An introduction to operator theory

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Contents

0.1 Purpose

Continuing on from an introduction to functional analysis, such as the one in [Bre10], the purpose of these notes is to delve deeper into the theory of operators on a Hilbert space, with [Sol18] serving as the main reference.

In [Sol18], the inner product on a Hilbert space is antilinear in the first argument and linear in the second argument. We adopt the opposite convention in these notes — an inner product on a Hilbert space is linear in the first argument and antilinear in the second argument. Note that [Bre10] uses the same convention for the inner product as these notes.

Chapter 1

The spectrum of a bounded operator

1.1 C*-algebras

Fundamental to the theory of bounded operators over a Hilbert space H is the concept of a **C*-algebra**. The definition of a C*-algebra requires a few preliminary definitions.

Definition 1.1.1. Let A be an associative algebra over $\mathbb{C}$ (or $\mathbb{R}$). We say that A is a **Banach algebra** if A is also a Banach space. That is, A is equipped with a norm $\|-\|$ which makes A complete — every Cauchy sequence in A converges with respect to the norm $\|-\|$. Additionally, if $x, y \in A$, the norm must satisfy

$$\|xy\| \leq \|x\|\|y\|. \tag{1.1}$$

Equation (1.1) ensures that the norm respects the algebraic structure of A, by rendering multiplication continuous. Let $m_x : A \to A$ be the operator which sends $y \in A$ to xy. If $\epsilon \in \mathbb{R}_{>0}$, choose $\delta = \epsilon/\|x\|$ and suppose that $\|y_1 - y_2\| < \delta$. Then,

$$\begin{aligned} \|m_x(y_1) - m_x(y_2)\| &= \|xy_1 - xy_2\| \\ &\leq \|x\|\|y_1 - y_2\| \\ &< \epsilon. \end{aligned}$$

Hence, m_x is a continuous operator on A for all $x \in A$.

Definition 1.1.2. Let A be a Banach algebra over $\mathbb{C}$. We say that A is a **Banach *-algebra** if A is equipped with a map $^* : A \to A$ which satisfies for all $x, y \in A$ and $\lambda \in \mathbb{C}$,

1. $(x^*)^* = x$ (Involution)
2. $(x + y)^* = x^* + y^*$
3. $(\lambda x)^* = \overline{\lambda} x^*$
4. $(xy)^* = y^* x^*$ (Anti-multiplicative)

The middle two properties of the map $^* : A \to A$ means that * is anti-linear (or conjugate linear).

Definition 1.1.3. Let A be a Banach *-algebra over $\mathbb{C}$. We say that A is a **C*-algebra** if for all $x \in A$, $\|x^* x\| = \|x\|^2$.

The defining property of a C*-algebra is that the involution $^* : A \to A$ is isometric (distance preserving). The first theorem we will state gives us the primary example of a C*-algebra.

Theorem 1.1.1. *Let H be a Hilbert space over $\mathbb{C}$ and $B(H)$ denote the Banach space of bounded linear operators $\phi : H \to H$. Then, $B(H)$ is a C*-algebra.*

Proof. Assume that H is a Hilbert space over $\mathbb{C}$ and that $B(H)$ is the Banach space of bounded linear operators from H to H. Let $\|-\|_H$ denote the norm on H and $\langle - . - \rangle$ denote the inner product on H. Define the map $^* : H \to B(H)$ by

$$\begin{array}{cccc} ^* : & B(H) & \to & B(H) \\ & h & \to & h^*. \end{array}$$

where h^* is the adjoint of h, which satisfies for all $\xi, \eta \in H$,

$$\langle h^*(\xi), \eta \rangle = \langle \xi, h(\eta) \rangle \tag{1.2}$$

To show: (a) $B(H)$ is a Banach algebra.

(b) $B(H)$ is a Banach *-algebra.

(c) $B(H)$ is a C*-algebra.

(a) Observe that $B(H)$ is an associative algebra over $\mathbb{C}$, where scalar multiplication and addition are defined as usual and multiplication is given by composition of linear operators, which we will denote by $\circ$. We also know that $B(H)$ is a Banach space when equipped with the operator norm

$$\|h\| = \sup_{\|x\|_H=1} \|h(x)\|_H$$

To show: (aa) If $f, g \in B(H)$, then $\|f \circ g\| \leq \|f\|\|g\|$.

(aa) Assume that $f, g \in B(H)$. Then, from the definition of the operator norm, we have

$$\begin{aligned} \|f \circ g\| &= \sup_{\|x\|_H=1} \|f(g(x))\|_H \\ &\leq \sup_{\|x\|_H=1} \|f\|\|g(x)\|_H \\ &= \|f\|\|g\|. \end{aligned}$$

Therefore, $B(H)$ is a Banach algebra.

(b) To show: (ba) The map * is an involution.

(bb) The map * is anti-linear.

(bc) The map * is anti-multiplicative.

(ba) Assume that $h \in B(H)$. By equation (1.2), $h^{**} \in B(H)$ must satisfy for all $\xi, \eta \in H$,

$$\langle h^{**}(\xi), \eta\rangle = \langle \xi, h^*(\eta)\rangle = \langle h(\xi), \eta\rangle.$$

Therefore, $h^{**}(\xi) = h(\xi)$ for all $\xi \in H$. So, $h^{**} = h$, revealing that $^* : B(H) \to B(H)$ is an involution.

(bb) Assume that $g, h \in B(H)$. Then, for all $\xi, \eta \in H$, we have

$$\begin{aligned} \langle (g+h)^*(\xi), \eta\rangle &= \langle \xi, (g+h)(\eta)\rangle \\ &= \langle \xi, g(\eta)\rangle + \langle \xi, h(\eta)\rangle \\ &= \langle (g^* + h^*)(\xi), \eta\rangle. \end{aligned}$$

So, $(g+h)^* = g^* + h^*$. Now assume that $\lambda \in \mathbb{C}$. Then,

$$\begin{aligned}\langle(\lambda h)^*(\xi),\eta\rangle &= \langle\xi,(\lambda h)(\eta)\rangle \\ &= \overline{\lambda}\langle\xi,h(\eta)\rangle \\ &= \overline{\lambda}\langle h^*(\xi),\eta\rangle \\ &= \langle(\overline{\lambda}h^*)(\xi),\eta\rangle.\end{aligned}$$

So, $(\lambda h)^* = \overline{\lambda}h^*$. This demonstrates that * is anti-linear.

(bc) We compute directly that for all $\xi, \eta \in H$,

$$\begin{aligned}\langle(g\circ h)^*(\xi),\eta\rangle &= \langle\xi,g(h(\eta))\rangle \\ &= \langle g^*(\xi),h(\eta)\rangle \\ &= \langle(h^*\circ g^*)(\xi),\eta\rangle.\end{aligned}$$

Therefore, $(g\circ h)^* = h^*\circ g^*$. Hence, the map * is anti-linear. So, $B(H)$ is a Banach *-algebra.

(c) To show: (ca) For all $h \in B(H)$, $\|h^*\circ h\| = \|h\|^2$.

(ca) Assume that $h \in B(H)$ and that $\|h\| > 0$ (the statement holds when $h = 0$). We have already shown that $\|h^*\circ h\| \leq \|h^*\|\|h\|$.

To show: (caa) $\|h^*\| = \|h\|$.

(caa) Observe that

$$\begin{aligned}\|h^*\circ h\| &= \sup_{\|\xi\|_H=1}\|h^*(h(\xi))\|_H \\ &= \sup_{\|\xi\|_H=1}\sup_{\|\eta\|_H=1}|\langle h^*(h(\xi)),\eta\rangle| \\ &\geq \sup_{\|\xi\|_H=1}|\langle h^*(h(\xi)),\xi\rangle| \\ &= \sup_{\|\xi\|_H=1}\|h(\xi)\|_H^2 \\ &= \|h\|^2.\end{aligned}$$

Therefore, $\|h\|^2 \leq \|h^*\|\|h\|$ and $\|h\| \leq \|h^*\|$. To establish the reverse inequality, we can interchange the roles of h and h^* in the above calculation so that

$$\begin{aligned}
\|h \circ h^*\| &= \sup_{\|\xi\|_H=1} \|h(h^*(\xi))\|_H \\
&= \sup_{\|\xi\|_H=1} \sup_{\|\eta\|_H=1} |\langle h(h^*(\xi)), \eta\rangle| \\
&\geq \sup_{\|\xi\|_H=1} |\langle h(h^*(\xi)), \xi\rangle| \\
&= \sup_{\|\xi\|_H=1} \|h^*(\xi)\|_H^2 \\
&= \|h^*\|^2.
\end{aligned}$$

So, $\|h^*\|^2 \leq \|h^*\|\|h\|$ and $\|h^*\| \leq \|h\|$. In tandem with $\|h\| \leq \|h^*\|$, we deduce that $\|h^*\| = \|h\|$.

(ca) Recall from part (caa) that $\|h\|^2 \leq \|h^* \circ h\|$ and from the beginning of part (ca) that $\|h^* \circ h\| \leq \|h^*\|\|h\|$. Since $\|h^*\| = \|h\|$, $\|h^* \circ h\| \leq \|h\|^2$ and consequently, $\|h\|^2 = \|h^* \circ h\|$ as required. □

Example 1.1.1. Here, we will give another example of a C*-algebra. Let X be a compact, Hausdorff space and $Cts(X, \mathbb{C})$ denote the space of continuous functions from X to $\mathbb{C}$. Then, $Cts(X, \mathbb{C})$ is a C*-algebra with scalar multiplication, addition and multiplication defined pointwise on $\mathbb{C}$. The norm on $Cts(X, \mathbb{C})$ is

$$\|f\| = \sup_{x \in X} |f(x)|.$$

and the map $^* : Cts(X, \mathbb{C}) \to Cts(X, \mathbb{C})$ is defined by the equation

$$f^*(x) = \overline{f(x)}.$$

1.2 Properties of the spectrum

Definition 1.2.1. Let H be a Hilbert space over $\mathbb{C}$ and $h \in B(H)$. The **resolvent set** of h is the set

$$\rho(h) = \{\lambda \in \mathbb{C} \mid \lambda I - h \text{ is injective and surjective}\}.$$

Here, $I : H \to H$ is the identity operator. The **spectrum** of h is the set $\sigma(h) = \mathbb{C} \backslash \rho(h)$.

Thus, if $h \in B(H)$ and $\lambda \in \rho(h)$, then the operator $(\lambda I - h)^{-1} : H \to H$ must exist. Since $\lambda I - h$ is bounded and surjective, the open mapping theorem tells us that $\lambda I - h$ is open. So, $\lambda I - h$ must be open and bijective. This means that it is a homeomorphism and therefore, $(\lambda I - h)^{-1} \in B(H)$.

We are interested in proving some topological properties of the spectrum $\sigma(h)$.

Lemma 1.2.1. *Let H be a Hilbert space over $\mathbb{C}$ and $g, h \in B(H)$. Then,*

$$\sigma(g \circ h) \cup \{0\} = \sigma(h \circ g) \cup \{0\}.$$

Proof. Assume that H is a Hilbert space over $\mathbb{C}$ and $g, h \in B(H)$. Assume that $\lambda \in \rho(h \circ g) - \{0\}$. We claim that

$$\frac{1}{\lambda}(I + g(\lambda I - hg)^{-1}h) \in B(H)$$

is the inverse operator to $\lambda I - gh$. Here, we abuse notation by writing $g \circ h$ as gh. We now compute directly that

$$\begin{aligned}
(\lambda I - gh)\Big(\frac{1}{\lambda}(I + g(\lambda I - hg)^{-1}h)\Big) &= \frac{1}{\lambda}((\lambda I - gh) + (\lambda I - gh)g(\lambda I - hg)^{-1}h) \\
&= \frac{1}{\lambda}((\lambda I - gh) + (\lambda g - ghg)(\lambda I - hg)^{-1}h) \\
&= \frac{1}{\lambda}((\lambda I - gh) + g(\lambda I - hg)(\lambda I - hg)^{-1}h) \\
&= I
\end{aligned}$$

and

$$\begin{aligned}
\Big(\frac{1}{\lambda}(I + g(\lambda I - hg)^{-1}h)\Big)(\lambda I - gh) &= \frac{1}{\lambda}((\lambda I - gh) + g(\lambda I - hg)^{-1}h(\lambda I - gh)) \\
&= \frac{1}{\lambda}((\lambda I - gh) + g(\lambda I - hg)^{-1}(\lambda h - hgh)) \\
&= \frac{1}{\lambda}((\lambda I - gh) + g(\lambda I - hg)^{-1}(\lambda I - gh)h) \\
&= I.
\end{aligned}$$

Since $\lambda I - gh$ has an inverse as demonstrated by the computations above, $\lambda \in \rho(gh) - \{0\}$. So, $\rho(hg) - \{0\} \subseteq \rho(gh) - \{0\}$. By reversing the roles of h and g in the above argument, we also conclude that $\rho(gh) - \{0\} \subseteq \rho(hg) - \{0\}$. So, $\rho(gh) - \{0\} = \rho(hg) - \{0\}$. Taking the complement of both sides then yields the desired statement. ☐

The next statement gives us important topological properties about the resolvent set and the spectrum.

Theorem 1.2.2. *Let H be a Hilbert space over $\mathbb{C}$ and $h \in B(H)$. Then,*

$$\{\lambda \in \mathbb{C} \mid |\lambda| > \|h\|\} \subseteq \rho(h).$$

Proof. Assume that H is a Hilbert space over $\mathbb{C}$ and $h \in B(H)$. Assume that $\lambda \in \mathbb{C}$ such that $|\lambda| > \|h\|$. We claim that the sum

$$\frac{1}{\lambda}\sum_{n=0}^{\infty} \lambda^{-n} h^n \tag{1.3}$$

converges in $B(H)$ and that it is equal to $(\lambda I - h)^{-1}$. To see that the sum in equation (1.3) converges, it suffices to show that its norm is finite. But,

$$\begin{aligned}
\|\frac{1}{\lambda}\sum_{n=0}^{\infty} \lambda^{-n} h^n\| &= \frac{1}{|\lambda|}\|\sum_{n=0}^{\infty} \lambda^{-n} h^n\| \\
&\leq \frac{1}{|\lambda|}\sum_{n=0}^{\infty} |\lambda|^{-n}\|h\|^n \\
&= \frac{C}{|\lambda|}
\end{aligned}$$

where $C = \sum_{n=0}^{\infty} |\lambda|^{-n}\|h\|^n \in \mathbb{R}_{>0}$. The sum $\sum_{n=0}^{\infty} |\lambda|^{-n}\|h\|^n$ is a convergent geometric series because $|\lambda| > \|h\|$.

Hence, the sum in equation (1.3) converges in $B(H)$ and is consequently, a well-defined operator in $B(H)$. For all $m \in \mathbb{Z}_{>0}$, let

$$S_m = \frac{1}{\lambda}\sum_{n=0}^{m} \lambda^{-n} h^n.$$

Then,

$$\begin{aligned}
S_m(\lambda I - h) &= \frac{1}{\lambda}(I + \lambda^{-1}h + \lambda^{-2}h^2 + \cdots + \lambda^{-m}h^m)(\lambda I - h) \\
&= \frac{1}{\lambda}((\lambda I - h) + (h - \lambda^{-1}h^2) + \cdots + (\lambda^{-m+1}h^m - \lambda^{-m}h^{m+1})) \\
&= \frac{1}{\lambda}(\lambda I - \lambda^{-m}h^{m+1}) \\
&= I - \lambda^{-m-1}h^{m+1}.
\end{aligned}$$

A similar calculation gives $(\lambda I - h)S_m = I - \lambda^{-m-1}h^{m+1}$. Now take the limit as $m \to \infty$. Observe that in $B(H)$,

$$\lim_{m\to\infty} \lambda^{-m-1}h^{m+1} = 0$$

because

$$\lim_{m\to\infty} (\frac{\|h\|}{|\lambda|})^{m+1} = 0.$$

So, in the limit as $m \to \infty$,

$$(\lambda I - h)\Big(\frac{1}{\lambda}\sum_{n=0}^{\infty} \lambda^{-n}h^n\Big) = \Big(\frac{1}{\lambda}\sum_{n=0}^{\infty} \lambda^{-n}h^n\Big)(\lambda I - h) = I.$$

Therefore,

$$\Big(\frac{1}{\lambda}\sum_{n=0}^{\infty} \lambda^{-n}h^n\Big) = (\lambda I - h)^{-1}$$

and subsequently, $\lambda \in \rho(h)$ and

$$\{\lambda \in \mathbb{C} \mid |\lambda| > \|h\|\} \subseteq \rho(h).$$

$\square$

Rewriting the conclusion of Theorem 1.2.2 in terms of the spectrum, we find that

$$\sigma(h) \subseteq \{\lambda \in \mathbb{C} \mid |\lambda| \leq \|h\|\} \tag{1.4}$$

So, for all $h \in B(H)$, $\sigma(h)$ is a bounded set. We now want to show that $\sigma(h)$ is a closed set.

Theorem 1.2.3. *Let H be a Hilbert space over $\mathbb{C}$, $h \in B(H)$ and $\lambda_0 \in \rho(h)$. Suppose that $\lambda \in \mathbb{C}$ such that*

$$|\lambda - \lambda_0| < \frac{1}{\|(\lambda_0 I - h)^{-1}\|}.$$

Then,

$$(\lambda I - h)^{-1} = \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n (\lambda_0 I - x)^{-n-1}.$$

and consequently, $\lambda \in \rho(h)$.

Proof. Assume that H is a Hilbert space over $\mathbb{C}$, $h \in B(H)$ and $\lambda_0 \in \rho(h)$. Assume that $\lambda \in \mathbb{C}$ such that

$$|\lambda - \lambda_0| < \frac{1}{\|(\lambda_0 I - h)^{-1}\|}.$$

To see that the sum

$$\sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n (\lambda_0 I - x)^{-n-1}$$

converges in $B(H)$, we must show that its norm is finite. We have

$$\begin{aligned}
\|\sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n (\lambda_0 I - h)^{-n-1}\| &\leq \sum_{n=0}^{\infty} |\lambda_0 - \lambda|^n \|(\lambda_0 I - h)^{-1}\|^{n+1} \\
&= \frac{1}{|\lambda_0 - \lambda|} \sum_{n=0}^{\infty} |\lambda_0 - \lambda|^{n+1} \|(\lambda_0 I - h)^{-1}\|^{n+1} \\
&= \frac{D}{|\lambda_0 - \lambda|}
\end{aligned}$$

for some $D \in \mathbb{R}_{>0}$. Hence, the sum $\sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n (\lambda_0 I - h)^{-n-1}$ converges in $B(H)$ and is a well-defined operator in $B(H)$. For all $m \in \mathbb{Z}_{>0}$, let

$$T_m = \sum_{n=0}^{m} (\lambda_0 - \lambda)^n (\lambda_0 I - h)^{-n-1}.$$

First, observe that

$$\lambda I - h = (\lambda_0 I - h)(I + (\lambda - \lambda_0)(\lambda_0 I - h)^{-1})$$

Then, a direct calculation yields

$$\begin{aligned}
T_m(\lambda I - h) &= \Big(\sum_{n=0}^{m} (\lambda_0 - \lambda)^n (\lambda_0 I - h)^{-n-1}\Big)((\lambda_0 I - h)(I + (\lambda - \lambda_0)(\lambda_0 I - h)^{-1})) \\
&= \Big(\sum_{n=0}^{m} (\lambda_0 - \lambda)^n (\lambda_0 I - h)^{-n}\Big)(I + (\lambda - \lambda_0)(\lambda_0 I - h)^{-1}) \\
&= \sum_{n=0}^{m} (\lambda_0 - \lambda)^n (\lambda_0 I - h)^{-n} - \sum_{n=0}^{m} (\lambda_0 - \lambda)^{n+1} (\lambda_0 I - h)^{-n-1} \\
&= I - (\lambda_0 - \lambda)^{m+1} (\lambda_0 I - h)^{-m-1}.
\end{aligned}$$

Note that by a similar computation,

$$(\lambda I - h)T_m = I - (\lambda_0 - \lambda)^{m+1}(\lambda_0 I - h)^{-m-1}$$

as well. Taking the limit as $m \to \infty$, we find that $(\lambda_0 - \lambda)^{m+1}(\lambda_0 I - h)^{-m-1} \to 0$ in $B(H)$ because its norm tends to 0 as $m \to \infty$. Hence,

$$(\lambda I - h)\Big(\sum_{n=0}^{\infty}(\lambda_0 - \lambda)^n(\lambda_0 I - h)^{-n-1}\Big) = \Big(\sum_{n=0}^{\infty}(\lambda_0 - \lambda)^n(\lambda_0 I - h)^{-n-1}\Big)(\lambda I - h) = I$$

and

$$\Big(\sum_{n=0}^{\infty}(\lambda_0 - \lambda)^n(\lambda_0 I - h)^{-n-1}\Big) = (\lambda I - h)^{-1}$$

which demonstrates that $\lambda \in \rho(h)$. $\square$

Theorem 1.2.3 tells us that if $h \in B(H)$, then $\rho(h)$ is an open subset of $\mathbb{C}$. In tandem with equation (1.4), the spectrum $\sigma(h)$ is a closed and bounded subset of $\mathbb{C}$ and is thus, compact.

Next, we will show that the spectrum of $h \in B(H)$ is always non-empty.

Theorem 1.2.4. *Let H be a Hilbert space over $\mathbb{C}$ and $h \in B(H)$. Then, $\sigma(h) \neq \emptyset$, the sequence $\{\|h^n\|^{1/n}\}_{n \in \mathbb{Z}_{>0}}$ converges in $\mathbb{R}$ and*

$$\lim_{n \to \infty} \|h^n\|^{\frac{1}{n}} = \sup_{\lambda \in \sigma(h)} |\lambda|.$$

Proof. Assume that H is a Hilbert space over $\mathbb{C}$ and $h \in B(H)$. Define

$$\alpha(h) = \inf_{n \in \mathbb{Z}_{>0}} \|h^n\|^{\frac{1}{n}}.$$

We will show that the sequence $\{\|h^n\|^{1/n}\}_{n \in \mathbb{Z}_{>0}}$ converges to $\alpha(h)$. Assume that $\epsilon \in \mathbb{R}_{>0}$. From the definition of infimum, there exists an index $n_\epsilon \in \mathbb{Z}_{>0}$ such that

$$\|h^{n_\epsilon}\|^{\frac{1}{n_\epsilon}} \leq \alpha(h) + \epsilon.$$

Take any $n \in \mathbb{Z}_{>0}$ and use the Euclidean algorithm to write $n = qn_\epsilon + r$, where $q \in \mathbb{Z}_{>0}$ and $r \in \{0, 1, \ldots, n_\epsilon - 1\}$. Then,

$$\begin{aligned}\|h^n\| &= \|h^{qn_\epsilon + r}\| \\ &\leq \|h^{n_\epsilon}\|^q \|h\|^r \\ &\leq (\alpha(h)+\epsilon)^{qn_\epsilon}\|h\|^r \\ &= (\alpha(h)+\epsilon)^{n-r}\|h\|^r.\end{aligned}$$

Taking the n^{th} root of both sides, we obtain the inequality

$$\|h^n\|^{\frac{1}{n}} \leq (\alpha(h)+\epsilon)^{1-\frac{r}{n}}\|h\|^{\frac{r}{n}}.$$

Consequently,

$$\alpha(h) \leq \liminf_{n\to\infty}\|h^n\|^{\frac{1}{n}} \leq \limsup_{n\to\infty}\|h^n\|^{\frac{1}{n}} \leq \alpha(h)+\epsilon.$$

This demonstrates that the sequence $\{\|h^n\|^{\frac{1}{n}}\}$ converges in $\mathbb{R}$ to $\alpha(h)$. The next step is to show that

$$\alpha(h) = \sup_{\lambda\in\sigma(h)}|\lambda|. \tag{1.5}$$

To show: (a) $\alpha(h) \geq \sup_{\lambda\in\sigma(h)}|\lambda|$.

(b) $\alpha(h) \leq \sup_{\lambda\in\sigma(h)}|\lambda|$.

(a) Suppose for the sake of contradiction that $\alpha(h) < |\lambda|$ for some $\lambda \in \sigma(h)$. By the root test, the series

$$\sum_{n=0}^{\infty}\frac{\|h^n\|}{|\lambda|^n}$$

in $\mathbb{R}$ is convergent. Therefore, the sum

$$\sum_{n=0}^{\infty}\lambda^{-n}h^n$$

converges in $B(H)$ and is a well-defined element of $B(H)$. By using similar arguments to Theorem 1.2.2 and Theorem 1.2.3, we deduce that

$$\sum_{n=0}^{\infty}\lambda^{-n}h^n = (I-\frac{h}{\lambda})^{-1}.$$

So, $\lambda I - h$ is invertible and $\lambda \in \rho(h)$. But this contradicts the fact that $\lambda \in \sigma(h)$. Therefore, $\alpha(h) \geq \sup_{\lambda \in \sigma(h)} |\lambda|$.

(b) We will divide this into two cases. First, we note that if $x, y \in B(H)$, then

$$\begin{aligned}
\alpha(xy) &= \lim_{n\to\infty} \|(xy)^n\|^{\frac{1}{n}} \\
&= \lim_{n\to\infty} \|x^n y^n\|^{\frac{1}{n}} \\
&\leq \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} \|y^n\|^{\frac{1}{n}} \\
&= \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} \lim_{n\to\infty} \|y^n\|^{\frac{1}{n}} \\
&= \alpha(x)\alpha(y).
\end{aligned}$$

Case 1: $\alpha(h) = 0$.

If $\alpha(h) = 0$, then h is not invertible because otherwise,

$$1 = \alpha(I) = \alpha(hh^{-1}) \leq \alpha(h)\alpha(h^{-1}) = 0.$$

So, $0 \in \sigma(h)$ and since $\alpha(h) \geq \sup_{\lambda \in \sigma(h)} |\lambda|$ from part (a),

$$\alpha(h) = \sup_{\lambda \in \sigma(h)} |\lambda| = 0.$$

Case 2: $\alpha(h) > 0$.

Assume that $\alpha(h) > 0$ and $\alpha(h) > \sup_{\lambda \in \sigma(h)} |\lambda|$. Since the spectrum $\sigma(h)$ is a compact subset of $\mathbb{C}$, there exists $r \in (0, \alpha(h))$ such that

$$\sigma(h) \subseteq \{\lambda \in \mathbb{C} \mid |\lambda| \leq r\}.$$

Let $D = \{\lambda \in \mathbb{C} \mid |\lambda| > r\}$. Then, $D \subseteq \rho(h)$. Let φ be a continuous linear functional on $B(H)$ and define the map

$$\begin{array}{rccc}
\psi: & D & \to & \mathbb{C} \\
& \lambda & \mapsto & \varphi((\lambda I - h)^{-1})
\end{array}$$

The map ψ is holomorphic due to the series expansion in Theorem 1.2.2. In particular, when $|\lambda| > \alpha(h)$,

$$\varphi((\lambda I - h)^{-1}) = \sum_{n=0}^{\infty} \lambda^{-n-1} \varphi(h^n).$$

The series $\sum_{n=0}^{\infty} \lambda^{-n-1} h^n$ converges in $B(H)$ because

$$\|\sum_{n=0}^{\infty} \lambda^{-n-1} h^n\| \leq \sum_{n=0}^{\infty} |\lambda|^{-n-1} \|h^n\|$$

and by applying the root test on $\sum_{n=0}^{\infty} \lambda^{-n} \|h^n\|$, we find that

$$\lim_{n\to\infty} \frac{\|h^n\|^{\frac{1}{n}}}{|\lambda|} = \frac{\alpha(h)}{|\lambda|} < 1.$$

Moreover, $\sum_{n=0}^{\infty} \lambda^{-n-1} \varphi(h^n) \in \mathbb{C}$ vanishes as $\lambda \to \infty$. To see why this is the case, replace λ by $\lambda\mu$ and take the limit as $|\mu| \to \infty$. We obtain for $|\mu| > 1$

$$\begin{aligned}
|\sum_{n=0}^{\infty} (\lambda\mu)^{-n-1} \varphi(h^n)| &\leq |\sum_{n=0}^{\infty} (\lambda\mu)^{-n-1} \|\varphi\| \|h^n\|| \\
&\leq \frac{\|\varphi\|}{|\mu|} \sum_{n=0}^{\infty} \frac{\|h^n\|}{|\lambda|^{n+1}} |\mu|^{-n} \\
&\leq \frac{\|\varphi\|}{|\mu|} \sum_{n=0}^{\infty} \frac{\|h^n\|}{|\lambda|^{n+1}} \\
&\to 0
\end{aligned}$$

as $|\mu| \to \infty$. Consequently, the function $f : \mathbb{C} \to \mathbb{C}$ defined by

$$f(\mu) = \begin{cases} 0, \text{ if } \mu = 0, \\ \varphi((\frac{1}{\mu} I - h)^{-1}), \text{ if } 0 < |\mu| < \frac{1}{r}. \end{cases}$$

is a holomorphic function on the set

$$B(0, 1/r) = \{\mu \in \mathbb{C} \mid |\mu| < \frac{1}{r}\}.$$

The Taylor expansion of f in the disk $B(0, 1/r)$ is

$$f(\mu) = \sum_{n=0}^{\infty} \mu^{n+1} \varphi(h^n).$$

Furthermore, if $\mu \in B(0, 1/r)$, then

$$\lim_{n\to\infty} \mu^{n+1} \varphi(h^n) = 0.$$

Now, we take $\lambda_0 \in \mathbb{C}$ such that $r < |\lambda_0| < \alpha(h)$. Then, $\frac{1}{\lambda_0} \in B(0, 1/r)$ and

$$\lim_{n\to\infty} \lambda_0^{-n-1}\varphi(h^n) = 0.$$

Let $B(H)^*$ denote the dual space of $B(H)$ and define for all $n \in \mathbb{Z}_{>0}$

$$\begin{array}{rccc} \rho_n : & B(H)^* & \to & \mathbb{C} \\ & \varphi & \mapsto & \lambda_0^{-n-1}\varphi(h^n) \end{array}$$

The family $\{\rho_n\}_{n\in\mathbb{Z}_{>0}}$ is a family of continuous linear functionals on $B(H)^*$. By the *uniform boundedness principle*, there exists a constant $M \in \mathbb{R}_{>0}$ such that

$$\begin{aligned} \sup_{n\in\mathbb{Z}_{>0}} |\lambda_0|^{-n-1}|\varphi(h^n)| &\leq \|\varphi\| \sup_{n\in\mathbb{Z}_{>0}} |\lambda_0|^{-n-1}\|h^n\| \\ &\leq M. \end{aligned}$$

Letting $N = M/\|\varphi\|$, we have $\sup_{n\in\mathbb{Z}_{>0}}|\lambda_0|^{-n-1}\|h^n\| \leq N$. For all $n \in \mathbb{Z}_{>0}$,

$$\|h^n\| \leq N|\lambda_0|^{n+1}$$

and from this inequality, we have

$$\begin{aligned} \alpha(h) &= \lim_{n\to\infty} \|h^n\|^{\frac{1}{n}} \\ &\leq \lim_{n\to\infty} N^{\frac{1}{n}}|\lambda_0|^{1+\frac{1}{n}} \\ &= |\lambda_0| < \alpha(h). \end{aligned}$$

This contradicts the assumption that $\alpha(h) > \sup_{\lambda\in\sigma(h)}|\lambda|$. Therefore, $\alpha(h) \leq \sup_{\lambda\in\sigma(h)}|\lambda|$.

Combining parts (a) and (b), we deduce that

$$\lim_{n\to\infty} \|h^n\|^{\frac{1}{n}} = \alpha(h) = \sup_{\lambda\in\sigma(h)} |\lambda|.$$

□

The quantity $\alpha(h)$ in Theorem 1.2.4 is the subject of the next definition.

Definition 1.2.2. Let H be a Hilbert space over $\mathbb{C}$ and $h \in B(H)$. Then, the **spectral radius** of h, denoted by $|\sigma(h)|$, is the quantity

$$|\sigma(h)| = \sup_{\lambda\in\sigma(h)} |\lambda|.$$

Lemma 1.2.5. *Let H be a Hilbert space over $\mathbb{C}$ and $x, y \in B(H)$. Then, $|\sigma(x)| \leq \|x\|$, $|\sigma(x^*)| = |\sigma(x)|$ and if x and y commute, then $|\sigma(xy)| \leq |\sigma(x)||\sigma(y)|$.*

Proof. Assume that H is a Hilbert space over $\mathbb{C}$ and $x, y \in B(H)$.

To show: (a) $|\sigma(x)| \leq \|x\|$.

(b) $|\sigma(x^*)| = |\sigma(x)|$.

(c) If x and y commute, then $|\sigma(xy)| \leq |\sigma(x)||\sigma(y)|$.

(a) From the definition of the spectral radius and Theorem 1.2.4, we have

$$\begin{aligned}
|\sigma(x)| &= \sup_{\lambda \in \sigma(x)} |\lambda| \\
&= \alpha(x) \\
&= \lim_{n \to \infty} \|x^n\|^{\frac{1}{n}} \\
&\leq \lim_{n \to \infty} \|x\| = \|x\|.
\end{aligned}$$

(b) We claim that

$$\sigma(x^*) = \{\overline{\lambda} \mid \lambda \in \sigma(x)\}.$$

Assume that $\beta \in \sigma(x^*)$. Then, $\beta I - x^*$ is not invertible. Observe that $(\beta I - x^*)^* = \overline{\beta} I - x$ and that $\overline{\beta} I - x$ is not invertible. Suppose for the sake of contradiction that $(\overline{\beta} I - x)^{-1}$ exists as an operator in $B(H)$. Then, $(\overline{\beta} I - x)^{-1}(\overline{\beta} I - x) = I$ and by applying * to both sides,

$$(\beta I - x^*)((\overline{\beta} I - x)^{-1})^* = I.$$

A similar argument also yields $((\overline{\beta} I - x)^{-1})^*(\beta I - x^*) = I$. This contradicts the assumption that $\beta \in \sigma(x^*)$. Since $\overline{\beta} I - x$ is not invertible, $\overline{\beta} \in \sigma(x)$ and

$$\beta \in \{\overline{\lambda} \mid \lambda \in \sigma(x)\}.$$

So, $\sigma(x^*) \subseteq \{\overline{\lambda} \mid \lambda \in \sigma(x)\}$.

For the reverse inclusion, assume that $\gamma \in \sigma(x)$. Then, $\gamma I - x$ is not invertible. By using a similar argument to before, the adjoint $(\gamma I - x)^* = \overline{\gamma} I - x^*$ is also not invertible. So, $\overline{\gamma} \in \sigma(x^*)$ and therefore,

$\sigma(x^*) = \{\overline{\lambda} \mid \lambda \in \sigma(x)\}$.

Hence,

$$\begin{aligned} |\sigma(x^*)| &= \sup_{\lambda \in \sigma(x^*)} |\lambda| \\ &= \sup_{\lambda \in \sigma(x)} |\overline{\lambda}| \\ &= \sup_{\lambda \in \sigma(x)} |\lambda| = |\sigma(x)|. \end{aligned}$$

(c) This was proven at the start of Theorem 1.2.4, part (b). □

For any bounded operator $h \in B(H)$, we can consider the polynomial

$$p(h) = \alpha_0 I + \alpha_1 h + \cdots + \alpha_n h^n \in B(H)$$

where $\alpha_i \in \mathbb{C}$ and ask what is the spectrum of $p(h)$? It turns out that the answer is particularly nice.

Theorem 1.2.6. *Let H be a Hilbert space over $\mathbb{C}$ and $h \in B(H)$. For $\lambda \in B(H)$, let*

$$p(\lambda) = \alpha_0 I + \alpha_1 h + \cdots + \alpha_n h^n$$

where $\alpha_i \in \mathbb{C}$ and $n \in \mathbb{Z}_{>0}$. Then

$$\sigma(p(h)) = p(\sigma(h)) = \{p(\lambda) \mid \lambda \in \sigma(h)\}.$$

Proof. Assume that H is a Hilbert space over $\mathbb{C}$ and $h \in B(H)$. Assume that $p(\lambda)$ is defined as above for all $\lambda \in B(H)$.

For $n = 0$, we have

$$\sigma(p(h)) = \sigma(\alpha_0 I) = \{\alpha_0\} = p(\sigma(h)).$$

So, assume that $n \in \mathbb{Z}_{>0}$.

To show: (a) $p(\sigma(h)) \subseteq \sigma(p(h))$.

(b) $\sigma(p(h)) \subseteq p(\sigma(h))$.

(a) Assume that $\lambda \in \sigma(h)$ so that $p(\lambda) \in p(\sigma(h))$.

To show: (aa) The operator $p(\lambda)I - p(h)$ is not invertible.

(aa) By definition,

$$\begin{aligned} p(\lambda)I - p(h) &= \sum_{i=0}^{n} \alpha_i(\lambda^i I - h^i) \\ &= \sum_{i=1}^{n} \alpha_i(\lambda^i I - h^i) \\ &= (\lambda I - h)\sum_{i=1}^{n} \alpha_i \sum_{j=1}^{i-1} \lambda^{i-j} h^{j-1}. \end{aligned}$$

Observe that $\lambda I - h$ and $\sum_{i=1}^{n} \alpha_i \sum_{j=1}^{i-1} \lambda^{i-j} h^{j-1}$ commute as operators in $B(H)$. So, if $p(\lambda)I - p(h)$ is invertible, then $\lambda I - h$ is also invertible, contradicting the assumption that $\lambda \in \sigma(h)$. So, $p(\lambda)I - p(h)$ is not invertible, $p(\lambda) \in \sigma(p(h))$ and $p(\sigma(h)) \subseteq \sigma(p(h))$.

(b) We will prove the contrapositive of this statement. Assume that $\mu \notin p(\sigma(h))$. Let $\lambda_1, \ldots, \lambda_n$ be the zeros of the polynomial $\mu - p(\lambda)$. We claim that all of $\lambda_1, \ldots, \lambda_n \notin \sigma(h)$. Suppose for the sake of contradiction that $\lambda_i \in \sigma(h)$ for some $i \in \{1, \ldots, n\}$. Then, $\mu - p(\lambda_i) = 0$ and $\mu \in p(\sigma(h))$, contradicting the assumption that $\mu \notin p(\sigma(h))$. So, $\lambda_i \notin \sigma(h)$.

Now, we factorise $\mu - p(\lambda)$ as

$$\mu - p(\lambda) = \gamma \prod_{i=1}^{n} (\lambda_i - \lambda)^{m_i}$$

where $m_1 + \cdots + m_n = n$ and $\gamma \in \mathbb{C} - \{0\}$. So, as operators in $B(H)$,

$$\mu I - p(h) = \gamma \prod_{i=1}^{n} (\lambda_i I - h)^{m_i}$$

Since $\lambda_i \notin \sigma(h)$ for all $i \in \{1, \ldots, n\}$, $\lambda_i I - h$ is invertible and consequently, $\mu I - p(h)$ is an invertible operator because it is the product of invertible operators. Therefore, $\mu \notin \sigma(p(h))$ and by the contrapositive, we have $\sigma(p(h)) \subseteq p(\sigma(h))$, which completes the proof. □

The astute reader will notice that Theorems 1.2.2, 1.2.3, 1.2.4 and 1.2.6 did not use anything special about $B(H)$. In fact, they can be generalised to unital C*-algebras — C*-algebras with a multiplicative unit. We will briefly sketch how this generalisation works.

Definition 1.2.3. Let A be a unital C*-algebra and $a \in A$. We say that a is **invertible** if there exists $b \in A$ such that $ab = ba = 1$, where $1 \in A$ is the identity element.

The **spectrum** of a, denoted by $\sigma(a)$, is the set

$$\sigma(a) = \{\lambda \in \mathbb{C} \mid \lambda 1 - a \text{ is not invertible}\}.$$

The theorems below are for a unital C*-algebra and are proved in almost the same manner as their specialisations to $B(H)$.

Theorem 1.2.7. *Let A be a unital C*-algebra and $a, b \in A$¿ Then,* $\sigma(ab) \cup \{0\} = \sigma(ba) \cup \{0\}$.

Theorem 1.2.8. *Let A be a unital C*-algebra and $a \in A$. For $\lambda \in A$, let*

$$p(\lambda) = \alpha_0 I + \alpha_1 h + \cdots + \alpha_n h^n$$

where $\alpha_i \in \mathbb{C}$ and $n \in \mathbb{Z}_{>0}$. Then

$$\sigma(p(a)) = p(\sigma(a)) = \{p(\lambda) \mid \lambda \in \sigma(a)\}.$$

Theorem 1.2.9. *Let A be a unital C*-algebra and $a \in A$. Then, $\sigma(a) \neq \emptyset$, the sequence $\{\|a^n\|^{1/n}\}_{n \in \mathbb{Z}_{>0}}$ converges in $\mathbb{R}$ and*

$$\lim_{n \to \infty} \|a^n\|^{\frac{1}{n}} = \sup_{\lambda \in \sigma(a)} |\lambda|.$$

Chapter 2

Continuous functional calculus

2.1 Normal operators

The concept of continuous functional calculus is central in the theory of operators. The rough idea behind continuous functional calculus is that one can apply continuous functions to bounded operators on a Hilbert space. Continuous functional calculus applies to *normal operators*, the subject of this particular section.

Definition 2.1.1. Let H be a Hilbert space over $\mathbb{C}$ and $x \in B(H)$. We say that x is **normal** if $xx^* = x^*x$. That is, x commutes with its adjoint x^*.

Before we prove various properties about normal operators, we will prove an identity we will use later. Called the *polarization formula*, it expresses the inner product of a complex Hilbert space in terms of its norm.

Theorem 2.1.1 (Polarization formula)**.** *Let H be a Hilbert space over $\mathbb{C}$. If $x, y \in H$ then*

$$\langle x, y \rangle = \frac{1}{4} \sum_{k=0}^{3} i^k \|x + i^k y\|^2.$$

Proof. Assume that H is a Hilbert space over $\mathbb{C}$ and $x, y \in H$. We compute directly that

$$\begin{aligned}
\frac{1}{4} \sum_{k=0}^{3} i^k \|x + i^k y\|^2 &= \frac{1}{4}(\|x + y\|^2 + i\|x + iy\|^2 - \|x - y\|^2 - i\|x - iy\|^2) \\
&= \frac{1}{2}(\langle x, y \rangle + \langle y, x \rangle + i\langle x, iy \rangle + i\langle iy, x \rangle) \\
&= \frac{1}{2}(\langle x, y \rangle + \langle y, x \rangle + \langle x, y \rangle - \langle y, x \rangle) = \langle x, y \rangle.
\end{aligned}$$

$\square$

Theorem 2.1.1 can be generalised to sesquilinear forms on H.

Theorem 2.1.2 (Polarization formula V2)**.** *Let H be a Hilbert space over $\mathbb{C}$ and $F : H \times H \to \mathbb{C}$ be a sesquilinear form (linear in the first argument and conjugate linear in the second argument). If $x, y \in H$ then*

$$F(x, y) = \frac{1}{4} \sum_{k=0}^{3} i^k F(x + i^k y, x + i^k y).$$

The proof of Theorem 2.1.2 is almost exactly the same as Theorem 2.1.1 and thus, is omitted. Using Theorem 2.1.2, we will prove an alternative characterisation of normal operators.

Theorem 2.1.3. *Let H be a Hilbert space over $\mathbb{C}$ and $x \in B(H)$. Then, x is normal if and only if for all $\eta \in H$, $\|x(\eta)\| = \|x^*(\eta)\|$.*

Proof. Assume that H is a Hilbert space over $\mathbb{C}$ and $x \in B(H)$.

To show: (a) If x is normal, then for all $\eta \in H$, $\|x(\eta)\| = \|x^*(\eta)\|$.

(b) If for all $\eta \in H$, $\|x(\eta)\| = \|x^*(\eta)\|$, then x is normal.

(a) Assume that x is normal so that $xx^* = x^*x$. If $\eta \in H$, then

$$\begin{aligned}
\|x(\eta)\|^2 &= \langle x(\eta), x(\eta) \rangle \\
&= \langle x^*x(\eta), \eta \rangle \\
&= \langle xx^*(\eta), \eta \rangle \\
&= \langle x^*(\eta), x^*(\eta) \rangle = \|x^*(\eta)\|^2.
\end{aligned}$$

(b) Assume that if $\eta \in H$, then $\|x(\eta)\| = \|x^*(\eta)\|$. Define two sesquilinear forms $F_1, F_2 : H \times H \to \mathbb{C}$ by

$$F_1(\eta, \xi) = \langle \eta, xx^*(\xi) \rangle \qquad \text{and} \qquad F_1(\eta, \xi) = \langle \eta, x^*x(\xi) \rangle$$

Since $\|x(\eta)\| = \|x^*(\eta)\|$, $F_1(\eta, \eta) = F_2(\eta, \eta)$. By Theorem 2.1.2, we have for all $\xi, \eta \in H$,

$$\begin{aligned}F_1(\eta,\xi) &= \frac{1}{4}\sum_{k=0}^{3} i^k F_1(\eta + i^k\xi, \eta + i^k\xi)\\ &= \frac{1}{4}\sum_{k=0}^{3} i^k F_2(\eta + i^k\xi, \eta + i^k\xi)\\ &= F_2(\eta,\xi).\end{aligned}$$

Therefore, if $\eta, \xi \in H$, then

$$\langle \eta, xx^*(\xi)\rangle = \langle \eta, x^*x(\xi)\rangle.$$

So, $xx^* = x^*x$ and x is normal. □

Self-adjoint and unitary operators are special types of normal operators.

Definition 2.1.2. Let H be a Hilbert space over $\mathbb{C}$ and $x \in B(H)$. We say that x is **self-adjoint** if $x = x^*$. We say that x is **unitary** if $xx^* = x^*x = I$, where $I : H \to H$ is the identity operator on H.

Lemma 2.1.4. *Let H be a Hilbert space over $\mathbb{C}$, $x \in B(H)$ be a normal operator, $\lambda \in \mathbb{C}$ and $\psi \in H$ such that $x(\psi) = \lambda\psi$. Then, $x^*(\psi) = \overline{\lambda}\psi$.*

Proof. Assume that H is a Hilbert space over $\mathbb{C}$ and $x(\psi) = \lambda\psi$, where $x \in B(H)$ is a normal operator. With $I \in B(H)$ denoting the identity operator, we claim that $\lambda I - x$ is a normal operator. Observe that if $\eta \in H$, then

$$\begin{aligned}\|(\lambda I - x)(\eta)\|^2 &= \langle \lambda\eta - x(\eta), \lambda\eta - x(\eta)\rangle\\ &= |\lambda|^2\|\eta\|^2 - \lambda\langle \eta, x(\eta)\rangle - \overline{\lambda}\langle x(\eta), \eta\rangle + \|x(\eta)\|^2\\ &= |\lambda|^2\|\eta\|^2 - \lambda\langle x^*(\eta), \eta\rangle - \overline{\lambda}\langle \eta, x^*(\eta)\rangle + \|x^*(\eta)\|^2\\ &= \langle \overline{\lambda}\eta - x^*(\eta), \overline{\lambda}\eta - x^*(\eta)\rangle\\ &= \|(\overline{\lambda}I - x^*)(\eta)\|^2 = \|(\lambda I - x)^*(\eta)\|^2.\end{aligned}$$

By Theorem 2.1.3, $\lambda I - x$ must be a normal operator. Moreover,

$$\|(\lambda I - x)(\psi)\| = \|(\overline{\lambda}I - x^*)(\psi)\| = 0.$$

So, $x^*(\psi) = \overline{\lambda}\psi$. □

For a normal operator, the eigenspaces for distinct eigenvalues are always orthogonal.

Lemma 2.1.5. *Let H be a Hilbert space over $\mathbb{C}$ and $x \in B(H)$ be a normal operator. Let $\lambda, \mu \in \sigma(x)$ be distinct eigenvalues of x. Then, the eigenspaces of x for λ and μ are orthogonal to each other.*

Proof. Assume that H is a Hilbert space over $\mathbb{C}$, $x \in B(H)$ is a normal operator and $\lambda, \mu \in \sigma(x)$ are two distinct eigenvalues for x. Suppose that $\varphi_\lambda, \varphi_\mu \in H$ are elements such that $x(\varphi_\lambda) = \lambda\varphi_\lambda$ and $x(\varphi_\mu) = \mu\varphi_\mu$.

To show: (a) $\langle \varphi_\lambda, \varphi_\mu \rangle = 0$.

(a) Observe that

$$\begin{aligned}
\lambda\langle \varphi_\lambda, \varphi_\mu \rangle &= \langle \lambda\varphi_\lambda, \varphi_\mu \rangle \\
&= \langle x(\varphi_\lambda), \varphi_\mu \rangle \\
&= \langle \varphi_\lambda, x^*(\varphi_\mu) \rangle \\
&= \langle \varphi_\lambda, \overline{\mu}\varphi_\mu \rangle = \mu\langle \varphi_\lambda, \varphi_\mu \rangle.
\end{aligned}$$

In the last line, we used Lemma 2.1.4. So, $(\lambda - \mu)\langle \varphi_\lambda, \varphi_\mu \rangle = 0$ and since $\lambda - \mu \neq 0$, we obtain $\langle \varphi_\lambda, \varphi_\mu \rangle = 0$ as required. □

Next, we will investigate the spectrum of normal and self-adjoint operators.

Theorem 2.1.6. *Let H be a Hilbert space over $\mathbb{C}$ and $x \in B(H)$. If x is normal, then the spectral radius $|\sigma(x)| = \|x\|$ and if x is self-adjoint, then $\sigma(x) \subset \mathbb{R}$.*

Proof. Assume that $x \in B(H)$. First, assume that x is a normal operator. We first observe that for all $n \in \mathbb{Z}_{>0}$, x^n is also normal. This is because x commutes with its adjoint x^*. Recalling that $\|x^*x\| = \|xx^*\| = \|x\|^2$, we have

$$\begin{aligned}
\|x^2\|^2 &= \|(x^2)^*(x^2)\| \\
&= \|x^*x^*xx\| \\
&= \|x^*xx^*x\| \\
&= \|(xx^*)^*x^*x\| \\
&= \|x^*x\|^2 = \|x\|^4.
\end{aligned}$$

Iterating this argument, we have $\|x^{2^n}\| = \|x\|^{2^n}$ for all $n \in \mathbb{Z}_{>0}$. Thus, the sequence $\{\|x^{2^n}\|^{2^{-n}}\}_{n\in\mathbb{Z}_{>0}}$ is a constant subsequence of $\{\|x^n\|^{\frac{1}{n}}\}$ which converges to $\|x\|$. So,

$$|\sigma(x)| = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = \|x\|.$$

Next, let x be a self-adjoint operator and $\lambda = \alpha + i\beta \in \sigma(x)$ with $\alpha, \beta \in \mathbb{R}$. For $n \in \mathbb{Z}_{>0}$, define the operator

$$x_n = x - (\alpha - in\beta)I.$$

Note that $i(n+1)\beta \in \sigma(x_n)$ because

$$i(n+1)\beta I - x_n = i(n+1)\beta I - x + (\alpha - in\beta)I = \lambda I - x$$

which is not invertible because $\lambda \in \sigma(x)$. Since $i(n+1)\beta \in \sigma(x_n)$, we have the inequality

$$\begin{aligned}
|i(n+1)\beta|^2 &= (n+1)^2\beta^2 \\
&\leq \sup_{\lambda\in\sigma(x_n)} |\lambda|^2 \\
&= |\sigma(x_n)|^2 \\
&\leq \|x_n\|^2 \\
&= \|(x_n)^* x_n\| \\
&= \|(x - (\alpha - in\beta)I)^*(x - (\alpha - in\beta)I)\| \\
&= \|(x - (\alpha + in\beta)I)(x - (\alpha - in\beta)I)\| \\
&= \|(x - \alpha I)^2 + n^2\beta^2 I\| \\
&\leq \|(x - \alpha I)^2\| + n^2\beta^2.
\end{aligned}$$

Since $(n+1)^2\beta^2 \leq \|(x-\alpha I)^2\| + n^2\beta^2$ for all $n \in \mathbb{Z}_{>0}$, $\beta = 0$. So, $\lambda \in \mathbb{R}$ and $\sigma(x) \subseteq \mathbb{R}$ as required. □

2.2 Continuous functional calculus on self-adjoint operators

Continuous functional calculus asserts that if $x \in B(H)$ is a self-adjoint operator, there exists an "isomorphism" between $Cts(\sigma(x), \mathbb{C})$ and $C^*(I, x)$ — the C* algebra generated by the operators I and x. This leads us to our next formal definition.

Definition 2.2.1. Let A and B be C*-algebras with involutions $*_A$ and $*_B$ respectively. A ***-isomorphism** is a bijective map $\phi : A \to B$ such that

1. If $a_1, a_2 \in A$, then $\phi(a_1 + a_2) = \phi(a_1) + \phi(a_2)$.
2. If $\lambda \in \mathbb{C}$ and $a \in A$, then $\phi(\lambda a) = \lambda\phi(a)$.
3. If $a_1, a_2 \in A$, then $\phi(a_1 a_2) = \phi(a_1)\phi(a_2)$.
4. If $a_1 \in A$, then $(\phi(a_1))^{*_B} = \phi((a_1)^{*_A})$.

If a map $\psi : A \to B$ satisfies the above four properties, we say that ψ is a ***-homomorphism**.

Theorem 2.2.1 (Continuous functional calculus)**.** *Let H be a Hilbert space over $\mathbb{C}$ and $x \in B(H)$ be a self-adjoint operator. Then, there exists a unique map*

$$\begin{array}{rccc} \Lambda : & Cts(\sigma(x), \mathbb{C}) & \to & B(H) \\ & f & \mapsto & f(x) \end{array}$$

such that if f is the polynomial function $f(\lambda) = a_0 + a_1\lambda + \cdots + a_n\lambda^n$, then

$$f(x) = a_0 I + a_1 x + \cdots + a_n x^n.$$

If $f \in Cts(\sigma(x), \mathbb{C})$, then $\|f(x)\| = \|f\|_\infty$ where

$$\|f\|_\infty = \sup_{\lambda\in\sigma(x)} |f(\lambda)|$$

*is the uniform norm on $Cts(\sigma(x), \mathbb{C})$. Moreover, Λ is a *-isomorphism between the C*-algebras $Cts(\sigma(x), \mathbb{C})$ and $C^*(I, x)$ — the C*-algebra generated by x and the identity map $I \in B(H)$.*

Proof. Assume that $x \in B(H)$ is a self-adjoint operator. Let $P(\sigma(x)) \subseteq Cts(\sigma(x), \mathbb{C})$ denote the set of polynomial functions from $\sigma(x)$ to $\mathbb{C}$. Also define the restriction map

$$\begin{array}{rccc} res : & \mathbb{C}[z] & \to & Cts(\sigma(x), \mathbb{C}) \\ & p(z) & \mapsto & p|_{\sigma(x)}. \end{array}$$

Informally, the map res takes a polynomial in $\mathbb{C}[z]$, thinks of it as a polynomial function from $\mathbb{C}$ to $\mathbb{C}$ and then restricts it to the spectrum $\sigma(x)$. If $p(z) \in \mathbb{C}[z]$, then

$$\begin{aligned} \|p(x)\| &= |\sigma(p(x))| \\ &= \sup_{\mu\in\sigma(p(x))} |\mu| \\ &= \sup_{\lambda\in\sigma(x)} |p(\lambda)| = \|res(p(z))\|_\infty. \end{aligned}$$

The first equality follows from two facts. Firstly, since x is self-adjoint, $p(x)$ must also be self-adjoint. Secondly, we apply Theorem 2.1.6. In the second last inequality, we used Theorem 1.2.6.

Thus, there exists a linear map $\Phi : P(\sigma(x)) \to B(H)$ such that

$$p(x) = \Phi(res(p(z)))$$

since $\|p(x)\| = \|res(p(z))\|_\infty$. Additionally, Φ is an isometry (distance/length-preserving).

By the Stone-Weierstrass theorem, the space $P(\sigma(x))$ of polynomial functions is dense in $Cts(\sigma(x), \mathbb{C})$ and the image $\Phi(P(\sigma(x)))$ is therefore dense in the C*-algebra $C^*(I, x)$. Since Φ is an isometry, it must extend uniquely to an isometry

$$\begin{array}{rccc} \Lambda : & Cts(\sigma(x), \mathbb{C}) & \to & C^*(I, x) \\ & f & \mapsto & f(x) \end{array}$$

From the construction of Φ, Λ must satisfy:

1. If f is the polynomial function $f(\lambda) = \sum_{i=0}^n a_i \lambda^i$, then

$$\Lambda(f(\lambda)) = f(x) = \sum_{i=0}^n a_i x^i.$$

2. If $f \in Cts(\sigma(x), \mathbb{C})$, then $\|f(x)\| = \|f\|_\infty$.

Tedious computations are required to check that Λ satisfies the properties in Definition 2.2.1 on elements of $P(\sigma(x))$. For elements in $Cts(\sigma(x), \mathbb{C})$, the properties in Definition 2.2.1 are also satisfied because we can approximate each element in $Cts(\sigma(x), \mathbb{C})$ with polynomial functions. Hence, Λ is an isometric *-isomorphism from $Cts(\sigma(x), \mathbb{C})$ to $C^*(I, x)$. $\square$

Theorem 2.2.1 gives meaning to applying a continuous function to an operator. The next few theorems are dedicated to proving various basic properties of the continuous functional calculus.

Theorem 2.2.2. *Let H be a Hilbert space over $\mathbb{C}$ and $x, y \in B(H)$ be commuting, self-adjoint operators. If $f \in Cts(\sigma(x), \mathbb{C})$ and $g \in Cts(\sigma(y), \mathbb{C})$, then $f(x)g(y) = g(y)f(x)$.*

Proof. Assume that $x, y \in B(H)$ are commuting, self-adjoint operators, $f \in Cts(\sigma(x), \mathbb{C})$ and $g \in Cts(\sigma(y), \mathbb{C})$. Let $\{f_n\}$ and $\{g_n\}$ be sequences of polynomials which uniformly approximate f and g respectively. Since f_n, g_n are polynomials for $n \in \mathbb{Z}_{>0}$ and $xy = yx$, we have

$$f_n(x)g_n(y) = g_n(y)f_n(x)$$

and by taking the limit as $n \to \infty$, we obtain $f(x)g(y) = g(y)f(x)$. □

The next theorem can be considered a generalisation of Theorem 1.2.6.

Theorem 2.2.3 (Spectral mapping theorem)**.** *Let H be a Hilbert space over $\mathbb{C}$ and $x \in B(H)$ be a self-adjoint operator. If $f \in Cts(\sigma(x), \mathbb{C})$, then*

$$\sigma(f(x)) = f(\sigma(x)) = \{f(\lambda) \mid \lambda \in \sigma(x)\}.$$

Proof. Assume that $x \in B(H)$ is a self-adjoint operator on the complex Hilbert space H. If $\mu \in \mathbb{C}$ then $\mu I - f(x) = (\mu - f)(x)$. Suppose that $\mu \notin \sigma(f(x))$. Then, $\mu I - f(x) = (\mu - f)(x)$ is an invertible operator. Hence, $\mu - f$ is an invertible element of $Cts(\sigma(x), \mathbb{C})$ which holds if and only if $\mu \notin f(\sigma(x))$ (the zero function is not invertible!). □

As a consequence of Theorem 2.2.3, we can show that there is only one nilpotent and self-adjoint operator.

Corollary 2.2.4. *Let H be a Hilbert space over $\mathbb{C}$ and $x \in B(H)$ be self-adjoint and nilpotent, with nilpotent meaning that there exists $n \in \mathbb{Z}_{>0}$ such that $x^n = 0$. Then, $x = 0$.*

Proof. Assume that $x \in B(H)$ is self-adjoint and nilpotent so that there exists $n \in \mathbb{Z}_{>0}$ such that $x^n = 0$. Then,

$$\{0\} = \sigma(x^n) = (\sigma(x))^n = \{\lambda^n \mid \lambda \in \sigma(x)\}.$$

by Theorem 2.2.3. So, $\sigma(x) = \{0\}$ and by Theorem 2.1.6,

$$|\sigma(x)| = \|x\| = 0.$$

So, $x = 0$. □

The final property of the continuous functional calculus is that it respects the composition of operators.

Theorem 2.2.5. *Let H be a Hilbert space over $\mathbb{C}$ and $x \in B(H)$ be self-adjoint. If $g \in Cts(\sigma(x), \mathbb{C})$ is real-valued, then the operator $g(x)$ is self-adjoint and if $f \in Cts(g(\sigma(x)), \mathbb{C})$, then $f(g(x)) = (f \circ g)(x)$.*

Proof. Assume that $x \in B(H)$ is a self-adjoint operator on a complex Hilbert space H. Assume that $g \in Cts(\sigma(x), \mathbb{C})$ is real-valued. Then, $g = \overline{g}$, where $\overline{g}(\lambda) = \overline{g(\lambda)}$ for all $\lambda \in \sigma(x)$.

By applying the *-isomorphism $\Lambda : Cts(\sigma(x), \mathbb{C}) \to C^*(I, x)$ from Theorem 2.2.1, we find that

$$g(x) = \Lambda(g) = \Lambda(\overline{g}) = (g(x))^*.$$

So, $g(x)$ must be a self-adjoint operator.

Next, assume that $f \in Cts(g(\sigma(x)), \mathbb{C})$. Consider the mapping

$$\begin{array}{rccc} \Omega: & Cts(g(\sigma(x)), \mathbb{C}) & \to & B(H) \\ & f & \mapsto & (f \circ g)(x). \end{array}$$

Notice that Ω is an isometry because

$$\begin{aligned} \|f\|_\infty &= \sup_{\lambda \in g(\sigma(x))} |f(\lambda)| \\ &= \sup_{\mu \in \sigma(x)} |f(g(\mu))| \\ &= |\sigma((f \circ g)(x))| \\ &= \|(f \circ g)(x)\|. \end{aligned}$$

Also note that if f is a polynomial function on $g(\sigma(x))$, then Ω maps polynomial functions to polynomials in $g(x)$ within $B(H)$. But, by the uniqueness of the continuous functional calculus (see Theorem 2.2.1), Ω must be the map $f \mapsto f(g(x))$. So, $(f \circ g)(x) = f(g(x))$ as operators in $B(H)$. $\square$

2.3 Positive operators and their square roots

Our first application of continuous functional calculus (see 2.2.1) is to give meaning to taking the square root of a *positive operator* defined on a complex Hilbert space. We obviously cannot do this for a general self-adjoint operator because the square root is not defined on all $\mathbb{R}$.

Definition 2.3.1. Let H be a Hilbert space over $\mathbb{C}$. We say that the operator $x \in B(H)$ is **positive** if x is self-adjoint and its spectrum $\sigma(x) \subseteq \mathbb{R}_{\geq 0}$. We write $x \geq 0$ to denote that x is a positive operator on H. The set of positive operators on H is denoted by $B(H)_+$.

By applying continuous functional calculus, we can define the square root of a positive operator.

Theorem 2.3.1. *Let H be a Hilbert space over $\mathbb{C}$ and $x \in B(H)$ be a positive operator. Then, there exists a unique positive operator $a \in B(H)$ such that $a^2 = a \circ a = x$.*

Proof. Assume that $x \in B(H)_+$ is a positive operator defined on a complex Hilbert space H. Since $\sigma(x) \subseteq \mathbb{R}_{\geq 0}$, the square root function

$$\begin{array}{rccc} f: & \sigma(x) & \to & \mathbb{R} \\ & \lambda & \mapsto & \lambda^{\frac{1}{2}} \end{array}$$

is continuous on $\sigma(x)$. Let $a = f(x)$. Since $f(\lambda)f(\lambda) = \lambda$, we can apply the *-isomorphism Λ from Theorem 2.2.1 to deduce that $a^2 = f(x)f(x) = x$.

To see that $a \in B(H)_+$, note that from Theorem 2.2.3, $f(\sigma(x)) = \sigma(f(x)) = \sigma(a)$.

$$\sigma(a) = \sigma(f(x)) = f(\sigma(x)) = \{\lambda^{\frac{1}{2}} \mid \lambda \in \sigma(x)\} \subseteq \mathbb{R}_{\geq 0}.$$

It remains to show uniqueness. Suppose that $b \in B(H)_+$ such that $b^2 = x$. Then, from Theorem 2.2.5, $b = f(g(b))$, where g is the function

$$\begin{array}{rccc} g: & \mathbb{R} & \to & \mathbb{R} \\ & \lambda & \mapsto & \lambda^2 \end{array}$$

So, $b = f(b^2) = f(x) = a$. Hence, $a \in B(H)_+$ is the unique positive operator satisfying $a^2 = x$. □

Theorem 2.3.1 tells us that every positive operator on H has a unique square root. The next theorem gives us a decomposition of self-adjoint operators in terms of positive operators. It should remind the reader of the Jordan decomposition theorem for a signed measure.

Theorem 2.3.2. *Let H be a Hilbert space over $\mathbb{C}$ and $x \in B(H)$ be a self-adjoint operator. Then, there exists a unique pair (a, b) of positive operators on H such that $x = a - b$ and $ab = 0$.*

Proof. Assume that $x \in B(H)$ is a self-adjoint operator. Consider the following two continuous complex-valued functions on the spectrum $\sigma(x)$:

$$f(\lambda) = \begin{cases} \lambda, & \text{if } \lambda \geq 0, \\ 0, & \text{otherwise.} \end{cases} \quad g(\lambda) = \begin{cases} 0, & \text{if } \lambda > 0, \\ -\lambda, & \text{otherwise.} \end{cases}$$

If $\lambda \in \sigma(x)$, then $f(\lambda)g(\lambda) = 0$ and $\lambda = f(\lambda) - g(\lambda)$. By applying the *-isomorphism Λ from Theorem 2.2.1 and setting $a = f(x)$ and $b = g(x)$, we deduce that $x = a - b$ and $ab = 0$ as required.

It remains to show uniqueness of the decomposition. Suppose that $c, d \in B(H)_+$ such that $x = c - d$ and $cd = 0$. Observe that c and d must commute because

$$dc = d^*c^* = (cd)^* = 0 = cd.$$

By using Theorem 2.2.2, we deduce that $c^{\frac{1}{2}}d^{\frac{1}{2}} = d^{\frac{1}{2}}c^{\frac{1}{2}} = 0$. This is because $(c^{\frac{1}{2}}d^{\frac{1}{2}})^2 = cd = 0$.

Now, $(c^{\frac{1}{2}} + d^{\frac{1}{2}})^2 = c + d$. Since $c + d$ is the square of a self-adjoint operator, it must be positive because

$$\sigma(c + d) = \sigma((c^{\frac{1}{2}} + d^{\frac{1}{2}})^2) = \sigma(c^{\frac{1}{2}} + d^{\frac{1}{2}})^2 \subseteq \mathbb{R}_{\geq 0}.$$

Also observe that $(c + d)^2 = (c - d)^2 = x^2$ so that $c + d$ is a square root of the positive operator x^2. But, from Theorem 2.3.1, $c + d$ is the unique square root of x^2. Therefore,

$$c = \frac{1}{2}((c + d) + (c - d)) = \frac{1}{2}((x^2)^{\frac{1}{2}} + x) = f(x)$$

and

$$d = \frac{1}{2}((c + d) - (c - d)) = \frac{1}{2}((x^2)^{\frac{1}{2}} - x) = g(x).$$

□

The operators a and b in Theorem 2.3.2 are called the positive and negative parts of the self-adjoint operator x respectively. We usually denote a by x^+ and b by x^-. Observe that $x^+, x^- \in C^*(x)$, where $C^*(x)$ is the C*-algebra generated by the operator x.

We will now like to prove equivalent characterisations of positive operators. For this, we need the following lemma.

Lemma 2.3.3. *Let H be a Hilbert space over $\mathbb{C}$ and $x \in B(H)$. Then, $x = 0$ if and only if for all $\xi \in H$, $\langle \xi, x(\xi) \rangle = 0$. Also, $x = x^*$ if and only if for all $\xi \in H$, $\langle \xi, x(\xi) \rangle \in \mathbb{R}$.*

Proof. Assume that $x \in B(H)$. For the first statement, assume that $x = 0$. Then, by positive definiteness of the inner product, if $\xi \in H$ then $\langle \xi, x(\xi) \rangle = 0$.

For the converse, assume that if $\xi \in H$ then $\langle \xi, x(\xi) \rangle = 0$. The map

$$\begin{array}{rccc} \phi: & H \times H & \to & \mathbb{C} \\ & (\xi, \eta) & \mapsto & \langle \xi, x(\eta) \rangle \end{array}$$

is a sesquilinear form and from an application of Theorem 2.1.2,

$$\begin{aligned} \phi(\xi, \eta) &= \frac{1}{4} \sum_{k=0}^{3} i^k \phi(\xi + i^k \eta, \xi + i^k \eta) \\ &= \frac{1}{4} \sum_{k=0}^{3} i^k \langle \xi + i^k \eta, x(\xi + i^k \eta) \rangle \\ &= 0. \end{aligned}$$

So, if $\xi, \eta \in H$ then $\langle \xi, x(\eta) \rangle = 0$. So, $x = 0$. This proves the first statement.

For the second statement, assume that $x = x^*$. If $\xi \in H$, then

$$\overline{\langle \xi, x(\xi) \rangle} = \langle x(\xi), \xi \rangle = \langle \xi, x(\xi) \rangle.$$

So, $\langle \xi, x(\xi) \rangle \in \mathbb{R}$. For the converse, assume that if $\xi \in H$ then $\langle \xi, x(\xi) \rangle \in \mathbb{R}$. Define the sesquilinear form

$$\begin{array}{rccc} \psi: & H \times H & \to & \mathbb{C} \\ & (\xi, \eta) & \mapsto & \langle x(\xi), \eta \rangle \end{array}$$

If $\xi \in H$ then

$$\phi(\xi, \xi) = \langle \xi, x(\xi) \rangle = \overline{\langle \xi, x(\xi) \rangle} = \langle x(\xi), \xi \rangle = \psi(\xi, \xi).$$

By Theorem 2.1.2, if $\xi, \eta \in H$, then $\phi(\xi, \eta) = \psi(\xi, \eta)$. So, $\langle \xi, x(\eta) \rangle = \langle x(\xi), \eta \rangle$ and $x = x^*$. □

Theorem 2.3.4. *Let H be a Hilbert space over $\mathbb{C}$ and $x \in B(H)$. The following statements are equivalent:*

1. *x is positive.*

2. *There exists a self-adjoint operator $y \in B(H)$ such that $x = y^2$.*

3. There exists an operator $z \in B(H)$ such that $x = z^*z$.

4. If $\xi \in H$, then $\langle \xi, x(\xi) \rangle \geq 0$.

Proof. Assume that $x \in B(H)$ is a bounded operator on H.

If x is positive then there exists a unique positive operator $y \in B(H)$ such that $x = y^2$ from Theorem 2.3.1. So, if the first statement is true, then the second statement is true.

The third statement follows from the second statement, since a positive operator is self-adjoint by definition.

Now suppose that there exists an operator $z \in B(H)$ such that $x = z^*z$. Assume that $\xi \in H$. Then,

$$\langle \xi, x(\xi) \rangle = \langle \xi, z^*z(\xi) \rangle = \|z(\xi)\|^2 \geq 0.$$

Finally, suppose that if $\xi \in H$, then $\langle \xi, x(\xi) \rangle \geq 0$. By Theorem 2.3.2, $x = x^+ - x^-$, where $x^+, x^- \in B(H)_+$ and $x^+x^- = 0$. With this decomposition, we have for all $\xi \in H$,

$$\begin{aligned} 0 &\leq \langle x^-(\xi), x(x^-(\xi)) \rangle \\ &= \langle \xi, x^-xx^-(\xi) \rangle \\ &= \langle \xi, x^-xx^-(\xi) \rangle \\ &= \langle \xi, x^-(x^+ - x^-)x^-(\xi) \rangle \\ &= -\langle \xi, (x^-)^3(\xi) \rangle. \end{aligned}$$

Let $f(\lambda) = \lambda^{3/2}$ for $\lambda \in \mathbb{R}_{\geq 0}$. Then, f is continuous and

$$\begin{aligned} -\langle \xi, (x^-)^3(\xi) \rangle &= -\langle x^-(\xi), (x^-)^2(\xi) \rangle \\ &= -\langle x^-(\xi), (x^-)^{\frac{1}{2}}(x^-)^{\frac{1}{2}}x^-(\xi) \rangle \\ &= -\langle (x^-)^{\frac{3}{2}}(\xi), (x^-)^{\frac{3}{2}}(\xi) \rangle \\ &= -\|f(x^-)(\xi)\|^2 \leq 0. \end{aligned}$$

Therefore, if $\xi \in H$, $\langle \xi, (x^-)^3(\xi) \rangle = 0$. So, $(x^-)^3 = 0$. From Corollary 2.2.4, $x^- = 0$ and $x = x^+ \in B(H)_+$. This completes the proof. $\square$

We will make a brief aside into the world of quantum physics. If $x \in B(H)$ and $\xi \in H$, the quantity $\langle \xi, x(\xi) \rangle$ is called the expectation value of x in the state determined by ξ. This terminology is usually applied when ξ is a unit vector. Theorem 2.3.4 states that an operator x is positive if and only if its expectation values are positive for all states $\xi \in H$.

Based on the notion of positivity, we can define a partial order relation on the elements of $B(H)$.

Definition 2.3.2. Let H be a Hilbert space over $\mathbb{C}$ and $x, y \in B(H)$. We say that x **dominates** y if $x - y \geq 0$ (alternatively, $x - y \in B(H)_+$). This is also denoted as $x \geq y$.

With the relation in Definition 2.3.2, we will prove that $B(H)$ is a poset. Recall the definition of a poset (partially ordered set) from [SS15, Page 7] First, we need to establish some properties about the relation in Definition 2.3.2.

Theorem 2.3.5. *Let H be a Hilbert space over $\mathbb{C}$. Define*

$$-B(H)_+ = \{-x \mid x \in B(H)_+\}$$

1. *If $x \in B(H)_+$ and $\lambda \in \mathbb{R}_{\geq 0}$, then $\lambda x \in B(H)_+$.*
2. *If $x, y \in B(H)_+$, then $x + y \in B(H)_+$.*
3. *$B(H)_+ \cap (-B(H)_+) = \{0\}$.*
4. *If $x \in B(H)_+$ and $y \in B(H)$, then $y^* x y \in B(H)_+$.*

Proof. Assume that H is a Hilbert space over $\mathbb{C}$.

(1) Assume that $x \in B(H)_+$ and $\lambda \in \mathbb{R}_{\geq 0}$. Then, $\sigma(x) \subseteq \mathbb{R}_{\geq 0}$ and

$$\begin{aligned} \sigma(\lambda x) &= \{\alpha \in \mathbb{C} \mid \alpha I - \lambda x \text{ is not invertible}\} \\ &= \{\lambda \beta \in \mathbb{C} \mid \beta I - x \text{ is not invertible}\} \subseteq \mathbb{R}_{\geq 0}. \end{aligned}$$

So, $\lambda x \in B(H)_+$.

(2) Assume that $x, y \in B(H)_+$. Then, from Theorem 2.3.4, if $\xi \in H$, then $\langle \xi, x(\xi) \rangle \geq 0$ and $\langle \xi, y(\xi) \rangle \geq 0$. So, $\langle \xi, (x + y)(\xi) \rangle \geq 0$ and consequently, $x + y \in B(H)_+$.

(3) Assume that $x \in B(H)_+ \cap (-B(H)_+)$. From Theorem 2.3.4, if $\xi \in H$ then $\langle \xi, x(\xi) \rangle \geq 0$ and $\langle \xi, -x(\xi) \rangle = -\langle \xi, x(\xi) \rangle \geq 0$. So, $\langle \xi, x(\xi) \rangle = 0$ and therefore, $x = 0$.

(4) Assume that $x \in B(H)_+$ and $y \in B(H)$. From Theorem 2.3.4, if $\xi \in H$ then

$$\langle \xi, y^* x y(\xi) \rangle = \langle y(\xi), x(y(\xi)) \rangle \geq 0.$$

since x is a positive operator. Hence, $y^* x y \in B(H)_+$. □

The fourth criterion in Theorem 2.3.5 is reminiscent of the criterion for a matrix $M_{n \times n}(\mathbb{R})$ to be *positive definite*. Next, we will show that $B(H)$ with the relation in Definition 2.3.2 is indeed a poset.

Theorem 2.3.6. *Let H be a Hilbert space over $\mathbb{C}$. The set $B(H)$, together with the relation in Definition 2.3.2, is a poset.*

Proof. Assume that H is a Hilbert space over $\mathbb{C}$.

To show: (a) If $x \in B(H)$ then $x \leq x$.

(b) If $x, y, z \in B(H)$, $x \leq y$ and $y \leq z$ then $x \leq z$.

(c) If $x, y \in B(H)$, $x \leq y$ and $y \leq x$ then $x = y$.

(a) Assume that $x \in B(H)$. Then, $x - x = 0$, whose spectrum is $\sigma(0) = \{0\} \subseteq \mathbb{R}_{\geq 0}$. So, $x - x \in B(H)_+$ and $x \leq x$.

(b) Assume that $x, y, z \in B(H)$, $x \leq y$ and $y \leq z$. Using Theorem 2.3.5, we deduce that

$$x - z = (x - y) + (y - z) \in B(H)_+$$

because $x - y, y - z \in B(H)_+$. Therefore, $x \leq z$.

(c) Assume that $x, y \in B(H)$, $x \leq y$ and $y \leq x$. Then, $x - y \in B(H)_+$ and $y - x \in B(H)_+$. So, $x - y \in B(H)_+ \cap (-B(H)_+) = \{0\}$. Therefore, $x = y$. □

Let us prove some more useful properties of the poset $(B(H), \leq)$.

Lemma 2.3.7. *Let H be a Hilbert space over $\mathbb{C}$ and $x \in B(H)$. Then, $x^* x \leq \|x\|^2 I$.*

Proof. Assume that $x \in B(H)$ and $\xi \in H$. Then,

$$\langle \xi, x^*x(\xi)\rangle = \langle x(\xi), x(\xi)\rangle \leq \|x\|^2\|\xi\|^2 = \langle \xi, \|x\|^2 I(\xi)\rangle.$$

So, if $\xi \in H$ then $\langle \xi, (\|x\|^2 I - x^*x)(\xi)\rangle \geq 0$. Hence, $x^*x \leq \|x\|^2 I$ as required. $\square$

Lemma 2.3.8. *Let H be a Hilbert space over $\mathbb{C}$ and $x \in B(H)$ be a positive operator. Then, $x \leq I$ if and only if $\|x\| \leq 1$.*

Proof. Assume that $x \in B(H)$ is a positive operator.

To show: (a) If $x \leq I$, then $\|x\| \leq 1$

(b) If $\|x\| \leq 1$, then $x \leq I$.

(a) Assume that $x \leq I$. Since x is positive, there exists $z \in B(H)$ such that $x = z^*z$ from Theorem 2.3.4. If $\xi \in H$ then

$$\begin{aligned}
\langle \xi, (I - x)(\xi)\rangle &= \langle \xi, \xi\rangle - \langle \xi, x(\xi)\rangle \\
&= \langle \xi, \xi\rangle - \langle \xi, z^*z(\xi)\rangle \\
&= \|\xi\|^2 - \|z(\xi)\|^2 \\
&\geq \|\xi\|^2 - \|z\|^2\|\xi\|^2 \\
&= (1 - \|z\|^2)\|\xi\|^2 \geq 0.
\end{aligned}$$

Hence, $\|z\|^2 \leq 1$. Recall from Theorem 1.1.1 that $\|z\|^2 = \|z^*z\| = \|x\|$. So, $\|x\| \leq 1$.

(b) Assume that $\|x\| \leq 1$. If $\xi \in H$ then

$$\begin{aligned}
\langle \xi, (I - x)(\xi)\rangle &= \|\xi\|^2 - \langle \xi, x(\xi)\rangle \\
&\geq \|\xi\|^2 - \|\xi\|\|x(\xi)\| \\
&\geq \|\xi\|^2 - \|x\|\|\xi\|^2 \\
&= (1 - \|x\|)\|\xi\|^2 \geq 0.
\end{aligned}$$

Thus, $I - x \in B(H)_+$ and $x \leq I$ as required. $\square$

Lemma 2.3.9. *Let H be a Hilbert space over $\mathbb{C}$ and $x, y \in B(H)_+$ such that $0 \leq x \leq y$. Assume that x has inverse x^{-1}. Then, y is invertible and its inverse y^{-1} satisfies $y^{-1} \leq x^{-1}$.*

Proof. Assume that $x, y \in B(H)_+$ such that $0 \leq x \leq y$ and x is invertible. Since x is positive, the spectrum $\sigma(x)$ is a compact subset of $\mathbb{R}_{\geq 0}$. Since x is invertible, $0 \notin \sigma(x)$. So, there exists $\delta \in \mathbb{R}_{>0}$ such that $\sigma(x) \subseteq [\delta, \infty)$.

The continuous functional calculus tells us that $x - \delta I \geq 0$. This is because $x - \delta I$ is self-adjoint (as x and I are self-adjoint) and

$$\begin{aligned}\sigma(x - \delta I) &= \{\lambda \in \mathbb{R} \mid (\lambda + \delta)I - x \text{ is not invertible}\} \\ &= \{\lambda \in \mathbb{R} \mid \lambda + \delta \in \sigma(x)\} \\ &\subseteq \{\lambda \in \mathbb{R} \mid \lambda + \delta \in [\delta, \infty)\} \\ &= [0, \infty).\end{aligned}$$

So, $x \geq \delta I$ and $y \geq \delta I$. So, $\sigma(y) \subseteq [\delta, \infty)$ and y must be invertible.

Since $x \leq y$, $y^{-\frac{1}{2}} x y^{-\frac{1}{2}} \leq I$. By Lemma 2.3.8, $\|y^{-\frac{1}{2}} x y^{-\frac{1}{2}}\| \leq 1$. Exploiting the fact that x and y^{-1} are self-adjoint, we must have

$$\|x^{\frac{1}{2}} y^{-\frac{1}{2}}\|^2 = \|(x^{\frac{1}{2}} y^{-\frac{1}{2}})^* x^{\frac{1}{2}} y^{-\frac{1}{2}}\| = \|y^{-\frac{1}{2}} x y^{-\frac{1}{2}}\| \leq 1.$$

Consequently,

$$\|y^{-\frac{1}{2}} x^{\frac{1}{2}}\| = \|(x^{\frac{1}{2}} y^{-\frac{1}{2}})^*\| = \|x^{\frac{1}{2}} y^{-\frac{1}{2}}\| \leq 1.$$

Now observe that

$$(y^{-\frac{1}{2}} x^{\frac{1}{2}})^* y^{-\frac{1}{2}} x^{\frac{1}{2}} \leq \|y^{-\frac{1}{2}} x^{\frac{1}{2}}\|^2 I \leq I.$$

From the LHS, we obtain $x^{\frac{1}{2}} y^{-1} x^{\frac{1}{2}} \leq I$. This means that the operator $I - x^{\frac{1}{2}} y^{-1} x^{\frac{1}{2}} \in B(H)_+$. Since $x^{-\frac{1}{2}}$ is self-adjoint because x^{-1} is self-adjoint and Theorem 2.3.4,

$$(x^{-\frac{1}{2}})^*(I - x^{\frac{1}{2}} y^{-1} x^{\frac{1}{2}}) x^{-\frac{1}{2}} = x^{-1} - y^{-1} \in B(H)_+$$

by Theorem 2.3.5. Hence, $x^{-1} \geq y^{-1}$ as required. $\square$

We will end this section with a very interesting application of positive operators.

Theorem 2.3.10. *Let H be a Hilbert space over $\mathbb{C}$ and $x \in B(H)$. Then, x is a $\mathbb{C}$-linear combination of four unitary operators.*

Proof. Assume that $x \in B(H)$. Observe that the operators $x + x^*$ and $-i(x - x^*)$ are self-adjoint. For the second operator, we have

$$(-i(x - x^*))^* = i(x^* - x) = -i(x - x^*).$$

Now observe that x is the $\mathbb{C}$-linear combination

$$x = \frac{1}{2}(x + x^*) - \frac{1}{2i}(-i(x - x^*)).$$

Now it suffices to show that any self-adjoint operator can be written as a $\mathbb{C}$-linear combination of two unitary operators. Without loss of generality, we can assume that the self-adjoint operator has norm 1.

Assume that $y \in B(H)$ is a self-adjoint operator with $\|y\| = 1$. Since y is self-adjoint, y^2 must be a positive operator from Theorem 2.2.3. It has norm $\|y^2\| = \|y^*y\| = \|y\|^2 = 1$. Therefore, $y^2 = y^*y \leq \|y\|^2 I = I$. Since $I - y^2 \in B(H)_+$, Theorem 2.3.1 tells us that there exists a unique positive operator $(I - y^2)^{\frac{1}{2}}$ such that

$$((I - y^2)^{\frac{1}{2}})^2 = I - y^2.$$

Now, we decompose y as

$$y = \frac{1}{2}\big((y + i(I - y^2)^{\frac{1}{2}}) + (y - i(I - y^2)^{\frac{1}{2}})\big)$$

Notice that the operators $y + i(I - y^2)^{\frac{1}{2}}$ and $y - i(I - y^2)^{\frac{1}{2}}$ are unitary. This is because the functions $f(\lambda) = \lambda^{\frac{1}{2}}$ and $g(\lambda) = (1 - \lambda^2)^{\frac{1}{2}}$ are continuous functions for $\lambda \in \sigma(y)$ and by Theorem 2.2.2, $y^{\frac{1}{2}}(1 - y^2)^{\frac{1}{2}} = (1 - y^2)^{\frac{1}{2}} y^{\frac{1}{2}}$. Consequently,

$$\begin{aligned}(y \pm i(I - y^2)^{\frac{1}{2}})(y \pm i(I - y^2)^{\frac{1}{2}})^* &= (y \pm i(I - y^2)^{\frac{1}{2}})(y \mp i(I - y^2)^{\frac{1}{2}}) \\ &= y^2 + (I - y^2) = I\end{aligned}$$

and

$$\begin{aligned}(y \pm i(I - y^2)^{\frac{1}{2}})^*(y \pm i(I - y^2)^{\frac{1}{2}}) &= (y \mp i(I - y^2)^{\frac{1}{2}})(y \pm i(I - y^2)^{\frac{1}{2}}) \\ &= y^2 + (I - y^2) = I.\end{aligned}$$

This yields the statement we are after. $\square$

2.4 Projection operators

One of the basic topics of Hilbert space theory is the notion of orthogonal projections onto closed subspaces of H. One of the fundamental theorems associated with this topic is

Theorem 2.4.1. *Let H be a Hilbert space and V be a closed subspace of H. Then, $H = V \oplus V^{\perp}$. In other words, for all $x \in H$, $x = y + z$, where $y \in V$ is the unique point in V which has minimal distance from x. Similarly, $z \in V^{\perp}$ is the unique point in $V^{\perp}$ which has minimal distance from x.*

The map $x \mapsto y$ is called a **projection operator**. With the goal of proving the *polar decomposition* for elements in $B(H)$, we will discuss a few aspects of projection operators in the next few sections.

Note that the projection operator as in Theorem 2.4.1 is not the definition of a projection operator that we will use. The definition we will use is motivated by the following result.

Theorem 2.4.2. *Let H be a Hilbert space and $P : H \to H$ be a bounded, linear operator which satisfies $P^2 = P$ and $P^* = P$. Then, P is the projection operator onto the closed subspace $\mathrm{im}(P)$ in the sense of Theorem 2.4.1.*

Proof. Assume that H is a Hilbert space and $P : H \to H$ is a bounded linear operator which satisfies $P^2 = P$ and $P^* = P$. The first part of the proof is to show that the image $\mathrm{im}(P)$ is a closed subspace of H.

To show: (a) The image $\mathrm{im}(P)$ is closed.

(a) Let $I : H \to H$ denote the identity operator on H. Consider the operator $I - P : H \to H$. We observe that $I - P$ satisfies the same properties as P. This is because

$$(I - P)^2 = I^2 - P - P + P^2 = I - P - P + P = I - P$$

and if $x, y \in H$ then

$$\begin{aligned}\langle (I - P)x, y\rangle &= \langle x, y\rangle - \langle P(x), y\rangle \\ &= \langle x, y\rangle - \langle x, P(y)\rangle \\ &= \langle x, (I - P)y\rangle.\end{aligned}$$

Furthermore, $\mathrm{im}(P) = \ker(I - P)$. To see why this is the case, suppose that $P(x) \in \mathrm{im}(P)$. Then, $(I - P)(Px) = Px - P^2x = 0$. So, $P(x) \in \ker(I - P)$ for all $x \in H$ and therefore, $\mathrm{im}(P) \subseteq \ker(I - P)$.

Conversely, suppose that $y \in \ker(I - P)$. Then, $(I - P)y = 0$ and consequently, $P(y) = y$. Hence, $y \in \mathrm{im}(P)$ and as a result, $\ker(I - P) \subseteq \mathrm{im}(P)$. Therefore, $\ker(I - P) = \mathrm{im}(P)$. Since $\ker(I - P)$ is a closed subspace of H, $\mathrm{im}(P)$ must also be closed as required.

We showed in part (a) that $I - P$ satisfies the same properties as P. Hence, we can apply the result of part (a), but to the operator $I - P$. So, $\mathrm{im}(I - P) = \ker P = (\mathrm{im}(P))^{\perp}$. Now assume that $x \in H$. Then,

$$x = P(x) + (x - P(x))$$

where $P(x) \in \mathrm{im}(P)$ and $x - P(x) \in \mathrm{im}(I - P) = (\mathrm{im}(P))^{\perp}$. Thus, P is the projection operator onto the closed subspace $\mathrm{im}(P)$, which completes the proof. □

Due to the characterisation in Theorem 2.4.2, the definition of a projection operator in [Sol18] is

Definition 2.4.1. Let H be a Hilbert space and $x \in B(H)$. We say that x is a **projection** if $x = x^2$ and $x = x^*$.

From now on, we will take Definition 2.4.1 as the definition of a projection operator. Another characterisation of a projection operator is given by considering its spectrum.

Theorem 2.4.3. *Let H be a Hilbert space over $\mathbb{C}$ and $x \in B(H)$. Then, x is a projection operator if and only if x is self-adjoint and the two continuous functions $\lambda \mapsto \lambda^2$ and $\lambda \mapsto \lambda$, defined for $\lambda \in \sigma(x)$, are equal.*

Proof. Assume that $x \in B(H)$. Define the functions $\phi, \psi \in Cts(\sigma(x), \mathbb{C})$ by $\phi(\lambda) = \lambda$ and $\psi(\lambda) = \lambda^2$.

To show: (a) If x is a projection operator, then x is self-adjoint and $\phi = \psi$ on the spectrum $\sigma(x)$.

(b) If x is self-adjoint and $\phi = \psi$ on the spectrum $\sigma(x)$, then x is a projection operator.

(a) Assume that x is a projection operator. Then, x is self-adjoint and $x^2 = x$. Applying the *-isomorphism Λ from Theorem 2.2.1 on both sides of

$x^2 = x$, we obtain $\psi = \phi$ on the spectrum $\sigma(x)$.

(b) Assume that x is a self-adjoint operator and that $\phi = \psi$ as functions in $Cts(\sigma(x), \mathbb{C})$. Applying the inverse map Λ^{-1} from Theorem 2.2.1, we find that $x = x^2$. Hence, x is a projection operator as required. $\square$

We note in the scenario of Theorem 2.4.2 that $\psi = \phi$ in $Cts(\sigma(x), \mathbb{C})$ if and only if $\sigma(x) \subseteq \{0, 1\}$. First suppose that $\psi = \phi$ and $\lambda \in \sigma(x)$. Then, $\psi(\lambda) = \phi(\lambda)$ and $\lambda^2 = \lambda$. So, $\lambda = 0$ or $\lambda = 1$ and $\sigma(x) \subseteq \{0, 1\}$.

Conversely, suppose that $\sigma(x) \subseteq \{0, 1\}$. From Theorem 2.2.3,

$$\sigma(x(I - x)) = \{\lambda - \lambda^2 \mid \lambda \in \sigma(x)\} = \{0\}.$$

So, $x - x^2 = 0$ in $B(H)$ and $\psi(x) = \phi(x)$.

Definition 2.4.2. Let H be a Hilbert space over $\mathbb{C}$ and $e_1, e_2 \in B(H)$ be projection operators. We say that e_1 and e_2 are **orthogonal** if $e_1 e_2 = 0$.

The reason why we use the (admittedly loaded) term orthogonal in the above definition lies with the following result.

Lemma 2.4.4. *Let H be a Hilbert space over $\mathbb{C}$ and $e_1, e_2 \in B(H)$ be projection operators. Then, e_1 and e_2 are orthogonal if and only if $im(e_1) = (im(e_2))^{\perp}$.*

Proof. Assume that $e_1, e_2 \in B(H)$ are projection operators. Then, e_1 and e_2 are orthogonal if and only if for $\xi, \eta \in H$,

$$0 = \langle e_1 e_2(\xi), \eta \rangle = \langle e_2(\xi), e_1(\eta) \rangle.$$

In turn, the above equation holds if and only if $\text{im}(e_1) = (\text{im}(e_2))^{\perp}$. $\square$

Definition 2.4.3. Let H be a Hilbert space over $\mathbb{C}$ and $x \in B(H)$. Define $l(x)$ to be the projection operator onto the closure $\overline{\text{im}(x)}$ and $r(x)$ to be the projection operator onto the orthogonal complement $(\ker x)^{\perp}$. The operator $l(x)$ is called the **left support** of x and $r(x)$ is called the **right support** of x.

Observe that $l(x)x = x$ because $\text{im}(x) \subseteq \overline{\text{im}(x)}$. Also, $xr(x) = x$ because by Theorem 2.4.1,

$$H = \ker x \oplus (\ker x)^{\perp}.$$

Let $h = h_1 + h_2 \in H$ where $h_1 \in \ker x$ and $h_2 \in (\ker x)^{\perp}$. Then,

$$xr(x)(h) = x(h_2) = x(h_1 + h_2) = x(h).$$

The left and right supports of a bounded operator have a useful connection via the adjoint. We will need to prove some properties of the orthogonal complement first. The most important property of the orthogonal complement is that for any subspace S of a Hilbert space H, $S^\perp$ is always a closed subspace of H.

Theorem 2.4.5. *Let H be a Hilbert space.*

(a) If S is a subspace of H then $S \subseteq (S^\perp)^\perp$.

(b) If S_1, S_2 are subspaces of H and $S_1 \subseteq S_2$ then $S_2^\perp \subseteq S_1^\perp$.

(c) If S is a closed subspace of H then $S = (S^\perp)^\perp$.

(d) If S is a subspace of H, then $\overline{S} = (S^\perp)^\perp$.

Proof. Assume that H is a Hilbert space.

(a) Assume that S is a subspace of H and that $s \in S$. If $t \in S^\perp$ then $\langle s, t\rangle = 0$. So, $s \in (S^\perp)^\perp$ and $S \subseteq (S^\perp)^\perp$.

(b) Assume that S_1, S_2 are subspaces of H such that $S_1 \subseteq S_2$. Then,

$$\begin{aligned} S_2^\perp &= \{x \in H \mid \langle x, v\rangle = 0 \text{ for } v \in S_2\} \\ &\subseteq \{x \in H \mid \langle x, v\rangle = 0 \text{ for } v \in S_1\} \\ &= S_1^\perp. \end{aligned}$$

(c) Assume that S is a closed subspace of H. Then, by Theorem 2.4.1, $H = S \oplus S^\perp$. We already have $S \subseteq (S^\perp)^\perp$ by part (a). It suffices to prove the reverse inclusion. Assume that $x \in (S^\perp)^\perp$. Then, $x = y + z$ where $y \in S$ and $z \in S^\perp$.

To show: (ca) $z = 0$.

(ca) Since $x \in (S^\perp)^\perp$ and $z \in S^\perp$, $\langle x, z\rangle = 0$. But,

$$\langle x, z\rangle = \langle y, z\rangle + \|z\|^2 = \|z\|^2 = 0.$$

Thus, $z = 0$ and $x = y \in S$. Hence, $(S^\perp)^\perp \subseteq S$ and $S = (S^\perp)^\perp$.

(d) Assume that S is a subspace of H. Then, $S \subseteq \overline{S}$. Applying part (b) twice, we deduce that $(S^\perp)^\perp \subseteq (\overline{S}^\perp)^\perp$. Since $\overline{S}$ is a closed subspace of H, $(\overline{S}^\perp)^\perp = \overline{S}$ from part (c). Thus, $(S^\perp)^\perp \subseteq \overline{S}$.

Now observe that from part (a), $\overline{S} \subseteq \overline{(S^\perp)^\perp} = (S^\perp)^\perp$. The last equality follows from the fact that $(S^\perp)^\perp$ is a closed subspace of H. So, $\overline{S} = (S^\perp)^\perp$ as required. □

Now we can properly prove the relation between the left and right supports.

Lemma 2.4.6. *Let H be a Hilbert space over $\mathbb{C}$ and $x \in B(H)$. Then, $l(x) = r(x^*)$ and $r(x) = l(x^*)$.*

Proof. Assume that $x \in B(H)$.

To show: (a) $l(x) = r(x^*)$.

(b) $r(x) = l(x^*)$.

(a) It suffices to show that $\mathrm{im}(x)^\perp = \ker x^*$. Note that if $\eta, \xi \in H$ then

$$\langle \eta, x(\xi) \rangle = 0 \text{ if and only if } \langle x^*(\eta), \xi \rangle = 0.$$

This demonstrates that $\mathrm{im}(x)^\perp = \ker x^*$. So, $l(x)$ is the projection onto the closed subspace

$$\overline{\mathrm{im}(x)} = (\mathrm{im}(x)^\perp)^\perp = (\ker x^*)^\perp$$

from Theorem 2.4.5. But, this is just $r(x^*)$. So, $l(x) = r(x^*)$.

(b) Part (a) gives $r(x) = r((x^*)^*) = l(x^*)$. □

If $x \in B(H)$ is a self-adjoint operator then Lemma 2.4.6 tells us that $l(x) = r(x^*) = r(x)$.

Definition 2.4.4. Let H be a Hilbert space and $x \in B(H)$ be self-adjoint. The common value of $l(x)$ and $r(x)$ is denoted by $s(x)$ and called the **support** of x.

The support of a self-adjoint operator satisfies the following property.

Theorem 2.4.7. *Let H be a Hilbert space over $\mathbb{C}$ and $x \in B(H)$. Then, $r(x) = s(x^*x)$ and $l(x) = s(xx^*)$.*

Proof. Assume that $x \in B(H)$. Notice that the operators x^*x and xx^* are self-adjoint. So, we are able to talk about the support of x^*x and xx^*.

To show: (a) $\ker(x) = \ker(x^*x)$.

(a) We have $\ker(x) \subseteq \ker(x^*x)$. For the reverse inclusion, suppose that $\xi \in \ker(x^*x)$. Then, $\langle \xi, x^*x(\xi) \rangle = 0 = \|x(\xi)\|^2$. So, $x(\xi) = 0$ and $\xi \in \ker(x)$.

Now, $r(x)$ is the projection operator onto the closed subspace $(\ker(x))^\perp$. From part (a), $(\ker(x))^\perp = (\ker(x^*x))^\perp$. Therefore, $r(x) = s(x^*x)$.

Similarly, $l(x)$ is the projection operator onto the closed subspace $\overline{\mathrm{im}(x)} = (\ker x^*)^\perp = (\ker xx^*)^\perp$. So, $l(x) = s(xx^*)$. $\square$

2.5 Partial isometries

The notion of a partial isometry is based on the following equivalent conditions.

Theorem 2.5.1. *Let H be a Hilbert space over $\mathbb{C}$ and $x \in B(H)$. Then, the following conditions are equivalent:*

(a) x^*x is a projection operator.

(b) $xx^*x = x$.

(c) $x^*xx^* = x^*$.

(d) xx^* is a projection operator.

Proof. Assume that $x \in B(H)$. For the first implication, assume that x^*x is a projection operator. Notice that if $e \in B(H)$ is a projection operator, $l(e)$ is a projection operator onto $\overline{\mathrm{im}(e)} = \mathrm{im}(e)$, as a consequence of Theorem 2.4.2. Hence, the support $s(e) = l(e) = e$.

Since x^*x is a projection operator, $x^*x = s(x^*x) = r(x)$. Hence,

$$xx^*x = xr(x) = x.$$

Next, assume that $xx^*x = x$. Taking the adjoint of both sides yields $x^*xx^* = x^*$.

Assume that $x^*xx^* = x^*$. To see that xx^* is a projection operator, note first that by direct computation, xx^* is self-adjoint. To see that xx^* is idempotent, observe that

$$(xx^*)^2 = x(x^*xx^*) = xx^*.$$

Thus, xx^* is a projection operator.

Finally, assume that xx^* is a projection operator. By Lemma 1.2.1, $\sigma(xx^*) \cup \{0\} = \sigma(x^*x) \cup \{0\}$. Since xx^* is a projection operator, $\sigma(xx^*) \subseteq \{0,1\}$. Since $\sigma(xx^*) \cup \{0\} = \sigma(x^*x) \cup \{0\}$ is true, $\sigma(xx^*) \subseteq \{0,1\}$ if and only if $\sigma(x^*x) \subseteq \{0,1\}$. Since x^*x is self-adjoint and $\sigma(x^*x) \subseteq \{0,1\}$, x^*x must be a projection operator as required. □

Definition 2.5.1. Let H be a Hilbert space over $\mathbb{C}$ and $x \in B(H)$. We say that x is a **partial isometry** if it satisfies any of the equivalent statements in Theorem 2.5.1.

The projection operator x^*x is called the **initial projection**, whereas the projection operator xx^* is called the **final projection**. Their respective images are called the **initial subspace** and **final subspace** of x.

The following theorem explains the origin of the name partial isometry.

Theorem 2.5.2. *Let H be a Hilbert space over $\mathbb{C}$ and $x \in B(H)$. The operator x is a partial isometry if and only if there exists a closed subspace $S \subseteq H$ such that*

$$\|x\xi\| = \begin{cases} \|\xi\|, & \text{if } \xi \in S, \\ 0, & \text{if } \xi \in S^\perp. \end{cases}$$

The subspace S is the initial subspace of x and xS is the final subspace of x.

Proof. Assume that $x \in B(H)$. First assume that x is a partial isometry. Then, x^*x is a projection operator and consequently, $\mathrm{im}(x^*x)$ is a closed subspace of H. Observe that if $\xi \in H$ then

$$\begin{aligned} \|x(x^*x(\xi))\|^2 &= \langle xx^*x\xi, xx^*x\xi\rangle \\ &= \langle x^*x\xi, x^*xx^*x\xi\rangle \\ &= \langle x^*x\xi, x^*x\xi\rangle = \|x^*x\xi\|^2. \end{aligned}$$

If $\eta \in \mathrm{im}(x^*x)^\perp$ then

$$\|x\eta\|^2 = \langle x\eta, x\eta\rangle = \langle \eta, x^*x\eta\rangle = 0.$$

Therefore,

$$\|x\xi\| = \begin{cases} \|\xi\|, \text{ if } \xi \in \mathrm{im}(x^*x), \\ 0, \text{ if } \xi \in \mathrm{im}(x^*x)^\perp. \end{cases}$$

For the converse, suppose that there exists a closed subspace $S \subseteq H$ such that

$$\|x\xi\| = \begin{cases} \|\xi\|, \text{ if } \xi \in S, \\ 0, \text{ if } \xi \in S^\perp. \end{cases}$$

To see that x is a partial isometry, we will show that $xx^*x = x$. By Theorem 2.4.1, $H = S \oplus S^\perp$. Assume that $\eta \in H$ and $\xi \in S$. Write $\eta = \eta_1 + \eta_2$, where $\eta_1 \in S$ and $\eta_2 \in S^\perp$. Then,

$$\begin{aligned} \langle \eta, x^*x(\xi)\rangle &= \langle \eta_1, x^*x(\xi)\rangle + \langle \eta_2, x^*x(\xi)\rangle \\ &= \langle x(\eta_1), x(\xi)\rangle + \langle x(\eta_2), x(\xi)\rangle \\ &= \langle x(\eta_1), x(\xi)\rangle \quad \text{(Theorem 2.1.2)} \\ &= \langle \eta_1, \xi\rangle \quad \text{(Theorem 2.1.2)} \\ &= \langle \eta, \xi\rangle. \end{aligned}$$

Thus, if $\xi \in S$ then $x^*x(\xi) = \xi$. This means that

$$xx^*x(\eta) = xx^*x(\eta_1 + \eta_2) = x(\eta_1) + 0 = x(\eta_1)$$

Since $\eta \in H$ was arbitrary, $xx^*x = x$ and x^*x must be a projection operator. So, x is a partial isometry. □

How is a partial isometry related to an isometry? Theorem 2.5.2 provides a convincing answer; if x is a partial isometry with initial subspace H then x is an isometry.

2.6 The polar decomposition

Let us foray briefly into the world of linear algebra. Let $A \in M_{n\times n}(\mathbb{C})$. Then, the polar decomposition states that $A = UP$, where $U \in GL_n(\mathbb{C})$ is unitary and $P \in M_{n\times n}(\mathbb{C})$ is a positive semi-definite Hermitian matrix. We will generalise this decomposition to the situation of a bounded linear operator x on a complex Hilbert space H, using the concepts discussed in the previous two sections.

Theorem 2.6.1. *Let H be a Hilbert space over $\mathbb{C}$ and $x \in B(H)$. Then there exists a unique pair of operators $v, a \in B(H)$ such that $x = va$, a is positive and $v^*v = s(a)$.*

Proof. Assume that $x \in H$. We will first prove the existence of a pair of operators $v, a \in B(H)$ which satisfy the conditions outlined above.

First observe that x^*x is a positive operator. To see why this is the case, let $\alpha : \sigma(x) \to \mathbb{C}$ be defined by $\alpha(\lambda) = \overline{\lambda}\lambda = |\lambda|^2$. Then, $\alpha \in Cts(\sigma(x), \mathbb{C})$ and by Theorem 2.2.3,

$$\sigma(x^*x) = \sigma(\alpha(x)) = \alpha(\sigma(x)) = \{|\lambda|^2 \mid \lambda \in \sigma(x)\} \subseteq \mathbb{R}_{\geq 0}.$$

Thus, x^*x is a positive operator and by Theorem 2.3.1, the square root $(x^*x)^{\frac{1}{2}}$ is also a positive operator. Define $a = (x^*x)^{\frac{1}{2}}$. If $\xi \in H$ then

$$\|a\xi\|^2 = \langle a\xi, a\xi\rangle = \langle \xi, a^2\xi\rangle = \langle \xi, x^*x\xi\rangle = \|x\xi\|^2.$$

Define the linear map

$$\begin{array}{rccc} v_{00}: & \mathrm{im}(a) & \to & \mathrm{im}(x) \\ & \eta = a\xi & \mapsto & x\xi \end{array}$$

We will show that v_{00} is well-defined. Suppose that $\eta = a\xi = a\xi'$ for $\xi, \xi' \in H$ with $\xi \neq \xi'$. Then,

$$\|x\xi - x\xi'\| = \|x(\xi - \xi')\| = \|a(\xi - \xi')\| = 0.$$

Thus, the linear map v_{00} is well-defined. By construction, it is an isometric map (distance-preserving). Thus, it extends uniquely to an isometry $v_0 : \overline{\mathrm{im}(a)} \to H$.

Now consider the map

$$v(\xi) = \begin{cases} v_0\xi, & \text{if } \xi \in \overline{\mathrm{im}(a)} \\ 0, & \text{if } \xi \in \mathrm{im}(a)^{\perp}. \end{cases}$$

This is a partial isometry (see Theorem 2.5.2) which satisfies $va = x$ by construction. Furthermore, the support $s(a) = l(a)$ is the projection operator onto $\overline{\mathrm{im}(a)}$.

The projection operator v^*v projects onto the closed subspace $\mathrm{im}(v^*v)$. But,

$$\begin{aligned}
\mathrm{im}(v^*v) &= \overline{\mathrm{im}(v^*v)} \\
&= (\mathrm{im}(v^*v)^{\perp})^{\perp} \quad \text{(Theorem 2.4.5)} \\
&= (\ker(v^*v))^{\perp} \\
&= (\ker(v))^{\perp} \\
&= (\mathrm{im}(a)^{\perp})^{\perp} \\
&= \overline{\mathrm{im}(a)} \quad \text{(Theorem 2.4.5).}
\end{aligned}$$

Thus, $v^*v = s(a)$, which proves existence.

In order to show uniqueness, suppose that $u, b \in B(H)$ such that $x = ub$, $b \in B(H)_+$ and $u^*u = s(b)$. Then,

$$x^*x = (va)^*va = av^*va = as(a)a = a^2.$$

But, we also have

$$x^*x = (ub)^*ub = bu^*ub = bs(b)b = b^2.$$

So, $a^2 = b^2$ and by the uniqueness of the square root, $a = b$. Since u is a partial isometry with initial projection $s(b) = s(a)$. The initial subspace is $\overline{\mathrm{im}(a)}$ and so, $u(\tau) = 0$ for $\tau \in \mathrm{im}(a)^{\perp}$. So, $u = v_0$ in $\mathrm{im}(a)^{\perp}$.

Now if $\eta \in H$ then

$$u(a(\eta)) = x(\eta) = v_0(a(\eta))$$

because $a = b$. Thus, $u = v_0$ on $\mathrm{im}(a)$. Consequently, $u = v$, proving uniqueness. $\square$

In light of the polar decomposition, we make the following definition.

Definition 2.6.1. Let H be a Hilbert space over $\mathbb{C}$ and $x \in B(H)$. By Theorem 2.6.1, $x = va$, where $a \in B(H)_+$ and $v \in B(H)$ satisfies $v^*v = s(a)$. The partial isometry v is called the **phase** of x and the positive operator a is called the **modulus/absolute value** of x. The absolute value of x is sometimes denoted by $|x|$.

According to [Sol18], the polar decomposition will play a major role in the sections that follow. Thus, we will prove a few properties satisfied by the polar decomposition.

Theorem 2.6.2. *Let H be a Hilbert space over $\mathbb{C}$ and $x \in B(H)$. Let $x = v|x|$ be the polar decomposition of x.*

(a) If x is self-adjoint then $|x| = f(x)$, where $f(\lambda) = |\lambda| \in Cts(\sigma(x), \mathbb{C})$.

(b) The phase v is unitary if and only if $\ker x = \{0\}$ and im(x) is dense in H.

(c) $x^* = v^*(v|x|v^*)$ where $s(v|x|v^*) = vv^*$.

(d) If x is self-adjoint then $|x| = x^+ + x^-$ and $v = s(x^+) - s(x^-)$, where x^+ and x^- are the positive and negative parts of x respectively.

Proof. Assume that $x = v|x| \in B(H)$, where $v|x|$ is the polar decomposition of x.

(a) Assume that x is self-adjoint. Then,

$$x^*x = (v|x|)^*v|x| = |x|v^*v|x| = |x|s(|x|)|x| = |x|^2.$$

Let $f \in Cts(\sigma(x), \mathbb{C})$ be such that $f(\lambda) = \overline{\lambda}\lambda = |\lambda|^2$. By Theorem 2.2.1, $f(x) = |x|^2$. Since $|x|$ is self-adjoint, $|x|^2$ is positive. By Theorem 2.3.1, $|x|^2$ has a unique square root. Hence, taking the square root of both sides gives $g(x) = |x|$, where $g \in Cts(\sigma(x), \mathbb{C})$ is given by $g(\lambda) = |\lambda|$.

(b) In one direction, assume that the phase v is unitary. Then, $vv^* = v^*v = I$. But, $v^*v = s(|x|) = I$. The projection operator $s(|x|)$ projects onto $\overline{\mathrm{im}|x|} = H$. This demonstrates that $\mathrm{im}|x|$ is a dense subspace of H. But, $|x| = v^*v|x| = v^*x$. Since $|x|$ is self-adjoint, $|x| = x^*v$. So, $\mathrm{im}|x| \subseteq \mathrm{im}(x^*)$ and

$$\overline{\mathrm{im}(x^*)} = H = (\mathrm{im}(x^*)^\perp)^\perp = \ker(x)^\perp.$$

Therefore, $\ker(x) = \{0\}$. Now observe that

$$\overline{\mathrm{im}(|x|)} = (\ker(|x|))^\perp = H.$$

So, $\ker(|x|) = 0$. Since $x^* = |x|v^*$,

$$\begin{aligned} \ker(x^*) &= \{\xi \in H \mid x^*(\xi) = 0\} \\ &= \{\xi \in H \mid |x|(v^*(\xi)) = 0\} \\ &\subseteq \{\xi \in H \mid |x|(\xi) = 0\} = \ker(|x|) = \{0\}. \end{aligned}$$

So, $\ker(x^*) = \{0\}$ and by Theorem 2.4.5,

$$\overline{\mathrm{im}(x)} = (\mathrm{im}(x)^\perp)^\perp = (\ker(x^*))^\perp = H.$$

For the other direction, assume that $\overline{\mathrm{im}(x)} = H$ and that $\ker(x) = \{0\}$. Since $x = v|x|$, $\mathrm{im}(x) \subseteq \mathrm{im}(v)$. So, the closure $\overline{\mathrm{im}(v)} = H$. Recall that v is a partial isometry by construction in Theorem 2.6.1. So, v^*v and vv^* are by definition projection operators.

Since $\overline{\mathrm{im}(v)} = H$, we can use Theorem 2.4.5 to deduce that $\ker(v^*) = \{0\}$. So, $\ker(vv^*) = \ker(v^*) = \{0\}$. Taking orthogonal complements and using Theorem 2.4.5 again, we deduce that vv^* projects onto the closed subspace

$$\mathrm{im}(vv^*) = \overline{\mathrm{im}(vv^*)} = (\ker(vv^*))^\perp = H.$$

Hence, $vv^* = I$. Now recall from Theorem 2.6.1 that the support $s(|x|) = v^*v$. But, $s(|x|)$ is the projection operator onto the subspace $\overline{\mathrm{im}(|x|)} = \ker(|x|)^\perp$.

If $\ker(x) = \{0\}$ then

$$\begin{aligned} \ker(|x|) &= \{\xi \in H \mid |x|(\xi) = 0\} \\ &\subseteq \{\xi \in H \mid v(|x|(\xi)) = 0\} = \ker(x) = \{0\}. \end{aligned}$$

So, $\ker(|x|) = \{0\}$ and by taking the orthogonal complement of both sides, $\overline{\mathrm{im}(|x|)} = H$. Thus, $v^*v = s(|x|) = I$.

(c) Using the support $s(|x|)$, we compute directly that

$$x^* = |x|v^* = (s(|x|)|x|)v^* = v^*(v|x|v^*).$$

Since $|x|$ is a positive operator, $v|x|v^*$ must also be a positive operator by Theorem 2.3.5. Since v is a partial isometry, we have from Theorem 2.5.1 that

$$(vs(|x|)v^*)(v|x|v^*) = (vv^*vv^*)(v|x|v^*) = v|x|v^*$$

because $(vv^*)^2 = vv^*$ and $vv^*v = v$. Also,

$$(v|x|v^*)(vs(|x|)v^*) = (v|x|v^*)(vv^*vv^*) = v|x|v^*.$$

Therefore, by the uniqueness of the decomposition in Theorem 2.4.1, $s(v|x|v^*) = vs(|x|)v^*$. But, $vs(|x|)v^* = vv^*$. Thus, the decomposition

$$x^* = v^*(v|x|v^*)$$

is the polar decomposition of the adjoint x^*.

(d) Assume that x is self-adjoint. By Theorem 2.3.2, there exists a unique pair (x^+, x^-) of positive operators such that $x^+x^- = 0$ and $x = x^+ - x^-$.

Consider the operator $s(x^+)x^-$. Since $x^+x^- = 0$, $\mathrm{im}(x^-) \subseteq \ker(x^+)$. But, the projection operator $s(x^+)$ projects onto the closed subspace $(\ker(x^+))^\perp$. Since $H = \ker(x^+) \oplus (\ker(x^+))^\perp$, $s(x^+)x^- = 0$. Similarly, since $x^-x^+ = 0$, $s(x^-)x^+ = 0$.

Now, we observe that

$$(s(x^+)-s(x^-))(x^++x^-) = s(x^+)x^++s(x^+)x^--s(x^-)x^+-s(x^-)x^- = x^+-x^- = x.$$

Since x^+ and x^- are both positive operators, $x^+ + x^-$ is also a positive operator by Theorem 2.3.5.

Now consider the operator $s(x^+)s(x^-)$. The projection $s(x^-)$ projects onto the subspace $(\ker(x^-))^\perp$. But, $(\ker(x^-))^\perp \subseteq (\mathrm{im}(x^+))^\perp$. Since $H = (\mathrm{im}(x^+))^\perp \oplus \overline{\mathrm{im}(x^+)}$ because $\overline{\mathrm{im}(x^+)} = (\mathrm{im}(x^+)^\perp)^\perp$, $s(x^+)s(x^-) = 0$. Similarly, $s(x^-)s(x^+) = 0$.

Since $s(x^+) - s(x^-)$ is self-adjoint,

$$(s(x^+) - s(x^-))^*(s(x^+) - s(x^-)) = (s(x^+) - s(x^-))^2 = s(x^+) + s(x^-).$$

To show: (da) The support $s(x^+ + x^-) = s(x^+) + s(x^-)$.

(da) We have by direct calculation

$$(x^+ + x^-)(s(x^+) + s(x^-)) = (s(x^+) + s(x^-))(x^+ + x^-) = x^+ + x^-.$$

We also check directly that $s(x^+) + s(x^-)$ is a projection operator. Thus, due to the uniqueness of the decomposition in Theorem 2.4.1, $s(x^+) + s(x^-) = s(x^+ + x^-)$.

Consequently, we have shown that the decomposition

$$x = (s(x^+) - s(x^-))(x^+ + x^-)$$

is a polar decomposition of x. By invoking the uniqueness of the polar decomposition in Theorem 2.6.1, we have $v = s(x^+) - s(x^-)$ and $|x| = x^+ + x^-$. □

2.7 Monotone convergence of operators in the strong topology

The purpose of this section is to show how the partial relation on $B(H)$, as seen in Definition 2.3.2, interacts with the strong topology on $B(H)$. The strong (operator) topology is defined by the family of seminorms $\{p_\xi\}_{\xi \in H}$ where $p_\xi(x) = \|x\xi\|$ for $x \in B(H)$.

Definition 2.7.1. Let $\{x_i\}_{i\in I}$ be a sequence in $B(H)$. We say that the sequence converges to $x \in B(H)$ in the strong topology if and only if for any $\xi \in H$, $x_i\xi \to x\xi$ as $i \to \infty$. In other words, the operators x_i converge to x pointwise on H.

The point here is that a uniformly bounded sequence of self-adjoint operators in $B(H)$ which is monotonically increasing according to the partial relation in Definition 2.3.2 converges in the strong topology.

Theorem 2.7.1. *Let H be a Hilbert space over $\mathbb{C}$ and $\{x_i\}_{i\in I}$ be a sequence of self-adjoint operators such that if $i \geq j$, then $x_i \geq x_j$. Assume that there exists $C \in \mathbb{R}_{>0}$ such that if $i \in I$ then $\|x_i\| \leq C$.*

Then, there exists a self-adjoint operator $x \in B(H)$ such that if $i \in I$ then $x \geq x_i$ and if $y \in B(H)$ is an operator satisfying $y \geq x_i$ for $i \in I$ then $y \geq x$. Moreover, $\{x_i\}_{i\in I}$ converges to x in the strong topology.

Proof. Assume that $\{x_i\}_{i\in I}$ is a sequence of self-adjoint operators satisfying the properties above. From Theorem 2.3.4 and Lemma 2.3.3, we deduce that if $\xi \in H$ then the sequence $\{\langle \xi, x_i(\xi)\rangle\}_{i\in I}$ in $\mathbb{R}$ is bounded and non-decreasing. Hence, it must converge to its supremum:

$$\lim_{i\to\infty} \langle \xi, x_i(\xi)\rangle = \sup_{i\in I} \langle \xi, x_i(\xi)\rangle.$$

Now observe that by the polarization identity (see Theorem 2.1.2), the sequence $\{\langle \xi, x_i(\eta)\rangle\}_{i\in I}$ in $\mathbb{C}$ must also converge. To see why this is the case, write

$$\langle \xi, x_j(\eta)\rangle = \frac{1}{4}\sum_{k=0}^{3} i^k \langle \xi + i^k\eta, x_j(\xi + i^k\eta)\rangle$$

for $j \in I$ and then take the limit of both sides as $j \to \infty$. Since the RHS converges in the limit, the LHS must converge as well.

Now let $F(\xi, \eta) = \lim_{i\to\infty}\langle \xi, x_i(\eta)\rangle$. This is a sesquilinear form which is bounded because

$$|F(\xi,\eta)| = |\lim_{i\to\infty}\langle \xi, x_i(\eta)\rangle| = \lim_{i\to\infty}|\langle \xi, x_i(\eta)\rangle| \leq C\|\xi\|\|\eta\|.$$

For $\eta \in H$, define the map

$$\begin{array}{rccc} \phi_\xi : & H & \to & \mathbb{C} \\ & \eta & \mapsto & \overline{F(\xi,\eta)} \end{array}$$

This is a continuous/bounded linear functional. By the Riesz representation theorem, there exists unique $\tau \in H$ such that

$$\overline{F(\xi,\eta)} = \lim_{i\to\infty}\langle \xi, x_i(\eta)\rangle = \langle \xi, \tau\rangle.$$

Consequently, there exists a unique operator $x \in B(H)$ such that

$$F(\xi,\eta) = \langle \xi, x(\eta)\rangle = \lim_{i\to\infty}\langle \xi, x_i(\eta)\rangle$$

for $\xi, \eta \in H$. Notice that $F(\xi,\xi) \in \mathbb{R}$ since it is the limit of a sequence in $\mathbb{R}$. Hence, x must be a self-adjoint operator.

To see that $x \geq x_i$ for $i \in I$, we use Theorem 2.3.4 and compute directly that

$$\langle \xi, x(\xi)\rangle = \lim_{j\to\infty}\langle \xi, x_j(\xi)\rangle = \sup_{j\in I}\langle \xi, x_j(\xi)\rangle \geq \langle \xi, x_i(\xi)\rangle.$$

Now assume that $y \in B(H)$ satisfies $y \geq x_i$ for $i \in I$. If $\xi \in H$ then

$$\langle \xi, y(\xi)\rangle \geq \sup_{i\in I}\langle \xi, x_i(\xi)\rangle = \langle \xi, x(\xi)\rangle.$$

So, $y \geq x$. Finally, to see that $\{x_i\}$ converges to x in the strong topology, we compute directly that for $\xi \in H$,

$$
\begin{aligned}
\|x(\xi) - x_i(\xi)\|^2 &= \|(x - x_i)\xi\|^2 \\
&= \|(x - x_i)^{\frac{1}{2}}(x - x_i)^{\frac{1}{2}}\xi\|^2 \qquad \text{(since } x \geq x_i\text{)} \\
&\leq \|(x - x_i)^{\frac{1}{2}}\|^2\|(x - x_i)^{\frac{1}{2}}\xi\|^2 \\
&= \|x - x_i\|\|(x - x_i)^{\frac{1}{2}}\xi\|^2 \qquad \text{(since } x - x_i \text{ is self-adjoint)} \\
&\leq (\|x\| + \|x_i\|)\|(x - x_i)^{\frac{1}{2}}\xi\|^2 \\
&\leq 2C\|(x - x_i)^{\frac{1}{2}}\xi\|^2 \\
&= 2C\langle (x - x_i)^{\frac{1}{2}}\xi, (x - x_i)^{\frac{1}{2}}\xi\rangle \\
&= 2C\langle \xi, (x - x_i)(\xi)\rangle \to 0
\end{aligned}
$$

as $i \to \infty$. $\square$

Chapter 3

Generalising the spectral theorem

3.1 Multiplication operators

Recall the spectral theorem for finite dimensional inner product spaces. Roughly, if m is a self-adjoint operator then

$$m = \sum_{i=1}^{N} \lambda_i E_i$$

where $\{E_1, \ldots, E_N\}$ consists of pairwise orthogonal projections and $\{\lambda_1, \ldots, \lambda_N\}$ are the real eigenvalues of m. Also, if f is a polynomial function then

$$f(m) = \sum_{i=1}^{N} f(\lambda_i) E_i.$$

To see why this is the case, note that if $f(\lambda) = \lambda^n$ then since the E_i are pairwise orthogonal,

$$f(m) = (\sum_{i=1}^{N} \lambda_i E_i)^n = \sum_{i=1}^{N} \lambda_i^n E_i.$$

We extend this to the rest of the polynomial functions by linearity. The ultimate goal of this chapter is to extend the spectral theorem to the case of Hilbert spaces. In our formulation of the spectral theorem, we will replace the linear combination with an appropriate integral. On the way to this goal, we will extend the class of functions for which functional calculus can be defined on.

Since we are working with integrals, measure theory will feature significantly in this chapter. We will begin with some definitions from measure theory.

Definition 3.1.1. Let $(X, \mu, \mathcal{A})$ be a measure space, consisting of a set X, a σ-algebra $\mathcal{A} \subset \mathcal{P}(X)$ and a measure $\mu : \mathcal{A} \to [-\infty, \infty]$. We say that $(X, \mu, \mathcal{A})$ is **semifinite** if for all measurable sets $B \subseteq X$ with $\mu(B) > 0$, there exists a measurable set $C \subseteq B$ such that $0 < \mu(C) < \infty$.

In line with [Sol18], we will denote a measure space by its set and its measure.

Definition 3.1.2. Let (X, μ) be a measure space. A function $f : X \to \mathbb{C}$ is **essentially bounded** if there exists $M \in \mathbb{R}$ such that the set

$$\{x \in X \mid |f(x)| > M\} = (|f| - M)^{-1}((0, \infty))$$

is a null set. The vector space of essentially bounded functions on X is denoted by $L^\infty(X, \mu)$.

The vector space $L^\infty(X, \mu)$ has the norm given by the *essential supremum*:

$$\|f\|_\infty = \inf\{M \in \mathbb{R} \mid (|f| - M)^{-1}((0, \infty)) \text{ is null}\}.$$

Essentially bounded functions can be thought of as functions which are bounded almost everywhere — bounded everywhere except for a subset of X with measure zero. An equivalent formulation of the essential supremum is

$$\|f\|_\infty = \inf\{M \in \mathbb{R}_{>0} \mid |f(x)| \leq M \text{ for almost all } x \in X\}. \tag{3.1}$$

Equation (3.1) is the definition of the essential supremum used in [Sol18]. Now we will prove another equivalent characterisation of the essential supremum.

Lemma 3.1.1. *Let (X, μ) be a measure space and $f \in L^\infty(X, \mu)$. Let*

$$\mathcal{C} = \{Y \subseteq X \mid \mu(X \backslash Y) = 0\}$$

denote the set of all conull sets in X. Then,

$$\|f\|_\infty = \inf_{Y \in \mathcal{C}} \sup_{x \in Y} |f(x)|.$$

Proof. Assume that (X, μ) is a measure space and $f \in L^\infty(X, \mu)$. Assume that $\mathcal{C}$ is the set of all conull sets in X. Define

$$S = \{\sup_{x \in Y} |f(x)| \mid Y \in \mathcal{C}\}$$

and

$$E = \{M \in \mathbb{R}_{>0} \mid |f(x)| \leq M \text{ for almost all } x \in X\}.$$

To show: (a) $\inf S \geq \inf E$.

(b) $\inf E \geq \inf S$.

(a) Assume that $s \in S$. Then, there exists a conull set $D \subseteq X$ such that $s = \sup_{x \in D} |f(x)|$ (and $\mu(X \backslash D) = 0$). So, $|f| \leq s$ on the set D, which has measure $\mu(D) = \mu(X) \neq 0$. So, $s \in E$, $S \subseteq E$ and consequently, $\inf S \geq \inf E$.

(b) Assume that $t \in E$. Then, there exists a null set $T \subseteq X$ such that $|f| \leq t$ on the conull set $X \backslash T$. So, $\sup_{x \in X \backslash T} |f(x)| \leq t$. Thus, we have demonstrates that if $t \in E$ then there exists $s \in S$ such that $s \leq t$. Therefore, $\inf E \geq \inf S$. □

The following fact about $L^\infty(X, \mu)$ is important in the discussion which follows:

Theorem 3.1.2. *Let (X, μ) be a measure space and $L^\infty(X, \mu)$ be the normed vector space of essentially bounded functions with norm given by the essential supremum in* (3.1). *Then, $L^\infty(X, \mu)$ is a C^*-algebra with involution given by $f \mapsto \overline{f}$, where $\overline{f}(x) = \overline{f(x)}$.*

An important fact used in the proof of Theorem 3.1.2 is that $\|f\|_\infty = \|\overline{f}\|_\infty$. Arguing in a similar manner to Theorem 1.1.1, we then obtain $\|\overline{f}f\|_\infty = \|f\|_\infty^2$.

An alternative way of thinking about essentially bounded functions is as multiplication operators on the Banach space $L^2(X, \mu)$ of square-integrable complex-valued functions.

Definition 3.1.3. Let (X, μ) be a *semifinite* measure space and $f \in L^\infty(X, \mu)$. The **multiplication operator** by f, denoted by M_f, is defined by

$$\begin{array}{rccc} M_f: & L^2(X,\mu) & \to & L^2(X,\mu) \\ & \psi(x) & \mapsto & f(x)\psi(x). \end{array}$$

Notice that M_f is a bounded operator on the Hilbert space $L^2(X,\mu)$. To see why this is the case, we compute directly that

$$\begin{aligned} \|M_f\| &= \sup_{\|\psi\|_2=1} \|M_f(\psi)\|_2 \\ &= \sup_{\|\psi\|_2=1} \Big(\int_X |M_f(\psi)|^2\, d\mu\Big)^{\frac{1}{2}} \\ &= \sup_{\|\psi\|_2=1} \Big(\int_X |f(x)\psi(x)|^2\, d\mu(x)\Big)^{\frac{1}{2}} \\ &\le \sup_{\|\psi\|_2=1} \|f\|_\infty \Big(\int_X |\psi(x)|^2\, d\mu(x)\Big)^{\frac{1}{2}} = \|f\|_\infty. \end{aligned}$$

Definitions 3.1.3 and 3.1.2 are intimately related by the following theorem.

Theorem 3.1.3. *Let (X,μ) be a semifinite measure space and $f \in L^\infty(X,\mu)$. Let $B(L^2(X,\mu))$ denote the C^*-algebra of bounded linear operators on the Hilbert space $L^2(X,\mu)$ (see Theorem 1.1.1). Then, the map*

$$\begin{array}{rccc} M: & L^\infty(X,\mu) & \to & B(L^2(X,\mu)) \\ & f & \mapsto & M_f \end{array}$$

*is an isometric *-isomorphism onto its image.*

Proof. Assume that (X,μ) is a semifinite measure space and $f \in L^\infty(X,\mu)$. Assume that $M : L^\infty(X,\mu) \to B(L^2(X,\mu))$ is the map defined as above. It is easy to check that if $f, g \in L^\infty(X,\mu)$ and $\lambda \in \mathbb{C}$ then $M(f+g) = M_f + M_g$, $M_{fg} = M_f M_g$ and $M_{\lambda f} = \lambda M_f$.

To see that M preserves involutions, observe that if $f \in L^\infty(X,\mu)$ and $\psi_1, \psi_2 \in L^2(X,\mu)$ then

$$\begin{aligned} \langle M_f(\psi_1), \psi_2\rangle &= \langle f\psi_1, \psi_2\rangle \\ &= \Big(\int_X f\psi_1\overline{\psi_2}\, d\mu\Big)^{\frac{1}{2}} \\ &= \Big(\int_X \psi_1\overline{\overline{f}\psi_2}\, d\mu\Big)^{\frac{1}{2}} \\ &= \langle \psi_1, M_{\overline{f}}(\psi_2)\rangle. \end{aligned}$$

So, $M(\overline{f}) = M_{\overline{f}} = (M_f)^*$ which shows that M preserves involutions. Thus, M is a *-homomorphism. Directly from the definition, we find that M is injective and surjective onto its image. It remains to show that M is isometric.

To show: (a) $\|M_f\| \leq \|f\|_\infty$.

(b) $\|f\|_\infty \leq \|M_f\|$.

(a) Assume that $\psi \in L^2(X, \mu)$. If $c \geq \|f\|_\infty$ then from equation (3.1), $|f(x)| \leq c$ for almost all $x \in X$. So,

$$\|f\psi\|_2^2 = \int_X |f|^2 |\psi|^2 \, d\mu \leq c^2 \|\psi\|_2^2.$$

So,

$$\|M_f\|^2 = \sup_{\|\psi\|_2 = 1} \|M_f(\psi)\|_2^2 \leq \sup_{\|\psi\|_2 = 1} c^2 \|\psi\|_2^2 = c^2.$$

Since $\|M_f\| \leq c$, we can take the infimum over all c satisfying $|f(x)| \leq c$ almost everywhere on both sides to deduce that $\|M_f\| \leq \|f\|_\infty$.

(b) Assume that $c < \|f\|_\infty$. Then, the set

$$Y = (|f| - c)^{-1}([0, \infty)) = \{x \in X \mid |f(x)| \geq c\}$$

has non-zero measure and is conull. Since the measure space (X, μ) is semifinite, there exists a measurable set $\Lambda \subseteq Y$ such that $0 < \mu(\Lambda) < \infty$. The key idea behind this step is that $\mu(Y)$ could be $+\infty$, but $\mu(\Lambda)$ is **guaranteed to be non-zero and finite**. Since $\Lambda \subseteq Y$, $|f| \geq c$ on Λ. Define

$$\phi = \frac{1}{\sqrt{\mu(\Lambda)}} \chi_\Lambda$$

where χ_Λ is the characteristic function on Λ. Then, $\phi \in L^2(X, \mu)$, whose norm is

$$\|\phi\|_2 = \Big(\int_X \Big| \frac{1}{\sqrt{\mu(\Lambda)}} \chi_\Lambda \Big|^2 \, d\mu \Big)^{\frac{1}{2}} = \frac{1}{\sqrt{\mu(\Lambda)}} \Big(\int_\Lambda \, d\mu \Big)^{\frac{1}{2}} = 1.$$

Now,

$$\|f\phi\|_2^2 = \int_X |\frac{1}{\sqrt{\mu(\Lambda)}}|^2 |f\chi_\Lambda|^2 \, d\mu = \frac{1}{\mu(\Lambda)} \int_\Lambda |f|^2 \, d\mu \geq c^2$$

since $|f| \geq c$ on Λ. Hence, $\|M_f\| \geq c$ and by taking the infimum over over all c satisfying $|f(x)| \leq c$ almost everywhere, we obtain $\|M_f\| \geq \|f\|_\infty$.

So, $M : L^\infty(X,\mu) \to B(L^2(X,\mu))$ is an isometric *-isomorphism. □

As with any bounded operator on a Hilbert space, we want to determine the spectrum of a multiplication operator.

Definition 3.1.4. Let (X,μ) be a measure space and $f : X \to \mathbb{C}$ be a measurable function. The **essential range** of f is defined by

$$V_{ess}(f) = \{\lambda \in \mathbb{C} \mid \text{If } U \text{ is an open neighbourhood of } \lambda \text{ then } \mu(f^{-1}(U)) > 0\}.$$

Let us highlight a few properties of the essential range before describing the spectrum of a multiplication operator.

Lemma 3.1.4. *Let (X,μ) be a measure space and $f, g : X \to \mathbb{C}$ be measurable functions. Suppose that $f(x) = g(x)$ for almost all $x \in X$. Then, $V_{ess}(f) = V_{ess}(g)$.*

Proof. Assume that (X,μ) is a measure space and $f, g : X \to \mathbb{C}$ are measurable functions. Assume that $f(x) = g(x)$ for almost all $x \in X$. Then, there exists a measurable set Y such that $f = g$ on Y and $\mu(X \backslash Y) = 0$ (Y is conull).

Note that if $A \subseteq \mathbb{C}$ is a Borel subset then

$$\begin{aligned}
\mu(f^{-1}(A)) &= \mu(f^{-1}(A) \cap Y) + \mu(f^{-1}(A) \cap Y^c) \\
&= \mu(f^{-1}(A) \cap Y) \\
&= \mu(\{x \in Y \mid f(x) \in A\}) \\
&= \mu(\{x \in Y \mid g(x) \in A\}) = \mu(g^{-1}(A)).
\end{aligned}$$

From Definition 3.1.4, we obtain $V_{ess}(f) = V_{ess}(g)$. □

Lemma 3.1.5. *Let (X,μ) be a measure space and $f : X \to \mathbb{C}$ be a measurable function. Then, the essential range $V_{ess}(f)$ is a closed subset of $\mathbb{C}$.*

Proof. Assume that (X, μ) is a measure space and $f : X \to \mathbb{C}$ is a measurable function.

To show: (a) $\overline{V_{ess}(f)} \subseteq V_{ess}(f)$.

(a) Assume that $\tau \in \overline{V_{ess}(f)}$. If $\epsilon \in \mathbb{R}_{>0}$ then $B(\tau, \epsilon) \cap V_{ess}(f) \neq \emptyset$, where $B(\tau, \epsilon)$ is the open ball centred at τ with radius ϵ. If we take $\lambda \in B(\tau, \epsilon) \cap V_{ess}(f)$ then $B(\tau, \epsilon)$ is an open neighbourhood of λ and consequently, $\mu(f^{-1}(B(\tau, \epsilon))) > 0$. Since the open balls form a basis for the metric topology on $\mathbb{C}$, we find that any open neighbourhood U of τ satisfies $U \cap V_{ess}(f) \neq \emptyset$ and subsequently, $\mu(f^{-1}(U)) > 0$. So, $\overline{V_{ess}(f)} \subseteq V_{ess}(f)$ and $V_{ess}(f)$ is a closed subset of $\mathbb{C}$. □

It turns out that the essential range completely characterises the spectrum of a multiplication operator.

Theorem 3.1.6. *Let (X, μ) be a semifinite measure space and $f \in L^\infty(X, \mu)$. Then, the spectrum of the multiplication operator $M_f \in B(L^2(X, \mu))$ is*

$$\sigma(M_f) = V_{ess}(f).$$

Proof. Assume that (X, μ) is a semifinite measure space and $f \in L^\infty(X, \mu)$.

To show: (a) $\sigma(M_f) \subseteq V_{ess}(f)$.

(b) $V_{ess}(f) \subseteq \sigma(M_f)$.

(a) We will prove the contrapositive of this statement. Assume that $\lambda \in \mathbb{C} \backslash V_{ess}(f)$. Then, there exists $r \in \mathbb{R}_{>0}$ such that the set

$$f^{-1}(B(\lambda, r)) = \{x \in X \mid |\lambda - f(x)| < r\}$$

has measure zero. This means that $|\lambda - f(x)| \geq r$ for almost all $x \in X$. Define the function $g : X \to \mathbb{C}$ by

$$g(x) = \frac{1}{\lambda - f(x)}.$$

Then, $|g(x)| \leq \frac{1}{r}$ for almost all $x \in X$. This means that $g \in L^\infty(X, \mu)$ and since g is the inverse of $\lambda - f$, the operator M_g is the inverse of $\lambda I - M_f$ in $B(L^2(X, \mu))$. So, $\lambda \notin \sigma(M_f)$.

(b) Assume that $\lambda \in V_{ess}(f)$. If $n \in \mathbb{Z}_{>0}$ then the set

$$f^{-1}(B(\lambda, \frac{1}{n})) = \{x \in X \mid |\lambda - f(x)| \leq \frac{1}{n}\}$$

has positive measure. Since (X, μ) is semifinite, there exists a measurable set X_n with finite and non-zero measure such that $X_n \subseteq f^{-1}(B(\lambda, \frac{1}{n}))$. So, $|\lambda - f(x)| \leq \frac{1}{n}$ for $x \in X_n$.

Now define $\psi_n = \frac{1}{\sqrt{\mu(X_n)}}\chi_{X_n}$. Then,

$$\|\psi_n\|_2 = \frac{1}{\mu(X_n)} \int_{X_n} d\mu = 1$$

and for all $n \in \mathbb{Z}_{>0}$ and $x \in X$,

$$|(\lambda - f(x))\psi_n(x)| \leq \frac{1}{n}|\psi_n(x)|.$$

So, $\|(\lambda I - M_f)\psi_n\|_2 \leq \frac{1}{n}$. Now suppose for the sake of contradiction that $y \in B(L^2(X, \mu))$ is the inverse of $\lambda I - M_f$. If $n \in \mathbb{Z}_{>0}$ then

$$\begin{aligned}
1 &= \|\psi_n\|_2 \\
&= \|y(\lambda I - M_f)\psi_n\|_2 \\
&\leq \|y\|\|(\lambda I - M_f)\psi_n\|_2 \\
&\leq \frac{\|y\|}{n} \to 0
\end{aligned}$$

as $n \to \infty$. This is a contradiction. So, $\lambda I - M_f$ is not invertible and $\lambda \in \sigma(M_f)$.

By combining parts (a) and (b), we deduce that $\sigma(M_f) = V_{ess}(f)$ as required. □

As a consequence of Theorem 3.1.6, Theorem 2.1.6 and the fact that the multiplication operator M_f is normal,

$$\|M_f\| = |\sigma(M_f)| = \sup\{|\lambda| \mid \lambda \in V_{ess}(f)\}.$$

In the next result, we give conditions for which $\lambda \in \mathbb{C}$ is an eigenvalue of a multiplication operator.

Lemma 3.1.7. *Let (X, μ) be a semifinite measure space and $f \in L^\infty(X, \mu)$. Then, $\lambda \in \mathbb{C}$ is an eigenvalue of the operator $M_f \in B(L^2(X, \mu))$ if and only if there exists a measurable set $Y \subseteq X$ such that $\mu(Y) > 0$ and $f(x) = \lambda$ for almost all $x \in Y$.*

Proof. Assume that (X, μ) is a semifinite measure space and $f \in L^\infty(X, \mu)$.

To show: (a) If there exists a measurable set $Y \subseteq X$ such that $\mu(Y) > 0$ and $f(x) = \lambda$ for almost all $x \in Y$ then $\lambda \in \mathbb{C}$ is an eigenvalue of M_f.

(b) If $\lambda \in \mathbb{C}$ is an eigenvalue of M_f then there exists a measurable set $Y \subseteq X$ such that $\mu(Y) > 0$ and $f(x) = \lambda$ for almost all $x \in Y$.

(a) Assume that $Y \subseteq X$ is a measurable set such that $\mu(Y) > 0$ and $f(x) = \lambda$ for almost all $x \in Y$. Since (X, μ) is a semifinite measure space, there exists a measurable set Y' such that $Y' \subseteq Y$ and $0 < \mu(Y') < \infty$. Define

$$\psi = \frac{1}{\sqrt{\mu(Y')}} \chi_{Y'} \in L^2(X, \mu).$$

Similarly to Theorem 3.1.6, $\|\psi\|_2 = 1$ and

$$M_f \psi = \frac{1}{\sqrt{\mu(Y')}} M_f(\chi_{Y'}) = \frac{1}{\sqrt{\mu(Y')}} f \chi_{Y'} = \lambda \frac{1}{\sqrt{\mu(Y')}} \chi_{Y'} = \lambda \psi$$

because $f(x) = \lambda$ for almost all $x \in Y$.

(b) Now assume that there exists a non-zero $\psi \in L^2(X, \mu)$ such that $M_f \psi = \lambda \psi$. Then, $f\psi = \lambda \psi$ almost everywhere. Define

$$Y = \{x \in X \mid \psi(x) \neq 0\}.$$

Since $f\psi = \lambda\psi$ almost everywhere, $f(x) = \lambda$ for almost all $x \in Y$. Finally, notice that since $\psi \neq 0$, $\mu(Y) > 0$. $\square$

In a similar vein to Theorem 2.2.5, if the multiplication operator M_f is self-adjoint and $g \in Cts(\sigma(M_f), \mathbb{C})$ then the map $g \mapsto M_{g \circ f}$ satisfies the conditions of Theorem 2.2.1. By uniqueness of the continuous functional calculus, we must have $g(M_f) = M_{g \circ f}$.

The next theorem gives us yet another decomposition of a bounded self-adjoint operator on a Hilbert space H. Its proof is very involved. In it, we use the following characterisation of a unitary operator.

Theorem 3.1.8. *Let H be a Hilbert space over $\mathbb{C}$ and $u \in B(H)$. Then, u is a unitary operator if and only if u is a surjective isometry.*

Proof. Assume that $u \in B(H)$.

To show: (a) If u is unitary then u is a surjective isometry.

(b) If u is a surjective isometry then u is a unitary operator.

(a) Assume that $u \in B(H)$ is unitary. Then, $uu^* = u^*u = I$, where $I \in B(H)$ is the identity operator on H. To see that u is surjective, assume that $h \in H$. Then, $h = u(u^*h)$. Hence, u is surjective. Observe that

$$\|u(h)\|^2 = \langle u(h), u(h)\rangle = \langle h, u^*u(h)\rangle = \langle h, h\rangle = \|h\|^2$$

So, u is an isometry.

(b) Assume that u is a surjective isometry. If $h \in H$ then $\|u(h)\|^2 = \|h\|^2$ and by using the inner product on H,

$$\langle u(h), u(h)\rangle = \langle h, u^*u(h)\rangle = \langle h, h\rangle.$$

So, $u^*u = I$. To see that $uu^* = I$, we first use the fact that u is surjective to deduce the existence of $g \in B(H)$ such that $u(g) = h$. Since u is an isometry, $g = u^*u(g) = u^*(h)$ and $uu^*(h) = u(g) = h$. Since $h \in H$ was arbitrary, $uu^* = I$. Hence, u is unitary. $\square$

Theorem 3.1.9. *Let H be a Hilbert space over $\mathbb{C}$ and $x \in B(H)$ be self-adjoint. There exists a semifinite measure space (X, μ), an essentially bounded measurable real-valued function $F \in L^\infty(X, \mu)$ and a unitary operator $u : L^2(X, \mu) \to H$ such that*

$$x = uM_Fu^*.$$

Proof. Assume that $x \in B(H)$. Define

$$A = \{f(x) \mid f \in Cts(\sigma(x), \mathbb{C})\} \subseteq \mathbb{C}.$$

If $\xi \in H$ then we also define

$$A\xi = \{f(x)\xi \mid f \in Cts(\sigma(x), \mathbb{C})\} \subseteq H.$$

Next, define

$$\mathcal{R} = \{\{\xi_i\}_{i\in I} \mid A\xi_i \perp A\xi_j \text{ for any } i, j \in I \text{ with } i \neq j\}$$

We define a partial order on $\mathcal{R}$ by insisting that $\{\xi_i\}_{i\in I} < \{\xi_j\}_{j\in J}$ if and only if $A\xi_i \subset A\xi_j$ as subspaces of H for all $i \in I$ and $j \in J$.

Since $\mathcal{R}$ is a partially ordered set, we can apply Zorn's lemma to obtain a maximal element $\{\xi_j\}_{j\in J}$ of $\mathcal{R}$.

To show: (a) The subspace $span(\bigcup_{j\in J} A\xi_j)$ is dense in H.

(a) Suppose for the sake of contradiction that $span(\bigcup_{j\in J} A\xi_j)$ is not dense in H. Then, there exists a non-zero vector $\xi \in H$ such that $\xi \in \overline{span(\bigcup_{j\in J} A\xi_j)}^{\perp}$. Without loss of generality, we can assume that ξ is a unit vector. If $f, g \in Cts(\sigma(x), \mathbb{C})$ and $j \in J$ then

$$\langle f(x)\xi, g(x)\xi_j\rangle = \langle \xi, f(x)^* g(x)\xi_j\rangle = \langle \xi, (\overline{f}g)(x)\xi_j\rangle = 0.$$

This calculation shows that the family $\{\xi_j\}_{j\in J} \cup \{\xi\}$ is an element of $\mathcal{R}$ which contains $\{\xi_j\}_{j\in J}$. However, this contradicts the fact that $\{\xi_j\}_{j\in J}$ is the maximal element of $\mathcal{R}$. So, $\overline{span(\bigcup_{j\in J} A\xi_j)}^{\perp} = 0$ and consequently, $span(\bigcup_{j\in J} A\xi_j)$ is a dense subspace of H.

Next, we will start building our semifinite measure space. Let $X = J \times \sigma(x)$, where J has the discrete topology. Let $\mathcal{A}$ be the family of subsets $Y \subset X$ such that for each $j \in J$,

$$Y_j = \{\lambda \in \sigma(x) \mid (j, \lambda) \in Y\}$$

is a Borel subset of $\sigma(x)$.

To show: (b) The set $\mathcal{A}$ is a σ-algebra.

(b) To show: (ba) If $A \in \mathcal{A}$ then $A^c \in \mathcal{A}$.

(bb) If $A_i \in \mathcal{A}$ for $i \in \mathbb{Z}_{>0}$ then $\bigcup_{i\in\mathbb{Z}_{>0}} A_i \in \mathcal{A}$.

(ba) Assume that $A \in \mathcal{A}$. Then, for each $j \in J$,

$$A_j = \{\lambda \in \sigma(x) \mid (j, \lambda) \in A\}$$

is a Borel subset of $\sigma(x)$. So, the complement

$$A_j^c = \{\lambda \in \sigma(x) \mid (j, \lambda) \in A^c\}$$

must be a Borel subset of $\sigma(x)$ for every $j \in J$. Therefore, $A^c \in \mathcal{A}$.

(bb) Assume that $A_i \in \mathcal{A}$ for $i \in \mathbb{Z}_{>0}$. Then, for each $j \in J$

$$(A_i)_j = \{\lambda \in \sigma(x) \mid (j, \lambda) \in A_i\}$$

is a Borel subset of $\sigma(x)$. Since the Borel σ-algebra on $\sigma(x)$ is closed under countable unions, we deduce that if $j \in J$ then the set

$$\bigcup_{i \in \mathbb{Z}_{>0}} (A_i)_j = \{\lambda \in \sigma(x) \mid (j, \lambda) \in \bigcup_{i \in \mathbb{Z}_{>0}} A_i\} = (\bigcup_{i \in \mathbb{Z}_{>0}} A_i)_j$$

is a Borel subset of $\sigma(x)$. Thus, $\bigcup_{i \in \mathbb{Z}_{>0}} A_i \in \mathcal{A}$ by the definition of $\mathcal{A}$.

(b) By combining parts (ba) and (bb), we deduce that $\mathcal{A}$ is a σ-algebra on $X = J \times \sigma(x)$.

Recall that the spectrum $\sigma(x)$ is a non-empty compact subset of $\mathbb{C}$. Hence, with the subspace topology from $\mathbb{C}$, it must be a LCH space. Since we have put the discrete topology on J, it must also be a LCH space. This is because if $j \in J$ then j is contained in the open set $\{j\}$. But, $\{j\}$ is also compact in J. Thus, $j \in \{j\} \subseteq \{j\}$ and J must be a LCH space. Subsequently, the product $J \times \sigma(x)$ is a LCH space.

For $j \in J$, define the map

$$\begin{array}{rccc} \phi_j : & Cts(\sigma(x), \mathbb{C}) & \to & \mathbb{C} \\ & f & \mapsto & \langle f(x)\xi_j, \xi_j \rangle \end{array}$$

Observe that ϕ_j is a positive linear functional. To see why this is the case, suppose that $f \in Cts(\sigma(x), \mathbb{C})$ such that $f(\lambda) \in \mathbb{R}_{\geq 0}$ for all $\lambda \in \sigma(x)$. By Theorem 2.2.3,

$$\sigma(f(x)) = f(\sigma(x)) = \{f(\lambda) \mid \lambda \in \sigma(x)\} \subseteq \mathbb{R}_{\geq 0}.$$

So, $f(x) \in B(H)_+$ is a positive operator. By Theorem 2.3.4, $\phi_j(f) = \langle f(x)\xi_j, \xi_j \rangle \geq 0$. So, ϕ_j must be a positive linear functional.

By the Riesz-Markov-Kakutani theorem (sometimes called the Riesz representation theorem), there exists a unique positive Radon measure μ_j on $\sigma(x)$ such that

$$\langle f(x)\xi_j, \xi_j \rangle = \int_{\sigma(x)} f \, d\mu_j.$$

Next, we define the measure $\mu : X \to [0, \infty]$ by

$$\mu(Y) = \sum_{j \in J} \mu_j(Y_j).$$

To see that this is a semifinite measure, notice that if $j \in J$ then μ_j is a finite measure. The function $h : \sigma(x) \to \mathbb{C}$ which maps λ to 1 is a continuous function from $\sigma(x)$ to $\mathbb{C}$. Hence,

$$\langle \xi_j, h(x)\xi_j \rangle = \int_{\sigma(x)} h(\lambda)\ d\mu_j(\lambda) = \int_{\sigma(x)} d\mu_j(x) = \mu_j(\sigma(x)) < \infty.$$

Consequently, μ is a semifinite measure and (X, μ) is a semifinite measure space.

It remains to define the operator $u \in B(L^2(X, \mu), H)$. We will take advantage of the fact that the space $C_c(X)$ of complex-valued continuous functions on X with compact support is dense in $L^2(X, \mu)$. Define the operator

$$\begin{array}{rccc} u : & C_c(X) & \to & H \\ & f & \mapsto & \sum_{j \in J} f(j, x)\xi_j \end{array}$$

Since f is compactly supported, the sum is finite. If $f \in C_c(X)$ then we compute directly that

$$\begin{aligned}
\|uf\|^2 &= \|\sum_{j \in J} f(j, x)\xi_j\|^2 \\
&= \langle \sum_{j \in J} f(j, x)\xi_j, \sum_{i \in J} f(i, x)\xi_i \rangle \\
&= \sum_{i,j \in J} \langle f(j, x)\xi_j, f(i, x)\xi_i \rangle \\
&= \sum_{j \in J} \langle f(j, x)\xi_j, f(j, x)\xi_j \rangle \quad (\{\xi_j\}_{j \in J} \text{ is a family of orthogonal vectors}) \\
&= \sum_{j \in J} \langle \xi_j, f(j, x)^* f(j, x)\xi_j \rangle \\
&= \sum_{j \in J} \langle \xi_j, |f|^2(j, x)\xi_j \rangle \\
&= \sum_{j \in J} \int_{\sigma(x)} |f|^2(j, \lambda)\ d\mu_j(\lambda) = \int_X |f|^2\ d\mu = \|f\|_2^2.
\end{aligned}$$

Since $C_c(X)$ is dense in $L^2(X, \mu)$, the above calculation shows that u extends to an isometry from $L^2(X, \mu)$ to H. Furthermore, the range of u contains the set $\bigcup_{j \in J} A\xi_j$, which is dense in H. Hence, u is surjective, preserves inner products and is thus, unitary by Theorem 3.1.8.

Now define $F : X \to \mathbb{C}$ by $F(j, \lambda) = \lambda$. This is a bounded and continuous function (and subsequently, a measurable function). It is also real-valued because x is self-adjoint and $\sigma(x) \subseteq \mathbb{R}$. Observe that if $f \in C_c(X)$ then

$$\begin{aligned} u(M_F f) &= u(Ff) \\ &= \sum_{j \in J} (Ff)(j, x)\xi_j \\ &= \sum_{j \in J} xf(j, x)\xi_j = x \sum_{j \in J} f(j, x)\xi_j = xuf. \end{aligned}$$

Since x and u are bounded/continuous operator and f is a continuous function, we have $uM_F\psi = xu\psi$ for $\psi \in L^2(X, \mu)$. Thus, $uM_F u^* = x$ as required. □

In light of Theorem 3.1.9, we make the following definition.

Definition 3.1.5. Let H_1 and H_2 be Hilbert spaces over $\mathbb{C}$, $x \in B(H_1)$ and $y \in B(H_2)$. We say that x and y are **unitarily equivalent** if there exists a unitary operator $u \in B(H_2, H_1)$ such that $x = uyu^*$.

Theorem 3.1.9 tells us that any self-adjoint operator $x \in B(H)$ is unitarily equivalent to a multiplication operator defined on an appropriate semifinite measure space. The main draw of unitarily equivalent operators is demonstrated by the following theorem:

Theorem 3.1.10. *Let H_1 and H_2 be Hilbert spaces over $\mathbb{C}$, $x \in B(H_1)$ and $y \in B(H_2)$. Suppose that x and y are unitarily equivalent. Then, $\sigma(x) = \sigma(y)$ and if $f \in Cts(\sigma(x), \mathbb{C})$ then $f(x) = uf(y)u^*$.*

Proof. Assume that $x \in B(H_1)$ and $y \in B(H_2)$ are unitarily equivalent operators. Then, there exists $u \in B(H_2, H_1)$ such that $x = uyu^*$. Observe that if $\lambda \in \mathbb{C}$ then

$$\lambda I - x = \lambda uu^* - uyu^* = u(\lambda I - y)u^*.$$

So, $\lambda I - x$ is invertible if and only if $\lambda I - y$ is invertible, revealing that $\sigma(x) = \sigma(y)$.

Now assume that $f \in Cts(\sigma(x), \mathbb{C})$. Consider the map

$$\begin{array}{rrcl} \phi: & Cts(\sigma(x), \mathbb{C}) = Cts(\sigma(y), \mathbb{C}) & \to & B(H) \\ & f & \mapsto & uf(y)u^* \end{array}$$

Note that ϕ is an isometry because by Theorem 3.1.8,

$$\begin{aligned} \|f\|_\infty^2 &= \|f(y)\|^2 \\ &= \sup_{\|z\|=1} \|f(y)(z)\|^2 \\ &= \sup_{\|z\|=1} \langle f(y)(z), f(y)(z) \rangle \\ &= \sup_{\|z\|=1} \langle (f(y)u^*)(z), (f(y)u^*)(z) \rangle \quad \text{(since } u^* \text{ is unitary and hence, surjective)} \\ &= \sup_{\|z\|=1} \langle (uf(y)u^*)(z), (uf(y)u^*)(z) \rangle \quad \text{(since } u \text{ is unitary)} \\ &= \|uf(y)u^*\|^2. \end{aligned}$$

Furthermore, if $f(\lambda) = \sum_{i=0}^n a_i \lambda^i$ is a polynomial function on $\sigma(x)$ then

$$uf(y)u^* = \sum_{i=0}^n a_i(uy^iu^*) = \sum_{i=0}^n a_i x^i = f(x)$$

By the uniqueness of the continuous functional calculus (see Theorem 2.2.1), ϕ must be the map $f \mapsto f(x)$. Thus, $f(x) = uf(y)u^*$. $\square$

3.2 Borel functional calculus

With Theorem 3.1.9, we will extend the continuous functional calculus (see Theorem 2.2.1) to bounded Borel functions on the spectrum of a bounded operator on a Hilbert space. Recall the definition of a Borel (measurable) function.

Definition 3.2.1. Let X and Y be topological spaces and $f : X \to Y$ be a function. Let σ_X and σ_Y be the σ-algebras generated by the open sets of X and the open sets of Y respectively. We say that f is a **Borel function** if for $V \in \sigma_Y$, the preimage $f^{-1}(V) \in \sigma_X$.

In the proof of the Borel functional calculus, we will use Dynkin's theorem, a well-known result in measure theory, which we will now set up and prove, following the exposition in [Sol18, Section A.2].

Definition 3.2.2. Let X be a set and $\mathcal{F} \subseteq \mathcal{P}(X)$. We say that $\mathcal{F}$ is a **π-system** if $\mathcal{F}$ is closed under finite intersections.

Definition 3.2.3. Let X be a set and $\mathcal{G} \subseteq \mathcal{P}(X)$. We say that $\mathcal{G}$ is a **λ-system** if $X \in \mathcal{G}$, $\mathcal{G}$ is closed under complements and if $\{\Delta_n\}_{n\in\mathbb{Z}_{>0}}$ is a sequence of pairwise disjoint elements in $\mathcal{G}$ then $\bigcup_{n=1}^{\infty} \Delta_n \in \mathcal{G}$.

Before we prove Dynkin's theorem, we will connect π-systems and λ-systems to σ-algebras.

Lemma 3.2.1. *Let X be a set and $\mathcal{F} \subseteq \mathcal{P}(X)$ be a λ-system. Then, $\mathcal{F}$ is a σ-algebra if and only if $\mathcal{F}$ is a π-system.*

Proof. Assume that X is a set and $\mathcal{F} \subseteq \mathcal{P}(X)$ is a λ-system.

To show: (a) If $\mathcal{F}$ is a σ-algebra then $\mathcal{F}$ is a π-system.

(b) If $\mathcal{F}$ is a π-system then $\mathcal{F}$ is a σ-algebra.

(a) Assume that $\mathcal{F}$ is a σ-algebra. Then, $\mathcal{F}$ is closed under countable intersections and hence, finite intersections. So, $\mathcal{F}$ is a π-system.

(b) Assume that $\mathcal{F}$ is a π-system so that it is closed under finite intersections. Since $\mathcal{F}$ is also a λ-system, it is closed under countable pairwise disjoint unions. Assume that $\{\Delta_n\}_{n\in\mathbb{Z}_{>0}}$ is a sequence of sets in $\mathcal{F}$. If $n \in \mathbb{Z}_{>0}$ then the set

$$\tilde{\Delta}_n = \Delta_n \backslash (\bigcup_{i=1}^{n-1} \Delta_i) = \Delta_n \cap (\bigcap_{i=1}^{n-1} \Delta_i)^c \in \mathcal{F}.$$

Since $\mathcal{F}$ is closed under countable pairwise disjoint unions,

$$\bigcup_{i=1}^{\infty} \Delta_i = \bigcup_{i=1}^{\infty} \tilde{\Delta}_i \in \mathcal{F}.$$

Hence, $\mathcal{F}$ is closed under countable unions and is thus, a σ-algebra. $\square$

Now we will state and prove Dynkin's theorem.

Theorem 3.2.2 (Dynkin)**.** *Let X be a set, $\mathcal{P} \subseteq \mathcal{P}(X)$ be a π-system and $\mathcal{L} \subseteq \mathcal{P}(X)$ be a λ-system, with $\mathcal{P} \subseteq \mathcal{L}$. Then, the σ-algebra generated by $\mathcal{P}$ is contained in $\mathcal{L}$.*

Proof. Assume that X is a set, $\mathcal{P}$ is a π-system and $\mathcal{L}$ is a λ-system. Assume that $\mathcal{P} \subseteq \mathcal{L}$.

Let $\lambda(\mathcal{P})$ be the smallest λ-system containing $\mathcal{P}$ and $\sigma(\mathcal{P})$ the smallest σ-algebra containing $\mathcal{P}$. Since $\mathcal{L}$ is a λ-system containing $\mathcal{P}$, $\lambda(\mathcal{P}) \subseteq \mathcal{L}$ by definition.

To show: (a) $\lambda(\mathcal{P}) = \sigma(\mathcal{P})$.

(a) Since every σ-algebra is a λ-system, $\lambda(\mathcal{P}) \subseteq \sigma(\mathcal{P})$. It remains to show that $\sigma(\mathcal{P}) \subseteq \lambda(\mathcal{P})$.

To show: (aa) $\lambda(\mathcal{P})$ is a π-system.

(aa) Assume that $\Delta \in \lambda(\mathcal{P})$ and define

$$\mathcal{L}_\Delta = \{Y \subseteq X \mid Y \cap \Delta \in \lambda(\mathcal{P})\}.$$

We will show that $\mathcal{L}_\Delta$ is a λ-system. First, we have $X \in \mathcal{L}_\Delta$ because $X \cap \Delta = \Delta \in \lambda(\mathcal{P})$.

Assume that $A \in \mathcal{L}_\Delta$ so that $A \cap \Delta \in \lambda(\mathcal{P})$. We compute directly that

$$\begin{aligned} A^c \cap \Delta &= (A^c \cup \Delta^c) \cap \Delta \\ &= (A \cap \Delta)^c \cap \Delta \\ &= ((A \cap \Delta) \cup \Delta^c)^c \in \lambda(\mathcal{P}) \end{aligned}$$

because $\lambda(\mathcal{P})$ is closed under disjoint unions and complements.

Let $\{A_i\}_{i \in \mathbb{Z}_{>0}}$ be a sequence of pairwise disjoint elements in $\mathcal{L}_\Delta$. Then,

$$(\bigcup_{i=1}^{\infty} A_i) \cap \Delta = \bigcup_{i=1}^{\infty} (A_i \cap \Delta) \in \lambda(\mathcal{P})$$

since $\lambda(\mathcal{P})$ is closed under countable disjoint unions and $A_i \cap \Delta$ is disjoint from $A_j \cap \Delta$ whenever $i \neq j$.

Thus, $\mathcal{L}_\Delta$ is a λ-system.

To show: (ab) If $\Gamma \in \mathcal{P}$ then $\mathcal{P} \subseteq \mathcal{L}_\Gamma$.

(ab) Assume that $\Gamma \in \mathcal{P}$. If $\Delta \in \mathcal{P}$ then $\Delta \cap \Gamma \in \mathcal{P} \subseteq \lambda(\mathcal{P})$ since $\mathcal{P}$ is closed under finite intersections. Hence, $\mathcal{P} \subseteq \mathcal{L}_\Gamma$.

(aa) Since $\mathcal{L}_\Gamma$ is a λ-system containing $\mathcal{P}$, $\lambda(\mathcal{P}) \subseteq \mathcal{L}_\Gamma$, which tells us that if $\Phi \in \lambda(\mathcal{P})$ and $\Gamma \in \mathcal{P}$ then $\Phi \cap \Gamma \in \lambda(\mathcal{P})$. Therefore, $\mathcal{P} \subseteq \mathcal{L}_\Phi$.

Now since $\mathcal{L}_\Phi$ is a λ-system containing $\mathcal{P}$, $\lambda(\mathcal{P}) \subseteq \mathcal{L}_\Phi$. Since this holds for any $\Phi \in \lambda(\mathcal{P})$, we deduce that $\lambda(\mathcal{P})$ is closed under finite intersections and is thus, a π-system.

Now since $\lambda(\mathcal{P})$ is both a λ-system and a π-system, it must be a σ-algebra which contains $\mathcal{P}$ by Lemma 3.2.1. So, $\sigma(\mathcal{P}) \subseteq \lambda(\mathcal{P})$, thereby showing that $\sigma(\mathcal{P}) = \lambda(\mathcal{P})$. Since the λ-system $\mathcal{L}$ contains $\mathcal{P}$,

$$\sigma(\mathcal{P}) = \lambda(\mathcal{P}) \subseteq \mathcal{L}.$$

□

A key ingredient we will need in the proof of Borel functional calculus is concept of a contraction mapping.

Definition 3.2.4. Let $(V_1, \|-\|_1)$ and $(V_2, \|-\|_2$ be normed vector spaces. We say that $\Phi : V_1 \to V_2$ is a **contraction** if for all $v_1 \in V_1$, $\|\Phi(v_1)\|_2 \leq \|v_1\|_1$.

Lemma 3.2.3. *Let H be a Hilbert space over $\mathbb{C}$ and B be a C*-algebra with unit 1_B. Let $\Phi : B \to B(H)$ be a unital *-homomorphism (unital means that $\Phi(1_B) = I$, where $I \in B(H)$ is the identity operator). Then, Φ is a contraction.*

Proof. Assume that H is a Hilbert space over $\mathbb{C}$ and B is a C*-algebra with unit 1_B. Assume that $\Phi : B \to B(H)$ is a unital *-homomorphism.

Let $b \in B$ and $\lambda \in \mathbb{C}$ such that $\lambda 1_B - b$ is invertible in B. By applying Φ, we deduce that $\lambda 1_B - \Phi(b)$ is an invertible operator in $B(H)$. By replacing b with b^*b, we find that

$$\sigma(\Phi(b^*b)) \subseteq \{\lambda \in \mathbb{C} \mid \lambda 1_B - b^*b \text{ is not invertible in } B\}$$

We need to prove an intermediate result before continuing. Let

$$S = \sup\{|\lambda| \mid \lambda 1_B - b^*b \text{ is not invertible in } B\}$$

To show: (a) $S \leq \|b^*b\|$.

(a) Suppose for the sake of contradiction that there exists

$$\lambda \in \{|\lambda| \mid \lambda 1_B - b^*b \text{ is not invertible in } B\}$$

such that $|\lambda| > \|b^*b\|$. Consider the following series in B:

$$\sum_{n=0}^{\infty} \lambda^{-n-1}(b^*b)^n.$$

It converges because

$$\begin{aligned}\|\sum_{n=0}^{\infty} \lambda^{-n-1}(b^*b)^n\| &\leq \sum_{n=0}^{\infty} \frac{\|b^*b\|^n}{|\lambda|^{n+1}} \\ &= \sum_{n=0}^{\infty} \frac{1}{|\lambda|}\frac{\|b^*b\|^n}{|\lambda|^n} < \infty\end{aligned}$$

which holds because $\frac{\|b^*b\|^n}{|\lambda|^n} < 1$ for $n \in \mathbb{Z}_{>0}$. Furthermore, a direct computation reveals that

$$(\lambda 1_B - b^*b)^{-1} = \sum_{n=0}^{\infty} \lambda^{-n-1}(b^*b)^n.$$

But this contradicts the assumption that $\lambda 1_B - b^*b$ is not invertible. So, $|\lambda| \leq \|b^*b\|$ and by taking the supremum, we deduce that $S \leq \|b^*b\|$.

Now we argue that for $b \in B$,

$$\begin{aligned}\|\Phi(b)\|^2 &= \|\Phi(b)^*\Phi(b)\| \\ &= \|\Phi(b^*b)\| = |\sigma(\Phi(b^*b))| \quad \text{(Theorem 2.1.6)} \\ &\leq S \leq \|b^*b\| = \|b\|^2.\end{aligned}$$

Think of the inequality $|\sigma(\Phi(b^*b))| \leq S$ as taking the spectral radii of both sides of the inclusion

$$\sigma(\Phi(b^*b)) \subseteq \{\lambda \in \mathbb{C} \mid \lambda 1_B - b^*b \text{ is not invertible in } B\}$$

Hence, $\|\Phi(b)\| \leq \|b\|$ and so, Φ is a contraction. □

The main point here is that if X is a compact Hausdorff space then the space of bounded Borel functions $Bor(X, \mathbb{C})$ is a C*-algebra in the same way as $Cts(X, \mathbb{C})$, referring back to Example 1.1.1. Here is the statement of Borel functional calculus.

Theorem 3.2.4 (Borel functional calculus)**.** *Let H be a Hilbert space over $\mathbb{C}$ and $x \in B(H)$ be a self-adjoint operator. Let $Bor(\sigma(x), \mathbb{C})$ denote the C^*-algebra of bounded Borel functions on $\sigma(x)$ (recall that $\sigma(x)$ is a compact subset of $\mathbb{C}$). Then, there exists a unique unital *-homomorphism*

$$\begin{array}{rccc} \Lambda_B : & Bor(\sigma(x), \mathbb{C}) & \to & B(H) \\ & f & \mapsto & f(x) \end{array}$$

such that

1. *If for all $\lambda \in \sigma(x)$, $f(\lambda) = \lambda$ then $f(x) = x$.*
2. *If $\{f_n\}_{n \in \mathbb{Z}_{>0}}$ is a uniformly bounded sequence in $Bor(\sigma(x), \mathbb{C})$ converging pointwise to $f : \sigma(x) \to \mathbb{C}$ then $f_n(x) \to f(x)$ as $n \to \infty$ in the strong topology.*
3. *The restriction $\Lambda_B|_{Cts(\sigma(x), \mathbb{C})} = \Lambda$, where Λ is the isomorphism in Theorem 2.2.1.*

Proof. Assume that H is a Hilbert space over $\mathbb{C}$ and $x \in B(H)$ is a self-adjoint operator. By Theorem 3.1.9, there exists a semifinite measure space (X, μ), $F \in L^\infty(X, \mu)$ and a unitary operator $u : L^2(X, \mu) \to H$ such that $x = uM_F u^*$. Notice that F is a Borel/measurable function on the measure space (X, μ).

By construction in Theorem 3.1.9, the range of F is $\sigma(x)$. If $f \in Bor(\sigma(x), \mathbb{C})$ then the composite $f \circ F$ is a bounded Borel/measurable function (with respect to the Borel σ-algebra on X). It is bounded because

$$\begin{aligned} \|f \circ F\| &\leq \|f\| \|F\| \\ &\leq \|f\| \sup_{(j,\lambda) \in X} |F(j, \lambda)| \\ &= \|f\| \sup_{\lambda \in \sigma(x)} |\lambda| \\ &= \|f\| |\sigma(x)| \leq \|f\| \|x\| < \infty. \end{aligned}$$

Thus, we define

$$f(x) = uM_{f \circ F} u^*.$$

We also define Λ_B by

$$\begin{array}{rccc} \Lambda_B : & Bor(\sigma(x), \mathbb{C}) & \to & B(H) \\ & f & \mapsto & f(x) = uM_{f \circ F} u^* \end{array}$$

It is straightforward to verify that Λ_B is a unital *-homomorphism. If $f(\lambda) = \lambda$ for all $\lambda \in \sigma(x)$ and $\xi \in H$ then

$$f(x)(\xi) = uM_{f\circ F}u^*(\xi) = u((f \circ F)u^*(\xi)) = u(Fu^*(\xi)) = uM_F u^*(\xi)$$

So, $f(x) = x$.

Assume that $\{f_n\}_{n\in\mathbb{Z}_{>0}}$ is a uniformly bounded sequence in $Bor(\sigma(x), \mathbb{C})$ which converges pointwise to f. Again, let $\xi \in H$ and set $\psi = u^*(\xi) \in L^2(X, \mu)$. Then,

$$\begin{aligned}
\|f_n(x)\xi - f(x)\xi\|^2 &= \|uM_{(f_n-f)\circ F}u^*\xi\|^2 \\
&= \|uM_{(f_n-f)\circ F}\psi\|^2 \\
&= \|M_{(f_n-f)\circ F}\psi\|^2 \quad \text{(since } u \text{ is isometric)} \\
&= \int_X |f_n(F(w)) - f(F(w))|^2 |\psi(w)|^2 \, d\mu(w) \\
&\to 0.
\end{aligned}$$

In the last step, we applied the dominated convergence theorem to the sequence of bounded Borel functions $\{f_n\}_{n\in\mathbb{Z}_{>0}}$. So, $\{f_n(x)\}$ must converge to $f(x)$ in the strong topology.

Note that Λ_B is also a contraction because

$$\begin{aligned}
\|f(x)\| &= \|uM_{f\circ f}u^*\| \\
&= \|M_{f\circ F}\| \\
&= \|f \circ F\|_\infty \quad \text{(Theorem 3.1.3)} \\
&= \inf_{Y\in\mathcal{C}} \sup_{y\in Y} |(f \circ F)(y)| \\
&\leq \inf_{Y\in\mathcal{C}} \sup_{y\in Y} |f(y)| = \|f\|_\infty.
\end{aligned}$$

Here, $\mathcal{C}$ is the set of conull sets in X and we used Lemma 3.1.1. Since Λ_B is a contractive unital *-homomorphism which maps the identity function to x and $Cts(\sigma(x), \mathbb{C}) \subseteq Bor(\sigma(x), \mathbb{C})$, we can use the uniqueness of the continuous functional calculus (see Theorem 2.2.1) to deduce that $\Lambda_B|_{Cts(\sigma(x),\mathbb{C})} = \Lambda$. $\square$

The uniqueness of the Borel functional calculus is established in a separate proof.

Theorem 3.2.5 (Uniqueness of Borel functional calculus). *The unital *-homomorphism $\Lambda_B : Bor(\sigma(x), \mathbb{C}) \to B(H)$ is unique.*

Proof. Suppose that $\Phi : Bor(\sigma(x), \mathbb{C}) \to B(H)$ is another unital *-homomorphism which satisfies the conditions of Theorem 3.2.4. Then, by Lemma 3.2.3, Φ must be contractive. By exploiting the uniqueness of the continuous functional calculus, we find that $\Phi|_{Cts(\sigma(x),\mathbb{C})} = \Lambda$, where Λ is the isomorphism in Theorem 2.2.1.

Let $\Delta \subseteq \sigma(x)$ be an open subset of $\sigma(x)$ and χ_Δ be its characteristic function. Choose a bounded sequence of continuous functions $\{f_n\}_{n\in\mathbb{Z}_{>0}}$ such that f_n converges pointwise to χ_Δ. Recall that χ_Δ is a simple function and hence, a Borel function from $\sigma(x)$ to $\mathbb{C}$. If $\xi \in H$ then

$$\chi_\Delta(x)\xi = \lim_{n\to\infty} f_n(x)\xi = \lim_{n\to\infty} \Phi(f_n)\xi = \Phi(\chi_\Delta)\xi$$

since $f_n(x) = \Phi(f_n)$ converges to $\chi_\Delta(x) = \Phi(\chi_\Delta)$ in the strong topology. Let $\mathcal{B}_{\sigma(x)}$ denote the Borel σ-algebra on $\sigma(x)$. Now define

$$\mathcal{L} = \{\Delta \in \mathcal{B}_{\sigma(x)} \mid \Phi(\chi_\Delta) = \chi_\Delta(x)\}$$

Notice that from the previous calculation, if Δ is an open set contained in $\sigma(x)$ then $\Delta \in \mathcal{L}$. The key observation here is that $\mathcal{L}$ is a λ-system.

To show: (a) $\mathcal{L}$ is a λ-system.

(a) Since $\sigma(x)$ is an open subset of $\sigma(x)$ with respect to the subspace topology on $\sigma(x)$, we must have $\sigma(x) \in \mathcal{L}$.

Now assume that $\Gamma \in \mathcal{L}$. Then,

$$\Phi(\chi_{\Gamma^c}) = \Phi(\chi_{\sigma(x)} - \chi_\Gamma) = \Phi(id_{\sigma(x)}) - \Phi(\chi_\Gamma) = I - \chi_\Gamma(x) = \chi_{\Gamma^c}(x).$$

So, $\Gamma^c \in \mathcal{L}$.

Finally assume that $\{\Delta_n\}$ is a sequence of pairwise disjoint element of $\mathcal{L}$. Define $D = \bigcup_{n=1}^\infty \Delta_n$. Then,

$$\chi_D = \sum_{n=1}^\infty \chi_{\Delta_n} = \lim_{N\to\infty} \sum_{n=1}^N \chi_{\Delta_n}.$$

Taking the sequence $\{\sum_{n=1}^N \chi_{\Delta_n}\}_{N\in\mathbb{Z}_{>0}}$ of functions in $Bor(\sigma(x), \mathbb{C})$, we have for $\xi \in H$

$$\Phi(\chi_D)\xi = \lim_{N\to\infty}\sum_{n=1}^{N}\Phi(\chi_{\Delta_n})\xi = \lim_{N\to\infty}\sum_{n=1}^{N}\chi_{\Delta_n}(x)\xi = \chi_D(x)\xi.$$

So, $\Phi(\chi_D) = \chi_D(x)$ and $D \in \mathcal{L}$. Hence, $\mathcal{L}$ is a λ-system.

Since the family of all open subsets of $\sigma(x)$ is a π-system, we can apply Dynkin's theorem (see Theorem 3.2.2) to deduce that $\mathcal{B}_{\sigma(x)} \subseteq \mathcal{L}$. Therefore, if $\Gamma \in \mathcal{B}_{\sigma(x)}$ then $\Phi(\chi_\Gamma) = \chi_\Gamma(x)$.

Recall that the simple functions from $\sigma(x)$ to $\mathbb{C}$ are finite $\mathbb{C}$-linear combinations of characteristic functions on Borel subsets of $\sigma(x)$. Since $\mathcal{B}_{\sigma(x)} \subseteq \mathcal{L}$ and Φ is linear,

$$\Phi(s) = s(x)$$

where $s : \sigma(x) \to \mathbb{C}$ is a simple function. The simple functions form a dense subset of $Bor(\sigma(x), \mathbb{C})$. Let $G \in Bor(\sigma(x), \mathbb{C})$. Then, there exists a bounded sequence of simple functions $\{\phi_n\}_{n\in\mathbb{Z}_{>0}}$ such that ϕ_n converges to G pointwise. Therefore, if $\xi \in H$ then

$$\begin{aligned}\Phi(G)\xi &= \Phi(\lim_{n\to\infty}\phi_n)\xi \\ &= \lim_{n\to\infty}\Phi(\phi_n)\xi \\ &= \lim_{n\to\infty}\phi_n(x)\xi = G(x)\xi = \Lambda_B(G)\xi.\end{aligned}$$

So, $\Phi = \Lambda_B$ as required. □

To top off this section, we will observe that the Borel functional calculus satisfies similar results to the continuous functional calculus. One of the most useful aspects of the continuous functional calculus is the spectral mapping theorem (see Theorem 2.2.3). In order to prove a similar statement for Borel functional calculus, we require the following lemma:

Lemma 3.2.6. *Let H be a Hilbert space over $\mathbb{C}$ and $x \in B(H)$ be a self-adjoint operator. Let $f \in Bor(\sigma(x), \mathbb{C})$. Then, $\sigma(f(x)) = V_{ess}(f)$.*

Proof. Assume that H is a Hilbert space over $\mathbb{C}$ and $x \in B(H)$ be a self-adjoint operator. Assume that $f \in Bor(\sigma(x), \mathbb{C})$.

By Theorem 3.2.4 and Theorem 3.1.9, there exists a semifinite measure space (X, μ), $F \in L^\infty(X, \mu)$ and a unitary operator $u : L^2(X, \mu) \to H$ such that

$$x = uM_Fu^* \qquad \text{and} \qquad f(x) = uM_{f\circ F}u^*.$$

By Theorem 3.1.10, $\sigma(f(x)) = \sigma(M_{f\circ F})$ which by Theorem 3.1.6 is

$$V_{ess}(f\circ F) = \{\lambda \in \mathbb{C} \mid \text{If } U \text{ is an open neighbourhood of } \lambda \text{ then } \mu((f\circ F)^{-1}(U)) > 0\}.$$

But, this is equal to

$$V_{ess}(f) = \{\lambda \in \mathbb{C} \mid \text{If } U \text{ is an open neighbourhood of } \lambda \text{ then } m(f^{-1}(U)) > 0\}$$

where m denotes the Lebesgue measure on $\sigma(x)$. This is because the preimage of measurable sets under the Borel functions f and F must be measurable themselves. Hence, $\sigma(f(x)) = V_{ess}(f)$. □

Here is the analogue of the spectral mapping theorem (see Theorem 2.2.3) for the Borel functional calculus.

Theorem 3.2.7. *Let H be a Hilbert space over $\mathbb{C}$ and $x \in B(H)$ be a self-adjoint operator. Let $f \in Bor(\sigma(x), \mathbb{C})$. Then,*

$$\sigma(f(x)) \subseteq \overline{f(\sigma(x))}.$$

Proof. Assume that H is a Hilbert space over $\mathbb{C}$ and $x \in B(H)$ be a self-adjoint operator. Assume that $f \in Bor(\sigma(x), \mathbb{C})$.

To show: (a) $\sigma(f(x)) \subseteq \overline{f(\sigma(x))}$.

(a) We will prove the contrapositive of this statement. Assume that $\lambda \notin \overline{f(\sigma(x))}$. Then, there exists $\epsilon \in \mathbb{R}_{>0}$ such that the ball

$$B(\lambda, \epsilon) = \{z \in \mathbb{C} \mid |z - \lambda| < \epsilon\}.$$

has empty intersection with $f(\sigma(x))$. This means that the $f(\sigma(x)) \subseteq B(\lambda, \epsilon)^c$. Taking the preimage with respect to f, we deduce that the spectrum $\sigma(x)$ is contained in

$$f^{-1}(B(\lambda, \epsilon)^c) = \{\gamma \in \sigma(x) \mid |f(\gamma) - \lambda| \geq \epsilon\}.$$

Since $\sigma(x) \subseteq f^{-1}(B(\lambda, \epsilon)^c)$, the set

$$\sigma(x) \backslash f^{-1}(B(\lambda, \epsilon)^c) = \{\gamma \in \sigma(x) \mid |f(\gamma) - \lambda| < \epsilon\}$$

must be a null set because it is empty. This demonstrates that $\lambda \notin V_{ess}(f) = \sigma(f(x))$ by Lemma 3.2.6. Therefore, $\sigma(f(x)) \subseteq \overline{f(\sigma(x))}$. □

We will also prove an analogue of Theorem 2.2.5 for Borel functional calculus.

Theorem 3.2.8. *Let H be a Hilbert space over $\mathbb{C}$ and $x \in B(H)$ be self-adjoint. If $g \in Bor(\sigma(x), \mathbb{C})$ is real-valued, then the operator $g(x)$ is self-adjoint and if $f \in Bor(g(\sigma(x)), \mathbb{C})$, then $f(g(x)) = (f \circ g)(x)$.*

Proof. Assume that H is a Hilbert space over $\mathbb{C}$ and $x \in B(H)$ is a self-adjoint operator. Assume that $g \in Bor(\sigma(x), \mathbb{C})$. Then, we repeat the argument in Theorem 2.2.5 to deduce that the operator $g(x)$ is self-adjoint.

Assume that $f \in Bor(g(\sigma(x)), \mathbb{C})$ and define

$$\begin{array}{rccc} \Omega_B: & Bor(g(\sigma(x)), \mathbb{C}) & \to & B(H) \\ & f & \mapsto & (f \circ g)(x). \end{array}$$

We must show that Ω_B satisfies the properties of Theorem 3.2.4.

Firstly, the restriction of Ω_B to $Cts(\sigma(x), \mathbb{C})$ is the map Ω in Theorem 2.2.5, which we know is the *-isomorphism defining continuous functional calculus by Theorem 2.2.5. Hence, Ω_B restricts to the continuous functional calculus on continuous functions.

Assume that $h \in Bor(g(\sigma(x)), \mathbb{C})$ such that $h(\lambda) = \lambda$ for $\lambda \in g(\sigma(x))$. Then, $(h \circ g)(x) = g(x)$ by definition of Ω_B.

Now assume that $\{f_n\}_{n \in \mathbb{Z}_{>0}}$ is a uniformly bounded sequence in $Bor(g(\sigma(x)), \mathbb{C})$, which converges pointwise to f. We will show that $(f_n \circ g)(x) \to (f \circ g)(x)$ in the strong topology.

By Theorem 3.1.9, there exists a semifinite measure space (X, μ), $F \in L^\infty(X, \mu)$ and a unitary operator $u : L^2(X, \mu) \to H$ such that $x = uM_F u^*$ so that by Theorem 3.2.4,

$$(f_n \circ g)(x) = uM_{(f_n \circ g) \circ F} u^* \qquad \text{and} \qquad (f \circ g)(x) = uM_{(f \circ g) \circ F} u^*$$

If $\xi \in H$ and $\psi = u^*(\xi) \in L^2(X, \mu)$ then

$$\begin{aligned}
\|(f_n \circ g)(x)\xi - (f \circ g)(x)\xi\|^2 &= \|uM_{(f_n \circ g)\circ F} u^*\xi - uM_{(f\circ g)\circ F} u^*\xi\|^2 \\
&= \|uM_{f_n \circ (g\circ F)} u^*\xi - uM_{f\circ(g\circ F)} u^*\xi\|^2 \\
&= \|uM_{(f_n - f)\circ(g\circ F)} u^*\xi\|^2 \\
&= \|uM_{(f_n - f)\circ(g\circ F)} \psi\|^2 \\
&= \|M_{(f_n - f)\circ(g\circ F)} \psi\|^2 \\
&= \int_X |f_n((g\circ F)(w)) - f((g\circ F)(w))|^2 |\psi(w)|^2 \, d\mu(w) \\
&\to 0
\end{aligned}$$

as $n \to \infty$ because $\{f_n\}$ converges pointwise to f. Therefore, $f_n(x) \to f(x)$ in the strong topology.

Since Ω_B satisfies all three properties in Theorem 3.2.4, we can invoke the uniqueness of the Borel functional calculus to deduce that $\Omega_B = \Phi_B$, where Φ_B is the unital *-homomorphism

$$\begin{array}{rccc}
\Phi_B : & Bor(g(\sigma(x)), \mathbb{C}) & \to & B(H) \\
& f & \mapsto & f(g(x)).
\end{array}$$

Thus, we conclude that $(f \circ g)(x) = f(g(x))$ as required. $\square$

3.3 Spectral measures

In order to generalise the spectral theorem to the scenario of Hilbert spaces, we want to replace the finite linear combinations which appear in the original spectral theorem for finite dimensional vector spaces with an appropriate integer. This leads to the concept of a spectral measure.

Definition 3.3.1. Let X be a set and $\mathcal{A}$ be a σ-algebra on X. Let H be a Hilbert space over $\mathbb{C}$ and $Proj(B(H))$ denote the set of all projection operators in $B(H)$. A **spectral measure** on X is a map $E : \mathcal{A} \to Proj(B(H))$ such that

1. $E(\emptyset) = 0$ and $E(X) = I$, where I is the identity operator on H.
2. If $\Delta_1, \Delta_2 \in \mathcal{A}$ then $E(\Delta_1 \cap \Delta_2) = E(\Delta_1)E(\Delta_2)$.
3. If $\{\Delta_i\}_{i \in \mathbb{Z}_{>0}}$ is a family of pairwise disjoint sets in $\mathcal{A}$ then $E(\bigcup_{n=1}^{\infty} \Delta_n) = \sum_{i=1}^{\infty} E(\Delta_i)$.

Here is one particular consequence of the first two properties of a spectral measure in Definition 3.3.1. Suppose that $\Delta_1, \Delta_2 \in \mathcal{A}$ are disjoint. Then,

$$E(\Delta_1)E(\Delta_2) = E(\Delta_1 \cap \Delta_2) = E(\emptyset) = 0.$$

So, the projection operators $E(\Delta_1)$ and $E(\Delta_2)$ are orthogonal to each other. More generally, the projection operators associated to disjoint sets in $\mathcal{A}$ must be orthogonal to each other. Because of this, the sum $\sum_{i=1}^{\infty} E(\Delta_i)$ in the third property is still a projection operator.

We also note that the sum $\sum_{i=1}^{\infty} E(\Delta_i)$ is the limit of the finite sums, taken with respect to the strong topology (recall Definition 2.7.1). In other words, if $\xi \in H$ then

$$\lim_{n\to\infty} (\sum_{i=1}^{n} E(\Delta_i))(\xi) = (\sum_{i=1}^{\infty} E(\Delta_i))(\xi).$$

Example 3.3.1. Let $x \in B(H)$ be a self-adjoint operator. Let $X = \sigma(x)$ and $\mathcal{A}$ be the Borel σ-algebra on $\sigma(x)$. The map

$$\begin{array}{rccc} E_x: & \mathcal{A} & \to & Proj(B(H)) \\ & \Delta & \mapsto & \chi_\Delta(x). \end{array}$$

is a spectral measure, where $\chi_\Delta \in Bor(\sigma(x), \mathbb{C})$ denotes the characteristic function on the set Δ.

How do we know that the operator $\chi_\Delta(x)$ is a projection operator? From Theorem 3.2.7,

$$\sigma(\chi_\Delta(x)) \subseteq \overline{\chi_\Delta(\sigma(x))} = \overline{\{0,1\}} = \{0,1\}.$$

By Theorem 3.2.8, since χ_Δ is real-valued, $\chi_\Delta(x)$ is a self-adjoint operator. So, by Theorem 2.4.3, we deduce that $\chi_\Delta(x)$ is a projection operator on H.

By considering the spectrums $\sigma(\chi_\emptyset(x)) \subseteq \{0\}$ and $\sigma(\chi_X(x)) \subseteq \{1\}$, $E_x(\emptyset) = \chi_\emptyset(x) = 0$ and $E_x(X) = \chi_X(x) = I$.

Next, assume that $\Delta_1, \Delta_2 \in \mathcal{A}$. By Theorem 3.2.4, we compute directly that

$$\begin{aligned} E_x(\Delta_1 \cap \Delta_2) &= \chi_{\Delta_1 \cap \Delta_2}(x) \\ &= \chi_{\Delta_1}(\chi_{\Delta_2}(x)) \\ &= (\chi_{\Delta_1} \circ \chi_{\Delta_2})(x) \\ &= E_x(\Delta_1)E_x(\Delta_2). \end{aligned}$$

Finally, assume that $\{\Delta_i\}_{i\in\mathbb{Z}_{>0}}$ is a family of pairwise disjoint sets in $\mathcal{A}$. Then,

$$\begin{aligned}E_x(\bigcup_{n=1}^{\infty}\Delta_n) &= \chi_{\bigcup_{n=1}^{\infty}\Delta_n}(x) \\ &= \sum_{n=1}^{\infty}\chi_{\Delta_n}(x) = \sum_{n=1}^{\infty}E_x(\Delta_n)\end{aligned}$$

where the second last equality follows from the fact that the $\Delta_i \cap \Delta_j = \emptyset$ whenever $i \neq j$. Therefore, E_x does indeed define a spectral measure on $X = \sigma(x)$.

Let E be a spectral measure on X, which has σ-algebra $\mathcal{A}$. If $\xi \in H$ then the map

$$\begin{array}{rccc}\langle\xi|E\xi\rangle : & \mathcal{A} & \to & \mathbb{R} \\ & \Delta & \mapsto & \langle E(\Delta)\xi, \xi\rangle\end{array}$$

is a finite positive measure on X. To see that $\langle\xi|E\xi\rangle$ is a finite positive measure, note that since $E(\Delta)$ is a projection operator, $\sigma(E(\Delta)) \subseteq \{0, 1\}$. So, $E(\Delta)$ is a positive operator and by Theorem 2.3.4, $\langle E(\Delta)\xi, \xi\rangle = \langle \xi, E(\Delta)\xi\rangle \geq 0$ for all $\xi \in H$.

To see that $\langle\xi|E\xi\rangle$ defines a measure, observe that

$$\langle\xi|E\xi\rangle(\emptyset) = \langle E(\emptyset)\xi, \xi\rangle = 0$$

because $E(\emptyset)$ is the zero operator by the definition of spectral measure. Furthermore, if $\{A_i\}_{i\in\mathbb{Z}_{>0}}$ is a collection of pairwise disjoint sets in $\mathcal{A}$ then

$$\begin{aligned}\langle\xi|E\xi\rangle(\bigcup_{i=1}^{\infty}A_i) &= \langle E(\bigcup_{i=1}^{\infty}A_i)\xi, \xi\rangle \\ &= \langle\sum_{i=1}^{\infty}E(A_i)\xi, \xi\rangle \\ &= \sum_{i=1}^{\infty}\langle E(A_i)\xi, \xi\rangle \\ &= \sum_{i=1}^{\infty}\langle\xi|E\xi\rangle(A_i).\end{aligned}$$

Hence, $\langle\xi|E\xi\rangle$ is a finite positive measure on X.

The integral of a function f with respect to the measure $\langle\xi|E\xi\rangle$ is written as

$$\int_X f\ d\langle\xi|E\xi\rangle \qquad \text{or} \qquad \int_X f(w)\ d\langle\xi|E(w)\xi\rangle.$$

Recall the notion of the total variation of a measure.

Definition 3.3.2. Let (X,μ) be a measure space with σ-algebra $\mathcal{A}$. The **total variation** of the measure μ is the quantity

$$\sup\sum_{n=1}^{N}|\mu(\Delta_n)|$$

where the supremum is taken over all finite partitions $\{\Delta_1,\ldots,\Delta_N\}$ of X into pairwise disjoint measurable sets.

By using the polarization formula (see Theorem 2.1.1), we define for $\xi,\eta\in H$

$$\begin{array}{rccc}\langle\xi|E\eta\rangle: & \mathcal{A} & \to & \mathbb{C}\\ & \Delta & \mapsto & \langle E(\Delta)\eta,\xi\rangle\end{array}$$

By the polarization formula, $\langle\xi|E\eta\rangle$ is a complex-valued measure. We also claim that $\langle\xi|E\eta\rangle$ has finite total variation. Suppose that $\{\Delta_1,\ldots,\Delta_n\}$ is a partition of X into pairwise disjoint measurable sets. For $i\in\{1,2,\ldots,n\}$, define

$$\lambda_i=\frac{\overline{\langle E(\Delta_i)\eta,\xi\rangle}}{|\langle E(\Delta_i)\eta,\xi\rangle|}\in\mathbb{C}$$

Then, $|\lambda_i|=1$ and

$$\sum_{i=1}^{n}|\langle\xi|E\eta\rangle(\Delta_i)|=\sum_{i=1}^{n}|\langle E(\Delta_i)\eta,\xi\rangle|=\sum_{i=1}^{n}\lambda_i\langle E(\Delta_i)\eta,\xi\rangle=\langle t(\eta),\xi\rangle$$

where $t=\sum_{i=1}^{n}\lambda E(\Delta_i)$. Now we compute the norm of t as

$$
\begin{aligned}
\|t\|^2 &= \|tt^*\| \\
&= \|\big(\sum_{i=1}^{n} \lambda_i E(\Delta_i)\big)\big(\sum_{j=1}^{n} \overline{\lambda_j} E(\Delta_j)\big)\| \\
&= \|\sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_i \overline{\lambda_j} E(\Delta_i) E(\Delta_j)\| \\
&= \|\sum_{i=1}^{n} |\lambda_i|^2 E(\Delta_i)\| \\
&= \|\sum_{i=1}^{n} E(\Delta_i)\| = \|E(X)\| = \|I\| = 1.
\end{aligned}
$$

Hence, by the Cauchy-Schwarz inequality,

$$
\sum_{i=1}^{n} |\langle \xi | E\eta \rangle(\Delta_i)| = \langle t(\eta), \xi \rangle \leq \|\xi\|\|t\|\|\eta\| = \|\xi\|\|\eta\|.
$$

This demonstrates that the measure $\langle \xi | E\eta \rangle$ has finite total variation. Now, if f is a bounded measurable/Borel function on X then the quantity

$$
F(\xi, \eta) = \int_X f(w)\, d\langle \xi | E(w)\eta \rangle
$$

is a sesquilinear form with respect to the variables ξ and η. It is also bounded by $\|\xi\|\|\eta\|\|f\|_\infty$. By the Riesz representation theorem, we can define a bounded operator $x_f \in B(H)$ such that

$$
\langle x_f \eta, \xi \rangle = \int_X f(w)\, d\langle \xi | E(w)\eta \rangle.
$$

The main theorem of this section concerns the operator x_f.

Theorem 3.3.1. *Let H be a Hilbert space over $\mathbb{C}$ and (X, μ) be a measure space. Define the map*

$$
\begin{array}{rccc}
\psi : & Bor(X, \mathbb{C}) & \to & B(H) \\
 & f & \mapsto & x_f
\end{array}
$$

*Then, ψ is a *-homomorphism from $Bor(X, \mathbb{C})$ to $B(H)$. If $\{f_n\}_{n \in \mathbb{Z}_{>0}}$ is a uniformly bounded sequence in $Bor(X, \mathbb{C})$ converging pointwise to f then $x_{f_n} \to x_f$ as $n \to \infty$ in the strong topology.*

Proof. Assume that H is a Hilbert space over $\mathbb{C}$ and (X, μ) be a measure space. Assume that ψ is the map defined as above. We claim that ψ is a well-defined map into $B(H)$.

To show: (a) If $f \in Bor(X, \mathbb{C})$ then $\|x_f\| \leq \|f\|_\infty$.

(a) Assume that $f \in Bor(X, \mathbb{C})$. By the definition of ψ and the Riesz representation theorem, we have

$$\begin{aligned}
\|x_f\| &= \sup_{\|\xi\|=1} \|x_f \xi\| \\
&= \sup_{\|\eta\|=1} \sup_{\|\xi\|=1} |\langle x_f \xi, \eta \rangle| \qquad \text{(Riesz representation theorem)} \\
&= \sup_{\|\eta\|=1} \sup_{\|\xi\|=1} \left| \int_X f(w)\, d\langle \eta | E(w) \xi \rangle \right| \\
&\leq \sup_{\|\eta\|=1} \sup_{\|\xi\|=1} \sup_{w \in X} |f(w)| \left| \int_X d\langle \eta | E(w) \xi \rangle \right| \\
&\leq \sup_{\|\eta\|=1} \sup_{\|\xi\|=1} (\sup_{w \in X} |f(w)|) \|\xi\| \|\eta\| = \|f\|_\infty.
\end{aligned}$$

Since $\|f\|_\infty < \infty$, we deduce that $x_f \in B(H)$ and the map ψ is well-defined. Next, we will show that ψ is a *-homomorphism.

To show: (b) If $f, g \in Bor(X, \mathbb{C})$ then $x_f x_g = x_{fg}$.

(c) If $f \in Bor(X, \mathbb{C})$ then $x_{\overline{f}} = x_f^*$.

(b) Since the set of simple functions are dense in $Bor(X, \mathbb{C})$, we can assume that f and g are simple functions on X. Write

$$f = \sum_{n=1}^{N} \alpha_n \chi_{\Delta_n} \qquad \text{and} \qquad g = \sum_{m=1}^{M} \beta_m \chi_{\Lambda_m}$$

where $\alpha_n, \beta_m \in \mathbb{C}$ and $\Delta_n, \Lambda_m \in \mathcal{A}$ where $\mathcal{A}$ is the σ-algebra of (X, μ). If $\xi, \eta \in H$ then

$$\begin{aligned}
\langle x_f\eta, \xi\rangle &= \int_X f(w)\ d\langle\xi|E(w)\eta\rangle \\
&= \int_X \sum_{n=1}^{N} \alpha_n \chi_{\Delta_n}(w)\ d\langle\xi|E(w)\eta\rangle \\
&= \sum_{n=1}^{N} \int_{\Delta_n} \alpha_n\ d\langle\xi|E(w)\eta\rangle \\
&= \sum_{n=1}^{N} \alpha_n \int_{\Delta_n} d\langle\xi|E(w)\eta\rangle = \sum_{n=1}^{N} \alpha_n \langle E(\Delta_n)\eta, \xi\rangle \\
&= \langle \sum_{n=1}^{N} \alpha_n E(\Delta_n)\eta, \xi\rangle.
\end{aligned}$$

So, we compute directly that

$$\begin{aligned}
x_f x_g &= (\sum_{n=1}^{N} \alpha_n E(\Delta_n))(\sum_{m=1}^{M} \beta_m E(\Lambda_m)) \\
&= \sum_{m,n} \alpha_n \beta_m E(\Delta_n) E(\Lambda_m) \\
&= \sum_{m,n} \alpha_n \beta_m E(\Delta_n \cap \Lambda_m) = x_{fg}.
\end{aligned}$$

Since the simple functions are dense in $Bor(X, \mathbb{C})$, we deduce that $x_{fg} = x_f x_g$ if $f, g \in Bor(X, \mathbb{C})$.

(c) By similar reasoning to part (b), we can assume that $f = \sum_{n=1}^{N} \alpha_n \chi_{\Delta_n}$. We compute directly that

$$\begin{aligned}
x_{\overline{f}} &= \sum_{n=1}^{N} \overline{\alpha_n} E(\Delta_n) \\
&= (\sum_{n=1}^{N} \alpha_n E(\Delta_n)^*)^* \\
&= (\sum_{n=1}^{N} \alpha_n E(\Delta_n))^* = x_f^*.
\end{aligned}$$

Since the simple functions are dense in $Bor(X, \mathbb{C})$, this result must also hold for all $f \in Bor(X, \mathbb{C})$.

The definition of x_f as an inner product also reveals that ψ respects addition and scalar multiplication. Hence, ψ is a *-homomorphism.

Now assume that $\{f_n\}_{n\in\mathbb{Z}_{>0}}$ is a uniformly bounded sequence of functions which converge pointwise to $f \in Bor(X, \mathbb{C})$.

To show: (d) The sequence $\{x_{f_n}\}$ in $B(H)$ converges to x_f in the strong topology.

(d) Assume that $\xi \in H$. Then,

$$\begin{aligned}
\|x_{f_n}\xi - x_f\xi\|^2 &= \langle (x_{f_n} - x_f)\xi, (x_{f_n} - x_f)\xi\rangle \\
&= \langle x_{f_n - f}\xi, x_{f_n - f}\xi\rangle \\
&= \langle x_{\overline{f_n - f}} x_{f_n - f}\xi, \xi\rangle = \langle x_{|f_n - f|^2}\xi, \xi\rangle \\
&= \int_X |f_n(w) - f(w)|^2 \, d\langle \xi | E(w)\xi\rangle \to 0
\end{aligned}$$

as $n \to \infty$. Here, we have used the dominated convergence theorem (this is allowed because $\{f_n\}$ is uniformly bounded). So, the sequence $\{x_{f_n}\}$ converges to x_f in the strong topology on $B(H)$. □

Theorem 3.3.1 bears a striking resemblance to Theorem 3.2.4, hinting that the Borel functional calculus can be constructing by using an appropriate spectral measure. The following theorem makes this remark precise. To simplify the notation, we will denote the operator x_f by

$$x_f = \int_X f(w) \, dE(w).$$

Theorem 3.3.2. *Let H be a Hilbert space over $\mathbb{C}$ and $x \in B(H)$ be a self-adjoint operator. Let $\mathcal{A}$ be the Borel σ-algebra on the spectrum $\sigma(x)$ and define the spectral measure $E_x : \mathcal{A} \to Proj(B(H))$ by*

$$\begin{array}{rccc} E_x : & \mathcal{A} & \to & Proj(B(H)) \\ & \Delta & \mapsto & \chi_\Delta(x). \end{array}$$

*Let Λ_B be the *-homomorphism from Theorem 3.2.4, which sends $f \in Bor(\sigma(x), \mathbb{C})$ to $f(x) \in B(H)$ and $x_f \in B(H)$ such that if $\eta, \xi \in H$ then*

$$\langle x_f \eta, \xi\rangle = \int_{\sigma(x)} f(w) \, d\langle \xi | E_x(w)\eta\rangle.$$

Let $\psi : Bor(\sigma(x), \mathbb{C}) \to B(H)$ denote the map $f \mapsto x_f$. Then, $\psi = \Lambda_B$ and E_x is the unique spectral measure such that

$$x = \int_{\sigma(x)} \lambda \, dE_x(\lambda).$$

Proof. Assume that $x \in B(H)$ is a self-adjoint operator.

To show: (a) If $f : \sigma(x) \to \mathbb{C}$ is a simple function then $\psi(f) = \Lambda_B(f)$.

(a) Assume that $f = \sum_{n=1}^{N} \lambda_n \chi_{\Delta_n}$ is a simple function, where $\lambda_n \in \mathbb{C}$ and $\Delta_n \in \mathcal{A}$. Then,

$$\begin{aligned} f(x) &= \sum_{n=1}^{N} \lambda_n \chi_{\Delta_n}(x) \\ &= \sum_{n=1}^{N} \lambda_n E_x(x) = x_f. \end{aligned}$$

The last equality in the above working is established via the same calculation done at the start of part (b) in Theorem 3.3.1. Hence, ψ and Λ_B agree on simple functions.

However, the set of simple functions is dense in $Bor(\sigma(x), \mathbb{C})$. Hence, $\Lambda_B = \psi$ on $Bor(\sigma(x), \mathbb{C})$. This means that if $f \in Bor(\sigma(x), \mathbb{C})$ then

$$f(x) = \int_{\sigma(x)} f(\lambda) \, dE_x(\lambda).$$

By Theorem 3.2.4, if $f(\lambda) = \lambda$ for all $\lambda \in \sigma(x)$ then $f(x) = x$ and

$$x = \int_{\sigma(x)} \lambda \, dE_x(\lambda).$$

Assume that E is another spectral measure such that $x = \int_{\sigma(x)} \lambda \, dE(\lambda)$.

To show: (b) $E_x = E$.

(b) Consider the map

$$f \in Bor(\sigma(x), \mathbb{C}) \mapsto \int_{\sigma(x)} f(\lambda) \, dE(\lambda) \in B(H).$$

By Theorem 3.3.1, the above map satisfies all of the conditions defining the Borel functional calculus in Theorem 3.2.4. By uniqueness, we have for $f \in Bor(\sigma(x), \mathbb{C})$,

$$\int_{\sigma(x)} f(\lambda)\, dE(\lambda) = f(x) = \int_{\sigma(x)} f(\lambda)\, dE_x(\lambda).$$

Now fix $\Delta \in \mathcal{A}$ and take $f = \chi_\Delta$ in the above equation. Then,

$$E(\Delta) = \int_\Delta dE(\lambda) = \int_\Delta dE_x(\lambda) = E_x(\Delta).$$

Therefore, $E = E_x$ as required. □

3.4 Holomorphic functional calculus

So far, we have defined the continuous and Borel functional calculus on $B(H)$. Recall that if $x \in B(H)$ then $\sigma(x)$ is a non-empty compact subset of $\mathbb{C}$. Thus, we can exploit the extra structure afforded by $\mathbb{C}$ (when compared to $\mathbb{R}$) and consider holomorphic functions from $\sigma(x)$ to $\mathbb{C}$. The goal of this section is to construct the holomorphic functional calculus on $B(H)$.

Let $x \in B(H)$ be an *arbitrary* operator. We do not assume x to be self-adjoint, unlike the situations for the continuous and Borel functional calculus. Let $Hol(\sigma(x), \mathbb{C})$ denote the algebra of functions which are holomorphic on an open neighbourhood containing $\sigma(x)$. Let $f \in Hol(\sigma(x), \mathbb{C})$ and Γ be a anticlockwise oriented closed curve which contains $\sigma(x)$ and is contained in the domain of holomorphy of f. Define

$$f(x) = \frac{1}{2\pi i} \oint_\Gamma f(\lambda)(\lambda I - x)^{-1}\, d\lambda. \tag{3.2}$$

The function we are integrating over takes values in the Banach space $B(H)$. That is, we define $f(x)$ with a Banach space-valued integral. This concept is formalised by the **Bochner integral**, which is developed and studied in [Coh13, Appendix E]. It can be shown that the value of the integral is independent of the choice of curve Γ.

Furthermore, $f(x)$ commutes with x because $x \in B(H)$ does not depend on λ and thus, passes through the integral. Another remark we will make is that the curve Γ cannot intersect the spectrum $\sigma(x)$. Otherwise, the operator $(\lambda I - x)^{-1}$ may not be well-defined.

For the proof of holomorphic functional calculus, we require the well-known resolvent identity.

Lemma 3.4.1 (Resolvent identity)**.** *Let H be a Hilbert space over $\mathbb{C}$ and $x \in B(H)$. Let $\lambda, \mu \in \rho(x)$. Then,*

$$(\lambda I - x)^{-1} - (\mu I - x)^{-1} = (\mu - \lambda)(\lambda I - x)^{-1}(\mu I - x)^{-1}.$$

Proof. Assume that $x \in B(H)$. Observe that the resolvent identity is satisfied trivially when $\lambda = \mu$. So, assume that $\lambda, \mu \in \rho(x)$ are distinct. Then, we compute directly that

$$\begin{aligned}
&\frac{1}{\mu - \lambda}\big((\lambda I - x)^{-1} - (\mu I - x)^{-1}\big)(\mu I - x)(\lambda I - x) \\
&= \frac{1}{\mu - \lambda}\big((\lambda I - x)^{-1}(\mu I - x) - I\big)(\lambda I - x) \\
&= \frac{1}{\mu - \lambda}\big((\lambda I - x)^{-1}(\mu I - \lambda I + \lambda I - x) - I\big)(\lambda I - x) \\
&= \frac{1}{\mu - \lambda}\big((\mu - \lambda)(\lambda I - x)^{-1} + I - I\big)(\lambda I - x) = I.
\end{aligned}$$

and

$$\begin{aligned}
&(\mu I - x)(\lambda I - x)\frac{1}{\mu - \lambda}\big((\lambda I - x)^{-1} - (\mu I - x)^{-1}\big) \\
&= (\mu I - x)\frac{1}{\mu - \lambda}\big(I - (\lambda I - x)(\mu I - x)^{-1}\big) \\
&= (\mu I - x)\frac{1}{\mu - \lambda}\big(I - (\lambda I + \mu I - \mu I - x)(\mu I - x)^{-1}\big) \\
&= (\mu I - x)\frac{1}{\mu - \lambda}\big(I - I + (\mu - \lambda)(\mu I - x)^{-1}\big) = I.
\end{aligned}$$

Hence, $\frac{1}{\mu-\lambda}((\lambda I - x)^{-1} - (\mu I - x)^{-1})$ is the inverse operator of $(\mu I - x)(\lambda I - x)$. So,

$$\frac{1}{\mu - \lambda}\big((\lambda I - x)^{-1} - (\mu I - x)^{-1}\big) = (\lambda I - x)^{-1}(\mu I - x)^{-1}$$

which gives the resolvent identity. □

Theorem 3.4.2 (Holomorphic functional calculus)**.** *Let H be a Hilbert space over $\mathbb{C}$ and $x \in B(H)$. Define the map*

$$\begin{array}{llll}
\Lambda_H : & Hol(\sigma(x), \mathbb{C}) & \to & B(H) \\
 & f & \mapsto & f(x) = \frac{1}{2\pi i} \oint_\Gamma f(\lambda)(\lambda I - x)^{-1}\, d\lambda
\end{array}$$

Then, Λ_H is a unital algebra homomorphism and if $f(\lambda) = a_0 + a_1\lambda + \cdots + a_n\lambda^n$ then

$$f(x) = a_0 I + a_1 x + \cdots + a_n x^n.$$

Proof. Assume that H is a Hilbert space over $\mathbb{C}$ and $x \in B(H)$. Assume that Λ_H is the map defined as above. First assume that $f(\lambda) = \lambda^m$ for $\lambda \in \sigma(x)$ and $m \in \mathbb{Z}_{\geq 0}$. Let Γ be a circle around 0 with radius $r > \|x\|$, traversed in the anticlockwise direction. Recalling equation (1.4)

$$\sigma(x) \subseteq \{\lambda \in \mathbb{C} \mid |\lambda| \leq \|x\|\},$$

we find that $\Gamma \cap \sigma(x) = \emptyset$. If $\lambda \in \Gamma$ then $\lambda \in \rho(x)$ and

$$(\lambda I - x)^{-1} = \sum_{n=0}^{\infty} \lambda^{-n-1} x^n$$

as in Theorem 1.2.2. Using equation (3.2), we have

$$\begin{aligned} f(x) &= \frac{1}{2\pi i} \oint_\Gamma f(\lambda)(\lambda I - x)^{-1} \, d\lambda \\ &= \frac{1}{2\pi i} \oint_\Gamma \sum_{n=0}^{\infty} \lambda^m \lambda^{-n-1} x^n \, d\lambda \\ &= \sum_{n=0}^{\infty} \Big(\frac{1}{2\pi i} \oint_\Gamma \lambda^{m-n-1} \, d\lambda \Big) x^n = \big(\frac{1}{2\pi i} \oint_\Gamma \frac{1}{\lambda} \, d\lambda \big) x^m = x^m. \end{aligned}$$

By linearity of the integral, we deduce that Λ_H is linear and that if $f(\lambda) = \sum_{i=0}^{n} a_i \lambda^i$ for $\lambda \in \sigma(x)$ then

$$f(x) = \sum_{i=0}^{n} a_i f(\lambda^i) = \sum_{i=0}^{n} a_i x^i \in B(H).$$

This also shows that Λ_H is unital because $\Lambda_H(id_{\sigma(x)}) = id_{\sigma(x)}(x) = x$, where $id_{\sigma(x)}$ is the identity map on $\sigma(x)$. Now let $f, g \in Hol(\sigma(x), \mathbb{C})$.

To show: (a) $\Lambda_H(f)\Lambda_H(g) = \Lambda_H(fg)$.

(a) Let U_f and U_g be the domains of holomorphy of f and g respectively. Let Γ and Γ' be curves around $\sigma(x)$ contained in $U_f \cap U_g$ with Γ' lying outside of Γ. By the resolvent identity in Lemma 3.4.1, we compute directly that

$$
\begin{aligned}
f(x)g(x) &= (\frac{1}{2\pi i})^2 \Big(\oint_\Gamma f(\lambda)(\lambda I - x)^{-1}\, d\lambda \Big)\Big(\oint_{\Gamma'} g(\mu)(\mu I - x)^{-1}\, d\mu \Big) \\
&= (\frac{1}{2\pi i})^2 \oint_\Gamma \oint_{\Gamma'} f(\lambda)g(\mu)(\lambda I - x)^{-1}(\mu I - x)^{-1}\, d\mu\, d\lambda \\
&= (\frac{1}{2\pi i})^2 \oint_\Gamma \oint_{\Gamma'} \frac{f(\lambda)g(\mu)}{\mu - \lambda}((\lambda I - x)^{-1} - (\mu I - x)^{-1})\, d\mu\, d\lambda \\
&= (\frac{1}{2\pi i})^2 \oint_\Gamma f(\lambda)(\lambda I - x)^{-1}(\oint_{\Gamma'} \frac{g(\mu)}{\mu - \lambda}\, d\mu)\, d\lambda \\
&\quad -(\frac{1}{2\pi i})^2 \oint_{\Gamma'} g(\mu)(\mu I - x)^{-1}(\oint_\Gamma \frac{f(\lambda)}{\mu - \lambda}\, d\lambda)\, d\mu \\
&= (\frac{1}{2\pi i})^2 \oint_\Gamma f(\lambda)(\lambda I - x)^{-1}(\oint_{\Gamma'} \frac{g(\mu)}{\mu - \lambda}\, d\mu)\, d\lambda \\
&= (\frac{1}{2\pi i})^2 \oint_\Gamma f(\lambda)g(\lambda)(\lambda I - x)^{-1}\, d\lambda = (fg)(x).
\end{aligned}
$$

To understand the second last equality, observe that $\mu \in \Gamma'$ and Γ lies inside Γ', due to how we set up the contours originally. Hence, the function

$$\frac{f(\lambda)}{\mu - \lambda}$$

is holomorphic in the region bounded by Γ and by Cauchy's theorem, the integral

$$\oint_\Gamma \frac{f(\lambda)}{\mu - \lambda}\, d\lambda = 0.$$

Hence, Λ_H is a unital algebra homomorphism as required. □

Just like its continuous and Borel counterparts, holomorphic functional calculus has a bevy of nice properties, which we will now prove, starting with an analogue of the spectral mapping theorem (see Theorem 2.2.3).

Theorem 3.4.3 (Spectral mapping for holomorphic functional calculus). *Let H be a Hilbert space over $\mathbb{C}$ and $x \in B(H)$. Let $f \in Hol(\sigma(x), \mathbb{C})$. Then, $\sigma(f(x)) = f(\sigma(x))$.*

Similarly to Theorem 3.4.2, x is not assumed to be self-adjoint.

Proof. Assume that $x \in B(H)$ and $\mu \in \mathbb{C} - f(\sigma(x))$. Define the function

$$
\begin{array}{rccc}
h_\mu : & \sigma(x) & \to & \mathbb{C} \\
& \lambda & \mapsto & (\mu - f(\lambda))^{-1}
\end{array}
$$

Then, $h_\mu \in Hol(\sigma(x), \mathbb{C})$ and by the holomorphic functional calculus,

$$h_\mu(x)(\mu I - f(x)) = (\mu I - f(x))h_\mu(x) = I.$$

Then, $\mu \notin \sigma(f(x))$. Therefore, $\sigma(f(x)) \subseteq f(\sigma(x))$.

Now assume that $\mu \in f(\sigma(x))$. Then, there exists $\lambda_0 \in \sigma(x)$ such that $\mu = f(\lambda_0)$. If $\lambda \in \sigma(x)$ then

$$\mu - f(\lambda) = f(\lambda_0) - f(\lambda) = (\lambda_0 - \lambda)\frac{f(\lambda_0) - f(\lambda)}{\lambda_0 - \lambda} = (\lambda_0 - \lambda)g(\lambda)$$

where

$$g(\lambda) = \begin{cases} \frac{f(\lambda_0)-f(\lambda)}{\lambda_0-\lambda}, & \text{if } \lambda \neq \lambda_0, \\ f'(\lambda_0), & \text{if } \lambda = \lambda_0. \end{cases} \in Hol(\sigma(x), \mathbb{C}).$$

Using the holomorphic functional calculus (see Theorem 3.4.2), we now apply the function $\lambda \mapsto \mu - f(\lambda)$ to the operator x to obtain

$$\mu I - f(x) = (\lambda_0 I - x)g(x).$$

Suppose for the sake of contradiction that $\mu I - f(x)$ is invertible. Then, $\lambda_0 I - x$ must also be invertible because since $\lambda_0 - \lambda$ and $g(\lambda)$ commute in $\mathbb{C}$, $\lambda_0 I - x$ and $g(x)$ must commute in $B(H)$. This contradicts the assumption that $\lambda_0 \in \sigma(x)$. Hence, $\mu I - f(x)$ is not invertible and $\mu \in \sigma(f(x))$. So, $f(\sigma(x)) \subseteq \sigma(f(x))$ and $f(\sigma(x)) = \sigma(f(x))$ as required. $\square$

Holomorphic functional calculus behaves well with composition of functions.

Theorem 3.4.4. *Let H be a Hilbert space over $\mathbb{C}$ and $x \in B(H)$. Let $g \in Hol(\sigma(x), \mathbb{C})$ and $f \in Hol(\sigma(g(x)), \mathbb{C})$. Then,*

$$f(g(x)) = (f \circ g)(x).$$

Proof. Assume that $x \in B(H)$, $g \in Hol(\sigma(x), \mathbb{C})$ and $f \in Hol(\sigma(g(x)), \mathbb{C})$. Let U_f and U_g be the domains of holomorphy of f and g respectively. Let Γ be a curve oriented anticlockwise in U_g surrounding $\sigma(x)$ and Γ' be a curve oriented anticlockwise in U_f surrounding $\sigma(g(x))$.

For $\mu \in \Gamma'$ and $\lambda \in \sigma(x)$, let $h_\mu(\lambda) = (\mu - g(\lambda))^{-1}$. Then, $h_\mu \in Hol(\sigma(x), \mathbb{C})$ because $\mu \notin \sigma(g(x)) = g(\sigma(x))$. From the holomorphic functional calculus (see Theorem 3.4.2), $h_\mu(x) = (\mu I - g(x))^{-1}$ and

$$
\begin{aligned}
f(g(x)) &= \frac{1}{2\pi i}\oint_{\Gamma'} f(\mu)(\mu I - g(x))^{-1}\, d\mu \\
&= \frac{1}{2\pi i}\oint_{\Gamma'} f(\mu)h_\mu(x)\, d\mu \\
&= \frac{1}{2\pi i}\oint_{\Gamma'} f(\mu)\big(\frac{1}{2\pi i}\oint_{\Gamma} \frac{1}{\mu - g(\lambda)}(\lambda I - x)^{-1}\, d\lambda\big)\, d\mu \\
&= \frac{1}{2\pi i}\oint_{\Gamma} \big(\frac{1}{2\pi i}\oint_{\Gamma'} \frac{f(\mu)}{\mu - g(\lambda)}\, d\mu\big)(\lambda I - x)^{-1}\, d\lambda \\
&= \frac{1}{2\pi i}\oint_{\Gamma} f(g(\lambda))(\lambda I - x)^{-1}\, d\lambda = (f\circ g)(x).
\end{aligned}
$$

□

Now let $D \subseteq \mathbb{C}$ be an open set and

$$\overline{D} = \{\overline{z} \mid z \in D\}.$$

Let $f \in Hol(D, \mathbb{C})$. Define $\tilde{f} : \overline{D} \to \mathbb{C}$ by $\tilde{f}(z) = \overline{f(\overline{z})}$. Then, $\tilde{f} \in Hol(\overline{D}, \mathbb{C})$. The point of this function is

Theorem 3.4.5. *Let H be a Hilbert space over $\mathbb{C}$ and $x \in B(H)$. Let $f \in Hol(\sigma(x), \mathbb{C})$. Then, $f(x)^* = \tilde{f}(x^*)$.*

Proof. Assume that $x \in B(H)$. Let Γ be a simple closed curve with anticlockwise orientation surrounding $\sigma(x)$ such that Γ is contained in the domain of holomorphy of f. Let $\overline{\Gamma}$ be the image of Γ under complex conjugation, with clockwise orientation and $\overline{\Gamma}'$ be the curve $\overline{\Gamma}$ with anticlockwise orientation. Then,

$$
\begin{aligned}
f(x)^* &= \Big(\frac{1}{2\pi i}\oint_{\Gamma} f(\lambda)(\lambda I - x)^{-1}\, d\lambda\Big)^* \\
&= -\frac{1}{2\pi i}\oint_{\Gamma} \overline{f}(\lambda)(\overline{\lambda} I - x^*)^{-1}\, d\overline{\lambda} \\
&= -\frac{1}{2\pi i}\oint_{\overline{\Gamma}} \overline{f}(\overline{\mu})(\mu I - x^*)^{-1}\, d\mu \\
&= \frac{1}{2\pi i}\oint_{\overline{\Gamma}'} \overline{f}(\overline{\mu})(\mu I - x^*)^{-1}\, d\mu \\
&= \frac{1}{2\pi i}\oint_{\overline{\Gamma}'} \tilde{f}(\mu)(\mu I - x^*)^{-1}\, d\mu = \tilde{f}(x^*).
\end{aligned}
$$

□

So far, we have discussed holomorphic functional calculus for an arbitrary operator $x \in B(H)$. Once we specialise to the case where $x \in B(H)$ is self-adjoint, we recover the continuous functional calculus.

Theorem 3.4.6 (CFC from HFC)**.** *Let H be a Hilbert space over $\mathbb{C}$ and $x \in B(H)$ be self-adjoint. Then, the *-homomorphism Λ_H in Theorem 3.4.2 is the restriction $\Lambda|_{Hol(\sigma(x),\mathbb{C})}$, where Λ is the isomorphism from Theorem 2.2.1.*

Proof. Assume that $x \in B(H)$ is self-adjoint. Assume that $f \in Hol(\sigma(x), \mathbb{C})$. The image $\Lambda_H(f) \in B(H)$ is normal. To see why this is the case, we compute using Theorem 3.4.5 that

$$\begin{aligned}
\Lambda_H(f)(\Lambda_H(f))^* &= f(x)f(x)^* \\
&= f(x)\tilde{f}(x^*) = f(x)\tilde{f}(x) \\
&= \tilde{f}(x^*)^*\tilde{f}(x) = \tilde{f}(x)^*\tilde{f}(x) \\
&= \tilde{f}(x)\tilde{f}(x)^* = (\Lambda_H(f))^*\Lambda_H(f).
\end{aligned}$$

So,

$$\|\Lambda_H(f)\| = |\sigma(\Lambda_H(f))| = \sup_{\lambda \in \sigma(x)} |f(\lambda)|.$$

Therefore, Λ_H extends uniquely to an isometry from $Cts(\sigma(x), \mathbb{C})$ to $B(H)$. By Theorem 3.4.2, Λ_H maps any polynomial p to $p(x) \in B(H)$. By uniqueness of Theorem 2.2.1, this isometry must coincide with Λ from Theorem 2.2.1.

Therefore, $\Lambda_H = \Lambda|_{Hol(\sigma(x),\mathbb{C})}$. $\square$

3.5 The exponential of an operator

One major application of the holomorphic functional calculus (Theorem 3.4.2) is to define the exponential of an operator. As usual, let H be a Hilbert space over $\mathbb{C}$ and $x \in B(H)$. The function $\exp : \mathbb{C} \to \mathbb{C}$ is entire. So, its restriction to the spectrum $\sigma(x)$ is holomorphic. By applying the map Λ_H from Theorem 3.4.2 and using the infinite series expansion of exp, we deduce that

$$\begin{aligned}
\exp(x) &= \frac{1}{2\pi i}\oint_\Gamma \exp(\lambda)(\lambda I - x)^{-1}\, d\lambda \\
&= \frac{1}{2\pi i}\oint_\Gamma \sum_{i=0}^{\infty} \frac{\lambda^i}{i!}(\lambda I - x)^{-1}\, d\lambda \\
&= \sum_{i=0}^{\infty} \frac{1}{2\pi i}\oint_\Gamma \frac{\lambda^i}{i!}(\lambda I - x)^{-1}\, d\lambda \\
&= \sum_{i=0}^{\infty} \frac{x^i}{i!}.
\end{aligned}$$

An important property of the exponential is that if $x, y \in B(H)$ commute then $\exp(x)\exp(y) = \exp(x+y)$. This follows from the computation

$$\begin{aligned}
\exp(x+y) &= \sum_{i=0}^{\infty} \frac{(x+y)^i}{i!} \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{i} \binom{i}{j}\frac{x^j y^{i-j}}{i!} \qquad (\text{since } xy = yx) \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{i} \frac{x^j y^{i-j}}{j!(i-j)!} \\
&= \sum_{j=0}^{\infty}\sum_{i=j}^{\infty} \frac{x^j y^{i-j}}{j!(i-j)!} = \sum_{j=0}^{\infty}\frac{x^j}{j!}\sum_{i=j}^{\infty}\frac{y^{i-j}}{(i-j)!} \\
&= \sum_{j=0}^{\infty}\frac{x^j}{j!}\sum_{k=0}^{\infty}\frac{y^k}{k!} \qquad (k = i-j) \\
&= \exp(x)\exp(y).
\end{aligned}$$

In the case where $x, y \in B(H)$ do not commute, there is a well-known formula which deals with this particular case. It is sometimes called the *Lie-Trotter formula.*

Theorem 3.5.1 (Lie-Trotter)**.** *Let H be a Hilbert space over $\mathbb{C}$ and $x, y \in B(H)$. Then,*

$$\exp(x+y) = \lim_{n\to\infty}(\exp(\frac{x}{n})\exp(\frac{y}{n}))^n.$$

Proof. Assume that H is a Hilbert space over $\mathbb{C}$ and $x, y \in B(H)$. For $n \in \mathbb{Z}_{>0}$, let

$$s_n = \exp(\frac{1}{n}(x+y)) \qquad \text{and} \qquad t_n = \exp(\frac{x}{n})\exp(\frac{y}{n}).$$

We are able to bound $\|s_n\|$ above as follows:

$$\begin{aligned}
\|s_n\| &= \|\exp(\frac{1}{n}(x+y))\| \\
&= \|\sum_{i=0}^{\infty} \frac{(x+y)^i}{n^i i!}\| \\
&\leq \sum_{i=0}^{\infty} \frac{\|(x+y)^i\|}{n^i i!} \\
&\leq \sum_{i=0}^{\infty} \frac{\|x+y\|^i}{n^i i!} \leq \sum_{i=0}^{\infty} \frac{(\|x\|+\|y\|)^i}{n^i i!} \\
&= \exp(\frac{\|x\|+\|y\|}{n}).
\end{aligned}$$

Similarly for $\|t_n\|$, we have

$$\begin{aligned}
\|t_n\| &= \|\exp(\frac{x}{n})\exp(\frac{y}{n})\| \\
&\leq \|\exp(\frac{x}{n})\|\|\exp(\frac{y}{n})\| \\
&\leq \exp(\frac{\|x\|}{n})\exp(\frac{\|y\|}{n}) = \exp(\frac{\|x\|+\|y\|}{n}).
\end{aligned}$$

Next, we write the quantity $\|s_n - t_n\|$ as:

$$\begin{aligned}
\|s_n - t_n\| &= \|\exp(\frac{1}{n}(x+y)) - \exp(\frac{x}{n})\exp(\frac{y}{n})\| \\
&= \|\sum_{i=0}^{\infty} \frac{(x+y)^i}{n^i i!} - (\sum_{j=0}^{\infty} \frac{x^j}{n^j j!})(\sum_{k=0}^{\infty} \frac{y^k}{n^k k!})\| \\
&= \|1 + \frac{1}{n}(x+y) + \sum_{i=2}^{\infty} \frac{(x+y)^i}{n^i i!} - 1 - \frac{x}{n} - \frac{y}{n} - (\sum_{j=1}^{\infty} \frac{x^j}{n^j j!})(\sum_{k=1}^{\infty} \frac{y^k}{n^k k!})\| \\
&= \|\sum_{i=2}^{\infty} \frac{(x+y)^i}{n^i i!} - (\sum_{j=1}^{\infty} \frac{x^j}{n^j j!})(\sum_{k=1}^{\infty} \frac{y^k}{n^k k!})\| \\
&= \frac{1}{n^2}\|\sum_{i=2}^{\infty} \frac{(x+y)^i}{n^{i-2} i!} - (\sum_{j=1}^{\infty} \frac{x^j}{n^{j-1} j!})(\sum_{k=1}^{\infty} \frac{y^k}{n^{k-1} k!})\|.
\end{aligned}$$

Critically, $\|s_n - t_n\| = C/n^2$, where $C \in \mathbb{R}_{>0}$ is some constant. Now, we have the identity

$$s_n^n - t_n^n = \sum_{r=0}^{n-1} s_n^r (s_n - t_n) t_n^{n-1-r}.$$

Using the identity, we bound the norm $\|s_n^n - t_n^n\|$ from above:

$$\begin{aligned}
\|s_n^n - t_n^n\| &\leq \sum_{r=0}^{n-1} \|s_n^r (s_n - t_n) t_n^{n-1-r}\| \\
&\leq \sum_{r=0}^{n-1} \|s_n^r\| \|s_n - t_n\| \|t_n^{n-1-r}\| \\
&= \frac{C}{n^2} \sum_{r=0}^{n-1} \|s_n^r\| \|t_n^{n-1-r}\| \\
&\leq \frac{C}{n^2} \sum_{r=0}^{n-1} (\exp(\frac{\|x\| + \|y\|}{n}))^{n-1} \\
&\leq \frac{C}{n^2} \sum_{r=0}^{n-1} (\exp(\frac{\|x\| + \|y\|}{n}))^{n} \\
&= \frac{C}{n} \exp(\|x\| + \|y\|).
\end{aligned}$$

The key finding here is that $\|s_n^n - t_n^n\| \to 0$ as $n \to \infty$. This means that $\lim_{n\to\infty} s_n^n = \lim_{n\to\infty} t_n^n$ as operators in $B(H)$. Assembling all of the previous results together, we obtain

$$\begin{aligned}
\lim_{n\to\infty} (\exp(\frac{x}{n}) \exp(\frac{y}{n}))^n &= \lim_{n\to\infty} t_n^n \\
&= \lim_{n\to\infty} s_n^n \\
&= \lim_{n\to\infty} \exp(x + y) = \exp(x + y).
\end{aligned}$$

□

One notable property of the exponential map is that if $x \in B(H)$ is *skew self-adjoint* then $\exp(x)$ is unitary. We will dedicate the rest of the section to proving this statement.

Definition 3.5.1. Let H be a Hilbert space over $\mathbb{C}$ and $x \in B(H)$. The operator $x \in B(H)$ is said to be **skew self-adjoint** if $x^* = -x$.

Theorem 3.5.2 (Spectrum of a skew self-adjoint operator)**.** *Let H be a Hilbert space over $\mathbb{C}$ and $x \in B(H)$ be a skew self-adjoint operator. Then, $\sigma(x) \subset i\mathbb{R}$, where*

$$i\mathbb{R} = \{i\lambda \mid \lambda \in \mathbb{R}\}.$$

Proof. Assume that H is a Hilbert space over $\mathbb{C}$ and $x \in B(H)$ be a skew self-adjoint operator. Observe that the operator $ix \in B(H)$ is self-adjoint. Indeed,

$$(ix)^* = -i(-x) = ix.$$

Let $f : \sigma(ix) \to \mathbb{C}$ be defined by $f(\lambda) = -i\lambda$. Then, $f \in Cts(\sigma(ix), \mathbb{C})$ and by the spectral mapping theorem (see Theorem 2.2.3),

$$\begin{aligned} f(\sigma(ix)) &= \{-i\lambda \mid \lambda \in \sigma(ix)\} \\ &= \sigma(f(ix)) = \sigma(x). \end{aligned}$$

But, since ix is self-adjoint, $\sigma(ix) \subset \mathbb{R}$. Therefore, $\sigma(x) \subset i\mathbb{R}$ as required. □

Unitary operators also have a neatly characterised spectrum. The proof requires the fact that the continuous functional calculus extends to normal operators. We will assume this for now and prove this assertion in a later section.

Theorem 3.5.3 (Spectrum of a unitary operator)**.** *Let H be a Hilbert space over $\mathbb{C}$ and $x \in B(H)$. Then, x is a unitary operator if and only if x is a normal operator and $\sigma(x) \subseteq \partial B(0,1)$, where*

$$\partial B(0,1) = \{z \in \mathbb{C} \mid |z| = 1\}.$$

Proof. Assume that H is a Hilbert space over $\mathbb{C}$ and $x \in B(H)$.

To show: (a) If x is unitary then x is normal and $\sigma(x) \subset \partial B(0,1)$.

(b) If x is a normal operator and $\sigma(x) \subset \partial B(0,1)$ then x is unitary.

(a) Assume that x is unitary. Then, $xx^* = x^*x = I$ and x is normal. Observe that

$$\|x\|^2 = \langle x, x\rangle = \langle I, x^*x\rangle = 1.$$

So, $\|x\| = 1$. Similarly, $\|x^*\| = 1$. This gives the inclusion

$$\sigma(x) \subseteq \{z \in \mathbb{C} \mid |z| \leq \|x\| = 1\} = B(0,1).$$

We also have $\sigma(x^*) \subset B(0,1)$. However,

$$\begin{aligned}
\sigma(x) &= \sigma((x^*)^{-1}) \\
&= \{\frac{1}{\lambda} \mid \lambda \in \sigma(x^*)\} \\
&\subset \{\frac{1}{\lambda} \mid \lambda \in B(0,1)\} \\
&= \{\lambda \in \mathbb{C} \mid |\lambda| \geq 1\}.
\end{aligned}$$

Note that the second equality holds in the above working because x^* is invertible and consequently, $0 \notin \sigma(x^*)$. Combining the two inclusions of $\sigma(x)$, we find that $\sigma(x) \subset \partial B(0,1)$ as required.

(b) Assume that x is a normal operator and $\sigma(x) \subseteq \partial B(0,1)$. Let $id : \sigma(x) \to \mathbb{C}$ denote the identity map and $\overline{id}$ denote the identity map composed with complex conjugation. Then, $id, \overline{id} \in Cts(\sigma(x), \mathbb{C})$. Notice that because $\sigma(x) \subseteq \partial B(0,1)$, $\overline{id}(\lambda)\ id(\lambda) = 1$ for all $\lambda \in \sigma(x)$. By the continuous functional calculus (Theorem 2.2.1) (for normal operators), we have

$$\begin{aligned}
xx^* &= \Lambda(id)\Lambda(id)^* \\
&= \Lambda(id)\Lambda(\overline{id}) \\
&= \Lambda(id\ \overline{id}) \\
&= \Lambda(1) = I.
\end{aligned}$$

By a similar argument, we also have $x^*x = I$. Hence, x is unitary. □

With Theorem 3.5.2 and Theorem 3.5.3, we can now prove the following theorem:

Theorem 3.5.4. *Let H be a Hilbert space over $\mathbb{C}$ and $x \in B(H)$ be skew self-adjoint. Then, the operator $\exp(x) \in B(H)$ is unitary.*

Proof. Assume that H is a Hilbert space over $\mathbb{C}$ and $x \in B(H)$. Assume that x is skew self-adjoint. By Theorem 3.5.2, $\sigma(x) \subset i\mathbb{R}$.

Since $\exp : \sigma(x) \to \mathbb{C}$ is continuous (actually holomorphic), the spectral mapping theorem (see Theorem 2.2.3) tells us that

$$\sigma(\exp(x)) = \exp(\sigma(x)) \subset \exp(i\mathbb{R}) = \partial B(0,1).$$

Since x is skew self-adjoint, $x^* = -x$. So, x and x^* commute and consequently,

$$\exp(x)(\exp(x))^* = \exp(x)\exp(x^*) = \exp(x^*)\exp(x) = (\exp(x))^* \exp(x).$$

So, $\exp(x)$ is a normal operator. By Theorem 3.5.3, we therefore find that $\exp(x)$ is a unitary operator. □

3.6 Fuglede's theorem and Putnam's theorem

Let $x, y \in B(H)$ and suppose that x and y commute. When does y commute with the adjoint x^*? Fuglede's theorem answers this question in the affirmative, provided that x is normal.

Theorem 3.6.1 (Fuglede's theorem)**.** *Let H be a Hilbert space over $\mathbb{C}$ and $x, y \in B(H)$ such that x and y commute ($xy = yx$). Assume that x is normal. Then, $yx^* = x^*y$.*

Proof. Assume that x and y are commuting elements of $B(H)$. Assume that x is a normal operator. Let $\lambda \in \mathbb{C}$ and define

$$x_\lambda = \lambda x^* - \overline{\lambda} x \in B(H).$$

A quick computation reveals that x_λ is skew self-adjoint. By Theorem 3.5.4, the operator $\exp(x_\lambda)$ must be unitary for $\lambda \in \mathbb{C}$.

Next, observe that if $\lambda, \mu \in \mathbb{C}$ then $x_\lambda x_\mu = x_\mu x_\lambda$. Therefore,

$$\exp(x_\lambda)\exp(x_\mu) = \exp(x_\lambda + x_\mu).$$

Moreover, $\exp(x_0) = I$ by definition and because x is normal,

$$\exp(x_\lambda) = \exp(\lambda x^*)\exp(-\overline{\lambda} x) = \exp(-\overline{\lambda} x)\exp(\lambda x^*).$$

Since x commutes with y, we compute that

$$\begin{aligned}
\exp(-\lambda x^*) y \exp(\lambda x^*) &= \exp(-\lambda x^*) \exp(\overline{\lambda} x) \exp(-\overline{\lambda} x) y \exp(\lambda x^*) \\
&= \exp(-\lambda x^*) \exp(\overline{\lambda} x) y \exp(-\overline{\lambda} x) \exp(\lambda x^*) \\
&= \exp(-\lambda x^* + \overline{\lambda} x) y \exp(-\overline{\lambda} x + \lambda x^*) \\
&= \exp(x_{-\lambda}) y \exp(x_\lambda).
\end{aligned}$$

Consider the function

$$\begin{array}{rccc} h: & \mathbb{C} & \to & B(H) \\ & \lambda & \mapsto & \exp(-\lambda x^*) y \exp(\lambda x^*). \end{array}$$

Then, h is a holomorphic function on all of $\mathbb{C}$ (entire). It is also bounded because

$$\|h(\lambda)\| = \|\exp(x_{-\lambda}) y \exp(x_\lambda)\| \leq \|y\|$$

because $\exp(x_{-\lambda})$ and $\exp(x_\lambda)$ are unitary operators and thus, have norm 1.

By Liouville's theorem, h must be constant. If $\lambda \in \mathbb{C}$ then

$$h(\lambda) = \exp(-\lambda x^*) y \exp(\lambda x^*) = h(0) = y.$$

Rewrite the above equation as $y \exp(\lambda x^*) = \exp(\lambda x^*) y$ and expand both sides as convergent power series:

$$\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} y (x^*)^n = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} (x^*)^n y.$$

The coefficients of both sides are equal. Looking at the coefficient where $n = 1$, we obtain $yx^* = x^*y$ as required. □

One remark about Theorem 3.6.1 is that if $x = y$ then Theorem 3.6.1 says that x is normal. If x is not normal then Theorem 3.6.1 fails even for the case where $x = y$.

The proof of Putnam's theorem relies on a corollary of Theorem 3.6.1.

Lemma 3.6.2. *Let H be a Hilbert space over $\mathbb{C}$ and $x_1, x_2, y \in B(H)$. Assume that x_1 and x_2 are normal and $yx_1 = x_2y$. Then, $yx_1^* = x_2^*y$.*

Proof. Assume that $x_1, x_2, y \in B(H)$ are the operators defined as above. The idea is to identify operators on the Hilbert space $H \oplus H$ with the space

$M_{2\times 2}(B(H))$ of 2×2 matrices whose elements are operators in $B(H)$. The identification is given by the following map

$$\begin{array}{rccc}\Omega: & B(H\oplus H) & \to & M_{2\times 2}(B(H)) \\ & z & \mapsto & \begin{pmatrix} z_1 & z_3 \\ z_2 & z_4 \end{pmatrix}\end{array}$$

where $z(h_1,h_2) = (z_1(h_1)+z_2(h_2), z_3(h_1)+z_4(h_2))$.

Now consider the operators $\tilde{x}, \tilde{y} \in B(H\oplus H)$, defined by

$$\tilde{x} = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \qquad \text{and} \qquad \tilde{y} = \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix}.$$

It is straightforward to check that $\tilde{x}$ is a normal operator and $\tilde{x}\tilde{y} = \tilde{y}\tilde{x}$. By Theorem 3.6.1, we deduce that $\tilde{x}^*\tilde{y} = \tilde{y}\tilde{x}^*$. This means that

$$\begin{pmatrix} 0 & 0 \\ x_2^* y & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ yx_1^* & 0 \end{pmatrix}$$

and subsequently, $x_2^* y = yx_1^*$. □

Before we state and prove Putnam's theorem, we need to define the notion of similar operators. Fortunately, the definition should be familiar from linear algebra.

Definition 3.6.1. Let H be a Hilbert space over $\mathbb{C}$ and $x_1, x_2 \in B(H)$. We say that x_1 and x_2 are **similar** if there exists an invertible operator $y \in B(H)$ such that $yx_1y^{-1} = x_2$.

Theorem 3.6.3 (Putnam's theorem)**.** *Let H be a Hilbert space over $\mathbb{C}$ and $x_1, x_2 \in B(H)$ be normal operators. If x_1 and x_2 are similar then they are unitarily equivalent. That is, $\sigma(x_1) = \sigma(x_2)$.*

Proof. Assume that x_1 and x_2 are normal operators which are similar. Then, there exists an invertible $y \in B(H)$ such that $yx_1y^{-1} = x_2$. So, $yx_1 = x_2 y$.

By Lemma 3.6.2, we must have $yx_1^* = x_2^* y$. Take the adjoint of both sides to obtain

$$x_1 y^* = y^* x_2.$$

Now multiply both sides on the right by y to obtain

$$x_1 y^* y = y^*(x_2 y) = y^*(yx_1).$$

This means that y^*y commutes with x_1. The key here is that y^*y is self-adjoint. Therefore, the operator $|y| = (y^*y)^{\frac{1}{2}}$ must also commute with x_1.

Let $y = u|y|$ be the polar decomposition of y (see Theorem 2.6.1). Since y and $|y|$ are both invertible, u is an invertible partial isometry.

From Theorem 3.1.8, u must be unitary and consequently,

$$ux_1u^* = u|y||y|^{-1}x_1|y||y|^{-1}u^{-1} = yx_1y^{-1} = x_2.$$

So, x_1 and x_2 are unitarily equivalent. $\square$

Chapter 4

Compact operators

4.1 Definition and properties of compact operators

In this chapter, we will shift gears and discuss compact operators on a Hilbert space. Following [Sol18, Chapter 5], we will define a compact operator as a "norm limit of finite dimensional operators". Note that in [Bre10] and [RS80], a different definition of a compact operator is usually used and is mainstream because it also allows us to define a compact operator between two different Banach spaces. Nonetheless, we will show that we can recover the mainstream definition of a compact operator from the one we will give.

Definition 4.1.1. Let H be a Hilbert space over $\mathbb{C}$. We say that an operator $x \in B(H)$ is **finite dimensional** if its range $\operatorname{im} x$ is a finite dimensional vector subspace of H.

We say that an operator $x \in B(H)$ is **compact** if there exists a sequence $\{x_n\}_{n\in\mathbb{Z}_{>0}}$ of finite dimensional operators such that

$$\lim_{n\to\infty} \|x_n - x\| = 0.$$

The set of finite dimensional operators and the set of compact operators will be denoted by $\mathcal{F}(H)$ and $B_0(H)$ respectively.

On the surface, $\mathcal{F}(H)$ and $B_0(H)$ are subsets of $B(H)$. In fact, they are *ideals* of $B(H)$. The definition of an ideal is quite similar to the one we are used to in ring theory.

Definition 4.1.2. Let H be a Hilbert space over $\mathbb{C}$ and $S \subseteq B(H)$. We say that S is an **ideal** in $B(H)$ if S is a vector subspace of $B(H)$ and if $x \in S$ and $y \in B(H)$ then $xy, yx \in S$.

If $x \in \mathcal{F}(H)$ then im x is a finite dimensional vector subspace of H. Therefore, we can find a basis $\{\psi_1, \ldots, \psi_N\}$ for im x. So for $\xi \in H$, the operator x takes the form

$$x\xi = \sum_{n=1}^{N} \alpha_n(\xi)\psi_n$$

where $\alpha_1, \ldots, \alpha_n$ are continuous linear functionals on H. By the Riesz representation theorem, $\alpha_i(\xi) = \langle \xi, \varphi_i \rangle$ for some $\varphi_1, \ldots, \varphi_n \in H$. So,

$$x = \sum_{n=1}^{N} |\psi_n\rangle\langle\varphi_n|.$$

Let us briefly recall bra-ket notation. If $\xi \in H$ then

$$x\xi = \sum_{n=1}^{N} |\psi_n\rangle\langle\varphi_n|\xi = \sum_{n=1}^{N} |\psi_n\rangle\langle\xi, \varphi_n\rangle = \sum_{n=1}^{N} \langle\xi, \varphi_n\rangle\psi_n = \sum_{n=1}^{N} \alpha_n(\xi)\psi_n.$$

With bra-ket notation, we can also write the adjoint of x as

$$x^* = \sum_{n=1}^{N} |\varphi_n\rangle\langle\psi_n|.$$

So, $x^* \in \mathcal{F}(H)$ and hence $\mathcal{F}(H)$ is closed under the involution operation on $B(H)$ (which is just the adjoint). Consequently, the ideal $B_0(H)$ is also closed under involution.

To see why this is the case, assume that $y \in B_0(H)$. Then, there exists a sequence $\{y_n\}$ in $\mathcal{F}(H)$ such that $y_n \to y$ as $n \to \infty$. This means that

$$y^* = \lim_{n\to\infty} (y_n)^*.$$

So, $y^* \in B_0(H)$ as required.

The next theorem provides the equivalence between Definition 4.1.2 and the usual definition of a compact operator present in [Bre10] and [RS80].

Theorem 4.1.1. *Let H be a Hilbert space over $\mathbb{C}$ and $x \in B(H)$. Then, x is compact if and only if for any bounded set $S \subseteq H$, the image $x(S)$ is pre-compact.*

Proof. Assume that H is a Hilbert space over $\mathbb{C}$ and $x \in B(H)$. Let

$$H_1 = \{\xi \in H \mid \|\xi\| \leq 1\}.$$

It suffices to prove that x is compact if and only if the image $x(H_1)$ is a pre-compact subset of H.

To show: (a) If x is compact then $x(H_1)$ is a pre-compact subset of H.

(b) If $x(H_1)$ is a pre-compact subset of H then x is compact.

(a) Assume that $x \in B(H)$ is compact and $\epsilon \in \mathbb{R}_{>0}$. Then, there exists a sequence $\{x_n\}$ in $\mathcal{F}(H)$ such that $x_n \to x$ as $n \to \infty$. In particular, there exists $k \in \mathbb{Z}_{>0}$ such that $\|x - x_k\| < \frac{\epsilon}{2}$.

Observe that since H_1 is bounded and $x_k \in B(H)$, the image $x_k(H_1)$ is a bounded subset of H. Since x_k is finite dimensional, $x_k(H_1)$ is contained in the finite dimensional subspace $x(H_1)$. Therefore, $x_k(H_1)$ is a pre-compact subset of H.

Subsequently, $x_k(H_1)$ has a finite $\frac{\epsilon}{2}$-net $\{\eta_1, \ldots, \eta_N\}$. We claim that this is also an ϵ-net for $x(H_1)$.

If $\xi \in H_1$ and $\ell \in \{1, 2, \ldots, N\}$ then

$$\|x\xi - \eta_\ell\| \leq \|x\xi - x_k\xi\| + \|x_k\xi - \eta_\ell\|.$$

By our construction of x_k, $\|x\xi - x_k\xi\| < \frac{\epsilon}{2}$. Since $x_k\xi \in x_k(H_1)$, there exists $\ell_0 \in \{1, 2, \ldots, n\}$ such that $\|x_k\xi - \eta_{\ell_0}\| < \frac{\epsilon}{2}$. Therefore,

$$\|x\xi - \eta_{\ell_0}\| \leq \|x\xi - x_k\xi\| + \|x_k\xi - \eta_{\ell_0}\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

So, $\{\eta_1, \ldots, \eta_N\}$ is an ϵ-net for $x(H_1)$. Hence, $x(H_1)$ is pre-compact.

(b) Assume that $x \in B(H)$ such that if $\epsilon \in \mathbb{R}_{>0}$ there exists a finite ϵ-net $\{\eta_1^\epsilon, \ldots, \eta_N^\epsilon\}$ for $x(H_1)$. Define p_ϵ to be the projection operator onto the finite dimensional subspace

$$span\{\eta_1^\epsilon, \ldots, \eta_N^\epsilon\}.$$

Let $x_\epsilon = p_\epsilon x$. Then, $x_\epsilon \in \mathcal{F}(H)$ and if $\xi \in H_1$ then there exists $k \in \{1, 2, \ldots, N\}$ such that $\|x\xi - \eta_k^\epsilon\| < \epsilon$. So,

$$\begin{aligned}\|x\xi - x_\epsilon \xi\| &= \|(I - p_\epsilon)x\xi\| \\ &\leq \|(I - p_\epsilon)(x\xi - \eta_k^\epsilon)\| + \|(I - p_\epsilon)\eta_k^\epsilon\| \\ &= \|(I - p_\epsilon)(x\xi - \eta_k^\epsilon)\| \\ &\leq \|I - p_\epsilon\|\|x\xi - \eta_k^\epsilon\| < \epsilon.\end{aligned}$$

To obtain the third equality, we used the fact that $p_\epsilon(\eta_k^\epsilon) = \eta_k^\epsilon$. To obtain the final inequality, we used the fact that $I - p_\epsilon$ is a projection operator onto the orthogonal complement of $span\{\eta_1^\epsilon, \ldots, \eta_N^\epsilon\}$ and consequently, that $\|I - p_\epsilon\| \leq 1$.

Hence, if $\epsilon \in \mathbb{R}_{>0}$ then there exists a finite dimensional operator x_ϵ such that $\|x\xi - x_\epsilon\xi\| < \epsilon$. Thus, $x \in B_0(H)$. □

An important consequence of Theorem 4.1.1 is that the eigenspace of a compact operator must be finite dimensional.

Lemma 4.1.2. *Let H be a Hilbert space over $\mathbb{C}$ and $x \in B_0(H)$. Let $\lambda \in \sigma(x)$ be a non-zero eigenvalue. Then, the eigenspace*

$$K = \{\xi \in H \mid x\xi = \lambda\xi\}$$

is a finite dimensional subspace of H.

Proof. Assume that $x \in B_0(H)$ and $\lambda \in \sigma(x)$ is a non-zero eigenvalue of x.

Suppose for the sake of contradiction that the eigenspace K is not finite dimensional. Define the bounded subset

$$K_1 = \{\xi \in K \mid \|\xi\| \leq 1\}.$$

Then, $x(K_1) = \{\xi \in K \mid \|\xi\| \leq |\lambda|\}$. But, since K is not finite dimensional, $x(K_1)$ cannot be pre-compact, despite being bounded. This contradicts Theorem 4.1.1. So, K must be finite dimensional. □

A consequence of Lemma 4.1.2 is that the identity operator I is not compact. Otherwise, the eigenspace of the eigenvalue 1 must be finite dimensional. However, the eigenspace in question is $H - \{0\}$, which is not finite dimensional. Nonetheless, there exists a sequence of finite dimensional operators converging to I in the strong topology, as the next theorem demonstrates.

Theorem 4.1.3. *Let H be a Hilbert space over $\mathbb{C}$. Then, there exists a net of finite dimensional projection operators which converge to the identity operator I in the strong topology.*

Proof. Assume that $\{\psi_k\}_{k\in K}$ is an orthonormal basis for H. Let F_k denote the set of finite subsets of K. This is a poset when equipped with the relation of inclusion. Let $J \in F_k$ and define the projection operator

$$p_J = \sum_{i\in J} |\psi_i\rangle\langle\psi_i|.$$

This is the projection onto $span\{\psi_j \mid j \in J\}$. Now observe that if $J \in F_K$ then $p_J \in \mathcal{F}(H)$ and

$$\|p_J\xi - \xi\|^2 = \sum_{i\in K\setminus J} |\langle\xi, \psi_i\rangle|^2.$$

We see that the quantity $\sum_{i\in K\setminus J}|\langle\xi, \psi_i\rangle|^2 \to 0$ as J grows larger (as we add more terms from K to J). Hence, $\{p_J\}_{J\in F_K}$ is a sequence of projection operators in $\mathcal{F}(H)$ which converge to the identity operator I. $\square$

4.2 Fredholm alternative and the Hilbert-Schmidt theorem

Roughly speaking, the Fredholm alternative tells us that operators of the form $I - x$ where $x \in B_0(H)$ is compact, behave like linear transformations on a finite dimensional vector space. In [Sol18] and [RS80], the Fredholm alternative is proved as a corollary of the *analytic Fredholm theorem* (see [RS80, Theorem VI.4]).

Theorem 4.2.1 (Analytic Fredholm theorem)**.** *Let D be an open, connected subset of $\mathbb{C}$. Let $f : D \to B(H)$ be a holomorphic function such that if $z \in D$ then $f(z)$ is a compact operator. Then, there are two situations which can occur:*

1. *If $z \in D$ then $I - f(z)$ is not an invertible operator.*
2. *The operator $I - f(z)$ is invertible for $z \in D - S$, where S is a discrete subset of D without an accumulation point in D.*

Proof. Assume that D is an open connected subset of $\mathbb{C}$. Assume that $f : D \to B(H)$ is the holomorphic function defined as above. It is enough to prove that the statement holds for a neighbourhood of any given $z_0 \in D$.

Fix $z_0 \in D$ and let $r \in \mathbb{R}_{>0}$ so that the set

$$D_r = \{z \in \mathbb{C} \mid |z - z_0| < r\}.$$

is contained in D and $z \in D_r$ implies that $\|f(z) - f(z_0)\| < \frac{1}{2}$. Since $f(z)$ is a compact operator, we can take a finite rank operator $y \in \mathcal{F}(H)$ which satisfies $\|f(z_0) - y\| < \frac{1}{2}$.

If $z \in D_r$ then the triangle inequality yields

$$\|f(z) - y\| \leq \|f(z) - f(z_0)\| + \|f(z_0) - y\| < 1.$$

Consequently, the operator $I - f(z) + y$ is invertible and the function

$$z \in D_r \mapsto (I - f(z) + y)^{-1}$$

is holomorphic.

Now write the operator y as

$$y = \sum_{n=1}^{N} |\psi_n\rangle\langle\varphi_n|$$

for some vectors $\varphi_1, \ldots, \varphi_N, \psi_1, \ldots, \psi_N \in H$, where the set $\{\psi_1, \ldots, \psi_N\}$ is linearly independent. For $z \in D_r$ and $n \in \{1, 2, \ldots, n\}$, define

$$\rho_n(z) = ((I - f(z) + y)^{-1})^* \varphi_n$$

and

$$g(z) = y(I - f(z) + y)^{-1}.$$

We expand the definition of $g(z)$ as

$$\begin{aligned} g(z) &= \sum_{n=1}^{N} |\psi_n\rangle\langle\varphi_n|(1 - f(z) + y)^{-1} \\ &= \sum_{n=1}^{N} |\psi_n\rangle\langle((1 - f(z) + y)^{-1})^* \varphi_n| \\ &= \sum_{n=1}^{N} |\psi_n\rangle\langle\rho_n(z)|. \end{aligned}$$

The primary consequence of this calculation is that $g(z)H \subset yH$.

Now we have

$$(I - g(z))(I - f(z) + y) = I - f(z) + y - y = I - f(z)$$

and since $I - f(z) + y$ is invertible for $z \in D_r$, $I - f(z)$ is invertible if and only if $I - g(z)$ is invertible. Also, $I - f(z)$ is injective if and only if $I - g(z)$ is injective.

Thus, it suffices to prove the analytic Fredholm theorem for $I - g(z)$. First, suppose that there exists $\varphi \in H - \{0\}$ such that $\varphi = g(z)\varphi$. Then, $\varphi \in g(z)H \subseteq yH$. By definition of y,

$$\varphi = \sum_{m=1}^{N} \beta_m \psi_m$$

and by definition of $g(z)$, we compute directly that

$$\begin{aligned}
\sum_{m=1}^{N} \beta_m \psi_m &= \varphi \\
&= g(z)\varphi \\
&= \sum_{n=1}^{N} |\psi_n\rangle\langle\rho_n(z)|\varphi \\
&= \sum_{n=1}^{N} |\psi_n\rangle\langle\varphi, \rho_n(z)\rangle \\
&= \sum_{n=1}^{N} \langle\varphi, \rho_n(z)\rangle\psi_n
\end{aligned}$$

By comparing coefficients, we find that if $n \in \{1, 2, \ldots, N\}$ then

$$\beta_n = \langle\varphi, \rho_n(z)\rangle = \langle\sum_{m=1}^{N} \beta_m \psi_m, \rho_n(z)\rangle = \sum_{m=1}^{N} \langle\psi_m, \rho_n(z)\rangle\beta_m.$$

The key idea is that due to the above equation, the quantity

$$d(z) = \det\Bigg(\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} - \begin{pmatrix} \langle\psi_1, \rho_1(z)\rangle & \ldots & \langle\psi_N, \rho_1(z)\rangle \\ \vdots & \ddots & \vdots \\ \langle\psi_1, \rho_N(z)\rangle & \ldots & \langle\psi_N, \rho_N(z)\rangle \end{pmatrix} \Bigg) = 0.$$

Note that the function d is a holomorphic function on D_r. So, we have proved that if the equation $\varphi = g(z)\varphi$ has a non-zero solution then $d(z) = 0$ for $z \in D_r$.

We will now prove the converse of the above statement. Assume that if $z \in D_r$ then $d(z) = 0$. Then, there exists non-zero $\beta_1, \ldots, \beta_N \in \mathbb{C}$ such that

$$\left(\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} - \begin{pmatrix} \langle \psi_1, \rho_1(z) \rangle & \ldots & \langle \psi_N, \rho_1(z) \rangle \\ \vdots & \ddots & \vdots \\ \langle \psi_1, \rho_N(z) \rangle & \ldots & \langle \psi_N, \rho_N(z) \rangle \end{pmatrix} \right) \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_N \end{pmatrix} = 0.$$

Consequently, $\beta_n = \sum_{m=1}^{N} \langle \psi_m, \rho_n(z) \rangle \beta_m$ and $\varphi = \sum_{m=1}^{N} \beta_m \psi_m$ is a non-zero solution to the equation $\varphi = g(z)\varphi$.

Before we proceed, we note that since d is holomorphic on D_r, its zeros must be isolated. This means that

$$S_r = \{z \in D_r \mid d(z) = 0\}$$

is either a discrete set with no accumulation points in D_r or $S_r = D_r$.

It remains to deal with the situation where $d(z) \neq 0$. Suppose that $\varphi, \xi \in H$ satisfy the equation $(I - g(z))\varphi = \xi$. Let $\varphi' = \varphi - \xi$. Substitute $\varphi = \varphi' + \xi$ into the equation $(I - g(z))\varphi = \xi$. We obtain

$$\varphi' + \xi - g(z)\varphi' - g(z)\xi = \xi.$$

Rearranging gives $\varphi' - g(z)\varphi' = g(z)\xi$. Thus, φ satisfies the equation $(I - g(z))\varphi = \xi$ if and only if φ' satisfies $(I - g(z))\varphi' = g(z)\xi$. If we vary ξ over H, $g(z)\xi$ varies over all of the image $g(z)H$. Therefore, we can solve $(I - g(z))\varphi = \xi$ for any $\xi \in H$ if and only if $d(z) \neq 0$.

We have established two different situations:

1. $d(z) = 0$ if and only if there exists $\varphi \in H - \{0\}$ such that $\varphi = g(z)\varphi$
2. $d(z) \neq 0$ if and only if for any $\xi \in H$, there exists φ such that $(I - g(z))\varphi = \xi$.

In the first scenario, we deduce that $I - g(z)$ is not invertible for any $z \in D_r$. In the second scenario, we find that $I - g(z)$ is invertible on the set

$$D_r \backslash S_r = \{z \in D_r \mid d(z) \neq 0\}$$

where we recall that the zero set S_r is discrete. This completes the proof. $\square$

The Fredholm alternative is now a special case of Theorem 4.2.1.

Theorem 4.2.2 (Fredholm alternative)**.** *Let H be a Hilbert space and $x \in B_0(H)$ be a compact operator. Then, either $I - x$ is invertible or there exists a non-zero $\psi \in H - \{0\}$ such that $\psi = x\psi$.*

Proof. Assume that H is a Hilbert space and $x \in B_0(H)$ is a compact operator on H. Let $D = \mathbb{C}$ and define the map f by

$$\begin{array}{rccc} f: & \mathbb{C} & \to & B(H) \\ & z & \mapsto & zx. \end{array}$$

Then, f is holomorphic on $\mathbb{C}$. By the analytic Fredholm theorem (see Theorem 4.2.1), either $I - f(z)$ is not invertible for any $z \in \mathbb{C}$ or $I - f(z)$ is invertible on $\mathbb{C} - S$, where $S \subseteq \mathbb{C}$ is a discrete set.

If we take $z = 1$ then we find that either $I - x$ is invertible or there exists $\psi \in H - \{0\}$ such that $\psi = x\psi$. $\square$

Here, we highlight that the Fredholm alternative is equivalent to the statement that $I - x$ is invertible if and only if $I - x$ is injective. We will give a standalone proof of this equivalent formulation of Theorem 4.2.2 which does not use Theorem 4.2.1.

Alternative proof of Theorem 4.2.2. Assume that H is a Hilbert space and $x \in B(H)$ is a compact operator. Let $F \in B(H)$ be an operator of finite rank such that $\|x - F\| < \epsilon$ for some $\epsilon \in \mathbb{R}_{>0}$. In particular, we can choose ϵ small enough such that $I - (x - F)$ is invertible (see Theorem 1.2.3). We compute directly that

$$I - x = I - (x - F) - F = \big(I - F(I - (x - F))^{-1}\big)(I - (x - F))$$

Notice that $F(I - (x - F))^{-1}$ is a finite rank operator. Observe that since $(I - (x - F))$ is invertible, $I - x$ is injective if and only if $\big(I - F(I - (x - F))^{-1}\big)$ is injective. Hence, it suffices to prove the Fredholm alternative for finite rank operators. To see why this is the case, suppose that the Fredholm alternative is satisfied for $F(I - (x - F))^{-1}$. Then,

$$\begin{array}{lll} I - x \text{ is invertible} & \text{if and only if} & I - F(I - (x - F))^{-1} \text{ is invertible} \\ & \text{if and only if} & I - F(I - (x - F))^{-1} \text{ is injective} \\ & \text{if and only if} & I - x \text{ is injective.} \end{array}$$

So, suppose that $G \in B(H)$ is a finite rank operator. Then, im G is a finite dimensional subspace of H and is thus, closed. Therefore,

$$H = \text{im } G \oplus (\text{im } G)^{\perp}.$$

Let $V = \text{im } G$. The key observation here is that the restriction $(I - G)|_V$ is an operator from V to V.

To show: (a) $I - G$ is injective if and only if $(I - G)|_V$ is injective.

(b) $I - G$ is surjective if and only if $(I - G)|_V$ is surjective.

(a) Assume that $(I - G)|_V$ is not injective. Then, there exists $v \in V - \{0\}$ such that $(I - G)(v) = 0$. Since $V \subseteq H$, $I - G$ is not injective.

Now assume that $I - G$ is not injective. Then, there exists $h \in H - \{0\}$ such that $(I - G)(h) = 0$. Since $H = V \oplus V^{\perp}$, $h = h_1 + h_2$, where $h_1 \in V$ and $h_2 \in V^{\perp}$. Consequently,

$$\begin{aligned}
(I - G)(h) &= (I - G)(h_1 + h_2) \\
&= h_1 + h_2 - Gh_1 - Gh_2 \\
&= (h_1 - G(h_1 + h_2)) + h_2 \\
&= (h_1 - G(h)) + h_2 = 0.
\end{aligned}$$

Notice that $h_1 - G(h) \in V$ and $h_2 \in V^{\perp}$. By uniqueness of Theorem 2.4.1, $h_2 = 0$ and $h_1 = G(h) = G(h_1 + h_2)$. This means that

$$h_1 - G(h_1) = (I - G)(h_1) = 0$$

where $h_1 \in V - \{0\}$. Hence, $(I - G)|_V$ is not injective.

(b) Assume that $(I - G)|_V$ is surjective. Assume that $y \in H$. Since $H = V \oplus V^{\perp}$, write $y = y_1 + y_2$, where $y_1 \in V$ and $y_2 \in V^{\perp}$.
The idea is that we want to solve the equation

$$(I - G)(x_1 + x_2) = y_1 + y_2$$

for $x_1 \in V$ and $x_2 \in V^{\perp}$. Rearranging yields

$$x_1 - G(x_1 + x_2) - y_1 = -x_2 + y_2.$$

The LHS is an element of V, whereas the RHS is an element of $V^\perp$. Thus, both sides must be equal to zero. This suggests that we take $x_2 = y_2$ and $x_1 = G(x_1 + x_2) + y_1$. Subsequently,

$$(I - G)(x_1) = G(x_2) + y_1 = G(y_2) + y_1.$$

Therefore, if $x_1 = G(y_2) + y_1$ and $x_2 = y_2$ then

$$(I - G)(x_1 + x_2) = G(y_2) + y_1 + y_2 - G(y_2) = y.$$

With this computation in mind, we can now complete the proof. Since $G(y_2) + y_1 \in \operatorname{im} G = V$, by surjectivity of $(I - G)|_V$, there exists $x_1 \in V$ such that $(I - G)(x_1) = G(y_2) + y_1$. So, $x_1 + y_2 \in H$ satisfies $(I - G)(x_1 + y_2) = y$ as required. Hence, $I - G$ is surjective.

Now assume that $I - G$ is surjective. Assume that $z \in V$. Then, there exists $x \in H$ such that $(I - G)(x) = z$. Since $H = V \oplus V^\perp$, write $x = x_1 + x_2$, where $x_1 \in V$ and $x_2 \in V^\perp$. So,

$$x_1 + x_2 - G(x_1) - G(x_2) = z$$

and

$$x_1 - G(x_1 + x_2) - z = -x_2$$

The LHS is an element of V and the RHS is an element of $V^\perp$. So, both expressions must be equal to zero. Consequently, $x_2 = 0$ and $(I - G)(x_1) = z$. Therefore, $(I - G)|_V$ is surjective.

To see why parts (a) and (b) yields the Fredholm alternative for G, note that because $(I - G)|_V : V \to V$ is a linear transformation on the finite dimensional vector space V, it is invertible if and only if it is injective. Hence,

$$\begin{aligned}
I - G \text{ is invertible} \qquad & \text{if and only if} \qquad (I - G)|_V \text{ is invertible} \\
& \text{if and only if} \qquad (I - G)|_V \text{ is injective} \\
& \text{if and only if} \qquad I - G \text{ is injective.}
\end{aligned}$$

The first line follows from both parts (a) and (b). The second line follows from the most recent observation about $(I - G)|_V$. The final line follows from part (a). Thus, the proof is complete. $\square$

The Fredholm alternative leads us to the spectral theorem for compact self-adjoint operators. First, the Riesz-Schauder theorem provides us with valuable information about the spectrum of a compact operator. The non-zero elements of the spectrum are all eigenvalues of the operator which accumulate at the origin.

Theorem 4.2.3 (Riesz-Schauder)**.** *Let H be a Hilbert space and $T \in B(H)$ be a compact operator. Then, $\sigma(T)$ is a discrete subset of $\mathbb{C}$, whose only accumulation point is at 0. Furthermore, any non-zero $\lambda \in \sigma(T)$ is a eigenvalue of finite multiplicity. That is, if $\lambda \in \sigma(T)$ then* $0 < \dim\ker \lambda I - T < \infty$.

Proof. Assume that H is a Hilbert space and $T \in B(H)$ be a compact operator. Assume that $\lambda \in \mathbb{C} - \{0\}$ and write

$$\lambda I - T = \lambda(I - \frac{1}{\lambda}T).$$

The map $z \mapsto zT$ is holomorphic on the open connected set $\mathbb{C} - \{0\}$, where $z = \frac{1}{\lambda}$. By the analytic Fredholm theorem (see Theorem 4.2.1), either $I - zT$ is not invertible on $\mathbb{C} - \{0\}$ or $I - zT$ is invertible on the complement of a discrete subset $\tilde{D}$ of $\mathbb{C} - \{0\}$.

We wish to rule out the first possibility. Recall from equation (1.4) that

$$\sigma(T) \subseteq \{\lambda \in \mathbb{C} \mid |\lambda| \leq \|T\|\}$$

and $\{\lambda \in \mathbb{C} \mid |\lambda| > \|T\|\} \subseteq \rho(T)$. If $\lambda \in \mathbb{C}$ satisfies $\lambda > \|T\|$ (or $z < \frac{1}{\|T\|}$) then $\lambda I - T$ is invertible and consequently, $I - zT$ is invertible. This rules out the first possibility.

Therefore, $(I - zT)^{-1}$ exists on $(\mathbb{C} - \{0\})\backslash\tilde{D}$. Notice that

$$\begin{aligned}
\sigma(T) &= \{\lambda \in \mathbb{C} \mid \lambda I - T \text{ is not invertible}\} \\
&= \{\lambda \in \mathbb{C} - \{0\} \mid I - \frac{1}{\lambda}T \text{ is not invertible}\} \\
&= \{\lambda \in \mathbb{C} - \{0\} \mid z = \frac{1}{\lambda} \in \tilde{D}\}
\end{aligned}$$

which is a discrete set with an accumulation point at $\lambda = 0$.

Now assume that $\lambda \in \sigma(T) - \{0\}$. By the Fredholm alternative (see Theorem 4.2.2), $I - zT$ is not invertible if and only if $\lambda I - T$ is not injective if and only if $\lambda I - T$ is not injective. This means that there exists $\psi \in H - \{0\}$ such that $T\psi = \lambda\psi$. So, λ is an eigenvalue and $0 < \dim\ker(\lambda I - T)$.

Suppose for the sake of contradiction that $\dim\ker(\lambda I - T) = \infty$. Since $\ker(\lambda I - T)$ is a closed subspace of H, we can take $\{\varphi_i\}_{i=1}^{\infty}$ to be an orthonormal basis of $\ker(\lambda I - T)$. Then, the sequence $\{\varphi_i\}_{i=1}^{\infty}$ is bounded, but $\{T\varphi_i\}_{i=1}^{\infty} = \{\lambda\varphi_i\}_{i=1}^{\infty}$ does not have a convergent subsequence because for distinct $i, j \in \mathbb{Z}_{>0}$,

$$\|T\varphi_i - T\varphi_j\| = \sqrt{2}|\lambda|.$$

This contradicts the fact that T is compact. Therefore, $0 < \dim\ker(\lambda I - T) < \infty$ and λ is a eigenvalue of T with finite multiplicity. □

Now we are ready to prove the spectral theorem for compact self-adjoint operators.

Theorem 4.2.4 (Hilbert-Schmidt)**.** *Let H be a Hilbert space and $A \in B(H)$ be a compact, self-adjoint operator. Then, there exists an orthonormal basis $\{\varphi_n\}_{n=1}^{\infty}$ of H consisting of eigenfunctions satisfying $A\varphi_n = \lambda_n\varphi_n$ for $\lambda_n \in \mathbb{R}$. Moreover, $\lambda_n \to 0$ as $n \to \infty$.*

Proof. Assume that H is a Hilbert space and $A \in B(H)$ is a compact self-adjoint operator. By Theorem 2.1.6, $\sigma(A) \subseteq \mathbb{R}$ and by the Riesz Schauder theorem (see Theorem 4.2.3), $\sigma(A)$ is a discrete subset of $\mathbb{C}$ with an accumulation point $0 \in \mathbb{C}$. This means that if $\lambda_1, \lambda_2, \ldots$ are the eigenvalues of A then $\lim_{n\to\infty} \lambda_n = 0$. Also, the eigenspaces of A all have finite non-zero multiplicity.

For each eigenvalue λ_i of A, choose an orthonormal basis for the set of eigenvectors with eigenvalue λ_i. Let $\{\varphi_i\}_{i=1}^{\infty}$ be the collection of all of these orthonormal bases. Then, $\{\varphi_i\}_{i=1}^{\infty}$ is an orthonormal set because eigenvectors corresponding to distinct eigenvalues must be orthogonal. Let

$$M = \overline{span\{\varphi_i\}_{i=1}^{\infty}}.$$

Let $\tilde{A} = A|_{M^{\perp}}$ be the restriction of A to the closed subspace $M^{\perp}$. Then, $\tilde{A}$ is compact and self-adjoint because it is the restriction of a compact self-adjoint operator.

This means that we can apply Theorem 4.2.3 to deduce that if $\mu \in \sigma(\tilde{A})$ then μ is an eigenvalue for $\tilde{A}$ and hence, an eigenvalue for A. In particular, any eigenvector $\psi \in M^\perp$ of $\tilde{A}$ must be an eigenvector of A and consequently, $\psi \in M$.

Since $\psi \in M \cap M^\perp$, $\psi = 0$. Thus, $\sigma(\tilde{A})$ cannot contain any non-zero eigenvalues and the spectral radius is therefore,

$$|\sigma(\tilde{A})| = \sup_{\lambda \in \sigma(\tilde{A})} |\lambda| = 0.$$

Since $\tilde{A}$ is self-adjoint, $\|\tilde{A}\| = |\sigma(\tilde{A})| = 0$. So, $\tilde{A} = 0$.

Now suppose for the sake of contradiction that $x \in M^\perp$ with $x \neq 0$. Then, $Ax = \tilde{A}x = 0$. Since $H = M \oplus M^\perp$, $x \in M$. Since $x \in M \cap M^\perp$, $x = 0$ which contradicts the assumption that $x \neq 0$.

Therefore, $M^\perp = \{0\}$ and $\overline{M} = M = H$ as required. □

An important corollary of Theorem 4.2.4 is that we can write a compact operator in a particular form.

Theorem 4.2.5. *Let H be a Hilbert space and $x \in B_0(H)$ be a compact operator. Then, there exists a finite or countably infinite set N and orthonormal sets $\{\psi_n\}_{n\in N}$ and $\{\varphi_n\}_{n\in N}$ in H and a sequence $\{\lambda_n\}_{n\in N}$ in $\mathbb{R}_{>0}$ such that*

$$x = \sum_{n\in N} \lambda_n |\varphi_n\rangle\langle\psi_n|.$$

Proof. Assume that H is a Hilbert space and $x \in B_0(H)$ is a compact operator on H. By using Theorem 2.6.1, we decompose x as

$$x = v|x| = v(x^*x)^{\frac{1}{2}}.$$

The key observation is that in the above polar decomposition, $|x| = (x^*x)^{\frac{1}{2}}$ is a compact, positive operator. By the spectral theorem (see Theorem 4.2.4), we obtain an orthonormal basis of H consisting of eigenvectors of $|x|$.

Let $\{\psi_n\}_{n\in N}$ be a subset of the orthonormal basis, consisting of eigenvectors of $|x|$ corresponding to non-zero eigenvalues. Then, $\{\psi_n\}_{n\in N}$ is an orthonormal basis for the orthogonal complement $(\ker|x|)^\perp$.

Let $\lambda_n \in \mathbb{R}_{>0}$ be the eigenvalue corresponding to ψ_n for $n \in N$. We claim that

$$|x| = \sum_{n\in N} \lambda_n |\psi_n\rangle\langle\psi_n|.$$

The sum on the RHS is well-defined because if $F \subseteq N$ is a finite subset of N then

$$\sum_{n\in F} \lambda_n |\psi_n\rangle\langle\psi_n| = f_F(|x|)$$

where f_F is defined as the continuous function

$$\begin{array}{rrcl} f_F : & \sigma(|x|) & \to & \mathbb{C} \\ & \lambda \in F & \mapsto & \lambda \\ & \lambda \notin F & \mapsto & 0 \end{array}$$

Now observe that f_F converges uniformly to the identity function id on $\sigma(|x|)$ as F grows larger as a subset of N. Hence, $f_F(|x|)$ converges uniformly to $|x|$ and consequently, $|x| = \sum_{n\in N} \lambda_n |\psi_n\rangle\langle\psi_n|$.

Recall from Theorem 2.6.1 that the partial isometry v maps the subspace $(\ker|x|)^\perp$ isometrically onto the closure $\overline{\operatorname{im} x}$. By setting $\varphi_n = v\psi_n$ for $n \in N$, we obtain another orthonormal set $\{\varphi_n\}_{n\in N}$ and subsequently,

$$x = v \sum_{n\in N} \lambda_n |\psi_n\rangle\langle\psi_n| = \sum_{n\in N} \lambda_n |\varphi_n\rangle\langle\psi_n|.$$

$\square$

Before we end this section, we will give an example of Theorem 4.2.4.

Example 4.2.1. Define the following operator on $L^2([0, 2\pi])$:

$$\begin{array}{rccc} T : & L^2([0,2\pi]) & \to & L^2([0,2\pi]) \\ & f(x) & \mapsto & \frac{i}{2}\Big(\int_0^x f(t)\, dt - \int_x^{2\pi} f(t)\, dt \Big) \end{array}$$

We will show that T is a compact, self-adjoint operator. To see that T is self-adjoint, observe that if $f, g \in L^2([0, 2\pi])$ then

$$
\begin{aligned}
\langle Tf, g\rangle_{L^2} &= \int_0^{2\pi} \overline{Tf}(x) g(x)\, dx \\
&= \int_0^{2\pi} \overline{\frac{i}{2}\Big(\int_0^x f(t)\, dt - \int_x^{2\pi} f(t)\, dt\Big)} g(x)\, dx \\
&= \int_0^{2\pi} -\frac{i}{2}\overline{\Big(\int_0^x f(t)\, dt - \int_x^{2\pi} f(t)\, dt\Big)} g(x)\, dx \\
&= -\frac{i}{2}\int_0^{2\pi}\int_0^x \overline{f(t)} g(x)\, dt\, dx + \frac{i}{2}\int_0^{2\pi}\int_x^{2\pi} \overline{f(t)} g(x)\, dt\, dx \\
&= \int_0^{2\pi} \frac{i}{2}\Big(\int_0^t g(x)\, dx - \int_t^{2\pi} g(x)\, dx\Big)\overline{f(t)}\, dt \\
&= \langle f, Tg\rangle_{L^2}.
\end{aligned}
$$

So, T must be self-adjoint. Note that we are working with integrable functions and thus, allowed to use Fubini's theorem in the second last line.

To see that T is compact, assume that $\{f_n\}$ is a bounded sequence in $L^2([0, 2\pi])$. We will apply the Arzela-Ascoli theorem to obtain a subsequence $\{f_{n_k}\}$ such that $\{Tf_{n_k}\}$ converges.

We will first show that the operator T is bounded. First, we compute for $f \in L^2([0, 2\pi])$ that

$$
\begin{aligned}
|Tf(x)| &= \Big|\frac{i}{2}\Big(\int_0^x f(t)\, dt - \int_x^{2\pi} f(t)\, dt\Big)\Big| \\
&\le |\frac{1}{2}\int_0^x f(t)\, dt| + |\frac{1}{2}\int_x^{2\pi} f(t)\, dt| \\
&\le \frac{1}{2}\int_0^x |f(t)|\, dt + \frac{1}{2}\int_x^{2\pi} |f(t)|\, dt \\
&= \frac{1}{2}\int_0^{2\pi} |f(t)|\, dt \\
&\le \frac{1}{2}\Big(\int_0^{2\pi} |f(t)|^2\, dt\Big)^{\frac{1}{2}}\Big(\int_0^{2\pi} 1\, dt\Big)^{\frac{1}{2}} \\
&= \frac{\sqrt{2\pi}}{2}\|f\|_{L^2} = \sqrt{\frac{\pi}{2}}\|f\|_{L^2}.
\end{aligned}
$$

Hence, the operator norm of T is

$$
\begin{aligned}
\|T\|^2 &= \sup_{\|f\|_{L^2}=1} \|Tf\|_{L^2}^2 \\
&= \sup_{\|f\|_{L^2}=1} \int_0^{2\pi} |Tf(x)|^2 \, dx \\
&\le \sup_{\|f\|_{L^2}=1} \int_0^{2\pi} \frac{\pi}{2} \|f\|_{L^2}^2 \, dx \\
&= \pi^2.
\end{aligned}
$$

So, $\|T\| \le \pi$.

To show: (a) $\{Tf_n\}$ is uniformly bounded.

(b) $\{Tf_n\}$ is equicontinuous.

(a) Since $\{f_n\}$ is bounded, there exists $C \in \mathbb{R}_{>0}$ such that $\|f_n\|_{L^2} < C$. If $n \in \mathbb{Z}_{>0}$ then

$$\|Tf_n\|_{L^2} \le \|T\| \|f_n\|_{L^2} < \pi C.$$

So, $\{Tf_n\}$ is a uniformly bounded sequence in $L^2([0, 2\pi])$.

(b) We compute that if $x, y \in [0, 2\pi]$ then

$$
\begin{aligned}
|(Tf)(x) - (Tf)(y)| &= \Big| \frac{i}{2}\Big(\int_0^x f(t)\, dt - \int_x^{2\pi} f(t)\, dt \Big) - \frac{i}{2}\Big(\int_0^y f(t)\, dt - \int_y^{2\pi} f(t)\, dt \Big) \Big| \\
&= \Big| \frac{1}{2} \int_y^x f(t)\, dt + \frac{1}{2} \int_y^x f(t)\, dt \Big| \\
&\le \int_y^x |f(t)|\, dt \\
&\le \Big(\int_0^{2\pi} |f(t)|^2 \, dt \Big)^{\frac{1}{2}} \Big(\int_y^x 1 \, dt \Big)^{\frac{1}{2}} \\
&= \|f\|_{L^2} |x - y|^{\frac{1}{2}}
\end{aligned}
$$

Assume that $\epsilon \in \mathbb{R}_{>0}$. Set $\delta = \epsilon^2/C^2$. If $|x - y| < \delta$ then

$$|(Tf_n)(x) - (Tf_n)(y)| \le \|f_n\|_{L^2} |x - y|^{\frac{1}{2}} < C \frac{\epsilon}{C} = \epsilon.$$

Since $n \in \mathbb{Z}_{>0}$ was arbitrary, we deduce that $\{Tf_n\}$ is an equicontinuous family of functions as required.

From parts (a) and (b), the conditions of the Arzela-Ascoli theorem are satisfied. Therefore, there exists a convergent subsequence $\{Tf_{n_k}\}$ of $\{Tf_n\}$. Consequently, T is a compact operator as required.

We will now use Theorem 4.2.4 on T in order to find an orthonormal basis for $L^2([0, 2\pi])$. Suppose that $\lambda \in \mathbb{C} - \{0\}$ and $f \in L^2([0, 2\pi])$ such that

$$(Tf)(x) = \frac{i}{2}\Big(\int_0^x f(t)\ dt - \int_x^{2\pi} f(t)\ dt\Big) = \lambda f(x).$$

We want to use the fundamental theorem of calculus. Let F be a function such that $F'(x) = f(x)$. Then,

$$\frac{i}{2}(2F(x) - F(0) - F(2\pi)) = \lambda f(x)$$

and if we differentiate both sides with respect to x, we find that

$$f'(x) - \frac{i}{\lambda} f(x) = 0.$$

So, $f(x) = Ce^{ix/\lambda}$, where $C \in \mathbb{R}_{>0}$. Let us substitute this expression for $f(x)$ back into the equation $(Tf)(x) = \lambda f(x)$. After some computation, we obtain

$$\frac{C\lambda}{2}(2e^{ix/\lambda} - 1 - e^{2\pi i/\lambda}) = \lambda C e^{ix/\lambda}.$$

Therefore, $1 + e^{2\pi i/\lambda} = 0$ and

$$\frac{2\pi i}{\lambda} = i\pi(1 + 2k)$$

for $k \in \mathbb{Z}$. So,

$$\lambda_k = \frac{2}{1 + 2k}.$$

and $f_k(x) = Ce^{(2k+1)ix/2}$. To determine the value of C, we must have $\|f_k\|_{L^2} = 1$ for $k \in \mathbb{Z}_{>0}$ in order for $\{f_k\}_{k\in\mathbb{Z}}$ to be an orthonormal basis for $L^2([0, 2\pi])$. So,

$$1 = \int_0^{2\pi} |f_k(x)|^2\ dx = 2\pi C^2.$$

Therefore, $C = \frac{1}{\sqrt{2\pi}}$ and by the spectral theorem, $\{f_k\}_{k\in\mathbb{Z}}$ is an orthonormal basis for $L^2([0, 2\pi])$, where

$$f_k(x) = \frac{1}{\sqrt{2\pi}} e^{(2k+1)ix/2}.$$

Chapter 5

The trace of an operator

5.1 Definition and properties

In this section, we will investigate a generalisation of the trace in linear algebra to certain operators on a Hilbert space.

Definition 5.1.1. Let H be a Hilbert space and $\{\xi_j\}_{j\in J}$ be an orthonormal basis of H. Let $t \in B(H)_+$ be a positive operator. The **trace** of t, denoted by $Tr_{\{\xi_j\}_{j\in J}}(t)$, is defined by

$$Tr_{\{\xi_j\}_{j\in J}}(t) = \sum_{j\in J} \langle t\xi_j, \xi_j\rangle \in [0, \infty].$$

We use slightly different notation from the definition of the trace in [Sol18, Section 6.1]. This is because the definition above depends on the choice of the orthonormal basis $\{\xi_j\}_{j\in J}$. Later, we will see that the trace is independent of the choice of orthonormal basis, which matches the fact that the trace of a square matrix is independent of the choice of bases.

Lemma 5.1.1. *Let H be a Hilbert space and $x \in B(H)$. Let $\{\xi_j\}_{j\in J}$ be an orthonormal basis. Then,*

$$Tr_{\{\xi_j\}_{j\in J}}(x^*x) = Tr_{\{\xi_j\}_{j\in J}}(xx^*).$$

Recall from Theorem 2.3.4 that x^*x and xx^* are positive operators and so, we can take the trace of these operators.

Proof. Assume that H is a Hilbert space and $\{\xi_j\}_{j\in J}$ is an orthonormal basis for H. Assume that $x \in B(H)$. For $i \in J$, we compute directly that

$$\begin{aligned}
\sum_{j\in J}\langle x^*\xi_j,\xi_i\rangle\langle x\xi_i,\xi_j\rangle &= \sum_{j\in J}\langle \xi_j, x\xi_i\rangle\langle x\xi_i,\xi_j\rangle \\
&= \langle \sum_{j\in J}\langle x\xi_i,\xi_j\rangle\xi_j, x\xi_i\rangle \\
&= \langle x\xi_i, x\xi_i\rangle = \langle x^*x\xi_i,\xi_i\rangle.
\end{aligned}$$

So, the trace of x^*x is

$$Tr_{\{\xi_j\}_{j\in J}}(x^*x) = \sum_{i\in J}\langle x^*x\xi_i,\xi_i\rangle = \sum_{i\in J}\sum_{j\in J}\langle x^*\xi_j,\xi_i\rangle\langle x\xi_i,\xi_j\rangle.$$

Since

$$\langle x^*\xi_j,\xi_i\rangle\langle x\xi_i,\xi_j\rangle = \langle \xi_j, x\xi_i\rangle\langle x\xi_i,\xi_j\rangle = |\langle x\xi_i,\xi_j\rangle|^2 \geq 0,$$

we can safely change the order of summation in the series expression for $Tr_{\{\xi_j\}_{j\in J}}(x^*x)$. Thus, we obtain

$$\begin{aligned}
Tr_{\{\xi_j\}_{j\in J}}(x^*x) &= \sum_{i\in J}\sum_{j\in J}\langle x^*\xi_j,\xi_i\rangle\langle x\xi_i,\xi_j\rangle \\
&= \sum_{j\in J}\sum_{i\in J}\langle x^*\xi_j,\xi_i\rangle\langle x\xi_i,\xi_j\rangle \\
&= \sum_{j\in J}\sum_{i\in J}\langle x\xi_i,\xi_j\rangle\langle x^*\xi_j,\xi_i\rangle \\
&= Tr_{\{\xi_j\}_{j\in J}}(xx^*).
\end{aligned}$$

□

Similarly to the trace in linear algebra, we expect the trace of a positive operator to be invariant under similarity transformations.

Lemma 5.1.2. *Let H be a Hilbert space and $u, t \in B(H)$ with u unitary and t positive. Let $\{\xi_j\}_{j\in J}$ be an orthonormal basis for H. Then,*

$$Tr_{\{\xi_j\}_{j\in J}}(utu^*) = Tr_{\{\xi_j\}_{j\in J}}(t).$$

Proof. Assume that H is a Hilbert space and $u, t \in B(H)$ with u unitary and t positive. Assume that $\{\xi_j\}_{j\in J}$ is an orthonormal basis for H.

Define $x = ut^{\frac{1}{2}} \in B(H)$. We compute directly that

$$x^*x = t^{\frac{1}{2}}u^*ut^{\frac{1}{2}} = t \qquad \text{and} \qquad xx^* = ut^{\frac{1}{2}}t^{\frac{1}{2}}u^* = utu^*.$$

By Lemma 5.1.1, we have

$$Tr_{\{\xi_j\}_{j\in J}}(t) = Tr_{\{\xi_j\}_{j\in J}}(x^*x) = Tr_{\{\xi_j\}_{j\in J}}(xx^*) = Tr_{\{\xi_j\}_{j\in J}}(utu^*).$$

□

With Lemma 5.1.2, we can prove that the trace of a positive operator is independent of the choice of orthonormal basis.

Theorem 5.1.3. *Let H be a Hilbert space and $\{\xi_j\}_{j\in J}$ and $\{\psi_j\}_{j\in J}$ be two different orthonormal bases for H. If $t \in B(H)_+$ is a positive operator then*

$$Tr_{\{\xi_j\}_{j\in J}}(t) = Tr_{\{\psi_j\}_{j\in J}}(t).$$

Proof. Assume that H is a Hilbert space and $\{\xi_j\}_{j\in J}$ and $\{\psi_j\}_{j\in J}$ are two different orthonormal bases for H. Assume that $t \in B(H)_+$ is a positive operator. Then, there exists a unitary operator $u \in B(H)$ such that $u\psi_j = \xi_j$ for $j \in J$. Consequently,

$$\begin{aligned} Tr_{\{\psi_j\}_{j\in J}}(t) &= \sum_{j\in J}\langle t\psi_j, \psi_j\rangle \\ &= \sum_{j\in J}\langle tu^*\xi_j, u^*\xi_j\rangle \\ &= \sum_{j\in J}\langle utu^*\xi_j, \xi_j\rangle \\ &= Tr_{\{\xi_j\}_{j\in J}}(utu^*) = Tr_{\{\xi_j\}_{j\in J}}(t) \end{aligned}$$

where in the last equality, we used Lemma 5.1.2. □

In light of Theorem 5.1.3, we can drop the notation established in the original definition of the trace. We will now simply write Tr to denote the trace of a positive operator.

Next, we will establish a few more familiar properties of the trace.

Lemma 5.1.4. *Let H be a Hilbert space, $\lambda \in \mathbb{R}_{>0}$ and $t, r \in B(H)_+$ be positive operators. Then,*

1. $Tr(\lambda t) = \lambda Tr(t)$

2. *If $t \geq r$ then $Tr(t) \geq Tr(r)$*

3. *$Tr(t + r) = Tr(t) + Tr(r)$*

Proof. Assume that H is a Hilbert space, $\lambda \in \mathbb{R}_{>0}$ and $t, r \in B(H)_+$. Let $\{\xi_j\}_{j\in J}$ be an orthonormal basis for H. We compute directly that

$$Tr(\lambda t) = \sum_{j\in J} \langle \lambda t\xi_j, \xi_j\rangle = \lambda \sum_{j\in J} \langle t\xi_j, \xi_j\rangle = \lambda Tr(t).$$

Next, assume that $t \geq r$ so that $t - r \in B(H)_+$. Then,

$$Tr(t - r) = \sum_{j\in J} \langle (t - r)\xi_j, \xi_j\rangle = \sum_{j\in J} \langle t\xi_j, \xi_j\rangle - \sum_{j\in J} \langle r\xi_j, \xi_j\rangle \in [0, \infty].$$

Hence, $Tr(t) \geq Tr(r)$. Finally, we have for *arbitrary* $t, r \in B(H)_+$,

$$Tr(t + r) = \sum_{j\in J} \langle (t + r)\xi_j, \xi_j\rangle = \sum_{j\in J} \langle t\xi_j, \xi_j\rangle + \sum_{j\in J} \langle r\xi_j, \xi_j\rangle = Tr(t) + Tr(r).$$

□

The final property of the trace we will prove in this section is that it is bounded below by the norm.

Theorem 5.1.5. *Let H be a Hilbert space and $t \in B(H)_+$ be a positive operator. Then, $Tr(t) \geq \|t\|$.*

Proof. Assume that H is a Hilbert space and $t \in B(H)_+$ is a positive operator. By Theorem 2.3.4, there exists a self-adjoint operator $r = t^{1/2}$ such that $r^2 = t$ and $\|t\| = \|r^*r\| = \|r\|^2$.

Assume that $\epsilon \in \mathbb{R}_{>0}$. Let $\psi \in H$ be such that $\|\psi\| = 1$ and

$$\|r\psi\| > \|r\| - \epsilon.$$

The existence of ψ is just from the definition of the operator norm on r. Now we compute that

$$\begin{aligned}
\langle t\psi, \psi\rangle &= \langle r^*r\psi, \psi\rangle \\
&= \langle r^*r\psi, \psi\rangle \\
&= \langle r\psi, r\psi\rangle = \|r\psi\|^2 \\
&> (\|r\| - \epsilon)^2 \\
&= \|r\|^2 - 2\epsilon\|r\| + \epsilon^2 \\
&= \|t\| - 2\epsilon\|t\|^{\frac{1}{2}} + \epsilon^2.
\end{aligned}$$

Now we can choose an orthonormal basis $\{\xi_j\}_{j\in J}$ of H such that $\psi = \xi_i$ for some $i \in J$. We can do this by using Gram-Schmidt orthogonalisation for instance. Consequently,

$$Tr(t) = \sum_{j\in J} \langle t\xi_j, \xi_j\rangle \geq \langle t\xi_i, \xi_i\rangle > \|t\| - 2\epsilon\|t\|^{\frac{1}{2}} + \epsilon^2.$$

Since $\epsilon \in \mathbb{R}_{>0}$ was arbitrary, we obtain $Tr(t) \geq \|t\|$. □

5.2 Trace class and Hilbert-Schmidt operators

The trace gives rise to two important classes of operators.

Definition 5.2.1. Let H be a Hilbert spaces. The set of **trace class operators**, denoted by $B_1(H)$, is defined by

$$B_1(H) = span\{x \in B(H)_+ \mid Tr(x) < \infty\}.$$

Definition 5.2.2. Let H be a Hilbert spaces. The set of **Hilbert-Schmidt operators**, denoted by $B_2(H)$, is defined by

$$B_2(H) = \{x \in B(H) \mid Tr(x^*x) < \infty\}.$$

Our first task with regards to trace class and Hilbert-Schmidt operators is to determine the inclusions amongst the sets $\mathcal{F}(H), B_1(H), B_2(H)$ and $B_0(H)$. We require a few lemmas to do this.

Lemma 5.2.1. *Let H be a Hilbert space and $a, b \in B(H)$. Then, the composite*

$$ab = \frac{1}{4}\sum_{i=0}^{3} i^k (b + i^k I)^* a(b + i^k I)$$

where $I \in B(H)$ is the identity operator.

Proof. The formula follows by expanding the expression $\frac{1}{4}\sum_{i=0}^{3} i^k (b + i^k I)^* a(b + i^k I)$ and then simplifying, which is a somewhat tedious process similar to Theorem 2.1.1 and Theorem 2.1.2. □

The next lemma we require gives an interesting criterion for an operator to be compact.

Lemma 5.2.2. *Let H be a Hilbert space and $x \in B(H)$ be such that $Tr(|x|^p) < \infty$ for some $p \in \mathbb{R}_{>0}$. Then, $x \in B_0(H)$ is compact.*

Proof. Assume that H is a Hilbert space and $x \in B(H)$ such that $Tr(|x|^p) < \infty$ for some $p \in \mathbb{R}_{>0}$.

Let $\{\psi_j\}_{j\in J}$ be an orthonormal basis for H and $\epsilon \in \mathbb{R}_{>0}$. Then, there exists a finite subset $J_\epsilon \subset J$ such that

$$\sum_{j\notin J_\epsilon} \langle |x|^p \psi_j, \psi_j \rangle < \epsilon.$$

Let p_ϵ be the projection operator onto $span\{\psi_j \mid j \in J_\epsilon\}$. We claim that the sequence of finite dimensional operators

$$\{|x|^{\frac{p}{2}} p_\epsilon\}_{\epsilon\in\mathbb{R}_{>0}}$$

converges to $|x|^{\frac{p}{2}}$ in the norm topology. Indeed, by Theorem 5.1.5, we have

$$\begin{aligned}
\||x|^{\frac{p}{2}} - |x|^{\frac{p}{2}} p_\epsilon\|^2 &= \||x|^{\frac{p}{2}}(I - p_\epsilon)\|^2 \\
&= \|(|x|^{\frac{p}{2}}(I - p_\epsilon))^* |x|^{\frac{p}{2}}(I - p_\epsilon)\| \\
&= \|(I - p_\epsilon)|x|^p(I - p_\epsilon)\| \\
&\le Tr((I - p_\epsilon)|x|^p(I - p_\epsilon)) \\
&= \sum_{j\notin J_\epsilon} \langle |x|^p \psi_j, \psi_j \rangle < \epsilon.
\end{aligned}$$

So, $|x|^{\frac{p}{2}}$ is the norm limit of a sequence of finite dimensional operators. Hence, $|x|^{\frac{p}{2}}$ is a compact operator. Now, $|x| = f(|x|^{\frac{p}{2}})$ where $f(\lambda) = \lambda^{\frac{2}{p}}$ for $\lambda \in \sigma(|x|^{\frac{p}{2}})$. Note that we can use the continuous functional calculus because $|x|^{\frac{p}{2}}$ is self-adjoint. Consequently, $|x|$ is a compact operator and since $B_0(H)$ is an ideal, we can use the polar decomposition of x (see Theorem 2.6.1) to find that

$$x = u|x| \in B_0(H).$$

□

Here is the first theorem pertaining to our new classes of operators.

Theorem 5.2.3. *Let H be a Hilbert space. Then, we have the following chain of inclusions*

$$\mathcal{F}(H) \subset B_1(H) \subset B_2(H) \subset B_0(H).$$

Moreover, each of the above subsets is an ideal in $B(H)$ which is self-adjoint (invariant under the adjoint). We also have

$$B_1(H) = \{x \in B(H) \mid Tr(|x|) < \infty\}.$$

Proof. Assume that H is a Hilbert space. It is easy to check that the sets $\mathcal{F}(H), B_1(H), B_2(H)$ and $B_0(H)$ are invariant under the adjoint. For instance, this means that if $x \in B_1(H)$ then $x^* \in B_1(H)$. We already know that $\mathcal{F}(H)$ and $B_0(H)$ are ideals of $B(H)$.

To show: (a) $B_1(H)$ is an ideal.

(b) $B_2(H)$ is an ideal.

(a) We claim that $B_1(H)$ is a right ideal in $B(H)$. Assume that $a \in B(H)_+$ such that $Tr(a) < \infty$ and that $b \in B(H)$.

To show: (aa) $ab \in B_1(H)$.

(aa) We know from Lemma 5.2.1 that

$$ab = \frac{1}{4}\sum_{k=0}^{3} i^k v_k^* a v_k$$

where $v_k = b + i^k I$. For $k \in \{0, 1, 2, 3\}$, we have

$$\begin{aligned}
Tr(v_k^* a v_k) &= Tr(v_k^* a^{\frac{1}{2}} a^{\frac{1}{2}} v_k) \\
&= Tr((a^{\frac{1}{2}} v_k)^* a^{\frac{1}{2}} v_k) \\
&= Tr(a^{\frac{1}{2}} v_k (a^{\frac{1}{2}} v_k)^*) \\
&= Tr(a^{\frac{1}{2}} v_k v_k^* a^{\frac{1}{2}}) \\
&\leq Tr(a^{\frac{1}{2}} \|v_k v_k^*\| a^{\frac{1}{2}}) \\
&= \|v_k\|^2 Tr(a) < \infty.
\end{aligned}$$

By linearity of the trace, $Tr(ab) < \infty$ and $ab \in B_1(H)$. So, $B_1(H)$ is a right ideal of $B(H)$.

(a) Since $B_1(H)$ is a right ideal which is invariant under the adjoint, it must also be a left ideal. If $x \in B_1(H)$ and $y \in B(H)$ then

$yx = (x^*y^*)^* \in B_1(H)$. So, $B_1(H)$ is an ideal of $B(H)$.

(b) We will show that $B_2(H)$ is a left ideal. Assume that $t \in B_2(H)$ and $s \in B(H)$.

To show: (ba) $st \in B_2(H)$.

(ba) Since $t \in B_2(H)$, $Tr(t^*t) < \infty$. Observe that $(st)^*st = t^*s^*st \le \|s\|^2 t^*t$ as positive operators. Consequently,

$$\|s\|^2 Tr(t^*t) = Tr(\|s\|^2 t^*t) > Tr((st)^*st)$$

and $Tr((st)^*st) < \infty$. So, $st \in B_2(H)$.

(b) Part (ba) tells us that $B_2(H)$ is a left ideal which is invariant under the adjoint. By similar reasoning to part (a), we deduce that $B_2(H)$ is also a right ideal and hence, an ideal in $B(H)$.

Next, we will prove that

$$B_1(H) = \{x \in B(H) \mid Tr(|x|) < \infty\}.$$

To show: (c) $\{x \in B(H) \mid Tr(|x|) < \infty\} \subseteq B_1(H)$.

(d) $B_1(H) \subseteq \{x \in B(H) \mid Tr(|x|) < \infty\}$.

(c) Assume that $x \in B(H)$ such that $Tr(|x|) < \infty$. Then, $|x| \in B_1(H)$ and by Theorem 2.6.1, we have $x = u|x|$. By part (a), $B_1(H)$ is an ideal of $B(H)$. So, $x \in B_1(H)$ and $\{x \in B(H) \mid Tr(|x|) < \infty\} \subseteq B_1(H)$.

(d) Now assume that $y \in B_1(H)$. Using the polar decomposition (see Theorem 2.6.1), write $y = v|y|$. Since $B_1(H)$ is an ideal, $|y| = v^*y \in B_1(H)$. By definition of $B_1(H)$,

$$|y| = \sum_{i=1}^{n} \alpha_i d_i$$

where $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ and $d_1, \ldots, d_n \in B(H)_+$ satisfying $Tr(d_i) < \infty$ for $i \in \{1, 2, \ldots, n\}$. As positive operators, we have

$$|y| \le \sum_{i=1}^{n} |\alpha_i| d_i.$$

So,

$$Tr(|y|) \leq \sum_{i=1}^{n} |\alpha_i| Tr(d_i) < \infty$$

and $B_1(H) \subseteq \{x \in B(H) \mid Tr(|x|) < \infty\}$.

Finally, we will prove the inclusions $\mathcal{F}(H) \subset B_1(H) \subset B_2(H) \subset B_0(H)$.

To show: (e) $\mathcal{F}(H) \subset B_1(H)$.

(f) $B_1(H) \subset B_2(H)$.

(g) $B_2(H) \subset B_0(H)$.

(e) Assume that $x \in \mathcal{F}(H)$. Since $\mathcal{F}$ is an ideal, $|x|$ is a finite dimensional positive operator. So, $|x|$ must have finite trace, $|x| \in B_1(H)$ and since $B_1(H)$ is an ideal, $x \in B_1(H)$ and $\mathcal{F}(H) \subseteq B_1(H)$.

(f) Assume that $y \in B_1(H)$. Then,

$$y^*y = |y|^2 = |y|^{\frac{1}{2}}(|y|^{\frac{1}{2}})^*(|y|^{\frac{1}{2}})|y|^{\frac{1}{2}} \leq \|y\||y|.$$

In the first equality, we used a result in Theorem 2.6.2. By taking the trace, we deduce that $Tr(y^*y) \leq \|y\|Tr(|y|) < \infty$. So, $y \in B_2(H)$ and $B_1(H) \subset B_2(H)$.

(g) Assume that $t \in B_2(H)$ so that $Tr(t^*t) < \infty$. By using the polar decomposition, we write $t = u|t|$ so that $t^*t = |t|^2$ (see Theorem 2.6.2). So, $Tr(|t|^2) < \infty$ and by Lemma 5.2.2, $t \in B_0(H)$ and $B_2(H) \subset B_0(H)$. This completes the proof. □

5.3 The Hilbert space $B_2(H)$

Theorem 5.2.3 tells us that $B_1(H)$ and $B_2(H)$ are ideals of $B(H)$. However, they actually possess more structure than that.

Theorem 5.3.1. *Let H be a complex Hilbert space. Then, $B_2(H)$ is a $\mathbb{C}$-vector space.*

Proof. Assume that H is a complex Hilbert space. Assume that $x, y \in B_2(H)$ and $\lambda \in \mathbb{C}$.

To show: (a) $\lambda x \in B_2(H)$

(b) $x + y \in B_2(H)$.

(a) To see that $\lambda x \in B_2(H)$, we compute directly that

$$Tr((\lambda x)^* \lambda x) = Tr(|\lambda|^2 x^* x) = |\lambda|^2 Tr(x^* x) < \infty.$$

So, $\lambda x \in B_2(H)$.

(b) Consider the expression $(x + y)^*(x + y) + (x - y)^*(x - y)$. Expanding, we obtain

$$(x + y)^*(x + y) + (x - y)^*(x - y) = 2x^* x + 2y^* y.$$

This means that as positive operators, $(x + y)^*(x + y) \leq 2x^* x + 2y^* y$. So,

$$Tr((x + y)^*(x + y)) \leq Tr(2x^* x + 2y^* y) = 2(Tr(x^* x) + Tr(y^* y)) < \infty.$$

Hence, $x + y \in B_2(H)$. So, $B_2(H)$ is a $\mathbb{C}$-vector space. $\square$

Here is an important observation about $B_2(H)$. If $x, y \in B_2(H)$ then one can verify that

$$y^* x = \frac{1}{4} \sum_{k=0}^{3} i^k (x + i^k y)^*(x + i^k y).$$

Since $B_2(H)$ is a vector space, $x + i^k y \in B_2(H)$. By taking traces of both sides, we find that

$$Tr(y^* x) = \frac{1}{4} \sum_{k=0}^{3} Tr((x + i^k y)^*(x + i^k y)) < \infty.$$

So, $y^* x \in B_2(H)$. This is crucial to the following theorem.

Theorem 5.3.2. *Let H be a Hilbert space. Then, $B_2(H)$ is a Hilbert space with inner product*

$$\langle x, y \rangle_{Tr} = Tr(y^* x).$$

Proof. Assume that H is a Hilbert space. If $x, y \in B_2(H)$ then $y^* x \in B_2(H)$ and so, the trace $\langle x, y \rangle_{Tr} = Tr(y^* x)$ is well-defined.

We will now show that $\langle -,-\rangle_{Tr}$ is an inner product on $B_2(H)$. Assume that $x_1, x_2, y_1 \in B_2(H)$. We compute directly that

$$\begin{aligned}\langle x_1 + x_2, y_1\rangle_{Tr} &= Tr((y_1)^*(x_1 + x_2)) \\ &= Tr((y_1)^* x_1 + (y_1)^* x_2) \\ &= Tr((y_1)^* x_1) + Tr((y_1)^* x_2) \\ &= \langle x_1, y_1\rangle_{Tr} + \langle x_2, y_1\rangle_{Tr}.\end{aligned}$$

Now assume that $\lambda \in \mathbb{C}$ and $\{\xi_j\}_{j\in J}$ is an orthonormal basis for H. Then,

$$\begin{aligned}\langle \lambda x_1, y_1\rangle_{Tr} &= Tr((y_1)^* \lambda x_1) \\ &= \sum_{j\in J} \langle \lambda (y_1)^* x_1 \xi_j, \xi_j\rangle \\ &= \lambda \sum_{j\in J} \langle (y_1)^* x_1 \xi_j, \xi_j\rangle \\ &= \lambda Tr((y_1)^* x_1) = \lambda \langle x_1, y_1\rangle_{Tr}.\end{aligned}$$

We also have

$$\begin{aligned}\langle x_1, y_1\rangle_{Tr} &= Tr((y_1)^* x_1) \\ &= \sum_{j\in J} \langle (y_1)^* x_1 \xi_j, \xi_j\rangle \\ &= \sum_{j\in J} \langle \xi_j, y_1 (x_1)^* \xi_j\rangle \\ &= \overline{\sum_{j\in J} \langle y_1 (x_1)^* \xi_j, \xi_j\rangle} \\ &= \overline{Tr(y_1 (x_1)^*)} = \overline{\langle y_1, x_1\rangle_{Tr}}\end{aligned}$$

Finally, we have the inequality

$$\|x\|_{Tr}^2 = \langle x, x\rangle_{Tr} = Tr(x^* x) \geq \|x^* x\| = \|x\|^2 \geq 0$$

for $x \in B_2(H)$. Also, $\|x\|_{Tr} = 0$ if and only if $\|x\| = 0$ if and only if $x = 0$. Hence, $\langle -,-\rangle_{Tr}$ is an inner product on $B_2(H)$.

Now suppose that $\{x_n\}_{n\in\mathbb{Z}_{>0}}$ is a Cauchy sequence in $B_2(H)$. Since $\|y\|_{Tr} \geq \|y\|$ for $y \in B_2(H)$, $\{x_n\}$ is a Cauchy sequence in $B(H)$ (equipped with the operator norm). Hence, $\{x_n\}$ must converge to some $x \in B(H)$.

Let $\{\xi_j\}_{j\in J}$ be an orthonormal basis of H. Since $x_n \to x$ in the norm topology, the sequence $\{x_n\}$ must also converge to x in the strong topology. For a finite subset $J_0 \subset J$, we have

$$\begin{aligned}
\sum_{j\in J_0} \|(x - x_n)\xi_j\|^2 &= \lim_{m\to\infty} \sum_{j\in J_0} \|(x_m - x_n)\xi_j\|^2 \\
&\leq \limsup_{m\to\infty} \sum_{j\in J} \|(x_m - x_n)\xi_j\|^2 \\
&= \limsup_{m\to\infty} \sum_{j\in J} \langle (x_m - x_n)^*(x_m - x_n)\xi_j, \xi_j\rangle \\
&= \limsup_{m\to\infty} Tr((x_m - x_n)^*(x_m - x_n)) \\
&= \limsup_{m\to\infty} \|x_m - x_n\|_{Tr}^2.
\end{aligned}$$

Therefore,

$$\|x - x_n\|_{Tr}^2 = \sup_{J_0\subset J,\ |J_0|<\infty} \sum_{j\in J_0} \|(x - x_n)\xi_j\|^2 \leq \limsup_{m\to\infty} \|x_m - x_n\|_{Tr}^2.$$

Hence, $x \in B_2(H)$ and $x_n \to x$ in $B_2(H)$. Hence, $B_2(H)$ is complete and $B_2(H)$ is a Hilbert space. □

The norm $\|-\|_{Tr}$ on $B_2(H)$ is referred to as the **Hilbert-Schmidt norm**.

The next theorem tells us how to construct an orthnormal basis for $B_2(H)$.

Theorem 5.3.3. *Let H be a Hilbert space and $\{\psi_i\}_{i\in I}$ be an orthonormal basis for H. Then, the set $\{|\psi_i\rangle\langle\psi_j|\}_{i,j\in I}$ is an orthonormal basis for the Hilbert space $B_2(H)$.*

Proof. Assume that H is a Hilbert space with orthonormal basis $\{\psi_i\}_{i\in I}$. To see that $\{|\psi_i\rangle\langle\psi_j|\}_{i,j\in I}$ is an orthonormal basis for $B_2(H)$, we compute directly that

$$
\begin{aligned}
\langle |\psi_i\rangle\langle\psi_j|, |\psi_p\rangle\langle\psi_q|\rangle_{Tr} &= Tr((|\psi_p\rangle\langle\psi_q|)^*|\psi_i\rangle\langle\psi_j|) \\
&= \sum_{k\in I}\langle\ (|\psi_p\rangle\langle\psi_q|)^*|\psi_i\rangle\langle\psi_j|\ \psi_k, \psi_k\rangle \\
&= \sum_{k\in I}\langle|\psi_i\rangle\langle\psi_j|\ \psi_k, |\psi_p\rangle\langle\psi_q|\ \psi_k\rangle \\
&= \sum_{k\in I}\langle|\psi_i\rangle\langle\psi_k, \psi_j\rangle, |\psi_p\rangle\langle\psi_k, \psi_q\rangle\rangle \\
&= \sum_{k\in I}\delta_{k,j}\delta_{k,q}\langle\psi_i, \psi_p\rangle = \delta_{j,q}\delta_{i,p}.
\end{aligned}
$$

So, $\{|\psi_i\rangle\langle\psi_j|\}_{i,j\in I}$ is an orthonormal basis for $B_2(H)$. □

5.4 The Banach algebra $B_1(H)$

Now we turn our attention to the trace class operators $B_1(H)$. We require a few lemmas for this purpose.

Lemma 5.4.1. *Let H be a Hilbert space.*

1. *If $x, y \in B_2(H)$ then $Tr(xy) = Tr(yx)$.*
2. *If $x \in B_1(H)$ and $y \in B(H)$ then $Tr(xy) = Tr(yx)$.*

Proof. Assume that H is a Hilbert space. First, assume that $x, y \in B_2(H)$. Then, we compute directly that

$$
\begin{aligned}
Tr(x^*y) &= \frac{1}{4}\sum_{k=0}^{3} i^k Tr((y+i^k x)^*(y+i^k x)) \\
&= \frac{1}{4}\sum_{k=0}^{3} i^k Tr((y+i^k x)(y+i^k x)^*) \\
&= \frac{1}{4}\sum_{k=0}^{3} i^k Tr((y^*-i^k x^*)^*(y^*-i^k x^*)) \\
&= \frac{1}{4}\sum_{k=0}^{3} i^k Tr(\big((-i^k)(i^k y^*+x^*)\big)^*(-i^k)(i^k y^*+x^*)) \\
&= \frac{1}{4}\sum_{k=0}^{3} i^k Tr((i^k y^*+x^*)^*(i^k y^*+x^*)) \\
&= \frac{1}{4}\sum_{k=0}^{3} i^k Tr((x^*+i^k y^*)^*(x^*+i^k y^*)) = Tr(yx^*).
\end{aligned}
$$

Since $x \in B_2(H)$, $x^* \in B_2(H)$. By replacing x with x^* in the above computation, we deduce that $Tr(xy) = Tr(yx)$ as required.

Now assume that $x \in B_1(H)$ and $y \in B(H)$. Then, x can be written as a linear combination of positive operators with finite trace. Assume that z is a positive operator with finite trace. Then, $z^{\frac{1}{2}}, z^{\frac{1}{2}}y$ and $yz^{\frac{1}{2}}$ are all elements of $B_2(H)$ and by the previous result, we have

$$
\begin{aligned}
Tr(zy) &= Tr(z^{\frac{1}{2}}(z^{\frac{1}{2}}y)) \\
&= Tr((z^{\frac{1}{2}}y)z^{\frac{1}{2}}) \\
&= Tr(z^{\frac{1}{2}}(yz^{\frac{1}{2}})) = Tr(yz).
\end{aligned}
$$

By linearity of the trace, $Tr(xy) = Tr(yx)$. □

Lemma 5.4.2. *Let H be a Hilbert space, $x \in B_1(H)$ and $y \in B(H)$. Then,*

$$|Tr(yx)| \le \|y\| Tr(|x|).$$

Proof. Assume that H is a Hilbert space, $x \in B_1(H)$ and $y \in B(H)$. Decompose x as $x = u|x|$ via the polar decomposition (see Theorem 2.6.1). Recall that

$$B_1(H) = \{x \in B(H) \mid Tr(|x|) < \infty\}.$$

So, $Tr(|x|) < \infty$ and $|x|^{\frac{1}{2}} \in B_2(H)$. Using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
|Tr(yx)|^2 &= |Tr(yu|x|^{\frac{1}{2}}|x|^{\frac{1}{2}})|^2 \\
&= |\langle |x|^{\frac{1}{2}}, |x|^{\frac{1}{2}}u^*y^*\rangle_{Tr}|^2 \\
&\leq \||x|^{\frac{1}{2}}\|_{Tr}^2\||x|^{\frac{1}{2}}u^*y^*\|_{Tr}^2 \qquad \text{(Cauchy Schwarz Inequality)} \\
&= Tr(|x|)Tr((yu|x|^{\frac{1}{2}})(|x|^{\frac{1}{2}}u^*y^*)) \\
&= Tr(|x|)Tr((|x|^{\frac{1}{2}}u^*y^*)(yu|x|^{\frac{1}{2}})) \quad \text{(see Lemma 5.4.1)} \\
&\leq Tr(\|yu\|^2|x|^{\frac{1}{2}}|x|^{\frac{1}{2}})Tr(|x|) \\
&\leq \|y\|^2 Tr(|x|)^2.
\end{aligned}$$

□

Now, we are ready to prove that $B_1(H)$ is a Banach algebra with the **trace norm**

$$\|x\|_1 = Tr(|x|).$$

Lemma 5.4.3. *Let H be a Hilbert space. Then, the trace norm $\|-\|_1$ defined as above is a norm on $B_1(H)$.*

Proof. Assume that H is a Hilbert space. First, assume that $\alpha \in \mathbb{C}$ and $x \in B_1(H)$. Then,

$$|\alpha x| = \big((\alpha x)^*\alpha x\big)^{\frac{1}{2}} = (|\alpha|^2 x^* x)^{\frac{1}{2}} = |\alpha||x|$$

and

$$\|\alpha x\|_1 = Tr(|\alpha x|) = |\alpha| Tr(|x|) = |\alpha|\|x\|_1.$$

Next, if $x \in B_1(H)$ then

$$\|x\|_1 = Tr(|x|) \geq \||x|\| = \|x\| \geq 0.$$

Also, $\|x\|_1 = 0$ if and only if $\|x\| = 0$ if and only if $x = 0$.

Finally, assume that $x, y \in B_1(H)$. Let $x + y = v|x + y|$ be the polar decomposition of $x + y$. By a direct computation, we obtain

$$\begin{aligned}
\|x+y\|_1 &= Tr(|x+y|) \\
&= Tr(v^*(x+y)) \\
&= Tr(v^*x) + Tr(v^*y) \\
&= |Tr(v^*x) + Tr(v^*y)| \\
&\leq |Tr(v^*x)| + |Tr(v^*y)| \\
&\leq \|v^*\|(Tr(|x|) + Tr(|y|)) \\
&\leq \|x\|_1 + \|y\|_1.
\end{aligned}$$

In the last inequality, we used the fact that $\|v^*\| \leq 1$ as v^* is a partial isometry. Therefore, the trace norm $\|-\|_1$ is a norm on $B_1(H)$. □

Our first task is to show that $B_1(H)$ is complete with respect to the norm $\|-\|_1$. The idea is to show that $B_1(H)$ is isomorphic to the dual space $B_0(H)^*$ via an extension of the trace to $B_1(H)$. Hence, we have to demonstrate how to extend the trace from $B(H)_+$ to $B_1(H)$.

First, we need the following lemma.

Lemma 5.4.4. *Let H be a Hilbert space. Then,*

$$B_1(H) \cap B(H)_+ = \{t \in B(H)_+ \mid Tr(t) < \infty\}.$$

Proof. Assume that H is a Hilbert space. We know by definition that

$$B_1(H) = span\{x \in B(H)_+ \mid Tr(x) < \infty\}.$$

So,

$$\{x \in B(H)_+ \mid Tr(x) < \infty\} \subseteq B_1(H) \cap B(H)_+.$$

Conversely, assume that $y \in B_1(H) \cap B(H)_+$. Then, $y = |y|$ (see Theorem 2.6.1) and

$$y \in \{x \in B(H) \mid Tr(|x|) < \infty\} = B_1(H).$$

The last equality in the above equation follows from Theorem 5.2.3. Therefore, $Tr(y) = Tr(|y|) < \infty$,

$$y \in \{x \in B(H)_+ \mid Tr(x) < \infty\}$$

and

$$B_1(H) \cap B(H)_+ = \{t \in B(H)_+ \mid Tr(t) < \infty\}.$$

$\square$

Now, we can extend the trace map defined on $B_1(H) \cap B(H)_+$ to $B_1(H)$.

Theorem 5.4.5. *Let H be a Hilbert space. Then, the trace map*

$$\begin{array}{rccc} Tr: & B_1(H) \cap B(H)_+ & \to & \mathbb{R}_{>0} \\ & x & \mapsto & Tr(x) \end{array}$$

extends uniquely to a linear functional on $B_1(H)$.

Proof. Assume that H is a Hilbert space. We know that $B_1(H) = span(B_1(H) \cap B(H)_+)$. So, it suffices to prove that an extension of the trace to $B_1(H)$ exists (linearity is for free).

Let $x \in B_1(H)$. By definition of $B_1(H)$, we can write $x = \sum_{i=1}^N \alpha_i x_i$, where $\alpha_i \in \mathbb{C}$ and $x_i \in B(H)_+ \cap B_1(H)$. Define

$$\begin{array}{rccc} Tr: & B_1(H) & \to & \mathbb{C} \\ & \sum_{i=1}^N \alpha_i x_i & \mapsto & \sum_{i=1}^N \alpha_i Tr(x_i). \end{array}$$

It is easy to see that restriction to $B(H)_+ \cap B_1(H)$ provides the original trace. To see that the trace above on $B_1(H)$ is the extension we are after, it suffices to check that if $\sum_{i=1}^N \alpha_i x_i = 0$ then $\sum_{i=1}^N \alpha_i Tr(x_i) = 0$.

To show: (a) If $\sum_{i=1}^N \alpha_i x_i = 0$ then $\sum_{i=1}^N Re(\alpha_i) Tr(x_i) = 0$.

(b) If $\sum_{i=1}^N \alpha_i x_i = 0$ then $\sum_{i=1}^N Im(\alpha_i) Tr(x_i) = 0$.

(a) Define

$$A = \{i \in \{1, 2, \ldots, N\} \mid Re(\alpha_i) \geq 0\}$$

and

$$B = \{i \in \{1, 2, \ldots, N\} \mid Re(\alpha_i) < 0\}$$

Then, $A \cup B = \{1, 2, \ldots, N\}$ and since $\sum_{i=1}^N \alpha_i x_i = 0$,

$$\sum_{i \in A} Re(\alpha_i) x_i = \sum_{i \in B} (-Re(\alpha_i)) x_i.$$

Note that both expressions above are linear combinations of positive operators with positive coefficients. Using linearity of the trace, we deduce that

$$\sum_{i\in A} Re(\alpha_i)Tr(x_i) = \sum_{i\in B}(-Re(\alpha_i))Tr(x_i)$$

and $\sum_{i=1}^{N} Re(\alpha_i)Tr(x_i) = 0$.

(b) This follows from exactly the same argument as in part (a). □

Now that we have extended the trace Tr to $B_1(H)$, we ask ourselves what is a formula for computing the trace of $x \in B_1(H)$?

Theorem 5.4.6. *Let H be a Hilbert space, $x \in B_1(H)$ and $\{\xi_i\}_{i\in I}$ be an orthonormal basis of H. Then, the sum $\sum_{i\in I}\langle x\xi_i, \xi_i\rangle$ is absolutely convergent and its sum is independent of the choice of basis $\{\xi_i\}_{i\in I}$.*

Proof. Assume that $x \in B_1(H)$ and $\{\xi_i\}_{i\in I}$ is an orthonormal basis for H. Using the polar decomposition of x, we write $x = u|x|^{\frac{1}{2}}|x|^{\frac{1}{2}}$. The point here is that

$$|x|^{\frac{1}{2}}, |x|^{\frac{1}{2}}u \in B_2(H).$$

By the Cauchy-Schwarz inequality, we have for $i \in I$

$$|\langle x\xi_i, \xi_i\rangle| = |\langle u|x|^{\frac{1}{2}}|x|^{\frac{1}{2}}\xi_i, \xi_i\rangle| = |\langle |x|^{\frac{1}{2}}\xi_i, |x|^{\frac{1}{2}}u^*\xi_i\rangle| \leq \||x|^{\frac{1}{2}}u^*\xi_i\|\||x|^{\frac{1}{2}}\xi_i\|.$$

Since $|x|^{\frac{1}{2}}, |x|^{\frac{1}{2}}u \in B_2(H)$,

$$\sum_{i\in I}\||x|^{\frac{1}{2}}\xi_i\|^2 = Tr(|x|) < \infty$$

and

$$\sum_{i\in I}\||x|^{\frac{1}{2}}u^*\xi_i\|^2 = Tr(u|x|u^*) < \infty$$

By Hölder's inequality,

$$\begin{aligned}\sum_{i\in I}|\langle x\xi_i, \xi_i\rangle| &\leq \sum_{i\in I}\||x|^{\frac{1}{2}}u^*\xi_i\|\||x|^{\frac{1}{2}}\xi_i\| \\ &\leq \Big(\sum_{i\in I}\||x|^{\frac{1}{2}}u^*\xi_i\|^2\Big)^{\frac{1}{2}}\Big(\sum_{i\in I}\||x|^{\frac{1}{2}}\xi_i\|^2\Big)^{\frac{1}{2}} \\ &< \infty.\end{aligned}$$

Therefore, the series $\sum_{i\in I}\langle x\xi_i,\xi_i\rangle$ is absolutely convergent. Now write $x=\sum_{k=1}^{N}\alpha_k x_k$, where $\alpha_1,\ldots,\alpha_N\in\mathbb{C}$ and the operators $x_1,\ldots,x_N\in B(H)_+\cap B_1(H)$. So,

$$\begin{aligned}\sum_{i\in I}\langle x\xi_i,\xi_i\rangle &= \sum_{i\in I}\langle\sum_{k=1}^{N}\alpha_k x_k\xi_i,\xi_i\rangle\\ &= \sum_{i\in I}\sum_{k=1}^{N}\alpha_k\langle x_k\xi_i,\xi_i\rangle\\ &= \sum_{k=1}^{N}\alpha_k\sum_{i\in I}\langle x_k\xi_i,\xi_i\rangle\end{aligned}$$

and by Theorem 5.1.3, the sum $\sum_{k=1}^{N}\alpha_k\sum_{i\in I}\langle x_k\xi_i,\xi_i\rangle$ is independent of the choice of orthonormal basis.

Finally, we note that if $x\in B_1(H)$ then the map $x\mapsto\sum_{i\in I}\langle x\xi_i,\xi_i\rangle$ is a linear functional equal to the trace on $B_1(H)\cap B(H)_+$. So, $Tr(x)=\sum_{i\in I}\langle x\xi_i,\xi_i\rangle$ for any $x\in B_1(H)$ and any orthonormal basis $\{\xi_i\}_{i\in I}$ of H. □

So, the same formula for the trace on $B_1(H)\cap B(H)_+$ works for the trace on $B_1(H)$. From here, the trace Tr as written generally refers to the trace on $B_1(H)$.

Now we can show that $B_1(H)$ is complete with respect to the norm $\|-\|_1$.

Theorem 5.4.7. *Let H be a Hilbert space and $x\in B_1(H)$. Define the map*

$$\begin{array}{rccc}\varphi_x: & B_0(H) & \to & \mathbb{C}\\ & y & \mapsto & Tr(xy)\end{array}$$

Then, $\varphi_x\in B_0(H)^*$ and the map $x\mapsto\varphi_x$ is a bijective isometry from $B_1(H)$ with the norm $\|-\|_1$ to $B_0(H)^*$ with the operator norm.

Proof. Assume that H is a Hilbert space, $x\in B_1(H)$ and φ_x is the map defined as above. By linearity of the trace (see Lemma 5.1.4), φ_x is a linear map. To see that φ_x is bounded, assume that $y\in B_0(H)$. Then,

$$|\varphi_x(y)|=|Tr(xy)|=|Tr(yx)|\le\|y\|Tr(|x|)=\|y\|\|x\|_1<\infty.$$

This uses Lemma 5.4.1 and Lemma 5.4.2. Therefore, if $x\in B_1(H)$ then $\varphi_x\in B_0(H)^*$.

To show: (a) If $\varphi \in B_0(H)^*$ then there exists $x \in B_1(H)$ such that $\varphi = \varphi_x$.

(b) φ_x is an isometry.

(a) Assume that $\varphi \in B_0(H)^*$. If $s \in B_2(H) \subset B_0(H)$ then

$$|\varphi(s)| \leq \|\varphi\|\|s\| \leq \|\varphi\|\|s\|_{Tr}.$$

Consequently, φ is a bounded linear functional on the Hilbert space $B_2(H)$. By the Riesz representation theorem, there exists $x \in B_2(H)$ such that

$$\varphi(s) = \langle s, x^* \rangle_{Tr}.$$

Now let $x = u|x|$ be the polar decomposition of x (see Theorem 2.6.1). Let $\{\psi_j\}_{j \in J}$ be an orthonormal basis for H and $J_0 \subset J$ be a finite subset of J. Let p_0 be the projection onto the closed subpsace $span\{\psi_j \mid j \in J_0\}$.

By Theorem 5.2.3, $x, p_0 \in B_2(H)$. So,

$$\begin{aligned}
\sum_{j \in J_0} \langle |x|\psi_j, \psi_j \rangle &= \Big| \sum_{j \in J_0} \langle |x|\psi_j, \psi_j \rangle \Big| \\
&= \Big| \sum_{j \in J} \langle p_0|x|\psi_j, \psi_j \rangle \Big| \\
&= |Tr(p_0|x|)| \\
&= |Tr((p_0 u^*)x)| \\
&= |Tr(x(p_0 u^*))| = |\langle p_0 u^*, x^* \rangle_{Tr}| \\
&= |\varphi(p_0 u^*)| \\
&\leq \|\varphi\|\|p_0 u^*\| \leq \|\varphi\|.
\end{aligned}$$

Taking the supremum over finite subsets $J_0 \subset J$, we deduce that $x \in B_1(H)$ (because $Tr(|x|) \leq \|\varphi\| < \infty$) and $\|x\|_1 \leq \|\varphi\|$.

Thus, if $s \in B_2(H)$ then

$$\varphi_x(s) = Tr(xs) = \langle s, x^* \rangle_{Tr} = \varphi(s).$$

Since $B_2(H)$ is a dense subspace of $B_0(H)$, we deduce that $\varphi = \varphi_x$ on $B_0(H)$.

(b) We compute directly that

$$\|\varphi_x\| = \sup_{\|y\|=1} |\varphi_x(y)| \leq \sup_{\|y\|=1} \|y\|\|x\|_1 = \|x\|_1.$$

From part (a), we have $\|x\|_1 \leq \|\varphi_x\|$. So, $\|x\|_1 = \|\varphi_x\|$ and φ_x is an isometry.

Combining parts (a) and (b), we deduce that the map $x \mapsto \varphi_x$ is a bijective isometry from $B_1(H)$ to $B_0(H)^*$. $\square$

Theorem 5.4.7 can be thought of as a "non-commutative analogue" of the well-known isomorphism $c_0^* \cong \ell^1$. We will now use Theorem 5.4.7 to prove the main result of this section.

Theorem 5.4.8. *Let H be a Hilbert space. Then, the ideal $B_1(H) \subset B(H)$ is a Banach algebra with the trace norm*

$$\|x\|_1 = Tr(|x|).$$

Proof. Assume that H is a Hilbert space. From Lemma 5.4.3, we know that $\|-\|_1$ is a norm on $B_1(H)$.

To show: (a) $B_1(H)$ is complete with respect to the trace norm.

(b) If $x, y \in B_1(H)$ then $\|xy\|_1 \leq \|x\|_1\|y\|_1$.

(a) We know that $B_1(H)$ is isometrically isomorphic to the dual space $B_0(H)^*$, as a consequence of Theorem 5.4.7. Since $B_0(H)$ is a closed subspace of $B(H)$, $B_0(H)$ is a Banach space and hence, $B_0(H)^*$ is also a Banach space. So, $B_1(H)$ is complete with respect to the trace norm.

(b) Assume that $x, y \in B_1(H)$. Let $xy = u|xy|$ be the polar decomposition of xy. We compute directly that

$$\begin{aligned}
\|xy\|_1 &= Tr(|xy|) \\
&= Tr(u^*xy) = |Tr(u^*xy)| \\
&\leq \|u^*x\|Tr(|y|) \quad \text{(Lemma 5.4.2)} \\
&\leq \|x\|Tr(|y|) \\
&\leq \|x\|_1 Tr(|y|) = \|x\|_1\|y\|_1.
\end{aligned}$$

By combining parts (a) and (b), we deduce that $B_1(H)$ is a Banach algebra with the trace norm $\|-\|_1$. $\square$

To finish this section, we note that there is a non-commutative analogue of the isometric isomorphism $(\ell^1)^* \cong \ell^\infty$.

Theorem 5.4.9. *Let H be a Hilbert space. For $y \in B(H)$, define*

$$\begin{array}{rccc} \psi_y : & B_1(H) & \to & \mathbb{C} \\ & x & \mapsto & Tr(yx). \end{array}$$

Then, $\psi_y \in B_1(H)^*$ and the map $y \mapsto \psi_y$ is a bijective isometry from $B(H)$ to $B_1(H)^*$.

Proof. Assume that H is a Hilbert space, $y \in B(H)$ and ψ_y is the map defined as above. By linearity of the trace, it is easy to check that ψ_y is a linear map. To see that ψ_y is bounded, we argue similarly to Theorem 5.4.7. If $x \in B_1(H)$ then

$$|\psi_y(x)| = |Tr(yx)| \le \|y\| Tr(|x|) = \|y\|\|x\|_1 < \infty.$$

So, $\psi_y \in B_1(H)^*$ and the above inequality shows us that $\|\psi_y\| \le \|y\|$.

Now, we will show that ψ_y is invertible. Assume that $\psi \in B_1(H)^*$. Define the sesquilinear form F by

$$\begin{array}{rccc} F : & H \times H & \to & \mathbb{C} \\ & (\eta, \xi) & \mapsto & \psi(|\eta\rangle\langle\xi|). \end{array}$$

In order for F to be well-defined, we must first show that if $\eta, \xi \in H$ then $|\eta\rangle\langle\xi| \in B_1(H)$.

To show: (a) If $\eta, \xi \in H$ then $|\eta\rangle\langle\xi| \in B_1(H)$.

(a) If $\eta = 0$ then $|\eta\rangle\langle\xi| = 0 \in B_1(H)$. So, assume that $\eta, \xi \in H$ with $\eta \neq 0$. Then,

$$\big||\eta\rangle\langle\xi|\big|^2 = \big(|\eta\rangle\langle\xi|\big)^*\big(|\eta\rangle\langle\xi|\big) = |\xi\rangle\langle\eta||\eta\rangle\langle\xi| = \|\eta\|^2|\xi\rangle\langle\xi| = \|\eta\|^2\|\xi\|^2|\zeta\rangle\langle\zeta|$$

where $\zeta = \xi/\|\xi\|$. Since $|\zeta\rangle\langle\zeta||\zeta\rangle\langle\zeta| = |\zeta\rangle\langle\zeta|$ (because $\|\zeta\| = 1$), $|\eta\rangle\langle\xi| = \|\eta\|\|\xi\||\zeta\rangle\langle\zeta|$ and

$$\big\||\eta\rangle\langle\xi|\big\|_1 = Tr\big(\big||\eta\rangle\langle\xi|\big|\big) = \|\eta\|\|\xi\|Tr(|\zeta\rangle\langle\zeta|) = \|\eta\|\|\xi\|.$$

Therefore, $|\eta\rangle\langle\xi| \in B_1(H)$ and consequently, F is well-defined.

Next, we show that F is bounded. We compute directly that

$$|F(\eta,\xi)| = |\psi(|\eta\rangle\langle\xi|)| \leq \|\psi\|\big\||\eta\rangle\langle\xi|\big\|_1 = \|\psi\|\|\eta\|\|\xi\|.$$

Hence, F is a bounded sesquilinear form on H. For $\xi \in H$, we define the bounded linear functional

$$\begin{array}{rccc} F_\xi : & H & \to & \mathbb{C} \\ & \eta & \mapsto & F(\eta,\xi). \end{array}$$

By the Riesz representation theorem, there exists $y \in B(H)$ such that

$$F_\xi(\eta) = \langle y\eta, \xi\rangle.$$

Now let $x \in B_1(H)$ so that the polar decomposition of x is $u|x|$. Since $B_1(H) \subset B_0(H)$, x is a compact operator. From Theorem 4.2.5 we can write

$$x = \sum_{n=1}^{\infty} \mu_n |\varphi_n\rangle\langle\psi_n|$$

and

$$|x| = \sum_{n=1}^{\infty} \mu_n |\psi_n\rangle\langle\psi_n|$$

where $\{\psi_n\}_{n\in\mathbb{Z}_{>0}}$ and $\{\varphi_n\}_{n\in\mathbb{Z}_{>0}}$ are orthonormal sets and $\{\mu_n\}$ is a sequence in $\mathbb{R}_{>0}$. By extending $\{\psi_n\}$ to an orthonormal basis for H, we can use the above equation to deduce that

$$Tr(|x|) = \sum_{n=1}^{\infty} \mu_n = \sum_{n=1}^{\infty} |\mu_n| < \infty.$$

The quantity $\sum_{n=1}^{\infty} \mu_n$ is finite because $x \in B_1(H)$. So, the sequence of partial sums of $\sum_{n=1}^{\infty} \mu_n |\varphi_n\rangle\langle\psi_n|$ must also converge with respect to the trace norm because

$$\big\||\varphi_n\rangle\langle\psi_n|\big\|_1 = \|\varphi_n\|\|\psi_n\| = 1.$$

Consequently,

$$
\begin{aligned}
Tr(yx) &= \sum_{n=1}^{\infty} \langle yx\psi_n, \psi_n\rangle \\
&= \sum_{n=1}^{\infty} \mu_n \langle y\varphi_n, \psi_n\rangle \\
&= \sum_{n=1}^{\infty} \mu_n F(\varphi_n, \psi_n) = \sum_{n=1}^{\infty} \mu_n \psi(|\varphi_n\rangle\langle\psi_n|) \\
&= \psi(\sum_{n=1}^{\infty} \mu_n |\varphi_n\rangle\langle\psi_n|) = \psi(x).
\end{aligned}
$$

We find that $\psi = \psi_y$ on all of $B_1(H)$. Therefore, the map $y \mapsto \psi_y$ is a bijection.

We finally show that $\|\psi_y\| \geq \|y\|$ so that the map $y \mapsto \psi_y$ is an isometry. Assume that $\epsilon \in \mathbb{R}_{>0}$. Then, there exists $\beta \in H$ such that $\|\beta\| = 1$ and $\|y\beta\| > \|y\| - \epsilon$. Define

$$
\phi = \frac{1}{\|y\beta\|}\|y\beta\|.
$$

Extend the orthonormal set $\{\phi\}$ to an orthonormal basis $\{\phi_j\}_{j\in J}$ of H. Now set $x = |\beta\rangle\langle\phi|$. Then,

$$
\|x\|_1 = \|\beta\|\|\phi\| = 1
$$

and

$$
\begin{aligned}
|\psi_y(x)| &= |Tr(yx)| \\
&= |\sum_{j\in J} \langle yx\phi_j, \phi_j\rangle| \\
&\geq |\langle yx\phi, \phi\rangle| \\
&= \langle y\beta, \phi\rangle = \frac{1}{\|y\beta\|}\langle y\beta, y\beta\rangle \\
&= \|y\beta\| > \|y\| - \epsilon.
\end{aligned}
$$

Since $\epsilon \in \mathbb{R}_{>0}$ was arbitrary, we find that $\|\psi_y\| \geq \|y\|$. Therefore, $\|y\| = \|\psi_y\|$ and the map $y \mapsto \psi_y$ defines an bijective isometry from $B(H)$ to $B_1(H)^*$. $\square$

5.5 Hilbert-Schmidt operators on L^2

Let (X, μ)be a σ-finite measure space such that the space $L^2(X, \mu)$ is separable. Let $k(x, y) \in L^2(X \times X, \mu \times \mu)$. By Fubini's theorem (see [Coh13, Theorem 5.2.2]), the function $y \mapsto k(x, y)$ is square-integrable with respect to the variable y. By subsequently integrating over x, we obtain

$$\int_X \Big(\int_X |k(x,y)|^2 \, d\mu(y) \Big) \, d\mu(x) = \|k\|_{L^2}^2 < \infty.$$

Using the inner product on $L^2(X, \mu)$, we find that if $\psi \in L^2(X, \mu)$ then

$$\int_X k(x,y)\psi(y) \, d\mu(y)$$

is well-defined for almost all $x \in X$ and a quick computation reveals that

$$\begin{aligned}\int_X \left| k(x,y)\psi(y) \, d\mu(y) \right|^2 d\mu(x) &\leq \int_X \int_X |k(x,y)|^2 \, d\mu(y) \|\psi\|_{L^2}^2 \, d\mu(x) \\ &= \|k\|_{L^2}^2 \|\psi\|_{L^2}^2 < \infty.\end{aligned}$$

Now define the map

$$\begin{array}{cccc} t_k : & L^2(X,\mu) & \to & L^2(X,\mu) \\ & \psi & \mapsto & (t_k\psi)(x) = \int_X k(x,y)\psi(y) \, d\mu(y) \end{array}$$

Then, t_k is a linear map, due to linearity of the integral. It is also bounded because

$$\|t_k\| = \sup_{\|\psi\|_{L^2}=1} \|t_k(\psi)\|_{L^2} \leq \|k\|_{L^2} \|\psi\|_{L^2} = \|k\|_{L^2}.$$

The bounded linear operator t_k is called an **integral operator** and the function $k(x, y)$ is the **integral kernel** of the operator t_k.

The next theorem demonstrates the connection of integral operators to Hilbert-Schmidt operators.

Theorem 5.5.1. *Let (X, μ) be a σ-finite measure space. Define the map*

$$\begin{array}{cccc} \Phi : & L^2(X \times X, \mu \times \mu) & \to & B(L^2(X,\mu)) \\ & k(x,y) & \mapsto & t_k \end{array}$$

Then, im $\Phi = B_2(L^2(X, \mu))$ and as an operator from $L^2(X \times X, \mu \times \mu)$ to $B_2(L^2(X, \mu))$, Φ is a unitary map.

Proof. Assume that (X, μ) is a σ-finite measure space and Φ is the map defined as above. Assume that $k \in L^2(X \times X, \mu \times \mu)$.

To show: (a) Φ is a isometry from $L^2(X \times X, \mu \times \mu)$ to $B_2(L^2(X, \mu))$.

(a) Let $\{\psi_i\}_{i \in I}$ be an orthonormal basis for $L^2(X, \mu)$. Then, $\{\varphi_i \otimes \overline{\varphi_j}\}_{i,j \in I}$ is an orthonormal basis for

$$L^2(X, \mu) \otimes L^2(X, \mu) \cong L^2(X \times X, \mu \times \mu).$$

For $i, j \in I$, there exists $\alpha_{i,j} \in \mathbb{C}$ such that

$$k = \sum_{i,j \in I} \alpha_{i,j} \ \varphi_i \otimes \overline{\varphi_j}$$

Let $\mathcal{I}$ denote the family of finite subsets of I. For $A \in \mathcal{I}$, define

$$k_A = \sum_{i,j \in A} \alpha_{i,j} \ \varphi_i \otimes \overline{\varphi_j}.$$

If $\psi \in L^2(X, \mu)$ and $x \in X$ then

$$(t_{k_A}\psi)(x) = \sum_{i,j \in A} \alpha_{i,j} \int_X \varphi_i(x)\overline{\varphi_j}(y)\psi(y) \ d\mu(y)$$

With bra-ket notation, the above equation can be simplified as

$$t_{k_A} = \sum_{i,j \in A} \alpha_{i,j} |\varphi_i\rangle\langle\varphi_j|.$$

So, $t_{k_A} \in \mathcal{F}(L^2(X, \mu)) \subset B_2(L^2(X, \mu))$ and

$$\|t_k - t_{k_A}\| = \|t_{k-k_A}\| \leq \|k - k_A\|_{L^2} \to 0$$

as the cardinality of A tends to infinity. Since t_k is the norm limit of finite dimensional operators, t_k must be a compact operator.

To see that $t_k \in B_2(L^2(X, \mu))$, we must compute the quantity $Tr(t_k^* t_k)$. If $s \in I$ then

$$\begin{aligned}
t_k\varphi_s &= \lim_{A\in\mathcal{I}} t_{k_A}\varphi_s \\
&= \lim_{A\in\mathcal{I}} \sum_{i,j\in A} \alpha_{i,j}|\varphi_i\rangle\langle\varphi_j|\varphi_s \\
&= \lim_{A\in\mathcal{I}} \sum_{i,j\in A} \alpha_{i,j}\langle\varphi_s,\varphi_j\rangle\varphi_i \\
&= \lim_{A\in\mathcal{I}} \sum_{i,j\in A} \alpha_{i,s}\varphi_i = \sum_{i\in I}\alpha_{i,s}\varphi_i
\end{aligned}$$

So,

$$\|t_k\varphi_s\|_{L^2}^2 = \langle t_k\varphi_s, t_k\varphi_s\rangle_{L^2} = \sum_{i\in I}|\alpha_{i,s}|^2$$

and

$$Tr(t_k^* t_k) = \sum_{s\in I}\|t_k\varphi_s\|_{L^2}^2 = \sum_{i,s\in I}|\alpha_{i,s}|^2 = \|k\|_{L^2}^2 < \infty.$$

Therefore, $t_k \in B_2(L^2(X,\mu))$ and im $\Phi = B_2(L^2(X,\mu))$. Moreover, the above equation tells us that $\|k\|_{L^2} = \|t_k\|_{Tr}$. So, Φ is also an isometry from $L^2(X\times X, \mu\times\mu)$ to $B_2(L^2(X,\mu))$.

The image of Φ contains the dense subset $\mathcal{F}(L^2(X,\mu))$. In conjunction with the fact that Φ is an isometry, we deduce that Φ must be unitary. $\square$

One consequence of Theorem 5.5.1 is that if $t \in B(L^2(X,\mu))$ is an operator which can be written as an integral operator with square-integrable integral kernel $k(x,y) \in L^2(X\times X, \mu\times\mu)$ then t is a Hilbert-Schmidt operator on $L^2(X,\mu)$ and is thus, compact by Theorem 5.2.3. Thus, Theorem 5.5.1 provides us with a method to determine if an operator is compact.

Chapter 6

Functional calculus for families of operators

6.1 Preliminary results on C*-algebras

In this chapter, we are interested in extending the continuous functional calculus to families of commuting self-adjoint operators and normal operators. The method presented in [Sol18, Chapter 7] relies on a few results on C*-algebras. These results are located in [Sol18, Appendix A.5.2], but for the sake of completeness, we will work through them here.

Much of the notation we have established for bounded operators on a Hilbert space extend to C*-algebras.

Definition 6.1.1. Let A be a C*-algebra. We say that an element $a \in A$ is **positive** if a is self-adjoint and $\sigma(a) \subseteq [0, \infty)$.

Let $a, b \in A$. We say that $a \leq b$ if and only if $b - a$ is a positive element of A.

Similarly to the case of bounded operators on a Hilbert space, every positive element of a C*-algebra has a unique square root.

The first preliminary result we need states that in any left ideal of a unital C*-algebra, we have an "approximation to the unit" lying entirely in the left ideal.

Theorem 6.1.1. *Let A be a unital C*-algebra (with unit $\mathbb{1}$) and let L be a left ideal of A. Let I be the set of pairs (n, F) such that $n \in \mathbb{Z}_{>0}$ and F is a finite subset of L. Then, we can define a partial order on I by stating that*

$(n, F) \preceq (n', F')$ if and only if $n \le n'$ and $F \subset F'$.

Then, there exists a sequence $\{e_i\}_{i \in I}$ of positive elements of L such that

1. *If $i \in I$ then $0 \le e_i \le \mathbb{1}$.*
2. *If $i \preceq j$ then $e_i \le e_j$.*
3. *If $a \in L$ then $\|a - ae_i\| \to 0$ as $i \to \infty$, with respect to the partial order $\preceq$ on I.*

Proof. Assume that A is a unital C*-algebra. Assume that L is a left ideal of A. Assume that $(I, \preceq)$ is the poset defined as above. For $i = (n, F) \in I$, define

$$v_i = \sum_{b \in F} b^* b \qquad \text{and} \qquad e_i = (\frac{1}{n}\mathbb{1} + v_i)^{-1} v_i.$$

Note that $v_i \in A$ is a positive element due to the analogous result of Theorem 2.3.5 as applied to C*-algebras. Since v_i is positive, the continuous functional calculus for C*-algebras applies so that in the above definitions,

$$e_i = f_n(v_i) \qquad \text{where} \qquad f_n(t) = \frac{t}{t + \frac{1}{n}}$$

for $t \in \sigma(v_i) \subset [0, \infty)$. Now observe that if $t \in [0, \infty)$ and $n \in \mathbb{Z}_{>0}$ then $0 \le f_n(t) \le 1$. So, $0 \le f_n(v_i) = e_i \le \mathbb{1}$.

Now let $i = (n, F)$ and $j = (n', F')$ be elements of I such that $i \preceq j$. Then, $v_i \le v_j$ because

$$v_j - v_i = \sum_{b \in F' - F} b^* b$$

is a positive element of A. By the analogous result to Lemma 2.3.9, we find that

$$(\frac{1}{n} + v_i)^{-1} \ge (\frac{1}{n} + v_j)^{-1}.$$

Next, we know that if $t \in [0, \infty)$ then

$$\frac{1}{n}(\frac{1}{n} + t)^{-1} \ge \frac{1}{n'}(\frac{1}{n'} + t)^{-1}$$

and subsequently, as elements of A,

$$\frac{1}{n}(\frac{1}{n}+v_j)^{-1} \geq \frac{1}{n'}(\frac{1}{n'}+v_j)^{-1}$$

With these two inequalities, we find that

$$\frac{1}{n}(\frac{1}{n}+v_i)^{-1} \geq \frac{1}{n}(\frac{1}{n}+v_j)^{-1} \geq \frac{1}{n'}(\frac{1}{n'}+v_j)^{-1}$$

and

$$\mathbb{1} - \frac{1}{n}(\frac{1}{n}+v_i)^{-1} \leq \mathbb{1} - \frac{1}{n'}(\frac{1}{n'}+v_j)^{-1}.$$

Now, we observe that

$$e_i = (\frac{1}{n}\mathbb{1}+v_i)^{-1}v_i = (\frac{1}{n}\mathbb{1}+v_i)^{-1}(\frac{1}{n}\mathbb{1}+v_i-\frac{1}{n}\mathbb{1}) = \mathbb{1} - \frac{1}{n}(\frac{1}{n}+v_i)^{-1}.$$

Similarly, $e_j = \mathbb{1} - \frac{1}{n'}(\frac{1}{n'}+v_j)^{-1}$ and $e_i \leq e_j$.

For the final assertion, assume that $a \in L$ and let $i = (n, F) \in I$ so that $a \in F$. Since $e_i = \mathbb{1} - \frac{1}{n}(\frac{1}{n}+v_i)^{-1}$ and

$$(\frac{1}{n}+t)^{-2}t \leq \frac{n}{4}$$

for $t \in [0, \infty)$, we have

$$\begin{aligned}
\sum_{b\in F}(b(\mathbb{1}-e_i))^*(b(\mathbb{1}-e_i)) &= (\mathbb{1}-e_i)(\sum_{b\in F} b^*b)(\mathbb{1}-e_i) \\
&= (\mathbb{1}-e_i)v_i(\mathbb{1}-e_i) \\
&= \frac{1}{n}(\frac{1}{n}+v_i)^{-1}\ v_i\ \frac{1}{n}(\frac{1}{n}+v_i)^{-1} \\
&= \frac{1}{n^2}(\frac{1}{n}+v_i)^{-2}v_i \leq \frac{1}{4n}\mathbb{1}.
\end{aligned}$$

Since $a \in F$,

$$(a(\mathbb{1}-e_i))^*(a(\mathbb{1}-e_i)) \leq \sum_{b\in F}(b(\mathbb{1}-e_i))^*(b(\mathbb{1}-e_i)) \leq \frac{1}{4n}\mathbb{1}.$$

Consequently,

$$\|a - ae_i\|^2 = \|a(\mathbb{1}-e_i)\|^2 = \|(a(\mathbb{1}-e_i))^*(a(\mathbb{1}-e_i))\| \leq \frac{1}{4n} \to 0$$

as $n \to \infty$. This completes the proof. $\square$

One particular application of Theorem 6.1.1 is to show that a closed ideal in a unital C*-algebra is self-adjoint.

Theorem 6.1.2. *Let A be a unital C*-algebra and J be a closed ideal in A. Then, J is self-adjoint.*

Proof. Assume that A is a unital C*-algebra and J is a closed ideal of A. By Theorem 6.1.1, there exists a sequence $\{e_i\}_{i\in I}$ such that

$$a = \lim_{i\in I} ae_i.$$

The limit is with respect to the norm topology on A. Since J is a two-sided ideal, $e_i a^* = (ae_i)^* \in J$. Since J is a closed subset of A, the limit

$$a^* = \lim_{i\in I} e_i a^* \in J.$$

So, J is a self-adjoint ideal. □

We know that if X is a Banach space and S is a closed subspace of X then the quotient X/S is also a Banach space, with quotient norm given by

$$\|x + S\| = \inf_{s\in S} \|x + s\|.$$

The next theorem shows us how to produce quotient C*-algebras.

Theorem 6.1.3. *Let A be a unital C*-algebra and J be a closed ideal of A. Then, the quotient space A/J with quotient norm is a C*-algebra,*

Proof. Assume that A is a unital C*-algebra and J is a closed ideal of A. By Theorem 6.1.2, J is a self-adjoint ideal of A. So, A/J is a unital *-algebra because the involution map * becomes well-defined from A/J to A/J. Furthermore, since J is a closed subspace, A/J is a Banach space with the quotient norm

$$\|a + J\| = \inf_{u\in J} \|a + u\|$$

for $a \in A$.

To show: (a) If $a \in A$ then $\|a^* + J\| = \|a + J\|$.

(b) If $a, b \in A$ then $\|(a + J)(b + J)\| \leq \|a + J\|\|b + J\|$.

(a) Assume that $a \in A$. Since J is self adjoint, we have

$$\|a^* + J\| = \|(a+J)^*\| = \|a+J\|.$$

(b) Assume that $a, b \in A$ and $\epsilon \in \mathbb{R}_{>0}$. By the definition of the quotient norm as an infimum, there exists $u, v \in J$ such that

$$\|a+u\| \leq \|a+J\| + \epsilon \qquad \text{and} \qquad \|b+v\| \leq \|b+J\| + \epsilon.$$

From this, we compute directly that

$$\begin{aligned}
\|ab+J\| &\leq \|ab + (av + ub + uv)\| \\
&= \|(a+u)(b+v)\| \\
&\leq \|a+u\|\|b+v\| \quad \text{(since } A \text{ is a C*-algebra)} \\
&\leq (\|a+J\| + \epsilon)(\|b+J\| + \epsilon) \\
&= \|a+J\|\|b+J\| + \epsilon(\|a+J\| + \|b+J\| + \epsilon).
\end{aligned}$$

Since $\epsilon \in \mathbb{R}_{>0}$ was arbitrary, we find that $\|ab+J\| \leq \|a+J\|\|b+J\|$.

To show: (c) $\|a+J\|^2 = \|a^*a+J\|$.

(c) By parts (b) and (a), we first have

$$\|a^*a + J\| \leq \|a^*+J\|\|a+J\| = \|a+J\|^2.$$

To establish the other inequality, let $\{e_i\}_{i\in I}$ be a sequence in J constructed from Theorem 6.1.1. Since $ae_i \in J$,

$$\|a+J\| \leq \|a - ae_i\|$$

for $i \in I$ and hence,

$$\|a+J\| \leq \inf_{i\in I} \|a - ae_i\|.$$

Now for any $u \in J$, we have

$$\|a+u\| \geq \|(a+u)(\mathbb{1} - e_i)\|$$

because $\|\mathbb{1} - e_i\| \leq 1$. Therefore,

$$\begin{aligned}
\|a+u\| &\geq \liminf_{i\in I} \|(a+u)(\mathbb{1} - e_i)\| \\
&= \liminf_{i\in I} \|(a - ae_i) + (u - ue_i)\| \\
&= \liminf_{i\in I} \|a - ae_i\| \geq \inf_{i\in I} \|a - ae_i\|.
\end{aligned}$$

In the second equality, we used the fact that $u - ue_i \to 0$ as I increases with respect to the partial order $\preceq$ in Theorem 6.1.1. If we take the infimum over all $u \in J$, we find that $\|a + J\| \geq \inf_{i\in I}\|a - ae_i\|$ and

$$\|a + J\| = \inf_{i\in I}\|a - ae_i\|.$$

Therefore,

$$\begin{aligned}\|a + J\|^2 &= \inf_{i\in I}\|a(\mathbb{1} - e_i)\|^2 \\ &= \inf_{i\in I}\|(\mathbb{1} - e_i)a^*a(\mathbb{1} - e_i)\| \\ &\leq \inf_{i\in I}\|a^*a(\mathbb{1} - e_i)\| \\ &= \|a^*a + J\|.\end{aligned}$$

By parts (a), (b) and (c), A/J is a C*-algebra. □

The next theorem we require is a generalisation of Lemma 3.2.3.

Theorem 6.1.4. *Let A and B be unital C*-algebras. Let $\Phi : A \to B$ be a unital *-homomorphism. Then, Φ is a contraction and $\Phi(A)$ is a closed unital C*-subalgebra of B. Moreover, if Φ is injective then Φ is isometric.*

Proof. Assume that A and B are unital C*-algebras. Let $1_A \in A$ be the unit in A and 1_B be the unit in B. Assume that $\Phi : A \to B$ is a unital *-homomorphism. Assume that $a \in A$.

To show: (a) $\sigma(\Phi(a)) \subseteq \sigma(a)$.

(a) Assume that $\lambda \in \rho(a)$ so that $\lambda 1_A - a \in A$ is invertible. By applying Φ to the equality $(\lambda 1_A - a)(\lambda 1_A - a)^{-1} = 1_A$, we find that

$$(\lambda 1_B - \Phi(a))(\lambda 1_B - \Phi(a))^{-1} = \Phi(1_A) = 1_B.$$

Similarly, $(\lambda 1_B - \Phi(a))^{-1}(\lambda 1_B - \Phi(a)) = 1_B$. So, $\lambda \in \rho(\Phi(a))$ and $\rho(a) \subseteq \rho(\Phi(a))$. By taking complements, we find that $\sigma(\Phi(a)) \subseteq \sigma(a)$.

Now suppose that $a \in A$ is self-adjoint. Then,

$$\|\Phi(a)\| = |\sigma(\Phi(a))| \leq |\sigma(a)| = \|a\|.$$

Now take an arbitrary element $b \in A$. Then, $b^*b \in A$ is self-adjoint. So, $\|\Phi(b^*b)\| \leq \|b^*b\|$. Since Φ is a unital *-homomorphism, we deduce that

$$\|\Phi(b)\|^2 = \|\Phi(b)^*\Phi(b)\| = \|\Phi(b^*b)\| \leq \|b^*b\| = \|b\|^2.$$

Hence, we have proved that if $b \in A$ then $\|\Phi(b)\| \leq \|b\|$. So, Φ is a contraction.

Now assume that Φ is injective. To show that Φ is an isometry, we can recycle the argument we used to show that Φ is a contraction. It suffices to prove that if $a \in A$ is self-adjoint then $\sigma(\Phi(a)) = \sigma(a)$.

To show: (b) If $a \in A$ is self-adjoint then $\sigma(a) \subseteq \sigma(\Phi(a))$.

(b) We already know that $\sigma(\Phi(a)) \subseteq \sigma(a)$. Suppose for the sake of contradiction that $\sigma(\Phi(a)) \neq \sigma(a)$. Then, there exists $f \in Cts(\sigma(a), \mathbb{C})$ such that $f \neq 0$, but $f = 0$ on $\sigma(\Phi(a))$. We know that $\Phi(f(a)) = f(\Phi(a))$ because this holds for polynomials and Φ is continuous (since it is a contraction).

However, since $f = 0$ on $\sigma(\Phi(a))$, $f(a) \in \ker \Phi$. Since Φ is injective, $f(a) = 0$ and $f = 0$ on $\sigma(a)$, which contradicts the assumption that $f \neq 0$ on $\sigma(a)$. Therefore, $\sigma(\Phi(a)) = \sigma(a)$.

By combining part (b) with the previous argument, we deduce that if Φ is injective then Φ is an isometry as required.

Finally, we return to the case where Φ may not be injective. We already know that the image $\Phi(A)$ is a unital C*-subalgebra of B because Φ is a unital *-homomorphism.

To show: (c) $\Phi(A)$ is a closed subset of B.

(c) The idea is to factorise Φ through the quotient C*-algebra $A/\ker\Phi$. There exists a unique unital *-homomorphism such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\pi"] \arrow[dr, "\Phi"'] & A/\ker\Phi \arrow[d, dashed, "\tilde{\Phi}"] \\ & B \end{tikzcd}$$

The key here is that $\tilde{\Phi}$ is an injective unital *-homomorphism from $A/\ker\Phi$ to B. By part (b), $\tilde{\Phi}$ is an isometry. Since the image of an isometry is closed,

$$\Phi(A) = \tilde{\Phi}(A/\ker\Phi)$$

is a closed subset of B. This completes the proof. $\square$

An important consequence of Theorem 6.1.4 is that the norm of a C*-algebra must be unique.

Theorem 6.1.5. *Let A be a unital C*-algebra and $\|-\|_1, \|-\|_2$ be two norms on A. If $a \in A$ then $\|a\|_1 = \|a\|_2$.*

Proof. Assume that A is a unital C*-algebra and $\|-\|_1, \|-\|_2$ be two norms on A. Let $id : A \to A$ be the identity map from $(A, \|-\|_1)$ to $(A, \|-\|_2)$. Then, id is injective and by Theorem 6.1.4, id is an isometry. Therefore, if $a \in A$ then

$$\|a\|_2 = \|id(a)\|_2 = \|a\|_1.$$

$\square$

For the purposes of the sections which follow in this chapter, we are particularly interested the unital C*-algebra $Cts(X, \mathbb{C})$, where X is a compact, Hausdorff topological space. Let J be a closed ideal in $Cts(X, \mathbb{C})$ and define

$$Y = \{x \in X \mid f(x) = 0 \text{ for } f \in J\}.$$

Note the eerie resemblance to the definition of an affine variety. By writing

$$Y = \bigcap_{f \in J} f^{-1}(\{0\}),$$

we find that Y is a closed subset of X. Next, we claim that $X \backslash Y$ is a LCH (locally compact Hausdorff) topological space. Let us first recall the definition of a LCH space.

Definition 6.1.2. Let (X, τ) be a topological space. We say that X is a **locally compact Hausdorff space** if X is Hausdorff and if $x \in X$ then there exists a compact set $K \subseteq X$ such that $x \in K$.

To see that $X \backslash Y$ is a LCH space, note that $X \backslash Y$ is a (topological) subspace of X and is hence, Hausdorff. Assume that $x_0 \in X \backslash Y$. Since X is compact and Hausdorff, X must be a normal topological space.

Now we can apply Urysohn's lemma to the disjoint closed sets Y and $\{x_0\}$. There exists a continuous function $f : [0, 1] \to X$ such that $f|_Y = 0$ and $f(x_0) = 1$. Now if N is a compact neighbourhood of x_0 in X then the set

$$N \cap \{x \in X \mid f(x) \geq \frac{1}{2}\}$$

qualifies as a compact neighbourhood of x_0 in $X\backslash Y$. Therefore, $X\backslash Y$ is a LCH space. This leads us to our next definition.

Definition 6.1.3. Let X be a compact Hausdorff space and J be a closed ideal of $Cts(X, \mathbb{C})$. Let

$$Y = \{x \in X \mid f(x) = 0 \text{ for } f \in J\}.$$

Define $C_0(X\backslash Y)$ to be the algebra

$$C_0(X\backslash Y) = \{f \in Cts(X, \mathbb{C}) \mid f(y) = 0 \text{ for } y \in Y\}.$$

Inheriting the norm from the C*-algebra $Cts(X, \mathbb{C})$, we find that $C_0(X\backslash Y)$ is itself a C*-algebra. Note that $C_0(X\backslash Y)$ is not a unital C*-algebra.

Theorem 6.1.6. *Let X be a compact Hausdorff space and J be a closed ideal of $Cts(X, \mathbb{C})$. Let*

$$Y = \{x \in X \mid f(x) = 0 \text{ for } f \in J\}.$$

Then, $C_0(X\backslash Y) = J$.

Proof. Assume that X is a compact Hausdorff space and J is a closed ideal of $Cts(X, \mathbb{C})$. Assume that Y is the set defined as above.

By definition of $C_0(X\backslash Y)$, we have $J \subseteq C_0(X\backslash Y)$.

To prove the reverse inclusion, we will invoke the Stone-Weierstrass theorem for LCH spaces. The key fact which allows this is that $C_0(X\backslash Y)$ is isomorphic to the algebra of continuous functions from $X\backslash Y$ to $\mathbb{C}$ which vanish at infinity. A function $f \in Cts(X\backslash Y, \mathbb{C})$ vanishes at infinity if for $\delta \in \mathbb{R}_{>0}$, the set

$$\{x \in X\backslash Y \mid |f(x)| \geq \delta\}$$

is a compact subset of $X\backslash Y$.

Since $J \subseteq C_0(X\backslash Y)$ is a closed ideal, it is a non-unital subalgebra of $C_0(X\backslash Y)$.

To show: (a) If $x \in X\backslash Y$ then there exists $f \in J$ such that $f(x) \neq 0$.

(b) If $x_1, x_2 \in X\backslash Y$ such that $x_1 \neq x_2$ then there exists $f \in J$ such that $f(x_1) \neq f(x_2)$.

(a) Suppose for the sake of contradiction that there exists $x \in X\backslash Y$ such that if $f \in J$ then $f(x) = 0$. By definition of Y, this means that $x \in Y$, which contradicts the assumption that $x \in X\backslash Y$. So, there exists $f \in J$ such that $f(x) = 0$.

(b) Assume that $x_1, x_2 \in X\backslash Y$ such that $x_1 \neq x_2$. By part (a), there exists $\tilde{f} \in J$ such that $\tilde{f}(x_1) \neq 0$. By applying Urysohn's lemma to the sets $\{x_1\}$ and $\{x_2\}$, we construct a continuous function $g \in Cts(X, \mathbb{C})$ such that $g(x_1) = 1$ and $g(x_2) = 0$. Now define $f = \tilde{f}g$. Since J is an ideal of $Cts(X, \mathbb{C})$, $f \in J$. Also, $f(x_1) \neq 0$ and $f(x_2) = 0$. So, $f(x_1) \neq f(x_2)$.

From parts (a) and (b), we can safely apply the Stone-Weierstrass theorem to the closed subalgebra J, in order to deduce that $J = \overline{J} = C_0(X\backslash Y)$. □

Finally, it turns out that the quotient space $Cts(X, \mathbb{C})/J$ has a neat characterisation.

Theorem 6.1.7. *Let X be a compact Hausdorff topological space and Y be a closed subset of X. Let $J = C_0(X\backslash Y)$ be the algebra of continuous functions on X which vanish on Y. Define the map*

$$\begin{array}{rccc} \beta : & Cts(X, \mathbb{C})/J & \mapsto & Cts(Y, \mathbb{C}) \\ & f + J & \mapsto & f|_Y \end{array}$$

*Then, β is a *-isomorphism.*

Proof. Assume that β is the map defined as above.

To show: (a) β is a well-defined unital *-homomorphism.

(a) Let $\pi : Cts(X, \mathbb{C}) \to Cts(X, \mathbb{C})/J$ be the canonical projection map and $R : Cts(X, \mathbb{C}) \to Cts(Y, \mathbb{C})$ be the map which sends $f \in Cts(X, \mathbb{C})$ to its restriction $f|_Y \in Cts(Y, \mathbb{C})$. Notice that if $g \in J = C_0(X\backslash Y)$ then $R(g) = 0$. By the universal property of the quotient, there exists a unique unital *-homomorphism β such that the following diagram commutes:

$$\begin{tikzcd}
Cts(X,\mathbb{C}) \arrow[r, "\pi"] \arrow[dr, "R"'] & Cts(X,\mathbb{C})/J \arrow[d, dashed, "\beta"] \\
 & Cts(Y,\mathbb{C})
\end{tikzcd}$$

This construction demonstrates that β is indeed a well-defined unital *-homomorphism.

To show: (b) β is injective.

(c) β is surjective.

(b) Observe that

$$\ker R = \{f \in Cts(X,\mathbb{C}) \mid f(y) = 0 \text{ for } y \in Y\} = J.$$

Hence, $\ker\beta = J + J = \{0\}$ and so, β is injective.

(c) Assume that $h \in Cts(Y,\mathbb{C})$. Observe that Y is a closed subset of the normal topological space X. By Tietze's extension theorem, there exists a continuous function $F \in Cts(X,\mathbb{C})$ such that $R(F) = F|_Y = f$. So, $\beta(F + J) = F|_Y = f$ and consequently, β is surjective.

By combining parts (a), (b) and (c), we deduce that β is a *-isomorphism. Note that by Theorem 6.1.4, β is also an isometry. □

6.2 Holomorphic functional calculus for commuting operators

Let H be a Hilbert space and $a_1,\dots,a_n \in B(H)$ be pairwise commuting operators. Define

$$\prod_{i=1}^{n}\sigma(a_i) = \sigma(a_1)\times\sigma(a_2)\times\cdots\times\sigma(a_n) \subseteq \mathbb{C}^n.$$

As a topological space, $\prod_{i=1}^{n}\sigma(a_i)$ is a compact Hausdorff space. Let $Hol(\prod_{i=1}^{n}\sigma(a_i),\mathbb{C})$ be the algebra of functions which are holomorphic on a neighbourhood of $\prod_{i=1}^{n}\sigma(a_i)$. Now define for $f \in Hol(\prod_{i=1}^{n}\sigma(a_i),\mathbb{C})$

$$f(a_1,\ldots,a_n)=(\frac{1}{2\pi i})^n\oint_{\Gamma_n}\cdots\oint_{\Gamma_1}f(\lambda_1,\ldots,\lambda_n)(\lambda_1 I-a_1)^{-1}\ldots(\lambda_n I-a_1)^{-1}\,d\lambda_1\ldots d\lambda_n. \tag{6.1}$$

Here, $\Gamma_1,\ldots,\Gamma_n$ are positively oriented curves in $\mathbb{C}$ such that if $i\in\{1,2,\ldots,n\}$ then Γ_i surrounds $\sigma(a_i)$ and $\prod_{i=1}^n\sigma(a_i)$ is contained in the domain of holomorphy of f. As in the one variable case, the value of the integral does not depend on the choice of curves $\Gamma_1,\ldots,\Gamma_n$.

The generalisation of Theorem 3.4.2 we will study here is

Theorem 6.2.1. *Let H be a Hilbert space and $a_1,\ldots,a_n\in B(H)$ be commuting self-adjoint operators. Define the map*

$$\begin{array}{cccc}\Lambda_{H,multi}: & Hol(\prod_{i=1}^n\sigma(a_i),\mathbb{C}) & \to & B(H)\\ & f & \mapsto & f(a_1,\ldots,a_n)\end{array}$$

where $f(a_1,\ldots,a_n)\in B(H)$ is given by equation (6.1). *Then, $\Lambda_{H,multi}$ is a unital *-homomorphism.*

Proof. By the linearity of the Bochner integral, we know that $\Lambda_{H,multi}$ is a linear map.

Next, let $k\in\{1,2,\ldots,n\}$ and $N\in\mathbb{Z}_{\geq 0}$. Let $f\in Hol(\prod_{i=1}^n\sigma(a_i),\mathbb{C})$.

To show: (a) If $f(\lambda_1,\ldots,\lambda_n)=\lambda_k^N$ then $f(a_1,\ldots,a_n)=a_k^N$ (recall that $f(a_1,\ldots,a_n)=\Lambda_{H,multi}(f)$).

(a) Choose the positively oriented curves $\Gamma_1,\ldots,\Gamma_n$ so that if $i\in\{1,2,\ldots,n\}$, the curve Γ_i lies in the set

$$\{z\in\mathbb{C}\mid |z|>\|a_i\|\}\subseteq\rho(a_i).$$

By a direct computation and Cauchy's theorem, we have

$$
\begin{aligned}
&(\frac{1}{2\pi i})^n \oint_{\Gamma_n} \cdots \oint_{\Gamma_1} \lambda_k^N (\lambda_1 I - a_1)^{-1} \ldots (\lambda_n I - a_n)^{-1}\, d\lambda_1 \ldots d\lambda_n \\
&= (\frac{1}{2\pi i})^n \oint_{\Gamma_n} \cdots \oint_{\Gamma_1} \lambda_k^N \Big(\sum_{i=0}^{\infty} \lambda_1^{-i-1} a_1^i \Big) \ldots \Big(\sum_{i=0}^{\infty} \lambda_n^{-i-1} a_n^i \Big)\, d\lambda_1 \ldots d\lambda_n \\
&= \prod_{j \neq k} \Big(\sum_{i=0}^{\infty} \frac{1}{2\pi i} \oint_{\Gamma_j} \lambda_j^{-i-1} a_j^i \, d\lambda_j \Big) \cdot \Big(\sum_{i=0}^{\infty} \frac{1}{2\pi i} \oint_{\Gamma_k} \lambda_k^{N-i-1} a_k^i \, d\lambda_j \Big) \\
&= \Big(\prod_{j \neq k} 1 \Big) a_k^N = a_k^N .
\end{aligned}
$$

Note that this computation proceeds analogously to the one in Theorem 3.4.2.

To show: (b) If $f, g \in Hol(\prod_{i=1}^n \sigma(a_i), \mathbb{C})$ then $f(a_1, \ldots, a_n) g(a_1, \ldots, a_n) = (fg)(a_1, \ldots, a_n)$.

(b) We claim that if $i, j \in \{1, 2, \ldots, n\}$ then the resolvents $(\lambda_i I - a_i)^{-1}$ and $(\lambda_j I - a_j)^{-1}$ commute. To see why this is case, note that $\lambda_i I - a_i$ and $\lambda_j I - a_j$ commute. So, $(\lambda_i I - a_i)^{-1}(\lambda_j I - a_j)^{-1}$ and $(\lambda_j I - a_j)^{-1}(\lambda_i I - a_i)^{-1}$ are both inverses to $(\lambda_i I - a_i)(\lambda_j I - a_j)$. Hence, the resolvents $(\lambda_i I - a_i)^{-1}$ and $(\lambda_j I - a_j)^{-1}$ commute.

Let $\Gamma_1, \ldots, \Gamma_n, \Gamma_1', \ldots, \Gamma_n'$ be positively oriented curves such that if $i \in \{1, 2, \ldots, n\}$ then Γ_i and Γ_i' surround the set $\sigma(a_i)$. Moreover, the sets

$$
\prod_{i=1}^n \Gamma_i = \{(b_1, \ldots, b_n) \in \mathbb{C}^n \mid b_i \in \Gamma_i\}
$$

and $\prod_{i=1}^n \Gamma_i'$ lie in the intersection of the domains of holomorphy of f and g. We also want the curve Γ_i' to lie outside of Γ_i. By Lemma 3.4.1,

$$
\begin{aligned}
&f(a_1,\dots,a_n)g(a_1,\dots,a_n)\\
&= \Big((\frac{1}{2\pi i})^n \oint_{\Gamma_n} \cdots \oint_{\Gamma_1} f(\lambda_1,\dots,\lambda_n) \prod_{i=1}^{n} (\lambda_i I - a_i)^{-1} \, d\lambda_1 \dots d\lambda_n \Big)\\
&\quad \cdot \Big((\frac{1}{2\pi i})^n \oint_{\Gamma'_n} \cdots \oint_{\Gamma'_1} g(\mu_1,\dots,\mu_n) \prod_{i=1}^{n} (\mu_i I - a_i)^{-1} \, d\mu_1 \dots d\mu_n \Big)\\
&= (\frac{1}{2\pi i})^{2n} \oint_{\Gamma_n} \cdots \oint_{\Gamma_1} \oint_{\Gamma'_n} \cdots \oint_{\Gamma'_1} f(\lambda_1,\dots,\lambda_n) g(\mu_1,\dots,\mu_n)\\
&\quad \prod_{i=1}^{n} \big((\lambda_i I - a_i)^{-1} (\mu_i I - a_i)^{-1} \big) \, d\lambda_1 \dots d\lambda_n d\mu_1 \dots d\mu_n\\
&= (\frac{1}{2\pi i})^{2n} \oint_{\Gamma_n} \cdots \oint_{\Gamma_1} \oint_{\Gamma'_n} \cdots \oint_{\Gamma'_1} f(\lambda_1,\dots,\lambda_n) g(\mu_1,\dots,\mu_n)\\
&\quad \prod_{i=1}^{n} \big(\frac{1}{\mu_i - \lambda_i} ((\lambda_i I - a_i)^{-1} - (\mu_i I - a_i)^{-1}) \big) \, d\lambda_1 \dots d\lambda_n d\mu_1 \dots d\mu_n.
\end{aligned}
$$

The idea is that if we expand the quantity

$$
\big(\frac{1}{\mu_i - \lambda_i} ((\lambda_i I - a_i)^{-1} - (\mu_i I - a_i)^{-1}) \big)
$$

then we obtain a linear combination of terms which are a product of some $(\lambda_i I - a_i)^{-1}$ and some $(\mu I - a_i)^{-1}$. Fortunately, we chose the curves $\Gamma_1, \dots, \Gamma'_n$ so that the integral of any term with at least one $(\mu_i I - a_i)^{-1}$ is equal to zero.

For instance, let $n = 3$ and consider the integral of one particular term in the expansion

$$
\begin{aligned}
&\oint_{\Gamma_3} \cdots \oint_{\Gamma'_1} f(\lambda_1,\lambda_2,\lambda_3) g(\mu_1,\mu_2,\mu_3)\\
&\quad \frac{1}{\mu_1 - \lambda_1} \frac{1}{\mu_2 - \lambda_2} \frac{1}{\mu_3 - \lambda_3} (\mu_1 I - a_1)^{-1} (\lambda_2 I - a_2)^{-1} (\lambda_3 I - a_3)^{-1} \, d\lambda_1 \dots d\mu_3\\
&= \oint_{\Gamma_3} \cdots \oint_{\Gamma'_1} \Big(\oint_{\Gamma_1} \frac{f(\lambda_1,\lambda_2,\lambda_3)}{\mu_1 - \lambda_1} \, d\lambda_1 \Big)\\
&\quad \frac{1}{\mu_2 - \lambda_2} \frac{1}{\mu_3 - \lambda_3} g(\mu_1,\mu_2,\mu_3) (\mu_1 I - a_1)^{-1} (\lambda_2 I - a_2)^{-1} (\lambda_3 I - a_3)^{-1} \, d\lambda_2 \dots d\mu_3.
\end{aligned}
$$

Observe that the map $\lambda \mapsto \frac{f(\lambda_1,\lambda_2,\lambda_3)}{\mu_1 - \lambda_1}$ is holomorphic in a contractible region containing the curve Γ_1. Note that $\lambda_1 \neq \mu_1$ because the curve Γ'_1 lies outside Γ_1. Therefore, the above integral is zero.

Returning to our expression for $f(a_1, \dots, a_n)g(a_1, \dots, a_n)$, we can eliminate all terms which contain at least one $(\mu_i I - a_i)^{-1}$. Therefore,

$$
\begin{aligned}
& f(a_1, \dots, a_n)g(a_1, \dots, a_n) \\
= & (\frac{1}{2\pi i})^{2n} \oint_{\Gamma_n} \cdots \oint_{\Gamma_1} \oint_{\Gamma'_n} \cdots \oint_{\Gamma'_1} f(\lambda_1, \dots, \lambda_n) g(\mu_1, \dots, \mu_n) \\
& \prod_{i=1}^{n} \big(\frac{1}{\mu_i - \lambda_i}((\lambda_i I - a_i)^{-1} - (\mu_i I - a_i)^{-1})\big) \, d\lambda_1 \dots d\lambda_n d\mu_1 \dots d\mu_n \\
= & (\frac{1}{2\pi i})^{2n} \oint_{\Gamma_n} \cdots \oint_{\Gamma_1} \oint_{\Gamma'_n} \cdots \oint_{\Gamma'_1} f(\lambda_1, \dots, \lambda_n) g(\mu_1, \dots, \mu_n) \\
& \prod_{i=1}^{n} \frac{1}{\mu_i - \lambda_i} \prod_{j=1}^{n} (\lambda_j I - a_j)^{-1} \, d\lambda_1 \dots d\mu_n \\
= & (\frac{1}{2\pi i})^{n} \oint_{\Gamma_n} \cdots \oint_{\Gamma_1} f(\lambda_1, \dots, \lambda_n) \Big((\frac{1}{2\pi i})^{n} \oint_{\Gamma'_n} \cdots \oint_{\Gamma'_1} \frac{g(\mu_1, \dots, \mu_n)}{\prod_{i=1}^{n}(\mu_i - \lambda_i)} \, d\mu_1 \dots d\mu_n \Big) \\
& \prod_{j=1}^{n} (\lambda_j I - a_j)^{-1} \, d\lambda_1 \dots d\lambda_n \\
= & (\frac{1}{2\pi i})^{n} \oint_{\Gamma_n} \cdots \oint_{\Gamma_1} f(\lambda_1, \dots, \lambda_n) g(\lambda_1, \dots, \lambda_n) \prod_{j=1}^{n} (\lambda_j I - a_j)^{-1} \, d\lambda_1 \dots d\lambda_n \\
= & (fg)(a_1, \dots, a_n).
\end{aligned}
$$

In the second last equality, we used the multi-dimensional version of Cauchy's integral formula to obtain the factor $g(\lambda_1, \dots, \lambda_n)$.

Part (b) in tandem with linearity of $\Lambda_{H,multi}$ reveals that $\Lambda_{H,multi}$ is a *-homomorphism. Part (a) tells us that $\Lambda_{H,multi}$ is unital. This completes the proof. $\square$

In the scenario of Theorem 6.2.1, a weaker version of the spectral mapping theorem applies.

Theorem 6.2.2. *Let H be a Hilbert space and $a_1, \dots, a_n \in B(H)$ be pairwise commuting operators. Let P be a polynomial in the polynomial ring $\mathbb{C}[x_1, \dots, x_n]$. Then,*

$$\sigma(P(a_1, \dots, a_n)) \subset P(\prod_{i=1}^{n} \sigma(a_i)) = \{P(\lambda_1, \dots, \lambda_n) \mid \lambda_i \in \sigma(a_i)\}.$$

Proof. Assume that H is a Hilbert space and $a_1, \dots, a_n \in B(H)$ are pairwise commuting operators. Assume that $P(x_1, \dots, x_n) \in \mathbb{C}[x_1, \dots, x_n]$.

To show: (a) If $\mu \notin P(\prod_{i=1}^n \sigma(a_i))$ then $\mu \notin \sigma(P(a_1, \dots, a_n))$.

(a) Assume that $\mu \notin P(\prod_{i=1}^n \sigma(a_i))$. Then, the function

$$\begin{array}{rccc} f: & \prod_{i=1}^n \sigma(a_i) & \to & \mathbb{C} \\ & (z_1, \dots, z_n) & \mapsto & \frac{1}{\mu - P(z_1, \dots, z_n)} \end{array}$$

is an element of the algebra $Hol(\prod_{i=1}^n \sigma(a_i), \mathbb{C})$. Also, the map $g : (z_1, \dots, z_n) \mapsto \mu - P(z_1, \dots, z_n)$ is an element of $Hol(\prod_{i=1}^n \sigma(a_i), \mathbb{C})$. By Theorem 6.2.1, we can use the fact that $\Lambda_{H,multi}$ is a unital *-homomorphism to deduce that

$$\Lambda_{H,multi}(f)\Lambda_{H,multi}(g) = \Lambda_{H,multi}(fg) = \Lambda_{H,multi}(1) = I.$$

By the same argument, we also have $\Lambda_{H,multi}(g)\Lambda_{H,multi}(f) = I$. Since $\Lambda_{H,multi}(f) = f(a_1, \dots, a_n)$ and $\Lambda_{H,multi}(g) = \mu - P(a_1, \dots, a_n)$, $\mu \in \rho(P(a_1, \dots, a_n))$. Therefore, $\sigma(P(a_1, \dots, a_n)) \subset P(\prod_{i=1}^n \sigma(a_i))$. $\square$

6.3 Continuous functional calculus for commuting self-adjoint operators

Theorem 6.2.2 plays a key role in establishing the continuous functional calculus for pairwise commuting self-adjoint operators.

Theorem 6.3.1. *Let H be a Hilbert space and $a_1, \dots, a_n \in B(H)$ be pairwise commuting self-adjoint operators. Then, there exists a unique unital *-homomorphism*

$$\begin{array}{rccc} \Lambda_{multi}: & Cts(\prod_{i=1}^n \sigma(a_i), \mathbb{C}) & \to & B(H) \\ & f & \mapsto & f(a_1, \dots, a_n) \end{array}$$

such that if $\pi_j : \prod_{i=1}^n \sigma(a_i) \to \sigma(a_j)$ is the projection operator onto the j^{th} component then

$$\pi_j(a_1, \dots, a_n) = a_j$$

for $j \in \{1, 2, \dots, n\}$.

Proof. Assume that H is a Hilbert space and $a_1, \dots, a_n \in B(H)$ be pairwise commuting self-adjoint operators. Define the map

$$\begin{array}{cccc} \Lambda_{poly}: & Poly(\prod_{i=1}^n \sigma(a_i), \mathbb{C}) & \to & B(H) \\ & \sum_{i_1,\dots,i_n} \alpha_{i_1,\dots,i_n} x_1^{i_1} \dots x_n^{i_n} & \mapsto & \sum_{i_1,\dots,i_n} \alpha_{i_1,\dots,i_n} a_1^{i_1} \dots a_n^{i_n}. \end{array}$$

Let $P(x_1, \dots, x_n) = \sum_{i_1,\dots,i_n} \alpha_{i_1,\dots,i_n} x_1^{i_1} \dots x_n^{i_n}$ be a polynomial function so that $\Lambda_{Poly}(P(x_1, \dots, x_n)) = P(a_1, \dots, a_n)$. Since $a_1, \dots, a_n$ are commuting self-adjoint operators, $P(a_1, \dots, a_n)$ is a normal operator.

So, we can use Theorem 6.2.2 to obtain

$$\begin{aligned} \|P(a_1, \dots, a_n)\| &= |\sigma(P(a_1, \dots, a_n))| \\ &\leq \sup\{|P(\lambda_1, \dots, \lambda_n)| \mid \lambda_i \in \sigma(a_i)\} = \|P\|_\infty \end{aligned}$$

The first equality follows from the fact that $P(a_1, \dots, a_n) \in B(H)$ is a normal operator. The second inequality uses Theorem 6.2.2.

It is easy to check that Λ_{Poly} is a unital *-homomorphism. For $j \in \{1, 2, \dots, n\}$, the projection map $\pi_j \in Poly(\prod_{i=1}^n \sigma(a_i), \mathbb{C})$ satisfies $\Lambda_{Poly}(\pi_j) = a_j$.

By the Stone-Weierstrass theorem and the estimate $\|P(a_1, \dots, a_n)\| \leq \|P\|_\infty$, we find that Λ_{Poly} uniquely extends to a unital *-homomorphism $\Lambda_{multi} : Cts(\prod_{i=1}^n \sigma(a_i), \mathbb{C}) \to B(H)$ which satisfies $\Lambda_{multi}|_{Poly(\prod_{i=1}^n \sigma(a_i), \mathbb{C})} = \Lambda_{Poly}$. This completes the proof. $\square$

There is one significant difference between Theorem 2.2.1 and Theorem 6.3.1. The *-isomorphism Λ in Theorem 2.2.1 is an isometry, whereas in Theorem 6.3.1, Λ_{multi} is merely a contraction mapping. The natural question which stems from this observation is: can we modify the statement of Theorem 6.3.1 so that we obtain an isometry from a subset of $Cts(\prod_{i=1}^n \sigma(a_i), \mathbb{C})$ to $B(H)$?

The answer to this question is yes and this is where our preliminary results on C*-algebras come into play. Let $X = \prod_{i=1}^n \sigma(a_i)$. Then, X is a compact Hausdorff topological space. Define

$$J = \{f \in Cts(X, \mathbb{C}) \mid f(a_1, \dots, a_n) = 0\}.$$

It is easy to check that J is an ideal in $Cts(X, \mathbb{C})$. By Theorem 6.1.6 and Theorem 6.1.7, if

$$Y = \{x \in X \mid f(x) = 0 \text{ for } f \in J\}$$

then $C_0(X\backslash Y) = J$ and $Cts(Y, \mathbb{C}) \cong Cts(X, \mathbb{C})/J$ as Banach *-algebras.

In the scenario of Theorem 6.3.1, $\ker \Lambda_{multi} = J$. So, the unital *-homomorphism Λ_{multi} factorises through the quotient $Cts(X, \mathbb{C})/J$.

By Theorem 6.1.7, we obtain a isometric unital *-homomorphism

$$\begin{array}{llcl} \Lambda_{C,multi}: & Cts(X, \mathbb{C})/J \cong Cts(Y, \mathbb{C}) & \to & B(H) \\ & f + J & \mapsto & f(a_1, \ldots, a_n) \end{array}$$

Definition 6.3.1. Let H be a Hilbert space and $a_1, \ldots, a_n$ be pairwise commuting self-adjoint operators. The isometric unital *-homomorphism $\Lambda_{C,multi} : Cts(Y, \mathbb{C}) \to B(H)$ is called the **continuous functional calculus** for the operators $a_1, \ldots, a_n$.

The set

$$Y = \{x \in X \mid f(x) = 0 \text{ for } f \in J = \ker \Lambda_{multi}\}$$

is called the **joint spectrum** of the operators $a_1, \ldots, a_n$ and is denoted by the symbol $\sigma(a_1, a_2, \ldots, a_n)$.

By the definition of the joint spectrum, a point $(\mu_1, \ldots, \mu_n) \in X\backslash\sigma(a_1, \ldots, a_n)$ if and only if there exists a function $f \in Cts(X, \mathbb{C})$ such that $f(\mu_1, \ldots, \mu_n) \neq 0$, but $f(a_1, \ldots, a_n) = 0$.

6.4 Functional calculi for normal operators

In this section, we will use the continuous functional calculus in Definition 6.3.1 to extend the continuous functional calculus in Theorem 2.2.1 to normal operators.

Let $x \in B(H)$ and define $Re(x) = \frac{1}{2}(x + x^*)$ and $Im(x) = \frac{1}{2i}(x - x^*)$. Then, $Re(x)$ and $Im(x)$ are self-adjoint operators and $x = Re(x) + iIm(x)$.

Lemma 6.4.1. *Let H be a Hilbert space and $x \in B(H)$. The operator x is a normal operator if and only if $Re(x)$ and $Im(x)$ commute with each other.*

Proof. Assume that H is a Hilbert space and $x \in B(H)$.

To show: (a) If x is a normal operator then $Re(x)$ and $Im(x)$ commute with each other.

(b) If $Re(x)$ and $Im(x)$ commute with each other then x is a normal operator.

(a) Assume that x is a normal operator. Then,

$$
\begin{aligned}
Re(x)Im(x) &= \frac{1}{2}(x+x^*)\frac{1}{2i}(x-x^*) \\
&= \frac{1}{2}\cdot\frac{1}{2i}(x+x^*)(x-x^*) \\
&= \frac{1}{2}\cdot\frac{1}{2i}(x^2-xx^*+x^*x-(x^*)^2) \\
&= \frac{1}{2}\cdot\frac{1}{2i}(x^2-x^*x+xx^*-(x^*)^2) \\
&= \frac{1}{2}\cdot\frac{1}{2i}(x^2+xx^*-x^*x-(x^*)^2) \\
&= \frac{1}{2}\cdot\frac{1}{2i}(x-x^*)(x+x^*) = Im(x)Re(x).
\end{aligned}
$$

So, $Re(x)$ and $Im(x)$ commute with each other.

(b) Assume that $Re(x)$ and $Im(x)$ commute with each other. Then, $Re(x)Im(x) = Im(x)Re(x)$ and expanding both sides, we find that

$$\frac{1}{2}\cdot\frac{1}{2i}(x^2-xx^*+x^*x-(x^*)^2) = \frac{1}{2}\cdot\frac{1}{2i}(x^2-x^*x+xx^*-(x^*)^2).$$

So, $x^*x - xx^* = xx^* - x^*x$ and by rearranging, we have $x^*x = xx^*$. So, x is normal. □

The key step to extending the continuous functional calculus in Theorem 2.2.1 to normal operators is to link a normal operator $x \in B(H)$ to the joint spectrum $\sigma(Re(x), Im(x))$.

Theorem 6.4.2. *Let H be a Hilbert space and $x \in B(H)$ be a normal operator. Then,*

$$\sigma(x) = \{a+ib \mid (a,b) \in \sigma(Re(x), Im(x))\}.$$

Proof. Assume that H is a Hilbert space and $x \in B(H)$ is a normal operator.

To show: (a) $\sigma(x) \subseteq \{a + ib \mid (a, b) \in \sigma(Re(x), Im(x))\}$.

(b) $\{a + ib \mid (a, b) \in \sigma(Re(x), Im(x))\} \subseteq \sigma(x)$.

(a) We will prove the contrapositive of this statement. Assume that $(c, d) \notin \sigma(Re(x), Im(x))$. Then, the function

$$\begin{array}{cccc} f: & \sigma(Re(x), Im(x)) & \to & \mathbb{C} \\ & (a, b) & \mapsto & \frac{1}{(c+id)-(a+ib)} \end{array}$$

is an element of $Cts(\sigma(Re(x), Im(x)), \mathbb{C})$. By the continuous functional calculus in Definition 6.3.1, we find that $f(Re(x), Im(x))$ is the inverse of the operator $(c + di)I - (Re(x) + iIm(x)) = (c + di)I - x$. Therefore, $c + di \in \rho(x)$ and $\sigma(x) \subseteq \{a + ib \mid (a, b) \in \sigma(Re(x), Im(x))\}$.

(b) We will also prove the contrapositive statement. Assume that $\lambda \in \rho(x)$ and define $r = Re(\lambda)$ and $s = Im(\lambda)$. Suppose for the sake of contradiction that $(r, s) \in \sigma(Re(x), Im(x))$.

Assume that $\epsilon \in \mathbb{R}_{>0}$. Let $f \in Cts(\sigma(Re(x), Im(x)), \mathbb{C})$ be such that $\|f\|_\infty \leq 1$, $f(r, s) = 1$ and the support of f is contained in the set

$$\{(p, q) \in \mathbb{C}^2 \mid (p - r)^2 + (q - s)^2 \leq \epsilon^2\}.$$

Write

$$f(Re(x), Im(x)) = f(Re(x), Im(x))\big((r + is)I - x\big)\big((r + is)I - x\big)^{-1}$$

so that

$$\|f(Re(x), Im(x))\| \leq \|f(Re(x), Im(x))\big((r + is)I - x\big)\|\|\big((r + is)I - x\big)^{-1}\|.$$

Now, observe that

$$(r + is)I - x = (r + is)I - (Re(x) + iIm(x)) = g(Re(x), Im(x))$$

where $g(p, q) = (r + is) - (p + iq)$. Using the continuous functional calculus in Definition 6.3.1, we find that

$$\begin{aligned} \|f(Re(x), Im(x))\big((r + is)I - x\big)\| &= \|f(Re(x), Im(x))g(Re(x), Im(x))\| \\ &= \|(fg)(Re(x), Im(x))\| \\ &\leq \|fg\|_\infty \leq \|f\|_\infty\|g\|_\infty \leq \epsilon. \end{aligned}$$

The final inequality follows from the fact if $(p, q) \in supp(f)$ then $|g(p, q)| \leq \epsilon$. So,

$$\|f(Re(x), Im(x))\| \leq \epsilon \|\big((r + is)I - x\big)^{-1}\|.$$

Since $f(r, s) = 1$, $\|f(Re(x), Im(x))\| \geq 1$. However, $\epsilon \in \mathbb{R}_{>0}$ is arbitrary. So, $\|f(Re(x), Im(x))\|$ is arbitrarily small, which contradicts the fact that $\|f(Re(x), Im(x))\| \geq 1$. So, $(r, s) \notin \sigma(Re(x), Im(x))$ and $\{a + ib \mid (a, b) \in \sigma(Re(x), Im(x))\} \subseteq \sigma(x)$. □

We will now use Theorem 6.4.2 to extend the continuous functional calculus to normal operators.

Theorem 6.4.3. *Let H be a Hilbert space and $x \in B(H)$ be a normal operator. Then, there exists a unique unital *-homomorphism*

$$\begin{array}{rccc} \Lambda_N : & Cts(\sigma(x), \mathbb{C}) & \to & B(H) \\ & f & \mapsto & f(x) \end{array}$$

*such that if $f(\lambda) = \lambda$ for $\lambda \in \sigma(x)$ then $f(x) = x$. Moreover, $f \mapsto f(x)$ is an isometric *-isomorphism from $Cts(\sigma(x), \mathbb{C})$ onto $C^*(x, I)$ — the C^*-algebra generated by x and the identity operator I.*

Proof. Assume that H is a Hilbert space and $x \in B(H)$ be a normal operator. By Theorem 6.4.2,

$$\sigma(x) = \{a + ib \mid (a, b) \in \sigma(Re(x), Im(x))\}.$$

To show: (a) $Cts(\sigma(x), \mathbb{C}) \cong Cts(\sigma(Re(x), Im(x)), \mathbb{C})$ as C*-algebras.

(a) Assume that $f \in Cts(\sigma(x), \mathbb{C})$. Define

$$\begin{array}{rccc} \widehat{f} : & \sigma(Re(x), Im(x)) & \to & \mathbb{C} \\ & (u, v) & \mapsto & f(u + iv) \end{array}$$

This function is well-defined because

$$\sigma(x) = \{a + ib \mid (a, b) \in \sigma(Re(x), Im(x))\}.$$

Now define the map

$$\begin{array}{rccc} \Phi : & Cts(\sigma(x), \mathbb{C}) & \to & Cts(\sigma(Re(x), Im(x)), \mathbb{C}) \\ & f & \mapsto & \widehat{f}. \end{array}$$

It is straightforward to check that Φ is a unital *-homomorphism. To see that Φ is an isometry, we have for $f \in Cts(\sigma(x), \mathbb{C})$

$$
\begin{aligned}
\|f\|_\infty &= \sup_{\lambda \in \sigma(x)} |f(\lambda)| \\
&= \sup_{(\lambda_1,\lambda_2)\in\sigma(Re(x),Im(x))} |f(\lambda_1 + i\lambda_2)| \\
&= \sup_{(\lambda_1,\lambda_2)\in\sigma(Re(x),Im(x))} |\widehat{f}(\lambda_1, \lambda_2)| \\
&= \|\widehat{f}\|_\infty = \|\Phi(f)\|_\infty.
\end{aligned}
$$

To see that Φ is invertible, assume that $g \in Cts(\sigma(Re(x), Im(x)), \mathbb{C})$. Define $g' : \sigma(x) \to \mathbb{C}$ by $g'(u + iv) = g(u, v)$. Then, the map $g \mapsto g'$ is the required inverse for Φ.

So, Φ defines an isometric *-isomorphism between the C*-algebras $Cts(\sigma(x), \mathbb{C})$ and $Cts(\sigma(Re(x), Im(x)), \mathbb{C})$.

Using the isometric *-homomorphism $\Lambda_{C,multi}$ in Definition 6.3.1, define $\Lambda_N : Cts(\sigma(x), \mathbb{C}) \to B(H)$ to be the composite

$$Cts(\sigma(x), \mathbb{C}) \xrightarrow{\Phi} Cts(\sigma(Re(x), Im(x)), \mathbb{C}) \xrightarrow{\Lambda_{C,multi}} B(H)$$

This is a isometric unital *-homomorphism which sends $f \in Cts(\sigma(x), \mathbb{C})$ to $\widehat{f}(Re(x), Im(x)) = f(x)$. If $f = id_{\sigma(x)}$ is the identity map on $\sigma(x)$ then $\Lambda_N(id_{\sigma(x)}) = \widehat{id_{\sigma(x)}}(Re(x), Im(x)) = x$ by Theorem 6.3.1. So, $f(x) = x$.

Now, polynomials in $Re(x)$ and $Im(x)$ span a dense subalgebra of $Cts(\sigma(Re(x), Im(x)), \mathbb{C}) \cong Cts(\sigma(x), \mathbb{C})$. By the Stone-Weierstrass theorem, the condition $f(x) = x$ determines Λ_N uniquely.

Finally, the image of Λ_N contains all polynomials in x and x^*, which is dense in the C*-algebra $C^*(x, I)$. Since the image of a *-homomorphism between C*-algebras is always closed by Theorem 6.1.4, we deduce that im $\Lambda_N = C^*(x, I)$. Consequently, Λ_N is an isometric *-isomorphism between $Cts(\sigma(x), \mathbb{C})$ and $C^*(x, I)$. □

With the continuous functional calculus for normal operators in Theorem 6.4.3, we are able to extend important theorems pertaining to the continuous functional calculus for self-adjoint operators to the case of normal operators. The proofs of such theorems are virtually the same as their counterparts for self-adjoint operators. We state them below.

Theorem 6.4.4 (Spectral mapping theorem). *Let H be a Hilbert space over $\mathbb{C}$ and $x \in B(H)$ be a normal operator. If $f \in Cts(\sigma(x), \mathbb{C})$, then*

$$\sigma(f(x)) = f(\sigma(x)) = \{f(\lambda) \mid \lambda \in \sigma(x)\}.$$

Theorem 6.4.5 (Composition). *Let H be a Hilbert space over $\mathbb{C}$ and $x \in B(H)$ be normal. If $g \in Cts(\sigma(x), \mathbb{C})$ then the operator $g(x)$ is normal and if $f \in Cts(\sigma(g(x)), \mathbb{C})$, then $f(g(x)) = (f \circ g)(x)$.*

Theorem 6.4.6 (Spectral theorem with multiplication operators). *Let H be a Hilbert space over $\mathbb{C}$ and $x \in B(H)$ be a normal operator. There exists a semifinite measure space (X, μ), an essentially bounded measurable real-valued function $F \in L^\infty(X, \mu)$ and a unitary operator $u : L^2(X, \mu) \to H$ such that*

$$x = uM_F u^*.$$

Theorem 6.4.7 (Borel functional calculus). *Let H be a Hilbert space over $\mathbb{C}$ and $x \in B(H)$ be a normal operator. Let $Bor(\sigma(x), \mathbb{C})$ denote the C^*-algebra of bounded Borel functions on $\sigma(x)$ (recall that $\sigma(x)$ is a compact subset of $\mathbb{C}$). Then, there exists a unique unital *-homomorphism*

$$\begin{array}{rccc} \Lambda_{B,N}: & Bor(\sigma(x), \mathbb{C}) & \to & B(H) \\ & f & \mapsto & f(x) \end{array}$$

such that

1. *If for all $\lambda \in \sigma(x)$, $f(\lambda) = \lambda$ then $f(x) = x$.*
2. *If $\{f_n\}_{n \in \mathbb{Z}_{>0}}$ is a uniformly bounded sequence in $Bor(\sigma(x), \mathbb{C})$ converging pointwise to $f : \sigma(x) \to \mathbb{C}$ then $f_n(x) \to f(x)$ as $n \to \infty$ in the strong topology.*
3. *The restriction $\Lambda_{B,N}|_{Cts(\sigma(x),\mathbb{C})} = \Lambda_N$, where Λ_N is the isomorphism in Theorem 6.4.3.*

We emphasise that Theorem 6.4.7 is the Borel functional calculus for normal operators, which extends Theorem 3.2.4.

Chapter 7

Unbounded operators

7.1 Graphs of unbounded operators

This chapter serves as an introduction to unbounded operators, which appear often in fields such as the analysis of PDEs and mathematical physics. One of the main reasons why unbounded operators are more difficult to analyse than their bounded counterparts is because there is a noticeable lack of algebraic structure on unbounded operators. Compare this to the space of bounded operators $B(H)$ on a Hilbert space H, which is a C*-algebra by Theorem 1.1.1.

The reference [Sol18] approaches unbounded operators by reducing questions about unbounded operator to questions about *bounded* operators. The main tool to achieve this is the z-transform, which was introduced by S.L Woronowicz in the more general context of C*-algebras (see [WN92]). We will follow the exposition of [Sol18] and develop some basic theory about unbounded operators before setting up the z-transform in the next chapter.

Let H be a Hilbert space. The product space $H \times H$ is a Hilbert space with inner product

$$\langle (\xi, \eta), (\psi, \phi) \rangle_{H \times H} = \langle \xi, \psi \rangle_H + \langle \eta, \phi \rangle_H.$$

From here, a linear operator T on the Hilbert space H (which is not necessarily bounded) will be defined on a vector subspace $D(T) \subset H$, which is called the **domain** of the operator T. So, T is a linear map $T : D(T) \to H$.

Most of the time, we will not assume that $D(T) = H$. Instead, we will assume that the vector subspace $D(T)$ is dense in H.

Definition 7.1.1. Let H be a Hilbert space and $T : D(T) \to H$. We say that T is **densely defined** if the vector subspace $D(T)$ is dense in H.

To summarise, the unbounded operators we will primarily work with are densely defined operators $T : D(T) \to H$. To understand unbounded operators, we will lean heavily on its graph.

Definition 7.1.2. Let H be a Hilbert space and $T : D(T) \to H$ be a linear operator. The **graph of the operator** T is the subspace

$$G(T) = \{(\psi, T\psi) \mid \psi \in D(T)\} \subseteq H \times H.$$

We say that T is **closed** if the subspace $G(T)$ is closed in $H \times H$.

As a simple example of the above definition, if $x \in B(H)$ (x is bounded) then $D(x) = H$ and x is a closed linear operator. In this scenario, we recall the closed graph theorem, which states that if x is a closed linear operator and $D(x) = H$ then x is bounded.

Here is an explicit characterisation of closed operators.

Lemma 7.1.1. *Let H be a Hilbert space and $T : D(T) \to H$ be a linear operator. Then, T is a closed operator if and only if for any sequence $\{\psi_n\}_{n \in \mathbb{Z}_{>0}}$ of elements in $D(T)$ satisfying $\lim_{n\to\infty} \psi_n = \psi$ and $\lim_{n\to\infty} T\psi_n = \phi$, $\psi \in D(T)$ and $T\psi = \phi$.*

Proof. Assume that H is a Hilbert space and $T : D(T) \to H$ be a linear operator.

To show: (a) If T is a closed operator then for any sequence $\{\psi_n\}_{n \in \mathbb{Z}_{>0}}$ of elements in $D(T)$ satisfying $\lim_{n\to\infty} \psi_n = \psi$ and $\lim_{n\to\infty} T\psi_n = \phi$, $\psi \in D(T)$ and $T\psi = \phi$.

(b) If for any sequence $\{\psi_n\}_{n \in \mathbb{Z}_{>0}}$ of elements in $D(T)$ satisfying $\lim_{n\to\infty} \psi_n = \psi$ and $\lim_{n\to\infty} T\psi_n = \phi$, $\psi \in D(T)$ and $T\psi = \phi$ then T is closed.

(a) Assume that T is a closed operator. Assume that $\{\psi_n\}_{n \in \mathbb{Z}_{>0}}$ is a sequence in $D(T)$ such that $\lim_{n\to\infty} \psi_n = \psi$ and $\lim_{n\to\infty} T\psi_n = \phi$. Observe that

$$\lim_{n \to \infty} (\psi_n, T\psi_n) = (\psi, \phi).$$

Since T is closed, $G(T)$ is a closed subspace of $H \times H$. So, $(\psi, \phi) \in G(T)$ and consequently, $\psi \in D(T)$ and $T\psi = \phi$.

(b) Assume that $\{\psi_n\}_{n \in \mathbb{Z}_{>0}}$ is a sequence in $D(T)$ such that if $\lim_{n \to \infty} \psi_n = \psi$ and $\lim_{n \to \infty} T\psi_n = \phi$ then $\psi \in D(T)$ and $T\psi = \phi$. Suppose that $\{(\phi_n, T\phi_n)\}_{n \in \mathbb{Z}_{>0}}$ is a sequence in $G(T)$ which converges to (ϕ, ρ). Then, $\lim_{n \to \infty} \phi_n = \phi$ and $\lim_{n \to \infty} T\phi_n = \rho$. By our assumption, this means that $\phi \in D(T)$ and $T\phi = \rho$. So, $(\phi, \rho) = (\phi, T\phi) \in G(T)$. Therefore, $G(T)$ is a closed subspace of $H \times H$. $\square$

As explained in [Sol18, Page 102], Lemma 7.1.1 is our replacement for the notion of continuity which accompanies bounded operators on a Hilbert space. The next theorem characterises subspaces of $H \times H$ which are graphs of linear operators.

Theorem 7.1.2. *Let H be a Hilbert space and $G \subseteq H \times H$ be a subspace. The subspace G is a graph of a linear operator if and only if for $\eta \in H - \{0\}$, $(0, \eta) \notin G$.*

Proof. Assume that H is a Hilbert space and $G \subseteq H \times H$ is a subspace.

To show: (a) If there exists a linear operator $T : D(T) \to H$ such that $G = G(T)$ then $(0, \eta) \notin G$ for $\eta \in H - \{0\}$.

(b) If for $\eta \in H - \{0\}$, $(0, \eta) \notin G$ then there exists a linear operator $T : D(T) \to H$ such that $G = G(T)$.

(a) Assume that there exists a linear operator $T : D(T) \to H$ such that $G = G(T)$. Suppose for the sake of contradiction that there exists $\eta \in H - \{0\}$ such that $(0, \eta) \in G$. Then, $T(0) = \eta \neq 0$. However, this contradicts the fact that $T(0) = 0$ (because T is linear). Hence, if $\eta \in H - \{0\}$ then $(0, \eta) \notin G$.

(b) Assume that if $\eta \in H - \{0\}$ then $(0, \eta) \notin G$. If $(\xi, \eta_1), (\xi, \eta_2) \in G$ then $\eta_1 = \eta_2$ because G is a vector subspace. Consequently, $G = G(T)$, where T is a map defined on the vector subspace

$$D(T) = \{\xi \in H \mid \text{There exists } \eta \in H \text{ such that } (\xi, \eta) \in G\} \subseteq H$$

by $T\xi = \eta$. To see that T is linear, assume that $\xi_1, \xi_2 \in D(T)$ so that there exists $\eta_1, \eta_2 \in H$ such that $(\xi_1, \eta_1), (\xi_2, \eta_2) \in G$. Since G is a vector subspace of $H \times H$, $(\xi_1 + \xi_2, \eta_1 + \eta_2) \in G$. So,

$$T(\xi_1 + \xi_2) = \eta_1 + \eta_2 = T(\xi_1) + T(\xi_2).$$

Therefore, T is a linear operator on H such that $G = G(T)$. $\square$

In light of Theorem 7.1.2, we make the following definition.

Definition 7.1.3. Let H be a Hilbert space. Vectors of the form $(0, \eta) \in H \times H$ are called **vertical**.

Next, we will define closable operators on a Hilbert space.

Definition 7.1.4. Let H be a Hilbert space and $T : D(T) \to H$ be a linear operator. We say that the operator T is **closable** if the closure $\overline{G(T)}$ is a graph of an operator. By Theorem 7.1.2, an operator T is closable if $\overline{G(T)}$ does not contain any non-zero vertical vectors.

If T is a closable operator then the operator whose graph is $\overline{G(T)}$ is called the **closure** of T and is denoted by $\overline{T}$.

Definition 7.1.5. Let H be a Hilbert space and $T : D(T) \to H$ and $S : D(S) \to H$ be linear operators on H. We say that S is an **extension** of T (or S contains T) if $G(T) \subset G(S)$. Equivalently, S is an extension of T if $D(T) \subset D(S)$ and if $\psi \in D(T)$ then $S\psi = T\psi$. We write $T \prec S$ to denote that S is an extension of T.

In particular, if T is closable then $T \prec \overline{T}$. We have the following characterisation of a closable operator.

Theorem 7.1.3. *Let H be a Hilbert space and $T : D(T) \to H$ is a linear operator. Then, the operator T is closable if and only if for any sequence $\{\psi_n\}_{n \in \mathbb{Z}_{>0}}$ in $D(T)$ with $\lim_{n\to\infty} \psi_n = 0$ and $\lim_{n\to\infty} T\psi_n = \phi$, $\phi = 0$.*

Proof. Assume that H is a Hilbert space and $T : D(T) \to H$ is a linear operator.

To show: (a) If T is a closable operator then for any sequence $\{\psi_n\}_{n \in \mathbb{Z}_{>0}}$ in $D(T)$ with $\lim_{n\to\infty} \psi_n = 0$ and $\lim_{n\to\infty} T\psi_n = \phi$, $\phi = 0$.

(b) If for any sequence $\{\psi_n\}_{n \in \mathbb{Z}_{>0}}$ in $D(T)$ with $\lim_{n\to\infty} \psi_n = 0$ and $\lim_{n\to\infty} T\psi_n = \phi$, $\phi = 0$ then T is a closable operator.

(a) Assume that T is a closable operator. Then, its closure $\overline{T} : D(\overline{T}) \to H$ has graph $G(\overline{T}) = \overline{G(T)}$. The closure $\overline{T}$ is a closed operator because $\overline{G(T)}$ is a closed subspace of $H \times H$. Assume that $\{\psi_n\}_{n\in\mathbb{Z}_{>0}}$ is a sequence in $D(T)$ such that $\lim_{n\to\infty} \psi_n = 0$ and $\lim_{n\to\infty} T\psi_n = \phi$. Since $(0, \phi) = \lim_{n\to\infty}(\psi_n, T\psi_n)$, $(0, \phi) \in \overline{G(T)}$. Since $\overline{G(T)} = G(\overline{T})$, we can use Theorem 7.1.2 to deduce that $\phi = 0$.

(b) Assume that if $\{\psi_n\}_{n\in\mathbb{Z}_{>0}}$ is a sequence in $D(T)$ with $\lim_{n\to\infty} \psi_n = 0$ and $\lim_{n\to\infty} T\psi_n = \phi$ then $\phi = 0$. By Theorem 7.1.2, it suffices to show that if $\eta \in H - \{0\}$ then $(0, \eta) \notin \overline{G(T)}$.

Suppose for the sake of contradiction that there exists $\eta \in H - \{0\}$ such that $(0, \eta) \in \overline{G(T)}$. Then, there exists a sequence $\{(\phi_n, T\phi_n)\}_{n\in\mathbb{Z}_{>0}}$ in $G(T)$ such that $\lim_{n\to\infty}(\phi_n, T\phi_n) = (0, \eta)$. By our assumption, $\eta = 0$ which contradicts the assumption that $\eta \in H - \{0\}$. Therefore, if $\eta \in H - \{0\}$ then $(0, \eta) \notin \overline{G(T)}$. □

The following characterisation of densely defined operators is quite similar to Theorem 7.1.2.

Theorem 7.1.4. *Let H be a Hilbert space and $T : D(T) \to H$ be a linear operator. Then, T is densely defined if and only if for $\xi \in H - \{0\}$, $(\xi, 0) \notin G(T)^\perp \subseteq H \times H$.*

Proof. Assume that H is a Hilbert space and $T : D(T) \to H$ be a linear operator.

To show: (a) If T is densely defined then if $\xi \in H - \{0\}$ then $(\xi, 0) \notin G(T)^\perp$.

(b) If for $\xi \in H - \{0\}$, $(\xi, 0) \notin G(T)^\perp$ then T is a densely defined operator.

(a) We will prove the contrapositive. Suppose that there exists $\gamma \in H - \{0\}$ such that $(\gamma, 0) \in G(T)^\perp$. If $(\xi, \eta) \in G(T)$ then

$$\langle \gamma, \xi \rangle_H = \langle (\gamma, 0), (\xi, \eta) \rangle_{H\times H} = 0.$$

Hence, $\gamma \in D(T)^\perp$ is non-zero. Since $D(T)^\perp \neq \{0\}$, Theorem 2.4.5 gives $\overline{D(T)} = (D(T)^\perp)^\perp \neq H$. So, T is not densely defined.

(b) We will also prove the contrapositive. Assume that T is not densely defined. Then, $\overline{D(T)} \neq H$ and consequently, there exists $\eta \in H - \{0\}$ such that $\eta \in D(T)^\perp$. By definition of the inner product on $H \times H$, $(\eta, 0) \in G(T)^\perp$. $\square$

Definition 7.1.6. Let H be a Hilbert space. Vectors of the form $(\eta, 0) \in H \times H$ are called **horizontal**.

If we combine Theorem 7.1.2 and Theorem 7.1.4 we obtain the following theorem.

Theorem 7.1.5. *Let H be a Hilbert space and G be a subspace of $H \times H$. Then, G is the graph of a closed, densely defined operator if and only if G is a closed subspace, does not contain non-zero vertical vectors and $G^\perp$ does not contain non-zero horizontal vectors.*

Now observe that if T is a closed operator then the graph $G(T) \subseteq H \times H$ is a closed subspace and is thus, a Hilbert space. Moreover, the map

$$\begin{array}{rccc} (id, T): & D(T) & \to & G(T) \\ & \psi & \mapsto & (\psi, T\psi) \end{array}$$

is bijective. Thus, we can define an inner product on $D(T)$ by

$$\langle \psi, \phi \rangle_{D(T)} = \langle (\psi, T\psi), (\phi, T\phi) \rangle_{H \times H}.$$

This results in the norm

$$\|\psi\|^2_{D(T)} = \|\psi\|^2_H + \|T\psi\|^2_H.$$

The above norm is called the **graph norm**. Since (id, T) is a bijection, $D(T)$ with the graph norm is a Hilbert space. Note that the graph norm can be defined for any linear operator, not just a closed operator.

Theorem 7.1.6. *Let H be a Hilbert space and $T : D(T) \to H$ be a linear operator. Then, T is closed if and only if the vector subspace $D(T) \subseteq H$ is complete with respect to the graph norm.*

Proof. Assume that H is a Hilbert space and $T : D(T) \to H$ is a linear operator.

To show: (a) If T is closed then $D(T)$ is complete with respect to the graph norm.

(b) If the vector subspace $D(T)$ is complete with respect to the graph norm then T is closed.

(a) Assume that T is closed. Then, the subspace $G(T) \subseteq H \times H$ is closed. Assume that $\{\psi_n\}_{n\in\mathbb{Z}_{>0}}$ is a Cauchy sequence with respect to the graph norm on $D(T)$. Assume that $\epsilon \in \mathbb{R}_{>0}$. Then, there exists $N \in \mathbb{Z}_{>0}$ such that if $m, n > N$ then

$$\|\psi_m - \psi_n\|_{D(T)} < \epsilon.$$

Squaring both sides, we have

$$\|\psi_m - \psi_n\|^2_{D(T)} = \|\psi_m - \psi_n\|^2_H + \|T\psi_m - T\psi_n\|^2_H < \epsilon^2.$$

But this means that

$$\|(\psi_m - \psi_n, T\psi_m - T\psi_n)\|^2_{H\times H} < \epsilon^2$$

So, the sequence $\{(\psi_n, T\psi_n)\}_{n\in\mathbb{Z}_{>0}}$ in $G(T)$ is a Cauchy sequence. Since $G(T)$ is a Hilbert space, the sequence $\{(\psi_n, T\psi_n)\}_{n\in\mathbb{Z}_{>0}}$ must converge to some $(\psi, T\psi) \in G(T)$. Therefore, $\{\psi_n\}_{n\in\mathbb{Z}_{>0}}$ converges to $\psi \in D(T)$ with respect to the graph norm. Hence, $D(T)$ is complete.

(b) Assume that the vector subspace $D(T) \subseteq H$ is complete with respect to the graph norm. We want to show that the vector subspace $G(T) \subseteq H \times H$ is closed. Suppose that $\{(\phi_n, T\phi_n)\}_{n\in\mathbb{Z}_{>0}}$ is a sequence in $G(T)$ which converges to $(\phi, \chi) \in H \times H$. Then, $\{(\phi_n, T\phi_n)\}_{n\in\mathbb{Z}_{>0}}$ is a Cauchy sequence.

Assume that $\epsilon \in \mathbb{R}_{>0}$. Then, there exists $N \in \mathbb{Z}_{>0}$ such that if $m, n > N$ then

$$\|(\phi_n - \phi_m, T\phi_n - T\phi_m)\|_{H\times H} < \epsilon.$$

Squaring both sides, we deduce that

$$\|\phi_n - \phi_m\|^2_H + \|T\phi_n - T\phi_m\|^2_H < \epsilon^2$$

So, $\|\phi_n - \phi_m\|_{D(T)} < \epsilon$, which means that the sequence $\{\phi_n\}_{n\in\mathbb{Z}_{>0}}$ is Cauchy with respect to the graph norm on $D(T)$. Since $D(T)$ is complete with respect to the graph norm, there exists $\rho \in D(T)$ such that

$$\lim_{n\to\infty} \|\phi_n - \rho\|_{D(T)} = 0.$$

So, there exists $M \in \mathbb{Z}_{>0}$ such that if $n > M$ then

$$\|\phi_n - \rho\|^2_{D(T)} = \|\phi_n - \rho\|^2_H + \|T\phi_n - T\rho\|^2_H < \frac{\epsilon^2}{4}.$$

So, $\|\phi_n - \rho\|_H < \epsilon/2$ and $\|T\phi_n - T\rho\|_H < \epsilon/2$. We want to show that $\phi = \rho$ and $\chi = T\rho$. Since $\{(\phi_n, T\phi_n)\}_{n\in\mathbb{Z}_{>0}}$ converges to (ϕ, χ), there exists $M' \in \mathbb{Z}_{>0}$ such that if $n > M'$ then

$$\|(\phi_n, T\phi_n) - (\phi, \chi)\|_{H\times H}^2 = \|\phi_n - \phi\|_H^2 + \|T\phi_n - \chi\|_H^2 < \frac{\epsilon^2}{4}.$$

So, $\|\phi_n - \phi\|_H < \epsilon/2$ and $\|T\phi_n - \chi\|_H < \epsilon/2$. If $L = \max(M, M')$ and $n > L$ then

$$\|\phi - \rho\|_H \le \|\phi - \phi_n\|_H + \|\phi_n - \rho\|_H < \epsilon$$

and

$$\|\chi - T\rho\|_H \le \|\chi - T\phi_n\|_H + \|T\phi_n - T\rho\|_H < \epsilon.$$

Since $\epsilon \in \mathbb{R}_{>0}$ was arbitrary, we deduce that $\phi = \rho$ and $\chi = T\rho$. Therefore, $(\phi, \chi) = (\rho, T\rho) \in G(T)$ and consequently, $G(T)$ is a closed subspace of $H \times H$. So, T is a closed operator. $\square$

7.2 The adjoint of an unbounded operator

Fundamental to our treatment of $B(H)$ is the adjoint, which acts as the involution map on the C*-algebra $B(H)$. We want to generalise this and define the adjoint of an unbounded operator.

Let T be a densely defined operator on H. Define the subspace G of $H \times H$ by

$$G = \{(\xi, \eta) \in H \times H \mid \text{If } \psi \in D(T) \text{ then } \langle \xi, T\psi\rangle = \langle \eta, \psi\rangle\}. \tag{7.1}$$

We claim that G is a graph of an operator on H. To see why this is the case, we will use Theorem 7.1.2. Assume that $(0, \eta) \in G$. If $\psi \in D(T)$ then $\langle 0, T\psi\rangle = \langle \eta, \psi\rangle = 0$. Therefore, $\eta \in D(T)^\perp$. But since T is densely defined, $\overline{D(T)} = (D(T)^\perp)^\perp = H$. Consequently, $D(T)^\perp = \{0\}$ and $\eta = 0$.

By Theorem 7.1.2, there exists a linear operator $T^* : D(T^*) \to H$ such that $G(T^*) = G$. The operator T^* is called the **adjoint** of T. By definition of G in equation (7.1), the domain $D(T^*)$ consists of vectors ξ such that the functional $\psi \mapsto \langle \xi, T\psi\rangle$ is bounded for $\psi \in D(T)$.

Definition 7.2.1. Let H be a Hilbert space and $T : D(T) \to H$ be a densely defined operator. We say that T is **self-adjoint** if $T = T^*$. We say that T is **symmetric** or **Hermitian** if $T \prec T^*$.

In particular, if T is self-adjoint then $D(T) = D(T^*)$. We now want to express the graph of the adjoint T^* in terms of $G(T)$.

Theorem 7.2.1. *Let H be a Hilbert space and $T : D(T) \to H$ be a densely defined operator. Let $I : H \to H$ be the identity operator on H. Then,*

$$G(T^*) = UG(T)^{\perp}$$

where

$$U = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

is an operator on $H \times H$ such that $U(\xi, \eta) = (\eta, -\xi)$.

Proof. Assume that H is a Hilbert space and $T : D(T) \to H$ is a densely defined operator. Assume that I is the identity operator on H and U is the operator on $H \times H$, defined as above.

To show: (a) The operator U is bounded and unitary.

(a) To see that U is bounded, we compute its operator norm as

$$\begin{aligned} \|U\|^2 &= \sup_{\|(\xi,\eta)\|=1} \|U(\xi, \eta)\|^2 \\ &= \sup_{\|(\xi,\eta)\|=1} \|(\eta, -\xi)\|^2 \\ &= \sup_{\|(\xi,\eta)\|=1} \langle (\eta, -\xi), (\eta, -\xi) \rangle \\ &= \sup_{\|(\xi,\eta)\|=1} (\|\eta\|_H^2 + \|\xi\|_H^2) \\ &= \sup_{\|(\xi,\eta)\|=1} \|(\xi, \eta)\|^2 = 1. \end{aligned}$$

So, $\|U\| = 1$ and U is a bounded operator on $H \times H$.

To see that U is unitary, we note that by the above computation, U is an isometry on $H \times H$. We also observe that U is surjective because if $(\xi, \eta) \in H \times H$ then $U(-\eta, \xi) = (\xi, \eta)$. Since U is a surjective isometry, we can use Theorem 3.1.8 to deduce that U is a unitary operator on $H \times H$.

Now by equation (7.1), $(\xi, \eta) \in G(T^*)$ if and only if for $\psi \in D(T)$, $\langle \xi, T\psi \rangle = \langle \eta, \psi \rangle$. Equivalently, $(\xi, \eta) \in G(T^*)$ if and only if

$$\langle (\xi, \eta), (T\psi, -\psi) \rangle = 0.$$

Since $U(\psi, T\psi) = (T\psi, -\psi)$, we find that the above equation holds if and only if $(\xi, \eta) \in (UG(T))^{\perp}$. Since U is unitary, $(UG(T))^{\perp} = UG(T)^{\perp}$. Consequently,

$$G(T^*) = UG(T)^{\perp} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} G(T)^{\perp}.$$

□

A major consequence of Theorem 7.2.1 is that $G(T^*)$ is a closed subspace of $H \times H$. To see why this is the case, we know that the orthogonal complement $G(T)^{\perp}$ is a closed subspace of $H \times H$. So, $G(T^*) = UG(T)^{\perp} = (U^{-1})^{-1}(G(T)^{\perp})$, which is closed because $U^{-1} = U^*$ is continuous.

Theorem 7.2.2. *Let H be a Hilbert space and $T : D(T) \to H$ be a densely defined operator. The operator T is closable if and only if T^* is densely defined.*

Proof. Assume that H is a Hilbert space and $T : D(T) \to H$ is a densely defined operator.

By Theorem 7.1.2, T is closable if and only if $\overline{G(T)}$ does not contain non-zero vertical vectors. This holds if and only if $U\overline{G(T)}$ does not contain non-zero horizontal vectors, where U is the unitary operator from Theorem 7.2.1. By Theorem 7.2.1,

$$G(T^*)^{\perp} = (UG(T)^{\perp})^{\perp} = U(G(T)^{\perp})^{\perp} = U\overline{G(T)}.$$

So, T is closable if and only if $G(T^*)^{\perp}$ does not contain any non-zero horizontal vectors. By Theorem 7.1.4, T is closable if and only if T^* is densely defined as required. □

Now if T is closable then by the above theorem, T^* is densely defined. So, we can construct the linear operator $T^{**} : D(T^{**}) \to H$. The graph of T^{**} is

$$G(T^{**}) = U(G(T^*)^{\perp}) = U(U(G(T)^{\perp})^{\perp}) = -\overline{G(T)} = \overline{G(T)}.$$

Hence, we have the following theorem

Theorem 7.2.3. *Let H be a Hilbert space and $T : D(T) \to H$ be a closable operator. Then, the adjoint of the adjoint operator T^*, denoted by T^{**}, is the closure of T. That is, $T^{**} = \overline{T}$.*

If we have two densely defined operators S and T on H such that $T \preceq S$ then their adjoints are related by the following theorem.

Theorem 7.2.4. *Let H be a Hilbert space and S, T be two densely defined operators on H such that $T \prec S$. Then, $S^* \prec T^*$.*

Proof. Assume that H is a Hilbert space and S, T be two densely defined operators on H. Assume that $T \prec S$. Then, $G(T) \subset G(S)$ and $G(S)^\perp \subset G(T)^\perp$. With U being the unitary operator in Theorem 7.2.1, we have

$$G(S^*) = UG(S)^\perp \subset UG(T)^\perp = G(T^*).$$

So, $S^* \prec T^*$ as required. □

An interesting consequence of Theorem 7.2.4 is that a self-adjoint operator does not have a proper symmetric extension. Assume that $T = T^*$ and there exists a linear, symmetric operator $S : D(S) \to H$ such that $T \prec S$. Since S is symmetric, $S \prec S^*$. Since $T \prec S$, $S^* \prec T^*$. So,

$$S \prec S^* \prec T^* = T \prec S$$

and $T = S$. But, since $T = T^*$, $D(T) = D(T^*)$. So, $D(S) = D(S^*)$, which contradicts the assumption that $D(S) \subset D(S^*)$.

7.3 Algebraic operations on unbounded operators

By definition of a C*-algebra, we are able to add and compose bounded operators on a Hilbert space H. In this section, we want to define addition and composition of linear (not necessarily bounded) operators on H. Unlike bounded operators, we also have to take into account the domains of the operators involved.

Definition 7.3.1. Let H be a Hilbert space and $T : D(T) \to H$ and $S : D(S) \to H$ be linear operators on H. Define the sum $T + S$ as an operator on the domain

$$D(T + S) = D(T) \cap D(S)$$

by $(T + S)(\xi) = T\xi + S\xi$ for $\xi \in D(T + S)$.

Define the composition ST on the domain

$$D(ST) = \{\xi \in D(T) \mid T\xi \in D(S)\}$$

by $(ST)(\eta) = S(T\eta)$ for $\eta \in D(ST)$.

This is similar to how sums and composites of functions are defined. If $S, T \in B(H)$ then $(S+T)^* = S^* + T^*$ and $(ST)^* = T^*S^*$. Unfortunately, this does not work with general unbounded linear operators because the sum or the composite of densely defined operators need not be densely defined. Also, the sum or composite of closed operations might not be closed or even closable. In the following theorems, we will discuss how close we can get to the relations $(S+T)^* = S^* + T^*$ and $(ST)^* = T^*S^*$ which hold for bounded linear operators.

Theorem 7.3.1. *Let H be a Hilbert space and $T : D(T) \to H$ be a closed operator. Let $x \in B(H)$. Then, the operator $T + x : D(T+x) \to H$ is closed.*

Proof. Assume that H is a Hilbert space, T is a closed operator on H and $x \in B(H)$.

Let $\{\psi_n\}_{n\in\mathbb{Z}_{>0}}$ be a sequence of elements in $D(T+x) = D(T) \cap D(x) = D(T)$ such that $\lim_{n\to\infty} \psi_n = \psi$ and $\lim_{n\to\infty}(T\psi_n + x\psi_n) = \phi$. We will show that $(\psi, \phi) \in G(T+x)$.

Observe that

$$\lim_{n\to\infty} T\psi_n = \phi - (\lim_{n\to\infty} x\psi_n) = \phi - x\psi.$$

Since T is closed, $(\psi, \phi - x\psi) \in G(T)$. So, $\psi \in D(T) = D(T+x)$ and $T\psi = \phi - x\psi$. Consequently, $(T+x)\psi = \phi$ and $(\psi, \phi) \in G(T+x)$. So, $G(T+x)$ is a closed subspace of $H \times H$ and $T + x$ is a closed operator. $\square$

A similar theorem holds for the composite of an unbounded operator with a bounded operator.

Theorem 7.3.2. *Let H be a Hilbert space and $T : D(T) \to H$ be a linear operator. Let $u \in B(H)$. Then, Tu is closed and if u is invertible then uT is closed.*

Proof. Assume that H is a Hilbert space, $T : D(T) \to H$ is a linear operator and $u \in B(H)$ be a bounded operator.

To show: (a) The operator Tu is closed.

(b) If u is invertible then uT is closed.

(a) The domain of the composite Tu is

$$D(Tu) = \{\xi \in D(u) = H \mid u\xi \in D(T)\}.$$

Let $\{\psi_n\}_{n\in\mathbb{Z}_{>0}}$ be a sequence in $D(Tu)$ such that $\lim_{n\to\infty} \psi_n = \psi$ and $\lim_{n\to\infty} Tu(\psi_n) = \rho$.

To show: (aa) $(\psi, \rho) \in G(Tu)$.

(aa) Notice that $\lim_{n\to\infty} T(u\psi_n) = \rho$ and $\lim_{n\to\infty} u\psi_n = u\psi$ because $u \in B(H)$. Since T is a closed operator, $(u\psi, \rho) \in G(T)$. So, $u\psi \in D(T)$ and $\rho = T(u\psi) = (Tu)(\psi)$. Therefore, $(\psi, \rho) = (\psi, (Tu)(\psi)) \in G(Tu)$.

(a) Part (aa) tells us that $G(Tu)$ is a closed subspace of $H \times H$. So, Tu is a closed operator.

(b) Assume that $u \in B(H)$ is invertible. Let $\{\gamma_n\}_{n\in\mathbb{Z}_{>0}}$ be a sequence of elements of $D(uT) = D(T)$ such that $\lim_{n\to\infty} \gamma_n = \gamma$ and $\lim_{n\to\infty}(uT)(\gamma_n) = \delta$.

We want to show that $(\gamma, \delta) \in G(uT)$. The sequence $\{T\gamma_n\}_{n\in\mathbb{Z}_{>0}}$ satisfies

$$\lim_{n\to\infty} T\gamma_n = \lim_{n\to\infty} u^{-1}((uT)(\gamma_n)) = u^{-1}\delta.$$

Since T is a closed operator, $(\gamma, u^{-1}\delta) \in G(T)$. So, $\gamma \in D(T)$ and $T\gamma = u^{-1}\delta$. So, $(uT)(\gamma) = \delta$ and $(\gamma, \delta) \in G(uT)$. Therefore, $G(uT)$ is a closed subspace of $H \times H$ and uT is closed as required. □

A consequence of Theorem 7.3.2 is that we can give a criterion for Tx to be a bounded operator.

Theorem 7.3.3. *Let H be a Hilbert space and $T : D(T) \to H$ be a closed operator. Let $x \in B(H)$ such that im $x \subseteq D(T)$. Then, $Tx \in B(H)$.*

Proof. Assume that H is a Hilbert space and $T : D(T) \to H$ is a closed linear operator. Assume that $x \in B(H)$ such that im $x \subseteq D(T)$. By Theorem 7.3.2, Tx is a closed operator. Since $D(Tx) = H$, we can used the closed graph theorem to find that Tx is bounded. □

In the next few theorems, we will investigate how the adjoint interacts with the sum and composite of unbounded operators, assuming that we can take the adjoint of the sum and composite of unbounded operators.

Theorem 7.3.4. *Let H be a Hilbert space and S, T be densely defined operators on H such that ST is densely defined. Then, $T^*S^* \prec (ST)^*$.*

Proof. Assume that H is a Hilbert space. Assume that S, T are densely defined operators on H such that the composite ST itself is densely defined.

To show: (a) $D(T^*S^*) \subset D((ST)^*)$.

(b) If $\eta \in D(T^*S^*)$ then $T^*S^*\eta = (ST)^*\eta$.

(a) Assume that $\eta \in D(T^*S^*)$. Then $S^*(\eta) \in D(T^*)$ and subsequently, the functional defined on $D(ST)$

$$\psi \mapsto \langle S^*(\eta), T\psi \rangle$$

is bounded. But, $\langle S^*\eta, T\psi \rangle = \langle \eta, ST\psi \rangle$. So, $\eta \in D((ST)^*)$. Therefore, $D(T^*S^*) \subset D((ST)^*)$.

(b) Continuing on from part (a), we also have $\langle S^*\eta, T\psi \rangle = \langle T^*S^*\eta, \psi \rangle$ for $\eta \in D(T^*S^*)$. We know from part (a) that $\eta \in D((ST)^*)$. Since $\langle T^*S^*\eta, \psi \rangle = \langle \eta, ST\psi \rangle = \langle (ST)^*\eta, \psi \rangle$ for $\psi \in D(ST)$, $(T^*S^*)(\eta) = (ST)^*\eta$ as required.

By combining parts (a) and (b), we deduce that $T^*S^* \prec (ST)^*$. $\square$

Theorem 7.3.5. *Let H be a Hilbert space and T be a densely defined operator on H. Let $x \in B(H)$. Then, $(xT)^* = T^*x^*$.*

Proof. Assume that H is a Hilbert space and $T : D(T) \to H$ is a densely defined operator on H. Assume that $x \in B(H)$. By the previous theorem, we have $T^*x^* \prec (xT)^*$. To see that $(xT)^* \prec T^*x^*$, assume that $\eta \in D((xT)^*)$. If $\xi \in D(T) = D(xT)$ then

$$\langle (xT)^*\eta, \xi \rangle = \langle \eta, (xT)\xi \rangle = \langle \eta, x(T\xi) \rangle = \langle x^*\eta, T\xi \rangle.$$

Hence, $x^*\eta \in D(T^*)$ and $\eta \in D(T^*x^*)$. By the above equation, we also have $(T^*x^*)\eta = (xT)^*\eta$. So, $(xT)^* \prec T^*x^*$, which establishes $(xT)^* = T^*x^*$. $\square$

Next, we will handle the adjoint of a sum of unbounded operators.

Theorem 7.3.6. *Let H be a Hilbert space and T, S be linear operators on H such that $T+S$ is densely defined. Then, T and S are densely defined and $T^* + S^* \prec (T+S)^*$.*

Proof. Assume that H is a Hilbert space and T, S be linear operators on H. Assume that $T+S$ is densely defined. Then, $D(T+S) = D(T) \cap D(S)$ is dense in H. Since $D(T+S) \subseteq D(T)$ and $D(T+S) \subseteq D(S)$, $D(T)$ and $D(S)$ are both dense subsets of H. Therefore, T and S are both densely defined.

To show: (a) $D(T^* + S^*) \subseteq D((T+S)^*)$.

(b) If $\eta \in D(T^* + S^*)$ then $(T^* + S^*)\eta = (T+S)^*\eta$.

(a) Assume that $\eta \in D(T^* + S^*)$. If $\psi \in D(T+S)$ then

$$\begin{aligned}\langle \eta, (T+S)\psi \rangle &= \langle \eta, T\psi \rangle + \langle \eta, S\psi \rangle \\ &= \langle T^*\eta, \psi \rangle + \langle S^*\eta, \psi \rangle \\ &= \langle (T^* + S^*)\eta, \psi \rangle.\end{aligned}$$

Since $\langle (T^* + S^*)\eta, \psi \rangle < \infty$, the functional $\psi \mapsto \langle \eta, (T+S)\psi \rangle$ must be bounded. Therefore, $\eta \in D((T+S)^*)$ and $D(T^* + S^*) \subseteq D((T+S)^*)$.

(b) From part (a), we know that

$$\langle (T+S)^*\eta, \psi \rangle = \langle \eta, (T+S)\psi \rangle = \langle (T^* + S^*)\eta, \psi \rangle.$$

So, $(T+S)^*\eta = (T^* + S^*)\eta$.

By combining parts (a) and (b), we deduce that $T^* + S^* \prec (T+S)^*$. $\square$

Theorem 7.3.7. *Let H be a Hilbert space and T be a densely defined operator on H. Let $x \in B(H)$. Then, $(T+x)^* = T^* + x^*$.*

Proof. Assume that H is a Hilbert space, $T : D(T) \to H$ is a densely defined linear operator and $x \in B(H)$. We know from the previous theorem that $T^* + x^* \prec (T+x)^*$.

To show: (a) $(T+x)^* \prec T^* + x^*$.

(a) Assume that $\rho \in D((T+x)^*)$. Then, the functional which maps $\psi \in D(T+x) = D(T)$ to $\langle \rho, (T+x)\psi \rangle$ is bounded. Therefore, the functional $\psi \mapsto \langle \rho, T\psi \rangle$ is also bounded because x is a bounded linear operator. Therefore, $\rho \in D(T^*)$.

Now, we compute directly that

$$\begin{aligned} \langle \rho, (T+x)\psi \rangle &= \langle \rho, T\psi \rangle + \langle \rho, x\psi \rangle \\ &= \langle T^*\rho, \psi \rangle + \langle x^*\rho, \psi \rangle \\ &= \langle (T^* + x^*)\rho, \psi \rangle. \end{aligned}$$

Since $\langle \rho, (T+x)\psi \rangle = \langle (T+x)^*\rho, \psi \rangle$, we find that if $\rho \in D((T+x)^*)$ then $(T^* + x^*)\rho = (T+x)^*\rho$. Consequently, $(T+x)^* \prec T^* + x^*$.

Because $T^* + x^* \prec (T+x)^*$ and $(T+x)^* \prec T^* + x^*$ from part (a), we deduce that $T^* + x^* = (T+x)^*$. □

7.4 Spectrum of a closed densely defined operator

As we know, the concept of a spectrum depends on invertibility. Hence, we first define what it means for an unbounded operator to be invertible.

Definition 7.4.1. Let H be a Hilbert space and $T : D(T) \to H$ be a linear operator. We say that T is **invertible** if T is a bijection from $D(T)$ onto H.

By the closed graph theorem, if T is a closed densely defined operator which is invertible then the inverse bijection $T^{-1} : H \to D(T)$ must be bounded. Fortunately, the concept of a spectrum does not differ from the spectrum of a bounded operator.

Definition 7.4.2. Let H be a Hilbert space and T be a closed densely defined operator on H. Let I be the identity operator on H. The **spectrum** of T, denoted by $\sigma(T)$, is the set

$$\sigma(T) = \{\lambda \in \mathbb{C} \mid \lambda I - T \text{ is not invertible}\}.$$

The **resolvent set** of T, denoted by $\rho(T)$, is

$$\rho(T) = \mathbb{C} - \sigma(T) = \{\lambda \in \mathbb{C} \mid \lambda I - T \text{ is invertible}\}.$$

Just like the spectrum of a bounded linear operator, the spectrum of a closed densely defined operator on H is also a closed subset of $\mathbb{C}$.

Theorem 7.4.1. *Let H be a Hilbert space and $T : D(T) \to H$ be a closed densely defined operator on H. Then, $\sigma(T)$ is a closed subset of $\mathbb{C}$*

Proof. Assume that H is a Hilbert space and T is a closed densely defined operator on H. We will prove that the resolvent set $\rho(T)$ is an open subset of $\mathbb{C}$.

Assume that $\lambda_0 \in \rho(T)$. Then, $\lambda_0 I - T$ is invertible and $(\lambda_0 I - T)^{-1} \in B(H)$. Recalling the argument in Theorem 1.2.3, if

$$|\lambda - \lambda_0| < \frac{1}{\|(\lambda_0 I - T)^{-1}\|}$$

then the series

$$\sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n (\lambda_0 I - T)^{-n-1}$$

converges in $B(H)$ to an operator r.

To show: (a) If $\psi \in D(\lambda I - T) = D(T)$ then $r(\lambda I - T)\psi = \psi$.

(b) If $\xi \in H$ then $r\xi \in D(T)$ and $(\lambda I - T)r\xi = \xi$.

(a) Assume that $\psi \in D(\lambda I - T) = D(T)$. Then,

$$(\lambda I - T)\psi = (\lambda - \lambda_0)\psi + (\lambda_0 I - T)\psi$$

and by direct computation,

$$\begin{aligned}
r(\lambda I - T)\psi &= \Big(\sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n (\lambda_0 I - T)^{-n-1}\Big)(\lambda I - T)\psi \\
&= -\sum_{n=0}^{\infty} (\lambda_0 - \lambda)^{n+1} (\lambda_0 I - T)^{-n-1}\psi + \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n (\lambda_0 I - T)^{-n}\psi \\
&= \psi.
\end{aligned}$$

(b) Assume that $\xi \in H$. If $n \in \mathbb{Z}_{>0}$ then $(\lambda_0 I - T)^{-n-1}\xi \in D(\lambda_0 I - T) = D(T)$. Now define

$$\xi_N = \sum_{n=0}^{N} (\lambda_0 - \lambda)^n (\lambda_0 I - T)^{-n-1}\xi.$$

Then, $\{\xi_N\}_{N\in\mathbb{Z}_{>0}}$ is a sequence of elements in $D(\lambda I - T)$ converging to $r\xi$. Now observe that

$$
\begin{aligned}
(\lambda I - T)\xi_N &= \big((\lambda-\lambda_0)I + (\lambda_0 I - T)\big)\xi_N \\
&= (\lambda-\lambda_0)\xi_N + \sum_{n=0}^{N}(\lambda_0-\lambda)^n(\lambda_0 I - T)^{-n}\xi \\
&= (\lambda-\lambda_0)\xi_N + (\lambda_0-\lambda)\sum_{n=0}^{N}(\lambda_0-\lambda)^{n-1}(\lambda_0 I - T)^{-n}\xi \\
&= (\lambda-\lambda_0)\xi_N + (\lambda_0-\lambda)\Big(\frac{1}{\lambda_0-\lambda}\xi + \sum_{n=1}^{N-1}(\lambda_0-\lambda)^{n-1}(\lambda_0 I - T)^{-n}\xi\Big) \\
&= (\lambda-\lambda_0)\xi_N + (\lambda_0-\lambda)\Big(\frac{1}{\lambda_0-\lambda}\xi + \sum_{n=0}^{N-1}(\lambda_0-\lambda)^{n}(\lambda_0 I - T)^{-n-1}\xi\Big) \\
&= (\lambda-\lambda_0)\xi_N + \xi + (\lambda_0-\lambda)\xi_N \\
&= \xi + (\lambda-\lambda_0(\xi_N - \xi_{N-1}) \to \xi
\end{aligned}
$$

as $N\to\infty$. So, $\{(\xi_N, (\lambda I - T)\xi_N)\}_{N\in\mathbb{Z}_{>0}}$ is a sequence in $G(\lambda I - T)$ which converges to $(r\xi, \xi)$. Since $\lambda I - T$ is closed, $(r\xi,\xi)\in G(\lambda I - T)$. So, $r\xi \in D(\lambda I - T)$ and $\xi = (\lambda I - T)r\xi$ as required.

By parts (a) and (b), we deduce that $\lambda I - T$ is invertible with inverse $r\in B(H)$. So, $\lambda\in\rho(T)$ and $\rho(T)$ is an open subset of $\mathbb{C}$. Therefore, $\sigma(T) = \mathbb{C} - \rho(T)$ is a closed subset of $\mathbb{C}$. $\square$

In [Sol18, Page 111], it is mentioned that one can construct closed densely defined operators T such that $\sigma(T)=\emptyset$ or $\sigma(T)=\mathbb{C}$. It is also explained that if u is a unitary operator then $\sigma(uTu^*)=\sigma(T)$. This mirrors Theorem 3.1.10.

In some references, the spectrum is defined as a subset of the Riemann sphere $\widehat{\mathbb{C}}$ and by definition, $\infty\in\sigma(T)$ whenever T is unbounded. This is sometimes called the *extended spectrum*. The extended spectrum is always a non-empty and compact subset of $\widehat{\mathbb{C}}$, regardless of whether the operator is bounded or not.

Chapter 8

The z-transform

8.1 Definition of the z-transform

In this section, we will define the z-transform of a closed and densely defined operator. We will approach the z-transform via a slew of preliminary results on closed operators.

Definition 8.1.1. Let H be a Hilbert space and T be a closed operator on H. A subspace $D \subset D(T)$ is a **core** for T if T is the closure of the restriction $T|_D$ of T to D.

If D is the core for the closed operator $T : D(T) \to H$ then $\overline{T|_D} = T$ and $\overline{G(T|_D)} = G(T)$. This show that the subspace $G(T|_D)$ is dense in $G(T)$.

From here, we will assume that T is closed and densely defined operator on a Hilbert space H. Recall from Theorem 7.2.1 that

$$G(T^*) = UG(T)^{\perp} = \{(T^*\phi, -\phi) \mid \phi \in D(T^*)\}$$

where

$$U = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

is an operator on $H \times H$ such that $U(\xi, \eta) = (\eta, -\xi)$. Since T is closed, $G(T)$ is a closed subspace of $H \times H$. So, $H \times H = G(T) \oplus G(T)^{\perp}$ by Theorem 2.4.1.

Theorem 8.1.1. *Let H be a Hilbert space and T be a closed, densely defined operator on H. If $\xi \in H$ then there exists a unique $\psi \in D(T^*T)$ such that*

$$\xi = (I + T^*T)\psi.$$

Moreover, $\|\psi\| \le \|\xi\|$.

Proof. Assume that H is a Hilbert space. Assume that T is a closed, densely defined operator on H. Assume that $\xi \in H$.

Since $H \times H = G(T) \oplus G(T)^\perp$, there exists $\psi \in D(T)$ and $\phi \in D(T^*)$ such that $(\xi, 0) = (\psi, T\psi) + (T^*\phi, -\phi)$. So, $\xi = \psi + T^*\phi$ and $0 = T\psi - \phi$. Consequently, $T\psi = \phi$, $\psi \in D(T^*T)$ and

$$\xi = \psi + T^*\phi = \psi + T^*T\psi = (I + T^*T)\psi.$$

To show: (a) $\|\psi\| \le \|\xi\|$

(a) We compute directly that

$$\begin{aligned}
\|\xi\|^2 &= \langle \xi, \xi \rangle \\
&= \langle (I + T^*T)\psi, (I + T^*T)\psi \rangle \\
&= \|\psi\|^2 + \langle \psi, T^*T\psi \rangle + \langle T^*T\psi, \psi \rangle + \|T^*T\psi\|^2 \\
&= \|\psi\|^2 + 2\|T\psi\|^2 + \|T^*T\psi\|^2 > \|\psi\|^2.
\end{aligned}$$

Finally, to see that $\psi \in D(T^*T)$ is unique, assume that there exists $\psi' \in D(T^*T)$ such that $\xi = (I + T^*T)\psi'$. Since $(I + T^*T)\psi = (I + T^*T)\psi'$, $(\psi - \psi') + T^*T(\psi - \psi') = 0$. By part (a), we have

$$\|\psi - \psi'\| \le 0.$$

So, $\psi = \psi'$ and $\psi \in D(T^*T)$ is therefore the unique element such that $(I + T^*T)\psi = \xi$. $\square$

Let us rewrite the statement of Theorem 8.1.1. If T is a closed, densely defined operator on a Hilbert space H then by Theorem 8.1.1, $D(T^*T) \neq \{0\}$ and the operator

$$\begin{array}{cccc} I + T^*T: & D(T^*T) & \to & H \\ & \psi & \mapsto & \psi + T^*T\psi \end{array}$$

is a bijection which does not decrease norms. That is, if $\psi \in D(T^*T) = D(I + T^*T)$ then $\|\psi\| \le \|(I + T^*T)\psi\|$.

Since $I + T^*T$ is a bijection, we can define its inverse $(I + T^*T)^{-1} : H \to D(T^*T)$. In fact, its inverse is actually bounded because by Theorem 8.1.1,

$$\|(I + T^*T)^{-1}\| = \sup_{\|\xi\|=1} \|(I + T^*T)^{-1}\xi\| \leq \sup_{\|\xi\|=1} \|\xi\| = 1.$$

This leads us to our next preliminary result.

Theorem 8.1.2. *Let H be a Hilbert space and T be a closed, densely defined operator on H. Then, the operator T^*T is closed and $D(T^*T)$ is a core for T.*

Proof. Assume that H is a Hilbert space and T is a closed, densely defined operator on H.

To show: (a) T^*T is a closed operator.

(b) The subspace $D(T^*T)$ is a core for T.

(a) We know that the inverse operator $(I + T^*T)^{-1} : H \to D(T^*T)$ is bounded. So, its graph $G((I + T^*T)^{-1})$ is a closed subspace of H. But,

$$G((I+T^*T)^{-1}) = \{(\xi, (I+T^*T)^{-1}\xi) \mid \xi \in H\} = \{((I+T^*T)\eta, \eta) \mid \eta \in D(T^*T)\}.$$

Now let $\phi : H \times H \to H \times H$ be the bounded linear operator defined by $\phi(\xi, \eta) = (\eta, \xi)$. Then,

$$G(I + T^*T) = \phi^{-1}(\{((I + T^*T)\eta, \eta) \mid \eta \in D(T^*T)\})$$

which is a closed subspace because ϕ is a continuous map. Hence, $I + T^*T$ is a closed operator and since $T^*T = (I + T^*T) + (-I)$, we can use Theorem 7.3.1 to show that T^*T must also be a closed operator.

(b) To see that $D(T^*T)$ is a core for T, it suffices to show that $\overline{G(T|_{D(T^*T)})} = G(T)$. Suppose for the sake of contradiction that $G(T|_{D(T^*T)})$ is not dense in $G(T)$. Then, there exists a non-zero $\phi \in D(T)$ such that $(\phi, T\phi) \in G(T|_{D(T^*T)})^{\perp}$.

Hence, if $\psi \in D(T^*T)$ then

$$\langle(\phi, T\phi), (\psi, T\psi)\rangle = \langle\phi, \psi\rangle + \langle T\phi, T\psi\rangle = 0.$$

So, $\langle \phi, (I + T^*T)\psi \rangle = 0$. By Theorem 8.1.1, $(I + T^*T)$ is a bijection from $D(T^*T)$ to H. So, $\phi \in H^\perp = \{0\}$ and $\phi = 0$. This contradicts the assumption that ϕ is non-zero. So, $G(T|_{D(T^*T)})$ is a dense subspace of $G(T)$ and $D(T^*T)$ is a core for T. □

We will proceed further with our analysis of the bounded operator $(I + T^*T)^{-1}$.

Theorem 8.1.3. *Let H be a Hilbert space and T be a closed, densely defined operator on H. Then, the bounded operator $(I + T^*T)^{-1}$ is positive.*

Proof. Assume that H is a Hilbert space and T is a closed, densely defined operator on H. To reiterate, the inverse operator $(I + T^*T)^{-1} \in B(H)$ satisfies $\|(I + T^*T)^{-1}\| \leq 1$. Now assume that $\xi \in H$ and $\psi = (I + T^*T)^{-1}\xi$. Then,

$$\begin{aligned}
\langle \xi, (I + T^*T)^{-1}\xi \rangle &= \langle \psi + T^*T\psi, (I + T^*T)^{-1}(I + T^*T)\psi \rangle \\
&= \langle \psi + T^*T\psi, \psi \rangle \\
&= \langle \psi, \psi \rangle + \langle T^*T\psi, \psi \rangle \\
&= \langle \psi, \psi \rangle + \langle T\psi, T\psi \rangle \\
&= \|\psi\|^2 + \|T\psi\|^2 \geq 0.
\end{aligned}$$

Therefore, $(I + T^*T)^{-1}$ is a positive operator. □

Theorem 8.1.3 allows us to construct the operator $(I + T^*T)^{-\frac{1}{2}}$ — the positive square root of the positive operator $(I + T^*T)^{-1}$. We claim that the image of $(I + T^*T)^{-\frac{1}{2}}$ is dense in H. To see why this is the case, recall from Theorem 8.1.2 that $D(T^*T)$ is a core for T. This means that $T = \overline{T|_{D(T^*T)}}$ and $D(T) = \overline{D(T^*T)}$. Taking the closure of both sides and noting that T is densely defined, we find that $\overline{D(T^*T)} = H$ and T^*T is densely defined.

We now have

$$\begin{aligned}
D(T^*T) &= (I + T^*T)^{-1}H \\
&= (I + T^*T)^{-\frac{1}{2}}(I + T^*T)^{-\frac{1}{2}}H \\
&\subseteq (I + T^*T)^{-\frac{1}{2}}H.
\end{aligned}$$

Since $D(T^*T)$ is dense in H, the image $(I + T^*T)^{-\frac{1}{2}}H$ must also be dense in H.

Theorem 8.1.4. *Let H be a Hilbert space and T be a closed, densely defined operator on H. Then,*

$$(I + T^*T)^{-\frac{1}{2}}H = D(T),$$

$T(I + T^*T)^{-\frac{1}{2}} \in B(H)$ *and* $\|T(I + T^*T)^{-\frac{1}{2}}\| \leq 1$.

Proof. Assume that H is a Hilbert space and T is a closed, densely defined operator on H.

To show: (a) $T(I + T^*T)^{-\frac{1}{2}} \in B(H)$.

(b) $\|T(I + T^*T)^{-\frac{1}{2}}\| \leq 1$.

(c) $(I + T^*T)^{-\frac{1}{2}}H = D(T)$.

(a) Assume that $\eta \in H$. Then, $(I + T^*T)^{-1}\eta \in D(T^*T) \subseteq D(T)$ and

$$\begin{aligned}
\|T(I + T^*T)^{-1}\eta\|^2 &= \langle T(I + T^*T)^{-1}\eta, T(I + T^*T)^{-1}\eta\rangle \\
&= \langle (I + T^*T)^{-1}\eta, T^*T(I + T^*T)^{-1}\eta\rangle \\
&\leq \langle (I + T^*T)^{-1}\eta, T^*T(I + T^*T)^{-1}\eta\rangle + \langle (I + T^*T)^{-1}\eta, \eta\rangle \\
&= \langle (I + T^*T)^{-1}\eta, (I + T^*T)(I + T^*T)^{-1}\eta\rangle \\
&= \langle (I + T^*T)^{-1}\eta, \eta\rangle \\
&= \langle (I + T^*T)^{-\frac{1}{2}}\eta, (I + T^*T)^{-\frac{1}{2}}\eta\rangle \\
&= \|(I + T^*T)^{-\frac{1}{2}}\eta\|^2.
\end{aligned}$$

If $\xi = (I + T^*T)^{-\frac{1}{2}}\eta$ then the above computation tells us that

$$\|T(I + T^*T)^{-\frac{1}{2}}\xi\| \leq \|\xi\|.$$

Now assume that $\zeta \in H$. Since the image $(I + T^*T)^{-\frac{1}{2}}H$ is dense in H, there exists a sequence $\{\zeta_n\}_{n\in\mathbb{Z}_{>0}}$ in $(I + T^*T)^{-\frac{1}{2}}H$ such that $\zeta_n \to \zeta$ as $n \to \infty$. So,

$$(I + T^*T)^{-\frac{1}{2}}\xi_n \to (I + T^*T)^{-\frac{1}{2}}\xi$$

as $n \to \infty$ because $(I + T^*T)^{-\frac{1}{2}} \in B(H)$.

We now claim that the sequence $\{T(I + T^*T)^{-\frac{1}{2}}\xi_n\}_{n\in\mathbb{Z}_{>0}}$ is Cauchy. Assume that $\epsilon \in \mathbb{R}_{>0}$. Since $\zeta_n \to \zeta$, there exists $N \in \mathbb{Z}_{>0}$ such that if $m, n > N$ then $\|\zeta_n - \zeta_m\| < \epsilon$ and

$$\begin{aligned}\|T(I+T^*T)^{-\frac{1}{2}}\xi_n - T(I+T^*T)^{-\frac{1}{2}}\xi_m\| &= \|T(I+T^*T)^{-\frac{1}{2}}(\xi_n - \xi_m)\| \\ &\le \|\xi_n - \xi_m\| < \epsilon.\end{aligned}$$

So, $\{T(I+T^*T)^{-\frac{1}{2}}\xi_n\}_{n\in\mathbb{Z}_{>0}}$ is a Cauchy sequence. Since H is a Hilbert space, the sequence $\{T(I+T^*T)^{-\frac{1}{2}}\xi_n\}_{n\in\mathbb{Z}_{>0}}$ must converge to some $s \in H$.

The key observation here is that the sequence

$$\{((I+T^*T)^{-\frac{1}{2}}\xi_n, T(I+T^*T)^{-\frac{1}{2}}\xi_n)\}_{n\in\mathbb{Z}_{>0}}$$

in $G(T)$ converges to $((I+T^*T)^{-\frac{1}{2}}\xi, s) \in H \times H$. Since T is a closed operator, $((I+T^*T)^{-\frac{1}{2}}\xi, s) \in G(T)$, $(I+T^*T)^{-\frac{1}{2}}\xi \in D(T)$ and $s = T(I+T^*T)^{-\frac{1}{2}}\xi$.

So, the image $(I+T^*T)^{-\frac{1}{2}}H \subseteq D(T)$. By Theorem 7.3.3, the composite $T(I+T^*T)^{-\frac{1}{2}} \in B(H)$.

(b) Since $\|T(I+T^*T)^{-\frac{1}{2}}\xi\| \le \|\xi\|$, we compute directly that

$$\|T(I+T^*T)^{-\frac{1}{2}}\| \le \sup_{\|\xi\|=1} \|\xi\| = 1.$$

(c) From part (a), we have $(I+T^*T)^{-\frac{1}{2}}H \subseteq D(T)$.

To show: (ca) $D(T) \subseteq (I+T^*T)^{-\frac{1}{2}}H$.

(ca) Assume that $\rho \in D(T)$. Recall from Theorem 8.1.2 that $D(T^*T)$ is a core for T. So, $\overline{D(T^*T)} = D(T)$ and there exists a sequence $\{\psi_n\}_{n\in\mathbb{Z}_{>0}}$ in $D(T^*T)$ such that $\lim_{n\to\infty}\psi_n = \rho$ and $\lim_{n\to\infty} T\psi_n = T\rho$.

For $n \in \mathbb{Z}_{>0}$, define $\phi_n = (I+T^*T)\psi_n$. To see that the sequence $\{(I+T^*T)^{-\frac{1}{2}}\phi_n\}_{n\in\mathbb{Z}_{>0}}$ converges, we compute directly that

$$\begin{aligned}\|(I+T^*T)^{-\frac{1}{2}}(\phi_n - \phi_m)\|^2 &= \langle (I+T^*T)^{-\frac{1}{2}}(\phi_n-\phi_m), (I+T^*T)^{-\frac{1}{2}}(\phi_n-\phi_m)\rangle \\ &= \langle (\phi_n-\phi_m), (I+T^*T)^{-1}(\phi_n-\phi_m)\rangle \\ &= \langle (I+T^*T)(\psi_n-\psi_m), \psi_n-\psi_m\rangle \\ &= \|\psi_n-\psi_m\|^2 + \|T\psi_n - T\psi_m\|^2 \\ &\to 0\end{aligned}$$

as $m, n \to \infty$. The above computation establishes that the sequence $\{(I + T^*T)^{-\frac{1}{2}}\phi_n\}_{n\in\mathbb{Z}_{>0}}$ is Cauchy and hence, converges in the Hilbert space H.

Finally, we have

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \psi_n \\
&= \lim_{n\to\infty} (I + T^*T)^{-1}\phi_n \\
&= \lim_{n\to\infty} (I + T^*T)^{-\frac{1}{2}}(I + T^*T)^{-\frac{1}{2}}\phi_n \\
&= (I + T^*T)^{-\frac{1}{2}} \lim_{n\to\infty} (I + T^*T)^{-\frac{1}{2}}\phi_n \\
&\in (I + T^*T)^{-\frac{1}{2}} H
\end{aligned}$$

So, $D(T) \subseteq (I + T^*T)^{-\frac{1}{2}} H$.

(c) Therefore, $D(T) = (I + T^*T)^{-\frac{1}{2}} H$. □

Now, we are able to state the most important definition in our analysis of unbounded operators.

Definition 8.1.2. Let H be a Hilbert space and $T : D(T) \to H$ be a closed, densely defined operator. Let $I : H \to H$ be the identity operator on H. The ***z*-transform** of T, denoted by the symbol z_T, is the operator

$$\begin{array}{rccc}
z_T : & H & \to & H \\
& \xi & \mapsto & T(I + T^*T)^{-\frac{1}{2}}\xi
\end{array}$$

The z-transform z_T well-defined by part (c) of Theorem 8.1.4 and z_T is a bounded linear operator on H which satisfies $\|z_T\| \le 1$ by part (b) of Theorem 8.1.4.

Note that since $\|z_T\| \le 1$, $z_T^* z_T \le I$ and consequently, the operator $I - z_T^* z_T$ is positive. To see why this is the case, we compute for $\xi \in H$ that

$$\begin{aligned}
\langle \xi, (I - z_T^* z_T)(\xi)\rangle &= \langle \xi, \xi\rangle - \langle \xi, z_T^* z_T \xi\rangle \\
&= \|\xi\|^2 - \|z_T\xi\|^2 \\
&\ge \|\xi\|^2(1 - \|z_T\|)^2 \ge 0.
\end{aligned}$$

By Theorem 2.3.4, $I - z_T^* z_T$ is a positive operator.

8.2 Properties of the z-transform

In order to understand the effectiveness of the z-transform, we will prove a variety of properties of the z-transform which demonstrate why the z-transform z_T contains information about a closed densely defined operator T.

Theorem 8.2.1. *Let H be a Hilbert space and $T : D(T) \to H$ be a closed, densely defined operator on H. Then,*

$$G(T) = \{((I - z_T^* z_T)^{\frac{1}{2}}\xi, z_T\xi) \mid \xi \in H\}.$$

Proof. Assume that H is a Hilbert space and $T : D(T) \to H$ is a closed densely defined operator on H. Recall from Theorem 8.1.4 that $(I + T^*T)^{-\frac{1}{2}}H = D(T)$. So, we can rewrite the graph $G(T)$ as

$$\begin{aligned} G(T) &= \{(\psi, T\psi) \mid \psi \in D(T)\} \\ &= \{((I + T^*T)^{-\frac{1}{2}}\xi, T(I + T^*T)^{-\frac{1}{2}}\xi) \mid \xi \in H\} \\ &= \{((I + T^*T)^{-\frac{1}{2}}\xi, z_T\xi) \mid \xi \in H\}. \end{aligned}$$

To show: (a) $(I + T^*T)^{-\frac{1}{2}} = (I - z_T^* z_T)^{\frac{1}{2}}$.

(a) If $\xi \in H$ then $(I + T^*T)^{-1}\xi \in D(T^*T)$ and

$$\begin{aligned} \|(I + T^*T)^{-\frac{1}{2}}\xi\|^2 &= \langle (I + T^*T)^{-\frac{1}{2}}\xi, (I + T^*T)^{-\frac{1}{2}}\xi\rangle \\ &= \langle (I + T^*T)^{-1}\xi, \xi\rangle \\ &= \langle (I + T^*T)^{-1}\xi, (I + T^*T)(I + T^*T)^{-1}\xi\rangle \\ &= \|(I + T^*T)^{-1}\xi\|^2 + \langle (I + T^*T)^{-1}\xi, T^*T(I + T^*T)^{-1}\xi\rangle \\ &= \|(I + T^*T)^{-1}\xi\|^2 + \|T(I + T^*T)^{-1}\xi\|^2 \\ &= \|(I + T^*T)^{-\frac{1}{2}}(I + T^*T)^{-\frac{1}{2}}\xi\|^2 + \|T(I + T^*T)^{-\frac{1}{2}}(I + T^*T)^{-\frac{1}{2}}\xi\|^2. \end{aligned}$$

By setting $\psi = (I + T^*T)^{-\frac{1}{2}}\xi$, we find that

$$\|\psi\|^2 = \|(I + T^*T)^{-\frac{1}{2}}\psi\|^2 + \|z_T\psi\|^2.$$

Notice that $\psi \in (I + T^*T)^{-\frac{1}{2}}H = D(T)$ by Theorem 8.1.4. Since $D(T)$ is a dense subspace of H (because T is densely defined), we find that the equation $\|\psi\|^2 = \|(I + T^*T)^{-\frac{1}{2}}\psi\|^2 + \|z_T\psi\|^2$ holds for any $\psi \in H$.

Now if $\psi, \phi \in H$ then Theorem 2.1.1 yields

$$\begin{aligned}
\langle \phi, \psi \rangle &= \frac{1}{4} \sum_{k=0}^{3} i^k \|\phi + i^k \psi\|^2 \\
&= \frac{1}{4} \sum_{k=0}^{3} i^k \big(\|(I + T^*T)^{-\frac{1}{2}} (\phi + i^k \psi)\|^2 + \|z_T(\phi + i^k \psi)\|^2 \big) \\
&= \frac{1}{4} \sum_{k=0}^{3} i^k \|(I + T^*T)^{-\frac{1}{2}} \phi + i^k (I + T^*T)^{-\frac{1}{2}} \psi\|^2 + \frac{1}{4} \sum_{k=0}^{3} i^k \|z_T \phi + i^k z_T \psi\|^2 \\
&= \langle (I + T^*T)^{-\frac{1}{2}} \phi, (I + T^*T)^{-\frac{1}{2}} \psi \rangle + \langle z_T \phi, z_T \psi \rangle \\
&= \langle \phi, (I + T^*T)^{-1} \psi \rangle + \langle \phi, z_T^* z_T \psi \rangle.
\end{aligned}$$

Thus, if $\psi \in H$ then

$$I\psi = \psi = ((I + T^*T)^{-1} + z_T^* z_T)\psi$$

and $I - z_T^* z_T = (I + T^*T)^{-1}$. Taking the positive square root of both sides, we find that $(I - z_T^* z_T)^{\frac{1}{2}} = (I + T^*T)^{-\frac{1}{2}}$ as required.

From part (a), we have

$$\begin{aligned}
G(T) &= \{(\psi, T\psi) \mid \psi \in D(T)\} \\
&= \{((I + T^*T)^{-\frac{1}{2}} \xi, z_T \xi) \mid \xi \in H\} \\
&= \{(I - z_T^* z_T)^{\frac{1}{2}} \xi, z_T \xi) \mid \xi \in H\}.
\end{aligned}$$

□

By Theorem 8.2.1, we find that the graph $G(T)$ can be expressed entirely with the z-transform z_T.

Theorem 8.2.2. *Let H be a Hilbert space and S, T be closed densely defined operators on H. If $z_S = z_T$ then $S = T$.*

Proof. Assume that H is a Hilbert space and S, T are two closed, densely defined operators on H. Assume that $z_S = z_T$. Then, by Theorem 8.2.1,

$$\begin{aligned}
G(T) &= \{(I - z_T^* z_T)^{\frac{1}{2}} \xi, z_T \xi) \mid \xi \in H\} \\
&= \{(I - z_S^* z_S)^{\frac{1}{2}} \xi, z_S \xi) \mid \xi \in H\} = G(S).
\end{aligned}$$

Therefore, $T = S$. □

Here is a useful property satisfied by the positive operator $I - z_T^* z_T$.

Theorem 8.2.3. *Let H be a Hilbert space and $T : D(T) \to H$ be a closed densely defined operator. Then, the positive operator $I - z_T^* z_T$ is injective.*

Proof. Assume that H is a Hilbert space and $T : D(T) \to H$ is a closed densely defined operator. We know that

$$\ker I - z_T^* z_T = (\text{im } I - z_T^* z_T)^\perp$$

By Theorem 8.2.1,

$$\text{im } I - z_T^* z_T = \text{im } (I + T^*T)^{-1} = D(T^*T).$$

By Theorem 8.1.2, $D(T^*T)$ is dense in H. Therefore,

$$\begin{aligned}(\ker I - z_T^* z_T)^\perp &= \big((\text{im } I - z_T^* z_T)^\perp\big)^\perp \\ &= \overline{\text{im } I - z_T^* z_T} \\ &= \overline{D(T^*T)} = H.\end{aligned}$$

Consequently, $\ker I - z_T^* z_T = \{0\}$ as required. □

It is useful to know when a bounded linear operator on H is the z-transform of a closed densely defined operator on H. To this end, we prove the following lemma.

Lemma 8.2.4. *Let H be a Hilbert space and $z \in B(H)$ be such that $\|z\| \leq 1$. If $f \in Cts([0,1], \mathbb{C})$ then*

$$f(z^*z)z^* = z^* f(zz^*) \qquad \textit{and} \qquad z f(z^*z) = f(zz^*)z.$$

Proof. Assume that H is a Hilbert space and $z \in B(H)$. Assume that $\|z\| \leq 1$ and $f \in Cts([0,1], \mathbb{C})$. Since $\|z\| \leq 1$,

$$\sigma(z^*z) \subseteq \{\lambda \in \mathbb{C} \mid \lambda \leq \|z^*z\|\} \subseteq \{\lambda \in \mathbb{C} \mid \lambda \leq 1\}.$$

Since z^*z is self-adjoint, its spectrum $\sigma(z^*z) \subseteq \mathbb{R}$. Therefore, $\sigma(z^*z) \subseteq [0,1]$. Similarly, $\sigma(zz^*) \subseteq [0,1]$.

By the Stone-Weierstrass theorem, let $\{f_n\}_{n \in \mathbb{Z}_{>0}}$ be a sequence of polynomials such that $f_n \to f$ uniformly on $[0,1]$ as $n \to \infty$. If $n \in \mathbb{Z}_{>0}$ then

$$f_n(z^*z)z^* = z^* f_n(zz^*).$$

By letting $n \to \infty$, we find that $f(z^*z)z^* = z^*f(zz^*)$. We also have for $n \in \mathbb{Z}_{>0}$

$$zf_n(z^*z) = f_n(zz^*)z.$$

By letting $n \to \infty$, we deduce that $zf(z^*z) = f(zz^*)z$. $\square$

Now, we will show when a bounded linear operator $z \in B(H)$ is a z-transform with the following major theorem.

Theorem 8.2.5. *Let H be a Hilbert space and $z \in B(H)$. Then, z is the z-transform of a closed densely defined operator T if and only if $\|z\| \leq 1$ and $\ker(I - z^*z) = \{0\}$.*

Proof. Assume that H is a Hilbert space and $z \in B(H)$.

To show: (a) If z is the z-transform of a closed densely defined operator T then $\|z\| \leq 1$ and $\ker(I - z^*z) = \{0\}$.

(b) If $\|z\| \leq 1$ and $\ker(I - z^*z) = \{0\}$ then z is the z-transform of a closed densely defined operator on H.

(a) Assume that $z = z_T$ for some closed densely defined operator $T : D(T) \to H$. By Theorem 8.1.4, $\|z\| \leq 1$. By Theorem 8.2.3, $\ker(I - z^*z) = \{0\}$.

(b) Assume that $\|z\| \leq 1$ and $\ker(I - z^*z) = \{0\}$. Define

$$G = \{((I - z^*z)^{\frac{1}{2}}\xi, z\xi) \mid \xi \in H\} \subseteq H \times H.$$

To show: (ba) There exists a closed densely defined operator T on H such that $G = G(T)$.

(ba) Define the operator U_z on $H \times H$ by

$$U_z = \begin{pmatrix} (I - z^*z)^{\frac{1}{2}} & -z^* \\ z & (I - zz^*)^{\frac{1}{2}} \end{pmatrix}$$

Notice that we can do this because $I - z^*z$ and $I - zz^*$ are positive operators on H. Explicitly, U_z is the operator

$$\begin{array}{llll} U_z : & H \times H & \to & H \times H \\ & (h_1, h_2) & \mapsto & ((I - z^*z)^{\frac{1}{2}}(h_1) - z^*(h_2), z(h_1) + (I - zz^*)^{\frac{1}{2}}(h_2)) \end{array}$$

Observe that U_z is bounded. To see why this is the case, we compute directly that

$$\begin{aligned}
\|U_z\|^2 &= \sup_{\|(h_1,h_2)\|=1} \|U_z(h_1,h_2)\|^2 \\
&= \sup_{\|(h_1,h_2)\|=1} \|(I-z^*z)^{\frac{1}{2}}(h_1) - z^*(h_2)\|^2 + \|z(h_1) + (I-zz^*)^{\frac{1}{2}}(h_2)\|^2 \\
&\leq \sup_{\|(h_1,h_2)\|=1} \left(\|(I-z^*z)^{\frac{1}{2}}(h_1)\| + \|z^*(h_2)\|\right)^2 + \left(\|z(h_1)\| + \|(I-zz^*)^{\frac{1}{2}}(h_2)\|\right)^2 \\
&\leq \sup_{\|(h_1,h_2)\|=1} \left(\|(I-z^*z)^{\frac{1}{2}}\|\|h_1\| + \|z^*\|\|h_2\|\right)^2 + \left(\|z\|\|h_1\| + \|(I-zz^*)^{\frac{1}{2}}\|\|h_2\|\right)^2
\end{aligned}$$

In the above computation, we have

$$\|(h_1,h_2)\|^2 = \langle (h_1,h_2),(h_1,h_2)\rangle = \|h_1\|^2 + \|h_2\|^2 = 1.$$

Consequently, $\|h_1\| \leq 1$ and $\|h_2\| \leq 1$. So,

$$\begin{aligned}
\|U_z\|^2 &\leq \sup_{\|(h_1,h_2)\|=1} \left(\|(I-z^*z)^{\frac{1}{2}}\|\|h_1\| + \|z^*\|\|h_2\|\right)^2 + \left(\|z\|\|h_1\| + \|(I-zz^*)^{\frac{1}{2}}\|\|h_2\|\right)^2 \\
&\leq \sup_{\|(h_1,h_2)\|=1} \left(\|(I-z^*z)^{\frac{1}{2}}\| + \|z^*\|\right)^2 + \left(\|z\| + \|(I-zz^*)^{\frac{1}{2}}\|\right)^2 < \infty.
\end{aligned}$$

So, $U_z \in B(H\times H)$. We now claim that U_z is unitary.

To show: (baa) U_z is unitary.

(baa) We compute directly that

$$\begin{aligned}
U_z^*U_z &= \begin{pmatrix} (I-z^*z)^{\frac{1}{2}} & z^* \\ -z & (I-zz^*)^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} (I-z^*z)^{\frac{1}{2}} & -z^* \\ z & (I-zz^*)^{\frac{1}{2}} \end{pmatrix} \\
&= \begin{pmatrix} (I-z^*z) + z^*z & z^*(I-zz^*)^{\frac{1}{2}} - (I-z^*z)^{\frac{1}{2}}z^* \\ (I-zz^*)^{\frac{1}{2}}z - z(I-z^*z)^{\frac{1}{2}} & zz^* + (I-zz^*) \end{pmatrix} \\
&= \begin{pmatrix} I & z^*(I-zz^*)^{\frac{1}{2}} - (I-z^*z)^{\frac{1}{2}}z^* \\ (I-zz^*)^{\frac{1}{2}}z - z(I-z^*z)^{\frac{1}{2}} & I \end{pmatrix}.
\end{aligned}$$

Define $f \in Cts([0,1],\mathbb{C})$ by $f(t) = \sqrt{1-t}$. By Lemma 8.2.4,

$$z^*(I-zz^*)^{\frac{1}{2}} = z^*f(zz^*) = f(z^*z)z^* = (I-z^*z)^{\frac{1}{2}}z^*$$

and

$$(I - zz^*)^{\frac{1}{2}} z = f(zz^*)z = zf(z^*z) = z(I - z^*z)^{\frac{1}{2}}.$$

Consequently,

$$U_z^* U_z = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

and by a similar computation,

$$\begin{aligned} U_z U_z^* &= \begin{pmatrix} (I - z^*z)^{\frac{1}{2}} & -z^* \\ z & (I - zz^*)^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} (I - z^*z)^{\frac{1}{2}} & z^* \\ -z & (I - zz^*)^{\frac{1}{2}} \end{pmatrix} \\ &= \begin{pmatrix} (I - z^*z) + z^*z & (I - z^*z)^{\frac{1}{2}} z^* - z^*(I - zz^*)^{\frac{1}{2}} \\ z(I - z^*z)^{\frac{1}{2}} - (I - zz^*)^{\frac{1}{2}} z & zz^* + (I - zz^*) \end{pmatrix} \\ &= \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \end{aligned}$$

So, U_z must be a unitary operator on $H \times H$.

(ba) The key reason for defining U_z is that

$$G = U_z(\{(\xi, 0) \mid \xi \in H\}).$$

Since G is the image of a closed subspace under a unitary operator, it must be a closed subspace of $H \times H$. By Theorem 7.1.5, it suffices to show that G does not contain non-zero vertical vectors and $G^\perp$ does not contain non-zero horizontal vectors.

To show: (bab) If $\xi \in H - \{0\}$ then $(0, \xi) \notin G$.

(bac) If $\xi \in H - \{0\}$ then $(\xi, 0) \notin G^\perp$.

(bab) We will prove the contrapositive of the given statement. Assume that $(0, \xi) \in G \subseteq H \times H$. By definition of G, there exists $\phi \in H$ such that

$$(0, \xi) = ((I - z^*z)^{\frac{1}{2}} \phi, z\phi).$$

Notice that $(I - z^*z)^{\frac{1}{2}} \phi = 0$. So, $(I - z^*z)\phi = 0$. By assumption, $\ker I - z^*z = \{0\}$. Therefore, $\phi = 0$ and $\xi = z(\phi) = 0$.

(bac) We will prove the contrapositive of the given statement. Assume that $(\xi, 0) \in G^\perp$. Note that

$$
\begin{aligned}
G^\perp &= (U_z(\{(\xi,0) \mid \xi \in H\}))^\perp \\
&= U_z(\{(\xi,0) \mid \xi \in H\})^\perp \\
&= U_z\{(0,\xi) \mid \xi \in H\} \\
&= \{(-z^*\xi, (I - zz^*)^{\frac{1}{2}}\xi) \mid \xi \in H\}.
\end{aligned}
$$

Since $(\xi, 0) \in G^\perp$, there exists $\varphi \in H$ such that

$$(\xi, 0) = (-z^*\varphi, (I - zz^*)^{\frac{1}{2}}\varphi).$$

So, $(I - zz^*)^{\frac{1}{2}}\varphi = 0$ and $(I - zz^*)\varphi = 0$. This means that $\varphi = zz^*\varphi$ If we compose by z^* on the left on both sides, we find that

$$z^*\varphi = z^*zz^*\varphi \qquad \text{and} \qquad (I - z^*z)(z^*\varphi) = 0.$$

Since $\ker I - z^*z = \{0\}$, $z^*\varphi = 0$ and so, $\xi = -z^*\varphi = 0$.

(ba) Part (bab) shows that G does not contain any non-zero vertical vectors. Part (bac) shows that $G^\perp$ does not contain any non-zero horizontal vectors. By Theorem 7.1.5, G must be the graph of a closed, densely defined operator $T : D(T) \to H$.

(b) Hence, $z = z_T$ as required. $\square$

There is one particular consequence of Theorem 8.2.5 we will like to highlight. In part (bac) of the proof, we were able to compute $G^\perp$ in terms of z. By repeating the same argument, we are able to express the orthogonal complement of a graph $G(T)^\perp$ in terms of the z-transform z_T.

Theorem 8.2.6. *Let H be a Hilbert space and $T : D(T) \to H$ be a closed densely defined operator. Then,*

$$G(T)^\perp = \{(-z_T^*\xi, (I - z_T z_T^*)^{\frac{1}{2}}\xi) \mid \xi \in H\}.$$

The z-transform behaves harmoniously with the adjoint.

Theorem 8.2.7. *Let H be a Hilbert space and $T : D(T) \to H$ be a closed, densely defined operator on H. Then, $z_{T^*} = z_T^*$.*

Proof. Assume that H is a Hilbert space and T is a closed densely defined operator on H. From Theorem 7.2.1, recall the bounded operator

$$U = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \in B(H \times H).$$

Using Theorem 8.2.6 and Theorem 8.2.1, we compute directly that

$$\begin{aligned} G(T^*) &= UG(T)^{\perp} \\ &= U\{(-z_T^*\xi, (I - z_T z_T^*)^{\frac{1}{2}}\xi) \mid \xi \in H\} \\ &= \{((I - z_T z_T^*)^{\frac{1}{2}}\xi, z_T^*\xi) \mid \xi \in H\} \\ &= \{((I - z_{T^*}^* z_{T^*})^{\frac{1}{2}}\xi, z_{T^*}\xi) \mid \xi \in H\} \end{aligned}$$

So, $z_{T^*} = z_T^*$. □

Finally, we show that the z-transforms of unitarily equivalent operators behave in exactly the expected way.

Theorem 8.2.8. *Let H and K be Hilbert space and $T : D(T) \to H$ be a closed densely defined operator on H. Let $u \in B(K, H)$ be a bounded, unitary operator from K to H and let $S = u^*Tu$, where the domain*

$$D(S) = \{\psi \in K \mid u\psi \in D(T)\} = \{u^*\eta \mid \eta \in D(T)\}.$$

Then, S is a closed, densely defined operator on K and $z_S = u^*z_Tu$.

Proof. Assume that H and K are Hilbert space and $T : D(T) \to H$ is a closed densely defined operator on H. Assume that $u \in B(K, H)$ is a bounded unitary operator and $S = u^*Tu$. We know that the domain of S is

$$D(S) = \{\psi \in K \mid u\psi \in D(T)\} = \{u^*\eta \mid \eta \in D(T)\}$$

where the rightmost equality follows from Theorem 3.1.8. Since $u : K \to H$ is a bounded unitary operator, $G(S)$ is a closed subspace of $H \times H$ and $D(S) = \{u^*\eta \mid \eta \in D(T)\}$ is a dense subspace of K. So, S is a closed, densely defined operator.

To see that $z_S = u^*z_Tu$, let $z = u^*z_Tu$. Then,

$$\begin{aligned} G(S) &= \{(\psi, S\psi) \mid \psi \in D(S)\} \\ &= \{(u^*\eta, u^*T\eta) \mid \eta \in D(T)\} \\ &= \{(u^*(I - z_T^* z_T)^{\frac{1}{2}}\xi, u^*z_T\xi) \mid \xi \in H\} \\ &= \{(u^*(I - z_T^* z_T)^{\frac{1}{2}}u\phi, u^*z_Tu\phi) \mid \phi \in K\} \\ &= \{((I - z^*z)^{\frac{1}{2}}\phi, z\phi) \mid \phi \in K\}. \end{aligned}$$

In the second last line, we used the fact that u is unitary and hence, surjective by Theorem 3.1.8. By Theorem 8.2.1, we find that

$$z_S = z = u^* z_T u.$$

□

8.3 Polar decomposition for closed densely defined operators

As our first significant application of the z-transform, we will extend the polar decomposition in Theorem 2.6.1 to closed densely defined operators. We begin by defining positivity for unbounded operators.

Definition 8.3.1. Let H be a Hilbert space and $T : D(T) \to H$ be a (not necessarily bounded) operator. We say that T is **positive** if for $\xi \in D(T)$,

$$\langle \xi, T\xi \rangle \geq 0.$$

Lemma 8.3.1. *Let H be a Hilbert space and $T : D(T) \to H$ be a closed densely defined operator on H. Then, T^*T is positive and self-adjoint.*

Proof. Assume that H is a Hilbert space and $T : D(T) \to H$ is a closed densely defined operator. To see that T^*T is positive, we compute directly that if $\xi \in D(T^*T)$ then

$$\langle \xi, T^*T\xi \rangle = \langle T\xi, T\xi \rangle = \|T\xi\|^2 \geq 0.$$

So, T^*T is positive.

To see that T^*T is self-adjoint, define $a = (I + T^*T)^{-1}$. By Theorem 8.1.3, a is a bounded positive operator and hence, self-adjoint. By Theorem 7.2.1,

$$G(a) = G(a^*) = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} G(a)^\perp.$$

Now the operator

$$V = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \in B(H \times H)$$

is a unitary operator. If we apply V to both sides of the equality with $G(a)$, we deduce that

$$VG(a) = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} G(a) = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} G(a)^{\perp}.$$

Now observe that

$$\begin{aligned}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} G(a) &= \{(a\xi, \xi) \mid \xi \in H\} \\ &= \{(\eta, (I + T^*T)\eta) \mid \eta \in D(T^*T)\} \\ &= G(I + T^*T).\end{aligned}$$

and

$$\begin{aligned}\begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} G(a)^{\perp} &= \{(-x, y) \in H \times H \mid (x, y) \in G(a)^{\perp}\} \\ &= \{(-x, y) \in H \times H \mid \text{If } \xi \in H \text{ then } \langle x, \xi\rangle + \langle y, a\xi\rangle = 0\} \\ &= \{(-x, y) \in H \times H \mid \text{If } \xi \in H \text{ then } \langle -x, -\xi\rangle + \langle y, a\xi\rangle = 0\} \\ &= \{(-x, y) \in H \times H \mid \text{If } \xi \in H \text{ then } \langle -x, -(I + T^*T)\xi\rangle + \langle y, \xi\rangle = 0\} \\ &= \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} G(I + T^*T)^{\perp}.\end{aligned}$$

Therefore,

$$G(I + T^*T) = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} G(I + T^*T)^{\perp} = G((I + T^*T)^*).$$

Therefore, the operator $I + T^*T$ is self-adjoint. By Theorem 7.3.7, we have

$$(T^*T)^* = ((I + T^*T) + (-I))^* = (I + T^*T)^* - I = I + T^*T - I = T^*T.$$

Therefore, T^*T is self-adjoint. □

We are able to tell when a closed densely defined operator is positive by examining its z-transform.

Lemma 8.3.2. *Let H be a Hilbert space and $T : D(T) \to H$ be a closed densely defined operator. Then, T is positive if and only if*

$$(I - z_T^* z_T)^{\frac{1}{2}} z_T \geq 0.$$

In particular, T is positive and self-adjoint if and only if $z_T \geq 0$.

Proof. Assume that H is a Hilbert space and $T : D(T) \to H$ is a closed densely defined operator.

To show: (a) T is positive if and only if $(I - z_T^* z_T)^{\frac{1}{2}} z_T \geq 0$.

(a) Assume that $\psi \in D(T)$. By Theorem 8.2.1, there exists $\xi \in H$ such that $\psi = (I - z_T^* z_T)^{\frac{1}{2}} \xi$. We compute directly that

$$\begin{aligned} \langle \psi, T\psi \rangle &= \langle (I - z_T^* z_T)^{\frac{1}{2}} \xi, T(I - z_T^* z_T)^{\frac{1}{2}} \xi \rangle \\ &= \langle (I - z_T^* z_T)^{\frac{1}{2}} \xi, T(I + T^*T)^{-\frac{1}{2}} \xi \rangle \\ &= \langle (I - z_T^* z_T)^{\frac{1}{2}} \xi, z_T \xi \rangle \\ &= \langle \xi, (I - z_T^* z_T)^{\frac{1}{2}} z_T \xi \rangle. \end{aligned}$$

By the above computation, we find that T is positive if and only if the bounded operator $(I - z_T^* z_T)^{\frac{1}{2}} z_T \geq 0$.

To show: (b) T is positive and self-adjoint if and only if $z_T \geq 0$.

(b) By Theorem 8.2.2, $T = T^*$ if and only if $z_T = z_T^*$. By using part (a), we find that T is positive and self-adjoint if and only if $z_T = z_T^*$ and if $\xi \in H$ then

$$\langle \xi, (I - z_T^* z_T)^{\frac{1}{2}} z_T \xi \rangle = \langle (I - z_T^* z_T)^{\frac{1}{4}} \xi, (I - z_T^* z_T)^{\frac{1}{4}} z_T \xi \rangle \geq 0.$$

We claim that im $(I - z_T^* z_T)^{\frac{1}{4}}$ is dense in H.

To show: (ba) im $(I - z_T^* z_T)^{\frac{1}{4}}$ is dense in H

(ba) Recall from Theorem 8.1.4 and Theorem 8.2.1 that

$$D(T) = (I + T^*T)^{-\frac{1}{2}} H = (I - z_T^* z_T)^{\frac{1}{2}} H.$$

So,

$$D(T) = (I - z_T^* z_T)^{\frac{1}{2}} H \subseteq (I - z_T^* z_T)^{\frac{1}{4}} H$$

and since T is densely defined, $(I - z_T^* z_T)^{\frac{1}{4}} H = \text{im } (I - z_T^* z_T)^{\frac{1}{4}}$ is dense in H.

(b) Thus, T is positive and self-adjoint if and only if $z_T = z_T^*$ and if $\varphi \in H$ then $\langle \varphi, z_T \varphi \rangle \geq 0$ by part (ba). So, T is positive and self-adjoint if and only if $z_T \geq 0$. $\square$

We are now ready to state and prove the polar decomposition for a closed densely defined operator.

Theorem 8.3.3 (Polar Decomposition V2)**.** *Let H be a Hilbert space and $T : D(T) \to H$ be a closed densely defined operator on H. Then, there exists a unique pair of operators (u, K) such that*

$$T = uK,$$

K is positive and self-adjoint and u^*u is the projection operator onto $\overline{im\ K}$.

Proof. Assume that H is a Hilbert space and $T : D(T) \to H$ is a closed densely defined operator on H.

The idea is to use the polar decomposition for bounded operators (see Theorem 2.6.1) to write $z_T = u|z_T|$.

To show: (a) $|z_T|$ is the z-transform of a closed densely defined operator.

(a) We will apply Theorem 8.2.5 to prove this statement.

To show: (aa) $\big\||z_T|\big\| \leq 1$.

(ab) $\ker(I - |z_T|^*|z_T|) = \{0\}$.

(aa) Using Theorem 2.6.1, we compute directly that

$$\begin{aligned}
\|z_T\|^2 &= \sup_{\|\xi\|=1} \|z_T\xi\|^2 \\
&= \sup_{\|\xi\|=1} \langle z_T\xi, z_T\xi\rangle \\
&= \sup_{\|\xi\|=1} \langle u|z_T|\xi, u|z_T|\xi\rangle \\
&= \sup_{\|\xi\|=1} \langle |z_T|\xi, u^*u|z_T|\xi\rangle \\
&= \sup_{\|\xi\|=1} \langle |z_T|\xi, s(|z_T|)|z_T|\xi\rangle \\
&= \sup_{\|\xi\|=1} \langle |z_T|\xi, |z_T|\xi\rangle = \big\||z_T|\big\|^2
\end{aligned}$$

So, $\big\||z_T|\big\| = \|z_T\| \leq 1$ by Definition 8.1.2.

(ab) Recall from Theorem 2.6.2 that

$$|z_T|^2 = z_T^* z_T.$$

Since $|z_T|$ is a positive operator and is thus, self-adjoint, we have

$$\ker(I - |z_T|^*|z_T|) = \ker(I - |z_T|^2) = \ker(I - z_T^* z_T) = \{0\}$$

by Theorem 8.2.3.

(a) By combining parts (aa) and (ab), we can then use Theorem 8.2.5 to show that there exists a closed densely defined operator $K : D(K) \to H$ such that $|z_T| = z_K$.

Since $|z_T|$ is positive (and self-adjoint), z_K is also positive (and self-adjoint). By Lemma 8.3.2, K is a positive and self-adjoint operator.

To show: (b) $T = uK$.

(b) To see that $D(K) = D(T)$, we use the fact that $|z_T|^2 = z_T^* z_T$ and Theorem 8.1.4 to compute that

$$D(K) = (I - z_K^* z_K)^{\frac{1}{2}} H = (I - |z_T|^*|z_T|)^{\frac{1}{2}} H = (I - z_T^* z_T)^{\frac{1}{2}} H = D(T).$$

Next, we establish the relation between the graphs $G(K)$ and $G(T)$. By Theorem 8.2.1,

$$\begin{aligned} G(K) &= \{((I - z_K^* z_K)^{\frac{1}{2}}\xi, z_K\xi) \mid \xi \in H\} \\ &= \{((I - |z_T|^*|z_T|)^{\frac{1}{2}}\xi, |z_T|\xi) \mid \xi \in H\} \\ &= \{((I - z_T^* z_T)^{\frac{1}{2}}\xi, |z_T|\xi) \mid \xi \in H\}. \end{aligned}$$

Therefore,

$$G(T) = \begin{pmatrix} I & 0 \\ 0 & u \end{pmatrix} G(K)$$

and consequently, $T = uK$.

Finally, recall from Theorem 2.6.1 that u^*u is the projection operator onto $\text{im } |z_T|$. Since

$$G(K) = \{((I - z_T^* z_T)^{\frac{1}{2}}\xi, |z_T|\xi) \mid \xi \in H\},$$

$\overline{\text{im } K = \text{im } |z_T|}$. Therefore, u^*u is the projection operator onto the closure $\overline{\text{im } K}$ as required.

Now we will prove that the decomposition $T = uK$ is unique. Suppose that (v, D) is another polar decomposition of T such that $T = vD$. Then,

$$T^*T = (uK)^*(uK) = K^*u^*uK = KK = K^2$$

and similarly, $T^*T = D^2$. By the definition of the z-transform (see Definition 8.1.2), we have

$$z_T = T(I + T^*T)^{-\frac{1}{2}} = T(I + K^2)^{-\frac{1}{2}} = T(I + D^2)^{-\frac{1}{2}}.$$

So, $uK(I + K^2)^{-\frac{1}{2}} = vD(I + D^2)^{-\frac{1}{2}}$ and $uz_K = vz_D$. Consequently,

$$z_K u^* u z_K = z_D v^* v z_D$$

but, $z_K u^* u z_K = z_K^2$ because im $z_K = \text{im } |z_T| = \text{im } K$. Similarly, $z_D v^* v z_D = z_D^2$. The above equation reduces to $z_K^2 = z_D^2$. So, $z_K = z_D$ and by Theorem 8.2.2, $K = D$.

Finally, to see that $u = v$, note that since $K = D$, u^*u and v^*v are both projection operators onto the closure $\overline{\text{im } K}$. So, $u^*u = v^*v$, which is equivalent to $s(z_K) = s(z_D)$. We can then repeat the argument in Theorem 2.6.1 to obtain $u = v$ as required. This proves uniqueness. $\square$

If T is a closed densely defined operator on H then the operator K in Theorem 8.3.3 is called the **modulus** or **absolute value** of T and is denoted by $|T|$. The partial isometry u is called the **phase** of T. These definitions are familiar from the polar decomposition for bounded linear operators in Theorem 2.6.1.

Chapter 9

Spectral theorems for unbounded operators

9.1 Continuous functional calculus for unbounded operators

Our second major application of z-transforms is to construct functional calculi and spectral theorems for certain unbounded operators. In this section, we will begin by constructing an analogue of the continuous functional calculus.

The idea is to start small and define a functional calculus for *bounded* continuous functions. The C*-algebra we will work with is

Definition 9.1.1. Define $Cts_b(\mathbb{R}, \mathbb{C})$ to be the set of bounded continuous functions from $\mathbb{R}$ to $\mathbb{C}$. The set $Cts_b(\mathbb{R}, \mathbb{C})$ is a C*-algebra with addition, multiplication and scalar multiplication defined pointwise on $\mathbb{C}$, the involution map defined by complex conjugation of functions and the norm given by the uniform norm

$$\|f\|_\infty = \sup_{x \in \mathbb{R}} |f(x)|.$$

Let H be a Hilbert space and $T : D(T) \to H$ be a self-adjoint operator. Our goal is to define a unital *-homomorphism from $Cts_b(\mathbb{R}, \mathbb{C})$ to $B(H)$ by using the z-transform $z_T \in B(H)$. Define the function

$$\begin{array}{rccc} \zeta : & \mathbb{R} & \to & \mathbb{C} \\ & t & \mapsto & \frac{t}{\sqrt{1+t^2}} \end{array}$$

Then, $\zeta \in Cts_b(\mathbb{R}, \mathbb{C})$. In particular, ζ is a homeomorphism from $\mathbb{R}$ to $(-1, 1)$. By definition of the z-transform (see Definition 8.1.2), we want to set up our unital *-homomorphism such that $\zeta(T) = z_T$.

Since T is self-adjoint, the z-transform z_T is self-adjoint and bounded. The map $f \mapsto (f \circ \zeta^{-1})(z_T)$ is a map from $Cts_b(\mathbb{R}, \mathbb{C})$ to $B(H)$.

We want this map to form the definition of the continuous functional calculus from $Cts_b(\mathbb{R}, \mathbb{C})$ to $B(H)$. A major complication presents itself here. The function $f \circ \zeta^{-1} \in Cts_b((-1, 1), \mathbb{C})$. However,

$$\sigma(z_T) \subseteq \{\lambda \in \mathbb{C} \mid |\lambda| \leq \|z_T\|\} \subseteq \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}.$$

Since z_T is self-adjoint, $\sigma(z_T) \subseteq [-1, 1]$. So, $f \circ \zeta^{-1} \notin Cts(\sigma(z_T), \mathbb{C})$ and we cannot blindly apply Theorem 2.2.1 to define $(f \circ \zeta^{-1})(z_T)$. Thus, the naive construction does not work.

Nonetheless, we can construct a unital *-homomorphism from $Cts_b(\mathbb{R}, \mathbb{C})$ to $B(H)$. We have to work much harder though.

Theorem 9.1.1. *Let H be a Hilbert space and $T : D(T) \to H$ be a self-adjoint operator. Define the function ζ by*

$$\begin{array}{cccc} \zeta : & \mathbb{R} & \to & \mathbb{C} \\ & t & \mapsto & \frac{t}{\sqrt{1+t^2}} \end{array}$$

*Then, there exists a unique unital *-homomorphism*

$$\begin{array}{cccc} \Phi_b : & Cts_b(\mathbb{R}, \mathbb{C}) & \to & B(H) \\ & f & \mapsto & f(T) \end{array}$$

such that $\zeta(T) = z_T$.

Proof. Assume that H is a Hilbert space and $T : D(T) \to H$ is a self-adjoint operator. Assume that ζ is the function defined as above.

The idea to rectifying our naive construction is to modify the composite $f \circ \zeta^{-1}$ slightly and define for $f \in Cts_b(\mathbb{R}, \mathbb{C})$

$$\begin{array}{cccc} \tilde{f} : & (-1, 1) & \mapsto & \mathbb{C} \\ & t & \mapsto & (f \circ \zeta^{-1})(t) \cdot (1 - t^2)^{\frac{1}{2}} \end{array}$$

The point of introducing the extra term $(1 - t^2)^{\frac{1}{2}}$ is that $\tilde{f}$ extends to a continuous function from the closed interval $[-1, 1]$ to $\mathbb{C}$. Hence, we can consider the continuous function $\tilde{f}$ as a function with closed domain $[-1, 1]$.

Now let $\xi \in D(T)$. By Theorem 8.2.1, there exists a unique $\eta \in H$ such that $\xi = (I - z_T^* z_T)^{\frac{1}{2}}\eta = (I + T^*T)^{-\frac{1}{2}}\eta$. Since $\sigma(z_T) \subseteq [-1, 1]$ and z_T is a bounded self-adjoint operator, we can use the continuous functional calculus in Theorem 2.2.1 to define the map

$$\begin{array}{rccc} \phi : & D(T) & \to & H \\ & \xi & \mapsto & \tilde{f}(z_T)\eta \end{array}$$

To show: (a) If $\xi \in D(T)$ then $\|\phi(\xi)\| < \infty$.

(a) First of all, the function $\tilde{f}$ is bounded. To see why this is the case, we compute directly that for $t \in [-1, 1]$,

$$\begin{aligned} (\overline{\tilde{f}}\tilde{f})(t) &= |\tilde{f}(t)|^2 \\ &= |f(\zeta^{-1}(t))|^2(1 - t^2) \\ &\leq \sup_{x \in \mathbb{R}} |f(x)|^2 \cdot (1 - t^2) \\ &= \|f\|_\infty^2 (1 - t^2) \leq \|f\|_\infty^2 < \infty. \end{aligned}$$

Since the continuous functional calculus in Theorem 2.2.1 is an isometry, if we apply the continuous functional calculus to the inequality $(\overline{\tilde{f}}\tilde{f})(t) \leq \|f\|_\infty^2 (1 - t^2)$, we obtain

$$\tilde{f}(z_T)^* \tilde{f}(z_T) \leq \|f\|_\infty^2 (I - z_T^* z_T).$$

Thus,

$$\begin{aligned} \|\phi(\xi)\|^2 &= \|\tilde{f}(z_T)\eta\|^2 \\ &= \langle \tilde{f}(z_T)\eta, \tilde{f}(z_T)\eta \rangle \\ &= \langle \eta, \tilde{f}(z_T)^* \tilde{f}(z_T)\eta \rangle \\ &\leq \|f\|_\infty^2 \langle \eta, (I - z_T^* z_T)\eta \rangle \\ &= \|f\|_\infty^2 \langle (I - z_T^* z_T)^{\frac{1}{2}}\eta, (I - z_T^* z_T)^{\frac{1}{2}}\eta \rangle \\ &= \|f\|_\infty^2 \langle \xi, \xi \rangle = \|f\|_\infty^2 \|\xi\|^2. \end{aligned}$$

So, $\|\phi\| \leq \|f\|_\infty < \infty$. This shows that the map ϕ is bounded and thus, proves part (a).

Since T is densely defined by assumption, $D(T)$ is dense in H. So, ϕ extends to an operator $\tilde{\phi} : H \to H$ such that $\|\tilde{\phi}\| \leq \|f\|_\infty$. Note that $\tilde{\phi} \in B(H)$. We now define $f(T) = \tilde{\phi}$.

To show: (b) The map $f \mapsto f(T)$ is a unital *-homomorphism.

(b) Assume that $f, g \in Cts_b(\mathbb{R}, \mathbb{C})$. We want to show that $(f+g)(T) = f(T) + g(T)$. By definition of $\tilde{\phi}$, we have for $\xi \in D(T)$

$$\begin{aligned}
(f+g)(T)(\xi) &= ((f+g) \circ \zeta^{-1})(z_T)(I - z_T^* z_T)^{\frac{1}{2}}(\eta) \\
&= (f \circ \zeta^{-1})(z_T)(I - z_T^* z_T)^{\frac{1}{2}}(\eta) + (g \circ \zeta^{-1})(z_T)(I - z_T^* z_T)^{\frac{1}{2}}(\eta) \\
&= \tilde{f}(z_T)(\eta) + \tilde{g}(z_T)(\eta) \\
&= f(T)(\xi) + g(T)(\xi)
\end{aligned}$$

where $\xi = (I - z_T^* z_T)^{\frac{1}{2}}\eta$. So, $(f+g)(T) = f(T) + g(T)$ on the dense subpsace $D(T)$. Therefore, $(f+g)(T) = f(T) + g(T)$ on H.

Next, let $\mathbb{1}$ be the unit of the C*-algebra $Cts_b(\mathbb{R}, \mathbb{C})$. That is, let $\mathbb{1}$ be the function which sends $x \in \mathbb{R}$ to 1. If $\xi \in D(T)$ then

$$\mathbb{1}(T)(\xi) = (\mathbb{1} \circ \zeta^{-1})(z_T)(I - z_T^* z_T)^{\frac{1}{2}}(\eta) = I(I - z_T^* z_T)^{\frac{1}{2}}(\eta) = I(\xi).$$

Since $\mathbb{1}(T)$ and I agree on the dense subspace $D(T)$, they must also be equal on the entire Hilbert space H.

Next, we show that $\overline{f}(T) = f(T)^*$. We compute directly that for $\xi \in D(T)$,

$$\begin{aligned}
\overline{f}(T)(\xi) &= (\overline{f} \circ \zeta^{-1})(z_T)(I - z_T^* z_T)^{\frac{1}{2}}(\eta) \\
&= (\overline{f \circ \zeta^{-1}})(z_T)(I - z_T^* z_T)^{\frac{1}{2}}(\eta) \\
&= \overline{\tilde{f}}(z_T)\eta = f(T)^*(\xi)
\end{aligned}$$

In the second equality, we used the fact that ζ^{-1} is a real-valued function. In the final equality, we used the fact that the continuous functional calculus in Theorem 2.2.1 is a unital *-isomorphism. Since $\overline{f}(T)$ and $f(T)^*$ agree on $D(T)$, they must also agree on H because $D(T)$ is a dense subspace of H.

Finally, we will show that $f(T)g(T) = (fg)(T)$. Observe that if $h \in Cts_b(\mathbb{R}, \mathbb{C})$ and $\xi \in H$ then by definition of $\tilde{h}$,

$$\tilde{h}(z_T)\xi = h(T)(1 - z_T^* z_T)^{\frac{1}{2}}\xi.$$

Using this identity, we find that if $\rho \in D(T^*T) = (I - z_T^* z_T)H$ then

$$\begin{aligned}
f(T)g(T)\rho &= f(T)g(z_T)(I - z_T^* z_T)\psi \\
&= f(T)g(T)(I - z_T^* z_T)^{\frac{1}{2}}(I - z_T^* z_T)^{\frac{1}{2}}\psi \\
&= f(T)\tilde{g}(z_T)(I - z_T^* z_T)^{\frac{1}{2}}\psi \\
&= f(T)(I - z_T^* z_T)^{\frac{1}{2}}\tilde{g}(z_T)\psi \quad \text{(see Theorem 2.2.2)} \\
&= \tilde{f}(z_T)\tilde{g}(z_T)\psi
\end{aligned}$$

where $\psi \in H$ satisfies $\rho = (I - z_T^* z_T)\phi$. The crucial observation in this computation is that if $t \in (-1, 1)$ then

$$\begin{aligned}
\tilde{f}(t)\tilde{g}(t) &= (f \circ \zeta^{-1})(t)(1 - t^2)^{\frac{1}{2}}(g \circ \zeta^{-1})(t)(1 - t^2)^{\frac{1}{2}} \\
&= \Big(((fg) \circ \zeta^{-1})(t)(1 - t^2)^{\frac{1}{2}}\Big)(1 - t^2)^{\frac{1}{2}} \\
&= \widetilde{fg}(t)(1 - t^2)^{\frac{1}{2}}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
f(T)g(T)\rho &= \tilde{f}(z_T)\tilde{g}(z_T)\psi \\
&= \widetilde{fg}(z_T)(I - z_T^* z_T)^{\frac{1}{2}}\psi \\
&= (fg)(T)(I - z_T^* z_T)^{\frac{1}{2}}(I - z_T^* z_T)^{\frac{1}{2}}\psi \\
&= (fg)(T)(I - z_T^* z_T)\psi = (fg)(T)\rho.
\end{aligned}$$

Therefore, $f(T)g(T) = (fg)(T)$ on $D(T^*T)$, which is a dense subspace of H by a consequence of Theorem 8.1.3. So, $f(T)g(T) = (fg)(T)$ on H. Consequently, we deduce that the map $f \mapsto f(T)$ is a unital *-homomorphism, which proves part (b).

To show: (c) $\zeta(T) = z_T$.

(c) By definition of the map $f \mapsto f(T)$, we compute directly that if $\xi \in D(T)$ and $\eta \in H$ satisfies $(I - z_T^* z_T)^{\frac{1}{2}}\eta = \xi$ then

$$\begin{aligned}
\zeta(T)(\xi) &= \tilde{\zeta}(z_T)(\eta) \\
&= (\zeta \circ \zeta^{-1})(z_T)(I - z_T^* z_T)^{\frac{1}{2}}(\eta) \\
&= z_T(I - z_T^* z_T)^{\frac{1}{2}}(\eta) = z_T(\xi).
\end{aligned}$$

Since $\zeta(T) = z_T$ on the dense subspace $D(T)$, $\zeta(T) = z_T$ on H.

Finally, we will show that the unital *-homomorphism $f \mapsto f(T)$ is unique.

To show: (d) The unital *-homomorphism $f \mapsto f(T)$ is unique.

(d) Suppose that $\Phi : Cts_b(\mathbb{R}, \mathbb{C}) \to B(H)$ is another unital *-homomorphism such that $\Phi(\zeta) = z_T$. Since Φ is a unital *-homomorphism, the values of Φ are uniquely determined by polynomials in ζ. By Lemma 3.2.3, Φ must be a contraction and is thus, continuous. This means that Φ is uniquely determined by its values on $\varphi(\zeta)$, where $\varphi \in Cts([-1, 1], \mathbb{C})$.

If $f \in Cts_b(\mathbb{R}, \mathbb{C})$ then define for $t \in \mathbb{R}$ and $n \in \mathbb{Z}_{>0}$

$$f_n(t) = \begin{cases} f(-n), & \text{if } t < -n, \\ f(t), & \text{if } -n \leq t \leq n, \\ f(n), & \text{if } t > n. \end{cases}$$

The idea is to show that if $\xi \in D(T)$ then $\Phi(f_n)\xi \to \Phi(f)\xi$ as $n \to \infty$. This means that $\Phi(f)$ is uniquely determined on $D(T)$, a dense subspace of H by values of Φ on functions f_n which have limits at $\pm\infty$.

Let us explain why this gives uniqueness of the continuous functional calculus. Since $f_n(t) \to f(t)$ as $n \to \infty$, there exists a constant $C \in \mathbb{R}_{>0}$ such that if $n \in \mathbb{Z}_{>0}$ then $\|\Phi(f_n)\| < C$. Now assume that $\xi \in H$ and $\epsilon \in \mathbb{R}_{>0}$. Since $D(T)$ is dense in H, we can select $\xi' \in D(T)$ such that $\|\xi - \xi'\| < \epsilon/3C$. We then argue that

$$\begin{aligned} \|\Phi(f_n)\xi - \Phi(f)\xi\| &\leq \|\Phi(f_n)\xi - \Phi(f_n)\xi'\| + \|\Phi(f_n)\xi' - \Phi(f)\xi'\| + \|\Phi(f)\xi' - \Phi(f)\xi\| \\ &\leq \|\Phi(f_n)\|\|\xi - \xi'\| + \|\Phi(f_n)\xi' - \Phi(f)\xi'\| + \|\Phi(f)\|\|\xi - \xi'\| \\ &< C\frac{\epsilon}{3C} + \|\Phi(f_n)\xi' - \Phi(f)\xi'\| + C\frac{\epsilon}{3C}. \end{aligned}$$

If we assume that $\Phi(f_n)\xi' \to \Phi(f)\xi'$ as $n \to \infty$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $n > N$ then $\|\Phi(f_n)\xi' - \Phi(f)\xi'\| < \epsilon/3$. Thus,

$$\|\Phi(f_n)\xi - \Phi(f)\xi\| < \epsilon$$

and $\Phi(f_n)$ must converge to $\Phi(f)$ on H. This shows that $\Phi(f)$ is uniquely determined.

To show: (da) If $\xi \in D(T)$ then $\Phi(f_n)\xi \to \Phi(f)\xi$ as $n \to \infty$.

(da) Let $g(t) = (1 - \zeta(t)^2)^{\frac{1}{2}}$. Since values of Φ on polynomials in z_T are uniquely determined, we have

$$\Phi(g^2) = (1 - \zeta(t)^2)(z_T) = I - z_T^* z_T$$

and by the uniqueness of positive square roots

$$\Phi(g) = (I - z_T^* z_T)^{\frac{1}{2}}.$$

Assume that $\xi \in D(T)$. By Theorem 8.2.1, there exists a unique $\eta \in H$ such that $\xi = (I - z_T^* z_T)^{\frac{1}{2}}\eta$. So, $\xi = \Phi(g)\eta$ and

$$\Phi(f_n)\xi = \Phi(f_n)\Phi(g)\eta = \Phi(f_n g)\eta.$$

Now observe that

$$\begin{aligned} |(f_n g)(t) - (fg)(t)| &\le \left|\big((f_n - f)g\big)(t)\right| \\ &\le \|f_n - f\|_\infty \sup_{|t|>n} |g(t)| \\ &\le 2\|f\|_\infty \sup_{|t|>n} |g(t)| \\ &\to 0 \end{aligned}$$

as $n \to \infty$. Thus, $f_n g \to fg$ uniformly on $\mathbb{R}$ and consequently, $\Phi(f_n g) \to \Phi(fg)$ in $B(H)$. So,

$$\lim_{n\to\infty} \Phi(f_n)\xi = \lim_{n\to\infty} \Phi(f_n g)\eta = \Phi(fg)\eta = \Phi(f)\xi.$$

This proves part (da).

(d) Thus, the continuous functional calculus $f \mapsto f(T)$ is unique, which completes the proof. □

We want to extend Theorem 9.1.1 to continuous real-valued functions on $\mathbb{R}$ (that is, functions in $Cts(\mathbb{R}, \mathbb{R})$). Fortunately, this can be done thanks to a specific result about the bounded continuous functional calculus we prove below.

Lemma 9.1.2. *Let $f \in Cts_b(\mathbb{R}, \mathbb{R})$ (f is a bounded real-valued function on $\mathbb{R}$) be such that if $t \in \mathbb{R}$ then $f(t) \neq 0$. Let H be a Hilbert space and $T : D(T) \to H$ be a self-adjoint operator. Then,*

$$\ker f(T) = \{0\}.$$

Proof. Assume that $f \in Cts_b(\mathbb{R}, \mathbb{R})$ and if $t \in \mathbb{R}$ then $f(t) \neq 0$. Assume that H is a Hilbert space and $T : D(T) \to H$ is a self-adjoint operator. In the setting of Theorem 9.1.1, define

$$a = f(T), \qquad b = \tilde{f}(z_T) \qquad \text{and} \qquad c = (I - z_T^* z_T)^{\frac{1}{2}}.$$

We first claim that $b \in B(H)$ has kernel $\ker b = \{0\}$.

To show: (a) $\ker b = \{0\}$.

(a) Invoking the spectral theorem in Theorem 3.1.9, we can assume that the z-transform z_T is a multiplication operator $M_g : L^2(X, \mu) \to L^2(X, \mu)$ for some semifinite measure space (X, μ) and $g \in L^\infty(X, \mu)$.

We know that $\ker(I - z_T^2) = \ker(I - z_T^* z_T) = \{0\}$ by Theorem 8.2.5. Then, ± 1 cannot be eigenvalues for $z_T = M_g$. By applying Lemma 3.1.7, the set

$$\{\omega \in X \mid g(\omega) = \pm 1\}$$

must have measure 0. Thus, $b = \tilde{f}(z_T)$ is the multiplication operator $M_{\tilde{f} \circ g}$. Now since $\tilde{f}(t) \neq 0$ for $t \in (-1, 1)$,

$$\mu(\{\omega \in X \mid (\tilde{f} \circ g)(\omega) = 0\}) = 0.$$

By another application of Lemma 3.1.7, we find that b has no non-trivial eigenvalues. Hence, $\ker b = \{0\}$ as required.

By definition of the functional calculus for T in Theorem 9.1.1, we have for $\eta \in H$

$$ac\eta = b\eta.$$

Since a, b and c are all bounded self-adjoint operators, we find that

$$ca = (ac)^* = b^* = b = ac.$$

So, $b = ac = ca$ and $\ker a \subset \ker b = \{0\}$. $\square$

Using Lemma 9.1.2, we can extend Theorem 9.1.1 as follows.

Theorem 9.1.3. *Let H be a Hilbert space and $T : D(T) \to H$ be a self-adjoint operator. If $f \in Cts(\mathbb{R}, \mathbb{R})$ then there exists a unique closed densely defined operator $f(T)$ such that $z_{f(T)} = (\zeta \circ f)(T)$, where ζ is the function*

$$\begin{array}{rccc} \zeta : & \mathbb{R} & \to & \mathbb{R} \\ & t & \mapsto & \frac{t}{\sqrt{1+t^2}} \end{array}$$

Moreover, $f(T)$ is self-adjoint.

Proof. Assume that H is a Hilbert space and $T : D(T) \to H$ is a self-adjoint operator. Assume that $f \in Cts(\mathbb{R}, \mathbb{R})$.

Now let $z = (\zeta \circ f)(T)$. Then, $\zeta \circ f \in Cts_b(\mathbb{R}, \mathbb{R})$ and

$$\|z\| \leq \|\zeta \circ f\|_\infty \leq \|\zeta\|_\infty = 1$$

because the unital *-homomorphism which maps $\zeta \circ f$ to $(\zeta \circ f)(T)$ in Theorem 9.1.1 is a contraction by Lemma 3.2.3.

Next, observe that

$$I - z^*z = g(T)$$

where $g(t) = 1 - \zeta(f(t))^2 > 0$ for $t \in \mathbb{R}$. Hence, we can use Lemma 9.1.2 to deduce that $\ker(I - z^*z) = \{0\}$.

By Theorem 8.2.5, z must be the z-transform of a unique closed densely defined operator, which we call $f(T)$.

To see that $f(T)$ is self-adjoint, we first observe that $z = z^*$ because the function $\zeta \circ f$ is real-valued and thus, equal to its complex conjugate. Next, we use Theorem 8.2.7 and the fact that $z = z^*$ in order to find that

$$z_{f(T)^*} = z^*_{f(T)} = z^* = z = z_{f(T)}.$$

By Theorem 8.2.2, we deduce that $f(T)$ is self-adjoint as required. $\square$

9.2 Borel functional calculus for unbounded operators

We begin with a simple observation. Let H be a Hilbert space and $T : D(T) \to H$ be a self-adjoint operator. Then, the z-transform $z_T \in B(H)$ is a self-adjoint operator such that $\sigma(z_T) \subseteq [-1, 1]$. In particular, we can apply Theorem 3.1.9 to z_T. This leads us to the following result.

Theorem 9.2.1. *Let H be a Hilbert space and $T : D(T) \to H$ be a self-adjoint operator on H. Then, there exists a semifinite measure space (X, μ), a measurable function $f : X \to \mathbb{R}$ and a unitary operator $u \in B(L^2(X, \mu), H)$ such that $T = uM_f u^*$. Moreover, the spectrum $\sigma(T) = V_{ess}(f)$.*

Proof. Assume that H is a Hilbert space and $T : D(T) \to H$ is a self-adjoint operator on H. By Theorem 8.2.7 and Theorem 8.2.2, the z-transform z_T is a self-adjoint operator.

By applying Theorem 3.1.9, we find that there exists a semifinite measure space (X, μ), a measurable function $F : X \to \mathbb{R}$ and a unitary operator $u \in B(L^2(X, \mu), H)$ such that $z_T = uM_F u^*$.

Now consider the operator $S = u^*Tu$. By Theorem 8.2.8, we have

$$z_S = u^* z_T u = u^*(uM_F u^*)u = M_F.$$

The domain of S is therefore given by (see Theorem 8.2.1)

$$\begin{aligned} D(S) &= \{(I - z_S^* z_S)^{\frac{1}{2}}\phi \mid \phi \in L^2(X, \mu)\} \\ &= \{(I - M_F^2)^{\frac{1}{2}}\phi \mid \phi \in L^2(X, \mu)\} \\ &= \{M_{\sqrt{1-F^2}}\phi \mid \phi \in L^2(X, \mu)\}. \end{aligned}$$

Since $\ker(I - z_S^* z_S) = \{0\}$, we can invoke Lemma 3.1.7 to find that $\sqrt{1 - F^2} \neq 0$ almost everywhere on X. Hence, the function

$$f = \frac{F}{\sqrt{1 - F^2}}$$

is well-defined and finite almost everywhere on X. Also, if $\psi = \sqrt{1 - F^2}\phi \in D(S)$ where $\phi \in L^2(X, \mu)$ then

$$\begin{aligned} S\psi &= z_S\phi \quad \text{(By Theorem 8.2.1)} \\ &= M_F\phi = F\phi \\ &= S\sqrt{1 - F^2}\phi \\ &= f\sqrt{1 - F^2}\phi = f\psi = M_f\psi. \end{aligned}$$

Therefore, $D(S) \subset D(M_f)$ and if $\psi \in D(S)$ then $S\psi = M_f\psi$. So, $S \prec M_f$ (M_f is an extension of S). Notice that both S and M_f are both self-adjoint operators. In particular, the latter is self-adjoint because the function $f : X \to \mathbb{R}$ is real-valued.

Since S cannot have a self-adjoint extension, we find that $S = M_f$ and $T = uSu^* = uM_f u^*$.

Now since T and M_f are unitarily equivalent operators, $\sigma(T) = \sigma(M_f)$. By Theorem 3.1.6, $\sigma(M_f) = V_{ess}(f)$ (see Definition 3.1.4). Hence, $\sigma(T) = \sigma(M_f)$ as required. $\square$

Analogously to bounded operators, Theorem 9.2.1 leads straight to an extension of the Borel functional calculus given in Theorem 3.2.4.

Theorem 9.2.2. *Let H be a Hilbert space and $T : D(T) \to H$ be a self-adjoint operator on H. Let $Bor(\mathbb{R}, \mathbb{C})$ denote the set of bounded Borel functions on $\mathbb{R}$. Then, there exists a unique unital *-homomorphism*

$$\begin{array}{rccc} \Phi_b : & Bor(\mathbb{R}, \mathbb{C}) & \to & B(H) \\ & g & \mapsto & g(T) \end{array}$$

such that if ζ is the function

$$\begin{array}{rccc} \zeta : & \mathbb{R} & \to & \mathbb{C} \\ & t & \mapsto & \frac{t}{\sqrt{1+t^2}} \end{array}$$

then $\zeta(T) = z_T$. Furthermore, if $\{g_n\}_{n \in \mathbb{Z}_{>0}}$ is a uniformly bounded sequence of Borel functions converging pointwise to g then $g_n(T) \to g(T)$ as $n \to \infty$ with respect to the strong topology.

Proof. Assume that H is a Hilbert space and $T : D(T) \to H$ be a self-adjoint operator on H. By Theorem 9.2.1, there exists a semifinite measure space (X, μ), a measurable function $f : X \to \mathbb{R}$ and a unitary operator $u \in B(L^2(X, \mu), H)$ such that $T = uM_f u^*$.

In particular, the proof of Theorem 3.1.9 suggests that we can assume X is a LCH space (locally compact Hausdorff), μ is a Borel measure on X and f is a Borel function. Now if $g \in Bor(\mathbb{R}, \mathbb{C})$ is a bounded Borel function on $\mathbb{R}$ then the composite $g \circ f$ is a bounded Borel function on X. Hence, we define $g(T) = \Phi_b(g)$ by

$$g(T) = uM_{g \circ f} u^*.$$

Arguing similarly to Theorem 3.2.4, we find that Φ_b is a contractive unital *-homomorphism. In particular, $\|g(T)\| \leq \|g\|_\infty$.

Assume that ζ is the function defined as above. Then,

$$\zeta(T) = \Phi_b(\zeta) = uM_{\frac{f}{\sqrt{1+f^2}}}u^*.$$

Recall from the proof of Theorem 9.2.1, we have $T = uM_f u^*$, where

$$f = \frac{F}{\sqrt{1 - F^2}}$$

and $z_T = uM_F u^*$. If we rearrange the above equation for F, we find that

$$F = \frac{f}{\sqrt{1 + f^2}}.$$

Consequently, $\zeta(T) = uM_F u^* = z_T$.

Next, let $\{g_n\}_{n \in \mathbb{Z}_{>0}}$ be a uniformly bounded sequence of Borel functions converging pointwise to g. If $\xi \in H$ then

$$\begin{aligned}
\|g_n(T)\xi - g(T)\xi\|^2 &= \|uM_{(g_n - g)\circ f}u^*\xi\|^2 \\
&= \|uM_{(g_n - g)\circ f}\psi\|^2 \\
&= \|M_{(g_n - g)\circ f}\psi\|^2 \quad \text{(since } u \text{ is isometric)} \\
&= \int_X |g_n(f(w)) - g(f(w))|^2 |\psi(w)|^2 \, d\mu(w) \\
&\to 0.
\end{aligned}$$

In the last line, we applied the dominated convergence theorem. So, $g_n(T) \to g(T)$ as $n \to \infty$ in the strong topology.

Finally, we will show that Φ_b is unique. First, note that Φ_b restricts to a unital *-homomorphism from $Cts_b(\mathbb{R}, \mathbb{C})$ to $B(H)$, which satisfies $\zeta(T) = z_T$. By uniqueness in Theorem 9.1.1, Φ_b must coincide with the unital *-homomorphism in Theorem 9.1.1 on $Cts_b(\mathbb{R}, \mathbb{C})$.

Arguing in a similar manner to Theorem 3.2.5, we find that Φ_b is unique, which completes the proof. □

9.3 Spectral measures for unbounded self-adjoint operators

As usual, let H be a Hilbert space and $T : D(T) \to H$ be a self-adjoint operator. The main goal of this section is to express T as an integral with respect to a particular spectral measure on $\mathbb{R}$. Previously, we have considered integrals of bounded Borel functions over spectral measures. This time, we will be dealing with arbitrary Borel functions.

Before we proceed, we recall some of the main definitions and constructions pertaining to spectral measures:

1. The definition of a spectral measure is given in Definition 3.3.1.

2. The total variation of a measure is defined in Definition 3.3.2.

3. Let H be a Hilbert space and $\eta, \xi \in H$. Let X be a set and $\mathcal{A}$ be a σ-algebra on X. We define the map

$$\begin{array}{rccc} \langle \xi | E\eta \rangle : & \mathcal{A} & \to & \mathbb{C} \\ & \Delta & \mapsto & \langle E(\Delta)\eta, \xi \rangle \end{array}$$

   This is a finite complex-valued measure on $(X, \mathcal{A})$, with finite total variation.

4. Let $f \in Bor(X, \mathbb{C})$ be a bounded Borel function. By using the Riesz representation theorem, we were able to define a bounded operator $x_f \in B(H)$ such that

$$\langle x_f \eta, \xi \rangle = \int_X f(w) \, d\langle \xi | E(w)\eta \rangle.$$

5. Finally, recall that we have a unital *-homomorphism given by Theorem 3.3.1.

We commence with the following lemma.

Lemma 9.3.1. *Let H be a Hilbert space and $\varphi, \psi \in H$. Let X be a set and $\mathcal{A}$ be a σ-algebra on X. Let E be a spectral measure on $(X, \mathcal{A})$. Let $g \in Bor(X, \mathbb{C})$. Then,*

$$\left| \int_X g \, d\langle \varphi | E\psi \rangle \right| \leq \|\varphi\| \Big(\int_X |g|^2 \, d\langle \psi | E\psi \rangle \Big)^{\frac{1}{2}}.$$

Proof. Assume that H is a Hilbert space and $\varphi, \psi \in H$. Let $|\langle \varphi | E\psi \rangle|$ be the total variation of $\langle \varphi | E\psi \rangle$.

Since $\langle \varphi | E\psi \rangle$ is a finite complex-valued measure, we can apply the Radon-Nikodym theorem. In particular, we have

$$
\begin{aligned}
\Big|\int_X g\ d\langle\varphi|E\psi\rangle\Big| &\le \int_X |g|\ d|\langle\varphi|E\psi\rangle| \\
&= \int_X |g|\ \frac{d|\langle\varphi|E\psi\rangle|}{d\langle\varphi|E\psi\rangle}\ d\langle\varphi|E\psi\rangle \\
&= \int_X ug\ d\langle\varphi|E\psi\rangle
\end{aligned}
$$

where $u : X \to \mathbb{C}$ is a measurable function with modulus 1 almost everywhere on X. This uses [Coh13, Corollary 4.2.6] as well as the fact that we can write $g \in Bor(X, \mathbb{C})$ as $g = p|g|$, where $p : X \to \mathbb{C}$ satisfies $\|p\|_\infty = 1$.

Hence, we have

$$
\begin{aligned}
\Big|\int_X g\ d\langle\varphi|E\psi\rangle\Big| &\le \int_X ug\ d\langle\varphi|E\psi\rangle \\
&= \langle x_{ug}\psi, \varphi\rangle \\
&= |\langle x_{ug}, \psi, \varphi\rangle \\
&\le \|\varphi\|\|x_{ug}\psi\|.
\end{aligned}
$$

It remains to compute the quantity $\|x_{ug}\psi\|$. By definition, we have

$$
\begin{aligned}
\|x_{ug}\psi\|^2 &= \langle x_{ug}\psi, x_{ug}\psi\rangle \\
&= \langle x_{|ug|^2}\psi, \psi\rangle \qquad \text{(See Theorem 3.3.1)} \\
&= \int_X |u(w)g(w)|^2\ d\langle\psi|E(w)\psi\rangle \\
&= \int_X |g(w)|^2\ d\langle\psi|E(w)\psi\rangle.
\end{aligned}
$$

Therefore,

$$
\Big|\int_X g\ d\langle\varphi|E\psi\rangle\Big| \le \|\varphi\|\|x_{ug}\psi\| = \|\varphi\|\Big(\int_X |g(w)|^2\ d\langle\psi|E(w)\psi\rangle\Big)^{\frac{1}{2}}.
$$

□

Soon, we will need the following computation, which is valid for bounded Borel functions $f, g \in Bor(X, \mathbb{C})$:

$$\int_X gf\ d\langle\varphi|E\psi\rangle = \langle x_f x_g\psi, \varphi\rangle = \langle x_g\psi, x_{\overline{f}}\varphi\rangle = \int_X g\ d\langle x_{\overline{f}}\varphi|E\psi\rangle.$$

Hence, we have the equality of complex-valued measures

$$\langle x_{\overline{f}}\varphi|E\psi\rangle = f\ d\langle\varphi|E\psi\rangle. \tag{9.1}$$

In the next theorem, f is now a general Borel function from X to $\mathbb{C}$. In order to define the integral of f, we must first have a domain which is dense in H.

Theorem 9.3.2. *Let H be a Hilbert space and $\psi \in H$. Let X be a set and $\mathcal{A}$ be a σ-algebra on X. Let E be a spectral measure on $(X, \mathcal{A})$. Let $f : X \to \mathbb{C}$ be a measurable function and define*

$$\mathscr{D}_f = \{\psi \in H \mid \int_X |f|^2\ d\langle\psi|E\psi\rangle < \infty\}.$$

Then, $\mathscr{D}_f$ is a dense subspace of H.

Proof. Assume that $n \in \mathbb{Z}_{>0}$. We define

$$\Lambda_n = \{\omega \in X \mid |f(\omega)| \leq n\}.$$

Now let $\psi \in E(\Lambda_n)H$. If $\Delta \subseteq X$ is measurable then

$$E(\Delta)\psi = E(\Delta)E(\Lambda_n)\psi = E(\Delta \cap \Lambda_n)\psi.$$

Hence, the spectral measure $\langle\psi|E\psi\rangle$ satisfies

$$\begin{aligned}
\langle\psi|E\psi\rangle(\Delta) &= \langle\psi|E(\Delta)\psi\rangle \\
&= \langle\psi|E(\Delta \cap \Lambda_n)\psi\rangle \\
&= \langle\psi|E\psi\rangle(\Delta \cap \Lambda_n).
\end{aligned}$$

We will now show that the images $E(\Lambda_n)H \subseteq \mathscr{D}_f$ for $n \in \mathbb{Z}_{>0}$.

To show: (a) If $n \in \mathbb{Z}_{>0}$ then the image $E(\Lambda_n)H \subseteq \mathscr{D}_f$.

(a) Assume that $n \in \mathbb{Z}_{>0}$. Assume that $f : X \to \mathbb{C}$ is a measurable function. Utilising the most recent finding, we have

$$\begin{aligned}
\int_X |f|^2\ d\langle\psi|E\psi\rangle &= \int_{\Lambda_n} |f|^2\ d\langle\psi|E\psi\rangle \\
&\leq n^2\|\psi\|^2 < \infty
\end{aligned}$$

In the second last inequality, we used the definition of Λ_n. Therefore, $E(\Lambda_n)H \subseteq \mathscr{D}_f$.

The characteristic functions $\{\chi_{\Lambda_n}\}_{n\in\mathbb{Z}_{>0}}$ converge to 1 pointwise as $n \to \infty$. Hence, $E(\Lambda_n)\psi \to \psi$ as $n \to \infty$. Thus, $\mathscr{D}_f$ is a dense subspace of H.

It remains to show that $\mathscr{D}_f$ is a vector subspace of H. Assume that $\psi, \varphi \in \mathscr{D}_f$. If $\Delta \in \mathcal{A}$ then

$$\begin{aligned}\|E(\Delta)(\psi+\varphi)\|^2 &\leq (\|E(\Delta)\psi\| + \|E(\Delta)\varphi\|)^2 \\ &\leq 2\|E(\Delta)\psi\|^2 + 2\|E(\Delta)\varphi\|^2.\end{aligned}$$

The last inequality follows from the AM-GM inequality. This means that

$$\langle \psi+\varphi | E(\psi+\varphi)\rangle \leq 2\langle\psi|E\psi\rangle + 2\langle\varphi|E\varphi\rangle.$$

and consequently,

$$\begin{aligned}\int_X |f|^2 \, d\langle\psi+\varphi|E(\varphi+\psi)\rangle &\leq 2\int_X |f|^2 \, d\langle\psi|E(\varphi)\rangle + 2\int_X |f|^2 \, d\langle\varphi|E(\psi)\rangle \\ &< \infty.\end{aligned}$$

Hence, $\psi + \varphi \in \mathscr{D}_f$. Now if $\lambda \in \mathbb{C}$ then

$$\begin{aligned}\int_X |f|^2 \, d\langle\lambda\varphi|E(\lambda\varphi)\rangle &\leq |\lambda|^2 \int_X |f|^2 \, d\langle\varphi|E(\varphi)\rangle \\ &< \infty.\end{aligned}$$

Hence, $\mathscr{D}_f$ is a dense vector subspace of H. □

With Theorem 9.3.2, we can now define an operator x_f from a measurable function $f : X \to \mathbb{C}$. We want the domain $D(x_f) = \mathscr{D}_f$ as in Theorem 9.3.2. Let $\psi \in D(x_f)$ and $\varphi \in H$. By Lemma 9.3.1 and the dominated convergence theorem, we find that

$$\Big|\int_X f \, d\langle\varphi|E\psi\rangle\Big| \leq \|\varphi\|\Big(\int_X |f|^2 \, d\langle\psi|E\psi\rangle\Big)^{\frac{1}{2}}.$$

Therefore, the map

$$\varphi \mapsto \int_X f \, d\langle\varphi|E\psi\rangle$$

is a bounded linear functional on H. By the Riesz representation theorem, there exists a unique $\eta \in H$ such that

$$\int_X f \, d\langle \varphi | E\psi \rangle = \langle \eta, \varphi \rangle.$$

Thus, we define x_f by

$$\begin{array}{rrcl} x_f : & \mathscr{D}_f = D(x_f) & \to & H \\ & \psi & \mapsto & \eta \end{array}$$

The expression $\int_X f \, d\langle \varphi | E\psi \rangle$ which defines η is linear in ψ. By Theorem 9.3.2, we find that x_f is a densely defined linear operator. Furthermore, if $f \in Bor(X, \mathbb{C})$ then this definition of x_f corresponds to the one in Theorem 3.3.1.

The operator x_f is also additive with respect to the measurable function f. To see what this means, assume that f_1 and f_2 are measurable functions from X to $\mathbb{C}$. Let $\psi \in D(x_{f_1}) \cap D(x_{f_2}) = \mathscr{D}_{f_1} \cap \mathscr{D}_{f_2}$. If $\varphi \in H$ then

$$\begin{aligned} \langle x_{f_1}\psi, \varphi \rangle + \langle x_{f_2}\psi, \varphi \rangle &= \int_X f_1 \, d\langle \varphi | E\psi \rangle + \int_X f_2 \, d\langle \varphi | E\psi \rangle \\ &= \int_X f_1 + f_2 \, d\langle \varphi | E\psi \rangle = \langle x_{f_1+f_2}\psi, \varphi \rangle. \end{aligned}$$

Hence, $x_{f_1} + x_{f_2} = x_{f_1+f_2}$. The next theorem is dedicated to proving more properties about x_f.

Theorem 9.3.3. *Let E be a spectral measure on the measurable space $(X, \mathcal{A})$. Then,*

1. *If $f : X \to \mathbb{C}$ is a measurable/Borel function then the operator x_f is closed.*

2. *If $\psi \in D(x_f)$ then*

$$\|x_f \psi\|^2 = \int_X |f|^2 \, d\langle \psi | E\psi \rangle.$$

3. *If $f, g : X \to \mathbb{C}$ are measurable functions then $x_f x_g \prec x_{fg}$ and $D(x_f x_g) = \mathscr{D}_g \cap \mathscr{D}_{fg}$*

4. *If $f : X \to \mathbb{C}$ is a measurable function then $x_f^* = x_{\overline{f}}$.*

Proof. Assume that E is a spectral measure on the measurable space $(X, \mathcal{A})$, where X is a set and $\mathcal{A}$ is a σ-algebra on X.

To show: (a) If $\psi \in D(x_f)$ then $\|x_f\psi\|^2 = \int_X |f|^2\, d\langle\psi|E\psi\rangle$.

(a) Assume that $\psi \in \mathscr{D}_f$. For $n \in \mathbb{Z}$, define

$$\Lambda_n = \{\omega \in X \mid |f(\omega)| \leq n\}.$$

Let χ_n be the characteristic function of Λ_n and define $f_n = \chi_n f$. Then, $f_n \in Bor(X, \mathbb{C})$ is bounded for $n \in \mathbb{Z}$ and $\mathscr{D}_{f-f_n} = \mathscr{D}_f$. We have

$$\begin{aligned}
\|x_f\psi - x_{f_n}\psi\|^2 &= \langle x_f\psi - x_{f_n}\psi, x_f\psi - x_{f_n}\psi\rangle \\
&= \langle x_{f-f_n}\psi, x_{f-f_n}\psi\rangle \\
&= \Big|\int_X (f - f_n)\, d\langle x_{f-f_n}\psi|E\psi\rangle\Big| \\
&\leq \|x_{f-f_n}\psi\|\Big(\int_X |f - f_n|^2\, d\langle\psi|E\psi\rangle\Big)^{\frac{1}{2}}.
\end{aligned}$$

By the dominated convergence theorem,

$$\|x_f\psi - x_{f_n}\psi\| \leq \Big(\int_X |f - f_n|^2\, d\langle\psi|E\psi\rangle\Big)^{\frac{1}{2}} \to 0$$

as $n \to \infty$. Since f_n is bounded, we know that by Theorem 3.3.1

$$\|x_{f_n}\psi\|^2 = \langle x_{|f_n|^2}\psi, \psi\rangle = \int_X |f_n|^2\, d\langle\psi|E\psi\rangle.$$

Since $\|x_f\psi - x_{f_n}\psi\| \to 0$, we obtain

$$\|x_f\psi\|^2 = \int_X |f|^2\, d\langle\psi|E\psi\rangle$$

as required.

To show: (b) If $f, g : X \to \mathbb{C}$ are measurable functions then $x_f x_g \prec x_{fg}$ and $D(x_f x_g) = \mathscr{D}_g \cap \mathscr{D}_{fg}$

(b) First assume that $g : X \to \mathbb{C}$ is measurable and $f \in Bor(X, \mathbb{C})$. Then, $\mathscr{D}_g \subseteq \mathscr{D}_{fg}$. If $\psi \in \mathscr{D}_g$ and $\varphi \in H$ then

$$\begin{aligned}
\langle x_f x_g \psi, \varphi \rangle &= \langle x_g \psi, x_{\overline{f}} \varphi \rangle \\
&= \int_X g \; d\langle x_{\overline{f}} \varphi | E \psi \rangle \\
&= \int_X fg \; d\langle \varphi | E \psi \rangle \qquad \text{(See equation (9.1))} \\
&= \langle x_{fg} \psi, \varphi \rangle .
\end{aligned}$$

We have shown that if $\psi \in \mathscr{D}_g$ and $f \in Bor(X, \mathbb{C})$ is bounded then $x_f x_g \psi = x_{fg} \psi$. Consequently,

$$\int_X |f|^2 \; d\langle x_g \psi | E x_g \psi \rangle = \|x_f x_g \psi\|^2 = \|x_{fg} \psi\|^2 = \int_X |fg|^2 \; d\langle \psi | E \psi \rangle .$$

The above equation holds for any bounded measurable function f. Thus, it also holds for any measurable function $f : X \to \mathbb{C}$ by a similar argument to part (a) using the dominated convergence theorem. The above equation tells us that for a measurable function f, $x_g \psi \in D(x_f) = \mathscr{D}_f$ if and only if $\psi \in \mathscr{D}_{fg}$. Therefore, $D(x_f x_g) = \mathscr{D}_{fg} \cap \mathscr{D}_g$ and because $x_f x_g \psi = x_{fg} \psi$ for $\psi \in \mathscr{D}_g$, $x_f x_g \prec x_{fg}$.

To show: (c) If $f : X \to \mathbb{C}$ is a measurable function then $x_f^* = x_{\overline{f}}$.

(c) Once again, assume that $f : X \to \mathbb{C}$ is a measurable function. We will use the bounded Borel functions $\{f_n\}_{n \in \mathbb{Z}_{>0}}$ constructed in part (a).

Assume that $\psi \in \mathscr{D}_f$ and $\varphi \in \mathscr{D}_{\overline{f}} = \mathscr{D}_f$.

To see that $\varphi \in D(x_f^*)$, we compute directly that

$$\begin{aligned}
\langle x_f \psi, \varphi \rangle &= \lim_{n \to \infty} \langle x_{f_n} \psi, \varphi \rangle \\
&= \lim_{n \to \infty} \langle \psi, x_{\overline{f_n}} \varphi \rangle \\
&= \langle \psi, x_{\overline{f}} \varphi \rangle .
\end{aligned}$$

So, $\varphi \in D(x_f^*)$. Furthermore, we also showed that $x_f^* \psi = x_{\overline{f}} \psi$. So, $x_{\overline{f}} \prec x_f^*$.

Now let $\xi \in D(x_f^*)$. By part (b) and the definition of f_n, we have $x_{f_n} = x_f x_{\chi_n}$. Since x_{χ_n} is self-adjoint, we have

$$x_{\chi_n} x_f^* \prec (x_f x_{\chi_n})^* = x_{f_n}^* = x_{\overline{f_n}} .$$

In particular, $x_{\chi_n} x_f^* \xi = x_{\overline{f_n}} \xi$ for $n \in \mathbb{Z}_{>0}$.

To see that $\xi \in D(x_{\overline{f}}) = \mathscr{D}_{\overline{f}}$, we compute directly that

$$\begin{aligned}
\int_X |f_n|^2 \, d\langle \xi | E \xi \rangle &= \|x_{\overline{f_n}} \xi\|^2 \\
&= \|x_{\chi_n} x_f^* \xi\|^2 \\
&= \int_X |\chi_n|^2 \, d\langle x_f^* \xi | E x_f^* \xi \rangle \\
&\leq \langle x_f^* \xi | E x_f^* \xi \rangle (X) < \infty
\end{aligned}$$

because $\langle x_f^* \xi | E x_f^* \xi \rangle$ is a finite measure. By the dominated convergence theorem, we must have

$$\int_X |f|^2 \, d\langle \xi | E \xi \rangle < \infty.$$

Therefore, $\xi \in D(x_{\overline{f}}) = \mathscr{D}_{\overline{f}}$ and $D(x_f^*) \subseteq D(x_{\overline{f}})$. In conjunction with the previous finding that $x_{\overline{f}} \prec x_f^*$, we deduce that $x_{\overline{f}} = x_f^*$.

To show: (d) If $f : X \to \mathbb{C}$ is measurable then the operator x_f is closed.

(d) By applying the result of part (c) to the measurable function $\overline{f} : X \to \mathbb{C}$, we find that $x_f = x_{\overline{f}}^*$. Since x_f is the adjoint of another densely defined linear operator, x_f must be closed as a consequence of Theorem 7.2.1. □

Here is the main result of this section.

Theorem 9.3.4. *Let H be a Hilbert space and $T : D(T) \to H$ be a self-adjoint operator. Then, there exists a unique spectral measure E_T on $\mathbb{R}$ such that*

$$T = \int_{\mathbb{R}} \lambda \, dE_T(\lambda).$$

Proof. Assume that H is a Hilbert space and $T : D(T) \to H$ is a self-adjoint operator. Let $X = \sigma(z_T)$ and $\mathcal{A}$ be the Borel σ-algebra on $\sigma(z_T)$. Recall that the spectral measure E_{z_T} is defined by

$$\begin{array}{rccc} E_{z_T} : & \mathcal{A} & \to & Proj(B(H)) \\ & \Delta & \mapsto & \chi_\Delta(z_T). \end{array}$$

where χ_Δ is the characteristic function on Δ. Since $\sigma(z_T) \subseteq [-1,1]$, we can treat E_{z_T} as a Borel spectral measure on the interval $[-1,1]$. Moreover, ± 1 are not eigenvalues of z_T because $\ker(I - z_T^* z_T) = \{0\}$. So, $E_{z_T}(\{-1,1\}) = 0$. Hence, E_{z_T} is a spectral measure on the open interval $(-1,1)$.

Define

$$S = \int_{-1}^{1} \frac{\mu}{\sqrt{1-\mu^2}}\, dE_{z_T}(\mu).$$

We claim that $S = T$.

To show: (a) $S = T$.

(a) First note that $S = x_f$, where $f(\mu) = \frac{\mu}{\sqrt{1-\mu^2}}$. Now set $g(\mu) = \sqrt{1-\mu^2}$. By Theorem 9.3.3, we have

$$Sx_g = x_f x_g \prec x_{fg} = \int_{-1}^{1} \mu\, dE_{z_T}(\mu) = z_T.$$

Now since g and fg are bounded on the open interval $(-1,1)$, $D(Sx_g) = \mathscr{D}_g \cap \mathscr{D}_{fg} = H$. In tandem with the fact that $Sx_g \prec z_T$, we deduce that $Sx_g = x_f x_g = z_T$.

By Theorem 3.3.2, we have $x_g = g(z_T) = (I - z_T^2)^{\frac{1}{2}}$. Hence, the equality $Sx_g = z_T$ becomes

$$\begin{aligned} S(I - z_T^2)^{\frac{1}{2}} &= Sx_g \\ &= z_T = T(I + T^*T)^{-\frac{1}{2}} \\ &= T(I - z_T^2)^{\frac{1}{2}}. \end{aligned}$$

Since $D(T) \subseteq D(S) = H$, we deduce that $T \prec S$. Since S and T are self-adjoint, $S = T$.

Now, let $\mathcal{R}$ be the Borel σ-algebra associated to $\mathbb{R}$. For $\Delta \in \mathcal{R}$, define

$$E_T(\Delta) = E_{z_T}(\zeta(\Delta))$$

where ζ is the homeomorphism

$$\begin{array}{rccc} \zeta : & \mathbb{R} & \to & (-1,1) \\ & t & \mapsto & \frac{t}{\sqrt{1+t^2}} \end{array}$$

In particular, E_T is the pushforward of E_{z_T} by ζ^{-1} and is thus, a spectral measure on $(\mathbb{R}, \mathcal{R})$. We have

$$\begin{aligned} T &= \int_{-1}^{1} \frac{\mu}{\sqrt{1-\mu^2}}\, dE_{z_T}(\mu) \\ &= \int_{-1}^{1} \zeta^{-1}(\mu)\, dE_{z_T}(\mu) \\ &= \int_{\mathbb{R}} \lambda\, dE_T(\lambda). \end{aligned}$$

To show: (b) The spectral measure E_T is unique.

(b) Assume that E is another Borel spectral measure on $\mathbb{R}$ such that

$$T = \int_{\mathbb{R}} \lambda\, dE(\lambda).$$

Then, $z_T = \zeta(T) = \int_{\mathbb{R}} \frac{\lambda}{\sqrt{1+\lambda^2}}\, dE(\lambda)$. Now let E' be the pushforward of E onto $(-1,1)$ by ζ. Then,

$$\begin{aligned} z_T &= \int_{\mathbb{R}} \frac{\lambda}{\sqrt{1+\lambda^2}}\, dE(\lambda) \\ &= \int_{\mathbb{R}} \zeta(\lambda)\, dE(\lambda) \\ &= \int_{-1}^{1} \mu\, dE'(\mu). \end{aligned}$$

By uniqueness of the spectral measure of z_T (see Theorem 3.3.2), we must have $E' = E_{z_T}$. Since ζ is a homeomorphism, the pushforward measures E and E_T must be equal. So, E_T is unique as required. $\square$

Let us note some specific consequences of Theorem 9.3.4. Since $T = \int_{\mathbb{R}} \lambda\, dE_T(\lambda)$, we can define a functional calculus for T via the map

$$f \mapsto f(T) = \int_{\mathbb{R}} f(\lambda)\, dE_T(\lambda). \tag{9.2}$$

Here, $f : \mathbb{R} \to \mathbb{C}$ is a (not necessarily bounded) Borel function. If f is bounded then we recover the functional calculus in Theorem 9.2.2.

Additionally, if $f, g : \mathbb{R} \to \mathbb{C}$ are Borel functions and g is bounded then $f(T)g(T) = (fg)(T)$. To see why this is the case, we compute directly that

$$D(f(T)g(T)) = \mathscr{D}_g \cap \mathscr{D}_{fg} = H \cap \mathscr{D}_{fg} = D((fg)(T)).$$

By Theorem 9.3.3, we have

$$g(T)f(T) \prec (gf)(T) = (fg)(T) = f(T)g(T).$$

So, $f(T)g(T) = (fg)(T)$. In the above equation, one cannot expect equality. For instance, if $f(T)$ is unbounded and $g = 0$ then the domain of $g(T)f(T)$ is $D(f(T))$ and the domain of $f(T)g(T)$ is all of H.

Let us reconcile Theorem 9.3.4 with Theorem 9.1.3, which says that $f(T)$ is the unique closed, densely defined operator whose z-transform is $(\zeta \circ f)(T)$, where $\zeta(x) = \frac{x}{\sqrt{1+x^2}}$.

Is $f(T)$ in Theorem 9.1.3 the same $f(T)$ defined in equation (9.2)?

Let E_T be the spectral measure of T in Theorem 9.3.4. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous real-valued function. With $f_* E_T$ denoting the pushforward of E_T by f, we have

$$\int_{\mathbb{R}} \lambda \; d(f_* E_T)(\lambda) = \int_{\mathbb{R}} f(\lambda) \; dE_T(\lambda) = f(T).$$

By uniqueness of the spectral measure of $f(T)$ in Theorem 9.3.4, we must have $E_{f(T)} = f_* E_T$. So,

$$\begin{aligned} z_{f(T)} &= \int_{\mathbb{R}} \zeta(u) \; dE_{f(T)}(\mu) \\ &= \int_{\mathbb{R}} (\zeta \circ f)(\lambda) \; dE_T(\lambda) \\ &= (\zeta \circ f)(T). \end{aligned}$$

Therefore, the z-transform of $f(T)$ in Theorem 9.3.4 is the operator whose z-transform is given in Theorem 9.1.3.

Chapter 10

Self-adjoint extensions of symmetric operators

10.1 The Cayley transform

One of the main questions about unbounded operators is whether symmetric/Hermitian operators have a self-adjoint extension. The idea here is that self-adjoint operators are crucial to several fields, such as quantum physics and partial differential equations. However, many operators which arise from problems turn out to be merely symmetric. This chapter is dedicated to developing a few results about self-adjoint extensions of operators, with the z-transform playing an important role.

First, we recall the definition of a symmetric/Hermitian operator.

Definition 10.1.1. Let H be a Hilbert space and $T : D(T) \to H$ be a densely defined operator. We say that T is **symmetric** or **Hermitian** if $T \prec T^*$.

Recall that the expression $T \prec T^*$ means that $D(T) \subset D(T^*)$ and $T = T^*$ on the dense subspace $D(T)$.

We want to define the Cayley transform, a useful tool for studying self-adjoint extensions. We need the following lemma about z-transforms to do this.

Lemma 10.1.1. *Let H be a Hilbert space and S, T be densely defined operators on H. Then, $T \prec S$ if and only if*

$$(I - z_S z_S^*)^{\frac{1}{2}} z_T = z_S (I - z_T^* z_T)^{\frac{1}{2}}.$$

Proof. Assume that H is a Hilbert space and S, T are two densely defined operators on H.

By Theorem 8.2.1, the graphs of S and T are

$$G(S) = \{((I - z_S^* z_S)^{\frac{1}{2}}\xi, z_S\xi) \mid \xi \in H\}$$

and

$$G(T) = \{((I - z_T^* z_T)^{\frac{1}{2}}\xi, z_T\xi) \mid \xi \in H\}.$$

The key idea is that we can express these graphs as $G(T) = U_{z_T}(H \oplus \{0\})$ and $G(S) = U_{z_S}(H \oplus \{0\})$. Here, U_{z_T} is a unitary operator on $H \times H$ defined by

$$U_{z_T} = \begin{pmatrix} (I - z_T^* z_T)^{\frac{1}{2}} & -z_T^* \\ z_T & (I - z_T z_T^*)^{\frac{1}{2}} \end{pmatrix}$$

The unitary operator U_{z_S} is defined similarly. Consequently, the statement $T \prec S$ is equivalent to saying that $G(T) \subset G(S)$. So,

$$U_{z_T}(H \oplus \{0\}) \subset U_{z_S}(H \oplus \{0\})$$

and

$$U_{z_S}^* U_{z_T}(H \oplus \{0\}) \subset H \oplus \{0\}.$$

Now, the only operators on $H \oplus H$ which preserve the subspace $H \oplus \{0\}$ are the ones with matrix form

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}.$$

So, $T \prec S$ if and only if the operator $U_{z_S}^* U_{z_T}$ takes on the above matrix form. Computing $U_{z_S}^* U_{z_T}$, we find that

$$\begin{aligned} U_{z_S}^* U_{z_T} &= \begin{pmatrix} (I - z_S^* z_S)^{\frac{1}{2}} & z_S^* \\ -z_S & (I - z_S z_S^*)^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} (I - z_T^* z_T)^{\frac{1}{2}} & -z_T^* \\ z_T & (I - z_T z_T^*)^{\frac{1}{2}} \end{pmatrix} \\ &= \begin{pmatrix} * & * \\ -z_S(I - z_T^* z_T)^{\frac{1}{2}} + (I - z_S z_S^*)^{\frac{1}{2}} z_T & * \end{pmatrix}. \end{aligned}$$

Therefore, $T \prec S$ if and only if $z_S(I - z_T^* z_T)^{\frac{1}{2}} = (I - z_S z_S^*)^{\frac{1}{2}} z_T$ as required. ∎

Before we develop the Cayley transform, we will introduce some notation used for handling partial isometries. It is useful to recall the notion of a partial isometry by consulting Theorem 2.5.1 and Theorem 2.5.2. In particular, Theorem 2.5.1 is the definition of a partial isometry.

Let $v \in B(H)$ be a partial isometry. By Theorem 2.5.2, there exists a closed subspace $S \subset H$ such that if $\xi \in S^{\perp}$ then $v\xi = 0$ and if $\eta \in S$ then $\|v\eta\| = \|\eta\|$.

Define $\mathring{v}$ to be the restriction $v|_S$. We think of $\mathring{v}$ as an operator on H with domain $D(\mathring{v}) = S$. For instance, if $x \in B(H)$ then the operator $x + \mathring{v}$ has domain $D(x + \mathring{v}) = D(x) \cap D(\mathring{v}) = S$. Also, we have the bijective correspondence

$$\begin{array}{ccc} \{\text{Partial isometries on } H\} & \leftrightarrow & \left\{\begin{array}{c}\text{Isometric operators defined}\\ \text{on closed subspaces of } H\end{array}\right\} \\ v & \mapsto & \mathring{v} \end{array}$$

In what follows, we let $T : D(T) \to H$ be a symmetric operator.

Definition 10.1.2. Let H be a Hilbert space and $T : D(T) \to H$ be a symmetric operator. A **self-adjoint extension** of T is a densely defined operator $S : D(S) \to H$ such that $T \prec S$ and $S = S^*$.

Recall that by Theorem 7.2.1, a self-adjoint operator must be closed because its graph is the graph of its adjoint, which is always a closed subspace of $H \times H$. By Theorem 7.2.2, a symmetric operator must be closable. Since $T \prec T^*$, the domain $D(T^*)$ is a dense subspace of H. So, T^* is densely defined and hence, T is closable.

Consequently, any self-adjoint extension of T also qualifies as an extension of the closure $\overline{T}$ because if S is a self-adjoint extension of T then $G(S)$ is a closed subspace of $H \times H$ which contains $G(T)$ and

$$G(\overline{T}) = \overline{G(T)} \subseteq G(S).$$

Without loss of generality, we may assume that T is a closed, symmetric operator. Since $T \prec T^*$, we can use Lemma 10.1.1 to deduce that

$$(I - z_T^* z_T)^{\frac{1}{2}} z_T = z_T^* (I - z_T^* z_T)^{\frac{1}{2}}.$$

We define the bounded operators w_+ and w_- by

$$w_{\pm} = z_T \pm i(I - z_T^* z_T)^{\frac{1}{2}}.$$

Lemma 10.1.2. *Let H be a Hilbert space and $T : D(T) \to H$ be a closed symmetric operator. Let $w_+, w_- \in B(H)$ be defined by*

$$w_\pm = z_T \pm i(I - z_T^* z_T)^{\frac{1}{2}}.$$

Then, w_+ and w_- are isometries.

Proof. Assume that H is a Hilbert space and $T : D(T) \to H$ is a closed, symmetric operator. Assume that $w_+, w_- \in B(H)$ are defined as above.

To see that w_+ and w_- are isometries, we compute directly from the definition that

$$\begin{aligned} w_\pm^* w_\pm &= (z_T^* \mp i(I - z_T^* z_T)^{\frac{1}{2}})(z_T \pm i(I - z_T^* z_T)^{\frac{1}{2}}) \\ &= z_T^* z_T \pm i z_T^*(I - z_T^* z_T)^{\frac{1}{2}} \mp i(I - z_T^* z_T)^{\frac{1}{2}} z_T + (I - z_T^* z_T) \\ &= z_T^* z_T + (I - z_T^* z_T) = I. \end{aligned}$$

Hence, w_+ and w_- are isometries. □

Let us make some more definitions with regards to w_+ and w_-.

Definition 10.1.3. Let H be a Hilbert space and $T : D(T) \to H$ be a closed, symmetric operator. Let w_+ and w_- be the isometries defined in Lemma 10.1.2. Let $\mathscr{W}_\pm = w_\pm H$.

The **deficiency subspaces** of T, denoted by $\mathscr{D}_\pm$, are defined by $\mathscr{D}_+ = \mathscr{W}_+^\perp$ and $\mathscr{D}_- = \mathscr{W}_-^\perp$.

The dimensions of the deficiency subspaces, denoted by $n_\pm$, are called the **deficiency indices** of T.

Here is a result regarding the deficiency subspaces $\mathscr{D}_+$ and $\mathscr{D}_-$.

Lemma 10.1.3. *Let $T : D(T) \to H$ be a closed, symmetric operator. Then, the deficiency subspaces $\mathscr{D}_\pm$ of T satisfy*

$$\mathscr{D}_\pm = \ker(T^* \mp iI).$$

Proof. Assume that $T : D(T) \to H$ is a closed symmetric operator. By definition, $\mathscr{D}_\pm = \mathscr{W}_\pm^\perp$. Thus, $\zeta \in \mathscr{D}_\pm$ if and only if

$$\langle \zeta, z_T \xi \pm i(I - z_T^* z_T)^{\frac{1}{2}} \xi \rangle = 0.$$

for any $\xi \in H$. By using Theorem 8.2.1, we find that the above equation reduces to

$$\langle \zeta, T\psi \pm i\psi \rangle = 0$$

for $\psi \in D(T)$. In turn, the above equation holds if and only if $\zeta \in D((T \pm iI)^*)$ and $(T \pm iI)^*\zeta = 0$. Using Theorem 7.3.7, we find that $(T \pm iI)^*\zeta = (T^* \mp iI)\zeta = 0$ and consequently, $\zeta \in \ker(T^* \mp iI)$.

Thus, we obtain $\mathscr{D}_\pm = \ker(T^* \mp iI)$. $\square$

In order to understand why we can define the Cayley transform in the first place, we require the following lemma, which shows that the Cayley transform arises from a partial isometry.

Theorem 10.1.4. *Let H be a Hilbert space and $T : D(T) \to H$ be a closed, symmetric operator. Let $c_T = w_- w_+^*$, where w_+ and w_- are the isometries defined in Lemma 10.1.2. Then, c_T is a partial isometry with initial subspace $\mathscr{W}_+$ and final subspace $\mathscr{W}_-$.*

Proof. Assume that H is a Hilbert space and $T : D(T) \to H$ is a closed symmetric operator. Assume that $c_T = w_- w_+^*$. To see that c_T is a partial isometry, we compute directly that

$$\begin{aligned} c_T(c_T)^*c_T &= w_- w_+^*(w_- w_+^*)^* w_- w_+^* \\ &= w_-(w_+^* w_+)(w_-^* w_-)w_+^* \\ &= w_- w_+^* = c_T. \end{aligned}$$

In the second last equality, we used Lemma 10.1.2. By Theorem 2.5.1, we deduce that c_T is a partial isometry.

Recall that the initial subspace of c_T is the image of $c_T^* c_T = w_+ w_+^*$ and the final subspace of c_T is the image of $c_T c_T^* = w_- w_-^*$. Hence, the initial subspace of c_T is $\mathscr{W}_+$ and the final subspace of c_T is $\mathscr{W}_-$. $\square$

Definition 10.1.4. Let H be a Hilbert space and $T : D(T) \to H$ be a closed symmetric operator. Let

$$w_\pm = z_T \pm i(I - z_T^* z_T)^{\frac{1}{2}}$$

and $c_T = w_- w_+^*$ be the partial isometry defined in Theorem 10.1.4. The operator $\mathring{c}_T : \mathscr{W}_+ \to H$ is called the **Cayley transform** of T.

As usual, we will prove some properties satisfied by the Cayley transform. First, we note that the graph of T can be recovered from the graph of $\mathring{c}_T$.

Theorem 10.1.5. *Let H be a Hilbert space and $T : D(T) \to H$ be a closed symmetric operator. Then,*

$$G(T) = \begin{pmatrix} -iI & iI \\ I & I \end{pmatrix} G(\mathring{c}_T).$$

Proof. Assume that H is a Hilbert space and $T : D(T) \to H$ is a closed symmetric operator.

By using the same argument in Lemma 10.1.3, we find that

$$\begin{aligned} G(\mathring{c}_T) &= \{(\theta, w_- w_+^* \theta) \in H \times H \mid \theta \in \mathscr{W}_+\} \\ &= \{(w_+\xi, w_- w_+^* w_+ \xi) \in H \times H \mid \xi \in H\} \\ &= \{(w_+\xi, w_-\xi) \in H \times H \mid \xi \in H\} \\ &= \{(T\psi + i\psi, T\psi - i\psi) \in H \times H \mid \psi \in D(T)\} \\ &= \begin{pmatrix} iI & I \\ -iI & I \end{pmatrix} G(T). \end{aligned}$$

By taking inverses, we find that

$$G(T) = \begin{pmatrix} -iI & iI \\ I & I \end{pmatrix} G(\mathring{c}_T).$$

□

The Cayley transform c_T provides a bijection from the set of closed symmetric operators to a particular subset of partial isometries.

Theorem 10.1.6. *Let H be a Hilbert space. Then, we have the bijection of sets*

$$\begin{array}{ccc} \{\text{Closed symmetric operators on } H\} & \leftrightarrow & \left\{ \begin{array}{c} \text{Partial isometries } c \in B(H) \\ \text{such that } \overline{(c - I)c^* H} = H \end{array} \right\} \\ T & \mapsto & c_T. \end{array}$$

Proof. Assume that H is a Hilbert space.

To show: (a) If $T : D(T) \to H$ is a closed symmetric operator then c_T is a partial isometry such that $\overline{(c_T - I)c_T^* H} = H$.

(b) If T, S are closed symmetric operators and $c_T = c_S$ then $T = S$.

(c) If $c \in B(H)$ is a partial isometry such that $\overline{(c-I)c^*H} = H$ then there exists a closed symmetric operator T such that $c = c_T$.

(a) Assume that T is a closed symmetric operator on H. By Theorem 10.1.4, we know that c_T is a partial isometry.

To show: (aa) $\overline{(c_T - I)c_T^*H} = H$.

(aa) We know that $G(T)$ is the graph of the densely defined operator T. By Theorem 7.1.2 and Theorem 7.1.4, we find that

$$G(T)^\perp \cap (H \oplus \{0\}) = G(T) \cap (\{0\} \oplus H) = \{0\}.$$

In particular, $G(T)^\perp \cap (H \oplus \{0\}) = \{0\}$ means that if $(\eta, 0) \in G(T)^\perp$ then $\eta = 0$. Using Theorem 10.1.5, we deduce that if

$$\left\langle (\eta, 0), \begin{pmatrix} -iI & iI \\ I & I \end{pmatrix} (\theta, \mathring{c_T}\theta) \right\rangle = 0$$

for $\theta \in D(\mathring{c_T})$ then $\eta = 0$. Simplifying the above equation, we find that if

$$\langle \eta, (\mathring{c_T} - I)\theta \rangle = 0$$

for $\theta \in D(\mathring{c_T}) = \mathscr{W}_+$ then $\eta = 0$. So, $((\mathring{c_T} - I)\mathscr{W}_+)^\perp = \{0\}$. However,

$$((\mathring{c_T} - I)\mathscr{W}_+)^\perp = ((c_T - I)c_T^*H)^\perp = \{0\}.$$

If we take the orthogonal complement of both sides of $((c_T - I)c_T^*H)^\perp = \{0\}$, we find that $\overline{(c_T - I)c_T^*H} = H$.

(a) By part (aa), we find that c_T is a partial isometry such that $\overline{(c_T - I)c_T^*H} = H$. This proves part (a).

(b) Assume that S and T are closed symmetric operators on H. Assume that $c_T = c_S$. By Theorem 10.1.5,

$$G(T) = \begin{pmatrix} -iI & iI \\ I & I \end{pmatrix} G(\mathring{c_T}) = \begin{pmatrix} -iI & iI \\ I & I \end{pmatrix} G(\mathring{c_S}) = G(S).$$

Hence, $T = S$ and the bijection described in the statement of the theorem is injective.

(c) Assume that $c \in B(H)$ is a partial isometry such that $\overline{(c-I)c^*H} = H$. Define the subspace $G \subseteq H \times H$ by

$$G = \begin{pmatrix} -iI & iI \\ I & I \end{pmatrix} G(\mathring{c}).$$

We want to show that G is the graph of a closed, densely defined operator. We will accomplish this by applying Theorem 7.1.5.

To show: (ca) G is a closed subspace of $H \times H$.

(cb) If $\eta \in H$ and $(0, \eta) \in G$ then $\eta = 0$.

(cc) If $\eta \in H$ and $(\eta, 0) \in G^\perp$ then $\eta = 0$.

(ca) Since c is a partial isometry, Theorem 2.5.2 tells us that there exists a closed subspace $S \subseteq H$ such that the restriction $c|_S$ is an isometry.

The graph of $\mathring{c}$ is given by

$$G(\mathring{c}) = \{(\xi, c(\xi)) \in H \times H \mid \xi \in S\} = S \times c(S).$$

Now since c is an isometry on the closed subspace S, the image $c(S)$ must also be closed. Hence, $G(\mathring{c}) = S \times c(S)$ is a closed subspace of $H \times H$.

(cb) Assume that $\eta \in H$ and $(0, \eta) \in G$. By definition of G, there exists $\theta \in D(\mathring{c}) = S$ such that

$$\begin{pmatrix} 0 \\ \eta \end{pmatrix} = \begin{pmatrix} -iI & iI \\ I & I \end{pmatrix} \begin{pmatrix} \theta \\ \mathring{c}\theta \end{pmatrix}.$$

So, $0 = (\mathring{c} - I)\theta$ and $\eta = (\mathring{c} + I)\theta$. This means that $\theta \in \ker(\mathring{c} - I)$ and subsequently that

$$\begin{aligned}
\theta &\in \ker(\mathring{c} - I) \\
&\subset \ker(c - I) \\
&\subset \ker(c^*(c - I)) \\
&= \ker(c(c^* - I)) = \big((c - I)c^*H\big)^\perp = \{0\}.
\end{aligned}$$

The second last equality follows from the fact that c is a partial isometry and the final equality follows from the fact that $\overline{(c-I)c^*H} = H$. Therefore,

$\theta = 0$ and $\eta = (\mathring{c} + I)\theta = 0$.

(cc) Assume that $\eta \in H$ and $(\eta, 0) \in G^{\perp}$. If $\theta \in D(\mathring{c}) = S$ then

$$\left\langle (\eta, 0), \begin{pmatrix} -iI & iI \\ I & I \end{pmatrix} (\theta, \mathring{c}\theta) \right\rangle = 0.$$

Again, this simplifies to

$$\langle \eta, (\mathring{c} - I)\theta \rangle = 0.$$

Therefore,

$$\eta \in ((\mathring{c} - I)S)^{\perp} = ((c - I)c^* H)^{\perp}$$

Since $H = \overline{(c - I)c^* H} = (((c - I)c^* H)^{\perp})^{\perp}$, we find that $\langle \eta, \rho \rangle = 0$ for arbitrary $\rho \in H$. So, $\eta = 0$.

(c) Parts (ca), (cb) and (cc) allow us to use Theorem 7.1.5 to demonstrate that there exists a closed densely defined operator T such that $G = G(T)$. Next, we must show that T is symmetric.

To show: (cd) $G(T) \subset G(T^*)$.

(cd) First we will work out the graph $G(T^*)$. By Theorem 7.2.1, it is

$$\begin{aligned} G(T^*) &= \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} G(T)^{\perp} \\ &= \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \left(\begin{pmatrix} -iI & iI \\ I & I \end{pmatrix} G(\mathring{c}) \right)^{\perp} \\ &= \left(\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} -iI & iI \\ I & I \end{pmatrix} G(\mathring{c}) \right)^{\perp} \\ &= \left(\begin{pmatrix} I & I \\ iI & -iI \end{pmatrix} G(\mathring{c}) \right)^{\perp}. \end{aligned}$$

The second last equality follows from the fact that the operator

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \in B(H \times H)$$

is unitary.

Now suppose that $(\xi, \eta) \in G(T)$. Then, there exists $\theta \in D(\mathring{c})$ such that

$$\begin{pmatrix}\xi \\ \eta\end{pmatrix} = \begin{pmatrix}-iI & iI \\ I & I\end{pmatrix}\begin{pmatrix}\theta \\ \mathring{c}\theta\end{pmatrix}.$$

To see that $(\xi, \eta) \in G(T^*)$, we compute directly that for $\theta' \in D(\mathring{c})$,

$$\begin{aligned}
&\Big\langle \begin{pmatrix}-iI & iI \\ I & I\end{pmatrix}\begin{pmatrix}\theta \\ \mathring{c}\theta\end{pmatrix}, \begin{pmatrix}I & I \\ iI & -iI\end{pmatrix}\begin{pmatrix}\theta' \\ \mathring{c}\theta'\end{pmatrix} \Big\rangle \\
&= \Big\langle \begin{pmatrix}\theta \\ \mathring{c}\theta\end{pmatrix}, \begin{pmatrix}iI & I \\ -iI & I\end{pmatrix}\begin{pmatrix}I & I \\ iI & -iI\end{pmatrix}\begin{pmatrix}\theta' \\ \mathring{c}\theta'\end{pmatrix} \Big\rangle \\
&= \Big\langle \begin{pmatrix}\theta \\ \mathring{c}\theta\end{pmatrix}, \begin{pmatrix}2iI & 0 \\ 0 & -2iI\end{pmatrix}\begin{pmatrix}\theta' \\ \mathring{c}\theta'\end{pmatrix} \Big\rangle \\
&= 2i(\langle \theta, \theta' \rangle - \langle \mathring{c}\theta, \mathring{c}\theta' \rangle) = 0.
\end{aligned}$$

The last equality follows from the fact that $\mathring{c}$ is an isometry on the closed subspace $D(\mathring{c}) = S$ and hence, preserves the inner product by the polarization formula in Theorem 2.1.1.

We conclude that $(\xi, \eta) \in G(T^*)$ and $G(T) \subset G(T^*)$.

(c) Part (cd) tells us that T is a closed symmetric operator. By definition of $G = G(T)$, we find that $c_T = c$. So, the bijection of sets in the statement of the theorem is surjective and the proof is complete. □

Next, we want to know how partial isometries behave under extensions.

Lemma 10.1.7. *Let H be a Hilbert space and $c_1, c_2 \in B(H)$ be partial isometries such that $\mathring{c}_1 \prec \mathring{c}_2$. Then,*

$$(c_1 - I)c_1^* H \subseteq (c_2 - I)c_2^* H.$$

Proof. Assume that H is a Hilbert space and $c_1, c_2 \in B(H)$ are partial isometries on H such that $\mathring{c}_1 \prec \mathring{c}_2$. By definition of an extension, we have

$$c_1^* H = D(\mathring{c}_1) \subset D(\mathring{c}_2) = c_2^* H.$$

Since $\mathring{c}_1 \prec \mathring{c}_2$, if $\xi \in D(\mathring{c}_1)$ then $(c_1 - I)\xi = (c_2 - I)\xi$. In tandem with the finding that $c_1^* H \subset c_2^* H$, we deduce that

$$(c_1 - I)c_1^* H \subseteq (c_2 - I)c_2^* H.$$

□

A particular consequence of Lemma 10.1.7 is that if $\mathring{c}_1 \prec \mathring{c}_2$ and $\overline{(c_1 - I)c_1^* H} = H$ then $\overline{(c_2 - I)c_2^* H} = H$.

The next theorem gives us a glimpse into why the Cayley transform is relevant to the study of self-adjoint extensions of symmetric operators.

Theorem 10.1.8. *Let H be a Hilbert space and $T : D(T) \to H$ be a closed symmetric operator. Let $T' : D(T') \to H$ be another operator. Then, T' is a closed symmetric extension of T if and only if $\mathring{c}_T \prec \mathring{c}_{T'}$.*

Proof. Assume that H is a Hilbert space and $T : D(T) \to H$ is a closed symmetric operator. Assume that $T' : D(T') \to H$ is another operator.

To show: (a) If T' is a closed symmetric extension of T then $\mathring{c}_T \prec \mathring{c}_{T'}$.

(b) If $\mathring{c}_T \prec \mathring{c}_{T'}$ then T' is a closed symmetric extension of T.

(a) Assume that T' is a symmetric operator and $T \prec T'$. From Theorem 10.1.5, we have

$$G(\mathring{c}_T) = \begin{pmatrix} -iI & iI \\ I & I \end{pmatrix} G(T).$$

Of course, a similar equality holds for $G(\mathring{c}_{T'})$. Since $G(T) \subset G(T')$,

$$\begin{pmatrix} -iI & iI \\ I & I \end{pmatrix} G(T) \subset \begin{pmatrix} -iI & iI \\ I & I \end{pmatrix} G(T').$$

Consequently, $G(\mathring{c}_T) \subset G(\mathring{c}_{T'})$ and $\mathring{c}_T \prec \mathring{c}_{T'}$.

(b) Assume that $\mathring{c}_T \prec \mathring{c}_{T'}$. By Theorem 10.1.6, $\overline{(c_T - I)c_T^* H} = H$. By Lemma 10.1.7, we find that $\overline{(c_{T'} - I)c_{T'}^* H} = H$.

Now since $G(\mathring{c}_T) \subset G(\mathring{c}_{T'})$, we can apply Theorem 10.1.5 to obtain $G(T) \subset G(T')$, which means that $T \prec T^*$ as required. □

The next theorem gives a decomposition which is key to describing self-adjoint operators on H.

Theorem 10.1.9. *Let H be a Hilbert space. Let $T : D(T) \to H$ be a closed symmetric operator with deficiency subspaces $\mathscr{D}_+$ and $\mathscr{D}_-$. Let*

$$\tilde{\mathscr{D}}_\pm = \{(\phi, \pm i\phi) \mid \phi \in \mathscr{D}_\pm\} \subseteq H \times H.$$

Then,

$$G(T^*) = G(T) \oplus \tilde{\mathscr{D}}_+ \oplus \tilde{\mathscr{D}}_-.$$

Proof. Assume that H is a Hilbert space and $T : D(T) \to H$ is a closed symmetric operator. Assume that $\tilde{\mathscr{D}}_+$ and $\tilde{\mathscr{D}}_-$ are the subspaces of $H \times H$ defined as above. We will show that the subspaces $G(T)$, $\tilde{\mathscr{D}}_+$ and $\tilde{\mathscr{D}}_-$ are all pairwise orthogonal.

To show: (a) $\tilde{\mathscr{D}}_+$ and $\tilde{\mathscr{D}}_-$ are pairwise orthogonal.

(b) $G(T)$ and $\tilde{\mathscr{D}}_\pm$ are pairwise orthogonal.

(a) To see that $\tilde{\mathscr{D}}_+$ and $\tilde{\mathscr{D}}_-$ are pairwise orthogonal, observe that if $\eta, \xi \in H$ then

$$\langle (\xi, i\xi), (\eta, -i\eta) \rangle = \langle \xi, \eta \rangle + \langle i\xi, -i\eta \rangle = 0.$$

Hence, the subspaces $\tilde{\mathscr{D}}_+$ and $\tilde{\mathscr{D}}_-$ are orthogonal.

(b) Recall from Lemma 10.1.3 that $\mathscr{D}_\pm = \ker(T^* \mp iI)$. Assume that $\psi \in D(T)$ and $\phi \in \mathscr{D}_\pm$. We compute directly that

$$\begin{aligned}
\langle (\psi, T\psi), (\phi, \pm i\phi) \rangle &= \langle \psi, \phi \rangle + \langle T\psi, \pm i\phi \rangle \\
&= \langle \psi, \phi \rangle \mp i \langle T\psi, \phi \rangle \\
&= \langle \psi, \phi \rangle \mp i \langle \psi, T^*\phi \rangle \\
&= \langle \psi, \phi \rangle \mp i \langle \psi, T^*\phi - (T^* \mp iI)\phi \rangle \\
&= \langle \psi, \phi \rangle \mp i \langle \psi, \pm i\phi \rangle = 0.
\end{aligned}$$

Therefore, $G(T)$ is orthogonal to both $\tilde{\mathscr{D}}_+$ and $\tilde{\mathscr{D}}_-$.

Now since T is a symmetric operator, $G(T) \subset G(T^*)$ and since the $\mathscr{D}_\pm$ are eigenspaces of T^* by Lemma 10.1.3, $\tilde{\mathscr{D}}_\pm \subset G(T^*)$. To see that $G(T^*) = G(T) \oplus \tilde{\mathscr{D}}_+ \oplus \tilde{\mathscr{D}}_-$, it suffices to show that any vector in $G(T^*)$ which is orthogonal to the direct sum $G(T) \oplus \tilde{\mathscr{D}}_+ \oplus \tilde{\mathscr{D}}_-$ is zero.

To show: (c) If $(\varphi, T^*\varphi) \in G(T^*)$ is orthogonal to the subspace $G(T) \oplus \tilde{\mathscr{D}}_+ \oplus \tilde{\mathscr{D}}_-$ then $\varphi = 0$.

(c) Assume that $(\varphi, T^*\varphi) \in G(T^*)$ is orthogonal to the subspace $G(T) \oplus \tilde{\mathscr{D}}_+ \oplus \tilde{\mathscr{D}}_-$. Since $(\varphi, T^*\varphi) \in G(T)^\perp$ by assumption, we have for $\psi \in D(T)$

$$\langle(\varphi, T^*\varphi), (\psi, T\psi)\rangle = \langle\varphi, \psi\rangle + \langle T^*\varphi, T\psi\rangle = 0.$$

Consequently, $T^*\varphi \in D(T^*)$ and $(T^*)^2\varphi = -\varphi$. This means that

$$\varphi \in D((T^*)^2 + I) = D((T^* + iI)(T^* - iI))$$

and $(T^* + iI)(T^* - iI)\varphi = 0$. Now set $\eta = (T^* - iI)\varphi$. By Lemma 10.1.3, $\eta \in \mathscr{D}_-$.

Next, we use the fact that $(\varphi, T^*\varphi) \in \tilde{\mathscr{D}}_-{}^\perp$ to compute directly for $\eta' \in \mathscr{D}_-$ that

$$\begin{aligned} i\langle\eta, \eta'\rangle &= i\langle(T^* - iI)\varphi, \eta'\rangle \\ &= \langle T^*\varphi, -i\eta'\rangle + i\langle -i\varphi, \eta'\rangle \\ &= \langle T^*\varphi, -i\eta'\rangle + \langle\varphi, \eta'\rangle \\ &= \langle(\varphi, T^*\varphi), (\eta', -i\eta')\rangle = 0. \end{aligned}$$

The last equality follows from the fact that $(\eta', -i\eta') \in \tilde{\mathscr{D}}_-$. Notice that if we set $\eta' = \eta$, we find that $\eta = 0$ and therefore, $T^*\varphi = i\varphi$. So, $\varphi \in \ker(T^* - iI) = \mathscr{D}_+$ and $(\varphi, T^*\varphi) \in \tilde{\mathscr{D}}_+$.

Since $(\varphi, T^*\varphi) \in \tilde{\mathscr{D}}_+$ was assumed to be orthogonal to $\tilde{\mathscr{D}}_+$, we find that $\varphi = 0$. This proves part (c) and completes the proof. □

Using Theorem 10.1.9, we will now give an explicit characterisation of self-adjoint operators T on a Hilbert space H.

Theorem 10.1.10. *Let H be a Hilbert space and $T : D(T) \to H$ be a closed symmetric operator. Then, the following are equivalent:*

1. *T is self-adjoint.*
2. *$\mathscr{D}_+ = \mathscr{D}_- = \{0\}$.*
3. *$n_+ = n_- = 0$*
4. *The Cayley transform $\mathring{c}_T$ of T is a unitary operator*

Proof. Assume that H is a Hilbert space and $T : D(T) \to H$ is a closed, symmetric operator.

If T is self-adjoint then $G(T^*) = G(T)$. By Theorem 10.1.9, we must have $\tilde{\mathscr{D}}_\pm = \{0\}$. Hence, $\mathscr{D}_\pm = \{0\}$.

If $\mathscr{D}_\pm = \{0\}$ then $n_\pm = \dim \mathscr{D}_\pm = 0$.

If $n_\pm = \dim \mathscr{D}_\pm = 0$ then since $\mathscr{D}_\pm = \mathscr{W}_\pm^\perp$, $\mathscr{W}_\pm = H$. So, the Cayley transform $\mathring{c_T} = c_T$, which is a bounded operator on H. Note that c_T is a partial isometry from the initial subspace $\mathscr{W}_+ = H$. So, c_T is an isometry. By Theorem 2.5.2, the final subspace of c_T is

$$H = \mathscr{W}_- = c_T \mathscr{W}_+ = \text{im } c_T.$$

So, c_T is a surjective isometry and by Theorem 3.1.8, the Cayley transform $\mathring{c_T} = c_T$ is a unitary operator.

Finally, suppose that the Cayley transform $\mathring{c_T}$ is a unitary operator. Then, $\mathring{c_T} = c_T$ is a surjective isometry on H by Theorem 3.1.8. By Theorem 2.5.2 and the fact that c_T is a partial isometry with initial subspace $\mathscr{W}_+$ and final subspace $\mathscr{W}_-$, we deduce that $\mathscr{W}_+ = H$ and

$$\mathscr{W}_- = c_T \mathscr{W}_+ = H$$

where the last equality is due to surjectivity of c_T. So, $\mathscr{D}_\pm = \mathscr{W}_\pm^\perp = \{0\}$ and by Theorem 10.1.9, $G(T^*) = G(T)$. So, T must be self-adjoint. □

From Theorem 10.1.10 and Theorem 10.1.8, we obtain criteria for T to have a self-adjoint extension.

Theorem 10.1.11. *Let H be a Hilbert space and $T : D(T) \to H$ be a closed symmetric operator. Then, the following are equivalent:*

1. *T has a self-adjoint extension.*
2. *There exists a unitary operator from $\mathscr{D}_+$ to $\mathscr{D}_-$.*
3. *$n_+ = n_-$.*

Proof. Assume that H is a Hilbert space and $T : D(T) \to H$ is a closed symmetric operator.

By Theorem 10.1.10, T has a self-adjoint extension if and only if its Cayley transform $\mathring{c}_T$ has a unitary extension. This means that $\mathring{c}_T : \mathscr{W}_+ \to H$ can be extended to a unitary operator defined on all of H. Note that $c_T \mathscr{W}_+ = \mathscr{W}_-$ by definition of the partial isometry c_T.

To see what this means, recall that $\mathscr{D}_\pm = \mathscr{W}_\pm^\perp$. So, $H = \mathscr{W}_\pm \oplus \mathscr{D}_\pm$. Hence, a unitary extension of $\mathring{c}_T$ is equivalent to defining any unitary operator from $\mathscr{D}_+$ to $\mathscr{D}_-$. Also, we have a unitary operator from $\mathscr{D}_+$ to $\mathscr{D}_-$ if and only if $n_+ = n_-$. □

An interesting aspect of Theorem 10.1.11 is that the number of self-adjoint extensions of T is exactly the number of unitary maps from $\mathscr{D}_+$ to $\mathscr{D}_-$. In most non-trivial cases, there are infinitely many unitary maps from $\mathscr{D}_+$ to $\mathscr{D}_-$ and thus, infinitely many self-adjoint extensions of T.

We remark that we can perform a similar analysis to the one in this section for symmetric operators T which are not necessarily closed. However, it is harder because we do not have the z-transform z_T at our disposal. The issue is that if T is not assumed to be closed then Theorem 10.1.9 does not hold. That is, the deficiency subspaces $\mathscr{D}_+$ and $\mathscr{D}_-$ can be zero, but $G(T) \neq G(T^*)$. When this happens, the closure $\overline{T}$ is self-adjoint and T is called *essentially self-adjoint.*

10.2 Krein and Friedrichs extensions of positive operators

Let H be a Hilbert space and $T : D(T) \to H$ be a densely defined, positive operator. Recall the definition of a positive operator from Definition 8.3.1 — if $\psi \in D(T)$ then $\langle \psi, T\psi \rangle \geq 0$. Subsequently,

$$\langle \psi, T\psi \rangle = \langle T\psi, \psi \rangle$$

and by the polarization formula in Theorem 2.1.1, if $\psi, \phi \in D(T)$ then

$$\langle T^*\psi, \phi \rangle = \langle \psi, T\phi \rangle = \langle T\psi, \phi \rangle.$$

We find that if $\psi \in D(T)$ then $T^*\psi = T\psi$. So, $T \prec T^*$ and T is symmetric.

In this section, we will study self-adjoint extensions of positive operators. For the same reason outlined in the previous section, we can assume that T

is closed.

Recall that if $x \in B(H)_+$ is positive then $\sigma(x) \subseteq \mathbb{R}_{\geq 0}$. A similar result holds for positive and self-adjoint unbounded operators.

Theorem 10.2.1. *Let H be a Hilbert space and $T : D(T) \to H$ be closed, positive and self-adjoint. Then, $\sigma(T) \subseteq \mathbb{R}_{\geq 0}$.*

Proof. Assume that H is a Hilbert space and $T : D(T) \to H$ is a closed, positive and self-adjoint operator.

To show: (a) $\ker(T + I) = \{0\}$.

(b) $\overline{\mathrm{im}(T + I)} = H$.

(c) The image $\mathrm{im}(T + I)$ is a closed subspace of H.

(a) Assume that $\psi \in \ker(T + I)$. Then, $T\psi + \psi = 0$ and $T\psi = -\psi$. Since T is positive,

$$-\|\psi\|^2 = \langle \psi, -\psi \rangle = \langle \psi, T\psi \rangle \geq 0.$$

So, $\psi = 0$ and $\ker(T + I) = \{0\}$.

(b) It suffices to show that $(\mathrm{im}(T + I))^\perp = \{0\}$. Assume that $\phi \in (\mathrm{im}(T + I))^\perp$. If $\psi \in D(T)$ then

$$\langle \phi, T\psi + \psi \rangle = \langle \phi, T\psi \rangle + \langle \phi, \psi \rangle = 0$$

So, $\langle \phi, T\psi \rangle = \langle -\phi, \psi \rangle$. This means that $\phi \in D(T^*)$ and $T\phi = T^*\phi = -\phi$. So, $\phi \in \ker(T + I) = \{0\}$ by part (a). So, $\phi = 0$ and $(\mathrm{im}(T + I))^\perp = \{0\}$.

(c) First observe that if $\psi \in D(T)$ then

$$\|(T + I)\psi\|^2 = \|T\psi\|^2 + \langle \psi, T\psi \rangle + \langle T\psi, \psi \rangle + \|\psi\|^2 \geq \|\psi\|^2.$$

To see that $\mathrm{im}(T + I)$ is closed, suppose that $\{(T + I)\psi_n\}_{n \in \mathbb{Z}_{>0}}$ is a Cauchy sequence in $\mathrm{im}(T + I)$. By the above estimate, we find that $\{\psi_n\}_{n \in \mathbb{Z}_{>0}}$ is a Cauchy sequence in H and thus, converges to some $\rho \in H$.

Now since T is closed, $T + I$ is also closed by Theorem 7.3.1. Since $\lim_{n \to \infty} \psi_n = \rho$, $\lim_{n \to \infty}(T + I)\psi_n = (T + I)\rho$ by Lemma 7.1.1.

Consequently, $\mathrm{im}(T+I)$ is a closed subspace of H.

By combining parts (b) and (c), we find that $\mathrm{im}(T+I) = H$. Together with part (a), we find that $T+I$ is a bijection from $D(T)$ to H. Moreover, the inverse operator $(T+I)^{-1}$ is continuous because if $\xi \in H$ then

$$\|(T+I)^{-1}\xi\|^2 \leq \|(T+I)(T+I)^{-1}\xi\|^2 = \|\xi\|^2.$$

In fact, $(T+I)^{-1}$ is actually a contraction.

By the above reasoning, we find that $-1 \notin \sigma(T)$. Now if $\lambda \in \mathbb{R}_{>0}$ then the operator $\frac{1}{\lambda}T$ is also closed, positive and self-adjoint. So, $-1 \notin \sigma(\frac{1}{\lambda}T)$ and consequently, $-\lambda \notin \sigma(T)$. So, $\sigma(T) \subseteq \mathbb{R}_{\geq 0}$. □

In order to proceed, we require a rather technical lemma.

Lemma 10.2.2. *Let H be a Hilbert space and K be a closed subspace of H. Let $a \in B(K)$, $b \in B(K, K^\perp)$ and $c \in B(K^\perp)$. Then, the operator*

$$\begin{pmatrix} a & b^* \\ b & c \end{pmatrix} \geq 0$$

in $B(H)$ if and only if $a \geq 0$ in $B(K)$ and if $\epsilon \in \mathbb{R}_{>0}$ then

$$c \geq b(a + \epsilon I_K)^{-1} b^*$$

in $B(K^\perp)$, where I_K is the identity operator on K.

Proof. Assume that H is a Hilbert space and K is a closed subspace of H. Assume that a, b and c are the operators defined as above. Define

$$f = \begin{pmatrix} a & b^* \\ b & c \end{pmatrix} \in B(H).$$

The matrix is written with respect to the basis $K, K^\perp$.

To show: (a) If $f \geq 0$ then $a \geq 0$ in $B(K)$ and if $\epsilon \in \mathbb{R}_{>0}$ then $c \geq b(a + \epsilon I_K)^{-1} b^*$ in $B(K^\perp)$.

(b) If $a \geq 0$ in $B(K)$ and if $\epsilon \in \mathbb{R}_{>0}$ then $c \geq b(a + \epsilon I_K)^{-1} b^*$ in $B(K^\perp)$ then $f \geq 0$ in $B(H)$.

(a) Assume that $f \geq 0$ in $B(H)$. Assume that $\xi \in K$. Then, $\langle (\xi, 0), f(\xi, 0) \rangle \geq 0$. Expanding the LHS, we have

$$\left\langle \begin{pmatrix} \xi \\ 0 \end{pmatrix}, \begin{pmatrix} a\xi \\ b\xi \end{pmatrix} \right\rangle \geq 0.$$

Using the fact that $H = K \oplus K^{\perp}$, we deduce that

$$\begin{aligned} \langle \xi + 0, a\xi + b\xi \rangle &= \langle \xi, a\xi + b\xi \rangle \\ &= \langle \xi, a\xi \rangle + \langle \xi, b\xi \rangle \\ &= \langle \xi, a\xi \rangle \geq 0. \end{aligned}$$

This means that $a \geq 0$ in $B(K)$.

Next, assume that $\epsilon \in \mathbb{R}_{>0}$. Let $p_K \in B(H)$ be the projection operator on the closed subspace K. Then, $\epsilon p_K \in B(H)$ is a positive operator and $\epsilon p_K \to 0$ as $\epsilon \to 0$ in the strong operator topology.

Define the operator $f_\epsilon \in B(H)$ by

$$f_\epsilon = \begin{pmatrix} a + \epsilon I_K & b^* \\ b & c \end{pmatrix}.$$

We claim that $f \geq 0$ if and only if $f_\epsilon \geq 0$ for $\epsilon \in \mathbb{R}_{>0}$.

To show: (aa) $f \geq 0$ if and only if $f_\epsilon \geq 0$ for $\epsilon \in \mathbb{R}_{>0}$.

(aa) Assume that $f \geq 0$. If $(\xi_1, \xi_2) \in K \oplus K^{\perp} = H$ then

$$\begin{aligned} \langle (\xi_1, \xi_2), f(\xi_1, \xi_2) \rangle &= \langle \xi_1 + \xi_2, a\xi_1 + b^*\xi_2 + b\xi_1 + c\xi_2 \rangle \\ &= \langle \xi_1, a\xi_1 \rangle + \langle \xi_1, b^*\xi_2 \rangle + \langle \xi_2, b\xi_1 \rangle + \langle \xi_2, c\xi_2 \rangle \\ &\geq 0. \end{aligned}$$

This inequality holds if and only if for $\epsilon \in \mathbb{R}_{>0}$,

$$\begin{aligned} \langle (\xi_1, \xi_2), f_\epsilon(\xi_1, \xi_2) \rangle &= \langle \xi_1, (a + \epsilon I_K)\xi_1 \rangle + \langle \xi_1, b^*\xi_2 \rangle + \langle \xi_2, b\xi_1 \rangle + \langle \xi_2, c\xi_2 \rangle \\ &= \langle (\xi_1, \xi_2), f(\xi_1, \xi_2) \rangle + \epsilon \|\xi_1\|^2 \geq 0. \end{aligned}$$

So, $f \geq 0$ if and only if $f_\epsilon \geq 0$ for $\epsilon \in \mathbb{R}_{>0}$.

(a) Using part (aa), we find that $f_\epsilon \geq 0$ for $\epsilon \in \mathbb{R}_{>0}$. By the previous computation in part (aa), if $(\xi_1, \xi_2) \in K \oplus K^{\perp} = H$ then

$$\langle \xi_1, (a+\epsilon I_K)\xi_1\rangle + \langle \xi_1, b^*\xi_2\rangle + \langle \xi_2, b\xi_1\rangle + \langle \xi_2, c\xi_2\rangle \geq 0.$$

Since $a \geq 0$ is a positive operator, $\sigma(a) \subseteq \mathbb{R}_{\geq 0}$ and $\mathbb{R}_{<0} \subseteq \rho(a)$. This means that if $\epsilon \in \mathbb{R}_{>0}$ then $-\epsilon I_K - a$ is invertible. So, $a + \epsilon I_K \in B(K)$ is invertible. Hence, if we set $\xi_1 = -(a+\epsilon I_K)^{-1}b^*\xi_2$ then

$$\begin{aligned}
&\langle -(a+\epsilon I_K)^{-1}b^*\xi_2, -(a+\epsilon I_K)(a+\epsilon I_K)^{-1}b^*\xi_2\rangle + \langle -(a+\epsilon I_K)^{-1}b^*\xi_2, b^*\xi_2\rangle \\
&\quad + \langle \xi_2, -b(a+\epsilon I_K)^{-1}b^*\xi_2\rangle + \langle \xi_2, c\xi_2\rangle \\
&= \langle (a+\epsilon I_K)^{-1}b^*\xi_2, b^*\xi_2\rangle - \langle (a+\epsilon I_K)^{-1}b^*\xi_2, b^*\xi_2\rangle \\
&\quad + \langle \xi_2, -b(a+\epsilon I_K)^{-1}b^*\xi_2\rangle + \langle \xi_2, c\xi_2\rangle \\
&= \langle \xi_2, (c - b(a+\epsilon I_K)^{-1}b^*)(\xi_2)\rangle \geq 0.
\end{aligned}$$

So, $c \geq b(a+\epsilon I_K)^{-1}b^*$ for $\epsilon \in \mathbb{R}_{>0}$.

(b) Assume that $a \geq 0$ in $B(K)$ and $c \geq b(a+\epsilon I_K)^{-1}b^*$ in $B(K^\perp)$ for $\epsilon \in \mathbb{R}_{>0}$. It suffices to show that $f_\epsilon \geq 0$ for $\epsilon \in \mathbb{R}_{>0}$.

Notice that

$$f_\epsilon = \begin{pmatrix} a+\epsilon I_K & b^* \\ b & c \end{pmatrix} = \begin{pmatrix} a+\epsilon I_K & b^* \\ b & b(a+\epsilon I_K)^{-1}b^* \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & c - b(a+\epsilon I_K)^{-1}b^* \end{pmatrix}.$$

By assumption, the operator

$$\begin{pmatrix} 0 & 0 \\ 0 & c - b(a+\epsilon I_K)^{-1}b^* \end{pmatrix}$$

on $K \oplus K^\perp = H$ is positive. Hence, it suffices to show that if $\epsilon \in \mathbb{R}_{>0}$ then

$$\begin{pmatrix} a+\epsilon I_K & b^* \\ b & b(a+\epsilon I_K)^{-1}b^* \end{pmatrix} \geq 0$$

on $B(H)$. Denote this operator by g_ϵ.

Let $(\xi_1, \xi_2) \in K \oplus K^\perp = H$. Then, $\langle (\xi_1, \xi_2), g_\epsilon(\xi_1, \xi_2)\rangle$ is equal to

$$\langle \xi_1, (a+\epsilon I_K)\xi_1\rangle + \langle \xi_1, b^*\xi_2\rangle + \langle \xi_2, b\xi_1\rangle + \langle \xi_2, b(a+\epsilon I_K)^{-1}b^*\xi_2\rangle.$$

By assumption, $b^* \in B(K^\perp, K)$. By Theorem 2.4.1, we can decompose $K^\perp$ as the direct sum

$$K^\perp = \ker b^* \oplus (\ker b^*)^\perp = \overline{\text{im } b} \oplus \ker b^*.$$

Note that since $b \in B(K, K^\perp)$, im $b = bK$. Hence, $K^\perp$ is the closure of the direct sum im $b \oplus \ker b^*$. So, it suffices to prove that

$$\langle \xi_1, (a + \epsilon I_K)\xi_1 \rangle + \langle \xi_1, b^*\xi_2 \rangle + \langle \xi_2, b\xi_1 \rangle + \langle \xi_2, b(a + \epsilon I_K)^{-1}b^*\xi_2 \rangle \geq 0$$

for $\xi_2 \in$ im $b \oplus \ker b^*$. To this end, assume that $\xi_2 = b\eta + \eta'$, where $\eta \in K$ and $\eta' \in \ker b^* = (\text{im } b)^\perp$. We compute directly that

$$\begin{aligned}
&\langle \xi_1, (a + \epsilon I_K)\xi_1 \rangle + \langle \xi_1, b^*\xi_2 \rangle + \langle \xi_2, b\xi_1 \rangle + \langle \xi_2, b(a + \epsilon I_K)^{-1}b^*\xi_2 \rangle \\
= &\langle \xi_1, (a + \epsilon I_K)\xi_1 \rangle + \langle \xi_1, b^*b\eta + b^*\eta \rangle + \langle b\eta + \eta', b\xi_1 \rangle \\
&+ \langle b\eta + \eta', b(a + \epsilon I_K)^{-1}b^*(b\eta + \eta') \rangle \\
= &\langle \xi_1, (a + \epsilon I_K)\xi_1 \rangle + \langle \xi_1, b^*b\eta \rangle + \langle b\eta + \eta', b\xi_1 \rangle \\
&+ \langle b\eta + \eta', b(a + \epsilon I_K)^{-1}b^*b\eta \rangle \\
= &\langle \xi_1, (a + \epsilon I_K)\xi_1 \rangle + \langle \xi_1, b^*b\eta \rangle + \langle b\eta, b\xi_1 \rangle + \langle b\eta, b(a + \epsilon I_K)^{-1}b^*b\eta \rangle \\
= &\langle \xi_1, (a + \epsilon I_K)\xi_1 + b^*b\eta \rangle + \langle b^*b\eta, \xi_1 + (a + \epsilon I_K)^{-1}b^*b\eta \rangle \\
= &\langle \xi_1, (a + \epsilon I_K)(\xi_1 + (a + \epsilon I_K)^{-1}b^*b\eta) \rangle + \langle b^*b\eta, \xi_1 + (a + \epsilon I_K)^{-1}b^*b\eta \rangle \\
= &\langle (a + \epsilon I_K)\xi_1, \xi_1 + (a + \epsilon I_K)^{-1}b^*b\eta \rangle + \langle b^*b\eta, \xi_1 + (a + \epsilon I_K)^{-1}b^*b\eta \rangle \\
= &\langle (a + \epsilon I_K)\xi_1 + b^*b\eta, \xi_1 + (a + \epsilon I_K)^{-1}b^*b\eta \rangle \\
= &\langle \xi_1 + (a + \epsilon I_K)^{-1}b^*b\eta, (a + \epsilon I_K)(\xi_1 + (a + \epsilon I_K)^{-1}b^*b\eta) \rangle \geq 0.
\end{aligned}$$

So, $\langle (\xi_1, \xi_2), g_\epsilon(\xi_1, \xi_2) \rangle \geq 0$ for $\xi_2 \in$ im $b \oplus \ker b^*$. Since $K^\perp$ is the closure of im $b \oplus \ker b^*$, $\langle (\xi_1, \xi_2), g_\epsilon(\xi_1, \xi_2) \rangle \geq 0$ must hold for $\xi_2 \in K^\perp$. Therefore, $g_\epsilon \geq 0$ for $\epsilon \in \mathbb{R}_{>0}$, which completes the proof. □

From Lemma 10.2.2, we obtain yet another technical result.

Lemma 10.2.3. *Let H be a Hilbert space and K be a closed subspace of H. Let $a \in B(K)$, $b \in B(K, K^\perp)$ and $c \in B(K^\perp)$. Then, the operator*

$$0 \leq \begin{pmatrix} a & b^* \\ b & c \end{pmatrix} \leq I$$

in $B(H)$ if and only if $0 \leq a \leq I_K$ and if $\epsilon \in \mathbb{R}_{>0}$ then

$$b(a + \epsilon I_K)^{-1}b^* \leq c \leq I_{K^\perp} - b((1 + \epsilon)I_K - a)^{-1}b^*.$$

Proof. Assume that H is a Hilbert space and K is a closed subspace of H. Assume that a, b and c are the operators defined as above. By Lemma 10.2.2, the operator

$$\begin{pmatrix} a & b^* \\ b & c \end{pmatrix} \geq 0$$

in $B(H)$ if and only if $a \geq 0$ in $B(K)$ and if $\epsilon \in \mathbb{R}_{>0}$ then

$$c \geq b(a + \epsilon I_K)^{-1} b^*$$

in $B(K^\perp)$, where I_K is the identity operator on K.

Now the operator

$$\begin{pmatrix} a & b^* \\ b & c \end{pmatrix} \leq I$$

if and only if

$$\begin{pmatrix} I_K - a & -b^* \\ -b & I_{K^\perp} - c \end{pmatrix} \geq 0.$$

By Lemma 10.2.2, this holds if and only if $a \leq I_K$ and if $\epsilon' \in \mathbb{R}_{>0}$ then

$$I_{K^\perp} - c \geq b(I_K - a + \epsilon' I_K)^{-1} b^*.$$

Rearranging the above equation, we obtain

$$c \leq I_{K^\perp} - b((1 + \epsilon) I_K - a)^{-1} b^*.$$

Hence, we have shown that

$$0 \leq \begin{pmatrix} a & b^* \\ b & c \end{pmatrix} \leq I$$

in $B(H)$ if and only if $0 \leq a \leq I_K$ and if $\epsilon \in \mathbb{R}_{>0}$ then

$$b(a + \epsilon I_K)^{-1} b^* \leq c \leq I_{K^\perp} - b((1 + \epsilon) I_K - a)^{-1} b^*.$$

□

Definition 10.2.1. Let H be a Hilbert space and T, S be positive, self-adjoint operators. We write $T \geq S$ if the bounded operators $(T + I)^{-1}, (S + I)^{-1}$ (see Theorem 10.2.1) satisfy $(T + I)^{-1} \leq (S + I)^{-1}$. That is, the bounded operator $(S + I)^{-1} - (T + I)^{-1}$ is positive.

In the next theorem, we show that positive self-adjoint extensions of a closed positive operator exist.

Theorem 10.2.4 (Krein and Friedrichs extensions)**.** *Let H be a Hilbert space and $T : D(T) \to H$ be a closed, densely defined, positive operator. Then, there exists positive self-adjoint operators T_K and T_F such that $T \prec T_K$ and $T \prec T_F$. Moreover, if $\tilde{T}$ is a positive self-adjoint operator then $\tilde{T}$ is an extension of T if and only if $T_K \le \tilde{T} \le T_F$.*

Proof. Assume that H is a Hilbert space and $T : D(T) \to H$ is a closed, densely defined, positive operator. Let $K = \mathrm{im}(T + I)$. Recall the following results we proved in Theorem 10.2.1:

1. The operator $T + I$ a bijection from $D(T)$ to K.
2. The inverse $(T + I)^{-1}$ is a contraction (and hence continuous).
3. The image K is closed subspace of H.

Let p_K and $p_{K^\perp}$ denote projection operators onto the closed subspaces K and $K^\perp$ respectively. Define

$$a = p_K(T + I)^{-1} \in B(K) \qquad \text{and} \qquad b = p_{K^\perp}(T + I)^{-1} \in B(K, K^\perp).$$

We will need the following result later.

To show: (a) $a(I_K - a) \ge b^*b$.

(a) Assume that $\zeta \in K$ and let $\xi = (T + I)^{-1}\zeta \in D(T)$. Since the projection operator p_K is self-adjoint by Definition 2.4.1,

$$\begin{aligned}
\langle \zeta, a\zeta \rangle &= \langle \zeta, p_K(T + I)^{-1}\zeta \rangle \\
&= \langle p_K\zeta, (T + I)^{-1}\zeta \rangle \\
&= \langle \zeta, (T + I)^{-1}\zeta \rangle \\
&= \langle \zeta, \xi \rangle \\
&= \langle (T + I)\xi, \xi \rangle \\
&= \langle \xi, (T + I)\xi \rangle \quad \text{(since } T + I \text{ is positive)} \\
&= \|\xi\|^2 + \langle \xi, T\xi \rangle \\
&\ge \|\xi\|^2 \\
&= \|a\zeta\|^2 + \|b\zeta\|^2.
\end{aligned}$$

In the last line, we used Pythagoras theorem. So, $a \ge a^*a + b^*b = a^2 + b^*b$ because a is self-adjoint. Rearranging, we deduce that

$$a - a^2 = a(I_K - a) \geq b^* b. \tag{10.1}$$

This proves part (a).

Let us explain how the proof proceeds from this point. The idea is to construct a bijection

$$\left\{ \begin{matrix} \text{Positive self-adjoint} \\ \text{extensions } \tilde{T} \text{ of } T \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{Operators } c \in B(K^\perp) \text{ such that} \\ c_F \leq c \leq c_K \end{matrix} \right\} \tag{10.2}$$

where $c_F, c_K \in B(K^\perp)$ are some operators which need to be constructed. Before we construct the bijection in equation (10.2), we will first construct the operators $c_F, c_K \in B(K^\perp)$.

To show: (b) If $\epsilon \in \mathbb{R}_{>0}$ then $b(a + \epsilon I_K)^{-1} b^* \leq I_{K^\perp} - b(I_K - a + \epsilon I_K)^{-1} b^*$.

(b) Assume that $\epsilon \in \mathbb{R}_{>0}$. We will first give an equivalent characterisation of the inequality

$$b(a + \epsilon I_K)^{-1} b^* \leq I_{K^\perp} - b(I_K - a + \epsilon I_K)^{-1} b^*. \tag{10.3}$$

First, equation (10.3) is equivalent to

$$b\big((a + \epsilon I_K)^{-1} + (I_K - a + \epsilon I_K)^{-1}\big) b^* \leq I_{K^\perp}.$$

Using the continuous functional calculus, the expression $(a + \epsilon I_K)^{-1} + (I_K - a + \epsilon I_K)^{-1}$ can be rewritten more simply as $f(a)$, where for $t \in [0, 1]$,

$$f(t) = \frac{1}{t + \epsilon} + \frac{1}{1 - t + \epsilon} = \frac{1 + 2\epsilon}{(t + \epsilon)(1 - t + \epsilon)}.$$

In particular, we can rewrite the previous equation as

$$(1 + 2\epsilon) b\big((a + \epsilon I_K)^{-1} (I_K - a + \epsilon I_K)^{-1}\big) b^* \leq I_{K^\perp}$$

and if we set $d = (a + \epsilon I_K)^{-1/2} (I_K - a + \epsilon I_K)^{-1/2}$, the expression simplifies to

$$(1 + 2\epsilon) b d^* d b^* \leq I_{K^\perp}. \tag{10.4}$$

In order to prove part (b), it suffices to show that equation (10.4) holds.

Recall that in part (a), we proved equation (10.1). Equation (10.1) can be expressed as $F(a) \geq b^*b$, where for $t \in [0,1]$,

$$F(t) = t(1-t).$$

The norm of F is

$$\|F\|_\infty = \sup_{t\in[0,1]} |F(t)| = \frac{1}{4}.$$

By Lemma 2.3.8, $b^*b \leq \frac{1}{4}I_K$. Subsequently, $2b^*b \leq 4b^*b \leq I_K$. Now, we have

$$\begin{aligned}(a+\epsilon I_K)(I_K - a + \epsilon I_K) &= a(I_K - a) + (\epsilon + \epsilon^2)I_K \\ &\geq b^*b + (\epsilon+\epsilon^2)I_K \\ &\geq b^*b + 2(\epsilon+\epsilon^2)b^*b \\ &= (1+2\epsilon)b^*b + \epsilon^2 b^*b \geq (1+2\epsilon)b^*b.\end{aligned}$$

The above inequality can be rewritten as $(1+2\epsilon)b^*b \leq d^{-1}(d^*)^{-1}$ and consequently, as $(1+2\epsilon)db^*bd^* \leq I_K$. By Lemma 2.3.8,

$$\|(1+2\epsilon)db^*bd^*\| \leq 1.$$

Therefore,

$$\begin{aligned}1 &\geq \|(1+2\epsilon)db^*bd^*\| \\ &= (1+2\epsilon)\|bd^*\|^2 \\ &= (1+2\epsilon)\|db^*\|^2 \\ &= (1+2\epsilon)\|bd^*db^*\|\end{aligned}$$

and by Lemma 2.3.8 again, we obtain equation (10.4) and consequently, equation (10.3). This proves part (b).

The key observation here is that if $\epsilon \to 0$ then the LHS of equation (10.3) is monotonically increasing with respect to the strong operator topology, whereas the RHS is monotonically decreasing. By Theorem 2.7.1, the LHS of equation (10.3) has a supremum in $B(K^\perp)$ and the RHS has an infimum in $B(K^\perp)$. Hence, we can define

$$c_F = \sup_{\epsilon>0} b(a+\epsilon I_K)^{-1}b^* \in B(K^\perp)$$

and

$$c_K = \inf_{\epsilon>0} I_{K^\perp} - b(I_K - a + \epsilon I_K)^{-1}b^* \in B(K^\perp).$$

Now we can construct the bijection in equation (10.2). Let $\tilde{T}$ be a positive and self-adjoint extension of T. Then, $T + I \prec \tilde{T} + I$ and consequently,

$$(\tilde{T} + I)^{-1}|_K = (T + I)^{-1}$$

Let us write $(\tilde{T} + I)^{-1}$ as a matrix with respect to the decomposition $H = K \oplus K^\perp$. Since $(\tilde{T} + I)^{-1}|_K = (T + I)^{-1}$, the matrix of $(\tilde{T} + I)^{-1}$ takes the form

$$(\tilde{T} + I)^{-1} = \begin{pmatrix} a & b^* \\ b & c \end{pmatrix}$$

for some $c \in B(K^\perp)$. Notice that b^* is the upper right element because $(\tilde{T} + I)^{-1}$ is self-adjoint. Since $(\tilde{T} + I)^{-1}$ is a positive contraction, we can use Lemma 2.3.8 to find that

$$0 \leq \begin{pmatrix} a & b^* \\ b & c \end{pmatrix} \leq I.$$

By Lemma 10.2.3, this holds if and only if for $\epsilon \in \mathbb{R}_{>0}$

$$b(a + \epsilon I_K)^{-1}b^* \leq c \leq I_{K^\perp} - b((1 + \epsilon)I_K - a)^{-1}b^*.$$

In turn, the above inequality holds if and only if $c_F \leq c \leq c_K$. Thus, the bijection in equation (10.2) maps $\tilde{T}$ to c.

We claim that the bijection $\tilde{T} \mapsto c$ in equation (10.2) is order-reversing, with respect to the partial orders (both denoted by $\leq$) on both sides of the bijection.

To prove the claim, suppose that $\tilde{T}$ and $\tilde{T}'$ are positive self-adjoint extensions of T and $c, c' \in B(K^\perp)$ are the corresponding bounded operators. By definition, $\tilde{T} \geq \tilde{T}'$ if and only if $(\tilde{T} + I)^{-1} \leq (\tilde{T}' + I)^{-1}$ as bounded operators. In matrix notation, this means that

$$\begin{pmatrix} a & b^* \\ b & c \end{pmatrix} \leq \begin{pmatrix} a & b^* \\ b & c' \end{pmatrix}$$

In turn, this holds if and only if

$$\begin{pmatrix} 0 & 0 \\ 0 & c' - c \end{pmatrix} \geq 0$$

if and only if $c' \geq c$. This proves the claim.

Now we will go from right to left in equation (10.2). Suppose that $c \in B(K^\perp)$ satisfies $c_F \leq c \leq c_K$. By Lemma 10.2.3, this holds if and only if

$$0 \leq \begin{pmatrix} a & b^* \\ b & c \end{pmatrix} \leq I.$$

For simplicity of notation, let

$$g_c = \begin{pmatrix} a & b^* \\ b & c \end{pmatrix}.$$

To show: (c) The image im g_c is dense in H.

(c) Using the fact that $H = K \oplus K^\perp$, we have

$$\begin{aligned} g_c H &= g_c(K \oplus K^\perp) \\ &\supset g_c(K \oplus \{0\}) \\ &= \{(a\zeta, b\zeta) \mid \zeta \in K\} = (T+I)^{-1}H = D(T). \end{aligned}$$

Since T is densely defined, $D(T)$ is dense in H. Therefore, the image $g_c H$ is dense in H.

By part (c), we find that

$$(\ker g_c)^\perp = \overline{\text{im } g_c} = H.$$

So, $\ker g_c = \{0\}$ and therefore, the inverse $g_c^{-1} : \text{im } g_c \to H$ is a closed, densely defined operator. Since g_c is self-adjoint by definition, g_c^{-1} must also be self-adjoint. Furthermore, by Lemma 2.3.8, $g_c^{-1} \geq I$ because $g_c \leq I$. Now define

$$\tilde{T} = g_c^{-1} - I.$$

Then, $\tilde{T}$ is positive and self-adjoint by Theorem 7.3.7. By construction, it is also a positive, self-adjoint extension of T. This finally demonstrates that we have the bijection in equation (10.2).

Finally, let T_K and T_F be the positive self-adjoint extensions of T corresponding to $c_K, c_F \in B(K^\perp)$. Then,

$$(T_K + I)^{-1} = \begin{pmatrix} a & b^* \\ b & c_K \end{pmatrix} \qquad \text{and} \qquad (T_F + I)^{-1} = \begin{pmatrix} a & b^* \\ b & c_F \end{pmatrix}.$$

By the bijection in equation (10.2), any positive extension $\tilde{T}$ of T must satisfy $T_K \leq \tilde{T} \leq T_F$ because the bijection is order reversing. Furthermore, any positive self-adjoint operator $\tilde{T}$ satisfying $T_K \leq \tilde{T} \leq T_F$ is an extension of T. $\square$

Let us summarise Theorem 10.2.4. A closed, densely defined, positive operator T always admits self-adjoint extensions. Amongst the self-adjoint extensions which are positive, there is a minimal and maximal positive, self-adjoint extension, with regards to the partial order on positive unbounded operators.

The minimal extension T_K is called the **Krein extension** of T and the maximal extension T_F is called the **Friedrichs extension** of T.

Chapter 11

One-parameter groups of unitary operators

11.1 Stone's theorem

One of the major applications of the theory of operators on a Hilbert space is to the study of representations of topological groups. This chapter aims to describe the basic results associated to the representation theory of the abelian (additive) group $\mathbb{R}$.

The notion of a one-parameter group of unitary operators is fundamental to this chapter.

Definition 11.1.1. Let H be a Hilbert space and $\{u_t\}_{t\in\mathbb{R}}$ be a collection of operators on H. We say that $\{u_t\}$ is a **strongly continuous one-parameter group of unitary operators** if

1. If $t \in \mathbb{R}$ then the operator u_t is unitary.
2. If $t, s \in \mathbb{R}$ then $u_{t+s} = u_t u_s$.
3. If $\psi \in H$ then the induced map

$$\begin{array}{rccc} ev_\psi : & \mathbb{R} & \to & H \\ & t & \mapsto & u_t\psi \end{array}$$

   is continuous.

In the above definition, we remark that the third condition is equivalent to saying that if $\psi \in H$ then ev_ψ is continuous at $0 \in \mathbb{R}$. This is precisely because of the second condition.

We will now give the archetypal example of a strong continuous one-parameter group of unitary operators.

Theorem 11.1.1. *Let H be a Hilbert space. Let $T : D(T) \to H$ be a self-adjoint operator. For $t \in \mathbb{R}$, define*

$$u_t = \exp(-itT).$$

Then, $\{u_t\}_{t\in\mathbb{R}}$ is a strongly continuous one-parameter group of unitary operators.

Proof. Assume that H is a Hilbert space and $T : D(T) \to H$ is self-adjoint. Assume that $\{u_t\}_{t\in\mathbb{R}}$ is the collection of operators defined as above.

First assume that $t \in \mathbb{R}$. To see that u_t is unitary, we will apply the Borel functional calculus in Theorem 9.2.2. For $t \in \mathbb{R}$, define

$$f_t(\lambda) = \exp(-it\lambda).$$

Then, $f_t \in Bor(\mathbb{R}, \mathbb{C})$. Let Φ_b be the unique unital *-homomorphism in Theorem 9.2.2 associated to the self-adjoint operator T. We compute directly that

$$\begin{aligned} u_t u_t^* &= \Phi_B(f_t)\Phi_B(f_t)^* \\ &= \Phi_B(f_t)\Phi_B(\overline{f_t}) \\ &= \Phi_B(f_t\overline{f_t}) = \Phi_B(1) \\ &= I. \end{aligned}$$

By a similar computation, we find that $u_t^* u_t = I$. So, u_t is a unitary operator.

Next, assume that $s, t \in \mathbb{R}$. Then,

$$\begin{aligned} u_{t+s} &= \Phi_B(f_{t+s}) \\ &= \Phi_B(\exp(-i(t+s)\lambda)) \\ &= \Phi_B(\exp(-it\lambda)\exp(-is\lambda)) \\ &= \Phi_B(\exp(-it\lambda))\Phi_B(\exp(-is\lambda)) \\ &= u_t u_s. \end{aligned}$$

Finally, assume that $\psi \in H$. To see that $ev_\psi : \mathbb{R} \to H$ is continuous, it suffices to show that ev_ψ is continuous at $0 \in \mathbb{R}$.

Now $\{f_t\}_{t\in\mathbb{R}}$ is a uniformly bounded sequence of Borel functions which converge pointwise to the constant function 1 as $t \to 0$. By Theorem 9.2.2, $f_t(T)(\psi) = u_t(\psi) \to \psi$ as $t \to 0$. So, ev_ψ must be continuous at $0 \in \mathbb{R}$.

We conclude that $\{u_t\}_{t\in\mathbb{R}}$ is a strongly continuous one-parameter group of unitary operators on H. $\square$

The example given in Theorem 11.1.1 is particularly important because Stone's theorem, which is the main theorem of this section, tells us that *any* strongly continuous one-parameter group of unitary operators arises from Theorem 11.1.1.

Before we formalise Stone's theorem, we will first demonstrate that if $u_t = \exp(-itT)$ is the unitary operator in Theorem 11.1.1 for $t \in \mathbb{R}$ then we can recover the self-adjoint operator T from $\{u_t\}_{t\in\mathbb{R}}$.

Theorem 11.1.2. *Let H be a Hilbert space and $T : D(T) \to H$ be a self-adjoint operator. For $t \in \mathbb{R}$, define*

$$u_t = \exp(-itT).$$

If $\psi \in D(T)$ then

$$\lim_{t\to 0} \frac{i}{t}(u_t\psi - \psi) = T\psi.$$

Furthermore, if the limit $\lim_{t\to 0} \frac{i}{t}(u_t\varphi - \varphi)$ exists then $\varphi \in D(T)$.

Proof. Assume that H is a Hilbert space and $T : D(T) \to H$ is a self-adjoint operator. Assume that $\{u_t\}_{t\in\mathbb{R}}$ is the collection of operators on H defined as above. By Theorem 11.1.1, $\{u_t\}_{t\in\mathbb{R}}$ is a strongly continuous one-parameter group of unitary operators.

Assume that $\psi \in D(T)$. Since T is self-adjoint, there exists $\xi \in H$ such that $\psi = (I + T^2)^{-\frac{1}{2}}$. By using the Borel functional calculus in Theorem 9.2.2, we can write

$$\frac{i}{t}(u_t\psi - \psi) = F_t(T)\xi$$

where for $t \in \mathbb{R}$, F_t is defined by

$$\begin{array}{rccc} F_t: & \mathbb{R} & \to & \mathbb{C} \\ & \lambda & \mapsto & \frac{i}{t}\frac{e^{-it\lambda}-1}{\lambda}\lambda(1+\lambda^2)^{-\frac{1}{2}}. \end{array}$$

It is straightforward to see that $\{F_t\}_{t\in\mathbb{R}_{>0}}$ is a uniformly bounded sequence of continuous functions. By taking the limit as $t \to 0$, we find that if $\lambda \in \mathbb{R}$ then

$$\begin{aligned} \lim_{t\to 0} F_t(\lambda) &= \lim_{t\to 0} i\frac{e^{-it\lambda}-1}{\lambda t}\lambda(1+\lambda^2)^{-\frac{1}{2}} \\ &= \lim_{t\to 0} i\frac{-i\lambda e^{-it\lambda}}{\lambda}\lambda(1+\lambda^2)^{-\frac{1}{2}} \\ &= \lim_{t\to 0} e^{-it\lambda}\lambda(1+\lambda^2)^{-\frac{1}{2}} \\ &= \frac{\lambda}{(1+\lambda^2)^{\frac{1}{2}}} \end{aligned}$$

Therefore, F_t converges pointwise to the function ζ, where we recall the definition of ζ from Theorem 9.2.2:

$$\begin{array}{rccc} \zeta: & \mathbb{R} & \to & \mathbb{C} \\ & \lambda & \mapsto & \frac{\lambda}{\sqrt{1+\lambda^2}} \end{array}$$

By Theorem 9.2.2, we must have

$$\lim_{t\to 0}\frac{i}{t}(u_t\psi - \psi) = \lim_{t\to 0} F_t(T)\xi = \zeta(T)\xi = z_T\xi = T\psi.$$

For the next claim, we define an operator $\tilde{T} : D(\tilde{H}) \to H$, where the domain

$$D(\tilde{T}) = \{\varphi \in H \mid \text{The limit } \lim_{t\to 0}\frac{i}{t}(u_t\varphi - \varphi) \text{ exists}\}$$

and $\tilde{T}$ is the map

$$\begin{array}{rccc} \tilde{T}: & D(\tilde{H}) & \to & H \\ & \varphi & \mapsto & \lim_{t\to 0}\frac{i}{t}(u_t\varphi - \varphi) \end{array}$$

Then, $\tilde{T}$ is a linear operator and by the previous computation, $T \prec \tilde{T}$. Since T is densely defined, $\tilde{T}$ must also be densely defined.

To see that $\tilde{T}$ is symmetric, assume that $\varphi_1, \varphi_2 \in D(\tilde{T})$. Then,

$$
\begin{aligned}
\langle \varphi_1, \tilde{T}\varphi_2\rangle &= \lim_{t\to 0}\langle \varphi_1, \frac{i}{t}(u_t\varphi_2 - \varphi_2)\rangle \\
&= \lim_{t\to 0}\big(\langle \varphi_1, \frac{i}{t}u_t\varphi_2\rangle - \langle \varphi_1, \frac{i}{t}\varphi_2\rangle\big) \\
&= \lim_{t\to 0}\big(\langle -\frac{i}{t}u_t^*\varphi_1, \varphi_2\rangle - \langle -\frac{i}{t}\varphi_1, \varphi_2\rangle\big) \\
&= \lim_{t\to 0}\big(\langle -\frac{i}{t}u_{-t}\varphi_1, \varphi_2\rangle - \langle -\frac{i}{t}\varphi_1, \varphi_2\rangle\big) \\
&= \lim_{t\to 0}\langle \frac{i}{-t}(u_{-t}\varphi_1 - \varphi_1), \varphi_2\rangle \\
&= \langle \tilde{T}\varphi_1, \varphi_2\rangle.
\end{aligned}
$$

So, $\tilde{T}$ is symmetric. In particular, it is a symmetric extension of T. Therefore, $T = \tilde{T}$ and $D(T) = D(\tilde{T})$. This completes the proof $\square$

One consequence of Theorem 11.1.2 is that a particular differential equation always has a global solution. This is not too surprising, given the definition of u_t as an exponential of an operator.

Theorem 11.1.3. *Let H be a Hilbert space and $T : D(T) \to H$ be a self-adjoint operator. Let $\psi_0 \in D(T)$. Then, the initial value problem*

$$i\frac{d\psi}{dt} = T\psi, \quad \psi(0) = \psi_0$$

has a unique solution $\psi : \mathbb{R} \to H$.

Proof. Assume that H is a Hilbert space and $T : D(T) \to H$ is a self-adjoint operator. Assume that $\psi_0 \in D(T)$. Define

$$
\begin{array}{cccc}
\psi : & \mathbb{R} & \to & H \\
 & t & \mapsto & e^{-itH}\psi_0
\end{array}
$$

Then, ψ is a continuous function such that $\psi(0) = \psi_0$. By Theorem 11.1.2, ψ is differentiable at $t = 0$ and

$$i\frac{d\psi}{dt}|_{t=0} = T\psi_0.$$

To show: (a) ψ is differentiable on all of $\mathbb{R}$.

(a) Recall that as a consequence of Theorem 9.3.3, if f, g are Borel functions from $\mathbb{R}$ to $\mathbb{C}$ and g is bounded then

$$g(T)f(T) \prec f(T)g(T).$$

Applying this to the functions $f(\lambda) = \lambda$ and $g = e^{-it\lambda}$, we deduce that if $t \in \mathbb{R}$ then $e^{-itT}T \prec Te^{-itT}$.

Moreover, we claim that if $s \in \mathbb{R}$ then $e^{-isT}(D(T)) \subset D(T)$. Indeed, if $e^{-isT}\varphi \in e^{-isT}(D(T))$ then with $u_t = e^{-itT}$, we compute directly that

$$\begin{aligned}\lim_{t\to 0} \frac{i}{t}(u_t e^{-isT}\varphi - e^{-isT}\varphi) &= \lim_{t\to 0} \frac{i}{t}(u_{t+s}\varphi - u_s\varphi) \\ &= Te^{-isT}(1+T^2)^{-\frac{1}{2}} = u_S T(1+T^2)^{-\frac{1}{2}}.\end{aligned}$$

In the above computation, we used a similar method of computation as in Theorem 11.1.2. Since the above limit exists, Theorem 11.1.2 tells us that $e^{-isT}\varphi \in D(T)$. So, $e^{-isT}(D(T)) \subset D(T)$.

Since $e^{-itT}T \prec Te^{-itT}$, $D(e^{-itT}T) \subset D(Te^{-itT})$. To see that $D(Te^{-itT}) \subset D(e^{-itT}T)$, we use the finding that $e^{-isT}(D(T)) \subset D(T)$ for any $s \in \mathbb{R}$ to conclude that

$$\begin{aligned}D(Te^{-itT}) &= \{\xi \in D(e^{-itT}) \mid e^{-itT}\xi \in D(T)\} \\ &= \{\xi \in H \mid e^{-itT}\xi \in D(T)\} \\ &= e^{itT}D(T) \subset D(T) \\ &= D(e^{-itT}T).\end{aligned}$$

The last equality follows from the fact that e^{-itT} is a unitary operator. So, $D(Te^{-itT}) = D(e^{-itT}T)$ and consequently, $Te^{-itT} = e^{-itT}T$ for $t \in \mathbb{R}$.

To see that ψ is differentiable on all of $\mathbb{R}$, we compute directly for $t \in \mathbb{R}$ that as $s \to 0$,

$$\begin{aligned}\frac{1}{s}\big(\psi(s+t) - \psi(t)\big) &= -ie^{-itT}\frac{1}{s}(\psi(s) - \psi(0)) \\ &\to -ie^{-itT}T\psi_0 = -iTe^{-itT}\psi_0 \\ &= -iT\psi(t).\end{aligned}$$

Hence, ψ is differentiable on all of $\mathbb{R}$ and solves the initial value problem given in the statement of the theorem.

It remains to show that ψ is unique. Assume that $\phi : \mathbb{R} \to H$ is another solution to the initial value problem. Define $f(t) = \|\Phi(t) - \psi(t)\|^2$. Then, $f(0) = 0$ and

$$\begin{aligned}
\frac{d}{dt}f(t) &= \frac{d}{dt}\langle \phi(t) - \psi(t), \phi(t) - \psi(t)\rangle \\
&= \langle -iT\phi(t) + iT\psi(t), \phi(t) - \psi(t)\rangle + \langle \phi(t) - \psi(t), -iT\phi(t) + iT\psi(t)\rangle \\
&= \langle -iT\phi(t) + iT\psi(t), \phi(t) - \psi(t)\rangle + \langle iT\phi(t) - iT\psi(t), \phi(t) - \psi(t)\rangle \\
&= 0.
\end{aligned}$$

Therefore, $f(t) = 0$ and $\psi(t) = \phi(t)$. This shows that ψ is the unique solution to the original initial value problem. □

Now we will embark on the proof of Stone's theorem. In fact, the techniques used in the proof of Theorem 11.1.3 will feature in the proof of Stone's theorem.

Theorem 11.1.4 (Stone's theorem)**.** *Let H be a Hilbert space and $(u_t)_{t\in\mathbb{R}}$ be a strongly continuous one-parameter group of unitary operators on H. Then, there exists a self-adjoint operator T on H such that if $t \in \mathbb{R}$ then $u_t = \exp(-itT)$.*

Proof. Assume that $(u_t)_{t\in\mathbb{R}}$ is a strongly continuous one-parameter group of unitary operators on a Hilbert space H.

We will construct the desired self-adjoint operator $T : D(T) \to H$ from scratch. Let $C_c^\infty(\mathbb{R}, \mathbb{C})$ denote the space of smooth, compact supported functions from $\mathbb{R}$ to $\mathbb{C}$. For $\varphi \in H$ and $f \in C_c^\infty(\mathbb{R}, \mathbb{C})$, define

$$\varphi_f = \int_{\mathbb{R}} f(t) u_t \varphi \, dt.$$

Let

$$\mathscr{D} = \text{span}\,\{\varphi_f \mid \varphi \in H, f \in C_c^\infty(\mathbb{R}, \mathbb{C})\}.$$

To show: (a) $\mathscr{D}$ is a dense subspace of H.

(a) Assume that $\varphi \in H$. Let $\{f_n\}_{n\in\mathbb{Z}_{>0}}$ be a sequence in $C_c^\infty(\mathbb{R}, \mathbb{C})$ such that

$$\text{supp}(f_n) \subset [-\frac{1}{n}, \frac{1}{n}] \qquad \text{and} \qquad \int_{\mathbb{R}} |f_n(t)| \, dt = 1.$$

We claim that the sequence $\{\varphi_{f_n}\}_{n\in\mathbb{Z}_{>0}}$ in $\mathscr{D}$ converges to φ. We compute directly that

$$\begin{aligned}\|\varphi_{f_n}-\varphi\| &= \|\int_{\mathbb{R}} f_n(t)u_t\varphi\ dt - \int_{\mathbb{R}} f_n(t)\varphi\ dt\| \\ &= \|\int_{\mathbb{R}} f_n(t)(u_t\varphi-\varphi)\ dt\| \\ &\le \int_{\mathbb{R}} |f_n(t)|\|u_t\varphi-\varphi\|\ dt \\ &\le \sup_{|t|\le\frac{1}{n}} \|u_t\varphi-\varphi\| \\ &\to 0\end{aligned}$$

as $n\to\infty$. Hence, $\varphi_{f_n}\to\varphi$ and $\mathscr{D}$ is a dense subspace of H.

Part (a) reveals that we would like to define our candidate self-adjoint operator on $\mathscr{D}$. For $s\in\mathbb{R}$ and $f\in C_c^\infty(\mathbb{R},\mathbb{C})$, we define f_s to be the function $t\mapsto f(t-s)$.

To show: (b) If $s,t\in\mathbb{R}$ then $u_s\varphi_f=\varphi_{f_s}$.

(c) $\lim_{s\to0}\frac{1}{s}(u_s\varphi_f-\varphi_f)=-\varphi_{f'}$.

(b) Assume that $s,t\in\mathbb{R}$. We compute directly that

$$\begin{aligned}u_s\varphi_f &= u_s\int_{\mathbb{R}} f(t)u_t\varphi\ dt \\ &= \int_{\mathbb{R}} f(t)u_su_t\varphi\ dt \\ &= \int_{\mathbb{R}} f(t)u_{s+t}\varphi\ dt \\ &= \int_{\mathbb{R}} f_s(x)u_x\varphi\ dx \qquad (x=s+t) \\ &= \varphi_{f_s}.\end{aligned}$$

(c) By using part (b), we compute directly that

$$\begin{aligned}
\frac{1}{s}(u_s\varphi_f - \varphi_f) &= \frac{1}{s}(\varphi_{f_s} - \varphi_f) \\
&= \frac{1}{s}\int_{\mathbb{R}} (f_s(t) - f(t))u_t\varphi \, dt \\
&= \int_{\mathbb{R}} \frac{f(t-s) - f(t)}{s} u_t\varphi \, dt \\
&\to \int_{\mathbb{R}} -f'(t)u_t\varphi \, dt \\
&= -\varphi_{f'}
\end{aligned}$$

as $s \to \infty$.

In light of part (c), we define the operator T_0 by

$$\begin{array}{rccc}
T_0 : & D(T_0) = \mathscr{D} & \to & H \\
& \phi & \mapsto & i \lim_{s\to 0} \frac{1}{s}(u_s\phi - \phi).
\end{array}$$

By part (c), we observe that im $T_0 \subseteq \mathscr{D}$ and if $t \in \mathbb{R}$ and $\varphi_f \in \mathscr{D}$ then

$$\begin{aligned}
T_0 u_t \varphi_f &= T_0 \varphi_{f_t} \\
&= i \lim_{s\to 0} \frac{1}{s}(u_s\varphi_{f_t} - \varphi_{f_t}) \\
&= i u_t \lim_{s\to 0} \frac{1}{s}(u_s\varphi_f - \varphi_f) \\
&= u_t T_0 \varphi_f.
\end{aligned}$$

So, $T_0 u_t = u_t T_0$.

To show: (d) T_0 is a symmetric operator.

(e) The closure $\overline{T_0}$ is self-adjoint.

(d) We argue in a similar fashion to Theorem 11.1.3. If $\phi, \varphi \in \mathscr{D}$ then

$$
\begin{aligned}
\langle \psi, T_0\phi \rangle &= \langle \psi, \frac{i}{s}\lim_{s\to 0}(u_s\phi - \phi)\rangle \\
&= \lim_{s\to 0}\frac{-i}{s}\langle \psi, u_s\phi - \phi\rangle \\
&= \lim_{s\to 0}\frac{-i}{s}\langle \psi, (u_s - I)\phi\rangle \\
&= \lim_{s\to 0}\frac{-i}{s}\langle (u_{-s} - I)\psi, \phi\rangle \\
&= i\langle \lim_{s\to 0}\frac{1}{-s}(u_{-s} - I)\psi, \phi\rangle \\
&= i\langle -iT_0\psi, \phi\rangle = \langle T_0\psi, \phi\rangle.
\end{aligned}
$$

Hence, T_0 is a symmetric operator.

(e) Since T_0 is a symmetric operator, its closure $\overline{T_0}$ is both closed and symmetric. By Lemma 10.1.3 and Theorem 10.1.10, it suffices to show that $\ker(T_0^* \pm iI) = \{0\}$.

Assume that $\eta \in \ker(T_0^* \pm iI)$. If $\phi \in D(T_0) = \mathscr{D}$ then

$$
\begin{aligned}
\frac{d}{dt}\langle \eta, u_t\phi\rangle &= \lim_{s\to 0}\langle \eta, \frac{1}{s}(u_{t+s} - u_t)\phi\rangle \\
&= \lim_{s\to 0}\langle \eta, u_t\frac{1}{s}(u_s - I)\phi\rangle \\
&= \langle \eta, u_t(-i)T_0\phi\rangle = i\langle \eta, T_0u_t\phi\rangle \\
&= i\langle T_0^*\eta, u_t\phi\rangle \\
&= i\langle \mp i\eta, u_t\phi\rangle = \pm\langle \eta, u_t\phi\rangle
\end{aligned}
$$

Now let $g : \mathbb{R} \to \mathbb{C}$ be the function $t \mapsto \langle \eta, u_t\phi\rangle$. The above computation tells us that $g' = \pm g$. Solving this ODE, we find that

$$
g(t) = g(0)e^{\pm t}
$$

for $t \in \mathbb{R}$. Now observe that

$$
|g(t)| \leq |\langle \eta, u_t\phi\rangle| \leq \|\eta\|\|u_t\phi\| = \|\eta\|\|\phi\|.
$$

The last equality follows from the fact that u_t is unitary. So, $g(t) = g(0)e^{\pm t}$ is uniformly bounded, which can only occur if $g = 0$. In particular, $g(0) = \langle \eta, \phi\rangle = 0$. Since $\phi \in \mathscr{D}$ was arbitrary and $\mathscr{D}$ is dense in H, we deduce that $\eta \in H^\perp = \{0\}$. So, $\ker(T_0^* \pm iI) = \{0\}$ and $\overline{T_0}$ is self-adjoint by

Theorem 10.1.10.

Now define $T = \overline{T_0}$. By part (e), T is a self-adjoint operator. By Theorem 11.1.1, $(e^{-itT})_{t\in\mathbb{R}}$ is a strong continuous one-parameter group of unitary operators on H. For $\phi \in \mathscr{D}$ and $t \in \mathbb{R}$, define

$$\begin{array}{rccc} \xi: & \mathbb{R} & \to & H \\ & t & \mapsto & u_t\phi - e^{-itT}\phi \end{array}$$

By definition of T_0, we find that

$$\frac{d}{dt}\xi(t) = -iT_0 u_t\phi + iTe^{-itT}\phi = -iH\xi(t).$$

Consequently,

$$\begin{aligned} \frac{d}{dt}\|\xi(t)\|^2 &= \frac{d}{dt}\langle \xi(t), \xi(t)\rangle \\ &= \langle \frac{d}{dt}\xi(t), \xi(t)\rangle + \langle \xi(t), \frac{d}{dt}\xi(t)\rangle \\ &= \langle -iT\xi(t), \xi(t)\rangle + \langle \xi(t), -iT\xi(t)\rangle = 0. \end{aligned}$$

Therefore, the map $t \mapsto \|\xi(t)\|$ is constant. But, $\xi(0) = \phi - \phi = 0$. So, $\xi(t) = 0$ for $t \in \mathbb{R}$ and consequently,

$$u_t\phi = e^{-itT}\phi$$

for $t \in \mathbb{R}$ and $\phi \in \mathscr{D}$. Since $\mathscr{D}$ is dense in H, we find that for $t \in \mathbb{R}$, $u_t = e^{-itT}$ as unitary operators on H. $\square$

In Theorem 11.1.4, the unique self-adjoint operator T such that $u_t = e^{-itT}$ for $t \in \mathbb{R}$ is called the **infinitesimal generator** of the group $(u_t)_{t\in\mathbb{R}}$.

11.2 Trotter formula

Let H be a Hilbert space and $T : D(T) \to H$ be a self-adjoint operator. Recall that the graph norm $\|-\|_T$ makes $D(T)$ into a Hilbert space. If $\psi \in D(T)$ then

$$\|\psi\|_T = \sqrt{\|\psi\|^2 + \|T\psi\|^2}.$$

We also recall from the proof of Theorem 11.1.3 that if $s \in \mathbb{R}$ then $e^{-isT}(D(T)) \subset D(T)$.

Lemma 11.2.1. *Let H be a Hilbert space and $T : D(T) \to H$ be a self-adjoint operator. Fix $\psi \in D(T)$ and define the function*

$$\begin{array}{rccc} \alpha : & \mathbb{R} & \to & D(T) \\ & t & \mapsto & e^{-itT}\psi. \end{array}$$

Then, α is continuous with respect to the graph norm $\|-\|_T$ on $D(T)$.

Proof. Assume that H is a Hilbert space and $T : D(T) \to H$ is a self-adjoint operator. Assume that for $\psi \in D(T)$, the function α is defined as above.

It suffices to show that α is continuous at zero with respect to the graph norm on $D(T)$. Recall from the proof of Theorem 11.1.3 that $Te^{-itT} = e^{-itT}T$. So,

$$\begin{aligned} \|\alpha(t) - \alpha(0)\|_T^2 &= \|e^{-itT}\psi - \psi\|_T^2 \\ &= \|e^{-itT}\psi - \psi\|^2 + \|Te^{-itT}\psi - T\psi\|^2 \\ &= \|e^{-itT}\psi - \psi\|^2 + \|e^{-itT}T\psi - T\psi\|^2 \\ &\to 0 \end{aligned}$$

as $t \to \infty$. Here, we use the fact that $(e^{-itT})_{t\in\mathbb{R}}$ is continuous at zero. Hence, α is continuous at zero and hence, continuous with respect to the graph norm. $\square$

The Trotter formula is very similar to Theorem 3.5.1.

Theorem 11.2.2 (Trotter formula)**.** *Let H be a Hilbert space and K, T be self-adjoint operators on H. Assume that $K + T$ is also self-adjoint. If $t \in \mathbb{R}$ then*

$$(\exp(-i\frac{t}{n}T)\exp(-i\frac{t}{n}K))^n \to e^{-it(T+K)}$$

as $n \to \infty$ in the strong operator topology.

Proof. Assume that H is a Hilbert space and K, T are self-adjoint operators on H. Assume that $K + T$ is also a self-adjoint operator. Define the map

$$\begin{array}{rccc} F : & \mathbb{R} - \{0\} & \to & B(H) \\ & t & \mapsto & \frac{1}{t}(e^{-itT}e^{-itK} - e^{-it(T+K)}) \end{array}$$

We claim that if $\xi \in H$ then the map Φ which sends $t \in \mathbb{R} - \{0\}$ to $F(t)\xi \in H$ is continuous on $\mathbb{R} - \{0\}$.

To show: (a) If $\xi \in H$ then $\Phi : \mathbb{R} - \{0\} \to H$ is continuous on all of $\mathbb{R} - \{0\}$.

(a) Assume that $\xi \in H$. Define the maps ψ_1 and ψ_2 by

$$\begin{array}{rccc} \psi_1 : & \mathbb{R} - \{0\} & \to & H \\ & t & \mapsto & \frac{1}{t} e^{-it(T+K)} \xi \end{array}$$

and

$$\begin{array}{rccc} \psi_2 : & \mathbb{R} - \{0\} & \to & H \\ & t & \mapsto & \frac{1}{t} e^{-itK} \xi \end{array}$$

Then, ψ_1 and ψ_2 are continuous functions on $\mathbb{R} - \{0\}$ and

$$\frac{1}{t} e^{-itT} e^{-itK} \xi = e^{-itT} \psi_2(t).$$

If $t, t' \in \mathbb{R} - \{0\}$ then

$$\begin{aligned} \|\frac{1}{t} e^{-itT} e^{-itK} \xi - \frac{1}{t} e^{-it'T} e^{-it'K} \xi\| &= \|e^{-itT} \psi_2(t) - e^{-it'T} \psi_2(t')\| \\ &\leq \|e^{-itT} \psi_2(t) - e^{-itT} \psi_2(t')\| + \|e^{-itT} \psi_2(t') - e^{-it'T} \psi_2(t')\| \\ &\leq \|\psi_2(t) - \psi_2(t')\| + \|e^{-i(t-t')T} \psi_2(t') - \psi_2(t')\| \\ &\to 0 \end{aligned}$$

as $t \to t'$. So, the map $t \mapsto \frac{1}{t} e^{-itT} e^{-itK} \xi$ is continuous on all of $\mathbb{R} - \{0\}$ and in tandem with the fact that $\psi_1(t)$ is continuous on $\mathbb{R} - \{0\}$, Φ must be continuous on $\mathbb{R} - \{0\}$.

We also observe that $\lim_{t \to \pm\infty} F(t)\xi = 0$. To see why this is the case, we compute directly for $\xi \in H$ that

$$\begin{aligned} \|F(t)\xi\| &\leq \|F(t)\| \|\xi\| \\ &= \|\frac{1}{t} (e^{-itT} e^{-itK} - e^{-it(T+K)})\| \|\xi\| \\ &\leq \frac{1}{t} (\|e^{-itT}\| \|e^{-itK}\| + \|e^{-it(T+K)}\|) \|\xi\| \\ &= \frac{2\|\xi\|}{t} \to 0 \end{aligned}$$

as $t \to \pm\infty$.

Now let $D = D(T+K) = D(T) \cap D(K)$. We claim that if $\psi \in D$ then $\lim_{t\to 0} F(t)\psi = 0$.

To show: (b) If $\psi \in D$ then $\lim_{t\to 0} F(t)\psi = 0$.

(b) Assume that $\psi \in D$. Then, we can write

$$F(t)\psi = e^{-itT}\frac{1}{t}(e^{-itK}\psi - \psi) + \frac{1}{t}(e^{-itT}\psi - \psi) - \frac{1}{t}(e^{-it(T+K)}\psi - \psi).$$

By Theorem 11.1.2, we deduce that

$$\begin{aligned}\lim_{t\to 0} F(t)\psi &= \lim_{t\to 0} e^{-itT}\frac{1}{t}(e^{-itK}\psi - \psi) + \lim_{t\to 0}\frac{1}{t}(e^{-itT}\psi - \psi) - \lim_{t\to 0}\frac{1}{t}(e^{-it(T+K)}\psi - \psi)\\ &= -iK\psi - iT\psi - (-i(T+K)) = 0.\end{aligned}$$

As a particular consequence of part (b), we can define $F(0)\psi = 0$ for $\psi \in D$. In this manner, we obtain a family of linear maps

$$\big(F(t) : D \to H\big)_{t\in\mathbb{R}}$$

such that the map Φ, which sends t to $F(t)\psi$ is continuous. Moreover, we have the limits

$$\lim_{t\to 0} F(t)\psi = \lim_{t\to\pm\infty} F(t)\psi = 0.$$

Consider the graph norm $\|-\|_{T+K}$ on $D(T+K) = D$. Since $\|-\|_{T+K} \geq \|-\|$, we find that the operators $F(t)$ are continuous from D with the graph norm $\|-\|_{T+K}$ to H. Thus, if $\psi \in D$ then the set

$$\{F(t)\psi \mid t \in \mathbb{R}\}$$

is bounded. By the uniform boundedness principle, there exists $M \in \mathbb{R}_{>0}$ such that

$$\|F(t)\| = \sup_{\|\psi\|_{T+K}=1} \|F(t)\psi\| \leq M.$$

For $\psi \in D$, let

$$C_\psi = \{e^{-is(T+K)}\psi \mid t \in [-1,1]\}.$$

By Lemma 11.2.1, the map $s \mapsto e^{-is(T+K)}\psi$ is a continuous function from $\mathbb{R}$ to $(D, \|-\|_{T+K})$ (we emphasise that D has the graph norm). Therefore, C_ψ

is a compact subset of $(D, \|-\|_{T+K})$ because it is the image of the compact interval $[-1, 1]$ under the continuous map $s \mapsto e^{-is(T+K)}\psi$.

Now assume $\epsilon \in \mathbb{R}_{>0}$ and let $\{\phi_1, \ldots, \phi_N\}$ be an $\epsilon/2M$-net in C_ψ. This means that if $\eta \in C_\psi$ then there exists $i \in \{1, 2, \ldots, N\}$ such that

$$\|\eta - \phi_i\|_{T+K} < \frac{\epsilon}{2M}.$$

Since $\lim_{t \to 0} F(t)\psi = 0$ for any $\psi \in D$, there exists $\delta \in \mathbb{R}_{>0}$ such that if $|t| < \delta$ then

$$\|F(t)\phi_j\| < \frac{\epsilon}{2}$$

for $j \in \{1, 2, \ldots, N\}$. So,

$$\begin{aligned}
\|F(t)\eta\| &\leq \|F(t)\eta - F(t)\phi_i\| + \|F(t)\phi_i\| \\
&\leq \|F(t)\|\|\eta - \phi_i\| + \|F(t)\phi_i\| \\
&\leq M\|\eta - \phi_i\| + \|F(t)\phi_i\| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Hence, we have shown that the functions

$$\big(\psi \mapsto F(t)\psi\big)_{t \in [-1,1]}$$

converge to zero uniformly on C_ψ as $t \to 0$. This means that if $\psi \in D$ is fixed then the quantity

$$\|F(t)e^{-is(T+K)}\psi\| \to 0$$

uniformly as $t \to 0$ for $s \in [-1, 1]$.

Recalling the proof of Theorem 3.5.1, we have the identity

$$s^n - t^n = \sum_{r=0}^{n-1} s^r(s - t)t^{n-1-r}.$$

for bounded operators $s, t \in B(H)$. So,

$$
\begin{aligned}
(e^{-i\frac{t}{n}T}e^{-i\frac{t}{n}K})^n\psi - e^{-it(T+K)}\psi &= (e^{-i\frac{t}{n}T}e^{-i\frac{t}{n}K})^n\psi - (e^{-i\frac{t}{n}(T+K)})^n\psi \\
&= \sum_{m=0}^{n-1}(e^{-i\frac{t}{n}T}e^{-i\frac{t}{n}K})^m(e^{-i\frac{t}{n}T}e^{-i\frac{t}{n}K} - e^{-i\frac{t}{n}(T+K)}) \\
&\quad (e^{-i\frac{t}{n}(T+K)})^{n-1-m}\psi.
\end{aligned}
$$

By the triangle inequality, we find that

$$
\begin{aligned}
\|(e^{-i\frac{t}{n}T}e^{-i\frac{t}{n}K})^n\psi - e^{-it(T+K)}\psi\| &\leq n \max_{0\leq m\leq n-1}\|(e^{-i\frac{t}{n}T}e^{-i\frac{t}{n}K})^m(e^{-i\frac{t}{n}T}e^{-i\frac{t}{n}K} - e^{-i\frac{t}{n}(T+K)}) \\
&\quad\ (e^{-i\frac{t}{n}(T+K)})^{n-1-m}\psi\| \\
&\leq n \max_{0\leq m\leq n-1}\|e^{-i\frac{t}{n}T}e^{-i\frac{t}{n}K}\|^m \\
&\quad\ \|(e^{-i\frac{t}{n}T}e^{-i\frac{t}{n}K} - e^{-i\frac{t}{n}(T+K)})(e^{-i\frac{t}{n}(T+K)})^{n-1-m}\psi\| \\
&= n \max_{0\leq m\leq n-1}\|(e^{-i\frac{t}{n}T}e^{-i\frac{t}{n}K} - e^{-i\frac{t}{n}(T+K)})(e^{-i\frac{t}{n}(T+K)})^{n-1-m}\psi\| \\
&= |t| \max_{0\leq m\leq n-1}\|F(\frac{t}{n})e^{-i\frac{t(n-1-m)}{n}(T+K)}\psi\| \\
&\leq |t| \max_{|s|<|t|}\|F(\frac{t}{n})e^{-is(T+K)}\psi\| \to 0
\end{aligned}
$$

as $n \to \infty$. This uses the previous finding that

$$\|F(t)e^{-is(T+K)}\psi\| \to 0$$

uniformly as $t \to 0$ for $s \in [-1, 1]$.

So, $(e^{-i\frac{t}{n}T}e^{-i\frac{t}{n}K})^n\psi \to e^{-it(T+K)}\psi$ as $n \to \infty$ for $\psi \in D$. Since $\|(e^{-i\frac{t}{n}T}e^{-i\frac{t}{n}K})^n\psi\| = 1$ for any $n \in \mathbb{R}_{>0}$, we finally conclude that

$$(\exp(-i\frac{t}{n}T)\exp(-i\frac{t}{n}K))^n\psi \to e^{-it(T+K)}\psi$$

for $\psi \in H$ because $D = D(T+K)$ is dense in H, as $T+K$ is by assumption, self-adjoint. This completes the proof. $\square$

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