement of Theorem 2.2.2). If $a \in A$ then

$$\begin{aligned}
\|(\psi_n \circ \varphi_n)(a) - \theta(a)\| &= \|b_n\tilde{\psi}_n(Y_n\varphi_n(a)Y_n)b_n - \theta(a)\| \\
&= \|b_n\tilde{\psi}_n(\tilde{\varphi}(1_A)^{\frac{1}{2}}\varphi_n(a)\tilde{\varphi}(1_A)^{\frac{1}{2}})b_n - \theta(a)\| \\
&= \|b_n(\tilde{\psi}_n \circ \tilde{\varphi}_n)(a)b_n - \theta(a)\| \\
&\leq \|b_n(\tilde{\psi}_n \circ \tilde{\varphi}_n)(a)b_n - b_n(\tilde{\psi}_n \circ \tilde{\varphi}_n)(a)\| \\
&\quad + \|b_n(\tilde{\psi}_n \circ \tilde{\varphi}_n)(a) - (\tilde{\psi}_n \circ \tilde{\varphi}_n)(a)\| \\
&\quad + \|(\tilde{\psi}_n \circ \tilde{\varphi}_n)(a) - \theta(a)\| \\
&\leq \|b_n(\tilde{\psi}_n \circ \tilde{\varphi}_n)(a)\|\|b_n - 1_B\| \\
&\quad + \|b_n - 1_B\|\|(\tilde{\psi}_n \circ \tilde{\varphi}_n)(a)\| + \|(\tilde{\psi}_n \circ \tilde{\varphi}_n)(a) - \theta(a)\| \\
&\to 0
\end{aligned}$$

in the limit over n. This completes the proof. □

Next, we provide two characterisations of unital weakly nuclear maps. Pay particular attention to the conditions required for each characterisation to hold.

Theorem 3.6.6. *Let A be a unital C*-algebra and N be a von Neumann algebra. Let $\theta : A \to N$ be a unital weakly nuclear map. Then, there exist u.c.p maps*

$$\varphi_n : A \to M_{k(n)}(\mathbb{C}) \qquad \textit{and} \qquad \psi_n : M_{k(n)}(\mathbb{C}) \to N$$

such that if $a \in A$ and $\eta \in N_*$ is a normal linear functional then

$$\eta((\psi_n \circ \varphi_n)(a)) \to \eta(\theta(a)).$$

Proof. Assume that A is a unital C*-algebra and N is a von Neumann algebra. Assume that $\theta : A \to N$ is a unital weakly nuclear map so that there exist c.c.p maps

$$\tilde{\varphi}_n : A \to M_{k(n)}(\mathbb{C}) \qquad \text{and} \qquad \tilde{\psi}_n : M_{k(n)}(\mathbb{C}) \to N$$

such that if $a \in A$ then $(\tilde{\psi}_n \circ \tilde{\varphi}_n)(a)$ converges to $\theta(a)$ in the ultraweak topology on N. By Theorem 3.6.4, there exist u.c.p maps $\varphi_n : A \to M_{k(n)}(\mathbb{C})$ such that if $a \in A$ then

$$\tilde{\varphi}_n(a) = \tilde{\varphi}(1_A)^{\frac{1}{2}} \varphi_n(a) \tilde{\varphi}(1_A)^{\frac{1}{2}}.$$

Using the fact that $\tilde{\psi}_n$ and $\tilde{\varphi}_n$ are contractive, positive maps, we have

$$0 \leq \tilde{\varphi}_n(1_A) \leq \|\tilde{\varphi}_n(1_A)\| 1_{M_{k(n)}(\mathbb{C})} \leq 1_{M_{k(n)}(\mathbb{C})}.$$

and

$$\tilde{\psi}_n(\tilde{\varphi}_n(1_A)) \leq \tilde{\psi}_n(1_{M_{k(n)}(\mathbb{C})}) \leq \|\tilde{\varphi}_n\| 1_N \leq 1_N.$$

Furthermore, the net $\{(\tilde{\psi}_n \circ \tilde{\varphi}_n)(1_A)\}_n$ converges in the ultraweak topology on N to $\theta(1_A) = 1_N$ because θ is unital. So, if we define

$$b_n = 1_N - \tilde{\psi}_n(\tilde{\varphi}_n(1_A))$$

then $\{b_n\}_n$ is a net of positive operators which converges in the ultraweak topology on N to 0.

Now let $\{\rho_n\}_n$ be an arbitrary net of states on $M_{k(n)}(\mathbb{C})$ and define linear maps ψ_n by

$$\begin{array}{rccc} \psi_n : & M_{k(n)}(\mathbb{C}) & \to & N \\ & T & \mapsto & \rho_n(T)b_n + \tilde{\psi}_n\big(\tilde{\varphi}_n(1_A)^{\frac{1}{2}} T \tilde{\varphi}_n(1_A)^{\frac{1}{2}}\big). \end{array}$$

Firstly, ψ_n is unital because

$$\psi_n(1_{M_{k(n)}(\mathbb{C})}) = 1 \cdot b_n + (\tilde{\psi}_n \circ \tilde{\varphi}_n)(1_A) = 1_N.$$

Since ψ_n is a sum of positive maps then ψ_n is itself a positive map. To see that ψ_n is completely positive, note that if $m \in \mathbb{Z}_{>0}$ and
$\tau = (T_{ij}) \in M_m(M_{k(n)}(\mathbb{C}))$ then

$$(\psi_n)^m(\tau) = (\rho_n(T_{ij}))\ diag(b_n) + (\tilde{\psi}_n)^m\big((\tilde{\varphi}_n(1_A)^{\frac{1}{2}} T_{ij} \tilde{\varphi}_n(1_A)^{\frac{1}{2}})\big)$$

which is the sum of two positive elements in $M_m(N)$ and is hence positive. Here, we used the fact that the states ρ_n are positive linear functionals by definition and are thus, completely positive by Example 3.1.2. So, ψ_n is a u.c.p map.

Now if $\eta \in N_*$ is a normal linear functional and $a \in A$ then

$$\begin{aligned} \eta((\psi_n \circ \varphi_n)(a)) &= \eta\big(\rho_n(\varphi_n(a))b_n + \tilde{\psi}_n\big(\tilde{\varphi}_n(1_A)^{\frac{1}{2}} \varphi_n(a) \tilde{\varphi}_n(1_A)^{\frac{1}{2}}\big)\big) \\ &= \rho_n(\varphi_n(a))\eta(b_n) + \eta((\tilde{\psi}_n \circ \tilde{\varphi}_n)(a)) \\ &\to \rho_n(\varphi_n(a))0 + \eta(\theta(a)) = \eta(\theta(a)) \end{aligned}$$

in the limit over n. Therefore, if $a \in A$ then the net $\{(\psi_n \circ \varphi_n)(a)\}_n$ converges ultraweakly to θ, which completes the proof. □

Theorem 3.6.7. *Let M and N be von Neumann algebras and $\theta : M \to N$ be a unital weakly nuclear map. Then, there exist normal u.c.p maps*

$$\varphi_n : M \to M_{k(n)}(\mathbb{C}) \qquad \text{and} \qquad \psi_n : M_{k(n)}(\mathbb{C}) \to N$$

such that if $a \in A$ and $\eta \in N_*$ is a normal linear functional then

$$\eta((\psi_n \circ \varphi_n)(a)) \to \eta(\theta(a)).$$

Proof. Assume that M and N are von Neumann algebras. Assume that $\theta : M \to N$ is a unital weakly nuclear map. Assume that $\epsilon \in \mathbb{R}_{>0}$, $\mathcal{F}$ is a finite subset of M and $\chi \subsetneq N^*$ is a finite set of normal linear functionals on N.

By Theorem 3.6.6, there exist u.c.p maps $\tilde{\varphi} : M \to M_n(\mathbb{C})$ and $\psi : M_n(\mathbb{C}) \to N$ such that if $m \in \mathcal{F}$ and $\eta \in \chi$ then

$$|\eta(\theta(m)) - \eta((\psi \circ \tilde{\varphi})(m))| < \epsilon.$$

By construction, ψ is a normal map. It remains to replace $\tilde{\varphi}$ with a normal u.c.p map. We turn to [BO08, Corollary 1.6.3] (which is a consequence of Arveson's extension theorem) to obtain a net of normal u.c.p maps $\{\varphi_\lambda\}_{\lambda \in \Lambda}$ such that if $m \in M$ then

$$\lim_{\lambda \in \Lambda} \|\varphi_\lambda(m) - \tilde{\varphi}(m)\| = 0.$$

Select $\mu \in \Lambda$ such that if $m \in M$ then

$$\|\varphi_\mu(m) - \tilde{\varphi}(m)\| < \frac{\epsilon}{2 \max_{\eta \in \chi} \|\eta\|}.$$

By replacing $\tilde{\varphi}$ with the normal u.c.p map φ_μ and using a standard ϵ argument, we are done. $\square$

3.7 Nuclearity and exactness via c.c.p maps

Now we are ready to examine nuclear C*-algebras from the point of view of nuclear maps.

Definition 3.7.1. Let A be a C*-algebra. We say that A is **nuclear** if the identity map $id_A : A \to A$ is a nuclear map.

We say that A is **exact** if there exists a faithful representation (π, H) such that the *-homomorphism $\pi : A \to B(H)$ is a nuclear map.

The notion of a nuclear C*-algebra adapts in a straightforward manner to the context of von Neumann algebras.

Definition 3.7.2. Let M be a von Neumann algebra. We say that M is **semidiscrete** if the identity map $id_M : M \to M$ is weakly nuclear.

The notion of an exact C*-algebra does not generalise to von Neumann algebras in the most straightforward way. Indeed, if we say that a von Neumann algebra M is exact if there exists a faithful representation (π, H) such that $\pi : A \to B(H)$ is a weakly nuclear map then M is a strongly closed *-subalgebra of $B(H)$ for some Hilbert space H and subsequently by Theorem 3.5.3, every von Neumann algebra is exact. Generalising the notion of an exact C*-algebra to von Neumann algebras is a delicate topic which is treated in [BO08, Chapter 14].

Nuclear C*-algebras are sometimes called *amenable* or alternatively, are said to have the *completely positive approximation property.* The completely positive approximation property refers to the diagrammatic descriptions of nuclear and exact C*-algebras which we state below.

A C*-algebra A is nuclear if there exist c.c.p maps $\varphi_n : A \to M_{k(n)}(\mathbb{C})$ and $\psi_n : M_{k(n)}(\mathbb{C}) \to A$ such that the following diagram approximately commutes:

$$\begin{tikzcd} A \arrow[rr, "id_A"] \arrow[dr, dashed, "\varphi_n"'] & & A \\ & M_{k(n)}(\mathbb{C}) \arrow[ur, dashed, "\psi_n"'] & \end{tikzcd}$$

A C*-algebra A is exact if there exist a faithful representation (π, H) and c.c.p maps $\varphi_n : A \to M_{k(n)}(\mathbb{C})$ and $\psi_n : M_{k(n)}(\mathbb{C}) \to B(H)$ such that the following diagram approximately commutes:

$$\begin{tikzcd} A \arrow[rr, "\pi"] \arrow[dr, dashed, "\varphi_n"'] & & B(H) \\ & M_{k(n)}(\mathbb{C}) \arrow[ur, dashed, "\psi_n"'] & \end{tikzcd}$$

Exact C*-algebras are sometimes called *nuclearly embeddable.* This is due to the following simple result on exact C*-algebras.

Theorem 3.7.1. *Let A be a C*-algebra. Then, A is exact if and only if there exists a C*-algebra B and an injective nuclear map $\iota : A \hookrightarrow B$.*

Proof. Assume that A is a C*-algebra. First assume that A is an exact C*-algebra. Then, there exists a faithful representation (π, H) of A such that $\pi : A \to B(H)$ is a nuclear map. Thus, π is an injective nuclear map from A into the C*-algebra $B(H)$.

Conversely, assume that there exists a C*-algebra B and an injective nuclear map $\iota : A \hookrightarrow B$. Let (π_u, H_u) be the universal representation of B

which is faithful. Then, $\pi_u : B \to B(H_u)$ is an injective *-homomorphism and by Theorem 3.5.5, $(\pi_u \circ \iota, H_u)$ is a representation of A such that the composite $\pi_u \circ \iota$ is a nuclear map. So, A is an exact C*-algebra as required. $\square$

The next basic result on nuclear and exact C*-algebra addresses the issue that the nuclearity of a linear map depends on its range.

Theorem 3.7.2. *Let A be a C*-algebra and (π, H) be a faithful representation of A. Then,*

1. *A is a nuclear C*-algebra if and only if the linear map $\pi : A \to \pi(A)$ is nuclear.*
2. *A is an exact C*-algebra if and only if the linear map $\pi : A \to B(H)$ is nuclear.*
3. *If A is a nuclear C*-algebra then A is an exact C*-algebra.*

Proof. Assume that A is a C*-algebra and that (π, H) is a faithful representation of A. First, assume that A is a nuclear C*-algebra. Then, the identity map $id_A : A \to A$ is a nuclear map. The linear map $\pi : A \to \pi(A)$ is a c.c.p map because it is an isometric *-homomorphism. By Theorem 3.5.5, the map $\pi = \pi \circ id_A$ from A to $\pi(A)$ is a nuclear map.

Conversely, assume that $\pi : A \to \pi(A)$ is a nuclear map. The inverse map $\pi^{-1} : \pi(A) \to A$ is a *-homomorphism and is thus, a c.c.p map itself. By Theorem 3.5.5, $id_A = \pi^{-1} \circ \pi$ is a nuclear map and consequently, A is a nuclear C*-algebra.

The second statement follows immediately from the definition of an exact C*-algebra. For the third statement, assume that A is a nuclear C*-algebra. By the first statement of the theorem, the linear map $\pi : A \to \pi(A)$ is a nuclear map. Now the inclusion map $\iota : \pi(A) \hookrightarrow B(H)$ is a *-homomorphism and is thus, a c.c.p map. By Theorem 3.5.5, the map $\pi : A \to B(H)$ is a nuclear map. By the second statement of the theorem, A is an exact C*-algebra. $\square$

In general, an exact C*-algebra is not nuclear. In the particular case of *separable factors*, the notion of semidiscreteness for von Neumann algebras happens to be equivalent to a plethora of other conditions, such as injectivity and hyperfiniteness. This is due to a fundamental theorem by Connes, which has important consequences for nuclear and exact

C*-algebras. For instance, one such consequence is [BO08, Theorem 9.3.3] which states that an injective von Neumann algebra is semidiscrete. We recall the notion of an injective von Neumann algebra from Theorem 3.3.8.

The notions of semidiscreteness and nuclearity are connected by the following main result of [BO08, Section 2.3].

Theorem 3.7.3. *Let A be a C*-algebra. If the double dual (or enveloping von Neumann algebra of A) A^{**} is semidiscrete then A is a nuclear C*-algebra.*

As usual, we require a few preliminary results for the proof of Theorem 3.7.3. The first one concerns unitization.

Theorem 3.7.4. *Let A be a non-unital C*-algebra. Let $\tilde{A}$ be the unitization of A.*

1. *A is a nuclear C*-algebra if and only if $\tilde{A}$ is a nuclear C*-algebra.*
2. *A is an exact C*-algebra if and only if $\tilde{A}$ is an exact C*-algebra.*

Proof. Assume that A is a non-unital C*-algebra and that $\tilde{A}$ is its unitization.

To show: (a) A is a nuclear C*-algebra if and only if $\tilde{A}$ is a nuclear C*-algebra.

(b) A is an exact C*-algebra if and only if $\tilde{A}$ is an exact C*-algebra.

(a) Assume that $\tilde{A}$ is nuclear. Then, the identity map $id_{\tilde{A}}$ on $\tilde{A}$ is a nuclear map and by Theorem 3.5.4, $id_A = id_{\tilde{A}}|_A$ is a nuclear map. So, A is nuclear.

Conversely, assume that A is nuclear so that the identity map id_A on A is nuclear. Then, there exist c.c.p maps

$$\varphi_n : A \to M_{k(n)}(\mathbb{C}) \qquad \text{and} \qquad \psi_n : M_{k(n)}(\mathbb{C}) \to A$$

such that if $a \in A$ then $\lim_n \|(\psi_n \circ \varphi_n)(a) - a\| = 0$.

Regard $M_{k(n)}(\mathbb{C})$ as a C*-subalgebra of the direct sum $M_{k(n)}(\mathbb{C}) \oplus \mathbb{C}$. By Theorem 3.6.1, there exist u.c.p maps $\tilde{\varphi}_n : \tilde{A} \to M_{k(n)}(\mathbb{C}) \oplus \mathbb{C}$ such that $\tilde{\varphi}_n|_A = \varphi_n$. Note that by [Pau02, Corollary 2.9], $\tilde{\varphi}_n$ is contractive.

Let $\iota : A \hookrightarrow \tilde{A}$ be the inclusion map. Since ι is a *-homomorphism then ι is a c.c.p map and the composite $\iota \circ \psi_n : M_{k(n)}(\mathbb{C}) \to \tilde{A}$ is a c.c.p map. By Theorem 3.6.2, there exist u.c.p maps

$$\tilde{\psi_n} : M_{k(n)}(\mathbb{C}) \oplus \mathbb{C} \to \tilde{A}$$

such that $\tilde{\psi_n}|_{M_{k(n)}(\mathbb{C})} = \iota \circ \psi_n$.

Finally, if $a \in A$ and $\lambda \in \mathbb{C}$ then

$$\begin{aligned}
\|(\tilde{\psi_n} \circ \tilde{\varphi}_n)(a + \lambda 1_{\tilde{A}}) - a - \lambda 1_{\tilde{A}}\| &= \|(\tilde{\psi_n} \circ \tilde{\varphi}_n)(a) + (\tilde{\psi_n} \circ \tilde{\varphi}_n)(\lambda 1_{\tilde{A}}) - a - \lambda 1_{\tilde{A}}\| \\
&= \|(\iota \circ \psi_n \circ \varphi_n)(a) + \lambda 1_{\tilde{A}} - a - \lambda 1_{\tilde{A}}\| \\
&= \|(\psi_n \circ \varphi_n)(a) - a\| \to 0
\end{aligned}$$

in the limit over n. By Theorem 3.5.10, we deduce that the identity map $id_{\tilde{A}}$ is a nuclear map. So, $\tilde{A}$ is a nuclear C*-algebra.

(b) Assume that $\tilde{A}$ is an exact C*-algebra. Then, there exists a faithful representation (π, H) of $\tilde{A}$ such that the *-homomorphism $\pi : \tilde{A} \to B(H)$ is a nuclear map. By Theorem 3.5.4, the representation $(\pi|_A, H)$ is faithful and $\pi|_A : A \to B(H)$ is a nuclear map. Therefore, A is an exact C*-algebra.

Conversely, assume that A is an exact C*-algebra so that there exists a faithful representation (π, H) such that the *-homomorphism $\pi : A \to B(H)$ is nuclear. Without loss of generality, assume that (π, H) is a non-degenerate representation of A. By Theorem 3.4.6, there exists a unique *-homomorphism $\tilde{\pi} : \tilde{A} \to B(H)$ such that $\tilde{\pi}|_A = \pi$. We claim that $\tilde{\pi}$ is injective.

To show: (ba) $\ker \tilde{\pi} = 0$.

(ba) Assume that $a \in \ker \tilde{\pi} \subsetneq \tilde{A}$. Then, $\tilde{\pi}(a) = 0$ on $B(H)$. This means that if $\xi \in H$ and $b \in A$ then

$$\pi(ab)\xi = \tilde{\pi}(a)(\pi(b)\xi) = 0.$$

Hence, if $b \in A$ then $ab \in \ker \pi = 0$. Consequently, $a = 0$ and $\tilde{\pi}$ is an injective *-homomorphism.

It remains to show that $\tilde{\pi} : \tilde{A} \to B(H)$ is nuclear. Assume that $\epsilon \in \mathbb{R}_{>0}$ and $\mathcal{F}$ is a finite subset of $\tilde{A}$. Let $\{u_\lambda\}_{\lambda \in \Lambda}$ be an approximate unit for A. By

Theorem 2.3.4, the net $\{\pi(u_\lambda)\}_{\lambda\in\Lambda}$ converges strongly to the identity operator id_H. So, there exists $\mu \in \Lambda$ such that if $\xi \in H$ then

$$\|b - bu_\mu\| < \frac{\epsilon}{3} \qquad \text{and} \qquad \|\xi - \pi(u_\mu)\xi\| < \frac{\epsilon}{3\max_{c\in\mathcal{F}}\|c\|}.$$

Subsequently, we have

$$\|id_H - \pi(u_\mu)\| = \sup_{\|\xi\|=1}\|\xi - \pi(u_\mu)\xi\| < \frac{\epsilon}{3\max_{c\in\mathcal{F}}\|c\|}.$$

The set

$$\mathcal{F}u_\mu = \{bu_\mu \mid b \in \mathcal{F}\} \subsetneq A$$

is a finite subset of A (because A is an ideal in $\tilde{A}$). Since π is nuclear then by Theorem 3.5.8, there exist $n \in \mathbb{Z}_{>0}$ and c.c.p maps

$$\varphi : A \to M_n(\mathbb{C}) \qquad \text{and} \qquad \psi : M_n(\mathbb{C}) \to B(H)$$

such that if $b \in \mathcal{F}$ then

$$\|(\psi\circ\varphi)(bu_\mu) - \pi(bu_\mu)\| = \|(\psi\circ\varphi)(bu_\mu) - \tilde{\pi}(b)\pi(u_\mu)\| < \frac{\epsilon}{3}$$

because $\tilde{\pi}|_A = \pi$. By Arveson's extension theorem in Theorem 3.3.7, there exists a c.c.p map $\tilde{\varphi} : \tilde{A} \to M_n(\mathbb{C})$ such that $\tilde{\varphi}|_A = \varphi$. Therefore, if $b \in \mathcal{F}$ then

$$\begin{aligned}
\|(\psi\circ\tilde{\varphi})(b) - \tilde{\pi}(b)\| &\leq \|(\psi\circ\tilde{\varphi})(b) - (\psi\circ\tilde{\varphi})(bu_\mu)\| + \|(\psi\circ\tilde{\varphi})(bu_\mu) - \tilde{\pi}(b)\pi(u_\mu)\| \\
&\quad + \|\tilde{\pi}(b)\pi(u_\mu) - \tilde{\pi}(b)\| \\
&\leq \|\psi\circ\tilde{\varphi}\|\|b - bu_\mu\| + \|(\psi\circ\varphi)(bu_\mu) - \tilde{\pi}(b)\pi(u_\mu)\| \\
&\quad + \|\tilde{\pi}(b)\|\|\pi(u_\mu) - id_H\| \\
&\leq \|b - bu_\mu\| + \|(\psi\circ\varphi)(bu_\mu) - \tilde{\pi}(b)\pi(u_\mu)\| + \|b\|\|\pi(u_\mu) - id_H\| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \|b\|\frac{\epsilon}{3\max_{c\in\mathcal{F}}\|c\|} < \epsilon.
\end{aligned}$$

By Theorem 3.5.8, we deduce that $\tilde{\pi} : \tilde{A} \to B(H)$ is a nuclear map. So, $\tilde{A}$ is an exact C*-algebra which completes the proof. $\square$

The next theorem is an application of the Hahn-Banach extension theorem.

Theorem 3.7.5. *Let A be a Banach space and $B(A)$ be the Banach space of bounded linear operators $T : A \to A$. Let $C \subsetneq B(A)$ be convex. Then, the point-weak and point-norm closures of C coincide.*

Proof. Assume that A is a Banach space and $B(A)$ is the Banach space of bounded linear maps $T : A \to A$. Assume that C is a convex subset of $B(A)$. We refer to the point-norm and point-weak topologies on C as PN and PW respectively.

To show: (a) $\overline{C}^{PN} \subseteq \overline{C}^{PW}$.

(b) $\overline{C}^{PW} \subseteq \overline{C}^{PN}$.

(a) Assume that $T \in \overline{C}^{PN}$. Then, there exists a net $\{T_i\}_{i\in I}$ in C such that if $a \in A$ then $T_i(a) \to T(a)$ in the limit over $i \in I$. If $\eta \in A^*$ then $\eta(T_i(a)) \to \eta(T(a))$. Hence, $T \in \overline{C}^{PW}$ and $\overline{C}^{PN} \subseteq \overline{C}^{PW}$.

(b) Assume that $T \in \overline{C}^{PW}$. Then, there exists a net $\{T_i\}_{i\in I}$ in C such that if $a \in A$ and $\eta \in A^*$ then $\eta(T_i(a)) \to \eta(T(a))$ in the limit over $i \in I$. Assume that $\mathcal{F} = \{a_1, \ldots, a_k\} \subsetneq A$ is finite and $\epsilon \in \mathbb{R}_{>0}$. We want to show that there exists $S \in C$ such that

$$\max_{j\in\{1,2,\ldots,k\}} \|S(a_j) - T(a_j)\| < \epsilon.$$

To this end, consider the net

$$\{T_i \oplus \cdots \oplus T_i\}_{i\in I} \subsetneq B(\bigoplus_{j=1}^{k} A).$$

If $j \in \{1, 2, \ldots, k\}$ then let $\iota_j : A \to \bigoplus_{j=1}^k A$ be the linear map

$$\iota_j(a) = 0 \oplus \cdots \oplus a \oplus \cdots \oplus 0.$$

In the above definition, the a appears in the j^{th} direct summand. Now if $\phi \in (\bigoplus_{j=1}^k A)^*$ then

$$\begin{aligned}
&|\phi(T_i(\alpha_1) \oplus \cdots \oplus T_i(\alpha_k)) - \phi(T(\alpha_1) \oplus \cdots \oplus T(\alpha_k))| \\
&\leq |\sum_{j=1}^{k} \big((\phi \circ \iota_j)(T_i(\alpha_j)) - (\phi \circ \iota_j)(T(\alpha_j))\big)| \\
&\leq \sum_{j=1}^{k} |(\phi \circ \iota_j)(T_i(\alpha_j)) - (\phi \circ \iota_j)(T(\alpha_j))| \to 0
\end{aligned}$$

in the limit over i. Here, we used the fact that if $j \in \{1, 2, \ldots, k\}$ then the composite $\phi \circ \iota_j \in A^*$. We conclude that the net $\{T_i \oplus \cdots \oplus T_i\}_{i \in I}$ converges in the point-weak topology to $\bigoplus_{j=1}^k T \in B(\bigoplus_{j=1}^k A)$. Now let

$$\mathcal{S} = \{T_i(a_1) \oplus \cdots \oplus T_i(a_k) \mid i \in I\}.$$

Then,

$$T(a_1) \oplus T(a_2) \oplus \cdots \oplus T(a_k) \in \bigoplus_{j=1}^{k} A$$

is an element of the weak closure $\overline{\mathcal{S}}^{WOT} \subsetneq \overline{conv(\mathcal{S})}^{WOT}$. By the separation version of the Hahn-Banach extension theorem,

$$\overline{conv(\mathcal{S})}^{WOT} = \overline{conv(\mathcal{S})}.$$

On the RHS, we have the norm closure of the convex hull $conv(S)$.

Before we proceed, let us quickly highlight how this argument works. Assume that X is a Banach space and that K is a weakly closed convex subset of X. Since the weak operator topology on X is weaker than the norm topology, K must be norm closed.

Conversely, assume that K is norm closed. Choose $x \in X \backslash K$. The singleton set $\{x\}$ is convex and compact in the norm topology on X. By the separation version of the Hahn-Banach theorem in [RS80, Theorem V.4], there exists a continuous real-valued linear functional $\tau : X \to \mathbb{R}$ and $t \in \mathbb{R}$ such that if $y \in K$ then

$$\tau(y) < t < \tau(x).$$

The preimage $\tau^{-1}((t, \infty))$ is a weakly open set which contains x and satisfies $\tau^{-1}((t, \infty)) \cap K = \emptyset$. It is weakly open because the weak operator topology in the weakest topology which makes the functionals in X^* continuous. So, $X \backslash K$ is weakly open and K is weakly closed. By arguing in a similar manner to Theorem 2.5.11, we deduce that $\overline{K}^{WOT} = \overline{K}$.

Returning to the proof, we have

$$T(a_1) \oplus \cdots \oplus T(a_k) \in \overline{conv(\mathcal{S})}$$

This means that there exists $S \in conv(\mathcal{S})$ such that

$$\|(T(a_1) \oplus \cdots \oplus T(a_k)) - S\| < \epsilon.$$

Since $S \in conv(\mathcal{S})$,

$$S = F(a_1) \oplus \cdots \oplus F(a_k)$$

where $F \in conv(\{T_i \mid i \in I\}) \subsetneq C$. Hence,

$$\|(T(a_1) \oplus \cdots \oplus T(a_k)) - S\| = \max_{j \in \{1,\ldots,k\}} \|T(a_j) - F(a_j)\| < \epsilon$$

and $T \in \overline{C}^{PN}$, which finishes the proof. $\square$

Definition 3.7.3. Let A be a C*-algebra and $\theta : A \to A$ be a completely positive map. We say that θ is **factorable** if there exist $n \in \mathbb{Z}_{>0}$ and c.p maps

$$\varphi : A \to M_n(\mathbb{C}) \qquad \text{and} \qquad \psi : M_n(\mathbb{C}) \to A$$

such that $\theta = \psi \circ \varphi$. We call the triple $(\varphi, \psi, M_n(\mathbb{C}))$ a **factorisation** of θ.

The point we will make in the next theorem is that the set of factorable maps on a C*-algebra is convex.

Theorem 3.7.6. *Let A be a C*-algebra and $\mathcal{F}$ be the set of factorable maps on A. Then, $\mathcal{F}$ is a convex set.*

Proof. Assume that A is a C*-algebra and that $\mathcal{F}$ is the set of factorable maps on A. Assume that $\theta_1, \theta_2 \in \mathcal{F}$ and that if $i \in \{1, 2\}$ then $(\varphi_i, \psi_i, M_{n_i}(\mathbb{C}))$ are the factorisations of θ_i.

Assume that $t \in (0, 1)$. We want to show that $t\theta_1 + (1-t)\theta_2 \in \mathcal{F}$. To this end, consider the maps

$$\begin{array}{rccc} \varphi_1 \oplus \varphi_2 : & A & \to & M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \\ & a & \mapsto & \varphi_1(a) \oplus \varphi_2(a) \end{array}$$

and

$$\begin{array}{rccc} \phi : & M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) & \to & A \\ & X \oplus Y & \mapsto & t\psi_1(X) + (1-t)\psi_2(Y). \end{array}$$

Since $\varphi_1, \varphi_2, \psi_1$ and ψ_2 are completely positive maps, $\varphi_1 \oplus \varphi_2$ and ϕ are completely positive maps too. Note that the composite $\phi \circ (\varphi_1 \oplus \varphi_2) = t\theta_1 + (1-t)\theta_2$.

However, $M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C})$ is technically not a matrix algebra. We can remedy this by arguing in a similar manner to Theorem 3.5.10. We have the inclusion map

$$\begin{array}{rccc} \iota : & M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) & \to & M_{n_1+n_2}(\mathbb{C}) \\ & X \oplus Y & \mapsto & \begin{pmatrix} X & \\ & Y \end{pmatrix} \end{array}$$

and the conditional expectation

$$\begin{array}{rccc} E : & M_{n_1+n_2}(\mathbb{C}) & \to & M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \\ & Z & \mapsto & p_1 Z p_1 + p_2 Z p_2 \end{array}$$

where p_1 and p_2 are pairwise orthogonal projections such that

$$M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) = p_1 M_{n_1+n_2}(\mathbb{C}) p_1 + p_2 M_{n_1+n_2}(\mathbb{C}) p_2.$$

Now observe that $\iota \circ (\varphi_1 \oplus \varphi_2)$ and $\phi \circ E$ are both completely positive maps satisfying

$$\phi \circ E \circ \iota \circ (\varphi_1 \oplus \varphi_2) = \phi \circ (\varphi_1 \oplus \varphi_2) = t\theta_1 + (1-t)\theta_2$$

because E is a conditional expectation. Therefore, $t\theta_1 + (1-t)\theta_2 \in \mathcal{F}$ and $\mathcal{F}$ is convex. $\square$

We note that the proof above can be adapted to show that the set of factorable maps with *contractive factorisations* is itself convex. We say that a factorisation $(\varphi, \psi, M_n(\mathbb{C}))$ of a c.p map θ is contractive if φ and ψ are both contractive.

We are now ready to prove Theorem 3.7.3.

Proof of Theorem 3.7.3. First assume that A is a unital C*-algebra. Assume that the double dual A^{**} is semidiscrete. Let F be a finite subset of A and χ be a finite subset of the dual space $A^* = (A^{**})_*$. Assume that $\epsilon \in \mathbb{R}_{>0}$.

We want to show that the identity map $id_A : A \to A$ is nuclear. Let $\mathcal{CF}$ be the set of contractive factorable maps on A. The idea is that by the proof of Theorem 3.7.6, $\mathcal{CF}$ is a convex subset of the Banach space $B(A)$ of bounded linear operators from A to A. By Theorem 3.7.5,

$$\overline{\mathcal{CF}}^{PW} = \overline{\mathcal{CF}}^{PN}$$

where PW and PN are the point-weak and point-norm topologies on $\mathcal{CF}$ respectively. So in order to prove that id_A is nuclear, it suffices to show that $id_A \in \overline{\mathcal{CF}}^{PW}$ because if $id_A \in \overline{\mathcal{CF}}^{PW}$ then $id_A \in \overline{\mathcal{CF}}^{PN}$ and consequently, there exists a contractive factorable map $\theta : A \to A$ such that if $a \in F$ then

$$\|a - \theta(a)\| < \epsilon.$$

Since $\theta \in \mathcal{CF}$, there exist c.c.p maps $\varphi : A \to M_n(\mathbb{C})$ and $\psi : M_n(\mathbb{C}) \to A$ such that $\theta = \psi \circ \varphi$. Hence if $a \in F$ then $\|a - (\psi \circ \varphi)(a)\| < \epsilon$ and by Theorem 3.5.8, we will be done.

To show: (a) There exist c.c.p maps $\varphi : A \to M_n(\mathbb{C})$ and $\psi : M_n(\mathbb{C}) \to A$ such that if $a \in F$ and $\eta \in \chi$ then

$$|\eta((\psi \circ \varphi)(a)) - \eta(a)| < \epsilon.$$

(a) Since A^{**} is semidiscrete, the identity map $id_{A^{**}}$ is a unital weakly nuclear map. By Theorem 3.6.6, there exist u.c.p maps

$$\varphi : A^{**} \to M_n(\mathbb{C}) \qquad \text{and} \qquad \psi' : M_n(\mathbb{C}) \to A^{**}$$

such that if $a \in F$ and $\eta \in \chi$ then

$$|\eta((\psi' \circ \varphi)(a)) - \eta(a)| < \epsilon.$$

By [Pau02, Corollary 2.9], φ and ψ' are both contractive. The idea here is to replace ψ' suitably so that it takes values in A rather than in A^{**}.

Recall from Theorem 3.3.1 that the following map is bijective:

$$\begin{array}{rccc} \mathcal{C}_{A^{**}}: & CP(M_n(\mathbb{C}), A^{**}) & \to & M_n(A^{**})_+ \\ & \varphi & \mapsto & \big(\varphi(e_{i,j})\big). \end{array}$$

Here, the set $\{e_{i,j} \mid i, j \in \{1, \ldots, n\}\}$ are the matrix units of $M_n(\mathbb{C})$. We also note that $M_n(A)_+$ is ultraweakly dense in $M_n(A^{**})_+$. Using this with the bijections $\mathcal{C}_{A^{**}}$ and $\mathcal{C}_A$, there exists a net of completely positive maps

$$\{\psi_\lambda : M_n(\mathbb{C}) \to A\}_{\lambda \in \Lambda}$$

such that if $X \in M_n(\mathbb{C})$ then $\psi_\lambda(X) \to \psi'(X)$ in the ultraweak topology on A^{**}. Note that if $\lambda \in \Lambda$ then the linear map ψ_λ is not necessarily contractive.

We observe that by construction, ψ' and φ are unital maps. So, the net

$$\{\psi_\lambda(1_{M_n(\mathbb{C})})\}_{\lambda\in\Lambda}$$

in A converges weakly to $\psi'(1_{M_n(\mathbb{C})}) = 1_A$. Hence in the net $\{\psi_\lambda\}_{\lambda\in\Lambda}$, we can select a completely positive map $\psi'' : M_n(\mathbb{C}) \to A$ such that if $a \in F$ and $\eta \in \chi$ then

$$|\eta((\psi'' \circ \varphi)(a)) - \eta(a)| < \epsilon \qquad \text{and} \qquad \|\psi''(1_{M_n(\mathbb{C})}) - 1_A\| < \epsilon.$$

Now if $X \in M_n(\mathbb{C})$ then define

$$\psi(X) = \frac{1}{\|\psi''(1_{M_n(\mathbb{C})})\|}\psi''(X).$$

Then, $\psi : M_n(\mathbb{C}) \to A$ is a completely positive map such that

$$\|\psi\| = \frac{\|\psi''\|}{\|\psi''(1_{M_n(\mathbb{C})})\|} = 1$$

by [Pau02, Corollary 2.9]. So, it is contractive and if $a \in F$ and $\eta \in \xi$ then

$$\begin{aligned}
|\eta((\psi \circ \varphi)(a)) - \eta(a)| &= \left|\frac{1}{\|\psi''(1_{M_n(\mathbb{C})})\|}\eta((\psi'' \circ \varphi)(a)) - \eta(a)\right| \\
&= \frac{1}{\|\psi''(1_{M_n(\mathbb{C})})\|}\left|\eta((\psi'' \circ \varphi)(a)) - \|\psi''(1_{M_n(\mathbb{C})})\|\eta(a)\right| \\
&\leq \frac{1}{\|\psi''(1_{M_n(\mathbb{C})})\|}\Big(|\eta((\psi'' \circ \varphi)(a)) - \eta(a)| \\
&\quad + |\eta(a) - \|\psi''(1_{M_n(\mathbb{C})})\|\eta(a)|\Big) \\
&\leq \frac{1}{\|\psi''(1_{M_n(\mathbb{C})})\|}\Big(|\eta((\psi'' \circ \varphi)(a)) - \eta(a)| \\
&\quad + \big|\|\psi''(1_{M_n(\mathbb{C})}) - 1_A\|\eta(a) + \eta(a) - \eta(a)\big|\Big) \\
&< \epsilon\frac{(1 + |\eta(a)|)}{\|\psi''(1_{M_n(\mathbb{C})})\|}.
\end{aligned}$$

This completes the proof in the unital case.

For the general case, assume that A is a non-unital C*-algebra. By Theorem 3.7.4, A is nuclear if and only if its unitization $\tilde{A}$ is also nuclear. Note that we have the isomorphism

$$(\tilde{A})^{**} \cong A^{**} \oplus \mathbb{C}.$$

To see why this is the case, assume that B is a C*-algebra and J is a closed two-sided ideal of B. Then, J^{**} is a weakly closed ideal of B^{**} and by Theorem 3.4.11, there exists a projection $p \in J^{**}$ such that $pB^{**} = B^{**}p = J^{**}$. So, we have the isomorphism of C*-algebras

$$B^{**} \cong pB^{**} \oplus (1_{B^{**}} - p)B^{**} \cong J^{**} \oplus (B/J)^{**}.$$

Hence, the isomorphism $(\tilde{A})^{**} \cong A^{**} \oplus \mathbb{C}$ is obtained by replacing B with $\tilde{A}$ and J with A. So if A^{**} is semidiscrete then $(\tilde{A})^{**}$ is semidiscrete and consequently, $\tilde{A}$ and A are nuclear C*-algebras. This completes the proof for the general case. $\square$

Next, we will explore various properties of nuclear and exact C*-algebras which are stated as exercises in [BO08, Exercise 2.3.1 - Exercise 2.3.15]. First, we will show that a hereditary C*-subalgebra of a nuclear C*-algebra is again nuclear. In particular, this means that any closed two-sided ideal of a nuclear C*-algebra is itself nuclear.

We have already used the fact that a closed two-sided ideal of a C*-algebra is a hereditary C*-subalgebra in Theorem 2.4.17 — this is [Mur90, Corollary 3.2.3]. Let us return to [Mur90] briefly and prove this fact.

Theorem 3.7.7. *Let A be a C*-algebra. Then, the map*

$$\begin{array}{cccc} F: & \{\textit{Closed left ideals in } A\} & \to & \{\textit{Hereditary C*-subalgebras of } A\} \\ & I & \mapsto & I \cap I^* \end{array}$$

is a bijection. In particular, if I_1 and I_2 are closed left ideals of A then $I_1 \subseteq I_2$ if and only if $I_1 \cap I_1^* \subseteq I_2 \cap I_2^*$.

Proof. Assume that A is a C*-algebra. Assume that F is the map defined as above.

To show: (a) If I is a closed left ideal of A then $F(I) = I \cap I^*$ is a hereditary C*-subalgebra of A.

(b) F is a bijection.

(a) Assume that I is a closed left ideal of A. Then, $F(I) = I \cap I^*$ is a C*-subalgebra of A. To see that $F(I)$ is hereditary, assume that $a \in A^+$, $b \in (I \cap I^*)^+$ and $a \leq b$. By Theorem 2.3.3, there exists an approximate unit $\{u_\lambda\}_{\lambda \in \Lambda}$ in the closed unit ball of I^+. By property 4 of Theorem 2.2.2,

$$0 \leq (1_{\tilde{A}} - u_\lambda)a(1_{\tilde{A}} - u_\lambda) \leq (1_{\tilde{A}} - u_\lambda)b(1_{\tilde{A}} - u_\lambda).$$

In order to show that $a \in I \cap I^*$, we will show that $a^{\frac{1}{2}} \in I \cap I^*$. We have

$$\begin{aligned}
\|a^{\frac{1}{2}} - a^{\frac{1}{2}}u_\lambda\|^2 &= \|a^{\frac{1}{2}}(1_{\tilde{A}} - u_\lambda)\|^2 \\
&= \|(1_{\tilde{A}} - u_\lambda)a(1_{\tilde{A}} - u_\lambda)\| \\
&\leq \|(1_{\tilde{A}} - u_\lambda)b(1_{\tilde{A}} - u_\lambda)\| \qquad \text{(by Theorem 2.2.5)} \\
&= \|b^{\frac{1}{2}} - b^{\frac{1}{2}}u_\lambda\|^2 \to 0
\end{aligned}$$

in the limit over Λ. Therefore, $a^{\frac{1}{2}} = \lim_\lambda a^{\frac{1}{2}}u_\lambda$ and since I is closed then $a^{\frac{1}{2}} \in I$. Therefore, a is a positive (and thus self-adjoint) element of I and $a \in I \cap I^*$. Therefore, $I \cap I^*$ is a hereditary C*-subalgebra. This shows that the map F is well-defined.

(b) To see that F is injective, we will show that if I_1 and I_2 are closed left ideals of A then $I_1 \subseteq I_2$ if and only if $F(I_1) \subseteq F(I_2)$. The "only if" direction of the statement is immediate. To obtain the reverse implication, assume that $F(I_1) \subseteq F(I_2)$ and $a \in I_1$. Let $\{v_\mu\}_{\mu \in M}$ be an approximate unit for $I_1 \cap I_1^*$. Since $a^*a \in I_1 \cap I_1^*$ then

$$\lim_\mu \|a - av_\mu\|^2 = \lim_\mu \|(1_{\tilde{A}} - v_\mu)a^*a(1_{\tilde{A}} - v_\mu)\| \leq \lim_\mu \|a^*a(1_{\tilde{A}} - v_\mu)\| = 0.$$

So $a = \lim_\mu av_\mu$ and since $I_1 \cap I_1^* \subseteq I_2$ then $a \in I_2$. So, $I_1 \subseteq I_2$.

Now observe that if $I_1 \cap I_1^* = F(I_1) = F(I_2) = I_2 \cap I_2^*$ then by the preceding statement, $I_1 = I_2$. So, F is injective.

To see that F is surjective, assume that B is a hereditary C*-subalgebra of A. Define

$$L(B) = \{a \in A \mid a^*a \in B\}.$$

Then, $L(B)$ is closed under scalar multiplication. To see that $L(B)$ is closed under left multiplication by A, assume that $a \in A$ and $b \in L(B)$. Then,

$$0 \leq (ab)^*ab = b^*a^*ab \leq \|a\|^2 b^*b \in B.$$

Since B is a hereditary C*-subalgebra then $(ab)^*ab \in B$ and consequently, $ab \in L(B)$. Now assume that $b_1, b_2 \in L(B)$. Then,

$$0 \le (b_1+b_2)^*(b_1+b_2) \le (b_1+b_2)^*(b_1+b_2)+(b_1-b_2)^*(b_1-b_2) = 2b_1^*b_1+2b_2^*b_2 \in B.$$

Therefore, $b_1 + b_2 \in L(B)$ and consequently, $L(B)$ is a left ideal of A. It is closed because B is closed.

It remains to show that $F(L(B)) = L(B) \cap L(B)^* = B$. Assume that $b \in B$. Then, $b^*b \in B$. So, $b \in L(B)$. Similarly, $b^* \in L(B)$ and subsequently, $b \in L(B) \cap L(B)^*$. So, $B \subseteq L(B) \cap L(B)^*$. Conversely, assume that $c \in L(B) \cap L(B)^*$. Write $c = c_1 + ic_2$ where

$$c_1 = \frac{1}{2}(c + c^*) \qquad \text{and} \qquad c_2 = \frac{1}{2i}(c - c^*).$$

Then, c_1 and c_2 are self-adjoint elements of $L(B) \cap L(B)^*$. If $i \in \{1, 2\}$ then write $c_i = f(c_i) - g(c_i)$ where $f, g \in Cts(\sigma(c_i), \mathbb{C})$ are defined by

$$f(x) = \max(x, 0) \qquad \text{and} \qquad g(x) = \max(-x, 0).$$

By Theorem 1.6.10, $f(c_i)$ and $g(c_i)$ are elements of the C*-subalgebra generated by c_i. Hence, they are positive elements of $L(B) \cap L(B)^*$.

So, $c \in L(B) \cap L(B)^*$ can be written as a linear combination of four positive elements in $L(B) \cap L(B)^*$. Hence, we can assume without loss of generality that $c \in L(B) \cap L(B)^*$ is positive. Then, $c^*c = c^2 \in B$ and by the continuous functional calculus in Theorem 1.6.10, $c = (c^2)^{\frac{1}{2}} \in B$. So,

$$F(L(B)) = L(B) \cap L(B)^* = B$$

and F is surjective. We conclude that F is a bijection as required. □

Using the bijection F in Theorem 3.7.7, we obtain

Theorem 3.7.8. *Let A be a C*-algebra and $I \subseteq A$ be a closed two-sided ideal of A. Then, I is a hereditary C*-algebra.*

Proof. Assume that A is a C*-algebra and $I \subseteq A$ is a closed two-sided ideal of A. Let F be the bijection in Theorem 3.7.7. Then, $F(I) = I \cap I^*$ is a hereditary C*-subalgebra of A. But by Theorem 1.7.4, $I = I^*$. So, $I \cap I^* = I$ is a hereditary C*-subalgebra of A. □

Now we are ready to prove that hereditary C*-subalgebras of nuclear C*-algebras are themselves nuclear.

Theorem 3.7.9. *Let A be a nuclear C*-algebra and $B \subseteq A$ be a hereditary C*-subalgebra of A. Then, B is a nuclear C*-algebra. In particular, if I is a closed two-sided ideal of A then I is a nuclear C*-algebra.*

Proof. Assume that A is a nuclear C*-algebra and that $B \subseteq A$ is a hereditary C*-subalgebra of A. The idea behind the proof is to make use of Theorem 3.5.12.

Since A is a nuclear C*-algebra then the identity map $id_A : A \to A$ is a nuclear map. Let $\iota : B \hookrightarrow A$ denote the inclusion map. Then, ι is a c.c.p map and by Theorem 3.5.5, $\iota = id_A \circ \iota$ is itself a nuclear map. Note that $\iota(B) = B$.

Let $\{u_\lambda\}_{\lambda \in \Lambda}$ be an approximate unit for B. If $\lambda \in \Lambda$ then define the map

$$\begin{array}{rccc} \Phi_\lambda : & A & \to & B \\ & a & \mapsto & u_\lambda a u_\lambda. \end{array}$$

To show: (a) If $\lambda \in \Lambda$ then Φ_λ is well-defined.

(b) Φ_λ is a c.c.p map.

(a) Assume that $a \in A$. By a similar argument outlined in Theorem 3.7.7, we may assume without loss of generality that $a \in A$ is positive. Then,

$$0 \leq u_\lambda a u_\lambda \leq \|a\| u_\lambda^2 = \|a\| u_\lambda \in B.$$

Since B is a hereditary C*-subalgebra then $u_\lambda a u_\lambda \in B$ and Φ_λ is a well-defined linear map.

(b) To see that Φ_λ is contractive, note that

$$\|\Phi_\lambda\| = \sup_{\|a\|=1} \|u_\lambda a u_\lambda\| \leq \sup_{\|a\|=1} \|u_\lambda\|^2 \|a\| = 1.$$

To see that Φ_λ is completely positive, assume that $n \in \mathbb{Z}_{>0}$ and $(a_{ij}) \in M_n(A)$ is positive. Then,

$$\Phi_\lambda^{(n)}((a_{ij})) = (u_\lambda a_{ij} u_\lambda) = diag(u_\lambda)\ (a_{ij})\ diag(u_\lambda).$$

By property 4 of Theorem 2.2.2, $\Phi_\lambda^{(n)}((a_{ij}))$ is a positive matrix in $M_n(B)$ and consequently, Φ_λ is a c.c.p map.

Now observe that if $b \in B$ then

$$
\begin{aligned}
\|\Phi_\lambda(b) - b\| &= \|u_\lambda b u_\lambda - b\| \\
&\leq \|u_\lambda\|\|b u_\lambda - b\| + \|u_\lambda b - b\| \\
&\leq \|b u_\lambda - b\| + \|u_\lambda b - b\| \to 0
\end{aligned}
$$

in the limit over Λ. Thus by Theorem 3.5.12, the map $\iota : B \to B$ is a nuclear map. But, this is just the identity map id_B. Therefore, B is a nuclear C*-algebra as required. □

On the topic of subalgebras, it is quite easy to see that a C*-subalgebra of an exact C*-algebra is again exact.

Theorem 3.7.10. *Let A be an exact C*-algebra and $B \subseteq A$ be a C*-subalgebra of A. Then, B is also an exact C*-algebra.*

Proof. Assume that A is an exact C*-algebra and B is a C*-subalgebra of A. Since A is exact then there exists a faithful representation (π, H) of A such that the *-homomorphism $\pi : A \to B(H)$ is a nuclear map.

Let $\iota : B \hookrightarrow A$ be the inclusion map. Then, ι is a c.c.p map and by Theorem 3.5.5, $(\pi \circ \iota, H)$ is a faithful representation of B such that $\pi \circ \iota : B \to B(H)$ is a nuclear map. So, B is exact. □

Although exactness passes to C*-subalgebras as exhibited by Theorem 3.7.10, the same cannot be said for nuclearity. The above proof breaks down because by Theorem 3.7.2, A is a nuclear C*-algebra if and only if there exists a faithful representation (π, H) of A such that $\pi : A \to \pi(A)$ is a nuclear map. The most one can say is that the composite $\pi \circ \iota : B \to \pi(A)$ is a nuclear map, which is not enough to conclude that the C*-subalgebra B of A is nuclear via Theorem 3.7.2. In general, a C*-subalgebra of a nuclear C*-algebra is not nuclear.

As explained in [BO08, Section 2.3], one of the main advantages of the class of exact C*-algebras is that they are defined by an *external* approximation property. If A is an exact C*-algebra then there exist a faithful representation (π, H) and c.c.p maps

$$
\varphi_n : A \to M_{k(n)}(\mathbb{C}) \qquad \text{and} \qquad \psi_n : M_{k(n)}(\mathbb{C}) \to B(H)
$$

such that if $a \in A$ then $\lim_n \|(\psi_n \circ \varphi_n)(a) - \pi(a)\| = 0$. The external approximation property means that the c.c.p maps ψ_n take values in $B(H)$, which is outside of the image $\pi(A)$. The external approximation property is simpler to verify in certain cases, with one instance of this phenomenon appearing in the recent proof of Theorem 3.7.10.

We will give another example of this by completing [BO08, Exercise 2.3.10]. First, we need the preliminary result outlined in [BO08, Exercise 2.3.9].

Theorem 3.7.11. *Let H be a Hilbert space and $A \subsetneq B(H)$ be a concretely represented exact C*-algebra. Then, there exist c.c.p maps*

$$\varphi_n : A \to M_{k(n)}(\mathbb{C}) \qquad \text{and} \qquad \psi_n : M_{k(n)}(\mathbb{C}) \to B(H)$$

such that if $a \in A$ then

$$\lim_n \|(\psi_n \circ \varphi_n)(a) - a\| = 0.$$

Proof. Assume that H is a Hilbert space and $A \subsetneq B(H)$ is a concretely represented exact C*-algebra. Then, there exists a faithful representation (π, K) of A such that the *-homomorphism $\pi : A \to B(K)$ is a nuclear map.

Now define the map ι by

$$\begin{array}{rccc} \iota : & \pi(A) & \to & B(H) \\ & \pi(a) & \mapsto & a \end{array}$$

Then, ι is a *-homomorphism and is thus, a c.c.p map. By Arveson's extension theorem in Theorem 3.3.7, there exists a c.c.p map $\tilde{\iota} : B(K) \to B(H)$ such that $\tilde{\iota}|_{\pi(A)} = \iota$.

The composite $\tilde{\iota} \circ \pi : A \to B(H)$ is a nuclear map by Theorem 3.5.5. So, there exist c.c.p maps

$$\varphi_n : A \to M_{k(n)}(\mathbb{C}) \qquad \text{and} \qquad \psi_n : M_{k(n)}(\mathbb{C}) \to B(H)$$

such that if $a \in A$ then

$$\lim_n \|(\psi_n \circ \varphi_n)(a) - (\tilde{\iota} \circ \pi)(a)\| = \lim_n \|(\psi_n \circ \varphi_n)(a) - a\| = 0.$$

$\square$

Now we state and prove [BO08, Exercise 2.3.10], which is a "local" formuation of exactness.

Theorem 3.7.12. *Let H be a Hilbert space and $A \subsetneq B(H)$ be a concretely represented C*-algebra which satisfies the following property: If $F \subsetneq A$ is a finite set and $\epsilon \in \mathbb{R}_{>0}$ then there exists an exact C*-algebra $B \subsetneq B(H)$ such that if $f \in F$ then there exists $b \in B$ such that*

$$\|b - f\| < \epsilon.$$

Then, A is also an exact C-algebra.*

Proof. Assume that H is a Hilbert space and $A \subsetneq B(H)$ is a concretely represented C*-algebra which satisfies the property described as above. Let $\iota : A \hookrightarrow B(H)$ be the inclusion map. Then, (ι, H) is a faithful representation of A. We must show that ι is a nuclear map.

To this end, assume that $F \subsetneq A$ is a finite set and $\epsilon \in \mathbb{R}_{>0}$. Then, there exists an exact C*-algebra $B \subsetneq B(H)$ such that if $f \in F$ then there exists $b \in B$ such that $\|b - f\| < \frac{\epsilon}{3}$.

By Theorem 3.7.11, there exist c.c.p maps

$$\varphi_n : B \to M_{k(n)}(\mathbb{C}) \qquad \text{and} \qquad \psi : M_{k(n)}(\mathbb{C}) \to B(H)$$

such that if $b \in B$ then

$$\lim_n \|(\psi_n \circ \varphi_n)(b) - b\| = 0.$$

Thus, there exist c.c.p maps $\varphi : B \to M_m(\mathbb{C})$ and $\psi : M_m(\mathbb{C}) \to B(H)$ such that if $b \in B$ then

$$\|(\psi \circ \varphi)(b) - b\| < \frac{\epsilon}{3}.$$

By Arveson's extension theorem in Theorem 3.3.7, we can extend φ to a c.c.p map $\tilde{\varphi} : B(H) \to M_m(\mathbb{C})$. Now if $f \in F$ then let $b_f \in B$ satisfy $\|b_f - f\| < \epsilon/3$. Then,

$$\begin{aligned}
\|\iota(f) - (\psi \circ \tilde{\varphi}|_A)(f)\| &= \|(\psi \circ \tilde{\varphi}|_A)(f) - f\| \\
&\leq \|(\psi \circ \tilde{\varphi}|_A)(f) - (\psi \circ \tilde{\varphi}|_B)(b_f)\| + \|(\psi \circ \tilde{\varphi}|_B)(b_f) - b_f\| \\
&\quad + \|b_f - f\| \\
&= \|(\psi \circ \tilde{\varphi})(f) - (\psi \circ \tilde{\varphi})(b_f)\| + \|(\psi \circ \varphi)(b_f) - b_f\| \\
&\quad + \|b_f - f\| \\
&\leq 2\|b_f - f\| + \|(\psi \circ \varphi)(b_f) - b_f\| \\
&< \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}$$

By Theorem 3.5.8, the map $\iota : A \hookrightarrow B(H)$ is a nuclear map. Therefore, A is an exact C*-algebra as required. $\square$

A similar result to Theorem 3.7.12 holds for nuclear C*-algebras.

Theorem 3.7.13. *Let A be a C*-algebra which satisfies the following property: If $F \subsetneq A$ is a finite set and $\epsilon \in \mathbb{R}_{>0}$ then there exists a nuclear C*-subalgebra $B \subsetneq A$ such that if $f \in F$ then there exists $b \in B$ such that*

$$\|b - f\| < \epsilon.$$

Then, A is also a nuclear C-algebra.*

Proof. Assume that A is a C*-algebra which satisfied the local property described in the statement of the theorem. Assume that $F \subsetneq A$ is finite and that $\epsilon \in \mathbb{R}_{>0}$. Then, there exists a nuclear C*-subalgebra $B \subsetneq A$ such that if $f \in F$ then there exists $b \in B$ such that $\|f - b\| < \frac{\epsilon}{3}$.

Since B is a nuclear C*-algebra then the identity map id_B is a nuclear map. Using a similar argument to Theorem 3.7.12, we find that there exist c.c.p maps $\varphi : B \to M_m(\mathbb{C})$ and $\psi : M_m(\mathbb{C}) \to B$ such that if $b \in B$ then

$$\|(\psi \circ \varphi)(b) - b\| < \frac{\epsilon}{3}.$$

By Arveson's extension theorem in Theorem 3.3.7, we can extend the c.c.p map $\varphi : B \to M_m(\mathbb{C})$ to a c.c.p map $\tilde{\varphi} : A \to M_m(\mathbb{C})$. Now if $f \in F$ then let $b_f \in B$ satisfy $\|b_f - f\| < \epsilon/3$. Then,

$$\begin{aligned}
\|f - (\psi \circ \tilde{\varphi})(f)\| &= \|(\psi \circ \tilde{\varphi})(f) - f\| \\
&\leq \|(\psi \circ \tilde{\varphi})(f) - (\psi \circ \tilde{\varphi})(b_f)\| + \|(\psi \circ \tilde{\varphi})(b_f) - b_f\| \\
&\quad + \|b_f - f\| \\
&\leq \|\psi \circ \tilde{\varphi}\| \|b_f - f\| + \|(\psi \circ \varphi)(b_f) - b_f\| + \|b_f - f\| \\
&\leq 2\|b_f - f\| + \|(\psi \circ \varphi)(b_f) - b_f\| \\
&< \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}$$

By Theorem 3.5.8, the identity map id_A is a nuclear map. Thus, A is a nuclear C*-algebra as required. $\square$

One major consequence of Theorem 3.7.13 is that under the right circumstance, the direct limit of a direct sequence of nuclear C*-algebras is again nuclear.

Theorem 3.7.14. *Let $\{(A_n, \varphi_n)\}_{n\in\mathbb{Z}_{>0}}$ be a direct sequence of C*-algebras. Let $\varinjlim A_n$ be its direct limit. Assume that if $n \in \mathbb{Z}_{>0}$ then A_n is a nuclear C*-algebra and the *-homomorphism $\varphi_n : A_n \to A_{n+1}$ is injective. Then, $\varinjlim A_n$ is also a nuclear C*-algebra.*

Proof. Assume that $\{(A_n, \varphi_n)\}_{n\in\mathbb{Z}_{>0}}$ is a direct sequence of C*-algebras. Assume that if $n \in \mathbb{Z}_{>0}$ then A_n is a nuclear C*-algebra and φ_n is an injective *-homomorphism. Assume that $\varinjlim A_n$ is the direct limit of the direct sequence $\{(A_n, \varphi_n)\}_{n\in\mathbb{Z}_{>0}}$.

Recall that

$$\prod_{n=1}^{\infty}{}' A_n = \left\{ (a_n)_{n\in\mathbb{Z}_{>0}} \in \prod_{n=1}^{\infty} A_n \;\middle|\; \begin{matrix} \text{There exists } N \in \mathbb{Z}_{>0} \text{ such that} \\ a_{\ell+1} = \varphi_\ell(a_\ell) \text{ for } \ell \geq N \end{matrix} \right\}$$

and that we have the canonical map

$$\begin{array}{rccc} \iota : & \prod_{n=1}^{\infty}{}' A_n & \to & \varinjlim A_n \\ & (a_n)_{n\in\mathbb{Z}_{>0}} & \mapsto & (a_n)_{n\in\mathbb{Z}_{>0}} + p^{-1}(\{0\}). \end{array}$$

The C*-seminorm p is given by

$$\begin{array}{rccc} p : & \prod_{n=1}^{\infty}{}' A_n & \to & \mathbb{R}_{\geq 0} \\ & (a_n)_{n\in\mathbb{Z}_{>0}} & \mapsto & \lim_{n\to\infty} \|a_n\|. \end{array}$$

If $n \in \mathbb{Z}_{>0}$ then we also have the natural map φ^n, which is the *-homomorphism

$$\begin{array}{rccc} \varphi^n : & A_n & \to & \varinjlim A_n \\ & a & \mapsto & \iota(\widehat{\varphi}^n(a)) = \iota\big((0, \ldots, 0, a, \varphi_n(a), \varphi_{n,n+2}(a), \ldots)\big). \end{array}$$

To show: (a) If $n \in \mathbb{Z}_{>0}$ then φ^n is an injective *-homomorphism.

(a) Assume that $n \in \mathbb{Z}_{>0}$ and $a \in \ker \varphi^n$ so that

$$\iota\big((0, \ldots, 0, a, \varphi_n(a), \varphi_{n,n+2}(a), \ldots)\big) = 0$$

in $\varinjlim A_n$. By construction of the direct limit $\varinjlim A_n$, this means that

$$p((0, \ldots, 0, a, \varphi_n(a), \varphi_{n,n+2}(a), \ldots)) = \lim_{m\to\infty} \|\varphi_{n,n+m}(a)\| = 0.$$

Recall that if $k \in \mathbb{Z}_{>0}$ then φ_k is an injective *-homomorphism and thus isometric by Theorem 1.6.4. Hence,

$$\lim_{m\to\infty} \|\varphi_{n,n+m}(a)\| = \lim_{m\to\infty} \|a\|$$

and subsequently we have $a = 0$. Thus, φ^n is an injective *-homomorphism.

Now we will show that $\varinjlim A_n$ is a nuclear C*-algebra. Assume that $F \subsetneq \varinjlim A_n$ is a finite set and $\epsilon \in \mathbb{R}_{>0}$. Recall that

$$\overline{\bigcup_{n=1}^{\infty} \varphi^n(A_n)} = \varinjlim A_n.$$

Hence if $f \in F$ then there exists $b_f \in \bigcup_{n=1}^{\infty} \varphi^n(A_n)$ such that $\|b_f - f\| < \epsilon$. Since the sequence of nuclear C*-algebras $\{\varphi^n(A_n)\}_{n\in\mathbb{Z}_{>0}}$ is increasing (with respect to inclusion) then there exists $\ell \in \mathbb{Z}_{>0}$ such that if $f \in F$ then $b_f \in \varphi^\ell(A_\ell)$. By assumption $\varphi^\ell(A_\ell)$ is a nuclear C*-subalgebra of $\varinjlim A_n$. By Theorem 3.7.13, we deduce that $\varinjlim A_n$ is nuclear as required. $\square$

Here are some more characterisations of nuclearity and exactness.

Theorem 3.7.15. *Let A and $\{A_n\}_{n\in I}$ be C*-algebras. Assume that there exist c.c.p maps*

$$\varphi_n : A \to A_n \qquad \text{and} \qquad \psi_n : A_n \to A$$

such that if $a \in A$ then $\lim_n \|(\psi_n \circ \varphi_n)(a) - a\| = 0$. If each of the A_n is nuclear (where $n \in I$) then A is a nuclear C-algebra.*

Proof. Assume that A and $\{A_n\}_{n\in I}$ are C*-algebras. Assume that there exist c.c.p maps $\varphi_n : A \to A_n$ and $\psi_n : A_n \to A$ such that if $a \in A$ then

$$\|(\psi_n \circ \varphi_n)(a) - a\| \to 0$$

in the limit over $n \in I$. Assume that if $n \in I$ then A_n is a nuclear C*-algebra. Then, the identity map $id_{A_n} : A_n \to A_n$ is a nuclear map. This means that there exist c.c.p maps

$$\phi_{n,m} : A_n \to M_{k(n,m)}(\mathbb{C}) \qquad \text{and} \qquad \rho_{n,m} : M_{k(n,m)}(\mathbb{C}) \to A_n$$

such that if $x \in A_n$ then $\lim_m \|(\rho_{n,m} \circ \phi_{n,m})(x) - x\| = 0$. Now the composites $\phi_{n,m} \circ \varphi_n : A \to M_{k(n,m)}(\mathbb{C})$ and $\psi_n \circ \rho_{n,m} : M_{k(n,m)}(\mathbb{C}) \to A$ are c.c.p maps such that if $a \in A$ then

$$\begin{aligned}
\|(\psi_n \circ \rho_{n,m} \circ \phi_{n,m} \circ \varphi_n)(a) - a\| &\le \|(\psi_n \circ \rho_{n,m} \circ \phi_{n,m} \circ \varphi_n)(a) - (\psi_n \circ \varphi_n)(a)\| \\
&\quad + \|(\psi_n \circ \varphi_n)(a) - a\| \\
&\le \|(\rho_{n,m} \circ \phi_{n,m})(\varphi_n(a)) - \varphi_n(a)\| \\
&\quad + \|(\psi_n \circ \varphi_n)(a) - a\| \to 0
\end{aligned}$$

where the limit is taken over all m and $n \in I$. Thus, id_A is a nuclear map and so, A is a nuclear C*-algebra as required. $\square$

Theorem 3.7.16. *Let H be a Hilbert space and $A \subsetneq B(H)$ be a C*-algebra. Assume that $\{A_n\}_{n\in I}$ are C*-algebras and that there exist c.c.p maps*

$$\varphi_n : A \to A_n \qquad \text{and} \qquad \psi : A_n \to A$$

such that if $a \in A$ then $\lim_n \|(\psi_n \circ \varphi_n)(a) - a\| = 0$. If each C-algebra A_n is exact then A is an exact C*-algebra.*

Proof. Assume that H is a Hilbert space and $A \subsetneq B(H)$ and $\{A_n\}_{n\in I}$ are C*-algebras. Assume that there exist c.c.p maps $\varphi_n : A \to A_n$ and $\psi_n : A_n \to A$ such that if $a \in A$ then $\|(\psi_n \circ \varphi_n)(a) - a\| \to 0$ in the limit over $n \in I$.

Assume that if $n \in I$ then A_n is an exact C*-algebra. Then, there exists a faithful representation (π_n, H_n) such that the *-homomorphism $\pi_n : A_n \to B(H_n)$ is a nuclear map. Now let $\iota : A \hookrightarrow B(H)$ be the inclusion map. We want to show that ι is a nuclear map.

Since π_n is a nuclear map then there exist c.c.p maps

$$\phi_{n,m} : A_n \to M_{k(n,m)}(\mathbb{C}) \qquad \text{and} \qquad \rho_{n,m} : M_{k(n,m)}(\mathbb{C}) \to B(H_n)$$

such that if $x \in A_n$ then $\lim_m \|(\rho_{n,m} \circ \phi_{n,m})(x) - \pi_n(x)\| = 0$. If $n \in I$ then define the map

$$\begin{array}{rccc} \alpha_n : & \pi_n(A_n) & \to & B(H) \\ & \pi_n(y) & \mapsto & (\iota \circ \psi_n)(y) = y. \end{array}$$

The map α_n is a c.c.p map. By Arveson's extension theorem in Theorem 3.3.7, there exists a c.c.p map $\tilde{\alpha}_n : B(H_n) \to B(H)$ such that $\tilde{\alpha}_n|_{\pi_n(A_n)} = \alpha_n$. So, the composites $\phi_{n,m} \circ \varphi_n : A \to M_{k(n,m)}(\mathbb{C})$ and $\tilde{\alpha}_n \circ \rho_{n,m} : M_{k(n,m)}(\mathbb{C}) \to B(H)$ are c.c.p maps and if $a \in A$ then

$$\begin{aligned} \|(\tilde{\alpha}_n \circ \rho_{n,m} \circ \phi_{n,m} \circ \varphi_n)(a) - \iota(a)\| &\leq \|(\tilde{\alpha}_n \circ \rho_{n,m} \circ \phi_{n,m} \circ \varphi_n)(a) - (\tilde{\alpha}_n \circ \pi_n \circ \varphi_n)(a)\| \\ &\quad + \|(\tilde{\alpha}_n \circ \pi_n \circ \varphi_n)(a) - \iota(a)\| \\ &= \|(\tilde{\alpha}_n \circ \rho_{n,m} \circ \phi_{n,m} \circ \varphi_n)(a) - (\tilde{\alpha}_n \circ \pi_n \circ \varphi_n)(a)\| \\ &\quad + \|(\iota \circ \psi_n \circ \varphi_n)(a) - \iota(a)\| \\ &\leq \|(\rho_{n,m} \circ \phi_{n,m})(\varphi_n(a)) - \pi_n(\varphi_n(a))\| \\ &\quad + \|(\psi_n \circ \varphi_n)(a) - a\| \\ &\to 0 \end{aligned}$$

where the limit is taken over m and $n \in I$. Hence, $\iota : A \hookrightarrow B(H)$ is a nuclear map and consequently, A is an exact C*-algebra as required. $\square$

The formuations of nuclearity, exactness and semidiscreteness asked for in [BO08, Exercise 2.3.1] are straightforward applications of Theorem 3.5.8 and Theorem 3.5.9. Hence, we will simply list them below without proof.

Theorem 3.7.17. *Let A be a C*-algebra. Then, A is a nuclear C*-algebra if and only if the following statement is satisfied: If $F \subsetneq A$ is a finite set and $\epsilon \in \mathbb{R}_{>0}$ then there exist $n \in \mathbb{Z}_{>0}$ and c.c.p maps $\varphi : A \to M_n(\mathbb{C})$ and $\psi : M_n(\mathbb{C}) \to A$ such that if $a \in F$ then*

$$\|(\psi \circ \varphi)(a) - a\| < \epsilon.$$

Theorem 3.7.18. *Let A be a C*-algebra. Then, A is an exact C*-algebra if and only if the following statement is satisfied: There exists a faithful representation (π, H) of A such that if $F \subsetneq A$ is a finite set and $\epsilon \in \mathbb{R}_{>0}$ then there exist $n \in \mathbb{Z}_{>0}$ and c.c.p maps $\varphi : A \to M_n(\mathbb{C})$ and $\psi : M_n(\mathbb{C}) \to B(H)$ such that if $a \in F$ then*

$$\|(\psi \circ \varphi)(a) - \pi(a)\| < \epsilon.$$

Theorem 3.7.19. *Let M be a von Neumann algebra. Then, M is a von Neumann algebra if and only if the following statement is satisfied: If $F \subsetneq M$ and $\chi \subsetneq M_*$ are finite sets and $\epsilon \in \mathbb{R}_{>0}$ then there exist $n \in \mathbb{Z}_{>0}$ and c.c.p maps $\varphi : M \to M_n(\mathbb{C})$ and $\psi : M_n(\mathbb{C}) \to M$ such that if $a \in F$ and $\eta \in \chi$ then*

$$|\eta((\psi \circ \varphi)(a)) - \eta(a)| < \epsilon.$$

Here is another simple condition for nuclearity which involves conditional expectations, courtesy of [BO08, Exercise 2.3.3].

Theorem 3.7.20. *Let A be a nuclear C*-algebra and $B \subsetneq A$ be a C*-subalgebra of A. Assume that there exists a conditional expectation $E : A \to B$. Then, B is also a nuclear C*-algebra.*

Proof. Assume that A is a nuclear C*-algebra and that $B \subsetneq A$ is a C*-subalgebra of A. Assume that $E : A \to B$ is a conditional expectation. By definition of a conditional expectation, E is a c.c.p map. So by Theorem 3.5.5, $E = E \circ id_A$ is a nuclear map. Applying Theorem 3.5.4, we deduce that

$$E|_B = id_B$$

is a nuclear map. Therefore, B is a nuclear C*-algebra. $\square$

There is also an analogue of Theorem 3.7.20 which applies to semidiscrete von Neumann algebras.

Theorem 3.7.21. *Let M be a semidiscrete von Neumann algebra and $N \subsetneq M$ be a von Neumann subalgebra of M. Assume that there exists a normal conditional expectation $E : M \to N$. Then, N is also a semidiscrete von Neumann algebra.*

Proof. Assume that M is a semidiscrete von Neumann algebra and that $N \subsetneq M$ is a von Neumann subalgebra of M. Assume that $E : M \to N$ is a normal conditional expectation. Since M is semidiscrete then the identity map $id_M : M \to M$ is weakly nuclear. So, there exist c.c.p maps

$$\varphi_n : M \to M_{k(n)}(\mathbb{C}) \qquad \text{and} \qquad \psi_n : M_{k(n)}(\mathbb{C}) \to M$$

such that if $m \in M$ and $\eta \in M_*$ then

$$\lim_n |\eta((\psi_n \circ \varphi_n)(m)) - \eta(m)| = 0.$$

Let $\iota : N \hookrightarrow M$ be the inclusion *-homomorphism which is a c.c.p map. By the above limit, ι is weakly nuclear. Furthermore, $id_N = E \circ \iota$. So if $n \in N$ and $\eta \in N_*$ then $\eta \circ E$ is a composite of normal maps and by Theorem 3.5.1, $\eta \circ E \in M_*$ and

$$\begin{aligned} &|\eta((E \circ \psi_\ell \circ \varphi_\ell \circ \iota)(n)) - \eta(n)| \\ &= |\eta((E \circ \psi_\ell \circ \varphi_\ell \circ \iota)(n)) - \eta((E \circ \iota)(n))| \\ &= |(\eta \circ E)((\psi_\ell \circ \varphi_\ell \circ \iota)(n)) - (\eta \circ E)(\iota(n))| \to 0 \end{aligned}$$

in the limit over ℓ. Hence, $id_N = E \circ \iota$ is a weakly nuclear map (because $N_* \subsetneq M_*$) and N is semidiscrete as required. □

The exercise in [BO08, Exercise 2.3.6] states that nuclearity and exactness passes to finite direct sums. We will only address nuclearity in the next result.

Theorem 3.7.22. *Let $\{A_i\}_{i=1}^k$ be a sequence of C*-algebras. Then, the direct sum $\bigoplus_{i=1}^k A_i$ is a nuclear C*-algebra if and only if $A_1, \ldots, A_k$ are nuclear C*-algebras.*

Proof. Assume that $\{A_i\}_{i=1}^k$ is a sequence of C*-algebras. First assume that the direct sum $B = \bigoplus_{i=1}^k A_i$ is a nuclear C*-algebra. Then, the identity map $id_B : B \to B$ is nuclear. If $i \in \{1, 2, \ldots, k\}$ then we have inclusion maps

$$\begin{array}{rccc} \iota_i : & A_i & \to & B \\ & a & \mapsto & (0, \ldots, 0, a, 0, \ldots, 0) \end{array}$$

and projection maps

$$\begin{array}{rccc} \pi_i : & B & \to & A_i \\ & (a_1, a_2, \ldots, a_k) & \mapsto & a_i. \end{array}$$

If $i \in \{1, 2, \ldots, k\}$ then ι_i and π_i are *-homomorphisms and are c.c.p maps. Furthermore, $id_{A_i} = \pi_i \circ id_B \circ \iota_i$. By Theorem 3.5.5, id_{A_i} is a nuclear map. We conclude that if $i \in \{1, 2, \ldots, k\}$ then A_i is a nuclear C*-algebra.

Conversely, assume that if $i \in \{1, 2, \ldots, k\}$ then A_i is a nuclear C*-algebra. Then, there exist an upwards direct set Λ_i and c.c.p maps

$$\varphi_{i,n} : A_i \to M_{k(i,n)}(\mathbb{C}) \qquad \text{and} \qquad \psi_{i,n} : M_{k(i,n)}(\mathbb{C}) \to A_i$$

where $n \in \Lambda_i$ and if $a \in A_i$ then

$$\lim_n \|(\psi_{i,n} \circ \varphi_{i,n})(a) - a\| = 0.$$

Now define the set $\Lambda = \Lambda_1 \times \cdots \times \Lambda_k$. Then, Λ is an upwards directed set. If $\lambda = (\lambda_1, \ldots, \lambda_k) \in \Lambda$ then define the c.c.p maps

$$\psi_\lambda = \psi_{1,\lambda_1} \oplus \psi_{2,\lambda_2} \oplus \cdots \oplus \psi_{n,\lambda_n}$$

and

$$\varphi_\lambda = \varphi_{1,\lambda_1} \oplus \varphi_{2,\lambda_2} \oplus \cdots \oplus \varphi_{n,\lambda_n}.$$

If $a = (a_1, a_2, \ldots, a_k) \in B$ then

$$\|(\psi_\lambda \circ \varphi_\lambda)(a) - a\| = \max_{i \in \{1, \ldots, k\}} \|(\varphi_{i,\lambda_i} \circ \psi_{i,\lambda_i})(a_i) - a_i\| \to 0$$

in the limit over $\lambda \in \Lambda$. By Theorem 3.5.10, we deduce that id_B is a nuclear map and that B is a nuclear C*-algebra as required. $\square$

Note that in general, an infinite direct sum (called a ℓ^∞-direct sum in [BO08]) of nuclear C*-algebras is in general not exact. That is, if $\{A_i\}_{i \in \mathbb{Z}_{>0}}$ is a sequence of nuclear C*-algebras then

$$\bigoplus_{i=1}^{\infty} A_i = \{(a_i)_{i \in \mathbb{Z}_{>0}} \mid a_i \in A_i, \sup_{i \in \mathbb{Z}_{>0}} \|a_i\| < \infty\}$$

need not be a nuclear C*-algebra. This holds even if $A_i = M_i(\mathbb{C})$.

3.8 More examples of nuclear C*-algebras

In this section, we give various examples of nuclear C*-algebras; nuclear in the sense of Definition 3.7.1. We follow [BO08, Section 2.4].

Theorem 3.8.1. *Let A be a finite dimensional C*-algebra. Then, A is a nuclear C*-algebra in the sense of Definition 3.7.1.*

Proof. Assume that A is a finite dimensional C*-algebra. Let I be an upwards directed set. If $n \in I$ then $\varphi_n = id_A$ and $\psi_n = id_A$ are c.c.p maps such that if $a \in A$ then

$$\lim_{n \in I} \|(\psi_n \circ \varphi_n)(a) - a\| = 0.$$

By Theorem 3.5.10, we deduce that A is a nuclear C*-algebra with respect to Definition 3.7.1. $\square$

Theorem 3.8.2. *Let A be an AF-algebra. Then, A is a nuclear C*-algebra in the sense of Definition 3.7.1.*

Proof. Assume that A is an AF-algebra. By Theorem 2.8.3, A is a direct limit of finite-dimensional C*-algebras. By Theorem 3.7.14, we deduce that A is a nuclear C*-algebra with respect to Definition 3.7.1. $\square$

Recall from the previous chapter that we proved abelian C*-algebras are nuclear in Theorem 2.13.17. The proof given in Theorem 2.13.17 was incredibly deep and required an entire section worth of preliminary results! We will see shortly that abelian C*-algebras are nuclear in the sense of Definition 3.7.1. The proof is much shorter and utilises a partition of unity.

Theorem 3.8.3. *Let A be an abelian C*-algebra. Then, A is a nuclear C*-algebra in the sense of Definition 3.7.1.*

Proof. Assume that A is an abelian C*-algebra. By Theorem 3.7.4, it suffices to prove that A is nuclear in the case where A is unital.

To this end, assume that A is a unital abelian C*-algebra. By Theorem 1.3.5, we may assume that $A = Cts(X, \mathbb{C})$ where X is a compact Hausdorff topological space.

In order to prove that A is nuclear, we will use Theorem 3.7.17. Assume that $F \subsetneq A$ is a finite set and $\epsilon \in \mathbb{R}_{>0}$. Take an open cover $\{U_1, \ldots, U_n\}$ of X such that if $f \in F$, $i \in \{1, 2, \ldots, n\}$ and $x, y \in U_i$ then

$$|f(x) - f(y)| < \epsilon.$$

If $i \in \{1, \ldots, n\}$ then let $y_i \in U_i$ and $\{\sigma_1, \ldots, \sigma_n\}$ be a partition of unity with respect to the open cover $\{U_1, \ldots, U_n\}$. First define φ by

$$\begin{array}{cccc} \varphi : & A & \to & \mathbb{C}^n \\ & f & \mapsto & (f(y_1), \ldots, f(y_n)). \end{array}$$

It is straightforward to check that φ is a unital *-homomorphism and thus a u.c.p map (which is contractive by [Pau02, Corollary 2.9]). Next define the map

$$\begin{array}{cccc} \psi : & \mathbb{C}^n & \to & A \\ & (d_1, \ldots, d_n) & \mapsto & \sum_{i=1}^n d_i \sigma_i. \end{array}$$

To see that ψ is positive, assume that $(r_1, \ldots, r_n) \in (\mathbb{R}_{\geq 0})^n$. We know that if $i \in \{1, 2, \ldots, n\}$ then $\sigma_i \in Cts(X, [0, 1])$ and has support contained in U_i. So if $x \in X$ then there exists $k_1, \ldots, k_j \in \{1, 2, \ldots, n\}$ such that $x \in U_{k_1} \cap \cdots \cap U_{k_j}$ and consequently,

$$\psi(r_1, \ldots, r_n)(x) = \sum_{i=1}^{n} r_i \sigma_i(x) = \sum_{\ell=1}^{j} r_{k_\ell} \sigma_{k_\ell}(x) \in \mathbb{R}_{\geq 0}.$$

This shows that ψ is a positive map. Now the range of ψ is A which is by assumption an abelian C*-algebra.

To show: (a) ψ is completely positive.

(a) Assume that $k \in \mathbb{Z}_{>0}$. By identifying $M_k(\mathbb{C}^n)$ as the direct sum of n copies of $M_k(\mathbb{C})$, we find that the map ψ_k is defined explicitly by

$$\begin{array}{cccc} \psi_k : & M_k(\mathbb{C}^n) \cong \bigoplus_{i=1}^n M_k(\mathbb{C}) & \to & M_k(A) \\ & T_1 \oplus \cdots \oplus T_n & \mapsto & \sum_{i=1}^n T_i \otimes \sigma_i \end{array}$$

where if $T \in M_k(\mathbb{C})$ and $\sigma \in A = Cts(X, \mathbb{C})$ then $T \otimes \sigma$ is an element of $M_k(Cts(X, \mathbb{C})) \cong Cts(X, M_k(\mathbb{C}))$, defined to be the matrix valued function

$$\begin{array}{cccc} T \otimes \sigma : & X & \to & M_k(\mathbb{C}) \\ & x & \mapsto & \sigma(x)T. \end{array}$$

If $T_1, \ldots, T_n \in M_k(\mathbb{C})$ are positive matrices and $i \in \{1, 2, \ldots, n\}$ then $T_i \otimes \sigma_i$ are positive functions (because im $\sigma_i \subseteq [0, 1]$) and consequently, ψ_k is a positive map. So ψ is completely positive. This proves part (a) of the proof.

To see that ψ is contractive, note that if $x \in X$ then

$$\psi(1, 1, \ldots, 1)(x) = \sum_{i=1}^{n} \sigma_i(x) = 1.$$

So ψ is a u.c.p map and hence contractive by [Pau02, Corollary 2.9] once again. Now if $f \in F$ and $x \in X$ then there exist $k_1, \ldots, k_j \in \{1, 2, \ldots, n\}$ such that $x \in U_{k_1} \cap \cdots \cap U_{k_j}$

$$\begin{aligned}
\|(\psi \circ \varphi)(f)(x) - f(x)\| &= \Big\| \sum_{i=1}^{n} f(y_i)\sigma_i(x) - (\sum_{i=1}^{n} \sigma_i(x))f(x) \Big\| \\
&= \Big\| \sum_{i=1}^{n} \big((f(x) - f(y_i))\sigma_i(x)\big) \Big\| \\
&\leq \Big\| \sum_{\ell=1}^{j} \big((f(x) - f(y_{k_\ell}))\sigma_{k_\ell}(x)\big) \Big\| \\
&\quad + \Big\| \sum_{i \notin \{k_1, \ldots, k_j\}} \big((f(x) - f(y_i))\sigma_i(x)\big) \Big\| \\
&< \Big\| \sum_{\ell=1}^{j} \epsilon \sigma_{k_\ell}(x) \Big\| + 0 = \epsilon.
\end{aligned}$$

Therefore if $f \in F$ then

$$\|(\psi \circ \varphi)(f) - f\| < \epsilon.$$

By Theorem 3.5.10 and Theorem 3.7.17, we deduce that A is a nuclear C*-algebra with respect to Definition 3.7.1. $\square$

The proof of Theorem 3.8.3 is why nuclearity of C*-algebras can be considered a non-commutative analogue of having a partition of unity, although note that in [BO08, Remark 2.4.3], it is stated that this analogy is not perfect.

An important consequence of the proof of Theorem 3.8.3 is

Theorem 3.8.4. *Let A be a nuclear C*-algebra. If $n \in \mathbb{Z}_{>0}$ then $M_n(A)$ is a nuclear C*-algebra.*

Here is an interesting example of a non-exact von Neumann algebra given in [BO08, Lemma 2.4.7].

Theorem 3.8.5. *Let $\{k(n)\}_{n\in\mathbb{Z}_{>0}}$ be a sequence in $\mathbb{Z}_{>0}$ such that $\lim_{n\to\infty} k(n) = \infty$. Let M be the von Neumann algebra*

$$M = \prod_{n\in\mathbb{Z}_{>0}} M_{k(n)}(\mathbb{C}) = \{(x_n) \mid x_n \in M_{k(n)}(\mathbb{C}),\ \sup_{n\in\mathbb{Z}_{>0}} \|x_n\| < \infty\}.$$

Then M is not an exact C-algebra.*

Proof. Assume that $\{k(n)\}_{n\in\mathbb{Z}_{>0}}$ is a sequence in $\mathbb{Z}_{>0}$ which tends to ∞. Assume that M is the von Neumann algebra defined as above.

We invoke [BO08, Corollary 3.7.12] and [BO08, Theorem 7.4.1], which shows that there exists a separable non-exact C*-algebra A and a sequence of representations

$$\{\pi_n : A \to M_{j(n)}(\mathbb{C})\}_{n\in\mathbb{Z}_{>0}}$$

such that $\bigoplus_{n\in\mathbb{Z}_{>0}} \pi_n : A \to \bigoplus_{n\in\mathbb{Z}_{>0}} M_{j(n)}(\mathbb{C})$ is a faithful representation of A.

Now take a representation $\pi_m : A \to M_{j(m)}(\mathbb{C})$. Since $\lim_{n\to\infty} k(n) = \infty$ then there exists $m' \in \mathbb{Z}_{>0}$ such that $j(m) < k(n)$. Take a non-unital embedding $\iota_m : M_{j(m)}(\mathbb{C}) \hookrightarrow M_{k(m')}(\mathbb{C})$. By considering the composite

$$\bigoplus_{m\in\mathbb{Z}_{>0}} \iota_m \circ \bigoplus_{n\in\mathbb{Z}_{>0}} \pi_n : A \to M$$

which is injective, we find that M contains a non-exact subalgebra and by Theorem 3.7.10, M is not exact as required. $\square$

3.9 C*-algebras of discrete groups

In this section, we will build important examples of C*-algebras from discrete groups — groups equipped with the discrete topology. Note that any group can be thought of as a discrete group because we can just add the discrete topology to it.

Definition 3.9.1. Let H be a Hilbert space. Define the **unitary group** of H by

$$U(H) = \{x \in B(H) \mid x^* = x^{-1}\} \subseteq B(H).$$

If H is a Hilbert space then $U(H)$ is a group equipped with composition as its binary operation. This justifies why $U(H)$ is called the unitary group of H.

Definition 3.9.2. Let G be a discrete group and H be a Hilbert space. A **unitary representation** of G is a group homomorphism $\pi : G \to U(H)$.

Definition 3.9.3. Let G be a discrete group. Define $\ell^2(G)$ to be the set

$$\ell^2(G) = \Big\{ f : G \to \mathbb{C} \,\Big|\, \sum_{\gamma \in G} |f(\gamma)|^2 < \infty \Big\}.$$

In a similar vein to the classical sequence space

$$\ell^2(\mathbb{C}) = \Big\{ (x_i)_{i \in \mathbb{Z}_{>0}} \,\Big|\, x_i \in \mathbb{C},\, \sum_{i=1}^{\infty} |x_i|^2 < \infty \Big\},$$

the set $\ell^2(G)$ is a Hilbert space when equipped with the inner product

$$\begin{array}{rccc} \langle -, - \rangle : & \ell^2(G) \times \ell^2(G) & \to & \mathbb{C} \\ & (f, g) & \mapsto & \sum_{\gamma \in G} f(\gamma)\overline{g(\gamma)}. \end{array}$$

If $\gamma \in G$ then let δ_γ be the map

$$\begin{array}{rccc} \delta_\gamma : & G & \to & \mathbb{C} \\ & g & \mapsto & \begin{cases} 1, & \text{if } g = \gamma, \\ 0, & \text{otherwise.} \end{cases} \end{array} \tag{3.6}$$

Then $\delta_\gamma \in \ell^2(G)$ and the set

$$\{\delta_\gamma \mid \gamma \in G\}$$

is an orthonormal basis for $\ell^2(G)$.

The importance of the Hilbert space $\ell^2(G)$ lies with the construction of a particularly important unitary representation of G — the *left regular representation.* If $\gamma \in G$ then define the map

$$\begin{array}{rccc} u_\gamma : & \ell^2(G) & \to & \ell^2(G) \\ & f & \mapsto & \big(\eta \mapsto f(\gamma\eta)\big) \end{array}$$

We will show below that $u_\gamma \in U(\ell^2(G))$. To see that u_γ is bounded, we compute directly that

$$
\begin{aligned}
\|u_\gamma\|^2 &= \sup_{\|f\|=1} \|u_\gamma(f)\|^2 \\
&= \sup_{\|f\|=1} \sum_{\eta\in G} |u_\gamma(f)(\eta)|^2 \\
&= \sup_{\|f\|=1} \sum_{\eta\in G} |f(\gamma\eta)|^2 \\
&= \sup_{\|f\|=1} \sum_{\eta\in G} |f(\eta)|^2 = 1.
\end{aligned}
$$

So $\|u_\gamma\| = 1$. The fact that u_γ is linear follows from direct computation as usual. Finally to see that $u_\gamma \in U(\ell^2(G))$, we compute directly that if $f, g \in \ell^2(G)$ then

$$
\begin{aligned}
\langle u_\gamma^*(f), g\rangle &= \langle f, u_\gamma(g)\rangle \\
&= \sum_{\eta\in G} f(\eta)\overline{u_\gamma(g)(\eta)} \\
&= \sum_{\eta\in G} f(\eta)\overline{g(\gamma\eta)} \\
&= \sum_{\eta\in G} f(\gamma^{-1}\eta)\overline{g(\eta)} \\
&= \langle u_{\gamma^{-1}}(f), g\rangle.
\end{aligned}
$$

So $u_\gamma^* = u_{\gamma^{-1}} = u_\gamma^{-1}$. Therefore we conclude that if $\gamma \in G$ then $u_\gamma \in U(\ell^2(G))$. We also note that if $\gamma, \rho \in G$ then $u_\gamma \circ u_\rho = u_{\gamma\rho}$. This ensures that the left regular representation defined below is a group homomorphism.

Definition 3.9.4. Let G be a discrete group. The **left regular representation** of G is the group homomorphism

$$
\begin{array}{rccc}
u: & G & \to & U(\ell^2(G)) \\
 & \gamma & \mapsto & u_\gamma.
\end{array}
$$

Note that we can also define the right regular representation of G. Explicitly it is given by the map

$$
\begin{array}{rccc}
v: & G & \to & U(\ell^2(G)) \\
 & \rho & \mapsto & v_\rho.
\end{array}
$$

In turn, if $\rho \in G$ then $v_\rho \in U(\ell^2(G))$ is the map

$$\begin{array}{rccc} v_\rho : & \ell^2(G) & \to & \ell^2(G) \\ & f & \mapsto & \big(\eta \mapsto f(\eta\rho^{-1})\big). \end{array}$$

The definitions of left and right regular representations of G are from [BO08]. Some references use a slightly different definition (see [Put19] and [Gol13]).

So far, we have only discussed unitary representations of a discrete group G. In order to see where the topic of C*-algebras enters the discussion, we need to recall the definition of a group ring.

Definition 3.9.5. Let G be a group. The **group ring** of G, denoted by $\mathbb{C}[G]$, is the set

$$\mathbb{C}[G] = \Big\{ \sum_{g\in G} \alpha_g g \;\Big|\; \alpha_g \in \mathbb{C},\ \text{all but finitely many } \alpha_g \text{ are zero} \Big\}.$$

Addition in $\mathbb{C}[G]$ is defined by

$$\sum_{g\in G} \alpha_g g + \sum_{g\in G} \beta_g g = \sum_{g\in G} (\alpha_g + \beta_g) g.$$

Multiplication in $\mathbb{C}[G]$ is defined by

$$\big(\sum_{g\in G} \alpha_g g\big)\big(\sum_{g\in G} \beta_g g\big) = \sum_{g,h\in G} \alpha_g \beta_h gh.$$

The group ring $\mathbb{C}[G]$ also has an involution given by

$$\big(\sum_{g\in G} \alpha_g g\big)^* = \sum_{g\in G} \overline{\alpha_g} g^{-1}.$$

The important part here is that if G is an arbitrary group then $\mathbb{C}[G]$ has an involution. This means that we can talk about *-homomorphisms from $\mathbb{C}[G]$ to another ring with involution — ring homomorphisms which additionally preserve the involution maps.

Now let G be a discrete group and $u : G \to U(\ell^2(G))$ be the left regular representation. We would like to extend the definition of u to the group ring $\mathbb{C}[G]$. Define the map u by

$$\begin{array}{rccc} u : & \mathbb{C}[G] & \to & B(\ell^2(G)) \\ & \sum_{g\in G} \alpha_g g & \mapsto & \sum_{g\in G} \alpha_g u_g. \end{array} \tag{3.7}$$

That is, if $\sum_{g\in G}\alpha_g g \in \mathbb{C}[G]$, $f \in \ell^2(G)$ and $\eta \in G$ then

$$u\big(\sum_{g\in G}\alpha_g g\big)(f)(\eta) = \sum_{g\in G}\alpha_g u_g(f)(\eta) = \sum_{g\in G}\alpha_g f(g\eta).$$

The definition above extends u from G to $\mathbb{C}[G]$. We will now demonstrate that $u : \mathbb{C}[G] \to B(\ell^2(G))$ is a *-homomorphism of rings with involution. First we show that u is well-defined. Assume that $\sum_{g\in G}\alpha_g g \in \mathbb{C}[G]$. To see that $\sum_{g\in G}\alpha_g u_g$ is a bounded linear operator on $\ell^2(G)$, we compute directly that

$$\begin{aligned}\|u(\sum_{g\in G}\alpha_g g)\|^2 &= \|\sum_{g\in G}\alpha_g u_g\|^2\\ &\le \Big(\sum_{g\in G}\|\alpha_g u_g\|\Big)^2\\ &= \Big(\sum_{g\in G}|\alpha_g|\|u_g\|\Big)^2\\ &= \Big(\sum_{g\in G}|\alpha_g|\Big)^2 < \infty.\end{aligned}$$

The last inequality follows from the definition of $\mathbb{C}[G]$ — all but finitely many of the scalars α_g are equal to zero. So $\sum_{g\in G}\alpha_g u_g$ is a bounded linear operator on $\ell^2(G)$ and the map $u : \mathbb{C}[G] \to B(\ell^2(G))$ is well-defined.

To see that u is a *-homomorphism of rings, we proceed with the usual computations:

$$\begin{aligned}u(\sum_{g\in G}\alpha_g g + \sum_{g\in G}\beta_g g) &= u(\sum_{g\in G}(\alpha_g+\beta_g)g) = \sum_{g\in G}(\alpha_g+\beta_g)u_g\\ &= u(\sum_{g\in G}\alpha_g g) + u(\sum_{g\in G}\beta_g g),\end{aligned}$$

$$\begin{aligned}u((\sum_{g\in G}\alpha_g g)^*) &= u(\sum_{g\in G}\overline{\alpha_g}g^{-1})\\ &= \sum_{g\in G}\overline{\alpha_g}u_{g^{-1}} = \sum_{g\in G}\overline{\alpha_g}u_g^*\\ &= \Big(\sum_{g\in G}\alpha_g u_g\Big)^* = \Big(u(\sum_{g\in G}\alpha_g g)\Big)^*\end{aligned}$$

and

$$
\begin{aligned}
u\big((\sum_{g\in G}\alpha_g g)(\sum_{g\in G}\beta_g g)\big) &= u(\sum_{g,h\in G}\alpha_g\beta_h gh) \\
&= \sum_{g,h\in G}\alpha_g\beta_h u_{gh} \\
&= \sum_{g,h\in G}\alpha_g\beta_h (u_g u_h) \\
&= u(\sum_{g\in G}\alpha_g g)u(\sum_{h\in G}\beta_h h).
\end{aligned}
$$

Thus, $u : \mathbb{C}[G] \to B(\ell^2(G))$ is a *-homomorphism of rings. Furthermore it is injective. Assume that $\sum_{g\in G}\alpha_g g, \sum_{h\in G}\beta_h h \in \mathbb{C}[G]$ satisfy

$$
u(\sum_{g\in G}\alpha_g g) = u(\sum_{h\in G}\beta_h h)
$$

so that $\sum_{g\in G}\alpha_g u_g = \sum_{h\in G}\beta_h u_h$. This means that if $\eta \in G$ then

$$
\sum_{g\in G}(\alpha_g - \beta_g)u_g(\delta_\eta) = 0
$$

and if $e_G \in G$ is the identity element of the discrete group G then

$$
0 = \sum_{g\in G}(\alpha_g - \beta_g)u_g(\delta_\eta)(e_G) = \sum_{g\in G}(\alpha_g - \beta_g)\delta_\eta(g) = \alpha_\eta - \beta_\eta.
$$

We conclude that if $\eta \in G$ then $\alpha_\eta = \beta_\eta$ and subsequently,

$$
\sum_{g\in G}\alpha_g g = \sum_{h\in G}\beta_h h.
$$

So u is injective and thus we have extended the left regular representation of G to the injective *-homomorphism of rings in equation (3.7).

As alluded to in [BO08, Section 2.5], the extension of the left regular representation of G can be applied to any unitary representation of G.

Theorem 3.9.1. *Let G be a discrete group. Then, the following map is a bijection:*

$$\begin{array}{ccc}
\{\textit{Unitary representations of } G\} & \longleftrightarrow & \left\{\begin{array}{c}\textit{*-homomorphisms of rings}\\ \textit{defined on } \mathbb{C}[G]\end{array}\right\} \\
\pi : G \to U(H) & \mapsto & \left(\sum_{g\in G}\alpha_g g \mapsto \sum_{g\in G}\alpha_g \pi(g)\right) \\
\psi|_G & \mapsfrom & \psi : \mathbb{C}[G] \to B(H)
\end{array}$$

We will omit the proof in order to get to the C*-algebra theory.

Definition 3.9.6. Let G be a discrete group. Let u be the left regular representation of G extended to $\mathbb{C}[G]$ as defined in equation (3.7). The **reduced C*-algebra** of G, denoted by $C^*_u(G)$, is defined to be the completion of $\mathbb{C}[G]$ with respect to the norm

$$\begin{array}{rccc}
\|-\|_r : & \mathbb{C}[G] & \to & \mathbb{R}_{\geq 0} \\
& x & \mapsto & \|u(x)\|_{B(\ell^2(G))} = \sup_{\|f\|_{\ell^2(G)}=1}\|u(x)(f)\|_{\ell^2(G)}.
\end{array}$$

Definition 3.9.7. Let G be a discrete group. The **full C*-algebra** (or universal C*-algebra) of G, denoted by $C^*(G)$, is defined to be the completion of $\mathbb{C}[G]$ with respect to the norm

$$\begin{array}{rccc}
\|-\|_u : & \mathbb{C}[G] & \to & \mathbb{R}_{\geq 0} \\
& x & \mapsto & \sup_\pi\|\pi(x)\|
\end{array}$$

where the supremum is taken over all cyclic *-representations $\pi : \mathbb{C}[G] \to B(H)$.

Example 3.9.1. In this example, we work with the discrete group $\mathbb{Z}$. We want to compute the reduced C*-algebra $C^*_u(\mathbb{Z})$. By definition

$$\ell^2(\mathbb{Z}) = \left\{ f : \mathbb{Z} \to \mathbb{C} \;\middle|\; \sum_{i\in\mathbb{Z}}|f(i)|^2 < \infty \right\} = \left\{ (x_i)_{i\in\mathbb{Z}} \;\middle|\; x_i \in \mathbb{C},\ \sum_{i\in\mathbb{Z}}|x_i|^2 < \infty \right\}.$$

As stated in [BO08], the Fourier transform yields the bijection

$$\begin{array}{rccc}
\mathcal{F} : & \ell^2(\mathbb{Z}) & \to & L^2(S^1, \mathcal{B}_{S^1}, m) \\
& (\dots, x_{-1}, x_0, x_1, \dots) & \mapsto & \sum_{n\in\mathbb{Z}} x_n e^{2\pi i n x}.
\end{array}$$

Here, $\mathcal{B}_{S^1}$ is the Borel σ-algebra on the circle S^1 and m is the associated Lebesgue measure. Let u be the left regular representation of $\mathbb{Z}$ as defined in equation (3.7). Let $\sum_{n\in\mathbb{Z}}\alpha_n n \in \mathbb{C}[\mathbb{Z}]$, $(x_i)_{i\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ and $y \in \mathbb{Z}$. We also represent the element $(x_i)_{i\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ by the function $f : \mathbb{Z} \to \mathbb{C}$ given by $f(i) = x_i$. Then

$$u(\sum_{n\in\mathbb{Z}} \alpha_n n)((x_i))(y) = \sum_{n\in\mathbb{Z}} \alpha_n u_n((x_i))(y) = \sum_{n\in\mathbb{Z}} \alpha_n f(y+n) = \sum_{n\in\mathbb{Z}} \alpha_n x_{y+n}.$$

In terms of sequences $u(\sum_{n\in\mathbb{Z}} \alpha_n n)$ is given explicitly by

$$\begin{array}{rccc} u(\sum_{n\in\mathbb{Z}} \alpha_n n) : & \ell^2(\mathbb{Z}) & \to & \ell^2(\mathbb{Z}) \\ & (x_i)_{i\in\mathbb{Z}} & \mapsto & \sum_{n\in\mathbb{Z}} \alpha_n (x_{i+n})_{i\in\mathbb{Z}} \end{array}$$

Using the Fourier transform $\mathcal{F}$, $u(\sum_{n\in\mathbb{Z}} \alpha_n n)$ becomes the bounded linear operator

$$\begin{array}{rccc} u(\sum_{n\in\mathbb{Z}} \alpha_n n) : & L^2(S^1, \mathcal{B}_{S^1}, m) & \to & L^2(S^1, \mathcal{B}_{S^1}, m) \\ & \sum_{m\in\mathbb{Z}} x_m e^{2\pi i m x} & \mapsto & \sum_{n\in\mathbb{Z}} \alpha_n \sum_{m\in\mathbb{Z}} x_m e^{2\pi i (m-n)x} \end{array}$$

This means that $u(\sum_{n\in\mathbb{Z}} \alpha_n n)$ is a polynomial of multiplication operators on $L^2(S^1, \mathcal{B}_{S^1}, m)$. More precisely, if $n \in \mathbb{Z}$ then define

$$\begin{array}{rccc} \mathfrak{m}_n : & L^2(S^1, \mathcal{B}_{S^1}, m) & \to & L^2(S^1, \mathcal{B}_{S^1}, m) \\ & e^{2\pi i \ell x} & \mapsto & e^{2\pi i (\ell-n)x} \end{array}.$$

We have

$$u(\sum_{n\in\mathbb{Z}} \alpha_n n) = \sum_{n\in\mathbb{Z}} \alpha_n \mathfrak{m}_n$$

where we recall that all but finitely many of the α_n are equal to zero so that the RHS is a well-defined element of $L^2(S^1, \mathcal{B}_{S^1}, m)$. By definition, the reduced C*-algebra $C_u^*(\mathbb{Z})$ is the completion of $\mathbb{C}[\mathbb{Z}]$ and the completion of the set of polynomials in $\mathfrak{m}_n$ is the set of continuous functions $Cts(S^1, \mathbb{C})$ by the Stone-Weierstrass theorem. Thus we have identified the reduced C*-algebra $C_u^*(\mathbb{Z})$ with $Cts(S^1, \mathbb{C})$.

An important property about the reduced C*-algebra of a discrete group is that it always admits a faithful tracial state.

Theorem 3.9.2. *Let G be a discrete group and $C_u^*(G)$ be its reduced C*-algebra. Let $u : C_u^*(G) \to B(\ell^2(G))$ be the left regular representation. Define the map*

$$\begin{array}{rccc} \tau : & C_u^*(G) & \to & \mathbb{C} \\ & x & \mapsto & \langle u(x)(\delta_{e_G}), \delta_{e_G} \rangle \end{array}$$

where $\langle -,-\rangle$ is the inner product on $\ell^2(G)$, e_G is the identity element of G and δ_{e_G} is defined as in equation (3.6)*. Then τ defines a faithful tracial state on $C_u^*(G)$.*

Proof. Assume that G is a discrete group and that $u : C_u^*(G) \to B(\ell^2(G))$ is the left regular representation. Assume that τ is the map defined as above. Then τ is linear because the inner product is linear in the first argument and u is linear.

To show: (a) τ is a tracial state.

(b) τ is faithful.

(a) To see that τ is a state, we will first show that τ is a positive linear functional. Since $\mathbb{C}[G]$ is dense in $C_u^*(G)$ then we may assume that $x = \sum_{g\in G} \alpha_g g \in \mathbb{C}[G]$. We find that

$$\begin{aligned}
\tau(x^*x) &= \langle u(x^*x)(\delta_{e_G}), \delta_{e_G}\rangle \\
&= \langle u(\sum_{g,h\in G} \overline{\alpha_g}\alpha_h g^{-1}h)(\delta_{e_G}), \delta_{e_G}\rangle \\
&= \sum_{\gamma\in G} u(\sum_{g,h\in G} \overline{\alpha_g}\alpha_h g^{-1}h)(\delta_{e_G})(\gamma)\overline{\delta_{e_G}(\gamma)} \\
&= u(\sum_{g,h\in G} \overline{\alpha_g}\alpha_h g^{-1}h)(\delta_{e_G})(e_G) \\
&= \delta_{e_G}(\sum_{g,h\in G} \overline{\alpha_g}\alpha_h g^{-1}h) = \sum_{g\in G} |\alpha_g|^2
\end{aligned}$$

which is positive. Hence τ is a positive linear functional. Next, we bound the norm of τ as follows

$$\begin{aligned}
\|\tau\| &= \sup_{\|x\|=1} |\tau(x)| \\
&= \sup_{\|u(x)\|_{B(\ell^2(G))}=1} |\langle u(x)(\delta_{e_G}), \delta_{e_G}\rangle| \\
&\leq \sup_{\|u(x)\|_{B(\ell^2(G))}=1} \|u(x)\|_{B(\ell^2(G))}\|\delta_{e_G}\|^2 = 1.
\end{aligned}$$

To obtain the reverse equality, we note that

$$\begin{aligned}
|\tau(e_G)| &= |\langle u(e_G)(\delta_{e_G}), \delta_{e_G}\rangle| \\
&= \Big|\sum_{\gamma\in G} u_{e_G}(\delta_{e_G})(\gamma)\overline{\delta(e_G)(\gamma)}\Big| \\
&= \Big|u_{e_G}(\delta_{e_G})(e_G)\Big| = 1
\end{aligned}$$

and

$$\|e_G\|^2 = \|u_{e_G}\|^2_{B(\ell^2(G))} = \sup_{\|f\|=1}\|u_{e_G}(f)\|^2 = \sup_{\|f\|=1}\sum_{\gamma\in G}|f(\gamma)|^2 = \sup_{\|f\|=1}\|f\|^2 = 1.$$

Therefore

$$1 = |\tau(e_G)| \leq \sup_{\|x\|=1}|\tau(x)| = \|\tau\|$$

and $\|\tau\| = 1$. So τ is a state.

Next we will show that τ is tracial. Assume that $y = \sum_{g\in G}\beta_g g \in \mathbb{C}[G]$. We compute directly that

$$\begin{aligned}
\tau(y^*y) &= \sum_{g\in G}|\beta_g|^2 = \delta_{e_G}(\sum_{g,h\in G}\beta_g\overline{\beta_h}gh^{-1}) \\
&= \sum_{\gamma\in G} u_{yy^*}(\delta_{e_G})(\gamma)\overline{\delta_{e_G}(\gamma)} \\
&= \langle u(yy^*)(\delta_{e_G}), \delta_{e_G}\rangle \\
&= \tau(yy^*).
\end{aligned}$$

Therefore τ defines a tracial state on $C^*_u(G)$.

(b) Assume that $z \in C^*_u(G)$ satisfies $\tau(z^*z) = 0$. Since $\mathbb{C}[G]$ is dense in $C^*_u(G)$, we may assume that $z = \sum_{g\in G}\zeta_g g \in \mathbb{C}[G]$. Then

$$\langle u(z^*z)\delta_{e_G}, \delta_{e_G}\rangle = \sum_{g\in G}|\zeta_g|^2 = 0.$$

We conclude that if $g \in G$ then $\zeta_g = 0$ and $z = 0$. So τ must be faithful. This completes the proof. $\square$

There is also an associated von Neumann algebra to a discrete group.

Definition 3.9.8. Let G be a discrete group. The **group von Neumann algebra** of G, denoted by $L(G)$, is the double commutant

$$L(G) = C_u^*(G)'' \subsetneq B(\ell^2(G)).$$

One characterisation of the group von Neumann algebra is given by [BO08, Theorem 6.1.4]; it turns out that if G is a discrete group then $L(G)$ is the commutant of the right regular representation $\rho : G \to B(\ell^2(G))$. That is,

$$L(G) = \rho(\mathbb{C}[G])' \qquad \text{and} \qquad L(G)' = \rho(\mathbb{C}[G])''.$$

We will provide another description of the group von Neumann algebra, as explained in [BO08, Section 2.5].

Definition 3.9.9. Let G be a discrete group and $T \in B(\ell^2(G))$. We say that T is **constant down the diagonals** if the following statement is satisfied: If $s, t, x, y \in G$ satisfies $ts^{-1} = yx^{-1}$ then

$$\langle T\delta_s, \delta_t\rangle = \langle T\delta_x, \delta_y\rangle.$$

Theorem 3.9.3. *Let G be a discrete group. Then*

$$L(G) = \{T \in B(\ell^2(G)) \mid T \textit{ is constant down the diagonals}\}.$$

Proof. Assume that G is a discrete group. First we will prove that

$$L(G) \subseteq \{T \in B(\ell^2(G)) \mid T \text{ is constant down the diagonals}\}.$$

Assume that $s, t, x, y \in G$ satisfy $ts^{-1} = yx^{-1}$. If $g \in G$ and $u : \mathbb{C}[G] \to B(\ell^2(G))$ is the left regular representation then

$$\begin{aligned}
\langle u_g(\delta_s), \delta_t\rangle &= \sum_{\gamma \in G} u_g(\delta_s)(\gamma)\overline{\delta_t(\gamma)} \\
&= u_g(\delta_s)(t) = \delta_s(gt) \\
&= \begin{cases} 1, & \text{if } g = st^{-1} = (ts^{-1})^{-1}, \\ 0, & \text{otherwise.} \end{cases} \\
&= \begin{cases} 1, & \text{if } g = xy^{-1} = (yx^{-1})^{-1}, \\ 0, & \text{otherwise.} \end{cases} \\
&= \langle u_g(\delta_x), \delta_y\rangle.
\end{aligned}$$

We conclude that if $g \in G$ then $u_g \in B(\ell^2(G))$ is constant down the diagonals. By linearity, all finite linear combinations of the u_g are constant down the diagonals. By density, the elements of the reduced C*-algebra $C_u^*(G)$ are constant down the diagonals. Finally by Theorem 2.5.7, we have

$$\overline{C_u^*(G)}^{WOT} = C_u^*(G)'' = L(G)$$

If $S \in L(G)$ then there exists a net $\{S_i\}_{i\in I}$ in $C_u^*(G)$ which weakly converges to S (converges in the weak operator topology on $B(\ell^2(G))$). Thus if $s, t, x, y \in G$ satisfies $ts^{-1} = yx^{-1}$ then

$$\langle S\delta_s, \delta_t\rangle = \lim_i \langle S_i\delta_s, \delta_t\rangle = \lim_i \langle S_i\delta_x, \delta_y\rangle = \langle S\delta_x, \delta_y\rangle.$$

Therefore, if $S \in L(G)$ then S is constant down the diagonals.

To prove the reverse inclusion, assume that $T \in B(\ell^2(G))$ is constant down the diagonals. Let $\rho : \mathbb{C}[G] \to B(\ell^2(G))$ be the right regular representation. If $s \in G$ then

$$\begin{aligned}
\langle (T\rho_s)\delta_g, \delta_h\rangle &= \sum_{\gamma\in G} (T\rho_s)(\delta_g)(\gamma)\overline{\delta_h(\gamma)} \\
&= (T\rho_s)(\delta_g)(h) = (T \circ \delta_g)(hs^{-1}) \\
&= (\rho_s T)(\delta_g)(h) \\
&= \sum_{\gamma\in G} (\rho_s T)(\delta_g)(\gamma)\overline{\delta_h(\gamma)} \\
&= \langle (\rho_s T)\delta_g, \delta_h\rangle.
\end{aligned}$$

Since $\{\delta_g \mid g \in G\}$ is an orthonormal basis for $\ell^2(G)$, we find that if $s \in G$ then $T\rho_s = \rho_s T$. By [BO08, Theorem 6.1.4],

$$T \in \rho(\mathbb{C}[G])' = L(G).$$

This completes the proof. $\square$

(TBA)

Chapter 4

On operators defined on a C*-algebra

4.1 The adjoint of a linear operator on a C*-algebra

There is a rich theory surrounding unbounded operators on a Hilbert space. A good introduction to unbounded operators on a Hilbert space is [Sol18]. If H is a Hilbert space then $B(H)$ is one of the archetypal examples of a C*-algebra. A natural question one might ask is whether the theory of operators on a Hilbert space can be generalised to an arbitrary C*-algebra. The goal of this chapter is to give an account of how such a generalisation is achieved. We primarily follow the references [WN92] and [Wor91].

Definition 4.1.1. Let A be a C*-algebra and $T : D(T) \to A$ be a linear operator defined on a vector subspace $D(T) \subseteq A$. We say that T is **densely defined** if $D(T)$ is dense in A.

The **adjoint operator** of a densely defined operator T, denoted by $T^\sharp : D(T^\sharp) \to A$, is defined by the following equivalence for arbitrary $x, y \in A$:

The elements $x \in D(T^\sharp)$ and $y = T^\sharp x$ if and only if for $a \in D(T)$,

$$(Ta)^* x = a^* T^\sharp x = a^* y.$$

In [WN92] and [Wor91], the adjoint operator is symbolised by T^*. We will use a different notation in order to not confuse the asterisk with the involution operation on A.

Let A be a C*-algebra and $T : D(T) \to A$ be a densely defined linear operator. To see that the adjoint operator $T^\sharp$ is linear, assume that $x_1, x_2 \in D(T^\sharp)$. If $a \in D(T)$ then

$$a^*(T^\sharp(x_1 + x_2)) = (Ta)^*(x_1 + x_2) = (Ta)^*x_1 + (Ta)^*x_2 = a^*(T^\sharp x_1 + T^\sharp x_2).$$

Since $D(T)$ is dense in A then $T^\sharp(x_1 + x_2) = T^\sharp x_1 + T^\sharp x_2$. So $T^\sharp : D(T^\sharp) \to A$ is linear operator.

Definition 4.1.2. Let A be a C*-algebra and $T : D(T) \to A$ be a densely defined linear operator. We say that T is closed if its graph

$$G(T) = \{(a, Ta) \mid a \in D(T)\}$$

is a closed subspace of $A \times A$.

Theorem 4.1.1. *Let A be a C*-algebra and $T : D(T) \to A$ be a densely defined linear operator. Then $T^\sharp : D(T^\sharp) \to A$ is a closed linear operator.*

Proof. Assume that A is a C*-algebra and that $T : D(T) \to A$ is a linear operator defined on a dense subspace $D(T) \subseteq A$. To see that the graph $G(T^\sharp)$ is a closed subspace of $A \times A$, assume that $(x, y) \in \overline{G(T^\sharp)}$. Then, there exists a sequence $\{(x_n, y_n)\}_{n \in \mathbb{Z}_{>0}}$ in $G(T^\sharp)$ such that

$$\lim_{n \to \infty} \|x_n - x\| = 0 \qquad \text{and} \qquad \lim_{n \to \infty} \|y_n - y\| = 0.$$

Note that if $n \in \mathbb{Z}_{>0}$ then $y_n = T^\sharp x_n$. We claim that $y = T^\sharp x$. If $a \in D(T)$ then

$$\begin{aligned}
\|(Ta)^*x - a^*y\| &\le \|(Ta)^*x - (Ta)^*x_n\| + \|(Ta)^*x_n - a^*y_n\| + \|a^*y_n - a^*y\| \\
&= \|(Ta)^*x - (Ta)^*x_n\| + \|a^*y_n - a^*y\| \\
&\le \|(Ta)^*\|\|x - x_n\| + \|a^*\|\|y_n - y\| \\
&\to 0
\end{aligned}$$

as $n \to \infty$. We deduce that if $a \in D(T)$ then $(Ta)^*x = a^*y$. So $y = T^\sharp x$ and $(x, y) \in G(T^\sharp)$. Therefore $G(T^\sharp)$ is a closed subspace of $A \times A$ and $T^\sharp$ is a closed linear operator. □

Theorem 4.1.2. *Let A be a C*-algebra and $T : D(T) \to A$ be a densely defined linear operator. If T is bounded then the adjoint operator $T^\sharp$ is also bounded.*

Proof. Assume that A is a C*-algebra and that

$$\|T\| = \sup_{a \in D(T)} \|Ta\| < \infty.$$

Let $x \in D(T^\sharp)$ and $y = T^\sharp x$. Since $D(T)$ is dense in A then there exists a sequence $\{a_n\}_{n \in \mathbb{Z}_{>0}}$ in $D(T)$ which converges to y. Now observe that

$$(Ty)^* x = \lim_{n \to \infty} (Ta_n)^* x = \lim_{n \to \infty} a_n^* y = y^* y.$$

In the above calculation, we used the fact that the involution * and T are continuous maps. So

$$\|T^\sharp x\|^2 = \|y\|^2 = \|y^* y\| \leq \|(Ty)^*\| \|x\| \leq \|T\| \|y\| \|x\|.$$

Therefore $\|y\| \leq \|T\| \|x\|$ and $\|T^\sharp\| \leq \|T\|$. So $T^\sharp : D(T^\sharp) \to A$ is a bounded operator. □

4.2 Bounded linear operators on a C*-algebra

Definition 4.2.1. Let A be a C*-algebra. We define $B(A)$ to be the set

$$B(A) = \{T : A \to A \mid \|T\| < \infty,\ T \text{ is linear}\}.$$

Note that $B(A)$ is a Banach algebra. In particular, the multiplication operation on $B(A)$ is given by the composition of bounded linear operators on A.

By Theorem 4.1.2, if $T \in B(A)$ then its adjoint $T^\sharp : D(T^\sharp) \to A$ is a bounded linear operator. However, there is no guarantee that $D(T^\sharp)$ is dense in A, let alone that $D(T^\sharp) = A$. This motivates the next definition.

Definition 4.2.2. Let A be a C*-algebra. We define the set

$$M(A) = \{T \in B(A) \mid T^\sharp \in B(A)\}.$$

This section is dedicated to proving some basic properties of $M(A)$. The next theorem is the first step towards this goal.

Theorem 4.2.1. *Let A be a C*-algebra. Then*

1. *$M(A)$ is a norm-closed subalgebra of $B(A)$.*

2. *If $T \in M(A)$ then $T^\sharp \in M(A)$.*

3. *$M(A)$, together with the adjoint as its involution operation, is a unital C^*-algebra.*

Proof. Let A be a C*-algebra. First we will show that $M(A)$ is a norm-closed subalgebra of $B(A)$.

To show: (a) $M(A)$ is closed under addition.

(b) $M(A)$ is closed under scalar multiplication.

(c) $M(A)$ is closed under multiplication.

(d) $M(A)$ is norm-closed.

(a) Assume that $S, T \in M(A)$ so that $S^\sharp, T^\sharp \in B(A)$. If $a, x \in A$ then

$$a^*((S+T)^\sharp x) = (S+T)(a)^* x = (Sa+Ta)^* x = (Sa)^* x + (Ta)^* x = a^*(S^\sharp x + T^\sharp x).$$

This means that $(S+T)^\sharp = S^\sharp + T^\sharp \in B(A)$ and consequently $S + T \in M(A)$.

(b) Assume that $\lambda \in \mathbb{C}$. If $a, x \in A$ then

$$a^*((\lambda T)^\sharp x) = (\lambda T)(a)^* x = (\lambda T a)^* x = \overline{\lambda}(Ta)^* x = a^*(\overline{\lambda} T^\sharp x).$$

So $(\lambda T)^\sharp = \overline{\lambda} T^\sharp \in B(A)$. Thus $\lambda T \in M(A)$.

(c) Again, we compute directly that if $a, x \in A$ then

$$a^*((ST)^\sharp x) = (ST)(a)^* x = S(T(a))^* x = T(a)^*(S^\sharp x) = a^*(T^\sharp S^\sharp x).$$

So $(ST)^\sharp = T^\sharp S^\sharp \in B(A)$ and $ST \in M(A)$ as required.

(d) Assume that $R \in \overline{M(A)}$ so that there exists a sequence $\{R_n\}_{n \in \mathbb{Z}_{>0}}$ in $M(A)$ such that $\lim_{n\to\infty} \|R_n - R\| = 0$. If $n \in \mathbb{Z}_{>0}$ then the adjoint operators $R_n^\sharp \in B(A)$. We want to show that $R^\sharp \in B(A)$.

Note that if $a \in A$ then

$$\|R^\sharp a - R_n^\sharp a\| = \|(R - R_n)^\sharp a\| \le \|R_n - R\| \to 0$$

as $n \to \infty$. So $R^\sharp a = \lim_{n\to\infty} R_n^\sharp a$ and so $R^\sharp \in B(A)$. Hence $R \in M(A)$ and $M(A)$ is norm-closed in $B(A)$.

To show: (e) If $T \in M(A)$ then $T^\sharp \in M(A)$.

(e) Assume that $T \in M(A)$ so that its adjoint $T^\sharp \in B(A)$. We want to show that the double adjoint $T^{\sharp\sharp} = (T^\sharp)^\sharp$ is an element of $B(A)$. It suffices to show that $D(T^{\sharp\sharp}) = A$. Using the definition of the adjoint, $x \in D(T^{\sharp\sharp})$ and $y = T^{\sharp\sharp}x$ if and only if for $z \in D(T^\sharp) = A$,

$$z^* y = (T^\sharp z)^* x.$$

Now assume that $z, a \in A$. We want to show that $T^{\sharp\sharp}a$ exists. Since $A = D(T) = D(T^\sharp)$ then

$$(Ta)^* z = a^*(T^\sharp z) \qquad \text{and} \qquad z^* Ta = (T^\sharp z)^* a = z^* T^{\sharp\sharp} a.$$

Now let $\{u_\lambda\}_{\lambda\in\Lambda}$ be an approximate unit of A. If $\lambda \in \Lambda$ then $u_\lambda Ta = u_\lambda T^{\sharp\sharp}a$ and

$$Ta - T^{\sharp\sharp}a = \lim_\lambda u_\lambda (Ta - T^{\sharp\sharp}a) = \lim_\lambda 0 = 0.$$

Therefore if $T^{\sharp\sharp}$ exists then $T^{\sharp\sharp} = T \in B(A)$. So we can set $T^{\sharp\sharp} = T$ and subsequently conclude that $T^\sharp \in M(A)$.

To see that the identity map $id_A : A \to A$ is an element of $M(A)$, observe that $a \in A = D(id_A^\sharp)$ and $b = (id_A)^\sharp a$ if and only if for $z \in A = D(id_A)$,

$$z^* a = (id_A(z))^* a = z^* b = z^*((id_A)^\sharp a).$$

By a similar argument to part (e), we find that $a = b$ and $(id_A)^\sharp = id_A \in B(A)$. Consequently the identity map $id_A \in M(A)$.

Finally to see that $M(A)$ is a unital C*-algebra, we compute directly that if $T \in M(A)$ then $\|T^\sharp T\| \le \|T^\sharp\|\|T\| \le \|T\|^2$ by Theorem 4.1.2 and

$$\begin{aligned}
\|T^\sharp T\| &= \sup_{\|a\|=1} \|T^\sharp T a\| \\
&= \sup_{\|a\|=1} \|a^*\|\|T^\sharp T a\| \\
&\geq \sup_{\|a\|=1} \|a^*(T^\sharp T a)\| \\
&= \sup_{\|a\|=1} \|(Ta)^* T a\| \\
&= \sup_{\|a\|=1} \|Ta\|^2 = \|T\|^2.
\end{aligned}$$

Therefore $\|T^\sharp T\| = \|T\|^2$ and $M(A)$ is a unital C*-algebra. □

By thinking of A acting on itself via left multiplication, we have the inclusion map

$$\begin{array}{rccc} \iota : & A & \hookrightarrow & M(A) \\ & a & \mapsto & (b \mapsto ab). \end{array} \tag{4.1}$$

Using the definition of the adjoint on $M(A)$, we note that if $a \in A$ then $\iota(a)^\sharp = \iota(a^*)$. That is, the adjoint of left multiplication by a is simply left multiplication by a^*.

Recall the definition of an essential ideal from Definition 2.9.2.

Theorem 4.2.2. *Let A be a C*-algebra and $\iota : A \hookrightarrow M(A)$ be the inclusion map in equation* (4.1). *With this inclusion, A is an essential ideal of $M(A)$.*

Proof. Assume that A is a C*-algebra and that $\iota : A \hookrightarrow M(A)$ is the inclusion map in equation (4.1).

To show: (a) A is an ideal of $M(A)$.

(a) Assume that $a, b \in A$. If $x \in A$ then

$$(\iota(a) + \iota(b))(x) = ax + bx = (a+b)x = \iota(a+b).$$

So $\iota(a) + \iota(b) = \iota(a+b)$. Next, assume that $T \in M(A)$. Then

$$(\iota(a)T)(x) = aTx = (T^\sharp a^*)^* x = \iota((T^\sharp a^*)^*)(x)$$

which means that $\iota(a)T = \iota((T^\sharp a^*)^*)$ in $M(A)$. By taking adjoints, we obtain $T^\sharp \iota(a^*) = \iota(T^\sharp a^*)$. So A is an ideal of $M(A)$.

To show: (b) A is an essential ideal.

(b) Assume that $T \in M(A)$ and that $TA = 0$. This means that if $a, x \in A$ then

$$(T\iota(a))(x) = \iota(Ta)(x) = T(a)x = 0.$$

Let $\{u_\lambda\}_{\lambda \in \Lambda}$ be an approximate unit for A. If $a \in A$ then

$$0 = \lim_\lambda T(a)u_\lambda = T(a).$$

Therefore $T = 0$ in $M(A)$ and consequently, A is an essential ideal in $M(A)$. □

The preceding theorem suggests that $M(A)$ is connected to the multiplier algebra of A, as seen in Theorem 2.9.2. In fact, we will prove that $M(A)$ is isomorphic to the multiplier algebra of A and as a consequence, justify the pernicious abuse of the notation $M(A)$. In the proof which follows, we let $\mathcal{M}(A)$ denote the multiplier algebra of A.

Theorem 4.2.3. *Let A be a C*-algebra. Then, the following map defines an isomorphism of C*-algebras:*

$$\begin{array}{cccc} \psi : & M(A) & \to & \mathcal{M}(A) \\ & T & \mapsto & \big(T,\ a \mapsto (T^\sharp a^*)^*\big) \end{array}$$

Proof. Assume that A is a C*-algebra and that ψ is the map defined as above. Let ι be the inclusion of A in $M(A)$ as in equation (4.1).

To show: (a) The map ψ is well-defined.

(a) Assume that $T \in M(A)$. Let $T' : A \to A$ be the map $a \mapsto (T^\sharp a^*)^*$. By the proof of Theorem 4.2.2, if $a, b \in A$ then

$$T(ab) = (T\iota(a))(b) = \iota(Ta)(b) = T(a)b,$$

$$\begin{aligned} T'(ab) &= (T^\sharp(b^*a^*))^* = (T^\sharp\iota(b^*)(a^*))^* \\ &= (\iota(T^\sharp b^*)(a^*))^* = ((T^\sharp b^*)a^*)^* \\ &= a(T^\sharp b^*)^* = aT'(b) \end{aligned}$$

and $aT(b) = (T^\sharp a^*)^* b = T'(a)b$. So $\psi(T) = (T, T') \in \mathcal{M}(A)$ and consequently ψ is well-defined.

To show: (b) ψ is a *-homomorphism.

(b) Assume that $S, T \in M(A)$ and $a \in A$. Then

$$\begin{aligned}
\psi(T+S)(a) &= ((T+S)(a), (T+S)'(a)) \\
&= (Ta + Sa, ((T+S)^\sharp(a^*))^*) \\
&= (Ta + Sa, (T^\sharp(a^*) + S^\sharp(a^*))^*) \\
&= (Ta + Sa, (T^\sharp(a^*))^* + (S^\sharp(a^*))^*) \\
&= \psi(T)(a) + \psi(S)(a).
\end{aligned}$$

If $\lambda \in \mathbb{C}$ then

$$\begin{aligned}
\psi(\lambda T)(a) &= ((\lambda T)(a), (\lambda T)'(a)) \\
&= (\lambda Ta, ((\lambda T)^\sharp(a^*))^*) \\
&= (\lambda Ta, (\overline{\lambda} T^\sharp(a^*))^*) \\
&= \lambda(Ta, (T^\sharp(a^*))^*) = \lambda\psi(T)(a).
\end{aligned}$$

We also have

$$\begin{aligned}
\psi(ST)(a) &= (ST(a), (ST)'(a)) \\
&= (ST(a), ((ST)^\sharp a^*))^*) \\
&= (ST(a), (T^\sharp S^\sharp(a^*))^*) \\
&= (ST(a), T'(S^\sharp(a^*)^*)) \\
&= (ST(a), T'S'(a)) = \psi(S)(a)\psi(T)(a)
\end{aligned}$$

where we recall the multiplication operation on the multiplier algebra $\mathcal{M}(A)$ from Theorem 2.9.2. To see that ψ respects involutions, first recall that the involution map on $\mathcal{M}(A)$ is given for $(L, R) \in \mathcal{M}(A)$

$$(L, R)^* = (R^\dagger, L^\dagger)$$

where $L^\dagger \in B(A)$ is defined by

$$\begin{array}{cccc}
L^\dagger : & A & \to & A \\
 & a & \mapsto & (L(a^*))^*.
\end{array}$$

So we compute directly that

$$
\begin{aligned}
\psi(T^\sharp)(a) &= (T^\sharp(a), (T^\sharp)'(a)) \\
&= (T^\sharp(a), ((T^\sharp)^\sharp(a^*))^*) \\
&= (T^\sharp(a), T(a^*)^*) \\
&= (T^\sharp(a), T^\dagger(a)) = ((T^\sharp(a)^*)^*, T^\dagger(a)) \\
&= (T'(a^*)^*, T^\dagger(a)) = ((T')^\dagger(a), T^\dagger(a)) \\
&= ((T')^\dagger, T^\dagger)(a) = (T, T')^*(a) \\
&= \psi(T)^*(a).
\end{aligned}
$$

We conclude that ψ is a *-homomorphism. This completes the proof of part (b).

To show: (c) ψ is injective.

(d) ψ is surjective.

(c) Assume that $S, T \in M(A)$ satisfies $\psi(S) = \psi(T)$. Then $(S, S') = (T, T')$ and $S = T$ as bounded operators on A. Therefore ψ is injective.

(d) Assume that $(L, R) \in \mathcal{M}(A)$. First, we must show that $L \in M(A)$. If $a, b \in A$ then

$$
R(a^*)b = a^*L(b) = L^\sharp(a)^*b.
$$

By taking adjoints, we find that if $a, b \in A$ then $b^*R(a^*)^* = b^*L^\sharp(a)$. By using an approximate unit for A, we find that $L^\sharp(a) = R(a^*)^*$. The map $a \mapsto R(a^*)^*$ is an element of $B(A)$. Therefore $L \in M(A)$ and

$$
R(a)b = (R(a)^*)^*b = (L^\sharp(a^*))^*b = L'(a)b.
$$

Thus $\psi(L) = (L, L') = (L, R)$ and so ψ is surjective.

By combining parts (a), (b), (c) and (d) of the proof we find that ψ is a *-isomorphism. This completes the proof. $\square$

Example 4.2.1. Let H be a Hilbert space. Recall that $B_0(H)$ is the space of compact operators on H. We claim that $M(B_0(H)) \cong B(H)$. First, observe that $B_0(H)$ is an essential ideal of $B(H)$. Assume that $u \in B(H)$ satisfies $uB_0(H) = 0$. If $\xi, \eta \in H$ then

$$
(u \circ |\xi\rangle\langle\xi|)\eta = u(\langle\eta, \xi\rangle\xi) = |u\xi\rangle\langle\xi| = 0.
$$

So $u\xi = 0$ by Theorem 1.5.2 and $B_0(H)$ is an essential ideal of $B(H)$.

By Theorem 4.2.3, we can think of the unital C*-algebra $M(B_0(H))$ as the multiplier algebra of $B_0(H)$. This will allow us to take advantage of the universal property in Theorem 2.9.3. In particular, the inclusion $\iota : B_0(H) \hookrightarrow M(B_0(H))$ extends to a unique injective *-homomorphism $\varphi : B(H) \to M(B_0(H))$.

We will now show that φ is surjective. Assume that $(L, R) \in M(B_0(H))$ and let $\nu \in H$ satisfy $\|\nu\| = 1$. Define the linear map

$$\begin{array}{rccc} u: & H & \to & H \\ & \xi & \mapsto & (L \circ |\xi\rangle\langle\nu|)(\nu). \end{array}$$

To see that u is bounded, we compute using Theorem 1.5.2 that if $\xi \in H$ then

$$\|u\xi\| \leq \|L\| \||\xi\rangle\langle\nu|\| \|\nu\| = \|L\| \|\xi\| \|\nu\|^2 = \|L\| \|\xi\| < \infty.$$

If $v \in B(H)$ then let $L_v : B(H) \to B(H)$ denote the left multiplication map (composition by v) and $R_v : B(H) \to B(H)$ denote the right multiplication map (precomposition by v). If $\alpha, \beta, \gamma \in H$ then

$$\begin{aligned} (L_u \circ |\alpha\rangle\langle\beta|)(\gamma) &= |u\alpha\rangle\langle\beta|)(\gamma) \\ &= \langle\beta, \gamma\rangle u\alpha \\ &= \langle\beta, \gamma\rangle (L \circ |\alpha\rangle\langle\nu|)(\nu) \\ &= \langle\beta, \gamma\rangle L\alpha = (L \circ |\alpha\rangle\langle\beta|)(\gamma). \end{aligned}$$

Therefore, $L_u = L$ on the ideal $B_0(H)$. Consequently

$$(\varphi(u) - (L, R))B_0(H) = 0 \qquad \text{and} \qquad \varphi(u) = (L, R).$$

So φ is surjective and hence, a *-isomorphism from $B(H)$ to $M(B_0(H))$.

4.3 Linear operators affiliated with a C*-algebra

In this section, we introduce the central definition of the paper [WN92] — the notion of a linear operator being affiliated with a C*-algebra. First, we need to understand the topological structure of $M(A)$ where A is a C*-algebra.

Definition 4.3.1. Let A be a C*-algebra. Let $\{a_\alpha\}_{\alpha\in I}$ be a net in $M(A)$. We say that $\{a_\alpha\}_{\alpha\in I}$ converges **almost uniformly** to 0 if for $x \in A$,

$$\lim_\alpha \|a_\alpha x\| = 0 \qquad \text{and} \qquad \lim_\alpha \|a_\alpha^\sharp x\| = 0.$$

The resulting topology on $M(A)$ is called the topology of **almost uniform convergence**.

Example 4.3.1. Let A be a C*-algebra and $\{u_\lambda\}_{\lambda\in\Lambda}$ be an approximate unit for A. Let $\iota : A \hookrightarrow M(A)$ denote the inclusion map. The net $\{\iota(u_\lambda)\}$ in $M(A)$ converges almost uniformly to the identity element of $M(A)$ which is the identity map $id_A : A \to A$. This is because if $x \in A$ then

$$\lim_\lambda \|\iota(u_\lambda)x - id_A(x)\| = \lim_\lambda \|u_\lambda x - x\| = 0$$

and

$$\lim_\lambda \|\iota(u_\lambda)^\sharp x - id_A^\sharp(x)\| = \lim_\lambda \|u_\lambda^* x - x\| = \lim_\lambda \|u_\lambda x - x\| = 0.$$

Now we will use Example 4.3.1 to prove the following important theorem relating A and $M(A)$.

Theorem 4.3.1. *Let A be a C*-algebra. Viewing A as an essential ideal of $M(A)$, we have the following equality:*

$$A = \left\{ x \in M(A) \;\middle|\; \begin{array}{c} \textit{If } \{a_\alpha\} \textit{ is a net in } M(A) \textit{ converging} \\ \textit{almost uniformly to } 0 \textit{ then } \lim_\alpha \|a_\alpha x\| = 0 \end{array} \right\}.$$

Moreover, A is dense in $M(A)$ (with respect to the topology of almost uniform convergence).

Proof. Assume that A is a C*-algebra. By definition of almost uniform convergence, we have the inclusion

$$A \subseteq \left\{ x \in M(A) \;\middle|\; \begin{array}{c} \text{If } \{a_\alpha\} \text{ is a net in } M(A) \text{ converging} \\ \text{almost uniformly to } 0 \text{ then } \lim_\alpha \|a_\alpha x\| = 0 \end{array} \right\}.$$

To prove the reverse inequality, assume that $x \in M(A)$ such that if $\{a_\alpha\}$ is a net in $M(A)$ which converges almost uniformly to 0 then $\lim_\alpha \|a_\alpha x\| = 0$. Let $\{u_\lambda\}_{\lambda\in\Lambda}$ be an approximate unit for A. If $\lambda \in \Lambda$ then define

$$a_\lambda = id_A - u_\lambda.$$

Then $\{a_\lambda\}_{\lambda\in\Lambda}$ is a net in $M(A)$ which converges almost uniformly to 0 by Example 4.3.1. So

$$\lim_\lambda \|a_\lambda x\| = \lim_\lambda \|(id_A - u_\lambda)x\| = \lim_\lambda \|x - u_\lambda x\| = 0.$$

Since A is an ideal in $M(A)$ then $u_\lambda x \in A$ and consequently, $x \in A$ because A is complete. Therefore

$$A = \left\{ x \in M(A) \;\middle|\; \begin{array}{c} \text{If } \{a_\alpha\} \text{ is a net in } M(A) \text{ converging} \\ \text{almost uniformly to 0 then } \lim_\alpha \|a_\alpha x\| = 0 \end{array} \right\}.$$

To see that A is dense in $M(A)$, assume that $y \in M(A)$. If $\lambda \in \Lambda$ then $u_\lambda y \in A$ and the net $\{u_\lambda y\}_{\lambda \in \Lambda}$ in A converges almost uniformly to $y \in M(A)$ by Example 4.3.1. So $\overline{A} = M(A)$ with respect to the topology of almost uniform convergence. □

Theorem 4.3.2. *Let A be a C*-algebra and $T \in B(A)$. If the adjoint operator $T^\sharp : D(T^\sharp) \to A$ is densely defined then $T \in M(A)$.*

Proof. Assume that A is a C*-algebra. Assume that $T \in B(A)$ such that its adjoint operator $T^\sharp : D(T^\sharp) \to A$ is densely defined. To see that $T \in M(A)$, we must show that $T^\sharp \in B(A)$. By Theorem 4.1.2, $T^\sharp$ is a bounded linear operator.

Since $D(T^\sharp)$ is dense in A then the continuous linear operator $T^\sharp$ can be extended to all of A. So $T^\sharp \in B(A)$ and $T \in M(A)$. □

Here is the important definition of affiliation.

Definition 4.3.2. Let A be a C*-algebra and $T : D(T) \to A$ be a densely defined linear operator on A. We say that T is **affiliated with** A and write $T \eta A$ if and only if there exists $z_T \in M(A)$ such that $\|z_T\| \leq 1$ and

$$G(T) = \left\{ \left((id_A - z_T^* z_T)^{\frac{1}{2}} a, z_T a \right) \mid a \in A \right\}$$

where we recall that $G(T) = \{(x, Tx) \mid x \in D(T)\}$ is the graph of T.

The element $z_T \in M(A)$ is called the **z-transform** of T.

Example 4.3.2. Let H be a Hilbert space and $A = B(H)$. Recall from Theorem 2.9.4 that $M(B(H)) = B(H)$ because $B(H)$ is a unital C*-algebra. We obviously have the inclusion

$$\begin{array}{ccc} H & \hookrightarrow & B(H) \\ \xi & \mapsto & (\nu \mapsto \xi\nu). \end{array}$$

If $T : D(T) \to H$ is a closed densely defined operator on H then $T \eta B(H)$. In fact, its z-transform z_T is given explicitly by

$$z_T = T(I + T^*T)^{-\frac{1}{2}} \in B(H) = M(B(H))$$

where I is the identity map on H and T^* is the adjoint of T as defined in [Sol18, Section 8.2]. See [Sol18, Chapter 9] for the full details on how the z-transform z_T is constructed.

If A is a C*-algebra and $T : D(T) \to A$ is a densely defined linear operator on A such that $T\eta A$ then T must be closed. To see why this is the case, we know that there exists $z_T \in M(A)$ such that the graph

$$G(T) = \left\{ \left((id_A - z_T^* z_T)^{\frac{1}{2}} a, z_T a \right) \mid a \in A \right\}.$$

The operators $(id_A - z_T^* z_T)^{\frac{1}{2}}$ and z_T are both bounded and thus continuous linear operators on A. In particular, they commute with limits. Since A is a closed subset of itself then $G(T)$ must be a closed subspace of $A \times A$ and T is a closed operator.

So far in these notes, we have said that the category of C*-algebras consists of C*-algebras as the objects and *-homomorphisms as the morphisms. In [Wor91] and [WN92], a more refined definition of the category of C*-algebras is used.

The objects in the category of C*-algebras are still C*-algebras. However, if A, B are C*-algebras then the set of morphisms, which we denote by $Mor(A, B)$, is defined by

$$Mor(A, B) = \left\{ \phi : A \to M(B) \;\middle|\; \begin{array}{c} \phi \text{ is a *-homomorphism,} \\ \phi(A)B \text{ is dense in } B \end{array} \right\}. \tag{4.2}$$

With the definition of morphisms in equation (4.2), one might wonder how composition is defined in the category of C*-algebras. The idea here is that if $\phi \in Mor(A, B)$ then ϕ admits a unique extension to a *-homomorphism from $M(A)$ to $M(B)$.

Theorem 4.3.3. *Let A and B be C*-algebras and $\phi \in Mor(A, B)$. Then, there exists a unique extension of ϕ to a *-homomorphism from $M(A)$ to $M(B)$.*

Proof. Assume that A and B are C*-algebras and that $\phi \in Mor(A, B)$. Then $\phi(A)B$ is a dense subset of B and its linear span $span_{\mathbb{C}}\, \phi(A)B$ is also dense in B. If $T \in M(A)$ then we define $\tilde{\phi}(T)$ by the map

$$\begin{array}{cccc} \tilde{\phi}(T) : & \phi(A)B & \to & B \\ & \phi(a)b = L_{\phi(a)(b)} & \mapsto & \phi(Ta)b = L_{\phi(Ta)b}. \end{array}$$

where $L_{\phi(a)(b)} \in B \subsetneq M(B)$ denotes left multiplication by $\phi(a)(b) \in B$ (see Theorem 4.2.2). The map $\tilde{\phi}(T)$ is then extended to all of $span_{\mathbb{C}}\, \phi(A)B$ by linearity.

To show: (a) $\tilde{\phi}|_A = \phi$.

(b) $\tilde{\phi}(T)$ is bounded.

(c) $\tilde{\phi}(T)$ is well-defined.

(a) If $a \in A$ then let $L_a \in M(A)$ denote left multiplication by a. If $a, c \in A$ and $b \in B$ then

$$\tilde{\phi}(L_c)(\phi(a)b) = \phi(L_c(a))b = \phi(ca)b = \phi(c)\phi(a)b = L_{\phi(c)}(\phi(a)b).$$

Since $\phi(A)B$ is dense in B then $\tilde{\phi}(L_c) = L_{\phi(c)}$. So $\tilde{\phi}|_A = \phi$.

(b) To see that $\tilde{\phi}(T)$ is bounded, let $\{u_\lambda\}_{\lambda\in\Lambda}$ be an approximate unit for A. If $a_1, \ldots, a_n \in A$ and $b_1, \ldots, b_n \in B$ then

$$\begin{aligned}
\|\tilde{\phi}(T)(\sum_{i=1}^n \phi(a_i)b_i)\| &= \|\sum_{i=1}^n \phi(Ta_i)b_i\| = \lim_\lambda \|\sum_{i=1}^n \phi(Tu_\lambda a_i)b_i\| \\
&= \lim_\lambda \|\phi(Tu_\lambda)\sum_{i=1}^n \phi(a_i)b_i\| \\
&\leq \|\phi\|\|T\|\|\sum_{i=1}^n \phi(a_i)b_i\| \leq \|T\|\|\sum_{i=1}^n \phi(a_i)b_i\|.
\end{aligned}$$

Therefore $\tilde{\phi}(T)$ is a bounded linear operator on $span_{\mathbb{C}}\, \phi(A)B$.

(c) Suppose that $a_1, a_2 \in A$ and $b_1, b_2 \in B$ satisfy $\phi(a_1)b_1 = \phi(a_2)b_2$. Arguing in a similar manner to Theorem 3.4.6, we have

$$\begin{aligned}
\tilde{\phi}(T)(\phi(a_1)b_1) &= \phi(Ta_1)b_1 \\
&= \lim_\lambda \phi(Tu_\lambda a_1)b_1 \\
&= \lim_\lambda \phi(Tu_\lambda)\phi(a_1)b_1 \\
&= \lim_\lambda \phi(Tu_\lambda)\phi(a_2)b_2 \\
&= \lim_\lambda \phi(Tu_\lambda a_2)b_2 = \tilde{\phi}(T)(\phi(a_2)b_2).
\end{aligned}$$

Hence $\tilde{\phi}(T)$ is well-defined.

Now since $span_{\mathbb{C}}\ \phi(A)B$ is dense in B then $\tilde{\phi}(T)$ extends to a bounded linear operator on B.

To show: (d) $\tilde{\phi}(T) \in M(B)$.

(d) We claim that $\tilde{\phi}(T)^\sharp = \tilde{\phi}(T^\sharp)$. If $a \in A$ and $b \in B$ then

$$\begin{aligned}
\tilde{\phi}(T^\sharp)(\phi(a)b) &= \phi(T^\sharp a)b = \lim_\lambda \phi(u_\lambda T^\sharp a)b \\
&= \lim_\lambda \phi((Tu_\lambda)^* a)b \\
&= \lim_\lambda \phi((Tu_\lambda)^*)\phi(a)b \\
&= \lim_\lambda \phi(Tu_\lambda)^\sharp \phi(a)b \\
&= \tilde{\phi}(T)^\sharp(\phi(a)b).
\end{aligned}$$

By density, we conclude that $\tilde{\phi}(T)^\sharp = \tilde{\phi}(T^\sharp) \in B(B)$. Therefore $\tilde{\phi}(T) \in M(B)$.

From here, it is straightforward to verify that $\tilde{\phi} : M(A) \to M(B)$ is a *-homomorphism (in particular, it respects multiplication and involution). Moreover, $\tilde{\phi}$ is unique by density, as $\phi(A)B$ is dense in B. $\square$

Now let A, B, C be C*-algebras, $\phi \in Mor(A, B)$ and $\psi \in Mor(B, C)$. By using Theorem 4.3.3, we can define composition in the category of C*-algebras by

$$\begin{array}{rccc}
\circ : & Mor(B,C) \times Mor(A,B) & \to & Mor(A,C) \\
& (\psi, \phi) & \mapsto & \big(a \mapsto \tilde{\psi}(\phi(a))\big)
\end{array}$$

Composition of morphisms in the category of C*-algebras is well-defined because if $\phi \in Mor(A, B)$ and $\psi \in Mor(B, C)$ then $\phi(A)B$ is dense in B and $\psi(B)C$ is dense in C. Consequently

$$(\psi \circ \phi)(A)C = \overline{\tilde{\psi}(\phi(A))\psi(B)C} = \overline{\tilde{\psi}(\phi(A)B)C} = \tilde{\psi}(B)C.$$

Taking closures, we find that $(\psi \circ \phi)(A)C$ is dense in C as required.

4.4 The z-transform is determined uniquely

Before we prove some more properties about affiliated elements of a C*-algebra and their z-transforms, there is one sticking point to address. If A is a C*-algebra and $T : D(T) \to A$ is densely defined linear operator such that $T \eta A$ then is the z-transform $z_T \in M(A)$ unique? In this section, we will show that z_T is determined uniquely by T by considering the graph $G(T)$. The approach we have here follows [Wor91, Chapter 2].

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