Category theory and internal structures

Brian Chan

August 22, 2022

Contents

0.1 Purpose

This document is a record of my notes on category theory. The primary reference we are following here is [Bou17], which after discussing the basics of category theory, proceeds to develop the theory behind *protomodular* and *Mal'tsev* categories. This document marks the second time I have recorded detailed notes on a particular reference, with the first set of notes being about functional analysis.

As mentioned at the start of [Bou17, Chapter 2], there are two main pathways one can take after gaining a basic grasp of category theory — enriched category theory and the theory of internal structures. The reference [Bou17] develops the theory in the latter direction.

Chapter 1

Category theory

1.1 Basic definitions and examples

The concept of a category crops up in a multitude of different fields, ranging from group theory to algebraic topology (via 2-categories and higher order category theory). Certain constructions such as taking quotients or pullbacks in different contexts/categories have an elegant and unified description in category theory. Additionally, there are many instances of adjoint pairs of functors which appear in fields such as representation theory (induction and restriction) and multilinear algebra (Hom-tensor adjunction). To put it simply, category theory is **very powerful and pervasive**.

We will start by defining categories. Instead of following [Bou17], we will define categories directly.

Definition 1.1.1. A **category** $\mathscr{C}$ is a triple, consisting of:

1. A class of **objects** $\mathrm{ob}(\mathscr{C})$
2. A class of **morphisms** (or arrows) between the objects $Hom(\mathscr{C})$. We say that the morphism $f : A \to B$ is an element of $Hom(A, B)$, which denotes the class of all morphisms from A to B. A is deemed the **source object** and B is the **target object** in this case.
3. A binary operation called **composition of morphisms**, defined by $\circ : Hom(B, C) \times Hom(A, B) \to Hom(A, C)$, $\circ(g, f) = g \circ f$.

Additionally, the composition of morphisms must satisfy the following two properties:

1. **Associativity**: $(f \circ g) \circ h = f \circ (g \circ h)$
2. **Identity**: For all objects $A \in ob(\mathscr{C})$, there exists a morphism $1_A : A \to A$ such that for all morphisms $f \in Hom(A, B)$, $f \circ 1_A = f$ and for all morphisms $g \in Hom(B, A)$, $1_A \circ g = g$.

Let us compare the definition of a category to the definitions of a graph and a reflexive graph given below.

Definition 1.1.2. A **graph** $\mathscr{C}$ is a pair, consisting of:

1. A class of **objects** $\mathrm{ob}(\mathscr{C})$
2. A class of **morphisms** (or arrows) between the objects $Hom(\mathscr{C})$. We say that the morphism $f : A \to B$ is an element of $Hom(A, B)$, which denotes the class of all morphisms from A to B. A is deemed the **source object** and B is the **target object** in this case.

A **reflexive graph** is a graph where each object X is associated to an identity morphism $1_X : X \to X$.

Hence, a category is a reflexive graph, equipped with the composition operation of morphisms. We will now give illustrative examples of categories.

Example 1.1.3. Here are the examples of categories which are most important to [Bou17].

The category **Set** is the category whose objects are sets and whose morphisms are functions between sets.

The category **Mon** is the category whose objects are monoids (groups without inversion) and whose morphisms are monoid homomorphisms.

The category **Grp** is the category whose objects are groups and whose morphisms are group homomorphisms.

Similarly, we have the category **Ab** of abelian groups and the category **CoM** of commutative monoids. Finally, the category **Top** is the category whose objects are topological spaces and whose morphisms are continuous functions.

Before we proceed, we will make a quick remark. Consider the category **Set**. The objects in **Set** are all the possible sets. However, this raises the issue of Russell's paradox — there is no "set of all sets". Indeed, we were careful to say a "class of objects" and not a "set of objects" in the definition of a category. This observation segues into a philosophical discussion about the foundations of category theory which we will not pursue here. For further details, consult the reference [Mur06] for a brief discussion.

Analogously to morphisms in a category, we can define the notion of morphisms between categories.

Definition 1.1.4. Let $\mathscr{C}, \mathscr{D}$ be categories. A **functor** $F : \mathscr{C} \to \mathscr{D}$ is a map which satisfies the following properties:

1. If $C \in ob(\mathscr{C})$, then $F(C) \in ob(\mathscr{D})$
2. If $C \in ob(\mathscr{C})$, then $F(1_C) = 1_{F(C)}$, where 1_C and $1_{F(C)}$ are the identity morphisms defined on C and $F(C)$ respectively.
3. If $X, Y, Z \in ob(\mathscr{C})$, $f \in Hom(X, Y)$ and $g \in Hom(Y, Z)$, then $F(g \circ f) = F(g) \circ F(f)$, where $F(g) \in Hom(F(Y), F(Z))$ and $F(f) \in Hom(F(X), F(Y))$.

A functor is quite literally a morphism of categories because it preserves the essential structures of a category — the identity morphism on every object and the composition operation.

Example 1.1.5. Let G be a group and

$$[G, G] = \{[g, h] = ghg^{-1}h^{-1} \mid g, h \in G\}$$

be the commutator subgroup of G. The quotient $G^{ab} = G/[G, G]$ is the abelianisation of G. We also have the projection map $\pi_G : G \to G^{ab}$, which is a group morphism.

The functor $ab : \mathbf{Grp} \to \mathbf{Ab}$ sends a group G to its abelianisation G^{ab} and a group morphism $f : G \to H$ to the group morphism $f^{ab} : G^{ab} \to H^{ab}$. The morphism f^{ab} is the unique group morphism which makes the below diagram commute, as a consequence of the universal property of the quotient:

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\pi_G"] & H \arrow[d, "\pi_H"] \\ G^{ab} \arrow[r, dashed, "f^{ab}"'] & H^{ab} \end{tikzcd}$$

Additionally, we also have a notion of maps between functors themselves.

Definition 1.1.6. Let $\mathscr{C}$ and $\mathscr{D}$ be categories. Let $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{C} \to \mathscr{D}$ be functors. A **natural transformation** $\alpha : F \to G$ is a family of morphisms

$$\{\alpha_A : F(A) \to G(A) \mid A \in \mathscr{C}\}$$

such that for every morphism $f : A \to A'$ in $\mathscr{C}$, the following diagram commutes:

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha_A"] & F(A') \arrow[d, "\alpha_{A'}"] \\ G(A) \arrow[r, "G(f)"] & G(A') \end{tikzcd}$$

If the α_A are all isomorphisms in $\mathscr{D}$, then α is said to be a **natural isomorphism**.

Example 1.1.7. Following on from the previous example, the projection map π is a natural transformation from the identity functor $id : \mathbf{Grp} \to \mathbf{Grp}$ to the abelianisation functor ab.

Our next definition focuses on the morphisms in a particular category.

Definition 1.1.8. Let $\mathscr{C}$ be a category. Suppose that we have the following diagram in $\mathscr{C}$:

$$\begin{tikzcd} U \arrow[r, "g"] & X \arrow[r, shift left, "h"] \arrow[r, shift right, "h'"'] & Y \arrow[r, "f"] & Z \end{tikzcd}$$

We say that g **equalizes the pair** (h, h') if $h \circ g = h' \circ g$. Furthermore, f **coequalizes the pair** (h, h') if $f \circ h = f \circ h'$.

We say that f is a **monomorphism** when the only pairs (h, h') which are coequalized by f are pairs of the form (h, h). Dually, g is an **epimorphism** when the only pairs (h, h') which are equalized by g are pairs of the form (h, h).

Let us apply this definition in the following lemma.

Lemma 1.1.1. *In the category of sets **Set**, a function (set morphism) $f : X \to Y$ is a monomorphism if and only if f is an injective function. Moreover, f is an epimorphism if and only if f is a surjective function.*

Proof. Assume that we have the following diagram in the category **Set**:

$$U \xrightarrow{g} X \underset{h'}{\overset{h}{\rightrightarrows}} Y \xrightarrow{f} Z$$

To show: (a) If g is surjective, then g is an epimorphism.

(b) If g is an epimorphism, then g is surjective.

(c) If f is injective, then f is a monomorphism.

(d) If f is a monomorphism, then f is injective.

(a) Assume that g is surjective and that $h \circ g = h' \circ g$. Since $g : U \to X$ is surjective, for $h \circ g = h' \circ g$ to hold, h and h' must agree on the image $g(U) = X$. Therefore, $h = h'$ which demonstrates that g is an epimorphism.

(b) We will prove this by contrapositive. Assume that g is not a surjective function. Then, there exists an element $x \in X$ such that $x \notin g(U)$. The key point here is that we can do anything with the element x. Define the functions $h, h' : X \to Y$ such that $h(x) = y_1$ and $h'(x) = y_2$ with $y_1 \neq y_2$ and $h(z) = h'(z)$ for all $z \in X - \{x\}$. By construction, $h \neq h'$, but $h \circ g = h' \circ g$. Hence, the function g equalizes the pair (h, h') with $h \neq h'$, which shows that g is not an epimorphism as required.

(c) Assume that f is an injective function and that $f \circ h = f \circ h'$. Then, for all $x \in X$, $f(h(x)) = f(h'(x))$ and since f is injective, $h(x) = h'(x)$ for all $x \in X$. Thus, f is a monomorphism.

(d) We will again prove the contrapositive statement. Assume that f is not an injective function. Then, there exists $y_1, y_2 \in Y$ with $y_1 \neq y_2$ such that $f(y_1) = f(y_2)$. Construct the functions $h, h' : X \to Y$ such that for some $x \in X$, $h(x) = y_1$, $h'(x) = y_2$ and for all $z \in X - \{x\}$, $h(z) = h'(z)$. Then by construction, $h \neq h'$, but $f \circ h = f \circ h'$. Therefore, f coequalizes the pair of functions (h, h') with $h \neq h'$, unveiling that f is not a monomorphism. $\square$

It is worth noting that 1.1.1 extends to the categories **Grp** and **Ab** by utilising a similar proof technique.

Definition 1.1.9. Let $\mathscr{C}$ be a category, A, B be objects in $\mathscr{C}$ and $f \in Hom(A, B)$ be a morphism. We say that f is an **isomorphism** if there exists a morphism $f^{-1} \in Hom(B, A)$ such that $f \circ f^{-1} = id_B$ and $f^{-1} \circ f = id_A$, where id denotes the identity morphism.

Lemma 1.1.2. *Let $\mathscr{C}$ be a category, A, B be objects in $\mathscr{C}$ and $f : A \to B$ be an isomorphism. Then, f is a monomorphism and an epimorphism.*

Proof. Assume that $f : A \to B$ is a morphism in the category $\mathscr{C}$. Suppose that $h, h' : B \to C$ are morphisms such that $h \circ f = h' \circ f$. By precomposing with the inverse map f^{-1}, we find that $h \circ (f \circ f^{-1}) = h' \circ (f \circ f^{-1})$ and consequently, $h = h'$. This shows that f equalizes the pair (h, h) for all $h \in Hom(B, C)$. Hence, f is an epimorphism.

Now assume that $g, g' : Z \to A$ are morphisms such that $f \circ g = f \circ g'$. By composing with f^{-1}, we find that $g = g'$ and hence, f coequalizes the pair (g, g) for all $g \in Hom(Z, A)$. This demonstrates that f is a monomorphism. $\square$

Interestingly, the converse of 1.1.2 does not hold. We will give an example from the category of monoids **Mon**. First, we need to prove the following lemma:

Lemma 1.1.3. *Let $f : M \to N$ be a monoid morphism. Then, f is a monomorphism if and only if f is injective.*

Proof. Assume that $f : M \to N$ is a monoid morphism.

To show: (a) If f is injective, then f is a monomorphism.

(b) If f is a monomorphism, then f is injective.

(a) Assume that f is injective and that $g, g' : P \to M$ are monoid morphisms themselves. Then, for all $p \in P$, $f(g(p)) = f(g'(p))$ and by injectivity, $g(p) = g'(p)$. Thus, f coequalizes the pair (g, g) and is a monomorphism.

(b) The proof is by contrapositive. Assume that f is not injective. Then, there exists $m_1, m_2 \in M$ such that $m_1 \neq m_2$, but $f(m_1) = f(m_2)$. Fix $p \in P$ and define monoid morphisms $g, g' : P \to M$ such that $g(e_P) = g'(e_P) = e_M$ where e_P is the identity in P, $g(p) = m_1$, $g'(p) = m_2$ and $g(x) = g'(x)$ for all $x \in P - \{e_P, p\}$. Then, $g \neq g'$, but $f \circ g = f \circ g'$, meaning that f coequalizes the pair (g, g') with $g \neq g'$. This demonstrates that f is not a monomorphism. $\square$

Now we will give an example to show that the converse of 1.1.2 is false.

Example 1.1.10. We will work in the category of monoids **Mon**. Consider the inclusion map $\iota : \mathbb{Z}_{\geq 0} \to \mathbb{Z}$, where $\mathbb{Z}_{\geq 0}$ and $\mathbb{Z}$ are monoid with the binary operation of addition. It is easily verified that ι is a monoid morphism which is injective. By 1.1.3, ι must be a monomorphism.

To see that ι is an epimorphism, let $g, h : \mathbb{Z} \to M$ be monoid morphisms such that $g \circ \iota = h \circ \iota$. By definition of the inclusion ι, we have $g(m) = h(m)$ for all $m \in \mathbb{Z}_{\geq 0}$. Since $g(-m) = \sum_{i=1}^{m} g(-1)$ for all $m \in \mathbb{Z}_{\geq 0}$, it suffices to show that $g(-1) = h(-1)$. We compute directly that (e_M is the identity element of M)

$$\begin{aligned} g(-1) &= g(-1)e_M \\ &= g(-1)h(0) \\ &= g(-1)h(1)h(-1) \\ &= g(-1)g(1)h(-1) \\ &= g(0)h(-1) \\ &= e_M h(-1) = h(-1). \end{aligned}$$

Thus, $g = h$, revealing that ι is an epimorphism. Now observe that ι is not an isomorphism because ι is not surjective.

The above example also demonstrates that in a given category, an epimorphism is not always surjective. Similarly, a monomorphism is not always injective. Given below are some more properties satisfied by monomorphisms and epimorphisms in $\mathscr{C}$. The proofs of these relations are omitted because they are simple applications of the definitions.

1. If f and g are monomorphisms in $\mathscr{C}$, then $g \circ f$ is also a monomorphism.

2. If $g \circ f$ is a monomorphism in $\mathscr{C}$, then f is also a monomorphism.

3. If f and g are epimorphisms in $\mathscr{C}$, then $g \circ f$ is also an epimorphism.

4. If $g \circ f$ is an epimorphism in $\mathscr{C}$, then g is also an epimorphism.

1.2 Equalizers and coequalizers

The definition of an epimorphism and monomorphism that we gave is a special case of the concept of an equalizer and a coequalizer respectively. We will make the necessary definitions below.

Definition 1.2.1. Let $\mathscr{C}$ be a category and consider the following diagram in $\mathscr{C}$:

$$\begin{tikzcd} I \arrow[r, "i"] & X \arrow[r, shift left, "h"] \arrow[r, shift right, "h'"'] & Y \end{tikzcd}$$

We say that the morphism $i : I \to X$ is an **equalizer** of the pair (h, h') if it equalizes the pair (h, h') and satisfies the following universal property: if $g : U \to X$ is a morphism which equalizes the pair (h, h') then there exists a unique morphism $\gamma : U \to I$ such that the the triangle in the below diagram commutes:

$$\begin{tikzcd} U \arrow[d, dashed, "\gamma"] \arrow[dr, "g"] & & \\ I \arrow[r, "i"] & X \arrow[r, shift left, "h"] \arrow[r, shift right, "h'"'] & Y \end{tikzcd}$$

The equalizer i of (h, h') is often denoted by $eq(h, h')$.

By reversing the arrows in the diagrams in the above definition, we obtain the definition of a coequalizer.

Definition 1.2.2. Let $\mathscr{C}$ be a category and consider the following diagram in $\mathscr{C}$:

$$\begin{tikzcd} X \arrow[r, shift left, "h"] \arrow[r, shift right, "h'"'] & Y \arrow[r, "q"] & Q \end{tikzcd}$$

We say that the morphism $q : Y \to Q$ is a **coequalizer** of the pair (h, h') if it coequalizes the pair (h, h') and satisfies the following universal property: for all morphisms $f : Y \to Z$ which coequalize the pair (h, h'), there exists a unique morphism $\phi : Q \to Z$ such that the the triangle in the below diagram commutes:

$$\begin{tikzcd} & & Z \\ X \arrow[r, shift left, "h"] \arrow[r, shift right, "h'"'] & Y \arrow[ur, "f"] \arrow[r, "q"'] & Q \arrow[u, dashed, "\phi"] \end{tikzcd}$$

The coequalizer q of (h, h') is often denoted by $coeq(h, h')$.

Example 1.2.3. As an example of equalizers and coequalizers, let us give a description of equalizers and coequalizers in the category of sets **Set**.

Suppose that we have the following diagram in **Set**:

$$\begin{tikzcd} I \arrow[r, "\iota"] & X \arrow[r, shift left, "h"] \arrow[r, shift right, "h'"'] & Y \end{tikzcd}$$

where I is the set

$$I = \{x \in X \mid h(x) = h'(x)\}$$

and $\iota : I \to X$ is the inclusion function. We will show that ι is the equalizer of the pair (h, h'). Suppose that $g : U \to X$ is a function which equalizes the pair (h, h'). The point here is that the image of g must be contained in I by the definition of the equalizer. Thus, we can define the unique map $g : U \to I$, which is just g, but with codomain restricted to I. Then, it is easy to check that the triangle in the following diagram commutes:

$$\begin{tikzcd} U \arrow[d, dashed, "g"'] \arrow[dr, "g"] & & \\ I \arrow[r, "\iota"] & X \arrow[r, shift left, "h"] \arrow[r, shift right, "h'"'] & Y \end{tikzcd}$$

Furthermore, ι equalizes the pair (h, h') because for all $i \in I$,

$$(h \circ \iota)(i) = h(i) = h'(i) = (h' \circ \iota)(i).$$

This is just from the definition of I. Therefore, ι is the equalizer of the pair of functions (h, h').

Now, we will find the coequalizer of the pair (h, h'). Let $\sim$ denote the smallest equivalence relation defined by setting $h(x) \sim h'(x)$ for all $x \in X$. This is an equivalence relation on Y and thus, we can define the quotient set $Y/\sim$, which is equipped with the usual projection map $\pi : Y \to Y/\sim$. We claim that π coequalizes the pair (h, h').

To see why this is the case, observe that for all $x \in X$,

$$(\pi \circ h)(x) = [h(x)] = [h'(x)] = (\pi \circ h')(x)$$

where $[h(x)] \in Y/\sim$ denotes the equivalence class with representative $h(x)$. So, π coequalizes (h, h'). Moreover, suppose that $g : Y \to Z$ is a morphism in **Set** which coequalizes (h, h') so that $g \circ h = g \circ h'$. Then, we define the function

$$\begin{array}{rccc} \phi : & Y/\sim & \to & Z \\ & [y] & \mapsto & g(y) \end{array}$$

To see why ϕ is well-defined, suppose that y_1 and y_2 are two representatives of the same equivalence class in $Y/\sim$. Then, since $g \circ h = g \circ h'$, $g(y_1) = g(y_2)$ by the definition of $\sim$. This ensures that ϕ is well-defined. Again, one can quickly check that ϕ makes the triangle in the below diagram commute:

$$\begin{tikzcd} & & Z \\ X \arrow[r, shift left, "h"] \arrow[r, shift right, "h'"'] & Y \arrow[ur, "g"] \arrow[r, "\pi"'] & Y/\sim \arrow[u, dashed, "\phi"] \end{tikzcd}$$

Also, ϕ is unique because it is entirely determined by the function g. Therefore, $\pi : Y \to Y/\sim$ is the coequalizer of the pair (h, h').

There are a multitude of examples of equalizers and coequalizers given in the reference [Lei16]. We will list a few of them below:

Example 1.2.4. Let $h, h' : X \to Y$ be morphisms in the category of topological spaces **Top**. Then, the equalizer of (h, h') is the inclusion map $\iota : I \to X$ where I is the topological space

$$I = \{x \in X \mid h(x) = h'(x)\}$$

equipped with the subspace topology from X. Similarly, $\pi : Y \to Y/\sim$ is the coequalizer of (h, h') where for all $x \in X$, $h(x) \sim h'(x)$ and $Y/\sim$ is equipped with the quotient topology.

Example 1.2.5. Let **k-Vect** be the category of k-vector spaces, where k is a field. Let $s, t : V \to W$ be linear transformations between the vector spaces V and W. Then, the equalizer of (s, t) is the inclusion map $\iota : \ker(t - s) \to V$.

Example 1.2.6. Let $u, v : G \to H$ be morphisms in the category of abelian groups **Ab**. The coequalizer of (u, v) is the group morphism $\psi : H \to H/\mathrm{im}(v - u)$. Notice that the image $\mathrm{im}(v - u)$ is just the cokernel $\mathrm{coker}(v - u)$.

Let us now prove some general properties satisfied by equalizers and coequalizers.

Theorem 1.2.1. *Let $\mathscr{C}$ be a category, A, B be objects in $\mathscr{C}$ and $h, h' \in Hom(A, B)$ be morphisms in $\mathscr{C}$. Then, any equalizer or coequalizer of (h, h') is unique up to isomorphism. Furthermore, any equalizer of (h, h') is a monomorphism and any coequalizer of (h, h') is a epimorphism.*

Proof. Assume that $\mathscr{C}$ is a category, A, B are objects in $\mathscr{C}$ and $h, h' \in Hom(A, B)$ be morphisms in $\mathscr{C}$.

To show: (a) Equalizers of (h, h') are unique up to isomorphism.

(b) Coequalizers of (h, h') are unique up to isomorphism.

(c) If $f : Z \to A$ is an equalizer of (h, h'), then f is a monomorphism.

(d) If $g : B \to C$ is a coequalizer of (h, h'), then g is an epimorphism.

(a) Suppose that Z is an object in $\mathscr{C}$ and $f_1, f_2 : Z \to A$ are equalizers of (h, h'). By the universal property of the equalizer, there exists a morphism $\rho : Z \to Z$ such that the triangle in the following diagram commutes:

$$\begin{tikzcd}
Z \arrow[d, dashed, "\rho"'] \arrow[dr, "f_2"] & & \\
Z \arrow[r, "f_1"] & A \arrow[r, shift left, "h"] \arrow[r, shift right, "h'"'] & B
\end{tikzcd}$$

But, another application of the universal property of the equalizer reveals the existence of another morphism $\phi : Z \to Z$ such that the triangle in the following diagram commutes:

$$\begin{tikzcd}
Z \arrow[d, dashed, "\phi"'] \arrow[dr, "f_1"] & & \\
Z \arrow[r, "f_2"] & A \arrow[r, shift left, "h"] \arrow[r, shift right, "h'"'] & B
\end{tikzcd}$$

Now, observe that from both diagrams, we have

$$\begin{aligned}
f_2 &= \rho \circ f_1 \\
&= \rho \circ (\phi \circ f_2) = (\rho \circ \phi) \circ f_2
\end{aligned}$$

and

$$
\begin{aligned}
f_1 &= \phi \circ f_2 \\
&= \phi \circ (\rho \circ f_1) = (\phi \circ \rho) \circ f_2.
\end{aligned}
$$

Thus, $\phi \circ \rho = \rho \circ \phi = id_Z$, which demonstrates that ρ and ϕ are both isomorphisms. Since $f_1 = \phi \circ f_2$, we deduce that f_1 and f_2 are in fact, equal up to isomorphism.

(b) Assume that C is an object in $\mathscr{C}$ and $g_1, g_2 : B \to C$ are coequalizers of (h, h'). Similarly to part (a), we apply the universal property of the coequalizer to obtain the morphisms $\phi, \rho : C \to C$, which make the triangles in the following diagram commute:

$$
\begin{tikzcd}
 & & C \\
A \arrow[r, shift left, "h"] \arrow[r, shift right, "h'"'] & B \arrow[ur, "g_2"] \arrow[r, "g_1"'] & C \arrow[u, dashed, "\phi"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & C \\
A \arrow[r, shift left, "h"] \arrow[r, shift right, "h'"'] & B \arrow[ur, "g_1"] \arrow[r, "g_2"'] & C \arrow[u, dashed, "\rho"]
\end{tikzcd}
$$

Arguing in exactly the same manner as the previous part, we find that ϕ and ρ are inverses of each other and thus, isomorphisms. Hence, g_1 and g_2 are equal up to isomorphism.

(c) Assume that $f : Z \to A$ is an equalizer of (h, h'). Assume that $x, y : E \to Z$ are morphisms such that $f \circ x = f \circ y$. This gives us the following commutative diagram:

$$
\begin{tikzcd}
E \arrow[d, shift right, "y"'] \arrow[d, shift left, "x"] \arrow[dr, "f \circ x"] & & \\
Z \arrow[r, "f"] & A \arrow[r, shift left, "h"] \arrow[r, shift right, "h'"'] & B
\end{tikzcd}
$$

The universal property of the equalizer now tells us that there exists a unique $u : E \to Z$ such that $f \circ x = f \circ u$. Therefore, $u = x = y$ and as a result, f must be a monomorphism.

(d) Assume that $g : B \to C$ is an equalizer of (h, h'). Assume that $p, q : C \to D$ are morphisms such that $p \circ g = q \circ g$. Then, the following diagram must commute:

$$\begin{tikzcd}
 & & D \\
A \arrow[r, shift left, "h"] \arrow[r, shift right, "h'"'] & B \arrow[r, "g"'] \arrow[ur, "p\circ g"] & C \arrow[u, shift right, "p"] \arrow[u, shift left, "q"']
\end{tikzcd}$$

Since g is a coequalizer of (h, h'), we can use the universal property of the coequalizer to deduce the existence of a morphism $v : C \to D$ such that $v \circ g = p \circ g$. Due to uniqueness, $v = p = q$ and consequently, g must be an epimorphism. $\square$

Next, we will define a special kind of epimorphism called a *split epimorphism*. [Bou17] explains that split epimorphisms are behind many "strong classification processes in algebra".

Definition 1.2.7. Let $\mathscr{C}$ be a category and $f : X \to Y$ be a morphism in $\mathscr{C}$. The morphism f is said to be a **split epimorphism** if there exists another morphism $s : Y \to X$ such that $f \circ s = id_Y$.

Let us prove some of the defining properties of split epimorphisms.

Theorem 1.2.2. *Let $\mathscr{C}$ be a category and $f : X \to Y$ be a split epimorphism so that there exists a morphism $s : Y \to X$ such that $f \circ s = id_Y$. Then, the composite $\theta = s \circ f$ is idempotent ($\theta^2 = \theta$). Furthermore, $s = eq(id_X, \theta)$ and $f = coeq(id_X, \theta)$.*

Proof. Assume that $\mathscr{C}$ is a category and $f : X \to Y$ be a split epimorphism. Then, there exists a morphism $s : Y \to X$ such that $f \circ s = id_Y$. Let $\theta = s \circ f$. Then,

$$\theta^2 = \theta \circ \theta = s \circ (f \circ s) \circ f = s \circ f = \theta.$$

Thus, θ is idempotent.

To show: (a) $s = eq(id_X, \theta)$.

(b) $f = coeq(id_X, \theta)$.

(a) To see that $s : Y \to X$ equalizes the pair (id_X, θ), we compute directly that

$$\theta \circ s = (s \circ f) \circ s = s \circ (f \circ s) = s = id_X \circ s.$$

Now suppose that $g : U \to X$ is a morphism in $\mathscr{C}$ which equalizes the pair (id_X, θ). Then, $g = \theta \circ g$. Now consider the composite $f \circ g : U \to Y$. It is unique because it is the composite of two unique morphisms. Additionally, we note that

$$s \circ (f \circ g) = \theta \circ g = g.$$

Thus, s is the equalizer of the pair (id_X, θ).

(b) To see that $f : X \to Y$ coequalizes the pair (id_X, θ), we observe that

$$f \circ \theta = (f \circ s) \circ f = id_Y \circ f = f \circ id_X.$$

Thus, f coequalizes (id_X, θ). Now suppose that $h : X \to W$ is a morphism such that h coequalizes (id_X, θ). Then, $h \circ \theta = h$. Consider the composite $h \circ s : Y \to W$. This is a unique morphism which satisfies

$$(h \circ s) \circ f = h \circ (s \circ f) = h \circ \theta = h.$$

Therefore, f is the coequalizer of (id_X, θ) as required. $\square$

As a result of 1.2.1, s must be a monomorphism and f must be an epimorphism. An isomorphism is a special case of a split epimorphism. If $f : X \to Y$ is an isomorphism in $\mathscr{C}$, then, it is a split epimorphism because $f \circ f^{-1} = id_Y$.

The next result can be thought of as a converse of 1.2.2. It shows that we can always construct a unique split epimorphism in a particular scenario.

Theorem 1.2.3. *Let $\mathscr{C}$ be a category and $\theta : X \to X$ be an idempotent morphism in $\mathscr{C}$. Let $s = eq(id_X, \theta)$ be a morphism from Y to X. Then, there exists a unique morphism $f : X \to Y$ such that $f \circ s = id_Y$ and $s \circ f = \theta$.*

Dually, if $f = coeq(id_X, \theta)$ is a morphism from X to Y, then there exists a unique morphism $s : Y \to X$ such that $f \circ s = id_Y$ and $s \circ f = \theta$.

Proof. Assume that $\mathscr{C}$ is a category and $\theta : X \to X$ is an idempotent morphism in $\mathscr{C}$. Let $s = eq(id_X, \theta)$, where s is a morphism from Y to X. By exploiting the universal property of the equalizer, there exists a unique morphism $f : X \to Y$ such that the triangle in the following diagram commutes:

$$\begin{tikzcd}
X \arrow[d, dashed, "f"'] \arrow[dr, "\theta"] & & \\
Y \arrow[r, "s"] & X \arrow[r, shift left, "id_X"] \arrow[r, shift right, "\theta"'] & X
\end{tikzcd}$$

Thus, $s \circ f = \theta$. To see that $f \circ s = id_Y$, we can use the fact that θ is idempotent, in tandem with $s \circ f = \theta$, to reveal that

$$s \circ (f \circ s) \circ f = s \circ f.$$

Since s is an equalizer, it is a monomorphism (see 1.2.1). Therefore, $(f \circ s) \circ f = f$ and consequently, $f \circ s = id_Y$ as required.

Now assume that $f = coeq(id_X, \theta)$, where f is a morphism from X to Y. From the universal property of the coequalizer, there exists a unique morphism $s : Y \to X$ such that the triangle in the below diagram commutes:

$$\begin{tikzcd}
& & X \\
X \arrow[r, shift left, "id_X"] \arrow[r, shift right, "\theta"'] & X \arrow[r, "f"'] \arrow[ur, "\theta"] & Y \arrow[u, dashed, "s"']
\end{tikzcd}$$

That is, $s \circ f = \theta$. Using the fact that θ is idempotent, we once again obtain $s \circ (f \circ s) \circ f = s \circ f$. From 1.2.1, f is an epimorphism because it is a coequalizer. Therefore, $s \circ (f \circ s) = s$ and so, $f \circ s = id_Y$ as required. $\square$

We observe that for a given category $\mathscr{C}$, the split epimorphisms (f, s), where s is the morphism whose existence is determined from f being a split epimorphism, form a category themselves. We will call this category $Pt(\mathscr{C})$. The morphisms in this category are given by pairs of morphisms (x, y) which form commutative squares of the form

$$\begin{tikzcd}
X \arrow[r, "x"] \arrow[d, shift right, "f"'] & X' \arrow[d, shift right, "f'"'] \\
Y \arrow[r, "y"] \arrow[u, shift right, "s"'] & Y' \arrow[u, shift right, "s'"']
\end{tikzcd}$$

The above commutative diagram ensures that split epimorphisms are mapped to split epimorphisms. Now suppose that $f \in Hom_{\mathscr{C}}(X, Y)$ and $s \in Hom_{\mathscr{C}}(Y, X)$. Finally, the functor $\P_{\mathscr{C}} : Pt(\mathscr{C}) \to \mathscr{C}$ sends a split epimorphism given by the pair (f, s) to Y which is the codomain of f. It is an exercise in the definitions to prove that $Pt(\mathscr{C})$ is a category and $\P_{\mathscr{C}}$ is a functor.

1.3 Reflexive relations

One of the most important constructions in the categories we are familiar with, such as **Set** and **Top**, is the quotient. The quotient is reliant on the notion of an equivalence relation. In this section, we will discuss the first step towards generalising the notion of an equivalence relation to category theory — the concept of reflexive graphs. As usual, we will begin with preliminary definitions:

Definition 1.3.1. Let $\mathscr{C}$ be a category and X be an object of $\mathscr{C}$. A **graph on the object** X is a pair of morphisms $d_0, d_1 \in Hom(G, X)$.

The graph (d_0, d_1) is said to be **reflexive** when d_0 and d_1 admit a common section. That is, there exists a morphism $s_0 \in Hom(X, G)$ such that $d_0 \circ s_0 = d_1 \circ s_0 = id_X$. In an abuse of notation, we often denote a reflexive graph by G, rather than specifying the pair of morphisms.

Another way of interpreting the idea of a reflexive graph on X is that the diagonal map $\Delta = (id_X, id_X) \in Hom(X, X \times X)$ factors through the morphism $(d_0, d_1) : G \to X \times X$. That is, there exists $s_0 \in Hom(X, G)$ such that the following diagram commutes:

$$\begin{tikzcd} & G \arrow[d, "{(d_0,d_1)}"] \\ X \arrow[ur, dashed, "s_0"] \arrow[r, "\Delta"'] & X \times X \end{tikzcd}$$

Definition 1.3.2. Let $\mathscr{C}$ be a category, X be an object in $\mathscr{C}$ and (d_0, d_1) be a reflexive graph on X, where $d_0, d_1 \in Hom(G, X)$. We say that (d_0, d_1) is a **reflexive relation** if d_0 and d_1 are both monomorphisms. Alternatively, the induced morphism $(d_0, d_1) : G \to X \times X$ is a monomorphism.

Why is this definition consistent with our usual concept of a reflexive relation on a set? Recall that a relation on a set X is just a subset of $X \times X$. More specifically, a relation on a set X is the subset

$$\Phi = \{(\phi_1(g), \phi_2(g)) \mid g \in G\} \subseteq X \times X$$

where $\phi_1, \phi_2 : G \to X$ are monomorphisms in **Set**. For a relation on X to be reflexive, the diagonal of X, which is defined by

$$\mathcal{D} = \{(x, x) \mid x \in X\}$$

must be a subset of Φ. This holds precisely when ϕ_1 and ϕ_2 have a common section $s_0 : X \to G$, a function which satisfies $\phi_1 \circ s_0 = \phi_2 \circ s_0 = id_X$. Moreover, we want (ϕ_1, ϕ_2) to be a monomorphism (or injective since we are dealing with sets) because we do not want an element in our relation to appear more than once.

Here are some well-known examples of reflexive graphs.

Example 1.3.3. Let $\mathscr{C}$ be a category and X be an object. Then, the **discrete reflexive relation** on X is given by the identity map id_X, represented by the following diagram:

$$\begin{tikzcd} X \arrow[r, shift left=2, "id_X"] \arrow[r, shift right=2, "id_X"'] & X \arrow[l, "id_X"'] \end{tikzcd}$$

The reflexive relation is given by the pair (id_X, id_X) and the appropriate section for both morphisms is in this case id_X. Since id_X is an isomorphism, it must also be a monomorphism (see 1.1.2). Hence, (id_X, id_X) is a reflexive relation on the object X. We commonly denote it by Δ_X.

Example 1.3.4. Let $\mathscr{C}$ be a category and X be an object. Then, the **indiscrete reflexive relation** on X is given by the following diagram:

$$\begin{tikzcd} X \times X \arrow[r, shift left=2, "\pi_2"] \arrow[r, shift right=2, "\pi_1"'] & X \arrow[l, "\Delta"'] \end{tikzcd}$$

Here, π_1 and π_2 are the canonical projection maps onto the first and second factors respectively and $\Delta = (id_X, id_X)$ is the diagonal map. This is a reflexive relation on X because the induced map $(\pi_1, \pi_2) : X \times X \to X \times X$ is just the identity morphism on $X \times X$, which is an isomorphism and hence, a monomorphism. The indiscrete reflexive relation is usually denoted by ∇_X.

Definition 1.3.5. Let $\mathscr{C}$ be a category and (d_0, d_1) be a reflexive graph on an object X, where $d_0, d_1 \in Hom(G, X)$. The **dual** of (d_0, d_1), often denoted by G^{op}, is the reflexive graph (d_1, d_0) on X.

For a given category $\mathscr{C}$, we will denote by $RGr(\mathscr{C})$ the category whose objects are reflexive graphs and whose morphisms are pairs of morphisms (α, β) which make the following diagram commute:

$$\begin{tikzcd}
G \arrow[r, shift left=2, "d_1^G"] \arrow[r, shift right=2, "d_0^G"'] \arrow[d, "\alpha"] & X \arrow[l] \arrow[d, "\beta"] \\
G' \arrow[r, shift left=2, "d_1^{G'}"] \arrow[r, shift right=2, "d_0^{G'}"'] & X' \arrow[l]
\end{tikzcd}$$

We will use $U_0 : RGr(\mathscr{C}) \to \mathscr{C}$ to denote a functor which maps a reflexive graph to its underlying object X. Finally, we denote by $Ref(\mathscr{C})$ the subcategory of $RGr(\mathscr{C})$ whose objects are the reflexive relations. As a demonstration of this category, we will prove the following lemma:

Lemma 1.3.1. *Let $\mathscr{C}$ be a category, X be an object and R be a reflexive graph on X. Then, there exists unique morphisms $d_R : R \to \nabla_X$ and $s_0^R : \Delta_X \to R$ in the category $RGr(\mathscr{C})$.*

Proof. Assume that X is an object in the category $\mathscr{C}$ and that R is a reflexive graph on X, depicted by the diagram below:

$$\begin{tikzcd}
R \arrow[r, shift left=3, "a_1"] \arrow[r, shift right=3, "a_0"'] & X \arrow[l, "r_0"']
\end{tikzcd}$$

We define $d_R : R \to \nabla_X$ to be the pair $((a_0, a_1), id_X)$, where id_X is the identity map on X and $(a_0, a_1) \in Hom(R, X \times X)$ is induced from the morphisms a_0 and a_1. To see that d_R is a morphism in the category $RGr(\mathscr{C})$, note that for all $r \in R$,

$$\pi_1 \circ (a_0, a_1)(r) = \pi_1(a_0(r), a_1(r)) = a_0(r) = (id_X \circ a_0)(r)$$

$$\pi_2 \circ (a_0, a_1)(r) = \pi_2(a_0(r), a_1(r)) = a_1(r) = (id_X \circ a_1)(r)$$

and for all $x \in X$,

$$\begin{aligned}
(a_0, a_1) \circ r_0(x) &= (a_0, a_1)(r_0(x)) \\
&= ((a_0 \circ r_0)(x), (a_1 \circ r_0)(x)) \\
&= (id_X(x), id_X(x)) \\
&= (x, x) = (\Delta \circ id_X)(x).
\end{aligned}$$

Indeed, the following diagram commutes:

$$
\begin{tikzcd}
R \arrow[r, shift left=2, "a_1"] \arrow[r, shift right=2, "a_0"'] \arrow[d, "{(a_0, a_1)}"'] & X \arrow[l, "r_0"] \arrow[d, "id_X"] \\
X \times X \arrow[r, shift left=2, "\pi_2"] \arrow[r, shift right=2, "\pi_1"'] & X \arrow[l, "\Delta"]
\end{tikzcd}
$$

This shows that $d_R : R \to \nabla_X$ is indeed a unique morphism in $RGr(\mathscr{C})$.

Next, we define the map $s_0^R : \Delta_X \to R$ by the pair of morphisms (r_0, id_X) (morphisms in $\mathscr{C}$). It is straightforward to check that this pair makes the following diagram commute and is thus, a unique morphism in $RGr(\mathscr{C})$:

$$
\begin{tikzcd}
X \arrow[r, shift left=2, "id_X"] \arrow[r, shift right=2, "id_X"'] \arrow[d, "r_0"'] & X \arrow[l, "id_X"] \arrow[d, "id_X"] \\
R \arrow[r, shift left=2, "a_1"] \arrow[r, shift right=2, "a_0"'] & X \arrow[l, "r_0"]
\end{tikzcd}
$$

$\square$

1.4 Pullbacks and pushouts

In this section, we introduce our second fundamental example of a limit/colimit in category theory — pullbacks and pushouts respectively. In particular, pullbacks appear in a lot of fields of mathematics, mainly under the guise of "precomposing" with a particular map. We will take the definition of pullbacks and pushouts from [Lei16, Section 5.1].

Definition 1.4.1. Let $\mathscr{C}$ be a category. Suppose that we have the following diagram in $\mathscr{C}$:

$$
\begin{tikzcd}
 & Y \arrow[d, "t"] \\
X \arrow[r, "s"] & Z
\end{tikzcd}
$$

A **pullback** of the above diagram is an object P of $\mathscr{C}$, together with morphisms $p_1 : P \to X$ and $p_2 : P \to Y$ such that the square below commutes.

$$\begin{tikzcd}
P \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d, "t"] \\
X \arrow[r, "s"] & Z
\end{tikzcd}$$

Furthermore, a pullback must satisfy the following universal property. For any commutative square of the below form in $\mathscr{C}$,

$$\begin{tikzcd}
A \arrow[r, "f_2"] \arrow[d, "f_1"] & Y \arrow[d, "t"] \\
X \arrow[r, "s"] & Z
\end{tikzcd}$$

there exists a unique morphism $f' : A \to P$ such that the two triangles in the below diagram commute:

$$\begin{tikzcd}
A \arrow[dr, dashed, "f'"] \arrow[drr, bend left, "f_2"] \arrow[ddr, bend right, "f_1"'] & & \\
& P \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d, "t"] \\
& X \arrow[r, "s"] & Z
\end{tikzcd}$$

Similarly to the relationship of the coequalizer to an equalizer, the definition of a pushout can be obtained from that of a pullback, by reversing the arrows.

Definition 1.4.2. Let $\mathscr{C}$ be a category. Suppose that we have the following diagram in $\mathscr{C}$:

$$\begin{tikzcd}
& Y \\
X & Z \arrow[l, "v"] \arrow[u, "u"']
\end{tikzcd}$$

A **pushout** of the above diagram is an object P of $\mathscr{C}$, together with morphisms $p_1 : X \to P$ and $p_2 : Y \to P$ such that the square below commutes.

$$\begin{tikzcd}
P & Y \arrow[l, "p_2"] \\
X \arrow[u, "p_1"] & Z \arrow[l, "v"] \arrow[u, "u"']
\end{tikzcd}$$

Furthermore, a pushout must satisfy the following universal property. For any commutative square of the below form in $\mathscr{C}$,

$$\begin{tikzcd} B & Y \arrow[l, "g_2"'] \\ X \arrow[u, "g_1"] & Z \arrow[l, "v"] \arrow[u, "u"'] \end{tikzcd}$$

there exists a unique morphism $g' : P \to B$ such that the two triangles in the below diagram commute:

$$\begin{tikzcd} B & & \\ & P \arrow[ul, dashed, "g'"'] & Y \arrow[l, "p_2"'] \arrow[ull, bend right, "g_2"'] \\ & X \arrow[u, "p_1"] \arrow[uul, bend left, "g_1"] & Z \arrow[l, "v"] \arrow[u, "u"'] \end{tikzcd}$$

We will give one example each of a pullback and a pushout.

Example 1.4.3. Let us work in the category of sets **Set**. Suppose that we have the following diagram in **Set**:

$$\begin{tikzcd} & Y \arrow[d, "t"] \\ X \arrow[r, "s"] & Z \end{tikzcd}$$

Let P be a subset of $X \times Y$ defined by

$$P = \{(x, y) \in X \times Y \mid s(x) = t(y)\}.$$

Let $\pi_X : P \to X$ and $\pi_Y : P \to Y$ be the usual projection maps. By the definition of P, it is easy to verify that the following square commutes:

$$\begin{tikzcd} P \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"] & Y \arrow[d, "t"] \\ X \arrow[r, "s"] & Z \end{tikzcd}$$

To see the universal property of a pullback is satisfied, suppose that we have the following commutative square:

$$\begin{tikzcd}
Q \arrow[r, "f_2"] \arrow[d, "f_1"] & Y \arrow[d, "t"] \\
X \arrow[r, "s"] & Z
\end{tikzcd}$$

We define the function $\alpha : Q \to P$ by

$$\alpha : Q \to P$$

$$q \mapsto (f_1(q), f_2(q)).$$

This is a well-defined unique map to P because $s \circ f_1 = t \circ f_2$. It remains to show that $f_1 = \pi_X \circ \alpha$ and $f_2 = \pi_Y \circ \alpha$. But this follows directly from the definition of α. Thus, the triple (P, π_X, π_Y) is a pullback in **Set**.

Example 1.4.4. For our example of a pushout, we will work in the category of topological spaces **Top**. Again, suppose we have the following diagram in **Top**:

$$\begin{tikzcd}
 & Y \\
X & Z \arrow[l, "v"] \arrow[u, "u"]
\end{tikzcd}$$

Consider the disjoint union $X \sqcup Y$ of the topological spaces X and Y. We define an equivalence relation on $X \sqcup Y$ by saying that if $x \in X$ and $y \in Y$, $x \sim y$ if and only if there exists a $z \in Z$ such that $v(z) = x$ and $u(z) = y$. In other words, $\sim$ is the smallest equivalence relation generated by pairs of the form $(v(z), u(z))$ for all $z \in Z$.

Next, we define $X \sqcup_Z Y$ to be the quotient topological space $(X \sqcup Y)/\sim$. There are continuous functions $\iota_X : X \to X \sqcup_Z Y$ and $\iota_Y : Y \to X \sqcup_Z Y$, defined by $\iota_X(x) = [x]$ and $\iota_Y(y) = [y]$. From the definition of $X \sqcup_Z Y$, it is straightforward to verify that the following diagram commutes:

$$\begin{tikzcd}
X \sqcup_Z Y & Y \arrow[l, "\iota_Y"] \\
X \arrow[u, "\iota_X"] & Z \arrow[l, "v"] \arrow[u, "u"]
\end{tikzcd}$$

To see that the universal property of the pushout is satisfied, suppose that we have the following commutative square:

$$\begin{array}{ccc} W & \xleftarrow{g_2} & Y \\ {\scriptstyle g_1}\uparrow & & \uparrow{\scriptstyle u} \\ X & \xleftarrow[v]{} & Z \end{array}$$

Define the map $\beta : X \sqcup_Z Y \to W$ by

$$\beta : X \sqcup_Z Y \to W$$

$$[x] \mapsto g_1(x)$$

$$[y] \mapsto g_2(y)$$

How do we know that β is a well-defined continuous map? In order to obtain the definition of β, we can exploit the universal property of the quotient in **Top**, which suggests that it is enough to construct a continuous map $\beta' : X \sqcup Y \to W$ such that $\beta'(u(z)) = \beta'(v(z))$ for all $z \in Z$. Fortunately, we have the homeomorphism

$$Cts(X \sqcup Y, W) \cong Cts(X, W) \times Cts(Y, W).$$

On the RHS, we have $g_1 \in Cts(X, W)$ and $g_2 \in Cts(Y, W)$. By the homeomorphism, the pair (g_1, g_2) induces the continuous function $\beta' : X \sqcup Y \to W$ which has the desired property because $g_1 \circ v = g_2 \circ u$. Thus, $\beta : X \sqcup_Z Y \to W$ is a well-defined continuous function. For all $x \in X$ and $y \in Y$,

$$\beta(\iota_X(x)) = \beta([x]) = g_1(x) \text{ and } \beta(\iota_Y(y)) = \beta([y]) = g_2(y).$$

It remains to show uniqueness. Suppose that $\beta^* : X \sqcup_Z Y \to W$ is another map which satisfies $\beta^* \circ \iota_X = g_1$ and $\beta^* \circ \iota_Y = g_2$. Then, a direct computation yields for all $x \in X$ and $y \in Y$,

$$\beta([x]) = \beta^*([x]) \text{ and } \beta([y]) = \beta^*([y]).$$

Since every element of $X \sqcup_Z Y$ is either $[y]$ for some $y \in Y$ or $[z]$ for some $z \in Z$, the above equalities reveal that $\beta = \beta^*$. Hence, we have uniqueness and therefore, the triple $(X \sqcup_Z Y, \iota_X, \iota_Y)$ is a pushout in **Top**.

It turns out that pushouts in **Top** play an important role in the construction of *finite CW-complexes*, which are the fundamental objects of study in algebraic topology. See [Mur21] for a brief discussion of this point.

The above example of a pushout in **Top** is also directly from [Mur21].

Pullbacks and pushouts indeed exist in the categories **Mon**, **CoM**, **Grp** and **Ab**. We will now turn our attention to proving various properties of pullbacks.

Lemma 1.4.1. *Let $\mathscr{C}$ be a category and $f : Y \to Z$ be a morphism in $\mathscr{C}$. Suppose that we have the following pullback square in $\mathscr{C}$:*

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d, "f"] \\ X \arrow[r, "g"] & Z \end{tikzcd}$$

If f is a monomorphism, then p_1 is also a monomorphism. Similar statements hold when f is an isomorphism or a split epimorphism. We say that monomorphisms, isomorphisms and split epimorphisms are stable under pullbacks.

Proof. Assume that $\mathscr{C}$ is a category and $f : Y \to Z$ is a morphism in $\mathscr{C}$. Consider the above pullback square involving f. Now suppose that f is a monomorphism and assume that $\alpha, \beta \in Hom(Q, P)$.

To show: (a) If $p_1 \circ \alpha = p_1 \circ \beta$, then $\alpha = \beta$.

(a) Assume that $p_1 \circ \alpha = p_1 \circ \beta$. By exploiting the commutativity of the pullback square, we find that upon composing both sides with g,

$$g \circ (p_1 \circ \alpha) = f \circ (p_2 \circ \alpha) = g \circ (p_1 \circ \beta) = f \circ (p_2 \circ \beta)$$

as morphisms from Q to Z. Since f is a monomorphism, we have $p_2 \circ \alpha = p_2 \circ \beta$. Next, we observe that α makes the two triangles in the below diagram commute:

$$\begin{tikzcd} Q \arrow[dr, "\alpha"] \arrow[drr, bend left, "p_2 \circ \alpha"] \arrow[ddr, bend right, "p_1 \circ \alpha"'] & & \\ & P \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d, "f"] \\ & X \arrow[r, "g"] & Z \end{tikzcd}$$

Since $p_1 \circ \alpha = p_1 \circ \beta$ by assumption and $p_2 \circ \alpha = p_2 \circ \beta$ as demonstrated previously, β must also make the two triangles commute, as depicted by the diagram

$$\begin{tikzcd}
Q \arrow[dr, "\beta"] \arrow[drr, bend left, "p_2 \circ \alpha"] \arrow[ddr, bend right, "p_1 \circ \alpha"'] & & \\
& P \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d, "f"] \\
& X \arrow[r, "g"] & Z
\end{tikzcd}$$

By the universal property of the pullback, the morphism which makes the two triangles commute must be unique. Therefore, $\alpha = \beta$.

Part (a) proves that p_1 is a monomorphism. Hence, monomorphisms are stable under pullbacks.

Now suppose that f is an isomorphism. Construct the following commutative square:

$$\begin{tikzcd}
X \arrow[r, "f^{-1} \circ g"] \arrow[d, "id_X"] & Y \arrow[d, "f"] \\
X \arrow[r, "g"] & Z
\end{tikzcd}$$

The universal property of the pullback tells us that there exists a unique morphism $h : X \to P$ such that the two triangles in the below diagram commute:

$$\begin{tikzcd}
X \arrow[dr, dashed, "h"] \arrow[drr, bend left, "f^{-1} \circ g"] \arrow[ddr, bend right, "id_X"'] & & \\
& P \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d, "f"] \\
& X \arrow[r, "g"] & Z
\end{tikzcd}$$

So, we have $p_1 \circ h = id_X$.

To show: (b) $h \circ p_1 = id_P$.

(b) Observe that

$$p_1 \circ (h \circ p_1) = (p_1 \circ h) \circ p_1 = id_X \circ p_1 = p_1 \circ id_P.$$

Since f is an isomorphism, it must be a monomorphism from 1.1.2. Therefore, $h \circ p_1 = id_P$ as required.

Part (b), in conjunction with the fact that $p_1 \circ h = id_X$, demonstrates that p_1 is an isomorphism in $\mathscr{C}$. Therefore, isomorphisms are stable under pullbacks.

Finally, suppose that f is a split epimorphism. Then, there exists $s : Z \to Y$ such that $f \circ s = id_Z$. Hence, we have the following commutative square:

$$\begin{tikzcd} X \arrow[r, "s \circ g"] \arrow[d, "id_X"] & Y \arrow[d, "f"] \\ X \arrow[r, "g"] & Z \end{tikzcd}$$

and by the universal property, there exists a unique morphism $t : X \to P$, which makes the two triangles in the following diagram commute:

$$\begin{tikzcd} X \arrow[dr, dashed, "t"] \arrow[drr, bend left, "s \circ g"] \arrow[ddr, bend right, "id_X"'] & & \\ & P \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d, "f"] \\ & X \arrow[r, "g"] & Z \end{tikzcd}$$

In particular, $p_1 \circ t = id_X$. Hence, p_1 is a split epimorphism. This reveals that split epimorphisms are stable under pullbacks. □

Let us look at a special case of the pullback — the pullback of the pair of morphisms (f, f), where $f \in Hom(X, Y)$. We will denote this pullback in a category $\mathscr{C}$ in the following specific manner:

$$\begin{tikzcd} R[f] \arrow[r, "p_1^f"] \arrow[d, "p_0^f"] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

We call the pair of morphisms (p_0^f, p_1^f) the **kernel pair** of f. By considering the commutative square

$$\begin{tikzcd} X \arrow[r, "id_X"] \arrow[d, "id_X"] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

we can use the universal property of the pullback to deduce the existence of a unique morphism $s_0^f : X \to R[f]$ such that the two triangles in the below diagram commute:

$$\begin{tikzcd}
X \arrow[dr, dashed, "s_0^f"] \arrow[drr, bend left, "id_X"] \arrow[ddr, bend right, "id_X"'] & & \\
& R[f] \arrow[r, "p_1^f"] \arrow[d, "p_0^f"] & X \arrow[d, "f"] \\
& X \arrow[r, "f"] & Y
\end{tikzcd}$$

The above diagram shows that any morphism $f \in Hom(X, Y)$ gives rise to a reflexive relation $R[f]$. More explicitly, the reflexive relation $R[f]$ is given by the kernel pair (p_0^f, p_1^f), with $p_0^f, p_1^f \in Hom(R[f], X)$. We also observe that f coequalizes the kernel pair (p_0^f, p_1^f).

The next lemma demonstrates when a pullback becomes an equalizer.

Lemma 1.4.2. *Let $\mathscr{C}$ be a category with pullbacks. Let $h, h' \in Hom(X, Y)$ be a pair of morphisms. Suppose that we have the pullback square*

$$\begin{tikzcd}
I \arrow[r, "i"] \arrow[d, "g"] & X \arrow[d, "{(h, h')}"] \\
Y \arrow[r, "\Delta"] & Y \times Y
\end{tikzcd}$$

where $\Delta : Y \to Y \times Y$ is the diagonal map. Then, $i = eq(h, h')$ is the equalizer of the pair of morphisms (h, h').

Proof. Assume that $\mathscr{C}$ is a category with pullbacks. Assume that $h, h' \in Hom(X, Y)$ are a pair of morphisms, which satisfy the pullback square in the statement of the lemma.

To see that i equalizes the pair (h, h'), the commutativity of the pullback square tells us that as morphisms from I to $Y \times Y$,

$$(h \circ i, h' \circ i) = (g, g).$$

So, $h \circ i = g = h' \circ i$, which reveals that i equalizes the pair (h, h'). Note that this also exposes the identity of g as the composite $h \circ i$.

To see that i is the equalizer of (h, h'), suppose that $k \in Hom(U, X)$ such that k equalizes (h, h') so that $h \circ k = h' \circ k$. This is enough to make the following square commute:

$$\begin{tikzcd}
U \arrow[r, "k"] \arrow[d, "h \circ k"] & X \arrow[d, "{(h,\, h')}"] \\
Y \arrow[r, "\Delta"] & Y \times Y
\end{tikzcd}$$

Now, we can use the universal property of the pullback to deduce the existence of a unique morphism $\phi : U \to I$ such that the two triangles in the below diagram commute:

$$\begin{tikzcd}
U \arrow[dr, dashed, "\phi"] \arrow[drr, bend left, "k"] \arrow[ddr, bend right, "h \circ k"'] & & \\
& I \arrow[r, "i"] \arrow[d, "g"] & X \arrow[d, "{(h,\, h')}"] \\
& Y \arrow[r, "\Delta"] & Y \times Y
\end{tikzcd}$$

The commutativity of the top triangle proves that i is the equalizer of the pair (h, h') as required. □

The next lemma answers the question of whether the category $Pt(\mathscr{C})$ of split epimorphisms in $\mathscr{C}$ has pullbacks. As one would expect, it turns out that the pullbacks are inherited from $\mathscr{C}$.

Lemma 1.4.3. *Let $\mathscr{C}$ be a category with pullbacks. Then, the category $Pt(\mathscr{C})$ also has pullbacks. Furthermore, the functor $\P_{\mathscr{C}} : Pt(\mathscr{C}) \to \mathscr{C}$ preserves pullbacks.*

Proof. Assume that $\mathscr{C}$ is a category with pullbacks. In order to communicate the proof properly, we will require some compact notation. We will abbreviate a split epimorphism in $\mathscr{C}$

$$\begin{tikzcd}
X \arrow[r, shift left, "f"] & Y \arrow[l, shift left, "s"]
\end{tikzcd}$$

as $X \leftrightarrow Y$. Suppose that we have the following diagram in $Pt(\mathscr{C})$:

$$\begin{tikzcd}
& X'' \leftrightarrow Y'' \arrow[d, "{a_1, b_1}"] \\
X' \leftrightarrow Y' \arrow[r, "{a_0, b_0}"] & X \leftrightarrow Y
\end{tikzcd}$$

Ignoring the morphisms which comprise the split epimorphisms in the above diagram, we find two separate diagrams in $\mathscr{C}$. Since $\mathscr{C}$ has pullbacks, we can take pullbacks of both diagrams to obtain the following diagram in $\mathscr{C}$:

$$\begin{tikzcd}
P, Q \arrow[r, "p_1{,}q_1"] \arrow[d, "p_0{,}q_0"] & X'' \leftrightarrow Y'' \arrow[d, "a_1{,}b_1"] \\
X' \leftrightarrow Y' \arrow[r, "a_0{,}b_0"] & X \leftrightarrow Y
\end{tikzcd}$$

This is a diagram in $\mathscr{C}$ because we do not know yet whether P and Q are related by a split epimorphism. To construct a split epimorphism between P and Q, first consider the following commutative square in $\mathscr{C}$:

$$\begin{tikzcd}
P \arrow[r, "f''\circ p_1"] \arrow[d, "f'\circ p_0"'] & Y'' \arrow[d, "b_1"] \\
Y' \arrow[r, "b_0"] & Y
\end{tikzcd}$$

By the universal property of the pullback, there exists a unique morphism $h : P \to Q$ such that the two triangles in the following diagram commute:

$$\begin{tikzcd}
P \arrow[dr, "h"] \arrow[drr, bend left, "f''\circ p_1"] \arrow[ddr, bend right, "f'\circ p_0"'] & & \\
& Q \arrow[r, "q_1"] \arrow[d, "q_0"'] & Y'' \arrow[d, "b_1"] \\
& Y' \arrow[r, "b_0"] & Y
\end{tikzcd}$$

By applying the universal property of the pullback again on the commutative square

$$\begin{tikzcd}
Q \arrow[r, "s''\circ q_1"] \arrow[d, "s'\circ q_0"'] & X'' \arrow[d, "a_1"] \\
X' \arrow[r, "a_0"] & X
\end{tikzcd}$$

we find that there exists a unique morphism $g : Q \to P$ such that the two triangles in the following diagram commute:

$$\begin{tikzcd}
Q \arrow[dr, "g"] \arrow[drr, bend left, "s''\circ q_1"] \arrow[ddr, bend right, "s'\circ q_0"'] & & \\
& P \arrow[r, "p_1"] \arrow[d, "p_0"'] & X'' \arrow[d, "a_1"] \\
& X' \arrow[r, "a_0"] & X
\end{tikzcd}$$

It remains to show that $h \circ g = id_Q$. We know from the diagrams above that $p_1 \circ g = s'' \circ q_1$ and $q_1 \circ h = f'' \circ p_1$. By combining these two equations, we find that $q_1 \circ (h \circ g) = q_1$. Similarly, $q_0 \circ (h \circ g) = q_0$. Notice that the morphism $h \circ g : Q \to Q$ makes the two triangles in the following diagram commute:

$$\begin{tikzcd}
Q \arrow[drr, bend left, "q_1"] \arrow[dr, "h \circ g"] \arrow[ddr, bend right, "q_0"'] & & \\
& Q \arrow[r, "q_1"] \arrow[d, "q_0"'] & Y'' \arrow[d, "b_1"] \\
& Y' \arrow[r, "b_0"] & Y
\end{tikzcd}$$

But, the identity map $id_Q : Q \to Q$ also makes the same diagram commute. Thus, by uniqueness from the universal property of the pullback, $id_Q = h \circ g$. This shows that $h : P \to Q$ is a split epimorphism and thus, the square in $Pt(\mathscr{C})$ commutes:

$$\begin{tikzcd}
P \leftrightarrow Q \arrow[r, "p_1, q_1"] \arrow[d, "p_0, q_0"] & X'' \leftrightarrow Y'' \arrow[d, "a_1, b_1"] \\
X' \leftrightarrow Y' \arrow[r, "a_0, b_0"] & X \leftrightarrow Y
\end{tikzcd}$$

This is indeed a pullback in $Pt(\mathscr{C})$ since we can use the universal property of the pullbacks (P, p_0, p_1) and (Q, q_0, q_1) in $\mathscr{C}$ to construct the appropriate morphisms in $Pt(\mathscr{C})$ (or pairs of morphisms in $\mathscr{C}$). Furthermore, the functor $\P_{\mathscr{C}}$ preserves pullbacks because it maps the above commutative square in $Pt(\mathscr{C})$ to the following commutative square in $\mathscr{C}$:

$$\begin{tikzcd}
Q \arrow[r, "q_1"] \arrow[d, "q_0"'] & Y'' \arrow[d, "b_1"] \\
Y' \arrow[r, "b_0"] & Y
\end{tikzcd}$$

This is a pullback in $\mathscr{C}$ by construction. $\square$

Unsurprisingly, a similar conclusion also holds for the category of reflexive graphs $RGr(\mathscr{C})$.

Lemma 1.4.4. *Let $\mathscr{C}$ be a category with pullbacks. Then, the category $RGr(\mathscr{C})$ also has pullbacks. Furthermore, the functor $U_0 : RGr(\mathscr{C}) \to \mathscr{C}$ preserves pullbacks.*

Proof. Assume that $\mathscr{C}$ is a category with pullbacks. We will denote a reflexive graph on X

$$\begin{tikzcd} G \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] & X \arrow[l, "s" description] \end{tikzcd}$$

by $G \rightleftharpoons X$. Suppose that we have the following diagram in $\mathscr{C}$:

$$\begin{tikzcd} & G'' \rightleftharpoons X'' \arrow[d, "a_1{,}b_1"] \\ G' \rightleftharpoons X' \arrow[r, "a_0{,}b_0"] & G \rightleftharpoons X \end{tikzcd}$$

Ignoring the morphisms which comprise each reflexive graph, we obtain two different graphs in $\mathscr{C}$. Since $\mathscr{C}$ has pullbacks, we can form the pullbacks of both graphs to obtain the commutative square

$$\begin{tikzcd} P, Q \arrow[r, "p_1{,}q_1"] \arrow[d, "p_0{,}q_0"] & G'' \rightleftharpoons X'' \arrow[d, "a_1{,}b_1"] \\ G' \rightleftharpoons X' \arrow[r, "a_0{,}b_0"] & G \rightleftharpoons X \end{tikzcd}$$

Similarly to 1.4.3, we have to show that P is a reflexive graph for Q. The procedure is almost the same as in 1.4.3. We apply the universal property of the pullback twice in $\mathscr{C}$ to deduce the existence of morphisms $h_1, h_2 : P \to Q$ such that the following diagrams commute in $\mathscr{C}$:

$$\begin{tikzcd} P \arrow[dr, dashed, "h_1"] \arrow[drr, bend left, "d'_0 \circ p_1"] \arrow[ddr, bend right, "d''_0 \circ p_0"'] & & \\ & Q \arrow[r, "q_1"] \arrow[d, "q_0"'] & X' \arrow[d, "b_1"] \\ & X'' \arrow[r, "b_0"] & X \end{tikzcd}$$

$$\begin{tikzcd} P \arrow[dr, dashed, "h_2"] \arrow[drr, bend left, "d'_1 \circ p_1"] \arrow[ddr, bend right, "d''_1 \circ p_0"'] & & \\ & Q \arrow[r, "q_1"] \arrow[d, "q_0"'] & X' \arrow[d, "b_1"] \\ & X'' \arrow[r, "b_0"] & X \end{tikzcd}$$

It remains to show that h_1 and h_2 have a common section (right inverse). Applying the universal property of the pullback once more, we obtain the following commutative diagram in $\mathscr{C}$:

$$\begin{tikzcd}
Q \arrow[drr, bend left, "s'\circ q_1"] \arrow[dr, dashed, "g"] \arrow[ddr, bend right, "s''\circ q_0"'] & & \\
& P \arrow[r, "p_1"] \arrow[d, "p_0"'] & G' \arrow[d, "a_1"] \\
& G'' \arrow[r, "a_0"] & G
\end{tikzcd}$$

Now the first two pullback diagrams tell us that $q_1 \circ h_2 = d_1' \circ p_1$ and $q_1 \circ h_1 = d_0' \circ p_1$. The third diagram tells us that $p_1 \circ g = s' \circ q_1$. By precomposing the first two relations with g, we find that $q_1 \circ (h_2 \circ g) = q_1$ and $q_1 \circ (h_1 \circ g) = q_1$. Similar statements hold for q_0 in place of q_1. So, the following diagram involving a pullback square commutes:

$$\begin{tikzcd}
Q \arrow[drr, bend left, "q_1"] \arrow[dr, dashed, "h_1\circ g"] \arrow[ddr, bend right, "q_0"'] & & \\
& Q \arrow[r, "q_1"] \arrow[d, "q_0"'] & X' \arrow[d, "b_1"] \\
& X'' \arrow[r, "b_0"] & X
\end{tikzcd}$$

By uniqueness of the pullback, $h_1 \circ g = h_2 \circ g = id_Q$. Thus, we obtain the following commutative square in $RGr(\mathscr{C})$:

$$\begin{tikzcd}
P \rightleftharpoons Q \arrow[r, "p_1,q_1"] \arrow[d, "p_0,q_0"] & G'' \rightleftharpoons X'' \arrow[d, "a_1,b_1"] \\
G' \rightleftharpoons X' \arrow[r, "a_0,b_0"] & G \rightleftharpoons X
\end{tikzcd}$$

This defines a pullback in $RGr(\mathscr{C})$ because we can use the universal property of the pullback in $\mathscr{C}$ for (P, p_0, p_1) and (Q, q_0, q_1) to construct the appropriate morphisms. Moreover, the functor U_0 preserves pullbacks because it sends the above pullback square in $RGr(\mathscr{C})$ to

$$\begin{tikzcd}
Q \arrow[r, "q_1"] \arrow[d, "q_0"'] & X' \arrow[d, "b_1"] \\
X'' \arrow[r, "b_0"] & X
\end{tikzcd}$$

This is a pullback square in $\mathscr{C}$ by construction. □

We also note from [Bou17, Exercise 1.5.6] that conveniently, the subcategory of reflexive relations $Ref(\mathscr{C})$ is stable under pullbacks in $RGr(\mathscr{C})$.

Our next task is to prove a few results which emphasise common situations where pullbacks arise from.

Lemma 1.4.5. *Let $\mathscr{C}$ be a category with pullbacks. Consider the following diagram in $\mathscr{C}$, which consists of adjacent commutative squares:*

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "h_1"] & A' \arrow[r, "\alpha'"] \arrow[d, "h_2"] & A'' \arrow[d, "h_3"] \\ B \arrow[r, "\beta"] & B' \arrow[r, "\beta'"] & B'' \end{tikzcd}$$

If both squares in the above diagram are pullback squares, then the outside square is also a pullback square. Furthermore, if the outside square and the right hand side square are pullbacks, then the left hand side square is also a pullback.

Proof. Assume that $\mathscr{C}$ is a category with pullbacks. Suppose that we have the following diagram shown in the statement of the lemma. Assume that the adjacent commutative squares are pullbacks. We want to show that the following square is a pullback square:

$$\begin{tikzcd} A \arrow[r, "\alpha'\circ\alpha"] \arrow[d, "h_1"] & A'' \arrow[d, "h_3"] \\ B \arrow[r, "\beta'\circ\beta"] & B'' \end{tikzcd}$$

Suppose that we are given the commutative square

$$\begin{tikzcd} X \arrow[r, "\gamma"] \arrow[d, "g"] & A'' \arrow[d, "h_3"] \\ B \arrow[r, "\beta'\circ\beta"] & B'' \end{tikzcd}$$

Then, the following diagram is also a commutative square

$$\begin{tikzcd} X \arrow[r, "\gamma"] \arrow[d, "\beta\circ g"] & A'' \arrow[d, "h_3"] \\ B' \arrow[r, "\beta'\circ\beta"] & B'' \end{tikzcd}$$

and by the universal property of the pullback, there exists a unique morphism $\phi : X \to A'$ such that the triangles in the following diagram commute:

$$\begin{tikzcd}
X \arrow[drr, bend left, "\gamma"] \arrow[dr, dashed, "\phi"] \arrow[ddr, bend right, "\beta\circ g"'] & & \\
& A' \arrow[r, "\alpha'"] \arrow[d, "h_2"'] & A'' \arrow[d, "h_3"] \\
& B' \arrow[r, "\beta'"] & B''
\end{tikzcd}$$

The bottom left triangle gives rise to the commutative square

$$\begin{array}{ccc}
X & \xrightarrow{\phi} & A' \\
\big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle h_2} \\
B & \xrightarrow{\beta} & B'
\end{array}$$

Since the left square is also a pullback square, we can again use the universal property of the pullback to deduce the existence of a unique morphism $\psi : X \to A$ such that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[drr, bend left, "\phi"] \arrow[dr, dashed, "\psi"] \arrow[ddr, bend right, "g"'] & & \\
& A \arrow[r, "\alpha"] \arrow[d, "h_1"'] & A' \arrow[d, "h_2"] \\
& B \arrow[r, "\beta"] & B'
\end{tikzcd}$$

Now observe that $h_1 \circ \psi = g$ and

$$(\alpha' \circ \alpha) \circ \psi = \alpha' \circ \phi = \gamma.$$

Therefore, the outer square is indeed a pullback square.

For the other direction, assume that the outside square and the right square are both pullbacks. Suppose that we have the following commutative diagram in $\mathscr{C}$:

$$\begin{array}{ccc}
Y & \xrightarrow{\delta} & A' \\
\big\downarrow{\scriptstyle j} & & \big\downarrow{\scriptstyle h_2} \\
B & \xrightarrow{\beta} & B'
\end{array}$$

Since the right square is commutative because it was assumed to be a pullback, we find that $\beta' \circ (\beta \circ j) = \beta' \circ (h_2 \circ \delta)$ and $(\beta' \circ \beta) \circ j = h_3 \circ (\alpha' \circ \delta)$. Using the universal property of the pullback on the outer square, there exists a unique morphism $\rho : Y \to A$ such that the following diagram commutes:

$$\begin{tikzcd}
Y \arrow[dr, dashed, "\rho"] \arrow[drr, bend left, "\alpha'\circ\delta"] \arrow[ddr, bend right, "j"'] & & \\
& A \arrow[r, "\alpha'\circ\alpha"] \arrow[d, "h_1"'] & A'' \arrow[d, "h_3"] \\
& B \arrow[r, "\beta'\circ\beta"] & B''
\end{tikzcd}$$

It suffices to show that $\alpha \circ \rho = \delta$. Note first that $\alpha \circ \rho$ makes the following diagram commute

$$\begin{tikzcd}
Y \arrow[dr, "\alpha\circ\rho"] \arrow[drr, bend left, "\alpha'\circ\delta"] \arrow[ddr, bend right, "\beta\circ j"'] & & \\
& A' \arrow[r, "\alpha'"] \arrow[d, "h_2"'] & A'' \arrow[d, "h_3"] \\
& B' \arrow[r, "\beta'"] & B''
\end{tikzcd}$$

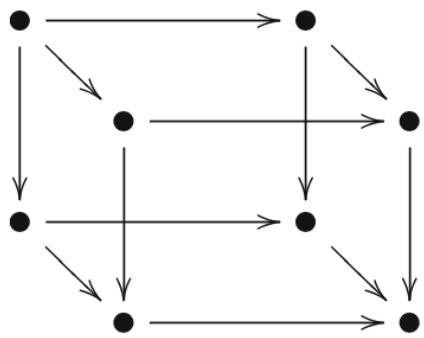
But $\delta : Y \to A'$ also makes the above diagram commute. Since the right hand side square is a pullback, we can use uniqueness of the universal property of pullbacks to deduce that $\alpha \circ \rho = \delta$. This demonstrates that $\rho : Y \to A$ is the unique morphism which makes the following diagram commute:

$$\begin{tikzcd}
Y \arrow[dr, dashed, "\rho"] \arrow[drr, bend left, "\delta"] \arrow[ddr, bend right, "j"'] & & \\
& A \arrow[r, "\alpha"] \arrow[d, "h_1"'] & A'' \arrow[d, "h_2"] \\
& B \arrow[r, "\beta"] & B''
\end{tikzcd}$$

This demonstrates that the LHS square is a pullback as required. $\square$

One application of 1.4.5 is to prove the Dice lemma, which comprises [Bou17, Corollary 1.6.3].

Lemma 1.4.6. *Let $\mathscr{C}$ be a category. Suppose that we have the commutative cube in $\mathscr{C}$ (the image is from [Bou17, Corollary 1.6.3]):*



Suppose further that the top, bottom and front faces of the cube are pullback squares. Then, the back face of the cube must also be a pullback square.

Proof. Assume that $\mathscr{C}$ is a category and that we have the commutative diagram in the statement of the lemma (evidently, the identities of the objects and morphisms in the diagram are unimportant).

Assume that the top, bottom and front faces of the cube are pullback squares. Then, from 1.4.5, the square consisting of both the front and top faces must be a pullback square because its constituent squares are pullbacks. But, by commutativity of the diagram, this means that the back and bottom squares together form a pullback. Since the bottom face is a pullback, another application of 1.4.5 reveals that the back face is a pullback (since the bottom face is to the right of the back face).

Notice that by an identical argument, if the top, bottom, front and left faces of the cube are all pullback squares, then the remaining two faces of the cube (the right and back faces) must also be pullbacks. $\square$

Here is a curious feature about the proof of 1.4.5. When we reached the point where it suffices to show that $\alpha \circ \rho = \delta$, it is tempting to write $\alpha' \circ (\alpha \circ \rho) = \alpha' \circ \delta$ and cancel out the α'. However, in 1.4.5, α' is not a monomorphism, which renders this line of reasoning invalid. On the flipside, if α' was a monomorphism, then this step works, producing the following lemma

Lemma 1.4.7. *Let $\mathscr{C}$ be a category with pullbacks. Consider the following diagram in $\mathscr{C}$, which consists of adjacent commutative squares:*

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, "h_1"] & A' \arrow[r, "\alpha'"] \arrow[d, "h_2"] & A'' \arrow[d, "h_3"] \\
B \arrow[r, "\beta"] & B' \arrow[r, "\beta'"] & B''
\end{tikzcd}$$

Suppose that α' is a monomorphism. If the outside square is a pullback, then the left hand side square is also a pullback.

Proof. Assume that $\mathscr{C}$ is a category with pullbacks and that the outside square is a pullback. Assume that α' is a monomorphism. Suppose that we have the following commutative diagram in $\mathscr{C}$:

$$\begin{tikzcd}
Y \arrow[r, "\delta"] \arrow[d, "j"] & A' \arrow[d, "h_2"] \\
B \arrow[r, "\beta"] & B'
\end{tikzcd}$$

Since the right square is commutative because it was assumed to be a pullback, we find that $\beta' \circ (\beta \circ j) = \beta' \circ (h_2 \circ \delta)$ and $(\beta' \circ \beta) \circ j = h_3 \circ (\alpha' \circ \delta)$. Using the universal property of the pullback on the outer square, there exists a unique morphism $\rho : Y \to A$ such that the following diagram commutes:

$$\begin{tikzcd}
Y \arrow[drr, bend left, "\alpha' \circ \delta"] \arrow[dr, dashed, "\rho"] \arrow[ddr, bend right, "j"'] & & \\
& A \arrow[r, "\alpha' \circ \alpha"] \arrow[d, "h_1"'] & A'' \arrow[d, "h_3"] \\
& B \arrow[r, "\beta' \circ \beta"] & B''
\end{tikzcd}$$

It suffices to show that $\alpha \circ \rho = \delta$. We know from the top triangle in the above diagram that $\alpha' \circ (\alpha \circ \rho) = \alpha' \circ \delta$. Since α' is a monomorphism, $\alpha \circ \rho = \delta$. Hence, the following diagram commutes

$$\begin{tikzcd}
Y \arrow[drr, bend left, "\delta"] \arrow[dr, dashed, "\rho"] \arrow[ddr, bend right, "j"'] & & \\
& A \arrow[r, "\alpha"] \arrow[d, "h_1"'] & A'' \arrow[d, "h_2"] \\
& B \arrow[r, "\beta"] & B''
\end{tikzcd}$$

and hence, the left hand side square is a pullback square. $\square$

1.5 Inverse image

Recall the notion of a reflexive relation from section 1.3. This was motivated by wanting to generalise the notion of an equivalence relation on sets to different categories. Before we proceed to define *internal equivalence relations* in a category $\mathscr{C}$, we will first discuss the notion of an inverse image in the category $RGr(\mathscr{C})$.

Definition 1.5.1. Let $\mathscr{C}$ be a category with pullbacks. From 1.4.4, the category $RGr(\mathscr{C})$ must also have pullbacks. Let G be a reflexive graph on Y given by

$$\begin{tikzcd} G \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] & Y \arrow[l, "s_0"' description] \end{tikzcd}$$

and $f : X \to Y$ be a morphism. The **inverse image** of G by f , which is denoted by $f^{-1}(G)$, is the following pullback in the category $RGr(\mathscr{C})$:

$$\begin{tikzcd} f^{-1}(G) \arrow[r, "(f_G, f)"] \arrow[d] & G \arrow[d, "d_G"] \\ \nabla_X \arrow[r, "\nabla f"] & \nabla_Y \end{tikzcd}$$

Recall that ∇_X and ∇_Y denote indiscrete reflexive relations and that the morphism $d_G : G \to \nabla_Y$ was constructed in 1.3.1.

Let us given an explicit description of the morphism $\nabla f : \nabla_X \to \nabla_Y$. This is just the pair of morphisms $((f, f), f)$ in $\mathscr{C}$. This observation is substantiated by the fact that the square (or rather three squares) in $\mathscr{C}$ commutes:

$$\begin{tikzcd} X \times X \arrow[r, shift left=2, "\pi_2"] \arrow[r, shift right=2, "\pi_1"'] \arrow[d, "(f, f)"'] & X \arrow[l, "\Delta" description] \arrow[d, "f"] \\ Y \times Y \arrow[r, shift left=2, "\pi_2"] \arrow[r, shift right=2, "\pi_1"'] & Y \arrow[l, "\Delta" description] \end{tikzcd}$$

By definition, the inverse image $f^{-1}(G)$ is a pullback in the category $RGr(\mathscr{C})$. Hence, $f^{-1}(G)$ defines a reflexive graph on the object X. From 1.3.1, the morphism in $RGr(\mathscr{C})$ from $f^{-1}(G)$ to ∇_X must be the morphism $d_{f^{-1}(G)}$ which is unique. What we are primarily interested in is the universal property satisfied by the inverse image.

Lemma 1.5.1. *Let $\mathscr{C}$ be a category with pullbacks. Let G be a reflexive graph on Y given by*

$$\begin{tikzcd} G \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] & Y \arrow[l, "s_0"'] \end{tikzcd}$$

and $f : X \to Y$ be a morphism. If Γ is a reflexive graph on X, then there exists a morphism $(f, \check{f}) : \Gamma \to G$ if and only if there exists a factorisation $\Gamma \to f^{-1}(G)$ above X.

Proof. Assume that $\mathscr{C}$ is a category with pullbacks and G is a reflexive graph on Y. Assume that $f : X \to Y$ is a morphism in $\mathscr{C}$ so that the inverse image $f^{-1}(G)$ can be formed as the pullback square

$$\begin{tikzcd} f^{-1}(G) \arrow[r, "(f_G{,}f)"] \arrow[d, "d_{f^{-1}(G)}"'] & G \arrow[d, "d_G"] \\ \nabla_X \arrow[r, "\nabla f"] & \nabla_Y \end{tikzcd}$$

Assume that Γ is a reflexive graph on X. First, assume that there exists a morphism $(\check{f}, f) : \Gamma \to G$. Then, the following diagram must commute:

$$\begin{tikzcd} \Gamma \arrow[r, "(\check{f}{,}f)"] \arrow[d, "d_\Gamma"] & G \arrow[d, "d_G"] \\ \nabla_X \arrow[r, "\nabla f"] & \nabla_Y \end{tikzcd}$$

We know that the left downwards pointing arrow in the above commutative square is d_Γ from 1.3.1, due to uniqueness. By exploiting the universal property of the pullback, there exists a morphism ϕ in $RGr(\mathscr{C})$ from Γ to $f^{-1}(G)$ such that the two triangles in the following diagram commute in $RGr(\mathscr{C})$:

$$\begin{tikzcd} \Gamma \arrow[dr, dashed, "\phi"] \arrow[drr, bend left, "(\check{f}{,}f)"] \arrow[ddr, bend right, "d_\Gamma"'] & & \\ & f^{-1}(G) \arrow[r, "(f_G{,}f)"] \arrow[d, "d_{f^{-1}(G)}"] & G \arrow[d, "d_G"] \\ & \nabla_X \arrow[r, "\nabla f"] & \nabla_Y \end{tikzcd}$$

So, $(\check{f}, f) : \Gamma \to G$ factors as the composite $(f_G, f) \circ \phi$.

Conversely, suppose that we have a morphism $\alpha : \Gamma \to f^{-1}(G)$ in the category $RGr(\mathscr{C})$ above X. Then, we can define a morphism in $RGr(\mathscr{C})$ from Γ to G by the composite $(f_G, f) \circ \alpha$. This completes the proof. $\square$

The next result demonstrates an important invariant of the inverse image.

Lemma 1.5.2. *Let $\mathscr{C}$ be a category with pullbacks. Let G be a reflexive graph on Y given by*

$$\begin{tikzcd} G \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] & Y \arrow[l, "s_0"'] \end{tikzcd}$$

and $f : X \to Y$ be a morphism. If G is a reflexive relation, then the inverse image $f^{-1}(G)$ is also a reflexive relation.

Proof. Assume that $\mathscr{C}$ is a category with pullbacks. Assume that G is a reflexive graph on Y and $f : X \to Y$ is a morphism. The inverse image $f^{-1}(G)$ satisfies the following commutative diagram in $RGr(\mathscr{C})$:

$$\begin{tikzcd} f^{-1}(G) \arrow[r, "(f_G, f)"] \arrow[d, "d_{f^{-1}(G)}"'] & G \arrow[d, "d_G"] \\ \nabla_X \arrow[r, "\nabla f"] & \nabla_Y \end{tikzcd}$$

The morphism $(f_G, f) : f^{-1}(G) \to G$ induces the following commutative diagram in $\mathscr{C}$:

$$\begin{tikzcd} f^{-1}(G) \arrow[r, shift left=2, "e_0"] \arrow[r, shift right=2, "e_1"'] \arrow[d, "f_G"'] & X \arrow[l, "t_0"'] \arrow[d, "f"] \\ G \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & Y \arrow[l, "s_0"'] \end{tikzcd}$$

In order to demonstrate that $f^{-1}(G)$ is a reflexive relation on X, we must show that $(e_0, e_1) : f^{-1}(G) \to X \times X$ is a monomorphism. To this end, suppose that $f_1, f_2 \in Hom_{\mathscr{C}}(Z, f^{-1}(G))$ and that $(e_0, e_1) \circ f_1 = (e_0, e_1) \circ f_2$.

To show: (a) $f_1 = f_2$.

(a) Since $f \circ e_0 = d_0 \circ f_G$ and $f \circ e_1 = d_1 \circ f_G$, we find that $(f, f) \circ (e_0, e_1) \circ f_1 = (f, f) \circ (e_0, e_1) \circ f_2$ and

$$(d_0, d_1) \circ (f_G, f_G) \circ f_1 = (d_0, d_1) \circ (f_G, f_G) \circ f_2.$$

But, G is a reflexive relation on Y. This means that the morphism $(d_0, d_1) \in Hom_{\mathscr{C}}(G, Y \times Y)$ is a monomorphism. In conjunction with the above equation, we obtain $f_G \circ f_1 = f_G \circ f_2$.

Recall how pullbacks in $RGr(\mathscr{C})$ were constructed in 1.4.4. We constructed two different pullback squares in $\mathscr{C}$ and then used the universal property of the pullback to turn these into a single pullback square in $RGr(\mathscr{C})$. Using this observation, we find that the following square in $\mathscr{C}$ is a pullback:

$$\begin{tikzcd}
f^{-1}(G) \arrow[r, "f_G"] \arrow[d, "(e_0,e_1)"'] & G \arrow[d, "(d_0,d_1)"] \\
X \times X \arrow[r, "(f,f)"] & Y \times Y
\end{tikzcd}$$

Notice that the morphism f_1 makes the following diagram commute:

$$\begin{tikzcd}
Z \arrow[dr, "f_1"] \arrow[drr, bend left, "f_G \circ f_2"] \arrow[ddr, bend right, "(e_0,e_1)\circ f_2"'] & & \\
& f^{-1}(G) \arrow[r, "f_G"] \arrow[d, "(e_0,e_1)"'] & G \arrow[d, "(d_0,d_1)"] \\
& X \times X \arrow[r, "(f,f)"] & Y \times Y
\end{tikzcd}$$

But, f_2 also makes the diagram commute. So, by uniqueness associated to the universal property of the pullback, $f_1 = f_2$.

Part (a) reveals that $(e_0, e_1) : f^{-1}(G) \to X \times X$ is a monomorphism in $\mathscr{C}$. Therefore, $f^{-1}(G)$ must be a reflexive relation on X. □

An important consequence of the proof of 1.5.2 is that a very similar proof technique can be used to prove the following result, which we will state below (but not prove).

Theorem 1.5.3. *Let $\mathscr{C}$ be a category with pullbacks. Then the subcategory $Ref(\mathscr{C})$ of $RGr(\mathscr{C})$ is stable under pullbacks in $RGr(\mathscr{C})$.*

We will end this section with another definition we require.

Definition 1.5.2. Let $\mathscr{C}$ be a category with pullbacks and R, S be reflexive relations on X. The **intersection** $R \cap S$ is the reflexive relation on X given by the diagonal of the pullback

$$\begin{tikzcd} R \cap S \arrow[r, "\iota_S"] \arrow[d, "\iota_R"] \arrow[dr, dashed, "d_{R\cap S}"] & S \arrow[d, "d_S"] \\ R \arrow[r, "d_R"] & \nabla_X \end{tikzcd}$$

where ι_S and ι_R are the inclusion morphisms.

As explained in [Bou17, Exercise 1.5.7], the inverse image along a morphism $f : X \to Y$ preserves the intersection of reflexive relations. This means that in the category $Ref(\mathscr{C})$, $f^{-1}(R \cap S) \cong f^{-1}(R) \cap f^{-1}(S)$.

1.6 Internal equivalence relations

We have now built up enough theory to describe our generalisation of an equivalence relation in category theory. After our definition of a reflexive relation (see 1.3.2), we briefly discussed why it is a valid generalisation of the usual notion of a reflexive relation on a set. We will take the opposite approach in this section and begin by motivating the ideas behind the definition of an *internal equivalence relation* before providing the definition.

Our starting point lies with the usual idea of an equivalence relation on a set.

Definition 1.6.1. Let X be a set. A relation $R \subseteq X \times X$ is said to be an **equivalence relation** on the set X if R satisfies the following three properties:

1. For all $x \in X$, $(x, x) \in R$.
2. For all $x, x' \in X$, if $(x, x') \in R$, then $(x', x) \in R$
3. For all $x, x', x'' \in X$, if $(x, x') \in R$ and $(x', x'') \in R$, then $(x, x'') \in R$.

If only the first property is satisfied in the above definition, then we recover the definition of a reflexive relation on a set. Before we proceed, we want to point out that instead of writing $(x, x') \in R$, [Bou17] writes xRx'. We will also adopt this notation.

The first step to generalising the above definition to category theory is to find a condition which condenses the second and third properties to a single property. This is precisely what the horn-filler condition does.

Lemma 1.6.1. *Let X be a set and R be a reflexive relation on X. Then, R is an equivalence relation if and only if R satisfies the horn-filler condition — for all $x, x', x'' \in X$, if xRx' and xRx'', then $x'Rx''$.*

Proof. Assume first that R is an equivalence relation on the set X. Assume that xRx' and xRx''. By symmetry, $x'Rx$ and by transitivity applied to $x'Rx$ and xRx'', $x'Rx''$. Hence, the horn-filler condition must be satisfied.

For the converse, suppose that R is a reflexive relation on X which satisfies the horn-filler condition. To see that symmetry is satisfied, assume that xRx'. Then, since xRx (because R is reflexive), we can apply the horn-filler condition to demonstrate that $x'Rx$. To see that transitivity is satisfied, assume that xRx' and $x'Rx''$. By symmetry, $x'Rx$ and by the horn-filler condition applied to $x'Rx$ and $x'Rx''$, we obtain xRx'' as required. □

The natural question to ask here is: how does the horn-filler condition generalise to category theory? Suppose that $\mathscr{C}$ is a category with pullbacks, X is an object in $\mathscr{C}$ and R is the following reflexive relation on X:

$$\begin{tikzcd} R \arrow[r, shift left=2, "d_0^R"] \arrow[r, shift right=2, "d_1^R"'] & X \arrow[l] \end{tikzcd}$$

The identity of the common section is not important here. Form the following pullback square from the pair (d_0^R, d_0^R).

$$\begin{tikzcd} R[d_0^R] \arrow[r, "d_0"] \arrow[d, "d_1"] & R \arrow[d, "d_0^R"] \\ R \arrow[r, "d_0^R"] & X \end{tikzcd}$$

This turns $R[d_0^R]$ into a reflexive relation on R, which is coequalized by d_0^R. This will become important in what follows. Now consider the following diagram in $\mathscr{C}$:

$$\begin{tikzcd} R[d_0^R] \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] \arrow[d, "d_2"'] & R \arrow[l] \arrow[d, "d_1^R"] \\ R \arrow[r, shift left=2, "d_0^R"] \arrow[r, shift right=2, "d_1^R"'] & X \arrow[l] \end{tikzcd}$$

This is a morphism (d_2, d_1^R) of reflexive relations. The two commutative squares reveal that $d_0^R \circ d_2 = d_1^R \circ d_0$ and $d_1^R \circ d_2 = d_1^R \circ d_1$. Let $s \in R[d_0^R]$ and suppose briefly that we are working in the category **Set** so that the usual definition of an equivalence relation on a set applies. Recall from the construction of the kernel pair that $(d_0^R \circ d_0)(s) = (d_0^R \circ d_1)(s)$. Hence, the following pairs are contained in our reflexive relation $R \subseteq X \times X$

$$((d_0^R \circ d_0)(s), (d_1^R \circ d_0)(s)), ((d_0^R \circ d_1)(s), (d_1^R \circ d_1)(s)), ((d_0^R \circ d_2)(s), (d_1^R \circ d_2)(s)).$$

But since $d_0^R \circ d_2 = d_1^R \circ d_0$ and $d_1^R \circ d_2 = d_1^R \circ d_1$, the third pair is just $((d_1^R \circ d_0)(s), (d_1^R \circ d_1)(s))$, which is an element of our reflexive relation. If we define $x = (d_0^R \circ d_0)(s)$, $x' = (d_1^R \circ d_0)(s)$ and $x'' = (d_1^R \circ d_1)(s)$, then we have demonstrated that if xRx' and xRx'', then $x'Rx''$ which is exactly the horn-filler condition.

It remains to construct the morphism $(d_2, d_1^R) : R[d_0^R] \to R$ in $Ref(\mathscr{C})$. One way to do this is via the pullback square given by the inverse image $(d_1^R)^{-1}(R)$. Explicitly, this yields the commutative diagram in $Ref(\mathscr{C})$

$$\begin{tikzcd}
R[d_0^R] \arrow[dr, dashed, "(\rho_1,\rho_2)"] \arrow[drr, bend left, "(d_2,d_1^R)"] \arrow[ddr, bend right, "d_{R[d_0^R]}"'] & & \\
& (d_1^R)^{-1}(R) \arrow[r, "(f,d_1^R)"] \arrow[d, "d_{(d_1^R)^{-1}(R)}"'] & R \arrow[d, "d_R"] \\
& \nabla_R \arrow[r, "\nabla_{d_1^R}"'] & \nabla_X
\end{tikzcd}$$

Recall the explicit description of the two unique morphisms $d_{R[d_0^R]}$ and $d_{(d_1^R)^{-1}(R)}$ from 1.3.1. We have

$$d_{R[d_0^R]} = ((d_0, d_1), id_R) \text{ and } d_{(d_1^R)^{-1}(R)} = ((e_0, e_1), id_R)$$

where e_0, e_1 are the morphisms constituting the reflexive relation $(d_1^R)^{-1}(R)$ on R. The main point here is that the second morphism in each pair is the identity id_R on R. By commutativity of the leftmost triangle, $\rho_2 = id_R$. This ensures that we can define the morphism $(d_2, d_1^R) : R[d_0^R] \to R$, by setting $d_2 = f \circ \rho_1$.

Therefore, if we have a unique morphism $(\rho_1, \rho_2) : R[d_0^R] \to (d_1^R)^{-1}(R)$ induced by the pullback square from $(d_1^R)^{-1}(R)$, then we can always

construct (d_2, d_1^R). In [Bou17], the first condition is written as
$R[d_0^R] \subset (d_1^R)^{-1}(R)$.

Now we can generalise the idea of an equivalence relation with the following important definition.

Definition 1.6.2. Let $\mathscr{C}$ be a category with pullbacks. Let X be an object in $\mathscr{C}$ and R be the following reflexive relation on X:

$$\begin{tikzcd} R \arrow[r, shift left=2, "d_0^R"] \arrow[r, shift right=2, "d_1^R"'] & X \arrow[l] \end{tikzcd}$$

We say that R is an **internal equivalence relation** if R satisfies
$R[d_0^R] \subset (d_1^R)^{-1}(R)$.

In order to parse the definition, we will go over some examples of internal equivalence relations.

Example 1.6.3. Let $\mathscr{C}$ be a category with pullbacks. Let X be an object in $\mathscr{C}$. We claim that the discrete reflexive relation Δ_X is an internal equivalence relation on X. The discrete reflexive relation is given by the diagram

$$\begin{tikzcd} X \arrow[r, shift left=3, "id_X"] \arrow[r, shift right=3, "id_X"'] & X \arrow[l, "id_X"'] \end{tikzcd}$$

We must show that $R[id_X] \subset id_X^{-1}(\Delta_X)$, where $id_X^{-1}(\Delta_X)$ is the inverse image formed from the pullback square

$$\begin{tikzcd} id_X^{-1}(\Delta_X) \arrow[r, "(f, id_X)"] \arrow[d, "d_{id_X^{-1}(\Delta_X)}"'] & \Delta_X \arrow[d, "(\Delta, id_X)"] \\ \nabla_X \arrow[r, "\nabla_{id_X}"'] & \nabla_X \end{tikzcd}$$

It suffices to construct a commutative square of the form

$$\begin{tikzcd} R[id_X] \arrow[r] \arrow[d, "d_{R[id_X]}"'] & \Delta_X \arrow[d, "(\Delta, id_X)"] \\ \nabla_X \arrow[r, "\nabla_{id_X}"'] & \nabla_X \end{tikzcd}$$

where $\Delta = (id_X, id_X)$ is the diagonal morphism from X to $X \times X$. Recall that $R[id_X]$ is formed as the pullback of the pair (id_X, id_X), giving the following commutative square in $\mathscr{C}$:

$$\begin{tikzcd}
R[id_X] \arrow[r, "p_0"] \arrow[d, "p_1"'] & X \arrow[d, "id_X"] \\
X \arrow[r, "id_X"'] & X
\end{tikzcd}$$

Since id_X is an isomorphism, 1.1.2 tells us that $p_0 = p_1$. It is not too hard to observe that the pair $(p_1, id_X) : R[id_X] \to \Delta_X$ is a morphism in $Ref(\mathscr{C})$ because the following diagram commutes in $\mathscr{C}$:

$$\begin{tikzcd}
R[id_X] \arrow[r, shift left=2, "p_0=p_1"] \arrow[r, shift right=2, "p_1"'] \arrow[d, "p_1"'] & X \arrow[l] \arrow[d, "id_X"] \\
X \arrow[r, shift left=2, "id_X"] \arrow[r, shift right=2, "id_X"'] & X \arrow[l]
\end{tikzcd}$$

Since the morphism $d_{R[id_X]} = ((p_1, p_1), id_X)$ (see 1.3.1), we find that the square in $Ref(\mathscr{C})$ commutes

$$\begin{tikzcd}
R[id_X] \arrow[r, "(p_1, id_X)"] \arrow[d, "d_{R[id_X]}"'] & \Delta_X \arrow[d, "(\Delta, id_X)"] \\
\nabla_X \arrow[r, "\nabla_{id_X}"'] & \nabla_X
\end{tikzcd}$$

Thus, by the universal property of the pullback, we obtain a unique morphism $(\phi_1, \phi_2) : R[id_X] \to id_X^{-1}(\Delta_X)$ such that the following diagram commutes

$$\begin{tikzcd}
R[id_X] \arrow[dr, dashed, "(\phi_1, \phi_2)"] \arrow[drr, bend left, "(p_1, id_X)"] \arrow[ddr, bend right, "d_{R[id_X]}"'] & & \\
& id_X^{-1}(\Delta_X) \arrow[r, "(f, id_X)"] \arrow[d, "d_{id_X^{-1}(\Delta_X)}"'] & \Delta_X \arrow[d, "(\Delta, id_X)"] \\
& \nabla_X \arrow[r, "\nabla_{id_X}"'] & \nabla_X
\end{tikzcd}$$

Therefore, $R[id_X] \subset id_X^{-1}(\Delta_X)$ and so, the discrete reflexive relation is in fact an internal equivalence relation.

Example 1.6.4. Following on from the previous example, we claim also that the indiscrete reflexive relation ∇_X on X is also an internal equivalence relation. Recall that the indiscrete reflexive relation is given by the diagram

$$\begin{tikzcd} X \times X \arrow[r, shift left=2, "\pi_1"] \arrow[r, shift right=2, "\pi_2"'] & X \arrow[l, "\Delta"'] \end{tikzcd}$$

We must show that $R[\pi_1] \subset \pi_2^{-1}(\nabla_X)$, where $\pi_2^{-1}(\nabla_X)$ is formed from the pullback

$$\begin{tikzcd} \pi_2^{-1}(\nabla_X) \arrow[r, "(f,\pi_2)"] \arrow[d, "d_{\pi_2}"'] & \nabla_X \arrow[d, "(id_{X\times X}, id_X)"] \\ \nabla_{X\times X} \arrow[r, "\nabla_{\pi_2}"'] & \nabla_X \end{tikzcd}$$

We want to construct a commutative square of the form

$$\begin{tikzcd} R[\pi_1] \arrow[r] \arrow[d, "d_{R[\pi_1]}"'] & \nabla_X \arrow[d, "(id_{X\times X}, id_X)"] \\ \nabla_{X\times X} \arrow[r, "\nabla_{\pi_2}"'] & \nabla_X \end{tikzcd}$$

The commutative square we are after is lying in plain sight

$$\begin{tikzcd} R[\pi_1] \arrow[r, "\nabla_{\pi_2}\circ d_{R[\pi_1]}"] \arrow[d, "d_{R[\pi_1]}"'] & \nabla_X \arrow[d, "(id_{X\times X}, id_X)"] \\ \nabla_{X\times X} \arrow[r, "\nabla_{\pi_2}"'] & \nabla_X \end{tikzcd}$$

The universal property of the pullback then gives us a unique morphism $(\phi_1, \phi_2) : R[\pi_1] \to \pi_2^{-1}(\nabla_X)$ such that the following diagram commutes:

$$\begin{tikzcd} R[\pi_1] \arrow[dr, dashed, "(\phi_1,\phi_2)"] \arrow[drr, bend left, "\nabla_{\pi_2}\circ d_{R[\pi_1]}"] \arrow[ddr, bend right, "d_{R[\pi_1]}"'] & & \\ & \pi_2^{-1}(\nabla_X) \arrow[r, "(f,\pi_2)"] \arrow[d, "d_{\pi_2}"'] & \nabla_X \arrow[d, "(id_{X\times X}, id_X)"] \\ & \nabla_{X\times X} \arrow[r, "\nabla_{\pi_2}"'] & \nabla_X \end{tikzcd}$$

Therefore, $R[\pi_1] \subset \pi_2^{-1}(\nabla_X)$ and consequently, the indiscrete reflexive relation ∇_X must be an internal equivalence relation.

We denote by $Equ(\mathscr{C})$ the subcategory of $Ref(\mathscr{C})$ whose objects are internal equivalence relations. Another remark we will make here is that by symmetry, an equivalent condition to $R[d_0^R] \subset (d_1^R)^{-1}(R)$ in the definition 1.6.2 is $R[d_1^R] \subset (d_0^R)^{-1}(R)$. Another equivalent formulation is that the intersection $R[d_0^R] \cap (d_1^R)^{-1}(R) \cong R[d_0^R]$.

Definition 1.6.5. Let $\mathscr{C}$ be a category with pullbacks and R and S denote reflexive relations on the objects X and Y respectively. Let $(\check{f}, f) : R \to S$ be a morphism in $Ref(\mathscr{C})$:

$$\begin{tikzcd}
R \arrow[r, shift left=2, "d_0^R"] \arrow[r, shift right=2, "d_1^R"'] \arrow[d, "\check{f}"'] & X \arrow[l] \arrow[d, "f"] \\
S \arrow[r, shift left=2, "d_0^S"] \arrow[r, shift right=2, "d_1^S"'] & Y \arrow[l]
\end{tikzcd}$$

We say that $(\check{f}, f)$ is **fibrant** if the square indexed by 0 is a pullback square.

It turns out that in the definition of fibrant, we do not care which square is a pullback, as the following lemma suggests

Lemma 1.6.2. *Let $\mathscr{C}$ be a category with pullbacks and R and S denote reflexive relations on the objects X and Y respectively. Let $(\check{f}, f) : R \to S$ be a morphism in $Ref(\mathscr{C})$. Then, $(\check{f}, f)$ is fibrant if and only if the square indexed by 1 is a pullback.*

Proof. Assume that $\mathscr{C}$ is a category with pullbacks and R and S are reflexive relations on the objects X and Y respectively. Assume that $(\check{f}, f) : R \to S$ is a morphism in $Ref(\mathscr{C})$, given by the diagram

$$\begin{tikzcd}
R \arrow[r, shift left=2, "d_0^R"] \arrow[r, shift right=2, "d_1^R"'] \arrow[d, "\check{f}"'] & X \arrow[l] \arrow[d, "f"] \\
S \arrow[r, shift left=2, "d_0^S"] \arrow[r, shift right=2, "d_1^S"'] & Y \arrow[l]
\end{tikzcd}$$

To show: (a) If $(\check{f}, f)$ is fibrant, then the square indexed by 1 is a pullback.

(b) If the square indexed by 1 is a pullback, then $(\check{f}, f)$ is fibrant.

(a) Suppose that we have the following commutative square in $\mathscr{C}$:

$$\begin{tikzcd}
T \arrow[r, "q_0"] \arrow[d, "q_1"] & X \arrow[d, "f"] \\
S \arrow[r, "d_1^S"] & Y
\end{tikzcd}$$

Then, the square below is also commutative:

$$\begin{tikzcd}
T \arrow[r, "q_0"] \arrow[d, "s_0^S \circ d_1^S \circ q_1"'] & X \arrow[d, "f"] \\
S \arrow[r, "d_0^S"] & Y
\end{tikzcd}$$

Here, s_0^S and s_0^R are the sections associated with the reflexive relations S and R respectively. Using the fact that $(\check{f}, f)$ is fibrant, we can use the pullback to deduce the existence of a unique morphism $\phi : T \to R$ such that the following diagram commutes:

$$\begin{tikzcd}
T \arrow[dr, dashed, "\phi"] \arrow[drr, bend left, "q_0"] \arrow[ddr, bend right, "s_0^S \circ d_1^S \circ q_1"'] & & \\
& R \arrow[r, "d_0^R"] \arrow[d, "\check{f}"'] & X \arrow[d, "f"] \\
& S \arrow[r, "d_0^S"'] & Y
\end{tikzcd}$$

Finally, we observe that the below diagram also commutes

$$\begin{tikzcd}
T \arrow[dr, dashed, "s_0^R \circ d_0^R \circ \phi"] \arrow[drr, bend left, "q_0"] \arrow[ddr, bend right, "s_0^S \circ d_1^S \circ q_1"'] & & \\
& R \arrow[r, "d_1^R"] \arrow[d, "\check{f}"'] & X \arrow[d, "f"] \\
& S \arrow[r, "d_1^S"'] & Y
\end{tikzcd}$$

The top triangle commutes because

$$d_1^R \circ (s_0^R \circ d_0^R \circ \phi) = d_0^R \circ \phi = q_0.$$

The bottom triangle commutes because

$$
\begin{aligned}
\check{f} \circ (s_0^R \circ d_0^R \circ \phi) &= (s_0^S \circ f) \circ d_0^R \circ \phi \\
&= s_0^S \circ f \circ q_0 \\
&= s_0^S \circ d_1^S \circ q_1.
\end{aligned}
$$

Therefore, the diagram reveals that the square indexed by 1 is a pullback.

(b) By repeating the argument in part (a) while interchanging the indices 0 and 1 (except for the sections), we also achieve the statement of part (b). $\square$

An additional component of 1.6.2 is that if f is a monomorphism, then $R \cong f^{-1}(S)$ or equivalently, $R \subset f^{-1}(S)$ (which is a result of the fibrant morphism $(\check{f}, f)$) and $f^{-1}(S) \subset R$. Unfortunately, **I am not sure how to prove this!** The full statement of 1.6.2 is from [Bou17, Exercise 1.6.9].

Example 1.6.6. Here is an important example of a fibrant morphism from [Bou17, Proposition 1.6.10]. Let R be an internal equivalence relation on X. The morphism (d_2, d_1^R) induced by the inclusion $R[d_0] \subset d_1^{-1}(R)$ is fibrant. That is, the commutative squares in the diagram below indexed by 0 (or 1) are pullbacks

$$
\begin{tikzcd}
R[d_0^R] \arrow[rr, shift left=2, "d_0"] \arrow[rr, shift right=2, "d_1"'] \arrow[d, "d_2"'] & & R \arrow[ll] \arrow[r, "d_0^R"] \arrow[d, "d_1^R"] & X \\
R \arrow[rr, shift left=2, "d_0^R"] \arrow[rr, shift right=2, "d_1^R"'] & & X \arrow[ll] &
\end{tikzcd}
$$

Example 1.6.6 will play a prominent role in the next lemma, which captures the essence of the symmetry condition associated to an equivalence relation.

Lemma 1.6.3. *Let $\mathscr{C}$ be a category with pullbacks and R be an internal equivalence relation on X. Then, there exists a unique morphism $\sigma_R : R \to R$ such that $d_0^R \circ \sigma_R = d_1^R$ and $d_1^R \circ \sigma_R = d_0^R$.*

Proof. Assume that $\mathscr{C}$ is a category with pullbacks and R is an internal equivalence relation on X. Then, there exists a fibrant morphism $(d_2, d_1^R) : R[d_0^R] \to R$ in $Ref(\mathscr{C})$, given by the following diagram in $\mathscr{C}$:

$$
\begin{tikzcd}
R[d_0^R] \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] \arrow[d, "d_2"'] & R \arrow[l] \arrow[r, "d_0^R"] \arrow[d, "d_1^R"] & X \\
R \arrow[r, shift left=2, "d_0^R"] \arrow[r, shift right=2, "d_1^R"'] & X \arrow[l] &
\end{tikzcd}
$$

Let us first use the pullback square associated with $R[d_0^R]$ in order to deduce the existence of a unique morphism $u : R[d_0^R] \to R[d_0^R]$ in $\mathscr{C}$ such that the following diagram commutes:

$$
\begin{tikzcd}
R[d_0^R] \arrow[dr, dashed, "u"] \arrow[drr, bend left, "d_1"] \arrow[ddr, bend right, "d_0"'] & & \\
& R[d_0^R] \arrow[r, "d_0"] \arrow[d, "d_1"'] & R \arrow[d, "d_0^R"] \\
& R \arrow[r, "d_0^R"'] & X
\end{tikzcd}
$$

Now we use the fact that (d_2, d_1^R) is fibrant, alongside the universal property of the pullback, to find a unique morphism $\beta : R \to R[d_0^R]$ such that the following diagram commutes:

$$
\begin{tikzcd}
R \arrow[dr, dashed, "\beta"] \arrow[drr, bend left, "id_R"] \arrow[ddr, bend right, "id_R"'] & & \\
& R[d_0^R] \arrow[r, "d_1"] \arrow[d, "d_2"'] & R \arrow[d, "d_1^R"] \\
& R \arrow[r, "d_1^R"'] & X
\end{tikzcd}
$$

Now consider the unique composite $d_2 \circ u \circ \beta : R \to R$. Then, we have

$$
\begin{aligned}
d_0^R \circ (d_2 \circ u \circ \beta) &= (d_1^R \circ d_0) \circ u \circ \beta \\
&= d_1^R \circ (d_1 \circ \beta) \\
&= d_1^R \circ id_R = d_1^R.
\end{aligned}
$$

and

$$
\begin{aligned}
d_1^R \circ (d_2 \circ u \circ \beta) &= (d_1^R \circ d_1) \circ u \circ \beta \\
&= (d_1^R \circ d_0) \circ \beta \\
&= d_0^R \circ (d_2 \circ \beta) \\
&= d_0^R \circ id_R = d_0^R.
\end{aligned}
$$

Thus, $\sigma_R = d_2 \circ u \circ \beta$ is the desired morphism. □

Similarly to 1.6.3, there is an appropriate characterisation of the transitivity property of an equivalence relation.

Lemma 1.6.4. *Let $\mathscr{C}$ be a category with pullbacks and R be an internal equivalence relation on X. Let $(R \times_X R, q_0, q_1)$ be the pullback of the morphisms $d_0^R, d_1^R : R \to X$ such that $d_0^R \circ q_0 = d_1^R \circ q_1$. Then, there exists a unique morphism $\tau : R \times_X R \to R$ such that $d_0^R \circ \tau = d_0^R \circ q_1$ and $d_1^R \circ \tau = d_1^R \circ q_0$.*

I attempted to prove this, but to no avail.

Why is it reasonable to assume that 1.6.4 holds? There is an alternative definition of 1.6.2 which is widely used in the literature. This alternative definition says roughly that R is an equivalence relation on X whenever R is a reflexive relation on X, which is further equipped with the unique morphisms $\sigma_R : R \to R$ and $\tau : R \times_X R \to R$ from 1.6.3 and 1.6.4 respectively. For instance, see [Bor94, Volume 2, Proposition 2.5.4] and [BG04, Page 167]. In particular, the latter reference was coauthored by the author of [Bou17]. With the interest of progressing further, we will adopt this alternative definition for the major theorem we will prove next.

Theorem 1.6.5. *Let $\mathscr{C}$ be a category with pullbacks and S be a reflexive relation on an object X. Let T be another object in $\mathscr{C}$. Then, the pairs of morphisms $(h, h') : T \to X \times X$ which can be factorised through the internal equivalence relation S, determine an equivalence relation on $Hom_{\mathscr{C}}(T, X)$ in the usual sense if and only if S is an internal equivalence relation.*

Proof. First assume that $\mathscr{C}$ is a category with pullbacks and S is an internal equivalence relation on an object X:

$$
S \underset{\xrightarrow[d_1]{}}{\overset{\xrightarrow{d_0}}{\xleftarrow{s_0}}} X
$$

From the definition of internal equivalence relation in [BG04, Page 167], there exists unique morphisms $\sigma : S \to S$ and $\tau : S \times_X S \to S$ such that for σ, $d_0 \circ \sigma = d_1$ and $d_1 \circ \sigma = d_0$. Furthermore, recall that $S \times_X S$ is the pullback

$$\begin{array}{ccc} S \times_X S & \xrightarrow{q_0} & S \\ \downarrow{\scriptstyle q_1} & & \downarrow{\scriptstyle d_1} \\ S & \xrightarrow{d_0} & X \end{array}$$

Then, τ must satisfy $d_1 \circ \tau = d_1 \circ q_1$ and $d_0 \circ \tau = d_0 \circ q_0$.

Assume that T is another object in $\mathscr{C}$. We claim that the pairs of morphisms which factor through S

$$\{(h, h') \in Hom_{\mathscr{C}}(T, X) \times Hom_{\mathscr{C}}(T, X) \mid h = d_0 \circ \phi, h' = d_1 \circ \phi \text{ where } \phi \in Hom_{\mathscr{C}}(T, S)\} \tag{1.1}$$

is an equivalence relation on $Hom_{\mathscr{C}}(T, X)$. Call this set H.

To see that reflexivity holds, we must show that $(h, h) \in H$ for all $h \in Hom_{\mathscr{C}}(T, X)$. Consider the morphism $s_0 \circ h \in Hom_{\mathscr{C}}(T, S)$. Then, $h = d_0 \circ (s_0 \circ h) = d_1 \circ (s_0 \circ h)$. By definition, $(h, h) \in H$ as required.

To see that symmetry holds, assume that $(h, h') \in H$ where $h, h' \in Hom_{\mathscr{C}}(T, X)$. Then, there exists $\alpha \in Hom_{\mathscr{C}}(T, S)$ such that $h = d_0 \circ \alpha$ and $h' = d_1 \circ \alpha$. By using the symmetry morphism $\sigma : S \to S$ (either from the definition or 1.6.3), $h' = d_0 \circ (\sigma \circ \alpha)$ and $h = d_1 \circ (\sigma \circ \alpha)$. Hence, $(h', h) \in H$, which proves symmetry.

To see that transitivity holds, assume that $(f, g), (g, h) \in H$. Then, there exists $\beta, \gamma \in Hom_{\mathscr{C}}(T, S)$ such that $f = d_0 \circ \beta$, $g = d_1 \circ \beta = d_0 \circ \gamma$ and $h = d_1 \circ \gamma$. Using the universal property of the pullback on

$$\begin{array}{ccc} S \times_X S & \xrightarrow{q_0} & S \\ \downarrow{\scriptstyle q_1} & & \downarrow{\scriptstyle d_1} \\ S & \xrightarrow{d_0} & X \end{array}$$

we find a unique morphism $u : T \to S \times_X S$ such that the following diagram commutes:

$$\begin{tikzcd}
T \arrow[drr, bend left, "\beta"] \arrow[dr, dashed, "u"] \arrow[ddr, bend right, "\gamma"'] & & \\
& S \times_X S \arrow[r, "q_0"] \arrow[d, "q_1"] & S \arrow[d, "d_1"] \\
& S \arrow[r, "d_0"] & X
\end{tikzcd}$$

Now we have $f = d_0 \circ (q_0 \circ u) = d_0 \circ (\tau \circ u)$ and
$h = d_1 \circ (q_1 \circ u) = d_1 \circ (\tau \circ u)$. Therefore, $(f, h) \in H$.

Thus, H defines an equivalence relation on $Hom_{\mathscr{C}}(T, X)$.

For the converse, assume that H (as defined in (1.1)) is an equivalence relation on $Hom_{\mathscr{C}}(T, X)$ in the usual sense of sets for any object T in $\mathscr{C}$. In order to emphasise the dependency on T, the set given in (1.1) will now be called H_T.

Firstly, to see that S is a reflexive relation, note that by reflexivity, $(id_X, id_X) \in H_X$. So, there exists $s \in Hom_{\mathscr{C}}(X, S)$ such that $id_X = d_0 \circ s = d_1 \circ s$. This shows that S is a reflexive relation on X.

To see that S is equipped with the symmetry morphism σ, observe that $(d_0, d_1) \in H_S$ because $d_0 = d_0 \circ id_S$ and $d_1 = d_1 \circ id_S$ and by symmetry $(d_1, d_0) \in H_S$. Hence, there exists a morphism $\sigma : S \to S$ such that $d_1 = d_0 \circ \sigma$ and $d_0 = d_1 \circ \sigma$ as required.

Finally, to see that S is equipped with the transitivity morphism τ, observe that $(d_0 \circ q_1, d_1 \circ q_1)$ and $(d_0 \circ q_0, d_1 \circ q_0)$ are both elements of $H_{S \times_X S}$. From the pullback square associated to $S \times_X S$, $d_1 \circ q_0 = d_0 \circ q_1$. By the transitivity of $H_{S \times_X S}$, $(d_0 \circ q_0, d_1 \circ q_1) \in H_{S \times_X S}$ and consequently, there exists $\tau : S \times_X S \to S$ such that $d_0 \circ q_0 = d_0 \circ \tau$ and $d_1 \circ q_1 = d_1 \circ \tau$.

Thus, S is an internal equivalence relation in the sense of [BG04, Page 167]. $\square$

Theorem 1.6.5 establishes a connection between internal equivalence relations and the definition of equivalence relations on sets that we are used to in 1.6.1.

1.7 More about equivalence relations

We dedicate this section to stating and proving various results about internal equivalence relations. The first result is from [Bou17, Exercise 1.6.11, Part a]. It warrants a mention since it is powerful.

Theorem 1.7.1. *Let $\mathscr{C}$ be a category with pullback. Then, the category of internal equivalence relations $Equ(\mathscr{C})$ is invariant under pullback. Specifically, if we have a pullback square in $Equ(\mathscr{C})$*

$$\begin{array}{ccc} T & \xrightarrow{\alpha} & U \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\ V & \xrightarrow{\delta} & W \end{array}$$

with U, V, W being internal equivalence relations, then T must be an internal equivalence relation.

In particular, since the intersection and inverse image are pullbacks by definition, internal equivalence relations must be preserved under intersection and inverse image.

Our next task is to demonstrate that $R[f]$ is an internal equivalence relation on X, where $f : X \to Y$ is any morphism in a category with pullbacks.

Theorem 1.7.2. *Let $\mathscr{C}$ be a category with pullbacks and $f : X \to Y$ be a morphism. Then, the reflexive relation $R[f]$ on X is formed by the following pullback square in $Ref(\mathscr{C})$:*

$$\begin{array}{ccc} R[f] & \xrightarrow{(\check{f},f)} & \Delta_Y \\ \downarrow{\scriptstyle d_{R[f]}} & & \downarrow{\scriptstyle s_0^Y} \\ \nabla_X & \xrightarrow{\nabla_f} & \nabla_Y \end{array}$$

In particular, $R[f]$ is an internal equivalence relation on X, referred to as the ***kernel equivalence relation*** *of f.*

Proof. Assume that $\mathscr{C}$ is a category with pullbacks and $f : X \to Y$ is a morphism in $\mathscr{C}$. First observe that the morphism $(\check{f}, f) : R[f] \to \Delta_Y$ makes the following diagram in $\mathscr{C}$ commute:

$$\begin{tikzcd}
R[f] \arrow[r, shift left=3, "p_0^f"] \arrow[r, shift right=3, "p_1^f"'] \arrow[d, "\check{f}"'] & X \arrow[l, "s_0^f" description] \arrow[d, "f"] \\
Y \arrow[r, shift left=3, "id_Y"] \arrow[r, shift right=3, "id_Y"'] & Y \arrow[l, "id_Y" description]
\end{tikzcd}$$

Hence, $\check{f} = f \circ p_0^f = f \circ p_1^f$. In turn, the following square in $Ref(\mathscr{C})$ also commutes:

$$\begin{tikzcd}
R[f] \arrow[r, "(f \circ p_0^f, f)"] \arrow[d, "((p_0^f, p_1^f), id_X)"'] & \Delta_Y \arrow[d, "(\Delta, id_Y)"] \\
\nabla_X \arrow[r, "((f,f),f)"] & \nabla_Y
\end{tikzcd}$$

For clarity, we expanded the definition of all the involved $Ref(\mathscr{C})$ morphisms. Again, $\Delta : Y \to Y \times Y$ is the diagonal map. To see that this defines a pullback square, suppose that we have the following commutative diagram in $Ref(\mathscr{C})$:

$$\begin{tikzcd}
Q \arrow[r, "(\check{g}, g)"] \arrow[d, "d_Q"'] & \Delta_Y \arrow[d, "s_0^Y"] \\
\nabla_X \arrow[r, "\nabla_f"] & \nabla_Y
\end{tikzcd}$$

Let $q_0, q_1 : Q \to X$ be the morphisms associated with the reflexive relation Q. By considering the morphism $(\check{g}, g) : Q \to \Delta_Y$, the commutative diagram it induces in $\mathscr{C}$ and the above commutative diagram in $Ref(\mathscr{C})$, we find that $g = f$ and $\check{g} = f \circ q_0 = f \circ q_1$. Now we use the fact that $R[f]$ is originally a pullback in $\mathscr{C}$ to deduce the existence of a unique morphism $\alpha : Q \to R[f]$ such that the following diagram commutes in $\mathscr{C}$:

$$\begin{tikzcd}
Q \arrow[dr, dashed, "\alpha"] \arrow[drr, bend left, "q_0"] \arrow[ddr, bend right, "q_1"'] & & \\
& R[f] \arrow[r, "p_0^f"] \arrow[d, "p_1^f"] & X \arrow[d, "f"] \\
& X \arrow[r, "f"] & Y
\end{tikzcd}$$

Thus, $(\alpha, id_X) : Q \to R[f]$ is the unique morphism in $Ref(\mathscr{C})$ which makes the following diagram commute:

$$\begin{tikzcd}
Q \arrow[dr, dashed, "(\alpha, id_X)"] \arrow[drr, bend left, "(f \circ q_0, f)"] \arrow[ddr, bend right, "((q_0^f, q_1^f), id_X)"'] & & \\
& R[f] \arrow[r, "(f \circ p_0^f, f)"] \arrow[d, "((p_0^f, p_1^f), id_X)"] & \Delta_Y \arrow[d, "(\Delta, id_Y)"] \\
& \nabla_X \arrow[r, "((f,f),f)"] & \nabla_Y
\end{tikzcd}$$

Hence, $R[f]$ is given by the following pullback square in $Ref(\mathscr{C})$:

$$\begin{tikzcd}
R[f] \arrow[r, "(\check{f}, f)"] \arrow[d, "d_{R[f]}"] & \Delta_Y \arrow[d, "s_0^Y"] \\
\nabla_X \arrow[r, "\nabla_f"] & \nabla_Y
\end{tikzcd}$$

$\square$

Here are two consequences of 1.7.2. Since ∇_X, ∇_Y and Δ_Y are internal equivalence relations and 1.7.1 holds, $R[f]$ must be an internal equivalence relation on X. Secondly, the inverse image $f^{-1}(\Delta_Y)$ is also given by the pullback square in $Ref(\mathscr{C})$:

$$\begin{tikzcd}
f^{-1}(\Delta_Y) \arrow[r, "(f_{\Delta_Y}, f)"] \arrow[d, "d_{R[f]}"] & \Delta_Y \arrow[d, "s_0^Y"] \\
\nabla_X \arrow[r, "\nabla_f"] & \nabla_Y
\end{tikzcd}$$

Thus, $R[f]$ and $f^{-1}(\Delta_Y)$ must factor through each other as internal equivalence relations, revealing that $R[f] \subset f^{-1}(\Delta_Y)$, $f^{-1}(\Delta_Y) \subset R[f]$ and subsequently $R[f] \cong f^{-1}(\Delta_Y)$. A question which directly stems from this is: what is the inverse image of the indiscrete reflexive relation ∇_Y?

Theorem 1.7.3. *Let $\mathscr{C}$ be a category with pullbacks and X, Y be objects in $\mathscr{C}$. Let $f : X \to Y$ be a morphism in $\mathscr{C}$. Then, $\nabla_X \cong f^{-1}(\nabla_Y)$.*

Proof. Assume that $\mathscr{C}$ is a category with pullbacks and $f : X \to Y$ is a morphism in $\mathscr{C}$. It suffices to show that the commutative square in $Ref(\mathscr{C})$

$$\begin{tikzcd}
\nabla_X \arrow[r, "\nabla_f"] \arrow[d, "d_{\nabla_X}"] & \nabla_Y \arrow[d, "d_{\nabla_Y}"] \\
\nabla_X \arrow[r, "\nabla_f"] & \nabla_Y
\end{tikzcd}$$

is a pullback square. In its most explicit form, the commutative diagram is

$$\begin{tikzcd}
\nabla_X \arrow[r, "((f,f),f)"] \arrow[d, "((\pi_{1,X},\pi_{2,X}),id_X)"'] & \nabla_Y \arrow[d, "((\pi_{1,Y},\pi_{2,Y}),id_Y)"] \\
\nabla_X \arrow[r, "((f,f),f)"] & \nabla_Y
\end{tikzcd}$$

where $\pi_{1,X}, \pi_{2,X} : X \times X \to X$ are projection maps onto the first and second factor respectively. Note that $(\pi_{1,X},\pi_{2,X}) = id_{X\times X}$ and similarly for Y.

Suppose that we have the following commutative diagram in $Ref(\mathscr{C})$:

$$\begin{tikzcd}
P \arrow[r, "(\check{p},p)"] \arrow[d, "d_P"] & \nabla_Y \arrow[d, "d_{\nabla_Y}"] \\
\nabla_X \arrow[r, "\nabla_f"] & \nabla_Y
\end{tikzcd}$$

Let $p_0, p_1 : P \to X$ be the pair of morphisms in $\mathscr{C}$ associated to the reflexive relation P. By commutativity of the above diagram, we find that $\check{p} = (f \circ p_0, f \circ p_1)$ and $p = f$. Now observe that $((p_0, p_1), id_X) : P \to \nabla_X$ is the unique morphism which makes the following diagram commute:

$$\begin{tikzcd}
P \arrow[dr, dashed, "((p_0,p_1),id_X)"] \arrow[drr, bend left, "((f\circ p_0,f\circ p_1),f)"] \arrow[ddr, bend right, "((p_0,p_1),id_X)"'] & & \\
& \nabla_X \arrow[r, "((f,f),f)"] \arrow[d, "(id_{X\times X},id_X)"] & \nabla_Y \arrow[d, "(id_{Y\times Y},id_Y)"] \\
& \nabla_X \arrow[r, "((f,f),f)"] & \nabla_Y
\end{tikzcd}$$

Thus, ∇_X forms the following pullback square

$$\begin{tikzcd}
\nabla_X \arrow[r, "\nabla_f"] \arrow[d, "d_{\nabla_X}"] & \nabla_Y \arrow[d, "d_{\nabla_Y}"] \\
\nabla_X \arrow[r, "\nabla_f"] & \nabla_Y
\end{tikzcd}$$

However, we also have the pullback square associated to the inverse image $f^{-1}(\nabla_Y)$:

$$\begin{tikzcd}
f^{-1}(\nabla_Y) \arrow[r, "(f_{\nabla_Y},f)"] \arrow[d, "d_{f^{-1}(\nabla_Y)}"] & \nabla_Y \arrow[d, "d_{\nabla_Y}"] \\
\nabla_X \arrow[r, "\nabla_f"] & \nabla_Y
\end{tikzcd}$$

Therefore, $f^{-1}(\nabla_Y)$ and ∇_X must factor through each other as reflexive relations on X. So, $f^{-1}(\nabla_Y) \cong \nabla_X$ as required. $\square$

The next theorem demonstrates when two kernel equivalence relations factor through each other.

Theorem 1.7.4. *Let $\mathscr{C}$ be a category with pullbacks. Suppose that $f : X \to Y$, $m : Y \to Z$ and $g = m \circ f : X \to Z$ are morphisms in $\mathscr{C}$. Then, $R[f] \subset R[g]$. Furthermore, if m is a monomorphism, then $R[f] \cong R[g]$.*

Proof. Assume that $\mathscr{C}$ is a category with pullbacks and f, m and $g = m \circ f$ are the morphisms defined as in the statement of the theorem. We notice that by our factorisation of g, the following diagram in $Ref(\mathscr{C})$ commutes:

$$\begin{tikzcd}
R[f] \arrow[r, "(f\circ p_0^f,f)"] \arrow[d, "((p_0^f,p_1^f),id_X)"'] & \Delta_Y \arrow[r, "(m,m)"] \arrow[d, "(\Delta,id_Y)"] & \Delta_Z \arrow[d, "(\Delta,id_Z)"] \\
\nabla_X \arrow[r, "((f,f),f)"] & \nabla_Y \arrow[r, "((m,m),m)"] & \nabla_Z
\end{tikzcd}$$

Note that the left side commutative square is the pullback constructed in 1.7.2. Focusing on the outer square and using the universal property of the pullback $R[g]$, we deduce the existence of a morphism $(\phi, id_X) : R[f] \to R[g]$ in $Ref(\mathscr{C})$ such that the following diagram commutes:

$$\begin{tikzcd}
R[f] \arrow[dr, dashed, "(\phi,id_X)"] \arrow[drr, bend left, "(g\circ p_0^f,g)"] \arrow[ddr, bend right, "((p_0^f,p_1^f),id_X)"'] & & \\
& R[g] \arrow[r, "(g\circ p_0^g,g)"] \arrow[d, "((p_0^g,p_1^g),id_X)"] & \Delta_Z \arrow[d, "(\Delta,id_Z)"] \\
& \nabla_X \arrow[r, "((g,g),g)"] & \nabla_Z
\end{tikzcd}$$

Therefore, $R[f] \subset R[g]$.

Now assume that m is a monomorphism. Our goal is to show that the RHS square in the diagram below

$$\begin{tikzcd}
R[f] \arrow[r, "(f\circ p_0^f, f)"] \arrow[d, "((p_0^f, p_1^f), id_X)"'] & \Delta_Y \arrow[r, "(m,m)"] \arrow[d, "(\Delta, id_Y)"] & \Delta_Z \arrow[d, "(\Delta, id_Z)"] \\
\nabla_X \arrow[r, "((f,f),f)"] & \nabla_Y \arrow[r, "((m,m),m)"] & \nabla_Z
\end{tikzcd}$$

is a pullback square. Suppose that we have the following commutative square in $Ref(\mathscr{C})$:

$$\begin{tikzcd}
Q \arrow[r, "(\check{q}, q)"] \arrow[d, "((q_0,q_1), id_Y)"'] & \Delta_Z \arrow[d, "(\Delta, id_Z)"] \\
\nabla_Y \arrow[r, "((m,m),m)"'] & \nabla_Z
\end{tikzcd}$$

From the above diagram, we must have $q = m$. Meanwhile, the morphism $(\check{q}, q) : Q \to \Delta_Z$ makes the following diagram in $\mathscr{C}$ commute:

$$\begin{tikzcd}
Q \arrow[r, shift left=2, "q_0"] \arrow[r, shift right=2, "q_1"'] \arrow[d, "\check{q}"'] & Y \arrow[l] \arrow[d, "m"] \\
Z \arrow[r, shift left=2, "id_Z"] \arrow[r, shift right=2, "id_Z"'] & Z \arrow[l, "id_Z"]
\end{tikzcd}$$

So, $\check{q} = m \circ q_0 = m \circ q_1$. Note that since m is a monomorphism, $q_0 = q_1$. From this, observe that the morphism $(q_0, id_Y) : Q \to \Delta_Y$ in $Ref(\mathscr{C})$ makes the following diagram commute:

$$\begin{tikzcd}
Q \arrow[dr, dashed, "(q_0, id_Y)"] \arrow[drr, bend left, "(m\circ q_0, m)"] \arrow[ddr, bend right, "((q_0,q_1), id_Y)"'] & & \\
& \Delta_Y \arrow[r, "(m,m)"] \arrow[d, "(\Delta, id_Y)"] & \Delta_Z \arrow[d, "(\Delta, id_Z)"] \\
& \nabla_Y \arrow[r, "((m,m),m)"] & \nabla_Z
\end{tikzcd}$$

The morphism $(q_0, id_Y) : Q \to \Delta_Y$ must be unique. To see why this is the case, suppose that there exists another morphism $(q_0', r) : Q \to \Delta_Y$ such that the above diagram commutes. Then, from the top triangle, $(m, m) \circ (q_0, id_Y) = (m, m) \circ (q_0', r)$. Since (m, m) is a monomorphism, $(q_0, id_Y) = (q_0', r)$ as required.

So, both squares of the commutative diagram below are pullbacks

$$\begin{tikzcd}
R[f] \arrow[r, "(f\circ p_0^f, f)"] \arrow[d, "((p_0^f, p_1^f), id_X)"'] & \Delta_Y \arrow[r, "(m,m)"] \arrow[d, "(\Delta, id_Y)"] & \Delta_Z \arrow[d, "(\Delta, id_Z)"] \\
\nabla_X \arrow[r, "((f,f),f)"] & \nabla_Y \arrow[r, "((m,m),m)"] & \nabla_Z
\end{tikzcd}$$

By 1.4.5, the outer square must also be a pullback in $Ref(\mathscr{C})$ and consequently, $R[g]$ must factor through $R[f]$. So, $R[g] \subset R[f]$, thereby proving that $R[f] \cong R[g]$ in the case where m is a monomorphism. $\square$

The proof of 1.7.4 suggests that a monomorphism satisfies particularly nice properties with regards to internal equivalence relations. This is formalised below.

Theorem 1.7.5. *Let $\mathscr{C}$ be a category with pullbacks and $f : X \to Y$ be a morphism in $\mathscr{C}$. Then, f is a monomorphism if and only if $R[f] \cong \Delta_X$.*

Proof. Assume that $\mathscr{C}$ is a category with pullbacks and $f : X \to Y$ is a morphism in $\mathscr{C}$.

To show: (a) If f is a monomorphism, then $R[f] \cong \Delta_X$.

(b) If $R[f] \cong \Delta_X$, then f is a monomorphism.

(a) This proceed similarly to the proof of 1.7.4. Suppose that we have the following commutative square in $Ref(\mathscr{C})$:

$$\begin{tikzcd}
Q \arrow[r, "(\check{q}, q)"] \arrow[d, "((q_0, q_1), id_X)"'] & \Delta_Y \arrow[d, "(\Delta, id_Y)"] \\
\nabla_X \arrow[r, "((f,f),f)"'] & \nabla_Y
\end{tikzcd}$$

From the above diagram, we must have $q = f$. Meanwhile, the morphism $(\check{q}, q) : Q \to \Delta_Z$ makes the following diagram in $\mathscr{C}$ commute:

$$\begin{tikzcd}
Q \arrow[r, shift left=4, "q_0"] \arrow[r, shift right=4, "q_1"'] \arrow[d, "\check{q}"'] & X \arrow[l] \arrow[d, "f"] \\
Y \arrow[r, shift left=4, "id_Y"] \arrow[r, shift right=4, "id_Y"'] & Y \arrow[l, "id_Y"']
\end{tikzcd}$$

So, $\check{q} = f \circ q_0 = f \circ q_1$. Note that since f is a monomorphism, $q_0 = q_1$. From this, observe that the morphism $(q_0, id_X) : Q \to \Delta_X$ in $Ref(\mathscr{C})$ makes the following diagram commute:

$$\begin{tikzcd}
Q \arrow[dr, dashed, "{(q_0,id_X)}"] \arrow[drr, bend left, "{(f\circ q_0,f)}"] \arrow[ddr, bend right, "{((q_0,q_1),id_X)}"'] & & \\
& \Delta_X \arrow[r, "{(f,f)}"] \arrow[d, "{(\Delta,id_X)}"] & \Delta_Y \arrow[d, "{(\Delta,id_Y)}"] \\
& \nabla_X \arrow[r, "{((f,f),f)}"] & \nabla_Y
\end{tikzcd}$$

The morphism $(q_0, id_X) : Q \to \Delta_Y$ must be unique. To see why this is the case, suppose that there exists another morphism $(q_0', r) : Q \to \Delta_X$ such that the above diagram commutes. Then, from the top triangle, $(f, f) \circ (q_0, id_X) = (f, f) \circ (q_0', r)$. Since (f, f) is a monomorphism, $(q_0, id_X) = (q_0', r)$ as required. Therefore, the square below in $Ref(\mathscr{C})$ is a pullback square

$$\begin{tikzcd}
\Delta_X \arrow[r, "{(f,f)}"] \arrow[d, "{(\Delta,id_X)}"] & \Delta_Y \arrow[d, "{(\Delta,id_Y)}"] \\
\nabla_X \arrow[r, "{((f,f),f)}"] & \nabla_Y
\end{tikzcd}$$

However, we also have the pullback square

$$\begin{tikzcd}
R[f] \arrow[r, "{(f_R,f)}"] \arrow[d, "d_{R[f]}"] & \Delta_Y \arrow[d, "{(\Delta,id_Y)}"] \\
\nabla_X \arrow[r, "{((f,f),f)}"] & \nabla_Y
\end{tikzcd}$$

Thus, $R[f]$ and Δ_X must factorise through each other, revealing that $R[f] \cong \Delta_X$.

(b) Now assume that $R[f] \cong \Delta_X$. Assume that $g, h : W \to X$ are morphisms which satisfy $f \circ g = f \circ h$. Since $R[f] \cong \Delta_X$, we have the following pullback square in $Ref(\mathscr{C})$:

$$\begin{tikzcd} \Delta_X \arrow[r, "{(f,f)}"] \arrow[d, "{(\Delta, id_X)}"] & \Delta_Y \arrow[d, "{(\Delta, id_Y)}"] \\ \nabla_X \arrow[r, "{((f,f),f)}"] & \nabla_Y \end{tikzcd}$$

and consequently, the following pullback square in $\mathscr{C}$:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\Delta"] & Y \arrow[d, "\Delta"] \\ X \times X \arrow[r, "{(f,f)}"] & Y \times Y \end{tikzcd}$$

Again, Δ is the usual diagonal morphism. By using the universal property of the pullback, there exists a unique morphism $\phi : W \to X$ such that the following diagram commutes:

$$\begin{tikzcd} W \arrow[dr, dashed, "\phi"] \arrow[drr, bend left, "f \circ g"] \arrow[ddr, bend right, "{(g,h)}"'] & & \\ & X \arrow[r, "f"] \arrow[d, "\Delta"] & Y \arrow[d, "\Delta"] \\ & X \times X \arrow[r, "{(f,f)}"] & Y \times Y \end{tikzcd}$$

From the bottom left triangle, we have

$$\Delta \circ \phi = (\phi, \phi) = (g, h)$$

and by uniqueness, $\phi = g = h$. This demonstrates that f is a monomorphism as required. $\square$

One particular corollary of 1.7.5 mentioned in [Bou17, Exercise 1.6.13] is that for all objects X in the category $\mathscr{C}$, $R[id_X] \cong \Delta_X$. We recall that id_X is the identity morphism on X.

The next lemma is a useful exercise in understanding the definition of the intersection of two reflexive relations.

Lemma 1.7.6. *Let $\mathscr{C}$ be a category with pullbacks. Consider the following factorisation of the morphisms $w : U \to X$ and $w' : U \to X'$ in $\mathscr{C}$:*

$$\begin{tikzcd} U \arrow[dr, dashed, "\phi"] \arrow[drr, bend left, "w'"] \arrow[ddr, bend right, "w"'] & & \\ & P \arrow[r, "h'"] \arrow[d, "h"'] & X' \arrow[d, "f'"] \\ & X \arrow[r, "f"] & Y \end{tikzcd}$$

Then, $R[\phi] \cong R[w] \cap R[w']$ as internal equivalence relations on U.

Proof. Assume that $\mathscr{C}$ is a category with pullbacks. Assume that the factorisation ϕ of the morphisms $w : U \to X$ and $w' : U \to X'$ in the commutative diagram in the statement of the lemma holds.

To show: (a) $R[\phi] \subset R[w] \cap R[w']$.

(b) $R[w] \cap R[w'] \subset R[\phi]$.

(a) It suffices to produce a pullback square in $Ref(\mathscr{C})$ of the form

$$\begin{tikzcd} R[\phi] \arrow[r] \arrow[d] & R[w'] \arrow[d] \\ R[w] \arrow[r] & \nabla_U \end{tikzcd}$$

We will let $d_0^w, d_1^w : R[w] \to U$ be the morphisms associated with the reflexive relation $R[w]$ on U. Similar definitions apply for $R[\phi]$ and $R[w']$. Using the universal property of the pullback on $R[w]$ in $\mathscr{C}$, there exists a unique morphism $\alpha : R[\phi] \to R[w]$ such that the following diagram commutes

$$\begin{tikzcd}
R[\phi] \arrow[drr, bend left, "d_0^\phi"] \arrow[dr, dashed, "\alpha"] \arrow[ddr, bend right, "d_1^\phi"'] & & \\
& R[w] \arrow[r, "d_0^w"] \arrow[d, "d_1^w"'] & U \arrow[d, "w"] \\
& U \arrow[r, "w"] & X
\end{tikzcd}$$

The outer square commutes because

$$w \circ d_0^\phi = h \circ (\phi \circ d_0^\phi) = h \circ (\phi \circ d_1^\phi) = w \circ d_1^\phi.$$

In a similar vein, we can apply the universal property of the pullback to $R[w']$ in $\mathscr{C}$ to deduce the existence of a unique morphism $\alpha' : R[\phi] \to R[w']$ such that the following diagram commutes:

$$\begin{tikzcd}
R[\phi] \arrow[drr, bend left, "d_0^\phi"] \arrow[dr, dashed, "\alpha'"] \arrow[ddr, bend right, "d_1^\phi"'] & & \\
& R[w'] \arrow[r, "d_0^{w'}"] \arrow[d, "d_1^{w'}"'] & U \arrow[d, "w'"] \\
& U \arrow[r, "w'"] & X
\end{tikzcd}$$

Using the morphisms α and α', we deduce that the following square in $Ref(\mathscr{C})$ is commutative:

$$\begin{tikzcd}
R[\phi] \arrow[r, "(\alpha', id_U)"] \arrow[d, "(\alpha, id_U)"'] & R[w'] \arrow[d, "((d_0^{w'}, d_1^{w'}), id_U)"] \\
R[w] \arrow[r, "((d_0^w, d_1^w), id_U)"'] & \nabla_U
\end{tikzcd}$$

By definition, it must factor through $R[w] \cap R[w']$:

$$\begin{tikzcd}
R[\phi] \arrow[drr, bend left, "(\alpha', id_U)"] \arrow[dr, dashed] \arrow[ddr, bend right, "(\alpha, id_U)"'] & & \\
& R[w] \cap R[w'] \arrow[r, "\iota'"] \arrow[d, "\iota"'] & R[w'] \arrow[d, "d_{R[w']}"] \\
& R[w] \arrow[r, "d_{R[w]}"] & \nabla_U
\end{tikzcd}$$

where ι and ι' are inclusion morphisms. This proves that
$R[\phi] \subset R[w] \cap R[w']$.

(b) Consider the following commutative diagram below:

$$\begin{tikzcd}
R[\phi] \arrow[r, "(\alpha', id_U)"] \arrow[d, "(\alpha, id_U)"] & R[w'] \arrow[r, "(w' \circ d_0^{w'}, w')"] \arrow[d, "d_{R[w']}"] & \Delta_{X'} \arrow[d, "(\Delta, id_{X'})"] \\
R[w] \arrow[r, "d_{R[w]}"] & \nabla_U \arrow[r, "((w', w'), w')"] & \nabla_{X'}
\end{tikzcd}$$

Notice that both the LHS and RHS squares are pullbacks. Hence, from 1.4.5, the outside square must also be a pullback square. Next, observe that the following square in $Ref(\mathscr{C})$ commutes:

$$\begin{tikzcd}
R[w] \cap R[w'] \arrow[r, "(w' \circ d_0^{w'}, w') \circ (\iota', id_U)"] \arrow[d, "(\iota, id_U)"] & \Delta_{X'} \arrow[d, "(\Delta, id_U)"] \\
R[w] \arrow[r, "((w', w'), w') \circ ((d_0^w, d_1^w), id_U)"'] & \nabla_{X'}
\end{tikzcd}$$

Therefore, $R[w] \cap R[w']$ must factor through $R[\phi]$ as follows:

$$\begin{tikzcd}
R[w] \cap R[w'] \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& R[\phi] \arrow[r] \arrow[d] & \Delta_{X'} \arrow[d, "d_{R[w']}"] \\
& R[w] \arrow[r, "d_{R[w]}"] & \nabla_{X'}
\end{tikzcd}$$

Hence, $R[w] \cap R[w'] \subset R[\phi]$ as required. So, parts (a) and (b) demonstrate that $R[\phi] \cong R[w] \cap R[w']$ as internal equivalence relations on U. $\square$

The next lemma is the content of [Bou17, Proposition 1.6.15]. It is labelled as very useful and is also one of the few propositions in [Bou17, Chapter 1] with an accompanying proof.

Lemma 1.7.7. *Let $\mathscr{C}$ be a category with pullbacks. Suppose that $(\tilde{f}, f) : R \to S$ is a morphism of equivalence relations, given by the commutative diagram below:*

$$\begin{tikzcd}
R \arrow[r, shift left=2, "d_0^R"] \arrow[r, shift right=2, "d_1^R"'] \arrow[d, "\tilde{f}"'] & X \arrow[l, "s_0^R"' description] \arrow[d, "f"] \\
S \arrow[r, shift left=2, "d_0^S"] \arrow[r, shift right=2, "d_1^S"'] & Y \arrow[l, "s_0^S"' description]
\end{tikzcd}$$

Then, the diagram given below is a pullback in the category $Equ(\mathscr{C})$:

$$\begin{tikzcd}
R \cap R[f] \arrow[r, "(\check{f}, f)"] \arrow[d, "j"] & \Delta_Y \arrow[d, "s_0^Y"] \\
R \arrow[r, "(\tilde{f}, f)"] & S
\end{tikzcd}$$

Here, $j : R \cap R[f] \to R$ is an inclusion of internal equivalence relations.

Proof. Assume that $\mathscr{C}$ is a category with pullbacks. Consider the following cube in $Equ(\mathscr{C})$:

$$\begin{tikzcd}
R \cap R[f] \arrow[rr] \arrow[dr] \arrow[dd] & & \Delta Y \arrow[dr, equal] \arrow[dd] & \\
& R[f] \arrow[rr] \arrow[dd] & & \Delta Y \arrow[dd] \\
R \arrow[rr] \arrow[dr] & & S \arrow[dr] & \\
& \nabla X \arrow[rr, "\nabla f"'] & & \nabla Y
\end{tikzcd}$$

The double line is just the identity morphism $(id_Y, id_Y) : \Delta_Y \to \Delta_Y$. Notice that the left and front faces of the cube are both pullbacks. Hence, from 1.4.5, the square formed from these two faces is also a pullback square. However, from the commutativity of the cube, this means that the square formed from the back and right faces is a pullback. Since $(id_Y, id_Y) : \Delta_Y \to \Delta_Y$ is a monomorphism (represented by the double line), we can apply 1.4.7 to deduce that the back face of the cube is a pullback as required. $\square$

We will finish this section by defining another type of equivalence relation.

Definition 1.7.1. Let $\mathscr{C}$ be a category with pullbacks. An internal equivalence relation R on X is called **effective** when there exists a morphism $f : X \to Y$ such that $R \cong R[f]$.

Example 1.7.2. We will work in the category of sets **Set**. Assume that T defines an internal equivalence relation on X:

$$\begin{tikzcd} T \arrow[r, shift left=2, "d_0^R"] \arrow[r, shift right=2, "d_1^R"'] & X \arrow[l, "s"'] \end{tikzcd}$$

We will show that T is an effective relation. Our first instinct is to form the quotient set X/T:

$$\begin{tikzcd} T \arrow[r, shift left=2, "t_0"] \arrow[r, shift right=2, "t_1"'] & X \arrow[l, "s"'] \arrow[r, "q"] & X/T \end{tikzcd}$$

Recall that in **Set**, the projection morphisms q is the coequalizer of d_0^R and d_1^R. Since we have the following pullback square in $\mathscr{C}$,

$$\begin{tikzcd} R[q] \arrow[r, "p_0^q"] \arrow[d, "p_1^q"] & X \arrow[d, "q"] \\ X \arrow[r, "q"] & X/T \end{tikzcd}$$

$T \subset R[q]$. We claim also that $R[q] \subset T$. Since T is an internal equivalence relation, we have the following pullback square in $\mathscr{C}$:

$$\begin{tikzcd} R[t_0] \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] \arrow[d, "d_2"'] & T \arrow[l] \arrow[d, "t_1"] \\ T \arrow[r, shift left=2, "t_0"] \arrow[r, shift right=2, "t_1"'] & X \arrow[l] \end{tikzcd}$$

Using the universal property of the pullback, we have the factorisation

$$\begin{tikzcd} R[q] \arrow[dr, dashed] \arrow[drr, bend left, "s \circ p_0^q"] \arrow[ddr, bend right, "s \circ p_0^q"'] & & \\ & R[t_0] \arrow[r, "d_1"] \arrow[d, "d_2"] & T \arrow[d, "t_1"] \\ & T \arrow[r, "t_1"] & X \end{tikzcd}$$

Thus, $R[q] \subset T$ and consequently, $T \cong R[q]$.

Equivalence relations in the categories **Grp** and **Mon** are also effective.The final lemma of this section states that effective equivalence relations are stable under the inverse image.

Lemma 1.7.8. *Let $\mathscr{C}$ be a category with pullbacks and S be an effective (internal) equivalence relation on Y. Let $f : X \to Y$ be a morphism. Then, the inverse image $f^{-1}(S)$ is also an effective equivalence relation.*

Proof. Assume that $\mathscr{C}$ is a category with pullbacks and S is an effective equivalence relation on the object X. Assume that $f : X \to Y$ is a morphism in $\mathscr{C}$. Since $f^{-1}(S)$ is a formed as a pullback in $Equ(\mathscr{C})$, it must be an internal equivalence relation as a result of 1.7.1.

Suppose that $g : Y \to Z$ is a morphism such that $S \cong R[g]$. Due to this, the following square is a pullback square in $Equ(\mathscr{C})$:

$$\begin{tikzcd} S \arrow[r, "(\check{g},g)"] \arrow[d, "((d_0^S,d_1^S),id_Y)"'] & \Delta_Z \arrow[d, "(\Delta,id_Y)"] \\ \nabla_Y \arrow[r, "((g,g),g)"] & \nabla_Z \end{tikzcd}$$

From this, we construct the following commutative diagram:

$$\begin{tikzcd} f^{-1}(S) \arrow[r, "(\check{f},f)"] \arrow[d, "d_{f^{-1}(S)}"'] & S \arrow[r, "(\check{g},g)"] \arrow[d, "d_S"'] & \Delta_Z \arrow[d, "(\Delta,id_Y)"] \\ \nabla_X \arrow[r, "((f,f),f)"] & \nabla_Y \arrow[r, "((g,g),g)"] & \nabla_Z \end{tikzcd}$$

Notice that both the LHS and RHS squares are pullbacks. So, by 1.4.5, the outer square is also a pullback. Therefore, from the definition of a kernel equivalence relation, $f^{-1}(S) \cong R[g \circ f]$ and $f^{-1}(S)$ is an effective equivalence relation. $\square$

1.8 The square construction

Let $\mathscr{C}$ be a category with pullback and R, S be internal equivalence relations on an object X in $\mathscr{C}$. Let $d_0^R, d_1^R : R \to X$ be the morphisms associated with the equivalence relation R and $s_0^R : X \to R$ be the associated section. Similarly, let $d_0^S, d_1^S : S \to X$ be the morphisms associated with the equivalence relation S and $s_0^S : X \to S$ be the associated section.

Define $\Delta_0^R : R \to X \times X$ by $\Delta_0^R = (d_0^R, d_1^R)$. The object $S \times S$ defines an equivalence relation on $X \times X$ with the associated pair of morphisms in $Hom_{\mathscr{C}}(S \times S, X \times X)$ given by

$$D_0^S = d_0^S \times d_0^S \text{ and } D_1^S = d_1^S \times d_1^S.$$

Taking the inverse image of the equivalence relation $S \times S$ on $X \times X$ along Δ_0^R produces the following pullback square in $Equ(\mathscr{C})$:

$$\begin{tikzcd}
(\Delta_0^R)^{-1}(S \times S) \arrow[r, "((\delta_0^S,\delta_1^S),\Delta_0^R)"] \arrow[d, "((\delta_0^R,\delta_1^R),id_R)"'] & S \times S \arrow[d, "((D_0^S,D_1^S),id_{X\times X})"] \\
\nabla_R \arrow[r, "((\Delta_0^R,\Delta_0^R),\Delta_0^R)"'] & \nabla_{X\times X}
\end{tikzcd}$$

By the universal property, there exists a unique morphism $(\sigma^S, \check{\sigma}) : S \times S \to (\Delta_0^R)^{-1}(S \times S)$ such that the following diagram commutes:

$$\begin{tikzcd}
S \times S \arrow[dr, dashed, "(\sigma^S,\check{\sigma})"] \arrow[drr, bend left, "id_{S\times S}"] \arrow[ddr, bend right, "((s_0^R\circ d_0^S,s_0^R\circ d_1^S),s_0^R)"'] & & \\
& (\Delta_0^R)^{-1}(S \times S) \arrow[r, "((\delta_0^S,\delta_1^S),\Delta_0^R)"] \arrow[d, "((\delta_0^R,\delta_1^R),id_R)"'] & S \times S \arrow[d, "((D_0^S,D_0^S),id_{X\times X})"] \\
& \nabla_R \arrow[r, "((\Delta_0^R,\Delta_0^R),\Delta_0^R)"'] & \nabla_{X\times X}
\end{tikzcd}$$

Observe that $\check{\sigma} = s_0^R$ and

$$\begin{aligned}
\delta_0^R \circ \sigma^S \circ s_0^S &= s_0^R \circ d_0^S \circ s_0^S \\
&= s_0^R \\
&= \delta_0^R \circ \sigma^R \circ s_0^R.
\end{aligned}$$

Here, $\delta_0^R \circ \sigma^R = \delta_1^R \circ \sigma^R$ is the identity morphism on R. In a similar vein, $\delta_1^R \circ \sigma^S \circ s_0^S = \delta_1^R \circ \sigma^S \circ s_0^S$. Therefore,

$$(\delta_0^R, \delta_1^R) \circ \sigma^S \circ s_0^S = (\delta_0^R, \delta_1^R) \circ \sigma^R \circ s_0^R$$

Since $(\Delta_0^R)^{-1}(S \times S)$ is a reflexive relation on R, the induced morphism $(\delta_0^R, \delta_1^R) : (\Delta_0^R)^{-1}(S \times S) \to R \times R$ is a monomorphism. So, $\sigma^S \circ s_0^S = \sigma^R \circ s_0^R$. The most interesting fact arising from this computation is that $(\Delta_0^R)^{-1}(S \times S)$ is also an equivalence relation on S, with morphisms δ_0^S, δ_1^S and common section σ^S. Furthermore, $(\Delta_0^R)^{-1}(S \times S)$ satisfies the following commutative diagram in $\mathscr{C}$:

$$\begin{tikzcd}
(\Delta_0^R)^{-1}(S \times S) \arrow[r, shift left=2, "\delta_0^S"] \arrow[r, shift right=2, "\delta_1^S"'] \arrow[d, shift right=3, "\delta_1^R"'] \arrow[d, shift left=3, "\delta_0^R"] & S \arrow[l, "\sigma^S"'] \arrow[d, shift right=3, "d_1^S"'] \arrow[d, shift left=3, "d_0^S"] \\
R \arrow[u, "\sigma^R"] \arrow[r, shift left=2, "d_0^R"] \arrow[r, shift right=2, "d_1^R"'] & X \arrow[l, "s_0^R"'] \arrow[u, "s_0^S"]
\end{tikzcd}$$

We will place particular emphasis on the fact that in the above diagram, the squares indexed by 0 and 1 both commute. **In addition,** we also have $d_0^S \circ \delta_1^S = d_1^R \circ \delta_0^R$ and $d_1^S \circ \delta_0^S = d_0^R \circ \delta_1^R$. To see why this is the case, the commutativity of our original inverse image diagram yields

$$\begin{aligned}
(D_0^S, D_1^S) \circ (\delta_0^S, \delta_1^S) &= (D_0^S \circ (\delta_0^S, \delta_1^S), D_1^S \circ (\delta_0^S, \delta_1^S)) \\
&= (d_0^S \circ \delta_0^S, d_0^S \circ \delta_1^S, d_1^S \circ \delta_0^S, d_1^S \circ \delta_1^S)
\end{aligned}$$

and

$$\begin{aligned}
(\Delta_0^R, \Delta_0^R) \circ (\delta_0^R, \delta_1^R) &= (\Delta_0^R \circ \delta_0^R, \Delta_1^R \circ \delta_1^R) \\
&= (d_0^R \circ \delta_0^R, d_1^R \circ \delta_0^R, d_0^R \circ \delta_1^R, d_1^R \circ \delta_1^R).
\end{aligned}$$

Definition 1.8.1. Let $\mathscr{C}$ be a category with pullback and R, S be internal equivalence relations on an object X in $\mathscr{C}$. The inverse image of the equivalence relation $S \times S$ on $X \times X$ along $\Delta_0^R = (d_0^R, d_1^R) : R \to X \times X$ is denoted by $R \square S = (\Delta_0^R)^{-1}(S \times S)$.

Example 1.8.2. This is the content of [Bou17, Exercise 1.6.19] and gives a concrete example of the square construction. Let $\mathscr{C}$ be a category with pullbacks and $R = R[f]$ be an internal equivalence relation on an object X, where $f : X \to Y$ is a morphism.

The **paraterminal map** $(\check{f}, f) : S \to \nabla_Y$ is the morphism in $Equ(\mathscr{C})$ which makes the following diagram in $\mathscr{C}$ commute:

$$\begin{tikzcd}
S \arrow[r, shift left=3, "d_1^S"] \arrow[r, shift right=3, "d_0^S"'] \arrow[d, "\check{f}"'] & X \arrow[l, "s_0^S"] \arrow[d, "f"] \\
Y \times Y \arrow[r, shift left=3, "\pi_2"] \arrow[r, shift right=3, "\pi_1"'] & Y \arrow[l, "\Delta"]
\end{tikzcd}$$

Note that $\check{f} = (f \circ d_0^S, f \circ d_1^S)$. We can describe the kernel equivalence relation of the paraterminal map via the following square construction:

$$\begin{tikzcd}
R[f]\square S \arrow[r, shift left=3, "\delta_0^S"] \arrow[r, shift right=3, "\delta_1^S"'] \arrow[d, shift right=3, "\delta_1^R"'] \arrow[d, shift left=3, "\delta_0^R"] & S \arrow[l] \arrow[d, shift right=3, "d_1^S"'] \arrow[d, shift left=3, "d_0^S"] \\
R[f] \arrow[r, shift left=3, "d_0^f"] \arrow[r, shift right=3, "d_1^f"'] \arrow[u] & X \arrow[l] \arrow[u] \arrow[d, "f"] \\
 & Y
\end{tikzcd}$$

Hence, the kernel equivalence relation is represented by the following diagram:

$$\begin{tikzcd}
R[f]\square S \arrow[r, shift left=3, "(\delta_0^S, d_0^S)"] \arrow[r, shift right=3, "(\delta_1^S, d_1^S)"'] & S \arrow[l] \arrow[r, "(\check{f}, f)"] & \nabla_Y
\end{tikzcd}$$

To see why this is the case, we have to show that $\check{f} \circ \delta_0^S = \check{f} \circ \delta_1^S$. We have

$$\begin{aligned}
\check{f} \circ \delta_0^S &= (f \circ d_0^S, f \circ d_1^S) \circ \delta_0^S \\
&= (f \circ d_0^S \circ \delta_0^S, f \circ d_1^S \circ \delta_0^S) \\
&= (f \circ d_0^f \circ \delta_0^R, f \circ d_0^f \circ \delta_1^R) \\
&= (f \circ d_1^f \circ \delta_0^R, f \circ d_1^f \circ \delta_1^R) \\
&= (f \circ d_0^S \circ \delta_1^S, f \circ d_1^S \circ \delta_1^S) \\
&= \check{f} \circ \delta_1^S.
\end{aligned}$$

Note that in the second last line, we used the fact that $f \circ d_0^f = f \circ d_1^f$.

In order to better understand 1.8.1, we will foray briefly into our flagship category **Set**. If we make the square construction in **Set** and obtain the following diagram:

$$\begin{tikzcd}
R\square S \arrow[r, shift left=3, "\delta_0^S"] \arrow[r, shift right=3, "\delta_1^S"'] \arrow[d, shift right=3, "\delta_1^R"'] \arrow[d, shift left=3, "\delta_0^R"] & S \arrow[l, "\sigma^S"'] \arrow[d, shift right=3, "d_1^S"'] \arrow[d, shift left=3, "d_0^S"] \\
R \arrow[u, "\sigma^R"] \arrow[r, shift left=3, "d_0^R"] \arrow[r, shift right=3, "d_1^R"'] & X \arrow[l, "s_0^R"'] \arrow[u, "s_0^S"']
\end{tikzcd}$$

The pair contained in our equivalence relation on X are $(d_0^S \circ \delta_0^S, d_1^S \circ \delta_0^S)$, $(d_0^S \circ \delta_1^S, d_1^S \circ \delta_1^S)$, $(d_0^R \circ \delta_0^R, d_1^R \circ \delta_0^R)$ and $(d_0^R \circ \delta_1^R, d_1^R \circ \delta_1^R)$. If we set $u = d_0^S \circ \delta_0^S$, $v = d_1^S \circ \delta_0^S$, $u' = d_0^S \circ \delta_1^S$ and $v' = d_1^S \circ \delta_1^S$, then $uSv, u'Sv', uRu'$ and vRv'. Hence, $R\square S$ is the subset $(u, v, u', v') \in X^4$ such that the relations $uSv, u'Sv', uRu'$ and vRv' hold. We represent this by the following diagram:

$$\begin{tikzcd}
u \arrow[r, dotted, "S"] \arrow[d, dotted, "R"] & v \arrow[d, dotted, "R"] \\
u' \arrow[r, dotted, "S"] & v'
\end{tikzcd}$$

In a general category with pullbacks $\mathscr{C}$, the notation uSv means that a pair of morphisms $(u, v) : T \to X \times X$ must factorise through the equivalence relation S (see [Bou17, Exercise 1.6.8] and 1.6.5), as exhibited by the following diagram in $\mathscr{C}$:

$$\begin{tikzcd}
& X \times X \\
T \arrow[ur, "{(u,v)}"] \arrow[r, dashed, "\phi"] & S \arrow[u, "{(d_0^S, d_1^S)}"']
\end{tikzcd}$$

The next lemma shows how the square construction gives rise to a fibrant morphism.

Lemma 1.8.1. *Let $\mathscr{C}$ be a category with pullbacks and (R, S) be any pair of equivalence relations on X. If $R \subset S$, then the following morphism in $Equ(\mathscr{C})$*

$$\begin{tikzcd}
R\Box S \arrow[rr, shift left=2, "\delta_1^S"] \arrow[rr, shift right=2, "\delta_0^S"'] \arrow[d, "(\delta_0^R{,}\delta_1^R)"'] & & S \arrow[ll] \arrow[d, "(d_0^S{,}d_1^S)"] \\
R\times R \arrow[rr, shift left=2, "d_1^R\times d_1^R"] \arrow[rr, shift right=2, "d_0^R\times d_0^R"'] & & X\times X \arrow[ll]
\end{tikzcd}$$

is fibrant.

Proof. Assume that $\mathscr{C}$ is a category with pullbacks and (R, S) is a pair of equivalence relations on X. Assume that $R \subset S$. We must show that the square indexed by 0

$$\begin{tikzcd}
R\Box S \arrow[r, "\delta_0^S"] \arrow[d, "(\delta_0^R{,}\delta_1^R)"] & S \arrow[d, "(d_0^S{,}d_1^S)"] \\
R\times R \arrow[r, "d_0^R\times d_0^R"'] & X\times X
\end{tikzcd}$$

is a pullback square. So, suppose we have the following commutative square:

$$\begin{tikzcd}
T \arrow[rr, shift left=2, "\alpha_1"] \arrow[rr, shift right=2, "\alpha_0"'] \arrow[d, "(\phi_1{,}\phi_2)"'] & & S \arrow[ll] \arrow[d, "(d_0^S{,}d_1^S)"] \\
R\times R \arrow[rr, shift left=2, "d_1^R\times d_1^R"] \arrow[rr, shift right=2, "d_0^R\times d_0^R"'] & & X\times X \arrow[ll]
\end{tikzcd}$$

Let $h = d_0^R \circ \phi_1$ and $h' = d_0^R \circ \phi_2$. Then, the pair $(h, h') : T \to X \times X$ factors through the equivalence relation S, from the top and right morphisms. Hence, we can write hSh'. Furthermore, define $\check{h} = d_1^R \circ \phi_1$ and $\check{h}' = d_1^R \circ \phi_2$. Then, $hR\check{h}$ and $h'R\check{h}'$. So, we have the following diagram:

$$\begin{tikzcd}
h \arrow[r, dotted, "S"] \arrow[d, dotted, "R"] & h' \arrow[d, dotted, "R"] \\
\check{h} & \check{h}'
\end{tikzcd}$$

Since $R \subset S$, the pairs $(h, \check{h})$ and $(h', \check{h}')$ both factor through S, giving rise to the following diagram:

$$\begin{tikzcd}
h \arrow[r, dotted, "S"] \arrow[d, dotted, "S"] & h' \arrow[d, dotted, "S"] \\
\check{h} & \check{h}'
\end{tikzcd}$$

Since $Hom_{\mathscr{C}}(T,X)$ is an equivalence relation in the set-theoretic sense (see 1.6.5), $\check{h}Sh, hSh', h'S\check{h}'$ and by transitivity, $\check{h}S\check{h}'$. Hence, the following diagram holds:

$$\begin{tikzcd}
h \arrow[r, dotted, "S"] \arrow[d, dotted, "R"] & h' \arrow[d, dotted, "R"] \\
\check{h} \arrow[r, dotted, "S"] & \check{h}'
\end{tikzcd}$$

This demonstrates that we obtain a factorisation of (h,h') through $R\square S$. Therefore, the commutative square indexed by 0 must be a pullback square as required. $\square$

1.9 Split graphs and split relations

Let $\mathscr{C}$ be a category and $f : X \to Y$ be a split epimorphism. Then, there exists $s : Y \to X$ such that $f \circ s = id_Y$. Consider the following commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "id_X"] \arrow[d, "s\circ f"] & X \arrow[d, "f"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

Then, there exists a unique morphism $s_1^f : X \to R[f]$ such that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[dr, dashed, "s_1^f"] \arrow[drr, bend left, "id_X"] \arrow[ddr, bend right, "s\circ f"'] & & \\
& R[f] \arrow[r, "p_1^f"] \arrow[d, "p_0^f"'] & X \arrow[d, "f"] \\
& X \arrow[r, "f"] & Y
\end{tikzcd}$$

The first lemma of this section describes some of the properties associated with the unique morphism s_1^f.

Lemma 1.9.1. *Let $\mathscr{C}$ be a category and $f : X \to Y$ be a split epimorphism and $s : Y \to X$ be the morphism such that $f \circ s = id_Y$. Let $s_1^f : X \to R[f]$ be the unique section of $p_1^f : R[f] \to X$ such that $p_0^f \circ s_1^f = s \circ f$. Then, the morphism $p_0^f \circ s_1^f : X \to X$ coequalizes the pair (p_0^f, p_1^f), $f = coeq(p_0^f, p_1^f)$ and the following square is a pullback square:*

$$\begin{tikzcd}
R[f] & X \arrow[l, "s_1^f"'] \\
X \arrow[u, "s_0^f"] & Y \arrow[l, "s"] \arrow[u, "s"']
\end{tikzcd}$$

Recall that $s_0^f : X \to R[f]$ is the unique section associated to the reflexive relation $R[f]$ on the object X.

Proof. Assume that $\mathscr{C}$ is a category and $f : X \to Y$ is a split epimorphism, with associated section $s : Y \to X$. Suppose that $s_1^f : X \to R[f]$ is the unique morphism which makes the following diagram commute:

$$\begin{tikzcd}
X \arrow[dr, dashed, "s_1^f"] \arrow[drr, bend left, "id_X"] \arrow[ddr, bend right, "s \circ f"'] & & \\
& R[f] \arrow[r, "p_1^f"] \arrow[d, "p_0^f"'] & X \arrow[d, "f"] \\
& X \arrow[r, "f"] & Y
\end{tikzcd}$$

To see that the composite $p_0^f \circ s_1^f : X \to X$ coequalizes the kernel pair (p_0^f, p_1^f), we compute directly from the definition that

$$\begin{aligned}
(p_0^f \circ s_1^f) \circ p_0^f &= (s \circ f) \circ p_0^f \\
&= (s \circ f) \circ p_1^f \quad (f \circ p_0^f = f \circ p_1^f) \\
&= (p_0^f \circ s_1^f) \circ p_1^f.
\end{aligned}$$

Thus, $p_0^f \circ s_1^f$ coequalizes the kernel pair (p_0^f, p_1^f).

To prove that f is in fact, the coequalizer of (p_0^f, p_1^f), we first recall from 1.2.2 that $f = coeq(id_X, s \circ f) = coeq(id_X, p_0^f \circ s_1^f)$. Assume that $g : X \to Z$ is a morphism which coequalizes (p_0^f, p_1^f). Then, $g \circ p_0^f = g \circ p_1^f$. By precomposing with s_1^f, we find that $g \circ p_0^f \circ s_1^f = g \circ p_1^f \circ s_1^f$ and by using the definition of s_1^f, we obtain $g = g \circ p_0^f \circ s_1^f$. Thus, g coequalizes the pair $(id_X, p_0^f \circ s_1^f)$. But, $f = coeq(id_X, p_0^f \circ s_1^f)$, which means that there exists a unique morphism $\alpha : Y \to Z$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, shift left, "id_X"] \arrow[r, shift right, "p_0^f \circ s_1^f"'] & X \arrow[r, "f"'] \arrow[dr, phantom, ""] \arrow[r, ""', phantom] \arrow[ur, "g", bend left=0, shift left=0, start anchor=north east, end anchor=south west, to path={}] & Y \arrow[u, dashed, "\alpha"] \\ & & \end{tikzcd}$$

$$\begin{tikzcd} & & Z \\ X \arrow[r, shift left, "id_X"] \arrow[r, shift right, "p_0^f \circ s_1^f"'] & X \arrow[ur, "g"] \arrow[r, "f"'] & Y \arrow[u, dashed, "\alpha"'] \end{tikzcd}$$

The main point here is that $\alpha : Y \to Z$ is also the unique morphism which makes the following diagram commute:

$$\begin{tikzcd} & & Z \\ R[f] \arrow[r, shift left, "p_0^f"] \arrow[r, shift right, "p_1^f"'] & X \arrow[ur, "g"] \arrow[r, "f"'] & Y \arrow[u, dashed, "\alpha"'] \end{tikzcd}$$

This demonstrates that $f = coeq(p_0^f, p_1^f)$.

For the last assertion, we begin with $p_0^f \circ s_1^f = s \circ f$. By first precomposing with s and then s_0^f, we first deduce that $p_0^f \circ s_1^f \circ s = s$ and consequently,

$$s_1^f \circ s = s_0^f \circ p_0^f \circ s_1^f \circ s = s_0^f \circ s.$$

Now suppose that we have the following commutative diagram:

$$\begin{tikzcd} T \arrow[r, "t_1"] \arrow[d, "t_2"] & X \arrow[d, "s_1^f"] \\ X \arrow[r, "s_0^f"] & R[f] \end{tikzcd}$$

We must show that there exists a unique morphism $\beta : T \to Y$ such that $s \circ \beta = t_1 = t_2$. Since $t_1 = t_2$, we will simply write t in place of t_1 and t_2. Since $s_0^f \circ t = s_1^f \circ t$, $t = p_0^f \circ s_1^f \circ t$. Similarly to the proof that $f = coeq(p_0^f, p_1^f)$, we recall from 1.2.2 that $s = eq(id_X, p_0^f \circ s_1^f)$. Hence, there exists a unique morphism $\beta : T \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} T \arrow[d, dashed, "\beta"'] \arrow[dr, "t"] & & \\ Y \arrow[r, "s"] & X \arrow[r, shift left, "id_X"] \arrow[r, shift right, "p_0^f \circ s_1^f"'] & X \end{tikzcd}$$

So, β also makes the following diagram commute:

$$\begin{tikzcd} T \arrow[d, dashed, "\beta"'] \arrow[dr, "t"] & & \\ Y \arrow[r, "s"] & X \arrow[r, shift left, "s_0^f"] \arrow[r, shift right, "s_1^f"'] & R[f] \end{tikzcd}$$

This completes the proof and also reveals that $s = eq(s_0^f, s_1^f)$. □

The point of 1.9.1 is that it is our first example of a *split reflexive relation.*

Definition 1.9.1. Let $\mathscr{C}$ be a category and G denote a reflexive graph/relation on the object X, which is given by the following diagram:

$$\begin{tikzcd} G \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] & X \arrow[l, "s_0"'] \end{tikzcd}$$

We say that G is a **split reflexive graph/relation** if there exists a morphism $s_1 : X \to G$ such that $d_1 \circ s_1 = id_X$ and $d_0 \circ s_1$ coequalizes the pair (d_0, d_1).

$$\begin{tikzcd} G \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] & X \arrow[l, "s_0"'] \arrow[l, bend right=60, "s_1"'] \end{tikzcd}$$

Lemma 1.9.1 shows that if $f : X \to Y$ is a split epimorphism, then $R[f]$ is a split reflexive (equivalence) relation. Interestingly, a converse statement also holds, which yields a characterisation of split equivalence relations. The remainder of this section is dedicated to proving this characterisation. First, we need to prove specific properties satisfied by split reflexive graphs.

Lemma 1.9.2. *Let $\mathscr{C}$ be a category with pullbacks and G be a split reflexive graph on the object X, as depicted by the diagram below:*

$$\begin{tikzcd} G \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] & X \arrow[l, "s_0"'] \arrow[l, bend right=60, "s_1"'] \end{tikzcd}$$

Then, the morphism $d_0 \circ s_1 : X \to X$ is idempotent. If $s = eq(id_X, d_0 \circ s_1) : I \to X$ and $q : X \to I$ is the unique map satisfying $q \circ s = d_0 \circ s_1$, then there exists a unique factorisation $\rho : G \to R[q]$. Finally, if G is a split reflexive relation, ρ is a monomorphism.

Proof. Assume that $\mathscr{C}$ is a category with pullbacks and G is a split reflexive graph on the object X. To see that the morphism $d_0 \circ s_1 : X \to X$ is idempotent, we compute directly that

$$\begin{aligned}(d_0 \circ s_1 \circ d_0) \circ s_1 &= (d_0 \circ s_1 \circ d_1) \circ s_1 \\ &= d_0 \circ s_1.\end{aligned}$$

This uses the fact that $d_0 \circ s_1$ coequalizes the pair (d_0, d_1).

Now assume that $s = eq(id_X, d_0 \circ s_1) : I \to X$ and $q : X \to I$ is the unique map satisfying $q \circ s = d_0 \circ s_1$. We know that q exists as a consequence of the universal property of the equalizer s, given by the diagram below:

$$\begin{tikzcd} X \arrow[d, dashed, "q"'] \arrow[dr, "d_0 \circ s_1"] & & \\ I \arrow[r, "s"] & X \arrow[r, shift left, "id_X"] \arrow[r, shift right, "d_0 \circ s_1"'] & X \end{tikzcd}$$

Since $d_0 \circ s_1$ coequalizes (d_0, d_1), $d_0 \circ s_1 \circ d_0 = d_0 \circ s_1 \circ d_1$. From the construction of q, we have $s \circ q \circ d_0 = s \circ q \circ d_1$. From 1.2.1, s is an equalizer and is subsequently, a monomorphism. So, $q \circ d_0 = q \circ d_1$. This means that the following square commutes:

$$\begin{tikzcd} G \arrow[r, "d_0"] \arrow[d, "d_1"'] & X \arrow[d, "q"] \\ X \arrow[r, "q"] & I \end{tikzcd}$$

Thus, there exists a unique morphism $\rho : G \to R[q]$ such that the following diagram commutes:

$$\begin{tikzcd} G \arrow[dr, dashed, "\rho"] \arrow[drr, bend left, "d_0"] \arrow[ddr, bend right, "d_1"'] & & \\ & R[q] \arrow[r, "p_0^q"] \arrow[d, "p_1^q"'] & X \arrow[d, "q"] \\ & X \arrow[r, "q"'] & I \end{tikzcd}$$

Finally, assume that G is a split reflexive relation on X. Then, the morphism $(d_0, d_1) : G \to X \times X$ is a monomorphism. Assume that $\alpha, \beta : Y \to G$ are morphisms which satisfy $\rho \circ \alpha = \rho \circ \beta$. By composing with p_0^q on both sides, we deduce that $d_0 \circ \alpha = d_0 \circ \beta$. Analogously, if we compose both sides by p_1^q, we obtain $d_1 \circ \alpha = d_1 \circ \beta$. Hence, $(d_0, d_1) \circ \alpha = (d_0, d_1) \circ \beta$. Since (d_0, d_1) is a monomorphism, we deduce that $\alpha = \beta$, which reveals that ρ is a monomorphism as required. □

Lemma 1.9.2 tells us that we have the inclusion $G \subset R[q]$. It is precisely the condition of an equivalence relation which yields the reverse inclusion and thus, the following theorem:

Theorem 1.9.3. *Let $\mathscr{C}$ be a category with pullbacks and R be a split equivalence relation on the object X, given by the following diagram:*

$$\begin{tikzcd} R \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] & X \arrow[l, "s_0"'] \arrow[l, bend right=60, "s_1"'] \end{tikzcd}$$

Let $s = eq(id_X, d_0 \circ s_1) : I \to X$ and $q : X \to I$ be the unique map satisfying $q \circ s = d_0 \circ s_1$. Then, $R \cong R[q]$. In particular, R must be effective.

Proof. Assume that $\mathscr{C}$ is a category with pullbacks and R is a split equivalence relation on X. Assume that s and q are the morphisms defined as above. We know from 1.9.2 that $R \subset R[q]$.

To show: (a) $R[q] \subset R$.

(a) Recall from 1.9.2 that $d_0 \circ s_1 = s \circ q$. Since $q \circ p_0^q = q \circ p_1^q$, we compose both sides with s to deduce that $d_0 \circ s_1 \circ p_0^q = d_0 \circ s_1 \circ p_1^q$. So, the following square commutes in $\mathscr{C}$:

$$\begin{tikzcd} R[q] \arrow[r, "s_1 \circ p_0^q"] \arrow[d, "s_1 \circ p_1^q"'] & R \arrow[d, "d_0"] \\ R \arrow[r, "d_0"] & X \end{tikzcd}$$

Hence, it must factorise through $R[d_0]$, yielding the following commutative diagram:

$$\begin{tikzcd}
R[q] \arrow[dr, dashed, "R(s_1)"] \arrow[drr, bend left, "s_1 \circ p_0^q"] \arrow[ddr, bend right, "s_1 \circ p_1^q"'] & & \\
& R[d_0] \arrow[r, "e_0"] \arrow[d, "e_1"'] & R \arrow[d, "d_0"] \\
& R \arrow[r, "d_0"'] & X
\end{tikzcd}$$

In turn, $R(s_1) : R[q] \to R[d_0]$ is the unique morphism which makes the following diagram commute:

$$\begin{tikzcd}
R[q] \arrow[r, dashed, "R(s_1)"] \arrow[d, shift right=2, "p_0^q"'] \arrow[d, shift left=2, "p_1^q"] & R[d_0] \arrow[r, "d_2"] \arrow[d, shift right=2, "e_0"'] \arrow[d, shift left=2, "e_1"] & R \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] \\
X \arrow[r, "s_1"] \arrow[u] & R \arrow[r, "d_1"] \arrow[u] & X \arrow[u]
\end{tikzcd}$$

Observe that the RHS square is a pullback square because R is an internal equivalence relation on X. Therefore, $R[q] \subset R$ and $R \cong R[q]$. $\square$

Hence, any split equivalence relation can be thought of as a kernel equivalence relation.

1.10 Fibres and split epimorphisms

Admittedly, the term "fibre" is used in a variety of contexts in mathematics. As the term is generally used, a fibre refers to a situation where something "small" induces an entire "structure" from it. We will give an example to illustrate the general idea.

Example 1.10.1. Let M be a smooth manifold of dimension m and $p \in M$. The tangent bundle TM on M is a vector bundle of rank m. There is an induced projection map

$$\pi : TM \to M$$

$$(p, v) \mapsto p$$

which maps a point $p \in M$ and an associated tangent vector $v \in T_pM$ to just the point p. The fibre of the point p is the preimage

$$\pi^{-1}(p) = \{(p, v) \in TM\} \cong T_pM.$$

This is a $\mathbb{R}$-vector space with dimension m. In this manner, every point p on the smooth manifold M, induces a real vector space — the tangent space T_pM. This illustrates the general idea behind the term "fibre".

The fibres we are concerned with are defined rigorously below.

Definition 1.10.2. Let $\mathscr{C}$ be a category with pullbacks and Y be an object in $\mathscr{C}$. The **fibre** above Y is the category $Pt_Y(\mathscr{C}) = ¶_{\mathscr{C}}^{-1}(Y)$. The objects in $Pt_Y(\mathscr{C})$ are the split epimorphisms with codomain Y and the morphisms are morphisms $x : X \to X'$ in $\mathscr{C}$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "x"] \arrow[d, shift right, "f"'] & X' \arrow[d, shift right, "f'"'] \\ Y \arrow[r, "id_Y"] \arrow[u, shift right, "s"'] & Y \arrow[u, shift right, "s'"'] \end{tikzcd}$$

Notice that the general concept of a "fibre" applies here too. An initially "small structure" (an object of $\mathscr{C}$) induces a "larger structure"; an entire category in this case!

Definition 1.10.3. Let $\mathscr{C}$ be a category with pullbacks and $y : Y \to Y'$ be a morphism in $\mathscr{C}$. The **base change functor** associated with y, denoted by y^*, is a functor $y^* : Pt_{Y'}(\mathscr{C}) \to Pt_Y(\mathscr{C})$, which sends the object $X' \in Pt_{Y'}(\mathscr{C})$ to $X \in Pt_Y(\mathscr{C})$. Here, X' and X must satisfy the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "x=s'\circ y\circ f"] \arrow[d, shift right, "f"'] & X' \arrow[d, shift right, "f'"'] \\ Y \arrow[r, "y"] \arrow[u, shift right, "s"'] & Y' \arrow[u, shift right, "s'"'] \end{tikzcd}$$

Now suppose that ϕ is a morphism between $X', Z' \in Pt_{Y'}(\mathscr{C})$ so that the following diagram commutes:

$$\begin{tikzcd} X' \arrow[r, "\phi"] \arrow[d, shift right, "f'"'] & Z' \arrow[d, shift right, "g'"'] \\ Y' \arrow[r, "id_{Y'}"] \arrow[u, shift right, "s'"'] & Y' \arrow[u, shift right, "t'"'] \end{tikzcd}$$

Notice that $\phi = t' \circ f'$. Then, $y^*(\phi) = y^*(t' \circ f') = t \circ f$, where $t \circ f : X \to Z$ makes the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "t\circ f"] \arrow[d, shift right, "f"'] & Z \arrow[d, shift right, "g"'] \\ Y \arrow[r, "id_Y"] \arrow[u, shift right, "s"'] & Y \arrow[u, shift right, "t"'] \end{tikzcd}$$

In order to understand the definition of $Pt_Y(\mathscr{C})$ better, we will delve into the claim made in [Bou17, Remark 1.6.25]. Let $(f, s) : X \to Y$ be a split epimorphism in a category $\mathscr{C}$ with pullbacks. We claim that $Pt_{(f,s)}(Pt_Y(\mathscr{C})) = Pt_X(\mathscr{C})$.

The objects in $Pt_{(f,s)}(Pt_Y(\mathscr{C}))$ are the split epimorphisms from some object in $Pt_Y(\mathscr{C})$ to the split epimorphism $(f, s) : X \to Y$, which is represented by the following diagram in $\mathscr{C}$:

$$\begin{tikzcd} X' \arrow[r, dashed, shift left, "g_1"] \arrow[d, shift right, "f_1"'] & X \arrow[l, dashed, shift left, "t_1"] \arrow[d, shift right, "f"'] \\ Y \arrow[r, "id_Y"] \arrow[u, shift right, "s_1"'] & Y \arrow[u, shift right, "s"'] \end{tikzcd}$$

The object of $Pt_{(f,s)}(Pt_Y(\mathscr{C}))$ in question is given by the dashed arrows. Clearly, this object is also a split epimorphism with codomain X and hence, an object in $Pt_X(\mathscr{C})$.

Let $(g_2, t_2) : X'' \to X$ denote another split epimorphism in $Pt_{(f,s)}(Pt_Y(\mathscr{C}))$. Since all the split epimorphisms in $Pt_Y(\mathscr{C})$ have the same codomain, the morphisms in $Pt_{(f,s)}(Pt_Y(\mathscr{C}))$ are determined by the domains of each object and are thus, morphisms in $\mathscr{C}$ which make the following diagram commute (in our specific example):

$$\begin{tikzcd} X' \arrow[r, dashed, shift left] \arrow[d, shift right, "g_1"'] & X'' \arrow[l, dashed, shift left] \arrow[d, shift right, "g_2"'] \\ X \arrow[r, "id_X"] \arrow[u, shift right, "t_1"'] & X \arrow[u, shift right, "t_2"'] \end{tikzcd}$$

But, the dashed arrows comprise a morphism in $Pt_X(\mathscr{C})$, hence justifying the statement $Pt_{(f,s)}(Pt_Y(\mathscr{C})) \subseteq Pt_X(\mathscr{C})$. By unpacking the definition of $Pt_X(\mathscr{C})$, it is not too difficult to argue the reverse inclusion, which then yields $Pt_{(f,s)}(Pt_Y(\mathscr{C})) = Pt_X(\mathscr{C})$.

The next few definitions we will make are standard definitions in category theory. We cite [Lei16] as a great reference for these definitions, complete with accompanying examples.

Definition 1.10.4. Let $\mathscr{C}$ be a category. We say that an object $I \in \mathscr{C}$ is **initial** if for all objects $A \in \mathscr{C}$, there exists a unique morphism $\phi_A : I \to A$. Dually, we say that an object $T \in \mathscr{C}$ is **terminal** if for all objects $A \in \mathscr{C}$, there exists a unique morphism $\alpha_A : A \to T$.

Example 1.10.5. Let $\mathscr{C}$ be a category and $Equ(\mathscr{C})$ be the category of internal equivalence relations in $\mathscr{C}$. Let $Equ_X(\mathscr{C})$ be the subcategory of internal equivalence relations on the object X. The result 1.3.1 tells us that the indiscrete reflexive relation ∇_X is a terminal object, whereas the discrete reflexive relation Δ_X is an initial object in $Equ_X(\mathscr{C})$.

Example 1.10.6. Let **Cat** denote the category whose objects are categories and whose morphisms are functors. The terminal object of **Cat** is the category denoted by **1**, which contains a single object and a single morphism (which must be the identity morphism).

Definition 1.10.7. A category $\mathscr{C}$ is said to be **finitely complete** if it has pullbacks and a terminal object.

Fortunately, our main examples of categories — **Set**, **Grp**, **Mon**, **CoM** and **Ab** — are finitely complete. For instance, the trivial group is a terminal object in **Grp** and the one-element set $\{*\}$ is terminal in **Set** (see [Lei16, Definition 2.1.7]).

Here is the most important definition to this section:

Definition 1.10.8. Let $\mathscr{C}, \mathscr{D}$ be categories and $H : \mathscr{C} \to \mathscr{D}$ be a functor. For all objects X, X' in $\mathscr{C}$, the functor H induces the mapping

$$H_{X,X'} : Hom_{\mathscr{C}}(X, X') \to Hom_{\mathscr{D}}(H(X), H(X'))$$

between Hom-sets. We say that H is **faithful** if $H_{X,X'}$ is an injective map for all pairs of objects X, X' in $\mathscr{C}$. We say that H is **full** if $H_{X,X'}$ is a surjective map for all pairs of objects X, X' in $\mathscr{C}$. Finally, H is **fully faithful** if $H_{X,X'}$ is a bijection for all pairs of objects X, X' in $\mathscr{C}$.

Example 1.10.9. Let $U : \mathbf{Grp} \to \mathbf{Set}$ be the forgetful functor, which maps a group to its underlying set and a group morphism to the underlying function between the sets. We claim that U is a faithful functor. Let G, H be groups. To see why the mapping

$$U_{G,H} : Hom_{\mathbf{Grp}}(G,H) \to Hom_{\mathbf{Set}}(U(G),U(H))$$

is injective, suppose that $\phi_1, \phi_2 \in Hom_{\mathbf{Grp}}(G,H)$ and $U_{G,H}(\phi_1) = U_{G,H}(\phi_2)$. Then, by definition of the forgetful functor U, ϕ_1 and ϕ_2 must agree on the underlying set $U(G)$ and hence, on G itself. So, $\phi_1 = \phi_2$ and $U_{G,H}$ must be injective. This demonstrates that U is a faithful functor.

On the other hand, to see that U is not a full functor, define $\alpha : U(G) \to U(H)$ by $\alpha(g) = h$ for all $g \in U(G)$ and with $h \neq e_H$ (e_H is the identity element of the group H). Since $\alpha(e_G) = h \neq e_H$ by definition, α can never define a group morphism between G and H. So, $U_{G,H}$ is not surjective and consequently, U is not full.

Definition 1.10.10. Let $\mathscr{C}$ and $\mathscr{D}$ be categories and $H : \mathscr{C} \to \mathscr{D}$ be a functor. We say that H is an **equivalence of categories** if H is fully faithful and there exists a functor $F : \mathscr{D} \to \mathscr{C}$ and a natural isomorphism $\eta : id_{\mathscr{D}} \to H \circ F$, which means that for all morphisms $f : A \to A'$ in $\mathscr{D}$, the following diagram must commute:

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[d, "\eta_A"] & & A' \arrow[d, "\eta_{A'}"] \\ (H \circ F)(A) \arrow[rr, "(H\circ F)(f)"'] & & (H \circ F)(A') \end{tikzcd}$$

with η_A and $\eta_{A'}$ being isomorphisms in $\mathscr{D}$.

Example 1.10.11. The following example was taken from [CSM95, Theorem 5.4]. Let G be a *connected and simply connected* Lie group and $\mathfrak{g} = T_{e_G}G$ be the associated Lie algebra, which is just the tangent space at the identity element $e_G \in G$. Let **LieGrpC** denote the category of connected and simply connected Lie groups and **LieAlg**$\mathbb{R}$ be the category of Lie algebras. Let $\Phi : G \to H$ be a Lie group morphism in **LieGrpC**. The Lie functor *Lie* is defined by

$$Lie : \mathbf{LieGrpC} \to \mathbf{LieAlg}\mathbb{R}$$

$$G \to \mathfrak{g}$$

$$\Phi \to Lie(\Phi) \text{ with } Lie(\Phi)(X) = \frac{d}{dt}\Phi(\exp(tX))|_{t=0}.$$

Here, $X \in \mathfrak{g}$. We recall that for all $t \in \mathbb{R}$, $\exp(tX) \in G$. Then, the Lie functor *Lie* is an equivalence of the categories **LieGrpC** and **LieAlg**$\mathbb{R}$. Examples of connected and simply connected Lie groups include $GL_n(\mathbb{R}), SL_n(\mathbb{R})$ and $SO(n)$, where $n \in \mathbb{Z}_{\geq 2}$.

The definition 1.10.10 provides a notion of an "isomorphism of categories". There are a few different definitions of an equivalence of categories which are used throughout the literature. In particular, we will state a further two definitions of an equivalence of categories, stemming from [Lei16, Section 1.3, Page 34].

Definition 1.10.12. Let $\mathscr{C}$ and $\mathscr{D}$ be categories and $H : \mathscr{C} \to \mathscr{D}$ be a functor. We say that H is an **equivalence of categories** if H is fully faithful and *essentially surjective*. Essentially surjective means that for all objects $D \in \mathscr{D}$, there exists an object $C \in \mathscr{C}$ such that $H(C) \cong D$.

Definition 1.10.13. Let $\mathscr{C}$ and $\mathscr{D}$ be categories. An **equivalence of categories** between $\mathscr{C}$ and $\mathscr{D}$ is a pair of functors $H : \mathscr{C} \to \mathscr{D}$ and $F : \mathscr{D} \to \mathscr{C}$, equipped with natural isomorphisms $\eta : id_{\mathscr{C}} \to F \circ H$ and $\epsilon : H \circ F \to id_{\mathscr{D}}$, where $id_{\mathscr{C}}$ is the identity functor on $\mathscr{C}$.

We claim that all three definitions 1.10.10, 1.10.12 and 1.10.13 are equivalent to each other.

Theorem 1.10.1. *Let $\mathscr{C}$ and $\mathscr{D}$ be categories. Then, the three definitions of an equivalence of categories — 1.10.10, 1.10.12 and 1.10.13 — are equivalent.*

Proof. Assume that $\mathscr{C}$ and $\mathscr{D}$ are categories.

To show: (a) If 1.10.10 is satisfied, then 1.10.12 is satisfied.

(b) If 1.10.12 is satisfied, then 1.10.13 is satisfied.

(c) If 1.10.13 is satisfied, then 1.10.10 is satisfied.

(a) Assume that $H : \mathscr{C} \to \mathscr{D}$ is a functor which satisfies 1.10.10. Then, there exists a functor $F : \mathscr{D} \to \mathscr{C}$ and a natural isomorphism $\eta : id_{\mathscr{D}} \to H \circ F$. Also, H is fully faithful. To see that H is essentially surjective, assume that D is an object in $\mathscr{D}$. Since η is a natural isomorphism, η_D defines an isomorphism between D and $(H \circ F)(D)$. Hence, $D \cong (H \circ F)(D) = H(F(D))$. This demonstrates that H is essentially surjective. So, H satisfies 1.10.12.

(b) Assume that $H : \mathscr{C} \to \mathscr{D}$ is fully faithful and essentially surjective so that 1.10.12 is satisfied.

We will first build a functor $F : \mathscr{D} \to \mathscr{C}$. Since H is essentially surjective, for all objects $D \in \mathscr{D}$, there exists an object $C \in \mathscr{C}$ such that $H(C) \cong D$ with corresponding isomorphism denoted by $\varphi_D : D \to H(C)$. On the objects of $\mathscr{D}$, we define $F(D) = C$. Consequently, φ_D is an isomorphism between D and $(H \circ F)(D)$.

Let $g : D \to D'$ be a morphism in $\mathscr{D}$ such that $F(D') = C'$. Since H is fully faithful, there exists a *unique* morphism $f : C \to C'$ such that the following diagram in $\mathscr{D}$ commutes:

$$\begin{tikzcd}
D \arrow[r, "g"] \arrow[d, "\varphi_D"] & D' \arrow[d, "\varphi_{D'}"] \\
(H \circ F)(D) \arrow[r, "H(f)"'] & (H \circ F)(D')
\end{tikzcd}$$

Subsequently, we define $F(g) = f$. To see that F defines a functor from $\mathscr{D}$ to $\mathscr{C}$, we first note that the following square commutes in $\mathscr{D}$:

$$\begin{tikzcd}
D \arrow[r, "id_D"] \arrow[d, "\varphi_D"] & D \arrow[d, "\varphi_D"] \\
(H \circ F)(D) \arrow[r, "id_{(H \circ F)(D)}"'] & (H \circ F)(D)
\end{tikzcd}$$

Note that $id_{(H \circ F)(D)} = H(id_{F(D)})$. By construction of F, $H(F(id_D))$ also makes the above diagram commute. By uniqueness, we must have $H(F(id_D)) = H(id_{F(D)})$. Since H is a faithful functor, we deduce that $F(id_D) = id_{F(D)}$.

For composition of morphisms, suppose that $\alpha \in Hom_{\mathscr{D}}(D, D')$ and $\beta \in Hom_{\mathscr{D}}(D', D'')$ with $F(D'') = C''$. Note that the following diagram in $\mathscr{D}$ commutes:

$$\begin{tikzcd}
D \arrow[r, "\beta \circ \alpha"] \arrow[d, "\varphi_D"] & D'' \arrow[d, "\varphi_{D''}"] \\
(H \circ F)(D) \arrow[r, "H(F(\beta \circ \alpha))"'] & (H \circ F)(D'')
\end{tikzcd}$$

But, we also have the following commutative diagram in $\mathscr{D}$:

$$\begin{tikzcd}
D \arrow[r, "\alpha"] \arrow[d, "\varphi_D"] & D' \arrow[r, "\beta"] \arrow[d, "\varphi_{D'}"] & D'' \arrow[d, "\varphi_{D''}"] \\
(H \circ F)(D) \arrow[r, "H(F(\alpha))"'] & (H \circ F)(D') \arrow[r, "H(F(\beta))"'] & (H \circ F)(D'')
\end{tikzcd}$$

Therefore, $H(F(\beta) \circ F(\alpha))$ and $H(F(\beta \circ \alpha))$ both make the same diagram commute. By uniqueness, $H(F(\beta \circ \alpha)) = H(F(\beta) \circ F(\alpha))$ and since H is faithful, we thus have $F(\beta) \circ F(\alpha) = F(\beta \circ \alpha)$. So, $F : \mathscr{D} \to \mathscr{C}$ is a functor and $\varphi : id_{\mathscr{D}} \to H \circ F$ is a natural isomorphism.

It remains to construct a natural isomorphism from $id_{\mathscr{C}}$ to $F \circ H$. Let $C \in \mathscr{C}$ be an object. Then, $\varphi_{H(C)}$ as defined before is an isomorphism between $H(C)$ and $(H \circ F \circ H)(C)$. Again, we lean on the assumption that H is fully faithful in order to deduce the existence of a unique morphism $\psi_C : C \to (F \circ H)(C)$ such that $H(\psi_C) = \varphi_{H(C)}$.

However, $\varphi_{H(C)}$ is an isomorphism. So, there exists a morphism $\gamma : (H \circ F \circ H)(C) \to H(C)$ such that

$$\gamma \circ \varphi_{H(C)} = id_{H(C)} \text{ and } \varphi_{H(C)} \circ \gamma = id_{(H \circ F \circ H)(C)}.$$

Using the fact that H is fully faithful again, there exists a unique morphism $\delta : (F \circ H)(C) \to C$ such that $H(\delta) = \gamma$. Now observe that

$$\begin{aligned}
H(\delta \circ \psi_C) &= H(\delta) \circ H(\psi_C) \\
&= \gamma \circ \varphi_{H(C)} \\
&= id_{H(C)} = H(id_C)
\end{aligned}$$

and similarly, $H(\psi_C \circ \delta) = H(id_{(F \circ H)(C)})$. Since H is faithful, $\delta \circ \psi_C = id_C$ and $\psi_C \circ \delta = id_{(F \circ H)(C)}$. We conclude that for all objects $C \in \mathscr{C}$, ψ_C is an isomorphism.

To see that ψ is a natural isomorphism between $id_{\mathscr{C}}$ and $F \circ H$, consider the following diagram in $\mathscr{C}$:

$$\begin{tikzcd}
C \arrow[r, "f"] \arrow[d, "\psi_C"] & C' \arrow[d, "\psi_{C'}"] \\
(F \circ H)(C) \arrow[r, "(F \circ H)(f)"'] & (F \circ H)(C')
\end{tikzcd}$$

By applying H to this diagram, we obtain a commutative diagram in $\mathscr{D}$. Since H is faithful, we deduce that the above diagram in $\mathscr{C}$ commutes. Hence, $\psi : id_{\mathscr{C}} \to F \circ H$ is a natural isomorphism and 1.10.13 is satisfied.

(c) Assume that 1.10.13 is satisfied so that there exists a pair of functors $H : \mathscr{C} \to \mathscr{D}$ and $F : \mathscr{D} \to \mathscr{C}$, equipped with natural isomorphisms $\eta : id_{\mathscr{C}} \to F \circ H$ and $\epsilon : H \circ F \to id_{\mathscr{D}}$. In order for H to satisfy 1.10.10, it suffices to show that H is fully faithful. Assume that $X, Y \in \mathscr{C}$. Then, the functor H induces the map between Hom-sets

$$H_{X,Y} : Hom_{\mathscr{C}}(X, Y) \to Hom_{\mathscr{D}}(H(X), H(Y)).$$

To see that H is faithful, we must show that $H_{X,Y}$ is an injective map. Since η is a natural isomorphism, we find that the function

$$(F \circ H)_{X,Y} : Hom_{\mathscr{C}}(X, Y) \to Hom_{\mathscr{C}}((F \circ H)(X), (F \circ H)(Y))$$

is bijective. Since this is a function between sets, $(F \circ H)_{X,Y}$ is both a monomorphism and an epimorphism. Observe that $(F \circ H)_{X,Y}$ is the composite $F_{H(X),H(Y)} \circ H_{X,Y}$. Since $(F \circ H)_{X,Y}$ is a monomorphism, $H_{X,Y}$ must also be a monomorphism and thus, injective.

To see that H is fully faithful, we use the natural isomorphism $\epsilon : H \circ F \to id_{\mathscr{D}}$, which unveils that the following induced map is bijective:

$$Hom_{\mathscr{D}}(H(X), H(Y)) \to Hom_{\mathscr{D}}((H \circ F \circ H)(X), (H \circ F \circ H)(Y))$$

But this map is the composite $H_{(F\circ H)(X),(F\circ H)(Y)} \circ F_{H(X),H(Y)}$. Arguing in a similar manner to before, we deduce that the map $H_{(H\circ F)(X),(H\circ F)(Y)}$ is an epimorphism. Applying the fact that H is faithful, we find that $H_{(F\circ H)(X),(F\circ H)(Y)}$ must define the bijection:

$$Hom_{\mathscr{C}}((F \circ H)(X), (F \circ H)(Y)) \to Hom_{\mathscr{D}}((H \circ F \circ H)(X), (H \circ F \circ H)(Y))$$

But, the LHS is isomorphic (as sets) to $Hom_{\mathscr{C}}(X, Y)$, whereas the RHS is isomorphic (as sets) to $Hom_{\mathscr{D}}(H(X), H(Y))$. Thus, $H_{X,Y}$ is a bijection and $Hom_{\mathscr{C}}(X, Y) \cong Hom_{\mathscr{D}}(H(X), H(Y))$, which demonstrates that H is fully faithful as required. So, H being fully faithful, together with the functor F and the natural isomorphism ϵ^{-1}, show that 1.10.10 is satisfied. This completes the proof. $\square$

Consider the pair of functors H and F in 1.10.13. According to 1.10.10, F is also an equivalence of categories, commonly referred to as the **inverse equivalence** of H. Now, we will prove some characteristic properties about equivalences of categories and fully faithful functors.

Lemma 1.10.2. *Let $\mathscr{C}$ and $\mathscr{D}$ be categories and $H : \mathscr{C} \to \mathscr{D}$ be an equivalence of categories (in the sense of 1.10.10) so that H is equipped with a functor $F : \mathscr{D} \to \mathscr{C}$ and a natural isomorphism $\eta : id_{\mathscr{D}} \to H \circ F$. Then, F must be unique up to a natural isomorphism.*

Proof. Assume that $\mathscr{C}$ and $\mathscr{D}$ are categories and $H : \mathscr{C} \to \mathscr{D}$ is an equivalence of categories in the sense of 1.10.10, equipped with a functor $F : \mathscr{D} \to \mathscr{C}$ and a natural isomorphism $\eta : id_{\mathscr{D}} \to H \circ F$. Then, from 1.10.1, there exists a natural isomorphism $\epsilon : id_{\mathscr{C}} \to F \circ H$. Assume that $F' : \mathscr{D} \to \mathscr{C}$ is another functor, with a natural isomorphism $\eta' : id_{\mathscr{D}} \to H \circ F'$. Then, the following diagram in $\mathscr{D}$ commutes:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\eta'_X"] & Y \arrow[d, "\eta'_Y"] \\
(H \circ F')(X) \arrow[r, "(H\circ F')(f)"'] & (H \circ F')(Y)
\end{tikzcd}$$

By applying the functor F to the above diagram, we find that the diagram in $\mathscr{C}$ commutes:

$$\begin{tikzcd}
F(X) \arrow[r, "F(f)"] \arrow[d, "F(\eta'_X)"] & F(Y) \arrow[d, "F(\eta'_Y)"] \\
(F \circ H \circ F')(X) \arrow[r, "(F\circ H\circ F')(f)"'] \arrow[d, "\epsilon^{-1}_{F'(X)}"] & (F \circ H \circ F')(Y) \arrow[d, "\epsilon^{-1}_{F'(Y)}"] \\
F'(X) \arrow[r, "F'(f)"'] & F'(Y)
\end{tikzcd}$$

Since functors preserve isomorphisms, we find that the composite $\epsilon^{-1}_{F'(X)} \circ F(\eta'_X)$ defines an isomorphism between $F(X)$ and $F'(X)$ for all objects $X \in \mathscr{D}$. Thus, the functors F and F' are related via a natural transformation. $\square$

Since injective and surjective functions between sets are preserved under composition, faithful, full and fully faithful functors must be stable under composition. The next lemma reveals further interactions between faithful and/or full functors and composition.

Lemma 1.10.3. *Let $\mathscr{C}, \mathscr{D}$ and $\mathscr{E}$ be categories. Let $H : \mathscr{C} \to \mathscr{D}$ and $H' : \mathscr{D} \to \mathscr{E}$ be functors. If $H' \circ H$ is faithful, then H is also a faithful functor. Furthermore, if $H' \circ H$ is fully faithful and H' is faithful, then H is fully faithful.*

Proof. Assume that $\mathscr{C}, \mathscr{D}$ and $\mathscr{E}$ are categories. Assume that $H : \mathscr{C} \to \mathscr{D}$ and $H' : \mathscr{D} \to \mathscr{E}$ are functors.

First, suppose that $H' \circ H$ is faithful. Then, for all objects $X, Y \in \mathscr{C}$, the induced map

$$(H' \circ H)_{X,Y} : Hom_{\mathscr{C}}(X, Y) \to Hom_{\mathscr{E}}((H' \circ H)(X), (H' \circ H)(Y))$$

is injective function between sets. Hence, it is a monomorphism. By noting that $(H' \circ H)_{X,Y} = H'_{H(X),H(Y)} \circ H_{X,Y}$, we deduce that $H_{X,Y}$ is a monomorphism and hence, an injective map. Thus, H must be a faithful functor.

Now assume that $H' \circ H$ is fully faithful and H' is faithful. Since $H' \circ H$ is fully faithful, the induced map $(H' \circ H)_{X,Y}$ must be bijective for all objects $X, Y \in \mathscr{C}$. By using the previous result, we deduce that H is a faithful functor. Also, since $(H' \circ H)_{X,Y}$ is an epimorphism, $H'_{H(X),H(Y)}$ is surjective and consequently, bijective because H' is faithful. Hence,

$$H_{X,Y} = {H'_{H(X),H(Y)}}^{-1} \circ (H' \circ H)_{X,Y}.$$

and consequently, $(H' \circ H)^{-1}_{X,Y} \circ H'_{H(X),H(Y)}$ defines an inverse function to $H_{X,Y}$. So, $H_{X,Y}$ is bijective, hence revealing that H is fully faithful. $\square$

Just like fully faithful functors, equivalences of categories are also stable under composition, as the following lemma demonstrates:

Lemma 1.10.4. *Let $\mathscr{C}, \mathscr{D}$ and $\mathscr{E}$ be categories and $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{E}$ be equivalences of categories in the sense of 1.10.10. Then, $G \circ F$ is also an equivalence of categories.*

Proof. Assume that $\mathscr{C}, \mathscr{D}$ and $\mathscr{E}$ are categories. Assume that $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{E}$ are equivalences of categories in the sense of 1.10.10. By using 1.10.1, F and G must be fully faithful and essentially surjective.

Since F and G are fully faithful, $G \circ F$ is also fully faithful. To see that $G \circ F$ is essentially surjective, let $E \in \mathscr{E}$ be an object. Since G is essentially surjective, there exists an object $D \in \mathscr{D}$ such that $G(D) = E$. Since F is essentially surjective, there exists an object $C \in \mathscr{C}$ such that $F(C) = D$. So, $E = G(F(C)) = (G \circ F)(C)$, thus demonstrating that $G \circ F$ is essentially surjective. Therefore, $G \circ F$ is fully faithful and essentially surjective. So, $G \circ F$ is an equivalence of categories from 1.10.1. $\square$

Next, we will combine the notions of fibres and equivalences of categories in order to give a criterion for which a base change functor is fully faithful.

Lemma 1.10.5. *Let $\mathscr{C}$ be a finitely complete category and $y : Y \to Y'$ be a morphism in $\mathscr{C}$. Then, the base change functor $y^* : Pt_{Y'}(\mathscr{C}) \to Pt_Y(\mathscr{C})$ is fully faithful if and only if any downwards pullback of split epimorphisms from the top right corner*

$$\begin{tikzcd} X \arrow[r, "x"] \arrow[d, shift right, "f"'] & X' \arrow[d, shift right, "f'"'] \\ Y \arrow[r, "y"] \arrow[u, shift right, "s"'] & Y' \arrow[u, shift right, "s'"'] \end{tikzcd}$$

doubles as an upward pushout from the bottom left corner of the above diagram.

Proof. Assume that $\mathscr{C}$ is a finitely complete category. Assume that $y : Y \to Y'$ is a morphism in $\mathscr{C}$.

First, assume that the base change functor $y^* : Pt_{Y'}(\mathscr{C}) \to Pt_Y(\mathscr{C})$ is fully faithful and that we have the following pullback of split epimorphisms:

$$\begin{tikzcd} X \arrow[r, "x"] \arrow[d, shift right, "f"'] & X' \arrow[d, shift right, "f'"'] \\ Y \arrow[r, "y"] \arrow[u, shift right, "s"'] & Y' \arrow[u, shift right, "s'"'] \end{tikzcd}$$

Suppose that we have the following commutative diagram in $\mathscr{C}$:

$$\begin{tikzcd} & & Z \\ X \arrow[r, "x"] \arrow[d, shift right, "f"'] \arrow[urr, bend left, "h"] & X' \arrow[d, shift right, "f'"'] & \\ Y \arrow[r, "y"] \arrow[u, shift right, "s"'] & Y' \arrow[u, shift right, "s'"'] \arrow[uur, bend right, "g"'] & \end{tikzcd}$$

We want to deduce the existence of a unique morphism $\phi : X' \to Z$ such that the two resulting triangles in the above diagram commute. The idea is to exploit the fact that y^* is fully faithful by considering the following commutative diagram of split epimorphisms:

$$
\begin{tikzcd}
 & & Y' \times Z \arrow[dl, phantom, ""] \arrow[ddl, bend left, "\pi_{Y'}"] \\
X \arrow[r, "x"] \arrow[d, shift right, "f"'] \arrow[urr, bend left, "(y \circ f, h)"] & X' \arrow[d, shift right, "f'"'] & \\
Y \arrow[r, "y"] \arrow[u, shift right, "s"'] & Y' \arrow[u, shift right, "s'"'] \arrow[uur, bend right, "(id_{Y'}, g)"] &
\end{tikzcd}
$$

Consider the morphism $\psi : (f, s) = y^*(f', s') \to y^*(\pi_{Y'}, (id_{Y'}, g))$ in the fibre $Pt_Y(\mathscr{C})$, which makes the following diagram commute:

$$
\begin{tikzcd}
X \arrow[r, "\psi = \beta \circ f"] \arrow[d, shift right, "f"'] & T \arrow[d, shift right, "\alpha"'] \\
Y \arrow[r, "id_Y"] \arrow[u, shift right, "s"'] & Y \arrow[u, shift right, "\beta"']
\end{tikzcd}
$$

Since y^* is fully faithful, there exists a unique morphism
$(f', \phi) : X' \to Y' \times Z$ in the fibre $Pt_{Y'}(\mathscr{C})$ such that the following diagram commutes:

$$
\begin{tikzcd}
X' \arrow[r, dashed, "(f', \phi)"] \arrow[d, shift right, "f'"'] & Y' \times Z \arrow[d, shift right, "\pi_{Y'}"'] \\
Y' \arrow[r, "id_{Y'}"] \arrow[u, shift right, "s'"'] & Y' \arrow[u, shift right, "(id_{Y'}, g)"']
\end{tikzcd}
$$

Thus, $(f', \phi) : X' \to Y' \times Z$ provides the desired unique factorisation by making the diagram below commute:

$$
\begin{tikzcd}
 & & Y' \times Z \arrow[ddl, bend left, "\pi_{Y'}"] \\
X \arrow[r, "x"] \arrow[d, shift right, "f"'] \arrow[urr, bend left, "(y \circ f, h)"] & X' \arrow[d, shift right, "f'"'] \arrow[ur, dashed, "(f', \phi)"] & \\
Y \arrow[r, "y"] \arrow[u, shift right, "s"'] & Y' \arrow[u, shift right, "s'"'] \arrow[uur, bend right, "(id_{Y'}, g)"] &
\end{tikzcd}
$$

For the converse, suppose that the downwards pullback emanating from the top right corner is an upwards pushout from the bottom left corner:

$$
\begin{tikzcd}
X \arrow[r, "x"] \arrow[d, shift right, "f"'] & X' \arrow[d, shift right, "f'"'] \\
Y \arrow[r, "y"] \arrow[u, shift right, "s"'] & Y' \arrow[u, shift right, "s'"']
\end{tikzcd}
$$

The proof that y^* is faithful (which is a proof of injectivity) is summarised in [Bou17, Lemma 1.6.29]. Admittedly, **I do not know how to write this part of the proof in my own words**! To see that y^* is full, suppose that we have a morphism $\psi : (f, s) \to (\overline{f}, \overline{s}) = y^*(\overline{f}', \overline{s}')$. Then, the following diagram commutes:

$$\begin{tikzcd}
 & \overline{X} \arrow[r, "\overline{x}"] & \overline{X}' \arrow[ddl, shift left, "\overline{f}'"] \\
X \arrow[ur, "\psi"] \arrow[r, "x"] \arrow[d, shift right, "f"'] & X' \arrow[d, shift right, "f'"'] & \\
Y \arrow[u, shift right, "s"'] \arrow[r, "y"] & Y' \arrow[u, shift right, "s'"'] \arrow[uur, shift left, "\overline{s}'"] &
\end{tikzcd}$$

Using the pushout from the bottom left corner, there exists a unique morphism $\phi : X' \to \overline{X}'$ such that the diagram commutes:

$$\begin{tikzcd}
 & \overline{X} \arrow[r, "\overline{x}"] & \overline{X}' \arrow[ddl, shift left, "\overline{f}'"] \\
X \arrow[ur, "\psi"] \arrow[r, "x"] \arrow[d, shift right, "f"'] & X' \arrow[ur, dashed] \arrow[d, shift right, "f'"'] & \\
Y \arrow[u, shift right, "s"'] \arrow[r, "y"] & Y' \arrow[u, shift right, "s'"'] \arrow[uur, shift left, "\overline{s}'"] &
\end{tikzcd}$$

The morphism ϕ is represented by the dashed arrow. Let us write the triangle on the RHS in more detail:

$$\begin{tikzcd}
X' \arrow[r, "\phi"] \arrow[d, shift right, "f'"'] & \overline{X}' \arrow[d, shift right, "\overline{f}'"'] \\
Y' \arrow[u, shift right, "s'"'] \arrow[r, "id_{Y'}"] & Y' \arrow[u, shift right, "\overline{s}'"']
\end{tikzcd}$$

From the definition of the base change functor, $\psi = y^*(\phi)$. This demonstrates that y^* is full. So, y^* is fully faithful. $\square$

We end this section by observing that we can generalise 1.10.5 even further; [Bou17, Corollary 1.6.30] provides a necessary and sufficient condition for which a base change functor is an equivalence of categories.

1.11 Different types of epimorphisms

The motivation for this section lies with 1.1.2. In any category $\mathscr{C}$, an isomorphism must be both a monomorphism and an epimorphism. We know that the converse is in general not true, as 1.1.10 serves as a counterexample. The idea here is that we want to make the converse to 1.1.2 true, with an appropriate modification. It turns out that the required modification leads to several refined notions of an epimorphism, which we will now define.

Definition 1.11.1. Let $\mathscr{C}$ be a category and $f : X \to Y$ be a morphism in $\mathscr{C}$. We say that f is an **extremal epimorphism** if for all factorisations of the form $f = m \circ f'$ with m being a monomorphism, m is in fact, an isomorphism.

The main point of extremal epimorphisms is that they provide the answer to our original problem — a partial converse to 1.1.2.

Theorem 1.11.1. *Let $\mathscr{C}$ be a category and $f : X \to Y$ be a morphism in $\mathscr{C}$. If f is both an extremal epimorphism and a monomorphism, it must be an isomorphism.*

Proof. Assume that $\mathscr{C}$ is a category and $f : X \to Y$ is an extremal epimorphism and a monomorphism in $\mathscr{C}$. Note that $f = f \circ id_X$, where $id_X : X \to X$ is the identity morphism on X and f is a monomorphism by assumption. Applying the fact that f is an extremal epimorphism, we deduce that f must be an isomorphism as required. $\square$

The next theorem identifies how the concepts of extremal epimorphisms and epimorphisms are related to each other.

Theorem 1.11.2. *Let $\mathscr{C}$ be a category* with equalizers. *Then, any extremal epimorphism is an epimorphism.*

Proof. Assume that $\mathscr{C}$ is a category with equalizers. Assume that $f : X \to Y$ is an extremal epimorphism in $\mathscr{C}$.

To show: (a) f is an epimorphism.

(a) Assume that $\alpha, \beta : Y \to Z$ are morphisms such that $\alpha \circ f = \beta \circ f$. We want to show that $\alpha = \beta$. Let $\gamma = eq(\alpha, \beta) : U \to Y$. By the universal property of the equalizer, there exists a unique morphism $\rho : X \to U$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[d, dashed, "\rho"'] \arrow[dr, "f"] & & \\ U \arrow[r, "\gamma"] & Y \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & Z \end{tikzcd}$$

So, $f = \gamma \circ \rho$. Since, γ is an equalizer, it must be a monomorphism (see 1.2.1). Using the fact that f is an extremal epimorphism, we deduce that γ is an isomorphism. Hence, γ is both a monomorphism and an epimorphism (see 1.1.2). Since $\alpha \circ \gamma = \beta \circ \gamma$, $\alpha = \beta$.

Therefore, f is an epimorphism as required. □

The next lemma reveals how extremal epimorphisms behave under composition.

Lemma 1.11.3. *Let $\mathscr{C}$ be a category with pullbacks and $f : X \to Y$, $g : Y \to Z$ be extremal epimorphisms. Then, $g \circ f$ is also an extremal epimorphism. Furthermore, if $h : Y \to W$ is a morphism in $\mathscr{C}$ such that $h \circ f$ is an extremal epimorphism, then h is also an extremal epimorphism.*

Proof. Assume that $\mathscr{C}$ is a category with pullbacks and $f : X \to Y$, $g : Y \to Z$ are extremal epimorphisms.

To show: (a) The composite $g \circ f$ is an extremal epimorphism.

(a) Suppose that we have the factorisation $g \circ f = m \circ j$, where $m : T \to Z$ is a monomorphism in $\mathscr{C}$ and $j \in Hom_{\mathscr{C}}(X, T)$. As standard practice, we represent this as the commutative square:

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "f"] & T \arrow[d, "m"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

Since $\mathscr{C}$ contains pullbacks, let us form the pullback square associated to the pair (m, g):

$$\begin{tikzcd} V \arrow[r, "p_1"] \arrow[d, "p_0"] & T \arrow[d, "m"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

From the universal property of the pullback, there exists a unique morphism $\phi : X \to V$ such that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[drr, bend left, "j"] \arrow[dr, dashed, "\phi"] \arrow[ddr, bend right, "f"'] & & \\
& V \arrow[r, "p_1"] \arrow[d, "p_0"] & T \arrow[d, "m"] \\
& Y \arrow[r, "g"] & Z
\end{tikzcd}$$

Notice that $f = p_0 \circ \phi$, where p_0 is a monomorphism because m was assumed to be a monomorphism (see lemma 1.4.1). Since f is an extremal epimorphism, p_0 must be an isomorphism. But, $g \circ p_0 = m \circ p_1$ and subsequently, $g = m \circ (p_1 \circ p_0^{-1})$. Using the fact that g is an extremal epimorphism and m is a monomorphism, m must be an isomorphism. This proves that $g \circ f$ is an extremal epimorphism.

Now assume that $h : Y \to W$ is a morphism in $\mathscr{C}$ such that $h \circ f$ is an extremal epimorphism.

To show: (b) The morphism h is an extremal epimorphism.

(b) Assume that $h = n \circ k$, where n is a monomorphism. Then, $h \circ f = n \circ (k \circ f)$. Since $h \circ f$ is an extremal epimorphism and n is a monomorphism, n must be an isomorphism. Consequently, h is an extremal epimorphism. $\square$

The example below provides us with a concrete example of extremal epimorphisms.

Example 1.11.2. We will work in the category **Set**. Suppose that $f : X \to Y$ is a morphism in **Set**. We claim that f is an extremal epimorphism if and only if f is surjective.

First assume that f is surjective and that $f = m \circ g$, where $m : Z \to Y$ is a monomorphism and $g \in Hom_{\mathbf{Set}}(X, Z)$. Recall that in **Set**, monomorphisms are precisely the injective functions and epimorphisms are precisely the surjective functions. So, m must be injective and since f is an epimorphism, m must also be an epimorphism and hence, surjective. So, m must be a bijective function and thus, an isomorphism in **Set**. Therefore, f is an extremal epimorphism.

For the converse, assume that f is an extremal epimorphism. Then, since **Set** has equalizers (see example 1.2.3), f must be an epimorphism and hence, a surjective map. Therefore, f is an extremal epimorphism if and only if f is surjective. In **Set**, extremal epimorphisms and epimorphism are exactly the same concept.

Notice that since the category **Grp** also has equalizers (see [Lei16, Example 5.1.14]), we can argue in exactly the same manner for **Set** in order to demonstrate that a group morphism $f : G \to H$ is an extremal epimorphism if and only if f is surjective (recall that lemma 1.1.1 can be adapted for **Grp**).

The next type of epimorphism is motivated by lemma 1.9.1.

Definition 1.11.3. Let $\mathscr{C}$ be a category with pullbacks and $f : X \to Y$ be a morphism in $\mathscr{C}$. We say that f is a **regular epimorphism** if it is the coequalizer of its kernel equivalence relation $R[f]$. That is, if we have the following diagram in $\mathscr{C}$,

$$\begin{tikzcd} R[f] \arrow[r, shift left=3, "p_0"] \arrow[r, shift right=3, "p_1"'] & X \arrow[l, "s_0"'] \arrow[r, "f"] & Y \end{tikzcd}$$

then $f \circ p_0 = f \circ p_1$.

Lemma 1.9.1 tells us that in a category $\mathscr{C}$ with pullbacks, a split epimorphism must be a regular epimorphism. It turns out that in our "ladder of inclusions", regular epimorphisms lie between split epimorphisms and extremal epimorphisms.

Lemma 1.11.4. *Let $\mathscr{C}$ be a category with pullbacks. If $f : X \to Y$ is a regular epimorphism in $\mathscr{C}$, then f is an extremal epimorphism.*

Proof. Assume that $\mathscr{C}$ is a category with pullbacks. Assume that $f : X \to Y$ is a regular epimorphism in $\mathscr{C}$. Suppose that $f = m \circ g$, where $g \in Hom_{\mathscr{C}}(X, T)$ and $m \in Hom_{\mathscr{C}}(T, Y)$ is a monomorphism.

To show: (a) m is an isomorphism.

(a) Suppose that the kernel equivalence relation $R[f]$ is given by the following diagram in $\mathscr{C}$:

$$\begin{tikzcd} R[f] \arrow[r, shift left=3, "p_0"] \arrow[r, shift right=3, "p_1"'] & X \arrow[l, "s_0"'] \arrow[r, "f"] & Y \end{tikzcd}$$

Since $m \circ g \circ p_0 = m \circ g \circ p_1$ and m is a monomorphism, $g \circ p_0 = g \circ p_1$, which means that g coequalizes $R[f]$. By the universal property of the coequalizer, there exists a unique morphism $\pi : Y \to T$ such that the following diagram commutes:

$$\begin{tikzcd}
 & & T \\
R[f] \arrow[r, shift left, "p_0"] \arrow[r, shift right, "p_1"'] & X \arrow[ur, "g"] \arrow[r, "f"'] & Y \arrow[u, dashed, "\pi"]
\end{tikzcd}$$

So, $f = m \circ g = (m \circ \pi) \circ f$. Since f is a coequalizer, it must be an epimorphism from theorem 1.2.1. So, $id_Y = m \circ \pi$ and subsequently, $m = m \circ (\pi \circ m)$. Since m is a monomorphism, we deduce that $id_T = \pi \circ m$. Thus, m is an isomorphism and f is an extremal epimorphism as required. $\square$

The next property we will prove about regular epimorphisms sets the stage for the next type of epimorphism we will introduce.

Lemma 1.11.5. *Let $\mathscr{C}$ be a category, $f : X \to Y$ be a regular epimorphism and $m : U \to V$ be a monomorphism such that f and m make the following square in $\mathscr{C}$ commute:*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "h"] & Y \arrow[d, "g"] \\
U \arrow[r, "m"] & V
\end{tikzcd}$$

Then, there exists a unique morphism $\phi : Y \to U$ such that the two triangles in the resulting diagram commute:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "h"] & Y \arrow[d, "g"] \arrow[dl, dashed, "\phi"'] \\
U \arrow[r, "m"] & V
\end{tikzcd}$$

Proof. Assume that $\mathscr{C}$ is a category, $f : X \to Y$ is a regular epimorphism and $m : U \to V$ is a monomorphism. Assume that we have the following commutative diagram:

$$\begin{tikzcd}
R[f] \arrow[r, shift left=2, "p_0"] \arrow[r, shift right=2, "p_1"'] & X \arrow[l] \arrow[r, "f"] \arrow[d, "h"] & Y \arrow[d, "g"] \\
 & U \arrow[r, "m"] & V
\end{tikzcd}$$

Since f is a regular epimorphism, $f = coeq(p_0, p_1)$. Using the fact that $f \circ p_0 = f \circ p_1$, in tandem with the commutative square, we deduce that $m \circ h \circ p_0 = m \circ h \circ p_1$ and as a result, $h \circ p_0 = h \circ p_1$ because m is a monomorphism. By the universal property of the coequalizer, there exists a unique morphism $\alpha : Y \to U$ such that the following diagram commutes:

$$\begin{tikzcd} & & U \\ R[f] \arrow[r, shift left, "p_0"] \arrow[r, shift right, "p_1"'] & X \arrow[ur, "h"] \arrow[r, "f"'] & Y \arrow[u, dashed, "\alpha"] \end{tikzcd}$$

Now observe that $(m \circ \alpha) \circ f = m \circ h = g \circ f$. Since f is a coequalizer, it must be an epimorphism (see theorem 1.2.1). Therefore, $m \circ \alpha = g$ and $\alpha : Y \to U$ is the unique morphism which makes the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"] & Y \arrow[d, "g"] \arrow[dl, dashed, "\alpha"'] \\ U \arrow[r, "m"] & V \end{tikzcd}$$

□

Interestingly, the next type of epimorphism we will introduce does not make an appearance in [Bou17]. We will now briefly follow the reference [Bor94, Volume 1, Section 4.3], which provides a detailed discussion on *strong epimorphisms.*

Definition 1.11.4. Let $\mathscr{C}$ be a category and $f : X \to Y$ be an epimorphism. We say that f is a **strong epimorphism** if for all commutative squares of the form

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"] & Y \arrow[d, "g"] \\ U \arrow[r, "m"] & V \end{tikzcd}$$

with m a monomorphism in $\mathscr{C}$, there exists a unique morphism $\alpha : Y \to U$ such that the two triangles in the diagram below commute:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"] & Y \arrow[d, "g"] \arrow[dl, dashed, "\alpha"'] \\ U \arrow[r, "m"] & V \end{tikzcd}$$

Lemma 1.11.5 demonstrates that a regular epimorphism must be a strong epimorphism. Our next lemma determines the place strong epimorphisms occupy on the “ladder of inclusions”.

Lemma 1.11.6. *Let $\mathscr{C}$ be a category and $f : X \to Y$ be a strong epimorphism. Then, f must be an extremal epimorphism.*

Proof. Assume that $\mathscr{C}$ is a category and $f : X \to Y$ is a strong epimorphism. Suppose that $f = m \circ g$, where $g \in Hom_{\mathscr{C}}(X, T)$ and $m \in Hom_{\mathscr{C}}(T, Y)$ is a monomorphism. This corresponds to the following commutative square:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "id_Y"] \\ T \arrow[r, "m"] & Y \end{tikzcd}$$

Since f is a strong epimorphism, there exists a unique morphism $\phi : Y \to T$ such that the two triangles in the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "id_Y"] \arrow[dl, dashed, "\phi"'] \\ T \arrow[r, "m"] & Y \end{tikzcd}$$

So, $m \circ \phi = id_Y$ and consequently, $m \circ (\phi \circ m) = m$. Since m is a monomorphism, we deduce that $\phi \circ m = id_T$. Hence, m is an isomorphism and f must be an extremal epimorphism. □

Strong epimorphisms satisfy very similar properties to that of extremal epimorphisms.

Lemma 1.11.7. *Let $\mathscr{C}$ be a category and $f : X \to Y$, $g : Y \to Z$ be strong epimorphisms. Then, $g \circ f$ is also a strong epimorphism. Furthermore, if $k : Y \to W$ and $\ell : X \to Y$ are morphisms in $\mathscr{C}$ such that $k \circ \ell$ is a strong epimorphism, then k is also a strong epimorphism.*

Proof. Assume that $\mathscr{C}$ is a category and $f : X \to Y$, $g : Y \to Z$ are strong epimorphisms.

To show: (a) The composite $g \circ f$ is a strong epimorphism.

(a) Suppose that we have the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "g\circ f"] \arrow[d, "h"] & Z \arrow[d, "j"] \\ Z \arrow[r, "m"] & W \end{tikzcd}$$

where $m : Z \to W$ is a monomorphism. Since composition satisfies associativity, the following square also commutes:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"] & Y \arrow[d, "j\circ g"] \\ Z \arrow[r, "m"] & W \end{tikzcd}$$

Since f is a strong epimorphism, there exists a unique morphism $\beta : Y \to Z$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"] & Y \arrow[d, "j\circ g"] \arrow[dl, dashed, "\beta"'] \\ Z \arrow[r, "m"] & W \end{tikzcd}$$

Now we write the bottom right triangle in the above diagram as a commutative square:

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "\beta"] & Z \arrow[d, "j"] \\ Z \arrow[r, "m"] & W \end{tikzcd}$$

Since g is a strong epimorphism, there exists a unique morphism $\gamma : Z \to Z$ such that the following diagram commutes:

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "\beta"] & Z \arrow[d, "j"] \arrow[dl, dashed, "\gamma"'] \\ Z \arrow[r, "m"] & W \end{tikzcd}$$

It remains to show that $\gamma \circ g \circ f = h$. Using the commutativity of the diagrams, we have

$$(\gamma \circ g) \circ f = \beta \circ f = h.$$

Hence, γ is the unique morphism such that the triangles in the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "g\circ f"] \arrow[d, "h"] & Z \arrow[d, "j"] \arrow[dl, dashed, "\gamma"'] \\ Z \arrow[r, "m"] & W \end{tikzcd}$$

So, $g \circ f$ is a strong epimorphism.

Now assume that $k : Y \to W$ and $\ell : X \to Y$ are morphisms in $\mathscr{C}$ such that $k \circ \ell$ is a strong epimorphism.

To show: (b) k is a strong epimorphism.

(b) Suppose that we have the following commutative diagram:

$$\begin{tikzcd} Y \arrow[r, "k"] \arrow[d, "p"] & W \arrow[d, "q"] \\ Z \arrow[r, "n"] & A \end{tikzcd}$$

Here, $n : Z \to A$ is a monomorphism. Upon precomposing with ℓ, we obtain the following commutative square:

$$\begin{tikzcd} X \arrow[r, "k\circ\ell"] \arrow[d, "p\circ\ell"] & W \arrow[d, "q"] \\ Z \arrow[r, "n"] & A \end{tikzcd}$$

Since $k \circ \ell$ is a strong epimorphism, there exists a unique morphism $\delta : W \to Z$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "k\circ\ell"] \arrow[d, "p\circ\ell"] & W \arrow[d, "q"] \arrow[dl, dashed, "\delta"'] \\ Z \arrow[r, "n"] & A \end{tikzcd}$$

Since $n \circ \delta = q$, $n \circ \delta \circ k = q \circ k = n \circ p$. So, $\delta \circ k = p$ because n is a monomorphism. This demonstrates that k is a strong epimorphism. $\square$

Theorem 1.11.8. *Let $\mathscr{C}$ be a category and $f : X \to Y$ be a morphism in $\mathscr{C}$. If f is both a strong epimorphism and a monomorphism, it must be an isomorphism.*

Proof. Assume that $\mathscr{C}$ is a category and $f : X \to Y$ is a strong epimorphism and a monomorphism in $\mathscr{C}$. Consider the following commutative square:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \Big\downarrow{\scriptstyle id_X} & & \Big\downarrow{\scriptstyle id_Y} \\ X & \xrightarrow{f} & Y \end{array}$$

Since f is a strong epimorphism, there exists a unique morphism $\beta : Y \to X$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "id_X"'] & Y \arrow[d, "id_Y"] \arrow[dl, dashed, "\beta"'] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Hence, f is an isomorphism as required. $\square$

Since strong epimorphisms bear a resemblance to extremal epimorphisms, a natural question which follows is why strong epimorphisms are useful in the first place. The next theorem tells us that strong epimorphisms are at their peak utility in finitely complete categories (recall definition 1.10.7).

Theorem 1.11.9. *Let $\mathscr{C}$ be a finitely complete category.*

(a) Let $f : X \to Y$ be a morphism which satisfies the diagonal property of 1.11.4. That is, for all commutative squares of the form

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \Big\downarrow{\scriptstyle h} & & \Big\downarrow{\scriptstyle g} \\ U & \xrightarrow{m} & V \end{array}$$

with m a monomorphism in $\mathscr{C}$, there exists a unique morphism $\alpha : Y \to U$ such that the two triangles in the diagram below commute:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] & Y \arrow[d, "g"] \arrow[dl, dashed, "\alpha"'] \\ U \arrow[r, "m"] & V \end{tikzcd}$$

Then, f is an epimorphism and thus, a strong epimorphism.

(b) Let $g : X \to Y$ be a morphism such that given any factorisation $g = m \circ k$ with m being a monomorphism, m must be an isomorphism. Then, g is an epimorphism and thus, an extremal epimorphism.

(c) An epimorphism in $\mathscr{C}$ is strong if and only if it is extremal.

Proof. Assume that $\mathscr{C}$ is a finitely complete category (it has a terminal object and pullbacks). Then, $\mathscr{C}$ must have equalizers (see [Bor94, Proposition 2.8.2]).

Part (a): Assume that $f : X \to Y$ is a morphism which satisfies the diagonal property of 1.11.4. To see that f is an epimorphism, suppose that $u \circ f = v \circ f$, where $u, v \in Hom_{\mathscr{C}}(Y, Z)$. Let $t = eq(u, v) : T \to Y$. By the universal property of the equalizer, there exists a unique morphism $\rho : X \to T$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[d, dashed, "\rho"'] \arrow[dr, "f"] & & \\ T \arrow[r, "t"] & Y \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & Z \end{tikzcd}$$

So, $f = t \circ \rho$. We can write this as the following commutative square:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\rho"] & Y \arrow[d, "id_Y"] \\ T \arrow[r, "t"] & Y \end{tikzcd}$$

Since f satisfies the diagonal property, there exists a unique morphism $\phi : Y \to T$ such that the two triangles in the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\rho"] & Y \arrow[d, "id_Y"] \arrow[dl, dashed, "\phi"'] \\ T \arrow[r, "t"] & Y \end{tikzcd}$$

So, $t \circ \phi = id_Y$ and since $u \circ t = v \circ t$ by assumption, $u = v$. Therefore, f is an epimorphism and consequently, a strong epimorphism.

Part (b): Assume that $g : X \to Y$ is a morphism such that given any factorisation $g = m \circ k$ with m being a monomorphism, m must be an isomorphism. To see that g is an epimorphism, suppose that $a \circ g = b \circ g$, where $a, b \in Hom_{\mathscr{C}}(Y, Z)$. Let $t = eq(a, b) : T \to Y$. Again, by the universal property of the equalizer, there exists a unique morphism $\nu : X \to T$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[d, dashed, "\nu"'] \arrow[dr, "g"] & & \\ T \arrow[r, "t"] & Y \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & Z \end{tikzcd}$$

So, $g = t \circ \nu$. Since t is an equalizer, it must be a monomorphism and hence, an isomorphism, due to our assumption on g. Since $a \circ t = b \circ t$, $a = b$ and so, g must be an epimorphism and thus, an extremal epimorphism.

Part (c): It suffices from lemma 1.11.6 to show that any extremal epimorphism is strong. Assume that $f : X \to Y$ is an extremal epimorphism. Assume that f satisfies the following commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "h"] & Y \arrow[d, "g"] \\
Z \arrow[r, "m"] & W
\end{tikzcd}$$

where m is a monomorphism. Form the pullback of the pair (m, g), which is given by the square

$$\begin{tikzcd}
P \arrow[r, "p_1"] \arrow[d, "p_0"] & Y \arrow[d, "g"] \\
Z \arrow[r, "m"] & W
\end{tikzcd}$$

By the universal property of the pullback, there exists a unique morphism $\alpha : X \to P$ such that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[dr, dashed, "\alpha"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "h"'] & & \\
& P \arrow[r, "p_1"] \arrow[d, "p_0"] & Y \arrow[d, "g"] \\
& Z \arrow[r, "m"] & W
\end{tikzcd}$$

Since m is a monomorphism, p_1 must also be a monomorphism by lemma 1.4.1. But $f = p_1 \circ \alpha$. Since f is an extremal epimorphism, p_1 must be an isomorphism. Now we compute directly that

$$m \circ (p_0 \circ p_1^{-1}) = g \circ (p_1 \circ p_1^{-1}) = g.$$

Therefore, $p_0 \circ p_1^{-1}$ is a morphism which makes the following diagram commute:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "h"'] & Y \arrow[d, "g"] \arrow[dl, dashed, "p_0 \circ p_1^{-1}"'] \\
Z \arrow[r, "m"] & W
\end{tikzcd}$$

One can check that $p_0 \circ p_1^{-1}$ is unique because f is an epimorphism and m is a monomorphism. Therefore, f must be a strong epimorphism as required. $\square$

We will now return to following the exposition of [Bou17] and define our final type of epimorphism.

Definition 1.11.5. Let $\mathscr{C}$ be a category with pullbacks. Let $f : X \to Y$ denote a regular epimorphism. We say that f is a **ps-regular epimorphism** if as a regular epimorphism, it is stable under pullback. That is, for all pullback squares of the form

$$\begin{tikzcd} P \arrow[r, "p_1"] \arrow[d, "p_0"] & X \arrow[d, "f"] \\ Z \arrow[r, "g"] & Y \end{tikzcd}$$

the morphism $p_0 : P \to Z$ is a regular epimorphism.

Lemma 1.9.1 and lemma 1.4.1 tell us that any split epimorphism must be a ps-regular epimorphism. Thus, our "ladder of inclusions" of epimorphisms is complete. In a category $\mathscr{C}$, we have

$$\text{split epi} \subset \text{ps-regular epi} \subset \text{regular epi} \subset \text{strong epi} \subset \text{extremal epi}. \quad (1.2)$$

In a finitely complete category, we have

$$\text{split epi} \subset \text{ps-regular epi} \subset \text{regular epi} \subset \text{strong epi} = \text{extremal epi} \subset \text{epi}. \quad (1.3)$$

So far, we have discussed how extremal and strong epimorphisms behave under composition. Regular epimorphisms are much less well behaved under composition. For instance, the composition of two regular epimorphisms need not be a regular epimorphism itself.

Lemma 1.11.10. *Let $\mathscr{C}$ be a category with pullbacks. If $g \circ f$ is a regular epimorphism and f is an epimorphism, then g is also a regular epimorphism.*

Proof. Assume that $\mathscr{C}$ is a category with pullbacks. Assume that $g \circ f : X \to Z$ is a regular epimorphism and $f : X \to Y$ is an epimorphism. Let (p_0^g, p_1^g) be the pair of morphisms associated to the kernel equivalence relation $R[g]$, as depicted by the diagram below:

$$\begin{tikzcd} R[g] \arrow[r, shift left=2, "p_0^g"] \arrow[r, shift right=2, "p_1^g"'] & Y \arrow[l] \arrow[rr, "g"] & & Z \end{tikzcd}$$

Suppose that $a : Y \to T$ is a morphism such that $a \circ p_0^g = a \circ p_1^g$. By definition of the kernel equivalence relation, the following diagram must commute:

$$\begin{tikzcd} R[g \circ f] \arrow[r, "f \circ p_1^{g \circ f}"] \arrow[d, "f \circ p_0^{g \circ f}"'] & Y \arrow[d, "g"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

Hence, $R[g \circ f]$ must factor through $R[g]$, yielding the following factorisation:

$$\begin{tikzcd} R[g \circ f] \arrow[dr, dashed, "R(f)"] \arrow[drr, bend left, "f \circ p_1^{g \circ f}"] \arrow[ddr, bend right, "f \circ p_0^{g \circ f}"'] & & \\ & R[g] \arrow[r, "p_1^g"] \arrow[d, "p_0^g"'] & Y \arrow[d, "g"] \\ & Y \arrow[r, "g"] & Z \end{tikzcd}$$

Since $a \circ p_0^g = a \circ p_1^g$, we can precompose both sides with $R(f)$ and use the commutativity of the above diagram to deduce that $(a \circ f) \circ p_0^{g \circ f} = (a \circ f) \circ p_1^{g \circ f}$. So, $a \circ f$ coequalizes the kernel equivalence relation $R[g \circ f]$. By the universal property of the coequalizer, there exists a unique morphism such that the following diagram commutes:

$$\begin{tikzcd} & & T \\ R[g \circ f] \arrow[r, shift left, "p_0^{g \circ f}"] \arrow[r, shift right, "p_1^{g \circ f}"'] & X \arrow[ur, "a \circ f"] \arrow[r, "g \circ f"'] & Z \arrow[u, dashed, "\phi"'] \end{tikzcd}$$

Since $(\phi \circ g) \circ f = a \circ f$ and f is an epimorphism, $\phi \circ g = a$. Therefore, ϕ is the unique morphism which makes the following diagram commute:

$$\begin{tikzcd} & & T \\ R[g] \arrow[r, shift left, "p_0^g"] \arrow[r, shift right, "p_1^g"'] & Y \arrow[ur, "a"] \arrow[r, "g"'] & Z \arrow[u, dashed, "\phi"'] \end{tikzcd}$$

Thus, $g = coeq(p_0^g, p_1^g)$ and is consequently a regular epimorphism.

$\square$

Lemma 1.11.11. *Let $\mathscr{C}$ be a category with pullbacks.*

(a) If f is a ps-regular epimorphism and g a regular epimorphism, then $g \circ f$ is also a regular epimorphism.

(b) If f and g are both ps-regular epimorphisms, then $g \circ f$ is also a ps-regular epimorphism.

Lemma 1.11.11 is directly from [Bou17, Exercise 1.7.14]. **I do not know a good proof of** lemma 1.11.11. However, it plays a major role in the next section. We will finish this section with some examples of the epimorphisms we have encountered in this section.

Example 1.11.6. In the category **Set**, ps-regular epimorphisms, regular epimorphisms, strong epimorphisms, extremal epimorphisms and epimorphisms all coincide with surjective functions. Similarly, in **Grp** and **Ab**, these epimorphisms all coincide with surjective group morphisms. Hence, the distinction in both of these categories is pointless!

Example 1.11.7. The distinction between different types of epimorphisms is necessary in **Top**. In this example, we will give an example of a regular epimorphism in **Top** which is not a ps-regular epimorphism.

Let X, Y be topological spaces and $f : X \to Y$ be a continuous function. We say that f is **cartesian** if the following property is satisfied: V is open in Y if and only if $f^{-1}(V)$ is open in X. It turns out from [Bou17, Exercise 1.7.12 (iii)] that regular epimorphisms in **Top** coincide with continuous, surjective, cartesian maps.

Let $X = \{a, b, c, d\}$, $Y = \{x, y, z\}$ and $Z = \{l, m, n\}$. We equip X, Y and Z with the topologies

$$\tau_X = \{\emptyset, X, \{a, b\}\}, \tau_Y = \{\emptyset, Y\} \text{ and } \tau_Z = \{\emptyset, Z, \{l, m\}\}.$$

We define continuous maps $f : X \to Y$ and $g : Z \to Y$ by

$$f(a) = x, f(b) = f(c) = y, f(d) = z$$

$$g(l) = x, g(m) = g(n) = z.$$

Observe that f is a continuous, surjective and cartesian function and is thus, a regular epimorphism. Now form the pullback $X \times_Y Z$:

$$\begin{tikzcd} X \times_Y Z \arrow[r, "\pi_Z"] \arrow[d, "\pi_X"'] & Z \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Recalling the definition of the pullback in **Top**, we have

$$X \times_Y Z = \{(\alpha, \beta) \in X \times Z \mid f(\alpha) = g(\beta)\} = \{(a, l), (d, m), (d, n)\}$$

with topology given by

$$\tau_{X \times_Y Z} = \{\emptyset, X \times_Y Z, \{(a, l)\}, \{(a, l), (d, m)\}\}.$$

The topology $\tau_{X \times_Y Z}$ is the coarsest (largest) topology ensuring that the projection maps π_X and π_Z are continuous. Now note that π_Z is not cartesian because the preimage $\pi^{-1}(\{l\}) = \{a, l\}$ is open in $X \times_Y Z$, but $\{l\}$ is not open in Z by definition of τ_Z. So, f is an example of a regular epimorphism which is not ps-regular. This example originates from [Bor94, Volume 2, Counterexample 2.4.5].

1.12 Barr-Kock Theorem

The Barr-Kock theorem gives us yet another important tool for deducing when particular squares in a category with pullbacks are pullback squares.

Lemma 1.12.1. *Let $\mathscr{C}$ be a category with pullbacks. Suppose that we have the following commutative diagram*

$$\begin{tikzcd} R[f'] \arrow[r, shift left=2, "p_0^{f'}"] \arrow[r, shift right=2, "p_1^{f'}"'] \arrow[d, "R(x)"] & X' \arrow[l] \arrow[r, "f'"] \arrow[d, "x"] & Y' \arrow[d, "y"] \\ R[f] \arrow[r, shift left=2, "p_0^{f}"] \arrow[r, shift right=2, "p_1^{f}"'] & X \arrow[l] \arrow[r, "f"] & Y \end{tikzcd}$$

where the RHS square is a pullback square. Then, both LHS commutative squares indexed by 0 *and* 1 *are also pullbacks.*

Proof. Assume that $\mathscr{C}$ is a category with pullbacks and that we are given the commutative diagram in the statement of the lemma. We construct the following cube in $\mathscr{C}$:

$$\begin{tikzcd}
R[f'] \arrow[rr, "p_1^{f'}"] \arrow[dr, "p_0^{f'}"'] \arrow[dd, "R(x)"'] & & X' \arrow[dr, "f'"] \arrow[dd, "x"' near start] & \\
& X' \arrow[rr, "f'" near start] \arrow[dd, "x"' near start] & & Y' \arrow[dd, "y"] \\
R[f] \arrow[rr, "p_1^f" near end] \arrow[dr, "p_0^f"'] & & X \arrow[dr, "f"] & \\
& X \arrow[rr, "f'"] & & Y
\end{tikzcd}$$

Notice that the bottom, front, top and right faces of the cube are all pullback squares. From lemma 1.4.6, the remaining two faces of the cube must also be pullback squares as required. $\square$

Lemma 1.12.2. *Let $\mathscr{C}$ be a category with pullbacks. Suppose that we have the following pullback square*

$$\begin{tikzcd}
X' \arrow[r, "q'"] \arrow[d, "m"] & Y' \arrow[d, "n"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

where q' is a ps-regular epimorphism and m is a monomorphism. Then, n must be a monomorphism.

Proof. Assume that $\mathscr{C}$ is a category with pullbacks and that we are given the pullback square in the statement of the lemma. Assume that q' is a ps-regular epimorphism and that m is a monomorphism. The idea is to write out the kernel equivalence relations of all the morphisms in the square, giving rise to the following diagram:

$$
\begin{tikzcd}[row sep=large, column sep=huge]
R[R(m)] \arrow[r, shift left=2, "R(p_1)"] \arrow[r, shift right=2, "R(p_0)"'] \arrow[d, shift right=3, "d_0^{R(m)}"'] \arrow[d, shift left=3, "d_1^{R(m)}"] & R[m] \arrow[l] \arrow[r, two heads, "R(q')"] \arrow[d, shift right=3, "d_0^m"'] \arrow[d, shift left=3, "d_1^m"] & R[n] \arrow[d, shift right=3, "d_0^n"'] \arrow[d, shift left=3, "d_1^n"] \\
R[q'] \arrow[u] \arrow[r, shift left=2, "p_1"] \arrow[r, shift right=2, "p_0"'] \arrow[d, tail, "R(m)"'] & X' \arrow[u] \arrow[l] \arrow[r, two heads, "q'"] \arrow[d, tail, "m"] & Y' \arrow[u] \arrow[d, "n"] \\
R[f] \arrow[r, shift left=2, "p_1"] \arrow[r, shift right=2, "p_0"'] & X \arrow[l] \arrow[r, "f"'] & Y
\end{tikzcd}
$$

By repeated application of Lemma 1.12.1, we find that since our original square was a pullback, all the commutative squares in the above diagram are pullbacks. Since q' is a ps-regular epimorphism, $R(q')$ must also be a ps-regular epimorphism. But, q' is also a regular epimorphism and thus, the coequalizer of the kernel equivalence relation $R[q']$ in the middle row. Therefore, $R(q')$ is the coequalizer of the upper row.

Now since m is a monomorphism, we can use Theorem 1.7.5 to deduce that d_0^m and d_1^m are both isomorphisms. Since isomorphsims are stable under pullbacks, $d_0^{R(m)}, d_1^{R(m)}, d_0^n, d_1^n$ are isomorphisms. Finally, by applying Theorem 1.7.5 to d_0^n and d_1^n, we find that n is a monomorphism as required. □

Now we have arrived at the namesake of this section — a powerful tool for finding pullback squares.

Theorem 1.12.3 (Barr-Kock theorem)**.** *Let $\mathscr{C}$ be a category with pullbacks. Suppose that we have the following diagram in $\mathscr{C}$:*

$$
\begin{tikzcd}[row sep=large, column sep=large]
R[q'] \arrow[r, shift left=2, "p_0^{q'}"] \arrow[r, shift right=2, "p_1^{q'}"'] \arrow[d, "R(x)"] & X' \arrow[l] \arrow[r, "q'"] \arrow[d, "x"] & Y' \arrow[d, "y"] \\
R[f] \arrow[r, shift left=2, "p_0^{f}"] \arrow[r, shift right=2, "p_1^{f}"'] & X \arrow[l] \arrow[r, "f"] & Y
\end{tikzcd}
$$

If any of the left hand side squares is a pullback square and q' is a ps-regular epimorphism then the right hand side square is a pullback square.

Proof. Assume that $\mathscr{C}$ is a category with pullbacks and that we have the above diagram in $\mathscr{C}$, where any of the LHS squares is a pullback square and q' is a ps-regular epimorphism.

We rely on a particular consequence of Lemma 1.7.7. Consider the pullback of y along f:

$$\begin{tikzcd}
\overline{X} \arrow[r, "\overline{f}"] \arrow[d, "\overline{x}"] & Y' \arrow[d, "y"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

A consequence of Lemma 1.7.7 is that the morphism $\phi : X' \to X$ is a monomorphism. This is [Bou17, Corollary 1.6.16].

Using ϕ, we construct the following cube in $\mathscr{C}$:

$$\begin{tikzcd}
R[q'] \arrow[rr, "p_0^{q'}"] \arrow[dr, "\phi \circ p_1^{q'}"] \arrow[dd, "R(x)"'] & & X' \arrow[dr, "q'"] \arrow[dd, "x"' near start] & \\
& \overline{X} \arrow[rr, "\overline{f}" near start] \arrow[dd, "\overline{x}"' near start] & & Y' \arrow[dd, "y"] \\
R[f] \arrow[rr, "p_0^f" near end] \arrow[dr, "p_1^f"'] & & X \arrow[dr, "f"] & \\
& X \arrow[rr, "f"] & & Y
\end{tikzcd}$$

Without loss of generality, assume that the following LHS square is a pullback square

$$\begin{tikzcd}
R[q'] \arrow[r, "p_0^{q'}"] \arrow[d, "R(x)"] & X' \arrow[d, "x"] \\
R[f] \arrow[r, "p_0^f"] & X
\end{tikzcd}$$

Then, the bottom, front and back faces of the cube are pullback squares. By Lemma 1.4.6, the top square of the cube must also be a pullback square.

Now since q' is a ps-regular epimorphism, $\phi \circ p_1^{q'}$ must also be a regular epimorphism. Since $p_1^{q'}$ is an epimorphism, we can use Lemma 1.11.10 to deduce that ϕ is an epimorphism. Since ϕ is a monomorphism and an epimorphism, it must be an isomorphism as required.

If instead the LHS square

$$\begin{tikzcd} R[q'] \arrow[r, "p_1^{q'}"] \arrow[d, "R(x)"] & X' \arrow[d, "x"] \\ R[f] \arrow[r, "p_1^f"] & X \end{tikzcd}$$

is a pullback square then we can interchange the roles of $p_0^{q'}$ and $p_1^{q'}$ in the above argument to obtain the same conclusion. □

We end this section by proving a useful corollary of Theorem 1.12.3.

Theorem 1.12.4. *Let $\mathscr{C}$ be a category with pullbacks. Assume that $f : X \to Y$ is a morphism such that the kernel equivalence relation $R[f]$ has a coequalizer $q : X \to Q$ which is a ps-regular epimorphism. Then, $R[q] \cong R[f]$ and the unique factorisation $m : Q \to Y$ such that $m \circ q = f$ is a monomorphism.*

Proof. Assume that $\mathscr{C}$ is a category with pullbacks. Assume that $f : X \to Y$ is a morphism such that the kernel equivalence relation $R[f]$ is coequalized by the ps-regular epimorphism $q : X \to Q$.

The fact that we can factorise $f = m \circ q$ follows from the universal property of the coequalizer, as seen by the diagram below:

$$\begin{tikzcd} & & Y \\ R[f] \arrow[r, shift left, "p_0^f"] \arrow[r, shift right, "p_1^f"'] & X \arrow[ur, "f"] \arrow[r, "q"'] & Q \arrow[u, dashed, "m"'] \end{tikzcd}$$

By Theorem 1.7.4, we deduce that $R[q] \subset R[m \circ q] = R[f]$. Since q coequalizes $R[f]$, we also have $R[f] \subset R[q]$. So, $R[f] \cong R[q]$.

Now consider the following diagram in $\mathscr{C}$:

$$
\begin{tikzcd}
R[q] \arrow[r, shift left=2, "p_0^q"] \arrow[r, shift right=2, "p_1^q"'] \arrow[d, "\cong"] & X \arrow[l] \arrow[r, "q'"] \arrow[d, "\cong"] & Q \arrow[d, "m"] \\
R[f] \arrow[r, shift left=2, "p_0^f"] \arrow[r, shift right=2, "p_1^f"'] & X \arrow[l] \arrow[r, "f"] & Y
\end{tikzcd}
$$

We can apply the Barr-Kock theorem (see Theorem 1.12.3) to find that the RHS square is a pullback square. By Lemma 1.12.2, m must be a monomorphism. □

1.13 Products and finitely complete categories

In this section, we will define the important constructions of the product and the coproduct in a category.

Definition 1.13.1. Let $\mathscr{C}$ be a category and X, Y be objects in $\mathscr{C}$. The **product** of X and Y is a triple $(X \times Y, \pi_X, \pi_Y)$ consisting of an object $X \times Y$ and two morphisms $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ (called projections) which satisfies the following universal property:

If we have two morphisms $f : W \to X$ and $g : W \to Y$ then there exists a unique morphism $\theta : W \to X \times Y$ such that the following diagram commutes:

$$
\begin{tikzcd}
 & X \times Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\
X & W \arrow[l, "f"] \arrow[r, "g"'] \arrow[u, dashed, "\theta"] & Y
\end{tikzcd}
$$

The coproduct is the dual construction of the product. It is obtained by reversing the arrows in the definition of the product.

Definition 1.13.2. Let $\mathscr{C}$ be a category and X, Y be objects in $\mathscr{C}$. The **coproduct** of X and Y is a triple $(X \sqcup Y, \iota_X, \iota_Y)$ consisting of an object $X \sqcup Y$ and two morphisms $\iota_X : X \to X \times Y$ and $\iota_Y : Y \to X \times Y$ which satisfies the following universal property:

If we have two morphisms $f : X \to W$ and $g : Y \to W$ then there exists a unique morphism $\alpha : X \sqcup Y \to W$ such that the following diagram commutes:

$$\begin{tikzcd} & X \sqcup Y \arrow[d, dashed, "\alpha"] & \\ X \arrow[ur, "\iota_X"] \arrow[r, "f"'] & W & Y \arrow[ul, "\iota_Y"'] \arrow[l, "g"] \end{tikzcd}$$

Example 1.13.3. We will describe the coproduct in **Grp**. Assume that G and H are groups. The **free product** of G and H, denoted by $G \star H$, is the set of all reduced words of the form

$$g_1 h_1 g_2 h_2 \dots g_k h_k$$

where $g_i \in G$ and $h_i \in H$ for $i \in \{1, 2, \dots, n\}$. The group operation on $G \star H$ is the concatenation of words, followed by reduction.

We will describe how to reduce a word. Suppose that $g_1 h_1 \dots g_k h_k \in G \star H$. If there exists $i \in \{1, 2, \dots k\}$ such that $g_i = e_G$ or $h_i = e_H$, where e_G and e_H are the identity elements of G and H respectively then we remove e_G or e_H from the word. If there is an instance of $g_j g_{j+1}$ or $h_i h_{i+1}$ then we reduce the word by considering the product $g_j g_{j+1}$ as one element of G (rather than two) or the product $h_i h_{i+1}$ as one element of H (rather than two).

The words $e_G, e_H \in G \star H$ are the empty words (words of length zero). This is the identity element $e_{G \star H}$ of $G \star H$.

Now define the inclusion maps $\iota_G : G \to G \star H$ and $\iota_H : H \to G \star H$ by $\iota_G(g) = g$ and $\iota_H(h) = h$. Suppose that we have the following diagram in **Grp**:

$$\begin{tikzcd} & G \star H & \\ G \arrow[ur, "\iota_G"] \arrow[r, "\phi_G"'] & K & H \arrow[ul, "\iota_H"'] \arrow[l, "\phi_H"] \end{tikzcd}$$

We want to construct a unique morphism $\psi : G \star H \to K$ such that the following diagram commutes:

$$\begin{tikzcd} & G \star H \arrow[d, dashed, "\psi"] & \\ G \arrow[ur, "\iota_G"] \arrow[r, "\phi_G"'] & K & H \arrow[ul, "\iota_H"'] \arrow[l, "\phi_H"] \end{tikzcd}$$

Define the map ψ by

$$\begin{array}{rccc} \psi: & G \star H & \to & K \\ & g_1h_1 \dots g_kh_k & \mapsto & \phi_G(g_1)\phi_H(h_1)\dots\phi_G(g_k)\phi_H(h_k) \end{array}$$

As a preliminary observation, we have

$$\psi(e_{G\star H}) = \phi_G(e_G) = \phi_H(e_H) = e_K.$$

To see that ψ is a group morphism, assume that $g_1h_1 \dots g_kh_k$ and $g'_1h'_1 \dots g'_lh'_l$ are two reduced words in $G \star H$. If the concatenation $g_1h_1 \dots g'_lh'_l$ is already a reduced word then

$$\begin{aligned} \psi(g_1h_1 \dots g_kh_kg'_1h'_1 \dots g'_lh'_l) &= \phi_G(g_1)\phi_H(h_1)\dots\phi_G(g'_l)\phi_H(h'_l) \\ &= (\phi_G(g_1)\phi_H(h_1)\dots\phi_G(g_k)\phi_H(h_k)) \\ &\quad\ (\phi_G(g'_1)\phi_H(h'_1)\dots\phi_G(g'_l)\phi_H(h'_l)) \\ &= \psi(g_1h_1 \dots g_kh_k)\psi(g'_1h'_1 \dots g'_lh'_l). \end{aligned}$$

If the concatenation $g_1h_1 \dots g'_lh'_l$ is not a reduced word then there are two cases which can occur.

Case 1: If e_G or e_H appears in our word, we remove it by the reduction process. Since $\phi_G(e_G) = \phi_H(e_H) = e_K$, any terms of the form $\phi_G(e_G)$ and $\phi_H(e_H)$ in $\psi(g_1h_1 \dots g'_lh'_l)$ are removed from the product.

Case 2: If g_ig_{i+1} or h_ih_{i+1} appears in our word, we consider them as a single element of G and H respectively in the word. Since ϕ_G and ϕ_H are group morphisms, $\phi_G(g_ig_{i+1}) = \phi_G(g_i)\phi_G(g_{i+1})$ and $\phi_H(h_ih_{i+1}) = \phi_H(h_i)\phi_H(h_{i+1})$. So, we simply rewrite the product $\psi(g_1h_1 \dots g'_lh'_l)$ by replacing $\phi_G(g_i)\phi_G(g_{i+1})$ with $\phi_G(g_ig_{i+1})$ and similarly for $\phi_H(h_i)\phi_H(h_{i+1})$.

These two cases show that ψ respects the reduction process in $G \star H$. Hence,

$$\psi(g_1h_1 \dots g_kh_kg'_1h'_1 \dots g'_lh'_l) = \psi(g_1h_1 \dots g_kh_k)\psi(g'_1h'_1 \dots g'_lh'_l)$$

even if the concatenation $g_1h_1 \dots g_kh_kg'_1h'_1 \dots g'_lh'_l$ is not a reduced word. So, ψ is a group morphism, which satisfies by direct computation, $\psi \circ \iota_G = \phi_G$ and $\psi \circ \iota_H = \phi_H$.

Finally, to see that ψ is a unique group morphism, assume that $\psi' : G \star H \to K$ is another group morphism such that $\psi' \circ \iota_G = \phi_G$ and

$\psi' \circ \iota_H = \phi_H$. If $g \in G$ and $h \in H$ then $\psi'(g) = \phi_G(g) = \psi(g)$ and $\psi'(h) = \phi_H(h) = \psi(h)$. Since ψ' and ψ are group morphisms, we conclude that $\psi' = \psi$ on all of $G \star H$. Hence, ψ must be unique.

The free product and the free product with amalgamation feature prominently in the Seifert Van-Kampen theorem, a useful tool for computing the fundamental group of a wide variety of topological spaces. See [Hat02, Section 1.2] for more information.

In a category $\mathscr{C}$ with terminal object, the product is a special case of the pullback. To see why this is the case, note that the universal property of the product implies that the following square is a pullback

$$\begin{tikzcd}
X \times Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"] & Y \arrow[d, "\alpha_Y"] \\
X \arrow[r, "\alpha_X"] & *
\end{tikzcd}$$

In the above diagram, $*$ is the terminal object in $\mathscr{C}$.

With products, we can give an equivalent definition of a finitely complete category.

Theorem 1.13.1. *Let $\mathscr{C}$ be a category. Then, $\mathscr{C}$ is a finitely complete category (has pullbacks and a terminal object) if and only if $\mathscr{C}$ has products and equalizers of any parallel pair of morphisms.*

Proof. Assume that $\mathscr{C}$ is a category.

To show: (a) If $\mathscr{C}$ is finitely complete then it has products and equalizers of any parallel pair of morphisms.

(b) If $\mathscr{C}$ has products and equalizers of any parallel pair of morphisms then it is finitely complete.

(a) Assume that $\mathscr{C}$ is finitely complete. Then, $\mathscr{C}$ has pullbacks and a terminal object. We already argued that in a category with a terminal object, the product is a special case of the pullback. Therefore, $\mathscr{C}$ must have products.

Assume that $h, h' : X \to Y$ is a (parallel) pair of morphisms. Since $\mathscr{C}$ has products, we can consider the following diagram in $\mathscr{C}$:

$$\begin{tikzcd} & X \arrow[d, "{(h,\, h')}"] \\ Y \arrow[r, "\Delta"] & Y \times Y \end{tikzcd}$$

where Δ is the diagonal map. Since $\mathscr{C}$ has pullbacks, we can form the pullback square of the above diagram:

$$\begin{tikzcd} I \arrow[r, "i"] \arrow[d, "g"] & X \arrow[d, "{(h,\, h')}"] \\ Y \arrow[r, "\Delta"] & Y \times Y \end{tikzcd}$$

By Lemma 1.4.2, $i = eq(h, h')$. Consequently, $\mathscr{C}$ has equalizers.

(b) Assume that $\mathscr{C}$ has products and equalizers. Let $*$ be the empty product in $\mathscr{C}$ — the product of zero objects in $\mathscr{C}$. By the universal property of the product, there exists a unique morphism from any object W to $*$. The remainder of the statement of the universal property, as applied to this situation, is vacuous. Therefore, $*$ is the terminal object in $\mathscr{C}$.

Now suppose we have the following diagram in $\mathscr{C}$:

$$\begin{tikzcd} & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

Since $\mathscr{C}$ has products, we can form the product $X \times Y$ and its projections π_X and π_Y. Since $\mathscr{C}$ has equalizers, let $e : P \to X \times Y$ be the equalizer of the composites $f \circ \pi_X$ and $g \circ \pi_Y$. We claim that the commutative square

$$\begin{tikzcd} P \arrow[r, "\pi_X \circ e"] \arrow[d, "\pi_Y \circ e"'] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

is the pullback square we are after. Suppose that we have the following commutative square in $\mathscr{C}$:

$$\begin{tikzcd} W \arrow[r, "p"] \arrow[d, "q"'] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

By the universal property of the equalizer, there exists a unique morphism $d : W \to P$ such that the following diagram commutes:

$$\begin{tikzcd}
W \arrow[d, dashed, "d"] \arrow[dr, "{(p,q)}"] & & \\
P \arrow[r, "e"] & X \times Y \arrow[r, shift left, "f\circ\pi_X"] \arrow[r, shift right, "g\circ\pi_Y"'] & Z
\end{tikzcd}$$

Consequently, d is the unique morphism which makes the following diagram commute:

$$\begin{tikzcd}
W \arrow[dr, dashed, "d"] \arrow[drr, bend left, "p"] \arrow[ddr, bend right, "q"'] & & \\
& P \arrow[r, "\pi_X\circ e"] \arrow[d, "\pi_Y\circ e"] & X \arrow[d, "f"] \\
& Y \arrow[r, "g"] & Z
\end{tikzcd}$$

Hence, $\mathscr{C}$ has pullbacks and subsequently, $\mathscr{C}$ is a finitely complete category. $\square$

Dually, we also have the notion of a finitely cocomplete category.

Definition 1.13.4. Let $\mathscr{C}$ be a category. We say that $\mathscr{C}$ is a **finitely cocomplete category** if it has pushouts and an initial object.

We also have a "dual version" of Theorem 1.13.1.

Theorem 1.13.2. *Let $\mathscr{C}$ be a category. Then, $\mathscr{C}$ is a finitely cocomplete category if and only if $\mathscr{C}$ has coproducts and coequalizers of any pair of parallel morphisms.*

We omit the proof here because it is quite long, especially given the fact that the theory we have established previously deals with mostly pullbacks and not pushouts. Fortunately, the proof proceed in a similar fashion to Theorem 1.13.1.

Chapter 2

Internal structures

2.1 Internal unitary magmas and internal monoids

In this chapter, we will introduce internal structures within a finitely complete category and discuss some of their properties.

Definition 2.1.1. Let $\mathscr{C}$ be a finitely complete category with terminal object $*$. An **internal unitary magma** is a triple (M, m, e) consisting of an object M of $\mathscr{C}$, a morphism $m : M \times M \to M$ giving rise to an internal binary operation and a morphism $e : * \to M$ giving rise to an internal unit.

Moreover, the morphisms m and e make the following diagram in $\mathscr{C}$ commute:

$$\begin{tikzcd}
M \arrow[r, "(id_M, e\circ\tau_M)"] \arrow[dr, "id_M"'] & M \times M \arrow[d, "m"] & M \arrow[l, "(e\circ\tau_M, id_M)"'] \arrow[dl, "id_M"] \\
& M &
\end{tikzcd}$$

Here, τ_M is the unique morphism from M to $*$ and id_M is the identity morphism on M.

We know that a monoid is a unitary magma with an associative binary operation. This leads us straight to the definition of an internal monoid.

Definition 2.1.2. Let $\mathscr{C}$ be a finitely complete category. An **internal monoid** is an internal unitary magma (M, m, e) in $\mathscr{C}$ such that the binary operation $m : M \times M \to M$ is associative. This means that the following diagram in $\mathscr{C}$ must commute:

$$
\begin{tikzcd}
M \times M \times M \arrow[r, "{(m, id_M)}"] \arrow[d, "{(id_M, m)}"] & M \times M \arrow[d, "m"] \\
M \times M \arrow[r, "m"] & M
\end{tikzcd}
$$

Definition 2.1.3. Let $\mathscr{C}$ be a finitely complete category and (M, m, e) and (M', m', e') be internal unitary magmas in $\mathscr{C}$. A **unitary magma homomorphism** is a map $f : (M, m, e) \to (M', m', e')$ which preserves the internal unit and the internal binary operation.

Similarly, a **monoid homomorphism** is a map from one internal monoid to another which preserves the internal unit and the internal binary operation (alongside the associativity).

We will use $\mathbf{UMg}(\mathscr{C})$ to denote the category of internal unitary magmas in $\mathscr{C}$ and $\mathbf{Mon}(\mathscr{C})$ to denote the category of internal monoids in $\mathscr{C}$. In an abuse of notation, we will use the same symbol $\mathcal{U}_{\mathscr{C}}$ to denote the forgetful functors from $\mathbf{UMg}(\mathscr{C})$ and $\mathbf{Mon}(\mathscr{C})$ to $\mathscr{C}$.

Example 2.1.4. The category $\mathbf{Mon}(\mathbf{Top})$ is the category of topological monoids — monoids $(M, m, 1)$ with a topology on M such that the binary operation m is continuous.

The famous Eckmann-Hilton argument tells us that an internal unitary magma in the category of unitary magmas **UMg** is actually a commutative monoid.

Theorem 2.1.1. *Let $(M, *, e)$ be an internal unitary magma in the category of unitary magmas **UMg**. Note that M is an object of **UMg** and is itself a unitary magma. Consequently, M has a binary operation, which we denote by $\cdot$. Then, the binary operations $*$ and $\cdot$ coincide and $(M, *, e)$ is a commutative monoid.*

Proof. Assume that $(M, *, e)$ is an internal unitary magma in **UMg**. Equivalently, $(M, *, e)$ is an object in the category $\mathbf{UMg}(\mathbf{UMg})$. Let $\cdot$ be the binary operation on M as a unitary magma and $e_\bullet \in M$ be the unit associated with $\cdot$.

To show: (a) If $a, b, c, d \in M$ then $(a \cdot b) * (c \cdot d) = (a * c) \cdot (b * d)$.

(a) Assume that $a, b, c, d \in M$. Using the fact that $* : M \times M \to M$ is a morphism in **UMg**, we compute directly that

$$
\begin{aligned}
(a \cdot b) * (c \cdot d) &= *(a \cdot b, c \cdot d) \\
&= *(a, c) \cdot *(b, d) \\
&= (a * c) \cdot (b * d).
\end{aligned}
$$

In particular, it is the second line where we used the fact that $* : M \times M \to M$ is a unitary magma morphism.

To show: (b) The unit e of $*$ and the unit $e_\bullet$ of $\cdot$ coincide.

(c) The operations $\cdot$ and $*$ both coincide.

(b) Using part (a), we compute directly that

$$
\begin{aligned}
e &= e * e \\
&= (e \cdot e_\bullet) * (e_\bullet \cdot e) \\
&= (e * e_\bullet) \cdot (e_\bullet * e) \\
&= e_\bullet \cdot e_\bullet = e_\bullet.
\end{aligned}
$$

(c) We use both part (a) and part (b) to argue that

$$
\begin{aligned}
a * b &= (a \cdot e_\bullet) * (e_\bullet \cdot b) \\
&= (a * e_\bullet) \cdot (e_\bullet * b) \quad \text{(Part (a))} \\
&= (a * e) \cdot (e * b) \quad \text{(Part (b))} \\
&= a \cdot b.
\end{aligned}
$$

Hence, $*$ and $\cdot$ are the same binary operation on M.

To show: (d) $*$ is commutative.

(e) $*$ is associative.

(d) By direct computation, we have

$$
\begin{aligned}
a * b &= (e_\bullet \cdot a) * (b \cdot e_\bullet) \\
&= (e_\bullet * b) \cdot (a * e_\bullet) \quad \text{(Part (a))} \\
&= (e * b) \cdot (a * e) \quad \text{(Part (b))} \\
&= b \cdot a = b * a
\end{aligned}
$$

where the last equality follows from part (c) of the proof. Hence, $*$ is commutative.

(e) To see that $*$ is associative, we also compute directly that

$$\begin{aligned} a * (b * c) &= (a * e) * (b * c) \\ &= (a * e) \cdot (b * c) \quad \text{(Part (c))} \\ &= (a \cdot b) * (e \cdot c) \quad \text{(Part (a))} \\ &= (a \cdot b) * (e_\bullet \cdot c) \quad \text{(Part (b))} \\ &= (a * b) * c. \end{aligned}$$

Hence, the internal unitary magma $(M, *, e)$ is a commutative monoid. $\square$

A major consequence of Theorem 2.1.1 is that the categories **UMg**(**UMg**), **UMg**(**Mon**), **Mon**(**UMg**) and **Mon**(**Mon**) are isomorphic to the category of commutative monoids **CoM**.

Our next task is to discuss the properties of the forgetful functor $\mathcal{U}_{\mathscr{C}}$. There are two important definitions here.

Definition 2.1.5. Let $\mathscr{C}$ and $\mathscr{D}$ be categories and $H : \mathscr{C} \to \mathscr{D}$ be a functor. We say that the functor H is **left exact** if H preserves finite limits (pullbacks, terminal objects, equalizers, products).

Here is a remark of caution. The term "left exact functor" also refers to a functor which preserves left exact sequences. Obviously, we will use the term "left exact functor" as defined above for these notes, but this is something to keep in mind when one peruses the literature on category theory.

Definition 2.1.6. Let $\mathscr{C}$ and $\mathscr{D}$ be categories and $H : \mathscr{C} \to \mathscr{D}$ be a functor. We say that the functor H is **conservative** if H satisfies the following property: If $g : A \to B$ is a morphism in $\mathscr{C}$ such that $H(g)$ is an isomorphism in $\mathscr{D}$ then g is an isomorphism in $\mathscr{C}$.

It is straightforward to see from the definition that the composite of left exact functors is a left exact functor. Conservative functors are also well behaved under composition.

Lemma 2.1.2. *Let $\mathscr{C}, \mathscr{D}$ and $\mathscr{E}$ be categories. Let $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{E}$ be functors.*

1. *If F and G are conservative functors then $G \circ F$ is also conservative.*

2. *If $G \circ F$ is a conservative functor then F is also a conservative functor.*

Proof. Assume that $\mathscr{C}, \mathscr{D}$ and $\mathscr{E}$ are categories. Assume that $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{E}$ are functors.

To show: (a) If F and G are conservative then $G \circ F$ is a conservative functor.

(b) If $G \circ F$ is a conservative functor then F is a conservative functor.

(a) Assume that F and G are conservative functors. Assume that f is a morphism in $\mathscr{C}$ such that $(G \circ F)(f)$ is an isomorphism in $\mathscr{E}$. Since G is a conservative functor, $F(f)$ must be an isomorphism in $\mathscr{D}$. Since F is a conservative functor, f must be an isomorphism. Consequently, $G \circ F$ is a conservative functor.

(b) Assume that $G \circ F$ is a conservative functor. Let f be a morphism in $\mathscr{C}$ such that $F(f)$ is an isomorphism. Then, $(G \circ F)(f)$ is an isomorphism in $\mathscr{E}$ and since $G \circ F$ is a conservative functor, f must be an isomorphism. Therefore, F is a conservative functor as required. □

If we have a left exact functor between two finitely complete categories, then the characterisation of a conservative functor can be weakened from isomorphisms to monomorphisms.

Theorem 2.1.3. *Let $\mathscr{C}$ and $\mathscr{D}$ be finitely complete categories. Let $H : \mathscr{C} \to \mathscr{D}$ be left exact. Then, H is a conservative functor if and only if H is "conservative on monomorphisms" — if f is a monomorphism such that $H(f)$ is an isomorphism then f is an isomorphism.*

Proof. Assume that $\mathscr{C}$ and $\mathscr{D}$ are finitely complete categories. Assume that $H : \mathscr{C} \to \mathscr{D}$ is a left exact functor.

To show: (a) If H is a conservative functor then H is conservative on monomorphisms.

(b) If H is conservative on monomorphisms then H is a conservative functor.

(a) Assume that H is a conservative functor. By definition of a conservative functor, H must be conservative on monomorphisms.

(b) Assume that H is conservative on monomorphisms. Assume that $f : X \to Y$ is a morphism in $\mathscr{C}$ such that $H(f)$ is an isomorphism in $\mathscr{D}$. Since $\mathscr{C}$ is finitely complete, we can consider the kernel equivalence relation $R[f]$ of f:

$$\begin{tikzcd} R[f] \arrow[r, shift left=2, "p_0^f"] \arrow[r, shift right=2, "p_1^f"'] & X \arrow[l, "s_0^f" description] \arrow[r, "f"] & Y \end{tikzcd}$$

Since H is a left exact functor, it maps the above equalizer in $\mathscr{C}$ to an equalizer in $\mathscr{D}$. Since $H(f)$ is an isomorphism, we can apply Theorem 1.7.5 to show that $H(s_0^f)$ is an isomorphism in $\mathscr{D}$.

Recall that $id_X = p_0^f \circ s_0^f$. Since id_X is an isomorphism, it must be a monomorphism and an epimorphism. Now monomorphisms are preserved under left cancellation. So, s_0^f is a monomorphism such that $H(s_0^f)$ is an isomorphism in $\mathscr{D}$.

Since H is conservative on monomorphisms, s_0^f is an isomorphism. By Theorem 1.7.5, we deduce that f is a monomorphism.

Since H is conservative on monomorphisms, f is a monomorphism and $H(f)$ is an isomorphism by assumption, we deduce that f is an isomorphism in $\mathscr{C}$. So, H is a conservative functor. $\square$

Example 2.1.7. This example is from [Bou17, Proposition 2.1.7]. Let $\mathscr{C}$ be a finitely complete category. The forgetful functors $\mathcal{U}_{\mathscr{C}} : \mathbf{UMg}(\mathscr{C}) \to \mathscr{C}$ and $\mathcal{U}_{\mathscr{C}} : \mathbf{Mon}(\mathscr{C}) \to \mathscr{C}$ are both left exact and conservative. In fact, the forgetful functors $\mathcal{U}_{\mathscr{C}}$ are faithful.

We have already talked about commutativity in this particular section. Now we will formally introduce commutativity in the context of internal unitary magmas and internal monoids.

Definition 2.1.8. Let $\mathscr{C}$ be a finitely complete category and (M, m, e) be an internal unitary magma (or an internal monoid). We say that (M, m, e) is **commutative** if the following diagram commutes:

$$\begin{tikzcd} M \times M \arrow[rr, "tw_{M,M}"] \arrow[dr, "m"'] & & M \times M \arrow[dl, "m"] \\ & M & \end{tikzcd}$$

Here, $tw_{M,M} : M \times M \to M \times M$ is the twisting isomorphism, which maps (m, m') to (m', m).

For this section and the next section, which deal with internal structures, we assume that **every category $\mathscr{C}$ is locally small**.

Definition 2.1.9. We say that a category $\mathscr{C}$ is **locally small** if for any pair of objects $X, X' \in \mathscr{C}$, the class of morphisms $Hom_{\mathscr{C}}(X, X')$ is actually a set.

Theorem 2.1.4. *Let $\mathscr{C}$ be a finitely complete category and (M, m, e) be an internal monoid in $\mathscr{C}$. Assume that $X \in \mathscr{C}$ is an object and $f : X \to X'$ is a morphism in $\mathscr{C}$. Then, $Hom_{\mathscr{C}}(X, M)$ is a monoid and*

$$Hom_{\mathscr{C}}(f, M) : Hom_{\mathscr{C}}(X', M) \to Hom_{\mathscr{C}}(X, M)$$

is a monoid morphism.

Proof. Assume that $\mathscr{C}$ is a finitely complete category and (M, m, e) is an internal monoid in $\mathscr{C}$. Assume that $X \in \mathscr{C}$ is an object and $f : X \to X'$ is a morphism in $\mathscr{C}$.

To show: (a) $Hom_{\mathscr{C}}(X, M)$ is a monoid.

(b) $Hom_{\mathscr{C}}(f, M)$ is a monoid morphism.

(a) To be clear, $Hom_{\mathscr{C}}(X, M)$ is the set of morphisms from X to M. We want to show that it is actually a monoid. First, we will define the binary operation on $Hom_{\mathscr{C}}(X, M)$ by

$$\begin{array}{rccc} \mu : & Hom_{\mathscr{C}}(X, M) \times Hom_{\mathscr{C}}(X, M) & \to & Hom_{\mathscr{C}}(X, M) \\ & (f, g) & \mapsto & \mu(f, g)(x) = m(f(x), g(x)). \end{array}$$

Since M is an internal monoid, the internal binary operation m is associative. Therefore, μ is also an associative binary operation.

Let $\iota : X \to M$ be the morphism in $\mathscr{C}$ which sends $x \in X$ to $e(*) \in M$, where $*$ is the terminal object in $\mathscr{C}$. To see that ι is the unit in $Hom_{\mathscr{C}}(X, M)$, we compute for $f, g \in Hom_{\mathscr{C}}(X, M)$ and $x \in X$ that

$$\mu(f, \iota)(x) = m(f(x), \iota(x)) = m(f(x), e(*)) = f(x)$$

and

$$\mu(\iota, g)(x) = m(\iota(x), g(x)) = m(e(*), g(x)) = g(x).$$

So, $\mu(f, \iota) = f$ and $\mu(\iota, g) = g$. Hence, ι is the unit in $Hom_{\mathscr{C}}(X, M)$. Consequently, $Hom_{\mathscr{C}}(X, M)$ is a monoid.

(b) Let μ' be the binary operation on $Hom_{\mathscr{C}}(X', M)$ and ι' be the unit in $Hom_{\mathscr{C}}(X', M)$. Explicitly, the map $Hom_{\mathscr{C}}(f, M)$ is defined by

$$\begin{array}{rccc} Hom_{\mathscr{C}}(f, M): & Hom_{\mathscr{C}}(X', M) & \to & Hom_{\mathscr{C}}(X, M) \\ & g' & \mapsto & g' \circ f \end{array}$$

First, we compute for $x \in X$ that

$$Hom_{\mathscr{C}}(f, M)(\iota')(x) = (\iota' \circ f)(x) = \iota'(f(x)) = e(*).$$

Therefore, $Hom_{\mathscr{C}}(f, M)(\iota') = \iota$. If $g', h' \in Hom_{\mathscr{C}}(X', M)$ then

$$\begin{aligned} Hom_{\mathscr{C}}(f, M)(\mu'(g', h'))(x) &= (\mu'(g', h') \circ f)(x) \\ &= \mu'(g', h')(f(x)) \\ &= m(g'(f(x)), h'(f(x))) \\ &= \mu(g' \circ f, h' \circ f)(x) \\ &= \mu(Hom_{\mathscr{C}}(f, M)(g'), Hom_{\mathscr{C}}(f, M)(h'))(x). \end{aligned}$$

Therefore, $Hom_{\mathscr{C}}(f, M)$ is a monoid morphism. □

Theorem 2.1.4 tells us that the functor $Hom_{\mathscr{C}}(-, M)$ factorises through the **Mon**, so that the following diagram commutes:

$$\begin{tikzcd} \mathscr{C}^{op} \arrow[r, dashed] \arrow[dr, "Hom_{\mathscr{C}}(-{,}M)"'] & \mathbf{Mon} \arrow[d, "U"] \\ & \mathbf{Set} \end{tikzcd}$$

Here, $U : \mathbf{Mon} \to \mathbf{Set}$ is the forgetful functor. Next, we will investigate a particular situation in Theorem 2.1.4, where the internal monoid in $\mathscr{C}$ is commutative.

Theorem 2.1.5. *Let $\mathscr{C}$ be a finitely complete category and (M, m, e) be an internal monoid in $\mathscr{C}$. Let X be an object in $\mathscr{C}$. Then, (M, m, e) is commutative if and only if the monoid $Hom_{\mathscr{C}}(X, M)$ is commutative.*

Proof. Assume that $\mathscr{C}$ is a finitely complete category and (M, m, e) is an internal monoid in $\mathscr{C}$. Assume that $X \in \mathscr{C}$ is an object.

To show: (a) If (M, m, e) is commutative then the monoid $Hom_{\mathscr{C}}(X, M)$ is commutative.

(b) If the monoid $Hom_{\mathscr{C}}(X, M)$ is commutative then the internal monoid (M, m, e) is commutative.

(a) Assume that the internal binary operation m is commutative. Let μ be the binary operation on $Hom_{\mathscr{C}}(X, M)$. If $f, g \in Hom_{\mathscr{C}}(X, M)$ and $x \in X$ then

$$\mu(f, g)(x) = m(f(x), g(x)) = m(g(x), f(x)) = \mu(g, f)(x).$$

So, $\mu(f, g) = \mu(g, f)$ and the binary operation μ on $Hom_{\mathscr{C}}(X, M)$ is commutative.

(b) Assume that the binary operation μ on $Hom_{\mathscr{C}}(X, M)$ is commutative. If $m_1, m_2 \in M$ then there exists $f_1, f_2 \in Hom_{\mathscr{C}}(X, M)$ such that $f_1(x) = m_1$ and $f_2(x) = m_2$ for some $x \in X$. So,

$$m(m_1, m_2) = m(f_1(x), f_2(x)) = \mu(f_1, f_2)(x) = \mu(f_2, f_1)(x) = m(m_2, m_1).$$

Therefore, the internal monoid (M, m, e) is commutative. $\square$

Consequently, if the internal monoid (M, m, e) is commutative then the functor $Hom_{\mathscr{C}}(-, M)$ factorises through the category **CoM**, so that the following diagram commutes:

$$\begin{tikzcd}
\mathscr{C}^{op} \arrow[r, dashed] \arrow[dr, "Hom_{\mathscr{C}}(-,M)"'] & \mathbf{CoM} \arrow[d, "U"] \\
 & \mathbf{Set}
\end{tikzcd}$$

Again, $U : \mathbf{CoM} \to \mathbf{Set}$ is the forgetful functor.

2.2 Internal groups

We want to distinguish two different ways of defining a group from a monoid. A group $(G, \cdot, 1)$ is a monoid such that every $g \in G$ has an inverse.

This definition actually requires extra data — the mapping $g \mapsto g^{-1}$. We can also define a group $(G, \cdot, 1)$ to be a monoid such that every $g \in G$ is invertible. Unlike the previous definition, invertibility is a property.

Although these definitions are equivalent, there is a subtle difference between the extra mapping and the property which we will now elucidate.

Theorem 2.2.1. *Let M be a monoid with binary operation $m : M \times M \to M$ and unit $e : 1 \to M$. Here, 1 is the trivial monoid with one element. Then, M is a group if and only if the following commutative square in **Set** is a pullback:*

$$\begin{array}{ccc} M \times M & \xrightarrow{m} & M \\ \downarrow{\scriptstyle p_0^M} & & \downarrow{\scriptstyle \tau_M} \\ M & \xrightarrow{\tau_M} & 1 \end{array}$$

where p_0^M is projection onto the first factor of $M \times M$.

We abuse notation in Theorem 2.2.1 by using 1 to represent the trivial monoid, its underlying singleton set and the element it contains.

Proof. Assume that M is a monoid with binary operation m and unit $e(1)$.

To show: (a) If M is a group then the above commutative square is a pullback square.

(b) If the above commutative square is a pullback square then M is a group.

(a) Assume that M is a group. For each $p \in M$, there exists $p^{-1} \in M$ such that $m(p, p^{-1}) = m(p^{-1}, p) = e(1)$. Suppose that we have the following commutative square in **Set**:

$$\begin{array}{ccc} N & \xrightarrow{f} & M \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \tau_M} \\ M & \xrightarrow{\tau_M} & 1 \end{array}$$

Define the function (morphism of sets)

$$\begin{array}{rccc} \phi : & N & \to & M \times M \\ & n & \mapsto & (g(n), g(n)^{-1} f(n)) \end{array}$$

This is a unique morphism of sets. It is easy to check that the following diagram commutes:

$$\begin{tikzcd}
N \arrow[drr, bend left, "f"] \arrow[dr, dashed, "\phi"] \arrow[ddr, bend right, "g"'] & & \\
& M \times M \arrow[r, "m"] \arrow[d, "p_0^M"] & M \arrow[d, "\tau_M"] \\
& M \arrow[r, "\tau_M"] & 1
\end{tikzcd}$$

Hence, the commutative square we began with must be a pullback square.

(b) Assume that the following commutative square in **Set** is a pullback square:

$$\begin{tikzcd}
M \times M \arrow[r, "m"] \arrow[d, "p_0^M"] & M \arrow[d, "\tau_M"] \\
M \arrow[r, "\tau_M"] & 1
\end{tikzcd}$$

Then, there exists a unique morphism $\alpha : M \to M \times M$ such that the following diagram commutes:

$$\begin{tikzcd}
M \arrow[drr, bend left, "e \circ \tau_M"] \arrow[dr, dashed, "\alpha"] \arrow[ddr, bend right, "id_M"'] & & \\
& M \times M \arrow[r, "m"] \arrow[d, "p_0^M"] & M \arrow[d, "\tau_M"] \\
& M \arrow[r, "\tau_M"] & 1
\end{tikzcd}$$

If $p \in M$ and $\alpha(p) = (i_1(p), i_2(p))$ then by commutativity of the above diagram, $i_1(p) = p = id_M(p)$ and $m(p, i_2(p)) = (e \circ \tau_M)(p) = e(1)$. It remains to show that $i_2(p)$ is the left inverse for p. Note that we also have $m(i_2(p), i_2(i_2(p))) = e(1)$.

To show: (ba) $m(e(1), i_2(i_2(p))) = m(e(1), p)$.

(ba) Abbreviating $m(a, b)$ as $a \cdot b$, we argue as follows

$$\begin{aligned}
e(1) \cdot p &= e(1) \cdot p \cdot e(1) \\
&= (p \cdot i_2(p)) \cdot p \cdot (i_2(p) \cdot i_2(i_2(p))) \\
&= (p \cdot i_2(p)) \cdot (p \cdot i_2(p)) \cdot i_2(i_2(p)) = e(1) \cdot i_2(i_2(p)).
\end{aligned}$$

(b) Using part (ba), we have

$$\begin{aligned}
e(1) &= i_2(p) \cdot i_2(i_2(p)) \\
&= (i_2(p) \cdot e(1)) \cdot i_2(i_2(p)) \\
&= i_2(p) \cdot (e(1) \cdot i_2(i_2(p))) \\
&= i_2(p) \cdot (e(1) \cdot p) \\
&= (i_2(p) \cdot e(1)) \cdot (e(1) \cdot p) \\
&= (i_2(p) \cdot p \cdot i_2(p)) \cdot (i_2(p) \cdot i_2(i_2(p)) \cdot p).
\end{aligned}$$

But

$$\begin{aligned}
(i_2(p) \cdot p \cdot i_2(p)) \cdot (i_2(p) \cdot i_2(i_2(p)) \cdot p) &= i_2(p) \cdot (p \cdot i_2(p)) \cdot (i_2(p) \cdot i_2(i_2(p))) \cdot p \\
&= i_2(p) \cdot p.
\end{aligned}$$

By equating the two expressions together, we find that
$i_2(p) \cdot p = m(i_2(p), p) = e(1)$. So, $i_2(p)$ is a left and right inverse for $p \in M$. Hence, every element of M is invertible and consequently, M is a group. $\square$

From Theorem 2.2.1, the map $i_2 = p_1^M \circ \alpha$ is the inverse mapping from M to M. Here, $\alpha : M \to M \times M$ is the unique group morphism defined in part (b) of Theorem 2.2.1 and $p_1^M : M \times M \to M$ denotes projection onto the right factor of $M \times M$.

The characterisation of a group in Theorem 2.2.1 leads us straight to the definition of an internal group in a finitely complete category.

Definition 2.2.1. Let $\mathscr{C}$ be a finitely complete category with terminal object $*$. An **internal group** in $\mathscr{C}$ is an internal monoid (M, m, e) such that the following commutative square in $\mathscr{C}$ is a pullback square:

$$\begin{tikzcd}
M \times M \arrow[r, "m"] \arrow[d, "p_0^M"] & M \arrow[d, "\tau_M"] \\
M \arrow[r, "\tau_M"] & *
\end{tikzcd}$$

Again, $p_0^M : M \times M \to M$ denotes projection onto the left factor.

An **internal abelian group** is an internal commutative monoid (M, m, e) such that the same square above is a pullback square in $\mathscr{C}$.

We denote the category of internal groups in $\mathscr{C}$ by $\mathbf{Grp}(\mathscr{C})$. The category of internal abelian groups in $\mathscr{C}$ is denoted by $\mathbf{Ab}(\mathscr{C})$.

In a similar vein to $\mathbf{UMg}(\mathscr{C})$ and $\mathbf{Mon}(\mathscr{C})$, the forgetful functors $\mathcal{U}_{\mathscr{C}} : \mathbf{Grp}(\mathscr{C}) \to \mathscr{C}$ and $\mathcal{U}_{\mathscr{C}} : \mathbf{Ab}(\mathscr{C}) \to \mathscr{C}$ are left exact and conservative.

Example 2.2.2. The category $\mathbf{Grp}(\mathbf{Top})$ is the category of topological groups — groups which are endowed with a topology which makes the binary operation and the inverse mapping continuous functions.

The Eckmann-Hilton argument can also be applied to internal groups.

Theorem 2.2.2. *Let $(G, *, e)$ be an internal group in the category of monoids **Mon**. Note that G is an object of **Mon** and is itself a monoid. Consequently, M has a binary monoid operation, which we denote by $\cdot$. Then, the binary operations $*$ and $\cdot$ coincide and $(G, *, e)$ is an abelian group.*

Proof. Assume that $(G, *, e)$ is an internal group in $\mathbf{UMg}$. Equivalently, $(G, *, e)$ is an object in the category $\mathbf{Grp}(\mathbf{Mon})$. Let $\cdot$ be the binary operation on G as a monoid and $e_\bullet \in M$ be the unit associated with $\cdot$.

To show: (a) If $a, b, c, d \in G$ then $(a \cdot b) * (c \cdot d) = (a * c) \cdot (b * d)$.

(a) Assume that $a, b, c, d \in G$. Using the fact that $* : G \times G \to G$ is a morphism in $\mathbf{Mon}$, we compute directly that

$$\begin{aligned}
(a \cdot b) * (c \cdot d) &= *(a \cdot b, c \cdot d) \\
&= *(a, c) \cdot *(b, d) \\
&= (a * c) \cdot (b * d).
\end{aligned}$$

In particular, it is the second line where we used the fact that $* : G \times G \to G$ is a monoid morphism.

To show: (b) The unit e of $*$ and the unit $e_\bullet$ of $\cdot$ coincide.

(c) The operations $\cdot$ and $*$ both coincide.

(b) Using part (a), we compute directly that

$$\begin{aligned} e &= e * e \\ &= (e \cdot e_\bullet) * (e_\bullet \cdot e) \\ &= (e * e_\bullet) \cdot (e_\bullet * e) \\ &= e_\bullet \cdot e_\bullet = e_\bullet. \end{aligned}$$

(c) We use both part (a) and part (b) to argue that

$$\begin{aligned} a * b &= (a \cdot e_\bullet) * (e_\bullet \cdot b) \\ &= (a * e_\bullet) \cdot (e_\bullet * b) \quad \text{(Part (a))} \\ &= (a * e) \cdot (e * b) \quad \text{(Part (b))} \\ &= a \cdot b. \end{aligned}$$

Hence, $*$ and $\cdot$ are the same binary operation on G.

To show: (d) $*$ is commutative.

(d) By direct computation, we have

$$\begin{aligned} a * b &= (e_\bullet \cdot a) * (b \cdot e_\bullet) \\ &= (e_\bullet * b) \cdot (a * e_\bullet) \quad \text{(Part (a))} \\ &= (e * b) \cdot (a * e) \quad \text{(Part (b))} \\ &= b \cdot a = b * a \end{aligned}$$

where the last equality follows from part (c) of the proof. Hence, $*$ is commutative and the internal group $(G, *, e)$ is an abelian group. $\square$

A major consequence of Theorem 2.2.2 is that the categories **Grp**(**Mon**) and **Ab**(**Mon**) are isomorphic to the category of abelian groups **Ab**. A similar argument also reveals that the categories **UMg**(**Grp**) and **Mon**(**Grp**) are isomorphic to **Ab**.

Before we continue, we would like to highlight a substantial application of the Eckmann-Hilton argument to algebraic topology.

Example 2.2.3. Let G be a topological group and e be the identity element of G. We want to prove that the fundamental group $\pi_1(G, e)$ is abelian. Our strategy is to exploit the fact that the binary operation $\mu : G \times G \to G$ is continuous. First, we require the following lemma about the fundamental group.

Lemma 2.2.3. *Let (X, x_0) and (Y, y_0) be pointed topological spaces. Then the fundamental groups $\pi_1(X \times Y, (x_0, y_0))$ and $\pi_1(X, x_0) \times \pi_1(Y, y_0)$ are isomorphic.*

Proof. Assume that (X, x_0) and (Y, y_0) are pointed topological spaces. Let $f : [0, 1] \to X \times Y$ be a loop with basepoint (x_0, y_0) so that $f(0) = f(1) = (x_0, y_0)$. Let $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ denote the canonical projection maps. These projections are continuous and hence, by applying the functor $\pi_1 : \mathbf{Top}_* \to \mathbf{Grp}$ which maps a pointed topological space to its fundamental group, we obtain the group morphisms

$$\begin{array}{rccc} \pi_{X\sharp} : & \pi_1(X \times Y, (x_0, y_0)) & \to & \pi_1(X, x_0) \\ & [f] & \mapsto & [\pi_X \circ f] \end{array}$$

and

$$\begin{array}{rccc} \pi_{Y\sharp} : & \pi_1(X \times Y, (x_0, y_0)) & \to & \pi_1(X, x_0) \\ & [g] & \mapsto & [\pi_Y \circ g] \end{array}$$

Now define

$$\begin{array}{rccc} \psi : & \pi_1(X \times Y, (x_0, y_0)) & \to & \pi_1(X, x_0) \times \pi_1(Y, y_0) \\ & [f] & \mapsto & ([\pi_X \circ f], [\pi_Y \circ f]) \end{array}$$

We claim that ψ is a group isomorphism. Since ψ is the product of the group morphisms $\pi_{X\sharp}$ and $\pi_{Y\sharp}$, ψ must be a group morphism itself.

To show: (a) ψ is well-defined.

(b) ψ is injective.

(c) ψ is surjective.

(a) Assume that $[f] = [g]$ in $\pi_1(X \times Y, (x_0, y_0))$. Then, there exists a homotopy $F : [0, 1] \times [0, 1] \to X \times Y$ rel $\{0, 1\}$ such that $F(0, t) = f(t)$, $F(1, t) = g(t)$, $F(s, 0) = f(0) = g(0)$ and $F(s, 1) = f(1) = g(1)$.

The composite $\pi_X \circ F : [0, 1] \times [0, 1] \to X$ continuous because it is the composite of continuous functions. Furthermore, $(\pi_X \circ F)(0, t) = (\pi_X \circ f)(t)$, $(\pi_X \circ F)(1, t) = (\pi_X \circ g)(t)$,

$$(\pi_X \circ F)(s, 0) = (\pi_X \circ f)(0) = (\pi_X \circ g)(0) = x_0$$

and

$$(\pi_X \circ F)(s, 1) = (\pi_X \circ f)(1) = (\pi_X \circ g)(1) = x_0.$$

So, $\pi_X \circ F$ is a homotopy rel $\{0, 1\}$ between $\pi_X \circ f$ and $\pi_X \circ g$. Similarly, $\pi_Y \circ F$ is a homotopy rel $\{0, 1\}$ between $\pi_Y \circ f$ and $\pi_Y \circ g$. Hence, $[\pi_X \circ f] = [\pi_X \circ g]$ in $\pi_1(X, x_0)$ and $[\pi_Y \circ f] = [\pi_Y \circ g]$ in $\pi_1(Y, y_0)$. So,

$$([\pi_X \circ f], [\pi_Y \circ f]) = ([\pi_X \circ g], [\pi_Y \circ g])$$

and consequently, ψ is well-defined.

(b) The kernel of ψ is given by

$$\ker \psi = \{[f] \in \pi_1(X \times Y, (x_0, y_0)) \mid \pi_X \circ f \simeq c_{x_0} \text{ and } \pi_Y \circ f \simeq c_{y_0}\}$$

where $c_{x_0} : [0, 1] \to X$ and $c_{y_0} : [0, 1] \to Y$ are the constant loops at x_0 and y_0 respectively.

The idea is to write the loop $f : [0, 1] \to X \times Y$ as $f(t) = (g(t), h(t))$, where $g : [0, 1] \to X$ and $h : [0, 1] \to Y$ are loops with basepoints x_0 and y_0 respectively. Let $M \in Cts([0, 1] \times [0, 1] \to X)$ be a homotopy between $\pi_X \circ f$ and c_{x_0} and $H \in Cts([0, 1] \times [0, 1] \to Y)$ be a homotopy between $\pi_Y \circ f$ and c_{y_0}. Then, $M(0, t) = g(t)$, $M(1, t) = c_{x_0}(t) = x_0$, $H(0, t) = h(t)$ and $H(1, t) = y_0$.

Now define the function

$$\begin{array}{rccc} F: & [0,1] \times [0,1] & \to & X \times Y \\ & (s,t) & \mapsto & (M(s,t), H(s,t)) \end{array}$$

The function F is continuous because it is the product of the homotopies M and H, which are continuous by definition. Furthermore, $F(0, t) = (g(t), h(t)) = f(t)$ and $F(1, t) = (x_0, y_0)$. This proves the homotopy equivalence $f \simeq c_{(x_0,y_0)}$ and so, $[f] = [c_{(x_0,y_0)}]$ as equivalence classes. So, $\ker \psi = \{0\}$ and ψ must be injective.

(c) Assume that $g : [0, 1] \to X$ and $h : [0, 1] \to Y$ are two loops with basepoints x_0 and y_0 respectively. Then, $(g(t), h(t)) : [0, 1] \to X \times Y$ is a loop with basepoint (x_0, y_0) and

$$\psi([(g, h)]) = ([\pi_X \circ (g, h)], [\pi_Y \circ (g, h)]) = ([g], [h])$$

Hence, ψ is surjective.

Combining parts (a) and (b) of the proof, we deduce that ψ defines a group isomorphism between $\pi_1(X \times Y, (x_0, y_0))$ and $\pi_1(X, x_0) \times \pi_1(Y, y_0)$. $\square$

Since the binary operation $\mu : (G \times G, (e, e)) \to (G, e)$ of G is continuous, it is a morphism in the category of pointed topological spaces $\mathbf{Top}_*$. Therefore, $\pi_1(\mu) = \mu_\sharp : \pi_1(G \times G, (e, e)) \to \pi_1(G, e)$ is a morphism in the category of groups $\mathbf{Grp}$. By precomposing with ψ^{-1} as defined in Lemma 2.2.3, we obtain the following binary operation on $\pi_1(G, e)$:

$$\begin{array}{rccc} \pi_1(\mu) \circ \psi^{-1} : & \pi_1(G, e) \times \pi_1(G, e) & \to & \pi_1(G, e) \\ & ([g], [h]) & \mapsto & [\mu \circ (g, h)]. \end{array} \tag{2.1}$$

Before we proceed, let us iron out the notation for the three binary operations involved. For $g, h \in G$, we will write the product $\mu(g, h)$ as gh. For the binary operation in equation (2.1), we will write

$$(\pi_1(\mu) \circ \psi^{-1})([g], [h]) \qquad \text{as} \qquad [g] \cdot [h].$$

Then, $([g] \cdot [h])(t) = [g(t)h(t)]$ in $\pi_1(G, e)$. For $[g], [h] \in \pi_1(G, e)$, let

$$g * h = \begin{cases} g(2t), & \text{if } t \in [0, 1/2], \\ h(2t - 1), & \text{if } t \in [1/2, 1]. \end{cases}$$

The original binary operation on the fundamental group $\pi_1(G, e)$ will be denoted by $\odot$. That is, $[g] \odot [h] = [g * h]$.

To show: (a) There exists $[k] \in \pi_1(G, e)$ such that if $[g] \in \pi_1(G, e)$ then $[g] \cdot [k] = [k] \cdot [g] = [g]$.

(b) If $[a], [b], [c], [d] \in \pi_1(G, e)$ then

$$([a] \cdot [b]) \odot ([c] \cdot [d]) = ([a] \odot [c]) \cdot ([b] \odot [d]). \tag{2.2}$$

(a) Let $c_e : [0, 1] \to G$ be the constant loop at e. Let $[g] \in \pi_1(G, e)$. We claim that $[c_e] \cdot [g] = [g]$ and $[g] \cdot [c_e] = [g]$.

By construction in equation (2.1), we deduce that $[c_e] \cdot [g] = [\mu \circ (c_e, g)]$. But, if $t \in [0, 1]$ then

$$(\mu \circ (c_e, g))(t) = \mu(c_e(t), g(t)) = c_e(t)g(t) = eg(t) = g(t).$$

This shows that $[\mu \circ (c_e, g)] = [g]$ in $\pi_1(G, e)$. Therefore, $[c_e] \cdot [g] = [g]$. Similarly,

$$(\mu \circ (g, c_e))(t) = \mu(g(t), c_e(t)) = g(t)c_e(t) = g(t)e = g(t).$$

So, $[g] \cdot [c_e] = [\mu \circ (g, c_e)] = [g]$. Thus, $[c_e] \in \pi_1(G, e)$ is the desired element — it is the unit of the binary operation $\cdot$. Note that $[c_e]$ is also the unit of the binary operation $\odot$.

(b) Assume that $[a], [b], [c], [d] \in \pi_1(G, e)$. We will compute the LHS and RHS of equation (2.2) separately. The RHS of equation (2.2) is

$$([a] \odot [c]) \cdot ([b] \odot [d]) = [a * c] \cdot [b * d] = \left[\begin{cases} a(2t)b(2t), & \text{if } t \in [0, 1/2], \\ c(2t-1)d(2t-1), & \text{if } t \in [1/2, 1]. \end{cases} \right]$$

Meanwhile, the LHS of equation (2.2) is

$$([a] \cdot [b]) \odot ([c] \cdot [d]) = [\mu \circ (a, b)] \odot [\mu \circ (c, d)] = \left[\begin{cases} a(2t)b(2t), & \text{if } t \in [0, 1/2], \\ c(2t-1)d(2t-1), & \text{if } t \in [1/2, 1]. \end{cases} \right]$$

Therefore, equation (2.2) is satisfied.

Using parts (a) and (b) of the proof, we now argue that if $[f], [g] \in \pi_1(G, e)$ then

$$\begin{aligned} [f] \odot [g] &= ([f] \cdot [c_e]) \odot ([c_e] \cdot [g]) \\ &= ([f] \odot [c_e]) \cdot ([c_e] \odot [g]) \quad \text{(Equation (2.2))} \\ &= [f] \cdot [g] = ([c_e] \odot [f]) \cdot ([g] \odot [c_e]) \\ &= ([c_e] \cdot [g]) \odot ([f] \cdot [c_e]) = [g] \odot [f]. \end{aligned}$$

This shows that $\pi_1(G, e)$ is an abelian group as required.

Returning to the more general scenario of internal groups, we want a characterisation of an internal group which runs parallel to the characterisation of an internal monoid in Theorem 2.1.4.

Theorem 2.2.4. *Let $\mathscr{C}$ be a finitely complete category and (M, m, e) be an internal monoid in $\mathscr{C}$. Then, (M, m, e) is an internal group if and only if for any object $X \in \mathscr{C}$, $Hom_{\mathscr{C}}(X, M)$ is a group and for any morphism $f : X \to X'$ in $\mathscr{C}$, the map*

$$Hom_{\mathscr{C}}(f, M) : Hom_{\mathscr{C}}(X', M) \to Hom_{\mathscr{C}}(X, M)$$

is a group morphism.

Proof. Assume that $\mathscr{C}$ is a finitely complete category and (M, m, e) is an internal monoid in $\mathscr{C}$. Assume that $X, X' \in \mathscr{C}$ are objects and $f : X \to X'$ is a morphism in $\mathscr{C}$.

To show: (a) If (M, m, e) is an internal group then $Hom_{\mathscr{C}}(X, M)$ is a group and $Hom_{\mathscr{C}}(f, M)$ is a group morphism.

(b) If $Hom_{\mathscr{C}}(X, M)$ is a group and $Hom_{\mathscr{C}}(f, M)$ is a group morphism then (M, m, e) is an internal group.

(a) Assume that (M, m, e) is an internal group. Then, the following diagram in $\mathscr{C}$ is a pullback square:

$$\begin{tikzcd}
M \times M \arrow[r, "m"] \arrow[d, "p_0^M"'] & M \arrow[d, "\tau_M"] \\
M \arrow[r, "\tau_M"] & *
\end{tikzcd}$$

where $*$ is the terminal object in $\mathscr{C}$. We know from Theorem 2.2.1 that we can use the universal property of the above pullback square to obtain the inverse mapping $(-)^{-1} : M \to M$ such that if $a \in M$ then $m(a, a^{-1}) = m(a^{-1}, a) = e(*)$.

We know from Theorem 2.1.4 that $Hom_{\mathscr{C}}(X, M)$ is a monoid, with binary operation

$$\begin{array}{rccc}
\mu : & Hom_{\mathscr{C}}(X, M) \times Hom_{\mathscr{C}}(X, M) & \to & Hom_{\mathscr{C}}(X, M) \\
& (f, g) & \mapsto & \mu(f, g)(x) = m(f(x), g(x)).
\end{array}$$

We claim that if $f \in Hom_{\mathscr{C}}(X, M)$ then the composite $(-)^{-1} \circ f \in Hom_{\mathscr{C}}(X, M)$ a multiplicative inverse to f. Observe that if $x \in X$ then

$$\begin{aligned}
\mu(f, (-)^{-1} \circ f)(x) &= m(f(x), ((-)^{-1} \circ f)(x)) \\
&= m(f(x), f(x)^{-1}) = e(*)
\end{aligned}$$

and

$$\begin{aligned}
\mu((-)^{-1} \circ f, f)(x) &= m(((-)^{-1} \circ f)(x), f(x)) \\
&= m(f(x)^{-1}, f(x)) = e(*).
\end{aligned}$$

Therefore, $(-)^{-1} \circ f$ is a multiplicative inverse to f. Consequently, $Hom_{\mathscr{C}}(X, M)$ is a group.

To see that the monoid morphism $Hom_{\mathscr{C}}(f, M)$ is a group morphism, we compute directly for $g \in Hom_{\mathscr{C}}(X', M)$ and $x \in X$ that

$$\begin{aligned} Hom_{\mathscr{C}}(f, M)((-)^{-1} \circ g)(x) &= ((-)^{-1} \circ g \circ f)(x) \\ &= (-)^{-1} \circ (g \circ f)(x) \\ &= ((g \circ f)(x))^{-1} \\ &= (Hom_{\mathscr{C}}(f, M)(g)(x))^{-1} \end{aligned}$$

So, $Hom_{\mathscr{C}}(f, M)$ is a group morphism.

(b) Conversely, assume that $Hom_{\mathscr{C}}(X, M)$ is a group and $Hom_{\mathscr{C}}(f, M)$ is a group morphism. Suppose that we have the following commutative square in $\mathscr{C}$:

$$\begin{tikzcd} Z \arrow[r, "z_1"] \arrow[d, "z_2"] & M \arrow[d, "\tau_M"] \\ M \arrow[r, "\tau_M"] & * \end{tikzcd}$$

Since $Hom_{\mathscr{C}}(X, M)$ is a group, there exists a unique multiplicative inverse for z_2, which we denote by z_2^{-1}. Define $\phi \in Hom_{\mathscr{C}}(X, M \times M)$ by $\phi = (z_2, \mu(z_2^{-1}, z_1))$. In a similar vein to Theorem 2.2.1, we find that the following diagram commutes:

$$\begin{tikzcd} Z \arrow[dr, dashed, "\phi"] \arrow[drr, bend left, "z_1"] \arrow[ddr, bend right, "z_2"'] & & \\ & M \times M \arrow[r, "m"] \arrow[d, "p_0^M"] & M \arrow[d, "\tau_M"] \\ & M \arrow[r, "\tau_M"] & 1 \end{tikzcd}$$

Hence, the square in $\mathscr{C}$ below is a pullback square:

$$\begin{tikzcd} M \times M \arrow[r, "m"] \arrow[d, "p_0^M"] & M \arrow[d, "\tau_M"] \\ M \arrow[r, "\tau_M"] & 1 \end{tikzcd}$$

and consequently, (M, m, e) is an internal group in $\mathscr{C}$ as required. $\square$

Just like Theorem 2.1.4, Theorem 2.2.4 can be reformulated as a diagram. In particular, Theorem 2.2.4 tells us that the functor $Hom_{\mathscr{C}}(-, M)$ factorises through the category **Grp**, so that the following diagram commutes:

$$\begin{tikzcd} \mathscr{C}^{op} \arrow[r, dashed] \arrow[dr, "Hom_{\mathscr{C}}(-,M)"'] & \mathbf{Grp} \arrow[d, "U"] \\ & \mathbf{Set} \end{tikzcd}$$

Again, U the forgetful functor. Also, we have an analogue of Theorem 2.2.4 which applies to abelian groups.

Theorem 2.2.5. *Let $\mathscr{C}$ be a finitely complete category and (M, m, e) be an internal monoid in $\mathscr{C}$. Then, (M, m, e) is an internal abelian group if and only if for any object $X \in \mathscr{C}$, $Hom_{\mathscr{C}}(X, M)$ is an abelian group.*

The proof of this is very similar to Theorem 2.1.5. Theorem 2.2.5 tells us that the functor $Hom_{\mathscr{C}}(-, M)$ factorises through the category **Ab**. So, the following diagram commutes:

$$\begin{tikzcd} \mathscr{C}^{op} \arrow[r, dashed] \arrow[dr, "Hom_{\mathscr{C}}(-,M)"'] & \mathbf{Ab} \arrow[d, "U"] \\ & \mathbf{Set} \end{tikzcd}$$

2.3 The Yoneda embedding

Recall the definition of a natural transformation from Definition 1.1.6.

Definition 2.3.1. Let $\mathscr{C}$ and $\mathscr{D}$ be categories. The **functor category** $\mathcal{F}(\mathscr{C}, \mathscr{D})$ is the category whose objects are functors $F : \mathscr{C} \to \mathscr{D}$ and morphism are natural transformations between functors.

The isomorphisms in the functor category $\mathcal{F}(\mathscr{C}, \mathscr{D})$ are the natural isomorphisms between functors from $\mathscr{C}$ to $\mathscr{D}$.

In the previous two sections, we worked with the contravariant functor

$$Hom_{\mathscr{C}}(-, X) : \mathscr{C}^{op} \to \mathbf{Set}.$$

for an object $X \in \mathscr{C}$. Here, $\mathscr{C}$ is an arbitrary category. This gives rise to the functor

$$
\begin{array}{llcc}
Y: & \mathscr{C} & \rightarrow & \mathcal{F}(\mathscr{C}^{op}, \mathbf{Set}) \\
 & X & \mapsto & Y(X) = Hom_{\mathscr{C}}(-, X) \\
 & f: X \rightarrow X' & \mapsto & Y(f)
\end{array} \tag{2.3}
$$

If $f : X \to X'$ is a morphism in $\mathscr{C}$ then $Y(f)$ is a natural transformation defined by the family of maps

$$\{Y(f)_A : Hom_{\mathscr{C}}(A, X) \to Hom_{\mathscr{C}}(A, X') \mid A \in \mathscr{C}\}$$

where we have for each object $A \in \mathscr{C}$ the morphism of sets

$$
\begin{array}{lccc}
Y(f)_A : & Hom_{\mathscr{C}}(A, X) & \rightarrow & Hom_{\mathscr{C}}(A, X') \\
 & g & \mapsto & f \circ g.
\end{array}
$$

For two functors $F : \mathscr{C} \to \mathscr{D}$ and $F' : \mathscr{C} \to \mathscr{D}$, we write $Nat(F, F')$ to denote the set of natural transformations from F to F'. This notation is adopted from [Mur16].

Lemma 2.3.1 (Yoneda lemma)**.** *Let $\mathscr{C}$ be a locally small category and $F : \mathscr{C}^{op} \to$ **Set** be a functor. Let $C \in \mathscr{C}$ be an object. Define the map*

$$
\begin{array}{lccc}
\Phi_{C,F} : & Nat(Y(C), F) & \rightarrow & F(C) \\
 & \alpha & \mapsto & \alpha_C(id_C)
\end{array}
$$

Explicitly, Y is the functor from equation (2.3), α_C is a morphism of sets from $Y(C)(C) = Hom_{\mathscr{C}}(C, C)$ to $F(C)$ and id_C is the identity map on the object C. Then, $\Phi_{C,F}$ is a bijection, which satisfies the following two properties:

*1. If $f : C \to C'$ is a morphism in $\mathscr{C}$ then the following square in **Set** commutes:*

$$
\begin{tikzcd}
Nat(Y(C), F) \arrow[r, "\Phi_{C,F}"] & F(C) \\
Nat(Y(C'), F) \arrow[u, "(-)\circ Y(f)"] \arrow[r, "\Phi_{C',F}"] & F(C') \arrow[u, "F(f)"']
\end{tikzcd}
$$

*2. If $\beta : F \to F'$ is a natural transformation then the following diagram in **Set** commutes:*

$$
\begin{tikzcd}
Nat(Y(C), F) \arrow[r, "\Phi_{C,F}"] \arrow[d, "\beta\circ(-)"'] & F(C) \arrow[d, "\beta_C"] \\
Nat(Y(C), F') \arrow[r, "\Phi_{C,F'}"] & F'(C)
\end{tikzcd}
$$

Proof. Assume that $\mathscr{C}$ is a locally small category and $C \in \mathscr{C}$ is an object. Assume that $F : \mathscr{C}^{op} \to \textbf{Set}$ is a functor.

To show: (a) The map $\Phi_{C,F}$ is surjective.

(b) The map $\Phi_{C,F}$ is injective.

(c) The first property in the statement of the lemma is satisfied.

(d) The second property in the statement of the lemma is satisfied.

(a) Assume that $X \in F(C)$ and D is an object in $\mathscr{C}^{op}$. Define the map $N(X)_D$ by

$$\begin{array}{rrcl} N(X)_D : & Y(C)(D) = Hom_{\mathscr{C}}(D, C) & \to & F(D) \\ & g & \mapsto & F(g)(X) \end{array}$$

Recall that F is a contravariant functor by assumption so that $F(g)$ is a morphism in **Set** from $F(C)$ to $F(D)$.

To show: (aa) $N(X) \in Nat(Y(C), F)$.

(aa) We will show that if $h : D \to D'$ is a morphism in $\mathscr{C}^{op}$ then the following diagram in **Set** commutes:

$$\begin{tikzcd} Y(C)(D') \arrow[r, "Y(C)(h)"] \arrow[d, "N(X)_{D'}"'] & Y(C)(D) \arrow[d, "N(X)_D"] \\ F(D') \arrow[r, "F(h)"] & F(D) \end{tikzcd}$$

Assume that $\xi \in Y(C)(D') = Hom_{\mathscr{C}}(D', C)$. We compute directly that

$$\begin{aligned} (N(X)_D \circ Y(C)(h))(\xi) &= (N(X)_D \circ Hom_{\mathscr{C}}(h, C))(\xi) \\ &= N(X)_D(\xi \circ h) \\ &= F(\xi \circ h)(X) \\ &= (F(h) \circ F(\xi))(X) \\ &= (F(h) \circ N(X)_{D'})(\xi). \end{aligned}$$

Hence, the above diagram in **Set** commutes and $N(X) \in Nat(Y(C), F)$.

(a) We claim that $\Phi_{C,F}(N(X)) = X$. Using the definitions of $\Phi_{C,F}$ and $N(X)$, we find that

$$\begin{aligned}\Phi_{C,F}(N(X)) &= N(X)_C(id_C) \\ &= F(id_C)(X) = id_{F(C)}(X) = X.\end{aligned}$$

Therefore, the map $\Phi_{C,F}$ is surjective.

(b) Assume that $\alpha, \beta \in Nat(Y(C), F)$ such that $\Phi_{C,F}(\alpha) = \Phi_{C,F}(\beta)$. Assume that $f \in Hom_{\mathscr{C}}(D, C)$ for some object $D \in \mathscr{C}$. By naturality of α, the following diagram in **Set** commutes:

$$\begin{tikzcd} Y(C)(C) \arrow[r, "Y(C)(f)"] \arrow[d, "\alpha_C"'] & Y(C)(D) \arrow[d, "\alpha_D"] \\ F(C) \arrow[r, "F(f)"] & F(D) \end{tikzcd}$$

We then have

$$\begin{aligned}(F(f) \circ \Phi_{C,F})(\alpha) &= F(f)(\alpha_C(id_C)) \\ &= (\alpha_D \circ Y(C)(f))(id_C) \\ &= \alpha_D(Hom_{\mathscr{C}}(f, C)(id_C)) \\ &= \alpha_D(id_C \circ f) = \alpha_D(f).\end{aligned}$$

Since $\Phi_{C,F}(\alpha) = \Phi_{C,F}(\beta)$ by assumption, $\alpha_C(id_C) = \beta_C(id_C)$. But, β is also a natural transformation between the functors $Y(C)$ and F. So, the following diagram in **Set** commutes:

$$\begin{tikzcd} Y(C)(C) \arrow[r, "Y(C)(f)"] \arrow[d, "\beta_C"'] & Y(C)(D) \arrow[d, "\beta_D"] \\ F(C) \arrow[r, "F(f)"] & F(D) \end{tikzcd}$$

If $f \in Hom_{\mathscr{C}}(D, C) = Y(C)(D)$ then

$$\begin{aligned}\alpha_D(f) &= \alpha_D(id_C \circ f)\\ &= \alpha_D(Hom_{\mathscr{C}}(f,C)(id_C))\\ &= (\alpha_D \circ Y(C)(f))(id_C)\\ &= F(f)(\alpha_C(id_C))\\ &= F(f)(\beta_C(id_C)) \quad (\text{since } \alpha_C(id_C) = \beta_C(id_C))\\ &= (\beta_D \circ Y(C)(f))(id_C)\\ &= \beta_D(Hom_{\mathscr{C}}(f,C)(id_C)) = \beta_D(f).\end{aligned}$$

Therefore, $\alpha_D = \beta_D$. Since the object $D \in \mathscr{C}$ was arbitrary, we deduce that $\alpha = \beta$ as natural transformations from $Y(C)$ to F. Therefore, $\Phi_{C,F}$ is injective.

Combining parts (a) and (b), we deduce that $\Phi_{C,F}$ is indeed a bijective map. Its inverse is given explicitly by

$$\begin{array}{rccc}\Phi_{C,F}^{-1}: & F(C) & \to & Nat(Y(C),F)\\ & X & \mapsto & N(X)\end{array}$$

where $N(X)$ is the natural transformation in parts (a) and (aa). Recall that it is defined by

$$\begin{array}{rccc}N(X)_D: & Y(C)(D) = Hom_{\mathscr{C}}(D,C) & \to & F(D)\\ & g & \mapsto & F(g)(X)\end{array}$$

(c) Now assume that $f : C \to C'$ is a morphism in $\mathscr{C}$. We want to show that the following square in **Set** commutes:

$$\begin{tikzcd} Nat(Y(C),F) \arrow[r, "\Phi_{C,F}"] & F(C) \\ Nat(Y(C'),F) \arrow[u, "(-)\circ Y(f)"] \arrow[r, "\Phi_{C',F}"] & F(C') \arrow[u, "F(f)"'] \end{tikzcd}$$

Assume that $\alpha \in Nat(Y(C'),F)$. We compute directly that

$$\begin{aligned}(\Phi_{C,F} \circ (-) \circ Y(f))(\alpha) &= \Phi_{C,F}(\alpha \circ Y(f))\\ &= (\alpha \circ Y(f))_C(id_C)\\ &= (\alpha_C \circ Y(f)_C)(id_C)\\ &= \alpha_C(Y(f)_C(id_C))\\ &= \alpha_C(f \circ id_C) = \alpha_C(f)\end{aligned}$$

and

$$
\begin{aligned}
(F(f) \circ \Phi_{C',F})(\alpha) &= F(f)(\alpha_{C'}(id_{C'})) \\
&= (F(f) \circ \alpha_{C'})(id_{C'}) \\
&= (\alpha_C \circ Y(C')(f))(id_{C'}) \quad \text{(Naturality of } \alpha) \\
&= \alpha_C(Hom_{\mathscr{C}}(f, C')(id_{C'})) \\
&= \alpha_C(f).
\end{aligned}
$$

So, the above diagram in **Set** is commutative.

(d) Assume that $\beta \in Nat(F, F')$. We want to show that the following diagram in **Set** commutes:

$$
\begin{array}{ccc}
Nat(Y(C), F) & \xrightarrow{\Phi_{C,F}} & F(C) \\
{\scriptstyle \beta\circ(-)}\big\downarrow & & \big\downarrow{\scriptstyle \beta_C} \\
Nat(Y(C), F') & \xrightarrow{\Phi_{C,F'}} & F'(C)
\end{array}
$$

Assume that $\chi \in Nat(Y(C), F)$. We compute directly that

$$
\begin{aligned}
(\beta_C \circ \Phi_{C,F})(\chi) &= \beta_C(\chi_C(id_C)) \\
&= (\beta_C \circ \chi_C)(id_C) \\
&= (\beta \circ \chi)_C(id_C) \\
&= \Phi_{C,F'}(\beta \circ \chi) \\
&= (\Phi_{C,F'} \circ \beta \circ (-))(\chi)
\end{aligned}
$$

Therefore, the above diagram in **Set** commutes. This completes the proof. □

In the Yoneda lemma (Lemma 2.3.1), the property proved in part (c) tells us that $\Phi_{C,F}$ is natural in the object $C \in \mathscr{C}$. Correspondingly, the property proved in part (d) tells us that $\Phi_{C,F}$ is natural with respect to the functor $F : \mathscr{C}^{op} \to$ **Set**.

We can now state the major theorem pertaining to the **Yoneda embedding**.

Theorem 2.3.2 (Yoneda embedding)**.** *Let $\mathscr{C}$ be a locally small category. The Yoneda embedding, which is the functor*

$$
\begin{array}{rccc}
Y: & \mathscr{C} & \to & \mathcal{F}(\mathscr{C}^{op}, \boldsymbol{Set}) \\
 & X & \mapsto & Y(X) = Hom_{\mathscr{C}}(-, X) \\
 & f: X \to X' & \mapsto & Y(f)
\end{array}
$$

is a fully faithful functor.

Proof. Assume that $\mathscr{C}$ is a locally small category and that the Yoneda embedding Y is the functor defined as above. Let X, X' be objects in $\mathscr{C}$. Then, the functor Y induces a mapping

$$Y_{X,X'} : Hom_{\mathscr{C}}(X, X') \to Hom_{\mathcal{F}(\mathscr{C}^{op},\mathbf{Set})}(Y(X), Y(X'))$$

Note that $Hom_{\mathcal{F}(\mathscr{C}^{op},\mathbf{Set})}(Y(X), Y(X')) = Nat(Y(X), Y(X'))$ and $Y_{X,X'}(f) = Y(f)$.

To show: (a) $Y_{X,X'}$ is bijective.

(a) By Lemma 2.3.1, it suffices to show that $Y_{X,X'}$ is the inverse to the bijection $\Phi_{X,Y(X')} : Nat(Y(X), Y(X')) \to Y(X')(X)$. Assume that $f \in Hom_{\mathscr{C}}(X, X')$. Then,

$$
\begin{aligned}
(\Phi_{X,Y(X')} \circ Y_{X,X'})(f) &= \Phi_{X,Y(X')}(Y(f)) \\
&= Y(f)_X(id_X) \\
&= f \circ id_X = f.
\end{aligned}
$$

Hence, $Y_{X,X'}$ is a bijection as required.

Part (a) shows that the Yoneda embedding is a fully faithful functor as required. □

2.4 Finite limits

The goal of the next few sections is to prove the following important fact about the Yoneda embedding in Theorem 2.3.2.

Theorem 2.4.1. *Let $\mathscr{C}$ be a finitely complete category and $Y : \mathscr{C} \to \mathcal{F}(\mathscr{C}^{op}, \boldsymbol{Set})$ denote the Yoneda embedding. Then, Y is a left exact functor.*

In the previous sections, we have used the term limit loosely to refer to the constructions of equalizers, pullbacks, terminal objects and products. In order to prove Theorem 2.4.1, we will need to pin down what a limit really is. The main reference we will follow for the next few sections is [Lei16, Chapters 5 and 6].

Definition 2.4.1. A category $\mathscr{C}$ is said to be **small** if the collection of all morphisms in $\mathscr{C}$ is a set.

Note that if a category $\mathscr{C}$ is small then the collection of objects $ob(\mathscr{C})$ is also a set because objects are in a one-to-one correspondence with identity morphisms.

Definition 2.4.2. Let $\mathscr{C}$ be a category and $\mathbf{I}$ be a small category. A functor $\mathbf{I} \to \mathscr{C}$ is called a **diagram** in $\mathscr{C}$ of shape $\mathbf{I}$.

Definition 2.4.3. Let $\mathscr{C}$ be a category, $\mathbf{I}$ be a small category and $D : \mathbf{I} \to \mathscr{C}$ be a diagram in $\mathscr{C}$. A **cone** on D is an object $A \in \mathscr{C}$, called the **vertex** of the cone, together with a family

$$\big(f_I : A \to D(I)\big)_{I \in \mathbf{I}}$$

of morphisms in $\mathscr{C}$ such that if $u : I \to J$ is a morphism in $\mathbf{I}$ then the triangle in $\mathscr{C}$ below commutes:

$$\begin{tikzcd} A \arrow[r, "f_I"] \arrow[dr, "f_J"'] & D(I) \arrow[d, "D(u)"] \\ & D(J) \end{tikzcd}$$

Before we proceed to the tantalising definition of a limit, let us first understand why the concept of a cone is relevant to the examples of limits we already know.

Example 2.4.4. Let $\mathscr{C}$ be a category and $\mathbf{P}$ be the small category depicted pictorially by

$$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

Let $D : \mathbf{P} \to \mathscr{C}$ denote the diagram in $\mathscr{C}$ which sends $\mathbf{P}$ to

$$\begin{tikzcd} & A_1 \arrow[d, "f"] \\ A_2 \arrow[r, "g"] & A_3 \end{tikzcd}$$

A cone on $D : \mathbf{P} \to \mathscr{C}$ is an object $V \in \mathscr{C}$ (the vertex), together with a family of morphisms $f_1 : V \to A_1$, $f_2 : V \to A_2$ and $f_3 : V \to A_3$ such that the following triangles in $\mathscr{C}$ commute:

$$\begin{tikzcd} V \arrow[r, "f_1"] \arrow[dr, "f_3"'] & A_1 \arrow[d, "f"] \\ & A_3 \end{tikzcd}$$

$$\begin{tikzcd} V \arrow[d, "f_2"] \arrow[dr, "f_3"] & \\ A_2 \arrow[r, "g"] & A_3 \end{tikzcd}$$

We can combine these two commutative triangles to find that the cone of D is the commutative square in $\mathscr{C}$

$$\begin{tikzcd} V \arrow[r, "f_1"] \arrow[d, "f_2"] & A_1 \arrow[d, "f"] \\ A_2 \arrow[r, "g"] & A_3 \end{tikzcd}$$

In the above example, we obtained the commutative square associated with a pullback from the definition of a cone, but we have yet to implement the universal property of the pullback. This is exactly what a limit does.

Definition 2.4.5. Let $\mathscr{C}$ be a category, $\mathbf{I}$ be a small category and $D : \mathbf{I} \to \mathscr{C}$ be a diagram in $\mathscr{C}$. A **limit** of D is a cone

$$\big(p_I : L \to D(I)\big)_{I \in \mathbf{I}}$$

such that if we have another cone $\big(f_I : V \to D(I)\big)_{I \in \mathbf{I}}$ on D then there exists a unique morphism $\tilde{f} : V \to L$ such that if $I \in \mathbf{I}$ then the following diagram in $\mathscr{C}$ commutes:

$$\begin{tikzcd} V \arrow[r, dashed, "\tilde{f}"] \arrow[dr, "f_I"'] & L \arrow[d, "p_I"] \\ & D(I) \end{tikzcd}$$

In a common abuse of notation, we refer to L as the limit of D. We write the limit L as $\varprojlim_{\mathbf{I}} D$.

Example 2.4.6. Let us return to the previous example. The previous cone of D is the following commutative square in $\mathscr{C}$:

$$\begin{tikzcd} V \arrow[r, "f_1"] \arrow[d, "f_2"] & A_1 \arrow[d, "f"] \\ A_2 \arrow[r, "g"] & A_3 \end{tikzcd}$$

A limit of the diagram $D : \mathbf{P} \to \mathscr{C}$ in $\mathscr{C}$ is another cone, which consists of an object $L \in \mathscr{C}$ and morphisms $p_j : L \to A_j$ for $j \in \{1, 2, 3\}$. This time, there exists a unique morphism $\tilde{f} : V \to L$ such that if $j \in \{1, 2, 3\}$ then $p_j \circ \tilde{f} = f_j$. This is equivalent to saying that the following diagram in $\mathscr{C}$ commutes:

$$\begin{tikzcd} V \arrow[dr, dashed, "\tilde{f}"] \arrow[drr, bend left, "f_1"] \arrow[ddr, bend right, "f_2"'] & & \\ & L \arrow[r, "p_2"] \arrow[d, "p_1"] & A_1 \arrow[d, "f"] \\ & A_2 \arrow[r, "g"] & A_3 \end{tikzcd}$$

The equation $p_3 \circ \tilde{f} = f_3$ is extraneous data and can be deduced from the above diagram. Indeed, we have

$$\begin{aligned} f_3 &= f \circ f_1 \\ &= f \circ (p_2 \circ \tilde{f}) \\ &= (f \circ p_2) \circ \tilde{f} \\ &= p_3 \circ \tilde{f}. \end{aligned}$$

Hence, we have shown that the limit of D is the pullback of the following diagram in $\mathscr{C}$:

$$\begin{tikzcd} & A_1 \arrow[d, "f"] \\ A_2 \arrow[r, "g"] & A_3 \end{tikzcd}$$

Therefore, the above example confirms that a pullback is a specific type of limit. In the next example, we will show that equalizers are also limits.

Example 2.4.7. Let $\mathscr{C}$ be a category and $\mathbf{E}$ be the small category depicted pictorially by

$$\bullet \rightrightarrows \bullet$$

Let $D : \mathbf{E} \to \mathscr{C}$ denote the diagram in $\mathscr{C}$ which sends $\mathbf{E}$ to

$$\begin{tikzcd} A_1 \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & A_2 \end{tikzcd}$$

A cone on $D : \mathbf{E} \to \mathscr{C}$ is an object $V \in \mathscr{C}$, together with morphisms $v_1 : V \to A_1$ and $v_2 : V \to A_2$ such that $f_1 \circ v_1 = v_2$ and $f_2 \circ v_1 = v_2$. Hence, a cone on D is the morphism $v_1 : V \to A_1$ which equalizes the pair (f_1, f_2).

A limit of the diagram $D : \mathbf{E} \to \mathscr{C}$ is another cone $L \in \mathscr{C}$ with accompanying morphisms $p_1 : L \to A_1$ and $p_2 : L \to A_2$ such that there exists a unique morphism $\tilde{f} : V \to L$ which makes the following diagram in $\mathscr{C}$ commute:

$$\begin{tikzcd} V \arrow[d, dashed, "\tilde{f}"'] \arrow[dr, "v_1"] & & \\ L \arrow[r, "p_1"] & A_1 \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & A_2 \end{tikzcd}$$

Similarly to the example of a pullback, the equation $p_2 \circ \tilde{f} = v_2$ is an extraneous condition, which can be determined from the commutative diagram above. We have

$$v_2 = f_2 \circ v_1 = f_2 \circ (p_1 \circ \tilde{f}) = (f_2 \circ p_1) \circ \tilde{f} = p_2 \circ \tilde{f}.$$

Therefore, the limit of the diagram D is the equalizer of the following diagram in $\mathscr{C}$:

$$\begin{tikzcd} A_1 \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & A_2 \end{tikzcd}$$

In the same way as the previous example, we will now show that the product is also a limit.

Example 2.4.8. Let $\mathscr{C}$ be a category and $\mathbf{T}$ be the small category depicted pictorially by

$$\bullet \qquad \bullet$$

Let $D : \mathbf{T} \to \mathscr{C}$ denote the diagram in $\mathscr{C}$ which sends $\mathbf{T}$ to

$$A_1 \qquad A_2$$

A cone on $D : \mathbf{T} \to \mathscr{C}$ is an object $V \in \mathscr{C}$, together with morphisms $v_1 : V \to A_1$ and $v_2 : V \to A_2$. There are no commutative diagrams to deal with here because there are no morphisms in $\mathbf{T}$.

A limit of the diagram $D : \mathbf{T} \to \mathscr{C}$ is another cone $L \in \mathscr{C}$ with accompanying morphisms $p_1 : L \to A_1$ and $p_2 : L \to A_2$ such that there exists a unique morphism $\tilde{f} : V \to L$ which makes the following diagram in $\mathscr{C}$ commute:

$$\begin{tikzcd}
& L \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
A_1 & V \arrow[l, "v_1"] \arrow[r, "v_2"'] \arrow[u, dashed, "\tilde{f}"] & A_2
\end{tikzcd}$$

Therefore, the limit of the diagram D is the product of the objects A_1 and A_2.

Finally, we will show that the terminal object can also be thought of as a limit.

Example 2.4.9. Let $\mathscr{C}$ be a category and $\mathbf{0}$ be the empty category, with no morphisms or objects. Let $D : \mathbf{0} \to \mathscr{C}$ denote the diagram in $\mathscr{C}$ which sends $\mathbf{0}$ to the empty subcategory of $\mathscr{C}$.

A cone on $D : \mathbf{0} \to \mathscr{C}$ is just an object $V \in \mathscr{C}$. This time, there are no accompanying morphisms because there are no objects in the empty category $D(\mathbf{0})$.

A limit of the diagram $D : \mathbf{0} \to \mathscr{C}$ is another cone/object $L \in \mathscr{C}$ such that there exists a unique morphism $t : V \to L$. Therefore, the limit of the diagram D is the terminal object $L \in \mathscr{C}$.

Let $D : \mathbf{I} \to \mathscr{C}$ be a diagram in $\mathscr{C}$. The universal property associated with a limit L of D can be interpreted as the bijective correspondence

$$\begin{array}{ccc} \{\text{Morphisms } A \to L\} & \leftrightarrow & \{\text{Cones on } D \text{ with vertex } A\} \\ g : A \to L & \mapsto & \big(p_I \circ g : A \to D(I)\big)_{I \in \mathbf{I}} \\ \tilde{f} : A \to L & \leftarrow\!\shortmid & \big(f_I : A \to D(I)\big)_{I \in \mathbf{I}} \end{array} \tag{2.4}$$

The maps $p_I : L \to D(I)$ are the morphisms accompanying the limit L. The universal property of the limit provides the direction from "right to left" in the above correspondence — from a cone on D with vertex A to a unique morphism $A \to L$.

So far, we have discussed limits in generality. In the next example, we will discuss limits in our most familiar (finitely complete) category — **Set**.

Example 2.4.10. Let $\mathbf{I}$ be a small category and $D : \mathbf{I} \to \mathbf{Set}$ be a diagram in **Set**. The limit $\lim\limits_{\mathbf{I}\longleftarrow} D$ satisfies

$$\begin{aligned} \varprojlim_{\mathbf{I}} D &\cong Hom_{\mathbf{Set}}(\{*\}, \varprojlim_{\mathbf{I}} D) \\ &\cong \{\text{Cones on } D \text{ with vertex } \{*\}\} \\ &\cong \{(x_I)_{I \in \mathbf{I}} \mid x_I \in D(I) \text{ and } (Du)(x_I) = x_J \text{ for } u : I \to J \text{ in } \mathbf{I}\} \end{aligned}$$

For clarity, $\{*\}$ is a set with cardinality 1. The second isomorphism is by the bijective correspondence in equation (2.4) and the third isomorphism is straight from the definition of the cone. We highlight the result in this example below, as it will be used later

$$\varprojlim_{\mathbf{I}} D \cong \{(x_I)_{I \in \mathbf{I}} \mid x_I \in D(I) \text{ and } (Du)(x_I) = x_J \text{ for } u : I \to J \text{ in } \mathbf{I}\}. \tag{2.5}$$

We will also need the following lemma, which tells us when two morphisms to a limit L are equal.

Lemma 2.4.2. *Let $\mathscr{C}$ be a category and $\boldsymbol{I}$ be a small category. Let $D : \boldsymbol{I} \to \mathscr{C}$ be a diagram and L be the limit on D, with accompanying morphisms $p_I : L \to D(I)$ for $I \in \boldsymbol{I}$. If $h, h' : A \to L$ are morphisms satisfying $p_I \circ h = p_I \circ h'$ for $I \in \boldsymbol{I}$ then $h = h'$.*

Proof. Assume that $\mathscr{C}$ is a category and $\mathbf{I}$ is a small category. Assume that $D : \mathbf{I} \to \mathscr{C}$ is a diagram and L be the limit on D.

Assume that h, h' are morphisms from A to L such that $p_I \circ h = p_I \circ h'$ for $I \in \mathbf{I}$. By the definition of the limit L, there exists a unique morphism $f : A \to L$ such that $p_I \circ f = p_I \circ h = p_I \circ h'$ for $I \in \mathbf{I}$. By uniqueness of f, we must have $h = h'$. $\square$

One can ask what happens when we have two different diagrams $D, D' : \mathbf{I} \to \mathscr{C}$ and a natural transformation $\alpha : D \to D'$. The natural transformation α induces a morphism between the limits on D and D'.

Theorem 2.4.3. *Let $\boldsymbol{I}$ be a small category and $\mathscr{C}$ be a category. Let $D, D' : \boldsymbol{I} \to \mathscr{C}$ be diagrams and $\alpha : D \to D'$ be a natural transformation. Let*

$$\big(p_I : \varprojlim_{\boldsymbol{I}} D \to D(I)\big)_{I \in \boldsymbol{I}} \qquad \text{and} \qquad \big(p'_I : \varprojlim_{\boldsymbol{I}} D' \to D'(I)\big)_{I \in \boldsymbol{I}}$$

be the limits of D and D' respectively. Then, there exists a unique morphism $\varprojlim_{\boldsymbol{I}} \alpha : \varprojlim_{\boldsymbol{I}} D \to \varprojlim_{\boldsymbol{I}} D'$ such that if $I \in \boldsymbol{I}$ then the following diagram in $\mathscr{C}$ commutes:

$$\begin{array}{ccc} \varprojlim_{\boldsymbol{I}} D & \xrightarrow{p_I} & D(I) \\ {\scriptstyle \varprojlim_{\boldsymbol{I}} \alpha}\downarrow & & \downarrow{\scriptstyle \alpha_I} \\ \varprojlim_{\boldsymbol{I}} D' & \xrightarrow{p'_I} & D'(I) \end{array}$$

Moreover, if we have two cones

$$\big(f_I : A \to D(I)\big)_{I \in \boldsymbol{I}} \qquad \text{and} \qquad \big(f'_I : A' \to D'(I)\big)_{I \in \boldsymbol{I}}$$

and a morphism $s : A \to A'$ which makes the following diagram commute for $I \in \boldsymbol{I}$

$$\begin{array}{ccc} A & \xrightarrow{f_I} & D(I) \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle \alpha_I} \\ A' & \xrightarrow{f'_I} & D'(I) \end{array}$$

then the square below also commutes:

$$\begin{array}{ccc} A & \xrightarrow{\overline{f}} & \varprojlim_{\boldsymbol{I}} D \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle \varprojlim_{\boldsymbol{I}} \alpha} \\ A' & \xrightarrow{\overline{f'}} & \varprojlim_{\boldsymbol{I}} D' \end{array}$$

Proof. Assume that $\mathbf{I}$ is a small category and $\mathscr{C}$ is a category. Assume that $D, D' : \mathbf{I} \to \mathscr{C}$ be diagrams and $\alpha : D \to D'$ be a natural transformation.

We will construct the unique morphism $\varprojlim_{\mathbf{I}} \alpha : \varprojlim_{\mathbf{I}} D \to \varprojlim_{\mathbf{I}} D'$. Making use of the natural transformation α_I, observe that we have the following cone on D':

$$\left(\alpha_I \circ p_I : \varprojlim_{\mathbf{I}} D \to D'(I)\right)_{I \in \mathbf{I}}$$

By the universal property of the limit $\varprojlim_{\mathbf{I}} D'$, there exists a unique morphism $\varprojlim_{\mathbf{I}} \alpha : \varprojlim_{\mathbf{I}} D \to \varprojlim_{\mathbf{I}} D'$ such that if $I \in \mathbf{I}$ then $p_I \circ \varprojlim_{\mathbf{I}} \alpha = \alpha_I \circ p'_I$. Thus, we obtain the following commutative diagram in $\mathscr{C}$:

$$\begin{tikzcd}
\varprojlim_{\mathbf{I}} D \arrow[r, "p_I"] \arrow[d, "\varprojlim_{\mathbf{I}} \alpha"'] & D(I) \arrow[d, "\alpha_I"] \\
\varprojlim_{\mathbf{I}} D' \arrow[r, "p'_I"] & D'(I)
\end{tikzcd}$$

Next, assume that we have the two cones

$$\left(f_I : A \to D(I)\right)_{I \in \mathbf{I}} \qquad \text{and} \qquad \left(f'_I : A' \to D'(I)\right)_{I \in \mathbf{I}}$$

and a morphism $s : A \to A'$ such that $f'_1 \circ s = \alpha_I \circ f_I$. By the universal property of the limits, we can construct unique morphisms $\overline{f} : A \to \varprojlim_{\mathbf{I}} D$ and $\overline{f'} : A' \to \varprojlim_{\mathbf{I}} D'$. Now observe that if $I \in \mathbf{I}$ then

$$\begin{aligned}
p'_I \circ (\varprojlim_{\mathbf{I}} \alpha) \circ \overline{f} &= (p'_I \circ \varprojlim_{\mathbf{I}} \alpha) \circ \overline{f} \\
&= \alpha_I \circ p_I \circ \overline{f} \\
&= \alpha_I \circ f_I = f'_I \circ s \\
&= p'_I \circ \overline{f'} \circ s.
\end{aligned}$$

By Lemma 2.4.2, we find that $\overline{f'} \circ s = \varprojlim_{\mathbf{I}} \alpha \circ \overline{f}$. So, the following diagram in $\mathscr{C}$ commutes

$$\begin{tikzcd}
A \arrow[r, "\overline{f}"] \arrow[d, "s"'] & \varprojlim_{\mathbf{I}} D \arrow[d, "\varprojlim_{\mathbf{I}} \alpha"] \\
A' \arrow[r, "\overline{f'}"] & \varprojlim_{\mathbf{I}} D'
\end{tikzcd}$$

as required. $\square$

2.5 The Yoneda embedding preserves finite limits

In the previous section, we have developed the theory of limits in order to prove Theorem 2.4.1 and established a few results about limits we will use in this section. Our next task is to define and prove a few results about *representable functors.*

Definition 2.5.1. Let $\mathscr{C}$ be a locally small category and $X : \mathscr{C}^{op} \to \mathbf{Set}$ be a (contravariant) functor. We say that X is **representable** if $X \cong Y(A)$ for some $A \in \mathscr{C}$. Here, Y is the Yoneda embedding in equation (2.3).

A **representation** of the functor X is a choice of object $A \in \mathscr{C}$ and an isomorphism from $Y(A)$ to X in $\mathcal{F}(\mathscr{C}^{op}, \mathbf{Set})$.

The Yoneda lemma (see Lemma 2.3.1) provides us with another characterisation of a representable **Set**-valued contravariant functor.

Lemma 2.5.1. *Let $\mathscr{C}$ be a locally small category and $X : \mathscr{C}^{op} \to \boldsymbol{Set}$ be a functor. Then, a representation of X consists of an object $A \in \mathscr{C}$ together with an element $u \in X(A)$ such that if $B \in \mathscr{C}$ and $x \in X(B)$ then there exists a unique morphism $\overline{x} : B \to A$ such that $X(\overline{x})(u) = x$.*

Proof. Assume that $\mathscr{C}$ is a locally small category and X is an object in the functor category $\mathcal{F}(\mathscr{C}^{op}, \mathbf{Set})$. Assume that $A \in \mathscr{C}$ and $u \in X(A)$.

By the definition of a representation of X, it suffices to show that the natural transformation

$$N(u) : Y(A) \to X$$

is a natural isomorphism if and only if for $B \in \mathscr{C}$ and $x \in X(B)$, there exists a unique morphism $\overline{x} : B \to A$ such that $X(\overline{x})(u) = x$. For the definition of $N(u)$, see the proof of Lemma 2.3.1.

Now observe that $N(u)$ is a natural isomorphism if and only if the morphism of sets

$$\begin{array}{rrcl} N(u)_B : & Y(A)(B) = Hom_{\mathscr{C}}(B, A) & \to & X(B) \\ & g & \mapsto & X(g)(u) \end{array}$$

is a bijection for each $B \in \mathscr{C}$. But, $N(u)_B$ is a bijection if and only if for $B \in \mathscr{C}$ and $x \in X(B)$, there exists a morphism $\overline{x} : B \to A$ in $\mathscr{C}$ such that

$$N(u)_B(\overline{x}) = X(\overline{x})(u) = x.$$

□

We will give an example of representations of a particular functor which is relevant to our goal.

Example 2.5.2. Let $\mathbf{I}$ be a small category, $\mathscr{C}$ be a category and $D : \mathbf{I} \to \mathscr{C}$ be a diagram. Let $A \in \mathscr{C}$ and $Cone(A, D)$ denote the set of cones on D with vertex A.

Define the functor

$$\begin{array}{rccc} Cone(-, D) : & \mathscr{C}^{op} & \to & \mathbf{Set} \\ & A & \mapsto & Cone(A, D) \\ & s : A \to B & \mapsto & Cone(s, D) : Cone(B, D) \to Cone(A, D) \end{array}$$

In particular, let us describe explicitly how the morphism of sets $Cone(s, D) : Cone(B, D) \to Cone(A, D)$ works. Suppose that we have cone

$$\big(f_I : B \to D(I)\big)_{I \in \mathbf{I}}$$

on D with vertex B. The map $Cone(s, D)$ sends each morphism f_I to $f_I \circ s$ and the vertex B to A. This works because

$$\big(f_I \circ s : A \to D(I)\big)_{I \in \mathbf{I}}$$

is a cone on D with vertex A.

What are representations of the functor $Cone(-, D)$? Using Lemma 2.5.1, we see that a representation of $Cone(-, D)$ consists of an object $A \in \mathscr{C}$, together with an element

$$u = \big(u_I : A \to D(I)\big)_{I \in \mathbf{I}} \in Cone(A, D)$$

such that if $B \in \mathscr{C}$ and

$$x = \big(x_I : A \to D(I)\big)_{I \in \mathbf{I}} \in Cone(B, D)$$

then there exists a unique morphism $\overline{x} : B \to A$ such that $Cone(\overline{x}, D)(u) = x$. This means that if $I \in \mathbf{I}$ then $u_I \circ \overline{x} = x_I$.

Therefore, a representation of a functor $Cone(-, D)$ is literally a limit on D with some vertex.

Another method of interpreting this example is that we have a bijective correspondence of sets

$$\begin{array}{ccc} Cone(A, D) & \leftrightarrow & Hom_{\mathscr{C}}(A, \varprojlim_{\mathbf{I}} D) \\ (f_I)_{I \in \mathbf{I}} & \mapsto & \overline{f} \\ (p_I \circ g)_{I \in \mathbf{I}} & \leftarrow\!\shortmid & g \end{array} \tag{2.6}$$

The morphisms $p_I : \varprojlim_{\mathbf{I}} D \to D(I)$ are the morphisms associated with the limit. The map $\overline{f}$ is the unique morphism formed by using the universal property of the limit. We find that equation (2.6) is the same as equation (2.4).

In our next lemma, we present the set $Cone(A, D)$ as a limit in **Set**.

Lemma 2.5.2. *Let $\boldsymbol{I}$ be a small category, $\mathscr{C}$ be a locally small category and $D : \boldsymbol{I} \to \mathscr{C}$ be a diagram. Let $A \in \mathscr{C}$. Define the functor*

$$\begin{array}{rccc} \mathscr{C}(A, D): & \boldsymbol{I} & \to & \boldsymbol{Set} \\ & I & \mapsto & Hom_{\boldsymbol{Set}}(A, D(I)) \\ & f & \mapsto & D(f) \circ (-) \end{array}$$

Then, we have the equality of sets

$$Cone(A, D) = \varprojlim_{\boldsymbol{I}} \mathscr{C}(A, D).$$

Proof. Assume that $\mathbf{I}$ is a small category, $\mathscr{C}$ is a locally small category and $D : \mathbf{I} \to \mathscr{C}$ is a diagram. Assume that $A \in \mathscr{C}$ and $\mathscr{C}(A, D)$ is the functor defined as above.

The key observation is that $\mathscr{C}(A, D)$ is a functor from a small category to **Set**. Hence, its limit is given by equation (2.5). Equation (2.5) tells us that $\varprojlim_{\mathbf{I}} \mathscr{C}(A, D)$ is a set consisting of families

$$\big(f_I : A \to D(I)\big)_{I \in \mathbf{I}}$$

such that if $u : I \to J$ is a morphism in $\mathbf{I}$ then

$$(\mathscr{C}(A, D)(u))(f_I) = f_J.$$

By definition of the functor $\mathscr{C}(A, D)$, the above equation tells us that $D(u) \circ f_I = f_J$ for any morphism $u : I \to J$ in $\mathbf{I}$. Therefore, the elements of

$\varprojlim_{\mathbf{I}} \mathscr{C}(A, D)$ are cones of D with vertex A. So, $\varprojlim_{\mathbf{I}} \mathscr{C}(A, D) = Cone(A, D)$ as required. $\square$

Our first major result of this section demonstrates that a particular **Set**-valued functor preserves limits. The functor in question is very similar to the output of the Yoneda embedding. Hence, it is wise to go over equation (2.3) before launching into the following theorem and proof.

Theorem 2.5.3. *Let $\mathscr{C}$ be a locally small category and $A \in \mathscr{C}$. Define the functor*

$$\begin{array}{rccc} \mathscr{C}(A, -): & \mathscr{C} & \to & \boldsymbol{Set} \\ & B & \mapsto & Hom_{\mathscr{C}}(A, B) \\ & f & \mapsto & f \circ (-) \end{array}$$

Then, the functor $\mathscr{C}(A, -)$ preserves limits.

Proof. Assume that $\mathscr{C}$ is a locally small category and $A \in \mathscr{C}$. Let $\mathbf{I}$ be a small category and $D : \mathbf{I} \to \mathscr{C}$ be a diagram. Suppose that D has a limit $\varprojlim_{\mathbf{I}} D$. We have the isomorphism of sets

$$\mathscr{C}(A, \varprojlim_{\mathbf{I}} D) \cong Cone(A, D) = \varprojlim_{\mathbf{I}} \mathscr{C}(A, D).$$

The first isomorphism follows from equation (2.6) and the second equality follows from Lemma 2.5.2. $\square$

The isomorphism $\mathscr{C}(A, \varprojlim_{\mathbf{I}} D) \cong \varprojlim_{\mathbf{I}} \mathscr{C}(A, D)$ in Theorem 2.5.3 tells us that if we feed a limit in $\mathscr{C}$ into the functor $\mathscr{C}(A, -)$ then we obtain a limit in **Set**. So, $\mathscr{C}(A, -)$ must preserve limits.

The Yoneda embedding maps from a category to a functor category. Hence, we must understand how limits behave in a functor category. Let $\mathscr{C}$ and $\mathscr{D}$ be categories. If $A \in \mathscr{C}$ is an object then there exists a functor

$$\begin{array}{rccc} ev_A : & \mathcal{F}(\mathscr{C}, \mathscr{D}) & \to & \mathscr{D} \\ & F & \mapsto & F(A) \\ & \alpha & \mapsto & \alpha_A \end{array}$$

called the **evaluation functor** at A. Given a diagram $D : \mathbf{I} \to \mathcal{F}(\mathscr{C}, \mathscr{D})$, we have for each $A \in \mathscr{C}$ the composite functor

$$\begin{array}{rccc} ev_A \circ D : & \mathbf{I} & \to & \mathscr{D} \\ & I & \mapsto & D(I)(A) \\ & \alpha & \mapsto & D(\alpha)_A \end{array}$$

In the next theorem, we will write the composite $ev_A \circ D$ as $D(-)(A)$.

Theorem 2.5.4. *Let $\boldsymbol{I}$ be a small category and $\mathscr{C}$ and $\boldsymbol{A}$ be locally small categories. Let $D : \boldsymbol{I} \to \mathcal{F}(\boldsymbol{A}, \mathscr{C})$ be a diagram and assume that if $A \in \boldsymbol{A}$ then the diagram $D(-)(A) : \boldsymbol{I} \to \mathscr{C}$ has a limit. Then, there exists a cone on D whose image under the evaluation functor ev_A is a limit on $D(-)(A)$ for each $A \in \boldsymbol{A}$. Moreover, a cone on D which satisfies the aforementioned property is a limit.*

Proof. Assume that $\mathbf{I}$ is a small category and $\mathscr{C}$ and $\mathbf{A}$ are locally small categories. Assume that $D : \mathbf{I} \to \mathcal{F}(\mathbf{A}, \mathscr{C})$ is a diagram and that if $A \in \mathbf{A}$ then the diagram $D(-)(A) : \mathbf{I} \to \mathscr{C}$ has a limit.

Suppose that

$$\big(p_{I,A} : L(A) \to D(I)(A)\big)_{I \in \mathbf{I}}$$

is such a limit.

To show: (a) There is a unique way of extending L to a functor on $\mathbf{A}$ such that $\big(p_I : L \to D(I)\big)_{I \in \mathbf{I}}$ is a cone on D.

(b) The cone $\big(p_I : L \to D(I)\big)_{I \in \mathbf{I}}$ on D is a limit.

(a) Let $f : A \to A'$ be a morphism in $\mathbf{A}$. By applying the functor, D, we obtain the natural transformation (morphism in $\mathcal{F}(\mathbf{A} \to \mathscr{C})$)

$$D(f) = D(-)(f) : D(-)(A) \to D(-)(A').$$

Now we can apply the construction in Theorem 2.4.3 to obtain a unique morphism $L(f) : L(A) \to L(A')$ such that if $I \in \mathbf{I}$ then the following diagram in $\mathscr{C}$ commutes:

$$\begin{tikzcd}
L(A) \arrow[r, "p_{I,A}"] \arrow[d, "L(f)"'] & D(I)(A) \arrow[d, "D(I)(f)"] \\
L(A') \arrow[r, "p_{I,A'}"] & D(I)(A')
\end{tikzcd}$$

It is tedious, but straightforward to check that L is a functor from $\mathbf{A}$ to $\mathscr{C}$. Commutativity of the above diagram in $\mathscr{C}$ tells us that if $I \in \mathbf{I}$ then the map

$$p_I : L \to D(I) \qquad \text{such that} \qquad (p_I)_A = p_{I,A}$$

is a natural transformation. Hence, we have constructed the following family of maps

$$\big(p_I : L \to D(I)\big)_{I \in \mathbf{I}}$$

in the functor category $\mathcal{F}(\mathbf{A}, \mathscr{C})$. We know that $L(A)$ is a cone on $D(-)(A)$ for any $A \in \mathbf{A}$. Hence, $\big(p_I : L \to D(I)\big)_{I \in \mathbf{I}}$ must be a cone on D.

(b) Now let $\big(q_I : X \to D(I)\big)_{I \in \mathbf{I}}$ be a cone on D in $\mathcal{F}(\mathbf{A}, \mathscr{C})$. If $A \in \mathbf{A}$ then we have a cone $\big(q_{I,A} : X(A) \to D(I)(A)\big)_{I \in \mathbf{I}}$ on $D(-)(A)$ in $\mathscr{C}$.

Since $\big(p_{I,A} : L(A) \to D(I)(A)\big)_{I \in \mathbf{I}}$ is a limit on $D(-)(A)$, there exists a unique map $\overline{q}_A : X(A) \to L(A)$ such that $p_{I,A} \circ \overline{q}_A = q_{I,A}$ for $I \in \mathbf{I}$.

It remains to prove that $\overline{q}_A$ is natural with respect to A. Since q_I is a natural transformation, if $f : A \to A'$ is a morphism in $\mathbf{A}$ then the following diagram commutes:

$$\begin{array}{ccc}
X(A) & \xrightarrow{\overline{q}_{I,A}} & D(I)(A) \\
{\scriptstyle X(f)}\big\downarrow & & \big\downarrow{\scriptstyle D(I)(f)} \\
X(A') & \xrightarrow{\overline{q}_{I,A'}} & D(I)(A')
\end{array}$$

Consequently by Theorem 2.4.3, the following diagram must commute:

$$\begin{array}{ccc}
X(A) & \xrightarrow{\overline{q}_{A}} & L(A) \\
{\scriptstyle X(f)}\big\downarrow & & \big\downarrow{\scriptstyle L(f)} \\
X(A') & \xrightarrow{\overline{q}_{A'}} & L(A')
\end{array}$$

So, $\overline{q} : X \to L$ is a unique natural transformation such that $p_I \circ \overline{q} = q_I$ because $p_{I,A} \circ \overline{q}_A = q_{I,A}$ for $A \in \mathbf{A}$. Therefore, $\big(p_I : L \to D(I)\big)_{I \in \mathbf{I}}$ is a limit on D as required. $\square$

We can now finally prove Theorem 2.4.1.

Proof of Theorem 2.4.1. It suffices to show that if $\mathscr{C}$ is a locally small category then the Yoneda embedding $Y : \mathscr{C} \to \mathcal{F}(\mathscr{C}^{op}, \mathbf{Set})$ preserves limits.

Assume that $\mathscr{C}$ is a locally small category and $D : \mathbf{I} \to \mathscr{C}$ is a diagram in $\mathscr{C}$. Let

$$\big(p_I : \varprojlim_{\mathbf{I}} D \to D(I)\big)_{I \in \mathbf{I}}$$

be a limit on D. If $A \in \mathscr{C}$ then observe that the composite functor $ev_A \circ Y = \mathscr{C}(A, -)$. By Theorem 2.5.3, $ev_A \circ Y$ preserves limits. Hence, for an object $A \in \mathbf{A}$,

$$\big((ev_A \circ Y)(p_I) : (ev_A \circ Y)(\lim_{\longleftarrow \mathbf{I}} D) \to (ev_A \circ Y)(D(I))\big)_{I \in \mathbf{I}}$$

is a limit on the diagram $ev_A \circ Y \circ D$. Now we can apply Theorem 2.5.4 to the diagram $Y \circ D$ in $\mathcal{F}(\mathscr{C}^{op}, \mathbf{Set})$, to find that

$$\big(Y(p_I) : Y(\lim_{\longleftarrow \mathbf{I}} D) \to Y(D(I))\big)_{I \in \mathbf{I}}$$

is a limit. Thus, the Yoneda embedding Y preserves limits. □

2.6 Embeddings for internal structures

In this chapter, we have addressed two main ideas — internal structures and the Yoneda embedding. In this final section, we will tie these two ideas together and study embeddings of internal structures into functor categories.

Let (M, m, e) be an internal unitary magma in the finitely complete, locally small category $\mathscr{C}$. Recall from Theorem 2.1.4 that the functor $Hom_{\mathscr{C}}(-, M)$ factorises through $\mathbf{UMg}$ so that the following diagram commutes:

$$\begin{tikzcd}
\mathscr{C}^{op} \arrow[r, dashed] \arrow[dr, "Hom_{\mathscr{C}}(-{,}M)"'] & \mathbf{UMg} \arrow[d, "U"] \\
& \mathbf{Set}
\end{tikzcd}$$

The forgetful functor $U : \mathbf{UMg} \to \mathbf{Set}$ induces a functor between functor categories

$$\begin{array}{rccc}
\mathcal{F}(\mathscr{C}^{op}, U) : & \mathcal{F}(\mathscr{C}^{op}, \mathbf{UMg}) & \to & \mathcal{F}(\mathscr{C}^{op}, \mathbf{Set}) \\
& F & \mapsto & U \circ F
\end{array}$$

The factorisation of $Hom_{\mathscr{C}}(-, M)$ through $\mathbf{UMg}$ can be restated with functor categories and the Yoneda embedding.

$$\begin{tikzcd}
\mathbf{UMg}(\mathscr{C}) \arrow[r, dashed, "\overline{Y}_{UMg}"] \arrow[d, "\mathcal{U}_{\mathscr{C}}"] & \mathcal{F}(\mathscr{C}^{op}, \mathbf{UMg}) \arrow[d, "\mathcal{F}(\mathscr{C}^{op}{,}U)"] \\
\mathscr{C} \arrow[r, "Y"] & \mathcal{F}(\mathscr{C}^{op}, \mathbf{Set})
\end{tikzcd}$$

Recall that the forgetful functor $\mathcal{U}_{\mathscr{C}} : \mathbf{UMg}(\mathscr{C}) \to \mathscr{C}$ is left exact, conservative and faithful. We claim that the induced functor $\overline{Y}_{UMg}$ is fully faithful and left exact.

First, let us show that $\overline{Y}_{UMg}$ is a faithful functor. Since $\mathcal{U}_{\mathscr{C}}$ and Y are faithful functors, $Y \circ \mathcal{U}_{\mathscr{C}} = \mathcal{F}(\mathscr{C}^{op}, U) \circ \overline{Y}_{UMg}$ is a faithful functor. By Lemma 1.10.3, the functor $\overline{Y}_{UMg}$ is faithful.

Next, we will show that $\overline{Y}_{UMg}$ is full. Let $(M, m, e), (M', m', e') \in \mathbf{UMg}(\mathscr{C})$ be internal unitary magmas in $\mathscr{C}$. Let $\theta : Hom_{\mathscr{C}}(-, M) \to Hom_{\mathscr{C}}(-, M')$ be a natural transformation — a morphism in the functor category $\mathcal{F}(\mathscr{C}^{op}, \mathbf{UMg})$. It suffices to construct a morphism $f : M \to M'$ such that $\overline{Y}_{UMg}(f) = \theta$.

The idea here is that the natural transformation θ has an underlying natural transformation $\theta_{Set} : Hom_{\mathscr{C}}(-, M) \to Hom_{\mathscr{C}}(-, M')$, which is a morphism in $\mathcal{F}(\mathscr{C}^{op}, \mathbf{Set})$. Since the Yoneda embedding Y is full (see Theorem 2.4.1), there exists a morphism of sets $f : M \to M'$ such that $Y(f) = \theta_{Set}$. To see that f is a morphism of internal unitary magmas, we will use the Yoneda embedding and the magma structure in Theorem 2.1.4.

First, we will show that f preserves internal units. That is, $f \circ e = e'$. Recall that $\iota_{A,M} \in Hom_{\mathscr{C}}(A, M)$ is the unit of $Hom_{\mathscr{C}}(A, M)$, which sends any $a \in A$ to the internal unit $e(*)$ where $*$ is the terminal object in $\mathscr{C}$. By applying the Yoneda embedding to $f \circ e$, we find that for $A \in \mathscr{C}$ and $g \in Hom_{\mathscr{C}}(A, *)$,

$$Y(f \circ e)_A(g) = (Y(f)_A \circ Y(e)_A)(g) = Y(f)_A(e \circ g) = Y(f)_A(\iota_{A,M})$$

Since $Y(f)_A = \theta_A$ is a morphism of unitary magmas, $Y(f)_A(\iota_{A,M}) = \iota_{A,M'} = Y(e')_A(g)$. If $A \in \mathscr{C}$ then $Y(f \circ e)_A = Y(e')_A$, $Y(f \circ e) = Y(e')$ and because Y is full, $f \circ e = e'$.

Next, we show that $f \circ m = m' \circ (f, f)$. Assume that $g = (g_1, g_2) \in Hom_{\mathscr{C}}(A, M \times M)$. Let μ and μ' denote the internal binary operations on $Hom_{\mathscr{C}}(A, M)$ and $Hom_{\mathscr{C}}(A, M')$ respectively. We then have

$$
\begin{aligned}
Y(f \circ m)_A(g_1, g_2) &= (Y(f)_A \circ Y(m)_A)(g_1, g_2) \\
&= Y(f)_A(m \circ (g_1, g_2)) \\
&= Y(f)_A(\mu(g_1, g_2)) \\
&= \mu'(Y(f)_A(g_1), Y(f)_A(g_2)) \\
&= m' \circ (Y(f)_A, Y(f)_A)(g_1, g_2) \\
&= (Y(m')_A \circ (Y(f)_A, Y(f)_A))(g_1, g_2) \\
&= Y(m' \circ (f, f))_A(g_1, g_2).
\end{aligned}
$$

Since $A \in \mathscr{C}$ and $g \in Hom_{\mathscr{C}}(A, M \times M)$ were arbitrary, $Y(m' \circ (f, f)) = Y(f \circ m)$ and by fullness of Y, $m' \circ (f, f) = f \circ m$. Therefore, f is an internal unitary magma satisfying $Y(f) = \theta$. So, $\overline{Y}_{UMg}$ is a full functor.

Finally, to see that $\overline{Y}_{UMg}$ is a left exact functor, note that the functors $\mathcal{U}_{\mathscr{C}}$ and Y are all left exact. So, the composite $Y \circ \mathcal{U}_{\mathscr{C}} = \mathcal{F}(\mathscr{C}^{op}, U) \circ \overline{Y}_{UMg}$ must be left exact. Consequently, $\overline{Y}_{UMg}$ is left exact.

So, $\overline{Y}_{UMg}$ is a fully faithful, left exact functor.

Since we have Theorem 2.1.5, Theorem 2.2.4 and Theorem 2.2.5, we can repeat the above argument for internal monoids and internal groups. We conclude that the induced functors

$$
\begin{array}{rccc}
\overline{Y}_{Mon}: & \mathbf{Mon}(\mathscr{C}) & \to & \mathcal{F}(\mathscr{C}^{op}, \mathbf{Mon}) \\
\overline{Y}_{CoM}: & \mathbf{CoM}(\mathscr{C}) & \to & \mathcal{F}(\mathscr{C}^{op}, \mathbf{CoM}) \\
\overline{Y}_{Grp}: & \mathbf{Grp}(\mathscr{C}) & \to & \mathcal{F}(\mathscr{C}^{op}, \mathbf{Grp}) \\
\overline{Y}_{Ab}: & \mathbf{Ab}(\mathscr{C}) & \to & \mathcal{F}(\mathscr{C}^{op}, \mathbf{Ab})
\end{array}
$$

are all fully faithful and left exact. In [Bou17, Exercise 2.3.4], the above functors are called *structure embeddings*.

The argument we outlined in this section to prove that $\overline{Y}_{UMg}$ is fully faithful and left exact is a special case of the *enriched Yoneda lemma*. A good reference for this is [Kel05, Section 2.4].

Chapter 3

The four major observations

3.1 Pointed categories

In this chapter, we want to better understand the relationships between the categories **Mon**, **CoM**, **Grp** and **Ab**. There are four major observations about this quartet of categories which we will investigate and describe in this chapter. First, we will point out a common feature of these categories in this section.

Definition 3.1.1. Let $\mathscr{C}$ be a category. We say that $\mathscr{C}$ is **pointed** if there exists an object $1 \in \mathscr{C}$ such that 1 is both a terminal object and an initial object. The object 1 is called a **zero object**.

The categories **Mon**, **CoM**, **Ab** and **Grp** are all pointed, with the zero object being the trivial monoid for **Mon** and **CoM** and the trivial group for **Grp** and **Ab**.

Example 3.1.2. Let $\mathbf{Set}_*$ be the category of pointed sets. The objects in $\mathbf{Set}_*$ are the pairs (X, x_0) consisting of a set X and a point $x_0 \in X$. The morphisms in $\mathbf{Set}_*$ are maps $f : (X, x_0) \to (Y, y_0)$ such that $f : X \to Y$ is a morphism of sets and $f(x_0) = y_0$.

The category $\mathbf{Set}_*$ is pointed, with the zero object being the pair $(\{*\}, *)$, where $\{*\}$ is the singleton set.

Let us establish some notation regarding the zero object of a pointed category.

Definition 3.1.3. Let $\mathscr{C}$ be a pointed category with zero object 1. Let $X \in \mathscr{C}$ be an object. The initial map from 1 to X will be denoted by α_X, whereas the terminal map from X to 1 is denoted by τ_X.

If $X, Y \in \mathscr{C}$ is any pair of objects then the composite $\alpha_Y \circ \tau_X : X \to Y$ is called the **zero map** between X and Y. It is usually denoted by $0_{X,Y}$.

One might ask whether the internal categories discussed in the previous chapter are pointed. It turns out that they are and we will prove this for the category $\mathbf{UMg}(\mathscr{C})$.

Theorem 3.1.1. *Let $\mathscr{C}$ be a finitely complete category with terminal object $*$. Then, the category of internal unitary magmas $\boldsymbol{UMg}(\mathscr{C})$ is pointed and finitely complete.*

Proof. Assume that $\mathscr{C}$ is a finitely complete category.

To show: (a) The category $\mathbf{UMg}(\mathscr{C})$ is finitely complete.

(b) The category $\mathbf{UMg}(\mathscr{C})$ is pointed.

(a) By Theorem 1.13.1, it suffices to show that $\mathbf{UMg}(\mathscr{C})$ has products and equalizers.

Assume that (M, m, e) and (N, n, f) are internal unitary magmas. Since $\mathscr{C}$ is finitely complete, we can construct the product $M \times N$ as an object in $\mathscr{C}$ which satisfies the universal property of products. Define the maps

$$\begin{array}{rccc} m \times n : & (M \times N) \times (M \times N) & \to & M \times N \\ & ((m_1, n_1), (m_2, n_2)) & \mapsto & (m(m_1, m_2), n(n_1, n_2)) \end{array}$$

and

$$\begin{array}{rccc} e \times f : & * & \mapsto & M \times N \\ & * & \mapsto & (e(*), f(*)) \end{array}$$

To see that $e \times f$ is an internal unit for $M \times N$, we compute directly that if $(x, y) \in M \times N$ then

$$\begin{aligned} (m \times n)((x, y), (e(*), f(*))) &= (m(x, e(*)), n(y, f(*))) \\ &= (x, y) = (m(e(*), x), n(f(*), y)) \\ &= (m \times n)((e(*), f(*)), (x, y)). \end{aligned}$$

Since, m, n, e and f are all morphisms in $\mathscr{C}$, $m \times n$ and $e \times f$ are also morphisms in $\mathscr{C}$. Hence, $m \times n$ is an internal binary operation on $M \times N$

and consequently, $M \times N$ is an internal unitary magma.

Next, let $h_1, h_2 : M \to N$ be two morphisms of internal unitary magmas. Since $\mathscr{C}$ is finitely complete, we can construct an equalizer of h_1 and h_2 in $\mathscr{C}$ so that we have the following diagram in $\mathscr{C}$:

$$\begin{tikzcd} E \arrow[r, "\eta"] & M \arrow[r, shift left, "h_1"] \arrow[r, shift right, "h_2"'] & N \end{tikzcd}$$

We will construct an internal binary operation and an internal unit on E by using the universal property of the equalizer. First, define the morphism in $\mathscr{C}$

$$\begin{array}{rccc} \phi: & E \times E & \to & M \\ & (a_1, a_2) & \mapsto & m(\eta(a_1), \eta(a_2)) \end{array}$$

Observe that

$$\begin{aligned} (h_1 \circ \phi)(a_1, a_2) &= h_1(m(\eta(a_1), \eta(a_2))) \\ &= n((h_1 \circ \eta)(a_1), (h_1 \circ \eta)(a_2)) \\ &= n((h_2 \circ \eta)(a_1), (h_2 \circ \eta)(a_2)) \\ &= h_2(m(\eta(a_1), \eta(a_2))) \\ &= (h_2 \circ \phi)(a_1, a_2). \end{aligned}$$

By the universal property of the equalizer, there exists a unique morphism $m_E : E \times E \to E$ in $\mathscr{C}$ such that the following diagram in $\mathscr{C}$ commutes:

$$\begin{tikzcd} E \times E \arrow[d, dashed, "m_E"'] \arrow[dr, "\phi"] & & \\ E \arrow[r, "\eta"] & M \arrow[r, shift left, "h_1"] \arrow[r, shift right, "h_2"'] & N \end{tikzcd}$$

Now since h_1 and h_2 are morphisms of internal unitary magmas,

$$h_1(e(*)) = f(*) = h_2(e(*)).$$

We can again apply the universal property of the equalizer to deduce the existence of a morphism $e_E : * \to E$ such that the following diagram in $\mathscr{C}$ commutes:

$$\begin{tikzcd}
* \arrow[d, dashed, "e_E"'] \arrow[dr, "e"] & & \\
E \arrow[r, "\eta"] & M \arrow[r, shift left, "h_1"] \arrow[r, shift right, "h_2"'] & N
\end{tikzcd}$$

To see that e_E is an internal unit for E, observe that if $x \in E$ then

$$\begin{aligned}
(\eta \circ m_E)(x, e_E(*)) &= \phi(x, e_E(*)) \\
&= m(\eta(x), \eta(e_E(*))) \\
&= m(\eta(x), e(*)) = \eta(x)
\end{aligned}$$

and similarly, $(\eta \circ m_E)(e_E(*), x) = \eta(x)$. Since η is an equalizer, it must be a monomorphism by Theorem 1.2.1 and consequently, $m_E(e_E(*), x) = m_E(x, e_E(*)) = x$. So, e_E defines an internal unit on E.

By commutativity of both diagrams, we have $e = \eta \circ e_E$ and $\phi = m \circ (\eta, \eta) = \eta \circ m_E$. Hence, η is a morphism of internal unitary magmas and E is an internal unitary magma.

It is straightforward but tedious to check that $(M \times N, m \times n, e \times f)$ and (E, m_E, e_E) respectively satisfy the universal properties of the product and equalizer in $\mathbf{UMg}(\mathscr{C})$. This works because $M \times N$ and E are the product and equalizer respectively in $\mathscr{C}$. Therefore, $\mathbf{UMg}(\mathscr{C})$ is a finitely complete category.

(b) Consider the terminal object $*$ in $\mathscr{C}$. By defining the morphisms

$$\begin{array}{rccc}
m_* : & * \times * & \to & * \\
& (*, *) & \mapsto & *
\end{array}$$

and

$$\begin{array}{rccc}
e_* : & * & \to & * \\
& * & \mapsto & *
\end{array}$$

we find that $(*, m_*, e_*)$ is an internal unitary magma in $\mathscr{C}$.

Now let (M, m, e) be another internal unitary magma in $\mathscr{C}$. Since $*$ is a terminal object in $\mathscr{C}$, there exists a unique morphism $\tau_M : M \to *$ in $\mathscr{C}$. To see that τ_M is a morphism of internal unitary magmas, we compute for $x, y \in M$ that

$$\tau_M(m(x,y)) = * = m_*(*,*) = m_*(\tau_M(x), \tau_M(y))$$

and $\tau_M(e(*)) = * = e_*(*)$. Therefore, $\tau_M : M \to *$ is a morphism of internal unitary magmas.

Now we will show that $e : * \to M$ is a unique morphism of internal unitary magmas. To see that it is a morphism of internal unitary magmas, we compute directly that

$$e(m_*(*,*)) = e(*) = m(e(*), e(*))$$

and $e(e_*(*)) = e(*)$. So, e is a morphism of internal unitary magmas. To see that e is unique, suppose that we have another morphism of internal unitary magmas $e' : * \to M$. Then, $e'(e_*(*)) = e'(*) = e(*)$ because e' preserves internal units. So, $e' = e$ and e must be unique.

Thus, $(*, m_*, e_*)$ is a zero object in the category $\mathbf{UMg}(\mathscr{C})$. So, $\mathbf{UMg}(\mathscr{C})$ is a pointed category. $\square$

A similar argument to the proof of Theorem 3.1.1 can be devised to show that the categories $\mathbf{Mon}(\mathscr{C}), \mathbf{CoM}(\mathscr{C}), \mathbf{Ab}(\mathscr{C})$ and $\mathbf{Grp}(\mathscr{C})$ are pointed and finitely complete.

3.2 Kernels, cokernels and exact sequences

In pointed categories, we are able to recover particular tools used in the category of groups — namely, the concepts of kernels, cokernels and exact sequences. First, we will like to generalise the kernel of a group morphism to morphisms in a pointed category.

In the category of groups **Grp**, we are used to talking about the kernel of a morphism $f : G \to H$ as a normal subgroup of G. In the context of category theory, **a kernel is instead a morphism**.

Theorem 3.2.1. *Let $f : G \to H$ be a group morphism. Then, the following commutative square in **Grp** is a pullback square:*

$$\begin{tikzcd} \ker f \arrow[r, "\iota"] \arrow[d, "\tau_K"] & G \arrow[d, "f"] \\ 1 \arrow[r, "\alpha_H"] & H \end{tikzcd}$$

Here, 1 denotes the trivial group, which is the zero object in ***Grp*** *and* $\iota : \ker f \hookrightarrow G$ *is the inclusion morphism.*

Proof. Assume that $f : G \to H$ is a group morphism. Assume that 1 is the trivial group and $\iota : \ker f \hookrightarrow G$ is the inclusion morphism. By definition of the kernel $\ker f$ and the fact that $\alpha_H \circ \tau_K$ is the zero map, the following square in **Grp** commutes:

$$\begin{tikzcd} \ker f \arrow[r, "\iota"] \arrow[d, "\tau_K"] & G \arrow[d, "f"] \\ 1 \arrow[r, "\alpha_H"] & H \end{tikzcd}$$

To see that the above square is a pullback square, suppose that the following square in **Grp** commutes:

$$\begin{tikzcd} Y \arrow[r, "\beta"] \arrow[d, "\tau_Y"] & G \arrow[d, "f"] \\ 1 \arrow[r, "\alpha_H"] & H \end{tikzcd}$$

By commutativity, if $y \in Y$ then $\beta(y) \in \ker f$. Hence, the following diagram must commute:

$$\begin{tikzcd} Y \arrow[dr, dashed, "\beta"] \arrow[drr, bend left, "\beta"] \arrow[ddr, bend right, "\tau_Y"'] & & \\ & \ker f \arrow[r, "\iota"] \arrow[d, "\tau_K"] & G \arrow[d, "f"] \\ & 1 \arrow[r, "\alpha_H"] & H \end{tikzcd}$$

To see that $\beta : Y \to \ker f$ is unique, suppose that $\gamma : Y \to \ker f$ is a group morphism which also makes the above diagram commute. Then, $\iota \circ \beta = \iota \circ \gamma$. Since the inclusion ι is an injective group morphism, it is a group monomorphism. So, $\beta = \gamma$.

Consequently, the commutative square

$$\begin{tikzcd} \ker f \arrow[r, "\iota"] \arrow[d, "\tau_K"] & G \arrow[d, "f"] \\ 1 \arrow[r, "\alpha_H"] & H \end{tikzcd}$$

is a pullback square in **Grp**. □

Theorem 3.2.1 provides us with the appropriate generalisation of a kernel to pointed categories.

Definition 3.2.1. Let $\mathscr{C}$ be a pointed category and $f : X \to Y$ be a morphism in $\mathscr{C}$. The **kernel** of f is a morphism $k_f : \ker f \to X$ in $\mathscr{C}$ such that the following commutative square in $\mathscr{C}$ is a pullback square:

$$\begin{tikzcd}
\ker f \arrow[r, "k_f"] \arrow[d, "\tau_K"] & X \arrow[d, "f"] \\
* \arrow[r, "\alpha_Y"] & Y
\end{tikzcd}$$

Here, $*$ is the zero object in $\mathscr{C}$.

To acquaint ourselves with the definition of a kernel in a pointed finitely complete category, we will first see what happens to the kernel when we replace the initial map $\alpha_Y : 1 \to Y$ with a different morphism.

Theorem 3.2.2. *Let $\mathscr{C}$ be a pointed finitely complete category. Let $f : X \to Y$ and $y : Y' \to Y$ be morphisms in $\mathscr{C}$ and suppose that we have the following pullback square in $\mathscr{C}$:*

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "x"] & Y' \arrow[d, "y"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

Then, there exists a unique morphism $k' : \ker f \to X'$ such that the following diagram in $\mathscr{C}$ commutes:

$$\begin{tikzcd}
\ker f \arrow[r, dashed, "k'"] \arrow[rr, bend left, "0"] \arrow[d, "\cong"] & X' \arrow[r, "x"] \arrow[d, "y"] & Y' \arrow[d, "y"] \\
\ker f \arrow[r, "k_f"] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

Here, $0 : \ker f \to Y'$ denotes the zero map.

Proof. Assume that $\mathscr{C}$ is a pointed finitely complete category. Assume that we have the following pullback square in $\mathscr{C}$:

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "x"] & Y' \arrow[d, "y"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

Observe that the composite $y \circ 0$ is the zero map from $\ker f$ to Y because the group morphism y must preserve units. So, the following diagram in $\mathscr{C}$ commutes:

$$\begin{tikzcd}
\ker f \arrow[r, "0"] \arrow[d, "k_f"] & Y' \arrow[d, "y"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

The morphism $k_f : \ker f \to X$ is the kernel of f. By the universal property of the pullback, there exists a unique morphism $k' : \ker f \to X'$ such that the following diagram in $\mathscr{C}$ commutes:

$$\begin{tikzcd}
\ker f \arrow[dr, dashed, "k'"] \arrow[drr, bend left, "0"] \arrow[ddr, bend right, "k_f"'] & & \\
& X' \arrow[r, "f'"] \arrow[d, "x"] & Y' \arrow[d, "y"] \\
& X \arrow[r, "f"] & Y
\end{tikzcd}$$

□

Theorem 3.2.2 tells us that $k' : \ker f \to X'$ is the kernel of f'. So, f and f' have the same "kernel object" $\ker f$.

The next theorem tells us that we can construct unique maps between kernel objects.

Theorem 3.2.3. *Let $\mathscr{C}$ be a pointed, finitely complete category. Let $f : X \to Y$ and $y : Y' \to Y$ be morphisms in $\mathscr{C}$. Suppose that the following square in $\mathscr{C}$ commutes:*

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "x"] & Y' \arrow[d, "y"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

Then, there exists a unique morphism $K(x) : \ker f' \to \ker f$ such that the LHS square of the following diagram commutes:

$$\begin{tikzcd}
\ker f' \arrow[r, "k_{f'}"] \arrow[d, dashed, "K(x)"] & X' \arrow[r, "x"] \arrow[d, "y"] & Y' \arrow[d, "y"] \\
\ker f \arrow[r, "k_f"] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

Moreover, if the RHS square is a pullback square then $K(x) : \ker f' \to \ker x$ is an isomorphism.

Proof. Assume that $\mathscr{C}$ is a pointed, finitely complete category. Assume that we have the following commutative square in $\mathscr{C}$:

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "x"] & Y' \arrow[d, "y"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Let $k_f : \ker f \to X$ and $k_{f'} : \ker f' \to X$ be the kernels of f and f' respectively. Using the commutative diagram above, we have

$$f \circ (x \circ k_{f'}) = (f \circ x) \circ k_{f'} = (y \circ f') \circ k_{f'} = y \circ 0_{\ker f', Y'} = 0_{\ker f', Y}$$

where $0_{\ker f', Y'} : \ker f' \to Y'$ is the zero map from $\ker f'$ to Y'. Hence, the following square in $\mathscr{C}$ commutes:

$$\begin{tikzcd} \ker f' \arrow[r, "x \circ k_{f'}"] \arrow[d, "\tau_{K'}"] & X \arrow[d, "f"] \\ * \arrow[r, "\alpha_Y"] & Y \end{tikzcd}$$

By the universal property of the pullback, there exists a unique morphism $K(x) : \ker f' \to \ker f$ such that the following diagram commutes:

$$\begin{tikzcd} \ker f' \arrow[dr, dashed, "K(x)"] \arrow[drr, bend left, "x \circ k_{f'}"] \arrow[ddr, bend right, "\tau_{K'}"'] & & \\ & \ker f \arrow[r, "k_f"] \arrow[d, "\tau_K"] & X \arrow[d, "f"] \\ & * \arrow[r, "\alpha_Y"] & Y \end{tikzcd}$$

Therefore, the diagram in $\mathscr{C}$ commutes:

$$\begin{tikzcd} \ker f' \arrow[r, "k_{f'}"] \arrow[d, dashed, "K(x)"] & X' \arrow[r, "f'"] \arrow[d, "x"] & Y' \arrow[d, "y"] \\ \ker f \arrow[r, "k_f"] & X \arrow[r, "f"] & Y \end{tikzcd}$$

Now assume that the RHS square is a pullback square in $\mathscr{C}$. The key is to form the following cube in $\mathscr{C}$:

$$
\begin{tikzcd}
\ker f' \arrow[rr, "\tau_{K'}"] \arrow[dr, "k_{f'}"'] \arrow[dd, "K(x)"'] & & * \arrow[dr, "\alpha_{Y'}"] \arrow[dd, "id"' near start] & \\
& X' \arrow[rr, "f'" near start] \arrow[dd, "x"' near start] & & Y' \arrow[dd, "y"] \\
\ker f \arrow[rr, "\tau_K" near end] \arrow[dr, "k_f"'] & & * \arrow[dr, "\alpha_Y"] & \\
& X \arrow[rr, "f"] & & Y
\end{tikzcd}
$$

By assumption, the top, front and bottom faces of the cube are pullback squares in $\mathscr{C}$. By the Dice lemma (see Lemma 1.4.6), the back face of the cube

$$
\begin{tikzcd}
\ker f' \arrow[r, "\tau_{K'}"] \arrow[d, "K(x)"'] & * \arrow[d, "id"] \\
\ker f \arrow[r, "\tau_K"] & *
\end{tikzcd}
$$

is a pullback square in $\mathscr{C}$. The identity map $id : * \to *$ is an isomorphism. By Lemma 1.4.1, $K(x)$ must also be an isomorphism as required. $\square$

Next, we will describe the kernel k_f of a morphism f as an equalizer. This is the definition Borceux uses for the kernel in [Bor94, Volume II, Definition 1.1.5].

Theorem 3.2.4. *Let $\mathscr{C}$ be a pointed, finitely complete category. Let $f : X \to Y$ be a morphism in $\mathscr{C}$ and $0_{X,Y} : X \to Y$ be the zero map. The kernel $k_f : \ker f \to X$ of f is the equalizer of f and $0_{X,Y}$. Consequently, k_f is a monomorphism.*

Proof. Assume that $\mathscr{C}$ is a pointed finitely complete category. Assume that $f : X \to Y$ is a morphism in $\mathscr{C}$ and $0_{X,Y} : X \to Y$ is the zero map.

To see that the kernel $k_f : \ker f \to X$ equalizes the pair $(f, 0_{X,Y})$, we compute directly from the definition of the kernel that

$$f \circ k_f = \alpha_Y \circ \tau_K = 0_{\ker f, Y} = 0_{X,Y} \circ k_f$$

where $\tau_K : X \to *$ is the unique terminal map from X to the zero object $* \in \mathscr{C}$ and $\alpha_Y : * \to Y$ is the unique initial map from $*$ to Y.

To see that k_f is the equalizer of f and $0_{X,Y}$, suppose that we have a morphism $\phi : Z \to X$ such that $f \circ \phi = 0_{X,Y} \circ \phi$. Then, the following square in $\mathscr{C}$ commutes:

$$
\begin{tikzcd}
Z \arrow[r, "\phi"] \arrow[d, "\tau_Z"'] & X \arrow[d, "f"] \\
* \arrow[r, "\alpha_Y"] & Y
\end{tikzcd}
$$

By the universal property of the pullback, there exists a unique morphism $\rho : Z \to \ker f$ such that the following diagram commutes:

$$
\begin{tikzcd}
Z \arrow[dr, dashed, "\rho"] \arrow[drr, bend left, "\phi"] \arrow[ddr, bend right, "\tau_Z"'] & & \\
& \ker f \arrow[r, "k_f"] \arrow[d, "\tau_K"] & X \arrow[d, "f"] \\
& * \arrow[r, "\alpha_Y"] & Y
\end{tikzcd}
$$

This means that the following diagram in $\mathscr{C}$ commutes:

$$
\begin{tikzcd}
Z \arrow[d, dashed, "\rho"] \arrow[dr, "\phi"] & & \\
\ker f \arrow[r, "k_f"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "0_{X,Y}"'] & Y
\end{tikzcd}
$$

So, the kernel k_f is the equalizer of f and $0_{X,Y}$. Moreover, by Theorem 1.2.1, k_f must be a monomorphism. $\square$

The cokernel of a morphism is defined dually to the kernel.

Definition 3.2.2. Let $\mathscr{C}$ be a pointed category and $f : X \to Y$ be a morphism in $\mathscr{C}$. The **cokernel** of f is a morphism $q_f : Y \to \operatorname{coker} f$ such that the following commutative square in $\mathscr{C}$ is a pushout square:

$$
\begin{tikzcd}
\operatorname{coker} f & Y \arrow[l, "q_f"'] \\
* \arrow[u, "\alpha_C"] & X \arrow[l, "\tau_X"] \arrow[u, "f"']
\end{tikzcd}
$$

Here, $*$ is the zero object in $\mathscr{C}$.

With the definition of a cokernel, we will prove the dual result of Theorem 3.2.4.

Theorem 3.2.5. *Let $\mathscr{C}$ be a pointed category. Let $f : X \to Y$ be a morphism in $\mathscr{C}$ and $0_{X,Y} : X \to Y$ be the zero map. The cokernel $q_f : Y \to$ coker f is the coequalizer of f and $0_{X,Y}$. Moreover, q_f is an epimorphism.*

Proof. Assume that $\mathscr{C}$ is a pointed category. Assume that $f : X \to Y$ is a morphism in $\mathscr{C}$ and $0_{X,Y} : X \to Y$ is the zero map.

To see that the cokernel $q_f : Y \to \text{coker } f$ coequalizes the pair $(f, 0_{X,Y})$, we use the definition of the cokernel to compute that

$$q_f \circ f = \alpha_C \circ \tau_X = 0_{X,\text{coker } f} = q_f \circ 0_{X,Y}.$$

To see that q_f is the coequalizer of f and $0_{X,Y}$, suppose that $\gamma : Y \to Z$ is a morphism in $\mathscr{C}$ such that $\gamma \circ f = \gamma \circ 0_{X,Y}$. Then, the following diagram in $\mathscr{C}$ commutes:

$$\begin{tikzcd}
Z & Y \arrow[l, "\gamma"'] \\
* \arrow[u, "\alpha_Z"] & X \arrow[l, "\tau_X"] \arrow[u, "f"']
\end{tikzcd}$$

By the universal property of the pushout, there exists a unique morphism $\delta : \text{coker } f \to Z$ such that the following diagram in $\mathscr{C}$ commutes:

$$\begin{tikzcd}
Z & & \\
& \text{coker } f \arrow[ul, dashed, "\delta"] & Y \arrow[l, "q_f"'] \arrow[ull, bend right, "\gamma"'] \\
& * \arrow[u, "\alpha_C"] \arrow[uul, bend left, "\alpha_Z"] & X \arrow[l, "\tau_X"] \arrow[u, "f"']
\end{tikzcd}$$

This means that the following diagram commutes:

$$\begin{tikzcd}
& & Z \\
X \arrow[r, shift left, "f"] \arrow[r, shift right, "0_{X,Y}"'] & Y \arrow[ur, "\gamma"] \arrow[r, "q_f"'] & \text{coker } f \arrow[u, dashed, "\delta"']
\end{tikzcd}$$

So, the cokernel q_f is the coequalizer of the morphisms f and $0_{X,Y}$. Since q_f is a coequalizer, it must be an epimorphism. □

As another example of a kernel, we will construct a kernel from the pullback square involving the product.

Lemma 3.2.6. *Let $\mathscr{C}$ be a pointed, finitely complete category. Let X and Y be objects in $\mathscr{C}$. Consider the following pullback square:*

$$\begin{tikzcd} X \times Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"] & Y \arrow[d, "\tau_Y"] \\ X \arrow[r, "\tau_X"] & * \end{tikzcd}$$

Then, there exists a unique section $j_X : X \to X \times Y$ of π_X such that the following square in $\mathscr{C}$ commutes:

$$\begin{tikzcd} X \times Y \arrow[r, "\pi_Y"] & Y \\ X \arrow[u, "j_X"] \arrow[r, "\tau_X"] & * \arrow[u, "\alpha_Y"'] \end{tikzcd}$$

Correspondingly, there exists a unique section $j_Y : Y \to X \times Y$ of π_Y such that the following square in $\mathscr{C}$ commutes:

$$\begin{tikzcd} X \times Y \arrow[r, "\pi_X"] & X \\ Y \arrow[u, "j_Y"] \arrow[r, "\tau_Y"] & * \arrow[u, "\alpha_X"'] \end{tikzcd}$$

Proof. Assume that $\mathscr{C}$ is a pointed, finitely complete category. Assume that X and Y are objects in $\mathscr{C}$. Then, the following square in $\mathscr{C}$ commutes:

$$\begin{tikzcd} X \arrow[r, "0_{X,Y}"] \arrow[d, "id_X"'] & Y \arrow[d, "\tau_Y"] \\ X \arrow[r, "\tau_X"] & * \end{tikzcd}$$

Here, id_X denotes the identity morphism on X. Using the universal property of the pullback, there exists a unique morphism $j_X : X \to X \times Y$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[dr, dashed, "j_X"] \arrow[drr, bend left, "0_{X,Y}"] \arrow[ddr, bend right, "id_X"'] & & \\ & X \times Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"] & Y \arrow[d, "\tau_Y"] \\ & X \arrow[r, "\tau_X"] & * \end{tikzcd}$$

So, $\pi_X \circ j_X = id_X$, which means that j_X is a unique section of π_X.
Moreover, $\pi_Y \circ j_X = 0_{X,Y} = \alpha_Y \circ \tau_X$ as required.

The unique section $j_Y : Y \to X \times Y$ of π_Y is constructed in a very similar manner. One can check directly from the commutative diagrams that $j_X = (id_X, 0_{X,Y})$ and $j_Y = (0_{Y,X}, id_Y)$. $\square$

Now, we will show that the section j_Y is the kernel of the projection map π_X.

Theorem 3.2.7. *Let $\mathscr{C}$ be a pointed, finitely complete category. Let X and Y be objects in $\mathscr{C}$. Suppose that we have the following pullback square in $\mathscr{C}$:*

$$\begin{tikzcd} X \times Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"] & Y \arrow[d, "\tau_Y"] \\ X \arrow[r, "\tau_X"] & * \end{tikzcd}$$

Let $j_X : X \to X \times Y$ and $j_Y : Y \to X \times Y$ be the sections constructed in Lemma 3.2.6. Then, j_Y is the kernel of the projection map $\pi_X : X \times Y \to X$. That is, the following commutative square in $\mathscr{C}$ is a pullback square:

$$\begin{tikzcd} Y \arrow[r, "j_Y"] \arrow[d, "\tau_Y"] & X \times Y \arrow[d, "\pi_X"] \\ * \arrow[r, "\alpha_X"] & X \end{tikzcd}$$

Correspondingly, the section j_X is the kernel of the projection map π_Y.

Proof. Assume that $\mathscr{C}$ be a pointed finitely complete category. Assume that X and Y are objects in $\mathscr{C}$. Assume that $j_X : X \to X \times Y$ and $j_Y : Y \to X \times Y$ are the sections of the projections π_X and π_Y respectively, which were constructed in Lemma 3.2.6.

Consider the following commutative diagram in $\mathscr{C}$:

$$\begin{tikzcd} Y \arrow[r, "j_Y"] \arrow[d, "\tau_Y"'] & X \times Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"] & Y \arrow[d, "\tau_Y"] \\ * \arrow[r, "\alpha_X"] & X \arrow[r, "\tau_X"] & * \end{tikzcd}$$

The outside square and the right hand side square of the above diagram are pullback squares. By Lemma 1.4.5, the LHS square is a pullback square. So, j_Y is the kernel of the projection map π_X.

A similar argument demonstrates that j_X is the kernel of the projection map π_Y. $\square$

With the kernel and cokernel, we are now able to define exact sequences.

Definition 3.2.3. Let $\mathscr{C}$ be a pointed category with zero object $*$. An **exact sequence** in $\mathscr{C}$ is a sequence

$$* \longrightarrow K \xrightarrow{k} X \xrightarrow{f} Y \longrightarrow *$$

where k is the kernel of f and f is the cokernel of k.

The above definition of an exact sequence is not the mainstream definition of an exact sequence. We will show in the next example how the definition we gave is connected to the mainstream definition.

Example 3.2.4. We will work in the pointed category **Grp**. An exact sequence in **Grp** is a sequence

$$1 \longrightarrow K \xrightarrow{k} X \xrightarrow{f} Y \longrightarrow 1$$

where k is the kernel of f and f is the cokernel of k. Here, 1 denotes the trivial group which is the zero object in **Grp**. Since k is the kernel of f, the following diagram in **Grp** is a pullback square:

$$\begin{tikzcd} \ker f \arrow[r, "k"] \arrow[d, "\tau_K"] & X \arrow[d, "f"] \\ 1 \arrow[r, "\alpha_Y"] & Y \end{tikzcd}$$

Since f is the cokernel of k, the following square in **Grp** is a pushout square:

$$\begin{tikzcd} \operatorname{coker} k & X \arrow[l, "f"'] \\ * \arrow[u, "\alpha_C"] & \ker f \arrow[l, "\tau_K"] \arrow[u, "k"'] \end{tikzcd}$$

So, $K = \ker f$ and $Y = \text{coker } k$. Moreover, since f is a coequalizer by Theorem 3.2.5, f must be an epimorphism in **Grp** and subsequently, a surjective group morphism.

Recall that in **Grp**, the kernel of f is the inclusion map $k : \ker f \hookrightarrow X$. Notice that $\text{im } k = \ker f$. Since f is surjective, $\text{im } f = Y$, which is the kernel of the terminal map $\tau_Y : Y \to 1$. Finally, $\ker k = \{e_X\}$, where e_X is the identity element of X. This is equal to the image of the initial map $\alpha_K : 1 \to K$.

Thus, we have shown that in **Grp**, the definition of an exact sequence we gave is equivalent to the mainstream definition of an exact sequence — the image of any morphism in the sequence is equal to the kernel of the next morphism in the sequence. In fact, our definition of an exact sequence in **Grp** is a short exact sequence in **Grp**.

The five lemma is a useful result concerning morphisms between exact sequences. We will state it for the category **Grp**. Since our definition of exact sequences corresponds to short exact sequences in **Grp**, we will generalise and state the five lemma for exact sequences in **Grp**, where we use the mainstream definition.

Lemma 3.2.8. *Suppose that the top and bottom rows of the following diagram in **Grp** are exact sequences:*

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, "f"] & B \arrow[r, "\beta"] \arrow[d, "g"] & C \arrow[r, "\xi"] \arrow[d, "h"] & D \arrow[r, "\delta"] \arrow[d, "k"] & E \arrow[d, "l"] \\
F \arrow[r, "\phi"] & G \arrow[r, "\gamma"] & H \arrow[r, "\eta"] & I \arrow[r, "\iota"] & J
\end{tikzcd}$$

Suppose that f, g, k and l are isomorphism. Then, h is also an isomorphism.

Proof. Assume that f, g, k and l are isomorphisms in the above diagram. Assume that the top and bottom rows of the above diagram are exact sequences in **Grp**.

To show: (a) h is injective.

(b) h is surjective.

(a) Consider the three commutative squares on the left:

$$\begin{array}{ccccccc}
A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \xrightarrow{\xi} & D \\
\Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle g} & & \Big\downarrow{\scriptstyle h} & & \Big\downarrow{\scriptstyle k} \\
F & \xrightarrow{\phi} & G & \xrightarrow{\gamma} & H & \xrightarrow{\eta} & I
\end{array}$$

Assume that $\Phi_C \in C$ such that $h(\Phi_C) = 0$. Then, $(\eta \circ h)(\Phi_C) = 0$ and by the commutativity of the square with corners, C, D, H and I,

$$(\eta \circ h)(\Phi_C) = (k \circ \xi)(\Phi_C) = 0.$$

Since k is an isomorphism, it must be injective. Consequently, $\xi(\Phi_C) = 0$ and by exactness of the top row, $\Phi_C \in \ker \xi = \text{im } \beta$. So, there exists $\Phi_B \in B$ such that $\beta(\Phi_B) = \Phi_C$.

By applying the morphism h to both sides, we deduce that $(h \circ \beta)(\Phi_B) = h(\Phi_C) = 0$. However, by the commutativity of the square with corners B, C, G and H,

$$(h \circ \beta)(\Phi_B) = (\gamma \circ g)(\Phi_B) = 0.$$

This means that $g(\Phi_B) \in \ker \gamma = \text{im } \phi$, by the exactness of the bottom row. So, there exists $\Phi_F \in F$ such that $\phi(\Phi_F) = g(\Phi_B)$.

Since f is also an isomorphism, it is surjective. So, there exists $\Phi_A \in A$ such that $f(\Phi_A) = \Phi_F$ and by commutativity of the square with corners A, B, F and G,

$$(\phi \circ f)(\Phi_A) = (g \circ \alpha)(\Phi_A) = g(\Phi_B).$$

Since g is injective, $\alpha(\Phi_A) = \Phi_B$. By exactness of the top row, $\Phi_B \in \text{im } \alpha = \ker \beta$. Therefore,

$$\beta(\Phi_B) = \Phi_C = 0.$$

So, $\Phi_C = 0$ and consequently, h is injective.

(b) Consider the three commutative squares on the right:

$$\begin{array}{ccccccc}
B & \xrightarrow{\beta} & C & \xrightarrow{\xi} & D & \xrightarrow{\delta} & E \\
\Big\downarrow{\scriptstyle g} & & \Big\downarrow{\scriptstyle h} & & \Big\downarrow{\scriptstyle k} & & \Big\downarrow{\scriptstyle l} \\
G & \xrightarrow{\gamma} & H & \xrightarrow{\eta} & I & \xrightarrow{\iota} & J
\end{array}$$

Assume that $\lambda_H \in H$. Since k is an isomorphism, it must be surjective. So, there exists $\lambda_D \in D$ such that $k(\lambda_D) = \eta(\lambda_H)$.

Next, we use the commutativity of the rightmost square (the square with corners D, E, I and J). In particular,

$$(\iota \circ k)(\lambda_D) = (l \circ \delta)(\lambda_D).$$

By exactness of the bottom row, $\text{im } \eta = \ker \iota$. This means that

$$0 = (\iota \circ \eta)(\lambda_H) = (\iota \circ k)(\lambda_D) = (l \circ \delta)(\lambda_D).$$

Since l is an isomorphism, it must be injective. Since $l(\delta(\lambda_D)) = 0$, $\delta(\lambda_D) = 0$. By exactness again, $\lambda_D \in \ker \delta = \text{im } \xi$. Thus, there exists $\lambda_C \in C$ such that $\xi(\lambda_C) = \lambda_D$.

Using commutativity of the middle square (with corners C, D, H and I), we have

$$(k \circ \xi)(\lambda_C) = (\eta \circ h)(\lambda_C)$$

and

$$k(\lambda_D) = \eta(\lambda_H) = (\eta \circ h)(\lambda_C).$$

Since η is a group morphism, $\eta(\lambda_H - h(\lambda_C)) = 0$. By exactness of the bottom row, $\lambda_H - h(\lambda_C) \in \ker \eta = \text{im } \gamma$. Thus, there exists $\lambda_G \in G$ such that

$$\gamma(\lambda_G) = \lambda_H - h(\lambda_C).$$

Since g is an isomorphism, it must be surjective. Thus, there exists $\lambda_B \in B$ such that $g(\lambda_B) = \lambda_G$. By using the commutativity of the square with corners B, C, G and H, we have

$$(\gamma \circ g)(\lambda_B) = \gamma(\lambda_G) = (h \circ \beta)(\lambda_B).$$

So,

$$(h \circ \beta)(\lambda_B) = \gamma(\lambda_G) = \lambda_H - h(\lambda_C)$$

Consequently,

$$\lambda_H = h(\beta(\lambda_B) + \lambda_C)$$

and h must be surjective.

Parts (a) and (b) together demonstrate that h is a bijective group morphism. Therefore, h is an isomorphism as required. $\square$

3.3 Observation A

In the following sections, we will introduce the observations about the categories **Mon**, **CoM**, **Grp** and **Ab** that we want to generalise. Observation A concerns the category of monoids **Mon**.

Example 3.3.1 (Observation A)**.** Let $M, N \in \mathbf{Mon}$ be monoids. If $(x, y) \in M \times N$ then

$$(x, y) = (x, e_N) \cdot (e_M, y) = (e_M, y) \cdot (x, e_N)$$

where e_M and e_N are the identity elements of M and N respectively and $\cdot$ is the monoid operation on the product $M \times N$.

We want to rephrase observation A with category theory. Observation A introduces the pair of monomorphisms in **Mon**.

$$(M, \cdot) \xrightarrow{j_M = (id_M, 0_{M,N})} (M, \cdot) \times (N, \cdot) \xleftarrow{j_N = (0_{N,M}, id_N)} (N, \cdot)$$

Recall that j_M and j_N are the sections constructed in Lemma 3.2.6. Observation A tells us that the monoid $M \times N$ is generated by the submonoids $j_M(M)$ and $j_N(N)$. Alternatively the only submonoid of $M \times N$ which contains $j_M(M)$ and $j_N(N)$ is $M \times N$ itself.

This conclusion is rewritten in [Bou17] as follows: any monomorphism $m : L \to X$ in **Mon** which produces the following factorisations

$$\begin{tikzcd} & L \arrow[d, "m"] & \\ M \arrow[ur, dashed] \arrow[r, "j_M"'] & M \times N & N \arrow[ul, dashed] \arrow[l, "j_N"] \end{tikzcd}$$

is necessarily an isomorphism. This motivates the definition we will now make.

Definition 3.3.2. Let $\mathscr{C}$ be a category and $u : U \to X$ and $v : V \to X$ be monomorphisms in $\mathscr{C}$. We say that the pair (u, v) is a **covering pair** if any monomorphism $m : Z \to Z$ which induces the factorisations

$$\begin{tikzcd}
 & Z \arrow[d, "m"] & \\
U \arrow[ur, dashed] \arrow[r, "u"'] & X & V \arrow[l, "v"] \arrow[ul, dashed]
\end{tikzcd}$$

is necessarily an isomorphism.

More generally, a pair of morphisms (f, f') with the same image X is called **jointly extremally epic** if any monomorphism $m : Z \to Z$ which induces the factorisations

$$\begin{tikzcd}
 & Z \arrow[d, "m"] & \\
Y \arrow[ur, dashed] \arrow[r, "f"'] & X & Y' \arrow[l, "f'"] \arrow[ul, dashed]
\end{tikzcd}$$

is necessarily an isomorphism.

Let us give our first example of a jointly extremally epic pair of morphisms.

Example 3.3.3. Let $\mathscr{C}$ be a category with coproducts. Let $X, Y \in \mathscr{C}$ be objects. The coproduct $X \sqcup Y$ has two canonical morphisms — the maps $\iota_X : X \to X \sqcup Y$ and $\iota_Y : Y \to X \sqcup Y$.

We claim that (ι_X, ι_Y) is jointly extremally epic. Suppose that $m : Z \to X \sqcup Y$ is a monomorphism which induces the following factorisations in $\mathscr{C}$:

$$\begin{tikzcd}
 & Z \arrow[d, "m"] & \\
X \arrow[ur, dashed, "\phi_X"] \arrow[r, "\iota_X"'] & X \sqcup Y & Y \arrow[l, "\iota_Y"] \arrow[ul, dashed, "\phi_Y"']
\end{tikzcd}$$

To see that m is an isomorphism, we use the universal property of the coproduct to obtain the unique morphism $\alpha : X \sqcup Y \to Z$ such that the following diagram in $\mathscr{C}$ commutes:

$$\begin{tikzcd}
 & X \sqcup Y \arrow[d, dashed, "\alpha"] & \\
X \arrow[ur, "\iota_X"] \arrow[r, "\phi_X"'] & Z & Y \arrow[l, "\phi_Y"] \arrow[ul, "\iota_Y"']
\end{tikzcd}$$

By commutativity of the two diagrams, we have

$$\iota_Y = m \circ \phi_Y = m \circ (\alpha \circ \iota_Y) = (m \circ \alpha) \circ \iota_Y.$$

Similarly, $\iota_X = (m \circ \alpha) \circ \iota_X$. Note that $m \circ \alpha$ makes the following diagram commute:

$$\begin{tikzcd}
 & X \sqcup Y \arrow[d, dashed, "m\circ\alpha"] & \\
X \arrow[ur, "\iota_X"] \arrow[r, "\iota_X"'] & X \sqcup Y & Y \arrow[ul, "\iota_Y"'] \arrow[l, "\iota_Y"]
\end{tikzcd}$$

However, the identity map $id_{X \sqcup Y}$ also makes the above diagram commute. By uniqueness, $m \circ \alpha = id_{X \sqcup Y}$. Subsequently,

$$m \circ (\alpha \circ m) = (m \circ \alpha) \circ m = m = m \circ id_Z.$$

Since m is a monomorphism, $\alpha \circ m = id_Z$. Therefore, α is the inverse morphism for m and m is an isomorphism. So, the pair (ι_X, ι_Y) is jointly extremally epic.

Here is an important observation regarding equalizers and jointly extremally epic pairs.

Theorem 3.3.1. *Let $\mathscr{C}$ be a category with equalizers. Let $f : X \to Z$ and $f' : Y \to Z$ be morphisms in $\mathscr{C}$. Suppose that the pair (f, f') is jointly extremally epic. Then, (f, f') is jointly epic; that is, a parallel pair of morphisms $h, h' : Z \to T$ are equal if and only if they are equalized by both f and f'.*

Proof. Assume that $\mathscr{C}$ is a category with equalizers. Assume that (f, f') is a jointly extremally epic pair of morphisms. Assume that $h, h' : Z \to T$ is a parallel pair of morphisms.

To show: (a) If $h = h'$ then the pair (h, h') is equalized by both f and f'.

(b) If the pair (h, h') is equalized by both f and f' then $h = h'$.

(a) Assume that $h = h'$. Then, $h \circ f = h' \circ f$ and $h \circ f' = h' \circ f'$ so that the pair (h, h') is equalized by both f and f'.

(b) Assume that the pair (h, h') is equalized by both f and f'. So, $h \circ f = h' \circ f$ and $h \circ f' = h' \circ f'$. Suppose that $e = eq(h, h')$ is the equalizer of h and h', where e is a morphism from E to Z. Then, we have the following commutative diagram in $\mathscr{C}$:

$$\begin{tikzcd} E \arrow[r, "e"] & Z \arrow[r, shift left, "h"] \arrow[r, shift right, "h'"'] & T \end{tikzcd}$$

By the universal property of the equalizer, there exists unique morphisms $e_X : X \to E$ and $e_Y : Y \to E$ such that the following diagrams in $\mathscr{C}$ commute:

$$\begin{tikzcd} X \arrow[d, dashed, "e_X"] \arrow[dr, "f"] & & \\ E \arrow[r, "e"] & Z \arrow[r, shift left, "h"] \arrow[r, shift right, "h'"'] & T \end{tikzcd}$$

$$\begin{tikzcd} Y \arrow[d, dashed, "e_Y"] \arrow[dr, "f'"] & & \\ E \arrow[r, "e"] & Z \arrow[r, shift left, "h"] \arrow[r, shift right, "h'"'] & T \end{tikzcd}$$

These two commutative diagrams can be combined to form the commutative diagram

$$\begin{tikzcd} & Z \arrow[d, "e"] & \\ X \arrow[ur, dashed, "e_X"] \arrow[r, "f"'] & Z & Y \arrow[ul, dashed, "e_Y"'] \arrow[l, "f'"] \end{tikzcd}$$

Since e is an equalizer, it must be a monomorphism by Theorem 1.2.1. Since the pair (f, f') is jointly extremally epic, e must therefore be an isomorphism and hence, both a monomorphism and an epimorphism.

Since $h \circ e = h' \circ e$ and e is an epimorphism, $h = h'$. This completes the proof. $\square$

Using covering pairs, observation A in Example 3.3.1 can be restated as follows: Let $M, N \in \mathbf{Mon}$ be monoids. Then, the pair of morphisms (j_M, j_N) is a covering pair. Recall that $j_M = (id_M, 0_{M,N})$ and $j_N = (0_{N,M}, id_N)$.

In fact, it is easy to see that observation A holds in the category **UMg** because we did not need to use associativity of the binary operation. Let us extend this reasoning to internal categories.

Theorem 3.3.2. *Let $\mathscr{C}$ be a category and $\mathscr{D}$ be a pointed finitely complete category. Then, the functor category $\mathcal{F}(\mathscr{C}, \mathscr{D})$ is a pointed finitely complete category.*

Proof. Assume that $\mathscr{C}$ is a category and $\mathscr{D}$ is a pointed finitely complete category.

To show: (a) The functor category $\mathcal{F}(\mathscr{C}, \mathscr{D})$ is finitely complete.

(b) The functor category $\mathcal{F}(\mathscr{C}, \mathscr{D})$ is pointed.

(a) By Theorem 1.13.1, it suffices to show that $\mathcal{F}(\mathscr{C}, \mathscr{D})$ has products and equalizers.

Let $F, G : \mathscr{C} \to \mathscr{D}$ be functors (objects in $\mathcal{F}(\mathscr{C}, \mathscr{D})$). We will define the product $F \times G$. The key idea is that $\mathscr{D}$ is a finitely complete category and thus, $\mathscr{D}$ has products and equalizers. Keeping this in mind, we define

$$\begin{array}{rccc} F \times G : & \mathscr{C} & \to & \mathscr{D} \\ & C & \mapsto & F(C) \times G(C) \\ & f : C \to C' & \mapsto & F(f) \times G(f) : F(C) \times G(C) \to F(C') \times G(C') \end{array}$$

It is straightforward to check that $F \times G$ is a functor (because F and G are both functors). We will show that $F \times G$ satisfies the universal property of the product. First, we define for an object $A \in \mathscr{C}$, the maps $\pi_F : F \times G \to F$ and $\pi_G : F \times G \to G$ by

$$\begin{array}{rccc} (\pi_F)_A : & F(A) \times G(A) & \to & F(A) \\ & (\alpha, \beta) & \mapsto & \alpha \end{array}$$

and

$$\begin{array}{rccc} (\pi_G)_A : & F(A) \times G(A) & \to & G(A) \\ & (\alpha, \beta) & \mapsto & \beta \end{array}$$

To show: (aa) π_F and π_G are natural transformations.

(aa) Assume that $f : A \to A'$ is a morphism in $\mathscr{C}$. We must show that $F(f) \circ (\pi_F)_A = (\pi_F)_{A'} \circ (F \times G)(f)$. From the definition of $F \times G$, we have for $(\alpha, \beta) \in F(A) \times G(A)$,

$$\begin{aligned}((\pi_F)_{A'} \circ (F \times G)(f))(\alpha, \beta) &= (\pi_F)_{A'} \circ (F(f) \times G(f))(\alpha, \beta)\\ &= (\pi_F)_{A'}(F(f)(\alpha), G(f)(\beta))\\ &= F(f)(\alpha)\\ &= (F(f) \circ (\pi_F)_A)(\alpha, \beta).\end{aligned}$$

So, $F(f) \circ (\pi_F)_A = (\pi_F)_{A'} \circ (F \times G)(f)$ and consequently, π_F is a natural transformation. In a similar fashion, one can show that $G(f) \circ (\pi_G)_A = (\pi_G)_{A'} \circ (F \times G)(f)$. So, π_G is also a natural transformation.

(a) Suppose that $\sigma_F : H \to F$ and $\sigma_G : H \to G$ are morphisms in the functor category $\mathcal{F}(\mathscr{C}, \mathscr{D})$ (natural transformations). Define the map $\theta : H \to F \times G$ for an object $A \in \mathscr{C}$ by

$$\begin{array}{rccc}\theta_A : & H(A) & \to & F(A) \times G(A)\\ & h & \mapsto & ((\sigma_F)_A(h), (\sigma_G)_A(h))\end{array}$$

To show: (ab) θ is a natural transformation.

(ab) Again, let $f : A \to A'$ be a morphism in $\mathscr{C}$. We compute directly for $h \in H(A)$ that

$$\begin{aligned}(\theta_{A'} \circ H(f))(h) &= \theta_{A'}(H(f)(h))\\ &= ((\sigma_F)_{A'}(H(f)(h)), (\sigma_G)_{A'}(H(f)(h)))\\ &= (((\sigma_F)_{A'} \circ H(f))(h), ((\sigma_G)_{A'} \circ H(f))(h))\\ &= ((F(f) \circ (\sigma_F)_A)(h), (G(f) \circ (\sigma_G)_A)(h))\\ &= (F \times G)(f)((\sigma_F)_A(h), (\sigma_G)_A(h))\\ &= ((F \times G)(f) \circ \theta_A)(h)\end{aligned}$$

In the third last equality, we used the assumption that σ_F and σ_G are natural transformations. Hence, $\theta_{A'} \circ H(f) = (F \times G)(f) \circ \theta_A$ and $\theta : H \to F \times G$ is a natural transformation.

(a) From the definitions, it is easy to check for $A \in \mathscr{C}$ that $(\pi_F)_A \circ \theta_A = (\sigma_F)_A$ and $(\pi_G)_A \circ \theta_A = (\sigma_G)_A$. Hence, the following diagram in $\mathcal{F}(\mathscr{C}, \mathscr{D})$ commutes:

$$\begin{tikzcd} & F \times G \arrow[dl, "\pi_F"'] \arrow[dr, "\pi_G"] & \\ F & H \arrow[l, "\sigma_F"] \arrow[r, "\sigma_G"'] \arrow[u, dashed, "\theta"] & G \end{tikzcd}$$

It is also simple to check that $\theta : H \to F \times G$ is the unique morphism in $\mathcal{F}(\mathscr{C}, \mathscr{D})$ which makes the above diagram commute. Hence, we have successfully defined the product $F \times G$ in $\mathcal{F}(\mathscr{C}, \mathscr{D})$.

Now, we will define equalizers in $\mathcal{F}(\mathscr{C}, \mathscr{D})$. Suppose that $F, G : \mathscr{C} \to \mathscr{D}$ are functors and $\gamma, \delta : F \to G$ is a pair of natural transformations. If $A \in \mathscr{C}$ is an object then we obtain a pair of morphisms in $\mathscr{D}$:

$$\begin{tikzcd} F(A) \arrow[r, shift left, "\gamma_A"] \arrow[r, shift right, "\delta_A"'] & G(A) \end{tikzcd}$$

Since $\mathscr{D}$ is a finitely complete category, it has equalizers. So, we form the equalizer of the pair of morphisms (γ_A, δ_A):

$$\begin{tikzcd} E(A) \arrow[r, "\epsilon_A"] & F(A) \arrow[r, shift left, "\gamma_A"] \arrow[r, shift right, "\delta_A"'] & G(A) \end{tikzcd}$$

Let $f : A \to A'$ be a morphism in $\mathscr{C}$. We want to define a corresponding morphism $E(f) : E(A) \to E(A')$ in $\mathscr{D}$. We claim that the composite

$$E(A) \xrightarrow{\epsilon_A} F(A) \xrightarrow{F(f)} F(A')$$

equalizes the pair of morphisms $(\gamma_{A'}, \delta_{A'})$. We compute directly that

$$\begin{aligned} \gamma_{A'} \circ (F(f) \circ \epsilon_A) &= (\gamma_{A'} \circ F(f)) \circ \epsilon_A \\ &= (G(f) \circ \gamma_A) \circ \epsilon_A \\ &= G(f) \circ (\gamma_A \circ \epsilon_A) \\ &= G(f) \circ (\delta_A \circ \epsilon_A) \\ &= \delta_{A'} \circ (F(f) \circ \epsilon_A). \end{aligned}$$

By the universal property of the equalizer, there exists a unique morphism $E(f) : E(A) \to E(A')$ such that the following diagram in $\mathscr{D}$ commutes:

$$\begin{tikzcd} E(A) \arrow[d, dashed, "E(f)"'] \arrow[dr, "F(f)\circ\epsilon_A"] & & \\ E(A') \arrow[r, "\epsilon_{A'}"] & F(A') \arrow[r, shift left, "\gamma_{A'}"] \arrow[r, shift right, "\delta_{A'}"'] & G(A') \end{tikzcd}$$

Now, we claim that $E : \mathscr{C} \to \mathscr{D}$ is a functor and $\epsilon : E \to F$ is a natural transformation.

To show: (ac) $E : \mathscr{C} \to \mathscr{D}$ is a functor.

(ad) $\epsilon : E \to F$ is a natural transformation.

(ac) Suppose that $A \in \mathscr{C}$ and $id_A : A \to A$ is the identity morphism on A. Then,

$$\epsilon_A \circ E(id_A) = F(id_A) \circ \epsilon_A = id_{F(A)} \circ \epsilon_A = \epsilon_A$$

Note also that $\epsilon_A \circ id_{E(A)} = \epsilon_A = F(id_A) \circ \epsilon_A$. By uniqueness, we must have $E(id_A) = id_{E(A)}$.

Next, assume that $p : A \to B$ and $q : B \to C$ are morphisms in $\mathscr{C}$. Then,

$$\epsilon_C \circ E(q \circ p) = F(q \circ p) \circ \epsilon_A = F(q) \circ (F(p) \circ \epsilon_A) = F(q) \circ (\epsilon_B \circ E(p)) = \epsilon_C \circ (E(q) \circ E(p)).$$

By uniqueness, we find that $E(q \circ p) = E(q) \circ E(p)$. Therefore, E is a functor and thus, an object in $\mathcal{F}(\mathscr{C}, \mathscr{D})$.

(ad) By construction, if $f : A \to A'$ is a morphism in $\mathscr{C}$ then $\epsilon_{A'} \circ E(f) = F(f) \circ \epsilon_A$. Therefore, $\epsilon : E \to F$ is a natural transformation — a morphism in $\mathcal{F}(\mathscr{C}, \mathscr{D})$.

(a) Next, we will show that $\epsilon : E \to F$ is the equalizer of the pair (γ, δ) in the functor category $\mathcal{F}(\mathscr{C}, \mathscr{D})$. Suppose that $\chi : H \to F$ is a natural transformation which satisfies $\gamma \circ \chi = \delta \circ \chi$. If $A \in \mathscr{C}$ is an object then by the universal property of the equalizer, there exists a unique morphism $\xi_A : H(A) \to E(A)$ in $\mathscr{D}$ such that the following diagram commutes:

$$\begin{tikzcd}
H(A) \arrow[d, dashed, "\xi_A"'] \arrow[dr, "\chi_A"] & & \\
E(A) \arrow[r, "\epsilon_A"] & F(A) \arrow[r, shift left, "\gamma_A"] \arrow[r, shift right, "\delta_A"'] & G(A)
\end{tikzcd}$$

Since $\epsilon_A \circ \xi_A = \chi_A$ for an arbitrary object $A \in \mathscr{C}$, $\epsilon \circ \xi = \chi$. It remains to show that ξ is a natural transformation.

We compute directly for a morphism $f : A \to A'$ in $\mathscr{C}$ that

$$\begin{aligned}\epsilon_{A'} \circ (\xi_{A'} \circ H(f)) &= (\epsilon_{A'} \circ \xi_{A'}) \circ H(f) \\ &= \chi_{A'} \circ H(f) \\ &= F(f) \circ \chi_A.\end{aligned}$$

Also,

$$\begin{aligned}\epsilon_{A'} \circ (E(f) \circ \xi_A) &= (\epsilon_{A'} \circ E(f)) \circ \xi_A \\ &= (F(f) \circ \epsilon_A) \circ \xi_A \\ &= F(f) \circ \chi_A.\end{aligned}$$

So, both $E(f) \circ \xi_A$ and $\xi_{A'} \circ H(f)$ make the following diagram in $\mathscr{D}$ commute:

$$\begin{tikzcd} H(A) \arrow[d, dashed, "E(f)\circ\xi_A"'] \arrow[dr, "F(f)\circ\chi_A"] & & \\ E(A') \arrow[r, "\epsilon_{A'}"] & F(A') \arrow[r, shift left, "\gamma_{A'}"] \arrow[r, shift right, "\delta_{A'}"'] & G(A') \end{tikzcd}$$

Once again by uniqueness, we must have $E(f) \circ \xi_A = \xi_{A'} \circ H(f)$. So, ξ is a natural transformation and consequently, ϵ is the equalizer of γ and δ in $\mathcal{F}(\mathscr{C}, \mathscr{D})$.

Since $\mathcal{F}(\mathscr{C}, \mathscr{D})$ has products and equalizers, we can use Theorem 1.13.1 to show that $\mathcal{F}(\mathscr{C}, \mathscr{D})$ is a finitely complete category.

(b) Let $*$ be the zero object in $\mathscr{D}$. We want to define a zero object in the functor category $\mathcal{F}(\mathscr{C}, \mathscr{D})$. Define the functor

$$\begin{array}{cccc} 0: & \mathscr{C} & \to & \mathscr{D} \\ & C & \mapsto & * \\ & f: C \to C' & \mapsto & id_* \end{array}$$

where $id_* : * \to *$ is the identity morphism on the terminal object $* \in \mathscr{D}$. Now let $H : \mathscr{C} \to \mathscr{D}$ be a functor. We will construct unique natural transformations $\tau_H : H \to 0$ and $\alpha_H : 0 \to H$.

Since $*$ is the terminal object in $\mathscr{D}$, for any object $A \in \mathscr{C}$, there exists a unique morphism $(\tau_H)_A : H(A) \to *$. To see that τ_H is a natural transformation, if $f : A \to A'$ is a morphism in $\mathscr{C}$ and $h \in H(A)$ then

$$
\begin{aligned}
(0(f) \circ (\tau_H)_A)(h) &= 0(f)(*) \\
&= id_*(*) = * \\
&= (\tau_H)_{A'}(H(f)(h)) = ((\tau_H)_{A'} \circ H(f))(h).
\end{aligned}
$$

Hence, τ_H is a unique natural transformation from H to 0.

Since $*$ is the zero object in $\mathscr{D}$, for any $A \in \mathscr{C}$, there exists a unique morphism $(\alpha_H)_A : 0(A) \to H(A)$. A similar argument as before reveals that α_H is also a natural transformation.

Therefore, $0 \in \mathcal{F}(\mathscr{C}, \mathscr{D})$ is a zero object.

By combining parts (a) and (b), we find that the functor category $\mathcal{F}(\mathscr{C}, \mathscr{D})$ is pointed and finitely complete. $\square$

The product of two functors, defined in Theorem 3.3.2, will play an important role in the next result.

Theorem 3.3.3. *Let $\mathscr{C}$ be a category. Then, the functor category $\mathcal{F}(\mathscr{C}, \boldsymbol{UMg})$ is pointed and finitely complete by Theorem 3.3.2. Let $H, K : \mathscr{C} \to \boldsymbol{UMg}$ be two functors and $H \times K$ be the product functor defined in Theorem 3.3.2. Let $\pi_H : H \times K \to H$ and $\pi_K : H \times K \to K$ be the natural transformations defined in Theorem 3.3.2. Let $j_H : H \to H \times K$ and $j_K : K \to H \times K$ be the unique sections constructed in Lemma 3.2.6. Then, (j_H, j_K) is a covering pair.*

Proof. Assume that $\mathscr{C}$ is a category. Suppose that $H, K : \mathscr{C} \to \mathbf{UMg}$ are functors and $H \times K$ is the product functor. Assume that $j_H : H \to H \times K$ and $j_K : K \to H \times K$ are the unique sections constructed in Lemma 3.2.6. Recall that $j_H = (id_H, 0_{H,K})$ and $j_K = (0_{K,H}, id_K)$.

Suppose that we have the following factorisation in $\mathcal{F}(\mathscr{C}, \mathbf{UMg})$:

$$
\begin{tikzcd}
& L \arrow[d, "m"] & \\
H \arrow[ur, dashed] \arrow[r, "j_H"'] & H \times K & K \arrow[ul, dashed] \arrow[l, "j_K"]
\end{tikzcd}
$$

where $m : L \to H \times K$ is a monomorphism.

To show: (a) $m : L \to H \times K$ is a natural isomorphism.

(a) Notice that if $A \in \mathscr{C}$ is an object then we have the following factorisation in **UMg**:

$$\begin{tikzcd}
 & & L(A) \arrow[d, "m_A"] & & \\
H(A) \arrow[urr, dashed] \arrow[r, "(j_H)_A"'] & & H(A) \times K(A) & & K(A) \arrow[ull, dashed] \arrow[ll, "(j_K)_A"]
\end{tikzcd}$$

Since m is a monomorphism, the morphism $m_A : L(A) \to H(A) \times K(A)$ in **UMg** is a monomorphism. Since $((j_H)_A, (j_K)_A)$ is a covering pair for $H(A) \times K(A)$, m_A must be an isomorphism. Since the object $A \in \mathscr{C}$ was arbitrary, we find that m must be a natural isomorphism as required

Hence, (j_H, j_K) is a covering pair for $H \times K$. □

Observe that in Theorem 3.3.3, we can freely replace the category **UMg** with **Mon**, **CoM**, **Ab** or **Grp** and obtain the same result. We can do this because observation A in Example 3.3.1 also applies to the pointed, finitely complete categories **UMg**, **CoM**, **Ab** and **Grp**.

Theorem 3.3.4. *Let $\mathscr{C}$ be a finitely complete category. By Theorem 3.1.1, **UMg**($\mathscr{C}$) is a pointed finitely complete category. Let (M, N) be a pair of internal unitary magmas in $\mathscr{C}$. Let $j_M : M \to M \times N$ and $j_N : N \to M \times N$ be the morphisms defined in Lemma 3.2.6. Then, (j_M, j_N) is a covering pair for the product $M \times N$.*

Proof. Assume that $\mathscr{C}$ is a finitely complete category. Recall the structure embedding functor $\overline{Y}_{UMg} : \mathbf{UMg}(\mathscr{C}) \to \mathcal{F}(\mathscr{C}^{op}, \mathbf{UMg})$, which is fully faithful and left exact.

Let M and N be internal unitary magmas in $\mathscr{C}$. Assume that $j_M : M \to M \times N$ and $j_N : N \to M \times N$ are the morphisms defined in Lemma 3.2.6. Suppose that we have the following factorisation in the category **UMg**($\mathscr{C}$):

$$\begin{tikzcd}
 & L \arrow[d, "m"] & \\
M \arrow[ur, dashed] \arrow[r, "j_M"'] & M \times N & N \arrow[ul, dashed] \arrow[l, "j_N"]
\end{tikzcd}$$

Here, $m : L \to M \times N$ is a monomorphism. Notice that $\overline{Y}_{UMg}(j_M) = j_{\overline{Y}_{UMg}(M)}$. By Theorem 3.3.3 and the fact that $\overline{Y}_{UMg}$ is left exact, $(j_{\overline{Y}_{UMg}(M)}, j_{\overline{Y}_{UMg}(N)})$ is a covering pair for $\overline{Y}_{UMg}(M) \times \overline{Y}_{UMg}(N)$ and $\overline{Y}_{UMg}(m)$ is a monomorphism in $\mathcal{F}(\mathscr{C}^{op}, \mathbf{UMg})$. By Theorem 3.3.3, $\overline{Y}_{UMg}(m)$ must be an isomorphism in $\mathcal{F}(\mathscr{C}^{op}, \mathbf{UMg})$.

Since the functor $\overline{Y}_{UMg}$ is fully faithful, m must be an isomorphism in $\mathbf{UMg}(\mathscr{C})$. Consequently, (j_M, j_N) is a covering pair for $M \times N$. □

Again, Theorem 3.3.4 still holds when we replace the internal category $\mathbf{UMg}(\mathscr{C})$ with $\mathbf{Mon}(\mathscr{C}), \mathbf{CoM}(\mathscr{C}), \mathbf{Ab}(\mathscr{C})$ or $\mathbf{Grp}(\mathscr{C})$.

3.4 Observations B and B'

Observations B and B' concern the category of commutative monoids **CoM**. Let us first deal with observation B.

Example 3.4.1 (Observation B)**.** Let $(M, +)$ be a commutative monoid. Then, the binary operation

$$\begin{array}{rccc} + : & M \times M & \to & M \\ & (m_1, m_2) & \mapsto & m_1 + m_2 \end{array}$$

is a morphism of commutative monoids. To see why this is the case, assume that $(m_1, m_2), (n_1, n_2) \in M \times M$. Using the commutativity of M, we compute directly that

$$\begin{aligned} +\big((m_1, m_2) + (n_1, n_2)\big) &= +(m_1 + n_1, m_2 + n_2) \\ &= (m_1 + n_1) + (m_2 + n_2) \\ &= (m_1 + m_2) + (n_1 + n_2) \\ &= +(m_1, m_2) + +(n_1, n_2). \end{aligned}$$

We want to emphasise that commutativity was used in the third equality in the above computation.

Now, let $f, f' : N \to M$ be a parallel pair of morphisms in **CoM**. Consider the composite

$$N \xrightarrow{(f,f')} M \times M \xrightarrow{\;+\;} M$$

This is a composite of commutative monoid morphisms and is thus, a commutative monoid morphism. This tells us that the sum $f + f'$, which maps $n \in N$ to $f(n) + f'(n) \in M$, is itself a morphism in **CoM**. This does not happen in **Mon**.

The key to rephrasing observation B with category theory is that the product of two commutative monoids $M \times N$ doubles as a coproduct.

Theorem 3.4.1. *Let M and N be commutative monoids. Let $j_M : M \to M \times N$ and $j_N : N \to M \times N$ denote the morphisms constructed in Lemma 3.2.6. Recall that $j_M = (id_M, 0_{M,N})$ and $j_N = (0_{N,M}, id_N)$. Then, the triple $(M \times N, j_M, j_N)$ is a coproduct in the category of commutative monoids **CoM**.*

Proof. Assume that M and N are commutative monoids. Assume that j_M and j_N are the morphisms defined above. Assume that $f : M \to L$ and $g : N \to L$ are morphisms in **CoM**.

The map

$$\begin{array}{rccc} \phi : & M \times N & \to & L \\ & (m, n) & \mapsto & f(m) + g(n) \end{array}$$

is a morphism in **CoM** (see Example 3.4.1) which makes the following diagram in **CoM** commute:

$$\begin{tikzcd} & M \times N \arrow[d, dashed, "\phi"] & \\ M \arrow[ur, "j_M"] \arrow[r, "f"'] & L & N \arrow[ul, "j_N"'] \arrow[l, "g"] \end{tikzcd}$$

It remains to show that ϕ is unique. Suppose that there exists another morphism $\psi : M \times N \to L$ such that $f = \psi \circ j_M$ and $g = \psi \circ j_N$. By Theorem 3.3.3, (j_M, j_N) is a covering pair. Since **CoM** has equalizers, we can apply Theorem 3.3.1 to deduce that the pair (j_M, j_N) is jointly epic.

Now, since the morphisms ψ and ϕ are both equalized by j_M and j_N, $\phi = \psi$ because (j_M, j_N) is jointly epic. Hence, ϕ must be unique. $\square$

In a similar vein to Theorem 3.3.3, we can devise a similar argument and use Theorem 3.4.1 to prove the following theorem.

Theorem 3.4.2. *Let $\mathscr{C}$ be a category. Then, the functor category $\mathcal{F}(\mathscr{C}, \boldsymbol{CoM})$ is pointed and finitely complete by Theorem 3.3.2. Let $H, K : \mathscr{C} \to \boldsymbol{CoM}$ be two functors and $H \times K$ be the product functor defined in Theorem 3.3.2. Let $j_H : H \to H \times K$ and $j_K : K \to H \times K$ be the unique morphisms constructed in Lemma 3.2.6. Then, $(H \times K, j_H, j_K)$ is a coproduct in the functor category $\mathcal{F}(\mathscr{C}, \boldsymbol{CoM})$.*

Finally, we can use Theorem 3.4.2 to extend observation B to internal categories.

Theorem 3.4.3. *Let $\mathscr{C}$ be a finitely complete category. We know that $\boldsymbol{CoM}(\mathscr{C})$ is a pointed finitely complete category. Let (M, N) be a pair of internal commutative monoids in $\mathscr{C}$. Let $j_M : M \to M \times N$ and $j_N : N \to M \times N$ be the morphisms defined in Lemma 3.2.6. Then, $(M \times N, j_M, j_N)$ is a coproduct in $\boldsymbol{CoM}(\mathscr{C})$.*

Proof. Assume that $\mathscr{C}$ is a finitely complete category. Recall the structure embedding functor $\overline{Y}_{CoM} : \mathbf{CoM}(\mathscr{C}) \to \mathcal{F}(\mathscr{C}^{op}, \mathbf{CoM})$, which is fully faithful and left exact.

Let M and N be internal commutative monoids in $\mathscr{C}$. Assume that $j_M : M \to M \times N$ and $j_N : N \to M \times N$ are the morphisms defined in Lemma 3.2.6. Suppose that we have the following diagram in the category $\mathbf{CoM}(\mathscr{C})$:

$$\begin{tikzcd} & M \times N & \\ M \arrow[ur, "j_M"] \arrow[r, "f"'] & L & N \arrow[ul, "j_N"'] \arrow[l, "g"] \end{tikzcd}$$

Since $\overline{Y}_{CoM}$ is left exact,

$$\overline{Y}_{CoM}(M \times N) = \overline{Y}_{CoM}(M) \times \overline{Y}_{CoM}(N).$$

Notice that $\overline{Y}_{CoM}(j_M) = j_{\overline{Y}_{CoM}(M)}$. By Theorem 3.4.2, there exists a unique morphism ψ such that

$$\psi \circ j_{\overline{Y}_{CoM}(M)} = \overline{Y}_{CoM}(f) \qquad \text{and} \qquad \psi \circ j_{\overline{Y}_{CoM}(N)} = \overline{Y}_{CoM}(g).$$

Since the functor $\overline{Y}_{CoM}$ is fully faithful, there exists a unique morphism ϕ in $\mathbf{CoM}(\mathscr{C})$ such that the following diagram commutes:

$$\begin{tikzcd}
 & M \times N \arrow[d, dashed, "\phi"] & \\
M \arrow[ur, "j_M"] \arrow[r, "f"'] & L & N \arrow[ul, "j_N"'] \arrow[l, "g"]
\end{tikzcd}$$

Hence, $(M \times N, j_M, j_N)$ is a coproduct in $\mathbf{CoM}(\mathscr{C})$. $\square$

Note that Theorem 3.4.3 also applies to the internal category $\mathbf{Ab}(\mathscr{C})$.

We turn to observation B'. We have already encountered observation B' — the Eckmann-Hilton argument in Theorem 2.1.1. In particular, the category $\mathbf{UMg}(\mathbf{UMg})$ coincides with $\mathbf{CoM}$. It is worth going through the proof of Theorem 2.1.1 again because we will generalise it to internal categories.

Theorem 3.4.4 (Observation B')**.** *Let $\mathscr{C}$ be a finitely complete category. Let (M, m, e) be an internal unitary magma in $\boldsymbol{UMg}(\boldsymbol{UMg}(\mathscr{C}))$. Then, (M, m, e) is a commutative monoid. Consequently, the category $\boldsymbol{UMg}(\boldsymbol{UMg}(\mathscr{C})) = \boldsymbol{CoM}(\mathscr{C})$.*

Proof. Assume that $\mathscr{C}$ is a finitely complete category and (M, m, e) is an internal unitary magma, which is an object in the category $\mathbf{UMg}(\mathbf{UMg}(\mathscr{C}))$. By Theorem 2.1.4, $Hom_{\mathscr{C}}(X, M)$ is a unitary magma for an object $X \in \mathscr{C}$.

Let $n : (M, m, e) \times (M, m, e) \to (M, m, e)$ be an internal binary operation with unit in $\mathbf{UMg}(\mathscr{C})$. Then, the map

$$\begin{array}{llcl} Hom_{\mathscr{C}}(X, n): & Hom_{\mathscr{C}}(X, M) \times Hom_{\mathscr{C}}(X, M) & \to & Hom_{\mathscr{C}}(X, M) \\ & (f, g) & \mapsto & Hom_{\mathscr{C}}(X, n)(f, g)(x) = n(f(x), g(x)) \end{array}$$

defines an internal binary operation on the unitary magma $Hom_{\mathscr{C}}(X, M)$, giving $Hom_{\mathscr{C}}(X, M)$ an internal unitary magma structure. By the Eckmann-Hilton argument in Theorem 2.1.1 and the original binary operation μ on $Hom_{\mathscr{C}}(X, M)$ in Theorem 2.1.4, we deduce that if $f, f' \in Hom_{\mathscr{C}}(X, M)$ and $x \in X$ then

$$m(f(x), f'(x)) = n(f(x), f'(x)).$$

Let $\pi_1 : M \times M \to M$ be the projection morphism onto the first factor and π_2 be the projection morphism onto the second factor. If $(m_1, m_2) \in M \times M$ then

$$
\begin{aligned}
m(m_1, m_2) &= m(\pi_1(m_1, m_2), \pi_2(m_1, m_2)) \\
&= n(\pi_1(m_1, m_2), \pi_2(m_1, m_2)) = n(m_1, m_2).
\end{aligned}
$$

Therefore, $m = n$. Moreover, the Eckmann-Hilton argument applied to the unitary magma $Hom_{\mathscr{C}}(X, M)$ shows that $Hom_{\mathscr{C}}(X, M)$ is a commutative monoid. By Theorem 2.1.5, we deduce that (M, m, e) is an internal commutative monoid in $\mathscr{C}$ as required. $\square$

3.5 Observation C

Our next observation concerns split epimorphisms in the category **Ab**.

Example 3.5.1 (Observation C)**.** In the category of abelian groups **Ab**, let $f : A \to B$ be a split epimorphism. Then, there exists a section $s : B \to A$ such that $f \circ s = id_B$, where id_B is the identity element on B.

We will use addition to represent the binary operation on A. If $a \in A$ then

$$a = s(f(a)) + (a - s(f(a))).$$

where $s(f(a)) \in s(B)$ and $a - s(f(a)) \in \ker f$. We claim that the above representation of a is unique.

Suppose that $a = s(b) + k$, where $b \in B$ and $k \in \ker f$. Then,

$$s(f(a)) - s(b) = k + a - s(f(a)).$$

Applying f to both sides, we find that

$$f(a) - b = 0$$

where $0 \in A$ is the identity element of A. Therefore, $f(a) = b$. Now, $a = s(f(a)) + (a - s(f(a))) = s(f(a)) + k$. So, $k = a - s(f(a))$. Hence, the decomposition

$$a = s(f(a)) + (a - s(f(a)))$$

is unique. This means that A is the direct sum $A = s(B) \oplus \ker f$.

Observation C in Example 3.5.1 tells us that the morphism in **Ab**

$$\begin{array}{rccc} \phi: & B \times \ker f & \to & A \\ & (b,k) & \mapsto & s(b)+k \end{array} \tag{3.1}$$

is an isomorphism. Let us construct this morphism explicitly.

First, let $j_B : B \to B \times \ker f$ and $j_K : \ker f \to B \times \ker f$ be the morphisms

$$j_B = (id_B, 0_{B,\ker f}) \qquad \text{and} \qquad j_K = (0_{\ker f,B}, id_{\ker f})$$

which were explicitly constructed in Lemma 3.2.6. Since **Ab** is a subcategory of **CoM**, Theorem 3.4.1 tells us that the triple $(B \times \ker f, j_B, j_K)$ is a coproduct in **CoM** and hence, a coproduct in **Ab**. Thus, there exists a unique morphism $\phi : B \times \ker f \to A$ such that the following diagram in **Ab** commutes:

$$\begin{tikzcd}
& B \times \ker f \arrow[d, dashed, "\phi"] & \\
B \arrow[ur, "j_B"] \arrow[r, "s"'] & A & \ker f \arrow[ul, "j_K"'] \arrow[l, "k_f"]
\end{tikzcd}$$

One can check by the commutativity of the above diagram that ϕ is the morphism given in equation (3.1).

Because ϕ is an isomorphism according to Observation C, we obtain a particular coproduct in **Ab**.

Theorem 3.5.1. *Let $f : A \to B$ be a split epimorphism in the category **Ab** so that there exists a morphism $s : B \to A$ such that $f \circ s = id_B$. Let $k_f : \ker f \to A$ be the kernel of f. Then, the triple (A, s, k_f) is a coproduct in **Ab**.*

Proof. Assume that $f : A \to B$ is a split epimorphism in the category **Ab**. Then, there exists a morphism $s : B \to A$ such that $f \circ s = id_B$. Assume that $k_f : \ker f \to A$ is the kernel of f.

Assume that $\alpha_1 : B \to C$ and $\alpha_2 : \ker f \to C$ are two morphisms in **Ab**. Define the morphisms $j_B : B \to B \times \ker f$ and $j_K : \ker f \to B \times \ker f$ by

$$j_B = (id_B, 0_{B,\ker f}) \qquad \text{and} \qquad j_K = (0_{\ker f,B}, id_{\ker f}).$$

Recall that these maps were constructed in Lemma 3.2.6. The category **Ab** is a subcategory of **CoM**. Thus, by Theorem 3.4.1, the triple

$(B \times \ker f, j_B, j_K)$ is a coproduct in $\mathbf{Ab}$.

By invoking the universal property of the coproduct, there exists a unique group morphism $\psi : B \times \ker f \to C$ such that the following diagram in $\mathbf{Ab}$ commutes:

$$\begin{tikzcd} & B \times \ker f \arrow[d, dashed, "\psi"] & \\ B \arrow[ur, "j_B"] \arrow[r, "\alpha_1"'] & C & \ker f \arrow[ul, "j_K"'] \arrow[l, "\alpha_2"] \end{tikzcd}$$

Now let ϕ denote the isomorphism in equation (3.1). Then, the composite $\psi \circ \phi^{-1} : A \to C$ is the unique group morphism making the following diagram in $\mathbf{Ab}$ commute:

$$\begin{tikzcd} & A \arrow[d, dashed, "\psi\circ\phi^{-1}"] & \\ B \arrow[ur, "s"] \arrow[r, "\alpha_1"'] & C & \ker f \arrow[ul, "k_f"'] \arrow[l, "\alpha_2"] \end{tikzcd}$$

In the above diagram, we used the fact that in our construction of ϕ, $s = \phi \circ j_B$ and $k_f = \phi \circ j_K$. Therefore, the triple (A, s, k_f) is a coproduct in $\mathbf{Ab}$. $\square$

As with Theorem 3.4.1 and Theorem 3.3.3, we can extend Theorem 3.5.1 to the functor category $\mathcal{F}(\mathscr{C}, \mathbf{Ab})$.

Theorem 3.5.2. *Let $\mathscr{C}$ be a category. Then, the functor category $\mathcal{F}(\mathscr{C}, \boldsymbol{Ab})$ is pointed and finitely complete by Theorem 3.3.2. Let $H, K : \mathscr{C} \to \boldsymbol{Ab}$ be functors and $f : H \to K$ be a split epimorphism in $\mathcal{F}(\mathscr{C}, \boldsymbol{Ab})$ so that there exists $s : K \to H$ such that $s \circ f = id_K$. Let $k_f : \ker f \to H$ be the kernel of f. Then, the triple (H, s, k_f) is a coproduct in $\mathcal{F}(\mathscr{C}, \boldsymbol{Ab})$.*

Proof. Assume that $\mathscr{C}$ is a category. Assume that $H, K : \mathscr{C} \to \mathbf{Ab}$ are functors and $f : H \to K$ is a split epimorphism in the functor category $\mathcal{F}(\mathscr{C}, \mathbf{Ab})$. Then, there exists a morphism $s : K \to H$ such that $f \circ s = id_K$.

Assume that $k_f : \ker f \to H$ is the kernel of f. This exists because $\mathcal{F}(\mathscr{C}, \mathbf{Ab})$ is a pointed finitely complete category by Theorem 3.3.2.

Assume that $p : K \to L$ and $q : \ker f \to L$ are morphisms in $\mathcal{F}(\mathscr{C}, \mathbf{Ab})$. For any object $A \in \mathscr{C}$, we have the following diagram in $\mathbf{Ab}$:

$$\begin{tikzcd}
& H(A) & \\
K(A) \arrow[ur, "s_A"] \arrow[r, "p_A"'] & L(A) & (\ker f)(A) \arrow[ul, "(k_f)_A"'] \arrow[l, "q_A"]
\end{tikzcd}$$

By Theorem 3.5.1, there exists a unique morphism $\phi_A : H(A) \to L(A)$ such that the diagram in **Ab** commutes:

$$\begin{tikzcd}
& H(A) \arrow[d, dashed, "\phi_A"] & \\
K(A) \arrow[ur, "s_A"] \arrow[r, "p_A"'] & L(A) & (\ker f)(A) \arrow[ul, "(k_f)_A"'] \arrow[l, "q_A"]
\end{tikzcd}$$

We claim that $\phi : H \to L$ is a natural transformation. Assume that $f : A \to A'$ is a morphism in $\mathscr{C}$. We compute directly that

$$\begin{aligned}
(\phi_{A'} \circ H(f)) \circ s_A &= \phi_{A'} \circ (H(f) \circ s_A) \\
&= \phi_{A'} \circ s_{A'} \circ K(f) \\
&= p_{A'} \circ K(f) \\
&= L(f) \circ p_A \\
&= (L(f) \circ \phi_A) \circ s_A.
\end{aligned}$$

By a similar argument,

$$(\phi_{A'} \circ H(f)) \circ (k_f)_A = (L(f) \circ \phi_A) \circ (k_f)_A.$$

Now observe that $\phi_{A'} \circ H(f)$ and $L(f) \circ \phi_A$ both make the following diagram in **Ab** commute:

$$\begin{tikzcd}
& H(A) \arrow[d, dashed] & \\
K(A) \arrow[ur, "s_A"] \arrow[r, "p_{A'} \circ K(f)"'] & L(A') & (\ker f)(A) \arrow[ul, "(k_f)_A"'] \arrow[l, "q_{A'} \circ K(f)"]
\end{tikzcd}$$

By uniqueness, we must have $\phi_{A'} \circ H(f) = L(f) \circ \phi_A$. So, $\phi : H \to L$ is the unique natural transformation which makes the following diagram in $\mathcal{F}(\mathscr{C}, \mathbf{Ab})$ commute:

$$\begin{tikzcd}
& H \arrow[d, dashed, "\phi"] & \\
K \arrow[ur, "s"] \arrow[r, "p"'] & L & \ker f \arrow[ul, "k_f"'] \arrow[l, "q"]
\end{tikzcd}$$

Therefore, the triple (H, s, k_f) is a coproduct in the functor category $\mathcal{F}(\mathscr{C}, \mathbf{Ab})$. $\square$

Again, we use Theorem 3.5.2 to demonstrate that observation C in Example 3.5.1 holds for internal abelian groups.

Theorem 3.5.3. *Let $\mathscr{C}$ be a finitely complete category. We know that $\boldsymbol{Ab}(\mathscr{C})$ is a pointed finitely complete category. Let $f : A \to B$ be a split epimorphism in $\boldsymbol{Ab}(\mathscr{C})$ so that there exists a morphism $s : B \to A$ such that $f \circ s = id_B$, where id_B is the identity morphism on B. Let $k_f : \ker f \to A$ be the kernel of f. Then, the triple (A, s, k_f) is a coproduct in $\boldsymbol{Ab}(\mathscr{C})$.*

Proof. Assume that $\mathscr{C}$ is a finitely complete category. Assume that $f : A \to B$ is a split epimorphism in $\mathbf{Ab}(\mathscr{C})$ with section given by $s : B \to A$. Assume that $k_f : \ker f \to A$ is the kernel of f.

Assume that we have the following diagram in $\mathbf{Ab}(\mathscr{C})$:

$$\begin{tikzcd} & A & \\ B \arrow[ur, "s"] \arrow[r, "p"'] & C & \ker f \arrow[ul, "k_f"'] \arrow[l, "q"] \end{tikzcd}$$

Recall that the structure embedding $\overline{Y}_{Ab} : \mathscr{C} \to \mathcal{F}(\mathscr{C}^{op}, \mathbf{Ab})$ is fully faithful and left exact. Since it is left exact, we can apply it to the previous diagram to obtain the following diagram in $\mathcal{F}(\mathscr{C}^{op}, \mathbf{Ab})$:

$$\begin{tikzcd} & \overline{Y}_{Ab}(A) & \\ \overline{Y}_{Ab}(B) \arrow[ur, "\overline{Y}_{Ab}(s)"] \arrow[r, "\overline{Y}_{Ab}(p)"'] & \overline{Y}_{Ab}(C) & \overline{Y}_{Ab}(\ker f) \arrow[ul, "\overline{Y}_{Ab}(k_f)"'] \arrow[l, "\overline{Y}_{Ab}(q)"] \end{tikzcd}$$

By Theorem 3.5.2, there exists a unique morphism $\phi : \overline{Y}_{Ab}(A) \to \overline{Y}_{Ab}(C)$ such that the following diagram commutes:

$$\begin{tikzcd} & \overline{Y}_{Ab}(A) \arrow[d, dashed, "\phi"] & \\ \overline{Y}_{Ab}(B) \arrow[ur, "\overline{Y}_{Ab}(s)"] \arrow[r, "\overline{Y}_{Ab}(p)"'] & \overline{Y}_{Ab}(C) & \overline{Y}_{Ab}(\ker f) \arrow[ul, "\overline{Y}_{Ab}(k_f)"'] \arrow[l, "\overline{Y}_{Ab}(q)"] \end{tikzcd}$$

Since the functor $\overline{Y}_{Ab}$ is fully faithful, there exists a unique morphism $\psi : A \to C$ such that the diagram in $\mathbf{Ab}(\mathscr{C})$ commutes:

$$\begin{tikzcd} & A \arrow[d, dashed, "\psi"] & \\ B \arrow[ur, "s"] \arrow[r, "p"'] & C & \ker f \arrow[ul, "k_f"'] \arrow[l, "q"] \end{tikzcd}$$

Therefore, (A, s, k_f) is a coproduct in the internal category $\mathbf{Ab}(\mathscr{C})$ as required. $\square$

3.6 Observation D

Our final observation deals with the category **Grp**. Without the commutativity afforded by **Ab**, observation C in Example 3.5.1 does not apply. Nonetheless, we can make a similar observation to observation C about **Grp**.

Example 3.6.1 (Observation D)**.** In the category of groups **Grp**, let $f : X \to Y$ be a split epimorphism. So, there exists a group morphism $s : Y \to X$ such that $f \circ s = id_Y$. If $x \in X$ then

$$x = sf(x) \cdot (s(f(x^{-1}))x).$$

This means that if X' is a subgroup of X which contains the subgroups $\ker f$ and $s(Y)$ then $X = X'$. By considering the subgroups of X as a lattice, we say that $X = s(Y) \bigvee \ker f$. That is, X is the supremum (join) of $s(Y)$ and $\ker f$.

As with the other observations, we rephrase Example 3.6.1 in terms of category theory.

Theorem 3.6.1. *Let $f : X \to Y$ be a split epimorphism in **Grp** so that there exists a group morphism $s : Y \to X$ such that $f \circ s = id_Y$. Let $k_f : \ker f \to X$ be the kernel of f. Then, the pair (s, k_f) is a covering pair.*

Proof. Assume that $f : X \to Y$ is a split epimorphism in **Grp**, which has unique section $s : Y \to X$. Assume that $k_f : \ker f \to X$ is the kernel of f.

Suppose that $m : Z \to X$ is a group monomorphism which induces two factorisations, making the following diagram in **Grp** commute:

$$\begin{tikzcd} & Z \arrow[d, "m"] & \\ Y \arrow[ur, dashed, "\gamma"] \arrow[r, "s"'] & X & \ker f \arrow[ul, dashed, "\kappa"'] \arrow[l, "k_f"] \end{tikzcd}$$

So, $m \circ \gamma = s$ and $m \circ \kappa = k_f$. We will now prove that m is an isomorphism using observation D in Example 3.6.1.

Assume that $x \in X$. By observation D,

$$x = sf(x) \cdot (s(f(x^{-1}))x)$$

where $sf(x) \in s(Y)$ and $s(f(x^{-1}))x \in \ker f$. Define the map

$$\begin{array}{rccc} n: & X & \to & Z \\ & x = sf(x) \cdot (s(f(x^{-1}))x) & \mapsto & \gamma(f(x))\kappa(s(f(x^{-1}))x). \end{array}$$

If $x \in X$ then

$$\begin{aligned}(m \circ n)(x) &= (m \circ n)(sf(x) \cdot (s(f(x^{-1}))x)) \\ &= m(\gamma(f(x))\kappa(s(f(x^{-1}))x)) \\ &= (m \circ \gamma)(f(x))(m \circ \kappa)(s(f(x^{-1}))x) \\ &= s(f(x))k_f(s(f(x^{-1}))x) \\ &= s(f(x))s(f(x^{-1}))x = x.\end{aligned}$$

In the second last equality, we used the fact that k_f is the inclusion morphism $\ker f \hookrightarrow X$.

So, $m \circ n = id_X$. Observe that $m \circ (n \circ m) = m = m \circ id_Z$. Since m is a monomorphism by assumption, $n \circ m = id_Z$. Thus, m is an isomorphism and the pair of morphisms (s, k_f) is a covering pair. $\square$

In the same fashion as the previous observations, we can generalise Theorem 3.6.1, first to the functor category $\mathcal{F}(\mathscr{C}, \mathbf{Grp})$ and then to the internal category $\mathbf{Grp}(\mathscr{D})$ for a finitely complete category $\mathscr{D}$. This time, we will omit the proofs because they follow in a similar fashion to how the previous observations were generalised.

Theorem 3.6.2. *Let $\mathscr{C}$ be a category. Then, the functor category $\mathcal{F}(\mathscr{C}, \boldsymbol{Grp})$ is pointed and finitely complete by Theorem 3.3.2. Let $H, K : \mathscr{C} \to \boldsymbol{Grp}$ be functors and $f : H \to K$ be a split epimorphism in $\mathcal{F}(\mathscr{C}, \boldsymbol{Grp})$ so that there exists $s : K \to H$ such that $s \circ f = id_K$. Let $k_f : \ker f \to H$ be the kernel of f. Then, the pair (s, k_f) is a covering pair.*

Theorem 3.6.3. *Let $\mathscr{C}$ be a finitely complete category. We know that $\boldsymbol{Grp}(\mathscr{C})$ is a pointed finitely complete category. Let $f : A \to B$ be a split epimorphism in $\boldsymbol{Grp}(\mathscr{C})$ so that there exists a morphism $s : B \to A$ such* *that $f \circ s = id_B$, where id_B is the identity morphism on B. Let $k_f : \ker f \to A$ be the kernel of f. Then, the pair (s, k_f) is a covering pair in **Grp**$(\mathscr{C})$.*

3.7 Natural structures

The main idea of this section concerns the following definition.

Definition 3.7.1. Let $\mathscr{C}$ be a finitely complete category. Let $U_{UMg} : \mathbf{UMg}(\mathscr{C}) \to \mathscr{C}$ be the forgetful functor. We say that any object $X \in \mathscr{C}$ is endowed with a **natural unitary magma structure** if there exists a section $S_{UMg} : \mathscr{C} \to \mathbf{UMg}(\mathscr{C})$ such that $U_{UMg} \circ S_{UMg} = id_{\mathscr{C}}$, where $id_{\mathscr{C}}$ is the identity functor on $\mathscr{C}$.

Similar definitions apply when we replace **UMg** with **Grp**, **Ab**, **CoM** and **Mon**. By the Eckmann-Hilton argument in Theorem 2.1.1 and Theorem 3.4.4, we can show that certain natural structures are equivalent to each other.

Theorem 3.7.1. *Let $\mathscr{C}$ be a finitely complete category. The following are equivalent:*

1. *Any object $X \in \mathscr{C}$ is endowed with a natural unitary magma structure.*
2. *Any object $X \in \mathscr{C}$ is endowed with a natural monoid structure.*
3. *Any object $X \in \mathscr{C}$ is endowed with a natural commutative monoid structure*
4. *The forgetful functors U_{UMg}, U_{Mon} and U_{CoM} are isomorphisms of categories.*

Proof. Assume that $\mathscr{C}$ is a finitely complete category. We will first show that the first three statements are equivalent to each other.

Assume that any object $X \in \mathscr{C}$ is endowed with a natural unitary magma structure. Then, there exists a functor $S_{UMg} : \mathscr{C} \to \mathbf{UMg}(\mathscr{C})$ such that $U_{UMg} \circ S_{UMg} = id_{\mathscr{C}}$, where $id_{\mathscr{C}}$ is the identity functor.

Set $S_{UMg}(X) = (X, m_X, e_X)$. We know that the forgetful functor U_{UMg} is left exact and conservative. Since $U_{UMg} \circ S_{UMg} = id_{\mathscr{C}}$, the functor S_{UMg} must be left exact and hence, it preserves products. Moreover, the map

$m_X : X \times X \to X$ is a unitary magma morphism. By the Eckmann-Hilton argument, (X, m_x, e_X) is an internal commutative monoid and any $X \in \mathscr{C}$ is endowed with a natural commutative monoid structure. Therefore, the first three statements are equivalent.

By definition of an isomorphism, the first statement is a consequence of the fourth statement. It remains to prove that the fourth statement follows from the first. Without loss of generality, suppose that any object $X \in \mathscr{C}$ is endowed with a natural unitary magma structure. Let (M, m, e) be an internal unitary magma in $\mathbf{UMg}(\mathscr{C})$. Then, $S_{UMg}(M) = (M, m_M, e_M)$, where a priori, the internal binary operation m_M and the internal unit e_M differ from m and e respectively. But, by the Eckmann-Hilton argument, $e = e_M$ and $m = m_M$. Therefore,

$$(S_{UMg} \circ U_{UMg})(M, m, e) = (M, m_M, e_M) = (M, m, e)$$

So, $S_{UMg} \circ U_{UMg} = id_{\mathbf{UMg}(\mathscr{C})}$. By assumption, $U_{UMg} \circ S_{UMg} = id_{\mathscr{C}}$. Therefore, the forgetful functor is an isomorphism of categories as required. □

Of course, we can repeat the above argument to prove

Theorem 3.7.2. *Let $\mathscr{C}$ be a finitely complete category. The following are equivalent:*

1. *Any object $X \in \mathscr{C}$ is endowed with a natural abelian group structure.*
2. *Any object $X \in \mathscr{C}$ is endowed with a natural group structure.*
3. *The forgetful functors U_{Grp} and U_{Ab} are isomorphisms of categories.*

Chapter 4

Unital and protomodular categories

4.1 Examples and basic properties

(TBA)

Bibliography

[Bor94] F. Borceux, *Handbook of Categorical Algebra*, Cambridge University Press, Cambridge Volumes 1-3, 1994.

[Bou17] D. Bourn, *From Groups to Categorial Algebra Introduction to Protomodular and Mal'tsev Categories*, Springer International Publishing, Birkhäuser, Compact Textbooks in Mathematics, 2017, ISBN 978-3-319-57218-5.

[BG04] D. Bourn, M. Gran, *Regular, Protomodular and Abelian Categories*, Categorical Foundations Special Topics in Order, Topology, Algebra, and Sheaf Theory, Cambridge University Press, Pages 165-212, 2004.

[CSM95] R. Carter, G. Segal, I. Macdonald, *Lectures on Lie Groups and Lie Algebras*, Cambridge University Press, London Mathematical Society Student Texts 32, Cambridge, 1995, ISBN 0-521-49922-4.

[Hat02] A. Hatcher, *Algebraic Topology*, Cambridge University Press, Available at: https://pi.math.cornell.edu/∼hatcher/AT/AT.pdf.

[Kel05] G. Kelly, *Basic Concepts Of Enriched Category Theory*, Reprints in Theory and Applications of Categories, No. 10, 2005, Available at: http://www.tac.mta.ca/tac/reprints/articles/10/tr10.pdf

[Lei16] T. Leinster, *Basic Category Theory*, 2016, Available at: https://arxiv.org/pdf/1612.09375.pdf.

[Mur06] D. Murfet, *Foundations for Category Theory*, 2006, Available at: http://therisingsea.org/notes/FoundationsForCategoryTheory.pdf.

[Mur16] D. Murfet, *MAST90068 - Lecture 6*, 2016, Available at: http://therisingsea.org/notes/mast90068/lecture6.pdf.

[Mur21] D. Murfet, *Lecture 7: Constructing topological spaces*, 2021, Available at: http://therisingsea.org/notes/mast30026/lecture7.pdf.