Notes on C*-algebras

Brian Chan

April 26, 2023

Contents

0.1 Purpose

These notes are for the study of the basic theory of C*-algebras. The content is typically encountered in a first graduate course in C*-algebras. The main reference for the first chapter of these notes is [Put19]. The main reference for the second chapter is [Mur90].

Chapter 1

An introduction to C*-algebras from [Put19]

1.1 Definition and examples

In this section, we will build up to the definition of a C*-algebra and then give some useful examples.

Definition 1.1.1. Let A be an associative algebra over $\mathbb{C}$ (or $\mathbb{R}$). We say that A is a **Banach algebra** if A is also a Banach space. That is, A is equipped with a norm $\|-\|$ which makes A complete — every Cauchy sequence in A converges with respect to the norm $\|-\|$. Additionally, if $x, y \in A$, the norm must satisfy

$$\|xy\| \leq \|x\|\|y\|. \tag{1.1}$$

Equation (1.1) ensures that the norm respects the algebraic structure of A, by rendering multiplication continuous. Let $m_x : A \to A$ be the operator which sends $y \in A$ to xy. If $\epsilon \in \mathbb{R}_{>0}$, choose $\delta = \epsilon/\|x\|$ and suppose that $\|y_1 - y_2\| < \delta$. Then,

$$\begin{aligned} \|m_x(y_1) - m_x(y_2)\| &= \|xy_1 - xy_2\| \\ &\leq \|x\|\|y_1 - y_2\| \\ &< \epsilon. \end{aligned}$$

Hence, m_x is a continuous operator on A for all $x \in A$.

Definition 1.1.2. Let A be a Banach algebra over $\mathbb{C}$. We say that A is a **Banach *-algebra** if A is equipped with a map $^* : A \to A$ which satisfies for all $x, y \in A$ and $\lambda \in \mathbb{C}$,

1. $(x^*)^* = x$ (Involution)
2. $(x+y)^* = x^* + y^*$
3. $(\lambda x)^* = \overline{\lambda} x^*$
4. $(xy)^* = y^* x^*$ (Anti-multiplicative)

The middle two properties of the map $^* : A \to A$ means that * is anti-linear (or conjugate linear). If $a \in A$ then the element a^* is called the **adjoint** of a.

Definition 1.1.3. Let A be a Banach *-algebra over $\mathbb{C}$. We say that A is a **C*-algebra** if for all $x \in A$, $\|x^* x\| = \|x\|^2$.

The first theorem we will state gives us the primary example of a C*-algebra.

Theorem 1.1.1. *Let H be a Hilbert space over $\mathbb{C}$ and $B(H)$ denote the Banach space of bounded linear operators $\phi : H \to H$. Then, $B(H)$ is a C*-algebra.*

Proof. Assume that H is a Hilbert space over $\mathbb{C}$ and that $B(H)$ is the Banach space of bounded linear operators from H to H. Let $\|-\|_H$ denote the norm on H and $\langle -.- \rangle$ denote the inner product on H. Define the map $^* : H \to B(H)$ by

$$\begin{array}{cccc} ^*: & B(H) & \to & B(H) \\ & h & \to & h^*. \end{array}$$

where h^* is the adjoint of h, which satisfies for all $\xi, \eta \in H$,

$$\langle h^*(\xi), \eta \rangle = \langle \xi, h(\eta) \rangle \tag{1.2}$$

To show: (a) $B(H)$ is a Banach algebra.

(b) $B(H)$ is a Banach *-algebra.

(c) $B(H)$ is a C*-algebra.

(a) Observe that $B(H)$ is an associative algebra over $\mathbb{C}$, where scalar multiplication and addition are defined as usual and multiplication is given by composition of linear operators, which we will denote by $\circ$. We also know that $B(H)$ is a Banach space when equipped with the operator norm

$$\|h\| = \sup_{\|x\|_H = 1} \|h(x)\|_H$$

To show: (aa) If $f, g \in B(H)$, then $\|f \circ g\| \leq \|f\| \|g\|$.

(aa) Assume that $f, g \in B(H)$. Then, from the definition of the operator norm, we have

$$\begin{aligned}
\|f \circ g\| &= \sup_{\|x\|_H = 1} \|f(g(x))\|_H \\
&\leq \sup_{\|x\|_H = 1} \|f\| \|g(x)\|_H \\
&= \|f\| \|g\|.
\end{aligned}$$

Therefore, $B(H)$ is a Banach algebra.

(b) To show: (ba) The map * is an involution.

(bb) The map * is anti-linear.

(bc) The map * is anti-multiplicative.

(ba) Assume that $h \in B(H)$. By equation (1.2), $h^{**} \in B(H)$ must satisfy for all $\xi, \eta \in H$,

$$\langle h^{**}(\xi), \eta \rangle = \langle \xi, h^*(\eta) \rangle = \langle h(\xi), \eta \rangle.$$

Therefore, $h^{**}(\xi) = h(\xi)$ for all $\xi \in H$. So, $h^{**} = h$, revealing that $^* : B(H) \to B(H)$ is an involution.

(bb) Assume that $g, h \in B(H)$. Then, for all $\xi, \eta \in H$, we have

$$\begin{aligned}
\langle (g+h)^*(\xi), \eta \rangle &= \langle \xi, (g+h)(\eta) \rangle \\
&= \langle \xi, g(\eta) \rangle + \langle \xi, h(\eta) \rangle \\
&= \langle (g^* + h^*)(\xi), \eta \rangle.
\end{aligned}$$

So, $(g+h)^* = g^* + h^*$. Now assume that $\lambda \in \mathbb{C}$. Then,

$$
\begin{aligned}
\langle (\lambda h)^*(\xi), \eta\rangle &= \langle \xi, (\lambda h)(\eta)\rangle \\
&= \overline{\lambda}\langle \xi, h(\eta)\rangle \\
&= \overline{\lambda}\langle h^*(\xi), \eta\rangle \\
&= \langle (\overline{\lambda} h^*)(\xi), \eta\rangle.
\end{aligned}
$$

So, $(\lambda h)^* = \overline{\lambda} h^*$. This demonstrates that * is anti-linear.

(bc) We compute directly that for all $\xi, \eta \in H$,

$$
\begin{aligned}
\langle (g\circ h)^*(\xi), \eta\rangle &= \langle \xi, g(h(\eta))\rangle \\
&= \langle g^*(\xi), h(\eta)\rangle \\
&= \langle (h^*\circ g^*)(\xi), \eta\rangle.
\end{aligned}
$$

Therefore, $(g\circ h)^* = h^*\circ g^*$. Hence, the map * is anti-linear. So, $B(H)$ is a Banach *-algebra.

(c) To show: (ca) For all $h \in B(H)$, $\|h^*\circ h\| = \|h\|^2$.

(ca) Assume that $h \in B(H)$ and that $\|h\| > 0$ (the statement holds when $h = 0$). We have already shown that $\|h^*\circ h\| \le \|h^*\|\|h\|$.

To show: (caa) $\|h^*\| = \|h\|$.

(caa) Observe that

$$
\begin{aligned}
\|h^*\circ h\| &= \sup_{\|\xi\|_H=1} \|h^*(h(\xi))\|_H \\
&= \sup_{\|\xi\|_H=1}\ \sup_{\|\eta\|_H=1} |\langle h^*(h(\xi)), \eta\rangle| \\
&\ge \sup_{\|\xi\|_H=1} |\langle h^*(h(\xi)), \xi\rangle| \\
&= \sup_{\|\xi\|_H=1} \|h(\xi)\|_H^2 \\
&= \|h\|^2.
\end{aligned}
$$

Therefore, $\|h\|^2 \le \|h^*\|\|h\|$ and $\|h\| \le \|h^*\|$. To establish the reverse inequality, we can interchange the roles of h and h^* in the above calculation so that

$$
\begin{aligned}
\|h \circ h^*\| &= \sup_{\|\xi\|_H=1} \|h(h^*(\xi))\|_H \\
&= \sup_{\|\xi\|_H=1} \sup_{\|\eta\|_H=1} |\langle h(h^*(\xi)), \eta\rangle| \\
&\geq \sup_{\|\xi\|_H=1} |\langle h(h^*(\xi)), \xi\rangle| \\
&= \sup_{\|\xi\|_H=1} \|h^*(\xi)\|_H^2 \\
&= \|h^*\|^2.
\end{aligned}
$$

So, $\|h^*\|^2 \leq \|h^*\|\|h\|$ and $\|h^*\| \leq \|h\|$. In tandem with $\|h\| \leq \|h^*\|$, we deduce that $\|h^*\| = \|h\|$.

(ca) Recall from part (caa) that $\|h\|^2 \leq \|h^* \circ h\|$ and from the beginning of part (ca) that $\|h^* \circ h\| \leq \|h^*\|\|h\|$. Since $\|h^*\| = \|h\|$, $\|h^* \circ h\| \leq \|h\|^2$ and consequently, $\|h\|^2 = \|h^* \circ h\|$ as required. □

Example 1.1.1. Here, we will give another important example of a C*-algebra. Let X be a compact, Hausdorff space and $Cts(X, \mathbb{C})$ denote the space of continuous functions from X to $\mathbb{C}$. Then, $Cts(X, \mathbb{C})$ is a C*-algebra with scalar multiplication, addition and multiplication defined pointwise on $\mathbb{C}$. The norm on $Cts(X, \mathbb{C})$ is

$$\|f\| = \sup_{x \in X} |f(x)|.$$

and the map $^* : Cts(X, \mathbb{C}) \to Cts(X, \mathbb{C})$ is defined by the equation

$$f^*(x) = \overline{f(x)}.$$

Example 1.1.2. The complex numbers $\mathbb{C}$ with addition, multiplication and complex conjugation is an example of a C*-algebra.

Example 1.1.3. As a special case of Theorem 1.1.1, if we set $H = \mathbb{C}^n$ for $n \in \mathbb{Z}_{>0}$, we find that the $\mathbb{C}$-algebra of $n \times n$ matrices $M_{n\times n}(\mathbb{C})$ is a C*-algebra.

Next, we define some specific types of C*-algebras.

Definition 1.1.4. Let A be a C*-algebra. We say that A is **unital** if as an associative algebra, A has a multiplicative unit which is usually denoted by 1_A.

Definition 1.1.5. Let A be a C*-algebra. We say that A is **commutative** if as an associative algebra, A is commutative. That is, if $a, b \in A$ then $ab = ba$.

One of the defining properties of a C*-algebra is that the involution map $^* : A \to A$ is isometric (distance preserving).

Theorem 1.1.2. *Let A be a C*-algebra. If $a \in A$ then $\|a\| = \|a^*\|$.*

Proof. Assume that A is a C*-algebra. Assume that $a \in A$. Then,

$$\|a\|^2 = \|a^*a\| \leq \|a^*\|\|a\|$$

and

$$\|a^*\|^2 = \|aa^*\| \leq \|a\|\|a^*\|.$$

The first equation shows that $\|a\| \leq \|a^*\|$ and the second shows that $\|a^*\| \leq \|a\|$. Hence, $\|a\| = \|a^*\|$. □

Next, we define some more terminology related to C*-algebras.

Definition 1.1.6. Let A be a C*-algebra.

1. We say that $a \in A$ is **self-adjoint** if $a^* = a$.
2. We say that $a \in A$ is **normal** if $a^*a = aa^*$.
3. We say that $a \in A$ is a **projection** if a is self-adjoint and $a^2 = a$.
4. We say that $a \in A$ is a **partial isometry** if a^*a is a projection.
5. We say that $a \in A$ is **positive** if there exists an element $b \in A$ such that $a = b^*b$. We write $a \geq 0$ to mean that a is positive.

Now let A be a unital C*-algebra.

1. We say that $a \in A$ is **unitary** if $a^*a = aa^* = 1_A$. That is a is invertible and $a^{-1} = a^*$.
2. We say that $a \in A$ is an **isometry** if $a^*a = 1_A$.

These definitions are consistent with those found in [Sol18], which focuses on the C*-algebra of bounded linear operators on a Hilbert space.

From a finite number of C*-algebras, we can construct a C*-algebra from them, which is called the direct sum.

Theorem 1.1.3 (Finite direct sum of C*-algebras). *Let $n \in \mathbb{Z}_{>0}$ and A_i be C*-algebras for $i \in \{1, 2, \dots, n\}$. Define the* ***direct sum*** *of the family $\{A_i\}_{i=1}^n$ as the associative algebra over $\mathbb{C}$*

$$\bigoplus_{i=1}^{n} A_i = \{(a_1, \dots, a_n) \mid a_i \in A_i \text{ for } i \in \{1, 2, \dots, n\}\}.$$

The algebraic operations of multiplication, scalar multiplication, addition and involution on $\bigoplus_{i=1}^n A_i$ are defined coordinate-wise. The norm on $\bigoplus_{i=1}^n A_i$ is given by

$$\|(a_1, \dots, a_n)\| = \max_{i \in \{1,2,\dots,n\}} \|a_i\|.$$

Then, $\bigoplus_{i=1}^n A_i$ is a C-algebra.*

Proof. Assume that $n \in \mathbb{Z}_{>0}$ and A_i are C*-algebras for $i \in \{1, 2, \dots, n\}$. Assume that the direct sum $A = \bigoplus_{i=1}^n A_i$ is defined as in the statement of the theorem.

To see that $\bigoplus_{i=1}^n A_i$ is a Banach algebra, first note that A is the direct sum of Banach spaces and is hence, a Banach space. Next, assume that $(a_1, \dots, a_n), (b_1, \dots, b_n) \in A$. We compute directly that

$$\begin{aligned}
\|(a_1, \dots, a_n)(b_1, \dots, b_n)\| &= \|(a_1 b_1, \dots, a_n b_n)\| \\
&= \max_{i \in \{1,2,\dots,n\}} \|a_i b_i\| \\
&\le \max_{i \in \{1,2,\dots,n\}} \|a_i\| \|b_i\| \\
&\le \big(\max_{i \in \{1,2,\dots,n\}} \|a_i\| \big) \big(\max_{j \in \{1,2,\dots,n\}} \|b_j\| \big) \\
&= \|(a_1, \dots, a_n)\| \|(b_1, \dots, b_n)\|.
\end{aligned}$$

Hence, A is a Banach algebra.

By definition of the involution map $* : A \to A$ pointwise, it is straightforward to check that A is a Banach *-algebra.

Finally, to see that A is a C*-algebra, we compute for $(a_1, \dots, a_n) \in A$ that

$$\begin{aligned}
\|(a_1,\dots,a_n)^*(a_1,\dots,a_n)\| &= \|(a_1^*,\dots,a_n^*)(a_1,\dots,a_n)\| \\
&= \|(a_1^*a_1,\dots,a_n^*a_n)\| \\
&= \max_{i\in\{1,2,\dots,n\}} \|a_i^*a_i\| \\
&= \max_{i\in\{1,2,\dots,n\}} \|a_i\|^2 \\
&= \|(a_1,\dots,a_n)\|^2.
\end{aligned}$$

Therefore, A is a C*-algebra. $\square$

Now, if we have a countable family of C*-algebras $\{A_i\}_{i=1}^{\infty}$ then the definition of the direct sum in Theorem 1.1.3 does not carry over to this situation. In this case, the direct sum is defined by

$$\bigoplus_{n=1}^{\infty} A_n = \{(a_1,a_2,\dots) \mid a_i \in A_i \text{ for } i \in \{1,2,\dots,n\}, \lim_{n\to\infty}\|a_n\| = 0\}.$$

Example 1.1.4. This example encapsulates [Put19, Exercise 1.2.1]. Consider the $\mathbb{C}$-algebra $\mathbb{C}^2 \cong \mathbb{C}\oplus\mathbb{C}$ with the operations of multiplication, addition and complex conjugation defined pointwise. We claim that $\mathbb{C}^2$ with the norm

$$\|(z_1,z_2)\| = |z_1| + |z_2|$$

is not a C*-algebra.

Consider the element $(1,1)\in\mathbb{C}^2$. Then,

$$\|(1,1)^*(1,1)\| = \|(1,1)\| = 1+1 = 2$$

and

$$\|(1,1)\|^2 = (1+1)^2 = 4.$$

So, $\|(1,1)\|^2 \neq \|(1,1)^*(1,1)\|$ and hence, $\mathbb{C}^2$ is not a C*-algebra with the above norm.

In fact, the only norm which makes $\mathbb{C}^2$ a C*-algebra is the one given in Theorem 1.1.3. In order to prove this, we will follow the outline given in [Put19, Exercise 1.2.1].

Suppose that $\|-\|$ is a norm on $\mathbb{C}^2$, which makes $\mathbb{C}^2$ into a C*-algebra.

To show: (a) $\|(1,0)\| = \|(1,1)\| = \|(0,1)\| = 1$.

(a) By the defining property of C*-algebras, we find that

$$\|(1,0)\| = \|(1,0)^*(1,0)\| = \|(1,0)\|^2.$$

Similarly, $\|(1,1)\| = \|(1,1)\|^2$ and $\|(0,1)\| = \|(0,1)\|^2$. Therefore, $\|(1,0)\| = \|(1,1)\| = \|(0,1)\| = 1$.

Now let $(z_1, z_2) \in \mathbb{C}^2$. Write $z_1 = |z_1|e^{i\theta_1}$ and $z_2 = |z_2|e^{i\theta_2}$, where $\theta_1, \theta_2 \in [0, 2\pi)$. Then, the element $(e^{-i\theta_1}, e^{-i\theta_2}) \in \mathbb{C}^2$ satisfies

$$(e^{-i\theta_1}, e^{-i\theta_2})^*(e^{-i\theta_1}, e^{-i\theta_2}) = (1,1) = (e^{-i\theta_1}, e^{-i\theta_2})(e^{-i\theta_1}, e^{-i\theta_2})^*.$$

So, $(e^{-i\theta_1}, e^{-i\theta_2})$ is a unitary element of $\mathbb{C}^2$ satisfying $(e^{-i\theta_1}, e^{-i\theta_2})(z_1, z_2) = (|z_1|, |z_2|)$

To show: (b) $\|(z_1, z_2)\| = \|(|z_1|, |z_2|)\|$.

(b) By the defining property of a C*-algebra, we have

$$\begin{aligned}
\|(z_1, z_2)\|^2 &= \|(z_1, z_2)^*(z_1, z_2)\| \\
&= \|(z_1, z_2)^*(e^{-i\theta_1}, e^{-i\theta_2})^*(e^{-i\theta_1}, e^{-i\theta_2})(z_1, z_2)\| \\
&= \|(|z_1|, |z_2|)^*(|z_1|, |z_2|)\| \\
&= \|(|z_1|, |z_2|)\|^2.
\end{aligned}$$

So, $\|(z_1, z_2)\| = \|(|z_1|, |z_2|)\|$.

Next, assume that $(a_1, a_2), (b_1, b_2) \in \mathbb{C}^2$ such that $\|(a_1, a_2)\| = \|(b_1, b_2)\| = 1$. Assume that $t \in [0, 1]$.

To show: (c) $\|t(a_1, a_2) + (1-t)(b_1, b_2)\| \leq 1$.

(c) We compute directly that

$$
\begin{aligned}
\|t(a_1,a_2)+(1-t)(b_1,b_2)\|^2 &= \|(ta_1+(1-t)b_1, ta_2+(1-t)b_2)\|^2 \\
&= \|(ta_1+(1-t)b_1, ta_2+(1-t)b_2)^* \\
&\quad (ta_1+(1-t)b_1, ta_2+(1-t)b_2)\| \\
&= \|(|ta_1+(1-t)b_1|^2, |ta_2+(1-t)b_2|^2)\| \\
&\leq \|\big((t|a_1|+(1-t)|b_1|)^2, (t|a_2|+(1-t)|b_2|)^2\big)\| \\
&\leq \|(t^2|a_1|^2, t^2|a_2|^2)\| + \|(t(1-t)|a_1||b_1|, t(1-t)|a_2||b_2|)\| \\
&\quad +\|((1-t)^2|b_1|^2, (1-t)^2|b_2|^2)\| \\
&= t^2\|(|a_1|^2, |a_2|^2)\| + t(1-t)\|(|a_1||b_1|, |a_2||b_2|)\| \\
&\quad +(1-t)^2\|(|b_1|^2, |b_2|^2)\| \\
&= t^2\|(a_1,a_2)^*(a_1,a_2)\| + t(1-t)\|(|a_1b_1|, |a_2b_2|)\| \\
&\quad +(1-t)^2\|(b_1,b_2)^*(b_1,b_2)\| \\
&\leq t^2\|(a_1,a_2)\|^2 + t(1-t)\|(|a_1|,|a_2|)\|\|(|b_1|,|b_2|)\| \\
&\quad +(1-t)^2\|(b_1,b_2)\|^2 \\
&\leq t^2 + t(1-t) + (1-t)^2 \quad \text{(by part (b))} \\
&= t + (1-t)^2 = 1 - t + t^2 \leq 1.
\end{aligned}
$$

This proves part (c). Now assume that $\alpha \in \mathbb{C}$ such that $|\alpha| \leq 1$.

To show: (d) $\|(1,\alpha)\| = 1$ and $\|(\alpha,1)\| = 1$.

(d) We compute directly that

$$
\begin{aligned}
\|(1,\alpha)\|^2 &= \|(1,\alpha)^*(1,\alpha)\| \\
&= \|(1,|\alpha|^2)\| \\
&= \||\alpha|^2(1,1) + (1-|\alpha|^2)(1,0)\| \\
&\leq 1 \quad \text{(by parts (a) and (c))}.
\end{aligned}
$$

The inequality $\|(\alpha,1)\| \leq 1$ follows from a similar computation.

Next, observe that

$$
\begin{aligned}
1 &= \|(1,0)\| \quad \text{(by part (a))} \\
&= \|(1,0)(1,\alpha)\| \\
&\leq \|(1,0)\|\|(1,\alpha)\| = \|(1,\alpha)\|.
\end{aligned}
$$

Therefore, $1 = \|(1, \alpha)\|$. Similarly, $1 = \|(\alpha, 1)\|$.

Now we put all of these observations together to obtain the required conclusion. Assume that $(\alpha, \beta) \in \mathbb{C}^2$ such that $0 < |\alpha| \leq |\beta|$. The norm of this element is by part (d)

$$\|(\alpha, \beta)\| = |\beta| \|(\frac{\alpha}{\beta}, 1)\| = |\beta|$$

Similarly, if $0 < |\beta| \leq |\alpha|$ then $\|(\alpha, \beta)\| = |\alpha|$. Furthermore, by part (a)

$$\|(\alpha, 0)\| = |\alpha| \|(1, 0)\| = |\alpha|$$

and $\|(0, \alpha)\| = |\alpha|$. Finally, $\|(0, 0)\| = 0$ by definition of the norm. Hence,

$$\|(\alpha, \beta)\| = \max(|\alpha|, |\beta|).$$

Consequently, the above norm is the only norm which makes $\mathbb{C}^2$ into a C*-algebra.

As usual with mathematical structures, we can define the notion of a map between C*-algebras.

Definition 1.1.7. Let A and B be C*-algebras with involution maps $*_A$ and $*_B$ respectively. A ***-homomorphism** is a function $\phi : A \to B$ such that

1. If $a_1, a_2 \in A$ then $\phi(a_1 + a_2) = \phi(a_1) + \phi(a_2)$.
2. If $\lambda \in \mathbb{C}$ and $a \in A$ then $\phi(\lambda a) = \lambda \phi(a)$.
3. If $a_1, a_2 \in A$ then $\phi(a_1 a_2) = \phi(a_1)\phi(a_2)$.
4. If $a_1 \in A$ then $\phi(a_1^{*_A}) = \phi(a_1)^{*_B}$.

If ϕ is bijective then we say that ϕ is a ***-isomorphism**.

If A and B are unital C*-algebras and $\phi : A \to B$ is a *-homomorphism such that $\phi(1_A) = 1_B$ then ϕ is called a **unital *-homomorphism**.

C*-algebras, together with the class of *-homomorphisms, form the category of C*-algebras.

1.2 The spectrum

The concept of the spectrum is used widely in linear algebra and for bounded operators on a Hilbert space. In this section, we will define the spectrum for elements in a C*-algebra. First, we prove the following important connection between the algebraic and topological structures on a C*-algebra.

Theorem 1.2.1. *Let A be a unital Banach *-algebra. If $a \in A$ and $\|a - 1_A\| < 1$ then a is invertible. Moreover, the set of invertible elements of A is open.*

Proof. Assume that A is a unital Banach *-algebra. First assume that $a \in A$ and $\|a - 1_A\| < 1$.

To show: (a) a is invertible.

(a) Consider the following infinite sum in A:

$$\sum_{n=0}^{\infty}(1_A - a)^n.$$

To see that this converges in A, we must show that its norm is finite. We compute directly that

$$\|\sum_{n=0}^{\infty}(1_A - a)^n\| \leq \sum_{n=0}^{\infty}\|1_A - a\|^n < \infty$$

because $\|1_A - a\| < 1$. Hence, $\sum_{n=0}^{\infty}(1_A - a)^n \in A$. Let $b = \sum_{n=0}^{\infty}(1_A - a)^n$. To see that $b = a^{-1}$, we compute directly that

$$\begin{aligned}
ab &= a\sum_{n=0}^{\infty}(1_A - a)^n \\
&= a\big(\lim_{N\to\infty}\sum_{n=0}^{N}(1_A - a)^n\big) \\
&= \lim_{N\to\infty} a\big(\sum_{n=0}^{N}(1_A - a)^n\big) \\
&= \lim_{N\to\infty}(1_A - (1_A - a))\big(\sum_{n=0}^{N}(1_A - a)^n\big) \\
&= \lim_{N\to\infty}\big(1_A - (1_A - a)^{N+1}\big) = 1_A.
\end{aligned}$$

A similar computation demonstrates that $ba = 1_A$. Therefore, $b = a^{-1}$ and a is invertible.

Next, assume that $a \in A$ is invertible. Select $x \in A$ such that $\|x\| < \frac{1}{2\|a^{-1}\|}$.

To show: (b) $a + x$ is invertible.

(b) Observe that

$$\begin{aligned}\|a^{-1}x\| &\leq \|a^{-1}\|\|x\| \\ &< \|a^{-1}\|\frac{1}{2\|a^{-1}\|} \\ &= \frac{1}{2} < 1.\end{aligned}$$

By the first part of the theorem, we deduce that $1_A + a^{-1}x$ is an invertible element of A. Hence,

$$a + x = a(1_A + a^{-1}x)$$

is an invertible element of A because it is the product of two invertible elements.

Let $I \subseteq A$ denote the set of invertible elements of A. By part (b), we find that the open ball

$$B(a, \frac{1}{2\|a^{-1}\|}) \subseteq I.$$

Hence, I is an open subset of A. □

Now we arrive at the familiar notion of the spectrum. Perhaps the easiest example of the spectrum arises from linear algebra. If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation then the spectrum of T is given by computing the roots of the characteristic polynomial of T.

Definition 1.2.1. Let A be a unital associative algebra. Let $a \in A$. The **spectrum** of a, denoted by $\sigma(a)$, is the set

$$\sigma(a) = \{\lambda \in \mathbb{C} \mid \lambda 1_A - a \text{ is not invertible}\}.$$

The **resolvent set** of a, denoted by $\rho(a)$, is the set $\rho(a) = \mathbb{C} - \sigma(a)$.

The **spectral radius** of a, denoted by $r(a)$, is the quantity

$$r(a) = \sup_{\lambda \in \sigma(a)} |\lambda|.$$

Note that the spectral radius of a is defined provided that $\sigma(a)$ is non-empty. Also, we allow the possibility that $\sigma(a) = \infty$.

Let us return to the world of linear algebra for a brief moment and work with the C*-algebra $M_{n\times n}(\mathbb{C})$. If $n \in \mathbb{Z}_{>0}$ and $a \in M_{n\times n}(\mathbb{C})$ then the following are equivalent:

1. a is invertible
2. The linear transformation $a : \mathbb{C}^n \to \mathbb{C}^n$ is injective.
3. The linear transformation $a : \mathbb{C}^n \to \mathbb{C}^n$ is surjective.
4. The determinant $\det(a) \neq 0$.

In particular, the characteristic polynomial of a is computed by using the determinant. In this very special case, the spectrum of a is indeed quite straightforward to compute. However, if H is an infinite dimensional Hilbert space then the above result completely falls apart for the C*-algebra $B(H)$ — the space of bounded linear operators on H. Additionally, a notion of the determinant does not exist for an arbitrary element of $B(H)$. So, one cannot expect to compute spectrums easily for the elements of $B(H)$.

Here are some fundamental results about the spectrum. First, we will prove that the spectrum is always a non-empty, compact subset of $\mathbb{C}$. We will follow [Sol18, Chapter 1].

Theorem 1.2.2. *Let A be a unital Banach *-algebra and $h \in A$. Then,*

$$\{\lambda \in \mathbb{C} \mid |\lambda| > \|h\|\} \subseteq \rho(h).$$

Proof. Assume that A is a unital Banach *-algebra and $h \in A$. Assume that $\lambda \in \mathbb{C}$ such that $|\lambda| > \|h\|$. We claim that the sum

$$\frac{1}{\lambda}\sum_{n=0}^{\infty} \lambda^{-n} h^n \tag{1.3}$$

converges in A and that it is equal to $(\lambda 1_A - h)^{-1}$. To see that the sum in equation (1.3) converges, it suffices to show that its norm is finite. But,

$$\begin{aligned}
\|\frac{1}{\lambda}\sum_{n=0}^{\infty}\lambda^{-n}h^n\| &= \frac{1}{|\lambda|}\|\sum_{n=0}^{\infty}\lambda^{-n}h^n\| \\
&\leq \frac{1}{|\lambda|}\sum_{n=0}^{\infty}|\lambda|^{-n}\|h\|^n \\
&= \frac{C}{|\lambda|}
\end{aligned}$$

where $C = \sum_{n=0}^{\infty}|\lambda|^{-n}\|h\|^n \in \mathbb{R}_{>0}$. The sum $\sum_{n=0}^{\infty}|\lambda|^{-n}\|h\|^n$ is a convergent geometric series because $|\lambda| > \|h\|$.

Hence, the sum in equation (1.3) converges in A and is consequently, a well-defined element in A. For all $m \in \mathbb{Z}_{>0}$, let

$$S_m = \frac{1}{\lambda}\sum_{n=0}^{m}\lambda^{-n}h^n.$$

Then,

$$\begin{aligned}
S_m(\lambda 1_A - h) &= \frac{1}{\lambda}(1_A + \lambda^{-1}h + \lambda^{-2}h^2 + \cdots + \lambda^{-m}h^m)(\lambda 1_A - h) \\
&= \frac{1}{\lambda}((\lambda 1_A - h) + (h - \lambda^{-1}h^2) + \cdots + (\lambda^{-m+1}h^m - \lambda^{-m}h^{m+1})) \\
&= \frac{1}{\lambda}(\lambda 1_A - \lambda^{-m}h^{m+1}) \\
&= 1_A - \lambda^{-m-1}h^{m+1}.
\end{aligned}$$

A similar calculation gives $(\lambda 1_A - h)S_m = 1_A - \lambda^{-m-1}h^{m+1}$. Now take the limit as $m \to \infty$. Observe that in A,

$$\lim_{m\to\infty}\lambda^{-m-1}h^{m+1} = 0$$

because

$$\lim_{m\to\infty}(\frac{\|h\|}{|\lambda|})^{m+1} = 0.$$

So, in the limit as $m \to \infty$,

$$(\lambda 1_A - h)\Big(\frac{1}{\lambda}\sum_{n=0}^{\infty}\lambda^{-n}h^n\Big) = \Big(\frac{1}{\lambda}\sum_{n=0}^{\infty}\lambda^{-n}h^n\Big)(\lambda 1_A - h) = 1_A.$$

Therefore,

$$\Big(\frac{1}{\lambda}\sum_{n=0}^{\infty}\lambda^{-n}h^{n}\Big) = (\lambda 1_A - h)^{-1}$$

and subsequently, $\lambda \in \rho(h)$ and

$$\{\lambda \in \mathbb{C} \mid |\lambda| > \|h\|\} \subseteq \rho(h).$$

$\square$

Rewriting the conclusion of Theorem 1.2.2 in terms of the spectrum, we find that

$$\sigma(h) \subseteq \{\lambda \in \mathbb{C} \mid |\lambda| \leq \|h\|\} \tag{1.4}$$

If $h \in A$ then $\sigma(h)$ is a bounded set. We now want to show that $\sigma(h)$ is a closed set.

Theorem 1.2.3. *Let A be a unital Banach *-algebra, $h \in B(H)$ and $\lambda_0 \in \rho(h)$. Suppose that $\lambda \in \mathbb{C}$ such that*

$$|\lambda - \lambda_0| < \frac{1}{\|(\lambda_0 1_A - h)^{-1}\|}.$$

Then,

$$(\lambda 1_A - h)^{-1} = \sum_{n=0}^{\infty}(\lambda_0 - \lambda)^n(\lambda_0 1_A - x)^{-n-1}.$$

and consequently, $\lambda \in \rho(h)$.

Proof. Assume that A is a unital Banach *-algebra, $h \in A$ and $\lambda_0 \in \rho(h)$. Assume that $\lambda \in \mathbb{C}$ such that

$$|\lambda - \lambda_0| < \frac{1}{\|(\lambda_0 1_A - h)^{-1}\|}.$$

To see that the sum

$$\sum_{n=0}^{\infty}(\lambda_0 - \lambda)^n(\lambda_0 1_A - x)^{-n-1}$$

converges in A, we must show that its norm is finite. We have

$$\|\sum_{n=0}^{\infty}(\lambda_0-\lambda)^n(\lambda_0 1_A-h)^{-n-1}\| \leq \sum_{n=0}^{\infty}|\lambda_0-\lambda|^n\|(\lambda_0 1_A-h)^{-1}\|^{n+1}$$
$$= \frac{1}{|\lambda_0-\lambda|}\sum_{n=0}^{\infty}|\lambda_0-\lambda|^{n+1}\|(\lambda_0 1_A-h)^{-1}\|^{n+1}$$
$$= \frac{D}{|\lambda_0-\lambda|}$$

for some $D \in \mathbb{R}_{>0}$. Hence, the sum $\sum_{n=0}^{\infty}(\lambda_0-\lambda)^n(\lambda_0 1_A-h)^{-n-1}$ converges in A and is a well-defined element in A. If $m \in \mathbb{Z}_{>0}$ then let

$$T_m = \sum_{n=0}^{m}(\lambda_0-\lambda)^n(\lambda_0 1_A-h)^{-n-1}.$$

First, observe that

$$\lambda 1_A - h = (\lambda_0 1_A - h)(1_A + (\lambda-\lambda_0)(\lambda_0 1_A-h)^{-1})$$

Then, a direct calculation yields

$$\begin{aligned}
T_m(\lambda 1_A - h) &= \Big(\sum_{n=0}^{m}(\lambda_0-\lambda)^n(\lambda_0 1_A-h)^{-n-1}\Big)((\lambda_0 1_A-h)(1_A+(\lambda-\lambda_0)(\lambda_0 1_A-h)^{-1}))\\
&= \Big(\sum_{n=0}^{m}(\lambda_0-\lambda)^n(\lambda_0 1_A-h)^{-n}\Big)(1_A+(\lambda-\lambda_0)(\lambda_0 1_A-h)^{-1})\\
&= \sum_{n=0}^{m}(\lambda_0-\lambda)^n(\lambda_0 1_A-h)^{-n} - \sum_{n=0}^{m}(\lambda_0-\lambda)^{n+1}(\lambda_0 1_A-h)^{-n-1}\\
&= 1_A - (\lambda_0-\lambda)^{m+1}(\lambda_0 1_A-h)^{-m-1}.
\end{aligned}$$

Note that by a similar computation,

$$(\lambda 1_A - h)T_m = 1_A - (\lambda_0-\lambda)^{m+1}(\lambda_0 1_A-h)^{-m-1}$$

as well. Taking the limit as $m \to \infty$, we find that
$(\lambda_0-\lambda)^{m+1}(\lambda_0 1_A-h)^{-m-1} \to 0$ in $B(H)$ because its norm tends to 0 as $m \to \infty$. Hence,

$$(\lambda 1_A - h)\Big(\sum_{n=0}^{\infty}(\lambda_0-\lambda)^n(\lambda_0 1_A-h)^{-n-1}\Big) = \Big(\sum_{n=0}^{\infty}(\lambda_0-\lambda)^n(\lambda_0 1_A-h)^{-n-1}\Big)(\lambda 1_A-h) = 1_A$$

and

$$\Big(\sum_{n=0}^{\infty}(\lambda_0-\lambda)^n(\lambda_0 1_A-h)^{-n-1}\Big)=(\lambda 1_A-h)^{-1}$$

which demonstrates that $\lambda\in\rho(h)$. $\square$

Theorem 1.2.3 tells us that if $h\in A$ then $\rho(h)$ is an open subset of $\mathbb{C}$. In tandem with equation (1.4), the spectrum $\sigma(h)$ is a closed and bounded subset of $\mathbb{C}$ and is thus, compact. Let us summarise this important finding as a theorem.

Theorem 1.2.4. *Let A be a unital Banach *-algebra and $a\in A$. Then, the spectrum $\sigma(a)$ is a compact subset of $\mathbb{C}$.*

Now, we will show that the spectrum is a non-empty subset of $\mathbb{C}$. This is exhibited by the following method of computing the spectral radius.

Theorem 1.2.5. *Let A be a unital Banach *-algebra and $h\in A$. Then, $\sigma(h)\neq\emptyset$, the sequence $\{\|h^n\|^{1/n}\}_{n\in\mathbb{Z}_{>0}}$ converges in $\mathbb{R}$ and*

$$\lim_{n\to\infty}\|h^n\|^{\frac{1}{n}}=r(h)=\sup_{\lambda\in\sigma(h)}|\lambda|.$$

Proof. Assume that A is a unital Banach *-algebra and $h\in A$. Define

$$\alpha(h)=\inf_{n\in\mathbb{Z}_{>0}}\|h^n\|^{\frac{1}{n}}.$$

We will show that the sequence $\{\|h^n\|^{1/n}\}_{n\in\mathbb{Z}_{>0}}$ converges to $\alpha(h)$. Assume that $\epsilon\in\mathbb{R}_{>0}$. From the definition of infimum, there exists an index $n_\epsilon\in\mathbb{Z}_{>0}$ such that

$$\|h^{n_\epsilon}\|^{\frac{1}{n_\epsilon}}\leq\alpha(h)+\epsilon.$$

Take any $n\in\mathbb{Z}_{>0}$ and use the Euclidean algorithm to write $n=qn_\epsilon+r$, where $q\in\mathbb{Z}_{>0}$ and $r\in\{0,1,\ldots,n_\epsilon-1\}$. Then,

$$\begin{aligned}\|h^n\|&=\|h^{qn_\epsilon+r}\|\\&\leq\|h^{n_\epsilon}\|^q\|h\|^r\\&\leq(\alpha(h)+\epsilon)^{qn_\epsilon}\|h\|^r\\&=(\alpha(h)+\epsilon)^{n-r}\|h\|^r.\end{aligned}$$

Taking the n^{th} root of both sides, we obtain the inequality

$$\|h^n\|^{\frac{1}{n}} \leq (\alpha(h)+\epsilon)^{1-\frac{r}{n}}\|h\|^{\frac{r}{n}}.$$

Consequently,

$$\alpha(h) \leq \liminf_{n\to\infty}\|h^n\|^{\frac{1}{n}} \leq \limsup_{n\to\infty}\|h^n\|^{\frac{1}{n}} \leq \alpha(h)+\epsilon.$$

This demonstrates that the sequence $\{\|h^n\|^{\frac{1}{n}}\}$ converges in $\mathbb{R}$ to $\alpha(h)$. The next step is to show that

$$\alpha(h) = r(h) = \sup_{\lambda\in\sigma(h)}|\lambda|. \tag{1.5}$$

To show: (a) $\alpha(h) \geq \sup_{\lambda\in\sigma(h)}|\lambda|$.

(b) $\alpha(h) \leq \sup_{\lambda\in\sigma(h)}|\lambda|$.

(a) Suppose for the sake of contradiction that $\alpha(h) < |\lambda|$ for some $\lambda \in \sigma(h)$. By the root test, the series

$$\sum_{n=0}^{\infty}\frac{\|h^n\|}{|\lambda|^n}$$

in $\mathbb{R}$ is convergent. Therefore, the sum

$$\sum_{n=0}^{\infty}\lambda^{-n}h^n$$

converges in A and is a well-defined element of A. By using similar arguments to Theorem 1.2.2 and Theorem 1.2.3, we deduce that

$$\sum_{n=0}^{\infty}\lambda^{-n}h^n = (1_A - \frac{h}{\lambda})^{-1}.$$

So, $\lambda 1_A - h$ is invertible and $\lambda \in \rho(h)$. But this contradicts the fact that $\lambda \in \sigma(h)$. Therefore, $\alpha(h) \geq \sup_{\lambda\in\sigma(h)}|\lambda|$.

(b) We will divide this into two cases. First, we note that if $x, y \in A$ then

$$
\begin{aligned}
\alpha(xy) &= \lim_{n\to\infty} \|(xy)^n\|^{\frac{1}{n}} \\
&= \lim_{n\to\infty} \|x^n y^n\|^{\frac{1}{n}} \\
&\le \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} \|y^n\|^{\frac{1}{n}} \\
&= \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} \lim_{n\to\infty} \|y^n\|^{\frac{1}{n}} \\
&= \alpha(x)\alpha(y).
\end{aligned}
$$

Case 1: $\alpha(h) = 0$.

If $\alpha(h) = 0$ then h is not invertible because otherwise,

$$1 = \alpha(1_A) = \alpha(hh^{-1}) \le \alpha(h)\alpha(h^{-1}) = 0.$$

So, $0 \in \sigma(h)$ and since $\alpha(h) \ge \sup_{\lambda\in\sigma(h)}|\lambda|$ from part (a),

$$\alpha(h) = \sup_{\lambda\in\sigma(h)} |\lambda| = 0.$$

Case 2: $\alpha(h) > 0$.

Assume that $\alpha(h) > 0$ and $\alpha(h) > \sup_{\lambda\in\sigma(h)}|\lambda|$. Since the spectrum $\sigma(h)$ is a compact subset of $\mathbb{C}$, there exists $r \in (0, \alpha(h))$ such that

$$\sigma(h) \subseteq \{\lambda \in \mathbb{C} \mid |\lambda| \le r\}.$$

Let $D = \{\lambda \in \mathbb{C} \mid |\lambda| > r\}$. Then, $D \subseteq \rho(h)$. Let φ be a continuous linear functional on A and define the map

$$
\begin{array}{rccc}
\psi : & D & \to & \mathbb{C} \\
& \lambda & \mapsto & \varphi((\lambda 1_A - h)^{-1})
\end{array}
$$

The map ψ is holomorphic due to the series expansion in Theorem 1.2.2. In particular, if $|\lambda| > \alpha(h)$ then we have the series expansion

$$\varphi((\lambda 1_A - h)^{-1}) = \sum_{n=0}^{\infty} \lambda^{-n-1}\varphi(h^n).$$

The series $\sum_{n=0}^{\infty} \lambda^{-n-1}h^n$ converges in A because

$$\|\sum_{n=0}^{\infty} \lambda^{-n-1}h^n\| \le \sum_{n=0}^{\infty}|\lambda|^{-n-1}\|h^n\|$$

and by applying the root test on $\sum_{n=0}^{\infty}\lambda^{-n}\|h^n\|$, we find that

$$\lim_{n\to\infty}\frac{\|h^n\|^{\frac{1}{n}}}{|\lambda|}=\frac{\alpha(h)}{|\lambda|}<1.$$

Moreover, $\sum_{n=0}^{\infty}\lambda^{-n-1}\varphi(h^n)\in\mathbb{C}$ vanishes as $\lambda\to\infty$. To see why this is the case, replace λ by $\lambda\mu$ and take the limit as $|\mu|\to\infty$. We obtain for $|\mu|>1$

$$\begin{aligned}
|\sum_{n=0}^{\infty}(\lambda\mu)^{-n-1}\varphi(h^n)| &\le |\sum_{n=0}^{\infty}(\lambda\mu)^{-n-1}\|\varphi\|\|h^n\||\\
&\le \frac{\|\varphi\|}{|\mu|}\sum_{n=0}^{\infty}\frac{\|h^n\|}{|\lambda|^{n+1}}|\mu|^{-n}\\
&\le \frac{\|\varphi\|}{|\mu|}\sum_{n=0}^{\infty}\frac{\|h^n\|}{|\lambda|^{n+1}}\\
&\to 0
\end{aligned}$$

as $|\mu|\to\infty$. Consequently, the function $f:\mathbb{C}\to\mathbb{C}$ defined by

$$f(\mu)=\begin{cases}0, \text{ if } \mu=0,\\ \varphi((\frac{1}{\mu}1_A-h)^{-1}), \text{ if } 0<|\mu|<\frac{1}{r}.\end{cases}$$

is a holomorphic function on the set

$$B(0,1/r)=\{\mu\in\mathbb{C}\mid|\mu|<\frac{1}{r}\}.$$

The Taylor expansion of f in the disk $B(0,1/r)$ is

$$f(\mu)=\sum_{n=0}^{\infty}\mu^{n+1}\varphi(h^n).$$

Furthermore, if $\mu\in B(0,1/r)$, then

$$\lim_{n\to\infty}\mu^{n+1}\varphi(h^n)=0.$$

Now, we take $\lambda_0\in\mathbb{C}$ such that $r<|\lambda_0|<\alpha(h)$. Then, $\frac{1}{\lambda_0}\in B(0,1/r)$ and

$$\lim_{n\to\infty}\lambda_0^{-n-1}\varphi(h^n)=0.$$

Let A^* denote the dual space of A and define for all $n\in\mathbb{Z}_{>0}$

$$\begin{array}{rccc}\rho_n: & A^* & \to & \mathbb{C}\\ & \varphi & \mapsto & \lambda_0^{-n-1}\varphi(h^n)\end{array}$$

The family $\{\rho_n\}_{n\in\mathbb{Z}_{>0}}$ is a family of continuous linear functionals on A^*. By the *uniform boundedness principle*, there exists a constant $M \in \mathbb{R}_{>0}$ such that

$$\begin{aligned}\sup_{n\in\mathbb{Z}_{>0}} |\lambda_0|^{-n-1}|\varphi(h^n)| \leq \|\varphi\| \sup_{n\in\mathbb{Z}_{>0}} |\lambda_0|^{-n-1}\|h^n\| \\ \leq M.\end{aligned}$$

Letting $N = M/\|\varphi\|$, we have $\sup_{n\in\mathbb{Z}_{>0}}|\lambda_0|^{-n-1}\|h^n\| \leq N$. Hence, if $n \in \mathbb{Z}_{>0}$ then

$$\|h^n\| \leq N|\lambda_0|^{n+1}$$

and from this inequality, we have

$$\begin{aligned}\alpha(h) &= \lim_{n\to\infty} \|h^n\|^{\frac{1}{n}} \\ &\leq \lim_{n\to\infty} N^{\frac{1}{n}}|\lambda_0|^{1+\frac{1}{n}} \\ &= |\lambda_0| < \alpha(h).\end{aligned}$$

This contradicts the assumption that $\alpha(h) > \sup_{\lambda\in\sigma(h)}|\lambda|$. Therefore, $\alpha(h) \leq \sup_{\lambda\in\sigma(h)}|\lambda|$.

Combining parts (a) and (b), we deduce that

$$\lim_{n\to\infty} \|h^n\|^{\frac{1}{n}} = \alpha(h) = r(h) = \sup_{\lambda\in\sigma(h)} |\lambda|.$$

□

Theorem 1.2.5 and Theorem 1.2.4 have particularly important consequences for C*-algebras, which we explore below.

Theorem 1.2.6. *Let A be a unital C*-algebra. Let $a \in A$ be normal. Then, the spectral radius $r(a) = \|a\|$.*

Proof. Assume that A is a unital C*-algebra and $a \in A$ is normal. If $n \in \mathbb{Z}_{>0}$ then a^n is also normal and

$$\|a^{2^n}\| = \|a\|^{2^n}.$$

To see why this is the case, we will proceed by induction. For the base case, if $n = 1$ then we compute directly that

$$\begin{aligned}\|a^2\|^2 &= \|(a^2)^*a^2\| \\ &= \|a^*(a^*a)a\| \\ &= \|a^*(aa^*)a\| \\ &= \|(a^*a)^*a^*a\| = \|a^*a\|^2 \\ &= \|a\|^4.\end{aligned}$$

So, $\|a^2\| = \|a\|^2$. This proves the base case. For the inductive hypothesis, assume that $\|a^{2^k}\| = \|a\|^{2^k}$ for some $k \in \mathbb{Z}_{>0}$. Then,

$$\begin{aligned}\|a^{2^{k+1}}\|^2 &= \|(a^{2^k})^2\|^2 \\ &= \|((a^{2^k})^2)^*(a^{2^k})^2\| \\ &= \|(a^*)^{2^k}(a^*)^{2^k}a^{2^k}a^{2^k}\| \\ &= \|(a^*)^{2^k}a^{2^k}(a^*)^{2^k}a^{2^k}\| \\ &= \|((a^*a)^{2^k})^*(a^*a)^{2^k}\| \quad \text{(since } a \text{ is normal)} \\ &= \|(a^*a)^{2^k}\|^2 = \|(a^{2^k})^*a^{2^k}\|^2 \\ &= \|a^{2^k}\|^4 = \|a\|^{2^{k+2}}.\end{aligned}$$

Hence, $\|a^{2^{k+1}}\| = \|a\|^{2^{k+1}}$, which completes the induction.

Now consider the sequence $\{\|a^n\|^{\frac{1}{n}}\}_{n\in\mathbb{Z}_{>0}}$ in $\mathbb{R}$. We know from Theorem 1.2.5 that this sequence converges to $r(a)$. However, the subsequence $\{\|a^{2^k}\|^{2^{-k}}\}_{k\in\mathbb{Z}_{>0}}$ is a constant sequence which converges to $\|a\|$. Therefore,

$$r(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} = \lim_{k\to\infty} \|a^{2^k}\|^{2^{-k}} = \|a\|.$$

□

It is remarked in [Put19] that Theorem 1.2.6 seems restrictive because it only applies to normal elements of a unital C*-algebra. However, we observe that Theorem 1.2.6 can be used to yield information about the norm of an *arbitrary* element of a unital C*-algebra. If A is a unital C*-algebra and $a \in A$ then $a^*a \in A$ is self-adjoint and by Theorem 1.2.6,

$$\|a\| = \|a^*a\|^{\frac{1}{2}} = r(a^*a)^{\frac{1}{2}}.$$

The next theorems demonstrate why the above observation is useful.

Theorem 1.2.7. *Let A and B be unital C*-algebras and $\phi : A \to B$ be a unital *-homomorphism. If $a \in A$ then $\|\phi(a)\| \leq \|a\|$. In particular, $\|\phi\| \leq 1$ and ϕ is called* ***contractive****.*

Proof. Assume that A and B are unital C*-algebras and $\rho : A \to B$ is a unital *-homomorphism.

First, assume that $b \in A$ is self-adjoint.

To show: (a) $\rho(b) \subseteq \rho(\phi(b))$.

(a) Assume that $\lambda \in \rho(b)$ so that $\lambda 1_A - b \in A$ is invertible. Then, there exists $(\lambda 1_A - b)^{-1} \in A$ such that

$$(\lambda 1_A - b)^{-1}(\lambda 1_A - b) = (\lambda 1_A - b)(\lambda 1_A - b)^{-1} = 1_A.$$

Applying the unital *-homomorphism ϕ to the above equation, we find that $\phi(\lambda 1_A - b) = \lambda 1_B - \phi(b)$ is invertible in B. So, $\lambda \in \rho(\phi(b))$ and $\rho(b) \subseteq \rho(\phi(b))$.

By part (a), $\sigma(\phi(b)) \subseteq \sigma(b)$. Since b and $\phi(b)$ are both self-adjoint, we have by Theorem 1.2.6

$$\|\phi(b)\| = \sup_{\lambda \in \sigma(\phi(b))} |\lambda| \leq \sup_{\lambda \in \sigma(b)} |\lambda| = \|b\|.$$

Now let $a \in A$ be an arbitrary element. We compute directly that

$$\begin{aligned} \|\phi(a)\| &= \|\phi(a^*a)\|^{\frac{1}{2}} \\ &\leq \|a^*a\|^{\frac{1}{2}} = \|a\|. \end{aligned}$$

Hence, $\phi : A \to B$ is contractive. □

Another surprising result is that the norm of a unital C*-algebra is unique.

Theorem 1.2.8. *Let A be a unital C*-algebra. Then, the norm on A is unique. That is, if a unital Banach *-algebra possesses a norm which turns it into a unital C*-algebra then this is the only such norm.*

Proof. Assume that A is a unital C*-algebra. By Theorem 1.2.6,

$$\|a\| = \|a^*a\|^{\frac{1}{2}} = (\sup\{|\lambda| \mid \lambda 1_A - a^*a \text{ is not invertible}\})^{\frac{1}{2}}.$$

The RHS of the above equation only depends on the algebraic structure of A. Hence, it uniquely determines the norm on A. □

To solidify the concepts introduced in this section, we will now work through some examples.

Example 1.2.1. We work in the C*-algebra $M_{2\times 2}(\mathbb{C})$. Let $t \in \mathbb{R}_{\geq 0}$ and

$$a = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

We will compute the spectrum, spectral radius and norm of a. The characteristic polynomial of a is

$$\det(\lambda I_2 - a) = (\lambda - 1)^2.$$

Here, I_2 is the 2×2 identity matrix. Hence, the spectrum $\sigma(a) = \{1\}$. By definition, the spectral radius of a is

$$r(a) = \sup_{\lambda \in \sigma(a)} |\lambda| = 1.$$

Computing the norm of a is more involved. Note that by a quick computation, $a^*a \neq aa^*$. Hence, Theorem 1.2.6 does not apply. Instead, we use the characterisation of the norm in Theorem 1.2.8.

We compute directly that

$$a^*a = \begin{pmatrix} 1 & t \\ t & t^2+1 \end{pmatrix}.$$

By computing the roots of the characteristic polynomial of a^*a, we find that

$$\sigma(a^*a) = \left\{ \frac{t^2 + 2 \pm t\sqrt{t^2+4}}{2} \right\}.$$

So,

$$r(a^*a) = \frac{t^2 + 2 + t\sqrt{t^2+4}}{2} = 1 + \frac{t(t + \sqrt{t^2+4})}{2}$$

and

$$\|a\| = \sqrt{r(a^*a)} = \sqrt{1 + \frac{t(t+\sqrt{t^2+4})}{2}}.$$

Example 1.2.2. Consider the $\mathbb{C}$-algebra of polynomials $\mathbb{C}[x]$. If $p(x) \in \mathbb{C}[x]$ is a non-constant polynomial then its degree is greater than zero and its spectrum is

$$\sigma(p(x)) = \mathbb{C}.$$

We claim that there is no norm on $\mathbb{C}[x]$ which makes it a C*-algebra. Suppose for the sake of contradiction that there exists a norm on $\mathbb{C}[x]$ which makes $\mathbb{C}[x]$ a C*-algebra. By Theorem 1.2.8, if $p(x) \in \mathbb{C}[x]$ is a non-constant polynomial then

$$\|p(x)\| = r(\overline{p(x)}p(x))^{\frac{1}{2}} = \infty.$$

This contradicts the fact that in a unital Banach *-algebra (or a C*-algebra), the spectral radius is bounded ($r(p(x)) \leq \|p(x)\|$) by Theorem 1.2.5. Hence, $\mathbb{C}[x]$ cannot be a C*-algebra.

Example 1.2.3. Now consider the $\mathbb{C}$-algebra of rational functions $\mathbb{C}(x)$ — the field of fractions of the polynomial ring $\mathbb{C}[x]$. If $p(x) \in \mathbb{C}(x)$ is non-constant then

$$\sigma(p(x)) = \emptyset.$$

Suppose for the sake of contradiction that there exists a norm on $\mathbb{C}(x)$ which makes $\mathbb{C}(x)$ a C*-algebra. If $p(x) \in \mathbb{C}(x)$ is non-constant then the spectrum $\sigma(p(x)) = \emptyset$, which contradicts the fact that the spectrum must be non-empty (see Theorem 1.2.5). Therefore, $\mathbb{C}(x)$ is also not a C*-algebra.

1.3 Commutative unital C*-algebras

The prototypical example of a commutative unital C*-algebra is $Cts(X, \mathbb{C})$ where X is a compact Hausdorff topological space. In this section, we will demonstrate that in a sense, this is the *only* example — every commutative unital C*-algebra is isomorphic to $Cts(X, \mathbb{C})$ for some compact Hausdorff space X. For the sake of clarity, let us define the term isomorphic in this context.

Definition 1.3.1. Let A and B be C*-algebras. We say that A and B are **isomorphic** if there exists an isometric *-isomorphism $\phi : A \to B$.

Actually, we can get away with omitting the term "isometric" in the above definition. We will eventually prove Theorem 1.6.4, which states that any injective *-homomorphism is isometric.

Definition 1.3.2. Let A be a $\mathbb{C}$-algebra. Define $\mathcal{M}(A)$ to be the set of non-zero $\mathbb{C}$-algebra homomorphisms from A to $\mathbb{C}$.

As explained in [Put19], the above notation originates from the fact that a $\mathbb{C}$-algebra homomorphism is a multiplicative linear map. We begin by proving some facts about the set $\mathcal{M}(A)$.

Theorem 1.3.1. *Let A be a commutative unital C*-algebra.*

1. *If $a \in A$ and $\phi \in \mathcal{M}(A)$ then $\phi(a) \in \sigma(a)$.*
2. *If $\phi \in \mathcal{M}(A)$ then $\|\phi\| = 1$.*
3. *If $a \in A$ and $\phi \in \mathcal{M}(A)$ then $\phi(a^*) = \overline{\phi(a)}$.*

Proof. Assume that A is a commutative unital C*-algebra. First assume that $a \in A$ and $\phi \in \mathcal{M}(A)$. Since ϕ is a non-zero $\mathbb{C}$-algebra homomorphism, the kernel $\ker \phi$ is a proper ideal of A.

We claim that if $a \in \ker \phi$ then a is not invertible. We prove the contrapositive statement. Assume that $a \in A$ is invertible so that there exists a^{-1} such that $aa^{-1} = a^{-1}a = 1_A$. By applying ϕ, we find that

$$\phi(a)\phi(a^{-1}) = \phi(1_A) = 1$$

where the last equality follows from the fact that ϕ is a $\mathbb{C}$-algebra homomorphism and thus, preserves multiplicative units. Therefore, $\phi(a) \neq 0$ and $a \notin \ker \phi$.

Now note that $\phi(a)1_A - a \in \ker \phi$ and by the previous claim, $\phi(a)1_A - a$ is not invertible in A. Consequently, $\phi(a) \in \sigma(a)$.

Next, we will show that $\|\phi\| = 1$.

To show: (a) $\|\phi\| \leq 1$.

(b) $\|\phi\| \geq 1$.

(a) Using the fact that $r(a) \leq \|a\|$, we compute directly that

$$\begin{aligned} \|\phi\| &= \sup_{\|a\|=1} |\phi(a)| \\ &\leq \sup_{\|a\|=1} r(a) \quad (\text{since } \phi(a) \in \sigma(a)) \\ &\leq \sup_{\|a\|=1} \|a\| = 1. \end{aligned}$$

(b) Consider the element $\phi(1_A) \in \mathbb{C}$. Since ϕ preserves multiplicative units, $\phi(1_A) = 1$. Now, $\|1\|^2 = \|\bar{1}1\| = \|1\|$. So, $\|1\| = 1$ and

$$\|\phi\| = \sup_{\|a\|=1} |\phi(a)| \geq |\phi(1_A)| = 1.$$

By combining parts (a) and (b), we deduce that $\|\phi\| = 1$.

For the final part of the proof, take $a \in A$ and write $a = b + ic$ where

$$b = \frac{1}{2}(a + a^*) \qquad \text{and} \qquad c = \frac{i(a^* - a)}{2}.$$

Note that b and c are both self-adjoint elements of A. So, it suffices to prove that if $b = b^*$ then $\phi(b) \in \mathbb{R}$. To this end, assume that $b \in A$ is self-adjoint.

For $t \in \mathbb{R}$, define

$$u_t = e^{itb} = \sum_{n=0}^{\infty} \frac{(itb)^n}{n!}$$

The norm of u_t is finite because

$$\begin{aligned}
\|u_t\| &= \|\sum_{n=0}^{\infty} \frac{(itb)^n}{n!}\| \\
&\leq \sum_{n=0}^{\infty} \frac{|t|^n \|b\|^n}{n!} \\
&= e^{|t|\|b\|} < \infty.
\end{aligned}$$

So, u_t is a well-defined element of A. Furthermore, one can verify that $u_{-t} = u_t^*$ and $u_t u_{-t} = 1_A$. So,

$$\|u_t\| = \|u_t u_t^*\|^{\frac{1}{2}} = \|u_t u_{-t}\| = \|1_A\| = 1.$$

Since $\|\phi\| = 1$ (from the second part of the theorem), if $t \in \mathbb{R}$ then

$$1 = \|\phi\| \geq |\phi(u_t)| = |e^{it\phi(b)}|.$$

Thus, $\phi(b) \in \mathbb{R}$ which completes the proof because if $a = b + ic \in A$ then

$$\phi(a^*) = \phi(b - ic) = \phi(b) - i\phi(c) = \overline{\phi(b) + i\phi(c)} = \overline{\phi(a)}.$$

$\square$

At this point, one might suspect that the set $\mathcal{M}(A)$ plays an important role in constructing the isomorphism alluded to in the beginning of this section. To elucidate this point, we need to better understand $\mathcal{M}(A)$ as a topological space, rather than just a set. Theorem 1.3.1 provides the first step towards this goal. In particular, the second statement in Theorem 1.3.1 reveals that

$$\mathcal{M}(A) \subseteq \{\psi \in A^* \mid \|\psi\| \leq 1\}.$$

Now recall that Banach-Alaoglu theorem, which states that the closed unit ball

$$\{\psi \in A^* \mid \|\psi\| \leq 1\} \subseteq A^*$$

is compact with respect to the weak-* topology on the dual space A^*. This suggests that we consider $\mathcal{M}(A)$ with the weak-* topology.

Here is a refresher on the weak-* topology.

Definition 1.3.3. Let X be a Banach space and X^* be its dual space. The **weak-* topology** on X^* is the weakest topology on X^* such that the evaluation maps

$$\begin{array}{rccc} ev_x : & X^* & \to & \mathbb{C} \\ & \varphi & \mapsto & \varphi(x) \end{array}$$

are continuous for $x \in X$.

Recall that a sequence $\{\phi_n\}_{n\in\mathbb{Z}_{>0}}$ in the dual space X^* converges to $\phi \in X^*$ in the weak-* topology if and only if for $x \in X$, the sequence $\{\phi_n(x)\}$ converges to $\phi(x)$ in $\mathbb{C}$.

Theorem 1.3.2. *Let A be a commutative unital C^*-algebra. The set $\mathcal{M}(A)$ is a compact subset of the closed unit ball*

$$\{\psi \in A^* \mid \|\psi\| \leq 1\} \subseteq A^*$$

with respect to the weak-* topology on A^*.

Proof. Assume that A is a commutative unital C*-algebra. As mentioned previously, Theorem 1.3.1 yields the inclusion

$$\mathcal{M}(A) \subseteq \{\psi \in A^* \mid \|\psi\| \leq 1\}.$$

By the Banach-Alaoglu theorem, the closed unit ball is compact with respect to the weak-* topology on A^*.

To show: (a) $\mathcal{M}(A)$ is a closed subset of the closed unit ball of A^*.

(a) Assume that $\phi \in \overline{\mathcal{M}(A)}$. Then, there exists a sequence $\{\phi_n\}_{n\in\mathbb{Z}_{>0}}$ which converges to ϕ with respect to the weak-* topology. This means that if $x \in X$ then the sequence $\{\phi_n(x)\}$ in $\mathbb{C}$ converges to $\phi(x)$.

By definition, ϕ is a linear functional on A. To see that ϕ is non-zero, we compute the norm of ϕ as

$$\begin{aligned}
\|\phi\| &= \sup_{\|x\|=1} |\phi(x)| \\
&= \sup_{\|x\|=1} \lim_{n\to\infty} |\phi_n(x)| \\
&= \lim_{n\to\infty} \sup_{\|x\|=1} |\phi_n(x)| \\
&= \lim_{n\to\infty} \|\phi_n\| = 1.
\end{aligned}$$

Finally, to see that ϕ is multiplicative, we have for $a, b \in A$ that

$$\begin{aligned}
\phi(ab) &= \lim_{n\to\infty} \phi_n(ab) \\
&= \lim_{n\to\infty} \phi_n(a)\phi_n(b) \\
&= \phi(a)\phi(b).
\end{aligned}$$

Therefore, $\phi \in \mathcal{M}(A)$ and $\overline{\mathcal{M}(A)} = \mathcal{M}(A)$. So, $\mathcal{M}(A)$ is a closed subset of the closed unit ball of A^* with respect to the weak-* topology.

Since $\mathcal{M}(A)$ is a closed subset of a compact set, it must be compact with respect to the weak-* topology as required. $\square$

To see that $\mathcal{M}(A)$ is Hausdorff, note that A^* with the weak-* topology is Hausdorff. Indeed, if $\phi, \psi \in A^*$ such that $\phi \neq \psi$ then there exists $x \in A$ such that

$$ev_x(\phi) = \phi(x) \neq \psi(x) = ev_x(\psi).$$

Hence, $\mathcal{M}(A)$ is a subspace of a Hausdorff space and is thus, Hausdorff. Therefore, $\mathcal{M}(A)$ is a compact Hausdorff topological space. Since we want to show that every commutative unital C*-algebra is of the form $Cts(X, \mathbb{C})$ for some compact Hausdorff space X, we will show that A as a commutative unital C*-algebra is isomorphic to $Cts(\mathcal{M}(A), \mathbb{C})$.

Theorem 1.3.3. *Let A be a commutative unital C*-algebra and $a \in A$. Then, the evaluation map*

$$\begin{array}{rccc} ev_a : & \mathcal{M}(A) & \to & \mathbb{C} \\ & \phi & \mapsto & \phi(a) \end{array}$$

is a continuous map from $\mathcal{M}(A)$ onto the spectrum $\sigma(a)$.

Proof. Assume that A is a commutative unital C*-algebra. Assume that $a \in A$. By definition of the weak-* topology on A^*, the evaluation map $ev_a : A^* \to \mathbb{C}$ is continuous. Furthermore, recall from Theorem 1.3.1 that if $\phi \in \mathcal{M}(A)$ then $ev_a(\phi) = \phi(a) \in \sigma(a)$.

To show: (a) If $\lambda \in \sigma(a)$ then there exists $\psi \in A^*$ such that $ev_a(\psi) = \lambda$.

(a) Assume that $\lambda \in \sigma(a)$. Define the set S by

$$S = \{\text{Ideals } I \subsetneq A \mid \lambda 1_A - a \in I\}.$$

Then, S is a poset when equipped with the partial order of inclusion. It is non-empty because $\lambda 1_A - a$ is an element of the ideal $(\lambda 1_A - a)$ generated by $\lambda 1_A - a$.

Now let $S' \subseteq S$ be a totally ordered subset of S. Define

$$J = \sum_{I \in S'} I.$$

Then, J is an ideal of A because it is the sum of ideals of A. Moreover if $I \in S'$ then $I \subseteq J$. Hence, J is an upper bound for the totally ordered set S'.

By Zorn's lemma, there exists a maximal element $I_{max} \in S$. This means that $\lambda 1_A - a$ is contained in the maximal proper ideal $I_{max} \subseteq A$. We now claim that I_{max} is closed.

To show: (aa) I_{max} is closed.

(aa) The closure $\overline{I_{max}}$ is an ideal of A such that $I_{max} \subseteq \overline{I_{max}}$. We claim that $\overline{I_{max}}$ is a proper ideal of A. To see why, it suffices to show that the unit $1_A \notin \overline{I_{max}}$.

Suppose for the sake of contradiction that $1_A \in \overline{I_{max}}$. Then, $\overline{I_{max}} = A$. Let V denote the set of invertible elements of A. Then, $V \subseteq \overline{I_{max}} = A$. Since

I_{max} is a proper ideal of A, it cannot contain any invertible elements of A (as otherwise, $1_A \in I_{max}$). Hence, $V \cap I_{max} = \emptyset$.

Now since V is an open subset of A, there exists $\epsilon \in \mathbb{R}_{>0}$ such that the open ball

$$B(1_A, \epsilon) = \{a \in V \mid \|1_A - a\| < \epsilon\} \subseteq V.$$

By assumption, $1_A \in \overline{I_{max}}$. So, $B(1_A, \epsilon) \cap I_{max} \neq \emptyset$. However, this means that $V \cap I_{max} \neq \emptyset$, which contradicts the fact that $V \cap I_{max} = \emptyset$. Therefore, $1_A \notin \overline{I_{max}}$ and $\overline{I_{max}}$ is a proper ideal of A.

In particular, $\overline{I_{max}} \in S$ and $I_{max} \subseteq \overline{I_{max}}$. Since I_{max} is maximal in S, $I_{max} = \overline{I_{max}}$ and consequently, I_{max} is a closed ideal of A.

(a) The idea behind part (aa) is that since I_{max} is a closed maximal ideal of A, the quotient A/I_{max} is both a Banach space and a field. Hence, A/I_{max} is a Banach *-algebra with quotient norm

$$\|a + I_{max}\| = \inf_{j \in I_{max}} \|a + j\|.$$

To show: (ab) As Banach *-algebras, $A/I_{max} \cong \mathbb{C}$.

(ab) Let $b + I_{max} \in A/I_{max}$. Since A/I_{max} is a Banach *-algebra, then by Theorem 1.2.5 the spectrum $\sigma(b + I_{max}) \neq \emptyset$. So, there exists $\mu \in \mathbb{C}$ such that $\mu 1_A - b - I_{max}$ is not invertible. Since A/I_{max} is a field, the only non-invertible element of A/I_{max} is $0 + I_{max}$. Therefore,

$$\mu 1_A - b - I_{max} = 0 + I_{max}$$

and $b + I_{max} = \mu 1_A + I_{max} = \mu(1_A + I_{max})$. This means that every element of A/I_{max} is a scalar multiple of the unit $1_A + I_{max} \in A/I_{max}$. Thus, we have the isomorphism of Banach *-algebras, $A/I_{max} \cong \mathbb{C}$.

(a) Following on from part (ab), consider the quotient map

$$\begin{array}{rccc} \pi : & A & \to & A/I_{max} \\ & b & \mapsto & b + I_{max} = \mu(1_A + I_{max}). \end{array}$$

This is a continuous map. Next, let $\delta : A/I_{max} \to \mathbb{C}$ be the isomorphism given by

$$\delta(b + I_{max}) = \delta(\mu(1_A + I_{max})) = \mu.$$

Then, the composite $\delta \circ \pi \in A^*$ and since $\lambda 1_A - a \in I_{max}$ and $\lambda \in \sigma(a)$ by assumption, then $\lambda 1_A + I_{max} = a + I_{max}$ and

$$(\delta \circ \pi)(a) = \delta(a + I_{max}) = \delta(\lambda 1_A + I_{max}) = \delta(\lambda(1_A + I_{max})) = \lambda.$$

So, $ev_a(\delta \circ \pi) = \lambda$ which demonstrates that the image of ev_a is the spectrum $\sigma(a)$ as required. $\square$

Of course, we do not expect the evaluation map in Theorem 1.3.3 to be injective. Under a particular hypothesis, the evaluation map becomes injective, as we will see in the next theorem.

Theorem 1.3.4. *Let A be a commutative unital C*-algebra. Assume that there exists $a \in A$ such that A is generated as a C*-algebra by the set $\{1_A, a, a^*\}$. Then, the evaluation map*

$$\begin{array}{rccc} ev_a : & \mathcal{M}(A) & \to & \mathbb{C} \\ & \phi & \mapsto & \phi(a) \end{array}$$

is a homeomorphism from $\mathcal{M}(A)$ to the spectrum $\sigma(a)$.

Proof. Assume that A is a commutative unital C*-algebra. Assume that there exists $a \in A$ such that A is generated as a C*-algebra by the set $\{1_A, a, a^*\}$.

By Theorem 1.3.3, the evaluation map ev_a is continuous and surjective onto the spectrum $\sigma(a)$.

To show: (a) ev_a is injective.

(a) Assume that $\phi, \psi \in \mathcal{M}(A)$ such that $\phi(a) = \psi(a)$. By Theorem 1.3.1, $\phi(a^*) = \overline{\phi(a)} = \overline{\psi(a)} = \psi(a^*)$. Since ϕ, ψ are non-zero $\mathbb{C}$-algebra homomorphisms, $\phi(1_A) = 1 = \psi(1_A)$. So, ϕ and ψ agree on the generating set $\{1_A, a, a^*\}$ for A. Hence, $\phi = \psi$ in $\mathcal{M}(A)$ and ev_a is injective. $\square$

Now, we are ready to prove our main result.

Theorem 1.3.5. *Let A be a commutative unital C*-algebra. Define the map*

$$\begin{array}{rccc} \Lambda : & A & \to & Cts(\mathcal{M}(A), \mathbb{C}) \\ & a & \mapsto & ev_a \end{array}$$

*where ev_a is the evaluation map in Theorem 1.3.3. Then, Λ is an isometric *-isomorphism from A to $Cts(\mathcal{M}(A), \mathbb{C})$.*

Proof. Assume that A is a commutative unital C*-algebra. Assume that Λ is the map defined as above. First observe that Λ is well-defined because if $a \in A$ then $\Lambda(a) = ev_a$ is a continuous function by definition of the weak-* topology on $\mathcal{M}(A)$.

It is straightforward to verify that Λ is a $\mathbb{C}$-algebra homomorphism. If $a_1, a_2 \in A$, $\mu \in \mathbb{C}$ and $\phi \in \mathcal{M}(A)$ then

$$\begin{aligned}
\Lambda(a_1 + a_2)(\phi) &= ev_{a_1+a_2}(\phi) \\
&= \phi(a_1 + a_2) \\
&= \phi(a_1) + \phi(a_2) \\
&= ev_{a_1}(\phi) + ev_{a_2}(\phi) = \Lambda(a_1)(\phi) + \Lambda(a_2)(\phi),
\end{aligned}$$

$$\begin{aligned}
\Lambda(\mu a_1)(\phi) &= ev_{\mu a_1}(\phi) \\
&= \phi(\mu a_1) \\
&= \mu\phi(a_1) \\
&= \mu ev_{a_1}(\phi) = \mu\Lambda(a_1)(\phi)
\end{aligned}$$

and

$$\begin{aligned}
\Lambda(a_1 a_2)(\phi) &= ev_{a_1 a_2}(\phi) \\
&= \phi(a_1 a_2) \\
&= \phi(a_1)\phi(a_2) \\
&= ev_{a_1}(\phi) ev_{a_2}(\phi) = \Lambda(a_1)(\phi)\Lambda(a_2)(\phi).
\end{aligned}$$

To see that Λ is a *-homomorphism, we have by Theorem 1.3.1

$$\begin{aligned}
\Lambda(a_1^*)(\phi) &= ev_{a_1^*}(\phi) \\
&= \phi(a_1^*) = \overline{\phi(a_1)} \\
&= \overline{ev_{a_1}(\phi)} \\
&= \overline{\Lambda(a_1)(\phi)} = \Lambda(a_1)^*(\phi).
\end{aligned}$$

Next, we will show that Λ is isometric.

To show: (a) If $a \in A$ then $\|\Lambda(a)\| = \|a\|$.

(a) Assume that $a \in A$. First assume that a is self-adjoint. Using Theorem 1.2.6, we compute directly that

$$\begin{aligned}\|a\| &= r(a)\\ &= \sup_{\lambda\in\sigma(a)} |\lambda|\\ &= \sup_{\phi\in\mathcal{M}(A)} |\phi(a)| \quad \text{(by Theorem 1.3.3)}\\ &= \|ev_a\| = \|\Lambda(a)\|.\end{aligned}$$

Now assume that $a \in A$ is an arbitrary element. Then,

$$\|a\| = \|a^*a\|^{\frac{1}{2}} = \|\Lambda(a^*a)\|^{\frac{1}{2}} = \|\Lambda(a)^*\Lambda(a)\|^{\frac{1}{2}} = \|\Lambda(a)\|.$$

Finally, we show that the image of Λ is $Cts(\mathcal{M}(A), \mathbb{C})$.

To show: (b) im $\Lambda = Cts(\mathcal{M}(A), \mathbb{C})$.

(b) The idea here is that since $\mathcal{M}(A)$ with the weak-* topology is a compact Hausdorff topological space, we could potentially use the Stone-Weierstrass theorem on $Cts(\mathcal{M}(A), \mathbb{C})$.

Here is how the theorem applies. The image of Λ is a unital *-subalgebra of $Cts(\mathcal{M}(A), \mathbb{C})$. To see that im Λ separates points, let $\phi, \psi \in \mathcal{M}(A)$ such that $\phi \neq \psi$. Then, there exists $a \in A$ such that $\phi(a) \neq \psi(a)$. This means that $\Lambda(a)(\phi) \neq \Lambda(a)(\psi)$. So, the image im Λ separates points.

Now we can apply the Stone-Weierstrass theorem to deduce that im Λ is dense in $Cts(\mathcal{M}(A), \mathbb{C})$. Since Λ is an isometry and A is complete, we find that im Λ is closed. Hence,

$$\text{im } \Lambda = \overline{\text{im } \Lambda} = Cts(\mathcal{M}(A), \mathbb{C}).$$

This completes the proof. □

Next, we will consider how Theorem 1.3.5 applies to a unital, not necessarily commutative C*-algebra B. Let $a \in B$ and A be the C*-subalgebra of B with generating set $\{1_B, a, a^*\}$. Observe that A is commutative whenever a and a^* commute with each other — that is, when a is normal. As a more formal statement, $a \in B$ is normal if and only if there exists a commutative unital C*-subalgebra $A \subseteq B$ such that $a \in A$.

If we apply Theorem 1.3.5 to A, we obtain the C*-algebra isomorphism $A \cong Cts(\mathcal{M}(A), \mathbb{C})$. Moreover, Theorem 2.1.8 yields a homeomorphism from $\mathcal{M}(A)$ with the weak-* topology to the spectrum $\sigma(a)$. However, by construction of A as a C*-subalgebra of B, the spectrum $\sigma(a)$ here is ill-defined. The problem is that if $c \in A$ then c could have an inverse in B, but not in A. That is, the invertibility of c depends on whether we consider c an element of A or an element of B.

We will prove below that in this special case, the above observation is not an issue — invertibility in A is equivalent to invertibility in B. Since the spectrum of c depends on whether we think of it in A or B, we introduce the notation

$$\sigma_A(c) = \{\lambda \in \mathbb{C} \mid \lambda 1_B - c \text{ is not invertible in } A\}.$$

and

$$\sigma_B(c) = \{\lambda \in \mathbb{C} \mid \lambda 1_B - c \text{ is not invertible in } B\}.$$

Note that in our case, $\sigma_B(c) \subseteq \sigma_A(c)$.

Theorem 1.3.6. *Let B be a unital C*-algebra and A be a commutative C*-subalgebra of B such that $1_B \in A$. Let $a \in A$. Then, a has an inverse in A if and only if it has an inverse in B. Consequently, $\sigma_A(a) = \sigma_B(a)$.*

Proof. Assume that B is a unital C*-algebra and A is a commutative C*-subalgebra of B such that $1_B \in A$. Assume that $a \in A$. We already know that $\sigma_B(a) \subseteq \sigma_A(a)$. This means that if a is invertible in A then it is invertible in B.

To show: (a) If a is invertible in B then it is invertible in A.

(a) Assume that a has an inverse in B. Its adjoint a^* must have inverse given by $(a^*)^{-1} = (a^{-1})^*$. Therefore, the self-adjoint element a^*a is invertible. By Theorem 1.3.3 and the first statement of Theorem 1.3.1, we have

$$\sigma_A(a^*a) = \{\phi(a^*a) \mid \phi \in \mathcal{M}(A)\}.$$

By the third statement of Theorem 1.3.1, we have

$$\phi(a^*a) = \phi(a^*)\phi(a) = \overline{\phi(a)}\phi(a) = |\phi(a)|^2 \in \mathbb{R}_{\geq 0}.$$

The second statement of Theorem 1.3.1 tells us that if $\phi \in \mathcal{M}(A)$ then $\|\phi\| = 1$ and

$$\frac{|\phi(a^*a)|}{\|a^*a\|} \leq \sup_{x\neq 0} \frac{|\phi(x)|}{\|x\|} = \|\phi\| = 1.$$

So, $|\phi(a^*a)| = \phi(a^*a) \leq \|a^*a\|$. By combining the previous two observations, we find that $0 \leq \phi(a^*a) \leq \|a^*a\|$ and

$$\sigma_B(a^*a) \subseteq \sigma_A(a^*a) \subseteq [0, \|a^*a\|].$$

Now a^*a is invertible in B. So, $0 \notin \sigma_B(a^*a)$. Thus, there exists $\delta \in \mathbb{R}_{>0}$ such that

$$\sigma_B(a^*a) \subseteq [\delta, \|a^*a\|].$$

Now consider the spectrum $\sigma_B(\|a^*a\|1_B - a^*a)$. We want to show that it is contained in a particular interval of $\mathbb{R}_{\geq 0}$. The key is to notice that if $\lambda \in \mathbb{C}$ then

$$\lambda 1_B - (\|a^*a\|1_B - a^*a) = -\big((\|a^*a\| - \lambda)1_B - a^*a\big).$$

Therefore,

$$\sigma_B(\|a^*a\|1_B - a^*a) \subseteq \{\|a^*a\| - \lambda \mid \lambda \in \sigma_B(a^*a)\} \subseteq [0, \|a^*a\| - \delta].$$

Now the element $\|a^*a\|1_B - a^*a$ is self-adjoint. By Theorem 1.2.6,

$$\big\|\|a^*a\|1_B - a^*a\big\| = r(\|a^*a\|1_B - a^*a) \leq \|a^*a\| - \delta < \|a^*a\|.$$

By the second part of Theorem 1.3.1, if $\phi \in \mathcal{M}(A)$ then

$$|\phi(\|a^*a\|1_B - a^*a)| \leq \big\|\|a^*a\|1_B - a^*a\big\| < \|a^*a\|.$$

But,

$$\phi(\|a^*a\|1_B - a^*a) = \|a^*a\| - |\phi(a)|^2.$$

Subsequently, $\phi(a) \neq 0$ for $\phi \in \mathcal{M}(A)$. By Theorem 1.3.3 and Theorem 1.3.1, we deduce that $0 \notin \sigma_A(a)$. Therefore, a has an inverse in A.

So, $a \in A$ has an inverse in B if and only if it has an inverse in A. The equality of spectra $\sigma_A(a) = \sigma_B(a)$ follows directly. □

Now we return to the situation where B is a unital C*-algebra and A is a commutative unital C*-subalgebra of B, generated by the set $\{1_B, a, a^*\}$ for some normal element $a \in B$. The next definition we make exploits the fact that we have an inverse to the isometric *-isomorphism in Theorem 1.3.5.

Definition 1.3.4. Let B be a unital C*-algebra and $a \in B$ be normal. Let A be the commutative unital C*-subalgebra of B generated by the set $\{1_B, a\}$. By Theorem 2.1.8, we have a homeomorphism $\mathcal{M}(A) \cong \sigma(a)$. Hence, we can identify $Cts(\mathcal{M}(A), \mathbb{C})$ with the space of continuous functions $Cts(\sigma(a), \mathbb{C})$.

Let $\Lambda : A \to Cts(\sigma(a), \mathbb{C})$ denote the isometric *-isomorphism in Theorem 1.3.5 and $f \in Cts(\sigma(a), \mathbb{C})$. Since Λ is surjective, we let $f(a)$ be the unique element of A such that $\Lambda(f(a)) = f$. That is, if $\phi \in \mathcal{M}(A)$ then $\phi(a) \in \sigma(a)$ and

$$\phi(f(a)) = ev_{f(a)}(\phi) = \Lambda(f(a))(\phi) = f(\phi(a)).$$

Using the above definition, we can restate Theorem 1.3.5 as follows:

Theorem 1.3.7 (Continuous functional calculus)**.** *Let B be a unital C*-algebra and $a \in B$ be normal. The map*

$$\begin{array}{rccc} \Lambda^{-1}: & Cts(\sigma(a), \mathbb{C}) \cong Cts(\mathcal{M}(A), \mathbb{C}) & \to & B \\ & f & \mapsto & f(a) \end{array}$$

*is an isometric *-isomorphism from $Cts(\sigma(a), \mathbb{C})$ to the C*-subalgebra of B generated by the set $\{1, a\}$. Moreover, if $f(z) = \sum_{k,l} a_{k,l} z^k \overline{z}^l$ is any polynomial in the variables z and $\overline{z}$ then*

$$\Lambda^{-1}(f(z)) = f(a) = \sum_{k,l} a_{k,l} a^k (a^*)^l.$$

Theorem 1.3.7 is the well-known continuous functional calculus on normal elements of a unital C*-algebra. In the reference [Sol18], the continuous functional calculus is done for the unital C*-algebra of bounded linear operators on a Hilbert space. The continuous functional calculus is first proved for self-adjoint operators and then after establishing the notion of a *joint spectrum*, it is extended to normal operators.

Here are some important consequences of Theorem 1.3.7.

Theorem 1.3.8. *Let B be a unital C*-algebra and $a \in B$ be normal. Then, a is self-adjoint if and only if $\sigma(a) \subseteq \mathbb{R}$.*

Proof. Assume that B is a unital C*-algebra and $a \in B$ is normal. Let $id_{\sigma(a)}$ denote the identity function on the spectrum $\sigma(a) \subseteq \mathbb{C}$. Then, $id_{\sigma(a)} \in Cts(\sigma(a), \mathbb{C})$ and by the continuous functional calculus in Theorem 1.3.7,

$$\Lambda^{-1}(id_{\sigma(a)}) = a.$$

So, a is self-adjoint if and only if $a = a^*$ if and only if $id_{\sigma(a)} = \overline{id_{\sigma(a)}}$ if and only if $\sigma(a) \subseteq \mathbb{R}$. $\square$

Theorem 1.3.9. *Let A and B be unital C*-algebras. Let $\phi : A \to B$ be a unital *-homomorphism. If $a \in A$ is normal then $\sigma(\phi(a)) \subseteq \sigma(a)$. Moreover, if $f \in Cts(\sigma(a), \mathbb{C})$ then $f(\phi(a)) = \phi(f(a))$.*

Proof. Assume that A and B are unital C*-algebras. Assume that $\phi : A \to B$ is a unital *-homomorphism.

To show: (a) If $a \in A$ is normal then $\sigma(\phi(a)) \subseteq \sigma(a)$.

(b) If $f \in Cts(\sigma(a), \mathbb{C})$ then $f(\phi(a)) = \phi(f(a))$.

(a) Assume that $a \in A$ is normal. If $\lambda \in \rho(a)$ then $\lambda 1_A - a$ is invertible and $\phi(\lambda 1_A - a) = \lambda 1_B - \phi(a)$ is invertible in B. Hence, $\rho(a) \subseteq \rho(\phi(a))$ and $\sigma(\phi(a)) \subseteq \sigma(a)$.

(b) Assume that $f \in Cts(\sigma(a), \mathbb{C})$. Let C be the C*-subalgebra of A generated by the set $\{1_A, a\}$ and D be the C*-subalgebra of B generated by the set $\{1_B, \phi(a)\}$. Then, the restriction $\phi|_C : C \to D$ is a unital *-homomorphism.

Using the notation in Theorem 1.3.7, let $\Lambda_C : C \to Cts(\sigma(a), \mathbb{C})$ and $\Lambda_D : D \to Cts(\sigma(\phi(a)), \mathbb{C})$ denote the continuous functional calculi on C and D respectively. To see that $\phi(f(a)) = f(\phi(a))$, assume that $\psi \in \mathcal{M}(D)$. Then $\psi \circ \phi \in \mathcal{M}(C)$ and

$$\begin{aligned}
\Lambda_D(\phi(f(a)))(\psi) &= ev_{\phi(f(a))}(\psi) = \psi(\phi(f(a))) \\
&= (\psi \circ \phi)(f(a)) \\
&= ev_{f(a)}(\psi \circ \phi) = \Lambda_C(f(a))(\psi \circ \phi) \\
&= f((\psi \circ \phi)(a)) \\
&= f(\psi(\phi(a))) = \Lambda_D(f(\phi(a)))(\psi).
\end{aligned}$$

Since Λ_D is injective and $\psi \in \mathcal{M}(D)$ was arbitrary, we must have $\phi(f(a)) = f(\phi(a))$. $\square$

Next, we would like to extend Theorem 1.2.7, which states that a unital *-homomorphism between C*-algebras is necessarily contractive. We will prove that if the unital *-homomorphism is also injective then it is necessarily an isometry.

Theorem 1.3.10. *Let A and B be unital C*-algebras and $\phi : A \to B$ be an injective, unital *-homomorphism. If $a \in A$ is normal then $\sigma(a) = \sigma(\phi(a))$.*

Proof. Assume that A and B are unital C*-algebras. Assume that $\phi : A \to B$ is an injective unital *-homomorphism. Assume that $a \in A$ is normal. We already have the inclusion $\sigma(\phi(a)) \subseteq \sigma(a)$.

Suppose for the sake of contradiction that $\sigma(\phi(a)) \subsetneq \sigma(a)$. Then, there exists a continuous function $f \in Cts(\sigma(a), \mathbb{C})$ such that $f \neq 0$, but if $\lambda \in \sigma(\phi(a))$ then $f(\lambda) = 0$.

Now by Theorem 1.3.9, $f(\phi(a)) = \phi(f(a))$. By the continuous functional calculus in Theorem 1.3.7 and the fact that the restriction $f|_{\sigma(\phi(a))} = 0$, $f(\phi(a)) = 0$. This contradicts the assumption that ϕ is injective. Therefore, $\sigma(\phi(a)) = \sigma(a)$. $\square$

Theorem 1.3.11. *Let A and B be unital C*-algebras. Let $\phi : A \to B$ be an injective *-homomorphism. Then, ϕ is an isometry.*

Proof. Assume that A and B are unital C*-algebras. Assume that $\phi : A \to B$ is an injective *-homomorphism.

To show: (a) If $a \in A$ is self-adjoint then $\|\phi(a)\| = \|a\|$.

(a) Assume that $a \in A$ is self-adjoint. By Theorem 1.2.6 and Theorem 1.3.10, we have

$$\|a\| = r(a) = \sup_{\lambda \in \sigma(a)} |\lambda| = \sup_{\lambda \in \sigma(\phi(a))} |\lambda| = r(\phi(a)) = \|\phi(a)\|.$$

Now assume that $a \in A$ is an arbitrary element. Then,

$$\|a\| = \|a^*a\|^{\frac{1}{2}} = \|\phi(a^*a)\|^{\frac{1}{2}} = \|\phi(a)^*\phi(a)\|^{\frac{1}{2}} = \|\phi(a)\|.$$

Hence, ϕ is an isometry. $\square$

Now recall Theorem 1.3.6, which states that if B is a unital C*-algebra and A is a commutative C*-subalgebra containing the unit then an element $a \in A$ is invertible in A if and only if it is invertible in B. We will now extend this result slightly, by removing the commutativity assumption.

Theorem 1.3.12 (Spectral permanence)**.** *Let B be a unital C*-algebra and A be a C*-subalgebra of B such that $1_B \in A$. If $a \in A$ then a is invertible in A if and only if a is invertible in B. That is, the spectrums $\sigma_A(a) = \sigma_B(a)$.*

Proof. Assume that B is a unital C*-algebra and A is a C*-subalgebra of B. Assume that $1_B \in A$. Assume that $a \in A$. If a is invertible in A then a is invertible in B since $A \subseteq B$.

To show: (a) If a has inverse $a^{-1} \in B$ then $a^{-1} \in A$.

(a) First assume that $a \in A$ is normal and has inverse $a^{-1} \in B$. The inclusion map $\iota : A \hookrightarrow B$ is an injective unital *-homomorphism. By Theorem 1.3.10, we find that $\sigma_A(a) = \sigma_B(a)$. In particular, $0 \notin \sigma_A(a)$ if and only if $0 \notin \sigma_B(a)$. So, a is invertible in A if and only if a is invertible in B.

Now assume that $a \in A$ is arbitrary. If a has an inverse $a^{-1} \in B$ then a^* also has an inverse in B. So, the normal element $a^*a \in A$ must have an inverse in B. But by the previous case, the inverse $(a^*a)^{-1} \in A$. Consequently,

$$a^{-1} = (a^*a)^{-1}(a^*a)a^{-1} = (a^*a)^{-1}a^* \in A.$$

Hence, $a^{-1} \in A$ which completes the proof. $\square$

A particularly useful consequence of the continuous functional calculus is the spectral mapping theorem, which yields a straightforward way of computing the spectrum of elements obtained by Theorem 1.3.7. We first need the following result.

Theorem 1.3.13. *Let X be a compact Hausdorff space and $f \in Cts(X, \mathbb{C})$. Then, the spectrum $\sigma(f) = im\ f$.*

Proof. Assume that X is a compact Hausdorff space and $f \in Cts(X, \mathbb{C})$. Let $\mathbb{1} \in Cts(X, \mathbb{C})$ be the function which sends $x \in X$ to 1. Then, $\mathbb{1}$ is the multiplicative unit in the C*-algebra $Cts(X, \mathbb{C})$.

First, assume that $\lambda \in \sigma(f)$. Then, $\lambda\mathbb{1} - f$ is not invertible and consequently, there exists $x \in X$ such that

$$\lambda - f(x) = (\lambda\mathbb{1} - f)(x) = 0.$$

So, $f(x) = \lambda$ and $\lambda \in \text{im } f$. This shows that $\sigma(f) \subseteq \text{im } f$.

Conversely, assume that $\lambda \in \text{im } f$. Then, there exists $y \in X$ such that $\lambda = f(y)$. So,

$$(\lambda\mathbb{1} - f)(y) = \lambda - f(y) = 0.$$

This means that the function $\frac{1}{(\lambda\mathbb{1}-f)}$ is not defined and hence, $\lambda\mathbb{1} - f$ is not invertible. So, $\lambda \in \sigma(f)$ and im $f \subseteq \sigma(f)$. We conclude that $\sigma(f) = \text{im } f$. □

Now we will state and prove the spectral mapping theorem.

Theorem 1.3.14 (Spectral mapping theorem)**.** *Let B be a unital C*-algebra and $a \in B$ be normal. If $f \in Cts(\sigma(a), \mathbb{C})$ then*

$$\sigma(f(a)) = f(\sigma(a)) = \{f(\lambda) \mid \lambda \in \sigma(a)\}.$$

Proof. Assume that B is a unital C*-algebra and $a \in B$ is normal. Assume that $f \in Cts(\sigma(a), \mathbb{C})$.

The point is that the spectrum $\sigma(a)$ is a compact Hausdorff space with the subspace topology induced from the usual topology on $\mathbb{C}$. Applying the known result to $f \in Cts(\sigma(a), \mathbb{C})$, we find that the spectrum of f is the image $f(\sigma(a))$.

Now let A be the C*-subalgebra of B generated by the set $\{1_B, a\}$. By the continuous functional calculus in Theorem 1.3.7, the map $f \mapsto f(a)$ is an isometric *-isomorphism from $Cts(\sigma(a), \mathbb{C})$ to A. So, $\sigma(f) = \sigma_A(f(a))$. Finally by Theorem 1.3.12, $\sigma_A(f(a)) = \sigma_B(f(a))$. Therefore,

$$\sigma_B(f(a)) = \sigma_A(f(a)) = \sigma(f) = f(\sigma(a))$$

where the last equality follows from Theorem 1.3.13. □

1.4 Positive elements

Let A be a C*-algebra. Recall that an element $a \in A$ is positive if there exists $b \in A$ such that $a = b^*b$. Note that by definition, any positive element is self-adjoint. In the last section, we proved Theorem 1.3.8, which states that a normal element $c \in A$ is self-adjoint if and only if $\sigma(c) \subseteq \mathbb{R}$. In this section, we will prove a similar characterisation of positive elements of a C*-algebra.

Theorem 1.4.1. *Let A be a unital C*-algebra and $a \in A$ be self-adjoint. For $x \in \mathbb{R}$, let $f(x) = \max\{x, 0\}$ and $g(x) = \max\{-x, 0\}$. Then, $f(a)$ and $g(a)$ are positive elements of A which satisfy*

1. $a = f(a) - g(a)$,
2. $af(a) = f(a)^2$,
3. $ag(a) = -g(a)^2$,
4. $f(a)g(a) = 0$.

Moreover, if $\sigma(a) \subseteq [0, \infty)$ then a is positive.

Proof. Assume that A is a unital C*-algebra and $a \in A$ is self-adjoint. Assume that f and g are the functions defined as above. Then, f and g restrict to continuous real-valued functions on the spectrum $\sigma(a)$, which is contained in $\mathbb{R}$ by Theorem 1.3.8. By definition of f and g, it is straightforward to check that if $x \in \sigma(a)$ then

1. $x = f(x) - g(x)$,
2. $xf(x) = f(x)^2$,
3. $xg(x) = -g(x)^2$,
4. $f(x)g(x) = 0$.

By the continuous functional calculus in Theorem 1.3.7, we obtain the required equations in A.

Next, observe that if $x \in \sigma(a)$ then $f(x) \geq 0$. Hence, the square root $\sqrt{f}$ is a continuous real-valued function which satisfies $(\sqrt{f})^2 = f$. So,

$$f(a) = (\sqrt{f}(a))^2 = (\sqrt{f}(a))^* \sqrt{f}(a).$$

Therefore, $f(a) \in A$ is a positive element. Similarly, $g(a) \in A$ is also a positive element. Finally, if $\sigma(a) \subseteq [0, \infty)$ then $g = 0$ and $a = f(a)$. Since $f(a) \in A$ is positive, a must be positive in this case. $\square$

Keeping the decomposition of Theorem 1.4.1 in mind, we prove the following characterisation of a spectrum contained in $[0, \infty)$.

Theorem 1.4.2. *Let A be a unital C*-algebra and $a \in A$ be self-adjoint. Then, the following are equivalent:*

1. $\sigma(a) \subseteq [0, \infty)$

2. *If* $t \geq \|a\|$ *then* $\|t1_A - a\| \leq t$

3. *There exists* $t \geq \|a\|$ *such that* $\|t1_A - a\| \leq t$.

Proof. Assume that A is a unital C*-algebra and $a \in A$ is self-adjoint. It is obvious that the second statement implies the third.

To show: (a) $\sigma(a) \subseteq [0, \infty)$ if and only if for $t \geq \|a\|$, $\|t1_A - a\| \leq t$.

(a) Since a is self-adjoint, $\sigma(a) \subseteq \mathbb{R}$ and its spectral radius $r(a) = \|a\|$. Therefore, $\sigma(a) \subseteq [-\|a\|, \|a\|]$.

Next, assume that $t \geq \|a\|$ and define the function

$$\begin{array}{rccc} f_t : & [-\|a\|, \|a\|] & \to & \mathbb{R} \\ & x & \mapsto & t - x \end{array}$$

Then, f_t is positive and monotonically decreasing. So, the norm of the restriction

$$\|f_t|_{\sigma(a)}\| = \sup_{x \in \sigma(a)} |f_t(x)| = \sup_{x \in \sigma(a)} |t - x| = t - \inf_{x \in \sigma(a)} x = f_t(\inf_{x \in \sigma(a)} x).$$

By the continuous functional calculus in Theorem 1.3.7,

$$\|t1_A - a\| = \|f_t|_{\sigma(a)}\| = f_t(\inf_{x \in \sigma(a)} x).$$

Finally, observe that $f_t(x) \leq t$ if and only if $x \geq 0$. Hence, $\sigma(a) \subseteq \mathbb{R}_{<0}$ if and only if $\|t1_A - a\| > t$. By taking the contrapositive statement, we find that $\sigma(a) \subseteq \mathbb{R}_{\geq 0}$ if and only if $\|t1_A - a\| \leq t$ as required. $\square$

One application of Theorem 1.4.2 is the fact that spectrums contained in $\mathbb{R}_{\geq 0}$ "add" in a sense made precise in the next theorem.

Theorem 1.4.3. *Let A be a unital C*-algebra and $a, b \in A$ be self-adjoint. Assume that $\sigma(a), \sigma(b) \subseteq [0, \infty)$. Then, $\sigma(a + b) \subseteq [0, \infty)$.*

Proof. Assume that A is a unital C*-algebra and $a, b \in A$ are self-adjoint. Assume that $\sigma(a), \sigma(b) \subseteq [0, \infty)$. Set $t = \|a\| + \|b\|$. Then,

$$\begin{aligned} \|t1_A - a - b\| &= \|(\|a\|1_A - a) + (\|b\|1_A - b)\| \\ &\leq \|\|a\|1_A - a\| + \|\|b\|1_A - b\| \\ &\leq \|a\| + \|b\| = t \quad \text{(by Theorem 1.4.2).} \end{aligned}$$

By another application of Theorem 1.4.2, we deduce that $\sigma(a+b) \subseteq [0,\infty)$ as required. $\square$

Before we proceed to our promised characterisation of positive elements, we need yet another theorem about the spectrum.

Theorem 1.4.4. *Let A be a unital C*-algebra A and $g,h \in A$. Then, $\sigma(gh) - \{0\} = \sigma(hg) - \{0\}$.*

Proof. Assume that A is a unital C*-algebra and $g,h \in A$. It suffices to show that if $\lambda \in \mathbb{C} - \{0\}$ such that $\lambda 1_A - gh$ is invertible then $\lambda 1_A - hg$ is also invertible.

To this end, assume that $\lambda \in \mathbb{C} - \{0\}$ such that $\lambda 1_A - gh$ is invertible. Define

$$x = \frac{1}{\lambda}1_A + \frac{1}{\lambda}h(\lambda 1_A - gh)^{-1}g.$$

We compute directly that

$$\begin{aligned} x(\lambda 1_A - hg) &= \big(\frac{1}{\lambda}1_A + \frac{1}{\lambda}h(\lambda 1_A - gh)^{-1}g\big)(\lambda 1_A - hg) \\ &= 1_A - \frac{1}{\lambda}hg + h(\lambda 1_A - gh)^{-1}g - \frac{1}{\lambda}h(\lambda 1_A - gh)^{-1}ghg \\ &= 1_A + h\big((\lambda 1_A - gh)^{-1} - \frac{1}{\lambda}1_A - \frac{1}{\lambda}(\lambda 1_A - gh)^{-1}gh\big)g \\ &= 1_A + h\big((\lambda 1_A - gh)^{-1} - \frac{1}{\lambda}(\lambda 1_A - gh)^{-1}(\lambda 1_A - gh) \\ &\quad - \frac{1}{\lambda}(\lambda 1_A - gh)^{-1}gh\big)g \\ &= 1_A \end{aligned}$$

and

$$\begin{aligned} (\lambda 1_A - hg)x &= (\lambda 1_A - hg)\big(\frac{1}{\lambda}1_A + \frac{1}{\lambda}h(\lambda 1_A - gh)^{-1}g\big) \\ &= 1_A + h(\lambda 1_A - gh)^{-1}g - \frac{1}{\lambda}hg - \frac{1}{\lambda}hgh(\lambda 1_A - gh)^{-1}g \\ &= 1_A + h\big((\lambda 1_A - gh)^{-1} - \frac{1}{\lambda}(\lambda 1_A - gh)(\lambda 1_A - gh)^{-1} \\ &\quad - \frac{1}{\lambda}gh(\lambda 1_A - gh)^{-1}\big)g \\ &= 1_A. \end{aligned}$$

Hence, $\lambda 1_A - hg$ is also invertible and $\sigma(gh) - \{0\} = \sigma(hg) - \{0\}$. $\square$

Now we are ready to prove our characterisation of positive elements.

Theorem 1.4.5. *Let A be a unital C*-algebra and $a \in A$ be self-adjoint. Then, a is positive if and only if $\sigma(a) \subseteq [0, \infty)$.*

Proof. Assume that A is a unital C*-algebra and $a \in A$ is self-adjoint. We already know that if $\sigma(a) \subseteq [0, \infty)$ then a is positive by Theorem 1.4.1.

Now assume that $a \in A$ is positive. Then, there exists $b \in A$ such that $a = b^*b$. Recall that in Theorem 1.4.1, we defined positive elements $f(a), g(a) \in A$. The idea is to consider the element $c = bg(a)$. We compute directly that

$$c^*c = g(a)^*b^*bg(a) = g(a)b^*bg(a) = g(a)ag(a) = g(a)(-g(a))g(a) = -g(a)^3.$$

Now write $c = d + ie$, where $d, e \in A$ are self-adjoint. Another computation yields

$$cc^* + c^*c = 2d^2 + 2e^2.$$

So, $cc^* = 2d^2 + 2e^2 - c^*c = 2d^2 + 2e^2 + g(a)^3$. Since the real-valued functions $2x^2$ and $g(x)$ are positive, we can use the spectral mapping theorem (see Theorem 1.3.14) to deduce that

$$\sigma(2d^2), \sigma(2e^2), \sigma(g(a)^3) \subseteq [0, \infty).$$

Consequently by Theorem 1.4.3,

$$\sigma(cc^*) = \sigma(2d^2 + 2e^2 + g(a)^3) \subseteq [0, \infty).$$

On the other hand, by Theorem 1.3.14

$$\sigma(c^*c) \subseteq (-\infty, 0].$$

Next, recall from Theorem 1.4.4 that

$$\sigma(c^*c) - \{0\} = \sigma(cc^*) - \{0\}.$$

Since $\sigma(c^*c) \subseteq (-\infty, 0]$ and $\sigma(cc^*) \subseteq [0, \infty)$, then $\sigma(c^*c) = \sigma(cc^*) = \{0\}$. Therefore, $c^*c = 0 = -g(a)^3$ and consequently, $g(a) = 0$. This means that the restricted function $g|_{\sigma(a)}$ is the zero function. Recalling from Theorem 1.4.1 that $g = \max\{-x, 0\}$, we conclude that $\sigma(a) \subseteq [0, \infty)$ as required. $\square$

A useful consequence of Theorem 1.4.5 is the connection between normal elements of a unital C*-algebra and the continuous functional calculus.

Theorem 1.4.6. *Let A be a unital C*-algebra and $a \in A$ be normal. If $f \in Cts(\sigma(a), \mathbb{R})$ then $f(a) \in A$ is self-adjoint. Moreover, if $f \in Cts(\sigma(a), \mathbb{R})$ is positive then $f(a) \in A$ is positive.*

Proof. Assume that A is a unital C*-algebra and $a \in A$ is normal. Assume that $f \in Cts(\sigma(a), \mathbb{R})$. By the spectral mapping theorem (see Theorem 1.3.14),

$$\sigma(f(a)) = f(\sigma(a)) = \{f(\lambda) \mid \lambda \in \sigma(a)\} \subseteq \mathbb{R}.$$

So, $f(a) \in A$ is self-adjoint by Theorem 1.3.8. If $f \in Cts(\sigma(a), \mathbb{R})$ is positive then $\sigma(f(a)) \subseteq \mathbb{R}_{\geq 0}$ and by Theorem 1.4.5, $f(a) \in A$ is normal as required. $\square$

The *polar decomposition* gives a method of decomposing an arbitrary element of a unital C*-algebra into the product of a positive element and a unitary element. This decomposition is analogous to computing the exponential form of a complex number. Here, we will consider the polar decomposition of invertible elements, in line with [Put19, Exercise 1.6.1]. We begin with a preliminary result.

Theorem 1.4.7. *Let A be a unital C*-algebra and $a \in A$. Let f be the function defined explicitly by*

$$\begin{array}{cccc} f: & \mathbb{R}_{\geq 0} & \to & \mathbb{R} \\ & x & \mapsto & x^{\frac{1}{2}}. \end{array}$$

Define $|a| = f(a^*a)$. If a is invertible then $|a|$ is invertible.

Proof. Assume that A is a unital C*-algebra and $a \in A$. Assume that f is the square root function defined as above. Define the function $g : \mathbb{C} \to \mathbb{C}$ by $g(z) = \overline{z}z = |z|^2$. By the continuous functional calculus in Theorem 1.3.7, $g(a) = a^*a$. Now by the spectral mapping theorem in Theorem 1.3.14,

$$\sigma(a^*a) = \sigma(g(a)) = g(\sigma(a)) = \{|\lambda|^2 \mid \lambda \in \sigma(a)\}.$$

By considering the restriction of f to $\sigma(a^*a) \subseteq \mathbb{R}_{\geq 0}$, we find that by another application of the spectral mapping theorem,

$$\sigma(f(a^*a)) = f(\sigma(a^*a)) = \{|\lambda| \mid \lambda \in \sigma(a)\}.$$

Hence, $0 \in \sigma(a)$ if and only if $0 \in \sigma(f(a^*a))$. Thus, if a is invertible then $|a|$ is also invertible. $\square$

The element $|a| \in A$ constructed in Theorem 1.4.7 is commonly referred to as the **modulus/absolute value** of a. The next theorem yields the second half of the polar decomposition.

Theorem 1.4.8. *Let A be a unital C*-algebra and $a \in A$. Let $|a| \in A$ be the element constructed in Theorem 1.4.7. If a is invertible then the element*

$$u = a|a|^{-1} \in A$$

is unitary.

Proof. Assume that A is a unital C*-algebra and $a \in A$. Assume that $|a| \in A$ is the element defined in Theorem 1.4.7. Assume that a is invertible. By Theorem 1.4.7, $|a| \in A$ is invertible. Hence, we can define the element $u = a|a|^{-1}$.

To show: (a) $u^*u = uu^* = 1_A$.

(a) Recall from Theorem 1.4.7 that

$$\sigma(|a|) = \{|\lambda| \mid \lambda \in \sigma(a)\}.$$

Since $\sigma(|a|) \subseteq \mathbb{R}_{\geq 0}$, we deduce by Theorem 1.4.5 that $|a|$ is positive and hence, self-adjoint. This means that

$$|a|^{-1} = (|a|^*)^{-1} = (|a|^{-1})^*.$$

So,

$$u^*u = (|a|^{-1})^*a^*a|a|^{-1} = |a|^{-1}(a^*a)|a|^{-1} = |a|^{-1}|a| = 1_A$$

and

$$uu^*a = a|a|^{-1}(|a|^{-1})^*a^*a = a|a|^{-1}(|a|^{-1}a^*a) = a|a|^{-1}|a| = a.$$

By multiplying both sides on the right by a^{-1}, we deduce that $uu^* = 1_A$.

By part (a), u is unitary. □

Theorem 1.4.8 establishes that if $a \in A$ is invertible then $a = u|a|$. This is referred to as the **polar decomposition** of a. The next theorem provides a criterion for the elements u and $|a|$ to commute.

Theorem 1.4.9. *Let A be a unital C*-algebra and $a \in A$ be invertible. Let u and $|a|$ be the elements defined in Theorem 1.4.8 and Theorem 1.4.7 respectively. Then, u and $|a|$ commute if and only if a is normal.*

Proof. Assume that A is a unital C*-algebra and $a \in A$ is invertible.

To show: (a) If a is normal then the elements u and $|a|$ commute.

(b) If u and $|a|$ commute then a is normal.

(a) Assume that $a \in A$ is normal. By the continuous functional calculus in Theorem 1.3.7, the elements a and $|a|$ must commute. Using this, we now have

$$\begin{aligned} |a|u &= |a|a|a|^{-1} \\ &= a|a||a|^{-1} \\ &= a = u|a|. \end{aligned}$$

Therefore, the elements u and $|a|$ commute.

(b) Assume that the elements u and $|a|$ commute. To see that a is normal, we compute directly that

$$\begin{aligned} a^*a &= (u|a|)^*u|a| \\ &= |a|u^*u|a| \quad (|a| \text{ is self-adjoint}) \\ &= |a|1_A|a| = |a|^2 \quad (\text{Theorem 1.4.8}) \end{aligned}$$

and

$$\begin{aligned} aa^* &= u|a|(u|a|)^* \\ &= u|a||a|u^* \quad (|a| \text{ is self-adjoint}) \\ &= u|a|^2u^* \\ &= |a|^2uu^* \\ &= |a|^2 1_A = |a|^2. \end{aligned}$$

Therefore, $a^*a = aa^*$ and a is normal. $\square$

1.5 Finite dimensional C*-algebras

Finite dimensional C*-algebras have a specific structure. This section is dedicated to proving the following theorem.

Theorem 1.5.1. *Let A be a unital, finite dimensional C*-algebra. Then, there exists positive integers $k, N_1, \ldots, N_k \in \mathbb{Z}_{>0}$ such that as C*-algebras,*

$$A \cong \bigoplus_{i=1}^{k} M_{N_i \times N_i}(\mathbb{C})$$

Moreover, k is unique and the integers $N_1, \ldots, N_k$ are unique up to permutation.

It is remarked in [Put19, Section 1.7] that Theorem 1.5.1 is valid without the C*-algebra being unital. In fact, it is a consequence of Theorem 1.5.1 that every finite dimensional C*-algebra is unital. This result will be proved later.

The proof of Theorem 1.5.1 revolves around the existence of projections in a finite dimensional C*-algebra. In general, C*-algebras may or may not have non-trivial projections. The reference [Put19, Section 1.7] gives the example of $Cts(X, \mathbb{C})$, where X is a compact Hausdorff space. It turns out that in this case, $Cts(X, \mathbb{C})$ has no non-trivial projections if and only if X is connected.

Let us start off with some notation pertaining to bounded operators on a Hilbert space.

Definition 1.5.1. Let H be a Hilbert space and $\xi, \eta \in H$. Define the map $|\xi\rangle\langle\eta|$ by

$$\begin{array}{rccc} |\xi\rangle\langle\eta| : & H & \to & H \\ & \zeta & \mapsto & \langle\zeta, \eta\rangle\xi \end{array}$$

This is the well-known bra-ket notation from quantum mechanics. In the next theorem, we prove various properties about the map $|\xi\rangle\langle\eta|$, which we will make use of later.

Theorem 1.5.2. *Let H be a Hilbert space, $\eta, \xi, \zeta, \omega \in H$ and $a \in B(H)$. Then,*

1. $\| |\xi\rangle\langle\eta| \| = \|\xi\| \|\eta\|$.
2. *If* $\eta \neq 0$ *then* $|\xi\rangle\langle\eta| H = span\{\xi\}$.
3. $(|\xi\rangle\langle\eta|)^* = |\eta\rangle\langle\xi|$.
4. $(|\xi\rangle\langle\eta|)(|\zeta\rangle\langle\omega|) = \langle\zeta, \eta\rangle(|\xi\rangle\langle\omega|)$.

5. $a|\xi\rangle\langle\eta| = |a\xi\rangle\langle\eta|$

6. $|\xi\rangle\langle\eta|a = |\xi\rangle\langle a^*\eta|$.

Proof. Assume that H is a Hilbert space, $\eta, \xi, \zeta, \omega \in H$ and $a \in B(H)$.

1. We compute directly that

$$\||\xi\rangle\langle\eta|\| = \sup_{\|\alpha\|=1} |\langle\alpha, \eta\rangle| \|\xi\| \leq \|\eta\|\|\xi\|.$$

We also have

$$\|\eta\|\|\xi\| = |\langle\frac{\eta}{\|\eta\|}, \eta\rangle| \|\xi\| \leq \sup_{\|\alpha\|=1} |\langle\alpha, \eta\rangle| \|\xi\| = \||\xi\rangle\langle\eta|\|.$$

Therefore, $\||\xi\rangle\langle\eta|\| = \|\eta\|\|\xi\|$.

2. Assume that $\eta \neq 0$. By the Hahn-Banach extension theorem, there exists a functional $\Gamma \in H^*$ such that $\Gamma(\eta) \neq 0$. By the Riesz representation theorem, there exists $\gamma \in H$ such that $\Gamma(\eta) = \langle\gamma, \eta\rangle \neq 0$. So, $|\xi\rangle\langle\eta|\gamma = \langle\gamma, \eta\rangle\xi$. Now if $a \in \mathbb{C}$ then

$$a\xi = \frac{a}{\langle\gamma, \eta\rangle}\langle\gamma, \eta\rangle\xi = \frac{a}{\langle\gamma, \eta\rangle}|\xi\rangle\langle\eta|\gamma = |\xi\rangle\langle\eta|(\frac{a}{\langle\gamma, \eta\rangle}\gamma).$$

Hence, $span\{\xi\} \subseteq |\xi\rangle\langle\eta|H$ and subsequently, $|\xi\rangle\langle\eta|H = span\{\xi\}$.

3. If $\alpha, \beta \in H$ then

$$\begin{aligned}
\langle(|\xi\rangle\langle\eta|)^*(\alpha), \beta\rangle &= \langle\alpha, |\xi\rangle\langle\eta|(\beta)\rangle \\
&= \langle\alpha, \langle\beta, \eta\rangle\xi\rangle \\
&= \langle\alpha, \xi\rangle\langle\eta, \beta\rangle \\
&= \langle\langle\alpha, \xi\rangle\eta, \beta\rangle \\
&= \langle|\eta\rangle\langle\xi|(\alpha), \beta\rangle.
\end{aligned}$$

So, $(|\xi\rangle\langle\eta|)^* = |\eta\rangle\langle\xi|$.

4. If $\alpha \in H$ then

$$\begin{aligned}(|\xi\rangle\langle\eta|)(|\zeta\rangle\langle\omega|)\alpha &= |\xi\rangle\langle\eta|(\langle\alpha,\omega\rangle\zeta) \\ &= \langle\alpha,\omega\rangle\langle\zeta,\eta\rangle\xi \\ &= \langle\zeta,\eta\rangle|\xi\rangle\langle\omega|\alpha.\end{aligned}$$

   Hence, $(|\xi\rangle\langle\eta|)(|\zeta\rangle\langle\omega|) = \langle\zeta,\eta\rangle|\xi\rangle\langle\omega|$.

5. If $\alpha \in H$ then

$$a|\xi\rangle\langle\eta|\alpha = a(\langle\alpha,\eta\rangle\xi) = \langle\alpha,\eta\rangle a\xi = |a\xi\rangle\langle\eta|\alpha.$$

6. If $\alpha \in H$ then

$$|\xi\rangle\langle\eta|a\alpha = \langle a\alpha,\eta\rangle\xi = \langle\alpha,a^*\eta\rangle\xi = |\xi\rangle\langle a^*\eta|\alpha.$$

□

The first prominent step towards our classification of finite dimensional C*-algebras is to prove that every normal element is a linear combination of projections.

Theorem 1.5.3. *Let A be a finite dimensional unital C*-algebra. If $a \in A$ is normal then a has finite spectrum and a is a linear combination of projections.*

Proof. Assume that A is a finite dimensional unital C*-algebra. Assume that $a \in A$ is normal.

To show: (a) The spectrum of a is finite.

(b) a can be written as a linear combination of projections.

(a) By the continuous functional calculus in Theorem 1.3.7, the C*-algebra $Cts(\sigma(a),\mathbb{C})$ is isomorphic to a C*-subalgebra of A, which is finite dimensional because A itself is finite dimensional. Consequently, $\sigma(a)$ must be finite.

(b) Let $\lambda \in \sigma(a)$ and define the function p_λ by

$$\begin{aligned} p_\lambda : \quad \sigma(a) &\to \mathbb{C} \\ \mu &\mapsto \begin{cases} 0, & \text{if } \mu \neq \lambda, \\ 1, & \text{if } \mu = \lambda. \end{cases} \end{aligned}$$

Then, $p_\lambda = \overline{p_\lambda}$ and $p_\lambda^2 = p_\lambda$ where we recall that multiplication in the C*-algebra $Cts(\sigma(a), \mathbb{C})$ is defined pointwise. By the continuous functional calculus in Theorem 1.3.7, $p_\lambda(a) \in A$ is self-adjoint and idempotent. Hence, it is a projection.

Notice that if $z \in \sigma(a)$ then

$$\sum_{\lambda \in \sigma(a)} \lambda p_\lambda(z) = z.$$

By another application of the continuous functional calculus, we have

$$\sum_{\lambda \in \sigma(a)} \lambda p_\lambda(a) = a.$$

Hence, a is a linear combination of projections as required. □

A consequence of the proof of Theorem 1.5.3 is that projections exist in finite dimensional unital C*-algebras. Studying this structure is key to the proof of our main result.

Definition 1.5.2. Let A be a C*-algebra. If $p, q \in A$ are projections then we define the relation $p \geq q$ if and only if $pq = q$.

Theorem 1.5.4. *Let A be a C*-algebra. The relation $\leq$ on the projections of A is a partial order.*

Proof. Assume that A is a C*-algebra. Reflexivity of the relation $\leq$ follows from the fact that a projection $p \in A$ is idempotent.

Next, assume that $p, q \in A$ are projections such that $p \leq q$ and $q \leq p$. Then, $qp = p$ and $pq = q$. Since p and q are self-adjoint, $pq = p = q$.

Finally to see that $\leq$ is transitive, assume that $p, q, r \in A$ are projections such that $p \leq q$ and $q \leq r$. Then, $qp = p$ and $rq = q$. So, $rp = rqp = qp = p$. Hence, $\leq$ defines a partial order on the set of projections in A. □

Under some conditions, there exist minimal non-zero projections with regards to the partial order $\leq$. This is established in the next theorem.

Theorem 1.5.5. *Let A be a C*-algebra and $p, q \in A$ be projections. Then,*

1. *$p \geq q$ if and only if $qAq \subseteq pAp$. In particular, if $pAp = qAq$ then $p = q$.*
2. *If pAp has finite dimension greater than 1 then there exists a projection $q \neq 0$ such that $p \geq q$.*
3. *If A is unital and finite dimensional then there exists non-zero projections which are minimal with respect to the order $\leq$.*

Proof. Assume that A is a C*-algebra and $p, q \in A$ are projections.

To show: (a) $p \geq q$ if and only if $qAq \subseteq pAp$.

(a) First assume that $p \geq q$. Then, $qAq = pqAqp \subseteq pAp$. Conversely, assume that $qAq \subseteq pAp$. Then, $q = qqq \in qAq = pqAqp \subseteq pAp$. So, there exists $a \in A$ such that $q = pap$. Thus, $pq = ppap = pap = q$ and subsequently, $q \leq p$.

Next, assume that $qAq = pAp$. By part (a), $q \leq p$ and $p \leq q$. Since $\leq$ is a partial order, $p = q$.

Before we proceed to proving the second statement, we first observe that pAp is a Banach *-algebra. Moreover, if A is finite dimensional then A is closed and pAp is a C*-subalgebra of A. It is also unital, with $p = ppp$ acting as a unit for the C*-subalgebra pAp.

To show: (b) If $a \in pAp$ is self-adjoint and $\sigma(a) = \{\lambda\}$ for some $\lambda \in \mathbb{C}$ then a is a scalar multiple of p.

(b) Assume that $a \in pAp$ is self-adjoint. Assume that there exists $\lambda \in \mathbb{C}$ such that $\sigma(a) = \{\lambda\}$. If $z \in \sigma(a)$ then the functions $f(z) = z$ and $g(z) = \lambda$ are equal. By the continuous functional calculus in Theorem 1.3.7,

$$a = f(a) = g(a) = \lambda p.$$

Hence, a is a scalar multiple of p.

Now suppose that pAp has dimension strictly greater than 1. By the contrapositive statement of part (b), there exists a self-adjoint element $b \in A$ such that $\sigma(b)$ contains at least two points. Let f be a surjective function from $\sigma(b)$ to $\{0, 1\}$. Since $\{0, 1\}$ is equipped with the discrete topology, f must be continuous.

By the continuous functional calculus, the element $f(b) = q$ is a non-zero projection. Since $q \in pAp$, we must have $pq = q$. So, $p \geq q$ as required.

Finally, assume that A is unital and finite dimensional. By Theorem 1.5.3, there exists a non-zero projection $p \in A$. The dimension of the C*-subalgebra pAp is at least one. If it is strictly greater than 1 then by part (b) of the proof, there exists a non-zero projection $q \in A$ such that $p \geq q$. By part (a), this holds if and only if $qAq \subseteq pAp$. Since $q \neq p$, qAq is strictly contained in pAp and $\dim qAq < \dim pAp$.

By iterating this procedure, we eventually obtain a non-zero projection $q' \in A$ such that $\dim q'Aq' = 1$. Hence, q' qualifies as a minimal projection with respect to the partial order $\leq$. □

One consequence of Theorem 1.5.5 is that if $p \in A$ is a non-zero minimal projection then $pAp = \mathbb{C}p$. Next, we want to now when a set of non-zero minimal projections are linearly independent.

Theorem 1.5.6. *Let A be a unital finite dimensional C*-algebra and $p_1, \ldots, p_K$ be non-zero minimal projections in A. Suppose that if $i, j \in \{1, 2, \ldots, K\}$ such that $i \neq j$ then $p_i A p_j = 0$. Then, $p_1, \ldots, p_K$ are linearly independent.*

Proof. Assume that A is a unital finite dimensional C*-algebra and $p_1, \ldots, p_K$ are non-zero minimal projections in A. Assume that $p_i A p_j = 0$ for $i \neq j$. Fix $i \in \{1, 2, \ldots, K\}$. Suppose for the sake of contradiction that

$$p_i = \sum_{j \neq i} \alpha_j p_j$$

for some $\alpha_j \in \mathbb{C}$. By Theorem 1.5.5, we have

$$\mathbb{C}p_i = p_i A p_i = p_i A \big(\sum_{j \neq i} \alpha_j p_j \big) \subseteq \sum_{j \neq i} p_i A p_j = 0.$$

This is a contradiction. Therefore, the minimal projections $p_1, \ldots, p_K$ are linearly independent. □

Theorem 1.5.6 tells us that a linearly independent set of non-zero minimal projections cannot contain more than $\dim A$ elements. Hence, we can select a maximal independent set of non-zero minimal projections and it will be finite.

Now suppose that A is a unital finite dimensional C*-algebra and $\{p_1, \ldots, p_K\}$ is a maximal set of independent minimal non-zero projections in A. Fix $i \in \{1, 2, \ldots, K\}$ and let $a, b \in Ap_i$. Then, $ap_i = a$ and $bp_i = b$. Observe that

$$b^*a = (bp_i)^*ap_i = p_ib^*ap_i \in p_iAp_i = \mathbb{C}p_i$$

as a consequence of the second statement in Theorem 1.5.5. Let $\langle a, b\rangle \in \mathbb{C}$ denote the scalar such that $b^*a = \langle a, b\rangle p_i$. We claim that $\langle -, -\rangle : Ap_i \times Ap_i \to \mathbb{C}$ defines an inner product on Ap_i.

First assume that $a, a', b, b' \in Ap_i$. Then,

$$\begin{aligned}\langle a + a', b\rangle p_i &= b^*(a + a')\\ &= b^*a + b^*a'\\ &= (\langle a, b\rangle + \langle a', b\rangle)p_i\end{aligned}$$

and

$$\begin{aligned}\langle a, b + b'\rangle p_i &= (b^* + (b')^*)a\\ &= b^*a + (b')^*a\\ &= (\langle a, b\rangle + \langle a, b'\rangle)p_i.\end{aligned}$$

If $\lambda \in \mathbb{C}$ then

$$\langle \lambda a, b\rangle p_i = b^*(\lambda a) = \lambda b^*a = \lambda\langle a, b\rangle p_i$$

and

$$\langle a, \lambda b\rangle p_i = (\lambda b)^*a = \overline{\lambda}b^*a = \overline{\lambda}\langle a, b\rangle p_i.$$

Hence, the map $\langle -, -\rangle$ is $\mathbb{C}$-bilinear. Also, if $a \in A$ then $\langle a, a\rangle = 0$ if and only if $a^*a = 0$ if and only if $a = 0$. Hence, $\langle -, -\rangle$ defines an inner product on Ap_i.

Since A is finite dimensional, Ap_i is also finite dimensional and thus, a closed subspace of A. We conclude that if $i \in \{1, 2, \ldots, K\}$ then Ap_i is a finite dimensional Hilbert space.

Next, we claim that if $i \in \{1, 2, \ldots, K\}$ then the subspace Ap_iA is a unital C*-subalgebra of A. Taking advantage of the fact that Ap_i is a Hilbert space, we can let B_i be an orthonormal basis for Ap_i. If $a \in Ap_i$ then

$$\sum_{b\in B_i} bb^*a = \sum_{b\in B_i} b\langle a, b\rangle p_i = \sum_{b\in B_i} \langle a, b\rangle bp_i = \sum_{b\in B_i} \langle a, b\rangle b = a.$$

In light of this computation, let $q_i = \sum_{b\in B_i} bb^* = \sum_{b\in B_i} bp_ib^*$. Then, $q_i \in Ap_iA$ is self-adjoint and from the previous computation, if $a \in Ap_i$ then $q_ia = a$.

We claim that q_i is the unit of Ap_iA. To this end, assume that $a, a' \in A$. We compute directly that

$$q_i(ap_ia') = (q_iap_i)a' = ap_ia'$$

and

$$(ap_ia')q_i = \big((q_i(a')^*p_i)a^*\big)^* = ((a')^*p_ia^*)^* = ap_ia'.$$

Therefore, Ap_iA is a C*-subalgebra of A with unit q_i.

Now if $k, \ell \in \{1, 2, \ldots, K\}$ with $k \neq \ell$ then $p_kAp_\ell = 0$. So, $(Ap_kA)(Ap_\ell A) = 0$. In particular, $q_kAp_\ell = 0$ and $q_kAp_\ell A = 0$, which means that the projections q_k are pairwise orthogonal (since $q_kq_\ell = 0$).

Now define

$$q = \sum_{k=1}^{K} q_k.$$

Then, q is a projection because each q_k is self-adjoint and $q^2 = q$ since the projections q_k are pairwise orthogonal. It is also a central projection. We claim that $q = 1_A$ in A.

Suppose for the sake of contradiction that $q \neq 1_A$. Then, there exists $b \in A$ such that $bq \neq b$. Hence, $b - bq$ is a non-zero element of the set

$$q^\perp = \{a \in A \mid qa = 0\}.$$

It is straightforward to verify that $q^\perp$ is a C*-subalgebra of A. Hence, $q^\perp$ must be finite dimensional. Since $q^\perp$ contains a non-zero element, then it must contain a non-zero minimal projection by the third part of Theorem 1.5.5. We denote this projection by p. Thus, if $k \in \{1, 2, \ldots, K\}$ then

$$pq_k \leq pq = 0.$$

If $k \in \{1, 2, \ldots, K\}$ then $pq_k = 0$. Therefore,

$$pAp_k \subseteq pAp_kA = pq_kAp_kA = 0$$

where the second last equality follows from the fact that q_k is the unit for the C*-subalgebra Ap_kA. The point here is that the projections $p, p_1, \ldots, p_K$ form an independent set of non-zero minimal projections by Theorem 1.5.6. However, this contradicts the maximality of the set $\{p_1, \ldots, p_K\}$. Therefore, $q = 1_A$. The primary consequence of this claim is that

$$A = \big(\sum_{k=1}^{K} q_k\big)A = \bigoplus_{k=1}^{K} q_kA = \bigoplus_{k=1}^{K} Ap_kA.$$

because $q_k \in Ap_kA$. Finally, recall that $B(Ap_i)$ is the C*-algebra of bounded linear operators on the Hilbert space Ap_i. For $i \in \{1, 2, \ldots, K\}$, define the map

$$\begin{array}{rccc} \pi_i: & A & \to & B(Ap_i) \\ & a & \mapsto & (b \mapsto ab). \end{array}$$

The fact that π_i is a *-homomorphism follows from routine computations. Observe that if $k \neq i$ then the restriction $\pi_i|_{Ap_kA} = 0$ because $(Ap_kA)Ap_i = A(p_kAp_i) = 0$.

Now we claim that π_i is an isomorphism from Ap_iA to $B(Ap_i)$. First, we show that π_i is injective. The key idea here is that the span of the orthonormal basis B_i is Ap_i by definition. Hence, $p_iA = (Ap_i)^*$ is the span of the adjoints of elements of B_i. Hence, $Ap_iA = (Ap_i)(Ap_i)^*$ is contained in the span of elements of the form bc^*, where $b, c \in B_i$. For $b, c, b_0, c_0 \in B_i$, let $\alpha_{b,c} \in \mathbb{C}$. Then,

$$\begin{aligned} \langle \pi_i\big(\sum_{b,c \in B_i} \alpha_{b,c} bc^*\big)c_0, b_0\rangle p_i &= \sum_{b,c \in B_i} \alpha_{b,c} \langle \pi_i\big(bc^*\big)c_0, b_0\rangle p_i \\ &= \sum_{b,c \in B_i} \alpha_{b,c} \langle bc^*c_0, b_0\rangle p_i \\ &= \sum_{b,c \in B_i} \alpha_{b,c} b_0^* bc^* c_0 = \alpha_{b_0,c_0} p_i. \end{aligned}$$

The last equality follows from the assumption that B_i is an orthonormal basis for the Hilbert space Ap_i. To see that the restriction $\pi_i|_{Ap_iA}$ is injective, assume that $\pi_k(\sum_{b,c\in B_i} \alpha_{b,c}bc^*) = 0$. By the previous computation, each coefficient $\alpha_{b,c} = 0$ for $b, c \in Ap_i$. Therefore, $\sum_{b,c\in B_i} \alpha_{b,c}bc^* = 0$ in Ap_iA and consequently, $\pi_i|_{Ap_iA}$ is injective.

To see that π_i is surjective, assume that $a, b \in Ap_i$ so that $ab^* \in Ap_iA$. If $c \in Ap_i$ then

$$\begin{aligned}
\pi_i(ab^*)c &= a(b^*c) \\
&= a(\langle c, b\rangle p_i) \\
&= \langle c, b\rangle ap_i = \langle c, b\rangle a \\
&= |a\rangle\langle b|c.
\end{aligned}$$

The above computation reveals that the rank one operator $|a\rangle\langle b|$ is in the image of π_i. So, π_i is surjective because the span of operators of the form $|a\rangle\langle b|$ is all of $B(Ap_i)$. Hence, $\pi_i|_{Ap_iA}$ is a *-isomorphism.

Let us summarise our findings pertaining to the structure of finite dimensional C*-algebras. This is [Put19, Theorem 1.7.8]. Recall that A is a finite dimensional C*-algebra and $p_1, \dots, p_K$ is a maximal set of independent minimal non-zero projections in A.

1. If $k \in \{1, 2, \dots, K\}$ then Ap_k is a finite dimensional Hilbert space with inner product given by $\langle a, b\rangle p_k = b^*a$ for $a, b \in Ap_k$.
2. If $k \in \{1, 2, \dots, K\}$ then Ap_kA is a unital C*-subalgebra of A, with unit $q_k = \sum_{b\in B_k} bb^*$.
3. $\bigoplus_{k=1}^{K} Ap_kA = A$
4. If $k \in \{1, 2, \dots, K\}$ then $\pi_k : A \to B(Ap_k)$ is a *-homomorphism. Moreover, if $i \neq k$ then $\pi_k|_{Ap_iA} = 0$ and $\pi_k|_{Ap_kA}$ is a *-isomorphism.

The next theorem describes the centres of the C*-algebra $M_{n\times n}(\mathbb{C})$.

Theorem 1.5.7. *Let $n \in \mathbb{Z}_{>0}$. The centre of $M_{n\times n}(\mathbb{C})$ is*

$$\{aI_n \mid a \in \mathbb{C}\}$$

where $I_n \in M_{n\times n}(\mathbb{C})$ is the $n \times n$ identity matrix. Furthermore, if $n_1, \dots, n_K \in \mathbb{Z}_{>0}$ then the centre of the direct sum $\bigoplus_{k=1}^{K} M_{n_k\times n_k}(\mathbb{C})$ is isomorphic to $\mathbb{C}^K$ and is spanned by the identity elements of the direct summands.

Proof. Assume that $n \in \mathbb{Z}_{>0}$. The case where $n = 1$ is trivial. So, assume that $n \geq 2$. Obviously, we can identify $M_{n\times n}(\mathbb{C})$ with the space of bounded linear transformations $B(\mathbb{C}^n)$. Assume that $a : \mathbb{C}^n \to \mathbb{C}^n$ is in the centre of $B(\mathbb{C}^n)$ and $\xi \in \mathbb{C}^n$ is non-zero. Note that

$$|a\xi\rangle\langle\xi| = a|\xi\rangle\langle\xi| = |\xi\rangle\langle\xi|a = |\xi\rangle\langle a^*\xi|.$$

Subsequently,

$$a\xi\langle\xi,\xi\rangle = \langle\xi, a^*\xi\rangle\xi.$$

We conclude that if $\xi \in \mathbb{C}^n$ is non-zero then there exists a scalar $r \in \mathbb{C}$ such that $a\xi = r\xi$. We want to show that the scalar r is independent of the choice of non-zero $\xi \in \mathbb{C}^n$.

Let $\eta \in \mathbb{C}^n$ such that the set $\{\xi, \eta\}$ is linearly independent. Then, there exists $r, s, t \in \mathbb{C}$ such that $a(\xi + \eta) = r(\xi + \eta)$, $a\xi = s\xi$ and $a\eta = t\eta$. So, $r\xi + r\eta = s\xi + t\eta$ and by linear independence, $r = s = t$.

Therefore, if $\xi \in \mathbb{C}^n$ is non-zero then $a\xi$ is a scalar multiple of ξ and this scalar is independent of ξ. Consequently, a is a multiple of the identity transformation $id_{\mathbb{C}^n} \in B(\mathbb{C}^n)$ as required.

The second statement follows from the first. □

Now we can finally put together a proof of Theorem 1.5.1.

Proof of Theorem 1.5.1. Assume that A is a unital, finite dimensional C*-algebra. By Theorem 1.5.6, we can select a maximal independent set of non-zero minimal projections, which has finite cardinality. Let $\{p_1, \ldots, p_K\}$ be such a set. We know that

$$A = \bigoplus_{k=1}^{K} Ap_kA.$$

Moreover, if $k \in \{1, 2, \ldots, K\}$ then the map

$$\begin{array}{rccc} \pi_k : & A & \to & B(Ap_k) \\ & a & \mapsto & (b \mapsto ab) \end{array}$$

is a *-isomorphism from the C*-subalgebra Ap_kA to $B(Ap_k)$. So, the direct sum

$$\bigoplus_{k=1}^{K} \pi_k : \ A \cong \bigoplus_{k=1}^{K} Ap_kA \longrightarrow \ \bigoplus_{k=1}^{K} B(Ap_k) \cong \bigoplus_{k=1}^{K} M_{n_k \times n_k}(\mathbb{C})$$

is a *-isomorphism. Here, $n_1, \ldots, n_K \in \mathbb{Z}_{>0}$ are finite because each Hilbert space Ap_k is finite dimensional for $k \in \{1, 2, \ldots, K\}$.

To see that $K, n_1, \ldots, n_K \in \mathbb{Z}_{>0}$ is unique, we first note that from Theorem 1.5.7, K is the dimension of the centre of $\bigoplus_{k=1}^{K} M_{n_k \times n_k}(\mathbb{C}) \cong A$. Hence, K must be unique. Next, recall that if $k, \ell \in \{1, 2, \ldots, K\}$ with $k \neq \ell$ then Ap_kA has unit q_k which satisfies $q_kAp_\ell A = 0$. By the *-isomorphism and Theorem 1.5.7, we find that n_k is the square root of the dimension of $q_kA = Ap_kA$. This completes the proof. □

We finish this section by stating [Put19, Exercise 1.7.1] as a theorem about partial isometries.

Theorem 1.5.8. *Let A be a C*-algebra and $e \in A$ be a partial isometry so that e^*e is a projection. Then, $ee^*e = e$ and ee^* is a projection.*

Proof. Assume that A is a C*-algebra. Assume that $e \in A$ is a partial isometry so that e^*e is a projection.

To see that $ee^*e = e$, consider the expression $(ee^*e - e)^*(ee^*e - e)$. By computing this element, we find that

$$\begin{aligned}(ee^*e - e)^*(ee^*e - e) &= (e^*ee^* - e^*)(ee^*e - e) \\ &= (e^*ee^*e)e^*e - e^*ee^*e - e^*ee^*e - e^*e \\ &= e^*e - e^*e - e^*e + e^*e = 0.\end{aligned}$$

By taking the norms of both sides, we find that

$$\|ee^*e - e\|^2 = \|(ee^*e - e)^*(ee^*e - e)\| = 0.$$

Therefore, $ee^*e = e$.

Next, to see that ee^* is also a projection (or e^* is a partial isometry), we simply note that $ee^*ee^* = (ee^*e)e^* = ee^*$. Since ee^* is also self-adjoint, we deduce that ee^* is a projection as required. □

1.6 Non-unital C*-algebras

The main idea we want to focus on in this section is that from a C*-algebra which is not necessarily unital, one can always construct a unique unital C*-algebra from it. This is stated more precisely in the following theorem.

Theorem 1.6.1. *Let A be a C*-algebra. Then, there exists a unique unital C*-algebra $\tilde{A}$ such that A is contained in $\tilde{A}$ as a closed two-sided ideal and the quotient $\tilde{A}/A \cong \mathbb{C}$.*

Proof. Assume that A is a C*-algebra. As a *vector space*, we define $\tilde{A} = \mathbb{C} \oplus A$. The next step is to endow $\tilde{A}$ with the structure necessary for it to be a C*-algebra — multiplication, involution and a suitable norm.

Multiplication: We define multiplication on $\tilde{A}$ by

$$(\lambda, a)(\mu, b) = (\lambda\mu, \lambda b + \mu a + ab).$$

To be clear, $\lambda, \mu \in \mathbb{C}$ and $a, b \in A$.

Involution: We define an involution map on $\tilde{A}$ by

$$(\lambda, a)^* = (\overline{\lambda}, a^*).$$

Again, $\lambda \in \mathbb{C}$ and $a \in A$. It is easy to check that multiplication and involution defined as above satisfies the required properties.

Before we move on to the problem of defining a norm on $\tilde{A}$, we first make the following observations:

1. It is straightforward to check that $(1, 0) \in \tilde{A}$ is the multiplicative unit for $\tilde{A}$.

2. Define the map

$$\begin{array}{rccc} \iota : & A & \to & \tilde{A} \\ & a & \mapsto & (0, a). \end{array}$$

   Then, ι is an injective *-homomorphism. Moreover, its image

$$\text{im } \iota = \{(0, a) \mid a \in A\}$$

   is a two-sided ideal in $\tilde{A}$. This shows that A is contained in $\tilde{A}$ as a closed two-sided ideal. Moreover by definition, the quotient $\tilde{A}/A$ is isomorphic to $\mathbb{C}$. For instance if $(\lambda, a) \in \tilde{A}$ then $[(\lambda, a)] = [(\lambda, 0)]$ in $\tilde{A}/A$.

Norm: Before we proceed, let us highlight a useful remark from [Put19]. One initial guess for a norm on $\tilde{A}$ is

$$\|(\lambda, a)\|_1 = |\lambda| + \|a\|.$$

With this norm, $\tilde{A}$ becomes a Banach *-algebra whose involution map is isometric. However, this norm does not satisfy the C*-algebra condition. If $(\lambda, a) \in \tilde{A}$ then

$$\begin{aligned}
\|(\lambda, a)\|_1^2 &= (|\lambda| + \|a\|)^2 \\
&= |\lambda|^2 + 2|\lambda|\|a\| + \|a\|^2 \\
&= |\lambda|^2 + 2|\lambda|\|a\| + \|a^*a\| \\
&= |\lambda|^2 + |\lambda|\|a\| + |\lambda|\|a^*\| + \|a^*a\|
\end{aligned}$$

and

$$\begin{aligned}
\|(\lambda, a)^*(\lambda, a)\|_1 &= \|(\overline{\lambda}, a^*)(\lambda, a)\|_1 \\
&= \|(|\lambda|^2, \overline{\lambda}a + \lambda a^* + a^*a)\|_1 \\
&= |\lambda|^2 + \|\overline{\lambda}a + \lambda a^* + a^*a\|.
\end{aligned}$$

One can see that both expression are not equal in general. By the triangle inequality, we only have $\|(\lambda, a)^*(\lambda, a)\|_1 \leq \|(\lambda, a)\|_1^2$ at best.

How do we proceed from here? The key tangential observation is that since A is a Banach space, we can study the related Banach space $B(A)$ of bounded linear operators on A. Define the map

$$\begin{array}{rccc}
\pi : & A & \to & B(A) \\
& a & \mapsto & (b \mapsto ab)
\end{array}$$

By definition of the operator norm, we have $\|\pi(a)\| \leq \|a\|$. For the reverse inequality, we note that

$$\|\pi(a)a^*\| = \|aa^*\| = \|a\|^2 = \|a\|\|a^*\|.$$

So, $\|a\|\|a^*\| \leq \|\pi(a)\|\|a^*\|$ and consequently, $\|a\| = \|\pi(a)\|$. In this way, we see that the norm of A is simply the operator norm on $B(A)$ acting on A

itself.

This observation suggests that we construct the required norm for $\tilde{A}$ by thinking of $\tilde{A}$ as acting on A. If $(\lambda, a) \in \tilde{A}$ then we define

$$\|(\lambda, a)\| = \sup\{|\lambda|, \|(\lambda, a)(0, b)\|, \|(0, b)(\lambda, a)\| \mid b \in A, \|b\| \leq 1\}. \tag{1.6}$$

A quick computation shows that $(\lambda, a)(0, b) = (0, \lambda b + ab)$ and $(0, b)(\lambda, a) = (0, \lambda b + ba)$. So, the norm appearing on the RHS of equation (1.6) is in fact, the norm on A. Here we implicitly use our embedding of A in $\tilde{A}$ via ι.

First we will deal with the existence of the expression in equation (1.6). By its definition and the triangle inequality, we have

$$\begin{aligned}
\|(\lambda, a)\| &= \sup\{|\lambda|, \|(\lambda, a)(0, b)\|, \|(0, b)(\lambda, a)\| \mid b \in A, \|b\| \leq 1\} \\
&= \sup\{|\lambda|, \|\lambda b + ab\|, \|\lambda b + ba\| \mid b \in A, \|b\| \leq 1\} \\
&\leq \sup_{b \in A, \|b\| \leq 1} (|\lambda|\|b\| + \|a\|\|b\|) \\
&= |\lambda| + \|a\| = \|(\lambda, a)\|_1.
\end{aligned}$$

Therefore, the supremum in equation (1.6) does indeed exist.

To show: (a) The map $\|-\|$ in equation (1.6) is a norm.

(a) Assume that $\mu \in \mathbb{C}$ and $(\lambda, a) \in \tilde{A}$. Then,

$$\begin{aligned}
\|\mu(\lambda, a)\| &= \|(\mu\lambda, \mu a)\| \\
&= \sup\{|\mu\lambda|, \|(\mu\lambda, \mu a)(0, b)\|, \|(0, b)(\mu\lambda, \mu a)\| \mid b \in A, \|b\| \leq 1\} \\
&= \sup\{|\mu\lambda|, \|\mu\lambda b + \mu ab\|, \|\mu\lambda b + \mu ba\| \mid b \in A, \|b\| \leq 1\} \\
&= |\mu|\|(\lambda, a)\|.
\end{aligned}$$

Next, assume that $(\sigma, c) \in \tilde{A}$. Then,

$$\begin{aligned}
\|(\lambda, a) + (\sigma, c)\| &= \|(\lambda + \sigma, a + c)\| \\
&= \sup\{|\lambda + \sigma|, \|(\lambda + \sigma)b + (a + c)b\|, \\
&\quad \|(\lambda + \sigma)b + b(a + c)\| \mid b \in A, \|b\| \le 1\} \\
&\le \sup\{|\lambda| + |\sigma|, \|\lambda b + ab\| + \|\sigma b + cb\|, \\
&\quad \|\lambda b + ba\| + \|\sigma b + bc\| \mid b \in A, \|b\| \le 1\} \\
&\le \|(\lambda, a)\| + \|(\sigma, c)\|.
\end{aligned}$$

By equation (1.6), if $(\lambda, a) = (0, 0)$ then $\|(\lambda, a)\| = 0$. For the converse statement, assume that $\|(\lambda, a)\| = 0$. Then,

$$\sup\{|\lambda|, \|\lambda b + ab\|, \|\lambda b + ba\| \mid b \in A, \|b\| \le 1\} = 0.$$

Firstly, we have $|\lambda| = 0$. Now if $b = a^*/\|a\|$ then

$$\|\lambda b + ab\| = \|\lambda b + ba\| = \frac{\|aa^*\|}{\|a\|} = \|a\| = 0.$$

Hence, $(\lambda, a) = (0, 0)$ as required. So, the map in equation (1.6) is indeed a norm.

To show: (b) $\|(0, a)\| = \|a\|$.

(c) $\|(\lambda, a)(\sigma, c)\| \le \|(\lambda, a)\|\|(\sigma, c)\|$.

(b) First observe that

$$\begin{aligned}
\|(0, a)\| &= \sup\{\|ab\|, \|ba\| \mid b \in A, \|b\| \le 1\} \\
&\le \sup_{b \in A, \|b\| \le 1} \|a\|\|b\| = \|a\|.
\end{aligned}$$

Since $\|(0, 0)\| = 0$ by equation (1.6), we may assume for the reverse inequality that $a \neq 0$. Then,

$$\begin{aligned}
\|(0, a)\| &= \sup\{\|ab\|, \|ba\| \mid b \in A, \|b\| \le 1\} \\
&\ge \frac{\|a^*a\|}{\|a\|} = \|a\|.
\end{aligned}$$

In the above computation, we set $b = a^*/\|a\|$. Hence, $\|(0, a)\| = \|a\|$.

A notable consequence of this above computation is that if $a \in A$ then

$$\|a\| = \frac{\|a^*a\|}{\|a\|} = \|a\frac{a^*}{\|a\|}\|. \tag{1.7}$$

(c) Observe that if $(\lambda, a), (\sigma, c) \in \tilde{A}$ then

$$\begin{aligned}
\|(\lambda, a)(\sigma, c)\| &= \|(\lambda\sigma, \lambda c + \sigma a + ac)\| \\
&= \sup\{|\lambda\sigma|, \|\lambda\sigma b + \lambda cb + \sigma ab + acb\|, \\
&\quad \|\lambda\sigma b + \lambda bc + \sigma ba + bac\| \mid b \in A, \|b\| \leq 1\} \\
&\leq \sup\{|\lambda|, \|\lambda b + ab\|, \|\lambda b + ba\| \mid b \in A, \|b\| \leq 1\} \\
&\quad \sup\{|\sigma|, \|\sigma b + cb\|, \|\sigma b + bc\| \mid b \in A, \|b\| \leq 1\} \\
&= \|(\lambda, a)\|\|(\sigma, c)\|.
\end{aligned}$$

Next, we will show that $\tilde{A}$ is complete with respect to the norm in equation (1.6). Let $\{(\lambda_n, a_n)\}_{n\in\mathbb{Z}_{>0}}$ be a Cauchy sequence in $\tilde{A}$. By equation (1.6), the sequence $\{\lambda_n\}_{n\in\mathbb{Z}_{>0}}$ in $\mathbb{C}$ is Cauchy and hence, converges to some $\lambda \in \mathbb{C}$.

We claim that $\{a_n\}_{n\in\mathbb{Z}_{>0}}$ is a Cauchy sequence in A. To this end, if $m, n \in \mathbb{Z}_{>0}$ then

$$\begin{aligned}
\|a_m - a_n\| &= \|(a_m - a_n)^*(a_m - a_n)\|^{\frac{1}{2}} \\
&\leq \|a_m - a_n\|^{\frac{1}{2}}\|(a_m - a_n)^*\|^{\frac{1}{2}} \\
&= \begin{cases} 0, \text{ if } a_m - a_n = 0, \\ \|a_m - a_n\|^{\frac{1}{2}}\|(a_m - a_n)\frac{(a_m-a_n)^*}{\|a_m-a_n\|}\|^{\frac{1}{2}}, \text{ if } a_m - a_n \neq 0. \end{cases} \\
&\leq \|a_m - a_n\|^{\frac{1}{2}} \sup_{b\in A, \|b\|\leq 1} \|(a_m - a_n)b\|^{\frac{1}{2}} \\
&\leq \|a_m - a_n\|^{\frac{1}{2}}\|(0, a_m - a_n)\|^{\frac{1}{2}}.
\end{aligned}$$

In the third line, we used equation (1.7). Consequently,

$$\begin{aligned}
\|a_m - a_n\| &\leq \|(0, a_m - a_n)\| \\
&\leq \|(\lambda_m, a_m) - (\lambda_n, a_n)\| + \|(\lambda_m - \lambda_n, 0)\| \\
&\leq \|(\lambda_m, a_m) - (\lambda_n, a_n)\| + |\lambda_m - \lambda_n| \to 0
\end{aligned}$$

as $m, n \to \infty$. To be clear, the second inequality follows from the triangle inequality and the last inequality follows straight from the definition of the norm on $\tilde{A}$. Therefore, $\{a_n\}_{n\in\mathbb{Z}_{>0}}$ is a Cauchy sequence in A. Hence, it must converge to some $a \in A$.

Now we claim that the sequence $\{(\lambda_n, a_n)\}_{n\in\mathbb{Z}_{>0}}$ converges to $(\lambda, a) \in A$. Assume that $\epsilon \in \mathbb{R}_{>0}$. Then, there exists $N_1, N_2 \in \mathbb{Z}_{>0}$ such that if $m > N_1$ and $n > N_2$ then

$$|\lambda_n - \lambda| < \frac{\epsilon}{2} \qquad \text{and} \qquad \|a_n - a\| < \frac{\epsilon}{2}.$$

So, if $m > \max\{N_1, N_2\}$ then

$$\begin{aligned}\|(\lambda_m, a_m) - (\lambda, a)\| &= \|(\lambda_m - \lambda, a_m - a)\| \\ &\leq \|(\lambda_m - \lambda, a_m - a)\|_1 \\ &= |\lambda_m - \lambda| + \|a_m - a\| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

Hence, the sequence $\{(\lambda_n, a_n)\}_{n\in\mathbb{Z}_{>0}}$ in $\tilde{A}$ converges to (λ, a). This demonstrates that $\tilde{A}$ is a unital Banach *-algebra.

Now we will show that $\tilde{A}$ is a unital C*-algebra. Again, assume that $(\lambda, a) \in \tilde{A}$. Then,

$$\begin{aligned}\|(\lambda, a)^*\| &= \|(\overline{\lambda}, a^*)\| \\ &= \sup\{|\overline{\lambda}|, \|\overline{\lambda}b + a^*b\|, \|\overline{\lambda}b + ba^*\| \mid b \in A, \|b\| \leq 1\} \\ &= \sup\{|\overline{\lambda}|, \|\overline{\lambda}b^* + a^*b^*\|, \|\overline{\lambda}b^* + b^*a^*\| \mid b \in A, \|b\| \leq 1\} \\ &= \sup\{|\lambda|, \|\lambda b + ba\|, \|\lambda b + ab\| \mid b \in A, \|b\| \leq 1\} \\ &= \|(\lambda, a)\|.\end{aligned}$$

Since the involution * in $\tilde{A}$ is isometric with respect to the norm in equation (1.6), we have

$$\|(\lambda, a)^*(\lambda, a)\| \leq \|(\lambda, a)^*\|\|(\lambda, a)\| = \|(\lambda, a)\|^2.$$

To obtain the reverse inequality, note that if $b \in A$ and $\|b\| \leq 1$ then

$$\begin{aligned}\|(\lambda,a)^*(\lambda,a)(0,b)\| &\geq \|(0,b)^*\|\|(\lambda,a)^*(\lambda,a)(0,b)\| \\ &\geq \|(0,b)^*(\lambda,a)^*(\lambda,a)(0,b)\| \\ &= \|(\lambda b+ab)^*(\lambda b+ab)\| \quad \text{(by part (b))} \\ &= \|\lambda b+ab\|^2 \quad (A \text{ is a C*-algebra}) \\ &= \|(\lambda,a)(0,b)\|^2 \quad \text{(by part (b))}.\end{aligned}$$

By replacing (λ, a) with $(\overline{\lambda}, a^*)$ and b with b^*, we deduce that $\|(\lambda,a)(\lambda,a)^*(0,b)^*\| \geq \|(\lambda,a)^*(0,b)^*\|^2$. Using the fact that the involution on $\tilde{A}$ is isometric, we find that

$$\|(0,b)(\lambda,a)(\lambda,a)^*\| \geq \|(0,b)(\lambda,a)\|^2.$$

Finally, we have

$$\begin{aligned}\|(\lambda,a)^*(\lambda,a)\| &= \sup\{|\lambda|^2, \|(\lambda,a)^*(\lambda,a)(0,b)\|, \|(0,b)(\lambda,a)^*(\lambda,a)\| \\ &\qquad \mid b \in A, \|b\| \leq 1\} \\ &\geq \sup\{|\lambda|^2, \|(\lambda,a)(0,b)\|^2, \|(0,b)(\lambda,a)\|^2 \mid b \in A, \|b\| \leq 1\} \\ &= \|(\lambda,a)\|^2.\end{aligned}$$

Therefore, $\tilde{A}$ is a unital C*-algebra as required.

Uniqueness: To see that $\tilde{A}$ is unique, assume that B is another unital C*-algebra which contains A as a closed two-sided ideal and the quotient $B/A \cong \mathbb{C}$. Let $1_B \in B$ be the unit of B. We define

$$\begin{array}{rccc}\rho: & \tilde{A} & \to & B \\ & (\lambda,a) & \mapsto & \lambda 1_B + a.\end{array}$$

It is straightforward to check that ρ is a unital *-homomorphism. Now let $q : B \to B/A$ be the quotient map. Since A is a closed two-sided ideal of B which is not all of B, then $1_B \notin A$ and subsequently, $q(1_B) \neq 0$.

To show: (d) ρ is injective.

(e) ρ is surjective.

(d) Assume that $(\lambda, a) \in \ker \rho$ so that $\rho(\lambda, a) = \lambda 1_B + a = 0$. Then, $\lambda 1_B = -a$ and by applying the quotient map q, we find that $\lambda q(1_B) = 0$ in $B/A \cong \mathbb{C}$. Since $q(1_B) \neq 0$, $\lambda = 0$ in $\mathbb{C}$ and $a = 0$ in A. Therefore, ρ is injective.

(e) Assume that $b \in B$. Note that the composite $q \circ \rho$ sends $(\lambda, a) \in \tilde{A}$ to $\lambda q(1_B) \in B/A \cong \mathbb{C}$. Hence, $q \circ \rho$ is surjective because if $\mu \in \mathbb{C}$ then

$$(q \circ \rho)(\frac{\mu}{q(1_B)}, 0) = \mu.$$

So, there exists $(\sigma, c) \in \tilde{A}$ such that

$$(q \circ \rho)(\sigma, c) = q(\sigma 1_B + c) = q(b).$$

This means that there exists $a' \in A \subseteq B$ such that $\sigma 1_B + c - a' = b$. Thus, $\rho(\sigma, c - a') = b$ and ρ is surjective.

By part (d) of the proof, ρ is an injective *-homomorphism. By Theorem 1.3.11, ρ is an isometry. In tandem with part (e), we conclude that ρ is a *-isomorphism from $\tilde{A}$ to B. Hence, $\tilde{A}$ is unique which completes the proof. □

The C*-algebra $\tilde{A}$ is referred to as the **unitization** of A. The next theorem tells us how $\tilde{A}$ as constructed in Theorem 1.6.1 is connected to A when A is a unital C*-algebra itself.

Theorem 1.6.2. *Let A be a unital C*-algebra. Let $\tilde{A}$ be the unital C*-algebra constructed in Theorem 1.6.1. Then, $\tilde{A} \cong \mathbb{C} \oplus A$ as C*-algebras.*

Proof. Assume that A is a unital C*-algebra. Assume that $\tilde{A}$ is the unital C*-algebra constructed in Theorem 1.6.1. Recall from Theorem 1.1.3 that $\mathbb{C} \oplus A$ is a C*-algebra with scalar multiplication, multiplication, addition and involution defined pointwise. It has norm given by

$$\|(\lambda, a)\|_D = \max\{|\lambda|, \|a\|\}.$$

In $\tilde{A}$, scalar multiplication, involution and addition are the same as $\mathbb{C} \oplus A$. However, multiplication is given by

$$(\lambda, a)(\mu, b) = (\lambda\mu, \lambda b + \mu a + ab).$$

The norm on $\tilde{A}$ is given explicitly by

$$\|(\lambda, a)\| = \sup\{|\lambda|, \|\lambda b + ab\|, \|\lambda b + ba\| \mid b \in A, \|b\| \leq 1\}.$$

The key is to observe how the norms $\|-\|_D$ and $\|-\|$ are related. In particular, since A is unital,

$$\|(\lambda, a - \lambda 1_A)\| = \sup\{|\lambda|, \|ab\|, \|ba\| \mid b \in A, \|b\| \leq 1\}.$$

By the triangle inequality, we obtain

$$\|(\lambda, a - \lambda 1_A)\| \leq \max\{|\lambda|, \|a\|\} = \|(\lambda, a)\|_D.$$

But, we also have

$$\begin{aligned}
\|(\lambda, a)\|_D &= \max\{|\lambda|, \|a\|\} \\
&= \max\{|\lambda|, \|a 1_A\|, \|1_A a\|\} \\
&\leq \sup\{|\lambda|, \|ab\|, \|ba\| \mid b \in A, \|b\| \leq 1\} \\
&= \|(\lambda, a - \lambda 1_A)\|.
\end{aligned}$$

Therefore, if $\lambda \in \mathbb{C}$ and $a \in A$ then $\|(\lambda, a)\|_D = \|(\lambda, a - \lambda 1_A)\|$. This suggests that we define the map

$$\begin{array}{cccc}
\varphi : & \mathbb{C} \oplus A & \to & \tilde{A} \\
& (\lambda, a) & \mapsto & (\lambda, a - \lambda 1_A)
\end{array}$$

We claim that φ is an isometric *-isomorphism. By our previous computation, φ is an isometry. Next, observe that if $(\lambda, a), (\mu, b) \in \mathbb{C} \oplus A$ and $\sigma \in \mathbb{C}$ then

$$\begin{aligned}
\varphi((\lambda, a) + (\mu, b)) &= \varphi(\lambda + \mu, a + b) \\
&= (\lambda + \mu, a + b - \lambda 1_A - \mu 1_A) \\
&= (\lambda, a - \lambda 1_A) + (\mu, b - \mu 1_A) \\
&= \varphi(\lambda, a) + \varphi(\mu, b),
\end{aligned}$$

$$\varphi((\lambda, a)^*) = \varphi(\overline{\lambda}, a^*) = (\overline{\lambda}, a^* - \overline{\lambda} 1_A) = \varphi(\lambda, a)^*,$$

$$\begin{aligned}
\varphi(\sigma(\lambda, a)) &= \varphi(\sigma\lambda, \sigma a) \\
&= (\sigma\lambda, \sigma a - \sigma\lambda 1_A) \\
&= \sigma(\lambda, a - \lambda 1_A) = \sigma\varphi(\lambda, a),
\end{aligned}$$

and

$$\begin{aligned}
\varphi((\lambda, a)(\mu, b)) &= \varphi(\lambda\mu, ab) \\
&= (\lambda\mu, ab - \lambda\mu 1_A) \\
&= (\lambda\mu, (\lambda b - \lambda\mu 1_A) + (\mu a - \lambda\mu 1_A) + (ab - \mu a - \lambda b + \lambda\mu 1_A)) \\
&= (\lambda, a - \lambda 1_A)(\mu, b - \mu 1_A) \\
&= \varphi(\lambda, a)\varphi(\mu, b).
\end{aligned}$$

Finally, to see that φ is bijective, observe that φ has an inverse given by

$$\begin{array}{rccc}
\varphi^{-1}: & \tilde{A} & \to & \mathbb{C} \oplus A \\
& (\lambda, a) & \mapsto & (\lambda, a + \lambda 1_A)
\end{array}$$

One can check by direct calculation that φ^{-1} is also a *-homomorphism. Consequently, φ is an isometric *-isomorphism and $\tilde{A} \cong \mathbb{C} \oplus A$ as C*-algebras. □

The unital C*-algebra $\tilde{A}$ constructed in Theorem 1.6.1 satisfies the following *universal property.* This is discussed in the reference [Ter20].

Theorem 1.6.3. *Let A be a C*-algebra. Then, the unital C*-algebra $\tilde{A}$ in Theorem 1.6.1 satisfies the following universal property: If B is a unital C*-algebra and $f : A \to B$ is a *-homomorphism then there exists a unique unital *-homomorphism $\tilde{f} : \tilde{A} \to B$ such that the following diagram commutes:*

$$\begin{tikzcd}
A \arrow[r, hook, "\iota"] \arrow[dr, "f"'] & \tilde{A} \arrow[d, dashed, "\tilde{f}"] \\
& B
\end{tikzcd}$$

Here, $\iota : A \to \tilde{A}$ denotes the inclusion map $a \mapsto (0, a)$.

Proof. Assume that A is a C*-algebra and that $\tilde{A}$ is the unital C*-algebra constructed in Theorem 1.6.1. Assume that B is a unital C*-algebra and that $f : A \to B$ is a *-homomorphism. Define the map $\tilde{f}$ by

$$\begin{array}{rccc}
\tilde{f}: & \tilde{A} & \to & B \\
& (\lambda, a) & \mapsto & \lambda 1_B + f(a)
\end{array}$$

A barrage of brief computations shows that $\tilde{f}$ is a unital *-homomorphism such that $\tilde{f} \circ \iota = f$, where $\iota : A \to \tilde{A}$ is the inclusion map $a \mapsto (0, a)$.

To see that $\tilde{f}$ is unique, assume that $g : \tilde{A} \to B$ is another unital *-homomorphism such that $g \circ \iota = f$. If $a \in A$ then

$$g(0,a) = (g \circ \iota)(a) = f(a) = \tilde{f}(0,a).$$

Since g is a unital *-homomorphism and $(1,0) \in \tilde{A}$ is the unit of $\tilde{A}$, we have $g(1,0) = 1_B$. Subsequently, if $\lambda \in \mathbb{C} - \{0\}$ and $a \in A$ then

$$\begin{aligned} g(\lambda, a) &= g(\lambda(1, \frac{1}{\lambda}a)) \\ &= \lambda g(1, \frac{1}{\lambda}a) \\ &= \lambda g((1,0) + (0, \frac{1}{\lambda}a)) \\ &= \lambda(g(1,0) + g(0, \frac{1}{\lambda}a)) \\ &= \lambda(1_B + f(\frac{1}{\lambda}a)) \\ &= \lambda 1_B + f(a) = \tilde{f}(\lambda, a). \end{aligned}$$

Therefore, $g = \tilde{f}$ which completes the proof. $\square$

We will use the universal property in Theorem 1.6.3 to extend Theorem 1.3.11.

Theorem 1.6.4. *Let A and B be C*-algebras and $\phi : A \to B$ be an injective *-homomorphism. Then, ϕ is an isometry.*

Proof. Assume that A and B are C*-algebras and $\phi : A \to B$ is an injective *-homomorphism. By the universal property of unitization in Theorem 1.6.3, there exists a unique unital *-homomorphism $\tilde{\phi} : \tilde{A} \to \tilde{B}$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, hook, "\iota_A"] \arrow[dr, "\iota_B \circ \phi"'] & \tilde{A} \arrow[d, dashed, "\tilde{\phi}"] \\ & \tilde{B} \end{tikzcd}$$

Here, ι_A and ι_B are the inclusion $A \hookrightarrow \tilde{A}$ and $B \hookrightarrow \tilde{B}$.

We claim that $\tilde{\phi}$ is injective. Explicitly, it is given by

$$\begin{array}{cccc} \tilde{\phi}: & \tilde{A} & \to & \tilde{B} \\ & (\lambda, a) & \mapsto & \lambda(1,0) + (0, \phi(a)) = (\lambda, \phi(a)). \end{array}$$

Assume that $(\mu, c) \in \ker \tilde{\phi}$. Then, $\tilde{\phi}(\mu, c) = (\mu, \phi(c)) = (0, 0)$. Then, $\mu = 0$ and $c = 0$ because ϕ is injective. So, $(\mu, c) = (0, 0)$ and $\tilde{\phi}$ is injective.

By Theorem 1.3.11, $\tilde{\phi}$ is an isometry. If $a \in A$ then

$$\|\phi(a)\| = \|(0, \phi(a))\| = \|\tilde{\phi}(0, a)\| = \|(0, a)\| = \|a\|.$$

Hence, ϕ is an isometry as required. $\square$

Returning to the exposition in [Put19], we are now interested in extending Theorem 1.3.5 and the continuous functional calculus in Theorem 1.3.7 to the case of non-unital C*-algebras. The most direct way to do this is to remove the assumption that the C*-algebra is unital in both theorems. Before we proceed, we remind ourselves of continuous functions which *vanish at infinity.* First recall that a topological space X is **locally compact** if for $x \in X$, there exists an open set U and a compact set K such that $x \in U \subseteq K$.

Definition 1.6.1. Let X be a locally compact Hausdorff space. We say that $f \in Cts(X, \mathbb{C})$ **vanishes at infinity** if for $\epsilon \in \mathbb{R}_{>0}$, there exists a compact subset $K \subseteq X$ such that if $x \in X \backslash K$ then $|f(x)| < \epsilon$. Equivalently, if $\epsilon \in \mathbb{R}_{>0}$ then the set

$$\{x \in X \mid |f(x)| \geq \epsilon\}$$

is compact. Now define

$$Cts_0(X, \mathbb{C}) = \{f \in Cts(X, \mathbb{C}) \mid f \text{ vanishes at infinity}\}.$$

The space $Cts_0(X, \mathbb{C})$ is a commutative C*-algebra, with multiplication, scalar multiplication, addition and involution defined pointwise. The norm on $Cts_0(X, \mathbb{C})$ is exactly the same as that of $Cts(X, \mathbb{C})$ — if $f \in Cts_0(X, \mathbb{C})$ then

$$\|f\| = \sup_{x \in X} |f(x)|.$$

There is a simple criterion to determine whether $Cts_0(X, \mathbb{C})$ is unital.

Theorem 1.6.5. *Let X be a locally compact Hausdorff space. Then, $Cts_0(X, \mathbb{C})$ is a unital C*-algebra if and only if X is compact.*

Proof. Assume that X is a locally compact Hausdorff space.

To show: (a) If $Cts_0(X, \mathbb{C})$ is a unital C*-algebra then X is compact.

(b) If X is compact then $Cts_0(X, \mathbb{C})$ is a unital C*-algebra.

(a) Assume that the C*-algebra $Cts_0(X, \mathbb{C})$ is unital. Let $1_X : X \to \mathbb{C}$ be the function defined by $1_X(x) = 1$ for $x \in X$. Then, 1_X is the multiplicative unit for $Cts_0(X, \mathbb{C})$.

Since $1_X \in Cts_0(X, \mathbb{C})$ by assumption, there exists a compact subset $K \subseteq X$ such that if $x \in X - K$ then $|1_X(x)| < \frac{1}{2}$. However, $1_X(x) = 1$ for arbitrary $x \in X$. Hence, $X - K = \emptyset$ and consequently, $X = K$ must be compact.

(b) Assume that X is compact. To see that $Cts_0(X, \mathbb{C})$ is unital, it suffices to show that $1_X \in Cts_0(X, \mathbb{C})$. Assume that $\epsilon \in \mathbb{R}_{>0}$. If $\epsilon > 1$ then $|1_X(x)| < \epsilon$ for $x \in X = X - \emptyset$. If $\epsilon \leq 1$ then $|1_X(x)| < \epsilon$ for $x \in \emptyset = X - X$. Since X and $\emptyset$ are compact sets, we deduce that $1_X \in Cts_0(X, \mathbb{C})$. So, $Cts_0(X, \mathbb{C})$ is a unital C*-algebra. □

If X happens to be a compact Hausdorff space then a particular C*-subalgebra of $Cts(X, \mathbb{C})$ can be related to functions which vanish at infinity in the following manner:

Theorem 1.6.6. *Let X be a compact Hausdorff space and $x_0 \in X$. Then, we have the isomorphism of C*-algebras*

$$\{f \in Cts(X, \mathbb{C}) \mid f(x_0) = 0\} \cong Cts_0(X - \{x_0\}, \mathbb{C}).$$

Proof. Assume that X is a compact Hausdorff space. Define the map

$$\begin{array}{cccc} r: & \{f \in Cts(X, \mathbb{C}) \mid f(x_0) = 0\} & \to & Cts_0(X - \{x_0\}, \mathbb{C}) \\ & \phi & \mapsto & \phi|_{X - \{x_0\}} \end{array}$$

The map r is restriction to $X - \{x_0\}$. To see that r is well-defined, assume that $\varphi \in Cts(X, \mathbb{C})$ and $\varphi(x_0) = 0$. Firstly, $X - \{x_0\}$ is locally compact because it is an open subset of the compact space X. Now assume that $\epsilon \in \mathbb{R}_{>0}$. Since φ is continuous, there exists an open neighbourhood U_0 of X such that $x_0 \in U_0$ and if $u \in U_0$ then $|f(u)| < \epsilon$.

Now the set $X - U_0$ is a closed subset of X, which is compact. Hence, $X - U_0$ is also compact. Observe that

$$\{x \in X \mid |f(x)| \geq \epsilon\} \subseteq X - U_0.$$

Since the set $\{x \in X \mid |f(x)| \geq \epsilon\}$ is closed, it must be compact by the above inclusion. Therefore, $f|_{X-\{x_0\}} \in Cts_0(X - \{x_0\}, \mathbb{C})$ and consequently, the map r is well-defined.

The fact that r is a *-homomorphism follows from its definition. It is also easy to see that r is injective. To see that r is surjective, assume that $g \in Cts_0(X - \{x_0\}, \mathbb{C})$. Define $\tilde{g} : X \to \mathbb{C}$ by $\tilde{g}(x) = g(x)$ if $x \neq x_0$ and $\tilde{g}(x_0) = 0$. Then, $\tilde{g} \in Cts(X, \mathbb{C})$ (see [Mur20]) and $r(\tilde{g}) = g$ by construction. So, r is surjective.

We conclude that r is a *-isomorphism and consequently, we obtain the isomorphism of C*-algebras

$$\{f \in Cts(X, \mathbb{C}) \mid f(x_0) = 0\} \cong Cts_0(X - \{x_0\}, \mathbb{C}).$$

□

Now we will prove our extension of Theorem 1.3.5.

Theorem 1.6.7. *Let A be a commutative C*-algebra and $\tilde{A}$ be the unital C*-algebra constructed in Theorem 1.6.1. Let $\pi : \tilde{A} \to \tilde{A}/A \cong \mathbb{C}$ be the canonical quotient map. Then, $\pi \in \mathcal{M}(\tilde{A})$. Moreover, the restriction of the *-isomorphism in Theorem 1.3.5*

$$\begin{array}{rccc} \Lambda: & \tilde{A} & \to & Cts(\mathcal{M}(\tilde{A}), \mathbb{C}) \\ & a & \mapsto & ev_a \end{array}$$

*to A is an isometric *-isomorphism from A to $Cts_0(\mathcal{M}(\tilde{A}) - \{\pi\}, \mathbb{C})$.*

Proof. Assume that A is a commutative C*-algebra. Since $(1, 0)$ is the multiplicative unit of $\tilde{A}$, $\pi((1, 0)) = 1 \neq 0$. So, π is non-zero and $\pi \in \mathcal{M}(\tilde{A})$.

Assume that Λ is the *-isomorphism defined as above on $\tilde{A}$. If $a \in A$ then

$$\Lambda(a)(\pi) = ev_a(\pi) = \pi(a) = 0.$$

Hence, the image of A under Λ is contained in the C*-subalgebra

$$\{f \in Cts(\mathcal{M}(\tilde{A}), \mathbb{C}) \mid f(\pi) = 0\} \cong Cts_0(\mathcal{M}(\tilde{A}) - \{\pi\}, \mathbb{C})$$

by Theorem 1.6.6. The proof that the restriction $\Lambda|_A$ is a *-homomorphism and an isometry follows what was done in Theorem 1.3.5. To see that it is injective, assume that $a_1, a_2 \in A$ and that $\Lambda(a_1) = \Lambda(a_2)$. Then, $ev_{a_1} = ev_{a_2}$ and if $f \in Cts(\mathcal{M}(\tilde{A}), \mathbb{C})$ then $f(a_1) = f(a_2)$. Hence, $a_1 = a_2$ and $\Lambda|_A$ is injective.

Finally to see that $\Lambda|_A$ is surjective, note that A has codimension 1 in $\tilde{A}$ since $\tilde{A}/A \cong \mathbb{C}$. Also, $Cts_0(\mathcal{M}(\tilde{A}) - \{\pi\}, \mathbb{C})$ has codimension 1 in $Cts(\mathcal{M}(\tilde{A}), \mathbb{C})$ by Theorem 1.6.6. Therefore, the restriction $\Lambda|_A$ is an isometric *-isomorphism from A to $Cts_0(\mathcal{M}(\tilde{A}) - \{\pi\}, \mathbb{C})$. □

Observe that we have the following important consequence of Theorem 1.6.7.

Theorem 1.6.8. *Let A be a commutative C*-algebra. Then, there exists a locally compact Hausdorff space X such that $A \cong Cts_0(X, \mathbb{C})$ as C*-algebras.*

Now we proceed to a generalisation of the continuous functional calculus in Theorem 1.3.7. As stated in [Put19, Page 39], there is a subtlety here which needs to be addressed. If B is a possibly non-unital C*-algebra and $a \in B$ is normal then we can consider the unital C*-algebra $\tilde{B}$ and then apply Theorem 1.3.7 to $\tilde{B}$. However, the image of the *-isomorphism in Theorem 1.3.7 is the C*-algebra generated by a and the multiplicative unit of $\tilde{B}$, which lies outside of B (since B is not assumed to be unital).

The way this issue is circumvented is to strengthen the conclusion of Theorem 1.3.7 by showing if $f \in Cts(\sigma(a), \mathbb{C})$ satisfies $f(0) = 0$ then $f(a) \in B$ lies in the C*-subalgebra generated by a. In fact, this is useful even if B is unital. Hence, our extension of Theorem 1.3.7 will be stated for unital C*-algebras.

Theorem 1.6.9. *Let B be a unital C*-algebra and $a \in B$ be normal. The *-isomorphism in Theorem 1.3.7 restricts to an isometric *-isomorphism from $Cts_0(\sigma(a) - \{0\}, \mathbb{C})$ to the C*-subalgebra of B generated by a.*

Proof. Assume that B is a unital C*-algebra and $a \in B$ be normal. We have two different cases to consider.

Case 1: $0 \notin \sigma(a)$.

First assume that $0 \notin \sigma(a)$. Then, a is invertible and

$$Cts_0(\sigma(a) - \{0\}, \mathbb{C}) = Cts(\sigma(a), \mathbb{C})$$

because $\sigma(a) - \{0\} = \sigma(a)$ is a compact Hausdorff space. Hence, we need to show that the C*-algebra generated by a and its unit is the same as the

C*-algebra generated by a alone.

To this end, let f be a continuous function on $\sigma(a) \cup \{0\}$ satisfying $f(0) = 0$ and $f(x) = 1$ for $x \in \sigma(a)$. Such a continuous function exists by Urysohn's lemma. Assume that $\epsilon \in \mathbb{R}_{>0}$. By Weierstrass's approximation theorem, there exists a polynomial $p(z, \overline{z})$ such that

$$|p(z, \overline{z}) - f(z)| < \frac{\epsilon}{2}.$$

Now $p(0, 0)$ yields the constant term of the polynomial $p(z, \overline{z})$. So, $p(z, \overline{z}) - p(0, 0)$ is a polynomial with no constant term which satisfies

$$|p(z, \overline{z}) - p(0, 0) - f(z)| = |p(z, \overline{z}) - p(0, 0) - f(z) - f(0)| < \epsilon.$$

By applying the map Λ^{-1} in Theorem 1.3.7, we deduce that $f(a) = 1_B$ and

$$\|p(a, a^*) - 1_B\| < \epsilon.$$

By Theorem 1.3.7, $p(a, a^*)$ is an element of the C*-subalgebra generated by a. Therefore, 1_B is also in the C*-subalgebra of B generated by a. This demonstrates that the C*-subalgebra generated by the set $\{1_B, a\}$ is exactly the C*-subalgebra generated by a.

By Theorem 1.6.6, Λ^{-1} maps $Cts_0(\sigma(a) - \{0\}, \mathbb{C})$ to the C*-subalgebra generated by a. Moreover, it is an isometric *-isomorphism by Theorem 1.3.7.

Case 2: $0 \in \sigma(a)$.

Assume that $0 \in \sigma(a)$. Let $f \in Cts_0(\sigma(a) - \{0\}, \mathbb{C})$. By the isomorphism in Theorem 1.6.6, we can think of f as a continuous function on $\sigma(a)$ satisfying $f(0) = 0$. Again, assume that $\epsilon \in \mathbb{R}_{>0}$. By Weierstrass's approximation theorem, there exists a polynomial $p(z, \overline{z})$ such that $|p(z, \overline{z}) - f(z)| < \epsilon/2$. By repeating the same argument as in Case 1, we find that $f(a)$ is approximated to within ϵ by $p(a, a^*)$ which is an element of the C*-subalgebra generated by only a. Hence, $f(a)$ is in fact, an element of the C*-subalgebra generated by a.

As in Case 1, we conclude that the restriction of Λ^{-1} to $Cts_0(\sigma(a) - \{0\}, \mathbb{C})$ is an isometric *-isomorphism from $Cts_0(\sigma(a) - \{0\}, \mathbb{C})$ to the C*-subalgebra generated by only a. $\square$

1.7 Ideals and Quotients

Let A be a C*-algebra and I be a subspace of A. What conditions on I are required for the quotient A/I to be a C*-algebra itself? Algebraically, if I is a two-sided ideal then A/I is a $\mathbb{C}$-algebra. Topologically, if I is a closed as a subset of A then A/I is a Banach space. This suggests that we focus on closed two-sided ideals I of A in order to answer our question.

We begin by formalising and proving the topological statement about the quotient A/I. First, we will recall a useful criterion for proving whether a normed vector space is a Banach space.

Definition 1.7.1. Let V be a normed vector space. A sequence $\{x_n\}_{n\in\mathbb{Z}_{>0}}$ is called **absolutely summable** if the quantity

$$\sum_{n=1}^{\infty}\|x_n\| < \infty.$$

The sequence $\{x_n\}_{n\in\mathbb{Z}_{>0}}$ is called **summable** if the sequence of partial sums $\{\sum_{n=1}^{N} x_n\}_{N\in\mathbb{Z}_{>0}}$ converges to some $x \in V$.

Theorem 1.7.1. *Let V be a normed vector space. Then, V is a Banach space if and only if every absolutely summable sequence is summable.*

Proof. Assume that V is a normed vector space.

To show: (a) If V is a Banach space then every absolutely summable sequence is summable.

(b) If every absolutely summable sequence is summable then V is a Banach space.

(a) Assume that V is a Banach space so that V is complete with respect to its norm. Assume that $\{x_n\}_{n\in\mathbb{Z}_{>0}}$ is an absolutely summable sequence in V. To see that the sequence of partial sums $\{\sum_{n=1}^{N} x_n\}_{N\in\mathbb{Z}_{>0}}$ converges, it suffices to show that it is a Cauchy sequence.

Now the sequence $\{\sum_{n=1}^{N}\|x_n\|\}_{N\in\mathbb{Z}_{>0}}$ converges in $\mathbb{R}$ to $\sum_{n=1}^{\infty}\|x_n\| < \infty$. So, it is a Cauchy sequence. Assume that $\epsilon \in \mathbb{R}_{>0}$. Then, there exists $N \in \mathbb{Z}_{>0}$ such that if $N_1, N_2 > N$ with $N_1 < N_2$ then

$$\Big|\sum_{n=1}^{N_1}\|x_n\| - \sum_{n=1}^{N_2}\|x_n\|\Big| = \sum_{n=N_1+1}^{N_2}\|x_n\| < \epsilon.$$

Therefore,

$$\|\sum_{n=1}^{N_1} x_n - \sum_{n=1}^{N_2} x_n\| = \|\sum_{n=N_1+1}^{N_2} x_n\| \leq \sum_{n=N_1+1}^{N_2} \|x_n\| < \epsilon.$$

So, the sequence $\{\sum_{n=1}^{N} x_n\}_{N\in\mathbb{Z}_{>0}}$ in V is Cauchy and hence, converges. Thus, the sequence $\{x_n\}_{n\in\mathbb{Z}_{>0}}$ is summable.

(b) Assume that in V, every absolutely summable sequence is summable. Assume that $\{x_n\}_{n\in\mathbb{Z}_{>0}}$ is a Cauchy sequence in V. To see that $\{x_n\}_{n\in\mathbb{Z}_{>0}}$ converges, it suffices to construct a convergent subsequence $\{x_{n_k}\}$.

Assume that $k \in \mathbb{Z}_{>0}$. Then, there exists $N_k \in \mathbb{Z}_{>0}$ such that if $m, n > N_k$ then

$$\|x_m - x_n\| < 2^{-k}.$$

Now define $n_k = \sum_{i=1}^{k} N_i$. If $i, j \in \mathbb{Z}_{>0}$ and $i < j$ then $n_i < n_j$. Moreover, if $k \in \mathbb{Z}_{>0}$ then by construction,

$$\|x_{n_k} - x_{n_{k+1}}\| < 2^{-k}.$$

We claim that the subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges. To this end, consider the sequence $\{x_{n_k} - x_{n_{k+1}}\}_{k\in\mathbb{Z}_{>0}}$ in V. This sequence is absolutely summable because

$$\sum_{k=1}^{\infty} \|x_{n_k} - x_{n_{k+1}}\| < \sum_{k=1}^{\infty} 2^{-k} = 1 < \infty.$$

By our assumption, we conclude that $\{x_{n_k} - x_{n_{k+1}}\}_{k\in\mathbb{Z}_{>0}}$ is summable, which means that the sequence $\{\sum_{k=1}^{L} x_{n_k} - x_{n_{k+1}}\}_{L\in\mathbb{Z}_{>0}}$ in V. But,

$$\sum_{k=1}^{L} (x_{n_k} - x_{n_{k+1}}) = x_{n_1} - x_{n_{L+1}}.$$

Consequently, the sequence $\{x_{n_k}\}$ converges and so does $\{x_n\}_{n\in\mathbb{Z}_{>0}}$. So, V is complete and hence, a Banach space. □

Theorem 1.7.2. *Let A be a Banach space and I be a closed subspace of A. Then, the quotient space A/I is a Banach space equipped with the norm*

$$\|a + I\| = \inf_{b\in I} \|a + b\|.$$

Proof. Assume that A is a Banach space and I is a closed subspace of A. For $a \in A$, we define

$$\|a + I\| = \inf_{b \in I} \|a + b\|.$$

To show: (a) The map $\|-\| : A/I \to \mathbb{R}_{\geq 0}$ is well-defined.

(a) Assume that $a_1 + I = a_2 + I$ in A/I. Then, $a_1 - a_2 \in I$ and

$$\begin{aligned}
\|a_1 + I\| &= \inf_{b \in I} \|a_1 + b\| \\
&= \inf_{b \in I} \|a_2 + (a_1 - a_2 + b)\| \\
&= \inf_{b \in I} \|a_2 + b\| = \|a_2 + I\|.
\end{aligned}$$

Hence, the map $\|-\| : A/I \to \mathbb{R}_{\geq 0}$ is well-defined.

Next, we show that $\|-\|$ defines a norm on A/I. If $\alpha \in \mathbb{C}$ then

$$\begin{aligned}
\|\alpha(a + I)\| &= \|\alpha a + I\| \\
&= \inf_{b \in I} \|\alpha a + b\| \\
&= \inf_{b \in I} \|\alpha a + \alpha b\| \\
&= |\alpha| \inf_{b \in I} \|a + b\| = |\alpha| \|a + I\|.
\end{aligned}$$

By definition of the map $\|-\|$, if $a + I \in A/I$ then $\|a + I\| \geq 0$. Observe that $\|a + I\| = 0$ if and only if

$$\inf_{b \in I} \|a + b\| = 0.$$

This means that there exists a sequence $\{b_n\}_{n \in \mathbb{Z}_{>0}}$ in I such that $\lim_{n \to \infty} \|a - b_n\| = 0$. Since I is a closed subspace of A, $a \in I$ and $a + I = 0$ in A/I.

Finally, if $a_1 + I, a_2 + I \in A/I$ then

$$\begin{aligned}
\|(a_1+I)+(a_2+I)\| &= \|(a_1+a_2)+I\| \\
&= \inf_{b\in I}\|a_1+a_2+b\| \\
&= \inf_{b\in I}\|a_1+b+a_2+b\| \\
&\leq \inf_{b\in I}\|a_1+b\| + \inf_{b\in I}\|a_2+b\| \\
&= \|a_1+I\| + \|a_2+I\|.
\end{aligned}$$

Hence, $\|-\|$ defines a norm on A/I.

To show: (b) A/I is a Banach space.

(b) We will use Theorem 1.7.1. Suppose that $\{a_n+I\}_{n\in\mathbb{Z}_{>0}}$ is an absolutely summable sequence in A/I. Then, the quantity

$$\sum_{n=1}^{\infty}\|a_n+I\| = \sum_{n=1}^{\infty}\inf_{b\in I}\|a_n+b\| < \infty.$$

If $n\in\mathbb{Z}_{>0}$ then we can select $b_n\in A$ such that $a_n-b_n\in I$ and $\|a_n+b_n\| < 2\inf_{b\in I}\|a_n+b\|$. We claim that the sequence $\{a_n+b_n\}_{n\in\mathbb{Z}_{>0}}$ in A is absolutely summable.

To see why this is the case, we compute directly that

$$\sum_{n=1}^{\infty}\|a_n+b_n\| \leq 2\sum_{n=1}^{\infty}\|a_n+b\| < \infty.$$

Since A is complete, the absolutely summable sequence $\{a_n+b_n\}$ is in turn summable by Theorem 1.7.1. Hence, the sequence $\{\sum_{n=1}^{N}(a_n+b_n)\}_{N\in\mathbb{Z}_{>0}}$ converges to some $c\in A$. Consequently,

$$\begin{aligned}
\|\Big(\sum_{n=1}^{N}(a_n+I)\Big)-(c+I)\| &\leq \|\Big(\sum_{n=1}^{N}a_n-c\Big)+\sum_{n=1}^{N}b_n\| \\
&= \|\Big(\sum_{n=1}^{N}(a_n+b_n)\Big)-c\| \\
&\to 0
\end{aligned}$$

as $N\to\infty$. Therefore, the sequence $\{a_n+I\}_{n\in\mathbb{Z}_{>0}}$ is summable. By Theorem 1.7.1, we deduce that A/I is complete and thus, a Banach space. $\square$

A natural application of Theorem 1.7.2 is to the case where A is a Banach algebra — a Banach *-algebra without an involution operation. If A is a Banach algebra and I is a closed, two sided ideal then the quotient A/I is simultaneously a $\mathbb{C}$-algebra and a Banach space. It is a Banach algebra itself because if $a+I, b+I \in A/I$ then

$$\begin{aligned}
\|(a+I)(b+I)\| &= \|ab+I\| \\
&= \inf_{j\in I}\|ab+j\| \\
&= \inf_{i,j\in I}\|ab+(ai+bj+ij)\| \\
&\leq \big(\inf_{i\in I}\|a+i\|\big)\big(\inf_{j\in I}\|b+j\|\big) \\
&= \|a+I\|\|b+I\|.
\end{aligned}$$

Now let A be a C*-algebra. In order for the quotient A/I to be a C*-algebra, there are two sticking points which must be addressed.

1. What is the involution operation on A/I? The obvious definition of an involution on A/I would be $(a+I)^* = a^*+I$, but in order for this map to be well-defined, the closed two-sided ideal I must be closed under involution.

2. Does the norm on A/I satisfy the C*-algebra condition?

Note that in order to answer the second question, we must have an answer to the first question. We will prove shortly that any closed two-sided ideal I of a C*-algebra is closed under involution. The reference [Sol18, Section A.5.2] proves this result for the special case of a unital C*-algebra. Our proof uses the following technical result.

Theorem 1.7.3. *Let A be a C*-algebra and $a \in A$. If $\epsilon \in \mathbb{R}_{>0}$ then there exists $f \in Cts([0,\infty),\mathbb{C})$ such that $e = f(a^*a) \in A$, is positive, $\|e\| \leq 1$ and $\|a - ae\| < \epsilon$.*

Proof. Assume that A is a C*-algebra and $a \in A$. First, recall that by Theorem 1.4.5, the spectrum $\sigma(a^*a) \subseteq [0,\infty)$ since a^*a is positive. Assume that $\epsilon \in \mathbb{R}_{>0}$. Define the function f by

$$\begin{array}{cccc}
f: & [0,\infty) & \to & \mathbb{C} \\
 & x & \mapsto & \frac{x}{x+\epsilon} = 1 - \frac{\epsilon}{x+\epsilon}.
\end{array}$$

Note that $f(0) = 0$. By Theorem 1.6.9 and Theorem 1.6.6, the element $e = f(a^*a)$ is contained in the C*-algebra generated by a^*a and hence, in A.

Now observe that if $t \in [0,\infty)$ then $f(t) \in (0,1)$. By the continuous functional calculus in Theorem 1.6.9,

$$\|e\| = \sup_{t\in[0,\infty)} |f(t)| \leq 1.$$

By the spectral mapping theorem in Theorem 1.3.14 and Theorem 1.4.5, $e = f(a^*a)$ is a positive element of A.

To see that $\|a - ae\| < \epsilon$, we turn to the unitization $\tilde{A}$ of A. Recalling that $(1,0) \in \tilde{A}$ is the multiplicative unit of $\tilde{A}$, we compute directly that

$$\begin{aligned}
\|a - ae\|^2 &= \|(0, a - ae)\|^2 \\
&= \|(0,a)\big((1,0) - (0,e)\big)\|^2 \\
&= \|(0,a)\big((1,0) - (0,f(a^*a))\big)\|^2 \\
&= \|\big((1,0) - (0,f(a^*a))\big)(0,a^*)(0,a)\big((1,0) - (0,f(a^*a))\big)\| \\
&= \|\big((1,0) - (0,f(a^*a))\big)(0,a^*a)\big((1,0) - (0,f(a^*a))\big)\| \\
&= \|g(a^*a)\| \leq \|g\|_\infty
\end{aligned}$$

where $g(t) = t(1 - f(t))^2$. The function $g(t)$ obtains it maximum at $t = \epsilon$ and $g(\epsilon) = \epsilon/4$. Therefore, $\|a - ae\| < \frac{\sqrt{\epsilon}}{2}$ and we are done. $\square$

The main theorem of this section uses Theorem 1.7.3.

Theorem 1.7.4. *Let A be a C*-algebra and I be a (topologically) closed, two-sided ideal. Then, I is closed under the involution on A and A/I with the quotient norm in Theorem 1.7.2 is a C*-algebra.*

Proof. Assume that A is a C*-algebra and I is a closed, two-sided ideal.

To show: (a) If $a \in I$ then $a^* \in I$.

(a) Assume that $a \in I$ and $\epsilon \in \mathbb{R}_{>0}$. By Theorem 1.7.3, there exists a sequence $\{e_i\}_{i\in\mathbb{Z}_{\geq 0}}$ in A such that

$$\|a - ae_i\| < \frac{\epsilon}{2^i}.$$

Each e_i is in the C*-algebra generated by a^*a. Since $a \in I$, then $a^*a \in I$ and the C*-algebra generated by a^*a is contained in I. Hence, if $i \in \mathbb{Z}_{\geq 0}$ then $e_i \in I$ and $\{ae_i\}_{i\in\mathbb{Z}_{\geq 0}}$ is a sequence in I which converges (in the norm topology on A) to a.

Now

$$\|e_i a^* - a^*\| = \|(a - ae_i)^*\| = \|a - ae_i\| < \frac{\epsilon}{2^i}.$$

Since $e_i \in I$ and I is an ideal, then $\{e_i a^*\}_{i \in \mathbb{Z}_{\geq 0}}$ is a sequence in I which converges (in the norm topology on A) to a^*. Since I is closed topologically, we deduce that $a^* \in I$.

To show: (b) If $a \in A$ then $\|(a + I)(a^* + I)\| = \|a^* a + I\| = \|a + I\|^2$.

(b) Again, assume that $a \in A$. By part (a), the involution operation on A/I given by $(a + I)^* = a^* + I$ is well-defined. Observe that

$$\begin{aligned}\|a^* + I\| &= \inf_{b \in I} \|a^* + b\| \\ &= \inf_{b \in I} \|a^* + b^*\| \\ &= \inf_{b \in I} \|(a + b)^*\| = \inf_{b \in I} \|a + b\| \\ &= \|a + I\|.\end{aligned}$$

This means that

$$\|a^* a + I\| \leq \|a + I\| \|a^* + I\| = \|a + I\|^2.$$

To see that the reverse inequality holds, we claim that

$$\|a + I\| = \inf\{\|a - ae\| \mid e \in I, e \geq 0, \|e\| \leq 1\}.$$

First, note that if $e \in I$ is positive and $\|e\| < 1$ then $\sigma((0, e)) \subseteq [0, 1]$. Here, we used Theorem 1.6.1 to consider the spectrum of an element of A in the unitization $\tilde{A}$. By the continuous functional calculus in Theorem 1.3.7 applied to the unitization $\tilde{A}$ and the spectral mapping Theorem 1.3.14, we find that the spectrum $\sigma((1, 0) - (0, e)) \subseteq [0, 1]$ and $\|(1, 0) - (0, e)\| = \|(1, -e)\| \leq 1$. So,

$$\|a + I\| = \inf_{b \in I} \|a + b\| \leq \inf\{\|a - ae\| \mid e \in I, e \geq 0, \|e\| \leq 1\}.$$

For the reverse inequality, let $b \in I$ and $\epsilon \in \mathbb{R}_{>0}$. By Theorem 1.7.3, there exists $e \in I$ such that e is positive, $\|e\| \leq 1$ and $\|b - be\| < \epsilon$. By the reverse triangle inequality, we have

$$\begin{aligned}
\|a+b\| &= \|(0,a+b)\| \\
&\geq \|(0,a+b)\|\|(1,0)-(0,e)\| \\
&\geq \|(0,a)\big((1,0)-(0,e)\big)\| - \|(0,b)\big((1,0)-(0,e)\big)\| \\
&= \|(0,a)\big((1,0)-(0,e)\big)\| - \|b-be\| \\
&\geq \|a-ae\| - \epsilon.
\end{aligned}$$

Since $\epsilon \in \mathbb{R}_{>0}$ was arbitrary, we deduce that

$$\inf_{b\in I}\|a+b\| \geq \inf\{\|a-ae\| \mid e \in I, e \geq 0, \|e\| \leq 1\}.$$

Combined with the previous inequality, we obtain equality. With part (b), we now compute for $e \in I$ positive with $\|e\| < 1$ that

$$\begin{aligned}
\|a-ae\|^2 &= \|(0,a)\big((1,0)-(0,e)\big)\|^2 \\
&= \|\big((1,0)-(0,e)\big)(0,a^*a)\big((1,0)-(0,e)\big)\| \\
&\leq \|\big((1,0)-(0,e)\big)\|\|(0,a^*a)\big((1,0)-(0,e)\big)\| \\
&\leq \|(0,a^*a)\big((1,0)-(0,e)\big)\| = \|a^*a - a^*ae\|.
\end{aligned}$$

Taking the infimum over all such e, we find that

$$\begin{aligned}
\|a+I\|^2 &= \inf_{b\in I}\|a+b\|^2 \\
&= \inf\{\|a-ae\|^2 \mid e \in I, e \geq 0, \|e\| \leq 1\} \\
&\leq \inf\{\|a^*a - a^*ae\| \mid e \in I, e \geq 0, \|e\| \leq 1\} \\
&= \inf_{b\in I}\|a^*a+b\| = \|a^*a+I\|.
\end{aligned}$$

Since we also have $\|a^*a+I\| \leq \|a+I\|^2$, then $\|a^*a+I\| = \|a+I\|^2$. Thus, A/I is a C*-algebra as required. □

A direct consequence of Theorem 1.7.4 is the following theorem regarding the projection map $A \to A/I$.

Theorem 1.7.5. *Let A be a C*-algebra and I be a closed two-sided ideal with $I \neq A$. Then, the projection map*

$$\begin{array}{rccc}
\pi: & A & \to & A/I \\
 & a & \mapsto & a+I
\end{array}$$

*is a *-homomorphism with norm 1.*

Proof. Assume that A is a C*-algebra and I is a closed two-sided ideal of A. The fact that the projection map $\pi : A \to A/I$ is a *-homomorphism follows from direct computation.

To see that $\|\pi\| = 1$, first observe that if $a \in A$ then

$$\|\pi(a)\| = \|a + I\| = \inf_{j \in I} \|a + j\| \leq \|a + 0\| = \|a\|.$$

So, $\|\pi\| \leq 1$. Alternatively, we could use Theorem 1.2.7.

To see that $\|\pi\| \geq 1$, let $b \in A - I$. Then, $\pi(b) = b + I \neq 0$ in A/I. If $x \in I$ then

$$\|b + I\| = \|(b + x) + I\| = \|\pi(b + x)\| \leq \|\pi\| \|b + x\|.$$

By taking the infimum over all $x \in I$, we deduce that

$$\|b + I\| \leq \|\pi\| \inf_{x \in I} \|b + x\| = \|\pi\| \|b + I\|.$$

So, $\|\pi\| \geq 1$. Hence, $\|\pi\| = 1$. □

It is remarked in [Put19, Page 42] that the projection map being a *-homomorphism is often useful. With Theorem 1.7.4, we will prove some more useful facts about *-homomorphisms. The first of these is [Put19, Exercise 1.9.3].

Theorem 1.7.6. *Let A and B be C*-algebras and $\phi : A \to B$ be a *-homomorphism. Then, the image $\phi(A)$ is closed and is a C*-subalgebra of B.*

Proof. Assume that A and B are C*-algebras and $\phi : A \to B$ is a *-homomorphism.

To show: (a) The image $\phi(A)$ is closed.

(a) Consider the kernel $\ker \phi$, which is a closed two-sided ideal. By the universal property of the quotient, there exists a unique *-homomorphism ϕ' such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\pi"] \arrow[dr, "\phi"'] & A/\ker\phi \arrow[d, dashed, "\tilde{\phi}"] \\ & B \end{tikzcd}$$

Observe that $\tilde{\phi}$ is injective. If $a+\ker\phi \in \ker\tilde{\phi}$ then $\tilde{\phi}(a+\ker\phi) = \phi(a) = 0$. Then, $a \in \ker\phi$ and $a+\ker\phi = \ker\phi$ in $A/\ker\phi$. So, $\tilde{\phi}$ is injective.

By Theorem 1.6.4, $\tilde{\phi}$ must be an isometry. To see that the image $\tilde{\phi}(A/\ker\phi)$ is closed, assume that $\{b_i\}_{i\in\mathbb{Z}_{>0}}$ is a sequence in $\tilde{\phi}(A/\ker\phi)$ which converges to some $b \in B$. If $i \in \mathbb{Z}_{>0}$ then there exists $a_i + \ker\phi \in A/\ker\phi$ such that $\tilde{\phi}(a_i+\ker\phi) = b_i$. Since $\{b_i\}_{i\in\mathbb{Z}_{>0}}$ converges, it must be Cauchy.

Assume that $\epsilon \in \mathbb{R}_{>0}$. Then, there exists $N \in \mathbb{Z}_{>0}$ such that if $m, n > N$ then

$$\|b_m - b_n\| < \epsilon.$$

Using the fact that $\tilde{\phi}$ is an isometry, we find that

$$\|a_m+\ker\phi-(a_n+\ker\phi)\| = \|\tilde{\phi}(a_m+\ker\phi)-\tilde{\phi}(a_n+\ker\phi)\| = \|b_m-b_n\| < \epsilon.$$

Hence, $\{a_i+\ker\phi\}$ is a Cauchy sequence in the C*-algebra $A/\ker\phi$. So, it must converge to some $a+\ker\phi \in A/\ker\phi$.

Now we claim that $\tilde{\phi}(a+\ker\phi) = b$. We argue that for $m \in \mathbb{Z}_{>0}$ large enough,

$$\begin{aligned}\|\tilde{\phi}(a+\ker\phi) - b\| &\leq \|\tilde{\phi}(a+\ker\phi) - \tilde{\phi}(a_m+\ker\phi)\| + \|\tilde{\phi}(a_m+\ker\phi) - b\| \\ &= \|(a+\ker\phi) - (a_m+\ker\phi)\| + \|b_m - b\| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

Since $\epsilon \in \mathbb{R}_{>0}$ was arbitrary, $\|\tilde{\phi}(a+\ker\phi) - b\| = 0$ and $\tilde{\phi}(a+\ker\phi) = b$. Therefore, $b \in \tilde{\phi}(A/\ker\phi)$ and the image $\tilde{\phi}(A/\ker\phi)$ is closed.

Subsequently, we deduce that $\phi(A) = \tilde{\phi}(A/\ker\phi)$ is closed. Furthermore, the image $\phi(A)$ is closed under scalar multiplication, multiplication, addition and involution. So, $\phi(A)$ is a C*-subalgebra of B as required. $\square$

1.8 Trace on a C*-algebra

By definition, a C*-algebra is a Banach space. By the Hahn-Banach extension theorem, a C*-algebra has plenty of linear functionals. It is often useful to consider linear functionals with extra properties. For instance, a $\mathbb{C}$-algebra homomorphism from a C*-algebra A to $\mathbb{C}$ is simply a linear functional on A which is multiplicative. These types of functionals featured prominently in our analysis of commutative unital C*-algebras.

However, as remarked in [Put19, Section 1.10], most C*-algebras do not have $\mathbb{C}$-algebra homomorphisms to $\mathbb{C}$. In order to deal with these C*-algebras, we need to consider maps which are somewhere in between a linear functional and a $\mathbb{C}$-algebra homomorphism. This gives rise to the notion of a *trace*.

Definition 1.8.1. Let A be a unital C*-algebra and $\phi : A \to \mathbb{C}$ be a linear functional. We say that ϕ is **positive** if for $a \in A$, $\phi(a^*a) \geq 0$.

A **trace** on A is a positive linear functional $\tau : A \to \mathbb{C}$ such that $\tau(1_A) = 1$ and if $a, b \in A$ then

$$\tau(ab) = \tau(ba).$$

The last property is referred to as the **trace property**. The trace is called **faithful** if $\tau(a^*a) = 0$ implies that $a = 0$.

Any $\mathbb{C}$-algebra homomorphism satisfies the trace property. If A is a commutative unital C*-algebra then every positive linear functional $\phi : A \to \mathbb{C}$ with $\phi(1_A) = 1$ is a trace.

In order to illustrate the concept of a trace, we will construct a trace on the C*-algebra of bounded linear operators on a finite-dimensional Hilbert space. Recall that if H is a Hilbert space then the C*-algebra of bounded linear operators on H is denoted by $B(H)$.

Theorem 1.8.1. *Let H be a Hilbert space with finite dimension $n \in \mathbb{Z}_{>0}$. Let $\{\xi_1, \ldots, \xi_n\}$ be an orthonormal basis for H. On the C*-algebra $B(H)$, define the map*

$$\begin{array}{rccc} \tau : & B(H) & \to & \mathbb{C} \\ & a & \mapsto & \frac{1}{n}\sum_{i=1}^{n}\langle a\xi_i, \xi_i\rangle. \end{array}$$

Then, τ is a faithful trace on $B(H)$, which is unique.

Before we delve into the proof, we note that by identifying $B(H)$ with the C*-algebra of matrices $M_{n\times n}(\mathbb{C})$, the map τ is expressed for $A = (a_{ij}) \in M_{n\times n}(\mathbb{C})$ as

$$\tau(A) = \frac{1}{n}\sum_{i=1}^{n} a_{ii} = \frac{1}{n}Tr(A).$$

Proof. Assume that H is a Hilbert space with finite dimension $n \in \mathbb{Z}_{>0}$. Assume that $\{\xi_1, \ldots, \xi_n\}$ is an orthonormal basis for H. Assume that τ is defined as above. By linearity of the inner product, we deduce that τ is a linear functional on $B(H)$.

To show: (a) τ is positive and if $id_H \in B(H)$ is the identity operator then $\tau(id_H) = 1$.

(b) If $a, b \in B(H)$ then $\tau(ab) = \tau(ba)$.

(c) τ is unique.

(d) τ is a faithful trace.

(a) Assume that $a \in B(H)$. We compute directly that

$$\begin{aligned}\tau(a^*a) &= \frac{1}{n}\sum_{i=1}^{n}\langle a^*a\xi_i, \xi_i\rangle \\ &= \frac{1}{n}\sum_{i=1}^{n}\langle a\xi_i, a\xi_i\rangle \\ &= \frac{1}{n}\sum_{i=1}^{n}\|a\xi_i\|^2 \geq 0.\end{aligned}$$

So, τ is a positive linear functional. Assume that $id_H \in B(H)$ is the identity operator. By the above computation,

$$\tau(id_H) = \tau(id_H^* id_H) = \frac{1}{n}\sum_{i=1}^{n}\|\xi_i\|^2 = 1.$$

(b) Since the linear span of rank one operators on H is $B(H)$, it suffices to prove the trace property for rank one operators a and b. We first observe that if $\xi, \eta \in H$ then

$$\begin{aligned}
\tau(|\xi\rangle\langle\eta|) &= \frac{1}{n}\sum_{i=1}^{n}\langle|\xi\rangle\langle\eta|)\xi_i, \xi_i\rangle \\
&= \frac{1}{n}\sum_{i=1}^{n}\langle\langle\xi_i, \eta\rangle\xi, \xi_i\rangle \\
&= \frac{1}{n}\sum_{i=1}^{n}\langle\xi_i, \eta\rangle\langle\xi, \xi_i\rangle \\
&= \frac{1}{n}\langle\sum_{i=1}^{n}\langle\xi, \xi_i\rangle\xi_i, \eta\rangle \\
&= \frac{1}{n}\langle\xi, \eta\rangle.
\end{aligned}$$

Now let $a = |\sigma_1\rangle\langle\eta_1|$ and $b = |\sigma_2\rangle\langle\eta_2|$, where $\sigma_1, \sigma_2, \eta_1, \eta_2 \in H$. Then,

$$ab = \langle\sigma_2, \eta_1\rangle|\sigma_1\rangle\langle\eta_2| \qquad \text{and} \qquad ba = \langle\sigma_1, \eta_2\rangle|\sigma_2\rangle\langle\eta_1|.$$

We compute directly that

$$\tau(ab) = \frac{1}{n}\langle\sigma_2, \eta_1\rangle\langle\sigma_1, \eta_2\rangle = \tau(ba).$$

Since the linear span of rank one operators on H is all of $B(H)$, τ satisfies the trace property and is therefore, a trace.

(c) Assume that φ is another trace on $B(H)$. Let $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$. We will first show that φ and τ agree on the rank one operator $|\xi_i\rangle\langle\xi_j|$.

Let $a = |\xi_i\rangle\langle\xi_i|$ and $b = |\xi_i\rangle\langle\xi_j|$. Then, $ab = |\xi_i\rangle\langle\xi_j| = b$ and $ba = 0$. We compute directly that

$$\phi(b) = \phi(ab) = \phi(ba) = 0 = \frac{1}{n}\langle\xi_i, \xi_j\rangle = \tau(b).$$

Now assume that $i, j \in \{1, 2, \ldots, n\}$, which may or may not be distinct. Observe that $b^*b = (|\xi_j\rangle\langle\xi_i|)(|\xi_i\rangle\langle\xi_j|) = |\xi_j\rangle\langle\xi_j|$ and $bb^* = |\xi_i\rangle\langle\xi_i|$. Since ϕ is a trace, we must have

$$\phi(|\xi_j\rangle\langle\xi_j|) = \phi(b^*b) = \phi(bb^*) = \phi(|\xi_i\rangle\langle\xi_i|).$$

Now observe that

$$1 = \phi(id_H) = \phi(\sum_{i=1}^{n} |\xi_i\rangle\langle\xi_i|) = n\phi(|\xi_1\rangle\langle\xi_1|).$$

So,

$$\phi(|\xi_i\rangle\langle\xi_i|) = \phi(|\xi_1\rangle\langle\xi_1|) = \frac{1}{n} = \tau(|\xi_i\rangle\langle\xi_i|).$$

Hence, we have shown that τ and ϕ agree on the set $\{|\xi_i\rangle\langle\xi_j| \mid i, j \in \{1, 2, \ldots, n\}\}$, which is a spanning set for $B(H)$. Hence, τ and ϕ agree on all of $B(H)$, rendering the trace τ unique.

(d) Assume that $a \in B(H)$ and $\tau(a^*a) = 0$. Then,

$$\frac{1}{n} \sum_{i=1}^{n} \|a\xi_i\|^2 = 0$$

and subsequently, if $i \in \{1, 2, \ldots, n\}$ then $\|a\xi_i\| = 0$. So, $a = 0$ and τ is a faithful trace on $B(H)$. □

Here are a few consequences of Theorem 1.8.1.

Theorem 1.8.2. *Let H be a finite-dimensional Hilbert space with dimension $\dim H \in \mathbb{Z}_{>0}$. Let τ be the unique trace on $B(H)$ constructed in Theorem 1.8.1. If p is a projection then $\dim pH = \tau(p) \dim H$.*

Proof. Assume that H is a finite dimensional Hilbert space. Assume that τ is the trace on $B(H)$ constructed in Theorem 1.8.1. Assume that p is a projection. We choose an orthonormal basis $\{\xi_1, \ldots, \xi_n\}$ for H in such a way that $\{\xi_1, \ldots, \xi_k\}$ is a basis for pH.

We compute directly that

$$\begin{aligned}
\tau(p) \dim H &= n \cdot \frac{1}{n} \sum_{i=1}^{n} \langle p\xi_i, \xi_i\rangle \\
&= \sum_{i=1}^{n} \langle p\xi_i, \xi_i\rangle = \sum_{i=1}^{k} \langle \xi_i, \xi_i\rangle \\
&= k = \dim pH.
\end{aligned}$$

□

Theorem 1.8.2 states that if the trace is applied to projections then it recovers the geometric notion of the dimension of the range. Theorem 1.8.1 also tells us that if $n \in \mathbb{Z}_{>1}$ then there is no non-zero *-homomorphism from $M_{n\times n}(\mathbb{C})$ to $\mathbb{C}$.

Theorem 1.8.3. *Let $n \in \mathbb{Z}_{>1}$. Then, there does not exist a non-zero *-homomorphism from $M_{n\times n}(\mathbb{C})$ to $\mathbb{C}$.*

Proof. Assume that $n \in \mathbb{Z}_{>1}$. We identify $M_{n\times n}(\mathbb{C})$ with $B(H)$, where H is a Hilbert space with finite dimension n. Let $\{\xi_1, \ldots, \xi_n\}$ be an orthonormal basis for H. Suppose for the sake of contradiction that $\alpha : B(H) \to \mathbb{C}$ is a non-zero *-homomorphism. Then, α is a trace.

By uniqueness in Theorem 1.8.1, $\alpha = \tau$ where τ is the trace constructed in Theorem 1.8.1. However, if $p \in B(H)$ is the projection operator onto the span of $\{\xi_1, \ldots, \xi_k\}$ (with $k < n$) then $\tau(p) = \tau(p^2) = \tau(p)^2$. But, by the computation in Theorem 1.8.2

$$\tau(p) = \frac{k}{n} \neq \frac{k^2}{n^2} = \tau(p)^2.$$

This yields the desired contradiction. So, there is no non-zero *-homomorphism from $M_{n\times n}(\mathbb{C})$ to $\mathbb{C}$. □

1.9 Representations of C*-algebras

As mentioned at the beginning of this document, the prototypical example of a C*-algebra is the space of bounded linear operators $B(H)$ on some Hilbert space H. This particular C*-algebra was studied extensively in [Sol18]. Keeping this example in mind, we would like to find the ways an arbitrary C*-algebra can act as bounded linear operators on H. This gives rise to representations of C*-algebras.

Definition 1.9.1. Let A be a *-algebra (a $\mathbb{C}$-algebra with an involution map). A **representation** of A is a pair (π, H), where H is a Hilbert space and $\pi : A \to B(H)$ is a *-homomorphism.

We also say that π is a representation of A on H. In this section, we will focus on the basic definitions and properties regarding representations of *-algebras. We will begin with the necessary definitions.

Definition 1.9.2. Let A be a *-algebra. We say that two representations of A, (π_1, H_1) and (π_2, H_2) are **unitarily equivalent** if there exists a unitary operator $u : H_1 \to H_2$ such that if $a \in A$ then

$$u\pi_1(a) = \pi_2(a)u.$$

If the representations (π_1, H_1) and (π_2, H_2) are unitarily equivalent then we write $(\pi_1, H_1) \sim_u (\pi_2, H_2)$ or $\pi_1 \sim_u \pi_2$.

As remarked in [Put19, Page 46], unitarily equivalent representations are considered to be the same. Before we press on with the definitions, we will formalise this remark and prove that unitary equivalence is an equivalence relation.

Theorem 1.9.1. *Let A be a *-algebra. The relation of unitary equivalence between two representations of A, denoted by $\sim_u$, is an equivalence relation.*

Proof. Assume that A is a *-algebra. For reflexivity, if (π_1, H_1) is a representation of A then $(\pi_1, H_1) \sim_u (\pi_1, H_1)$ because if $a \in A$ and $id_{H_1} \in B(H_1)$ is the identity operator then

$$id_{H_1}\pi_1(a) = \pi_1(a)id_{H_1}.$$

For symmetry, assume that $(\pi_1, H_1) \sim_u (\pi_2, H_2)$. Then, there exists a unitary operator $u : H_1 \to H_2$ such that if $a \in A$ then $u\pi_1(a) = \pi_2(a)u$. Its inverse $u^{-1} : H_2 \to H_1$ is also unitary,

$$\pi_1(a) = u^{-1}\pi_2(a)u \qquad \text{and} \qquad \pi_1(a)u^{-1} = u^{-1}\pi_2(a).$$

So, $(\pi_2, H_2) \sim_u (\pi_1, H_1)$.

For transitivity, assume that $(\pi_1, H_1) \sim_u (\pi_2, H_2)$ and $(\pi_2, H_2) \sim_u (\pi_3, H_3)$. Then, there exists unitary operators $s : H_1 \to H_2$ and $t : H_2 \to H_3$ such that if $a \in A$ then $s\pi_1(a) = \pi_2(a)s$ and $t\pi_2(a) = \pi_3(a)t$. The composite $ts : H_1 \to H_3$ is a unitary operator which satisfies

$$ts\pi_1(a) = t\pi_2(a)s = \pi_3(a)ts.$$

Hence, $(\pi_1, H_1) \sim_u (\pi_3, H_3)$ and $\sim_u$ is therefore an equivalence relation. □

A major operation one can perform on representations of a fixed *-algebra is to take the direct sum of representations.

Definition 1.9.3. Let A be a *-algebra and $\{(\pi_i, H_i)\}_{i\in I}$ be a family of representations of A. The **direct sum** of the representations $\{(\pi_i, H_i)\}_{i\in I}$ is given by the pair

$$(\bigoplus_{i\in I} \pi_i, \bigoplus_{i\in I} H_i)$$

where

$$\bigoplus_{i\in I} H_i = \{(\xi_i)_{i\in I} \mid \sum_{i\in I} \|\xi_i\|^2 < \infty\}$$

and $\bigoplus_{i\in I} \pi_i$ is the *-homomorphism

$$\begin{array}{rccc} \bigoplus_{i\in I} \pi_i : & A & \to & B(\bigoplus_{i\in I} H_i) \\ & a & \mapsto & \big((\xi_i)_{i\in I} \mapsto (\pi_i(a)\xi_i)_{i\in I}\big) \end{array}$$

For a refresher on the direct sum of Hilbert spaces, a good reference is [Con90, Chapter I, §6]. The next few definitions mirror foundational definitions in representation theory.

Definition 1.9.4. Let A be a *-algebra and (π, H) be a representation of A. A subspace $N \subseteq H$ is called **invariant** if for $a \in A$, $\pi(a)N \subset N$.

Definition 1.9.5. Let A be a *-algebra and (π, H) be a representation of A. We say that the representation (π, H) is **non-degenerate** if the following statement is satisfied: If $\xi \in H$, $a \in A$ and $\pi(a)\xi = 0$ then $\xi = 0$.

Otherwise, the representation is called **degenerate**.

Definition 1.9.6. Let A be a *-algebra and (π, H) be a representation of A. Let $\xi \in H$. We say that ξ is **cyclic** if the linear subspace $\pi(A)\xi$ is dense in H. We say that the representation (π, H) is **cyclic** if there exists a cyclic vector $\xi \in H$.

Observe that if A is *-algebra and (π, H) is a cyclic representation of A then (π, H) must be non-degenerate. Assume that $\xi \in H$ is the cyclic vector, $\eta \in H$, $a \in A$ and $\pi(a)\eta = 0$. Using the inner product on H, we have

$$0 = \langle \pi(a)\eta, \xi\rangle = \langle \eta, \pi(a)^*\xi\rangle = \langle \eta, \pi(a^*)\xi\rangle.$$

So, $\eta \in (\pi(A)\xi)^\perp$ and since $\pi(A)\xi$ is dense in H, $(\pi(A)\xi)^\perp = \{0\}$. Therefore, $\eta = 0$ and the representation (π, H) is cyclic as required.

Definition 1.9.7. Let A be a *-algebra and (π, H) be a representation of A. The representation (π, H) is called **irreducible** if the only closed invariant subspaces of H are the zero subspace 0 and H. Otherwise, the representation is called **reducible**.

When dealing with representations of *-algebras, we are generally interested in invariant subspaces which are closed topologically. The following theorem tells us when a closed subspace is invariant.

Theorem 1.9.2. *Let A be a *-algebra and (π, H) be a representation of A. Let $N \subseteq H$ be a closed subspace. Then, N is invariant if and only if the orthogonal complement $N^\perp$ is invariant.*

Proof. Assume that A is a *-algebra and (π, H) be a representation of A. Assume that $N \subseteq H$ is a closed subspace.

To show: (a) If N is invariant then $N^\perp$ is invariant.

(b) If $N^\perp$ is invariant then N is invariant.

(a) Assume that the closed subspace N is invariant. This means that if $a \in A$ then $\pi(a)N \subset N$. To see that $N^\perp$ is invariant, assume that $a \in A$ and $\eta \in N^\perp$. If $\xi \in H$ then

$$\langle \pi(a)\eta, \xi\rangle = \langle \eta, \pi(a)^*\xi\rangle = \langle \eta, \pi(a^*)\xi\rangle = 0$$

because $\pi(a^*)\xi \in N$ and $\eta \in N^\perp$. Hence, $\pi(a)\eta \in N^\perp$ and the subspace $N^\perp$ is invariant.

(b) Assume that $N^\perp$ is invariant. If we apply part (a), we find that $(N^\perp)^\perp = N$ is invariant because N is closed. □

In the scenario of Theorem 1.9.2, we can define two further representations of A, by restricting the operators to either N or $N^\perp$. More precisely in the case of N, the Hilbert space N together with the *-homomorphism

$$\begin{array}{rccc} \pi|_N : & A & \to & B(N) \\ & a & \mapsto & \pi(a)|_N \end{array}$$

is a representation of A. Unsurprisingly, the following two representations of A are unitarily equivalent

$$(\pi, H) \sim_u (\pi|_N, N) \oplus (\pi|_{N^\perp}, N^\perp).$$

The following result is a direct consequence of the definitions introduced.

Theorem 1.9.3. *Let A be a unital *-algebra with multiplicative unit 1_A and (π, H) be a representation of A. The representation (π, H) is non-degenerate if and only if $\pi(1_A) = id_H$ where id_H is the identity operator on H.*

Proof. Assume that A is a unital *-algebra and (π, H) is a representation of A.

To show: (a) If (π, H) is non-degenerate then $\pi(1_A) = id_H$.

(b) If $\pi(1_A) = id_H$ then (π, H) is non-degenerate.

(a) Assume that the representation (π, H) is non-degenerate. Assume that $\xi \in H - \{0\}$ is non-zero. Since the representation (π, H) is non-degenerate, there exists $a \in A$ such that $\pi(a)\xi \neq 0$. But,

$$0 = \pi(a)\xi - \pi(a)\xi = \pi(1_A)\pi(a)\xi - id_H(\pi(a)\xi).$$

Since $\pi(a)\xi \neq 0$, we deduce that $\pi(1_A) = id_H$ as required.

Since the representation (π, H) is non-degenerate, then $\pi(a) = 0$.

(b) Assume that $\pi(1_A) = id_H$. Assume that $\xi \in H$ satisfies $\pi(a)\xi = 0$ for $a \in A$. Then,

$$\|\xi\|^2 = \langle \xi, \xi \rangle = \langle \pi(1_A)\xi, \pi(1_A)\xi \rangle = 0.$$

So, $\xi = 0$ and the representation (π, H) is non-degenerate. □

Here is a useful characterisation of non-degenerate representations.

Theorem 1.9.4. *Let A be a *-algebra and (π, H) be a representation of A. Then, (π, H) is non-degenerate if and only if $\overline{\pi(A)H} = H$.*

Proof. Assume that A is a *-algebra and (π, H) is a representation of A.

To show: (a) If (π, H) is non-degenerate then $\overline{\pi(A)H} = H$.

(b) If $\overline{\pi(A)H} = H$ then (π, H) is non-degenerate.

(a) Assume that (π, H) is non-degenerate. Since $H = (\pi(A)H)^\perp \oplus \overline{\pi(A)H}$, it suffices to show that $(\pi(A)H)^\perp = \{0\}$. To this end, assume that $\eta \in (\pi(A)H)^\perp$. If $a \in A$ and $\xi \in H$ then

$$\langle \pi(a^*)\xi, \eta \rangle = \langle \xi, \pi(a)\eta \rangle = 0.$$

So, $\pi(a)\eta = 0$ and since (π, H) is non-degenerate, $\eta = 0$. Hence, $(\pi(A)H)^\perp = \{0\}$ and $\overline{\pi(A)H} = H$.

(b) Assume that $\overline{\pi(A)H} = H$. Assume that $\xi \in H$ satisfies $\pi(a)\xi = 0$ for $a \in A$. Since $\overline{\pi(A)H} = H$ and $H = (\pi(A)H)^\perp \oplus \overline{\pi(A)H}$, $(\pi(A)H)^\perp = \{0\}$. If $\pi(b)\eta \in \pi(A)H$ then

$$\langle \pi(b)\eta, \xi\rangle = \langle \eta, \pi(b^*)\xi\rangle = 0.$$

So, $\xi \in (\pi(A)H)^\perp = \{0\}$ and $\xi = 0$. Hence, (π, H) is a non-degenerate representation of A. $\square$

When dealing with representations of *-algebras, we can focus on non-degenerate representations, as evidenced by the following theorem:

Theorem 1.9.5. *Let A be a *-algebra and (π, H) be a representation of A. Then, (π, H) is the direct sum of a non-degenerate representation and the zero representation.*

Proof. Assume that A is a *-algebra and (π, H) is a representation of A. Consider the closure $\overline{\pi(A)H}$, where $\pi(A)H$ is the subspace

$$\pi(A)H = \{\pi(a)\xi \mid a \in A, \xi \in H\}.$$

Then, H can be written as the direct sum

$$H = (\pi(A)H)^\perp \oplus \overline{\pi(A)H}.$$

We claim that $(\pi|_{(\pi(A)H)^\perp}, (\pi(A)H)^\perp)$ is the zero representation. If $a \in A$, $\xi \in (\pi(A)H)^\perp$ and $\eta \in H$ then

$$\langle \pi(a)\xi, \eta\rangle = \langle \xi, \pi(a^*)\eta\rangle = 0.$$

We deduce that if $a \in A$ and $\xi \in (\pi(A)H)^\perp$ then

$$\pi(a)\xi = \pi(a)|_{(\pi(A)H)^\perp}\xi = \pi|_{(\pi(A)H)^\perp}(a)\xi = 0.$$

Thus, the representation $(\pi|_{(\pi(A)H)^\perp}, (\pi(A)H)^\perp)$ is the zero representation.

To see that the representation $(\pi|_{\overline{\pi(A)H}}, \overline{\pi(A)H})$ is non-degenerate, it suffices to show that

$$\overline{\pi|_{\overline{\pi(A)H}}(A)\overline{\pi(A)H}} = \overline{\pi(A)H}$$

by Theorem 1.9.4. Hence, it suffices to show that

$$\pi(A)\overline{\pi(A)H} = \pi(A)H.$$

We already have the inclusion $\pi(A)\overline{\pi(A)H} \subseteq \pi(A)H$. For the reverse inclusion, assume that $\pi(a)\xi \in \pi(A)H$. Write $\xi = \xi_1 + \xi_2$, where $\xi_1 \in \overline{\pi(A)H}$ and $\xi_2 \in (\pi(A)H)^\perp$. Then, $\pi(a)\xi = \pi(a)\xi_1 \in \pi(A)\overline{\pi(A)H}$ because $\pi(a)\xi_2 = 0$ as found previously. Therefore, $\pi(A)H \subseteq \pi(A)\overline{\pi(A)H}$

and $\pi(A)H = \pi(A)\overline{\pi(A)H}$. Consequently by Theorem 1.9.4, the representation $(\pi|_{\overline{\pi(A)H}}, \overline{\pi(A)H})$ is non-degenerate.

Finally, we observe that as representations of A,

$$(\pi, H) \sim_u (\pi|_{(\pi(A)H)^\perp}, (\pi(A)H)^\perp) \oplus (\pi|_{\overline{\pi(A)H}}, \overline{\pi(A)H}).$$

□

Next, we link irreducible representations to cyclic representations.

Theorem 1.9.6. *Let A be a *-algebra and (π, H) be a non-degenerate representation of A. Then, (π, H) is irreducible if and only if every non-zero vector is cyclic.*

Proof. Assume that A is a *-algebra and (π, H) is a non-degenerate representation of A.

To show: (a) If (π, H) is irreducible then every non-zero vector is cyclic.

(b) If every non-zero vector of H is cycilc then (π, H) is irreducible.

(a) Assume that (π, H) is an irreducible representation. Assume that $\xi \in H - \{0\}$ and let

$$\pi(A)\xi = \{\pi(a)\xi \mid a \in A\}.$$

Then, $\pi(A)\xi$ is an invariant subspace of H and its closure $\overline{\pi(A)\xi}$ is a closed invariant subspace of H. Since (π, H) is irreducible, either $\overline{\pi(A)\xi} = 0$ or $\overline{\pi(A)\xi} = H$. Suppose for the sake of contradiction that $\overline{\pi(A)\xi} = 0$. Then, $\pi(A)\xi = 0$ and the representation (π, H) is degenerate. This contradicts our original assumption that (π, H) is non-degenerate. So, $\overline{\pi(A)\xi} = H$ and ξ is cyclic.

(b) We will prove the contrapositive statement. Assume that the representation (π, H) is reducible. Then, there exists a closed invariant subspace $N \subsetneq H$. Select a non-zero $\eta \in N$. If $a \in A$ then $\pi(a)\eta \in N$ since N is invariant. Since N is closed and $N \neq H$, the subspace $\pi(A)\eta$ cannot be dense in H. So, $\eta \in H - \{0\}$ is not a cyclic vector. □

We have another, more useful criterion, for a representation to be irreducible.

Theorem 1.9.7. *Let A be a *-algebra and (π, H) be a non-degenerate representation. Then, (π, H) is irreducible if and only if the only positive operators which commute with its image are scalars.*

Proof. Assume that A is a *-algebra and (π, H) is a non-degenerate representation.

To show: (a) If the only positive operators which commute with operators in the image $\pi(A)$ are scalars then (π, H) is irreducible.

(b) If (π, H) is an irreducible representation then the only positive operators which commute with operators in the image $\pi(A)$ are scalars.

(a) We will prove the contrapositive statement. Assume that the representation (π, H) is reducible. Then, there exists a non-trivial closed invariant subspace $N \subseteq H$. Now let $p \in B(H)$ be the projection operator onto N. Since p is a projection, it satisfies $p = p^* = p^2$ and is positive. Moreover, p is not a scalar operator because both N and $N^\perp$ are non-zero subspaces of H.

Now assume that $a \in A$. We will show that the operator $\pi(a)$ commutes with p. There are two different cases to consider:

Case 1: $\xi \in N$.

Assume that $\xi \in N$. Since N is invariant, then $\pi(a)\xi \in N$ and

$$(p\pi(a))(\xi) = p(\pi(a)\xi) = \pi(a)\xi = (\pi(a)p)\xi.$$

Case 2: $\xi \in N^\perp$.

Assume that $\xi \in N^\perp$. By Theorem 1.9.2, $N^\perp$ is also an invariant subspace of H and $\pi(a)\xi \in N^\perp$. So,

$$(p\pi(a))(\xi) = p(\pi(a)\xi) = 0 = \pi(a)(0) = \pi(a)(p\xi) = (\pi(a)p)(\xi).$$

By combining both cases, we deduce that as operators on H, p is a non-scalar, positive operator which commutes with $\pi(a)$ for $a \in A$.

(b) Again, we proceed by proving the contrapositive statement. Assume that h is a positive non-scalar operator on H which commutes with every element of $\pi(A)$. We claim that the spectrum $\sigma(h)$ contains at least two points.

Suppose for the sake of contradiction that the spectrum $\sigma(h)$ contains a single point. By the continuous functional calculus in Theorem 1.3.7, h must be a scalar operator. This contradicts our assumption that h is a non-scalar operator. So, $\sigma(h)$ must contain at least two points.

By Urysohn's lemma applied to $\sigma(h)$ (which is a normal topological space with the subspace topology inherited from $\mathbb{C}$), we can construct two non-zero functions $f, g \in Cts(\sigma(h), \mathbb{C})$ such that if $x \in \sigma(h)$ then $f(x)g(x) = 0$. By using the continuous functional calculus in Theorem 1.3.7, the operator $f(h) \in B(H)$ is non-zero because the function f is non-zero on $\sigma(h)$.

Now let $N = \overline{\text{im } f(h)}$. Then, N is a non-zero subspace of H. However by the same reasoning as before, $g(h) \in B(H)$ is a non-zero operator which is zero on im $f(h)$ and hence, on N. This means that N is a proper subspace of H.

Next, we claim that $f(h)$ commutes with the operators in $\pi(A)$.

To show: (ba) If $a \in A$ then $\pi(a)f(h) = f(h)\pi(a)$.

(ba) Assume that $a \in A$ and $\epsilon \in \mathbb{R}_{>0}$. By the Weierstrass theorem, there exists a polynomial function $p \in Cts(\sigma(h), \mathbb{C})$ such that $\|p - f\|_\infty < \epsilon$. By Theorem 1.3.7, $\|p(h) - f(h)\| < \epsilon$.

Since $\pi(a)$ commutes with h by assumption, $\pi(a)$ must also commute with $p(h)$. By the standard ϵ argument, we have

$$\begin{aligned}
\|\pi(a)f(h) - f(h)\pi(a)\| &= \|\pi(a)f(h) - \pi(a)p(h) + p(h)\pi(a) - f(h)\pi(a)\| \\
&\leq \|\pi(a)f(h) - \pi(a)p(h)\| + \|f(h)\pi(a) - p(h)\pi(a)\| \\
&\leq 2\|\pi(a)\|\|f(h) - p(h)\| \\
&< 2\|\pi(a)\|\epsilon.
\end{aligned}$$

Since $\epsilon \in \mathbb{R}_{>0}$ was arbitrary, we deduce that $f(h)$ commutes with the operators in $\pi(A)$.

Our final claim is that if $a \in A$ then N is invariant under $\pi(a)$. Since $N = \overline{\text{im } f(h)}$, then it suffices to show that the image im $f(h)$ is invariant.

If $\xi \in H$ and $a \in A$ then by part (ba),

$$\pi(a)(f(h)\xi) = (\pi(a)f(h))\xi = (f(h)\pi(a))\xi \in \mathrm{im}\ h.$$

We conclude that N is a non-trivial closed, invariant subspace of H. So, (π, H) is a reducible representation of A as required. $\square$

1.10 Representations of matrix C*-algebras

Complementary to the theory developed in the previous section, we will this time investigate representations of the matrix C*-algebra $M_{n\times n}(\mathbb{C})$ where $n \in \mathbb{Z}_{>0}$.

We first need to prove various properties about $M_{n\times n}(\mathbb{C})$. The next few results originate from [Put19, Exercise 1.9.2].

Theorem 1.10.1. *Let $n \in \mathbb{Z}_{>1}$ and $A = M_{n\times n}(\mathbb{C})$. Let $I \subseteq A$ be a right ideal. Then, the set*

$$I\mathbb{C}^n = \{a\xi \mid a \in I, \xi \in \mathbb{C}^n\}$$

is a subspace of $\mathbb{C}^n$.

Proof. Assume that $n \in \mathbb{Z}_{>1}$ and $A = M_{n\times n}(\mathbb{C})$. Assume that $I \subseteq A$ is a right ideal. Assume that $I\mathbb{C}^n$ is defined as above. To see that $I\mathbb{C}^n$ is closed under scalar multiplication, assume that $a \in I$, $\xi \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$. Then,

$$\lambda a\xi = a(\lambda\xi) \in I\mathbb{C}^n.$$

Checking that $I\mathbb{C}^n$ is closed under addition is more tricky. Assume that $a, b \in I$ and $\xi, \eta \in \mathbb{C}^n$. If at least one of ξ or η is the zero vector in $\mathbb{C}^n$ then $a\xi + b\eta \in I\mathbb{C}^n$ by inspection. So, assume that both $\xi, \eta \neq 0$. Let $\widehat{\xi} = \frac{\xi}{\|\xi\|}$ and $\widehat{\eta} = \frac{\eta}{\|\eta\|}$. Then

$$a\xi + b\eta = a\|\xi\|\widehat{\xi} + b\|\eta\|\widehat{\eta}.$$

The key here is to realise that

$$|\widehat{\xi}\rangle\langle\widehat{\eta}|\widehat{\eta} = \langle\widehat{\eta}, \widehat{\eta}\rangle\widehat{\xi} = \widehat{\xi}.$$

Therefore, $a\xi + b\eta$ further simplifies to

$$a\|\xi\|\widehat{\xi} + b\|\eta\|\widehat{\eta} = a\|\xi\||\widehat{\xi}\rangle\langle\widehat{\eta}|\widehat{\eta} + b\|\eta\|\widehat{\eta} = \big(a\|\xi\||\widehat{\xi}\rangle\langle\widehat{\eta}| + b\|\eta\|\big)\widehat{\eta}.$$

For $\lambda \in \mathbb{C}$, define $D_\lambda = diag[\lambda, \ldots, \lambda] \in M_{n\times n}(\mathbb{C})$. Then,

$$a\xi + b\eta = \big(aD_{\|\xi\|}|\widehat{\xi}\rangle\langle\widehat{\eta}| + bD_{\|\eta\|}\big)\widehat{\eta}.$$

Since $a, b \in I$, $aD_{\|\xi\|}|\widehat{\xi}\rangle\langle\widehat{\eta}| + bD_{\|\eta\|} \in I$ (because I is a right ideal) and $a\xi + b\eta \in I\mathbb{C}^n$ as a result.

Hence, $I\mathbb{C}^n$ is a vector subspace of $\mathbb{C}^n$. □

There is a correspondence between right ideals of $M_{n\times n}(\mathbb{C})$ and vector subspaces of $\mathbb{C}^n$. This is illustrated by the following theorem.

Theorem 1.10.2. *Let $n \in \mathbb{Z}_{>1}$ and $A = M_{n\times n}(\mathbb{C})$. Then, there is a bijection of sets*

$$\begin{array}{ccc} \Big\{\textit{Right ideals of } A\Big\} & \longleftrightarrow & \Big\{\textit{Vector subspaces of } \mathbb{C}^n\Big\} \\ I & \mapsto & I\mathbb{C}^n \end{array}$$

where

$$I\mathbb{C}^n = \{a\xi \mid a \in I, \xi \in \mathbb{C}^n\}.$$

Proof. Assume that $n \in \mathbb{Z}_{>1}$ and $A = M_{n\times n}(\mathbb{C})$. Let Θ denote the set function $I \mapsto I\mathbb{C}^n$, where $I\mathbb{C}^n$ is defined as above. By Theorem 1.10.1, $I\mathbb{C}^n$ is a vector subspace of $\mathbb{C}^n$.

Firstly, to see that Θ is injective, assume that $I \in \ker\Theta$ so that $I\mathbb{C}^n = \{0\}$. Then, $I = 0$ as a right ideal in A. Therefore, Θ is injective.

To see that Θ is surjective, assume that V is a vector subspace of $\mathbb{C}^n$. Let $\{v_1, v_2, \ldots, v_k\}$ be a basis for V, where $k \in \{1, 2, \ldots, n\}$. For $i \in \{1, 2, \ldots, k\}$, write $v_i = (v_{i,1}, v_{i,2}, \ldots, v_{i,k})$. Let $V_i \in M_{n\times n}(\mathbb{C})$ be the matrix whose i^{th} column is v_i^T. The remaining entries of V_i are zeros. Let $e_i \in \mathbb{C}^n$ be the vector with a 1 in the i^{th} position and zeros elsewhere. If $i \in \{1, 2, \ldots, k\}$ then $V_i e_i = v_i$.

If we let $(V_1, V_2, \ldots, V_k)$ be the right ideal generated by the matrices $V_1, V_2, \ldots, V_k$ then $\Theta((V_1, V_2, \ldots, V_k)) = V$. So, Θ is surjective and consequently, bijective. □

The next result requires a definition.

Definition 1.10.1. Let A be a C*-algebra. We say that A is **simple** if the only two closed two-sided ideals of A are the zero ideal 0 and A itself.

We will now prove that the matrix C*-algebra $M_{n\times n}(\mathbb{C})$ is a simple C*-algebra.

Theorem 1.10.3. *Let $n \in \mathbb{Z}_{>1}$ and $A = M_{n\times n}(\mathbb{C})$. Then, A is simple.*

Proof. Assume that $n \in \mathbb{Z}_{>1}$ and $A = M_{n\times n}(\mathbb{C})$. Suppose for the sake of contradiction that I is a non-trivial ideal of A. By using the bijection Θ in Theorem 1.10.2, we obtain a vector subspace $I\mathbb{C}^n$ of $\mathbb{C}^n$.

Assume that $a \in I$ and $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n$ so that $a\xi = (\xi'_1, \xi'_2, \dots, \xi'_n) \in I\mathbb{C}^n - \{0\}$. Let ξ'_k be the first non-zero element of the vector $a\xi$. For $i \in \{1, 2, \dots, n\}$, let $W_i \in M_{n\times n}(\mathbb{C})$ be the matrix whose ik entry is $\frac{1}{\xi'_k}$ and whose remaining entries are zeros. If $i \in \{1, 2, \dots, n\}$ then

$$W_i a\xi = e_i$$

where $e_i \in \mathbb{C}^n$ is the vector with a 1 in the i^{th} position and zeros elsewhere. Crucially, I is both a left and right sided ideal. So, $W_i a \in I$ and subsequently, $e_i = W_i a\xi \in I\mathbb{C}^n$ for $i \in \{1, 2, \dots, n\}$. So, $I\mathbb{C}^n = \mathbb{C}^n$, which means that $\Theta(I) = \Theta(A)$. Here, Θ is the bijection in Theorem 1.10.2.

Since Θ is injective, $I = A$. However, this contradicts the assumption that I is a non-trivial ideal of A. Therefore, A does not have any non-trivial two-sided ideals. So, A is a simple C*-algebra. □

Let us assume that $n \in \mathbb{Z}_{\geq 1}$ and $A = M_{n\times n}(\mathbb{C})$. We have the representation $(\rho, \mathbb{C}^n)$ of A, where ρ is the *-homomorphism given by matrix multiplication:

$$\begin{array}{rccc} \rho : & A & \to & B(\mathbb{C}^n) = M_{n\times n}(\mathbb{C}) \\ & a & \mapsto & (\xi \mapsto a\xi). \end{array}$$

We will return to this representation later. Let (π, H) be a non-degenerate representation of A. If $i, j \in \{1, 2, \dots, n\}$ then let $e_{i,j} \in A$ be the matrix with a 1 in the ij position and zeros elsewhere.

We highlight the following result about simple C*-algebras.

Theorem 1.10.4. *Let A be a simple unital C*-algebra and B be a unital C*-algebra. Let $\phi : A \to B$ be a unital *-homomorphism. Then, ϕ is injective.*

Proof. Assume that A is a simple unital C*-algebra and B is a unital C*-algebra. Assume that $\phi : A \to B$ is a unital *-homomorphism. Then,

$\phi(1_A) = 1_B$. In particular, the *-homomorphism ϕ is non-zero.

The kernel $\ker\phi$ is a closed two-sided ideal of A. Since A is simple, then either $\ker\phi = \{0\}$ or $\ker\phi = H$. If $\ker\phi = H$ then ϕ is the zero map, which contradicts the fact that $\phi(1_A) = 1_B$. Hence, $\ker\phi = \{0\}$ and ϕ is injective. $\square$

Since (π, H) is a non-degenerate representation of $A = M_{n\times n}(\mathbb{C})$, then by Theorem 1.9.3 π is a unital *-homomorphism. Utilising Theorem 1.10.4, we deduce that π is injective. In particular, $\pi(e_{1,1}) \neq 0$.

Next, let $\xi \in \pi(e_{1,1})H$ be a unit vector. Then, there exists $\eta \in H$ such that $\xi = \pi(e_{1,1})\eta$. Note that if $i \in \{1, 2, \ldots, n\}$ then $e_{i,1}e_{1,1} = e_{i,1}$. We claim that the set

$$\{\xi, \pi(e_{2,1})\xi, \ldots, \pi(e_{n,1})\xi\} \subset H$$

is an orthonormal set. Firstly, if $i \in \{2, 3, \ldots, n\}$ then

$$\begin{aligned}
\|\pi(e_{i,1})\xi\|^2 &= \langle \pi(e_{i,1})\xi, \pi(e_{i,1})\xi \rangle \\
&= \langle \xi, \pi(e_{i,1})^*\pi(e_{i,1})\xi \rangle \\
&= \langle \xi, \pi(e_{1,i})\pi(e_{i,1})\xi \rangle \\
&= \langle \xi, \pi(e_{1,1})\xi \rangle \\
&= \langle \xi, \xi \rangle = \|\xi\|^2 = 1.
\end{aligned}$$

By a similar computation, if $i, j \in \{1, 2, \ldots, n\}$ are distinct then

$$\begin{aligned}
\langle \pi(e_{i,1})\xi, \pi(e_{j,1})\xi \rangle &= \langle \xi, \pi(e_{i,1})^*\pi(e_{j,1})\xi \rangle \\
&= \langle \xi, \pi(e_{1.i})\pi(e_{j,1})\xi \rangle \\
&= \langle \xi, 0\xi \rangle = 0.
\end{aligned}$$

Hence, $\{\xi, \pi(e_{2,1})\xi, \ldots, \pi(e_{n,1})\xi\}$ is an orthonormal set in H. We would like to find the span of this set. We claim that

$$\pi(A)\xi = span\{\xi, \pi(e_{2,1})\xi, \ldots, \pi(e_{n,1})\xi\}.$$

We certainly have the inclusion $span\{\xi, \pi(e_{2,1})\xi, \ldots, \pi(e_{n,1})\xi\} \subseteq \pi(A)\xi$ because $\pi(e_{1,1})\xi = \xi$. Now assume that

$$\sum_{i=1}^{n} \lambda_i \pi(e_{1,1})\xi \in span\{\xi, \pi(e_{2,1})\xi, \ldots, \pi(e_{n,1})\xi\}$$

where $\lambda_i \in \mathbb{C}$ for $i \in \{1, 2, \ldots, n\}$. Then,

$$\sum_{i=1}^{n} \lambda_i \pi(e_{1,1})\xi = \pi(\sum_{i=1}^{n} \lambda_i e_{1,1})\xi \in \pi(A)\xi.$$

Therefore,

$$\pi(A)\xi = span\{\xi, \pi(e_{2,1})\xi, \ldots, \pi(e_{n,1})\xi\} \tag{1.8}$$

as required. Note that $\pi(A)\xi$ is a finite dimensional subspace of the Hilbert space H and is thus, closed. We now have enough to state our first main result.

Theorem 1.10.5. *Let $n \in \mathbb{Z}_{\geq 1}$ and $A = M_{n\times n}(\mathbb{C})$. Let (π, H) be a non-degenerate representation of A. Let $\xi \in \pi(e_{1,1})H$ be a unit vector. Then, the restriction $(\pi|_{\pi(A)\xi}, \pi(A)\xi)$ is unitarily equivalent to $(\rho, \mathbb{C}^n)$ where we recall that ρ is the *-homomorphism*

$$\begin{array}{rccc} \rho: & A & \to & B(\mathbb{C}^n) = M_{n\times n}(\mathbb{C}) \\ & a & \mapsto & (\xi \mapsto a\xi). \end{array}$$

Proof. Assume that $A = M_{n\times n}(\mathbb{C})$ and (π, H) is a non-degenerate representation of A. Assume that $\xi \in \pi(e_{1,1})H$ is a unit vector.

To see that the representations $(\pi|_{\pi(A)\xi}, \pi(A)\xi)$ and $(\rho, \mathbb{C}^n)$ are unitarily equivalent, it suffices to show that there exists a unitary operator $u : \pi(A)\xi \to \mathbb{C}^n$ such that if $a \in A$ then

$$u\pi|_{\pi(A)\xi}(a) = \rho(a)u.$$

Recall from equation (1.8) that $\pi(A)\xi$ is the $\mathbb{C}$-span of the set $\{\xi, \pi(e_{2,1})\xi, \ldots, \pi(e_{n,1})\xi\}$. Define the map u by

$$\begin{array}{rccc} u: & \pi(A)\xi & \to & \mathbb{C}^n \\ & \pi(e_{i,1})\xi & \mapsto & E_i = (0, \ldots, 0, 1, 0, \ldots, 0) \end{array}$$

where $i \in \{1, 2, \ldots, n\}$ and E_i is the n-tuple with a 1 in the i^{th} position and zeros elsewhere. To see that u is unitary, observe that if $i, j \in \{1, 2, \ldots, n\}$ then

$$\delta_{ij} = \langle E_i, E_j\rangle = \langle u(\pi(e_{i,1})\xi), u(\pi(e_{j,1})\xi)\rangle = \langle \pi(e_{i,1})\xi, \pi(e_{j,1})\xi\rangle.$$

By linearity of the inner product, we find that if $\xi_1, \xi_2 \in \pi(A)\xi$ then $\langle u\xi_1, u\xi_2\rangle = \langle \xi_1, \xi_2\rangle$. Hence, u is a unitary map.

Now assume that $\eta \in \pi(A)\xi$. Then, there exists $\gamma_1, \ldots, \gamma_n \in \mathbb{C}$ such that

$$\eta = \sum_{i=1}^{n} \gamma_i \pi(e_{i,1})\xi.$$

We compute directly that

$$\begin{aligned}
(\rho(a)u)(\eta) &= (\rho(a)u)(\sum_{i=1}^{n} \gamma_i \pi(e_{i,1})\xi) \\
&= \rho(a)\big(\sum_{i=1}^{n} \gamma_i E_i\big) \\
&= a\big(\sum_{i=1}^{n} \gamma_i E_i\big) = \sum_{i=1}^{n} \gamma_i a(E_i)
\end{aligned}$$

and

$$\begin{aligned}
(u\pi(a))(\eta) &= (u\pi(a))(\sum_{i=1}^{n} \gamma_i \pi(e_{i,1})\xi) \\
&= u\big(\sum_{i=1}^{n} \gamma_i \pi(ae_{i,1})\xi\big) \\
&= u\big(\sum_{i=1}^{n}\sum_{j=1}^{n} \gamma_i a_{j,i} \pi(e_{i,1})\xi\big) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \gamma_i a_{j,i} E_i = \sum_{i=1}^{n} \gamma_i a(E_i).
\end{aligned}$$

Hence, if $a \in A$ then $u\pi|_{\pi(A)\xi}(a) = \rho(a)u$. So, the representations $(\pi|_{\pi(A)\xi}, \pi(A)\xi)$ and $(\rho, \mathbb{C}^n)$ are unitarily equivalent as required. $\square$

Theorem 1.10.5 tells us that our non-degenerate representation of A is unitarily equivalent to the "matrix multiplication representation" of A, provided that we restrict our representation appropriately. This suggests that we can build the non-degenerate representation (π, H) by using a direct sum of matrix multiplication representations. It turns out that this is indeed the case and the clue here is that we took $\xi \in \pi(e_{1,1})H$ to be an arbitrary unit vector. This in turn, suggests that we consider an orthonormal basis for the Hilbert space $\pi(e_{1,1})H$.

Let $\mathcal{B}$ be an orthonormal basis for $\pi(e_{1,1})H$. We claim that if $\eta, \xi \in \mathcal{B}$ are distinct then $\pi(A)\xi$ and $\pi(A)\eta$ are orthogonal subspaces to each other. Recall that

$$\pi(A)\xi = span\{\xi, \pi(e_{2,1})\xi, \dots, \pi(e_{n,1})\xi\}.$$

A similar statement holds for $\pi(A)\eta$. If $i, j \in \{1, 2, \dots, n\}$ then

$$\begin{aligned}
\langle \pi(e_{i,1})\xi, \pi(e_{j,1})\eta \rangle &= \langle \xi, \pi(e_{i,1})^* \pi(e_{j,1})\eta \rangle \\
&= \langle \xi, \pi(e_{1,i}e_{j,1})\eta \rangle \\
&= \begin{cases} 0, \text{ if } i \neq j, \\ \langle \xi, \pi(e_{1,1})\eta \rangle, \text{ if } i = j. \end{cases} \\
&= \begin{cases} 0, \text{ if } i \neq j, \\ \langle \xi, \eta \rangle = 0, \text{ if } i = j. \end{cases}
\end{aligned}$$

Therefore, $\pi(A)\xi$ and $\pi(A)\eta$ are orthogonal subspaces to each other.

Next, we can decompose H as the direct sum $H = \pi(A)\xi \oplus (\pi(A)\xi)^{\perp}$. Since $\pi(A)\eta \subseteq (\pi(A)\xi)^{\perp}$, then $\pi(A)\eta$ is a closed subspace of $(\pi(A)\xi)^{\perp}$ because it is finite dimensional. So, H decomposes further as

$$H = \pi(A)\xi \oplus \pi(A)\eta \oplus \big((\pi(A)\xi)^{\perp} \cap (\pi(A)\eta)^{\perp}\big).$$

Since the subspaces $\pi(A)\eta$ are all mutually orthogonal for $\eta \in \mathcal{B}$, we can repeat the above argument to obtain

$$H = \Big(\bigoplus_{\eta \in \mathcal{B}} \pi(A)\eta\Big) \oplus \bigcap_{\eta \in \mathcal{B}} (\pi(A)\eta)^{\perp}.$$

Now if $\delta \in \bigcap_{\eta \in \mathcal{B}} (\pi(A)\eta)^{\perp}$ then $\langle \delta, \eta \rangle = 0$ for $\eta \in \mathcal{B}$. Since $\mathcal{B}$ is an orthonormal basis, then $\delta = 0$. So,

$$H = \bigoplus_{\eta \in \mathcal{B}} \pi(A)\eta$$

as required. With this, we can now state the main theorem of this section.

Theorem 1.10.6. *Let $n \in \mathbb{Z}_{\geq 1}$ and $A = M_{n \times n}(\mathbb{C})$. Let (π, H) be a non-degenerate representation of A. Let $\mathcal{B}$ be an orthonormal basis for $\pi(e_{1,1})H$. Then, (π, H) is unitarily equivalent to the direct sum*

$$\bigoplus_{\eta\in\mathcal{B}}(\rho,\mathbb{C}^n)$$

*where ρ is the *-homomorphism*

$$\begin{array}{rccc} \rho: & A & \to & B(\mathbb{C}^n) = M_{n\times n}(\mathbb{C}) \\ & a & \mapsto & (\xi \mapsto a\xi). \end{array}$$

Proof. Assume that $n \in \mathbb{Z}_{\geq 1}$ and $A = M_{n\times n}(\mathbb{C})$. Assume that (π, H) is a non-degenerate representation of A and $\mathcal{B}$ is an orthonormal basis for $\pi(e_{1,1})H$. By Theorem 1.10.5, if $\xi \in \mathcal{B}$ then

$$(\pi|_{\pi(A)\xi}, \pi(A)\xi) \sim_u (\rho, \mathbb{C}^n).$$

By taking direct sums over all elements of $\mathcal{B}$, we find that

$$(\bigoplus_{\xi\in\mathcal{B}}\pi|_{\pi(A)\xi}, \bigoplus_{\xi\in\mathcal{B}}\pi(A)\xi) \sim_u \bigoplus_{\xi\in\mathcal{B}}(\rho, \mathbb{C}^n).$$

But, we know that $H = \bigoplus_{\eta\in\mathcal{B}}\pi(A)\eta$. Therefore,

$$(\pi, H) \sim_u \bigoplus_{\xi\in\mathcal{B}}(\rho, \mathbb{C}^n).$$

□

Thus, Theorem 1.10.6 demonstrates that any non-degenerate representation of $M_{n\times n}(\mathbb{C})$ can be built directly from the matrix multiplication representation $(\rho, \mathbb{C}^n)$. In this sense, the representation $(\rho, \mathbb{C}^n)$ is the only representation of $M_{n\times n}(\mathbb{C})$ one needs to know in order to understand arbitrary non-degenerate representations of $M_{n\times n}(\mathbb{C})$.

In fact, $(\rho, \mathbb{C}^n)$ is irreducible, as proven below.

Theorem 1.10.7. *Let $n \in \mathbb{Z}_{\geq 1}$ and $A = M_{n\times n}(\mathbb{C})$. Then, the matrix multiplication representation $(\rho, \mathbb{C}^n)$ is irreducible.*

Proof. Assume that $n \in \mathbb{Z}_{\geq 1}$ and $A = M_{n\times n}(\mathbb{C})$. Since $(\rho, \mathbb{C}^n)$ is a non-degenerate representation of A, it suffices to prove that every non-zero vector of $\mathbb{C}^n$ is cyclic by Theorem 1.9.6.

Assume that $\eta = (\eta_1, \ldots, \eta_n) \in \mathbb{C}^n - \{0\}$. If $i, j \in \{1, 2, \ldots, n\}$ then let $e_{i,j} \in A$ denote the matrix with a 1 in the ij position and zeros elsewhere.

Let η_k be the first non-zero element of the n-tuple η. If $i \in \{1, 2, \ldots, n\}$ then

$$\rho(\frac{1}{\eta_k} e_{i,k})\eta = \frac{1}{\eta_k} e_{i,k}\eta = E_i$$

where $E_i \in \mathbb{C}^n$ is the n-tuple with a 1 in the i^{th} position and zeros elsewhere. Therefore,

$$\mathbb{C}^n = span\{E_1, E_2, \ldots, E_n\} \subseteq \rho(A)\xi.$$

So, $\rho(A)\xi = \mathbb{C}^n$ and ξ is a cyclic vector as required. By Theorem 1.9.6, the representation $(\rho, \mathbb{C}^n)$ is irreducible. $\square$

1.11 The Gelfand-Naimark-Segal construction

In the last few sections, we defined representations of C*-algebras, investigated a few of their properties and in the case of the last section, delved into a particular example of a representation. Here, we are interested in constructing a representation of an arbitrary C*-algebra.

To motivate this sections, let us consider the parallel situation for groups. Groups naturally occur as symmetries and one often looks for ways that abstract groups act as symmetries. The simplest way this occurs is to consider a group G acting on itself via left multiplication. The main result stemming from this is Cayley's theorem (see [DF04, Section 4.2]).

As remarked in [Put19, Section 1.12], we will apply a similar line of thinking to producing representations of a C*-algebra. The multiplication operation on a C*-algebra allows one to think of its elements as linear transformations acting on the C*-algebra itself. The problem here is that C*-algebras are not generally Hilbert spaces. The GNS construction (Gelfand-Naimark-Segal) produces the desired inner product on the C*-algebra by using the linear functionals on the C*-algebra.

The key property for the functionals we consider in the GNS construction is positivity, which was introduced in the context of traces. We briefly recall that a linear function ϕ on a C*-algebra A is positive if for $a \in A$, $\phi(a^*a) \geq 0$.

Definition 1.11.1. Let A be a unital C*-algebra and ϕ be a linear functional on A. We say that ϕ is a **state** if ϕ is positive and $\phi(1_A) = 1$.

We have the following simple characterisation of states.

Theorem 1.11.1. *Let A be a unital C*-algebra and ϕ be a linear functional on A. If $\phi(1_A) = \|\phi\| = 1$ then ϕ is a state.*

Proof. Assume that A is a unital C*-algebra and ϕ is a linear functional on A. Assume that $\phi(1_A) = \|\phi\| = 1$.

To show: (a) ϕ is positive.

(a) We will first show that if $a \in A$ is self-adjoint then $\phi(a) \in \mathbb{R}$. So, assume that $a \in A$ is self-adjoint. Suppose for the sake of contradiction that $Im(\phi(a)) \neq 0$. Without loss of generality, we may assume that $Im(\phi(a)) > 0$. Select $r \in \mathbb{R}_{>0}$ satisfying the inequality

$$0 < r < Im(\phi(a)) \leq \|a\|.$$

Then, define the function f by

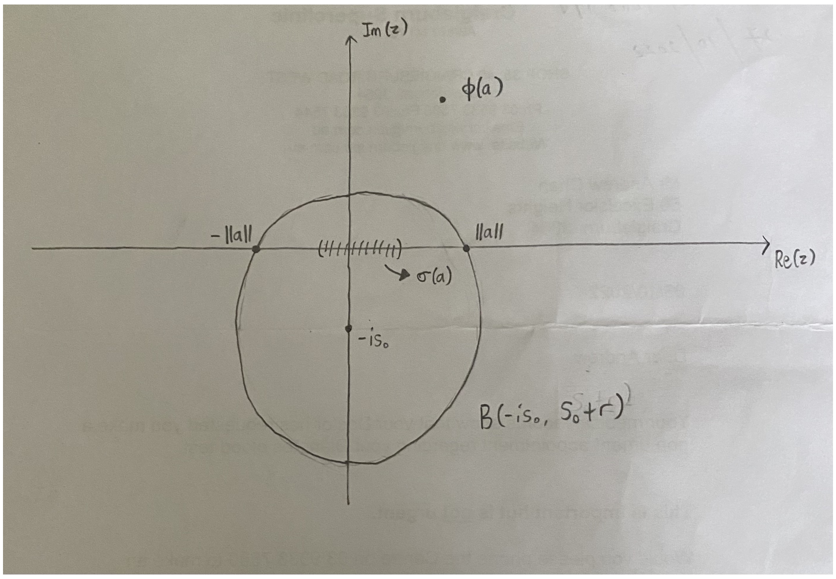
$$\begin{aligned} f: \ \mathbb{R} &\to \mathbb{R} \\ s &\mapsto \sqrt{\|a\|^2 + s^2} - s. \end{aligned}$$

Observe that $f(0) > r$. Observe that the limit

$$\begin{aligned} \lim_{s \to \infty} f(s) &= \lim_{s \to \infty} (\sqrt{\|a\|^2 + s^2} - s) \\ &= \lim_{s \to \infty} \frac{\|a\|^2 + s^2 - s^2}{\sqrt{\|a\|^2 + s^2} + s} \\ &= \lim_{s \to \infty} \frac{\|a\|^2}{\sqrt{\|a\|^2 + s^2} + s} \\ &= 0. \end{aligned}$$

Hence, there exists $s_0 \in \mathbb{R}_{>0}$ such that $f(s_0) = r$.

The argument we make here is geometric in nature. Here is a diagram of the situation.



Let $B(-is_0, s_0 + r)$ be the open ball centred at $-is_0$ with radius $s_0 + r$. Note that $s_0 + r = s_0 + f_0 = \sqrt{\|a\|^2 + s_0^2}$. So, $[-\|a\|, \|a\|] \subseteq B(-is_0, s_0 + r)$. Since a is self-adjoint, then the spectrum $\sigma(a) \subseteq [-\|a\|, \|a\|]$. So, $\sigma(a) \subseteq B(-is_0, s_0 + r)$.

From the diagram, we also see that $\phi(a) \notin B(-is_0, s_0 + r)$ because by construction,

$$r = (s_0 + r) - s_0 < Im(\phi(a))$$

With this information, we now compute

$$\begin{aligned}
|\phi(a - is_0 1_A)| &= |\phi(a) - is_0\phi(1_A)| \\
&= |\phi(a) - is_0| \quad (\text{since } \phi(1_A) = 1) \\
&> r + s_0 = \sqrt{\|a\|^2 + s_0^2}.
\end{aligned}$$

Note that by the diagram,

$$\|a - is_0 1_A\| \leq \sqrt{\|a\|^2 + s_0^2}.$$

So,

$$\phi(\frac{a - is_0 1_A}{\sqrt{\|a\|^2 + s_0^2}}) > 1$$

which contradicts the assumption that $\|\phi\| = 1$. So, $Im(\phi(a)) = 0$ and $\phi(a) \in \mathbb{R}$.

Now let $a \in A$ be positive. By the previous part, $\phi(a) \in \mathbb{R}$. It suffices to show that $\phi(a) \geq 0$. By Theorem 1.4.2, $\|\|a\|1_A - a\| \leq \|a\|$. Since $\phi(1_A) = 1 = \|\phi\|$, we have

$$\begin{aligned} |\phi(\|a\|1_A - a)| &= |\|a\| - \phi(a)| \\ &= \|a\||1 - \phi(\frac{a}{\|a\|})| \\ &\leq \|a\|. \end{aligned}$$

So, $\phi(a) \geq 0$ as required. □

Next, we prove some more properties of positive linear functionals.

Theorem 1.11.2. *Let A be a unital C*-algebra and ϕ be a positive linear functional.*

1. *If $a, b \in A$ then $|\phi(b^*a)|^2 \leq \phi(a^*a)\phi(b^*b)$.*
2. *If $a \in A$ then $\phi(a^*) = \overline{\phi(a)}$.*
3. *$\phi(1_A) = \|\phi\|$*
4. *If $a, b \in A$ then $\phi(b^*a^*ab) \leq \|a\|^2\phi(b^*b)$.*

Proof. Assume that A is a unital C*-algebra and ϕ is a positive linear functional. Assume that $a, b \in A$ and $\lambda \in \mathbb{C}$. Then,

$$0 \leq \phi((\lambda a + b)^*(\lambda a + b)) = |\lambda|^2\phi(a^*a) + \overline{\lambda}\phi(a^*b) + \lambda\phi(b^*a) + \phi(b^*b). \quad (1.9)$$

Since $|\lambda|^2\phi(a^*a), \phi(b^*b) \in \mathbb{R}_{\geq 0}$, the sum

$$\overline{\lambda}\phi(a^*b) + \lambda\phi(b^*a) \in \mathbb{R}.$$

Now set $\lambda = i$ and $b = 1_A$. Then, $i\phi(a) - i\phi(a^*) \in \mathbb{R}$. On the other hand, if we set $\lambda = 1$ and $b = 1_A$ then $\phi(a^*) + \phi(a) \in \mathbb{R}$. Now let $\phi(a) = \alpha + \beta i$ and $\phi(a^*) = \gamma + \delta i$. By the two equations, we find that $\alpha - \gamma = 0$ and $\beta + \delta = 0$. So,

$$\phi(a^*) = \gamma + \delta i = \alpha - \beta i = \overline{\phi(a)}.$$

To prove the first statement, note that by equation (1.9),

$$\begin{aligned} -\overline{\lambda}\phi(a^*b) - \lambda\phi(b^*a) &= -\overline{\lambda\phi(b^*a)} - \lambda\phi(b^*a) \\ &= -2Re(\lambda\phi(b^*a)) \\ &\leq |\lambda|^2\phi(a^*a) + \phi(b^*b). \end{aligned}$$

There are two cases to consider.

Case 1: $\phi(a^*a) = 0$.

Assume that $\phi(a^*a) = 0$. Then, $-Re(\lambda\phi(b^*a)) \leq \phi(b^*b)$. Since this holds for arbitrary $\lambda \in \mathbb{C}$, then $\phi(b^*a) = 0$. So, the inequality $|\phi(b^*a)|^2 \leq \phi(a^*a)\phi(b^*b)$ holds in this case.

Case 2: $\phi(a^*a) \neq 0$.

Assume that $\phi(a^*a) \neq 0$. Select $z \in \mathbb{C}$ such that $|z| = 1$ and $z\phi(b^*a) = |\phi(b^*a)|$. If $\lambda = -z\phi(a^*a)^{-\frac{1}{2}}\phi(b^*b)^{\frac{1}{2}}$ then

$$\begin{aligned} -2Re(\lambda\phi(b^*a)) &= 2Re(z\phi(a^*a)^{-\frac{1}{2}}\phi(b^*b)^{\frac{1}{2}}\phi(b^*a)) \\ &= 2\phi(a^*a)^{-\frac{1}{2}}\phi(b^*b)^{\frac{1}{2}}|\phi(b^*a)| \\ &\leq |\lambda|^2\phi(a^*a) + \phi(b^*b) \\ &= \phi(b^*b) + \phi(b^*b) = 2\phi(b^*b). \end{aligned}$$

So, $|\phi(b^*a)| \leq \phi(a^*a)^{\frac{1}{2}}\phi(b^*b)^{\frac{1}{2}}$. Squaring both sides, we obtain the first statement of the theorem.

Next, we show that $\phi(1_A) = \|\phi\|$. Since $\|1_A\| = 1$, we have by definition of the operator norm $|\phi(1_A)| \leq \|\phi\|$.

Now let $b = 1_A$ in the inequality $|\phi(b^*a)|^2 \leq \phi(a^*a)\phi(b^*b)$. Then, $|\phi(a)|^2 \leq \phi(a^*a)\phi(1_A)$. So,

$$\begin{aligned} |\phi(a)| &\leq \phi(a^*a)^{\frac{1}{2}}\phi(1_A)^{\frac{1}{2}} \\ &\leq \|\phi\|^{\frac{1}{2}}\|a^*a\|^{\frac{1}{2}}\phi(1_A)^{\frac{1}{2}} \\ &= \|\phi\|^{\frac{1}{2}}\|a\|\phi(1_A)^{\frac{1}{2}}. \end{aligned}$$

Taking the supremum over all $a \in A$ satisfying $\|a\| = 1$, we deduce that $\|\phi\| \leq \phi(1_A)^{\frac{1}{2}} \leq \phi(1_A)$. So, $\phi(1) = \|\phi\|$ as required.

To the prove the fourth and final statement, define the map $\psi : A \to \mathbb{C}$ by $\psi(c) = \phi(b^*cb)$. Then, ψ is a positive linear functional and by the third part, $\|\psi\| = \psi(1) = \phi(b^*b)$. Therefore,

$$\phi(b^*a^*ab) = \psi(a^*a) \leq \|\psi\|\|a^*a\| = \phi(b^*b)\|a\|^2.$$

□

Now we begin the GNS construction. Let A be a unital C*-algebra and ϕ be a state on A. Define

$$N_\phi = \{a \in A \mid \phi(a^*a) = 0\}.$$

To see that N_ϕ is a left ideal in A, assume that $a, b \in N_\phi$. Then,

$$\begin{aligned} \phi((a+b)^*(a+b)) &= \phi(a^*a + a^*b + b^*a + b^*b) \\ &= \phi(a^*a) + \phi(a^*b + b^*a) + \phi(b^*b) \\ &= \phi(a^*b + b^*a). \end{aligned}$$

By the first statement in Theorem 1.11.2,

$$|\phi(a^*b + b^*a)| \leq |\phi(a^*b)| + |\phi(b^*a)| \leq 2\phi(a^*a)^{\frac{1}{2}}\phi(b^*b)^{\frac{1}{2}} = 0.$$

Hence, $\phi((a+b)^*(a+b)) = 0$ and $a + b \in N_\phi$.

Now assume that $c \in A$. Then,

$$\begin{aligned} \phi((ca)^*ca) &= \phi(a^*c^*ca) \\ &\leq \|c\|^2\phi(a^*a) = 0. \end{aligned}$$

The inequality follows from the fourth statement in Theorem 1.11.2. Hence, $ca \in N_\phi$ and N_ϕ is a left ideal of A.

To see that N_ϕ is closed, suppose that $\{a_n\}_{n \in \mathbb{Z}_{>0}}$ is a sequence in N_ϕ which converges to some $a \in A$. Since ϕ is continuous,

$$\phi(a^*a) = \phi(\lim_{n \to \infty} a_n^* a_n) = \lim_{n \to \infty} \phi(a_n^* a_n) = 0.$$

So, $a \in N_\phi$ and N_ϕ is therefore a closed left ideal of A.

Next, we take the quotient A/N_ϕ. The next task is to define an inner product on A/N_ϕ. Define the map

$$\begin{array}{rccc} \langle -,- \rangle : & A/N_\phi \times A/N_\phi & \to & \mathbb{C} \\ & (a+N_\phi, b+N_\phi) & \mapsto & \phi(b^*a). \end{array}$$

First, we have to show that the above map is well-defined. Assume that $a_1 + N_\phi = a_2 + N_\phi$. Then, $\phi((a_1-a_2)^*(a_1-a_2)) = 0$ and by the first statement of Theorem 1.11.2,

$$|\phi(b^*(a_1-a_2))|^2 \leq \phi(b^*b)\phi((a_1-a_2)^*(a_1-a_2)) = 0$$

for $b \in A$. Hence, if $b \in A$ then $\phi(b^*(a_1-a_2)) = 0$ and consequently,

$$\begin{aligned} \langle a_1 + N_\phi, b + N_\phi \rangle &= \phi(b^*a_1) \\ &= \phi(b^*(a_1-a_2)) + \phi(b^*a_2) \\ &= \phi(b^*a_2) = \langle a_2 + N_\phi, b + N_\phi \rangle. \end{aligned}$$

By a similar argument, the map $\langle -,- \rangle$ is well-defined in the second argument. So, the map $\langle -,- \rangle$ is well-defined overall. It is straightforward to check that $\langle -,- \rangle$ is linear in the first argument and antilinear in the second argument.

To see that the map $\langle -,- \rangle$ is a non-degenerate inner product, assume that $\langle a + N_\phi, a + N_\phi \rangle = 0$. This holds if and only if $\phi(a^*a) = 0$ if and only if $a \in N_\phi$. So, $a + N_\phi = N_\phi$ in A/N_ϕ. Thus, $\langle a + N_\phi, a + N_\phi \rangle = 0$ if and only if $a + N_\phi = N_\phi$.

Hence, $\langle -,- \rangle$ defines an inner product on A/N_ϕ. Now define $\mathcal{H}_\phi$ to be the completion of the inner product space A/N_ϕ. Then, $\mathcal{H}_\phi$ is a Hilbert space.

Next, we need to define a *-homomorphism on $\mathcal{H}_\phi$. To do this, we will first define the *-homomorphism on A/N_ϕ and then use the universal property of the completion to extend it to a *-homomorphism on $\mathcal{H}_\phi$.

Let $b + N_\phi \in A/N_\phi$. Define the map π_ϕ on A by

$$\pi_\phi(a)(b + N_\phi) = ab + N_\phi$$

First, we show that if $a \in A$ then $\pi_\phi(a)$ is a well-defined and bounded operator. First, assume that $b_1 + N_\phi = b_2 + N_\phi$ in A/N_ϕ. Then, $b_1 - b_2 \in N_\phi$. Since N_ϕ is a left ideal of A, $ab_1 - ab_2 \in N_\phi$ and $ab_1 + N_\phi = ab_2 + N_\phi$. So, $\pi_\phi(a)(b_1 + N_\phi) = \pi_\phi(a)(b_2 + N_\phi)$ and $\pi_\phi(a)$ is well-defined.

To see that $\pi_\phi(a)$ is bounded, we compute directly that

$$\begin{aligned}
\|\pi_\phi(a)\|^2 &= \sup_{\|b+N_\phi\|=1} \|ab + N_\phi\|^2 \\
&= \sup_{\|b+N_\phi\|=1} \phi(b^*a^*ab) \\
&\leq \sup_{\|b+N_\phi\|=1} \|a\|^2\phi(b^*b) \quad \text{(by Theorem 1.11.2)} \\
&= \|a\|^2.
\end{aligned}$$

Hence, $\pi_\phi(a)$ is a bounded operator on A/N_ϕ.

Next, we show that π_ϕ is a *-homomorphism. The fact that π_ϕ is linear and multiplicative follows from direct computations. To see that π_ϕ preserves adjoints, assume that $a, b, c \in A$. Then,

$$\begin{aligned}
\langle \pi_\phi(a^*)(b + N_\phi), c + N_\phi \rangle &= \langle a^*b + N_\phi, c + N_\phi \rangle \\
&= \phi(c^*a^*b) = \phi((ac)^*b) \\
&= \langle b + N_\phi, ac + N_\phi \rangle \\
&= \langle b + N_\phi, \pi_\phi(a)(c + N_\phi) \rangle \\
&= \langle \pi_\phi(a)^*(b + N_\phi), c + N_\phi \rangle.
\end{aligned}$$

Since $b, c \in A$ was arbitrary, we deduce that $\pi_\phi(a)^* = \pi_\phi(a^*)$. So, π_ϕ is a *-homomorphism. If $a \in A$ then by the universal property of the completion, $\pi_\phi(a)$ extends to a bounded operator on $\mathcal{H}_\phi$ (the completion of A/N_ϕ). Therefore, π_ϕ defines a representation of A on $\mathcal{H}_\phi$.

Finally, define the vector

$$u_\phi = 1_A + N_\phi \in A/N_\phi \subseteq \mathcal{H}_\phi.$$

We claim that u_ϕ is a cyclic vector for the representation $(\pi_\phi, \mathcal{H}_\phi)$ with norm 1.

Assume that $b + N_\phi \in A/N_\phi$. Then, $b + N_\phi = \pi_\phi(b)(u_\phi)$ and $A/N_\phi \subseteq \pi_\phi(A)u_\phi$. Since A/N_ϕ is dense in $\mathcal{H}_\phi$, then $\pi_\phi(A)u_\phi$ is dense in $\mathcal{H}_\phi$ and so, u_ϕ is a cyclic vector.

Since ϕ is a state, $\phi(1_A) = 1$ and

$$\|u_\phi\| = \|1_A + N_\phi\| = \phi(1_A) = 1.$$

This completes the GNS construction. We summarise it below with the following definition.

Definition 1.11.2. Let A be a unital C*-algebra and ϕ be a state on A. The triple $(\pi_\phi, \mathcal{H}_\phi, u_\phi)$ is called the **GNS representation** of ϕ. This is a representation of A, as shown previously.

In summary, the GNS construction takes a state ϕ on a unital C*-algebra A and produces a representation $(\pi_\phi, \mathcal{H}_\phi)$ and a unit cyclic vector u_ϕ. The next theorem shows that this process can be reversed — from a representation with a unit cyclic vector, we can construct a state on A.

Theorem 1.11.3. *Let A be a unital C*-algebra and (π, H) be a representation of A with unit cyclic vector ξ. Then, the map*

$$\begin{array}{rccc} \phi: & A & \to & \mathbb{C} \\ & a & \mapsto & \langle \pi(a)\xi, \xi\rangle \end{array}$$

is a state on A. Moreover, the GNS representation of ϕ is unitarily equivalent to (π, H).

Proof. Assume that A is a unital C*-algebra. Assume that (π, H) is a representation of A and that $\xi \in H$ is a unit cyclic vector of this representation. Assume that ϕ is the linear functional defines as above.

To show: (a) ϕ is a state.

(a) First observe that ϕ is a positive linear functional because if $a \in A$ then

$$\phi(a^*a) = \langle \pi(a^*a)\xi, \xi\rangle = \langle \pi(a)\xi, \pi(a)\xi\rangle = \|\pi(a)\xi\|^2 \geq 0.$$

Now, if $a \in A$ then

$$\begin{aligned} \phi(a) &= \langle \pi(a)\xi, \xi\rangle \\ &= \langle \pi(1_A a)\xi, \xi\rangle = \langle \pi(1_A)\pi(a)\xi, \xi\rangle \\ &= \langle \pi(a)\xi, \pi(1_A)\xi\rangle. \end{aligned}$$

This means that if $a \in A$ then

$$0 = \langle \pi(a)\xi, \pi(1_A)\xi - \xi \rangle.$$

Since $\xi \in H$ is cyclic, the subspace $\pi(A)\xi$ is dense in H. So, $(\pi(A)\xi)^{\perp} = \{0\}$ and $\pi(1_A)\xi = \xi$. Therefore,

$$\phi(1_A) = \langle \pi(1_A)\xi, \xi \rangle = \|\xi\|^2 = 1.$$

Therefore, ϕ is a state.

Next, we show that (π, H) is unitarily equivalent to the GNS representation $(\pi_\phi, H_\phi, \xi_\phi)$ of ϕ. Define the map $u : \pi(A)\xi \to A/N_\phi$ by

$$\begin{array}{rccc} u: & \pi(A)\xi & \to & A/N_\phi \\ & \pi(a)\xi & \mapsto & a + N_\phi. \end{array}$$

We will use $\langle -, - \rangle_\phi$ to represent the inner product on H_ϕ constructed in the GNS representation. To see that u is isometric, we compute directly that

$$\begin{aligned} \langle \pi(a)\xi, \pi(a)\xi \rangle &= \langle \pi(a^*a)\xi, \xi \rangle \\ &= \phi(a^*a) = \langle a + N_\phi, a + N_\phi \rangle_\phi \\ &= \langle u(\pi(a)\xi), u(\pi(a)\xi) \rangle_\phi. \end{aligned}$$

We also have $u(\xi) = u(\pi(1_A)\xi) = 1_A + N_\phi = \xi_\phi$. By the universal property of the Hilbert space H (which is complete), u extends to a unitary operator $\tilde{u} : H \to H_\phi$.

To see that $\tilde{u}\pi(a)\tilde{u}^* = \pi_\phi(a)$ for $a \in A$, assume that $h \in H_\phi$ so that there exists a sequence $\{h_n + N_\phi\}_{n \in \mathbb{Z}_{>0}}$ which converges to h. Then,

$$\begin{aligned} \pi_\phi(a)(h) &= \pi_\phi(a)(\lim_{n \to \infty}(h_n + N_\phi)) \\ &= \lim_{n \to \infty} \pi_\phi(a)(h_n + N_\phi) \\ &= \lim_{n \to \infty}(ah_n + N_\phi) = ah \end{aligned}$$

and

$$
\begin{aligned}
\tilde{u}\pi(a)\tilde{u}^*(h) &= \tilde{u}\pi(a)\tilde{u}^*(\lim_{n\to\infty}(h_n + N_\phi)) \\
&= \lim_{n\to\infty}\big(u\pi(a)u^*(h_n + N_\phi)\big) \\
&= \lim_{n\to\infty} u\pi(a)(\pi(h_n)\xi) \\
&= \lim_{n\to\infty}(ah_n + N_\phi) = ah.
\end{aligned}
$$

Hence, the representations (π, H) and $(\pi_\phi, H_\phi, \xi_\phi)$ are unitarily equivalent. $\square$

The GNS construction, in tandem with Theorem 1.11.3, tells us that if A is a unital C*-algebra then there is a bijection of sets

$$
\begin{array}{ccc}
\{\text{States on } A\} & \longleftrightarrow & \left\{\begin{array}{c}\text{Representations of } A \\ \text{with unit cyclic vector}\end{array}\right\} \\
\phi & \mapsto & (\pi_\phi, H_\phi, \xi_\phi) \\
(a \mapsto \langle \pi(a)\xi, \xi\rangle) & \leftarrow\!\shortmid & (\pi, H, \xi)
\end{array}
$$

The next question we will answer is this: where do the irreducible representations of A map to under the above bijection? As remarked in [Put19], irreducible representations of A map to the "extreme points" in the set of states. Theses states are often called *pure states.*

Theorem 1.11.4. *Let A be a unital C*-algebra and ϕ be a state on A. The GNS representation $(\pi_\phi, H_\phi, \xi_\phi)$ is irreducible if and only if ϕ is not a non-trivial convex combination of two other states — if there exists states ϕ_0, ϕ_1 and $t \in (0,1)$ such that $\phi = t\phi_0 + (1-t)\phi_1$ then $\phi = \phi_0 = \phi_1$.*

Proof. Assume that A is a unital C*-algebra and ϕ is a state on A.

To show: (a) If ϕ is not a non-trivial convex combination of two other states then the GNS representation of ϕ is irreducible.

(b) If the GNS representation of ϕ is irreducible then ϕ is not a non-trivial convex combination of two other states.

(a) We will prove the contrapositive of this statement. Assume that the GNS representation $(\pi_\phi, H_\phi, \xi_\phi)$ is reducible. Then, there exists a non-trivial closed invariant subspace $\mathcal{N} \subseteq H_\phi$.

Since $H_\phi = \mathcal{N} \oplus \mathcal{N}^\perp$, we can decompose $\xi_\phi = \xi_0 + \xi_1$, where $\xi_0 \in \mathcal{N}$ and $\xi_1 \in \mathcal{N}^\perp$. We claim that $\xi_0, \xi_1 \neq 0$.

Suppose for the sake of contradiction that $\xi_1 = 0$. Then, $\xi_\phi = \xi_0 \in \mathcal{N}$. Consequently, $\pi(A)\xi_\phi \subseteq \pi(A)\mathcal{N} \subseteq \mathcal{N}$. However, since ξ_ϕ is a cyclic vector, the subspace $\pi(A)\xi_\phi$ is dense in H_ϕ. Therefore, $\mathcal{N} = \overline{\mathcal{N}} = H_\phi$, which contradicts the assumption that $\mathcal{N}$ is a non-trivial closed, invariant subspace of H_ϕ. So, $\xi_1 \neq 0$. The argument that $\xi_0 \neq 0$ is similar and uses Theorem 1.9.2.

If $a \in A$ and $i \in \{0, 1\}$ then define

$$\phi_i(a) = \|\xi_i\|^{-2} \langle \pi_\phi(a)\xi_i, \xi_i \rangle.$$

We claim that ϕ_i is a state. If $a \in A$ then

$$\phi_i(a^*a) = \|\xi_i\|^{-2} \|\pi_\phi(a)\xi_i\|^2 \geq 0.$$

So, ϕ_i is a positive linear functional. Now since $\mathcal{N}$ and $\mathcal{N}^\perp$ are both closed invariant subspaces of H_ϕ,

$$\begin{aligned}
\phi_i(1_A) &= \|\xi_i\|^{-2} \langle \pi_\phi(1_A)\xi_i, \xi_i \rangle \\
&= \|\xi_i\|^{-2} \langle \pi_\phi(1_A)\xi_i, \xi_i + \xi_{1-i} \rangle \\
&= \|\xi_i\|^{-2} \langle \pi_\phi(1_A)\xi_i, \xi_\phi \rangle \\
&= \|\xi_i\|^{-2} \langle \xi_i, \pi_\phi(1_A)\xi_\phi \rangle \\
&= \|\xi_i\|^{-2} \langle \xi_i, \xi_\phi \rangle \\
&= \|\xi_i\|^{-2} \|\xi_i\|^2 = 1.
\end{aligned}$$

Hence, ϕ_0 and ϕ_1 are both states.

Next, we claim that $\phi = \|\xi_0\|^2 \phi_0 + \|\xi_1\|^2 \phi_1$. We compute directly that if $a \in A$ then

$$\begin{aligned}
\phi(a) &= \langle \pi_\phi(a)\xi_\phi, \xi_\phi \rangle \\
&= \langle \pi_\phi(a)(\xi_0 + \xi_1), \xi_0 + \xi_1 \rangle \\
&= \langle \pi_\phi(a)\xi_0, \xi_0 \rangle + \langle \pi_\phi(a)\xi_0, \xi_1 \rangle + \langle \pi_\phi(a)\xi_1, \xi_0 \rangle + \langle \pi_\phi(a)\xi_1, \xi_1 \rangle \\
&= \langle \pi_\phi(a)\xi_0, \xi_0 \rangle + \langle \pi_\phi(a)\xi_1, \xi_1 \rangle \\
&= \|\xi_0\|^2 \phi_0(a) + \|\xi_1\|^2 \phi_1(a).
\end{aligned}$$

Finally, we claim that $\phi_0 \neq \phi_1$. Let $C = \min\{\|\xi_0\|, \|\xi_1\|\} > 0$. Since the vector ξ_ϕ is cyclic, there exists $a \in A$ such that

$$\|\pi_\phi(a)\xi_\phi - \xi_0\| < 2^{-1}C.$$

Now, $\pi_\phi(a)\xi_\phi = \pi_\phi(a)\xi_0 + \pi_\phi(a)\xi_1$, where $\pi_\phi(a)\xi_0 \in \mathcal{N}$ and $\pi_\phi(a)\xi_1 \in \mathcal{N}^\perp$. Then,

$$\begin{aligned}
\|\pi_\phi(a)\xi_1\|^2 &= \langle \pi_\phi(a)\xi_1, \pi_\phi(a)\xi_1\rangle \\
&= \langle \pi_\phi(a)\xi_\phi, \pi_\phi(a)\xi_1\rangle \\
&= \langle \pi_\phi(a)\xi_\phi - \xi_0, \pi_\phi(a)\xi_1\rangle \\
&\leq \|\pi_\phi(a)\xi_0 - \xi_0\|\|\pi_\phi(a)\xi_1\|.
\end{aligned}$$

and

$$\begin{aligned}
\|\pi_\phi(a)\xi_0 - \xi_0\|^2 &= \langle \pi_\phi(a)\xi_0 - \xi_0, \pi_\phi(a)\xi_0 - \xi_0\rangle \\
&= \langle \pi_\phi(a)\xi_0 + \pi_\phi(a)\xi_1 - \xi_0, \pi_\phi(a)\xi_0 - \xi_0\rangle \\
&= \langle \pi_\phi(a)\xi_\phi - \xi_0, \pi_\phi(a)\xi_0 - \xi_0\rangle \\
&\leq \|\pi_\phi(a)\xi_\phi - \xi_0\|\|\pi_\phi(a)\xi_0 - \xi_0\|.
\end{aligned}$$

Hence, we have the inequalities

$$\|\pi_\phi(a)\xi_1\| \leq 2^{-1}C \qquad \text{and} \qquad \|\pi_\phi(a)\xi_0 - \xi_0\| \leq 2^{-1}C.$$

From both inequalities we obtain the upper bounds

$$\begin{aligned}
|\phi_0(a) - 1| &= |\|\xi_0\|^{-2}\langle \pi_\phi(a)\xi_0, \xi_0\rangle - 1| \\
&= |\|\xi_0\|^{-2}\langle \pi_\phi(a)\xi_0, \xi_0\rangle - \|\xi_0\|^{-2}\|\xi_0\|^2| \\
&= \|\xi_0\|^{-2}|\langle \pi_\phi(a)\xi_0 - \xi_0, \xi_0\rangle| \\
&\leq \|\xi_0\|^{-1}\|\pi_\phi(a)\xi_0 - \xi_0\| \\
&< \|\xi_0\|^{-1}2^{-1}C \leq \frac{1}{2}
\end{aligned}$$

and

$$\begin{aligned}
|\phi_1(a)| &= |\|\xi_1\|^{-2}\langle \pi_\phi(a)\xi_1, \xi_1\rangle| \\
&\leq \|\xi_1\|^{-1}\|\pi_\phi(a)\xi_1\| \\
&< \|\xi_1\|^{-1}2^{-1}C \leq \frac{1}{2}.
\end{aligned}$$

From these upper bounds, we deduce that $\phi_0(a) \neq \phi_1(a)$. Hence, the state ϕ is a non-trivial convex combination of the states ϕ_0 and ϕ_1.

(b) We will prove the contrapositive of this statement. Suppose that there exists states ϕ_0, ϕ_1 and $t \in (0,1)$ such that $\phi = t\phi_0 + (1-t)\phi_1$. Define the bilinear form

$$\begin{array}{rccc} B: & A/N_\phi \times A/N_\phi & \to & \mathbb{C} \\ & (a+N_\phi, b+N_\phi) & \mapsto & t\phi_0(b^*a). \end{array}$$

Observe that $B(a+N_\phi, b+N_\phi)$ is bounded above as follows:

$$\begin{aligned} |B(a+N_\phi, b+N_\phi)|^2 &= t^2|\phi_0(b^*a)|^2 \\ &\leq |t\phi_0(a^*a)||t\phi_0(b^*b)| \quad \text{(by Theorem 1.11.2)} \\ &\leq |\phi_0(a^*a)||\phi_0(b^*b)| \\ &= \|a+N_\phi\|^2\|b+N_\phi\|^2. \end{aligned}$$

By a similar argument to Theorem 1.11.3, we find that the bilinear form B is well-defined on A/N_ϕ. Since it is also bounded (and hence, continuous), B extends to a bounded bilinear form $\tilde{B} : H_\phi \times H_\phi \to \mathbb{C}$.

Since H_ϕ is a Hilbert space, there exists a unique positive operator $h \in B(H_\phi)$ such that if $a, b \in A$ then

$$t\phi_0(b^*a) = B(a+N_\phi, b+N_\phi) = \langle h(a+N_\phi), b+N_\phi \rangle.$$

By Theorem 1.9.7, it suffices to show that h commutes with $\pi_\phi(a)$ for $a \in A$ and that h is not a scalar multiple of the identity operator id_{H_ϕ}.

To show: (ba) h is not a scalar multiple of id_{H_ϕ}.

(bb) If $a \in A$ then $\pi_\phi(a)h = h\pi_\phi(a)$.

(ba) Suppose for the sake of contradiction that $h = \lambda id_{H_\phi}$ for some $\lambda \in \mathbb{C}$. Then,

$$t\phi_0(b^*a) = \lambda\langle a+N_\phi, b+N_\phi \rangle.$$

If $b = 1_A$ then $t\phi_0(a) = \lambda\langle a+N_\phi, 1_A+N_\phi \rangle = \lambda\phi(a)$. Thus, ϕ is a multiple of ϕ_0. However, ϕ and ϕ_0 are both states. So, $\phi(1_A) = \phi_0(1_A) = 1$. Therefore, $\phi = \phi_0$. Since $\phi = t\phi_0 + (1-t)\phi_1$, then $\phi_1 = \phi$. This contradicts the assumption that $\phi_0 \neq \phi_1$. Hence, h is not a scalar multiple of id_{H_ϕ}.

(bb) Assume that $a, b, c \in A$. Then,

$$\begin{aligned}
\langle \pi_\phi(a)h(b + N_\phi), c + N_\phi \rangle &= \langle h(b + N_\phi), \pi_\phi(a)^*(c + N_\phi) \rangle \\
&= \langle h(b + N_\phi), a^*c + N_\phi \rangle \\
&= t\phi_0((a^*c)^*b) = t\phi_0(c^*ab) \\
&= \langle h(ab + N_\phi), c + N_\phi \rangle \\
&= \langle h\pi_\phi(a)(b + N_\phi), c + N_\phi \rangle
\end{aligned}$$

Since $b, c \in A$ were arbitrary, we deduce that if $a \in A$ then $h\pi_\phi(a) = \pi_\phi(a)h$.

(b) By combining parts (ba) and (bb), we find that by Theorem 1.9.7, the GNS representation $(\pi_\phi, H_\phi, \xi_\phi)$ is reducible as required. □

One of the most important applications of the GNS construction is the following strong theorem:

Theorem 1.11.5. *Let A be a C*-algebra. Then, there exists a Hilbert space H and a C*-subalgebra $B \subset B(H)$ such that $A \cong B$ as C*-algebras.*

Put simply, every unital C*-algebra is isomorphic to a C*-algebra of operators, reinforcing the notion that the space of bounded linear operators $B(H)$ is, in this sense, the prototypical example of a C*-algebra.

In order to prove Theorem 1.11.5, we require the following preliminary result.

Theorem 1.11.6. *Let A be a C*-algebra and $a \in A$ be self-adjoint. Then, there exists an irreducible representation π of A such that $\|\pi(a)\| = \|a\|$.*

Proof. We will first prove Theorem 1.11.6 for unital C*-algebras. Assume that A is a unital C*-algebra and $a \in A$ is self-adjoint. The idea is to use the GNS construction to construct the required representation of A.

However, we first need a state on A to do this. Let B be the C*-subalgebra of A generated by the set $\{1_A, a\}$. Then, B is commutative and unital. Recall that $\mathcal{M}(B)$ is the set of non-zero $\mathbb{C}$-algebra homomorphisms from B to $\mathbb{C}$. By Theorem 2.1.8, $\mathcal{M}(B)$ is homeomorphic to the spectrum $\sigma(a)$, which is a compact subset of $\mathbb{R}$.

Since a is self-adjoint, its spectral radius $r(a)$ is equal to $\|a\|$ by Theorem 1.2.6. Now choose $\phi_0 \in \mathcal{M}(B)$ such that

$$|\phi_0(a)| = \sup_{x \in \sigma(a)} |x| = \|a\|.$$

Note that $\phi_0(1_A) = 1$. By the Hahn-Banach extension theorem, we obtain a linear functional ϕ on A which extends ϕ_0 and has $\|\phi\| = \|\phi_0\|$. Hence,

$$\phi(1_A) = \phi_0(1_A) = 1 = \|\phi_0\| = \|\phi\|.$$

By Theorem 1.11.1, ϕ is a state on A. Note that by construction of ϕ, $|\phi(a)| = |\phi_0(a)| = \|a\|$.

Now let $(\pi_\phi, H_\phi, \xi_\phi)$ be the GNS representation of ϕ. Since ξ_ϕ is a unit vector, we have the inequality

$$\|\pi_\phi(a)\| = \|\pi_\phi(a)\|\|\xi_\phi\|^2 \geq |\langle \pi_\phi(a)\xi_\phi, \xi_\phi\rangle| = |\phi(a)| = \|a\|.$$

Recall from the GNS construction that if $a, b \in A$ then π_ϕ is defined by

$$\pi_\phi(a)(b + N_\phi) = ab + N_\phi.$$

In particular, $\pi_\phi(1_A) = id_{H_\phi}$ (the identity operator on H_ϕ). So, π_ϕ is a unital *-homomorphism and by Theorem 1.2.7, π_ϕ is a contraction which means that $\|a\| \leq \|\pi_\phi(a)\|$. Therefore, $\|a\| = \|\pi_\phi(a)\|$.

It remains to show that the representation $(\pi_\phi, H_\phi, \xi_\phi)$. The idea is to use Theorem 1.11.4. Let S be the set of states ϕ on A satisfying $|\phi(a)| = |\phi_0(a)| = \|a\|$. Let $\mathcal{B}$ be the closed unit ball in the dual space A^*. Then, $S \subseteq \mathcal{B}$.

To show: (a) S is a convex set.

(b) S is closed with respect to the weak *-topology on A^*.

(a) Assume that $\alpha, \beta \in S$ and $t \in [0, 1]$. Then, the convex combination $t\alpha + (1-t)\beta$ is a positive linear functional on A which satisfies

$$t\alpha(1_A) + (1-t)\beta(1_A) = t + 1 - t = 1.$$

Hence, $t\alpha + (1-t)\beta$ is a state on A. Moreover,

$$\|a\| = t\|a\| + (1-t)\|a\| = t|\alpha(a)| + (1-t)|\beta(a)|.$$

Hence, $t\alpha + (1-t)\beta \in S$ and S is a convex set.

(b) Assume that $\{\alpha_n\}_{n \in \mathbb{Z}_{>0}}$ is a sequence in S which converges to some $\alpha \in A^*$ with respect to the weak *-topology on A^*. We want to show that $\alpha \in S$.

We find that

$$|\alpha(a)| = |\lim_{n\to\infty} \alpha_n(a)| = \lim_{n\to\infty} \|a\| = \|a\|.$$

To see that α is a state on A, note that

$$\alpha(1_A) = \lim_{n\to\infty} \alpha_n(1_A) = \lim_{n\to\infty} 1 = 1.$$

If $d \in A$ then

$$\alpha(d^*d) = \lim_{n\to\infty} \alpha_n(d^*d) \geq 0.$$

So, α is a state on A satisfying $|\alpha(a)| = \|a\|$. Therefore, $\alpha \in S$ and S is closed with respect to the weak *-topology.

Since the closed unit ball $\mathcal{B}$ is compact with respect to the weak *-topology on A (Banach-Alaoglu), S is also compact by part (b). Note also that S is non-empty because by construction, $\phi \in S$. Due to parts (a) and (b), we can apply the Krein-Milman theorem (see [Zim90, Section 2.3]) to deduce that S is the closure of the convex hull of the extreme points of S.

In particular, S has extreme points. So, we can choose $\phi \in S$ so that ϕ is one of the extreme points of S. It is straightforward to show from this that ϕ is extreme among the set of states — it cannot be written as a convex combination of states. By Theorem 1.11.4, we deduce that $(\pi_\phi, H_\phi, \xi_\phi)$ is an irreducible representation of A.

For the general case, assume that A is a C*-algebra and $a \in A$ is self-adjoint. Then, there exists an irreducible representation $(\pi_\phi, H_\phi, \xi_\phi)$ of the unitization $\tilde{A}$ such that $\|\pi_\phi(a)\| = \|a\|$. Since A is a closed two-sided ideal of $\tilde{A}$, the restriction $(\pi_\phi|_A, H_\phi)$ defines a representation of A with $\|\pi_\phi(a)\| = \|a\|$.

Note that the GNS representation $(\pi_\phi, H_\phi, \xi_\phi)$ of $\tilde{A}$ is non-degenerate because it is a cyclic representation. By Theorem 1.9.6, every non-zero vector of $\tilde{A}$ is cyclic. In particular, every non-zero vector of $A \subseteq \tilde{A}$ is cyclic.

Since the restricted representation $(\pi_\phi|_A, H_\phi)$ is also non-degenerate, then by Theorem 1.9.6, it is an irreducible representation as required. $\square$

Now we prove Theorem 1.11.5.

Proof of Theorem 1.11.5. Assume that A is a C*-algebra. Let

$$B = \{a \in A \,|\, \|a\| \leq 1\}$$

denote the closed unit ball in A. By Theorem 1.11.6, if $a \in B$ then there exists an irreducible representation (π_a, H_a) such that $\|\pi_a(a^*a)\| = \|a^*a\|$. Since $B(H_a)$ and A are both C*-algebras,

$$\|\pi_a(a)\| = \|\pi_a(a^*a)\|^{\frac{1}{2}} = \|a^*a\|^{\frac{1}{2}} = \|a\|.$$

Taking the direct sum over all $a \in B$, we obtain the *-homomorphism

$$\begin{array}{rccc} \bigoplus_{a\in B}\pi_a : & A & \to & \bigoplus_{a\in B} B(H_a) \\ & d & \mapsto & (\pi_a(d))_{a\in B} \end{array}$$

where on the RHS, we have a direct sum of C*-algebras. If $d \in A$ then

$$\begin{aligned} \|\big(\bigoplus_{a\in B}\pi_a\big)(d)\| &= \|(\pi_a(d))_{a\in B}\| \\ &= \sup_{a\in B}\|\pi_a(d)\| \\ &= \sup_{a\in B}\left(\|\pi_a(\frac{d}{\|d\|})\|\|d\|\right) \\ &\geq \|\pi_{\frac{d}{\|d\|}}(\frac{d}{\|d\|})\|\|d\| = \|d\|. \end{aligned}$$

Next, we will show that $\bigoplus_{a\in B}\pi_a$ is injective. Suppose for the sake of contradiction that there exists $k \in A - \{0\}$ such that $\big(\bigoplus_{a\in B}\pi_a\big)(k) = (0)_{a\in B}$. For clarity, $(0)_{a\in B} \in \bigoplus_{a\in B} B(H_a)$ is a sequence of zero operators. By the above inequality, we have

$$0 = \|\big(\bigoplus_{a\in B}\pi_a\big)(k)\| \geq \|k\|.$$

So, $k = 0$. This contradicts the assumption that k is non-zero. So, $\ker \bigoplus_{a\in B}\pi_a = \{0\}$ and $\bigoplus_{a\in B}\pi_a$ is an injective *-homomorphism.

By Theorem 1.6.4, $\bigoplus_{a\in B}\pi_a$ is an isometric *-homomorphism and an isometric *-isomorphism onto its image $\mathcal{I}$. By Theorem 1.7.6, $\mathcal{I}$ is a C*-subalgebra of $\bigoplus_{a\in B} B(H_a) \cong B(\bigoplus_{a\in B} H_a)$. This proves Theorem 1.11.5. $\square$

1.12 An introduction to von Neumann algebras

Recall Theorem 1.6.8, which states that if A is a commutative C*-algebra then there exists a locally compact Hausdorff space X such that $A \cong Cts_0(X, \mathbb{C})$ as C*-algebras. In this sense, the theory of C*-algebras can be thought of as a theory of "non-commutative topology".

Von Neumann algebras are a specific class of C*-algebras which happen to be the setting for "non-commutative measure theory". The reason for this is because abelian von Neumann algebras are, up to isomorphism, of the form $L^\infty(X, \mu)$, where (X, μ) is a measure space. This section serves as a brief introduction to the rich theory of von Neumann algebras, with [Mur90, Chapter 4] serving as the main reference. The main result we cover is the well-known *double commutant theorem*.

Let H be a Hilbert space. There are a wide variety of topologies on the space of bounded linear operators $B(H)$. The most important one for this section is the *strong operator topology*, which we will introduce in a more general context.

Definition 1.12.1. Let V and W be Banach spaces and $B(V, W)$ be the Banach space of bounded linear operators $T : V \to W$. The **strong operator topology** on $B(V, W)$ is the $\mathcal{F}$-weak topology, where $\mathcal{F}$ is the set

$$\mathcal{F} = \{ev_v : B(V, W) \to W \mid v \in V\}$$

and $ev_v(T) = T(v)$. That is, the strong operator topology on $B(V, W)$ is the weakest topology on $B(V, W)$ which makes the evaluation maps in $\mathcal{F}$ continuous.

Note that the ordinary norm topology on $B(V, W)$ also makes the evaluation maps in $\mathcal{F}$ continuous. Hence, the strong operator topology is weaker than the norm topology. Here is how convergence is expressed in the strong operator topology.

Theorem 1.12.1. *Let V and W be Banach spaces and $B(V, W)$ denote the Banach space of bounded linear operators $T : V \to W$. Then, a sequence A_n in $B(V, W)$ converges to an operator A in the strong operator topology if and only if for all $v \in V$, $A_n v \to Av$ in W.*

Proof. Assume that V and W are Banach spaces and $B(V, W)$ is the Banach space of bounded linear operators.

To show: (a) If a sequence $A_n \to A$ in the strong operator topology on $B(V, W)$ then for all $v \in V$, $A_n v \to Av$ in W.

(b) If for all $v \in V$, $A_n v \to Av$ in W then $A_n \to A$ in the strong operator topology on $B(V, W)$.

(a) Assume that $\{A_n\}_{n \in \mathbb{Z}_{>0}}$ is a sequence in $B(V, W)$ which converges to $A \in B(V, W)$ in the strong operator topology. Assume that $\epsilon \in \mathbb{R}_{>0}$. Then, there exists $N \in \mathbb{Z}_{>0}$ such that if $n > N$ then

$$A_n - A \in L_{v,0,\epsilon} = \{T \in B(V, W) \mid \|Tv\| < \epsilon\}.$$

So, $\|(A_n - A)v\| < \epsilon$ and consequently, $A_n v \to Av$ in W.

(b) Assume that if $v \in V$ then $A_n v \to Av$ in W. Assume that $\epsilon \in \mathbb{R}_{>0}$. Then, there exists $N \in \mathbb{Z}_{>0}$ such that if $n > N$ then $\|A_n v - Av\| < \epsilon$. This means that

$$A_n - A \in L_{v,0,\epsilon} = \{T \in B(V, W) \mid \|Tv\| < \epsilon\}.$$

Since $L_{v,0,\epsilon}$ is a basic open set in the space $B(V, W)$ with the strong operator topology, we deduce that $A_n \to A$ in the strong operator topology. $\square$

Using Theorem 1.12.1, we will provide an example illustrating the difference between the strong operator topology and the norm topology.

Example 1.12.1. Let $V = W = \ell^2(\mathbb{C})$. For $n \in \mathbb{Z}_{>0}$, define

$$\begin{array}{rccc} L_n : & \ell^2(\mathbb{C}) & \to & \ell^2(\mathbb{C}) \\ & (x_1, x_2, \dots) & \mapsto & (x_{n+1}, x_{n+2}, \dots) \end{array}$$

We will show that the sequence L_n converges to zero in the strong operator topology, but does not converge in the norm topology.

If $x = (x_1, x_2, \dots) \in \ell^2(\mathbb{C})$ then

$$\|L_n x\|^2 = \sum_{i=n+1}^{\infty} |x_i|^2 \to 0$$

as $n \to \infty$. Hence, $L_n x \to 0x$ and consequently, $L_n \to 0$ in the strong operator topology.

To see that L_n does not converge to zero in the norm topology, observe that

$$\begin{aligned}\|L_n\| &= \sup_{\|x\|=1} \|L_n x\| \\ &\geq \|L_n(0,0,\dots,1,0,\dots)\| \\ &= \|(1,0,0,\dots)\| = 1.\end{aligned}$$

Since $\|L_n\| \geq 1$ for $n \in \mathbb{Z}_{>0}$, L_n can never converge to zero in the norm topology.

If H is a Hilbert space then $B(H)$ is a topological vector space. So, the operations of addition and scalar multiplication are continuous with respect to the strong operator topology on $B(H)$. However, multiplication and involution are not in general strongly continuous.

Example 1.12.2. Let H be an infinite-dimensional Hilbert space with orthonormal basis $\{e_n\}_{n=1}^{\infty}$. If $n \in \mathbb{Z}_{>0}$ then let $u_n = |e_1\rangle\langle e_n|$. If $x \in H$ then

$$u_n(x) = \langle x, e_n\rangle e_1.$$

and

$$\lim_{n\to\infty} \|u_n(x)\| = \lim_{n\to\infty} |\langle x, e_n\rangle| = 0$$

because if we write $x = \sum_{i=1}^{\infty} \langle x, e_i\rangle e_i$ then all but finitely many of the coefficients $\langle x, e_i\rangle$ are zero.

Now, $u_n^* = |e_n\rangle\langle e_1|$ and if $x \in X$ then

$$\lim_{n\to\infty} \|u_n^*(x)\| = \lim_{n\to\infty} |\langle x, e_1\rangle|.$$

This clearly does not converge in the strong operator topology on $B(H)$ because if $x = e_1$ then $\lim_{n\to\infty}\|u_n^*(x)\| = 1$. Hence, the involution operation on $B(H)$ is not strongly continuous.

So far, we have mentioned that the strong operator topology on $B(H)$ behaves badly in some aspects. Next, we will discuss a useful feature of the strong operator topology. Let $B(H)_{sa}$ denote the set of self-adjoint operators on H. Then, $(B(H)_{sa}, \leq)$ is a poset, where $x \leq y$ if and only if $y - x \geq 0$ or alternatively, if and only if $y - x$ is a positive operator. This is elaborated on further in [Sol18, Section 3.1].

We will prove that if we have a sequence $\{u_i\}_{i\in I}$ of self-adjoint operators which is increasing (with respect to the partial order $\leq$) and bounded above then it must strongly converge.

Theorem 1.12.2. *Let H be a Hilbert space over $\mathbb{C}$ and $\{x_i\}_{i\in I}$ be a sequence of self-adjoint operators such that if $i \geq j$, then $x_i \geq x_j$. Assume that there exists $C \in \mathbb{R}_{>0}$ such that if $i \in I$ then $\|x_i\| \leq C$.*

Then, there exists a self-adjoint operator $x \in B(H)$ such that if $i \in I$ then $x \geq x_i$ and if $y \in B(H)$ is an operator satisfying $y \geq x_i$ for $i \in I$ then $y \geq x$. Moreover, $\{x_i\}_{i\in I}$ converges to x in the strong topology.

Proof. Assume that $\{x_i\}_{i\in I}$ is a sequence of self-adjoint operators satisfying the properties above. If $\xi \in H$ then the sequence $\{\langle \xi, x_i(\xi)\rangle\}_{i\in I}$ in $\mathbb{R}$ is bounded and non-decreasing. Hence, it must converge to its supremum:

$$\lim_{i\to\infty} \langle \xi, x_i(\xi)\rangle = \sup_{i\in I} \langle \xi, x_i(\xi)\rangle.$$

Now observe that by the polarization identity, the sequence $\{\langle \xi, x_i(\eta)\rangle\}_{i\in I}$ in $\mathbb{C}$ must also converge. To see why this is the case, write

$$\langle \xi, x_j(\eta)\rangle = \frac{1}{4}\sum_{k=0}^{3} i^k \langle \xi + i^k\eta, x_j(\xi + i^k\eta)\rangle$$

for $j \in I$ and then take the limit of both sides as $j \to \infty$. Since the RHS converges in the limit, the LHS must converge as well.

Now let $F(\xi,\eta) = \lim_{i\to\infty}\langle \xi, x_i(\eta)\rangle$. This is a sesquilinear form which is bounded because

$$|F(\xi,\eta)| = |\lim_{i\to\infty}\langle \xi, x_i(\eta)\rangle| = \lim_{i\to\infty}|\langle \xi, x_i(\eta)\rangle| \leq C\|\xi\|\|\eta\|.$$

For $\eta \in H$, define the map

$$\begin{array}{rccc} \phi_\xi : & H & \to & \mathbb{C} \\ & \eta & \mapsto & \overline{F(\xi,\eta)} \end{array}$$

This is a continuous/bounded linear functional. By the Riesz representation theorem, there exists unique $\tau \in H$ such that

$$\overline{F(\xi,\eta)} = \lim_{i\to\infty}\langle \xi, x_i(\eta)\rangle = \langle \xi, \tau\rangle.$$

Consequently, there exists a unique operator $x \in B(H)$ such that

$$F(\xi,\eta) = \langle \xi, x(\eta)\rangle = \lim_{i\to\infty}\langle \xi, x_i(\eta)\rangle$$

for $\xi, \eta \in H$. Notice that $F(\xi, \xi) \in \mathbb{R}$ since it is the limit of a sequence in $\mathbb{R}$. Hence, x must be a self-adjoint operator.

To see that $x \geq x_i$ for $i \in I$, we compute directly that

$$\langle \xi, x(\xi) \rangle = \lim_{j \to \infty} \langle \xi, x_j(\xi) \rangle = \sup_{j \in I} \langle \xi, x_j(\xi) \rangle \geq \langle \xi, x_i(\xi) \rangle.$$

Now assume that $y \in B(H)$ satisfies $y \geq x_i$ for $i \in I$. If $\xi \in H$ then

$$\langle \xi, y(\xi) \rangle \geq \sup_{i \in I} \langle \xi, x_i(\xi) \rangle = \langle \xi, x(\xi) \rangle.$$

So, $y \geq x$. Finally, to see that $\{x_i\}$ converges to x in the strong topology, we find that if $\xi \in H$ then

$$\begin{aligned}
\|x(\xi) - x_i(\xi)\|^2 &= \|(x - x_i)\xi\|^2 \\
&= \|(x - x_i)^{\frac{1}{2}}(x - x_i)^{\frac{1}{2}}\xi\|^2 \qquad \text{(since } x \geq x_i\text{)} \\
&\leq \|(x - x_i)^{\frac{1}{2}}\|^2 \|(x - x_i)^{\frac{1}{2}}\xi\|^2 \\
&= \|x - x_i\| \|(x - x_i)^{\frac{1}{2}}\xi\|^2 \qquad \text{(since } x - x_i \text{ is self-adjoint)} \\
&\leq (\|x\| + \|x_i\|) \|(x - x_i)^{\frac{1}{2}}\xi\|^2 \\
&\leq 2C \|(x - x_i)^{\frac{1}{2}}\xi\|^2 \\
&= 2C \langle (x - x_i)^{\frac{1}{2}}\xi, (x - x_i)^{\frac{1}{2}}\xi \rangle \\
&= 2C \langle \xi, (x - x_i)(\xi) \rangle \to 0
\end{aligned}$$

as $i \to \infty$. □

Note that by multiplying by -1, Theorem 1.12.2 also tells us that a non-increasing sequence of self-adjoint operators which is bounded below must be strongly convergent.

Now suppose that H is a Hilbert space and that $\{p_i\}_{i \in I}$ is a sequence of projections in $B(H)$, which strongly converges to an operator p. Firstly, p is self-adjoint by similar reasoning to Theorem 1.12.2. To see that p is idempotent, we compute directly that if $\xi, \eta \in H$ then

$$\begin{aligned}
\langle p\xi, \eta\rangle &= \langle \lim_{i\to\infty} p_i\xi, \eta\rangle \\
&= \lim_{i\to\infty} \langle p_i\xi, \eta\rangle \\
&= \lim_{i\to\infty} \langle p_i^2\xi, \eta\rangle \\
&= \lim_{i\to\infty} \langle p_i\xi, p_i\eta\rangle \\
&= \langle p\xi, p\eta\rangle = \langle p^2\xi, \eta\rangle.
\end{aligned}$$

Therefore, $p = p^2$ and p is a projection.

Recall that if V is a normed vector space then a sequence $\{x_n\}_{n\in\mathbb{Z}_{>0}}$ is summable if the sequence of partial sums $\{\sum_{n=1}^N x_n\}_{N\in\mathbb{Z}_{>0}}$ converges to some $x \in V$. We will extend this definition to locally convex spaces.

Definition 1.12.2. Let X be a locally convex space and $\{x_i\}_{i\in I}$ be a sequence in X. The sequence $\{x_i\}_{i\in I}$ is **summable** to a point $x \in X$ if the net $\{\sum_{i\in F} x_i\}_{F\subseteq I,\, F \text{ finite}}$ converges to x. We write $x = \sum_{i\in I} x_i$. Note that F runs over all non-empty finite subsets of I.

Here is a result regarding sequences of pairwise orthogonal projections.

Theorem 1.12.3. *Let H be a Hilbert space and $\{p_i\}_{i\in I}$ be a sequence of projections which are pairwise orthogonal. That is, if $i, j \in I$ are distinct then $p_ip_j = 0$. Then, $\{p_i\}_{i\in I}$ is summable to a projection p with respect to the strong operator topology on $B(H)$. Moreover, p satisfies for $\xi \in H$*

$$\|p\xi\| = (\sum_{i\in I} \|p_i\xi\|^2)^{\frac{1}{2}}$$

and if $p = id_H$ then the map

$$\begin{array}{ccc} H & \to & \bigoplus_{i\in I} p_i(H) \\ \xi & \mapsto & (p_i\xi)_{i\in I} \end{array}$$

is a unitary operator.

Proof. Assume that H is a Hilbert space and $\{p_i\}_{i\in I}$ is a sequence of projections which are pairwise orthogonal. If F is a non-empty finite subset of I then $\sum_{i\in F} p_i$ is itself a projection because the projections $\{p_i\}_{i\in I}$ are pairwise orthogonal.

So, the sequence $\{\sum_{i\in F} p_i\}_{F\subseteq I,\, F \text{ finite}}$ is increasing and bounded above. By Theorem 1.12.2, it must strongly converge to some projection p.

Alternatively, the sequence $\{p_i\}_{i\in I}$ is strongly summable. Moreover, if $\xi \in H$ then

$$\|p\xi\|^2 = \lim_F \|\sum_{i\in F} p_i\xi\|^2 = \lim_F \sum_{i\in F}\|p_i\xi\|^2 = \sum_{i\in I}\|p_i\xi\|^2.$$

The second equality follows from the fact that the projections $\{p_i\}_{i\in I}$ are pairwise orthogonal.

Now consider the case where $p = id_H$ (p is the identity operator on H). Define the map

$$\begin{array}{cccc} \Delta : & \bigoplus_{i\in I} p_i(H) & \to & H \\ & (p_i\xi)_{i\in I} & \mapsto & \sum_{i\in I} p_i\xi = \xi. \end{array}$$

The map Δ is bijective. To see that Δ is a unitary operator, we compute directly that if $\xi, \eta \in H$ then

$$\langle (p_i\xi)_{i\in I}, \Delta^*\eta\rangle = \langle \Delta\big((p_i\xi)_{i\in I}\big), \eta\rangle = \langle \xi, \eta\rangle$$

and

$$\langle (p_i\xi)_{i\in I}, \Delta^{-1}\eta\rangle = \langle (p_i\xi)_{i\in I}, (p_i\eta)_{i\in I}\rangle = \sum_{i\in I}\langle p_i\xi, p_i\eta\rangle = \sum_{i,j\in I}\langle p_i\xi, p_j\eta\rangle = \langle \xi, \eta\rangle.$$

So, $\Delta^* = \Delta^{-1}$ and consequently, Δ is a unitary operator as required. □

Here is the most important definition to this section.

Definition 1.12.3. Let A be a $\mathbb{C}$-algebra and C be a subset of A. The **commutant** of C, denoted by C', is defined as the set

$$C' = \{a \in A \mid \text{If } c \in C \text{ then } ac = ca\}.$$

Let us prove the following useful properties satisfied by the commutant.

Theorem 1.12.4. *Let A be a $\mathbb{C}$-algebra and C be a subset of A.*

1. *The commutant C' is a subalgebra of A.*
2. *$C \subseteq C''$.*
3. *$C' = C'''$.*
4. *If A is a normed algebra then the commutant C' is closed.*

5. *If A is a *-algebra and C is closed under involution then C' is a *-subalgebra of A.*

Proof. Assume that A is a $\mathbb{C}$-algebra and C is a subset of A. Assume that $a, b \in C'$ and $\lambda \in \mathbb{C}$. If $c \in C$ then $(\lambda a)c = a(\lambda c)$, $(a+b)c = c(a+b)$ and $(ab)c = acb = c(ab)$. Hence, C' is a subalgebra of A.

If $c \in C$ then c commutes with C' by assumption. Hence, $C \subseteq C''$. In particular, this means that $C' \subseteq C'''$. On the other hand, if $k \in C'''$ then k commutes with the double commutant C''. Since $C \subseteq C''$, k must commute with C. Hence, $k \in C'$ and $C''' \subseteq C'$. Subsequently, we have $C''' = C'$.

Now assume that A is a normed algebra. To see that C' is closed, assume that $\{c'_n\}_{n \in \mathbb{Z}_{>0}}$ is a sequence in C' which converges to some $c' \in A$. If $c \in C$ then

$$\begin{aligned}
\|cc' - c'c\| &= \|cc' - cc'_n + c'_n c - c'c\| \\
&\leq \|cc' - cc'_n\| + \|c'_n c - c'c\| \\
&\leq \|c\|\|c' - c'_n\| + \|c'_n - c'\|\|c\| \\
&< 2\|c\|(\frac{\epsilon}{2\|c\|}) = \epsilon.
\end{aligned}$$

Since $\epsilon \in \mathbb{R}_{>0}$ was arbitrary, we deduce that $cc' = c'c$ and $c' \in C'$. Hence, C' is closed.

Finally, assume that A is a *-algebra and that C is closed under involution. We know from the first part that A is a $\mathbb{C}$-subalgebra of A. If $c' \in C'$ and $c \in C$ then the adjoint $c^* \in C$ and

$$(c')^* c = (c')^*(c^*)^* = (c^* c')^* = (c' c^*)^* = c(c')^*.$$

Hence, C' is a *-subalgebra of A as required. □

As we will see shortly, the double commutant theorem is a consequence of the following theorem.

Theorem 1.12.5. *Let H be a Hilbert space and A a *-subalgebra of $B(H)$. Assume that $id_H \in A$. Then, A is strongly dense in its double commutant A''. That is, A is dense in A'' with respect to the strong operator topology on $B(H)$.*

Proof. Assume that H is a Hilbert space and A is a *-subalgebra of $B(H)$. Assume that $id_H \in A$. If $\xi \in H$ then define

$$K = \overline{\{v(\xi) \mid v \in A\}}.$$

Then, K is a closed vector subspace of H. Let $p \in B(H)$ denote the projection onto K.

To show: (a) $p \in A'$.

(a) Assume that $\eta \in H$. Then, $p\eta \in K$. Hence, there exists a sequence $\{v_n\xi\}_{n\in\mathbb{Z}_{>0}}$ in K which converges to $p\eta$. Now, if $w \in A$ and $n \in \mathbb{Z}_{>0}$ then $wv_n\xi \in K$. Since w is a bounded operator, the sequence $\{wv_n\xi\}_{n\in\mathbb{Z}_{>0}}$ converges to $wp\eta \in K$.

In particular, $wp\eta = pwp\eta$ and since $\eta \in H$ was arbitrary, we deduce that if $w \in A$ then $wp = pwp$.

Now if $w \in A$ and $\xi, \eta \in H$ then

$$\begin{aligned}
\langle wp\eta, \xi\rangle &= \langle pwp\eta, \xi\rangle \\
&= \langle \eta, pw^*p\xi\rangle \\
&= \langle \eta, w^*p\xi\rangle \quad \text{(since } A \text{ is a *-algebra)} \\
&= \langle pw\eta, \xi\rangle.
\end{aligned}$$

So, $p \in A'$.

Assume that $u \in A''$. Since $p \in A'$ by part (a), $pu = up$. Recall that by assumption, the identity operator $id_H \in A$. Hence, $\xi \in \{v\xi \mid v \in A\} \subseteq K$ and

$$u\xi = up\xi = pu\xi \in K.$$

Assume that $\epsilon \in \mathbb{R}_{>0}$. Since $u\xi \in K$, then there exists $v \in A$ such that $\|u\xi - v\xi\| < \epsilon$. This means that

$$v \in \{w \in B(H) \mid \|w\xi - u\xi\| < \epsilon\}.$$

We recognise that the set $\{w \in B(H) \mid \|w\xi - u\xi\| < \epsilon\}$ is a basic open set in the strong operator topology on $B(H)$. Consequently, every open set of $B(H)$ containing $u \in A''$ contains an element of A. Hence, A'' is the strong closure of A. $\square$

Now we introduce the notion of a von Neumann algebra.

Definition 1.12.4. Let H be a Hilbert space. A **von Neumann algebra** on H is a strongly closed *-subalgebra of $B(H)$.

Notice that if A is a von Neumann algebra then A is strongly closed and hence, closed with respect to the norm topology. Hence, A is also a C*-algebra.

One example of a von Neumann algebra is $B(H)$ for a Hilbert space H. We will give another example from [Put19, Example 1.13.3].

Example 1.12.3. Let (X, μ) be a measure space. The space of essentially bounded functions $L^\infty(X, \mu)$ acts on the Hilbert space $L^2(X, \mu)$ as multiplication operators (see [Sol18, Section 4.1]). It turns out that $L^\infty(X, \mu)$ is a commutative von Neumann algebra.

A useful result which gives us some more examples of von Neumann algebras is that the commutant of a *-algebra on a Hilbert space is a von Neumann algebra.

Theorem 1.12.6. *Let H be a Hilbert space and A be a *-subalgebra of $B(H)$. Then, the commutant A' is a von Neumann algebra.*

Proof. Assume that H is a Hilbert space and A is a *-subalgebra of $B(H)$. By Theorem 1.12.4, the commutant A' is a *-subalgebra of $B(H)$.

To show: (a) A' is strongly closed.

(a) Assume that $\{a'_n\}_{n \in \mathbb{Z}_{>0}}$ is a sequence in A' which strongly converges to some $a' \in B(H)$. To see that $a' \in A'$, assume that $a \in A$ and $\epsilon \in \mathbb{R}_{>0}$. Then, there exists $N \in \mathbb{Z}_{>0}$ such that if $n > N$ and $\xi \in H$ then

$$\|a'_n(\xi) - a'(\xi)\| < \frac{\epsilon}{2\|a\|}.$$

By taking the supremum over all $\xi \in H$ with $\|\xi\| = 1$, we obtain the inequality $\|a'_n - a'\| < \frac{\epsilon}{2\|a\|}$. Hence,

$$\begin{aligned}
\|a'a - aa'\| &= \|a'a - a'_n a + aa'_n - aa'\| \\
&\leq 2\|a\|\|a' - a'_n\| \\
&< 2\|a\|(\frac{\epsilon}{2\|a\|}) = \epsilon.
\end{aligned}$$

Since $\epsilon \in \mathbb{R}_{>0}$ was arbitrary, we deduce that $a'a = aa'$ and $a' \in A'$ as required.

Since A' is strongly closed by part (a), we deduce that A' is a von Neumann algebra. $\square$

Now we will state and prove the important double commutant theorem, attributed to von Neumann himself.

Theorem 1.12.7 (Double commutant theorem)**.** *Let H be a Hilbert space and A be a *-subalgebra of $B(H)$. Assume that the identity operator $id_H \in A$. Then, A is a von Neumann algebra on H if and only if $A = A''$.*

Proof. Assume that H is a Hilbert space and A is a *-subalgebra of $B(H)$. Assume that the identity operator $id_H \in A$. If A is a von Neumann algebra then it is strongly closed. By Theorem 1.12.5, we have $A = \overline{A} = A''$.

On the other hand, if $A = A''$ then by Theorem 1.12.5, $\overline{A} = A'' = A$. So, A is strongly closed and is therefore, a von Neumann algebra. $\square$

Chapter 2

Topics from [Mur90]

2.1 The Gelfand representation for abelian Banach algebras

In this section, we follow [Mur90, Section 1.3]. The goal of this section is to make precise the *Gelfand representation*, a way to represent an abelian Banach algebra as an algebra of continuous functions on an appropriate LCH (locally compact Hausdorff) space.

Note that we already accomplished this for commutative C*-algebras, as seen from Theorem 1.3.5 and Theorem 1.6.7. Thus, we will recycle results from section 1.3 if their proofs also work for abelian Banach algebras.

We will commence by proving a few preliminary results.

Definition 2.1.1. Let A be a $\mathbb{C}$-algebra. An ideal $I \subseteq A$ is called **modular** if there exists an element $u \in A$ such that if $a \in A$ then

$$a - au \in I \quad \text{and} \quad a - ua \in I.$$

Theorem 2.1.1. *Let A be a Banach algebra and I be a modular ideal. If I is a proper ideal then its closure $\overline{I}$ is also a proper ideal. If I is a maximal ideal then it is closed (topologically).*

Proof. Assume that A is a Banach algebra and I is a modular ideal. Then, there exists $u \in A$ such that if $a \in A$ then $a - au \in I$ and $a - ua \in I$. First, assume that I s a proper ideal of A.

To show: (a) If $b \in I$ then $\|u - b\| \geq 1$.

(a) Suppose for the sake of contradiction that there exists $b \in I$ such that $\|u - b\| < 1$. Consider the unitization $\tilde{A}$ of A (which is a Banach algebra by the construction in Theorem 1.6.1) and let $v = 1_{\tilde{A}} - u + b$.

We claim that $v \in \tilde{A}$ is invertible. Consider the sum

$$\sum_{n=0}^{\infty} \|(u-b)^n\|.$$

Note that $(u-b)^0 = 1_{\tilde{A}}$. This is a convergent sum in $\mathbb{R}$ because $\|u - b\| < 1$. By Theorem 1.7.1, the sequence $\{(u-b)^n\}_{n=0}^{\infty}$ in $\tilde{A}$ is summable. Thus, the sequence $\{\sum_{n=0}^{N}(u-b)^n\}_{N=1}^{\infty}$ converges to an element of $\tilde{A}$. We call this element $\sum_{n=0}^{\infty}(u-b)^n$.

To see that $\sum_{n=0}^{\infty}(u-b)^n$ is the inverse of v, we compute directly that

$$\begin{aligned} v\Big(\sum_{n=0}^{\infty}(u-b)^n\Big) &= \lim_{N\to\infty} v\Big(\sum_{n=0}^{N}(u-b)^n\Big) \\ &= \lim_{N\to\infty} (1_{\tilde{A}} - (u-b))\Big(\sum_{n=0}^{N}(u-b)^n\Big) \\ &= \lim_{N\to\infty} (1_{\tilde{A}} - (u-b)^{N+1}) = 1_{\tilde{A}}. \end{aligned}$$

Similarly, $\big(\sum_{n=0}^{\infty}(u-b)^n\big)v = 1_{\tilde{A}}$. Hence, $v^{-1} = \big(\sum_{n=0}^{\infty}(u-b)^n\big)$ and v is invertible in $\tilde{A}$.

By Theorem 1.6.1, A is a closed two-sided ideal of $\tilde{A}$. So, $Av \subseteq A$. Since v is invertible in $\tilde{A}$, $A = Av^{-1}v \subseteq Av$. Thus, $A = Av$. Now if $a \in A$ then

$$av = (a - au) + ab \in I.$$

Thus, $A = Av \subseteq I$. However, this contradicts the assumption that I is a proper ideal of A. We conclude that if $b \in I$ then $\|u - b\| \geq 1$.

Now define

$$B(u, \frac{1}{2}) = \{a \in A \mid \|u - a\| < \frac{1}{2}\}.$$

By part (a), $B(u, \frac{1}{2}) \cap I = \emptyset$. Hence, $u \in A$ but $u \notin \overline{I}$. Therefore, $\tilde{I}$ is a proper ideal of A.

Finally, if I is a maximal ideal of A then $\tilde{I}$ is a proper ideal of A containing I. So, $I = \overline{I}$ and I is closed. $\square$

We briefly remark that if L is a left ideal of a Banach algebra A then it is called modular if there exists $u \in A$ such that if $a \in A$ then $a - au \in L$. Theorem 2.1.1 carries over for modular left ideals.

Theorem 2.1.2. *Let A be a commutative Banach algebra and $I \subseteq A$ be a modular maximal ideal of A. Then, A/I is a field.*

Proof. Assume that A is a commutative Banach algebra. Assume that I is a modular maximal ideal of A. Let $u \in A$ be such that if $a \in A$ then $a - au \in I$ and $a - ua \in I$. Then, I is closed by Theorem 2.1.1. Thus, we can form the quotient Banach algebra A/I. It is abelian and unital, with multiplicative unit $u + I \in A/I$.

Let $\pi : A \to A/I$ be the canonical projection map. Then, π is a $\mathbb{C}$-algebra homomorphism. So, if J is an ideal of A/I then by the correspondence principle, the preimage $\pi^{-1}(J)$ is an ideal of A containing I. Since I is a maximal ideal, then either $\pi^{-1}(J) = I$ or $\pi^{-1}(J) = A$. Therefore, either $J = A/I$ or $J = 0$.

We conclude that A/I and 0 are the only ideals of A/I. So, A/I is a field. $\square$

Now recall from section 1.3 that if A is a $\mathbb{C}$-algebra then $\mathcal{M}(A)$ is the set of non-zero $\mathbb{C}$-algebra homomorphisms from A to $\mathbb{C}$. In [Mur90], the elements of $\mathcal{M}(A)$ are referred to as *characters*.

By Theorem 1.3.1, if A is a commutative unital C*-algebra then every element of $\mathcal{M}(A)$ has norm 1. A close examination of the proof of Theorem 1.3.1 shows that this conclusion also extends to the case where A is a commutative unital Banach algebra.

A relevant fact about $\mathcal{M}(A)$ which we have not proved yet is a bijection from $\mathcal{M}(A)$ to the set of maximal ideals of A.

Theorem 2.1.3. *Let A be a unital abelian Banach algebra. Then, there is a bijection of sets*

$$\begin{array}{rccc} K: & \mathcal{M}(A) & \to & \{\textit{Maximal ideals of } A\} \\ & \tau & \mapsto & \ker \tau \end{array}$$

Moreover, $\mathcal{M}(A) \neq \emptyset$.

Proof. Assume that A is a unital abelian Banach algebra. Assume that K is the function of sets defined as above. First, we will show that K is well-defined.

Assume that $\tau \in \mathcal{M}(A)$ so that $K(A) = \ker \tau$. Then, $\ker \tau$ is a closed ideal of A. Since $\tau \in \mathcal{M}(A)$, τ is not the zero map. So, $\ker \tau$ is a proper ideal of A. Now if $a \in A$ and 1_A is the unit of A then

$$\tau(a - \tau(a)1_A) = \tau(a) - \tau(a)\tau(1_A) = \tau(a) - \tau(a) = 0.$$

We conclude that if $a \in A$ then $a - \tau(a)1_A \in \ker \tau$. This means that $\ker \tau + \mathbb{C}1_A = A$ as ideals. To see that $\ker \tau$ is a maximal ideal of A, suppose for the sake of contradiction that I is a proper ideal of A which contains $\ker \tau$. We claim that $I \subseteq \ker \tau$. Assume that $\alpha \in I$. Since $\ker \tau + \mathbb{C}1_A = A$, there exists $\lambda \in \mathbb{C}$ and $\kappa \in \ker \tau$ such that

$$\alpha = \kappa + \lambda 1_A.$$

Now $\lambda 1_A = \alpha - \kappa \in I$. Hence, $1_A \in I$ and $I = A$. This contradicts the assumption that I is proper. Hence, $\ker \tau$ is a maximal ideal and the map K is well-defined.

To show: (a) K is injective.

(b) K is surjective.

(a) Assume that $\tau_1, \tau_2 \in \mathcal{M}(A)$ such that $K(\tau_1) = K(\tau_2)$. Then, $\ker \tau_1 = \ker \tau_2$. Note that if $a \in A$ then $a - \tau_2(a)1_A \in \ker \tau_2$ and

$$\tau_1(a - \tau_2(a)1_A) = \tau_1(a) - \tau_2(a) = 0.$$

So, $\tau_1 = \tau_2$ and K is injective.

(b) Assume that I is a maximal ideal of A. Since A is unital, I is also a modular ideal. By Theorem 2.1.1, I is a closed (two-sided) ideal. This means that we can form the quotient Banach algebra A/I. By Theorem 2.1.2, A/I is a field with unit $1_A + I$.

In particular, A/I is a unital Banach algebra where every non-zero element is invertible. Now we claim that $A/I = \mathbb{C}(1_A + I)$. Suppose for the sake of contradiction that $A/I \neq \mathbb{C}(1_A + I)$. Then, there exists $a + I \in A/I$ such that $a + I \notin \mathbb{C}(1_A + I)$. In particular, $a + I$ is non-zero. If $\lambda \in \mathbb{C}$ then

$(\lambda 1_A - a) + I$ is a non-zero element of A/I and is thus invertible. Consequently, the spectrum $\sigma(a+I) = \emptyset$ which contradicts the fact that $\sigma(a+I)$ must be non-empty (because $a+I$ is non-zero and A/I is a unital Banach algebra).

Hence, $A/I = \mathbb{C}(1_A + I)$ and $A = I \oplus \mathbb{C}1_A$. Now define the map

$$\begin{array}{rccc} \phi: & A = I \oplus \mathbb{C}1_A & \to & \mathbb{C} \\ & a + \lambda 1_A & \mapsto & \lambda. \end{array}$$

Then, ϕ is a non-zero $\mathbb{C}$-algebra homomorphism such that $K(\phi) = \ker \phi = I$. So, K is surjective.

By parts (a) and (b) of the proof, we find that K is a bijection. Note that since A is unital, it has maximal ideals. By the bijection K, $\mathcal{M}(A) \neq \emptyset$. $\square$

Now we will use Theorem 2.1.3 to generalise part 1 of Theorem 1.3.1.

Theorem 2.1.4. *Let A be an abelian Banach algebra and $a \in A$. If A is unital then*

$$\sigma(a) = \{\tau(a) \mid \tau \in \mathcal{M}(A)\}.$$

Furthermore, if A is non-unital then

$$\sigma(a) = \{\tau(a) \mid \tau \in \mathcal{M}(A)\} \cup \{0\}.$$

Proof. Assume that A is an abelian Banach algebra and $a \in A$. First, assume that A is unital with multiplicative unit 1_A. By the proof of part 1 of Theorem 1.3.1, we have the inclusion

$$\{\tau(a) \mid \tau \in \mathcal{M}(A)\} \subseteq \sigma(a).$$

For the reverse inclusion, assume that $\lambda \in \sigma(a)$. Then, $\lambda 1_A - a$ is not invertible in A and the ideal $I = (\lambda 1_A - a)A$ is a proper ideal of A. Hence, I must be contained in a maximal ideal I_{max}. By the bijection established in Theorem 2.1.3, $I_{max} = \ker \phi$ for some $\phi \in \mathcal{M}(A)$.

Now observe that $\phi(\lambda 1_A - a) = 0$. We find that $\phi(a) = \lambda$. Hence, $\sigma(a) \subseteq \{\tau(a) \mid \tau \in \mathcal{M}(A)\}$ and subsequently,

$$\sigma(a) = \{\tau(a) \mid \tau \in \mathcal{M}(A)\}.$$

For the more general case, assume that A is non-unital. Let $\sigma_{\tilde{A}}(a)$ denote the spectrum of $a \in \tilde{A}$, where $\tilde{A}$ is the unitization of A. By the previous case, we have

$$\sigma_{\tilde{A}}(a) = \{\tau(a) \mid \tau \in \mathcal{M}(\tilde{A})\}.$$

By the universal property of unitization in Theorem 1.6.3, we have

$$\mathcal{M}(\tilde{A}) = \{\tilde{\tau} \mid \tau \in \mathcal{M}(A)\} \cup \{\tau_\infty\}$$

where $\tilde{\tau}$ is the unique homomorphism extending τ from the universal property and $\tau_\infty((\lambda, b)) = \lambda$ for $\lambda \in \mathbb{C}$ and $b \in A$. Since $a \in A$, we find that

$$\sigma(a) = \sigma_{\tilde{A}}(a) = \{\tau(a) \mid \tau \in \mathcal{M}(\tilde{A})\} = \{\beta(a) \mid \beta \in \mathcal{M}(A)\} \cup \{0\}$$

because $\tau_\infty((0, a)) = 0$ as required. □

By Theorem 2.1.4, if A is an abelian Banach algebra then $\mathcal{M}(A)$ is contained in the closed unit ball of the dual space A^*. This is analogous to Theorem 1.3.2. The space $\mathcal{M}(A)$ with the weak *-topology is called the *character space* of A or the *spectrum* of A.

The proof of Theorem 1.3.2 readily extends to the case where A is an abelian *unital* Banach algebra. Here is what happens in the non-unital case.

Theorem 2.1.5. *Let A be an abelian Banach algebra. Then, $\mathcal{M}(A)$ is a locally compact Hausdorff space when equipped with the weak *-topology from A^*.*

Proof. Assume that A is an abelian Banach algebra. If $\tau \in \mathcal{M}(A)$ then $\|\tau\| \leq 1$ by the proof of Theorem 1.3.1. In this case, $\mathcal{M}(A)$ is still contained in the closed unit ball of A^*.

By the same argument in Theorem 1.3.2, $\mathcal{M}(A) \cup \{0\}$ is a compact subset of A^* with respect to the weak *-topology. By the one point compactification, $\mathcal{M}(A)$ is a locally compact Hausdorff space as required. □

If A is an abelian Banach algebra and $a \in A$ then we can again define the evaluation map

$$\begin{array}{rccc} ev_a : & \mathcal{M}(A) & \to & \mathbb{C} \\ & \tau & \mapsto & \tau(a) \end{array}$$

By definition of the weak *-topology on $\mathcal{M}(A)$, the evaluation map ev_a is continuous. We claim that $ev_a \in Cts_0(\mathcal{M}(A), \mathbb{C})$. That is, the evaluation map on a vanishes at infinity.

Assume that $\epsilon \in \mathbb{R}_{>0}$. We want to show that the set

$$\mathcal{M}(A)_\epsilon = \{\tau \in \mathcal{M}(A) \mid |\tau(a)| \geq \epsilon\}$$

is compact with respect to the weak *-topology on $\mathcal{M}(A)$ and A^*. By the Banach-Alaoglu theorem, it suffices to show that $\mathcal{M}(A)_\epsilon$ is closed. To this end, assume that $\{\tau_n\}_{n\in\mathbb{Z}_{>0}}$ is a sequence in $\mathcal{M}(A)_\epsilon$ which converges to $\tau \in \mathcal{M}(A)$ in the weak *-topology. Then, $\tau_n(a) \to \tau(a)$ in $\mathbb{C}$ as $n \to \infty$. To see that $|\tau(a)| \geq \epsilon$, note that if $m \in \mathbb{Z}_{>0}$ then there exists $N_m \in \mathbb{Z}_{>0}$ such that if $n > N_m$ then

$$|\tau_n(a) - \tau(a)| < \frac{1}{2^m}.$$

So,

$$|\tau(a)| \geq \big||\tau(a) - \tau_n(a)| - |-\tau_n(a)|\big| > \epsilon - \frac{1}{2^m}.$$

Since $m \in \mathbb{Z}_{>0}$ was arbitrary, we deduce that $|\tau(a)| \geq \epsilon$. Hence, $\mathcal{M}(A)_\epsilon$ is closed with respect to the weak *-topology and the evaluation map $ev_a \in Cts_0(\mathcal{M}(A), \mathbb{C})$.

The evaluation map ev_a is called the **Gelfand transform** of a. Now we are able to extend Theorem 1.3.5 and prove the *Gelfand representation.*

Theorem 2.1.6 (Gelfand representation)**.** *Let A be an abelian Banach algebra and assume that $\mathcal{M}(A)$ is non-empty. Define the map*

$$\begin{array}{rccc} \Lambda: & A & \to & Cts_0(\mathcal{M}(A), \mathbb{C}) \\ & a & \mapsto & ev_a. \end{array}$$

Then, Λ is a norm-decreasing $\mathbb{C}$-algebra homomorphism (and hence, a homomorphism of Banach algebras) with $r(a) = \|ev_a\|_\infty$. Furthermore, if A is unital then $ev_a(\mathcal{M}(A)) = \sigma(a)$ and if A is non-unital then $\sigma(a) = ev_a(\mathcal{M}(A)) \cup \{0\}$.

Proof. Assume that A is an abelian Banach algebra and that $\mathcal{M}(A)$ is non-empty. Assume that Λ is the map defined as above. By Theorem 2.1.4, we have

$$ev_a(\mathcal{M}(A)) = \sigma(a)$$

if A is unital and $ev_a(\mathcal{M}(A)) \cup \{0\} = \sigma(a)$ if A is non-unital. Recalling the definition of the spectral radius, we have for $a \in A$,

$$\|ev_a\|_\infty = \sup_{\|\tau\|=1} |\tau(a)| = \sup_{\lambda \in \sigma(a)} |\lambda| = r(a).$$

This means that

$$\|\Lambda(a)\| = \|ev_a\|_\infty = r(a) \leq \|a\|$$

and consequently, $\|\Lambda\| \leq 1$. The fact that Λ is a $\mathbb{C}$-algebra homomorphism follows from direct computation. $\square$

The Gelfand representation Λ in Theorem 2.1.6 allows us to define the notion of a radical.

Definition 2.1.2. Let A be an abelian Banach algebra. Let Λ be the Gelfand representation in Theorem 2.1.6. The **radical** of A, denoted by $Rad(A)$, is defined as

$$Rad(A) = \ker \Lambda = \{a \in A \mid \Lambda(a) = ev_a = 0\}.$$

The abelian Banach algebra A is said to be **semisimple** if the radical $Rad(A) = \ker \Lambda = \{0\}$.

One consequence of the Gelfand representation in Theorem 2.1.6 is that we can prove further properties about the spectral radius.

Theorem 2.1.7. *Let A be a Banach algebra. Let $a, b \in A$ satisfy $ab = ba$. Then, $r(a + b) \leq r(a) + r(b)$ and $r(ab) \leq r(a)r(b)$.*

Proof. Assume that A is a Banach algebra. Assume that $a, b \in A$ such that $ab = ba$. Without loss of generality, we may assume that A is abelian and unital (if not, pass to the unitization $\tilde{A}$ and consider the commutative Banach subalgebra generated by the set $\{1_{\tilde{A}}, a, b\}$).

By the Gelfand representation in Theorem 2.1.6, we have

$$r(a + b) = \|ev_{a+b}\|_\infty = \|ev_a + ev_b\|_\infty \leq \|ev_a\|_\infty + \|ev_b\|_\infty = r(a) + r(b).$$

and

$$r(ab) = \|ev_{ab}\|_\infty = \|ev_a ev_b\|_\infty \leq \|ev_a\|_\infty \|ev_b\|_\infty = r(a)r(b).$$

$\square$

It is mentioned in [Mur90] that proofs of Theorem 2.1.6 which do not use the Gelfand representation are much more complicated that the proof we gave. In fact, in [Sol18, Section 1.2], the inequality $r(ab) \leq r(a)r(b)$ is mentioned as one of the properties of spectral radius, but is not proved.

Returning to Theorem 2.1.7, we note that if $a, b \in A$ do not commute then the inequalities do not necessarily hold. As an example, let $A = M_{2\times 2}(\mathbb{C})$,

$$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad b = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then, $r(a) = r(b) = 0$. But, $r(a+b) = 1$ and $r(ab) = 1$. Therefore, Theorem 2.1.7 does not hold in this general case.

Now we will state and prove the appropriate generalisation of Theorem 2.1.8.

Theorem 2.1.8. *Let A be a unital Banach algebra generated by the set $\{1_A, a\}$ where $a \in A$. Then, A is abelian and the evaluation map $ev_a \in Cts_0(\mathcal{M}(A), \mathbb{C})$ is a homeomorphism from $\mathcal{M}(A)$ to the spectrum $\sigma(a)$.*

Proof. Assume that A is a unital Banach algebra generated by the set $\{1_A, a\}$ for some $a \in A$. Since 1_A and a commute with each other, A must be abelian. By Theorem 2.1.4, the evaluation map ev_a is a bijection from the compact space $\mathcal{M}(A)$ to the compact Hausdorff space $\sigma(a) \subseteq \mathbb{C}$. Hence, ev_a is a homeomorphism. $\square$

We end this section with a particular application of the material, taken from [Mur90, Example 1.3.1].

Example 2.1.1. Let $\ell^1(\mathbb{Z})$ denote the set

$$\ell^1(\mathbb{Z}) = \{f : \mathbb{Z} \to \mathbb{C} \mid \sum_{n=-\infty}^{\infty} |f(n)| < \infty\}.$$

We know that $\ell^1(\mathbb{Z})$ is a Banach space, with scalar multiplication and addition defined pointwise. The norm of $\ell^1(\mathbb{Z})$ is given by

$$\|f\|_1 = \sum_{n=-\infty}^{\infty} |f(n)|.$$

If $f, g \in \ell^1(\mathbb{Z})$ and $m \in \mathbb{Z}$ then we define the *convolution* of f and g by

$$(f * g)(m) = \sum_{n=-\infty}^{\infty} f(m-n)g(n).$$

We claim that $f * g \in \ell^1(\mathbb{Z})$. A quick computation reveals that

$$\begin{aligned}
\|f * g\|_1 &= \sum_{m=-\infty}^{\infty} |(f * g)(m)| \\
&= \sum_{m=-\infty}^{\infty} |\sum_{n=-\infty}^{\infty} f(m-n)g(n)| \\
&\leq \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} |f(m-n)g(n)| \\
&= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} |f(m-n)g(n)| \\
&= \sum_{n=-\infty}^{\infty} |g(n)| \sum_{m=-\infty}^{\infty} |f(m-n)| = \|f\|_1 \|g\|_1 < \infty.
\end{aligned}$$

Hence, $f * g \in \ell^1(\mathbb{Z})$.

Next, we claim that $\ell^1(\mathbb{Z})$ is an abelian unital Banach algebra with multiplication defined by convolution. The established bound $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$ shows that multiplication is continuous in $\ell^1(\mathbb{Z})$. Let $\chi_{\{0\}}$ denote the characteristic function on the set $\{0\}$. Then, $\chi_{\{0\}} \in \ell^1(\mathbb{Z})$ and

$$\begin{aligned}
(f * \chi_{\{0\}})(m) &= \sum_{n=-\infty}^{\infty} f(m-n)\chi_{\{0\}}(n) \\
&= f(m-0) = f(m).
\end{aligned}$$

Similarly, if $m \in \mathbb{Z}$ then $(\chi_{\{0\}} * f)(m) = f(m)$. So, $\chi_{\{0\}}$ is the multiplicative unit for $\ell^1(\mathbb{Z})$.

Finally, to see that $\ell^1(\mathbb{Z})$ is abelian, we compute directly that if $f, g \in \ell^1(\mathbb{Z})$ and $m \in \mathbb{Z}$ then

$$\begin{aligned}
(f * g)(m) &= \sum_{n=-\infty}^{\infty} f(m-n)g(n) \\
&= \sum_{m-n=-\infty}^{\infty} f(m-(m-n))g(m-n) \\
&= \sum_{n=-\infty}^{\infty} f(m-n)g(n) = (g * f)(m).
\end{aligned}$$

So, $\ell^1(\mathbb{Z})$ is an abelian unital Banach algebra. Hence, we can look at its Gelfand representation in Theorem 2.1.6. First, observe that if $f \in \ell^1(\mathbb{Z})$ then

$$f = \sum_{n=-\infty}^{\infty} f(n)(\chi_{\{1\}})^n.$$

where $(\chi_{\{1\}})^n$ is the convolution of $\chi_{\{1\}}$ with itself n times for n positive. In fact, one can show that if $n \in \mathbb{Z}$ then $(\chi_{\{1\}})^n = \chi_{\{n\}}$. Let $U = \{z \in \mathbb{C} \mid |z| = 1\}$ and $z \in U$. We define τ_z by

$$\begin{array}{rccc}
\tau_z : & \ell^1(\mathbb{Z}) & \to & \mathbb{C} \\
 & f & \mapsto & \sum_{n=-\infty}^{\infty} f(n)z^n.
\end{array}$$

Note that $\tau_z \in \mathcal{M}(\ell^1(\mathbb{Z}))$. In particular, if $f, g \in \ell^1(\mathbb{Z})$ and $z \in U$ then

$$\begin{aligned}
\tau_z(f * g) &= \sum_{n=-\infty}^{\infty} (f * g)(n)z^n \\
&= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} f(n-m)g(m)z^n \\
&= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} f(n)g(m)z^{m+n} \\
&= \big(\sum_{n=-\infty}^{\infty} f(n)z^n \big) \big(\sum_{m=-\infty}^{\infty} g(m)z^m \big) \\
&= \tau_z(f)\tau_z(g)
\end{aligned}$$

Thus, we have a map given by

$$\begin{array}{rccc}
\Gamma : & U & \to & \mathcal{M}(\ell^1(\mathbb{Z})) \\
 & z & \mapsto & \tau_z.
\end{array}$$

Next, we claim that Γ is a bijection. To see that Γ is injective, assume that $z_1, z_2 \in U$ such that $\tau_{z_1} = \tau_{z_2}$. If $f \in \ell^1(\mathbb{Z})$ then

$$\sum_{n=-\infty}^{\infty} f(n)z_1^n = \sum_{n=-\infty}^{\infty} f(n)z_1^n.$$

In particular, if $f = \chi_{\{1\}}$ then $z_1 = z_2$. So, Γ is injective.

To see that Γ is surjective, assume that $\phi \in \mathcal{M}(\ell^1(\mathbb{Z}))$. First, observe that if $f = \sum_{n=-\infty}^{\infty} f(n)(\chi_{\{1\}})^n \in \ell^1(\mathbb{Z})$ then

$$\phi(f) = \phi(\sum_{n=-\infty}^{\infty} f(n)(\chi_{\{1\}})^n) = \sum_{n=-\infty}^{\infty} f(n)\phi(\chi_{\{1\}}^n) = \sum_{n=-\infty}^{\infty} f(n)\phi(\chi_{\{1\}})^n.$$

Since $\phi \in \mathcal{M}(\ell^1(\mathbb{Z}))$, then $\|\phi\| = 1$ and $|\phi(\chi_{\{1\}})| \leq 1$. We also have

$$1 = |\phi(\chi_{\{0\}})| = |\phi(\chi_{\{1\}})\phi(\chi_{\{1\}})^{-1}| = |\phi(\chi_{\{1\}})||\phi(\chi_{\{-1\}})|.$$

Since $|\phi(\chi_{\{-1\}})| \leq 1$, then $|\phi(\chi_{\{1\}})| \geq 1$ and consequently, $|\phi(\chi_{\{1\}})| = 1$. So, $\phi(\chi_{\{1\}}) \in U$ and

$$\phi = \tau_{\phi(\chi_{\{1\}})} = \Gamma(\phi(\chi_{\{1\}})).$$

Thus, Γ is surjective and hence, a bijection.

Let us now take a step further and prove that Γ is in fact, a homeomorphism. Since U is a compact subset of $\mathbb{C}$ and $\mathcal{M}(\ell^1(\mathbb{Z}))$ is a Hausdorff space, it suffices to show that Γ is continuous.

To this end, it suffices to show that if $f \in \ell^1(\mathbb{Z})$ then the composite $ev_f \circ \Gamma : U \to \mathbb{C}$ is continuous. This is straightforward to see because if $z \in U$ then $(ev_f \circ \Gamma)(z) = \tau_z(f)$ is the uniform limit of the sequence of continuous functions of z

$$\Big\{ \sum_{|n| \leq N} f(n)z^n \Big\}_{N \in \mathbb{Z}_{>0}}.$$

In turn, this holds because the sum $\sum_{n=-\infty}^{\infty} |f(n)z^n| = \|f\|_1 < \infty$. So, Γ is a homeomorphism and U can be identified with $\mathcal{M}(\ell^1(\mathbb{Z}))$ as topological spaces.

If $f \in \ell^1(\mathbb{Z})$ then the Gelfand transform of f is the evaluaton map $ev_f : \mathcal{M}(\ell^1(\mathbb{Z})) \to \mathbb{C}$. By the homeomorphism Γ, the Gelfand transform of f is a continuous function $\widehat{f} : U \to \mathbb{C}$ such that

$$\widehat{f}(z) = (ev_f \circ \Gamma)(z) = \tau_z(f) = \sum_{n=-\infty}^{\infty} f(n)z^n.$$

We recognise that $\widehat{f}$ is similar to a Fourier transform. Indeed, the coefficients $f(n)$ are given by

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} \widehat{f}(e^{it})e^{-imt}\,dt &= \frac{1}{2\pi}\int_0^{2\pi} \sum_{n=-\infty}^{\infty} f(n)e^{int}e^{-imt}\,dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} \sum_{n=-\infty}^{\infty} f(n)e^{i(n-m)t}\,dt \\
&= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \int_0^{2\pi} f(n)e^{i(n-m)t}\,dt \\
&= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} 2\pi f(n)\delta_{m,n} = f(m).
\end{aligned}$$

We conclude that the set of Gelfand transforms of $\ell^1(\mathbb{Z})$ is the set of functions $h \in Cts(U, \mathbb{C})$ whose Fourier series is absolutely convergent.

2.2 More on positive operators on unital C*-algebras

In this section, we will prove various results about positive operators on unital C*-algebras. The relevant sections we will follow are [Mur90, Section 2.2] and [Sol18, Section 3.1]. First, we begin with [Mur90, Remark 2.2.1].

Example 2.2.1. Let $A = Cts_0(X, \mathbb{C})$, where X is a LCH space. Let A_{sa} be the set of real-valued functions in A. We can turn A_{sa} into a poset given by the relation $f \leq g$ if and only if $f(x) \leq g(x)$ for $x \in X$.

In this case, the function $f \in A$ is positive if and only if there exists $g \in A$ such that $f = \overline{g}g$. Also, f has a unique positive square root, which is the function $x \mapsto \sqrt{f(x)}$. In this section, we want to generalise this situation to an arbitrary C*-algebra.

First, we will prove the existence of a unique square root of a positive element. Usually, the proof is done using the continuous functional calculus. However, the proof we will give relies on the Gelfand representation of a commutative C*-algebra in Theorem 1.6.7.

Definition 2.2.1. Let A be a C*-algebra. We define A^+ to be the set of positive elements of A.

Theorem 2.2.1. *Let A be a C*-algebra and $a \in A^+$. Then, there exists a unique element $b \in A^+$ such that $b^2 = a$.*

Proof. Assume that A is a C*-algebra and $a \in A^+$. Define B to be the C*-subalgebra generated by the set $\{a\}$. Since a is positive, it is self-adjoint. So, B is a commutative C*-algebra.

The idea is to apply the Gelfand representation in Theorem 1.6.7 to B. As a C*-algebra, B is isomorphic to $Cts_0(X, \mathbb{C})$ for some locally compact Hausdorff space X. Let Λ denote the isometric *-isomorphism from B to $Cts_0(X, \mathbb{C})$. Since a is positive, $\Lambda(a)$ is a positive function in $Cts_0(X, \mathbb{C})$.

By the previous remark, $\Lambda(a)$ has a unique square root, which we denote by $\Lambda(b) \in Cts_0(X, \mathbb{C})$. By applying the inverse map, we obtain a positive element $b \in A$ such that

$$b^2 = \Lambda^{-1}(\Lambda(b))^2 = \Lambda^{-1}(\Lambda(b)^2) = \Lambda^{-1}(\Lambda(a)) = a.$$

Now we will address the uniqueness of b. Assume that there exists $c \in A^+$ such that $c^2 = a$. Then, c must commute with a. Since $b \in B$ and B is the C*-subalgebra generated by $\{a\}$, c must also commute with b, as b is the limit of polynomials in a.

Now let C be the C*-subalgebra of A generated by the set $\{b, c\}$. Then, C is a commutative C*-algebra. By another application of Theorem 1.6.7, let $\Lambda_C : C \to Cts_0(Y, \mathbb{C})$ be the isometric *-isomorphism defining the Gelfand representation of C. Here, Y is a LCH space.

Observe that $\Lambda_C(a) = \Lambda_C(b)^2 = \Lambda_C(c)^2$. So, $\Lambda_C(b)$ and $\Lambda_C(c)$ are positive square roots of $\Lambda_C(a) \in Cts_0(Y, \mathbb{C})$ because b and c are positive. By the previous remark again, we deduce that by uniqueness, $\Lambda_C(b) = \Lambda_C(c)$ and so, $b = c$. This proves uniqueness. □

The square root of a positive element a is denoted by $a^{\frac{1}{2}}$.

Definition 2.2.2. Let A be a unital C*-algebra and $a, b \in A$. We say that $a \leq b$ if the element $b - a \in A^+$. That is, the element $b - a$ is positive.

With the relation in Definition 2.2.2, we will prove that the pair $(A, \leq)$ is a poset. Let us first prove some more properties about the relation in Definition 2.2.2.

Theorem 2.2.2. *Let A be a unital C*-algebra. Define*

$$-A^+ = \{-x \mid x \in A^+\}.$$

1. *If $x \in A^+$ and $\lambda \in \mathbb{R}_{\geq 0}$ then $\lambda x \in A^+$.*
2. *If $x, y \in A^+$ then $x + y \in A^+$.*
3. *$A^+ \cap (-A^+) = \{0\}$*
4. *If $x \in A^+$ and $y \in A$ then $y^* x y \in A^+$.*

Proof. Assume that A is a unital C*-algebra.

(1) Assume that $x \in A^+$ and $\lambda \in \mathbb{R}_{\geq 0}$. Then,

$$\begin{aligned} \sigma(\lambda x) &= \{\alpha \in \mathbb{C} \mid \alpha 1_A - \lambda x \text{ is not invertible}\} \\ &= \{\lambda \beta \in \mathbb{C} \mid \beta 1_A - x \text{ is not invertible}\} \subseteq \mathbb{R}_{\geq 0}. \end{aligned}$$

Hence, $\lambda x \in A^+$.

(2) This follows from Theorem 1.4.3 and Theorem 1.4.5.

(3) Assume that $x \in A^+$ and $x \in -A^+$. Then, the spectrum $\sigma(x) \subseteq [0, \infty)$. By the spectral mapping theorem in Theorem 1.3.14, $\sigma(-x) \subseteq (-\infty, 0]$. But since $x \in -A^+$, $-x \in A^+$ and $\sigma(-x) \subseteq [0, \infty)$. Consequently, $\sigma(-x) = \{0\}$. This means that the spectral radius $r(-x) = 0$ and by Theorem 1.2.6, $\|-x\| = r(-x) = 0$. So, $x = 0$ and we conclude that $A^+ \cap (-A^+) = \{0\}$.

(4) Assume that $x \in A^+$ and $y \in A$. By Theorem 1.4.2, there exists $t \geq \|x\|$ such that $\|t 1_A - x\| \leq t$. Now consider $t\|y\|^2 \in \mathbb{R}_{\geq 0}$. Then, $t\|y\|^2 \geq \|y^* x y\|$ and

$$\left\| t\|y\|^2 1_A - y^* x y \right\| \leq t\|y\|^2 + \|y^* x y\| \leq t\|y\|^2 + t\|y\|^2 = 2t\|y\|^2.$$

By Theorem 1.4.2, we deduce that $\sigma(y^* x y) \subseteq [0, \infty)$ and consequently, $y^* x y \in A^+$ as required. □

Now we are ready to prove that $(A, \leq)$ is a poset.

Theorem 2.2.3. *Let A be a unital C*-algebra and $\leq$ be the relation defined in Definition 2.2.2. Then, the pair $(A, \leq)$ is a poset.*

Proof. Assume that A is a unital C*-algebra. Assume that $\leq$ is the relation defined in Definition 2.2.2.

To show: (a) If $x \in A$ then $x \leq x$.

(b) If $x, y, z \in A$, $x \leq y$ and $y \leq z$ then $x \leq z$.

(c) If $x, y \in A$, $x \leq y$ and $y \leq x$ then $x = y$.

(a) Assume that $x \in A$. Then, $x - x = 0 \in A^+$. So, $x \leq x$.

(b) Assume that $x, y, z \in A$, $x \leq y$ and $y \leq z$. Then, $y - x, z - y \in A^+$ and since A^+ is closed under addition,

$$z - x = (z - y) + (y - x) \in A^+.$$

So, $x \leq z$.

(c) Assume that $x, y \in A$, $x \leq y$ and $y \leq x$. Then, $y - x, x - y \in A^+$. This means that $y - x \in A^+ \cap (-A^+)$. But, $A^+ \cap (-A^+) = \{0\}$ by Theorem 2.2.2. Hence, $y - x = 0$ and $x = y$.

Consequently, the pair $(A, \leq)$ is a poset. □

We will prove some more properties about positive elements in a unital C*-algebra. The next few results originate from [Mur90, Theorem 2.2.5].

Theorem 2.2.4. *Let A be a unital C*-algebra. Let $a, b, c \in A$. If $a \leq b$ then $c^*ac \leq c^*bc$.*

Proof. Assume that A is a unital C*-algebra. Assume that $a, b, c \in A$ and $a \leq b$. Then, $b - a \in A^+$ and by part 4 of Theorem 2.2.2,

$$c^*bc - c^*ac = c^*(b - a)c \in A^+.$$

Therefore, $c^*ac \leq c^*bc$. □

Theorem 2.2.5. *Let A be a unital C*-algebra. Let $a, b \in A$ such that $0 \leq a \leq b$. Then, $\|a\| \leq \|b\|$.*

Proof. Assume that A is a unital C*-algebra. Assume that $a, b \in A$ and $0 \leq a \leq b$. First, we claim that $b \leq \|b\| 1_A$. Let C_b be the C*-subalgebra of A generated by the set $\{1_A, b\}$. Then, C_b is a commutative C*-algebra and thus, we can consider its Gelfand representation from Theorem 1.6.7.

Let $\Lambda_b : C_b \to Cts_0(X_b, \mathbb{C})$ be the isometric *-isomorphism from C_b to $Cts_0(X_b, \mathbb{C})$, where X_b is a locally compact Hausdorff space. Since Λ_b is isometric, then

$$\sup_{x \in X} |\Lambda_b(b)(x)| = \|\Lambda_b(b)\| = \|b\|.$$

Now let $\mathbb{1}$ be the multiplicative unit of $Cts_0(X_b, \mathbb{C})$ (the constant function 1). Then, the function $\|b\|\mathbb{1} - \Lambda_b(b)$ is positive and by applying the inverse map Λ_b^{-1}, we deduce that the element $\|b\| 1_A - b \in C_b$ is positive. So, $\|b\| 1_A \geq b$.

Since $0 \leq a \leq b$ by assumption, then $a \leq \|b\| 1_A$. Next, let C_a be the C*-subalgebra of A generated by the set $\{1_A, a\}$. Again, C_a is a commutative C*-algebra and thus, we let $\Lambda_a : C_a \to Cts_0(X_a, \mathbb{C})$ denote the Gelfand representation of C_a. Then, the function $\|b\|\mathbb{1} - \Lambda_a(a)$ is positive and if $x \in X_a$ then

$$0 \leq \Lambda_a(a)x \leq \|b\|x.$$

By taking the absolute value and then the supremum over all $x \in X_a$, we find that $\|a\| = \|\Lambda_a(a)\| \leq \big\| \|b\|\mathbb{1} \big\|$. Hence, $(\|b\| - \|a\|)\mathbb{1}$ is a positive function and consequently, $\|b\| \geq \|a\|$. $\square$

Theorem 2.2.6. *Let A be a unital C*-algebra and $a, b \in A$ be positive invertible elements of A. If $a \leq b$ then $0 \leq b^{-1} \leq a^{-1}$.*

Proof. Assume that A is a unital C*-algebra. Assume that $a, b \in A$ are positive invertible elements of A.

To show: (a) If $c \geq 1_A$ then c is invertible.

(a) Assume that $c \in A$ such that $c \geq 1_A$. Let C_c denote the C*-subalgebra generated by the set $\{1_A, c\}$ and $\Lambda_c : C_c \to Cts_0(X_c, \mathbb{C})$ be the Gelfand representation of C_c. Here, X_c is a locally compact Hausdorff space.

Since $c \geq 1_A$, then the function $\Lambda_c(c) - \mathbb{1}$ is positive. Here, $\mathbb{1}$ is the multiplicative unit of $Cts_0(X_c, \mathbb{C})$. Since $0 \leq c$, the function $\Lambda_c(c)$ is real-valued and positive. If $x \in X_c$ then $\Lambda_c(c)(x) \geq 1$. Thus, we can define the inverse function $\Lambda_c(c)^{-1}$ by

$$\begin{array}{rccc} \Lambda_c(c)^{-1}: & X_c & \to & \mathbb{C} \\ & x & \mapsto & \frac{1}{\Lambda_c(c)(x)}. \end{array}$$

Then, $\Lambda_c(c)^{-1} \in Cts_0(X_c, \mathbb{C})$ and $\Lambda_c(c)\Lambda_c(c)^{-1} = \mathbb{1}$. By applying the inverse map Λ_c^{-1}, we find that c is an invertible element of A, with inverse given by

$$c^{-1} = \Lambda_c^{-1}(\Lambda_c(c)^{-1}).$$

Note that by construction, we also have $c^{-1} \leq 1_A$.

Now assume that $a \leq b$. Since a is invertible, then

$$1_A = a^{-\frac{1}{2}}aa^{-\frac{1}{2}} \leq a^{-\frac{1}{2}}ba^{-\frac{1}{2}}.$$

By part (a), we deduce that

$$a^{\frac{1}{2}}b^{-1}a^{\frac{1}{2}} \leq 1_A.$$

Consequently, $b^{-1} \leq a^{-\frac{1}{2}}a^{-\frac{1}{2}} = a^{-1}$ as required. □

2.3 Approximate units

Theorem 1.7.3 can be strengthened, giving rise to the powerful notion of an **approximate unit**. It is powerful because every C*-algebra has an approximate unit.

Definition 2.3.1. Let A be a C*-algebra. An **approximate unit** is an increasing net $\{u_\lambda\}_{\lambda \in L}$ of positive elements in the closed unit ball of A (L is a upwards directed set) such that if $a \in A$ then

$$a = \lim_\lambda au_\lambda = \lim_\lambda u_\lambda a.$$

Let A be a C*-algebra and A^+ be the poset of positive elements in A. Let A_1^+ denote the set of positive elements in A with norm less than 1. Then, A_1^+ is a poset, inheriting the relation from A^+. We will prove that the set A_1^+ is upwards directed.

Theorem 2.3.1. *Let A be a C*-algebra and A_1^+ be the poset of positive elements of A with norm less than 1. Let $a, b \in A_1^+$. Then, there exists $c \in A_1^+$ such that $a \leq c$ and $b \leq c$.*

Proof. Assume that A is a C*-algebra and A_1^+ is the poset of positive elements of A with norm less than 1. If $a \in A^+$ then $1_{\tilde{A}} + a$ is invertible in the unitization $\tilde{A}$ (since $\sigma_A(a) = \sigma_{\tilde{A}}(a)$) and $a(1_{\tilde{A}} + a)^{-1} = 1_{\tilde{A}} - (1_{\tilde{A}} + a)^{-1}$.

To show: (a) If $a, b \in A^+$ and $a \leq b$ then $a(1_{\tilde{A}} + a)^{-1} \leq b(1_{\tilde{A}} + b)^{-1}$.

(a) Assume that $a, b \in A^+$ and $a \leq b$. Then, $1_{\tilde{A}} + a \leq 1_{\tilde{A}} + b$ in the unitization $\tilde{A}$. By Theorem 2.2.6, $(1_{\tilde{A}} + b)^{-1} \leq (1_{\tilde{A}} + a)^{-1}$. Therefore,

$$a(1_{\tilde{A}} + a)^{-1} = 1_{\tilde{A}} - (1_{\tilde{A}} + a)^{-1} \leq 1_{\tilde{A}} - (1_{\tilde{A}} + b)^{-1} = b(1_{\tilde{A}} + b)^{-1}$$

as required.

Next, we claim that if $a \in A^+$ then $a(1_{\tilde{A}} + a)^{-1} \in A_1^+$. Let C be the C*-subalgebra of $\tilde{A}$ generated by the set $\{1_{\tilde{A}}, a\}$ and $\Lambda : C \to Cts_0(X, \mathbb{C})$ denote the Gelfand representation of C, where X is a locally compact Hausdorff space. Since Λ is isometric, we have

$$\|a(1_{\tilde{A}} + a)^{-1}\| = \|\frac{\Lambda(a)}{\mathbb{1} + \Lambda(a)}\|_\infty < 1.$$

where $\mathbb{1} \in Cts_0(X, \mathbb{C})$ is the unit. Obviously, $a(1_{\tilde{A}} + a)^{-1}$ is still positive.

Now assume that $a, b \in A_1^+$. Define $a' = a(1_{\tilde{A}} - a)^{-1}$ and $b' = b(1_{\tilde{A}} - b)^{-1}$. Then,

$$a'(1_{\tilde{A}} + a')^{-1} = a(1_{\tilde{A}} - a)^{-1}(1_{\tilde{A}} + a(1_{\tilde{A}} - a)^{-1})^{-1} = a$$

and similarly $b'(1_{\tilde{A}} + b')^{-1} = b$. Now define $c = (a' + b')(1_{\tilde{A}} + a' + b')^{-1}$. Then, $c \in A_1^+$ (by part (a)), $a' \leq a' + b'$ and $b' \leq a' + b'$. By part (a), we deduce that

$$a = a'(1_{\tilde{A}} + a')^{-1} \leq (a' + b')(1_{\tilde{A}} + a' + b')^{-1} = c$$

and $b \leq c$. Therefore, the poset A_1^+ is upwards directed. □

Now we will proceed to prove the existence of an approximate unit.

Theorem 2.3.2. *Let A be a C*-algebra. Let $\Lambda = A_1^+$ be the set of positive elements of A with norm less than 1, which is an upwards directed poset by Theorem 2.3.1. If $\lambda \in \Lambda$ then define $u_\lambda = \lambda$. Then, $\{u_\lambda\}_{\lambda \in \Lambda}$ is an approximate unit for A (sometimes called the canonical approximate unit).*

Proof. Assume that A is a C*-algebra and that Λ is defined as above. By Theorem 2.3.1, $\{u_\lambda\}_{\lambda\in\Lambda}$ is an increasing net of positive elements contained in the closed unit ball of A. Hence, it suffices to show that if $a \in A$ then $a = \lim_\lambda u_\lambda a$. Since Λ linearly spans A, we may assume that $a \in \Lambda$.

Assume that $\epsilon \in \mathbb{R}_{>0}$ and $C^*(a)$ is the C*-subalgebra of the unitization $\tilde{A}$ generated by the set $\{1_{\tilde{A}}, a\}$. Using Theorem 1.6.7, let $\varphi : C^*(a) \to Cts_0(X, \mathbb{C})$ denote the Gelfand representation of $C^*(a)$, where X is a compact Hausdorff space. If $f = \varphi(a)$ then

$$K = \{\omega \in X \mid |f(\omega)| \geq \epsilon\} \subseteq X$$

is compact. By Urysohn's lemma, we can construct a continuous function $g : X \to [0, 1]$ with compact support such that if $\omega \in K$ then $g(\omega) = 1$. Let $\delta \in \mathbb{R}_{>0}$ such that $\delta < 1$ and $1 - \delta < \epsilon$. Then,

$$\|f - \delta g f\|_\infty \leq \|f\|_\infty \|\mathbb{1} - \delta g\|_\infty = \|a\|\|\mathbb{1} - \delta g\|_\infty < \epsilon.$$

Now define $\lambda_0 = \varphi^{-1}(\delta g)$. Then, $\lambda_0 \in \Lambda$ and by the above inequality,

$$\|a - u_{\lambda_0} a\| < \epsilon.$$

To see that $\lim_\lambda u_\lambda a = a$, let $\lambda \in \Lambda$ such that $\lambda \geq \lambda_0$. Then, $1_{\tilde{A}} - u_\lambda \leq 1_{\tilde{A}} - u_{\lambda_0}$ and by Theorem 2.2.4,

$$a(1_{\tilde{A}} - u_\lambda)a = a^*(1_{\tilde{A}} - u_\lambda)a \leq a^*(1_{\tilde{A}} - u_{\lambda_0})a = a(1_{\tilde{A}} - u_{\lambda_0})a.$$

Therefore,

$$\begin{aligned}
\|a - u_\lambda a\|^2 &= \|(1_{\tilde{A}} - u_\lambda)^{\frac{1}{2}}(1_{\tilde{A}} - u_\lambda)^{\frac{1}{2}} a\|^2 \\
&\leq \|(1_{\tilde{A}} - u_\lambda)^{\frac{1}{2}}\|^2 \|(1_{\tilde{A}} - u_\lambda)^{\frac{1}{2}} a\|^2 \\
&= \|1_{\tilde{A}} - u_\lambda\| \|(1_{\tilde{A}} - u_\lambda)^{\frac{1}{2}} a\|^2 \\
&\leq \|1_{\tilde{A}}\| \|(1_{\tilde{A}} - u_\lambda)^{\frac{1}{2}} a\|^2 \quad \text{(Theorem 2.2.5)} \\
&= \|(1_{\tilde{A}} - u_\lambda)^{\frac{1}{2}} a\|^2 \\
&= \|a(1_{\tilde{A}} - u_\lambda)a\| \\
&\leq \|a(1_{\tilde{A}} - u_{\lambda_0})a\| \quad \text{(Theorem 2.2.5)} \\
&\leq \|a\|\|a - u_{\lambda_0} a\| < \epsilon.
\end{aligned}$$

So, $\lim_\lambda u_\lambda a = a$. □

The rest of this section is dedicated to proving useful properties about approximate units.

Theorem 2.3.3. *Let A be a C*-algebra and I be a closed left ideal of A. Then, there exists an increasing net $\{u_\lambda\}_{\lambda \in L}$ of positive elements contained in the closed unit ball of I such that if $a \in I$ then $a = \lim_\lambda au_\lambda$.*

Proof. Assume that A is a C*-algebra and I is a closed left ideal of A. Define $B = I \cap I^*$. Then, B is a C*-algebra and thus, we can use Theorem 2.3.2 to construct an approximate unit $\{u_\lambda\}_{\lambda \in L}$. If $a \in I$ then $a^*a \in B$. So,

$$\lim_\lambda a^*a(1_{\tilde{B}} - u_\lambda) = \lim_\lambda (a^*a - a^*au_\lambda) = 0.$$

Now observe that

$$\begin{aligned}
\lim_\lambda \|a - au_\lambda\|^2 &= \lim_\lambda \|a(1_{\tilde{B}} - u_\lambda)\|^2 \\
&= \lim_\lambda \|(1_{\tilde{B}} - u_\lambda)a^*a(1_{\tilde{B}} - u_\lambda)\| \\
&\leq \lim_\lambda \|(1_{\tilde{B}} - u_\lambda)\| \|a^*a(1_{\tilde{B}} - u_\lambda)\| \\
&\leq \lim_\lambda \|a^*a(1_{\tilde{B}} - u_\lambda)\| = 0.
\end{aligned}$$

So, $a = \lim_\lambda au_\lambda$. □

Theorem 2.3.4. *Let A be a C*-algebra and (φ, H) be a non-degenerate representation of A. Let $\{u_\lambda\}_{\lambda \in \Lambda}$ be an approximate unit for A. Then, $\{\varphi(u_\lambda)\}_{\lambda \in \Lambda}$ is an approximate unit for the image $\varphi(A)$ which converges strongly to the identity operator id_H.*

Proof. Assume that A is a C*-algebra and (φ, H) is a non-degenerate representation of A. Assume that $\{u_\lambda\}_{\lambda \in \Lambda}$ is an approximate unit for A. Since φ is continuous, the sequence $\{\varphi(u_\lambda)\}_{\lambda \in \Lambda}$ is an approximate unit for the C*-algebra $\varphi(A)$.

If $b \in \varphi(A)$ then

$$\lim_\lambda b\varphi(u_\lambda) = \lim_\lambda \varphi(u_\lambda)b = b.$$

Since (φ, H) is non-degenerate, we can use Theorem 1.9.4 to show that $\overline{\varphi(A)H} = H$. If $\xi \in H$ then

$$\lim_\lambda (b\varphi(u_\lambda))\xi = \lim_\lambda (\varphi(u_\lambda)b)\xi = b\xi.$$

Since this holds for arbitrary $b\xi \in \varphi(A)H$, we deduce that if $\psi \in H$ then

$$\lim_\lambda \varphi(u_\lambda)\psi = \psi.$$

Therefore, the sequence $\{\varphi(u_\lambda)\}_{\lambda\in\Lambda}$ converges strongly to the identity operator id_H. $\square$

Theorem 2.3.5. *Let A be a C*-algebra and τ be a bounded linear functional on A. The following are equivalent:*

1. *τ is positive.*
2. *If $\{u_\lambda\}_{\lambda\in L}$ is an approximate unit for A then $\|\tau\| = \lim_\lambda \tau(u_\lambda)$.*
3. *There exists $\{u_\lambda\}_{\lambda\in L}$ is an approximate unit for A then $\|\tau\| = \lim_\lambda \tau(u_\lambda)$.*

Proof. Assume that A is a C*-algebra and τ is a bounded linear functional on A. Without loss of generality, assume that $\|\tau\| = 1$.

First assume that τ is positive and $\{u_\lambda\}_{\lambda\in L}$ is an approximate unit for A. By the construction in Theorem 2.3.2, u_λ is positive (and self-adjoint) for $\lambda \in L$ and the sequence $\{\tau(u_\lambda)\}_{\lambda\in L}$ is increasing in $\mathbb{R}$. Hence, it must converge to its supremum. Since $\|\tau\| = 1$, $\sup_{\lambda\in L}\tau(u_\lambda) \leq 1$. Consequently, $\lim_\lambda \tau(u_\lambda) \leq 1$.

Now assume that $a \in A$ and $\|a\| \leq 1$. Then,

$$|\tau(u_\lambda a)|^2 \leq \tau(u_\lambda^2)\tau(a^*a) = \tau(u_\lambda)\tau(a^*a) \leq \tau(u_\lambda)\|\tau\|\|a^*a\| \leq \lim_\lambda \tau(u_\lambda).$$

By taking the limit of $\lambda \in L$, we deduce that $|\tau(a)|^2 \leq \lim_\lambda \tau(u_\lambda)$. Taking the supremum over all $a \in A$ with $\|a\| \leq 1$, we find that $1 \leq \lim_\lambda \tau(u_\lambda)$. Therefore, $1 = \lim_\lambda \tau(u_\lambda)$.

It is obvious that the second statement implies the third. Finally, assume that there exists an approximate unit $\{u_\lambda\}_{\lambda\in L}$ such that $1 = \lim_\lambda \tau(u_\lambda)$. First, assume that $a \in A$ is self-adjoint and $\|a\| \leq 1$.

To show: (a) $\tau(a) \in \mathbb{R}$.

(a) Assume that $\tau(a) = \alpha + i\beta$, where $\alpha, \beta \in \mathbb{R}$. We may assume further that $\beta \leq 0$. If $n \in \mathbb{Z}_{>0}$ then

$$\begin{aligned}
|\tau(a - inu_\lambda)|^2 &\leq \|\tau\|^2\|a - inu_\lambda\|^2 \\
&= \|a - inu_\lambda\|^2 \\
&= \|(a + inu_\lambda)(a - inu_\lambda)\| \\
&= \|a^2 + n^2u_\lambda^2 - in(au_\lambda - u_\lambda a)\| \\
&\leq 1 + n^2 + n\|au_\lambda - u_\lambda a\|.
\end{aligned}$$

Taking the limit over $\lambda \in L$, we find that by our assumption

$$\begin{aligned}
|\tau(a) - in|^2 &= \lim_\lambda |\tau(a - inu_\lambda)|^2 \\
&\leq \lim_\lambda (1 + n^2 + n\|au_\lambda - u_\lambda a\|) \\
&= 1 + n^2.
\end{aligned}$$

So, $|\alpha + i(\beta - n)|^2 \leq 1 + n^2$. By expanding the LHS of the inequality and then rearranging, we find that

$$-2\beta n \leq 1 - \beta^2 - \alpha^2$$

Since $\beta \leq 0$ and the above inequality holds for $n \in \mathbb{Z}_{>0}$, then $\beta = 0$. Hence, $\tau(a) \in \mathbb{R}$.

Finally, assume that a is positive and $\|a\| \leq 1$. Then, $u_\lambda - a$ is self-adjoint and $\|u_\lambda - a\| \leq \|u_\lambda\| \leq 1$ by Theorem 2.2.5. By part (a), $\tau(u_\lambda - a) \in \mathbb{R}$ and subsequently, $\tau(u_\lambda - a) \leq \|\tau\|\|u_\lambda - a\| \leq 1$. Taking the limit over $\lambda \in L$, we find that

$$1 \geq \lim_\lambda \tau(u_\lambda - a) = \lim_\lambda(\tau(u_\lambda) - \tau(a)) = 1 - \tau(a).$$

Hence, $\tau(a) \geq 0$ and τ must be positive. This completes the proof. □

An intriguing consequence of Theorem 2.3.5 is the following result.

Theorem 2.3.6. *Let A be a C*-algebra and τ, τ' be positive linear functionals on A. Then, $\|\tau + \tau'\| = \|\tau\| + \|\tau'\|$.*

Proof. Assume that A is a C*-algebra. Assume that τ and τ' are positive linear functionals on A. Let $\{u_\lambda\}_{\lambda \in \Lambda}$ be an approximate unit for A. By Theorem 2.3.5, $\|\tau\| = \lim_\lambda \tau(u_\lambda)$ and $\|\tau'\| = \lim_\lambda \tau'(u_\lambda)$. So,

$$\|\tau + \tau'\| = \lim_\lambda(\tau(u_\lambda) + \tau'(u_\lambda)) = \|\tau\| + \|\tau'\|.$$

□

Theorem 2.3.7. *Let A be a unital C*-algebra and $\tau : A \to \mathbb{C}$ be a bounded linear functional. Then, τ is positive if and only if $\tau(1_A) = \|\tau\|$.*

Proof. Assume that A is a unital C*-algebra. Assume that $\tau : A \to \mathbb{C}$ is a bounded linear functional. Let $\{u_\lambda\}_{\lambda \in L}$ be an approximate unit for A. By Theorem 2.3.5, τ is positive if and only if

$$\|\tau\| = \lim_\lambda \tau(u_\lambda) = \lim_\lambda \tau(1_A u_\lambda) = \tau(1_A).$$

$\square$

Our next use of approximate units is to generalise certain results contained in Theorem 1.11.2.

Theorem 2.3.8. *Let A be a C*-algebra and $\tau : A \to \mathbb{C}$ be a positive linear functional.*

1. *If $a \in A$ then $\tau(a^*a) = 0$ if and only for $b \in A$, $\tau(ba) = 0$*
2. *If $a, b \in A$ then $\tau(b^*a^*ab) \leq \|a^*a\|\tau(b^*b)$.*
3. *If $a \in A$ then $\tau(a^*) = \overline{\tau(a)}$.*
4. *If $a \in A$ then $|\tau(a)|^2 \leq \|\tau\|\tau(a^*a)$.*

Proof. Assume that A is a C*-algebra and $\tau : A \to \mathbb{C}$ is a positive linear functional. The map

$$\begin{array}{rccc} \langle -, - \rangle : & A \times A & \to & \mathbb{C} \\ & (a, b) & \mapsto & \tau(b^*a) \end{array}$$

defines a sesquilinear form on A and thus, satisfies the Cauchy-Schwarz inequality. If $a \in A$ satisfies $\tau(a^*a) = 0$ then this holds if and only if for $b \in A$,

$$|\tau(ba)|^2 \leq \tau(a^*a)\tau(bb^*) = 0$$

if and only if $\tau(ba) = 0$.

For the second statement, assume that $a, b \in A$. Observe that if $\tau(b^*b) = 0$ then by the first part, $\tau(b^*a^*ab) = 0$ and the inequality is trivially satisfied. So, assume that $\tau(b^*b) > 0$. Define the function

$$\begin{array}{rccc} \rho : & A & \to & \mathbb{C} \\ & c & \mapsto & \frac{\tau(b^*cb)}{\tau(b^*b)}. \end{array}$$

It is easy to verify that ρ is a positive linear functional. Now let $\{u_\lambda\}_{\lambda\in L}$ be an approximate unit for A. By Theorem 2.3.5,

$$\|\rho\| = \lim_\lambda \rho(u_\lambda) = \lim_\lambda \frac{\tau(b^*u_\lambda b)}{\tau(b^*b)} = \lim_\lambda \frac{\tau(b^*b)}{\tau(b^*b)} = 1.$$

Hence,

$$\frac{\tau(b^*a^*ab)}{\tau(b^*b)} = \rho(a^*a) \leq \|a^*a\|$$

which gives the desired inequality. For the third statement, assume that $a \in A$. Then,

$$\tau(a^*) = \lim_\lambda \tau(a^*u_\lambda) = \lim_\lambda \overline{\tau(u_\lambda a)} = \overline{\tau(a)}.$$

Note that the second last equality in the above equation follows from the fact that $\langle -, - \rangle$ is a sesquilinear form on A. For the final statement, we argue directly that

$$\begin{aligned} |\tau(a)|^2 &= \lim_\lambda |\tau(u_\lambda a)|^2 \\ &\leq \lim_\lambda |\tau(u_\lambda^2)||\tau(a^*a)| \\ &\leq \|\tau\|\tau(a^*a). \end{aligned}$$

□

2.4 Enveloping C*-algebras

In the next few sections, we follow the exposition in [Mur90, Chapter 6]. We are particularly interested in the construction of various C*-algebras.

Definition 2.4.1. Let A be a *-algebra. A **C*-seminorm** on A is a seminorm $p : A \to \mathbb{R}_{\geq 0}$ such that if $a, b \in A$ then

$$p(ab) \leq p(a)p(b), \qquad p(a^*) = p(a) \qquad \text{and} \qquad p(a^*a) = p(a)^2.$$

If in addition p is a norm then p is called a **C*-norm**.

Example 2.4.1. Assume that A is a *-algebra and B is a C*-algebra. Let $\varphi : A \to B$ be a *-homomorphism. Define the map

$$\begin{aligned} p : \ A &\to \mathbb{R}_{\geq 0} \\ a &\mapsto \|\varphi(a)\|. \end{aligned}$$

We claim that p defines a C*-seminorm on A. Since $\|-\|$ is a norm on B, it is straightforward to check that if $\lambda \in \mathbb{C}$ and $a, b \in A$ then

$$p(\lambda a) = |\lambda| p(a) \qquad \text{and} \qquad p(a+b) \leq p(a) + p(b).$$

Now since B is a C*-algebra and φ is a *-homomorphism by assumption, then

$$p(ab) \leq p(a)p(b), \qquad p(a^*a) = p(a)^2 \qquad \text{and} \qquad p(a^*) = p(a).$$

So, p is a C*-seminorm on A.

Now assume that φ is injective. To see that p is a C*-norm, it suffices to show that if $p(a) = 0$ then $a = 0$. If $p(a) = 0$ then $\|\varphi(a)\| = 0$. So, $\varphi(a) = 0$ in B and since φ is injective, $a = 0$ in A. Therefore, we have shown that if φ is injective then p is a C*-norm.

The concept of a C*-seminorm plays an important role in the construction of the enveloping C*-algebra. The first step to this construction is given by the following theorem.

Theorem 2.4.1. *Let A be a *-algebra and p be a C*-seminorm on A. Let $N = p^{-1}(\{0\})$. Then, N is a self-adjoint ideal of A and we can consider the quotient *-algebra A/N. The map $\|-\| : A/N \to \mathbb{R}_{\geq 0}$ given by $\|a + N\| = p(a)$ defines a C*-norm on A/N.*

Proof. Assume that A is a *-algebra and p is a C*-seminorm on A. Assume that N is the preimage

$$p^{-1}(\{0\}) = \{a \in A \mid p(a) = 0\}.$$

To show: (a) $p^{-1}(\{0\})$ is a self-adjoint ideal.

(a) Assume that $a, b \in N$. Then, $p(a+b) \leq p(a) + p(b) = 0$. Hence, $a + b \in N$. Now assume that $c \in A$. Since p is a C*-seminorm, $p(ca) \leq p(c)p(a) = 0$. Hence, $ca \in N$. Finally, we also have $p(a^*) = p(a) = 0$. So, $a^* \in N$ and consequently, N is a self-adjoint ideal of A.

Now consider the quotient *-algebra A/N and assume that the map $\|-\| : A/N \to \mathbb{R}_{\geq 0}$ is defined as above.

To show: (b) $\|-\|$ is a C*-norm.

(b) It suffices to check that $\|a+N\|=0$ if and only if $a+N=0+N$. Firstly, if $a+N=0+N$ then $a \in N$ and $\|a+N\|=p(a)=0$. Conversely, if $\|a+N\|=0$ then $p(a)=0$ and $a \in N$. Hence, $\|-\|$ is a C*-norm on A/N. □

In the scenario of Theorem 2.4.1, let B denote the Banach space completion of the normed vector space $(A/N, \|-\|)$. We claim that B is in fact, a C*-algebra.

We need to define multiplication and involution on B. Let $b \in B$. Then, there exists a sequence $\{a_n\}_{n\in\mathbb{Z}_{>0}}$ in A such that $a_n \to b$ as $n \to \infty$. Since $\|a\| = \|a^*\|$ for $a \in A$, the sequence $\{a_n^*\}_{n\in\mathbb{Z}_{>0}}$ also converges. We define the adjoint of b as the limit

$$b^* = \lim_{n\to\infty} a_n^*.$$

Now let $b' \in B$ so that there exists a sequence $\{a'_n\}_{n\in\mathbb{Z}_{>0}}$ which converges to b'. The product bb' is defined by

$$bb' = \big(\lim_{n\to\infty} a_n\big)\big(\lim_{n\to\infty} a'_n\big) = \lim_{m\to\infty}\lim_{n\to\infty} a_m a'_n.$$

Observe that

$$\|bb'\| = \lim_{m\to\infty}\lim_{n\to\infty}\|a_m a'_n\| \le \lim_{m\to\infty}\|a_m\| \lim_{n\to\infty}\|a'_n\| = \|b\|\|b'\|.$$

We also have

$$\|b^*\| = \lim_{n\to\infty}\|a_n^*\| = \lim_{n\to\infty}\|a_n\| = \|b\|$$

and

$$\|b\|^2 = \lim_{n\to\infty}\|a_n\|^2 = \lim_{n\to\infty}\|a_n^* a_n\| = \|b^* b\|.$$

Thus, B is a C*-algebra.

Definition 2.4.2. Let A be a *-algebra and p be a C*-seminorm on A. Let $N = p^{-1}(\{0\})$ and $\|-\| : A/N \to \mathbb{R}_{\ge 0}$ be the C*-norm given by $\|a+N\| = p(a)$. The C*-algebra B constructed as above is called the **enveloping C*-algebra** of the pair (A, p).

The reason for the name "enveloping" is because if we define the map

$$\begin{array}{rccc} \iota : & A & \to & B \\ & a & \mapsto & a+N \end{array}$$

then the image $i(A)$ is a dense *-subalgebra of B. It is dense due to the universal property of completion. The map ι is sometimes called the *canonical map* from A to B.

If p is a C*-norm then the enveloping C*-algebra B is referred to as the **C*-completion of** A. In this case, A is a dense *-subalgebra of B.

2.5 Direct limit of C*-algebras

We will use the construction of the enveloping C*-algebra to define the direct limit of a *sequence of C*-algebras*. We first begin with the definition of a direct sequence of C*-algebras.

Definition 2.5.1. Let $\{A_n\}_{n\in\mathbb{Z}_{>0}}$ be a sequence of C*-algebras and

$$\{\varphi_n : A_n \to A_{n+1}\}_{n\in\mathbb{Z}_{>0}}$$

be a sequence of *-homomorphisms. The sequence of pairs $\{(A_n, \varphi_n)\}_{n\in\mathbb{Z}_{>0}}$ is called a **direct sequence of C*-algebras**.

If $\{(A_n, \varphi_n)\}_{n\in\mathbb{Z}_{>0}}$ is a direct sequence of C*-algebras then the infinite product

$$\prod_{k=1}^{\infty} A_k = \{(a_1, a_2, \dots) \mid \text{If } i \in \mathbb{Z}_{>0} \text{ then } a_i \in A_i\}.$$

is a *-algebra with the operations of scalar multiplication, addition and multiplication defined pointwise. Now define

$$A' = \left\{ (a_k)_{k\in\mathbb{Z}_{>0}} \in \prod_{k=1}^{\infty} A_k \;\middle|\; \begin{matrix} \text{There exists } N \in \mathbb{Z}_{>0} \text{ such that} \\ a_{\ell+1} = \varphi_\ell(a_\ell) \text{ for } \ell \geq N \end{matrix} \right\}.$$

Then, A' is a *-subalgebra of $\prod_{k=1}^{\infty} A_k$. Notice that we have stuck to talking about *-algebras for now. At the moment, it is impractical to give $\prod_{k=1}^{\infty} A_k$ a norm, since there are issues with convergence. As we will see shortly, this is remedied when we work with A'.

Recall that if $i \in \mathbb{Z}_{>0}$ then φ_i is a *-homomorphism and is thus, contractive. This means that if $(a_k)_{k\in\mathbb{Z}_{>0}} \in A'$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $i \geq N$ then $a_{i+1} = \varphi_i(a_i)$ and consequently, $\|a_{i+1}\| \leq \|a_i\|$ for $i \geq N$. Hence, the sequence $\{\|a_k\|\}_{k\in\mathbb{Z}_{>0}}$ in $\mathbb{R}$ is decreasing and bounded below. So, it must converge. Next, we define the map p by

$$\begin{array}{rccc} p: & A' & \to & \mathbb{R}_{\geq 0} \\ & (a_k)_{k\in\mathbb{Z}_{>0}} & \mapsto & \lim_{k\to\infty}\|a_k\| \end{array}$$

Observe that p defines a C*-seminorm on A'. The fact that p is a seminorm follows from the properties of a norm. The fact that p is a C*-seminorm follows from the fact that each A_i is a C*-algebra.

Now we proceed to defining the direct limit of the direct sequence $\{(A_n, \varphi_n)\}_{n\in\mathbb{Z}_{>0}}$.

Definition 2.5.2. Let $\{(A_n, \varphi_n)\}_{n\in\mathbb{Z}_{>0}}$ be a direct sequence of C*-algebras. The **direct limit** of the sequence $\{(A_n, \varphi_n)\}_{n\in\mathbb{Z}_{>0}}$ is the enveloping C*-algebra of the pair (A', p), where

$$A' = \left\{ (a_k)_{k\in\mathbb{Z}_{>0}} \in \prod_{k=1}^{\infty} A_k \;\middle|\; \begin{array}{c}\text{There exists } N \in \mathbb{Z}_{>0} \text{ such that} \\ a_{\ell+1} = \varphi_\ell(a_\ell) \text{ for } \ell \geq N \end{array} \right\}.$$

and $p : A' \to \mathbb{R}_{\geq 0}$ is defined by $p((a_k)_{k\in\mathbb{Z}_{>0}}) = \lim_{k\to\infty}\|a_k\|$. The direct limit is usually denoted by $\varinjlim A_k$.

Assume that $n, m \in \mathbb{Z}_{>0}$ with $n \leq m$. We define $\varphi_{n,n} = id_{A_n}$. The *-homomorphism $\varphi_{n,m} : A_n \to A_m$ is defined as the composite

$$\varphi_{n,m} = \varphi_{m-1} \circ \varphi_{m-2} \circ \cdots \circ \varphi_n.$$

If $b \in A_n$ then we define the element $\widehat{\varphi}^n(b) \in A'$ to be the sequence

$$\widehat{\varphi}^n(b) = (0, \ldots, 0, b, \varphi_n(b), \varphi_{n,n+2}(b), \varphi_{n,n+3}(b), \ldots).$$

Note that in the above definition, the first $n-1$ entries of the sequence are zeros. Now let $\iota : A' \to \varinjlim A_k$ be the canonical map. If $n \in \mathbb{Z}_{>0}$ then define the map

$$\begin{array}{rccc} \varphi^n: & A_n & \to & \varinjlim A_k \\ & a & \mapsto & \iota(\widehat{\varphi}^n(a)) \end{array}$$

Since ι and $\widehat{\varphi}^n$ are both *-homomorphisms, then φ^n is a *-homomorphism for $n \in \mathbb{Z}_{>0}$. We also observe that if $b \in A_n$ then

$$\begin{aligned} \varphi^{n+1}(\varphi_n(b)) &= \iota(\widehat{\varphi}^{n+1}(\varphi_n(b))) \\ &= \iota((0, \ldots, 0, \varphi_n(b), \varphi_{n,n+2}(b), \varphi_{n,n+3}(b), \ldots)) \\ &= \iota((0, \ldots, 0, b, \varphi_n(b), \varphi_{n,n+2}(b), \varphi_{n,n+3}(b), \ldots)) \\ &= \varphi^n(b). \end{aligned}$$

Let us justify the third equality. The difference of the two sequences in the direct limit $\varinjlim A_n$ is

$$(0, \dots, 0, \varphi_n(b), \varphi_{n,n+2}(b), \dots) - (0, \dots, 0, b, \varphi_n(b), \varphi_{n,n+2}(b), \dots)$$
$$= (0, \dots, 0, -b, 0, 0, \dots).$$

Note that the $-b$ appears in the n^{th} position. The norm of its equivalence class in $\varinjlim A_n$ is 0. Hence, $\varphi^{n+1}(\varphi_n(b)) = \varphi^n(b)$ in $\varinjlim A_n$. The map φ^n is called the **natural map** from A_n to $\varinjlim A_n$.

Theorem 2.5.1. *Let $\{(A_n, \varphi_n)\}_{n\in\mathbb{Z}_{>0}}$ be a direct sequence of C*-algebras and $A = \varinjlim A_n$ be its direct limit. If $n \in \mathbb{Z}_{>0}$ then let $\varphi^n : A_n \to A$ denote the natural map from A_n to A.*

1. *The sequence of C*-algebras $\{\varphi^n(A_n)\}_{n\in\mathbb{Z}_{>0}}$ is increasing.*
2. *The union $\bigcup_{n\in\mathbb{Z}_{>0}} \varphi^n(A_n)$ is a dense *-subalgebra of A.*
3. *If $a \in A_n$ then $\|\varphi^n(a)\| = \lim_{k\to\infty}\|\varphi_{n,n+k}(a)\|$.*

Proof. Assume that $n \in \mathbb{Z}_{>0}$. We already showed previously that $\varphi^{n+1} \circ \varphi_n = \varphi^n$. So,

$$\varphi^n(A_n) = (\varphi^{n+1})(\varphi_n(A_n)) \subseteq \varphi^{n+1}(A_{n+1}).$$

Thus, the sequence $\{\varphi^n(A_n)\}_{n\in\mathbb{Z}_{>0}}$ is increasing with respect to inclusion. Also note that $\varphi^n(A_n)$ is a C*-algebra by Theorem 1.7.6.

To see that the union $\bigcup_{n\in\mathbb{Z}_{>0}} \varphi^n(A_n)$ is a dense *-subalgebra of A, assume that $(a_1, a_2, \dots) \in A$. Consider the sequence

$$\{\varphi^n(a_n)\}_{n\in\mathbb{Z}_{>0}}$$

in the union $\bigcup_{n\in\mathbb{Z}_{>0}} \varphi^n(A_n)$. Note that if $n \in \mathbb{Z}_{>0}$ then

$$\varphi^n(a_n) = \iota(\widehat{\varphi}^n(a_n)) = (a_1, \dots, a_n, \varphi_n(a_n), \varphi_{n,n+2}(a_n), \dots)$$

in A. Therefore,

$$\lim_{n\to\infty} \|(a_1, a_2, \dots) - \varphi^n(a_n)\| = 0$$

and the sequence $\{\varphi^n(a_n)\}_{n\in\mathbb{Z}_{>0}}$ converges to $(a_1, a_2, \dots)$. Thus, the union $\bigcup_{n\in\mathbb{Z}_{>0}} \varphi^n(A_n)$ is a dense C*-subalgebra of A.

Finally by the definition of the natural map φ^n,

$$\|\varphi^n(a)\| = \|\iota((0,\ldots,0,a,\varphi_n(a),\varphi_{n,n+2}(a),\ldots))\| = \lim_{k\to\infty} \|\varphi_{n,n+k}(a)\|.$$

□

The direct limit of a direct sequence of C*-algebras satisfies the following universal property given in [Mur90, Theorem 6.1.2].

Theorem 2.5.2. *Let $\{(A_n,\varphi_n)\}_{n\in\mathbb{Z}_{>0}}$ be a direct sequence of C*-algebras and $A = \varinjlim A_n$ denote its direct limit. For $n \in \mathbb{Z}_{>0}$, let $\varphi^n : A_n \to A$ be the natural map.*

1. *Assume that $\epsilon \in \mathbb{R}_{>0}$, $a \in A_n$ and $b \in A_m$ satisfy $\varphi^n(a) = \varphi^m(b)$. Then, there exists $k \in \mathbb{Z}_{>0}$ such that $k \geq \max(m,n)$ and $\|\varphi_{n,k}(a) - \varphi_{m,k}(b)\| < \epsilon$.*
2. *If B is a C*-algebra and there exists a *-homomorphism $\psi^n : A_n \to B$ such that the following diagram commutes for each $n \in \mathbb{Z}_{>0}$*

$$\begin{tikzcd} A_n \arrow[r, "\varphi_n"] \arrow[dr, "\psi^n"'] & A_{n+1} \arrow[d, "\psi^{n+1}"] \\ & B \end{tikzcd}$$

*then there exists a unique *-homomorphism $\psi : A \to B$ such that if $n \in \mathbb{Z}_{>0}$ then the following diagram commutes:*

$$\begin{tikzcd} A_n \arrow[r, "\varphi^n"] \arrow[dr, "\psi^n"'] & A \arrow[d, dashed, "\psi"] \\ & B \end{tikzcd}$$

Proof. Assume that $\{(A_n,\varphi_n)\}_{n\in\mathbb{Z}_{>0}}$ is a direct sequence of C*-algebras and $A = \varinjlim A_n$ is its direct limit.

Assume that $\epsilon \in \mathbb{R}_{>0}$, $a \in A_n$, $b \in A_m$ and $\varphi^n(a) = \varphi^m(b)$. By Theorem 2.5.1,

$$\|\varphi^n(a)\| = \|\varphi^m(b)\| = \lim_{k\to\infty} \|\varphi_{n,n+k}(a)\|.$$

Thus, there exists $k \geq \max(m,n)$ such that

$$\|\varphi_{n,k}(a) - \varphi_{m,k}(b)\| < \epsilon.$$

Now assume that B is a C*-algebra and $\{\psi^n : A_n \to B\}_{n \in \mathbb{Z}_{>0}}$ is a sequence of *-homomorphisms satisfying $\psi^n = \psi^{n+1} \circ \varphi_n$. The idea is to first define our required map on the C*-subalgebra $\bigcup_{n \in \mathbb{Z}_{>0}} \varphi^n(A_n)$. Again, assume that $a \in A_n$ and $b \in A_m$ satisfy $\varphi^n(a) = \varphi^m(b)$.

To show: (a) $\psi^n(a) = \psi^m(a)$.

(a) By the first part of this theorem, there exists $k \geq \max(m, n)$ such that

$$\|\varphi_{n,k}(a) - \varphi_{m,k}(b)\| < \epsilon.$$

So,

$$\begin{aligned}
\|\psi^n(a) - \psi^m(b)\| &= \|\psi^k(\varphi_{n,k}(a)) - \psi^k(\varphi_{m,k}(b))\| \\
&\leq \|\psi^k\| \|\varphi_{n,k}(a) - \varphi_{m,k}(b)\| \\
&< \epsilon.
\end{aligned}$$

Since $\epsilon \in \mathbb{R}_{>0}$ was arbitrary, we find that $\psi^n(a) = \psi^n(b)$ as required.

Now we define the map ψ by

$$\begin{array}{rccc}
\psi : & \bigcup_{n \in \mathbb{Z}_{>0}} \varphi^n(A_n) & \to & B \\
& \varphi^n(a) & \mapsto & \psi^n(a).
\end{array}$$

Part (a) of the proof demonstrates that ψ is well-defined. To see that ψ is a *-homomorphism, assume that $a \in A_n$, $b \in A_m$ and $\lambda \in \mathbb{C}$. Assume without loss of generality that $n \geq m$. Then,

$$\psi(\varphi^n(a)^*) = \psi(\varphi^n(a^*)) = \psi^n(a^*) = \psi^n(a)^* = \psi(\varphi^n(a))^*,$$

$$\psi(\lambda\varphi^n(a)) = \psi(\varphi^n(\lambda a)) = \psi^n(\lambda a) = \lambda\psi^n(a) = \lambda\psi(\varphi^n(a)),$$

$$\begin{aligned}
\psi(\varphi^n(a) + \varphi^m(b)) &= \psi(\varphi^n(a) + \varphi^n(\varphi_{m,n}(b))) \\
&= \psi(\varphi^n(a + \varphi_{m,n}(b))) \\
&= \psi^n(a + \varphi_{m,n}(b)) \\
&= \psi^n(a) + \psi^n(\varphi_{m,n}(b)) \\
&= \psi(\varphi^n(a)) + \psi^m(b) = \psi(\varphi^n(a)) + \psi(\varphi^m(b))
\end{aligned}$$

and

$$\begin{aligned}
\psi(\varphi^n(a)\varphi^m(b)) &= \psi(\varphi^n(a)\varphi^n(\varphi_{m,n}(b))) \\
&= \psi(\varphi^n(a\varphi_{m,n}(b))) \\
&= \psi^n(a\varphi_{m,n}(b)) \\
&= \psi^n(a)\psi^n(\varphi_{m,n}(b)) \\
&= \psi(\varphi^n(a))\psi^m(b) = \psi(\varphi^n(a))\psi(\varphi^m(b)).
\end{aligned}$$

So, ψ is a *-homomorphism. By Theorem 2.5.1, $\bigcup_{n\in\mathbb{Z}_{>0}} \varphi^n(A_n)$ is dense in the direct limit A. Therefore, ψ extends to a *-homomorphism from A to B which satisfies $\psi \circ \varphi^n = \psi^n$ for $n \in \mathbb{Z}_{>0}$.

To see that ψ is unique, suppose that $\phi : A \to B$ is another *-homomorphism such that if $n \in \mathbb{Z}_{>0}$ then $\phi \circ \varphi^n = \psi^n$. Then, $\phi = \psi$ on $\bigcup_{n\in\mathbb{Z}_{>0}} \varphi^n(A_n)$, which is dense in A. Therefore, $\phi = \psi$. $\square$

We finish this section with a consequence of Theorem 2.5.2.

Theorem 2.5.3. *Let A be a C*-algebra and let $\{A_n\}_{n\in\mathbb{Z}_{>0}}$ be an increasing sequence of C*-subalgebras of A. Assume that*

$$\overline{\bigcup_{n\in\mathbb{Z}_{>0}} A_n} = A.$$

Let $\iota_n : A_n \to A_{n+1}$ denote the inclusion map. Then, $A \cong \varinjlim A_n$ as C-algebras, where $\varinjlim A_n$ is the direct limit of the direct sequence $\{(A_n, \iota_n)\}_{n\in\mathbb{Z}_{>0}}$.*

Proof. Assume that A is a C*-algebra and $\{A_n\}_{n\in\mathbb{Z}_{>0}}$ is an increasing sequence of C*-subalgebras of A. Assume that

$$\overline{\bigcup_{n\in\mathbb{Z}_{>0}} A_n} = A.$$

Assume that $\varinjlim A_n$ and ι_n are defined as above. Let $i^n : A_n \to A$ denote the inclusion map. If $n \in \mathbb{Z}_{>0}$ then $i^n = i^{n+1} \circ \iota_n$. By the universal property of the direct limit in Theorem 2.5.2, there exists a unique *-homomorphism $\phi : \varinjlim A_n \to A$ such that the following diagram commutes for $n \in \mathbb{Z}_{>0}$:

$$\begin{tikzcd}
A_n \arrow[r, "\iota^n"] \arrow[dr, "i^n"'] & \varinjlim A_n \arrow[d, dashed, "\phi"] \\
& A
\end{tikzcd}$$

Here, $\iota^n : A_n \to \varinjlim A_n$ is the natural map. To see that ϕ is surjective. Assume that $a \in A$. Since $\bigcup_{n\in\mathbb{Z}_{>0}} A_n$ is dense in A, there exists a sequence $\{a_n\}_{n\in\mathbb{Z}_{>0}}$ such that $a_n \in A_n$ and $\lim_{n\to\infty}\|a_n - a\| = 0$. Since the sequence $\{\|a_n\|\}_{n\in\mathbb{Z}_{>0}}$ in $\mathbb{R}$ converges, the sequence $\{\|\iota^n(a_n)\|\}_{n\in\mathbb{Z}_{>0}}$ must also converge. Hence, the sequence $\{\iota^n(a_n)\}_{n\in\mathbb{Z}_{>0}}$ in $\varinjlim A_n$ converges to some $\tilde{a} \in \varinjlim A_n$. Thus,

$$\phi(\tilde{a}) = \phi(\lim_{n\to\infty} \iota^n(a_n)) = \lim_{n\to\infty} \phi(\iota^n(a_n)) = \lim_{n\to\infty} i^n(a_n) = i^n(a) = a.$$

Therefore, ϕ is surjective. To see that ϕ is injective, assume that $k \in \varinjlim A_n$ satisfies $\phi(k) = 0$. Let $k = (k_1, k_2, \dots)$ and assume that $\epsilon \in \mathbb{R}_{>0}$. By construction of the direct limit $\varinjlim A_n$, there exists $k' \in A'$ such that $\|k - k'\| < \epsilon$. We recall the definition of A' from Definition 2.5.2. If $k' = (k'_1, k'_2, \dots)$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $n, m \geq N$ then $k'_m = k'_n$.

Now consider $k'_N \in A_N$. In A we have

$$k'_N = i^N(k'_N) = \phi(\iota^N(k'_N)) = \phi(k) = 0.$$

Hence, $k' = 0$ in the direct limit $\varinjlim A_n$ and since $\epsilon \in \mathbb{R}_{>0}$ was arbitrary, $k = 0$. So, ϕ is injective and subsequently, a *-isomorphism from $\varinjlim A_n$ to A. $\square$

2.6 The multiplier algebra

In section 1.6, we studied the unitization of a C*-algebra. To summarise, unitization is a method of constructing a unital C*-algebra from an arbitrary C*-algebra. The original C*-algebra is then an ideal of the unitization. In this section, we will briefly study another construction of a unital C*-algebra — the **multiplier algebra** of a C*-algebra.

The multiplier algebra of a C*-algebra satisfies a universal property, akin to the unitization in Theorem 1.6.3. Roughly speaking, the universal property satisfied by the multiplier algebra states that the multiplier algebra of a C*-algebra A is the largest unital C*-algebra which contains A.

There are multiple ways to construct the multiplier algebra. We will follow [Mur90, Section 2.1] and construct the multiplier algebra through the use of *double centralisers.*

Definition 2.6.1. Let A be a C*-algebra. A **double centraliser** is a pair (L, R) of bounded linear maps on A such that if $a, b \in A$ then

$$L(ab) = L(a)b, \qquad R(ab) = aR(b) \qquad \text{and} \qquad R(a)b = aL(b).$$

The set of double centralisers of A is denoted by $M(A)$.

Here is a basic example of a double centraliser.

Example 2.6.1. Let A be a C*-algebra and $c \in A$. Define the linear maps L_c and R_c on A by $L_c(a) = ca$ and $R_c(a) = ac$. To see that L_c and R_c are bounded, observe that $\|L_c\| \leq \|c\|$ and $\|R_c\| \leq \|c\|$. We also have

$$\|c\| = \|c\frac{c^*}{\|c\|}\| \leq \sup_{\|a\|=1} \|L_c(a)\| = \|L_c\|.$$

Similarly, $\|R_c\| = \|c\|$. By direct computation, the pair (L_c, R_c) is a double centraliser of A. So, $(L_c, R_c) \in M(A)$.

Our goal is to show that the set $M(A)$ becomes a C*-algebra, when equipped with the necessary operations and norm. We will first work on defining a viable norm on $M(A)$.

Theorem 2.6.1. *Let A be a C*-algebra and (L, R) be a double centraliser on A. Then, $\|L\| = \|R\|$.*

Proof. Assume that A is a C*-algebra. Assume that (L, R) is a double centraliser on A. If $b \in A$ then

$$\begin{aligned}
\|L(b)\| &= \sup_{\|a\|=1} \|aL(b)\| \\
&= \sup_{\|a\|=1} \|R(a)b\| \\
&\leq \sup_{\|a\|=1} \|R\|\|a\|\|b\| \\
&= \|R\|\|b\|.
\end{aligned}$$

Taking the supremum over all $b \in A$ with $\|b\| = 1$ yields the inequality $\|L\| \leq \|R\|$. Similarly,

$$\begin{aligned}
\|R(b)\| &= \sup_{\|a\|=1} \|R(b)a\| \\
&= \sup_{\|a\|=1} \|aL(b)\| \\
&\leq \sup_{\|a\|=1} \|a\|\|L\|\|b\| \\
&= \|L\|\|b\|.
\end{aligned}$$

Taking the supremum over all $b \in A$ with $\|b\| = 1$ yields the inequality $\|R\| \leq \|L\|$. So, $\|L\| = \|R\|$. $\square$

In light of Theorem 2.6.1, if $(L, R) \in M(A)$ then we define the norm of (L, R) as

$$\|(L, R)\| = \|L\| = \|R\|.$$

Now we proceed to defining the operations on the set $M(A)$.

Addition: If $(L_1, R_1), (L_2, R_2) \in M(A)$ then define

$$(L_1, R_1) + (L_2, R_2) = (L_1 + L_2, R_1 + R_2).$$

If $a, b \in A$ then

$$(L_1 + L_2)(ab) = L_1(ab) + L_2(ab) = L_1(a)b + L_2(a)b = (L_1 + L_2)(a)b,$$

$$(R_1 + R_2)(ab) = R_1(ab) + R_2(ab) = aR_1(b) + aR_2(b) = a(R_1 + R_2)(b)$$

and

$$(R_1 + R_2)(a)b = R_1(a)b + R_2(a)b = aL_1(b) + aL_2(b) = a(L_1 + L_2)(b).$$

So, $(L_1, R_1) + (L_2, R_2) \in M(A)$.

Scalar multiplication: If $\lambda \in \mathbb{C}$ then we define

$$\lambda(L_1, R_1) = (\lambda L_1, \lambda R_1).$$

If $a, b \in A$ then

$$(\lambda L_1)(ab) = \lambda L_1(ab) = \lambda L_1(a)b = (\lambda L_1)(a)b,$$

$$(\lambda R_1)(ab) = \lambda R_1(ab) = \lambda a R_1(b) = a(\lambda R_1)(b)$$

and

$$(\lambda R_1)(a)b = \lambda R_1(a)b = \lambda a L_1(b) = a(\lambda L_1)(b).$$

So, $\lambda(L_1, R_1) \in M(A)$.

Multiplication: We define

$$(L_1, R_1) \cdot (L_2, R_2) = (L_1 \circ L_2, R_2 \circ R_1).$$

If $a, b \in A$ then

$$(L_1 \circ L_2)(ab) = L_1(L_2(ab)) = L_1(L_2(a)b) = (L_1 \circ L_2)(a)b,$$

$$(R_2 \circ R_1)(ab) = R_2(R_1(ab)) = R_2(aR_1(b)) = a(R_2 \circ R_1)(b),$$

and

$$(R_2 \circ R_1)(a)b = R_2(R_1(a))b = R_1(a)L_2(b) = a(L_1 \circ L_2)(b).$$

Hence, $(L_1 \circ L_2, R_2 \circ R_1) \in M(A)$.

Involution: Firstly, if $L : A \to A$ is a bounded linear operator then we define

$$\begin{array}{rccc} L^* : & A & \to & A \\ & a & \mapsto & (L(a^*))^* \end{array}$$

Then, L^* is also a bounded linear map on A which satisfies for $a \in A$

$$L^{**}(a) = (L^*(a^*))^* = (L(a))^{**} = L(a)$$

and

$$\begin{aligned} (L_1 \circ L_2)^*(a) &= ((L_1 \circ L_2)(a^*))^* \\ &= L_1(L_2(a^*))^* \\ &= L_1^*(L_2(a^*)^*) \\ &= (L_1^* \circ L_2^*)(a). \end{aligned}$$

In fact, it is easy to check that the map $L \mapsto L^*$ is isometric. If $(L, R) \in M(A)$ then we define

$$(L, R)^* = (R^*, L^*).$$

If $a, b \in A$ then

$$R^*(ab) = (R(b^*a^*))^* = (b^*R(a^*))^* = R^*(a)b,$$

$$L^*(ab) = (L(b^*a^*))^* = (L(b^*)a^*)^* = aL^*(b)$$

and

$$L^*(a)b = (L(a^*))^*b = (b^*L(a^*))^* = (R(b^*)a^*)^* = aR^*(b).$$

Therefore, $(R^*, L^*) \in M(A)$. The computations which check that the map $(L, R) \mapsto (L, R)^*$ satisfies the properties of an involution is suppressed here.

Theorem 2.6.2. *Let A be a C*-algebra. Let $M(A)$ denote the set of double centralisers on A. With the operations, norm and involution defined as above, $M(A)$ is a C*-algebra.*

Proof. Assume that A is a C*-algebra. Assume that $M(A)$ is the set of double centralisers on A. Let $B(A)$ denote the space of bounded linear operators on A. Since A is complete, $B(A)$ is a Banach space when equipped with the operator norm.

To show: (a) $M(A)$ is a closed vector subspace of $B(A) \oplus B(A)$.

(b) $M(A)$ is a Banach *-algebra.

(c) If $T = (L, R) \in M(A)$ then $\|T^*T\| = \|T\|^2$.

(a) The direct sum $B(A) \oplus B(A)$ is a Banach space when equipped with the norm

$$\|(S, T)\| = \max(\|S\|, \|T\|).$$

We already know that $M(A)$ is a vector subspace of $B(A) \oplus B(A)$. To see that $M(A)$ is closed, assume that $(L, R) \in \overline{M(A)}$ so that there exists a sequence of double centralisers $\{(L_n, R_n)\}_{n \in \mathbb{Z}_{>0}}$ which converges to (L, R). Then, $L_n \to L$ and $R_n \to R$ as $n \to \infty$. If $a, b \in A$ then

$$L(ab) = \lim_{n\to\infty} L_n(ab) = \lim_{n\to\infty} L_n(a)b = L(a)b,$$

$$R(ab) = \lim_{n\to\infty} R_n(ab) = \lim_{n\to\infty} aR_n(b) = aR(b),$$

and

$$R(a)b = \lim_{n\to\infty} R_n(a)b = \lim_{n\to\infty} aL_n(b) = aL(b).$$

Therefore, $(L, R) \in M(A)$ and $M(A)$ is a closed vector subspace of $B(A) \oplus B(A)$. This means that $M(A)$ is itself a Banach space.

(b) To see that $M(A)$ is a Banach *-algebra, assume that $(L_1, R_1), (L_2, R_2) \in M(A)$. Then, $(L_1, R_1) \cdot (L_2, R_2) = (L_1 \circ L_2, R_2 \circ R_1)$ and

$$\|(L_1 \circ L_2, R_2 \circ R_1)\| = \|L_1 \circ L_2\| \leq \|L_1\|\|L_2\| = \|(L_1, R_1)\|\|(L_2, R_2)\|.$$

Hence, $M(A)$ is a Banach *-algebra.

(c) Assume that $T = (L, R) \in M(A)$. If $a \in A$ satisfies $\|a\| = 1$ then

$$\|L(a)\|^2 = \|L(a)^*L(a)\| = \|L^*(a^*)L(a)\| = \|a^*R^*(L(a))\| \leq \|R^*L\| = \|T^*T\|.$$

Taking the supremum over all such a, we find that $\|T\|^2 \leq \|T^*T\|$. We also have $\|T^*T\| \leq \|T^*\|\|T\| = \|T\|^2$. Therefore, $\|T^*T\| = \|T\|^2$ and consequently, $M(A)$ is a C*-algebra as required. □

If A is a C*-algebra then the C*-algebra $M(A)$ in Theorem 2.6.2 is called the **multiplier algebra**. It is unital because if $id_A : A \to A$ is the identity map on A then (id_A, id_A) is the multiplicative unit of $M(A)$.

Now define the map

$$\begin{array}{rccc} \iota: & A & \to & M(A) \\ & a & \mapsto & (L_a, R_a) \end{array}$$

where the maps L_a and R_a are defined by $L_a(b) = ab$ and $R_a(b) = ba$. By the previous example, we find that ι is a isometric *-homomorphism. So, we are able to identify A as a C*-subalgebra of $M(A)$. Additionally, we can also identify A as an ideal of $M(A)$.

Before we state the universal property of the multiplier algebra, we require the following definition.

Definition 2.6.2. Let A be a C*-algebra and I be a closed ideal of A. We say that I is **essential** in A if the following statement is satisfied: If $a \in A$ and $aI = 0$ then $a = 0$.

We claim that if A is a C*-algebra then A is an essential ideal in its multiplier algebra $M(A)$. Assume that $(L, R) \in M(A)$ satisfies $(L, R)A = 0$. Let $(L_a, R_a) \in A$ for $a \in A$. Then,

$$(L, R)(L_a, R_a) = (L \circ L_a, R_a \circ R) = (0, 0).$$

Here, 0 denotes the zero map on A. If $b \in A$ then $(L \circ L_a)(b) = L(ab) = 0$ and $(R_a \circ R)(b) = R(b)a = 0$. Since a was arbitrary, $R(b) = 0$. But, we also have $R(b)a = bL(a) = 0$. So, $L(a) = 0$ and consequently, $(L, R) = (0, 0)$. Therefore, A is an essential ideal of $M(A)$.

Theorem 2.6.3. *Let A be a C*-algebra and I be a closed ideal in A. Let $\iota_I : I \to M(I)$ denote the inclusion map. Then, there exists a unique *-homomorphism $\varphi : A \to M(I)$ extending ι_I. Furthermore, if I is an essential ideal of A then φ is injective.*

Proof. Assume that A is a C*-algebra. Assume that I is a closed ideal of A. Assume that ι_I is the inclusion of I into its multiplier algebra $M(I)$. Define the map φ by

$$\begin{array}{rrcl} \varphi : & A & \to & M(I) \\ & a & \mapsto & (L_a|_I, R_a|_I) \end{array}$$

Here, $L_a|_I$ is the restriction of the left multiplication map $L_a : A \to A$ to the closed ideal I. Similarly, $R_a|_I$ is the restriction of the right multiplication map $R_a : A \to A$ to I. It is easy to check that φ is a *-homomorphism which extends ι_I.

To prove uniqueness, assume that there exists another *-homomorphism $\psi : A \to M(I)$ which extends ι_I. If $b \in I$ and $a \in A$ then

$$\varphi(a)b = \varphi(ab) = ab = \psi(ab) = \psi(a)b.$$

This shows that if $a \in A$ then $(\varphi(a) - \psi(a))I = 0$. Since I is an essential ideal of $M(I)$, then $\varphi(a) = \psi(a)$. So, $\varphi = \psi$.

Finally, assume that I is an essential ideal of A. Assume that $a \in \ker \varphi$. Then, $L_a|_I$ is the zero map on I. This means that $aI = 0$ and since I is essential in A, $a = 0$. Hence, φ is injective. □

Every closed ideal of a C*-algebra is itself a C*-algebra. Hence, Theorem 2.6.3 states that $M(I)$ is the largest unital C*-algebra containing I as an essential ideal.

Theorem 2.6.4. *Let A be a C*-algebra. If A is unital then $M(A) = A$.*

Proof. Assume that A is a unital C*-algebra. We know that $M(A)$ contains A as an essential ideal. To see that $M(A) \subseteq A$, notice that if $a \in A$ then

$$(1_A - 1_{M(A)})a = a - 1_{M(A)}a = 0.$$

So, $(1_A - 1_{M(A)})A = 0$ and since A is an essential ideal of $M(A)$, $1_A - 1_{M(A)}$. Hence, $1_A = 1_{M(A)}$ and $M(A) \subseteq A$. Consequently, $M(A) = A$. □

2.7 Uniformly hyperfinite algebras

In this section, we follow [Mur90, Section 6.2]. We will first set up the relevant theory before defining uniformly hyperfinite algebras. Our first task is to characterise finite-dimensional simple C*-algebras.

Definition 2.7.1. Let A be a C*-algebra. We say that A is **liminal** if for every non-zero irreducible representation (π, H), the image $\pi(A) = B_0(H)$, where $B_0(H) \subseteq B(H)$ is the space of compact operators on H.

We recall that if H is a Hilbert space and $x : H \to H$ is a bounded operator on H then x is **compact** if there exists a sequence of $\{x_n\}_{n\in\mathbb{Z}_{>0}}$ of finite dimensional (or finite rank) operators such that

$$\lim_{n\to\infty} \|x_n - x\| = 0.$$

Theorem 2.7.1. *Let A be a C*-algebra. If A is finite dimensional then A is liminal.*

Proof. Assume that A is a finite dimensional C*-algebra. Let (π, H) be a non-zero, non-degenerate irreducible representation of A. If $\xi \in H - \{0\}$ then ξ is a cyclic vector (see Theorem 1.9.6) and

$$\overline{\pi(A)\xi} = H.$$

Since A is finite dimensional, the subspace $\pi(A)\xi$ is also finite dimensional and hence, closed. So, $\pi(A)\xi = H$ and H must be finite dimensional. Hence, $\pi(A) \subseteq B_0(H) = B(H)$ and by [Mur90, Theorem 2.4.9], $B_0(H) \subseteq \pi(A)$. So, $\pi(A) = B_0(H)$ and A is liminal. □

[Mur90, Theorem 2.4.9] states that if (π, H) is an irreducible representation of a C*-algebra A and $\pi(A) \cap B_0(H) \neq \emptyset$ then $B_0(H) \subseteq \pi(A)$. Now we will use Theorem 2.7.1 to characterise finite dimensional simple C*-algebras.

Theorem 2.7.2. *A non-zero finite dimensional C*-algebra is simple if and only if there exists $n \in \mathbb{Z}_{>0}$ such that A is isomorphic to the matrix algebra $M_{n\times n}(\mathbb{C})$.*

Proof. We already know that if $n \in \mathbb{Z}_{>0}$ then the matrix C*-algebra $M_{n\times n}(\mathbb{C})$ is simple by Theorem 1.10.3. So, assume that A is a non-zero finite dimensional simple C*-algebra.

By Theorem 2.7.1, A is a liminal C*-algebra. That is, if (π, H) is a non-zero irreducible representation of A then $\pi(A) = B_0(H)$ and subsequently, H is finite dimensional (because A is finite dimensional). Moreover, the kernel $\ker \pi$ is a closed ideal of A. Since A is simple, $\ker \pi = 0$.

Thus, if $n = \dim H$ then A is isomorphic as a C*-algebra to $K(H) = B(H)$ (as H is finite dimensional). In turn, $B(H)$ is isomorphic to $M_{n\times n}(\mathbb{C})$ as required. $\square$

Here is another useful characterisation of simple C*-algebras.

Theorem 2.7.3. *Let A be a C*-algebra. If any surjective *-homomorphism $\pi : A \to B$ onto a non-zero C*-algebra B is also injective then A is a simple C*-algebra.*

Proof. Assume that A is a C*-algebra. Suppose that any surjective *-homomorphism from A to a non-zero C*-algebra B is injective. Let I be a closed ideal of A and suppose for the sake of contradiction that I is a proper ideal of A. Then, the projection map

$$\begin{array}{rccc} \pi : & A & \to & A/I \\ & a & \mapsto & a+I \end{array}$$

is a surjective *-homomorphism onto A/I which is a non-zero C*-algebra (non-zero because I is proper). By our assumption, π must also be injective. However, this means that $A = I$ which contradicts the assumption that I is a proper ideal of A.

So, I is not a proper ideal and is hence, either 0 or A. This means that A is a simple C*-algebra as required. $\square$

Next, we require a few technical results pertaining to projections.

Theorem 2.7.4. *Let A be a unital C*-algebra and $p, q \in A$ be projections satisfying $\|q - p\| < 1$. Then, there exists a unitary element $u \in A$ such that $q = upu^*$ and $\|1_A - u\| \leq \sqrt{2}\|q - p\|$.*

Proof. Assume that A is a unital C*-algebra and $p, q \in A$ are projections satisfying $\|p - q\| < 1$. Consider the element $v = 1_A - p - q + 2qp$.

To show; (a) v is invertible.

(a) The idea here is to consider the elements v^*v and vv^*. By direct computation, $v^* = 1_A - p - q + 2pq$ and

$$v^*v = vv^* = 1_A - (q-p)^2.$$

This means that v is normal. Since $\|q-p\| < 1$, then $\|(q-p)^2\| = \|q-p\|^2 < 1$ (by C*-algebra condition). Since the spectrum $\sigma((q-p)^2) \subseteq [0, \|(q-p)^2\|]$, $1 \notin \sigma((q-p)^2)$. Therefore, $v^*v = 1_A - (q-p)^2$ is invertible. Since v is normal, then by the spectral mapping theorem in Theorem 1.3.14,

$$\sigma(v^*v) = \{|\lambda|^2 \mid \lambda \in \sigma(v)\}.$$

Since $0 \notin \sigma(v^*v)$, $0 \notin \sigma(v)$. Therefore, v is invertible.

Recall the polar decomposition of v from Theorem 1.4.7, Theorem 1.4.8 and Theorem 1.4.9. In particular, $v = u|v|$, where $|v|$ is invertible (by Theorem 1.4.7) and $u = v|v|^{-1}$ is unitary (by Theorem 1.4.8).

To show: (b) $q = upu^*$.

(c) $\|1_A - u\| \leq \sqrt{2}\|q-p\|$.

(b) First, we observe that

$$vp = (1_A - p - q + 2qp)p = p - p - qp + 2qp = qp$$

and

$$qv = q(1_A - p - q + 2qp) = q - qp - q + 2qp = qp.$$

This means that $v^*vp = (v^*q)v = pqv = pv^*v$. Since p commutes with v^*v, it must also commute with $|v| = (v^*v)^{\frac{1}{2}}$ and $|v|^{-1}$. Now, we have

$$up = v|v|^{-1}p = vp|v|^{-1} = qv|v|^{-1} = qu$$

and consequently, $q = upu^*$.

(c) Let $Re(v)$ denote the real part of the operator v. Then,

$$Re(v) = \frac{1}{2}(v + v^*) = 1_A - p - q + qp + pq = 1_A - (q-p)^2 = |v|^2$$

and

$$Re(u) = \frac{1}{2}(u+u^*) = \frac{1}{2}(v+v^*)|v|^{-1} = |v|.$$

Hence, we have the upper bound

$$\begin{aligned}
\|1_a - u\|^2 &= \|(1_A - u^*)(1_A - u)\| \\
&= \|21_A - u - u^*\| \\
&= 2\|1_A - Re(u)\| \\
&= 2\|1_A - |v|\| \le 2\|1_A - |v|^2\|.
\end{aligned}$$

The inequality follows from the fact that if $t \in [0,1]$ then $1 - t \le 1 - t^2$ (see Theorem 1.3.7). Now since $1_A - |v|^2 = 1_A - v^*v = (q-p)^2$, we have

$$\|1_A - u\|^2 \le 2\|1_A - |v|^2\| = 2\|q-p\|^2.$$

By taking square roots, we are done. $\square$

Theorem 2.7.5. *Let A be a C*-algebra and $a \in A$ be self-adjoint. Suppose that $\|a - a^2\| < \frac{1}{4}$. Then, there exists a projection $p \in A$ such that $\|a - p\| < \frac{1}{2}$.*

Proof. Assume that A is a C*-algebra and $a \in A$ is self-adjoint. By Theorem 1.6.7, we may assume that A is abelian and that $A = Cts_0(X, \mathbb{C})$ where X is a locally compact Hausdorff space.

Consider the absolute value of the function a. Notice that $\frac{1}{2} \notin \text{im } |a|$. Otherwise, there exists $x \in X$ such that $|a(x)| = \frac{1}{2}$. Since a is self-adjoint, it is a real-valued function. If $a(x) = \frac{1}{2}$ then

$$|a(x) - a^2(x)| = |\frac{1}{2} - \frac{1}{4}| = \frac{1}{4}$$

which contradicts the assumption that $\|a - a^2\| < \frac{1}{4}$. If $a(x) = -\frac{1}{2}$ then

$$|a(x) - a^2(x)| = |-\frac{1}{2} - \frac{1}{4}| = \frac{3}{4}$$

which again contradicts the assumption that $\|a - a^2\| < \frac{1}{4}$. Hence, $\frac{1}{2} \notin \text{im } |a|$.

Now let $S = |a|^{-1}((\frac{1}{2}, \infty))$. Then, S is an open subset of X. It is also compact because

$$S = \{x \in X \mid |a(x)| \geq \frac{1}{2}\}$$

and X is a LCH space. Now consider the characteristic function χ_S. Since χ_S is self-adjoint and idempotent, it is a projection in A. If $x \notin S$ then

$$|a(x) - \chi_S(x)| < \frac{1}{2} + 0 = \frac{1}{2}.$$

If $x \in S$ then first observe that by the reverse triangle inequality,

$$\frac{1}{4} > \|a - a^2\|_\infty \geq |\|a\|_\infty - \|a^2\|_\infty| = |\|a\|_\infty - \|a\|_\infty^2|.$$

Consequently, $\|a\|_\infty \in [0, \frac{1}{2}(1+\sqrt{2})] - \{\frac{1}{2}\}$. If $\|a\|_\infty \in [0, \frac{1}{2})$ then $S = \emptyset$ and the result is satisfied trivially. On the other hand, if $\|a\|_\infty \in (\frac{1}{2}, \frac{1}{2}(1+\sqrt{2})]$ and $x \in S$ then

$$|a(x) - \chi_S(x)| = |1 - a(x)| < |\frac{1}{2}(1+\sqrt{2}) - 1| < \frac{1}{2}.$$

Therefore, $\|a - \chi_S\|_\infty < \frac{1}{2}$ as required. $\square$

In a previous section, we defined the concept of a trace on a unital C*-algebra. The notion of a trace generalises to an arbitrary C*-algebra. In [Mur90], positive linear functionals which are traces are called **tracial**. We already know one example of a tracial positive linear functional — from Theorem 1.8.1. We will give another example from [Mur90, Remark 6.2.2]. Before we do this, here is how the notion of a state generalises to an arbitrary C*-algebra.

Definition 2.7.2. Let A be a C*-algebra. A **state** on A is a positive linear functional $\phi : A \to \mathbb{C}$ such that $\|\phi\| = 1$.

Example 2.7.1. Let H be an infinite dimensional Hilbert space and $B_0(H)$ denote the space of bounded compact operators on H. We will show that there does not exist a tracial state on $B_0(H)$.

Suppose for the sake of contradiction that $\tau : B_0(H) \to \mathbb{C}$ is a tracial state on $B_0(H)$. The idea is to show that all rank one projections on H are unitarily equivalent. Let $p, q \in B_0(H)$ be rank one projections. Then, there exists unit vectors $\xi, \psi \in H$ such that

$$p = |\xi\rangle\langle\xi| \qquad \text{and} \qquad q = |\psi\rangle\langle\psi|.$$

Since ξ and ψ are both unit vectors, there exists a unitary operator $u \in B(H)$ such that $u\xi = \psi$. Therefore,

$$q = |\psi\rangle\langle\psi| = |u\xi\rangle\langle u\xi| = u|\xi\rangle\langle\xi|u^* = upu^*.$$

The second last equality follows from Theorem 1.5.2. Hence, p and q are unitarily equivalent. So,

$$\tau(q) = \tau(upu^*) = \tau(u^*(up)) = \tau(p).$$

Every rank one projection takes the same value under τ. Let $t \in \mathbb{C}$ be this value. Since τ is a positive linear functional, $t \in \mathbb{R}_{\geq 0}$. Now let $n \in \mathbb{Z}_{>0}$ and $\{e_i\}_{i \in \mathbb{Z}_{>0}}$ be an orthonormal basis for H. Define

$$p_n = \sum_{i=1}^{n} |e_i\rangle\langle e_i|.$$

Then, p_n is a sum of n rank one projections and

$$\tau(p_n) = \sum_{i=1}^{n} \tau(|e_i\rangle\langle e_i|) = nt.$$

However, since $\{e_i\}_{i \in \mathbb{Z}_{>0}}$ is an orthonormal basis, p_n is also a projection. So, $\tau(p_n) \leq \|\tau\|\|p_n\| \leq 1$. This means that $nt \leq 1$ and $n \leq 1/t$ for arbitrary $n \in \mathbb{Z}_{>0}$. This gives the required contradiction. Hence, $B_0(H)$ does not admit a tracial state.

Recall that if A is a C*-algebra and $\phi : A \to \mathbb{C}$ is a positive linear functional then

$$N_\phi = \{a \in A \mid \phi(a^*a) = 0\}$$

is a left ideal of A. This was derived as part of the GNS construction. If ϕ is tracial then

$$\begin{aligned} N_\phi &= \{a \in A \mid \phi(a^*a) = 0\} \\ &= \{a \in A \mid \phi(aa^*) = 0\} \\ &= \{a^* \mid a \in N_\phi\} = N_\phi^*. \end{aligned}$$

So, N_ϕ is a closed ideal of A.

Theorem 2.7.6. *Let A be a simple C*-algebra and $\tau : A \to \mathbb{C}$ be a non-zero tracial positive linear functional. Then, τ is faithful.*

Proof. Assume that A is a simple C*-algebra. Assume that τ is a non-zero tracial positive linear functional on A. By the previous discussion,

$$N_\tau = \{a \in A \mid \tau(a^*a) = 0\}$$

is a closed ideal of A. Since A is simple, N_τ is either the zero ideal or A. Since τ is non-zero, $N_\tau \neq A$. Therefore, $N_\tau = 0$ and τ is faithful. $\square$

We will now list a particular application of Theorem 2.7.6 for the purpose of motivating the definition of a uniformly hyperfinite algebra.

Example 2.7.2. Let $(A_n)_{n\in\mathbb{Z}_{>0}}$ be an increasing sequence of C*-subalgebras of a C*-algebra A such that

$$A = \overline{\bigcup_{n=1}^{\infty} A_n}.$$

Suppose further that A is unital with multiplicative unit 1_A and that if $n \in \mathbb{Z}_{>0}$ then $1_A \in A_n$. We claim that if there exists a unique tracial state τ_n on A_n then there exists a unique tracial state τ on A.

First, note that if $m, n \in \mathbb{Z}_{>0}$ and $m \leq n$ then the restriction $\tau_n|_{A_m} = \tau_m$ by uniqueness of the tracial state τ_m on A_m. Keeping this in mind, we define the map

$$\begin{array}{rccc} \tau : & \bigcup_{n\in\mathbb{Z}_{>0}} A_n & \to & \mathbb{C} \\ & a \in A_m & \mapsto & \tau_m(a). \end{array}$$

By the most recent remark, τ is a well-defined map. It is straightforward to check that τ is a linear map. Since each τ_n is a state, then

$$|\tau(a)| = |\tau_m(a)| \leq \|\tau_m\|\|a\| = \tau_m(1_A)\|a\| = \|a\|.$$

The second last equality follows from Theorem 1.11.2. Hence, τ is norm decreasing.

Since A is the closure (with respect to the norm topology) of $\bigcup_{n\in\mathbb{Z}_{>0}} A_n$, we can extend τ to a bounded linear functional on all of A. To see that τ is positive, assume that $a \in A$. Then, there exists a sequence $\{a_n\}_{n\in\mathbb{Z}_{>0}}$ in $\bigcup_n A_n$ such that $a_n \in A_n$ and $\lim_{n\to\infty}\|a_n - a\| = 0$. So,

$$\tau(a^*a) = \tau(\lim_{n\to\infty} a_n^* a_n) = \lim_{n\to\infty} \tau(a_n^* a_n) = \lim_{n\to\infty} \tau_n(a_n^* a_n) \geq 0$$

and τ is a positive linear functional on A. To see that τ is a state, note that

$$\tau(1_A) = \tau_n(1_A) = 1$$

since $1_A \in A_n$ for $n \in \mathbb{Z}_{>0}$. To see that τ is tracial, assume that $a, b \in A$. Then,

$$\tau(ab) = \lim_{n\to\infty} \tau(a_n b_n) = \lim_{n\to\infty} \tau_n(a_n b_n) = \lim_{n\to\infty} \tau(b_n a_n) = \tau(ba).$$

So, τ defines a tracial state on A. Finally, the fact that τ is unique follows from the fact that τ_m is the unique tracial state on A_m for $m \in \mathbb{Z}_{>0}$.

Definition 2.7.3. A **uniformly hyperfinite algebra** or **UHF algebra** is a unital C*-algebra A which has an increasing sequence $\{A_n\}_{n\in\mathbb{Z}_{>0}}$ of finite dimensional simple C*-subalgebras such that $1_A \in A_m$ for $m \in \mathbb{Z}_{>0}$ and the union $\bigcup_{n=1}^{\infty} A_n$ is dense in A.

Let us unpack the definition of a UHF algebra with what we know so far. Let A be a UHF algebra with increasing sequence $\{A_n\}_{n=1}^{\infty}$. By definition, if $n \in \mathbb{Z}_{>0}$ then A_n is finite dimensional and simple. By Theorem 2.7.2, A_n is isomorphic as a C*-algebra to the matrix algebra $M_{k\times k}(\mathbb{C})$ for some $k \in \mathbb{Z}_{>0}$. By Theorem 1.8.1, each A_n admits a unique tracial state. Hence, A also admits a unique tracial state.

We claim that A is a simple C*-algebra. To see why this is the case, we have to prove a few results first. Fortunately, most of the work has already been done.

Theorem 2.7.7. *Let $\mathcal{S}$ be a non-empty set of simple C*-subalgebras of a C*-algebra A. Suppose that the set $\mathcal{S}$ is upwards-directed — that is, if $B, C \in \mathcal{S}$ then there exists $D \in \mathcal{S}$ such that $B \subseteq D$ and $C \subseteq D$. Suppose also that the union $\bigcup_{T\in\mathcal{S}} T$ is dense in A. Then, A is a simple C*-algebra.*

Proof. Assume that A is a C*-algebra and $\mathcal{S}$ is a non-empty set of simple C*-subalgebras of A. Assume that $\mathcal{S}$ is upwards-directed and that the union $\bigcup_{T\in\mathcal{S}} T$ is dense in A.

We will use Theorem 2.7.3 to prove that A is simple. Assume that B is a non-zero C*-algebra and $\pi : A \to B$ is a surjective *-homomorphism. If $C \in \mathcal{S}$ then the restriction $\pi|_C$ is either the zero map or a surjective *-homomorphism on its non-zero image. In the latter case, $\pi|_C$ is injective because C is simple. Hence, $\pi|_C$ is an isometry.

By assumption, the map π is not the zero map on $\bigcup_{T\in\mathcal{S}} T$. Since $\mathcal{S}$ is upwards-directed, π cannot be the zero map on any non-zero $C \in \mathcal{S}$.

Otherwise, if $\pi|_C = 0$ then the restriction $\pi|_D = 0$ where $D \in \mathcal{S}$ such that $D \subseteq C$. Using the fact that $\mathcal{S}$ is upwards-directed, there exists $E \in \mathcal{S}$ such that $C \subseteq E$. Since E is a simple C*-algebra, the closed ideal $\ker \pi|_E$ is either 0 or E. However, $C \subseteq \ker \pi|_E$ and C is non-zero. So, $\ker \pi|_E = E$ and $\pi|_E = 0$. By iterating this argument, we find that π is the zero map on the union $\bigcup_{T \in \mathcal{S}} T$.

Therefore, π is an isometry on $\bigcup_{T \in \mathcal{S}} T$. Since the union $\bigcup_{T \in \mathcal{S}} T$ is dense in A, π must also be an isometry on A by continuity. Hence, π is injective on A and by Theorem 2.7.3, A is a simple C*-algebra. $\square$

Theorem 2.7.8 is an application of Theorem 2.7.7.

Theorem 2.7.8. *Let $\{(A_n, \varphi_n)\}_{n \in \mathbb{Z}_{>0}}$ be a direct sequence of simple C*-algebras. Then, the direct limit $\varinjlim A_n$ is also a simple C*-algebra.*

Proof. Assume that $\{(A_n, \varphi_n)\}_{n \in \mathbb{Z}_{>0}}$ is a direct sequence of simple C*-algebras. If $n \in \mathbb{Z}_{>0}$ then let $\varphi^n : A_n \to \varinjlim A_n$ denote the natural map. Define

$$\mathcal{S} = \{\varphi^n(A_n) \mid n \in \mathbb{Z}_{>0}\}.$$

Recall from Theorem 2.5.1 that $\mathcal{S}$ is an upwards-directed set whose union $\bigcup_{n=1}^{\infty} \varphi^n(A_n)$ is dense in $\varinjlim A_n$. By Theorem 2.7.7, we deduce that the direct limit $\varinjlim A_n$ is a simple C*-algebra. $\square$

Theorem 2.7.9. *Let A be a UHF algebra. Then, A is a simple C*-algebra.*

Proof. Assume that A is a UHF algebra. By Theorem 2.5.3, A is the direct limit of the direct sequence of simple C*-algebras $\{(A_n, \iota_n)\}_{n \in \mathbb{Z}_{>0}}$ where $\iota_n : A_n \hookrightarrow A_{n+1}$ is the inclusion map. By Theorem 2.7.8, we find that A is a simple C*-algebra as required. $\square$

In [Gli59], Glimm proved that there are uncountably many UHF algebras which are not isomorphic to each other as C*-algebras. We will now prove this fact. Let $n, d \in \mathbb{Z}_{>0}$. Define the unital *-homomorphism

$$\begin{array}{rccc} \varphi : & M_{n \times n}(\mathbb{C}) & \to & M_{dn \times dn}(\mathbb{C}) \\ & a & \mapsto & \begin{pmatrix} a & & & \\ & a & & \\ & & \ddots & \\ & & & a \end{pmatrix}. \end{array}$$

We call φ the canonical map from $M_{n\times n}(\mathbb{C})$ to $M_{dn\times dn}(\mathbb{C})$. Let $\mathcal{S}$ denote the set of all functions $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$. If $s \in \mathcal{S}$ then define the function

$$\begin{array}{rccc} s! : & \mathbb{Z}_{>0} & \to & \mathbb{Z}_{>0} \\ & n & \mapsto & s(1)s(2)\dots s(n). \end{array}$$

If $n \in \mathbb{Z}_{>0}$ then let $\varphi_n : M_{s!(n)\times s!(n)}(\mathbb{C}) \to M_{s!(n+1)\times s!(n+1)}(\mathbb{C})$ denote the canonical map. Define M_s to be the direct limit of the sequence

$$\{(M_{s!(n)\times s!(n)}(\mathbb{C}), \varphi_n)\}_{n\in\mathbb{Z}_{>0}}.$$

By construction, M_s is the direct limit of a sequence of finite dimensional simple algebras. Hence, it is a UHF algebra. Now let $\mathcal{P} \subseteq \mathbb{Z}_{>0}$ denote the set of prime numbers. If $s \in \mathcal{S}$ then we define

$$\begin{array}{rccc} \epsilon_s : & \mathcal{P} & \to & \mathbb{Z}_{>0} \cup \{\infty\} \\ & p & \mapsto & \sup\{m \in \mathbb{Z}_{>0} \mid \ p^m \text{ divides some } s!(n)\ \}. \end{array}$$

The point of ϵ_s is that it can tell us when M_s and M_t are isomorphic where $s, t \in \mathcal{S}$.

Theorem 2.7.10. *Let $s, s' \in \mathcal{S}$ and assume that $M_s \cong M_{s'}$ as C*-algebras. Then, $\epsilon_s = \epsilon_{s'}$.*

Proof. Assume that $s, s' \in \mathcal{S}$ and that M_s is isomorphic to $M_{s'}$ as C*-algebras. Let $\pi : M_s \to M_{s'}$ be a *-isomorphism. Let τ and τ' be the unique tracial states of M_s and $M_{s'}$ respectively. If $n \in \mathbb{Z}_{>0}$ then let $\varphi^n : M_{s!(n)\times s!(n)}(\mathbb{C}) \to M_s$ and $\psi^n : M_{s'!(n)\times s'!(n)}(\mathbb{C}) \to M_{s'}$ be the natural maps.

First, observe that the composite $\tau' \circ \pi$ is a tracial state on M_s. By uniqueness, we have $\tau' \circ \pi = \tau$.

To show: (a) $\epsilon_s \leq \epsilon_{s'}$.

(a) In order to prove this statement, we will show that if $n \in \mathbb{Z}_{>0}$ then there exists $m \in \mathbb{Z}_{>0}$ such that $s!(n)$ divides $s'!(m)$. This is enough because if $p \in \mathbb{Z}_{>0}$ is prime and $k \in \mathbb{Z}_{>0}$ such that p^k divides $s!(n)$ then from the statement we want to prove, p^k must also divide $s'!(m)$ and subsequently, $\epsilon_s(p) \leq \epsilon_s(p')$.

So, assume that $n \in \mathbb{Z}_{>0}$ and q is a rank one projection in $M_{s!(n)\times s!(n)}(\mathbb{C})$. Consider the composite $\tau \circ \varphi^n : M_{s!(n)\times s!(n)}(\mathbb{C}) \to \mathbb{C}$. It is a tracial state on

$M_{s!(n)\times s!(n)}(\mathbb{C})$. By uniqueness, the composite $\tau \circ \varphi^n$ is the tracial state given in Theorem 1.8.1 and

$$\tau(\varphi^n(q)) = \frac{1}{s!(n)}.$$

Since π and φ^n are both *-homomorphisms, $\pi(\varphi^n(q)) \in M_{s'}$ is a projection. Now we use the fact that the *-subalgebra

$$\bigcup_{k\in\mathbb{Z}_{>0}} \psi^k(M_{s'!(k)\times s'!(k)}(\mathbb{C}))$$

is dense in $M_{s'}$ to obtain a positive integer $m \in \mathbb{Z}_{>0}$ and a self-adjoint element $a \in M_{s'!(m)\times s'!(m)}(\mathbb{C})$ such that

$$\|\pi(\varphi^n(q)) - \psi^m(a)\| < \frac{1}{8} \qquad \text{and} \qquad \|\pi(\varphi^n(q)) - \psi^m(a^2)\| < \frac{1}{8}.$$

So,

$$\begin{aligned}
\|a - a^2\| &= \|\psi^m(a) - \psi^m(a^2)\| \\
&\le \|\psi^m(a) - \pi(\varphi^n(q))\| + \|\pi(\varphi^n(q)) - \psi^m(a^2)\| \\
&< \frac{1}{8} + \frac{1}{8} = \frac{1}{4}.
\end{aligned}$$

By Theorem 2.7.5, there exists a projection r in $M_{s'!(m)\times s'!(m)}(\mathbb{C})$ such that $\|a - r\| < \frac{1}{2}$. So, $\psi^m(r)$ is a projection in $M_{s'}$ satisfying

$$\begin{aligned}
\|\pi(\varphi^n(q)) - \psi^m(r)\| &\le \|\pi(\varphi^n(q)) - a\| + \|a - r\| \\
&< \frac{1}{8} + \frac{1}{2} < 1.
\end{aligned}$$

By Theorem 2.7.4, the projections $\pi(\varphi^n(q))$ and $\psi^m(r)$ in $M_{s'}$ are unitarily equivalent. Since τ' is the unique tracial state on $M_{s'}$,

$$\tau'(\psi^m(r)) = \tau'(\pi(\varphi^n(q))) = \tau(\varphi^n(q)) = \frac{1}{s!(n)}.$$

But, $\tau' \circ \psi^m$ is a tracial state on $M_{s'!(m)\times s'!(m)}(\mathbb{C})$, which must be unique. Since r is a projection in $M_{s'!(m)\times s'!(m)}(\mathbb{C})$, there exists $d \in \mathbb{Z}_{>0}$ such that

$$\tau'(\psi^m(r)) = \frac{d}{s'!(m)}.$$

Hence, $\frac{d}{s'!(m)} = \frac{1}{s!(n)}$ and $ds!(n) = s'!(m)$ as required. This proves part (a).

The inequality $\epsilon_{s'} \leq \epsilon_s$ follows by a symmetric argument to that used in part (a). Hence, $\epsilon_s = \epsilon_{s'}$. $\square$

Theorem 2.7.11. *There exists an uncountable number of UHF algebras which are not isomorphic to each other.*

Proof. If $n \in \mathbb{Z}_{>0}$ then let p_n denote the n^{th} prime number. If $s \in \mathcal{S}$ then we define

$$\begin{array}{rccc} \overline{s}: & \mathbb{Z}_{>0} & \to & \mathbb{Z}_{>0} \\ & n & \mapsto & p_n^{s(n)}. \end{array}$$

Then, $\epsilon_{\overline{s}}(p_n) = s(n)$. Now let $s' \in \mathcal{S}$. If $\epsilon_{\overline{s}} = \epsilon_{\overline{s'}}$ then

$$p_n^{s(n)} = \epsilon_{\overline{s}}(p_n) = \epsilon_{\overline{s'}}(p_n) = p_n^{s'(n)}$$

and $s = s'$.

Now consider the family of UHF algebras $\{M_{\overline{s}}\}_{s \in \mathcal{S}}$. This is an uncountable family because $\mathcal{S}$ itself is uncountable (it is isomorphic as sets to $\mathbb{R}$). If $s, t \in \mathcal{S}$ are distinct then $\epsilon_{\overline{s}} \neq \epsilon_{\overline{t}}$ and by Theorem 2.7.10, $M_{\overline{s}}$ is not isomorphic to $M_{\overline{t}}$. $\square$

We will now investigate an application of UHF algebras to the theory of von Neumann algebras.

Definition 2.7.4. Let H be a Hilbert space. A **factor** on H is a von Neumann algebra A on H such that $A \cap A' = \mathbb{C}id_H$ (recall that A' is the commutant of A).

A basic example of a factor is once again $B(H)$. Our application will give an example of an infinite dimensional factor which is not isomorphic to $B(H)$ for any Hilbert space H.

Definition 2.7.5. Let H be a Hilbert space and A be a von Neumann algebra on H. We say that A is **hyperfinite** if there exists a weakly dense C*-subalgebra W of A such that W is a UHF algebra and whose unit is id_H.

Example 2.7.3. We claim that if H is a separable Hilbert space then $B(H)$ is hyperfinite. This is easy to see if H is finite dimensional. To prove this for the infinite dimensional case, let A be an infinite dimensional UHF algebra and (φ, H) be a non-zero irreducible representation of A.

Since A is a UHF algebra, it is simple by Theorem 2.7.9. By Theorem 2.7.3, the *-homomorphism $\varphi : A \to B(H)$ defines a *-isomorphism from A to its image $\varphi(A)$. Hence, $\varphi(A)$ is a UHF algebra.

Now let $x \in H - \{0\}$. By Theorem 1.9.6, x is a cyclic vector for the representation (φ, H). So,

$$H = \overline{\varphi(A)x}.$$

Now since A is infinite dimensional, H must also be infinite dimensional. Also, since A is separable, H must also be a separable Hilbert space. Since (φ, H) is irreducible, then by Theorem 1.9.7, $\varphi(A)' = \mathbb{C}id_H$ and the double commutant $\varphi(A)'' = B(H)$. Therefore, $\varphi(A)$ is a UHF subalgebra of $B(H)$ which contains the identity map id_H and is weakly dense in $B(H)$ (see [Mur90, Theorem 4.2.5]). So, $B(H)$ is a hyperfinite algebra.

Now we will give our example of a factor in the following theorem.

Theorem 2.7.12. *Let A be a UHF algebra with unique tracial state τ. Let $(\varphi_\tau, H_\tau, \xi_\tau)$ be the GNS representation of A. Then, the von Neumann algebra $\varphi_\tau(A)''$ is a hyperfinite factor which admits a faithful tracial state.*

Proof. Assume that A is a UHF algebra with unique tracial state τ. Assume that $(\varphi_\tau, H_\tau, \xi_\tau)$ is the GNS representation of A. Let $B = \varphi_\tau(A)$.

To show: (a) B is hyperfinite.

(a) Since A is a UHF algebra, it is simple by Theorem 2.7.9. By Theorem 2.7.3, φ_τ defines a *-isomorphism from A to B. Therefore, B is a UHF algebra. By [Mur90, Theorem 4.2.5], B is weakly dense in its double commutant B''.

Since the GNS representation $(\varphi_\tau, H_\tau, \xi_\tau)$ is cyclic, it is non-degenerate. Since A is unital, then by Theorem 1.9.3, $\varphi_\tau(1_A) = id_{H_\tau}$. Hence, $id_{H_\tau} \in B \subseteq B''$ and consequently, B'' is a hyperfinite algebra.

Now we will construct a faithful tracial state on B''. Assume that $u, u' \in B$. Then, there exists $a, a' \in A$ such that $\varphi_\tau(a) = u$ and $\varphi_\tau(a') = u'$. So,

$$
\begin{aligned}
\langle uu'(\xi_\tau), \xi_\tau \rangle &= \langle \varphi_\tau(aa')(\xi_\tau), \xi_\tau \rangle \\
&= \langle \varphi_\tau(aa')(1_A + N_\tau), 1_A + N_\tau \rangle \\
&= \langle aa' + N_\tau, 1_A + N_\tau \rangle \\
&= \tau(1_A aa') = \tau(aa') = \tau(a'a) \\
&= \langle a'a + N_\tau, 1_A + N_\tau \rangle \\
&= \langle \varphi_\tau(a'a)(1_A + N_\tau), 1_A + N_\tau \rangle \\
&= \langle u'u(\xi_\tau), \xi_\tau \rangle .
\end{aligned}
$$

Since B is weakly dense in B'', the above identity also holds for $u, u' \in B''$. Now define

$$
\begin{array}{rccc}
\omega : & B'' & \to & \mathbb{C} \\
& u & \mapsto & \langle u(\xi_\tau), \xi_\tau \rangle .
\end{array}
$$

Then, ω is a tracial state on B''.

To show: (b) ω is faithful.

(b) Assume that $u \in B''$ satisfies $\omega(u^*u) = 0$. Then,

$$\omega(u^*u) = \langle u^*u(\xi_\tau), \xi_\tau \rangle = \langle u(\xi_\tau), u(\xi_\tau) \rangle = \|u(\xi_\tau)\|^2 = 0.$$

Hence, $u(\xi_\tau) = 0$. We claim that this implies that $u = 0$. If $v \in B$ then

$$\|uv(\xi_\tau)\|^2 = \langle v^*u^*uv(\xi_\tau), \xi_\tau \rangle = \langle vv^*u^*u(\xi_\tau), \xi_\tau \rangle$$

where the last equality follows from the tracial condition on ω. Since $u(\xi_\tau) = 0$, then $\|uv(\xi_\tau)\|^2 = 0$. From this, we find that $uB\xi_\tau = u\varphi_\tau(A)\xi_\tau = 0$. Since ξ_τ is a cyclic vector, $\overline{\varphi_\tau(A)\xi_\tau} = H$ and consequently, $u = 0$. Therefore, ω is a faithful tracial state on B''.

To show: (c) B'' is a factor.

(c) By Theorem 1.12.6, B'' is a von Neumann algebra. Let $p \in B' \cap B''$ be a projection. Define

$$
\begin{array}{rccc}
\omega' : & B'' & \to & \mathbb{C} \\
& u & \mapsto & \omega(pu)
\end{array}
$$

Then, ω' defines a weakly continuous trace on B''. If ω' is restricted to the UHF algebra B, which has a unique tracial state we denote by ω_B, then there exists $t \in \mathbb{C}$ such that if $v \in B$ then

$$\omega'(v) = t\omega(v).$$

Since B is weakly dense in B'' and ω and ω' are both weakly continuous, we deduce that $\omega' = t\omega$ as functionals on B''. Hence,

$$\omega'(id_{H_\tau}) = \langle p(\xi_\tau), \xi_\tau \rangle = \langle t\xi_\tau, \xi_\tau \rangle = t\omega(id_{H_\tau}).$$

and $\omega(p) = t$. Now consider $\omega'(id_{H_\tau} - p)$. We have

$$\omega'(id_{H_\tau} - p) = \omega(p - p^2) = \omega(p - p) = 0.$$

However,

$$\omega'(id_{H_\tau} - p) = t\omega(id_{H_\tau} - p) = \omega(p)\omega(id_{H_\tau} - p).$$

We conclude that either $\omega(p) = 0$ or $\omega(id_{H_\tau} - p) = 0$. Since the tracial state ω is faithful, we deduce that either $p = 0$ or $p = id_{H_\tau}$. Consequently, the only projections in the von Neumann algebra $B' \cap B''$ are trivial.

Since a von Neumann algebra is the closed linear span of its projections, we deduce that $B' \cap B'' = \mathbb{C}id_{H_\tau}$. So, B'' is a factor as required. □

Now suppose that in the statement of Theorem 2.7.12, we let A be an infinite dimensional UHF factor. Then, $\varphi_\tau(A)''$ is a hyperfinite factor which is not *-isomorphic to $B(H)$ for any Hilbert space H. This is because if H is infinite dimensional then $B(H)$ does not admit a faithful tracial state.

We can weaken the definition of a UHF algebra slightly in order to obtain AF-algebras (approximately finite).

Definition 2.7.6. Let A be a C*-algebra. We say that A is an **AF-algebra** if A contains an increasing sequence $\{A_n\}_{n=1}^\infty$ of finite dimensional C*-subalgebras such that the union $\bigcup_{n=1}^\infty A_n$ is dense in A.

Example 2.7.4. Suppose that A is a direct limit of a direct sequence $\{(A_n, \varphi_n)\}_{n=1}^\infty$ of C*-algebras where the A_n are finite dimensional. Then, A is an AF-algebra by definition of the direct limit (see Theorem 2.5.1).

By Theorem 2.5.3, any AF-algebra is isomorphic as a C*-algebra to a direct limit of finite dimensional C*-algebras.

A useful property about AF-algebras is that they are stable under taking closed ideals and quotients.

Theorem 2.7.13. *Let A be an AF-algebra and I be a closed ideal of A. Then, I and A/I are AF-algebras.*

Proof. Assume that A is an AF-algebra. Assume that I is a closed ideal of A.

To show: (a) A/I is an AF-algebra.

(b) I is an AF-algebra.

(a) Let $\pi : A \to A/I$ denote the projection map. Since A is a UHF algebra, it contains a increasing sequence $\{A_n\}_{n\in\mathbb{Z}_{>0}}$ of finite dimensional C*-subalgebras such that $\bigcup_{n=1}^{\infty} A_n$ is dense in A. By Theorem 1.7.6, the sequence $\{\pi(A_n)\}_{n\in\mathbb{Z}_{>0}}$ is an increasing sequence of finite dimensional C*-subalgebras of A/I. Moreover,

$$\overline{\bigcup_{n\in\mathbb{Z}_{>0}} \pi(A_n)} = \pi(\overline{\bigcup_{n\in\mathbb{Z}_{>0}} A_n}) = \pi(A) = A/I.$$

Therefore, A/I is an AF-algebra.

(b) If $n \in \mathbb{Z}_{>0}$ then define $I_n = I \cap A_n$. Then, the sequence $\{I_n\}_{n\in\mathbb{Z}_{>0}}$ is an increasing sequence of finite dimensional C*-subalgebras. We need to show that

$$\overline{\bigcup_{n\in\mathbb{Z}_{>0}} I_n} = I.$$

Let $J = \overline{\bigcup_{n\in\mathbb{Z}_{>0}} I_n}$. Define

$$\begin{array}{rccc} \varphi: & A/J & \to & A/I \\ & a+J & \mapsto & a+I \end{array}$$

Then, φ is a well-defined *-homomorphism.

To show: (ba) φ is isometric.

(ba) By part (a), it suffices to prove that φ is isometric on each C*-subalgebra in the increasing sequence

$$\{(A_n + J)/J\}_{n\in\mathbb{Z}_{>0}}$$

because the increasing union $\bigcup_n (A_n + J)/J$ is a dense *-subalgebra of A/J. If $n \in \mathbb{Z}_{>0}$ then let $\psi : (A_n + J)/J \to A_n/(A_n \cap J)$ and

$\theta : (A_n + I)/I \to A_n/(A_n \cap I)$ be *-isomorphisms (the second isomorphism theorem). Let $\iota_n : (A_n + I)/I \hookrightarrow A/I$ denote the inclusion.

Observe that $I_n = A_n \cap I = A_n \cap J$ and that the restriction of φ to $(A_n + J)/J$ is simply the composite $\iota_n \circ \theta^{-1} \circ \psi$. Notably, the restriction of φ is a composite of isometries. Hence, φ is an isometry on $(A_n + J)/J$ and consequently, an isometry on A/J.

(b) Since φ is isometric, we conclude that $J = I$. Hence, I is an AF-algebra as required. □

A useful consequence of Theorem 2.7.13 is the following theorem:

Theorem 2.7.14. *Let A be a C*-algebra. Then, A is an AF-algebra if and only if its unitization $\tilde{A}$ is an AF-algebra.*

2.8 Tensor products of C*-algebras

We begin by recalling the tensor product for the category of $\mathbb{C}$-vector spaces. If H and K are $\mathbb{C}$-vector spaces then their **algebraic tensor product** is the $\mathbb{C}$-vector space

$$H \otimes K = \operatorname{span}\{x \otimes y \mid x \in H, y \in K\}.$$

The primary use of tensor products is to turn bilinear (or multilinear) maps into linear maps via its universal property.

Theorem 2.8.1. *Let H and K be $\mathbb{C}$-vector spaces. Let ϕ be the $\mathbb{C}$-bilinear map*

$$\begin{array}{rccc} \phi : & H \times K & \to & H \otimes K \\ & (h,k) & \mapsto & h \otimes k. \end{array}$$

Let V be another $\mathbb{C}$-vector space and $f : H \times K \to V$ be a $\mathbb{C}$-bilinear map. Then, there exists a unique $\mathbb{C}$-linear map $\overline{f} : H \otimes K \to V$ such that the following diagram commutes:

$$\begin{tikzcd} H \times K \arrow[r, "\phi"] \arrow[dr, "f"'] & H \otimes K \arrow[d, dashed, "\overline{f}"] \\ & V \end{tikzcd}$$

Here is a quick application of Theorem 2.8.1.

Example 2.8.1. Let H and K be $\mathbb{C}$-vector spaces. Let $\tau : H \to \mathbb{C}$ and $\rho : K \to \mathbb{C}$ be linear functionals. Define the map

$$\begin{array}{rccc} \tau \times \rho : & H \times K & \to & \mathbb{C} \\ & (h,k) & \mapsto & \tau(h)\rho(k) \end{array}$$

The map $\tau \times \rho$ is $\mathbb{C}$-bilinear. By the universal property in Theorem 2.8.1, there exists a unique linear functional $\tau \otimes \rho : H \otimes K \to \mathbb{C}$ such that the following diagram commutes:

$$\begin{tikzcd} H \times K \arrow[r, "\phi"] \arrow[dr, "\tau\times\rho"'] & H \otimes K \arrow[d, dashed, "\tau\otimes\rho"] \\ & \mathbb{C} \end{tikzcd}$$

In particular, $\tau \otimes \rho$ is defined by $(\tau \otimes \rho)(h \otimes k) = \tau(h)\rho(k)$.

Now suppose that $\sum_{j=1}^{n} x_j \otimes y_j = 0$ where $x_j \in H$ and $y_j \in K$. We claim that if the set $\{y_1, y_2, \ldots, y_n\}$ is linearly independent in K then $x_1 = \cdots = x_n = 0$.

If $j \in \{1, 2, \ldots, n\}$ then let ρ_j be the linear functional defined by $\rho_j(y_i) = \delta_{i,j}$ where $\delta_{i,j}$ is the Kronecker delta. If $\tau : H \to \mathbb{C}$ is a linear functional then

$$\begin{aligned} 0 &= (\tau \otimes \rho_j)(\sum_{i=1}^{n} x_i \otimes y_i) \\ &= \sum_{i=1}^{n} \tau(x_i)\rho_j(y_i) \\ &= \sum_{i=1}^{n} \tau(x_i)\delta_{i,j} = \tau(x_j). \end{aligned}$$

So, $\tau(x_j) = 0$ for arbitrary $\tau \in H^*$. Therefore, if $j \in \{1, 2, \ldots, n\}$ then $x_j = 0$. Analogously, if the set $\{x_1, \ldots, x_n\}$ is linearly independent in H then $y_1 = \cdots = y_n = 0$.

Now if H and K are normed vector space then there are multiple different norms one can equip the tensor product $H \otimes K$ with. As we will see shortly, this makes the business of constructing tensor products of C*-algebras quite complicated. First, we will see that the tensor product of two Hilbert spaces is a relatively simple affair.

Theorem 2.8.2. *Let H and K be Hilbert spaces. Then, there exists a unique inner product $\langle -, - \rangle$ on the tensor product $H \otimes K$, defined by*

$$\begin{array}{rccc} \langle -, - \rangle : & (H \otimes K) \times (H \otimes K) & \to & \mathbb{C} \\ & \langle x \otimes y, x' \otimes y' \rangle & \mapsto & \langle x, x' \rangle \langle y, y' \rangle. \end{array}$$

Proof. Assume that H and K are Hilbert spaces. First observe that if $\tau : H \to \mathbb{C}$ and $\phi : K \to \mathbb{C}$ are conjugate-linear maps then there exists a unique conjugate-linear map

$$\begin{array}{rccc} \tau \otimes \phi : & H \otimes K & \to & \mathbb{C} \\ & (h, k) & \mapsto & \tau(h)\phi(k). \end{array}$$

Here is how the map $\tau \otimes \phi$ is constructed. The composites $\overline{\tau}$ and $\overline{\phi}$ are linear maps. By Theorem 2.8.1, we obtain a unique linear functional $\overline{\tau} \otimes \overline{\phi}$ and then set $\tau \otimes \phi = \overline{\overline{\tau} \otimes \overline{\phi}}$.

Now assume that $x \in H$ and $y \in K$. Let $\phi_x : H \to \mathbb{C}$ be the conjugate-linear functional defined by $h \mapsto \langle x, h \rangle$. Similarly, let $\phi_y : K \to \mathbb{C}$ be the conjugate-linear functional defined by $k \mapsto \langle y, k \rangle$. Then, there exists a unique conjugate-linear functional $\phi_x \otimes \phi_y : H \otimes K \to \mathbb{C}$ defined by

$$(\phi_x \otimes \phi_y)(h \otimes k) = \phi_x(h)\phi_y(k) = \langle x, h \rangle \langle y, k \rangle.$$

Next, let X be the vector space of conjugate-linear functionals on $H \otimes K$ and define

$$\begin{array}{rccc} M' : & H \times K & \to & X \\ & (h, k) & \mapsto & \phi_h \otimes \phi_k \end{array}$$

Then, M' is a $\mathbb{C}$-bilinear map. By Theorem 2.8.1, there exists a unique linear map $M : H \otimes K \to X$ such that $M(h \otimes k) = \phi_h \otimes \phi_k$.

Now, we define a sesquilinear form on $H \otimes K$ by

$$\begin{array}{rccc} \langle -, - \rangle : & (H \otimes K) \times (H \otimes K) & \to & \mathbb{C} \\ & \langle x \otimes y, x' \otimes y' \rangle & \mapsto & M(x \otimes y)(x' \otimes y'). \end{array}$$

Since the maps M and $\phi_x \otimes \phi_y$ are all unique, the map $\langle -, - \rangle$ must also be unique. It is straightforward to check that $\langle -, - \rangle$ defines a sesquilinear form on $H \otimes K$.

To see that $\langle -, - \rangle$ is an inner product on $H \otimes K$, it suffices to show that if $z \in H \otimes K$ and $\langle z, z \rangle = 0$ then $z = 0$. So, assume that $z \in H \otimes K$ and

$\langle z, z\rangle = 0$. Then, $z = \sum_{j=1}^{n}(x_j \otimes y_j)$. Now let $\{e_i\}_{i\in\mathbb{Z}_{>0}}$ be an orthonormal basis for K. Then, there exists $x_1', \ldots, x_n'$ such that

$$z = \sum_{j=1}^{n}(x_j' \otimes e_j).$$

Now we compute directly that

$$\begin{aligned}
\langle z, z\rangle &= \langle \sum_{j=1}^{n}(x_j' \otimes e_j), \sum_{j=1}^{n}(x_j' \otimes e_j)\rangle \\
&= \sum_{i,j=1}^{n}\langle x_i' \otimes e_i, x_j' \otimes e_j\rangle \\
&= \sum_{i,j=1}^{n}\langle x_i', x_j'\rangle\langle e_i, e_j\rangle \\
&= \sum_{i=1}^{n}\|x_i'\|^2 = 0.
\end{aligned}$$

Hence, if $i \in \{1, 2, \ldots, n\}$ then $\|x_i'\| = 0$ and $x_i' = 0$. Consequently, $z = 0$ and $\langle -, -\rangle$ is the unique inner product on $H \otimes K$ as required. $\square$

In the scenario of Theorem 2.8.2, we regard $H \otimes K$ as a pre-Hilbert space, equipped with its unique inner product.

Definition 2.8.1. Let H and K be Hilbert spaces and $H \otimes K$ be the pre-Hilbert space with the unique inner product in Theorem 2.8.2. The completion of $H \otimes K$, denoted by $H \hat{\otimes} K$, is called the **Hilbert space tensor product** of H and K.

Here are some easy consequences of Theorem 2.8.2. If $x \in H$ and $y \in K$ then

$$\|x \otimes y\| = \|x\|\|y\|.$$

Theorem 2.8.3. *Let H and K be Hilbert spaces. Let E_1 and E_2 be orthonormal bases for H and K respectively. Then,*

$$E_1 \otimes E_2 = \{x \otimes y \mid x \in E_1, y \in E_2\}$$

is an orthonormal basis for $H \hat{\otimes} K$.

If H' and K' are closed vector subspaces of H and K respectively then the inclusion map $\iota : H' \otimes K' \to H\hat{\otimes}K$ is isometric where $H' \otimes K'$ inherits its inner product from $H \otimes K$. Hence, we can consider $H'\hat{\otimes}K'$ as a closed vector subspace of $H\hat{\otimes}K$.

Before we move onto the next result regarding the Hilbert space tensor product, we recall the following result about bounded linear operators on a Hilbert space.

Theorem 2.8.4. *Let H be a Hilbert space and $x \in B(H)$. Then, x is a $\mathbb{C}$-linear combination of four unitary operators.*

Theorem 2.8.5. *Let H and K be Hilbert space. Let $u \in B(H)$ and $v \in B(K)$. Then, there exists a unique operator $u\hat{\otimes}v \in B(H\hat{\otimes}K)$ such that if $x \in H$ and $y \in K$ then*

$$(u\hat{\otimes}v) = u(x) \otimes v(y).$$

We also have $\|u\hat{\otimes}v\| = \|u\|\|v\|$.

Proof. Assume that H and K are Hilbert space. Assume that $u \in B(H)$ and $v \in B(K)$. Then, the map $(u, v) \mapsto u \otimes v$ is $\mathbb{C}$-bliinear. By Theorem 2.8.1, there exists a unique $\mathbb{C}$-linear map $u \otimes v : H \otimes K \to H \otimes K$, defined by $(u \otimes v)(x \otimes y) = u(x) \otimes v(y)$.

We would like to extend $u \otimes v$ to $H\hat{\otimes}K$ by using the universal property of completion. In order to do this, we first need to show that $u \otimes v$ is bounded. By Theorem 2.8.4, we may assume without loss of generality that u and v are unitary operators. If $z \in H \otimes K$ then we write $z = \sum_{j=1}^{n}(x_j \otimes k_j)$ where $\{k_1, \dots, k_n\}$ is an orthogonal subset of K. We have

$$\begin{aligned}
\|(u \otimes v)(z)\|^2 &= \|(u \otimes v)(\sum_{j=1}^{n}(x_j \otimes k_j))\|^2 \\
&= \|\sum_{j=1}^{n}(u(x_j) \otimes v(k_j))\|^2 = \sum_{j=1}^{n}\|u(x_j) \otimes v(k_j)\|^2 \\
&= \sum_{j=1}^{n}\|u(x_j)\|^2\|v(k_j)\|^2 = \sum_{j=1}^{n}\|x_j\|^2\|k_j\|^2 \\
&= \sum_{j=1}^{n}\|x_j \otimes k_j\|^2 = \|z\|^2.
\end{aligned}$$

The third equality follows from the fact that the set $\{v(k_1), v(k_2), \ldots, v(k_n)\}$ is orthogonal. The third last equality follows from the fact that u and v are isometries (because they were assumed to be unitary). Finally, the last equality follows from the fact that if $i, j \in \{1, 2, \ldots, n\}$ are distinct then $x_i \otimes k_i$ is orthogonal to $x_j \otimes k_j$.

We deduce that $\|u \otimes v\| = 1$. Since the linear map $u \otimes v$ is bounded for $u \in B(H)$ and $v \in B(K)$ we can apply the universal property of completion in order to extend $u \otimes v$ to a bounded linear map $u \hat{\otimes} v$ on the Hilbert space $H \hat{\otimes} K$.

To show: (a) If $u \in B(H)$ and $v \in B(K)$ then $\|u \hat{\otimes} v\| = \|u\|\|v\|$.

(a) Define the maps

$$\begin{array}{rccc} \iota_H : & B(H) & \to & B(H \hat{\otimes} K) \\ & u & \mapsto & u \hat{\otimes} id_K \end{array}$$

and

$$\begin{array}{rccc} \iota_K : & B(K) & \to & B(H \hat{\otimes} K) \\ & v & \mapsto & id_H \hat{\otimes} v \end{array}$$

Then, ι_H and ι_K are injective $\mathbb{C}$-linear maps. We claim that ι_H and ι_K are in fact, *-homomorphisms. If $x \in H$, $y \in K$ and $u_1, u_2 \in B(H)$ then

$$\begin{aligned} \iota_H(u_1 u_2)(x \otimes y) &= (u_1 u_2 \hat{\otimes} id_K)(x \otimes y) \\ &= u_1(u_2(x)) \hat{\otimes} y \\ &= (u_1 \hat{\otimes} id_K)(u_2(x) \otimes y) \\ &= \iota_H(u_1)\iota_H(u_2)(x \otimes y). \end{aligned}$$

Hence, $\iota_H(u_1 u_2) = \iota_H(u_1)\iota_H(u_2)$ on $H \otimes K$. Since $H \otimes K$ is dense in $H \hat{\otimes} K$, $\iota_H(u_1 u_2) = \iota_H(u_1)\iota_H(u_2)$ on $H \hat{\otimes} K$. Similarly, if $v_1, v_2 \in B(K)$ then $\iota_K(v_1 v_2) = \iota_K(v_1)\iota_K(v_2)$. To show that ι_H preserves adjoints, first assume that $x_1, x_2 \in H$, $y_1, y_2 \in K$, $u \in B(H)$ and $v \in B(K)$. Then,

$$\begin{aligned} \langle (u \hat{\otimes} v)^*(x_1 \otimes y_1), x_2 \otimes y_2 \rangle &= \langle x_1 \otimes y_1, (u \hat{\otimes} v)(x_2 \otimes y_2) \rangle \\ &= \langle x_1 \otimes y_1, u(x_2) \otimes v(y_2) \rangle \\ &= \langle x_1, u(x_2) \rangle \langle y_1, v(y_2) \rangle \\ &= \langle u^*(x_1), x_2 \rangle \langle v^*(y_1), y_2 \rangle \\ &= \langle (u^* \hat{\otimes} v^*)(x_1 \otimes y_1), x_2 \otimes y_2 \rangle. \end{aligned}$$

So, $(u\hat{\otimes}v)^* = u^*\hat{\otimes}v^*$ on $H \otimes K$ and subsequently, on $H\hat{\otimes}K$. Consequently,

$$\iota_H(u^*) = u^*\hat{\otimes}id_K = (u\hat{\otimes}id_K)^* = \iota_H(u)^*.$$

Of course, we also have $\iota_K(v^*) = \iota_K(v)^*$. Therefore, ι_H and ι_K are injective *-homomorphisms and are thus, isometries by Theorem 1.6.4. Hence,

$$\|u\hat{\otimes}v\| = \|(u\hat{\otimes}id_K)(id_H\hat{\otimes}v)\| \leq \|\iota_H(u)\|\|\iota_K(v)\| = \|u\|\|v\|.$$

To prove the reverse inequality, assume that $\epsilon \in \mathbb{R}_{>0}$. Assume without loss of generality that $u, v \neq 0$. Then, there exists unit vectors $x \in H$ and $y \in K$ such that

$$\|u(x)\| > \|u\| - \epsilon > 0 \qquad \text{and} \qquad \|v(y)\| > \|v\| - \epsilon > 0.$$

This means that

$$\|(u\hat{\otimes}v)(x \otimes y)\| = \|u(x)\|\|v(y)\| > (\|u\| - \epsilon)(\|v\| - \epsilon).$$

Since $\epsilon \in \mathbb{R}_{>0}$ was arbitrary, we obtain the desired inequality. So, $\|u\hat{\otimes}v\| = \|u\|\|v\|$. □

So far, we understand how to construct Hilbert spaces by using the tensor product (in the category of $\mathbb{C}$-vector spaces) and then completing to a Hilbert space. In our quest to define the tensor product of C*-algebras, we now have to define a suitable notion of multiplication and involution.

We will first address multiplication. Let A and B be $\mathbb{C}$-algebras. Since A and B are $\mathbb{C}$-vector spaces, we can construct the tensor product $A \otimes B$. We will construct multiplication on $A \otimes B$ so that $A \otimes B$ is itself a $\mathbb{C}$-algebra. If $a \in A$ then let L_a denote left multiplication by a and let X be the vector space of linear maps on $A \otimes B$. Then, if $a \in A$ and $b \in B$ then $L_a \otimes L_b \in X$ and the map $(a, b) \mapsto L_a \otimes L_b$ is $\mathbb{C}$-bilinear.

By Theorem 2.8.1, there exists a unique linear map M, defined explicitly as

$$\begin{array}{rccc} M: & A \otimes B & \to & X \\ & a \otimes b & \mapsto & L_a \otimes L_b. \end{array}$$

Now consider the map

$$\begin{array}{rccc} \cdot: & (A \otimes B)^2 & \to & A \otimes B \\ & (c, d) & \mapsto & M(c)(d). \end{array} \tag{2.1}$$

This map is $\mathbb{C}$-bilinear and since M is unique, it defines the unique multiplication on $A \otimes B$. Explicitly, if $a, a' \in A$ and $b, b' \in B$ then

$$(a \otimes b) \cdot (a' \otimes b') = M(a \otimes b)(a' \otimes b') = aa' \otimes bb'.$$

Definition 2.8.2. Let A and B be $\mathbb{C}$-algebras. The tensor product of $\mathbb{C}$-vector spaces $A \otimes B$, equipped with the bilinear multiplication in equation (2.1), is a $\mathbb{C}$-algebra called the **algebra tensor product** of A and B.

Next, we address involution. Let A and B be *-algebras so that $A \otimes B$ is a $\mathbb{C}$-algebra. We will define involution on $A \otimes B$ so that $A \otimes B$ becomes a *-algebra. The obvious way to do this is to define

$$(a \otimes b)^* = a^* \otimes b^*$$

for $a \in A$ and $b \in B$. It is straightforward to check that this satisfies all the properties demanded by an involution. However, we have to show that involution is well-defined. We do this by showing that if $\sum_{j=1}^{n}(a_j \otimes b_j) = 0$ then $\sum_{j=1}^{n}(a_j^* \otimes b_j^*) = 0$.

To this end, assume that $\sum_{j=1}^{n}(a_j \otimes b_j) = 0$. Let $\{c_1, \ldots, c_m\}$ be a linearly independent set in B with the same linear span as the set $\{b_1, \ldots, b_n\}$. If $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, m\}$ then there exists scalars $\lambda_{i,j} \in \mathbb{C}$ such that

$$b_i = \sum_{j=1}^{m} \lambda_{i,j} c_j.$$

Therefore,

$$\begin{aligned} 0 &= \sum_{i=1}^{n}(a_i \otimes b_i) \\ &= \sum_{i=1}^{n}(a_i \otimes \sum_{j=1}^{m} \lambda_{i,j} c_j) \\ &= \sum_{i=1}^{n}(\sum_{j=1}^{m} \lambda_{i,j} a_i \otimes c_j) \end{aligned}$$

So, $\sum_{j=1}^{m} \lambda_{i,j} a_i = 0$ in A since the set $\{c_1, c_2, \ldots, c_m\}$ is linearly independent. Consequently, $\sum_{j=1}^{m} \overline{\lambda_{i,j}} a_i^* = 0$ and

$$
\begin{aligned}
\sum_{i=1}^{n}(a_i^* \otimes b_i^*) &= \sum_{i=1}^{n}(a_i^* \otimes \sum_{j=1}^{m} \overline{\lambda_{i,j}} c_j^*) \\
&= \sum_{i=1}^{n}(\sum_{j=1}^{m} \overline{\lambda_{i,j}} a_i^* \otimes c_j^*) \\
&= \sum_{i=1}^{n}(0 \otimes c_j^*) = 0.
\end{aligned}
$$

Definition 2.8.3. Let A and B be *-algebras. The algebra tensor product $A \otimes B$, equipped with the involution above, is a *-algebra called the ***-algebra tensor product** of A and B.

In a similar manner to before, if A' and B' are *-subalgebras of A and B respectively then we can regard $A' \otimes B'$ as a *-subalgebra of $A \otimes B$.

Here is how we can form *-homomorphisms on *-algebra tensor products.

Theorem 2.8.6. *Let A, B, C, D be *-algebras. Let $\phi : A \to B$ and $\psi : C \to D$ be *-homomorphisms. Then, the map*

$$
\begin{array}{rccc}
\phi \otimes \psi : & A \otimes C & \to & B \otimes D \\
& a \otimes c & \mapsto & \phi(a) \otimes \psi(c)
\end{array}
$$

*is a *-homomorphism.*

Proof. Assume that A, B, C and D are *-algebras. Assume that $\phi : A \to B$ and $\psi : C \to D$ are *-homomorphisms. By Theorem 2.8.1, $\phi \otimes \psi$ is a linear map. Assume that $a, a' \in A$ and $c, c' \in C$. Then,

$$
\begin{aligned}
(\phi \otimes \psi)((a \otimes c)(a' \otimes c')) &= (\phi \otimes \psi)(aa' \otimes cc') \\
&= \phi(aa') \otimes \psi(cc') \\
&= \phi(a)\phi(a') \otimes \psi(c)\psi(c') \\
&= (\phi(a) \otimes \psi(c))(\phi(a') \otimes \psi(c')) \\
&= (\phi \otimes \psi)(a \otimes c)(\phi \otimes \psi)(a' \otimes c').
\end{aligned}
$$

We also have

$$
(\phi \otimes \psi)((a \otimes c)^*) = (\phi \otimes \psi)(a^* \otimes c^*) = \phi(a)^* \otimes \psi(c)^* = (\phi \otimes \psi)(a \otimes c)^*.
$$

Therefore, $\phi \otimes \psi$ is a *-homomorphism from $A \otimes C$ to $B \otimes D$. $\square$

At this point, we have set up enough machinery to discuss tensor products of C*-algebras. Previously, we mentioned that a tensor product of C*-algebras could have more than one possible norm. The next result will be used to demonstrate this claim.

Theorem 2.8.7. *Let A and B be C*-algebras. Let (φ, H) and (ψ, K) be representations of A and B respectively. Then, there exists a unique *-homomorphism $\pi : A \otimes B \to B(H\hat{\otimes}K)$ such that if $a \in A$ and $b \in B$ then*

$$\pi(a \otimes b) = \varphi(a)\hat{\otimes}\psi(b).$$

Furthermore, if φ and ψ are injective then π is injective.

Proof. Assume that A and B are C*-algebras with representations (φ, H) and (ψ, K) respectively. Define the maps

$$\begin{array}{rccc} \varphi' : & A & \to & B(H\hat{\otimes}K) \\ & a & \mapsto & \varphi(a)\hat{\otimes}id_K \end{array}$$

and

$$\begin{array}{rccc} \psi' : & B & \to & B(H\hat{\otimes}K) \\ & b & \mapsto & id_H\hat{\otimes}\psi(b). \end{array}$$

By Theorem 2.8.6, φ' and ψ' are *-homomorphisms. Furthermore, the images $\varphi'(A)$ and $\psi'(B)$ commute. Now define the map

$$\begin{array}{rccc} \alpha : & A \times B & \to & B(H\hat{\otimes}K) \\ & (a,b) & \mapsto & \varphi'(a)\psi'(b). \end{array}$$

Then, α is a bilinear map. By the universal property of the tensor product in Theorem 2.8.1, there exists a unique linear map $\pi : A \otimes B \to B(H\hat{\otimes}K)$ defined by

$$\pi(a \otimes b) = \alpha(a,b) = \varphi'(a)\psi'(b) = \varphi(a)\hat{\otimes}\psi(b).$$

Since φ' and ψ' are both *-homomorphisms, we deduce that π is also a *-homomorphism as required.

Now, assume that φ and ψ are both injective. Assume that $z \in \ker \pi$. Then, $z = \sum_{j=1}^{n}(a_j \otimes b_j)$, where $a_j \in A$ and $b_j \in B$. We may also assume that $\{b_1, b_2, \ldots, b_n\}$ is linearly independent. Since ψ is injective, then the set $\{\psi(b_1), \psi(b_2), \ldots, \psi(b_n)\}$ is linearly independent.

We have

$$\pi(z) = \sum_{j=1}^{n} (\varphi(a_j) \hat{\otimes} \psi(b_j)) = 0.$$

To show: (a) If $j \in \{1, 2, \ldots, n\}$ then $\varphi(a_j) = 0$.

(a) Assume that $h \hat{\otimes} k \in H \hat{\otimes} K$. Then, we can construct an orthonormal set $\{e_1, e_2, \ldots, e_m\}$ such that

$$\mathbb{C}\varphi(a_1)(h\hat{\otimes}k) + \cdots + \mathbb{C}\varphi(a_n)(h\hat{\otimes}k) \subseteq \mathbb{C}e_1 + \cdots + \mathbb{C}e_m.$$

Then, there exists $\lambda_{i,j} \in \mathbb{C}$ such that

$$\varphi(a_i)(h\hat{\otimes}k) = \sum_{j=1}^{m} \lambda_{i,j} e_j.$$

Now, if $h'\hat{\otimes}k' \in H\hat{\otimes}K$ then

$$\begin{aligned}
0 &= \sum_{i=1}^{n} (\varphi(a_i)(h\hat{\otimes}k) \hat{\otimes} \psi(b_i)(h'\hat{\otimes}k')) \\
&= \sum_{i=1}^{n} (\sum_{j=1}^{m} \lambda_{i,j} e_m \hat{\otimes} \psi(b_i)(h'\hat{\otimes}k')) \\
&= (\sum_{j=1}^{m} e_m \hat{\otimes} \sum_{i=1}^{n} \lambda_{i,j} \psi(b_i)(h'\hat{\otimes}k')).
\end{aligned}$$

Therefore, $\sum_{i=1}^{n} \lambda_{i,j}\psi(b_i)(h'\hat{\otimes}k') = 0$ and since the element $h'\hat{\otimes}k' \in H\hat{\otimes}K$ is arbitrary, then $\sum_{i=1}^{n} \lambda_{i,j}\psi(b_i) = 0$. By linear independence of $\{\psi(b_1), \ldots, \psi(b_n)\}$, if $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, m\}$ then $\lambda_{i,j} = 0$. Thus, if $i \in \{1, 2, \ldots, n\}$ then $\varphi(a_i) = 0$ as required.

By part (a) and the fact that φ is injective, we find that $a_1 = \cdots = a_n = 0$. Hence, $z = \sum_{j=1}^{n} (a_j \otimes b_j) = 0$ and π is injective as required. □

The *-homomorphism π in Theorem 2.8.7 is usually denoted as $\varphi\hat{\otimes}\psi$.

Definition 2.8.4. Let A be a C*-algebra and $S(A)$ denote the set of states of A. If $\phi \in S(A)$ then let $(\pi_\phi, H_\phi, \xi_\phi)$ be the GNS representation associated to A. Then, the direct sum of all GNS representations

$$(\bigoplus_{\phi \in S(A)} \pi_\phi, \bigoplus_{\phi \in S(A)} H_\phi)$$

is called the **universal representation** of A. Note that the universal representation is faithful — the *-homomorphism $\bigoplus_{\phi \in S(A)} \pi_\phi$ is injective.

Now let A and B be C*-algebras with universal representations (φ, H) and (ψ, K) respectively. By Theorem 2.8.7, there exists a unique injective *-homomorphism $\pi : A \otimes B \to B(H \hat{\otimes} K)$ such that $\pi(a \otimes b) = \varphi(a) \hat{\otimes} \psi(b)$. Define the function

$$\begin{array}{rccc} \|-\|_* : & A \otimes B & \to & \mathbb{R}_{\geq 0} \\ & c & \mapsto & \|\pi(c)\|. \end{array}$$

Since the norm on $B(H \hat{\otimes} K)$ satisfies the property of a C*-algebra, $\|-\|_*$ defines a C*-norm on $A \otimes B$.

Definition 2.8.5. Let A and B be C*-algebras with universal representations (φ, H) and (ψ, K) respectively. Let $\pi : A \otimes B \to B(H \hat{\otimes} K)$ denote the unique injective *-homomorphism constructed in Theorem 2.8.7. Then, the C*-norm $\|-\|_*$ on $A \otimes B$ is called the **spatial C*-norm**.

One of the basic properties of the spatial C*-norm is that if $a \in A$ and $b \in B$ then

$$\|a \otimes b\|_* = \|\pi(a \otimes b)\| = \|\varphi(a) \hat{\otimes} \psi(b)\| = \|\varphi(a)\| \|\psi(b)\| = \|a\| \|b\|$$

where in the last equality, we used Theorem 1.6.4.

Definition 2.8.6. Let A and B be C*-algebras. The C*-completion of $A \otimes B$ with respect to the spatial norm $\|-\|_*$ is called the **spatial tensor product** of A and B and is denoted by $A \otimes_* B$.

We will now demonstrates that there can be more than one C*-norm on $A \otimes B$.

Definition 2.8.7. Let A and B be C*-algebras and γ be a C*-norm on $A \otimes B$. The C*-completion of $A \otimes B$ with respect to γ will be denoted by $A \otimes_\gamma B$.

Theorem 2.8.8. *Let A and B be C*-algebras and γ be a C*-norm on $A \otimes B$. If $a' \in A$ and $b' \in B$ then define*

$$\begin{array}{rccc} \iota_{b'} : & A & \to & A \otimes_\gamma B \\ & a & \mapsto & a \otimes b' \end{array}$$

and

$$\begin{array}{rccc} \iota_{a'} : & B & \to & A \otimes_\gamma B \\ & b & \mapsto & a' \otimes b \end{array}$$

Then, $\iota_{a'}$ and $\iota_{b'}$ are both continuous maps.

Proof. Assume that A and B are C*-algebras. Assume that γ is a C*-norm on $A \otimes B$. Assume that $a' \in A$, $b' \in B$ and that the maps $\iota_{a'}$ and $\iota_{b'}$ are defined as above. Since A, B and $A \otimes_\gamma B$ are Banach spaces, the *closed graph theorem* applies.

Hence, to show that $\iota_{a'}$ and $\iota_{b'}$ are continuous, it suffices to show that their graphs are closed.

To show: (a) If $\{a_n\}_{n \in \mathbb{Z}_{>0}}$ is a sequence converging to 0 in A and the sequence $\{\iota_{b'}(a_n)\}_{n \in \mathbb{Z}_{>0}}$ converges to $c \in A \otimes_\gamma B$ then $c = 0$.

(a) Assume that $\{a_n\}_{n \in \mathbb{Z}_{>0}}$ is a sequence which converges to 0 in A. Assume that $\{\iota_{b'}(a_n)\}_{n \in \mathbb{Z}_{>0}}$ converges to $c \in A \otimes_\gamma B$. Note that we can assume this without loss of generality because every Banach space is a topological vector space and addition is itself continuous in a topological vector space.

Furthermore, we can assume that if $n \in \mathbb{Z}_{>0}$ then a_n and b' are positive (if not, we can simply replace a_n with $a_n^* a_n$ and b' by $(b')^* b'$). Hence, the sequence $\{\iota_{b'}(a_n)\}_{n \in \mathbb{Z}_{>0}}$ converges to $c \in A \otimes_\gamma B$ which is positive.

Now let τ be a positive linear functional on $A \otimes_\gamma B$. The composite $\rho = \tau \circ \iota_{b'}$ is a positive linear functional on A and hence, continuous. So,

$$\tau(c) = \tau(\lim_{n \to \infty} \iota_{b'}(a_n)) = \lim_{n \to \infty} (\tau \circ \iota_{b'})(a_n) = \lim_{n \to \infty} \rho(a_n) = \rho(0) = 0.$$

So, $\tau(c) = 0$ for any positive functional τ on $A \otimes_\gamma B$. Therefore, $c = 0$ and consequently, the graph of $\iota_{b'}$ is closed.

A similar argument to part (a) establishes that the graph of $\iota_{a'}$ is closed. By the closed graph theorem, $\iota_{a'}$ and $\iota_{b'}$ are both continuous. □

The next result is integral to the construction of our next C*-norm.

Theorem 2.8.9. *Let A and B be non-zero C*-algebras and γ be a C*-norm on $A \otimes B$. Let (π, H) be a non-degenerate representation of $A \otimes_\gamma B$. Then, there exists unique *-homomorphisms $\varphi : A \to B(H)$ and $\psi : B \to B(H)$ such that if $a \in A$ and $b \in B$ then*

$$\pi(a \otimes b) = \varphi(a)\psi(b) = \psi(b)\varphi(a).$$

Moreover, the representations (φ, H) and (ψ, H) are both non-degenerate.

Proof. Assume that A and B are non-zero C*-algebras and that $A \otimes_\gamma B$ is defined as above. Assume that (π, H) is a non-degenerate representation of $A \otimes_\gamma B$.

Let $H_0 = \pi(A \otimes B)H$. If $z \in H_0$ and $i \in \{1, 2, \ldots, n\}$ then there exists $a_i \in A$, $b_i \in B$ and $x_i \in H$ such that

$$z = \sum_{i=1}^{n} \pi(a_i \otimes b_i)(x_i).$$

We want to define a well-defined map on H_0. We will deal with whether this mysterious map is well-defined first before defining it. Suppose that $z \in H_0$ has two expressions of the above form:

$$z = \sum_{i=1}^{n} \pi(a_i \otimes b_i)(x_i) = \sum_{j=1}^{m} \pi(c_j \otimes d_j)(y_j).$$

By using Theorem 1.7.3, let $\{v_\mu\}_{\mu \in M}$ be an approximate unit for B. If $a \in A$ then

$$\pi(a \otimes v_\mu)(z) = \sum_{i=1}^{n} \pi(a \otimes v_\mu)\pi(a_i \otimes b_i)(x_i) = \sum_{i=1}^{n} \pi(aa_i \otimes v_\mu b_i)(x_i).$$

But, we also have

$$\pi(a \otimes v_\mu)(z) = \sum_{j=1}^{m} \pi(ac_j \otimes v_\mu d_j)(y_j).$$

By taking the limit with respect to the variable μ, we find that

$$\begin{aligned}
\lim_\mu \pi(a \otimes v_\mu)(z) &= \lim_\mu \sum_{i=1}^{n} \pi(aa_i \otimes v_\mu b_i)(x_i) \\
&= \sum_{i=1}^{n} \pi(\lim_\mu (aa_i \otimes v_\mu b_i))(x_i) \\
&= \sum_{i=1}^{n} \pi(aa_i \otimes b_i)(x_i)
\end{aligned}$$

where the last equality follows from the fact that tensoring with aa_i is continuous (see Theorem 2.8.8). Therefore,

$$\sum_{i=1}^{n} \pi(aa_i \otimes b_i)(x_i) = \sum_{j=1}^{m} \pi(ac_j \otimes d_j)(y_j).$$

Therefore, if $a \in A$ then the map

$$\begin{array}{cccc} \varphi(a): & H_0 & \to & H_0 \\ & z = \sum_{i=1}^{n} \pi(a_i \otimes b_i)(x_i) & \mapsto & \sum_{i=1}^{n} \pi(aa_i \otimes b_i)(x_i) \end{array}$$

is well-defined. Since $\varphi(a)(z) = \lim_\mu \pi(a \otimes v_\mu)(z)$, we deduce that $\varphi(a)$ is a linear map on H_0. To see that $\varphi(a)$ is bounded, we first use Theorem 2.8.8 to show that there exists $M \in \mathbb{Z}_{>0}$ depending on a such that

$$\|\pi(a \otimes b)\| \leq M\|b\|.$$

Hence,

$$\begin{aligned} \|\varphi(a)(z)\| &= \|\lim_\mu \pi(a \otimes v_\mu)(z)\| \\ &\leq \lim_\mu \|\pi(a \otimes v_\mu)\| \|z\| \\ &\leq \lim_\mu M\|v_\mu\| \|z\| \\ &\leq M\|z\|. \end{aligned}$$

So, $\|\varphi(a)\| \leq M$. This means that $\varphi(a)$ is a bounded linear map on H_0. Since the representation (π, H) of $A \otimes_\gamma B$ is non-degenerate, H_0 is dense in H by Theorem 1.9.4. Therefore, we can extend $\varphi(a)$ to a bounded linear map on H, which we also denote by $\varphi(a)$ as an abuse of notation.

By arguing in nearly the same fashion as before, if $b \in B$ then the map

$$\begin{array}{cccc} \psi(b): & H_0 & \to & H_0 \\ & z = \sum_{i=1}^{n} \pi(a_i \otimes b_i)(x_i) & \mapsto & \sum_{i=1}^{n} \pi(a_i \otimes bb_i)(x_i) \end{array}$$

is a well-defined bounded linear operator on H_0, which extends to a bounded linear operator on H, which we also denote by $\psi(b)$.

Now consider the maps $\varphi : A \to B(H)$ and $\psi : B \to B(H)$, given by $a \mapsto \varphi(a)$ and $b \mapsto \psi(b)$.

To show: (a) φ is a *-homomorphism.

(b) $\pi(a \otimes b) = \varphi(a)\psi(b) = \psi(b)\varphi(a)$.

(a) Since H_0 is dense in H is suffices to check that φ is a *-homomorphism by using elements of H_0. Let $a, a' \in A$. If $z = \sum_{i=1}^{n} \pi(a_i \otimes b_i)(x_i) \in H_0$ then

$$\begin{aligned}
\varphi(a+a')(z) &= \lim_{\mu} \pi((a+a') \otimes v_\mu)(z) \\
&= \lim_{\mu} \Big(\pi(a \otimes v_\mu)(z) + \pi(a' \otimes v_\mu)(z) \Big) \\
&= \varphi(a)(z) + \varphi(a')(z).
\end{aligned}$$

If $\alpha \in \mathbb{C}$ then

$$\varphi(\alpha a)(z) = \lim_{\mu} \pi(\alpha a \otimes v_\mu)(z) = \alpha \lim_{\mu} \pi(a \otimes v_\mu)(z) = \alpha \varphi(a)(z).$$

We also have

$$\begin{aligned}
\varphi(a^*)(z) &= \lim_{\mu} \pi(a^* \otimes v_\mu)(z) \\
&= \lim_{\mu} \pi((a \otimes v_\mu^*)^*)(z) \\
&= \lim_{\mu} \pi(a \otimes v_\mu^*)^*(z) \\
&= \lim_{\mu} \pi(a \otimes v_\mu)^*(z) \quad \text{(by Theorem 2.3.2)} \\
&= \varphi(a)(z)^*.
\end{aligned}$$

and

$$\begin{aligned}
\varphi(aa')(z) &= \lim_{\mu} \pi((aa') \otimes v_\mu)(z) \\
&= \lim_{\mu} \lim_{\mu'} \pi((aa') \otimes v_\mu v_{\mu'})(z) \\
&= \big(\lim_{\mu} \pi(a \otimes v_\mu)(z) \big) \big(\lim_{\mu'} \pi(a' \otimes v_{\mu'})(z) \big) \\
&= \varphi(a)(z) \varphi(a')(z).
\end{aligned}$$

Since $z \in H_0$ was arbitrary, φ is a *-homomorphism. Note that by similar computations, ψ is also a *-homomorphism.

(b) Let $\{u_\lambda\}_{\lambda \in L}$ be an approximate unit for A. If $z \in H_0$ then

$$
\begin{aligned}
\varphi(a)\psi(b)(z) &= \varphi(a)(\lim_{\lambda} \pi(u_\lambda \otimes b)(z)) \\
&= \lim_{\lambda} \lim_{\mu} \pi(a \otimes v_\mu)\pi(u_\lambda \otimes b)(z) \\
&= \lim_{\lambda} \lim_{\mu} \pi(au_\lambda \otimes v_\mu b)(z) \\
&= \lim_{\lambda} \pi(au_\lambda \otimes b)(z) \\
&= \pi(a \otimes b)(z).
\end{aligned}
$$

By a similar computation, we also have $\psi(b)\varphi(a)(z) = \pi(a \otimes b)(z)$.

Now, we will address the uniqueness of φ and ψ. Assume that there exists *-homomorphisms $\varphi' : A \to B(H)$ and $\psi' : B \to B(H)$ such that if $a \in A$ and $b \in B$ then

$$\pi(a \otimes b) = \varphi'(a)\psi'(b) = \psi'(b)\varphi'(a).$$

The idea is to use the approximate units $\{u_\lambda\}_{\lambda \in L}$ and $\{v_\mu\}_{\mu \in M}$. First, we need to show that the representations (φ', H) and (ψ', H) are non-degenerate.

To this end, assume that $z \in H$ satisfies $\varphi'(a)(z) = 0$. By assumption, if $b \in B$ then

$$\pi(a \otimes b)(z) = \psi'(b)\varphi'(a)(z) = 0.$$

This holds for arbitrary $a \in A$ and $b \in B$. So, $\pi(A \otimes B)z = 0$ and since the representation (π, H) is non-degenerate, $z = 0$. Therefore, (φ', H) is a non-degenerate representation of A. By a similar argument, the representations $(\psi', H), (\psi, H)$ and (φ, H) are all non-degenerate.

Since the representations (ψ', H) and (φ', H) are non-degenerate, the sequences $\{\psi'(v_\mu)\}_{\mu \in M}$ and $\{\varphi'(u_\lambda)\}_{\lambda \in L}$ both strongly converge to the identity operator id_H by Theorem 2.3.4. Now, we have

$$\lim_{\mu} \pi(a \otimes v_\mu) = \lim_{\mu} \varphi'(a)\psi'(v_\mu) = \varphi'(a)$$

But, $\lim_\mu \pi(a \otimes v_\mu) = \varphi(a)$. So, $\varphi'(a) = \varphi(a)$ and $\varphi' = \varphi$. An analogous argument reveals that $\psi' = \psi$. Hence, ψ and φ are unique. $\square$

The maps φ and ψ in Theorem 2.8.9 will be referred to as π_A and π_B respectively.

Theorem 2.8.10. *Let A and B be C*-algebras. Let γ be a C*-seminorm on $A \otimes B$. If $a \in A$ and $b \in B$ then*

$$\gamma(a \otimes b) \leq \|a\|\|b\|.$$

Proof. Assume that A and B are C*-algebras. Assume that γ is a C*-seminorm on $A \otimes B$. Let $\delta = \max(\gamma, \|-\|_*)$. Then, δ is a C*-norm on $A \otimes B$ because the spatial norm $\|-\|_*$ is itself a C*-norm on $A \otimes B$. Hence, we can form the C*-algebra $A \otimes_\delta B$.

Now let (π, H) denote the universal representation of $A \otimes_\delta B$. Then, (π, H) is faithful and non-degenerate. By Theorem 2.8.9, if $a \in A$ and $b \in B$ then $\pi(a \otimes b) = \pi_A(a)\pi_B(b)$. Since π is isometric, we have

$$\delta(a \otimes b) = \|\pi(a \otimes b)\| \leq \|\pi_A(a)\|\|\pi_B(b)\| \leq \|a\|\|b\|.$$

Therefore, $\gamma(a \otimes b) \leq \|a\|\|b\|$. $\square$

Now we will begin the construction of our next important C*-norm. Let A and B be C*-algebras. Let Γ be the set of all C*-norms on $A \otimes B$. If $c \in A \otimes B$ then define

$$\|c\|_{\max} = \sup_{\gamma \in \Gamma} \gamma(c).$$

By Theorem 2.8.10, if $c = \sum_{j=1}^{n}(a_j \otimes b_j)$ and $\gamma \in \Gamma$ then

$$\gamma(c) \leq \sum_{j=1}^{n} \gamma(a_j \otimes b_j) \leq \sum_{j=1}^{n} \|a_j\|\|b_j\|.$$

Hence, $\|c\|_{\max} < \infty$. Furthermore, it is a C*-norm on $A \otimes B$.

Definition 2.8.8. Let A and B be C*-algebras. The norm $\|-\|_{\max}$ defined above is called the **maximal C*-norm** on $A \otimes B$. The C*-completion of the tensor product $A \otimes B$ with respect to the maximal C*-norm $\|-\|_{\max}$ is called the **maximal tensor product** of A and B and is denoted by $A \otimes_{\max} B$.

The name "maximal tensor product" is not just for show. In fact, the maximal tensor product satisfies the following universal property:

Theorem 2.8.11. *Let A, B and C be C*-algebras. Let $\varphi : A \to C$ and $\psi : B \to C$ be *-homomorphisms such that the image $\varphi(A)$ commutes with $\psi(B)$. Then, there exists a unique *-homomorphism $\pi : A \otimes_{\max} B \to C$ such that if $a \in A$ and $b \in B$ then*

$$\pi(a \otimes b) = \varphi(a)\psi(b).$$

Proof. Assume that A, B and C are C*-algebras. Assume that $\varphi : A \to C$ and $\psi : B \to C$ are *-homomorphisms such that $\varphi(A)$ commutes with $\psi(B)$.

By Theorem 2.8.1, there exists a unique *-homomorphism $\pi : A \otimes B \to C$ such that if $a \in A$ and $b \in B$ then

$$\pi(a \otimes b) = \varphi(a)\psi(b).$$

Now consider the map

$$\begin{array}{rccc} \gamma : & A \otimes B & \to & \mathbb{R}_{\geq 0} \\ & c & \mapsto & \|\pi(c)\|. \end{array}$$

Then, γ is a C*-seminorm on $A \otimes B$ (seminorm because π may not be injective). By Theorem 2.8.10, $\gamma(c) \leq \delta(c) \leq \|c\|_{\max}$ because δ is a C*-norm on $A \otimes B$. Therefore, $\|\pi(c)\| \leq \|c\|_{\max}$ and π is a norm-decreasing *-homomorphism on $A \otimes B$.

Consequently, we can extend π to a norm-decreasing *-homomorphism on the maximal tensor product $A \otimes_{\max} B$. $\square$

Now, we arrive at the definition of a class of C*-algebras which has been the subject of much research.

Definition 2.8.9. Let A be a C*-algebra. We say that A is **nuclear** if for all C*-algebras B, the tensor product $A \otimes B$ has exactly one C*-norm.

Here is an important remark we will use shortly. Let A be a *-algebra equipped with a complete C*-norm $\|-\|$. We claim that this is the only C*-norm on A. Let γ be another C*-norm on A and B be the completion of A with respect to γ. Define the inclusion map

$$\begin{array}{rccc} \iota : & (A, \|-\|) & \to & (B, \gamma) \\ & a & \mapsto & a. \end{array}$$

Then, ι is an injective *-homomorphism and thus, isometric. Hence, if $a \in A$ then $\gamma(a) = \|a\|$ and as a result, $\gamma = \|-\|$.

Example 2.8.2. Let $n \in \mathbb{Z}_{\geq 1}$. We claim that the C*-algebra $M_{n\times n}(\mathbb{C})$ is nuclear. Assume that A is a C*-algebra. To see that the tensor product $M_{n\times n}(\mathbb{C}) \otimes A$ has exactly one C*-norm, it suffices by the previous remark to show that $M_{n\times n}(\mathbb{C})\otimes A$ admits a complete C*-norm.

Define the map

$$\begin{array}{rccc} \pi': & M_{n\times n}(\mathbb{C})\times A & \to & M_{n\times n}(A) \\ & ((\lambda_{i,j}),a) & \mapsto & (\lambda_{i,j}a). \end{array}$$

It is straightforward to verify that π' is a bilinear map. By the universal property in Theorem 2.8.1, there exists a unique linear map $\pi : M_{n\times n}(\mathbb{C})\otimes A \to M_{n\times n}(A)$ such that

$$\pi((\lambda_{i,j})\otimes a) = (\lambda_{i,j}a).$$

Next, we show that π is a *-homomorphism. Assume that $(\lambda_{i,j})\in M_{n\times n}(\mathbb{C})$ and $a\in A$. Then,

$$\begin{aligned} \pi(((\lambda_{i,j})\otimes a)^*) &= \pi((\overline{\lambda_{j,i}})\otimes a^*) \\ &= (\overline{\lambda_{j,i}}a^*) \\ &= (\lambda_{i,j}a)^* = \pi((\lambda_{i,j})\otimes a)^* \end{aligned}$$

and π is also multiplicative by direct computation. Therefore, π is a *-homomorphism. We now claim that π is actually a *-isomorphism.

To show: (a) π is surjective.

(b) π is injective.

(a) Assume that $X=(x_{ij})\in M_{n\times n}(A)$. If $i,j\in\{1,2,\ldots,n\}$ then let $e_{i,j}\in M_{n\times n}(\mathbb{C})$ be the matrix unit with a 1 in the ij position and zeros elsewhere. Then, $\pi(e_{i,j}\otimes x_{i,j})$ is the matrix with $x_{i,j}\in A$ in the ij position and zeros elsewhere. By linearity of π, we have

$$\pi(\sum_{i=1}^{n}\sum_{j=1}^{n}(e_{i,j}\otimes x_{i,j})) = X.$$

So, π is surjective.

(b) Assume that $(\lambda_{i,j})\otimes a\in\ker\pi$ so that

$$\pi((\lambda_{i,j})\otimes a) = (\lambda_{i,j}a) = 0.$$

Then either $a = 0$ or if $i, j \in \{1, 2, \ldots, n\}$ then $\lambda_{i,j} = 0$ and the matrix $(\lambda_{i,j})$ is the zero matrix. In either case, $(\lambda_{i,j}) \otimes a = 0$ and π must be injective.

By parts (a) and (b), we deduce that π is a *-isomorphism. Now, we define the map $\|-\|$ by

$$\begin{array}{rccc} \|-\|: & M_{n\times n}(\mathbb{C}) \otimes A & \to & \mathbb{C} \\ & (\lambda_{i,j}) \otimes a & \mapsto & \|\pi((\lambda_{i,j}) \otimes a)\| \end{array}$$

Since $M_{n\times n}(A)$ is a C*-algebra with its norm (see [Mur90, Theorem 3.4.2]), $\|-\|$ defines a C*-norm on the tensor product $M_{n\times n}(\mathbb{C}) \otimes A$. It is complete because π is a *-isomorphism and is thus, isometric. By the preceding remark, $\|-\|$ is the only C*-norm on $M_{n\times n}(\mathbb{C}) \otimes A$. Since A was arbitrary, we deduce that $M_{n\times n}(\mathbb{C})$ is a nuclear C*-algebra.

Recalling the structure of finite-dimensional C*-algebras from Theorem 1.5.1, we suspect that in light of above example, every finite-dimensional C*-algebra must be nuclear. We prove this in Theorem 2.8.12 below.

Theorem 2.8.12. *Let A be a finite-dimensional C*-algebra. Then, A is a nuclear C*-algebra.*

Proof. Assume that A is a finite-dimensional C*-algebra. By Theorem 1.5.1, there exists $k \in \mathbb{Z}_{>0}$ and $n_1, \ldots, n_k \in \mathbb{Z}_{>0}$ such that

$$A \cong \bigoplus_{i=1}^{k} M_{n_i\times n_i}(\mathbb{C}).$$

as C*-algebras. Let B be a C*-algebra and define the map

$$\begin{array}{rccc} \phi: & A \times B & \to & \bigoplus_{i=1}^{k}(M_{n_i\times n_i}(\mathbb{C}) \otimes B) \\ & (a, b) = ((a_1, \ldots, a_k), b) & \mapsto & (a_1 \otimes b, \ldots, a_k \otimes b) \end{array}$$

where if $i \in \{1, 2, \ldots, k\}$ then $a_i \in M_{n_i\times n_i}(\mathbb{C})$. It is easy to verify that ϕ is a bilinear map. By Theorem 2.8.1, there exists a unique linear map ψ, defined explicitly by

$$\begin{array}{rccc} \psi: & A \otimes B & \to & \bigoplus_{i=1}^{k}(M_{n_i\times n_i}(\mathbb{C}) \otimes B) \\ & a \otimes b = (a_1, \ldots, a_k) \otimes b & \mapsto & (a_1 \otimes b, \ldots, a_k \otimes b). \end{array}$$

By direct computation, ψ is in fact, a *-homomorphism. To see that ψ is injective, assume that $(a_1, \ldots, a_k) \otimes b \in \ker \psi$ so that $\psi((a_1, \ldots, a_k) \otimes b) = 0$. Then, either $b = 0$ or if $i \in \{1, \ldots, k\}$ then $a_i = 0$. In either case, $(a_1, \ldots, a_k) \otimes b = 0$. So, ψ is injective.

To see that ψ is surjective, assume that

$$(a_1 \otimes b_1, \ldots, a_k \otimes b_k) \in \bigoplus_{i=1}^{k} (M_{n_i \times n_i}(\mathbb{C}) \otimes B).$$

If $i \in \{1, 2, \ldots, k\}$ then let $e_i \in \bigoplus_{i=1}^{k} M_{n_i \times n_i}(\mathbb{C}) \cong A$ be the tuple with the identity matrix in the i^{th} position and zero matrices elsewhere. Since ψ is linear, we have

$$\psi(\sum_{i=1}^{k} (a_i e_i \otimes b_i)) = (a_1 \otimes b_1, \ldots, a_k \otimes b_k).$$

Hence, ψ is a *-isomorphism. Now define the map

$$\begin{array}{rccc} \|-\| : & A \otimes B & \to & \mathbb{R}_{\geq 0} \\ & a \otimes b = (a_1, \ldots, a_k) \otimes b & \mapsto & \|\psi((a_1, \ldots, a_k) \otimes b)\|. \end{array}$$

Since the direct sum $\bigoplus_{i=1}^{k} (M_{n_i \times n_i}(\mathbb{C}) \otimes B)$ is a C*-algebra, then $\|-\|$ must be a C*-norm on $A \otimes B$. It is also complete because ψ is a *-isomorphism and thus, isometric. Hence, $\|-\|$ is the only C*-norm on $A \otimes B$ and hence, A is a nuclear C*-algebra. $\square$

Now, we will show that there are many examples of C*-algebras with the next few results.

Theorem 2.8.13. *Let A be a C*-algebra and $\mathcal{S}$ be a non-empty, upwards directed set of C*-subalgebras of A. Suppose that the union $\bigcup_{T \in \mathcal{S}} T$ is dense in A and that every element of $\mathcal{S}$ is nuclear. Then, A is a nuclear C*-algebra.*

Proof. Assume that A is a C*-algebra and $\mathcal{S}$ is defined as above. Assume that every C*-subalgebra of A in $\mathcal{S}$ is nuclear.

Assume that B is a C*-algebra and that β and γ are C*-norms on $A \otimes B$. We will show that $\beta = \gamma$. Let $C = (\bigcup_{T \in \mathcal{S}} T) \otimes B$, where we regard $T \otimes B$ as a *-subalgebra of $A \otimes B$ for $T \in \mathcal{S}$. Then, C is a C*-subalgebra of $A \otimes B$ and is dense in $A \otimes_\beta B$ and $A \otimes_\gamma B$ (see Theorem 2.8.8).

We assumed that if $T \in \mathcal{S}$ then T is nuclear. Since $T \otimes B$ only has one C*-norm, $\beta = \gamma$ on $T \otimes B$. Since $T \in \mathcal{S}$ was arbitrary, $\beta = \gamma$ on C. Hence, the identity map $id_C : (C, \beta) \to (C, \gamma)$ is bounded and by density of C, extends to a *-homomorphism $\pi : A \otimes_\beta B \to A \otimes_\gamma B$.

Now assume that $a \in A$ and $b \in B$. Then, there exists a sequence $\{a_n\}_{n\in\mathbb{Z}_{>0}}$ in $\bigcup_{T\in\mathcal{S}} T$ such that $\lim_{n\to\infty} a_n = a$. By Theorem 2.8.8,

$$\lim_{n\to\infty}(a_n \otimes b) = a \otimes b.$$

Therefore,

$$\pi(a\otimes b) = \lim_{n\to\infty} \pi(a_n \otimes b) = \lim_{n\to\infty}(a_n \otimes b) = a \otimes b$$

where the convergence in the last equality is with respect to the C*-norm γ. We conclude that π is the identity map on $A \otimes B$. Consequently, if $c \in A \otimes B$ then

$$\gamma(c) = (\gamma \circ \pi)(c) = \beta(c).$$

So, $\beta = \gamma$ on $A \otimes B$ and thus, A is a nuclear C*-algebra. □

As a consequence of Theorem 2.8.13, if H is a Hilbert space then the C*-algebra of compact operators $B_0(H)$ is nuclear. See [Mur90, Example 6.3.2].

Theorem 2.8.14. *Let A be an AF-algebra. Then, A is a nuclear C*-algebra.*

Proof. Assume that A is an AF-algebra. Then, there exists an increasing sequence of finite-dimensional C*-algebras $\{B_n\}_{n\in\mathbb{Z}_{>0}}$ such that $\bigcup_{i=1}^{n} B_n$ is dense in A.

The set $\{B_n \mid n \in \mathbb{Z}_{>0}\}$ of C*-subalgebras of A is upwards directed and each element of the set is a nuclear C*-algebra by Theorem 2.8.12. By Theorem 2.8.13, we find that A is a nuclear C*-algebra as required. □

In particular, since UHF algebras are AF-algebras by definition, UHF algebras are also nuclear C*-algebras. By Theorem 2.7.11, we now have uncountably many examples of nuclear C*-algebras.

2.9 More on irreducible representations and pure states

Let A and B be C*-algebras and $\|-\|_{\max}$ denote the maximal C*-norm on the tensor product $A \otimes B$. By construction, $\|-\|_{\max}$ is the largest of all possible C*-norms. From this, one might wonder if there is a *smallest* possible C*-norm on $A \otimes B$. It turns out that the spatial C*-norm $\|-\|_*$ is the least C*-norm on $A \otimes B$, but the proof of this is highly technical and makes use of results about irreducible representations of C*-algebras and pure states. With the interest of following [Mur90, Section 6.4] in the next section, this section is dedicated to stating and proving the preliminary results we need in the following section.

We will follow [Mur90, Section 5.1]. In these notes, the GNS construction was previously done for a **unital C*-algebra**. We note here that the construction generalises in a straightforward fashion to an arbitrary C*-algebra (see [Mur90, Section 3.4]). However, we will have to do something different to recover the unit cyclic vector associated to the GNS representation of an arbitrary C*-algebra.

Theorem 2.9.1. *Let A be a C*-algebra and $\tau : A \to \mathbb{C}$ be a state on A. Let (φ_τ, H_τ) denote the GNS representation of A. Then, there exists a unique unit cyclic vector $\xi_\tau \in A$ such that if $a \in A$ then*

$$\tau(a) = \langle a + N_\tau, \xi_\tau \rangle \qquad \text{and} \qquad \varphi_\tau(a)\xi_\tau = a + N_\tau.$$

Proof. Assume that A is a C*-algebra and that τ is a state on A. Assume that (φ_τ, H_τ) is the GNS representation of A. Define the map

$$\begin{array}{rccc} \rho_0 : & A/N_\tau & \to & \mathbb{C} \\ & a + N_\tau & \mapsto & \tau(a). \end{array}$$

Obviously, ρ_0 is linear. To see that ρ_0 is well-defined, assume that $a + N_\tau = b + N_\tau$. Then, $\tau((a-b)^*(a-b)) = 0$. If $c \in A$ then by the Cauchy-Schwarz inequality,

$$\begin{aligned} |\tau(c^*(a-b))|^2 &= |\langle a - b + N_\tau, c + N_\tau \rangle|^2 \\ &\leq \|a - b + N_\tau\|^2 \|c + N_\tau\|^2 \\ &= \tau((a-b)^*(a-b)) \|c + N_\tau\|^2 = 0. \end{aligned}$$

Therefore, $\langle a - b + N_\tau, c + N_\tau \rangle = \tau(c^*(a-b)) = 0$ and subsequently, $a - b + N_\tau = N_\tau$ and $a + N_\tau = b + N_\tau$ as required. So, ρ_0 is well-defined.

To see that ρ_0 is bounded, recall that since τ is a state, $\|\tau\| = 1$. Let $\pi : A \to A/N_\tau$ denote the projection *-homomorphism. Then, π is bounded (recall that $\|\pi\| = 1$) and surjective. By the open mapping theorem, it is an open map. So, there exists $r \in \mathbb{R}_{>0}$ such that

$$B_{A/N_\tau}(0,1) \subseteq \pi(B_A(0,r))$$

where $B_{A/N_\tau}(0,1)$ is the open ball centred at $0 = N_\tau$ with radius 1 and $B_A(0,r)$ is the open ball centred at $0 \in A$ with radius r. By increasing $r \in \mathbb{R}_{>0}$ as necessary, we may assume that

$$\overline{B}_{A/N_\tau}(0,1) \subseteq \pi(B_A(0,r))$$

where $\overline{B}_{A/N_\tau}(0,1)$ is the closed unit ball of A/N_τ. Now assume that $a + N_\tau \in \overline{B}_{A/N_\tau}(0,1)$. Then, $\|a\| < r$ and

$$|\rho_0(a + N_\tau)| = |\tau(a)| \leq \|\tau\|\|a\| < r.$$

By taking the supremum over all element of the closed ball $\overline{B}_{A/N_\tau}(0,1)$, we deduce that $\|\rho_0\| < r$. Hence, ρ_0 is bounded.

Recall from the GNS construction that H_τ is the completion of A/N_τ. Since ρ_0 is a bounded linear functional on A/N_τ, we can extend it to a bounded linear functional on H_τ, which we will call ρ. Since H_τ is a Hilbert space, we can apply the Riesz representation theorem to find a unique element $\xi_\tau \in H_\tau$ such that if $y \in H_\tau$ then

$$\rho(y) = \langle y, \xi_\tau \rangle.$$

Hence, if $a \in A$ then

$$\tau(a) = \rho_0(a + N_\tau) = \rho(a + N_\tau) = \langle a + N_\tau, \xi_\tau \rangle.$$

If in addition $b \in A$ then

$$\begin{aligned}
\langle b + N_\tau, \varphi_\tau(a)\xi_\tau \rangle &= \langle \varphi_\tau(a^*)(b + N_\tau), \xi_\tau \rangle \\
&= \langle a^*b + N_\tau, \xi_\tau \rangle \\
&= \tau(a^*b) = \langle b + N_\tau, a + N_\tau \rangle.
\end{aligned}$$

Since this hold for arbitrary $b + N_\tau \in A/N_\tau$, it must also hold in H_τ. Therefore, $\varphi_\tau(a)\xi_\tau = a + N_\tau$. Note from this that the image $\varphi_\tau(A)\xi_\tau = A/N_\tau$ which is dense in H_τ. This means that ξ_τ is a cyclic vector for the GNS representation of A.

Finally, to see that ξ_τ is a unit vector, let $\{u_\lambda\}_{\lambda \in L}$ be an approximate unit for A. Then, the family $\{\varphi_\tau(u_\lambda)\}_{\lambda \in L}$ is an approximate unit for $\varphi_\tau(A)$ and thus, it must converge strongly to the identity operator id_{H_τ} by Theorem 2.3.4. So,

$$\begin{aligned} \|\xi_\tau\|^2 &= \langle \xi_\tau, \xi_\tau \rangle \\ &= \lim_\lambda \langle \varphi_\tau(u_\lambda)(\xi_\tau), \xi_\tau \rangle \\ &= \lim_\lambda \tau(u_\lambda) \\ &= \|\tau\| = 1. \end{aligned}$$

Note that the second last equality follows from Theorem 2.3.5. This completes the proof. $\square$

For the next result, we make a quick definition. If ρ and τ are positive linear functionals on a C*-algebra A then we say that $\rho \leq \tau$ if $\tau - \rho$ is itself a positive linear functional on A.

Theorem 2.9.2. *Let A be a C*-algebra. Let $\tau : A \to \mathbb{C}$ be a state and $\rho : A \to \mathbb{C}$ be a positive linear functional. Let $(\varphi_\tau, H_\tau, \xi_\tau)$ be the GNS representation of A associated to the state τ. Then, there exists a unique operator $v \in \varphi_\tau(A)'$ such that $0 \leq v \leq id_{H_\tau}$ and if $a \in A$ then*

$$\rho(a) = \langle \varphi_\tau(a) v \xi_\tau, \xi_\tau \rangle.$$

Proof. Assume that A is a C*-algebra. Assume that τ is a state on A and ρ is a positive linear functional on A. Assume that $(\varphi_\tau, H_\tau, \xi_\tau)$ is the GNS representation of A associated to the state τ.

First, define the map σ by

$$\begin{array}{cccc} \sigma : & A/N_\tau \times A/N_\tau & \to & \mathbb{C} \\ & (a + N_\tau, b + N_\tau) & \mapsto & \langle a + N_\tau, b + N_\tau \rangle = \rho(b^* a). \end{array}$$

First, we claim that σ is well-defined. Assume that $a_1, a_2 \in A$ satisfy $a_1 + N_\tau = a_2 + N_\tau$. Arguing in a similar fashion to the GNS construction and using Theorem 2.3.8, we deduce that if $b \in A$ then $\tau(b^*(a_1 - a_2)) = 0$. Subsequently,

$$\sigma(a_1 - a_2 + N_\tau, b + N_\tau) = \rho(b^*(a_1 - a_2)) \leq \tau(b^*(a_1 - a_2)) = 0.$$

Therefore, $\sigma(a_1 + N_\tau, b + N_\tau) = \sigma(a_2 + N_\tau, b + N_\tau)$. By a similar argument for the second argument, we deduce that σ is well-defined. We also see that σ is a sesquilinear form because ρ is linear.

Next, to see that σ is bounded, we use the Cauchy-Schwarz inequality to deduce that

$$\sigma(a+N_\tau, b+N_\tau)^2| \leq \|a+N_\tau\|^2\|b+N_\tau\|^2 = \rho(a^*a)\rho(b^*b) \leq \tau(a^*a)\tau(b^*b).$$

By taking the supremum over all $a, b \in A$ such that $\|a+N_\tau\| = \|b+N_\tau\| = 1$, we find that $\|\sigma\| \leq 1$ because τ is a state and hence, satisfies $\|\tau\| = 1$. Since σ is a bounded sesquilinear form on A/N_τ and H_τ is the completion of A/N_τ, we can extend σ to a sesquilinear form on H_τ which has norm less than or equal to 1. We will abuse notation and call the sesquilinear form σ.

Assume that $\psi \in H_\tau$. By applying the Riesz representation theorem to each of the linear functionals $\xi \mapsto \sigma(\xi, \psi)$, we obtain an operator $v \in B(H_\tau)$ such that if $\xi \in H_\tau$ then

$$\sigma(\xi, \psi) = \langle v\xi, \psi\rangle.$$

To be clear, the inner product on the RHS is the inner product from H_τ. Observe that if $\xi, \psi \in H_\tau$ then

$$\begin{aligned}\|v\|^2 &= \sup_{\|\xi\|=1} \|v\xi\|^2 \\ &= \sup_{\|\xi\|=1} \langle v\xi, v\xi\rangle \\ &\leq \sup_{\|\xi\|=1} \sigma(\xi, v\xi) \\ &\leq \sup_{\|\xi\|=1} \|\sigma\|\|\xi\|\|v\xi\| \\ &\leq \sup_{\|\xi\|=1} \|v\|\|\xi\|^2 = \|v\|.\end{aligned}$$

Therefore, $\|v\| \leq 1$ and consequently, $v \leq id_{H_\tau}$. Next, to see that v is positive, we will show that if $a \in A$ then $\langle v(a+N_\tau), a+N_\tau\rangle \geq 0$. So, assume that $a \in A$. Then,

$$\begin{aligned}\langle v(a+N_\tau), a+N_\tau\rangle &= \sigma(a+N_\tau, a+N_\tau) \\ &= \rho(a^*a) \geq 0.\end{aligned}$$

Therefore, v is positive. Now, we will show that $v \in \varphi_\tau(A)'$. Assume that $a, b, c \in A$. Then,

$$
\begin{aligned}
\langle \varphi_\tau(a)v(b+N_\tau), c+N_\tau\rangle &= \langle v(b+N_\tau), \varphi_\tau(a^*)(c+N_\tau)\rangle \\
&= \langle v(b+N_\tau), a^*c+N_\tau\rangle \\
&= \sigma(b+N_\tau, a^*c+N_\tau) = \rho(c^*ab) \\
&= \sigma(ab+N_\tau, c+N_\tau) \\
&= \langle v(ab+N_\tau), c+N_\tau\rangle \\
&= \langle v\varphi_\tau(a)(b+N_\tau), c+N_\tau\rangle.
\end{aligned}
$$

Since this holds for arbitrary $b, c \in A$, we deduce that if $a \in A$ then $\varphi_\tau(a)v = v\varphi_\tau(a)$ because A/N_τ is dense in H_τ. Therefore, $v \in \varphi_\tau(A)'$. Now let $\{u_\lambda\}_{\lambda\in L}$ be an approximate unit for A. If $b \in B$ then

$$
\begin{aligned}
\rho(u_\lambda b) &= \sigma(b+N_\tau, u_\lambda+N_\tau) \\
&= \sigma(b+N_\tau, \varphi_\tau(u_\lambda)\xi_\tau) \\
&= \langle v(b+N_\tau), \varphi_\tau(u_\lambda)\xi_\tau\rangle \\
&= \langle \varphi_\tau(u_\lambda)v(b+N_\tau), \xi_\tau\rangle \\
&= \langle v\varphi_\tau(u_\lambda b)\xi_\tau, \xi_\tau\rangle.
\end{aligned}
$$

Taking the limit over $\lambda \in L$, we obtain $\rho(b) = \langle v\varphi_\tau(b)\xi_\tau, \xi_\tau\rangle$. Finally, to see that v is unique, suppose that there exists $w \in \varphi_\tau(A)'$ such that if $a \in A$ then

$$
\rho(a) = \langle \varphi_\tau(a)w\xi_\tau, \xi_\tau\rangle = \langle \varphi_\tau(a)v\xi_\tau, \xi_\tau\rangle.
$$

If $a, b \in A$ then

$$
\begin{aligned}
\langle w(b+N_\tau), a+N_\tau\rangle &= \langle w(b+N_\tau), \varphi_\tau(a)\xi_\tau\rangle \\
&= \langle \varphi_\tau(a^*)w(b+N_\tau), \xi_\tau\rangle \\
&= \langle \varphi_\tau(a^*)v(b+N_\tau), \xi_\tau\rangle \\
&= \langle v(b+N_\tau), a+N_\tau\rangle.
\end{aligned}
$$

Since A/N_τ is dense in H_τ, we deduce that $w = v$. □

It is remarked in [Mur90, Page 142] that when faced with an arbitrary representation of a C*-algebra, we can always reduce to the case of a non-degenerate representation by Theorem 1.9.5.

The next main result we will prove is that a non-degenerate representation can be written as a direct sum of cyclic representations. We require the following preliminary definition and result.

Definition 2.9.1. Let H be a Hilbert space and A be a *-subalgebra of $B(H)$. We say that A is **non-degenerate** on H if $\overline{AH} = H$.

Theorem 2.9.3. *Let H be a Hilbert space and A be a non-degenerate *-subalgebra of $B(H)$. If $\xi \in H$ then $\xi \in \overline{A\xi}$.*

Proof. Assume that H is a Hilbert space and A is a non-degenerate *-subalgebra of $B(H)$. Assume that $\xi \in H$ and let P be the projection operator onto the closed subspace $\overline{A\xi}$. If $T \in A$ then $PTP = TP$ and by taking adjoints, we find that $PT^* = PT^*P$. Since A is self-adjoint, if $T \in A$ then

$$PT = TP = PTP.$$

We conclude that $P \in A'$. Now, if $\xi' = P\xi$ and $\xi'' = (id_H - P)\xi$ then $\xi = \xi' + \xi''$. However,

$$T\xi' = TP\xi = PT\xi = T\xi.$$

Therefore, if $T \in A$ then $T\xi'' = 0$. Since A is non-degenerate, $\xi'' = 0$ and $\xi = \xi' \in \overline{A\xi}$. □

Theorem 2.9.4. *Let A be a C*-algebra and (φ, H) be a non-degenerate representation of A. Then, (φ, H) can be written as a direct sum of cyclic representations of A.*

Proof. Assume that A is a C*-algebra and (φ, H) is a non-degenerate representation of A. If $x \in H$ then define $H_x = \overline{\varphi(A)x}$. Let Λ be the set whose elements are non-empty subsets S of $H - \{0\}$ such that if $x, y \in S$ then the closed subspaces H_x and H_y are pairwise orthogonal.

Our strategy is to apply Zorn's lemma to Λ which is a poset when equipped with the relation of inclusion. To see that Λ is non-empty, assume that $x \in H - \{0\}$. Then, $H = H_x \oplus H_x^\perp$. Let $y \in H_x^\perp$ be non-zero. Then, H_y is orthogonal to H_x, $\{x, y\} \in \Lambda$ and hence, Λ is non-empty.

Now assume that $\mathcal{S}$ is a totally ordered subset of Λ. Consider the set

$$\bigcup_{S \in \mathcal{S}} S.$$

If $V \in \mathcal{S}$ then $V \subseteq \bigcup_{S \in \mathcal{S}} S$. Furthermore, it is easy to check that $\bigcup_{S \in \mathcal{S}} S \in \Lambda$. Hence, $\mathcal{S}$ has an upper bound.

By Zorn's lemma, Λ has a maximal element — we will denote this set by $\mathcal{M}$. We will show that $H = \bigoplus_{x \in \mathcal{M}} H_x$.

To show: (a) $(\bigcup_{x \in \mathcal{M}} H_x)^\perp = \{0\}$.

(a) Suppose for the sake of contradiction that there exists $y \in (\bigcup_{x \in \mathcal{M}} H_x)^\perp$ such that $y \neq 0$. If $a, b \in A$ and $x \in \mathcal{M}$ then

$$\langle \varphi(b^* a)x, y \rangle = \langle \varphi(a)x, \varphi(b)y \rangle = 0.$$

We conclude that if $x \in \mathcal{M}$ then H_x is orthogonal to H_y. Since (φ, H) is a non-degenerate representation, then by Theorem 2.9.3, $y \in \overline{\varphi(A)y} = H_y$. So, $\mathcal{M} \cup \{y\} \in \Lambda$ which contradicts the maximality of $\mathcal{M}$. Therefore, $y = 0$ and $(\bigcup_{x \in \mathcal{M}} H_x)^\perp = \{0\}$.

Hence, $H = \bigoplus_{x \in \mathcal{M}} H_x$ and the restricted representation $(\varphi|_{H_x}, H_x)$ of A is cyclic with cyclic vector x. Therefore, (φ, H) is the direct sum of representations (φ_x, H_x) for $x \in \mathcal{M}$. □

Our next result concerns unitary equivalence.

Theorem 2.9.5. *Let A be a C*-algebra. Let (φ_1, H_1) and (φ_2, H_2) be representations of A with cyclic vectors x_1 and x_2 respectively. Then, there exists a unitary map $u : H_1 \to H_2$ such that $x_2 = u(x_1)$ and $\varphi_2(a) = u\varphi_1(a)u^*$ for $a \in A$ if and only if*

$$\langle \varphi_1(a)(x_1), x_1 \rangle = \langle \varphi_2(a)(x_2), x_2 \rangle$$

for $a \in A$.

Proof. Assume that A is a C*-algebra. Assume that (φ_1, H_1) and (φ_2, H_2) are representations of A with cyclic vectors x_1 and x_2 respectively. First suppose that there exists a unitary map $u : H_1 \to H_2$ such that $x_2 = u(x_1)$ and if $a \in A$ then $\varphi_2(a) = u\varphi_1(a)u^*$.

If $a \in A$ then

$$\begin{aligned}
\langle \varphi_2(a)(x_2), x_2 \rangle &= \langle u\varphi_1(a)u^*(x_2), x_2 \rangle \\
&= \langle \varphi_1(a)u^*(x_2), u^*(x_2) \rangle \\
&= \langle \varphi_1(a)u^{-1}(x_2), u^{-1}(x_2) \rangle \\
&= \langle \varphi_1(a)(x_1), x_1 \rangle.
\end{aligned}$$

Conversely, assume that if $a \in A$ then $\langle \varphi_1(a)(x_1), x_1\rangle = \langle \varphi_2(a)(x_2), x_2\rangle$. Define the map u_0 by

$$\begin{array}{rccc} u_0: & \varphi_1(A)x_1 & \to & H_2 \\ & \varphi_1(a)x_1 & \mapsto & \varphi_2(a)x_2 \end{array}$$

To see that u_0 is well-defined, suppose that $a, b \in A$ satisfy $\varphi_1(a)x_1 = \varphi_1(b)x_1$. If $c \in A$ then

$$\begin{aligned} \langle u_0(\varphi_1(a)(x_1)), \varphi_2(c)(x_2)\rangle &= \langle \varphi_2(a)(x_2), \varphi_2(c)(x_2)\rangle \\ &= \langle \varphi_1(c^*a)(x_1), x_1\rangle \\ &= \langle \varphi_1(c^*b)(x_1), x_1\rangle \\ &= \langle \varphi_2(c^*b)(x_2), x_2\rangle \\ &= \langle u_0(\varphi_1(b)(x_1)), \varphi_2(c)(x_2)\rangle. \end{aligned}$$

Since x_2 is a cyclic vector for (φ_2, H_2), then $\overline{\varphi_2(A)x_2} = H_2$ and consequently, $u_0(\varphi_1(a)(x_1)) = u_0(\varphi_1(b)(x_1))$. So, u_0 is well-defined.

To see that u_0 is an isometry, observe that if $a \in A$ then

$$\begin{aligned} \langle u_0(\varphi_1(a)(x_1)), u_0(\varphi_1(a)(x_1))\rangle &= \langle \varphi_2(a)(x_2), \varphi_2(a)(x_2)\rangle \\ &= \langle \varphi_2(a^*a)(x_2), x_2\rangle \\ &= \langle \varphi_1(a^*a)(x_1), x_1\rangle \\ &= \langle \varphi_1(a)(x_1), \varphi_1(a)(x_1)\rangle. \end{aligned}$$

Hence, u_0 is an isometry from $\varphi_1(A)x_1$ to H_2. Since $\varphi_1(A)x_1$ is dense in H_1, we can extend u_0 to an isometric linear map $u : H_1 \to H_2$. Notice that the image im $u = \overline{\varphi_2(A)x_2} = H_2$. SInce u is surjective, u must be unitary.

To see that $u(x_1) = x_2$, first observe that if $a, b \in A$ then

$$u\varphi_1(a)\varphi_1(b)x_1 = \varphi_2(ab)x_2 = \varphi_2 u(\varphi_1(b)(x_1)).$$

So, if $a \in A$ then $u\varphi_1(a) = \varphi_2(a)u$. In particular,

$$\varphi_2(a)u(x_1) = u\varphi_1(a)(x_1) = \varphi_2(a)(x_2).$$

Therefore, if $a \in A$ then $\varphi_2(a)(u(x_1) - x_2) = 0$. Since the representation (φ_2, H_2) is non-degenerate, we deduce that $u(x_1) = x_2$ as required. $\square$

We have already encountered pure states in the context of GNS representations of unital C*-algebras. Let us define pure states for arbitrary C*-algebras.

Definition 2.9.2. Let A be a C*-algebra and τ be a state on A. We say that τ is **pure** if it has the following property: If ρ is a positive linear functional on A such that $\rho \leq \tau$ then there exists $t \in [0,1]$ such that $\rho = t\tau$.

We denote the set of pure states on A by $PS(A)$.

Here is a characterisation of pure states.

Theorem 2.9.6. *Let A be a C*-algebra and τ be a state on A. Then*

1. *τ is pure if and only if its associated GNS representation $(\varphi_\tau, H_\tau, \xi_\tau)$ of A is irreducible.*
2. *If A is commutative then $PS(A) = \mathcal{M}(A)$.*

Proof. Assume that A is a C*-algebra and τ is a state on A. Let $(\varphi_\tau, H_\tau, \xi_\tau)$ be the GNS representation associated to τ. We will first prove that τ is pure if and only if $(\varphi_\tau, H_\tau, \xi_\tau)$ is irreducible.

Assume that τ is a pure state and $v \in \varphi_\tau(A)'$ such that $0 \leq v \leq id_{H_\tau}$.

To show: (a) $v \in \mathbb{C} id_{H_\tau}$.

(a) Similarly to Theorem 1.11.4, define the map

$$\begin{array}{rccc} \rho : & A & \to & \mathbb{C} \\ & a & \mapsto & \langle \varphi_\tau(a) v(\xi_\tau), \xi_\tau \rangle \end{array}$$

Then, ρ is a linear functional. To see that ρ is positive, note that if $a \in A$ then

$$\begin{aligned} \rho(a^*a) &= \langle \varphi_\tau(a^*a) v(\xi_\tau), \xi_\tau \rangle \\ &= \langle \varphi_\tau(a) v(\xi_\tau), \varphi_\tau(a)(\xi_\tau) \rangle \\ &= \langle v\varphi_\tau(a)(\xi_\tau), \varphi_\tau(a)(\xi_\tau) \rangle \geq 0 \end{aligned}$$

where the inequality follows from the assumption that v is a positive operator. To see that $\rho \leq \tau$, we have by Theorem 2.9.1,

$$
\begin{aligned}
\rho(a) &= \langle \varphi_\tau(a)v(\xi_\tau), \xi_\tau\rangle \\
&= \langle v\varphi_\tau(a)(\xi_\tau), \xi_\tau\rangle \\
&= \langle v(a+N_\tau), \xi_\tau\rangle \\
&\leq \|v\|\langle a+N_\tau, \xi_\tau\rangle \\
&\leq \langle a+N_\tau, \xi_\tau\rangle = \tau(a).
\end{aligned}
$$

Hence, ρ is a positive linear functional satisfying $\rho \leq \tau$. Since τ is pure, there exists $t \in [0,1]$ such that $\rho = t\tau$. We claim that $v = t \cdot id_{H_\tau}$. If $a, b \in A$ then

$$
\begin{aligned}
\langle v(a+N_\tau), b+N_\tau\rangle &= \langle v\varphi_\tau(a)(\xi_\tau), \varphi_\tau(b)(\xi_\tau)\rangle \\
&= \langle v\varphi_\tau(b^*a)(\xi_\tau), \xi_\tau\rangle \\
&= \rho(b^*a) = t\tau(b^*a) \\
&= \langle t(b^*a+N_\tau), \xi_\tau\rangle \\
&= \langle t(a+N_\tau), b+N_\tau\rangle.
\end{aligned}
$$

Since A/N_τ is dense in H_τ, we conclude that $v = t \cdot id_{H_\tau}$. So, $v \in \mathbb{C}id_{H_\tau}$.

From part (a), we conclude that $\varphi_\tau(A)' = \mathbb{C}id_{H_\tau}$. By Theorem 1.9.7, we deduce that the GNS representation $(\varphi_\tau, H_\tau, \xi_\tau)$ is irreducible.

Now assume that the GNS representation $(\varphi_\tau, H_\tau, \xi_\tau)$ is irreducible. Let $\rho : A \to \mathbb{C}$ be a positive linear functional such that $\rho \leq \tau$. By Theorem 2.9.2, there exists a unique operator $v \in \varphi_\tau(A)'$ such that $0 \leq v \leq id_{H_\tau}$ and if $a \in A$ then

$$\rho(a) = \langle \varphi_\tau(a)v\xi_\tau, \xi_\tau\rangle.$$

By Theorem 1.9.7, $\varphi_\tau(A)' = \mathbb{C}id_{H_\tau}$. Hence, there exists $t \in [0,1]$ such that $v = tid_{H_\tau}$. So, $\rho = t\tau$ and τ is a pure state.

Next, assume that A is commutative. Assume that τ is a pure state on A. We will show that $\tau \in \mathcal{M}(A)$. Certainly, τ is a linear functional. It is non-zero because $\|\tau\| = 1$ by definition of a state. By Theorem 2.3.8, if $a \in A$ then $\tau(a^*) = \overline{\tau(a)}$.

To see that τ is multiplicative, we know from part (a) of the proof that $\varphi_\tau(A)' = \mathbb{C}id_{H_\tau}$. Since A is commutative, $\varphi_\tau(A) \subseteq \varphi_\tau(A)'$. So, $\varphi_\tau(A)$

consists of scalar operators and $B(H_\tau) \subseteq \varphi_\tau(A)'$. Consequently, $B(H_\tau) = \varphi_\tau(A)' = \mathbb{C}id_{H_\tau}$ and if $a, b \in A$ then

$$\begin{aligned}
\tau(ab) &= \langle \varphi_\tau(ab)(\xi_\tau), \xi_\tau \rangle \\
&= \langle \varphi_\tau(b)(\xi_\tau), \xi_\tau \rangle \\
&= \varphi_\tau(a) \langle \varphi_\tau(b)(\xi_\tau), \xi_\tau \rangle \\
&= \varphi_\tau(a) \langle \xi_\tau, \xi_\tau \rangle \langle \varphi_\tau(b)(\xi_\tau), \xi_\tau \rangle \\
&= \langle \varphi_\tau(a)(\xi_\tau), \xi_\tau \rangle \langle \varphi_\tau(b)(\xi_\tau), \xi_\tau \rangle = \tau(a)\tau(b).
\end{aligned}$$

So, τ is multiplicative and $\tau \in \mathcal{M}(A)$. We conclude that $PS(A) \subseteq \mathcal{M}(A)$.

Finally, assume that $\tau \in \mathcal{M}(A)$. Let ρ be a positive linear functional on A such that $\rho \leq \tau$. If $\tau(a) = 0$ then $\tau(a^*a) = 0$ and $\rho(a^*a) = 0$. Since ρ is positive, we have

$$|\rho(a)| \leq \rho(a^*a)^{\frac{1}{2}} = 0.$$

So, $\rho(a) = 0$ and $\ker \tau \subseteq \ker \rho$. Hence, there exists $t \in \mathbb{R}$ such that $\rho = t\tau$. Now pick $a \in A$ such that $\tau(a) = 1$. Then, $\tau(a^*a) = 1$ and

$$0 \leq \rho(a^*a) = t\tau(a^*a) = t \leq \tau(a^*a) = 1.$$

So, $t \in [0, 1]$ and $\tau \in PS(A)$. So, $\mathcal{M}(A) \subseteq PS(A)$ and consequently, $PS(A) = \mathcal{M}(A)$. □

Recall that the GNS construction takes a state τ on a C*-algebra A and produces a representation of A. In the next result, we go in the opposite direction. We begin with a representation of A which has a unit cyclic vector and then produce a state on A. This is similar to what was done in Theorem 1.11.4.

Theorem 2.9.7. *Let A be a C*-algebra and (φ, H) be a representation of A. Let x be a unit cyclic vector for (φ, H). Then, the function*

$$\begin{array}{rccc} \tau: & A & \to & \mathbb{C} \\ & a & \mapsto & \langle \varphi(a)(x), x \rangle \end{array}$$

is a state of A and (φ, H) is unitarily equivalent to the GNS representation $(\varphi_\tau, H_\tau, \xi_\tau)$. Moreover, if (φ, H) is an irreducible representation then τ is pure.

Proof. Assume that A is a C*-algebra and (φ, H) is a cyclic representation of A, with unit cyclic vector x. Assume that τ is defined as above. It is straightforward to check that τ is a positive linear functional.

To see that $\|\tau\| = 1$, assume that $\{u_\lambda\}_{\lambda \in L}$ is an approximate unit for A. Then,

$$\|\tau\| = \lim_\lambda \tau(u_\lambda) = \lim_\lambda \langle \varphi(u_\lambda)(x), x \rangle = \langle x, x \rangle = 1$$

because the sequence $\{\varphi(u_\lambda)(x)\}_{\lambda \in L}$ strongly converges to id_H. Therefore, τ is a state on A.

Now let $(\varphi_\tau, H_\tau, \xi_\tau)$ be the GNS representation associated to A. To see that $(\varphi_\tau, H_\tau, \xi_\tau)$ is unitarily equivalent to (φ, H), we observe that if $a \in A$ then

$$\langle \varphi_\tau(a)(\xi_\tau), \xi_\tau \rangle = \tau(a) = \langle \varphi(a)(x), x \rangle.$$

By Theorem 2.9.5, $(\varphi_\tau, H_\tau, \xi_\tau)$ is unitarily equivalent to (φ, H). Finally, if (φ, H) is an irreducible representation then $(\varphi_\tau, H_\tau, \xi_\tau)$ is also irreducible and by Theorem 2.9.6, τ is a pure state as required. $\square$

Recall that in the specific case where A is unital, the pure states of A are the extreme points of the set of states of A, which is a convex set (see Theorem 1.11.4).

Theorem 2.9.8. *Let A be a C*-algebra. Define*

$$S = \{\phi \in A^* \mid \phi \textit{ is positive and } \|\phi\| \leq 1\}.$$

Then, S is a convex and weak- compact set. Moreover, the extreme points of S are the zero functional 0 and the pure states of A.*

Proof. Assume that A is a C*-algebra. Assume that S is the set of positive linear and norm-decreasing functionals on A, defined as above.

To show: (a) S is a convex set.

(b) S is a weak-* compact set.

(a) Assume that $\phi, \psi \in S$ and $t \in [0, 1]$. Then, $t\phi + (1-t)\psi$ is a positive linear functional with norm

$$\|t\phi + (1-t)\psi\| \leq t\|\phi\| + (1-t)\|\psi\| \leq 1.$$

So, $t\phi + (1-t)\psi \in S$ and S is a convex set.

(b) Note that S is a subset of the closed unit ball of A^*, which is a weak-* compact set by the Banach-Alaoglu theorem. Hence, it suffices to show that S is weak-* closed. Assume that $\{\phi_n\}_{n\in\mathbb{Z}_{>0}}$ is a sequence in S which converges to some $\phi \in A^*$ with respect to the weak-* topology. To see that $\phi \in S$, first note that ϕ is contained in the closed unit ball of A^* so that $\|\phi\| \leq 1$.

To see that ϕ is positive, observe that if $a \in A$ then

$$\phi(a^*a) = \lim_{n\to\infty} \phi_n(a^*a) \geq 0$$

because each ϕ_n is positive. Therefore, $\phi \in S$ and we conclude that S is a weak-* closed set.

By parts (a) and (b), we deduce that S is a convex and weak-* compact subset of A^*.

To show: (c) 0 is an extreme point of S.

(d) If $\tau \in PS(A)$ then τ is an extreme point of S.

(c) Assume that there exists $\alpha, \beta \in S$ and $t \in (0,1)$ such that $0 = t\alpha + (1-t)\beta$. If $a \in A$ then

$$0 \geq -t\alpha(a^*a) = (1-t)\beta(a^*a) \geq 0.$$

Therefore, $\alpha = \beta = 0$ and 0 is an extreme point of S.

(d) Assume that $\tau \in PS(A)$. Assume that $\tau = t\gamma + (1-t)\delta$, where $t \in (0,1)$ and $\gamma, \delta \in S$. Then, $t\gamma \leq \tau$ and since τ is a pure state, there exists $t' \in [0,1]$ such that $t\gamma = t'\tau$. Since γ and δ are positive linear functionals, we have

$$1 = \|\tau\| = t\|\gamma\| + (1-t)\|\delta\|$$

by Theorem 2.3.6. Therefore, $\|\gamma\| = \|\delta\| = 1$ and

$$t = t\|\gamma\| = t'\|\tau\| = t'.$$

Therefore, $\gamma = \tau$ and $(1-t)\delta = \tau - t\gamma = (1-t)\tau$. So, $\delta = \tau$ as well and consequently, τ is an extreme point of S.

Finally, assume that ρ is a non-zero extreme point of S. We will demonstrate that ρ is a pure state. First, observe that

$$\rho = \|\rho\|(\frac{\rho}{\|\rho\|}) + (1 - \|\rho\|)0$$

and $0, \rho/\|\rho\| \in S$. Since ρ is a non-zero extreme point of S, we conclude that $\|\rho\| = 1$. Now assume that τ is a non-zero positive linear functional on A such that $\tau < \rho$. Then, $\|\tau\| \in (0, 1)$ and

$$\begin{aligned}
\|\tau\|(\frac{\tau}{\|\tau\|}) + (1 - \|\tau\|)(\frac{\rho - \tau}{\|\rho - \tau\|}) &= \tau + (1 - \|\tau\|)(\frac{\rho - \tau}{\|\rho - \tau\|}) \\
&= \tau + (1 - \|\tau\|)(\frac{\rho - \tau}{\|\rho\| - \|\tau\|}) \\
&= \tau + (1 - \|\tau\|)(\frac{\rho - \tau}{1 - \|\tau\|}) \\
&= \rho.
\end{aligned}$$

Since $\rho > \tau$, then $(\rho - \tau)/\|\rho - \tau\| \in S$. Consequently, $\rho = \tau/\|\tau\|$ and thus, ρ is a pure state on A as required. □

One particular consequence of Theorem 2.9.8 is that by the Krein-Milman theorem, the set S in Theorem 2.9.8 is the closed convex hull of its extreme points, which in this case are the zero functional and the pure states on A.

We end this section by noting the following generalisation of Theorem 1.11.6, which is [Mur90, Theorem 5.1.12].

Theorem 2.9.9. *Let A be a C*-algebra and $a \in A$ be arbitrary. Then, there exists an irreducible representation (φ, H) of A such that $\|a\| = \|\varphi(a)\|$.*

Proof. Assume that A is a C*-algebra and $a \in A$. Then, a^*a is self-adjoint and by Theorem 1.11.6, there exists an irreducible representation (φ, H) of A such that $\|a^*a\| = \|\varphi(a^*a)\|$. Since A and $B(H)$ are C*-algebras,

$$\|a\| = \|a^*a\|^{\frac{1}{2}} = \|\varphi(a^*a)\|^{\frac{1}{2}} = \|\varphi(a)\|.$$

□

2.10 The spatial C*-norm is minimal

Now we will embark on the long proof that the spatial C*-norm on a tensor product of C*-algebras $A \otimes B$ is the smallest C*-norm. Along the way, we will prove a significant number of useful results we will depend on in this section. First, there is a canonical approximate unit on the tensor product $A \otimes B$.

Theorem 2.10.1. *Let A and B be C*-algebras with approximate units $\{u_\lambda\}_{\lambda \in L}$ and $\{v_\mu\}_{\mu \in M}$ respectively. Let γ be a C*-norm on $A \otimes B$. Then, $A \otimes_\gamma B$ admits an approximate unit of the form $\{u_\delta \otimes v_\delta\}_{\delta \in D}$.*

Proof. Assume that A and B are C*-algebras. Assume that $\{u_\lambda\}_{\lambda \in L}$ and $\{v_\mu\}_{\mu \in M}$ are approximate units for A and B respectively. Assume that γ is a C*-norm on $A \otimes B$.

We first impose a partial order on $L \times M$. If $(\lambda, \mu), (\lambda', \mu') \in L \times M$ then $(\lambda, \mu) \leq (\lambda', \mu')$ if and only if $\lambda \leq \lambda'$ and $\mu \leq \mu'$. Then, $L \times M$ is reflexive, transitive and upwards-directed. In particular, it is upwards-directed because L and M are both upwards-directed.

If $(\lambda, \mu) \in L \times M$ then set $u'_{(\lambda,\mu)} = u_\lambda$ and $v'_{(\lambda,\mu)} = v_\mu$. We claim that $\{u'_{(\lambda,\mu)} \otimes v'_{(\lambda,\mu)}\}_{(\lambda,\mu) \in L \times M}$ is an approximate unit for $A \otimes B$. Assume that $z \in A \otimes B$. If $i \in \{1, 2, \ldots, n\}$ then there exists $a_i \in A$ and $b_i \in B$ such that

$$z = \sum_{i=1}^{n} (a_i \otimes b_i).$$

We compute directly that

$$\begin{aligned} \gamma\big(z(u'_{(\lambda,\mu)} \otimes v'_{(\lambda,\mu)}) - z\big) &= \gamma\big(\sum_{i=1}^{n} (a_i u'_{(\lambda,\mu)} \otimes b_i v'_{(\lambda,\mu)}) - \sum_{i=1}^{n} (a_i \otimes b_i)\big) \\ &= \gamma\big(\sum_{i=1}^{n} (a_i u_\lambda \otimes b_i v_\mu) - \sum_{i=1}^{n} (a_i \otimes b_i)\big) \\ &\to \gamma\big(\sum_{i=1}^{n} (a_i \otimes b_i) - \sum_{i=1}^{n} (a_i \otimes b_i)\big) = 0 \end{aligned}$$

where the limit is taken over the variables λ and μ. Note that in the limit above, we implicitly used Theorem 2.8.8. Therefore, $\{u'_{(\lambda,\mu)} \otimes v'_{(\lambda,\mu)}\}_{(\lambda,\mu) \in L \times M}$ is an approximate unit for $A \otimes B$. □

The next results concerns direct sums of representations.

Theorem 2.10.2. *Let A and B be C*-algebras. Let $(\varphi_\lambda, H_\lambda)_{\lambda \in L}$ and $(\psi_\mu, K_\mu)_{\mu \in M}$ be families of representations of A and B respectively. Let $(\varphi, H) = \bigoplus_{\lambda \in L} (\varphi_\lambda, H_\lambda)$ and $(\psi, K) = \bigoplus_{\mu \in M} (\psi_\mu, K_\mu)$. If $c \in A \otimes B$ then*

$$\|(\varphi\hat{\otimes}\psi)(c)\| = \sup_{\lambda\in L,\mu\in M} \|(\varphi_\lambda\hat{\otimes}\psi_\mu)(c)\|.$$

Proof. Assume that A and B are C*-algebras. Define

$$\begin{array}{cccc} \tilde{u}: & H \times K & \to & \bigoplus_{\lambda\in L,\mu\in M}(H_\lambda\hat{\otimes}K_\mu) \\ & \big((x_\lambda)_{\lambda\in L},(y_\mu)_{\mu\in M}\big) & \mapsto & (x_\lambda\otimes y_\mu)_{\lambda,\mu} \end{array}$$

Then, $\tilde{u}$ is $\mathbb{C}$-bilinear and by Theorem 2.8.1, there exists a unique linear map, defined by

$$\begin{array}{cccc} u': & H \otimes K & \to & \bigoplus_{\lambda\in L,\mu\in M}(H_\lambda\hat{\otimes}K_\mu) \\ & (x_\lambda)_{\lambda\in L}\otimes(y_\mu)_{\mu\in M} & \mapsto & (x_\lambda\otimes y_\mu)_{\lambda,\mu} \end{array}$$

It is clear by definition that u' is surjective. To see that u' is bounded, observe that

$$\|u'((x_\lambda)_{\lambda\in L}\otimes(y_\mu)_{\mu\in M})\| = \|(x_\lambda\otimes y_\mu)_{\lambda,\mu}\| = \|(x_\lambda)_{\lambda\in L}\|\|(y_\mu)_{\mu\in M}\|.$$

Thus, u' is a unitary operator and since $H\otimes K$ is dense in $H\hat{\otimes}K$, we can extend u' to a unitary operator u defined on $H\hat{\otimes}K$.

Next, observe that if $a\otimes b\in A\otimes B$ and $(x_\lambda)_{\lambda\in L}\otimes(y_\mu)_{\mu\in M}\in H\otimes K$ then

$$\begin{aligned}
& u^*\big(\bigoplus_{\lambda\in L,\mu\in M}(\varphi_\lambda\hat{\otimes}\psi_\mu)(a\otimes b)\big)u((x_\lambda)_{\lambda\in L}\otimes(y_\mu)_{\mu\in M}) \\
&= u^*\big(\bigoplus_{\lambda\in L,\mu\in M}(\varphi_\lambda\hat{\otimes}\psi_\mu)(a\otimes b)\big)\big((x_\lambda\otimes y_\mu)_{\lambda,\mu}\big) \\
&= u^*\big(\bigoplus_{\lambda\in L,\mu\in M}(\varphi_\lambda(a)\otimes\psi_\mu(b))\big)\big((x_\lambda\otimes y_\mu)_{\lambda,\mu}\big) \\
&= u^*\big(\bigoplus_{\lambda\in L,\mu\in M}\varphi_\lambda(a)(x_\lambda)\otimes\psi_\mu(b)(y_\mu)\big) \\
&= \varphi(a)((x_\lambda)_{\lambda\in L})\otimes\psi(b)((y_\mu)_{\mu\in M}) \\
&= (\varphi\hat{\otimes}\psi)(a\otimes b)((x_\lambda)_{\lambda\in L}\otimes(y_\mu)_{\mu\in M}).
\end{aligned}$$

Since u is unitary, we conclude that if $c\in A\otimes B$ then

$$\|(\varphi\hat{\otimes}\psi)(c)\| = \|\bigoplus_{\lambda\in L,\mu\in M}(\varphi_\lambda\hat{\otimes}\psi_\mu)(c)\| = \sup_{\lambda\in L,\mu\in M}\|(\varphi_\lambda\hat{\otimes}\psi_\mu)(c)\|.$$

$\square$

We can apply Theorem 2.10.2 to the spatial C*-norm to obtain the following result.

Theorem 2.10.3. *Let A and B be non-zero C*-algebras. Let $c \in A \otimes B$. Then,*

$$\|c\|_* = \sup_{\tau \in S(A), \rho \in S(B)} \|(\varphi_\tau \hat{\otimes} \varphi_\rho)(c)\|.$$

Proof. Assume that A and B are non-zero C*-algebras and $c \in A \otimes B$. Let (φ, H) and (ψ, H) be the universal representations of A and B respectively. By definition of the spatial C*-norm,

$$\|c\|_* = \|(\varphi \hat{\otimes} \psi)(c)\|.$$

By the definition of the universal representation and Theorem 2.10.2, we have

$$\|c\|_* = \|(\varphi \hat{\otimes} \psi)(c)\| = \sup_{\tau \in S(A), \rho \in S(B)} \|(\varphi_\tau \hat{\otimes} \varphi_\rho)(c)\|.$$

$\square$

Theorem 2.10.4. *Let A and B be C*-algebras. Let τ and ρ be states on A and B respectively. Then, $\tau \otimes \rho$ is continuous on $A \otimes B$ with respect to the spatial C*-norm.*

Proof. Assume that A and B are C*-algebras. Assume that $\tau \in S(A)$, $\rho \in S(B)$ and $a \otimes b \in A \otimes B$. Let $(\varphi_\tau, H_\tau, \xi_\tau)$ and $(\varphi_\rho, H_\rho, \xi_\rho)$ be the GNS representations associated to τ and ρ respectively. Then,

$$\begin{aligned}
\langle (\varphi_\tau \hat{\otimes} \varphi_\rho)(a \otimes b)(\xi_\tau \otimes \xi_\rho), \xi_\tau \otimes \xi_\rho \rangle &= \langle (\varphi_\tau(a) \otimes \varphi_\rho(b))(\xi_\tau \otimes \xi_\rho), \xi_\tau \otimes \xi_\rho \rangle \\
&= \langle \varphi_\tau(a)\xi_\tau \otimes \varphi_\rho(b)(\xi_\rho), \xi_\tau \otimes \xi_\rho \rangle \\
&= \langle \varphi_\tau(a)\xi_\tau, \xi_\tau \rangle \langle \varphi_\rho(b)\xi_\rho, \xi_\rho \rangle \\
&= \tau(a)\rho(b) = (\tau \otimes \rho)(a \otimes b)
\end{aligned}$$

Since $a \otimes B \in A \otimes B$ was arbitrary, we conclude that if $c \in A \otimes B$ then

$$(\tau \otimes \rho)(c) = \langle (\varphi_\tau \hat{\otimes} \varphi_\rho)(c)(\xi_\tau \otimes \xi_\rho), \xi_\tau \otimes \xi_\rho \rangle.$$

By Theorem 2.10.3, we first have

$$\|(\varphi_\tau \hat{\otimes} \varphi_\rho)(c)\| \leq \sup_{\tau \in S(A), \rho \in S(B)} \|(\varphi_\tau \hat{\otimes} \varphi_\rho)(c)\| = \|c\|_*.$$

Therefore,

$$\begin{aligned}|(\tau \otimes \rho)(c)| &\leq \|(\varphi_\tau \hat{\otimes} \varphi_\rho)(c)\| \|\xi_\tau \otimes \xi_\rho\|^2 \\ &= \|(\varphi_\tau \hat{\otimes} \varphi_\rho)(c)\| \\ &\leq \|c\|_*.\end{aligned}$$

So, $\tau \otimes \rho$ is continuous with respect to the spatial C*-norm on $A \otimes B$. $\square$

Before we move onto the next result, we state the following remark in [Mur90, Remark 6.4.2]. Let A and B be C*-algebras. Let (φ, H) and (φ', H') be unitarily equivalent representations of A and (ψ, K) and (ψ', K') be unitarily equivalent representations of B. Then, there exists a unitary map $u : H \hat{\otimes} K \to H' \hat{\otimes} K'$ such that if $c \in A \otimes B$ then

$$(\varphi' \hat{\otimes} \psi')(c) = u(\varphi \hat{\otimes} \psi)(c)u^*.$$

Theorem 2.10.5. *Let A and B be C*-algebras. Let (φ, H) and (ψ, K) be representations of A and B respectively. Let $c \in A \otimes B$. Then,*

$$\|(\varphi \hat{\otimes} \psi)(c)\| \leq \|c\|_*.$$

Proof. Assume that A and B are C*-algebras. Assume that (φ, H) and (ψ, K) are representations of A and B respectively. We would like to assume that the representations (φ, H) and (ψ, K) are non-degenerate.

To this end, let $H' = \overline{\varphi(A)H}$ and $K' = \overline{\psi(B)K}$. Then,

$$H' \hat{\otimes} K' = \overline{(\varphi \hat{\otimes} \psi)(A \otimes B)(H \hat{\otimes} K)}$$

and if $a, a' \in A$, $b, b' \in B$, $h \in H$ and $k \in K$ then

$$\begin{aligned}(\varphi \hat{\otimes} \psi)(a \otimes b)(\varphi(a')(h) \otimes \psi(b')(k)) &= (\varphi(a) \otimes \psi(b))(\varphi(a')(h) \otimes \psi(b')(k)) \\ &= \varphi(aa')(h) \otimes \psi(bb')(k) \\ &= (\varphi|_{H'} \hat{\otimes} \psi|_{K'})(a \otimes b)(\varphi(a')(h) \otimes \psi(b')(k)).\end{aligned}$$

Since $a \otimes b \in A \otimes B$ and $\varphi(a')(h) \otimes \psi(b')(k) \in H' \hat{\otimes} K'$ were arbitrary, we deduce that if $c \in A \otimes B$ then

$$(\varphi \hat{\otimes} \psi)(c)|_{H' \hat{\otimes} K'} = (\varphi|_{H'} \hat{\otimes} \psi|_{K'})(c).$$

By Theorem 1.9.5,

$$\|(\varphi\hat{\otimes}\psi)(c)\| = \|(\varphi\hat{\otimes}\psi)(c)|_{H'\hat{\otimes}K'}\| = \|(\varphi|_{H'}\hat{\otimes}\psi|_{K'})(c)\|$$

Hence, we may assume without loss of generality that the representations (φ, H) and (ψ, K) are non-degenerate. By Theorem 2.9.4, we can write (φ, H) and (ψ, K) as the direct sum of cyclic representations. Next, recall from the proof of Theorem 2.9.6 that each non-zero cyclic representation of A or B is unitarily equivalent to a GNS representation associated to some state on A or B.

By replacing the cyclic representations in the direct sums comprising (φ, H) and (ψ, K) with unitarily equivalent representations if necessary (this invokes the previous remark), we may assume that

$$(\varphi, H) = \bigoplus_{\lambda\in\Lambda}(\varphi_{\tau_\lambda}, H_{\tau_\lambda}, \xi_{\tau_\lambda})$$

for some indexing set Λ and states $\tau_\lambda \in S(A)$ for $\lambda \in \Lambda$. Similarly,

$$(\psi, K) = \bigoplus_{\mu\in M}(\varphi_{\rho_\mu}, H_{\rho_\mu}, \xi_{\rho_\mu}).$$

Now if $c \in A \otimes B$ then by Theorem 2.10.2,

$$\|(\varphi\hat{\otimes}\psi)(c)\| = \sup_{\lambda\in\Lambda,\mu\in M}\|(\varphi_{\tau_\lambda}\hat{\otimes}\varphi_{\rho_\mu})(c)\|.$$

By Theorem 2.10.3, we have

$$\|(\varphi\hat{\otimes}\psi)(c)\| \leq \sup_{\tau\in S(A),\rho\in S(B)}\|(\varphi_\tau\hat{\otimes}\varphi_\rho)(c)\| = \|c\|_*.$$

□

Next, we need [Mur90, Remark 6.4.3]. Let $p \in M_{n\times n}(\mathbb{C})$ be a rank-one projection. Then, write $p = x \otimes x$ where $x \in \mathbb{C}^n$. Let $\{e_1, \ldots, e_n\}$ be the canonical orthonormal basis of $\mathbb{C}^n$. Then, there exists $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ such that

$$x = \sum_{i=1}^{n} \lambda_i e_i.$$

Now $e_i \otimes e_j \in M_{n\times n}(\mathbb{C})$ is the matrix with a 1 in the i, j entry and zeros elsewhere. So,

$$p = \sum_{i,j=1}^{n} \lambda_i \overline{\lambda_j}(e_i \otimes e_j) = (\lambda_i \overline{\lambda_j})_{i,j}.$$

The next result we need concerns the tensor product of positive linear functionals.

Theorem 2.10.6. *Let A and B be C*-algebras. Let $\tau : A \to \mathbb{C}$ and $\rho : B \to \mathbb{C}$ be positive linear functionals. Then, the linear functional $\tau \otimes \rho$ on $A \otimes B$ is positive.*

Proof. Assume that A and B are C*-algebras. Assume that τ and ρ are positive linear functionals on A and B respectively. Assume that $c = \sum_{j=1}^{n}(a_j \otimes b_j) \in A \otimes B$. We compute directly that

$$(\tau \otimes \rho)(c^* c) = (\tau \otimes \rho)(\sum_{i,j=1}^{n}(a_i^* a_j \otimes b_i^* b_j) = \sum_{i,j=1}^{n} \tau(a_i^* a_j)\rho(b_i^* b_j).$$

Now define the matrix $u = (\rho(b_i^* b_j))_{i,j} \in M_{n\times n}(\mathbb{C})$. If $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ then

$$\sum_{i,j=1}^{n} \rho(b_i^* b_j)\overline{\lambda_i}\lambda_j = \rho\Big((\sum_{i=1}^{n} \lambda_i b_i)^*(\sum_{i=1}^{n} \lambda_i b_i)\Big) \geq 0$$

because ρ is positive. We conclude that u is a positive element of $M_{n\times n}(\mathbb{C})$. So, it can be diagonalised and written as

$$u = \sum_{j=1}^{n} t_j p_j$$

where $t_1, \ldots, t_n \in \mathbb{R}_{>0}$ and $p_1, \ldots, p_n$ are rank-one projections in $M_{n\times n}(\mathbb{C})$.

To see that $(\tau \otimes \rho)(c^* c) \geq 0$, it suffices to show that if $p = (p_{i,j})$ is a rank-one projection in $M_{n\times n}(\mathbb{C})$ then $\sum_{i,j=1}^{n} \tau(a_i^* a_j)p_{i,j} \geq 0$. By [Mur90, Remark 6.4.3], if $i, j \in \{1, 2, \ldots, n\}$ then $p_{i,j} = \overline{\lambda_i}\lambda_j$ for some $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$. So,

$$\begin{aligned}\sum_{i,j=1}^{n} \tau(a_i^* a_j)p_{i,j} &= \sum_{i,j=1}^{n} \tau(a_i^* a_j)\overline{\lambda_i}\lambda_j \\ &= \tau\Big((\sum_{i=1}^{n} \lambda_i a_i)^*(\sum_{j=1}^{n} \lambda_j a_j)\Big) \geq 0\end{aligned}$$

because τ is positive. Therefore,

$$(\tau \otimes \rho)(c^*c) = \sum_{i,j=1}^{n} \tau(a_i^*a_j)\rho(b_i^*b_j) \geq 0$$

and $\tau \otimes \rho$ defines a positive linear functional on $A \otimes B$. $\square$

The point of Theorem 2.10.6 is that in the next result, we want to use two states τ and ρ on C*-algebras A and B to obtain a unique state on $A \otimes_\gamma B$.

Theorem 2.10.7. *Let A and B be C*-algebras and γ be a C*-norm on $A \otimes B$. Let $\tau \in S(A)$ and $\rho \in S(B)$. If $\tau \otimes \rho$ is continuous with respect to γ then $\tau \otimes \rho$ extends uniquely to a state ω on $A \otimes_\gamma B$.*

Proof. Assume that A and B are C*-algebras. Assume that γ is a C*-norm on $A \otimes B$, $\tau \in S(A)$ and $\rho \in S(B)$. Assume that the linear functional $\tau \otimes \rho$ is continuous with respect to γ. Since $A \otimes B$ is dense in $A \otimes_\gamma B$, we can extend $\tau \otimes \rho$ to a unique continuous linear functional ω on $A \otimes_\gamma B$.

To show: (a) ω is positive.

(b) $\|\omega\| = 1$.

(a) Assume that $c \in A \otimes_\gamma B$. Then, there exists a sequence $\{c_n\}_{n \in \mathbb{Z}_{>0}}$ in $A \otimes B$ such that $\lim_{n\to\infty} \gamma(c - c_n) = 0$. So,

$$\omega(c^*c) = \lim_{n\to\infty} \omega(c_n^*c_n) = \lim_{n\to\infty} (\tau \otimes \rho)(c_n^*c_n) \geq 0$$

where the last inequality follows from Theorem 2.10.6. Hence, ω is positive.

(b) By Theorem 2.10.1, let $\{u_\lambda \otimes v_\lambda\}_{\lambda \in L}$ be an approximate unit for $A \otimes B$, where $\{u_\lambda\}_{\lambda \in L}$ and $\{v_\lambda\}_{\lambda \in L}$ are approximate units for A and B respectively. By Theorem 2.3.5, we have

$$\begin{aligned}
\|\omega\| &= \lim_\lambda \omega(u_\lambda \otimes v_\lambda) \\
&= \lim_\lambda (\tau \otimes \rho)(u_\lambda \otimes v_\lambda) \\
&= \lim_\lambda \tau(u_\lambda)\rho(v_\lambda) \\
&= \|\tau\|\|\rho\| = 1.
\end{aligned}$$

By parts (a) and (b), ω is the unique state on $A \otimes_\gamma B$ which extends the continuous linear functional $\tau \otimes \rho$ on $A \otimes B$. $\square$

In Theorem 2.10.7, we denote the state ω by $\tau \otimes_\gamma \rho$.

Theorem 2.10.8. *Let A and B be C*-algebras. Let γ be a C*-norm on $A \otimes B$. Let $\tau \in S(A)$ and $\rho \in S(B)$ such that the linear functional $\tau \otimes \rho$ is continuous with respect to γ. Then, there exists a unitary map $u : H_\tau \hat{\otimes} H_\rho \to H_{\tau \otimes_\gamma \rho}$ such that if $c \in A \otimes B$ then*

$$\varphi_{\tau \otimes_\gamma \rho}(c) = u(\varphi_\tau \hat{\otimes} \varphi_\rho)(c)u^*.$$

Proof. Assume that A and B are C*-algebras. Let $\omega = \tau \otimes_\gamma \rho$ and $\pi = \varphi_\tau \hat{\otimes} \varphi_\rho$. Let ξ_τ and ξ_ρ be the unit cyclic vectors corresponding to the GNS representations of τ and ρ respectively. Let $y = \xi_\tau \otimes \xi_\rho \in H_\tau \hat{\otimes} H_\rho$.

Let $(\varphi_\omega, H_\omega, \xi_\omega)$ be the GNS representation associated to the state ω constructed as an extension of $\tau \otimes \rho$ in Theorem 2.10.7. If $a \otimes b \in A \otimes B$ then

$$\begin{aligned}
\langle \varphi_\omega(c)(\xi_\omega), \xi_\omega \rangle &= \omega(a \otimes b) \\
&= (\tau \otimes \rho)(a \otimes b) = \tau(a)\rho(b) \\
&= \langle \varphi_\tau(a)(\xi_\tau), \xi_\tau \rangle \langle \varphi_\rho(b)(\xi_\rho), \xi_\rho \rangle \\
&= \langle \varphi_\tau(a)(\xi_\tau) \otimes \varphi_\rho(b)(\xi_\rho), \xi_\tau \otimes \xi_\rho \rangle \\
&= \langle (\varphi_\tau \hat{\otimes} \varphi_\rho)(a \otimes b)(\xi_\tau \otimes \xi_\rho), \xi_\tau \otimes \xi_\rho \rangle \\
&= \langle \pi(a \otimes b)(y), y \rangle.
\end{aligned}$$

Now let $H_0 = \varphi_\omega(A \otimes B)\xi_\omega$ and $K_0 = \pi(A \otimes B)(y)$. Define the map

$$\begin{array}{rccc}
u_0 : & K_0 & \to & H_0 \\
& \pi(c)(y) & \mapsto & \varphi_\omega(c)(\xi_\omega).
\end{array}$$

To see that u_0 is well-defined, assume that $c_1, c_2 \in A \otimes B$ such that $\pi(c_1)(y) = \pi(c_2)(y)$. Then,

$$\langle \varphi_\omega(c_1)(\xi_\omega), \xi_\omega \rangle = \langle \pi(c_1)(y), y \rangle = \langle \pi(c_2)(y), y \rangle = \langle \varphi_\omega(c_2)(\xi_\omega), \xi_\omega \rangle$$

and since ξ_ω is cyclic, $\varphi_\omega(c_1)(\xi_\omega) = \varphi_\omega(c_2)(\xi_\omega)$. To see that u_0 is isometric, we compute directly that if $c \in A \otimes B$ then

$$\begin{aligned}
\|u_0(\pi(c)(y))\|^2 &= \|\varphi_\omega(c)(\xi_\omega)\|^2 \\
&= \langle \varphi_\omega(c)(\xi_\omega), \varphi_\omega(c)(\xi_\omega) \rangle \\
&= \langle \varphi_\omega(c^*c)(\xi_\omega), \xi_\omega \rangle \\
&= \langle \pi(c^*c)(y), y \rangle = \|\pi(c)(y)\|^2.
\end{aligned}$$

So, u_0 is a well-defined isometric linear map and since K_0 is dense in $H_\tau \hat{\otimes} H_\rho$ and H_0 is dense in H_ω, we can extend u_0 uniquely to a unitary map $u : H_\tau \hat{\otimes} H_\rho \to H_\omega$.

Finally, if $c, d \in A \otimes B$ and $\varphi_\omega(d)(\xi_\omega) \in H_0$ then

$$\begin{aligned} u\pi(c)u^*(\varphi_\omega(d)(\xi_\omega)) &= u\pi(c)\big(\pi(d)(y)\big) \\ &= u(\pi(cd)(y)) = \varphi_\omega(cd)(\xi_\omega) \\ &= \varphi_\omega(c)\big(\varphi_\omega(d)(\xi_\omega)\big). \end{aligned}$$

Since $u\pi(c)u^*$ and $\varphi_\omega(c)$ agree on H_0 and H_0 is dense in H_ω, $u\pi(c)u^* = \varphi_\omega(c)$ on H_ω. □

Next, we will use Theorem 2.10.7 and Theorem 2.10.8 to give an alternative characterisation of the spatial C*-norm.

Theorem 2.10.9. *Let A and B be non-zero C*-algebras. Suppose that $c \in A \otimes B$. Then,*

$$\|c\|_*^2 = \sup_{\substack{\tau \in S(A) \\ \rho \in S(B)}} \sup_{\substack{d \in A \otimes B \\ (\tau\otimes\rho)(d^*d) > 0}} \frac{(\tau \otimes \rho)(d^* c^* c d)}{(\tau \otimes \rho)(d^* d)}.$$

Proof. Assume that A and B are non-zero C*-algebras. Let ω be a state on the spatial tensor product $A \otimes_* B$. If $(\varphi_\omega, H_\omega, \xi_\omega)$ is the GNS representation of $A \otimes_* B$ associated to ω and $c, d \in A \otimes B$ then

$$\begin{aligned} \sup_{\|d+N_\omega\| \neq 0} \frac{\|\varphi_\omega(c)(d+N_\omega)\|^2}{\|d+N_\omega\|^2} &= \sup_{\|d+N_\omega\| \neq 0} \frac{\|cd+N_\omega\|^2}{\|d+N_\omega\|^2} \\ &= \sup_{\|d+N_\omega\| \neq 0} \frac{\omega(d^*c^*cd)}{\omega(d^*d)} \\ &= \sup_{\omega(d^*d)>0} \frac{\omega(d^*c^*cd)}{\omega(d^*d)}. \end{aligned}$$

Since $\varphi_\omega(A \otimes B)\xi_\omega$ is dense in H_ω and $d + N_\omega = \varphi_\omega(d)\xi_\omega$, we can use Theorem 2.9.1 to deduce that

$$\|\varphi_\omega(c)\|^2 = \sup_{\omega(d^*d)>0} \frac{\omega(d^*c^*cd)}{\omega(d^*d)}.$$

By Theorem 2.10.3, we have

$$\|c\|_* = \sup_{\substack{\tau \in S(A) \\ \rho \in S(B)}} \|(\varphi_\tau \hat{\otimes} \varphi_\rho)(c)\|.$$

By Theorem 2.10.4, if $\tau \in S(A)$ and $\rho \in S(B)$ then the tensor product $\tau \otimes \rho : A \otimes B \to \mathbb{C}$ is continuous with respect to the spatial C*-norm $\|-\|_*$ on $A \otimes B$. Furthermore, by Theorem 2.10.7, we can construct the state $\tau \otimes_{\|-\|_*} \rho$ as the unique extension of $\tau \otimes \rho$ to $A \otimes_* B$. Hence,

$$\begin{aligned}
\|c\|_*^2 &= \sup_{\tau \in S(A),\, \rho \in S(B)} \|(\varphi_\tau \hat{\otimes} \varphi_\rho)(c)\|^2 \\
&= \sup_{\tau \in S(A),\, \rho \in S(B)} \|\varphi_{\tau \otimes_{\|-\|_*} \rho}(c)\|^2 \quad \text{(by Theorem 2.10.8)} \\
&= \sup_{\substack{\tau \in S(A) \\ \rho \in S(B)}} \sup_{\substack{d \in A \otimes B \\ (\tau \otimes \rho)(d^* d) > 0}} \frac{(\tau \otimes \rho)(d^* c^* c d)}{(\tau \otimes \rho)(d^* d)}.
\end{aligned}$$

This completes the proof. □

Next, we will use the characterisation of the spatial C*-norm in Theorem 2.10.9 to prove a result involving tensor products of C*-algebras and unitizations.

Theorem 2.10.10. *Let A and B be C*-algebras. Let $\tilde{B}$ be the unitization of B. Then, the restriction of the spatial C*-norm on $A \otimes \tilde{B}$ to $A \otimes B$ is the spatial C*-norm on $A \otimes B$.*

Proof. Assume that A and B are C*-algebras. Assume that $\tilde{B}$ is the unitization of B. Let γ be the restriction of the spatial C*-norm on $A \otimes \tilde{B}$ to $A \otimes B$. By Theorem 2.10.9, if $c \in A \otimes B$ then

$$\gamma(c)^2 = \sup_{\substack{\tau \in S(A) \\ \rho \in S(\tilde{B})}} \sup_{\substack{d \in A \otimes \tilde{B} \\ (\tau \otimes \rho)(d^* d) > 0}} \frac{(\tau \otimes \rho)(d^* c^* c d)}{(\tau \otimes \rho)(d^* d)}$$

and

$$\|c\|_*^2 = \sup_{\substack{\tau \in S(A) \\ \rho \in S(B)}} \sup_{\substack{d \in A \otimes B \\ (\tau \otimes \rho)(d^* d) > 0}} \frac{(\tau \otimes \rho)(d^* c^* c d)}{(\tau \otimes \rho)(d^* d)}$$

where $\|-\|_*$ is the spatial C*-norm on $A \otimes B$. Now observe that if $\rho \in S(B)$ then there exists a unique extension $\tilde{\rho}$ of ρ such that $\tilde{\rho} \in S(\tilde{B})$. Thus, if $c \in A \otimes B$ then $\gamma(c) \geq \|c\|_*$.

To see that $\gamma(c) \leq \|c\|_*$, let (φ, H) and (ψ, K) denote the universal representations of A and $\tilde{B}$ respectively. Let $\psi|_B$ denote the restriction of the *-homomorphism ψ to B. By the definition of the spatial norm on $A \otimes B$ and Theorem 2.8.7,

$$\gamma(c) = \|(\varphi \hat{\otimes} \psi)(c)\| = \|(\varphi \hat{\otimes} \psi|_B)(c)\| \leq \|c\|_*.$$

Thus, if $c \in A \otimes B$ then $\gamma(c) \leq \|c\|_*$. So, $\gamma = \|-\|_*$ on $A \otimes B$. □

The next result tells us that we can extend a C*-norm on $A \otimes B$ to $A \otimes \tilde{B}$.

Theorem 2.10.11. *Let A and B be C*-algebras with B non-unital. Let γ be a C*-norm on $A \otimes B$. Then, there exists a C*-norm on $A \otimes \tilde{B}$ which extends γ.*

Proof. Assume that A and B are C*-algebras, with B non-unital. Assume that γ is a C*-norm on $A \otimes B$. Let (π, H) be a faithful non-degenerate representation of the C*-algebra $A \otimes_\gamma B$. By Theorem 2.8.9, there exists unique *-homomorphisms $\pi_A : A \to B(H)$ and $\pi_B : B \to B(H)$ such that if $a \in A$ and $b \in B$ then

$$\pi(a \otimes b) = \pi_A(a)\pi_B(b) = \pi_B(b)\pi_A(a).$$

Since π is injective, π_A and π_B are both injective. Since π_B is an injective *-homomorphism. By the universal property of unitization in Theorem 1.6.3, there exists a unique unital *-homomorphism $\pi'_B : \tilde{B} \to B(H)$ such that if $\iota : B \hookrightarrow \tilde{B}$ denotes the inclusion map then

$$\pi'_B \circ \iota = \pi_B.$$

To show: (a) π'_B is injective.

(a) Assume that $(\lambda, b) \in \ker \pi'_B$ so that $\pi'_B((\lambda, b)) = 0$. Then,

$$\pi'_B((\lambda, b)) = \lambda \pi'_B((1, 0)) + \pi'_B(0, b) = \lambda id_{B(H)} + \pi_B(b) = 0.$$

Suppose for the sake of contradiction that $\lambda \neq 0$. Then, $\pi_B(-\lambda^{-1}b) = id_{B(H)}$. Since π_B is injective, we deduce that $-\lambda^{-1}b \in B$ is a unit for B. However, this contradicts the assumption that B is non-unital. Therefore, $\lambda = 0$ and consequently, $b = 0$. So, $(\lambda, b) = (0, 0)$ and π'_B is injective.

Since the images $\pi_A(A)$ and $\pi_B(B)$ commute, the images $\pi_A(A)$ and $\pi'_B(\tilde{B})$ must also commute. By invoking the universal property in Theorem 2.8.1, there exists a unique *-homomorphism $\pi' : A \otimes \tilde{B} \to B(H)$ such that $\pi'|_{A\otimes B} = \pi$.

By injectivity of π, if $c \in A \otimes_\gamma B$ then $\|\pi(c)\| = \gamma(c)$ because π is isometric by Theorem 1.6.4. At this point, it suffices to show that π' is injective, as in this case, the map $c \mapsto \|\pi'(c)\|$ becomes a C*-norm on $A \otimes \tilde{B}$ extending γ.

To show: (b) π' is injective.

(b) Assume that $d \in \ker \pi'$. If $c \in A \otimes B$ then $dc \in A \otimes B$ and $\pi(dc) = 0$. Since π is injective, $dc = 0$. Now let $\theta = \pi_A \hat{\otimes} \pi'_B$. Then,

$$\theta(d)\theta(c) = \theta(dc) = (\pi_A \hat{\otimes} \pi'_B)(dc) = 0$$

where the last equality follows from Theorem 2.8.7 and the fact that π_A and π'_B are both injective. Since this holds for arbitrary $c \in A \otimes B$, then $\theta(d) = 0$ on the subspace

$$K_0 = \theta(A \otimes B)(H \hat{\otimes} H).$$

Since (π, H) is a non-degenerate representation of $A \otimes_\gamma B$, then (π_A, H) and (π_B, H) are both non-degenerate representations of A and B respectively by Theorem 2.8.9. By Theorem 1.9.4,

$$\overline{\pi_A(A)H} = H \qquad \text{and} \qquad \overline{\pi_B(B)H} = H.$$

By definition of θ,

$$\theta(A \otimes B)(H \hat{\otimes} H) = (\pi_A \hat{\otimes} \pi'_B)(A \otimes B)(H \hat{\otimes} H) = \pi_A(A)(H) \hat{\otimes} \pi'_B(B)(H).$$

Hence, K_0 is dense in $H \hat{\otimes} H$ and $\theta(d) = 0$ on $H \hat{\otimes} H$. Since θ is injective by Theorem 2.8.7, $d = 0$ and π' is injective. This completes the proof. □

Theorem 2.10.12. *Let A and B be C*-algebras. Let $u \in \tilde{A}$ and $v \in \tilde{B}$ be unitary elements. If γ is a C*-norm on $A \otimes B$ then the unique *-isomorphism*

$$\begin{array}{rccc} \pi: & A \otimes B & \to & A \otimes B \\ & a \otimes b & \mapsto & uau^* \otimes vbv^* \end{array}$$

*is an isometry. Note that in the definition of π, we regard $A \otimes B$ as a *-subalgebra of $\tilde{A} \otimes \tilde{B}$.*

Proof. Assume that A and B are C*-algebras and that π is the *-isomorphism on $A \otimes B$ defined as above. Note that the inverse of π is the map $a \otimes b \mapsto u^*au \otimes v^*bv$. By symmetry, it suffices to show that π is norm-decreasing.

To show: (a) If γ is a C*-norm on $A \otimes B$ and $c \in A \otimes B$ then $\gamma(\pi(c)) \leq \gamma(c)$.

(a) Assume that γ is a C*-norm on $A \otimes B$. By Theorem 2.10.1, let $\{u_\lambda \otimes v_\lambda\}_{\lambda \in L}$ be an approximate unit for $A \otimes_\gamma B$, where $\{u_\lambda\}_{\lambda \in L}$ and $\{v_\lambda\}_{\lambda \in L}$ are approximate units for A and B respectively.

If $\lambda \in L$ then let $w_\lambda = u_\lambda \otimes v_\lambda$. Let $w = u \otimes v$ so that if $c \in A \otimes B$ then $\pi(c) = wcw^*$. If $a \in A$ and $b \in B$ then

$$uau^* = \lim_\lambda uu_\lambda au_\lambda u^* \qquad \text{and} \qquad vbv^* = \lim_\lambda vv_\lambda bv_\lambda v^*.$$

So,

$$\begin{aligned} w(a \otimes b)w^* &= uau^* \otimes vbv^* \\ &= \lim_\lambda (uu_\lambda au_\lambda u^* \otimes vv_\lambda bv_\lambda v^*) \\ &= \lim_\lambda ww_\lambda(a \otimes b)w_\lambda w^* \end{aligned}$$

in $A \otimes_\gamma B$. Therefore,

$$\pi(c) = wcw^* = \lim_\lambda ww_\lambda cw_\lambda w^* = \lim_\lambda \pi(w_\lambda cw_\lambda).$$

Finally, we argue that

$$\begin{aligned} \gamma(\pi(c)) &= \gamma(\lim_\lambda \pi(w_\lambda cw_\lambda)) \\ &= \lim_\lambda \gamma(ww_\lambda cw_\lambda w^*) \\ &\leq \sup_{\lambda \in L} \gamma(ww_\lambda)\gamma(c)\gamma(w_\lambda w^*) \\ &\leq \sup_{\lambda \in L} \|uu_\lambda\|\|vv_\lambda\|\gamma(c)\|u_\lambda u^*\|\|v_\lambda v^*\| \\ &\leq \gamma(c). \end{aligned}$$

This completes the proof. $\square$

Within the onslaught of these results, we now present a definition that is the key to proving that abelian C*-algebras are nuclear and that the spatial C*-norm is minimal.

Definition 2.10.1. Let A and B be C*-algebras and γ be a C*-norm on $A \otimes B$. Let $PS(A)$ denote the topological space of pure states on A, where $PS(A)$ is endowed with the weak-* topology from A^*. Define

$$S_\gamma = \left\{ (\tau, \rho) \in PS(A) \times PS(B) \;\middle|\; \begin{array}{c} \tau \otimes \rho \text{ is continuous} \\ \text{on } A \otimes B \text{ with respect to } \gamma \end{array} \right\}. \tag{2.2}$$

The next few results are dedicated to proving properties about S_γ.

Theorem 2.10.13. *Let A and B be C*-algebras. Let γ be a C*-norm on $A \otimes B$. Let $PS(A)$ and $PS(B)$ denote the topological spaces of pure states on A and B respectively, where the topology is the weak-* topology induced from A^* and B^*. Then, S_γ is closed in $PS(A) \times PS(B)$.*

Moreover, if $u \in \tilde{A}$ and $v \in \tilde{B}$ are unitary elements and $(\tau, \rho) \in S_\gamma$ then $(\tau^u, \rho^v) \in S_\gamma$ where $\tau^u(a) = \tau(uau^*)$ and $\rho^v(b) = \rho(vbv^*)$ (recall that A is a closed two-sided ideal of $\tilde{A}$ and so, $uau^* \in A$. Similarly, $vbv^* \in B$).

Proof. Assume that A and B are C*-algebras. Assume that γ is a C*-norm on $A \otimes B$. Assume that $u \in \tilde{A}$ and $v \in \tilde{B}$ are unitary elements. If π is the *-isomorphism from Theorem 2.10.12 then

$$\tau^u \otimes \rho^v = (\tau \otimes \rho)\pi.$$

By Theorem 2.10.12, π is continuous with respect to γ. Since $\tau \otimes \rho \in S_\gamma$, it is continuous with respect to γ. So, $\tau^u \otimes \rho^v \in S_\gamma$.

To see that S_γ is closed, assume that $\{(\tau_n, \rho_n)\}_{n \in \mathbb{Z}_{>0}}$ is a sequence in S_γ which converges to some $(\tau, \rho) \in PS(A) \times PS(B)$ with respect to the topology on $PS(A) \times PS(B)$. If $n \in \mathbb{Z}_{>0}$ and $c \in A \otimes B$ then

$$|(\tau_n \otimes \rho_n)(c)| \leq \|\tau_n \otimes \rho_n\|\gamma(c) = \|\tau_n\|\|\rho_n\|\gamma(c) = \gamma(c).$$

Now let $d = \sum_{i=1}^n (a_i \otimes b_i) \in A \otimes B$ with $\|d\| \leq 1$. Then,

$$\begin{aligned}(\tau \otimes \rho)(d) &= (\tau \otimes \rho)(\sum_{i=1}^{n}(a_i \otimes b_i)) \\ &= \sum_{i=1}^{n} \tau(a_i)\rho(b_i) \\ &= \lim_{m\to\infty} \sum_{i=1}^{n} \tau_m(a_i)\rho_m(b_i) \\ &= \lim_{m\to\infty} (\tau_m \otimes \rho_m)(d).\end{aligned}$$

So, if $d \in A \otimes B$ and $\|d\| = 1$ then

$$|(\tau \otimes \rho)(d)| = \lim_{m\to\infty} |(\tau_m \otimes \rho_m)(d)| \leq \lim_{m\to\infty} \gamma(d) = \gamma(d).$$

Therefore, $\tau \otimes \rho$ is continuous with respect to γ and subsequently, $\tau \otimes \rho \in S_\gamma$. So, S_γ is a closed subset of $PS(A) \otimes PS(B)$. □

In the next result, we will construct a particular decomposition of a state on $A \otimes_\gamma B$.

Theorem 2.10.14. *Let A and B be C*-algebras. Let γ be a C*-norm on $A \otimes B$ and ω be a state on $A \otimes_\gamma B$. Let $(\pi, H_\omega, \xi_\omega)$ be the GNS representation of $A \otimes_\gamma B$ associated to ω. Using Theorem 2.8.9, let $\pi_A : A \to B(H_\omega)$ and $\pi_B : B \to B(H_\omega)$ be the unique *-homomorphism such that if $a \in A$ and $b \in B$ then*

$$\pi(a \otimes b) = \pi_A(a)\pi_B(b) = \pi_B(b)\pi_A(a).$$

Define the states ω_A and ω_B on A and B respectively by

$$\omega_A(a) = \langle \pi_A(a)(\xi_\omega), \xi_\omega\rangle \qquad \text{and} \qquad \omega_B(b) = \langle \pi_B(b)(\xi_\omega), \xi_\omega\rangle.$$

If $(\tau, \rho) \in S_\gamma$ and $\omega = \tau \otimes_\gamma \rho$ then $\tau = \omega_A$ and $\rho = \omega_B$.

Proof. Assume that A and B are C*-algebras. Assume that γ is a C*-norm on $A \otimes B$. Assume that ω is a state on $A \otimes_\gamma B$. Assume that $\pi, \pi_A, \pi_B, \omega_A$ and ω_B are defined as above. Then, ω_A and ω_B are states by Theorem 2.9.7.

Let $\{u_\lambda\}_{\lambda \in L}$ be an approximate unit for A. Since (π_A, H_ω) is non-degenerate by Theorem 2.8.9, then $\xi_\omega = \lim_\lambda \pi_A(u_\lambda)(\xi_\omega)$ by Theorem 2.3.4. Hence, if $b \in B$ then

$$
\begin{aligned}
\omega_B(b) &= \langle \pi_B(b)(\xi_\omega), \xi_\omega\rangle \\
&= \lim_\lambda \langle \pi_B(b)\pi_A(u_\lambda)(\xi_\omega), \xi_\omega\rangle \\
&= \lim_\lambda \langle \pi(u_\lambda \otimes b)(\xi_\omega), \xi_\omega\rangle \\
&= \lim_\lambda \omega(u_\lambda \otimes b).
\end{aligned}
$$

By a similar argument if $\{v_\mu\}_{\mu\in M}$ is an approximate unit for B and $a \in A$ then

$$\omega_A(a) = \lim_\mu \omega(a \otimes v_\mu).$$

Assume that $(\tau, \rho) \in S_\gamma$ and $\omega = \tau \otimes_\gamma \rho$. Then,

$$\omega_A(a) = \lim_\mu \omega(a \otimes v_\mu) = \lim_\mu \tau(a)\rho(v_\mu) = \lim_\mu \tau(a)\|\rho\| = \lim_\mu \tau(a) = \tau(a)$$

by Theorem 2.3.5. By a similar argument, $\omega_B = \rho$. □

Theorem 2.10.15. *Let A and B be C*-algebras. Suppose that either one of A or B is commutative. Let γ be a C*-norm on $A \otimes B$ and $(\tau, \rho) \in S_\gamma$. Then, $\tau \otimes_\gamma \rho$ (see Theorem 2.10.7) is a pure state of $A \otimes_\gamma B$.*

Proof. Assume that A and B are C*-algebras and that γ is a C*-norm on $A \otimes B$. Without loss of generality, we may assume that A is abelian. Assume that $(\tau, \rho) \in S_\gamma$ and define $\omega = \tau \otimes_\gamma \rho$. By the construction in Theorem 2.10.7, ω defines a state on $A \otimes_\gamma B$.

Let (π, H, ξ) denote the GNS representation of $A \otimes_\gamma B$ associated to ω. Let $\pi_A : A \to B(H)$ and $\pi_B : B \to B(H)$ denote the unique *-homomorphisms (see Theorem 2.8.9) satisfying for $a \in A$ and $b \in B$

$$\pi(a \otimes b) = \pi_A(a)\pi_B(b) = \pi_B(b)\pi_A(a).$$

Define $K = \overline{\pi_A(A)\xi}$. Then, K is a closed vector space of H, which is invariant for $\pi_A(A)$. Define the map

$$
\begin{array}{rccc}
\psi : & A & \to & B(K) \\
 & a & \mapsto & \pi_A(a)|_K.
\end{array}
$$

Since π_A is a *-homomorphism, then ψ is also a *-homomorphism. By definition of K, ξ is a unit cyclic vector for the representation (ψ, K). Now observe that

$$\langle \psi(a)(\xi), \xi \rangle = \langle \pi_A(a)(\xi), \xi \rangle = \tau(a)$$

by Theorem 2.10.14. Subsequently by Theorem 2.9.5, the representation (ψ, K) of A is unitarily equivalent to the GNS representation $(\varphi_\tau, H_\tau, \xi_\tau)$. Since τ is a pure state by assumption, the GNS representation $(\varphi_\tau, H_\tau, \xi_\tau)$ is irreducible. Therefore, (ψ, K) is an irreducible representation of A.

By Theorem 1.9.7, the commutant $\psi(A)' = \mathbb{C} id_K$. Invoking the assumption that A is abelian, we find that $\psi(A) \subseteq \psi(A)'$. Thus, if $a \in A$ then there exists a scalar $\lambda \in \mathbb{C}$ such that $\psi(a) = \lambda id_K$. In particular, we can deduce the identity of this scalar by noticing that

$$\tau(a) = \langle \psi(a)(\xi), \xi \rangle = \langle \lambda \xi, \xi \rangle = \lambda.$$

Therefore, if $a \in A$ then $\psi(a) = \tau(a) id_K$.

To show: (a) If $a \in A$ and $b \in B$ then $\pi_A(a)\pi_B(b) = \tau(a)\pi_B(b)$.

(a) Assume that $a \in A$ and $b \in B$. It suffices to prove the identity $\pi_A(a)\pi_B(b) = \tau(a)\pi_B(b)$ on the subspace

$$\pi_A(A)\pi_B(B)(\xi) = \pi(A \otimes B)(\xi)$$

because ξ is a cyclic vector for the GNS representation (π, H, ξ) of $A \otimes_\gamma B$. Assume that $a' \in A$ and $b' \in B$. Then,

$$\begin{aligned}
\pi_A(a)\pi_B(b)\big(\pi_A(a')\pi_B(b')\xi\big) &= \pi_A(a)\pi_A(a')\pi_B(b)\pi_B(b')(\xi) \\
&= \pi_A(a')\pi_A(a)\pi_B(b)\pi_B(b')(\xi) \\
&= \pi_A(a')\pi_B(b)\pi_B(b')\pi_A(a)(\xi) \\
&= \pi_A(a')\pi_B(b)\pi_B(b')\pi_A(a)|_K(\xi) \\
&= \tau(a)\pi_A(a')\pi_B(b)\pi_B(b')(\xi) \\
&= \tau(a)\pi_B(b)\big(\pi_A(a')\pi_B(b')(\xi)\big).
\end{aligned}$$

So, $\pi_A(a)\pi_B(b) = \tau(a)\pi_B(b)$.

From part (a), we obtain the equality

$$\pi(A \otimes B) = \pi_A(A)\pi_B(B) = \pi_B(B).$$

Therefore, ξ is a unit cyclic vector for the non-degenerate representation (π_B, H). By Theorem 2.10.14, if $b \in B$ then

$$\rho(b) = \langle \pi_B(b)(\xi), \xi \rangle.$$

Arguing in a similar manner to before, we deduce that the representations (π_B, ξ) and $(\varphi_\rho, H_\rho, \xi_\rho)$ are unitarily equivalent. Since ρ is a pure state on B by assumption, the GNS representation $(\varphi_\rho, H_\rho, \xi_\rho)$ is irreducible. Therefore, (π_B, ξ) is an irreducible representation of B. By Theorem 1.9.7,

$$\pi(A \otimes B)' = \pi_B(B)' = \mathbb{C}id_H$$

and consequently, (π, H, ξ) is an irreducible representation of $A \otimes_\gamma B$. By Theorem 2.9.7, the state ω on $A \otimes_\gamma B$ is pure. $\square$

Theorem 2.10.16. *Let A and B be C*-algebras. Let γ be a C*-norm on $A \otimes B$ and ω be a pure state on $A \otimes_\gamma B$ such that the state ω_A on A is pure (see Theorem 2.10.14). Then, $(\omega_A, \omega_B) \in S_\gamma$ and $\omega = \omega_A \otimes_\gamma \omega_B$.*

Theorem 2.10.16 can be thought of as a partial converse to Theorem 2.10.14.

Proof. Assume that A and B are C*-algebras. Assume that γ is a C*-norm on $A \otimes B$. Assume that ω is a pure state on $A \otimes_\gamma B$ such that the state ω_A from Theorem 2.10.14 on A is also pure.

To simplify the notation we will use, let $(\pi, H, \xi) = (\varphi_\omega, H_\omega, \xi_\omega)$ be the GNS representation of $A \otimes_\gamma B$ associated to ω. Let $\tau = \omega_A$ and $\rho = \omega_B$. Define the subspace $K = \overline{\pi_A(A)\xi}$. Let ψ be the *-homomorphism

$$\begin{array}{rccc} \psi : & A & \to & B(K) \\ & a & \mapsto & \pi_A(a)|_K. \end{array}$$

By a similar argument to that of Theorem 2.10.15, the vector ξ is a cyclic vector for the representation (ψ, K). Now if $a \in A$ then

$$\langle \psi(a)(\xi), \xi \rangle = \langle \pi_A(a)|_K(\xi), \xi \rangle = \tau(a).$$

Therefore, the representations (ψ, K) and $(\varphi_\tau, H_\tau, \xi_\tau)$ are unitarily equivalent. Since τ is pure by assumption, the GNS representation $(\varphi_\tau, H_\tau, \xi_\tau)$ is irreducible by Theorem 2.9.6. Now let $p : H \to K$ denote the projection onto the closed subspace K of H. Since K is an invariant subspace for $\pi_A(A)$, if $a \in A$ and $\mu \in H$ then

$$p\pi_A(a)(\mu) = \pi_A(a)(\mu) = \pi_A(a)p(\mu).$$

So, $p \in \pi_A(A)'$. Now let q be a projection in $p\pi_A(A)'p$. Then, the image $q(H)$ is a closed vector space of K. Furthermore, it is an invariant subspace for (ψ, K). Since (ψ, K) is an irreducible representation of A, either $q(H) = 0$ or $q(H) = K$. This means that either $q = 0$ or $q = p$.

We conclude that the von Neumann algebra $p\pi_A(A)'p$ contains only scalar projections (scalars of the projection p). Recalling the fact that a von Neumann algebra is the closed linear span of its projections, we deduce that $p\pi_A(A)'p = \mathbb{C}p$.

By Theorem 2.8.9, $\pi_B(B) \subseteq \pi_A(A)'$. If $b \in B$ then there exists a scalar $\lambda \in \mathbb{C}$ such that $p\pi_B(b)p = \lambda p$. Using Theorem 2.10.14, we find that

$$\begin{aligned}
\rho(b) &= \langle \pi_B(b)(\xi), \xi \rangle \\
&= \langle \pi_B(b)p(\xi), p(\xi) \rangle \\
&= \langle p\pi_B(b)p(\xi), \xi \rangle \\
&= \langle \lambda\xi, \xi \rangle = \lambda.
\end{aligned}$$

Therefore, $p\pi_B(b)p = \rho(b)p$. Now if $a \in A$ then

$$\begin{aligned}
\omega(a \otimes b) &= \langle \pi(a \otimes b)(\xi), \xi \rangle \\
&= \langle \pi_A(a)\pi_B(b)(\xi), \xi \rangle \\
&= \langle \pi_A(a)\pi_B(b)p(\xi), p(\xi) \rangle \\
&= \langle p\pi_A(a)\pi_B(b)p(\xi), \xi \rangle \\
&= \langle \pi_A(a)(\rho(b)\xi), \xi \rangle \\
&= \rho(b)\tau(a).
\end{aligned}$$

So, ω extends the functional $\tau \otimes \rho$ to a linear functional on $A \otimes_\gamma B$. By Theorem 2.10.7, we deduce that $\omega = \tau \otimes_\gamma \rho$.

Finally, we show that $(\tau, \rho) = (\omega_A, \omega_B) \in S_\gamma$. First, we use Theorem 2.10.8 to obtain a unitary map $u : H_\tau \hat{\otimes} H_\rho \to H$ such that if $c \in A \otimes B$ then

$$\pi(c) = u(\varphi_\tau \hat{\otimes} \varphi_\rho)(c)u^*.$$

Suppose for the sake of contradiction that ρ is not pure. Then, the associated GNS representation $(\varphi_\rho, H_\rho, \xi_\rho)$ of B is not irreducible and hence, there exists a non-trivial closed vector subspace L of H_ρ, which is invariant for $\varphi_\rho(B)$. Define $L' = H_\tau \hat{\otimes} L$. Then, L' is a non-trivial closed vector subspace of $H_\tau \hat{\otimes} H_\rho$, which is invariant for $(\varphi_\tau \hat{\otimes} \varphi_\rho)(A \otimes B)$.

Now define $L'' = u(L')$. Then, L'' is a non-trivial closed vector subspace of H, which is invariant for $\pi(A \otimes_\gamma B)$. To see why this is the case, let $c \in A \otimes B$. Then,

$$\pi(c)L'' = u(\varphi_\tau \hat{\otimes} \varphi_\rho)(c)u^* u(L') = u(\varphi_\tau \hat{\otimes} \varphi_\rho)(c)(L') \subseteq L''.$$

Hence, L'' is invariant for $\pi(A \otimes B)$ and hence, also for $\pi(A \otimes_\gamma B)$. This contradicts the assumption that ω is a pure state on $A \otimes_\gamma B$. So, ρ is pure and $(\tau, \rho) \in S_\gamma$ as required. □

Now we have reached one of the two pinnacles of this section.

Theorem 2.10.17. *Let A be an abelian C*-algebra. Then, A is a nuclear C*-algebra.*

Proof. Assume that A is an abelian C*-algebra and B is an arbitrary C*-algebra. Let γ be a C*-norm on $A \otimes B$. Let $\omega \in PS(A \otimes_\gamma B)$ and let (π, H, ξ) be the GNS representation of $A \otimes_\gamma B$ associated to ω. Let $\tau = \omega_A$ and $\rho = \omega_B$ where ω_A and ω_B are defined as in Theorem 2.10.14.

Now let π_A and π_B be the unique *-homomorphisms constructed in Theorem 2.8.9. Since ω is a pure state, the GNS representation (π, H, ξ) is irreducible. Since A is abelian, we deduce that

$$\pi_A(A) \subseteq \pi(A \otimes_\gamma B)' = \mathbb{C} id_H.$$

Therefore, if $a \in A$ then there exists $\lambda \in \mathbb{C}$ such that $\pi_A(a) = \lambda id_H$. Arguing in a similar fashion to Theorem 2.10.15, we find that if $a \in A$ then $\pi_A(a) = \tau(a) id_H$. Since π_A is a *-homomorphism, τ is a multiplicative state on A. By Theorem 2.9.6, we deduce that τ is a pure state on A.

By Theorem 2.10.16, we deduce that ρ is a pure state of B, $(\tau, \rho) \in S_\gamma$ and $\omega = \tau \otimes_\gamma \rho$. Next, Theorem 2.10.12 tells us that if $c \in A \otimes B$ then

$$\|\pi(c)\| = \|(\varphi_\tau \hat{\otimes} \varphi_\rho)(c)\|.$$

We can relate this to the C*-norm γ by observing that if $c \in A \otimes B$ then

$$\gamma(c) = \sup_{\omega \in PS(A \otimes_\gamma B)} \|\varphi_\omega(c)\|.$$

This is due to Theorem 2.9.9 and Theorem 2.9.7. By Theorem 2.10.8,

$$\gamma(c) = \sup_{\omega \in PS(A \otimes_\gamma B)} \|\varphi_\omega(c)\| = \sup_{(\tau,\rho) \in S_\gamma} \|(\varphi_\tau \hat{\otimes} \varphi_\rho)(c)\|.$$

The key idea behind this proof is that if we show that
$S_\gamma = PS(A) \times PS(B)$ then

$$\gamma(c) = \sup_{\substack{\tau \in PS(A) \\ \rho \in PS(B)}} \|(\varphi_\tau \hat{\otimes} \varphi_\rho)(c)\|$$

and since the RHS of the above equation is independent of γ, we can conclude that $A \otimes B$ has a unique C*-norm.

To show: (a) $S_\gamma = PS(A) \times PS(B)$.

(a) Suppose for the sake of contradiction that $S_\gamma \neq PS(A) \times PS(B)$. By Theorem 2.10.13, S_γ is a closed subset of $PS(A) \times PS(B)$. By the product topology on $PS(A) \times PS(B)$, there exists a pair of weak-* open sets $U \subseteq PS(A)$ and $V \subseteq PS(B)$ such that $S_\gamma \cap (U \times V) = \emptyset$.

For the next part, we want to assume that U and V are unitarily equivalent. In order to justify this, let $u \in \tilde{A}$ and $v \in \tilde{B}$ be unitary elements, where $\tilde{A}$ and $\tilde{B}$ are the unitizations of A and B respectively. Define

$$U^u = \{\tau^u \mid \tau \in U\} \qquad \text{and} \qquad V^v = \{\rho^v \mid \rho \in V\}$$

where we adopt the notation from Theorem 2.10.13. Define $U' = \bigcup_u U^u$ and $V' = \bigcup_v V^v$. Then, U' and V' are unitarily invariant sets. That is, if $\tau \in U'$ and $u' \in \tilde{A}$ is unitary then $\tau^{u'} \in U'$. Similarly, if $\rho \in V'$ and $v' \in \tilde{B}$ is unitary then $\rho^{v'} \in V'$.

We claim that U' and V' are weak-* open. To see why this is the case, it suffices to show that if $u \in \tilde{A}$ and $v \in \tilde{B}$ are unitary then U^u and V^v are weak-* open in $PS(A)$ and $PS(B)$ respectively.

To this end, assume that $u \in \tilde{A}$ is unitary. Let $\{\tau_n\}_{n \in \mathbb{Z}_{>0}}$ be a sequence in $(U^u)^c$ which weakly converges to $\tau \in PS(A)$. This means that if $a \in A$ then $\lim_{n\to\infty} \tau_n(a) = \tau(a)$. In particular,

$$\lim_{n\to\infty} (\tau_n)^{u^*}(a) = \lim_{n\to\infty} \tau_n(u^* a u) = \tau(u^* a u) = \tau^{u^*}(a).$$

Since $\tau_n = ((\tau_n)^{u^*})^u$ and $\tau_n \notin U^u$ for $n \in \mathbb{Z}_{>0}$, then $(\tau_n)^{u^*} \notin U$. By assumption, U is a weak-* open subset of $PS(A)$. Hence its complement U^c

is weak-* closed and consequently, $\tau^{u^*} \notin U$. Therefore, $\tau = ((\tau)^{u^*})^u \notin U^u$. We conclude that $(U^u)^c$ is weak-* closed and subsequently, that U^u is weak-* open in $PS(A)$. By an analogous argument, V^v is weak-* open in $PS(B)$.

Therefore, U' and V' are unitarily invariant and weak-* open subsets of $PS(A)$ and $PS(B)$ respectively. Moreover, $S_\gamma \cap (U' \times V') = \emptyset$ by the contrapositive of Theorem 2.10.13. Hence, if U and V are not initially unitarily invariant, we can always replace them by U' and V' respectively.

Returning to the problem at hand, we may assume without loss of generality that U and V are unitarily invariant. Then, the complements $S_A = PS(A)\backslash U$ and $S_B = PS(B)\backslash V$ are weak-* closed and unitarily invariant sets in $PS(A)$ and $PS(B)$ respectively. By assumption, $S_A \neq PS(A)$ and $S_B \neq PS(B)$. Hence, the orthogonal complements $S_A^\perp$ and $S_B^\perp$ are non-zero closed ideals (of A and B respectively) and hence, contain non-zero positive elements. Let $a \in S_A^\perp$ and $b \in S_B^\perp$ be such positive elements.

If $(\tau, \rho) \in S_\gamma$ then because $S_\gamma \cap (U \times V) = \emptyset$, either $\tau \notin U$ or $\rho \notin V$ because U and V are unitarily invariant. In either case,

$$(\tau \otimes_\gamma \rho)(a \otimes b) = \tau(a)\rho(b) = 0$$

because $a \in S_A^\perp$ and $b \in S_B^\perp$. Now by Theorem 2.9.9, there exists an irreducible representation (φ, H) of $A \otimes_\gamma B$ such that $\gamma(a \otimes b) = \|\varphi(a \otimes b)\|$. By the proof of Theorem 1.11.6, the representation (φ, H) is the GNS representation associated to some state ω on $A \otimes_\gamma B$. By Theorem 2.9.7, ω is a pure state on $A \otimes_\gamma B$ such that

$$\omega(a \otimes b) = \|\varphi(a \otimes b)\| = \gamma(a \otimes b).$$

By the first part of the proof of this theorem, we deduce that there exists pure states $\tau \in PS(A)$ and $\rho \in PS(B)$ such that $(\tau, \rho) \in S_\gamma$ and $\omega = \tau \otimes_\gamma \rho$. By the previous finding and our construction of a and b, we deduce that

$$\omega(a \otimes b) = (\tau \otimes_\gamma \rho)(a \otimes b) = 0 = \gamma(a \otimes b).$$

So, $a \otimes b = 0$ and either $a = 0$ or $b = 0$. In either case, this contradicts the assumption that $a \in S_A^\perp$ and $b \in S_B^\perp$ are non-zero.

Part (a) shows that $S_\gamma = PS(A) \times PS(B)$ and therefore,

$$\gamma(c) = \sup_{\substack{\tau \in PS(A) \\ \rho \in PS(B)}} \|(\varphi_\tau \hat{\otimes} \varphi_\rho)(c)\|$$

is the only C*-norm on $A \otimes B$. Therefore, A is a nuclear C*-algebra. $\square$

Before we move on, we first recall the notion of a partition of unity. Let X be a compact Hausdorff space, $n \in \mathbb{Z}_{>0}$ and $U_1, \ldots, U_n$ be open subsets of X such that $X = U_1 \cup U_2 \cup \cdots \cup U_n$. Then, there exists continuous functions $h_1, \ldots, h_n \in Cts(X, [0,1])$ such that if $i \in \{1, 2, \ldots, n\}$ then

$$supp(h_i) = \overline{\{x \in X \mid h_i(x) \neq 0\}} \subseteq U_i.$$

and $\sum_{i=1}^{n} h_i = \mathbb{1}$ where $\mathbb{1}$ is the constant function which sends $x \in X$ to 1.

Definition 2.10.2. Let Ω be a LCH space and X be a Banach space. Define $Cts_0(\Omega, X)$ to be the Banach space of all continuous functions $g : \Omega \to X$ such that the continuous map $\omega \mapsto \|g(\omega)\|$ vanishes at infinity. The operations on $Cts_0(\Omega, X)$ are defined pointwise and its norm is the supremum norm.

If $f \in Cts_0(\Omega, \mathbb{C})$ and $x \in X$ then we define $fx \in Cts_0(\Omega, X)$ by $(fx)(\omega) = f(\omega)x$.

We remark here that if Ω is a LCH space and X is a C*-algebra then $Cts_0(\Omega, X)$ is also a C*-algebra, with multiplication and involution defined pointwise.

Theorem 2.10.18. *Let Ω be a LCH space and X be a Banach space. Then, $Cts_0(\Omega, X)$ is the closed linear span of the set*

$$\{fx \mid f \in Cts_0(\Omega, \mathbb{C}),\ x \in X\}.$$

Proof. Assume that Ω is a LCH space and that X is a Banach space. Let $\tilde{\Omega}$ be the one-point compactification of Ω. The point at infinity of $\tilde{\Omega}$ is denoted by ∞.

Assume that $g \in Cts_0(\Omega, X)$. Define the function $\tilde{g} : \tilde{\Omega} \to X$ by

$$\tilde{g}(\omega) = \begin{cases} g(\omega), & \text{if } \omega \in \Omega, \\ 0, & \text{if } \omega = \infty. \end{cases}$$

Since g is continuous and vanishes at infinity, the function $\tilde{g}$ is continuous. Now assume that $\epsilon \in \mathbb{R}_{>0}$. Since $\tilde{\Omega}$ is a compact Hausdorff space, the image

$\tilde{g}(\tilde{\Omega})$ is compact and hence, totally bounded. So, it is precompact and thus, there exists $x_1, \ldots, x_n \in \tilde{g}(\tilde{\Omega})$ such that if $j \in \{1, 2, \ldots, n\}$ and

$$U_j = \{\omega \in \tilde{\Omega} \mid \|\tilde{g}(\omega) - x_j\| < \epsilon\}$$

then $\tilde{\Omega} = U_1 \cup \cdots \cup U_n$. We also note that $U_1, \ldots, U_n$ are all open subsets of $\tilde{\Omega}$. This means that we can construct a partition of unity. So, there exists $h_1, \ldots, h_n \in Cts(\tilde{\Omega}, [0, 1])$ such that if $j \in \{1, 2, \ldots, n\}$ then the support $supp(h_j) \subseteq U_j$ and $\sum_{i=1}^{n} h_i = \mathbb{1}$.

If $\omega \in \tilde{\Omega}$ then there exists distinct $j_1, \ldots, j_k \in \{1, 2, \ldots, n\}$ such that if $i \in \{1, 2, \ldots, k\}$ then $\omega \in U_{j_i}$. So,

$$\begin{aligned}
\|\tilde{g}(\omega) - \sum_{i=1}^{n} h_i(\omega)x_i\| &= \|\tilde{g}(\omega)\sum_{i=1}^{n} h_i(\omega) - \sum_{i=1}^{n} h_i(\omega)x_i\| \\
&= \|\sum_{i=1}^{n} h_i(\omega)(\tilde{g}(\omega) - x_i)\| \\
&\leq \sum_{i=1}^{n} h_i(\omega)\|\tilde{g}(\omega) - x_i\| \\
&= \sum_{i\in\{j_1,\ldots,j_k\}} h_i(\omega)\|\tilde{g}(\omega) - x_i\| + \sum_{i\notin\{j_1,\ldots,j_k\}} h_i(\omega)\|\tilde{g}(\omega) - x_i\| \\
&< \sum_{i\in\{j_1,\ldots,j_k\}} h_i(\omega)\epsilon + \sum_{i\notin\{j_1,\ldots,j_k\}} h_i(\omega)\|\tilde{g}(\omega) - x_i\| \\
&= \sum_{i\in\{j_1,\ldots,j_k\}} h_i(\omega)\epsilon \leq \sum_{i=1}^{n} h_i(\omega)\epsilon = \epsilon.
\end{aligned}$$

The second last inequality follows from the fact that if $i \notin \{j_1, \ldots, j_k\}$ then $\omega \notin U_i$ and $h_i(\omega) = 0$ because ω lies outside the support of h_i.

In particular, if $\omega = \infty$ then

$$\|\tilde{g}(\infty) - \sum_{i=1}^{n} h_i(\omega)x_i\| = \|\sum_{i=1}^{n} h_i(\omega)x_i\| = 0.$$

Hence, if $\epsilon \in \mathbb{R}_{>0}$ then $\|\sum_{i=1}^{n} h_i(\omega)x_i\| \leq \epsilon$. If $i \in \{1, 2, \ldots, n\}$ then define $f_i = h_i|_{\Omega}$. Then, $f_i \in Cts_0(\Omega, \mathbb{C})$ and if $\omega \in \Omega$ then

$$\begin{aligned}\|g(\omega) - \sum_{i=1}^{n} f_i(\omega)x_i\| &= \|\tilde{g}(\omega) - \sum_{i=1}^{n} h_i(\omega)x_i\| + \|\tilde{g}(\infty) - \sum_{i=1}^{n} h_i(\infty)x_i\| \\ &\leq \epsilon + \epsilon = 2\epsilon.\end{aligned}$$

By taking the supremum over all $\omega \in \Omega$, we find that $g = \sum_{i=1}^{n} f_i x_i$. $\square$

We need one more result before we can prove that the spatial C*-norm on a tensor product of C*-algebras is the smallest one. This requires the following construction. Let Ω be a locally compact Hausdorff space and A be a C*-algebra. Define the map

$$\begin{array}{rccc} B: & Cts_0(\Omega, \mathbb{C}) \times A & \to & Cts_0(\Omega, A) \\ & (f, a) & \mapsto & fa. \end{array}$$

Then, B is a $\mathbb{C}$-bilinear map. By the universal property of the tensor product in Theorem 2.8.1, there exists a unique linear map $\pi : Cts_0(\Omega, \mathbb{C}) \otimes A \to Cts_0(\Omega, A)$ such that if $f \in Cts_0(\Omega, \mathbb{C})$ and $a \in A$ then

$$\pi(f \otimes a) = fa. \tag{2.3}$$

The map in equation (2.3) is called the **canonical map** from $Cts_0(\Omega, \mathbb{C}) \otimes A$ to $Cts_0(\Omega, A)$.

Theorem 2.10.19. *Let Ω be a LCH space and A be a C*-algebra. Let π denote the canonical map in equation* (2.3)*. Then, π extends uniquely to a *-isomorphism from $Cts_0(\Omega, \mathbb{C}) \otimes_* A$ to $Cts_0(\Omega, X)$, where $\otimes_*$ denotes the spatial tensor product.*

Proof. Assume that Ω is a locally compact Hausdorff space and that A is a C*-algebra. Assume that π is the canonical map defined in equation (2.3). To see that π is a *-homomorphism in this case, assume that $f, g \in Cts_0(\Omega, \mathbb{C})$, $a, b \in A$ and $\omega \in \Omega$. Then,

$$\begin{aligned}\pi((f \otimes a) \cdot (g \otimes b))(\omega) &= \pi(fg \otimes ab)(\omega) \\ &= (fg)ab(\omega) \\ &= f(\omega)g(\omega)ab = f(\omega)ag(\omega)b \\ &= (fa)(\omega)(gb)(\omega) \\ &= \pi(f \otimes a)(\omega) \cdot \pi(g \otimes b)(\omega)\end{aligned}$$

and

$$\begin{aligned}
\pi((f \otimes a)^*)(\omega) &= \pi(\overline{f} \otimes a^*)(\omega) \\
&= (\overline{f} a^*)(\omega) \\
&= \overline{f}(\omega) a^* = \overline{f(\omega)} a^* \\
&= (f(\omega)a)^* = ((fa)(\omega))^* \\
&= (\pi(f \otimes a)(\omega))^*.
\end{aligned}$$

So, the canonical map π is a *-homomorphism. Next, we claim that π is injective.

To show: (a) π is an injective *-homomorphism.

(a) Assume that $c \in \ker \pi$. Then, we can write $c = \sum_{i=1}^{n} (f_i \otimes a_i)$, where $f_i \in Cts_0(\Omega, \mathbb{C})$ and the $a_i \in A$ are linearly independent. We compute directly that

$$\pi(c) = \pi(\sum_{i=1}^{n} (f_i \otimes a_i)) = \sum_{i=1}^{n} (f_i a_i) = 0.$$

If $\omega \in \Omega$ then $\sum_{i=1}^{n} f_i(\omega) a_i = 0$. Since the set $\{a_1, \ldots, a_n\}$ is linearly independent, we deduce that $f_1 = \cdots = f_n = 0$. Therefore, $c = 0$ and π is injective.

Now define the map $\|-\|$ by

$$\begin{array}{cccc}
\pi : & Cts_0(\Omega, \mathbb{C}) \otimes A & \to & \mathbb{R}_{\geq 0} \\
& c & \mapsto & \|\pi(c)\|.
\end{array}$$

Since π is an injective *-homomorphism, the map $\|-\|$ defines a C*-norm on $Cts_0(\Omega, \mathbb{C}) \otimes A$. However, $Cts_0(\Omega, \mathbb{C})$ is an abelian C*-algebra and is thus, nuclear by Theorem 2.10.17. Hence, if $c \in Cts_0(\Omega, \mathbb{C}) \otimes A$ then

$$\|c\| = \|\pi(c)\| = \|c\|_*.$$

This means that we can extend π uniquely to an isometric *-homomorphism π' on the spatial tensor product $Cts_0(\Omega, \mathbb{C}) \otimes_* A$. Finally, to see that π' is surjective, we observe that

$$\{fa \mid f \in Cts_0(\Omega, \mathbb{C}), a \in A\} \subseteq \text{im } \pi'$$

By Theorem 2.10.18, we deduce that π' is surjective and consequently, a *-isomorphism from $Cts_0(\Omega, \mathbb{C}) \otimes_* A$ to $Cts_0(\Omega, X)$. □

Now let A and B be *-algebras and $\theta : A \otimes B \to B \otimes A$ be the unique linear map defined by $\theta(a \otimes b) = b \otimes a$. Then, θ is a *-isomorphism. The key observation we make here is that if $A \otimes B$ admits a unique C*-norm then $B \otimes A$ also admits a unique C*-norm. This observation is used in the proof of the second main theorem of the section.

Theorem 2.10.20. *Let A and B be C*-algebras. Then, the spatial C*-norm $\|-\|_*$ is the smallest C*-norm on the tensor product $A \otimes B$.*

Proof. Assume that A and B are C*-algebras. Let γ be a C*-norm on $A \otimes B$. If B is non-unital then by Theorem 2.10.11, we can extend γ to a C*-norm on $A \otimes \tilde{B}$. By Theorem 2.10.10, the spatial C*-norm on $A \otimes \tilde{B}$ restricts to the spatial C*-norm on $A \otimes B$. Therefore, it suffices to prove the theorem in the case where B is unital. Hence, assume that B is a unital C*-algebra. Recall the definition of S_γ from equation (2.2).

To show: (a) $S_\gamma = PS(A) \times PS(B)$.

(a) Suppose for the sake of contradiction that $S_\gamma \neq PS(A) \times PS(B)$. By the proof of Theorem 2.10.17, there exists weak-* closed unitarily invariant subsets $S_A \subsetneq PS(A)$ and $S_B \subsetneq PS(B)$ such that

$$S_\gamma \subseteq (S_A \times PS(B)) \cup (PS(A) \times S_B).$$

Furthermore, $S_A^\perp$ and $S_B^\perp$ contain non-zero positive elements a_0 and b_0 respectively. Now if $(\tau, \rho) \in S_\gamma$ then because

$$S_\gamma \cap \big((PS(A) \backslash S_A) \times (PS(B) \backslash S_B)\big) = \emptyset$$

either $\tau \in S_A$ or $\rho \in S_B$. In either case,

$$(\tau \otimes_\gamma \rho)(a_0 \otimes b_0) = \tau(a_0)\rho(b_0) = 0.$$

Now let C be the C*-subalgebra generated by the set $\{1_B, b_0\}$. Then, C is abelian and hence nuclear by Theorem 2.10.17. Since $A \otimes C$ has a unique C*-norm, $\gamma = \|-\|_*$ on $A \otimes C$. Therefore, the spatial tensor product $A \otimes_* C$ can be considered a C*-subalgebra of $A \otimes_\gamma B$.

Now let $\tau \in PS(A)$ and $\rho \in PS(C)$ such that $\tau(a_0) = \|a_0\| > 0$ and $\rho(b_0) = \|b_0\| > 0$ by Theorem 1.11.6. Now by Theorem 2.10.4 and Theorem 2.10.15, $\tau \otimes \rho$ is continuous with respect to the spatial C*-norm and consequently, extends to a pure state ω' on $A \otimes_* C$. Now by [Mur90, Theorem 5.1.13], ω' extends to a pure state ω on $A \otimes_\gamma B$.

Now let ω_A and ω_B be the states on A and B respectively, defined in Theorem 2.10.14. If $a \in A$ and $\{v_\mu\}_{\mu \in M}$ is an approximate unit then

$$\omega_A(a) = \lim_\mu \omega(a \otimes v_\mu) = \omega(a \otimes 1_B) = (\tau \otimes \rho)(a \otimes 1_B) = \tau(a)\rho(1_B) = \tau(a).$$

Consequently, $\tau = \omega_A$ is a pure state on A. By Theorem 2.10.16, $(\omega_A, \omega_B) \in S_\gamma$ and $\omega = \omega_A \otimes_\gamma \omega_B$. Therefore,

$$\omega(a_0 \otimes b_0) = (\tau \otimes \rho)(a_0 \otimes b_0) = \tau(a_0)\rho(b_0) = \|a_0\|\|b_0\| > 0.$$

However, since $(\omega_A, \omega_B) \in S_\gamma$, then

$$\omega(a_0 \otimes b_0) = \omega_A(a_0)\omega_B(b_0) = 0.$$

This contradicts the fact that $\omega(a_0 \otimes b_0) > 0$. Therefore, $S_\gamma = PS(A) \times PS(B)$.

Next, observe that by Theorem 2.9.8 and the Krein-Milman theorem, the states of a C*-algebra are weak-* limits of nets of convex combinations of the extreme points — the zero functional and the pure states. Now let τ and ρ be positive linear functionals on A and B respectively, which are convex combinations of the zero functional and pure states. Then, there exists scalars

$$t_1, \dots, t_n, s_1, \dots, s_m \in \mathbb{R}_{\geq 0}$$

and functionals

$$\tau_1, \dots, \tau_n \in \{0\} \cup PS(A) \qquad \text{and} \qquad \rho_1, \dots, \rho_m \in \{0\} \cup PS(B)$$

such that

$$\tau = \sum_{i=1}^{n} t_i \tau_i \qquad \text{and} \qquad \rho = \sum_{j=1}^{m} s_j \rho_j.$$

Now we have

$$\tau \otimes \rho = \sum_{i=1}^{n} \sum_{j=1}^{m} t_i s_j (\tau_i \otimes \rho_j).$$

By part (a), $S_\gamma = PS(A) \times PS(B)$. So, if $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, m\}$ then $\tau_i \otimes \rho_j$ is continuous with respect to γ. Therefore, $\tau \otimes \rho$ is also continuous with respect to γ.

Now suppose that τ and ρ are arbitrary states of A and B respectively. Then, there exists nets $\{\tau_\lambda\}_{\lambda \in L}$ and $\{\rho_\mu\}_{\mu \in M}$ of positive linear functionals on A and B respectively such that $\{\tau_\lambda\}$ weakly converges to τ and $\{\rho_\mu\}$ weakly converges to ρ. Moreover, if $\lambda \in L$ and $\mu \in M$ then $\|\tau_\lambda\|, \|\rho_\mu\| \leq 1$ and $\tau_\lambda \otimes \rho_\mu$ is continuous with respect to γ.

By Theorem 2.10.7, $\tau_\lambda \otimes \rho_\mu$ uniquely extends to a state on $A \otimes_\gamma B$, which is a positive linear functional of norm $\|\tau_\lambda\|\|\rho_\mu\|$. So, if $c \in A \otimes B$ then

$$|(\tau_\lambda \otimes \rho_\mu)(c)| \leq \|\tau_\lambda\|\|\rho_\mu\|\gamma(c) \leq \gamma(c).$$

Now if $c \in A \otimes B$ then $(\tau \otimes \rho)(c) = \lim_{\lambda,\mu}(\tau_\lambda \otimes \rho_\mu)(c)$. So, $|(\tau \otimes \rho)(c)| \leq \gamma(c)$ and hence, $\tau \otimes \rho$ is continuous with respect to γ.

Now let D be the unitization of $A \otimes_\gamma B$, $\tau \in S(A)$ and $\rho \in S(B)$. Let ω be the unique state on D extending $\tau \otimes_\gamma \rho$. This uses the previous observation and the universal property of the unitization. Now if $d \in D$ then the linear functional

$$\begin{array}{rccc} \omega^d : & D & \to & \mathbb{C} \\ & c & \mapsto & \omega(d^* c d) \end{array}$$

is positive. If $c \in D$ and 1_D is the multiplicative unit of D then $\gamma(c^*c)1_D - c^*c \geq 0$ by Theorem 2.2.5. Therefore,

$$\omega^d(\gamma(c^*c)1_D - c^*c) = \gamma(c^*c)\omega^d(1_D) - \omega^d(c^*c) \geq 0.$$

Now if $\omega(d^*d) = \omega^d(1_D) > 0$ then

$$\gamma(c)^2 = \gamma(c^*c) = \frac{\omega^d(c^*c)}{\omega^d(1_D)} = \frac{\omega(d^*c^*cd)}{\omega(d^*d)} \geq 0.$$

By Theorem 2.10.9, we deduce that if $c \in A \otimes B$ then

$$
\begin{aligned}
\|c\|_*^2 &= \sup_{\substack{\tau\in S(A)\\ \rho\in S(B)}} \sup_{\substack{d\in A\otimes B\\ (\tau\otimes\rho)(d^*d)>0}} \frac{(\tau\otimes\rho)(d^*c^*cd)}{(\tau\otimes\rho)(d^*d)} \\
&\leq \sup_{\substack{\tau\in S(A)\\ \rho\in S(B)}} \sup_{\substack{d\in D\\ \omega(d^*d)>0}} \frac{\omega(d^*c^*cd)}{\omega(d^*d)} \\
&= \sup_{\substack{\tau\in S(A)\\ \rho\in S(B)}} \sup_{\substack{d\in D\\ \omega(d^*d)>0}} \gamma(c)^2 = \gamma(c)^2.
\end{aligned}
$$

Therefore, the spatial C*-norm $\|-\|_*$ is the smallest norm on $A \otimes B$ as required. □

We finish this long section with two corollaries of Theorem 2.10.20.

Theorem 2.10.21. *Let A and B be C*-algebras. Let γ be a C*-norm on $A \otimes B$. If $a \in A$ and $b \in B$ then $\gamma(a \otimes b) = \|a\|\|b\|$.*

Proof. Assume that A and B are C*-algebras. Assume that γ is a C*-norm on $A \otimes B$. If $a \in A$ and $b \in B$ then

$$\|a\|\|b\| = \|a \otimes b\|_* \leq \gamma(a \otimes b) \leq \|a\|\|b\|.$$

Recall that the first equality follows from the original definition of the spatial C*-norm. The final inequality follows from Theorem 2.8.10. □

Theorem 2.10.22. *Let A and B be C*-algebras. Let (φ, H) and (ψ, K) be faithful representations of A and B respectively. If $c \in A \otimes B$ then*

$$\|(\varphi\hat{\otimes}\psi)(c)\| = \|c\|_*.$$

Proof. Assume that A and B are. C*-algebras. Assume that (φ, H) and (ψ, K) are faithful representations of A and B respectively. Define

$$
\begin{array}{rccc}
\gamma: & A\otimes B & \to & \mathbb{R}_{\geq 0} \\
 & c & \mapsto & \|(\varphi\hat{\otimes}\psi)(c)\|.
\end{array}
$$

By Theorem 2.8.7, $\varphi\hat{\otimes}\psi$ is an injective *-homomorphism. By Theorem 1.6.4, γ is a C*-norm on $A \otimes B$. By Theorem 2.10.20, if $c \in A \otimes B$ then $\gamma(c) \geq \|c\|_*$.

But by Theorem 2.10.3,

$$\gamma(c) = \|(\varphi \hat{\otimes} \psi)(c)\| \leq \sup_{\substack{\tau \in S(A) \\ \rho \in S(B)}} \|(\varphi_\tau \hat{\otimes} \varphi_\rho)(c)\| = \|c\|_*.$$

Hence, $\gamma = \|-\|_*$. $\square$

Chapter 3

Topics from [BO08]

3.1 Completely positive maps and Stinespring's theorem

As explained in [BO08, Section 1.5], completely positive maps form the foundations of C*-approximation theory. In fact, nuclear C*-algebras are defined using completely positive maps in [BO08, Definition 2.3.1]. This section is dedicated to outlining the basic theory behind completely positive maps, following the treatment in [BO08, Section 1.5].

In order to define completely positive maps, we must first define the objects they map from.

Definition 3.1.1. Let A be a unital C*-algebra. An **operator system** E is a closed self-adjoint subspace of A such that $1_A \in E$.

By definition, a unital C*-algebra is an operator system.

Theorem 3.1.1. *Let E be an operator system. Then, E is spanned by its positive elements. That is, E is spanned by the set*

$$\{e \in E \mid e \text{ is positive}\}.$$

Proof. Assume that E is an operator system. Then, E is a closed self-adjoint subspace of a unital C*-algebra A such that $1_A \in E$. Assume that $e \in E$. Then, e can be written as

$$e = \frac{1}{2}(e + e^*) + i\big(\frac{1}{2i}(e - e^*)\big).$$

In particular, e is a linear combination of two self-adjoint elements of E. Hence, it suffices to prove that a self-adjoint element of E can be written as a linear combination of positive elements in E.

So, assume that $s \in E$ is self-adjoint. By Theorem 1.2.6, the spectral radius $r(s) = \|s\|$. If $t > \|s\|$ then $s + t1_A$ and $-s + t1_A$ are positive elements of E and

$$s = \frac{1}{2}(s + t1_A) - \frac{1}{2}(-s + t1_A).$$

Thus, any self-adjoint element in E is a linear combination of two positive elements in E. We conclude that any element in E is a linear combination of four positive elements in E. So, E is spanned by its positive elements. □

Before we define completely positive maps, we must first explain the structure of the matrix algebra $M_{n\times n}(A)$ as a C*-algebra, where $n \in \mathbb{Z}_{>0}$ and A is a C*-algebra. We already made use of this fact when we proved that the matrix algebra $M_{n\times n}(\mathbb{C})$ is a nuclear C*-algebra.

If $n \in \mathbb{Z}_{>0}$ and A is a *-algebra then $M_{n\times n}(A)$ is a *-algebra where scalar multiplication, addition and multiplication is defined analogously to $M_{n\times n}(\mathbb{C})$. Involution in $M_{n\times n}(A)$ is given by $(a_{ij})^* = (a^*_{ji})$.

Definition 3.1.2. Let A and B be *-algebras and $\varphi : A \to B$ be a *-homomorphism. The **inflation** of φ, which is also denoted by φ, is the *-homomorphism

$$\begin{array}{rccc} \varphi : & M_{n\times n}(A) & \to & M_{n\times n}(B) \\ & (a_{ij}) & \mapsto & (\varphi(a_{ij})) \end{array}$$

Now if H is a Hilbert space and $n \in \mathbb{Z}_{>0}$ then we define the Hilbert space

$$H^{(n)} = \bigoplus_{i=1}^{n} H.$$

If $u \in M_{n\times n}(B(H))$ then define the map $\psi : M_{n\times n}(B(H)) \to B(H^{(n)})$, where $\psi(u)$ is given explicitly by

$$\begin{array}{rccc} \psi(u) : & H^{(n)} & \to & H^{(n)} \\ & (x_1, \dots, x_n) & \mapsto & \left(\sum_{j=1}^{n} u_{1j}(x_j), \dots, \sum_{j=1}^{n} u_{nj}(x_j)\right) \end{array}$$

It is straightforward to verify that ψ is a *-isomorphism. We call ψ the **canonical *-isomorphism** of $M_{n\times n}(B(H))$ onto $B(H^{(n)})$.

Definition 3.1.3. Let H be a Hilbert space and $n \in \mathbb{Z}_{>0}$. Let $v \in B(H^{(n)})$ and $\psi : M_{n\times n}(B(H)) \to B(H^{(n)})$ be the canonical *-isomorphism. Let $u \in M_{n\times n}(B(H))$ be such that $\psi(u) = v$. Then, u is called the **operator matrix** of v.

To define a norm on $M_{n\times n}(B(H))$ which makes it a C*-algebra, we set

$$\begin{array}{rccc} \|{-}\| : & M_{n\times n}(B(H)) & \to & \mathbb{R}_{\geq 0} \\ & u & \mapsto & \|\psi(u)\|. \end{array}$$

Theorem 3.1.2. *Let H be a Hilbert space and $n \in \mathbb{Z}_{>0}$. If $i, j \in \{1, 2, \ldots, n\}$ and $u \in M_{n\times n}(B(H))$ then*

$$\|u_{ij}\| \leq \|u\| \leq \sum_{k,l=1}^{n} \|u_{kl}\|.$$

Proof. Assume that H is a Hilbert space and $n \in \mathbb{Z}_{>0}$. Let ψ denote the canonical *-isomorphism from $M_{n\times n}(B(H))$ to $B(H^{(n)})$. If $i, j \in \{1, 2, \ldots, n\}$ and $u = (u_{ij}) \in M_{n\times n}(B(H))$ then

$$\begin{aligned} \|u\| &= \|\psi(u)\| \\ &= \sup_{\|(x_1,\ldots,x_n)\|=1} \|\psi(u)(x_1, \ldots, x_n)\| \\ &= \sup_{\|(x_1,\ldots,x_n)\|=1} \|\big(\sum_{j=1}^{n} u_{1j}(x_j), \ldots, \sum_{j=1}^{n} u_{nj}(x_j)\big)\| \\ &= \sup_{\|(x_1,\ldots,x_n)\|=1} \max_{k\in\{1,2,\ldots,n\}} \|\sum_{j=1}^{n} u_{kj}(x_j)\| \\ &\geq \sup_{\|(x_1,\ldots,x_n)\|=1} \|\sum_{j=1}^{n} u_{ij}(x_j)\| \geq \sup_{\|x_j\|=1} \|u_{ij}(x_j)\| = \|u_{ij}\|. \end{aligned}$$

We also compute directly that

$$
\begin{aligned}
\|u\| = \|\psi(u)\| &= \sup_{\|(x_1,\dots,x_n)\|=1} \|\psi(u)(x_1,\dots,x_n)\| \\
&= \sup_{\|(x_1,\dots,x_n)\|=1} \|\big(\sum_{j=1}^n u_{1j}(x_j),\dots,\sum_{j=1}^n u_{nj}(x_j)\big)\| \\
&= \sup_{\|(x_1,\dots,x_n)\|=1} \max_{k\in\{1,2,\dots,n\}} \|\sum_{j=1}^n u_{kj}(x_j)\| \\
&\le \sup_{\|(x_1,\dots,x_n)\|=1} \max_{k\in\{1,2,\dots,n\}} \sum_{l=1}^n \|u_{kl}(x_l)\| \\
&\le \sup_{\|(x_1,\dots,x_n)\|=1} \sum_{k,l=1}^n \|u_{kl}(x_l)\| = \sum_{k,l=1}^n \|u_{kl}\|.
\end{aligned}
$$

□

Theorem 3.1.3. *Let A be a C*-algebra and $n \in \mathbb{Z}_{>0}$. Then, there exists a unique norm on $M_{n\times n}(A)$ such that $M_{n\times n}(A)$ is a C*-algebra.*

Proof. Assume that A is a C*-algebra and $n \in \mathbb{Z}_{>0}$. Let (φ, H) denote the universal representation of A, which is faithful. Since $\varphi : A \to B(H)$ is injective, its inflation $\varphi : M_{n\times n}(A) \to M_{n\times n}(B(H))$ is also injective. Now define the map

$$
\begin{array}{rccc}
\|-\|: & M_{n\times n}(A) & \to & \mathbb{R}_{\ge 0} \\
 & a & \mapsto & \|\varphi(a)\|.
\end{array}
$$

It is straightforward to verify that $\|-\|$ is a norm on $M_{n\times n}(A)$. If $a \in A$ then

$$\|a\|^2 = \|\varphi(a)\|^2 = \|\varphi(a)^*\varphi(a)\| = \|\varphi(a^*a)\| = \|a^*a\|.$$

To see that $M_{n\times n}(A)$ is complete with respect to its norm, let $\{a_m\}_{m\in\mathbb{Z}_{>0}}$ be a Cauchy sequence in $M_{n\times n}(A)$. If $\epsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $k, l > N$ then

$$\|a_k - a_l\| = \|\varphi(a_k) - \varphi(a_l)\| < \epsilon.$$

By Theorem 3.1.2, if $i, j \in \{1, 2, \dots, n\}$ then

$$\|\varphi((a_k)_{ij}) - \varphi((a_l)_{ij})\| < \epsilon.$$

Therefore, the sequence $\{\varphi((a_k)_{ij})\}_{k\in\mathbb{Z}_{>0}}$ is a Cauchy sequence in $B(H)$ and hence converges to $b_{ij} \in B(H)$. By Theorem 1.7.6, there exists $a_{ij} \in A$ such that $\varphi(a_{ij}) = b_{ij}$. Now let $\Lambda = (a_{ij}) \in M_{n\times n}(A)$. We claim that the sequence $\{a_m\}_{m\in\mathbb{Z}_{>0}}$ converges to Λ. We compute directly that if $\epsilon \in \mathbb{R}_{>0}$ then

$$\begin{aligned}
\|a_k - \Lambda\| &= \|\varphi(a_k) - \varphi(\Lambda)\| \\
&= \|\varphi(a_k - \Lambda)\| \\
&\leq \sum_{l,m=1}^{n} \|\big(\varphi(a_k - \Lambda)\big)_{lm}\| \\
&= \sum_{l,m=1}^{n} \|\varphi((a_k)_{lm}) - \varphi(a_{lm})\| \\
&< \sum_{l,m=1}^{n} \frac{\epsilon}{n^2} = \epsilon.
\end{aligned}$$

We conclude that $M_{n\times n}(A)$ is complete with respect to the norm $\|-\|$ defined as above. Therefore, $\|-\|$ makes $M_{n\times n}(A)$ into a C*-algebra. By Theorem 1.2.8, it is the unique norm which does this. $\square$

Since injective *-homomorphisms are isometric, if A is a C*-algebra, $n \in \mathbb{Z}_{>0}$, $a = (a_{ij}) \in M_{n\times n}(A)$ and $i, j \in \{1, 2, \ldots, n\}$ then

$$\|a_{ij}\| \leq \|a\| \leq \sum_{k,l=1}^{n} \|a_{kl}\|.$$

This follows directly from Theorem 3.1.2.

Definition 3.1.4. Let A and B be C*-algebras. A linear map $\varphi : A \to B$ is **positive** if for a positive element $a \in A$, $\varphi(a)$ is a positive element of B.

Let E be an operator system and B be a C*-algebra. We say that the linear map $\varphi : E \to B$ is **completely positive** if for $n \in \mathbb{Z}_{>0}$ the map

$$\begin{array}{rccc}
\varphi_n : & M_{n\times n}(E) & \to & M_{n\times n}(B) \\
& (a_{ij}) & \mapsto & (\varphi(a_{ij}))
\end{array}$$

is positive. The set of completely positive maps from E to B is denoted by $CP(E, B)$.

If A is a unital C*-algebra and $E \subseteq A$ is an operator system then $M_{n\times n}(E)$ inherits positivity from $M_{n\times n}(A)$. We say that $\Lambda \in M_{n\times n}(E)$ is positive if and only if Λ is positive in $M_{n\times n}(A)$.

The term “completely positive” is commonly abbreviated as c.p, “unital completely positive” is abbreviated as u.c.p and “contractive completely positive” is abbreviated as c.c.p.

Example 3.1.1. Let A and B be C*-algebras and $\pi : A \to B$ be a *-homomorphism. Then, π is completely positive because if $n \in \mathbb{Z}_{>0}$ then its inflations $\pi : M_{n\times n}(A) \to M_{n\times n}(B)$ are *-homomorphisms themselves and hence, preserve positivity.

More generally, let H be a Hilbert space and $\varphi : A \to B(H)$ be a map of the form $\varphi(a) = V^*\pi(a)V$, where $\pi : A \to B(H')$ is a *-homomorphism and $V : H \to H'$ is an operator. We claim that φ is a completely positive map. Assume that $n \in \mathbb{Z}_{>0}$ and $\Lambda = (\lambda_{ij}) \in M_{n\times n}(A)$ is positive. Then, there exists $\Gamma = (\gamma_{ij}) \in M_{n\times n}(A)$ such that $\Lambda = \Gamma^*\Gamma$. If $i, j \in \{1, 2, \dots, n\}$ then

$$\big(\varphi_n(\Lambda)\big)_{ij} = V^*\pi(\lambda_{ij})V = V^*\pi(\sum_{k=1}^{n} \gamma_{ki}^*\gamma_{kj})V = \sum_{k=1}^{n} V^*\pi(\gamma_{ki}^*\gamma_{kj})V.$$

Then, $\varphi_n(\Lambda) = \Delta^*\Delta$ where $\Delta_{kj} = \pi(\gamma_{kj})V \in M_{n\times n}(B(H, H'))$. Making use of the isomorphism $\psi : M_{n\times n}(B(H)) \to B(H^{(n)})$, assume that $x = (x_1, \dots, x_n) \in H^{(n)}$. Then,

$$\langle(\psi \circ \varphi_n)(\Lambda)x, x\rangle = \langle\psi(\Delta^*\Delta)x, x\rangle = \|\psi(\Delta)x\|^2 \geq 0.$$

This means that $(\psi \circ \varphi_n)(\Lambda)$ is a positive element of $B(H^{(n)})$. By composing with the *-homomorphism ψ^{-1}, we deduce that $\varphi_n(\Lambda)$ is positive. Since $n \in \mathbb{Z}_{>0}$ was arbitrary, we deduce that φ is completely positive.

Example 3.1.2. Let A be a unital C*-algebra and $E \subseteq A$ be an operator system. Let $f : E \to \mathbb{C}$ be a positive linear functional. We claim that f is completely positive. Assume that $n \in \mathbb{Z}_{>0}$. The idea is to take advantage of the isomorphism $M_{n\times n}(\mathbb{C}) \cong B(\mathbb{C}^n)$ and the fact that $\mathbb{C}^n$ is a Hilbert space.

Assume that $A = (a_{ij}) \in M_{n\times n}(E)$ is positive. Assume that $x = (x_1, \dots, x_n) \in \mathbb{C}^n$. Then,

$$
\begin{aligned}
\langle f_n(A)x, x\rangle &= \left\langle \begin{pmatrix} f(a_{11}) & \dots & f(a_{1n}) \\ \vdots & \ddots & \vdots \\ f(a_{n1}) & \dots & f(a_{nn}) \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right\rangle \\
&= \left\langle \begin{pmatrix} \sum_{i=1}^n f(a_{1i})x_i \\ \vdots \\ \sum_{i=1}^n f(a_{ni})x_i \end{pmatrix}, \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right\rangle \\
&= \sum_{i,j=1}^n f(a_{ji})x_i\overline{x_j} = f(\sum_{i,j=1}^n a_{ji}x_i\overline{x_j}) = f(x^*Ax).
\end{aligned}
$$

Since A is positive, there exists $B = (b_{ij}) \in M_{n\times n}(E)$ such that $A = B^*B$. Then, we have

$$\langle f_n(A)x, x\rangle = f(x^*B^*Bx) = f((Bx)^*Bx) = f(\|(Bx)^*\|^2) \geq 0.$$

Hence, the map $f_n : M_{n\times n}(E) \to M_{n\times n}(\mathbb{C})$ is positive for arbitrary $n \in \mathbb{Z}_{>0}$. So, f is a c.p map.

Here is an example of a map which is not c.p.

Example 3.1.3. Let $n \in \mathbb{Z}_{>0}$ and $\varphi : M_{n\times n}(\mathbb{C}) \to M_{n\times n}(\mathbb{C})$ be the adjoint map $A \mapsto A^*$. We claim that φ is not completely positive. First, we observe that φ is positive. Assume that $A = B^*B \in M_{n\times n}(\mathbb{C})$ is a positive element. Then,

$$\varphi(B^*B) = (B^*B)^* = B^*B = A.$$

Hence, φ is positive. To see that φ is not completely positive, we will give a counterexample.

Let $n = 2$. Then, $M_{2\times 2}(M_{2\times 2}(\mathbb{C})) \cong M_{4\times 4}(\mathbb{C})$ as C*-algebras. Let

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \in M_{4\times 4}(\mathbb{C}).$$

If $x = (x_1, x_2, x_3, x_4) \in \mathbb{C}^4$ then

$$\langle Ax, x\rangle = (x_1 + x_4)(\overline{x_1} + \overline{x_4}) = |x_1 + x_4|^2 \geq 0.$$

So, A is a positive element of $M_{4\times 4}(\mathbb{C})$. However,

$$\varphi_2(A) = \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^* & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^* \\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}^* & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}^* \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

If $x = (x_1, x_2, x_3, x_4) \in \mathbb{C}^4$ then

$$\langle \varphi_2(A)x, x \rangle = |x_1|^2 + x_2\overline{x_3} + x_3\overline{x_2} + |x_4|^2.$$

If $x = (0, 1, -1, 0)$ then $\langle \varphi_2(A), x, x \rangle = -2$. Therefore, $\varphi_2(A)$ is not a positive element of $M_{4\times 4}(\mathbb{C})$. So, φ is not a completely positive map.

Stinespring's dilation theorem characterises completely positive maps to $B(H)$, where H is a Hilbert space.

Theorem 3.1.4. *Let A be a unital C^*-algebra, H be a Hilbert space and $\varphi : A \to B(H)$ be a c.p map. Then, there exists a representation $(\pi, \widehat{H})$ and an operator $V : H \to \widehat{H}$ such that if $a \in A$ then*

$$\varphi(a) = V^*\pi(a)V.$$

Moreover, $\|V^*V\| = \|\varphi(1_A)\|$.

Proof. Assume that A is a unital C*-algebra and H is a Hilbert space. Assume that $\varphi : A \to B(H)$ is a completely positive map. We begin by defining a sesquilinear form on the algebraic tensor product $A \otimes H$.

Define

$$\begin{array}{cccc} \langle -, - \rangle : & (A \otimes H)^2 & \to & \mathbb{C} \\ & (\sum_j b_j \otimes \eta_j, \sum_i a_i \otimes \xi_i) & \mapsto & \sum_{i,j} \langle \varphi(a_i^* b_j)\eta_j, \xi_i \rangle_H \end{array}$$

To be clear, $\langle -, - \rangle_H$ is the inner product on H.

To show: (a) $\langle -, - \rangle$ defines a positive semidefinite sesquilinear form.

(a) Assume that $\lambda \in \mathbb{C}$. Then,

$$\begin{aligned}\langle \lambda \sum_j b_j \otimes \eta_j, \sum_i a_i \otimes \xi_i \rangle &= \langle \sum_j (\lambda b_j) \otimes \eta_j, \sum_i a_i \otimes \xi_i \rangle \\ &= \sum_{i,j} \langle \varphi(a_i^* \lambda b_j)\eta_j, \xi_i \rangle_H \\ &= \lambda \sum_{i,j} \langle \varphi(a_i^* b_j)\eta_j, \xi_i \rangle_H \\ &= \lambda \langle \sum_j b_j \otimes \eta_j, \sum_i a_i \otimes \xi_i \rangle\end{aligned}$$

and

$$\begin{aligned}\langle \sum_j b_j \otimes \eta_j, \lambda \sum_i a_i \otimes \xi_i \rangle &= \langle \sum_j b_j \otimes \eta_j, \sum_i (\lambda a_i) \otimes \xi_i \rangle \\ &= \sum_{i,j} \langle \varphi((\lambda a_i)^* b_j)\eta_j, \xi_i \rangle_H \\ &= \overline{\lambda} \sum_{i,j} \langle \varphi(a_i^* b_j)\eta_j, \xi_i \rangle_H \\ &= \overline{\lambda} \langle \sum_j b_j \otimes \eta_j, \sum_i a_i \otimes \xi_i \rangle.\end{aligned}$$

Now assume that $b_1, b_2 \in A$ and $\eta_1, \eta_2 \in H$. Then,

$$\begin{aligned}\langle (b_1 \otimes \eta_1) + (b_2 \otimes \eta_2), \sum_i a_i \otimes \xi_i \rangle &= \sum_i \sum_{j=1}^{2} \langle \varphi(a_i^* b_j)\eta_j, \xi_i \rangle_H \\ &= \sum_i \big(\langle \varphi(a_i^* b_1)\eta_1, \xi_i \rangle_H + \langle \varphi(a_i^* b_2)\eta_2, \xi_i \rangle_H \big) \\ &= \sum_i \langle \varphi(a_i^* b_1)\eta_1, \xi_i \rangle_H + \sum_i \langle \varphi(a_i^* b_2)\eta_2, \xi_i \rangle_H \\ &= \langle b_1 \otimes \eta_1, \sum_i a_i \otimes \xi_i \rangle + \langle b_2 \otimes \eta_2, \sum_i a_i \otimes \xi_i \rangle.\end{aligned}$$

By a similar computation, we also have

$$\langle \sum_i a_i \otimes \xi_i, (b_1 \otimes \eta_1) + (b_2 \otimes \eta_2) \rangle = \langle \sum_i a_i \otimes \xi_i, b_1 \otimes \eta_1 \rangle + \langle \sum_i a_i \otimes \xi_i, b_2 \otimes \eta_2 \rangle.$$

Hence, $\langle -, - \rangle$ is a sesquilinear form. Next, assume that $\sum_{i=1}^{n} a_i \otimes \xi_i \in A \otimes H$. Then,

$$\langle\sum_{i=1}^{n} a_i \otimes \xi_i, \sum_{i=1}^{n} a_i \otimes \xi_i\rangle = \sum_{i,j=1}^{n} \langle\varphi(a_i^* a_j)\xi_j, \xi_i\rangle_H$$

$$= \left\langle \varphi_n((a_i^* a_j)) \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}, \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} \right\rangle_{H^{(n)}}$$

$$= \left\langle \varphi_n(\alpha^* \alpha) \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}, \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} \right\rangle_{H^{(n)}}$$

where

$$\alpha = \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \in M_{n\times n}(A).$$

In the above computation, we recall that the inner product on the direct sum of Hilbert spaces $H^{(n)}$ is given by

$$\langle(g_1, \dots, g_n), (h_1, \dots, h_n)\rangle_{H^{(n)}} = \sum_{i=1}^{n} \langle g_i, h_i\rangle_H.$$

Since φ is completely positive, φ_n is a positive map. Therefore,

$$\langle\sum_{i=1}^{n} a_i \otimes \xi_i, \sum_{i=1}^{n} a_i \otimes \xi_i\rangle = \left\langle \varphi_n(\alpha^* \alpha) \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}, \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} \right\rangle_{H^{(n)}} \geq 0.$$

Finally, assume that $a \otimes \xi = 0$ in $A \otimes H$. By definition of $\langle -, - \rangle$, we compute directly that $\langle a \otimes \xi, a \otimes \xi\rangle = 0$. Hence, we conclude that $\langle -, - \rangle$ is a positive semidefinite sesquilinear form on $A \otimes H$.

Now define

$$\mathcal{N} = \{u \in A \otimes H \mid \langle u, u\rangle = 0\rangle.$$

Since $\langle -, - \rangle$ is a sesquilinear form on $A \otimes H$, it must satisfy the Cauchy-Schwarz inequality. If $u, v \in A \otimes H$ then

$$|\langle u, v\rangle|^2 \leq \langle u, u\rangle\langle v, v\rangle.$$

Thus,

$$\mathcal{N} = \{u \in A \otimes H \mid \text{If } v \in A \otimes H \text{ then } \langle u, v \rangle = 0\}. \tag{3.1}$$

Using equation (3.1), it is straightforward to check that $\mathcal{N}$ is a vector subspace of $A \otimes H$. To see that $\mathcal{N}$ is closed, assume that $\{u_n\}_{n \in \mathbb{Z}_{>0}}$ is a sequence in $\mathcal{N}$ which converges to $u \in A \otimes H$. If $v \in A \otimes H$ then

$$\langle u, v \rangle = \langle \lim_{n \to \infty} u_n, v \rangle = \lim_{n \to \infty} \langle u_n, v \rangle = 0.$$

Hence, $u \in \mathcal{N}$ and $\mathcal{N}$ is a closed subspace of $A \otimes H$.

Now consider the quotient space $(A \otimes H)/\mathcal{N}$. Define the map $\langle -, - \rangle'$ by

$$\begin{array}{rccc} \langle -, - \rangle' : & ((A \otimes H)/\mathcal{N})^2 & \to & \mathbb{C} \\ & ([\sum_j b_j \otimes \eta_j], [\sum_i a_i \otimes \xi_i]) & \mapsto & \langle \sum_j b_j \otimes \eta_j, \sum_i a_i \otimes \xi_i \rangle. \end{array}$$

Here, $[\sum_j b_j \otimes \eta_j]$ refers to the equivalence class of $\sum_j b_j \otimes \eta_j \in A \otimes H$ in $(A \otimes H)/\mathcal{N}$. By construction, the pair $((A \otimes H)/\mathcal{N}, \langle -, - \rangle')$ is an inner product space. Analogously to the GNS construction, let $\widehat{H}$ be the completion of $(A \otimes H)/\mathcal{N}$ with respect to the inner product $\langle -, - \rangle'$. Then, $\widehat{H}$ is a Hilbert space by construction.

Following the notation in [BO08, Theorem 1.5.3], let $(\sum_i a_i \otimes \xi_i)^\wedge$ be the element in $\widehat{H}$ corresponding to $\sum_i a_i \otimes \xi_i \in H$. Define

$$\begin{array}{rccc} V : & H & \to & \widehat{H} \\ & x & \mapsto & (1_A \otimes x)^\wedge. \end{array}$$

Then, V is a linear operator. To see that V is bounded, we compute directly that

$$\begin{aligned} \|V\|^2 &= \sup_{\|x\|_H = 1} \|V(x)\|^2 \\ &= \sup_{\|x\|_H = 1} \|(1_A \otimes x)^\wedge\|^2 \\ &= \sup_{\|x\|_H = 1} \langle [1_A \otimes x], [1_A \otimes x] \rangle' \\ &= \sup_{\|x\|_H = 1} \langle \varphi(1_A^* 1_A) x, x \rangle_H \\ &\leq \sup_{\|x\|_H = 1} \|\varphi(1_A)\| \|x\|^2 = \|\varphi(1_A)\|. \end{aligned}$$

In fact, we claim that the reverse inequality also holds. We have

$$
\begin{aligned}
\|\varphi(1_A)\|^2 &= \sup_{\|x\|_H=1} \|\varphi(1_A)(x)\|_H^2 \\
&= \sup_{\|x\|_H=1} \langle \varphi(1_A)(x), \varphi(1_A)(x)\rangle_H \\
&= \sup_{\|x\|_H=1} \langle [1_A \otimes x], [1_A \otimes \varphi(1_A)(x)]\rangle' \\
&= \sup_{\|x\|_H=1} \langle (1_A \otimes x)^\wedge, (1_A \otimes \varphi(1_A)(x))^\wedge\rangle \\
&\leq \sup_{\|x\|_H=1} \|(1_A \otimes x)^\wedge\| \|(1_A \otimes \varphi(1_A)(x))^\wedge\| \\
&= \sup_{\|x\|_H=1} \|V(x)\| \|V(\varphi(1_A)(x))\| \\
&\leq \sup_{\|x\|_H=1} \|V\|^2 \|x\|_H \|\varphi(1_A)(x)\|_H = \|V\|^2 \|\varphi(1_A)\|.
\end{aligned}
$$

So, $\|\varphi(1_A)\| \leq \|V\|^2$ and consequently, $\|V^*V\| = \|V\|^2 = \|\varphi(1_A)\|$.

For the next step, we define the map

$$
\begin{array}{rccc}
\pi : & A & \to & B(\widehat{H}) \\
& a & \mapsto & \Big((\sum_i b_i \otimes \eta_i)^\wedge \mapsto (\sum_i ab_i \otimes \eta_i)^\wedge \Big)
\end{array}
$$

To show: (b) π is a *-homomorphism.

(b) Since $(A \otimes H)/\mathcal{N}$ is dense in $\widehat{H}$, it suffices to check that π is a *-homomorphism on $(A \otimes H)/\mathcal{N}$. It is straightforward to check that π is a linear map. To see that π is a *-homomorphism, assume that $a \in A$ and

$$
(\sum_i b_i \otimes \eta_i)^\wedge, (\sum_j c_j \otimes \xi_j)^\wedge \in \widehat{H}.
$$

We compute directly that

$$\begin{aligned}
\langle \pi(a)^*(\sum_i b_i \otimes \eta_i)^\wedge, (\sum_j c_j \otimes \xi_j)^\wedge\rangle &= \langle (\sum_i b_i \otimes \eta_i)^\wedge, \pi(a)(\sum_j c_j \otimes \xi_j)^\wedge\rangle \\
&= \langle (\sum_i b_i \otimes \eta_i)^\wedge, (\sum_j ac_j \otimes \xi_j)^\wedge\rangle \\
&= \sum_{i,j} \langle \varphi(c_j^* a^* b_i)\eta_i, \xi_j\rangle_H \\
&= \langle (\sum_i a^* b_i \otimes \eta_i)^\wedge, (\sum_j c_j \otimes \xi_j)^\wedge\rangle \\
&= \langle \pi(a^*)(\sum_i b_i \otimes \eta_i)^\wedge, (\sum_j c_j \otimes \xi_j)^\wedge\rangle
\end{aligned}$$

and if $b \in A$ then

$$\pi(ab)(\sum_j c_j \otimes \xi_j)^\wedge = (\sum_j abc_j \otimes \xi_j)^\wedge = \pi(a)(\sum_j bc_j \otimes \xi_j)^\wedge = \pi(a)\pi(b)(\sum_j c_j \otimes \xi_j)^\wedge.$$

Therefore, π is a *-homomorphism.

By part (b), the pair $(\pi, \widehat{H})$ is a representation of A. Now assume that $x, y \in H$ and $a \in A$. Then,

$$\begin{aligned}
\langle V^*\pi(a)V(x), y\rangle_H &= \langle \pi(a)V(x), V(y)\rangle_H \\
&= \langle \pi(a)(1_A \otimes x)^\wedge, (1_A \otimes y)^\wedge\rangle \\
&= \langle (a \otimes x)^\wedge, (1_A \otimes y)^\wedge\rangle \\
&= \langle \varphi(1_A^* a)x, y\rangle_H \\
&= \langle \varphi(a)x, y\rangle_H.
\end{aligned}$$

Finally, we conclude that if $a \in A$ then $V^*\pi(a)V = \varphi(a)$. □

In the statement of Theorem 3.1.4, we note that $\|\varphi(1_A)\| = \|\varphi\|$. This was stated in [BO08, Theorem 1.5.3]. However, this fact is not obvious.

Definition 3.1.5. Let A be a unital C*-algebra and H be a Hilbert space. Let $\varphi : A \to B(H)$ be a c.p map. The triplet $(\pi, \widehat{H}, V)$ in Theorem 3.1.4 is called a **Stinespring dilation** of φ.

If φ is a u.c.p map then

$$V^*V = V^*\pi(1_A)V = \varphi(1_A) = id_{B(H)}.$$

In this case, V is an isometry and $VV^* \in B(\widehat{H})$ is a projection operator. We call VV^* the **Stinespring projection**.

In general, there are many different Stinespring dilations. It is explained in [BO08] that we can always choose a *minimal* Stinespring dilation in the following manner.

Definition 3.1.6. Let A be a unital C*-algebra and H be a Hilbert space. Let $\varphi : A \to B(H)$ be a c.p map and $(\pi, \widehat{H}, V)$ be a Stinespring dilation of φ. The Stinespring dilation $(\pi, \widehat{H}, V)$ is called **minimal** if the subspace $\pi(A)VH$ is dense in $\widehat{H}$.

Note that in the proof of Theorem 3.1.4, the Stinespring dilation we constructed is minimal because $\pi(A)VH = (A \otimes H)/\mathcal{N}$ which is dense in $\widehat{H}$.

Theorem 3.1.5. *Let A be a unital C*-algebra and H be a Hilbert space. Let $\pi : A \to B(H)$ be a c.p map and $(\pi, \widehat{H}, V)$ be a minimal Stinespring dilation. Then, $(\pi, \widehat{H}, V)$ is unique up to unitary equivalence.*

Proof. Assume that $\varphi : A \to B(H)$ is a c.p map and $(\pi, \widehat{H}, V)$ be a minimal Stinespring dilation of φ. Suppose that (π', H', W) is another minimal Stinespring dilation of φ. If $a \in A$ then

$$\varphi(a) = W^*\pi'(a)W = V^*\pi(a)V.$$

Since the Stinespring dilations $(\pi, \widehat{H}, V)$ and (π', H', W) are both minimal, the subspaces $\pi(A)VH$ and $\pi'(A)WH$ are both dense in $\widehat{H}$ and H' respectively. Define the map

$$\begin{array}{rrcl} u: & \pi(A)VH & \to & \pi'(A)WH \\ & \pi(a)V(x) & \mapsto & \pi'(a)W(x). \end{array}$$

To see that u is isometric, assume that $a \in A$ and $x \in X$. Then,

$$\begin{aligned} \langle u(\pi(a)V(x)), u(\pi(a)V(x))\rangle &= \langle \pi'(a)W(x), \pi'(a)W(x)\rangle \\ &= \langle W^*\pi'(a^*a)W(x), x\rangle \\ &= \langle \varphi(a^*a)(x), x\rangle \\ &= \langle V^*\pi(a^*a)V(x), x\rangle \\ &= \langle \pi(a)V(x), \pi(a)V(x)\rangle. \end{aligned}$$

By the universal property of completeness, we can extend u to a unitary operator $\tilde{u} : \widehat{H} \to H'$. Now observe that if $a \in A$ and $\pi(b)V(x) \in \pi(A)VH$ then

$$\tilde{u}\pi(a)\big(\pi(b)V(x)\big) = \pi'(ab)W(x) = \pi'(a)\big(\pi'(b)W(x)\big) = \pi'(a)\tilde{u}\big(\pi(b)V(x)\big).$$

Since $\pi(A)VH$ is dense in $\widehat{H}$, we deduce that $\tilde{u}\pi(a) = \pi'(a)\tilde{u}$ on $\widehat{H}$. Therefore, the Stinespring dilations $(\pi, \widehat{H}, V)$ and (π', H', W) are unitarily equivalent as required. □

As explained in [BO08, Remark 1.5.4], Stinespring's theorem also holds for non-unital C*-algebras. We also claim that Stinespring's theorem is a generalisation of the GNS construction. Let A be a unital C*-algebra and τ be a state on A. Since τ is a state, it is a positive linear functional on A and is thus, completely positive.

By identifying $\mathbb{C}$ with $B(\mathbb{C})$, let (π, H, V) be a minimal Stinespring dilation of τ. If $a \in A$ then $\tau(a) = V^*\pi(a)V$. Define $\xi = V(1) \in H$. If $\lambda \in \mathbb{C}$ then $V(\lambda) = \lambda\xi$ and by Theorem 3.1.4 and Theorem 1.11.2,

$$\|V\|^2 = \|V^*V\| = |\tau(1_A)| = \|\tau\| = 1.$$

So, $\|V\| = 1$ and

$$\|V\| = \sup_{|\lambda|=1} \|\lambda\xi\| = \|\xi\| = 1.$$

Hence, ξ is a unit vector in H. Moreover, if $a \in A$ then

$$\langle \pi(a)\xi, \xi\rangle = \langle \pi(a)V(1), V(1)\rangle = \langle V^*\pi(a)V(1), 1\rangle = \langle \tau(a)1, 1\rangle = \tau(a).$$

Finally, to see that ξ is a cyclic vector, observe that since (π, H, V) is a minimal Stinespring dilation of τ, the subspace $\pi(A)V\mathbb{C}$ is dense in H. Hence, $\overline{\pi(A)\xi} = \overline{\pi(A)V\mathbb{C}} = H$ and ξ is a unit cyclic vector for the representation (π, H). This connects Stinespring's theorem to the GNS construction.

The next result we prove is an analogue of Theorem 2.9.2, applied to a minimal Stinespring dilation.

Theorem 3.1.6. *Let A be a unital C*-algebra and H be a Hilbert space. Let $\varphi : A \to B(H)$ be a c.c.p map and $(\pi, \widehat{H}, V)$ be the minimal Stinespring dilation of φ. Then, there exists a *-homomorphism*

$$\rho : \varphi(A)' \to \pi(A)' \subseteq B(\widehat{H})$$

such that if $a \in A$ and $x \in \varphi(A)'$ then

$$\varphi(a)x = V^*\pi(a)\rho(x)V.$$

Proof. Assume that $\varphi : A \to B(H)$ is a c.c.p map and $(\pi, \widehat{H}, V)$ is a minimal Stinespring dilation of φ. Define the map $\rho : \varphi(A)' \to B(\widehat{H})$ by

$$\begin{array}{rccc} \rho(x): & \pi(A)VH & \to & \pi(A)VH \\ & \sum_i \pi(a_i)V\xi_i & \mapsto & \sum_i \pi(a_i)Vx\xi_i \end{array}$$

Note that $\rho(x)$ is a linear operator.

To show: (a) If $x \in \varphi(A)'$ then $\rho(x)$ is well-defined and bounded.

(a) Assume that $x \in \varphi(A)'$. Assume that $\sum_i \pi(a_i)V\xi_i = \sum_j \pi(b_j)V\mu_j$ in $\pi(A)VH$. If $c \in A$ and $\lambda \in H$ then

$$\begin{aligned}
\langle \rho(x)\big(\sum_i \pi(a_i)V\xi_i\big), \pi(c)V\lambda\rangle &= \langle \sum_i \pi(a_i)Vx\xi_i, \pi(c)V\lambda\rangle \\
&= \langle \sum_i V^*\pi(c^*a_i)Vx\xi_i, \lambda\rangle \\
&= \langle \sum_i \varphi(c^*a_i)x\xi_i, \lambda\rangle = \langle \sum_i x\varphi(c^*a_i)\xi_i, \lambda\rangle \\
&= \langle \sum_i \pi(a_i)V\xi_i, \pi(c)Vx^*\lambda\rangle \\
&= \langle \sum_j \pi(b_j)V\mu_j, \pi(c)Vx^*\lambda\rangle \\
&= \langle \sum_j xV^*\pi(c^*b_j)V\mu_j, \lambda\rangle = \langle \sum_j \varphi(c^*b_j)x\mu_j, \lambda\rangle \\
&= \langle \sum_j \pi(b_j)x\mu_j, \pi(c)V\lambda\rangle \\
&= \langle \rho(x)\big(\sum_j \pi(b_j)V\mu_j\big), \pi(c)V\lambda\rangle.
\end{aligned}$$

Since $\pi(A)VH$ is dense in $\widehat{H}$, we conclude that

$$\rho(x)\big(\sum_i \pi(a_i)V\xi_i\big) = \rho(x)\big(\sum_j \pi(b_j)V\mu_j\big).$$

Therefore, $\rho(x)$ is well-defined. To see that $\rho(x)$ is bounded, let $\xi = [\xi_1, \ldots, \xi_n]^T \in H^n$. Let $diag(x)$ be the $n \times n$ matrix whose diagonal elements are x and non-diagonal elements are zeros. Then,

$$
\begin{aligned}
\|\rho(x) \sum_i \pi(a_i) V \xi_i\|^2 &= \langle \sum_i \pi(a_i) V x \xi_i, \sum_j \pi(a_j) V x \xi_j \rangle \\
&= \sum_{i,j} \langle x^* V^* \pi(a_j^* a_i) V x \xi_i, \xi_j \rangle \\
&= \sum_{i,j} \langle x^* \varphi(a_j^* a_i) x \xi_i, \xi_j \rangle \\
&= \langle diag(x)^* \varphi_n((a_j^* a_i)) diag(x) \xi, \xi \rangle_{H^n} \\
&= \langle diag(x)^* diag(x) \varphi_n((a_j^* a_i)) \xi, \xi \rangle_{H^n} \\
&\leq \|x\|^2 \langle \varphi_n((a_j^* a_i)) \xi, \xi \rangle_{H^n} \\
&= \|x\|^2 \sum_{i,j} \langle \varphi(a_j^* a_i) \xi_i, \xi_j \rangle \\
&= \|x\|^2 \|\sum_i \pi(a_i) V \xi_i\|^2.
\end{aligned}
$$

Therefore, $\|\rho(x)\| \leq \|x\|$ and $\rho(x)$ is a bounded operator.

By part (a), if $x \in \varphi(A)'$ then $\rho(x)$ can be extended to a bounded linear operator on $\widehat{H}$ because $\overline{\pi(A)VH} = \widehat{H}$.

To show: (b) ρ is a *-homomorphism.

(c) $\rho(\varphi(A)') \subseteq \pi(A)'$.

(d) If $a \in A$ and $x \in \varphi(A)'$ then $\varphi(a)x = V^* \pi(a) \rho(x) V$.

(b) It is straightforward to check that ρ is a linear operator. Now assume that $x, y \in \varphi(A)'$ and $\sum_i \pi(a_i) V \xi_i \in \pi(A)VH$. Then,

$$
\begin{aligned}
\rho(xy)\big(\sum_i \pi(a_i) V \xi_i\big) &= \sum_i \pi(a_i) V xy \xi_i \\
&= \rho(x)\big(\sum_i \pi(a_i) V y \xi_i\big) \\
&= \rho(x)\rho(y)\big(\sum_i \pi(a_i) V \xi_i\big).
\end{aligned}
$$

We also have

$$\begin{aligned}
\langle \rho(x)^*\big(\sum_i \pi(a_i)V\xi_i\big), \sum_j \pi(b_j)V\mu_j\rangle &= \langle \sum_i \pi(a_i)V\xi_i, \rho(x)\big(\sum_j \pi(b_j)V\mu_j\big)\rangle \\
&= \langle \sum_i \pi(a_i)V\xi_i, \rho(x)\big(\sum_j \pi(b_j)V\mu_j\big)\rangle \\
&= \langle \sum_i \pi(a_i)V\xi_i, \sum_j \pi(b_j)Vx\mu_j\rangle \\
&= \sum_{i,j} \langle x^*V^*\pi(b_j^*a_i)V\xi_i, \mu_j\rangle \\
&= \sum_{i,j} \langle x^*\varphi(b_j^*a_i)\xi_i, \mu_j\rangle \\
&= \sum_{i,j} \langle \varphi(b_j^*a_i)x^*\xi_i, \mu_j\rangle \\
&= \langle \sum_i \pi(a_i)Vx^*\xi_i, \sum_j \pi(b_j)V\mu_j\rangle \\
&= \langle \rho(x^*)\big(\sum_i \pi(a_i)V\xi_i\big), \sum_j \pi(b_j)V\mu_j\rangle.
\end{aligned}$$

Since $\pi(A)VH$ is dense in $\widehat{H}$, we conclude that if $x, y \in \varphi(A)'$ then $\rho(xy) = \rho(x)\rho(y)$ and $\rho(x^*) = \rho(x)^*$. So, ρ is a *-homomorphism.

(c) Assume that $\pi(a)V\xi \in \pi(A)VH$, $b \in A$ and $x \in \varphi(A)'$. Then,

$$\begin{aligned}
(\rho(x)\pi(b))\big(\pi(a)V\xi\big) &= \rho(x)\big(\pi(ba)V\xi\big) \\
&= \pi(ba)Vx\xi = \pi(b)\pi(a)Vx\xi \\
&= (\pi(b)\rho(x))\big(\pi(a)V\xi\big).
\end{aligned}$$

Hence, $\pi(b)\rho(x) = \rho(x)\pi(b)$ in $B(\widehat{H})$ because $\pi(A)VH$ is dense in $\widehat{H}$. So, $\rho(\varphi(A)') \subseteq \pi(A)'$.

(d) Assume that $a \in A$ and $x \in \varphi(A)'$. Recall from the construction in Theorem 3.1.4 that $\pi : A \to B(\widehat{H})$ is a unital *-homomorphism. If $\xi \in H$ then

$$V^*\pi(a)\rho(x)V\xi = V^*\pi(a)\rho(x)\pi(1_A)V\xi = V^*\pi(a)\pi(1_A)Vx\xi = \varphi(a)x\xi.$$

This completes the proof. □

3.2 Multiplicative domains and conditional expectations

(TBA)

Bibliography

[BO08] NP. Brown, N. Ozawa. *C*-algebras and finite dimensional approximations*, 2008, American Mathematical Society, Providence RI, ISBN: 978-0-8218-4381-9, MR2391387.

[Con90] J. Conway, *A course in functional analysis*, 1990, Springer-Verlag, New York, ISBN 0-387-97245-5, MR1070713.

[DF04] D. Dummit, R. Foote. *Abstract Algebra*, John Wiley and Sons Inc, 3rd Edition, 2004. ISBN 0-471-43334-9. MR2286236.

[Gli59] J. Glimm, *On a certain class of operator algebras*, Trans. Amer. Math. **95** (1960) 318-340, MR0112057.

[Mur20] D. Murfet, *Tutorial 4: One-point compactification*, 2020, Available at: http://therisingsea.org/notes/mast30026/tutorial4-2020.pdf.

[Mur90] G. J. Murphy, *C*-Algebras and Operator Theory*, Boston, Academic Press Inc, 1990, ISBN: 0-12-511360-9, MR1074574.

[Put19] I. Putnam, *Lecture Notes on C*-algebras*, January 3rd 2019, Available at: https://www.math.uvic.ca/faculty/putnam/ln/C*-algebras.pdf.

[Sol18] P. Sołtan, *A Primer on Hilbert Space Operators*, 2018, Switzerland, Birkhäuser, ISBN 978-3-319-92061-0.

[Ter20] Y. Terasawa, *About Unitization of C*-algebras*, July 28th 2020, Available at: http://www.math.nagoya-u.ac.jp/ richard/teaching/s2020/Terasawa.pdf.

[Zim90] R. J. Zimmer, *Essential Results of Functional Analysis*, Chicago Lectures in Mathematics, University of Chicago Press, 1990, ISBN: 0-226-98337-4, MR1045444.