Wedge product matrices and applications

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September 8, 2022

Overview

1. Wedge product matrices
2. Quasideterminants
3. Eigenvector-eigenvalue identity
4. Smith normal form
5. Orbits of principal congruence subgroups

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Wedge product matrices:

- R commutative ring, $A \in M_{n\times n}(R)$, $k \in \{1,2,\ldots,n\}$
- $T_{\binom{n}{k}} := \{ S \subseteq \{1,2,\ldots,n\} \mid |S| = k \}$
- In the free R-module $\Lambda^k(R^n)$,

$$\boxed{Ae_J = Ae_{j_1} \wedge \ldots \wedge Ae_{j_k} = \sum_{I \in T_{\binom{n}{k}}} (\Lambda^k(A))_{I,J}\, e_I}$$

where $J = \{j_1, \ldots, j_k\} \subseteq T_{\binom{n}{k}}$.

- The matrix $\Lambda^k(A) \in M_{\binom{n}{k}\times\binom{n}{k}}(R)$.

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Properties of wedge product matrices:

- $\boxed{\Lambda^k(AB) = \Lambda^k(A)\,\Lambda^k(B)}$

- $(\Lambda^k(A))_{I,J} = \det(A_{I,J})$ where $I, J \in T_{\binom{n}{k}}$.

- $\Lambda^1(A) = A$ and $\Lambda^n(A) = (\det(A))$

- $\Lambda^k(I_n) = I_{\binom{n}{k}}$ where I_n is the $n \times n$ identity matrix.

- $\boxed{\det(\Lambda^k(A)) = \det(A)^{\binom{n-1}{k-1}}}$

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Adjugate matrices:

- From Laplace expansion:

$$\det(A) = \sum_{L \in T_{\binom{n}{k}}} (\Lambda^k(A))_{L,H} \overbrace{\boxed{\underbrace{S_{L,\{1,2,\dots,n\}} \; S_{H,\{1,\dots,n\}}}_{\pm 1} \; (\Lambda^{n-k}(A))_{L^c,H^c}}}^{\boxed{(\Upsilon^{n-k}(A))_{H,L}}}$$

- $$\boxed{\Lambda^k(A)\,\Upsilon^{n-k}(A) = \Upsilon^{n-k}(A)\,\Lambda^k(A) = \det(A)\, I_{\binom{n}{k}}.}$$

- Υ^{n-k} shares similar properties to Λ^k:

Quasideterminants:

- An analogue of the determinant for square matrices in a $\boxed{\text{non-commutative ring.}}$

- Molev: Quasideterminants used to factorise the quantum determinant

- In a commutative ring, quasideterminants are connected to elements of the adjugate matrix.

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Gel'fand Retakh Molev definition:

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 3 & -4 \\ 7 & 6 & -3 \end{pmatrix}$$

$$|A|_{2,3} = -4 - \begin{pmatrix} 0 & 3 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 7 & 6 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

$$= -\frac{37}{19}$$

$$\boxed{|A|_{j,\ell} = a_{j,\ell} - r_j^{\ell} (A^{j\ell})^{-1} c_{\ell}^{j}}$$

- In a commutative ring,

$$\boxed{|A|_{j,\ell} \left(\Upsilon^{n-1}(A)\right)_{\ell,j} = \det(A).}$$

My general definition:

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 3 & -4 \\ 7 & 6 & -3 \end{pmatrix}$$

$$|A|_{\{1,2\},\{1,3\}} = \begin{pmatrix} 2 & 1 \\ 0 & -4 \end{pmatrix} - \begin{pmatrix} -1 \\ 3 \end{pmatrix} 6^{-1} \begin{pmatrix} 7 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{19}{6} & \frac{1}{2} \\ -\frac{7}{2} & -\frac{5}{2} \end{pmatrix}.$$

$$\boxed{|A|_{J,L} = A_{J,L} - A_{J,L^c} (A_{J^c,L^c})^{-1} A_{J^c,L}, \quad J, L \in T_{\binom{n}{k}}}$$

- In a commutative ring,

$$\boxed{\det(|A|_{J,L}) \left(\Upsilon^{n-k}(A)\right)_{L,J} = \det(A).}$$

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Some properties of quasideterminants:

- $A \in M_{n \times n}(R)$, R commutative ring

$$\boxed{\det(A) = \frac{1}{n} \begin{vmatrix} & A & \\ |A|_{j,1} \; |A|_{j,2} & \cdots & |A|_{j,n} \\ & A & \end{vmatrix}, \quad j \in \{1, 2, \ldots, n\}}$$

- If $J = \{j_1, \ldots, j_k\}$ and $L = \{l_1, \ldots, l_k\}$ are elements of $T_{\binom{n}{k}}$ then

$$\boxed{\det(|A|_{J,L}) = \begin{vmatrix} |A|^{-1}_{j_1,l_1} & |A|^{-1}_{j_1,l_2} & \cdots & |A|^{-1}_{j_1,l_k} \\ \vdots & \vdots & \ddots & \vdots \\ |A|^{-1}_{j_k,l_1} & |A|^{-1}_{j_k,l_2} & \cdots & |A|^{-1}_{j_k,l_k} \end{vmatrix}^{-1}}$$

Bump, Hoffstein:

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- Minimal parabolic Eisenstein series:

$$\boxed{E_{v_1, v_2}(\tau) = \sum_{\gamma \in \Gamma_\infty(3) \backslash \Gamma(3)} \kappa(\gamma) I_{v_1, v_2}(\gamma \tau)}$$

- Fourier coefficients of Eisenstein series computed by particular representatives of $\Gamma_\infty(3) \backslash \Gamma(3)$.

$\sigma = \mathbb{Z}[e^{2\pi i/3}]$

$$\Gamma(3) = \{ A \in SL_3(\sigma) \mid A \equiv I_3 \bmod 3\sigma \}$$

$$\Gamma_\infty(3) = \Gamma(3) \cap \underline{U}$$

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Invariants of the coset $\Gamma_\infty(3) \cdot A$:

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \in \Gamma(3)$$

- Λ^1 invariants:

$$\boxed{A_1 = g, \quad B_1 = h, \quad C_1 = i}$$

- Λ^2 invariants:

$$\boxed{A_2 = dh - eg, \quad B_2 = di - fg, \quad C_2 = ei - fh}$$

$$\boxed{\mathrm{Inv}(A) = (A_1, B_1, C_1, A_2, B_2, C_2) \in \mathbb{Z}^6}$$

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Invariant conditions:

$$\boxed{\begin{array}{ll} (I1) & A_1 \equiv A_2 \equiv B_1 \equiv B_2 \equiv 0 \mod 3\sigma \\ (I2) & C_1 \equiv C_2 \equiv 1 \mod 3\sigma \\ (I3) & \gcd(A_1, B_1, C_1) = \gcd(A_2, B_2, C_2) = 1 \\ (I4) & A_1 C_2 - B_1 B_2 + A_2 C_1 = 0 \end{array}}$$

Theorem: If $A, B \in \Gamma(3)$ then

$$\underline{\Gamma_\infty(3) \cdot A = \Gamma_\infty(3) \cdot B} \iff \underline{\mathrm{Inv}(A) = \mathrm{Inv}(B)}$$

The connection to Bump and Hoffstein:

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- For $A \in \Gamma(3)$, define

$$\boxed{{}^{\iota}A = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} (A^{-1})^{T} \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}}$$

- Invariants of A (BH): Bottom rows of A and ${}^{\iota}A$

$$\boxed{\Lambda^{2}(A) = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix} ({}^{\iota}A) \begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix}}$$

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From invariants to a representative:

$\mathcal{O} = \mathbb{Z}[\omega]$, $\omega = e^{2\pi i/3}$

$(A_1, B_1, C_1, A_2, B_2, C_2) = (-3+6\omega, -3, -2-3\omega, -6+3\omega, 3-6\omega, 4+3\omega)$
satisfies the invariant conditions. (Either $A_1 \neq 0$ or $B_1 \neq 0$)

$$\downarrow$$

$$X = \begin{pmatrix} 1 & 0 & 2\omega \\ & 1 & 1+2\omega \\ & & 1 \end{pmatrix} \begin{pmatrix} -1-\omega & & \\ & 1+\omega & \\ & & 1+\omega \end{pmatrix} \boxed{\begin{pmatrix} 2+2\omega & 1+6\omega & \\ 3+2\omega & 4+8\omega & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1+\omega & -1-\omega \\ & 3 & -3-\omega \end{pmatrix} \begin{pmatrix} 1+2\omega & \omega & \\ 2+3\omega & \omega & \\ & & 1 \end{pmatrix}}$$

$X \in \Gamma(3)$

$$\boxed{\mathrm{Inv}(X) = (A_1, B_1, C_1, A_2, B_2, C_2)}$$

Similar to
Bruhat decomposition

The End

Thank you for listening