Snake Lemma

Brian Chan

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0.1 Statement and proof of the Snake lemma

In this document, we prove the Snake lemma for an abelian category $\mathscr{C}$. The Freyd-Mitchell embedding theorem states that any abelian category embeds as a full subcategory of the category of R-modules **R-Mod** for some unital ring R. This means that we are allowed to use elements of R-modules to prove the Snake lemma.

The full statement of the Snake lemma is given below. Due to its importance, we state it as a theorem:

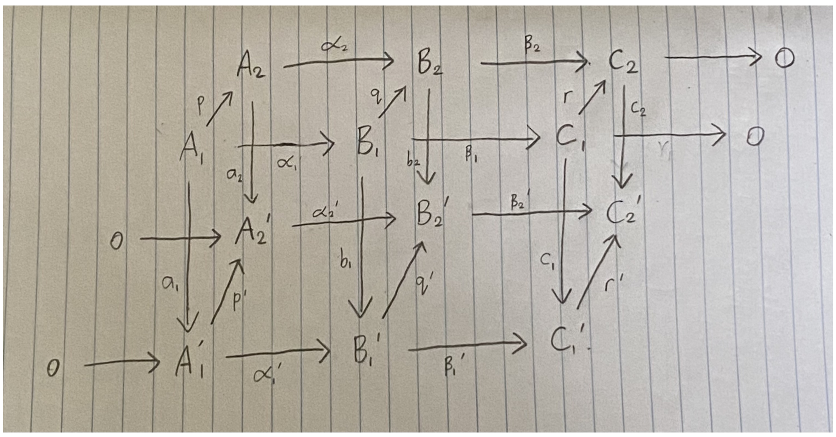
Theorem 0.1.1 (Snake lemma)**.** *Let $\mathscr{C}$ be an abelian category. Assume that we have the following diagram in $\mathscr{C}$:*

$$\begin{tikzcd} & M' \arrow[r, "m_1"] \arrow[d, "f'"] & M \arrow[r, "m_2"] \arrow[d, "f"] & M'' \arrow[r] \arrow[d, "f''"] & 0 \\ 0 \arrow[r] & N' \arrow[r, "n_1"] & N \arrow[r, "n_2"] & N'' & \end{tikzcd}$$

where the top and bottom rows are exact sequences. Then, there exists a morphism $\partial : \ker f'' \to coker\ f'$ such that the following sequence is a long exact sequence:

$$\ker f' \longrightarrow \ker f \longrightarrow \ker f'' \xrightarrow{\partial} coker\ f' \longrightarrow coker\ f \longrightarrow coker\ f''$$

If $m_1 : M' \to M$ is injective then the induced map $\ker f' \to \ker f$ is injective. If $n_2 : N \to N''$ is surjective then the induced map $coker\ f \to coker\ f''$ is unique. Moreover, the connecting morphism ∂ is natural in the following sense: If we have the following diagram in $\mathscr{C}$:



where the top and bottom rows are exact and connecting morphisms $\partial_1 : \ker c_2 \to$ *coker* a_1 *and* $\partial_2 : \ker c_1 \to$ *coker* a_1 *then the following square commutes:*

$$\begin{array}{ccc} \ker c_1 & \xrightarrow{\partial_1} & \textit{coker } a_1 \\ \downarrow & & \downarrow \\ \ker c_2 & \xrightarrow{\partial_2} & \textit{coker } a_2 \end{array}$$

Proof. Assume that $\mathscr{C}$ is an abelian category. First, assume that we have the following commutative diagram in $\mathscr{C}$:

$$\begin{tikzcd} & M' \arrow[r, "m_1"] \arrow[d, "f'"] & M \arrow[r, "m_2"] \arrow[d, "f"] & M'' \arrow[r] \arrow[d, "f''"] & 0 \\ 0 \arrow[r] & N' \arrow[r, "n_1"] & N \arrow[r, "n_2"] & N'' & \end{tikzcd}$$

We will divide the proof into multiple sections:

1: Constructing induced maps between kernels

Assume that $x' \in \ker f'$. By commutativity of the diagram, we have

$$(f \circ m_1)(x') = (n_1 \circ f')(x') = 0$$

in N. So, $m_1(x') \in \ker f$. Hence, the restriction $m_1|_{\ker f'}$ is a morphism in $\mathscr{C}$ from $\ker f'$ to $\ker f$. Arguing in exactly the same way, we find that the restriction $m_2|_{\ker f}$ is a morphism in $\mathscr{C}$ from $\ker f$ to $\ker f''$.

Thus, we have the sequence

$$\ker f' \xrightarrow{m_1|_{\ker f'}} \ker f \xrightarrow{m_2|_{\ker f}} \ker f''$$

In order to build the long exact sequence we want, we will now show that im $m_1|_{\ker f'} = \ker m_2|_{\ker f}$.

To show: (a) im $m_1|_{\ker f'} \subseteq \ker m_2|_{\ker f}$.

(b) $\ker m_2|_{\ker f} \subseteq$ im $m_1|_{\ker f'}$.

(a) Assume that $x' \in \ker f'$ so that $m_1(x') \in$ im $m_1|_{\ker f'}$. By applying $m_2|_{\ker f}$, we find that

$$m_2|_{\ker f}(m_1(x')) = m_2(m_1(x')) = 0$$

since $m_2 \circ m_1 = 0$. Therefore, im $m_1|_{\ker f'} \subseteq \ker m_2|_{\ker f}$.

(b) Assume that $x \in \ker m_2|_{\ker f}$. Then, $x \in \ker m_2 = \text{im } m_1$. Thus, there exists $x' \in M'$ such that $m_1(x') = x$. Since $x \in \ker f$,

$$f(x) = 0 = (f \circ m_1)(x') = (n_1 \circ f')(x').$$

Now, $n_1(f'(x')) = 0$. By exactness, n_1 is injective. Therefore, $f'(x') = 0$ and $x' \in \ker f'$. Thus, $m_1|_{\ker f'}(x') = m_1(x') = x$ and $\ker m_2|_{\ker f} \subseteq \text{im } m_1|_{\ker f'}$.

By combining parts (a) and (b), we deduce that the sequence

$$\ker f' \xrightarrow{m_1|_{\ker f'}} \ker f \xrightarrow{m_2|_{\ker f}} \ker f''$$

is exact.

2. Constructing induced maps between cokernels

Let $\pi_f : N \to N/\text{im } f$ denote the projection morphism. Similar definitions apply for $\pi_{f'}$ and $\pi_{f''}$. Consider the composite of morphisms

$$N' \xrightarrow{n_1} N \xrightarrow{\pi_f} N/\text{im } f = \text{coker } f.$$

We claim that im $f' \subseteq \ker(\pi_f \circ n_1)$.

To show: (c) im $f' \subseteq \ker(\pi_f \circ n_1)$.

(c) Assume that $x' \in M'$ so that $f'(x') \in \text{im } f'$. By commutativity of the diagram, we have

$$\begin{aligned}(\pi_f \circ n_1)(f'(x')) &= (\pi_f \circ (n_1 \circ f'))(x') \\ &= (\pi_f \circ f \circ m_1)(x') \\ &= f(m_1(x')) + \text{im } f = \text{im } f.\end{aligned}$$

Thus, im $f' \subseteq \ker(\pi_f \circ n_1)$.

Due to part (c), we can invoke the universal property of the quotient to obtain a unique morphism $\phi_1 : \text{coker } f' \to \text{coker } f$ which makes the following diagram commute:

$$\begin{tikzcd} N' \arrow[r, "\pi_{f'}"] \arrow[dr, "\pi_f \circ n_1"'] & \text{coker } f' \arrow[d, dashed, "\phi_1"] \\ & \text{coker } f \end{tikzcd}$$

By a similar argument, there exists a unique morphism
$\phi_2 : \text{coker } f \to \text{coker } f''$ such that the following diagram commutes:

$$\begin{tikzcd} N \arrow[r, "\pi_{f}"] \arrow[dr, "\pi_{f''} \circ n_2"'] & \text{coker } f \arrow[d, dashed, "\phi_2"] \\ & \text{coker } f \end{tikzcd}$$

Thus, we have the following sequence:

$$\text{coker } f' \xrightarrow{\phi_1} \text{coker } f \xrightarrow{\phi_2} \text{coker } f''$$

We now want to show that $\text{im } \phi_1 = \ker \phi_2$.

To show: (d) $\text{im } \phi_1 \subseteq \ker \phi_2$.

(e) $\ker \phi_2 \subseteq \text{im } \phi_1$.

(d) Assume that $y' + \text{im } f' \in \text{coker } f'$ so that

$$\phi_1(y' + \text{im } f') = n_1(y') + \text{im } f \in \text{im } \phi_1.$$

Then,

$$\begin{aligned} \phi_2(n_1(y') + \text{im } f) &= ((\phi_2 \circ \pi_f) \circ n_1)(y') \\ &= (\pi_{f''} \circ n_2 \circ n_1)(y') = \text{im } f''. \end{aligned}$$

So, $n_1(y') + \text{im } f \in \ker \phi_2$ and $\text{im } \phi_1 \subseteq \ker \phi_2$.

(e) Assume that $y + \text{im } f \in \ker \phi_2$. Then,

$$\begin{aligned} \phi_2(y + \text{im } f) &= (\phi_2 \circ \pi_f)(y) \\ &= (\pi_{f''} \circ n_2)(y) \\ &= n_2(y) + \text{im } f'' = \text{im } f''. \end{aligned}$$

Therefore, $n_2(y) \in \text{im } f''$. So, there exists $x'' \in M''$ such that $f''(x'') = n_2(y)$. By exactness, the morphism $m_2 : M \to M''$ is surjective. So, there exists $x \in M$ such that $m_2(x) = x''$.

Next, observe that

$$n_2(y) = f''(x'') = (f'' \circ m_2)(x) = n_2(f(x)).$$

So, $n_2(y - f(x)) = 0$ and $y - f(x) = \ker n_2 = \text{im } n_1$ by exactness. Thus, there exists $y' \in N'$ such that $n_1(y') = y - f(x)$.

We now compute directly that

$$\begin{aligned} \phi_1(y' + \text{im } f') &= (\phi_1 \circ \pi_{f'})(y') \\ &= (\pi_f \circ n_1)(y') \\ &= \pi_f(y - f(x)) \\ &= y - f(x) + \text{im } f = y + \text{im } f. \end{aligned}$$

Thus, $\ker \phi_2 \subseteq \text{im } \phi_1$.

By combining parts (d) and (e), we deduce that the sequence

$$\text{coker } f' \xrightarrow{\phi_1} \text{coker } f \xrightarrow{\phi_2} \text{coker } f''$$

is exact.

3: Constructing the connecting morphism

So far, we have constructed two exact sequences

$$\ker f' \xrightarrow{m_1|_{\ker f'}} \ker f \xrightarrow{m_2|_{\ker f}} \ker f'' \qquad \text{coker } f' \xrightarrow{\phi_1} \text{coker } f \xrightarrow{\phi_2} \text{coker } f''$$

We will now construct a connecting morphism $\partial : \ker f'' \to \text{coker } f'$ and show that ∂ converts the above two exact sequences into one long exact sequence.

Assume that $x'' \in \ker f''$. Since m_2 is surjective, there exists $x \in M$ such that $m_2(x) = x''$. By commutativity of the diagram, we have

$$0 = (f'' \circ m_2)(x) = (n_2 \circ f)(x).$$

So, $f(x) \in \ker n_2 = \text{im } n_1$ by exactness. Thus, there exists $\epsilon_{x'',x} \in N'$ such that $n_1(\epsilon_{x'',x}) = f(x)$.

We claim that $\epsilon_{x'',x} \in N'$ is the unique element of N' satisfying $n_1(\epsilon_{x'',x}) = f(x)$. Assume that $y' \in N'$ is another element such that $n_1(y') = f(x)$. Then, $n_1(y') = n_1(\epsilon_{x'',x'})$ and since n_1 is injective, $y' = \epsilon_{x'',x}$. So, $\epsilon_{x'',x} \in N'$ is unique.

Let us pause the argument briefly so we can explain what is going on. Starting with $x'' \in \ker f''$, we were able to construct a unique $\epsilon_{x'',x} \in N'$. Notice that $\epsilon_{x'',x}$ depends on a choice of $x \in M$ in the preimage $m_2^{-1}(\{x''\})$. If we choose another $x_2 \in m_2^{-1}(\{x''\})$ such that $x_2 \neq x$ then we obtain another element $\epsilon_{x'',x_2} \in N'$, distinct from $\epsilon_{x'',x}$.

The crucial claim which allows us to define the connecting morphism ∂ is that in the cokernel coker f', $\epsilon_{x'',x} + \text{im } f' = \epsilon_{x'',x_2} + \text{im } f'$.

To show: (f) If $x, x_2 \in m_2^{-1}(\{x''\})$ then $\epsilon_{x'',x} + \text{im } f' = \epsilon_{x'',x_2} + \text{im } f'$.

(f) Assume that $x, x_2 \in m_2^{-1}(\{x''\})$ so that $m_2(x) = m_2(x_2) = x''$. By construction, the elements $\epsilon_{x'',x}, \epsilon_{x'',x_2} \in N'$ satisfy

$$n_1(\epsilon_{x'',x}) = f(x) \qquad \text{and} \qquad n_1(\epsilon_{x'',x_2}) = f(x_2)$$

Observe that $m_2(x - x_2) = 0$. So, $x - x_2 \in \ker m_2 = \text{im } m_1$ and there exists $\chi' \in M'$ such that

$$m_1(\chi') = x - x_2.$$

By applying f to both sides, we find that

$$\begin{aligned}(n_1 \circ f')(\chi') &= (f \circ m_1)(\chi') \\ &= f(x - x_2) = f(x) - f(x_2) \\ &= n_1(\epsilon_{x'',x}) - n_1(\epsilon_{x'',x_2}) \\ &= n_1(\epsilon_{x'',x} - \epsilon_{x'',x_2}).\end{aligned}$$

Since n_1 is injective, $f'(\chi') = \epsilon_{x'',x} - \epsilon_{x'',x_2}$. By applying the projection morphism $\pi_{f'}$, we deduce that

$$\text{im } f' = (\epsilon_{x'',x} - \epsilon_{x'',x_2}) + \text{im } f'$$

Consequently, $\epsilon_{x'',x} + \text{im } f' = \epsilon_{x'',x_2} + \text{im } f'$ in coker f'.

By part (f), the following map is well-defined.

$$\begin{array}{rccc} \partial : & \ker f'' & \to & \text{coker } f' \\ & x'' & \mapsto & \epsilon_{x'',x} + \text{im } f' \end{array}$$

where $x \in m_2^{-1}(\{x''\})$. By viewing $\mathscr{C}$ as a full subcategory of **R-Mod** for some unital ring R, we will now show that ∂ is a morphism in $\mathscr{C}$.

To show: (g) If $x_1'', x_2'' \in \ker f''$ then $\partial(x_1'' + x_2'') = \partial(x_1'') + \partial(x_2'')$.

To show: (h) If $\rho \in R$ and $x'' \in \ker f''$ then $\partial(\rho x'') = \rho\partial(x'')$.

(g) Assume that $x_1'', x_2'' \in M''$. Then, $\partial(x_1'') = \epsilon_{x_1'',x_1} + \text{im } f'$ and $\partial(x_2'') = \epsilon_{x_2'',x_2} + \text{im } f'$, where $n_1(\epsilon_{x_1'',x_1}) = f(x_1)$ and $n_1(\epsilon_{x_2'',x_2}) = f(x_2)$.

Now consider $x_1'' + x_2'' \in \ker f''$. Since $x_1 + x_2 \in m_2^{-1}(\{x_1'' + x_2''\})$, we can construct a unique element $\epsilon_{x_1''+x_2'',x_1+x_2} \in N'$ such that

$$n_1(\epsilon_{x_1''+x_2'',x_1+x_2}) = f(x_1 + x_2) = f(x_1) + f(x_2) = n_1(\epsilon_{x_1'',x_1} + \epsilon_{x_2'',x_2}).$$

Since n_1 is injective, $\epsilon_{x_1''+x_2'',x_1+x_2} = \epsilon_{x_1'',x_1} + \epsilon_{x_2'',x_2}$. Therefore,

$$\begin{aligned} \partial(x_1'' + x_2'') &= \epsilon_{x_1''+x_2'',x_1+x_2} + \text{im } f' \\ &= (\epsilon_{x_1'',x_1} + \text{im } f') + (\epsilon_{x_2'',x_2} + \text{im } f') \\ &= \partial(x_1'') + \partial(x_2''). \end{aligned}$$

(h) Assume that $\rho \in R$ and $x'' \in \ker f''$. Then, $\partial(x'') = \epsilon_{x'',x} + \text{im } f'$, where $n_1(\epsilon_{x'',x}) = f(x)$. Consider the element $\rho x'' \in \ker f''$. Since $\rho x \in m_2^{-1}(\{\rho x''\})$, we can construct the unique element $\epsilon_{\rho x'',\rho x} \in N'$ such that

$$n_1(\epsilon_{\rho x'',\rho x}) = f(\rho x) = \rho f(x) = n_1(\rho\epsilon_{x'',x}).$$

Since n_1 is injective, we deduce that $\epsilon_{\rho x'',\rho x} = \rho\epsilon_{x'',x}$. Consequently,

$$\begin{aligned} \partial(\rho x'') &= \epsilon_{\rho x'',\rho x} + \text{im } f' \\ &= \rho\epsilon_{x'',x} + \text{im } f' \\ &= \rho(\epsilon_{x'',x} + \text{im } f') = \rho\partial(x''). \end{aligned}$$

By combining parts (g) and (h) of the proof, we deduce that ∂ is a well-defined morphism in $\mathscr{C}$.

4. The connecting morphism satisfies the exactness conditions

Next, we will demonstrate that im $m_2|_{\ker f} = \ker \partial$ and im $\partial = \ker \phi_1$.

To show: (i) im $m_2|_{\ker f} \subseteq \ker \partial$.

(j) $\ker \partial \subseteq$ im $m_2|_{\ker f}$.

(k) im $\partial \subseteq \ker \phi_1$.

(l) $\ker \phi_1 \subseteq$ im ∂.

(i) Assume that $x'' \in$ im $m_2|_{\ker f}$. Then, there exists $x \in \ker f$ such that $m_2(x) = x''$. Now, $\partial(x'') = \epsilon_{x'',x} +$ im f', where $\epsilon_{x'',x} \in N'$ satisfies $n_1(\epsilon_{x'',x}) = f(x) = 0$. Since n_1 is injective, $\epsilon_{x'',x} = 0$ in N' and $\partial(x'') =$ im f'. Therefore, im $m_2|_{\ker f} \subseteq \ker \partial$.

(j) Assume that $x'' \in \ker \partial \subseteq \ker f''$. Then,

$$\partial(x'') = \epsilon_{x'',x} + \text{im } f' = \text{im } f'$$

for some $x \in m_2^{-1}(\{x''\})$. Then, $\epsilon_{x'',x} \in$ im f' and there exists $x' \in M'$ such that $f'(x') = \epsilon_{x'',x}$. Now,

$$\begin{aligned} f(x) &= n_1(\epsilon_{x'',x}) \\ &= (n_1 \circ f')(x') \\ &= (f \circ m_1)(x'). \end{aligned}$$

Therefore, $x - m_1(x') \in \ker f$ and

$$\begin{aligned} m_2(x - m_1(x')) &= m_2(x) - (m_2 \circ m_1)(x') \\ &= m_2(x) = x''. \end{aligned}$$

Therefore, $x'' \in$ im $m_2|_{\ker f}$ and $\ker \partial \subseteq$ im $m_2|_{\ker f}$.

(k) Assume that $n' +$ im $f' \in$ im ∂. Then, there exists $x'' \in \ker f''$ such that $\partial(x'') = n' +$ im f'. If we apply ϕ_1, we find that

$$\phi_1(n' + \text{im } f') = (\phi_1 \circ \pi_{f'})(n') = (\pi_f \circ n_1)(n').$$

Since $\partial(x'') = n' + \text{im } f'$, n' must satisfy $n_1(n') = f(x)$ for some $x \in m_2^{-1}(\{x''\})$. So,

$$\phi_1(n' + \text{im } f') = (\pi_f \circ n_1)(n') = (\pi_f \circ f)(x) = \text{im } f.$$

Therefore, $n' + \text{im } f' \in \ker \phi_1$ and $\text{im } \partial \subseteq \ker \phi_1$.

(l) Conversely to part (k) of the proof, assume that $n' + \text{im } f' \in \ker \phi_1$. Then,

$$\begin{aligned} \phi_1(n' + \text{im } f') &= n_1(n') + \text{im } f \\ &= \text{im } f. \end{aligned}$$

Therefore, $n_1(n') \in \text{im } f$ and there exists $x \in M$ such that $f(x) = n_1(n')$. Now, $f(x) \in \text{im } n_1 = \ker n_2$. Taking advantage of this, we have

$$(n_2 \circ f)(x) = (f'' \circ m_2)(x) = 0.$$

Therefore, $m_2(x) \in \ker f''$. Now observe that $\partial(m_2(x)) = \epsilon_{m_2(x),x} + \text{im } f'$, where $\epsilon_{m_2(x),x} \in N'$ satisfies

$$n_1(\epsilon_{m_2(x),x}) = f(x) = n_1(n').$$

Since n_1 is injective, $\epsilon_{m_2(x),x} = n'$ and

$$\partial(m_2(x)) = n' + \text{im } f'.$$

So, $n' + \text{im } f' \in \text{im } \partial$ and $\ker \phi_1 \subseteq \text{im } \partial$.

By combining parts (i), (j), (k) and (l), we deduce that the following sequence is exact:

$$\ker f' \xrightarrow{m_1|_{\ker f'}} \ker f \xrightarrow{m_2|_{\ker f}} \ker f'' \xrightarrow{\partial} \text{coker } f' \xrightarrow{\phi_1} \text{coker } f \xrightarrow{\phi_2} \text{coker } f''$$

5. Induced map between kernels is injective

Assume that $m_1 : M' \to M$ is injective. We want to show that $m_1|_{\ker f'}$ is injective. Assume that $x_1', x_2' \in \ker f'$ such that $m_1|_{\ker f'}(x_1') = m_1|_{\ker f'}(x_2')$. Then, $m_1(x_1') = m_1(x_2')$ and since m_1 is injective, $x_1' = x_2'$. Hence, the induced morphism $m_1|_{\ker f'} : \ker f' \to \ker f$ is injective.

6. Induced map between cokernels is surjective

Assume that $n_2 : N \to N''$ is surjective. We want to show that $\phi_2 : \text{coker } f \to \text{coker } f''$ is surjective. Assume that $y'' + \text{im } f'' \in \text{coker } f''$. Then, $y'' \in N''$ and there exists $y \in N$ such that $n_2(y) = y''$.

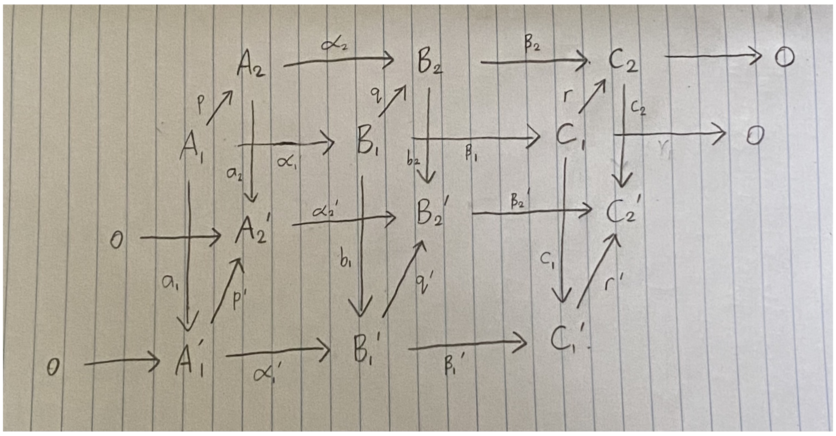
We compute directly that

$$\begin{aligned}\phi_2(y + \text{im } f) &= (\phi_2 \circ \pi_f)(y)\\ &= (\pi_{f''} \circ n_2)(y)\\ &= n_2(y) + \text{im } f'' = y'' + \text{im } f''.\end{aligned}$$

Therefore, the induced morphism $\phi_2 : \text{coker } f \to \text{coker } f''$ is surjective as required.

7. Naturality of the connecting morphism

Now suppose that we have the following commutative diagram in $\mathscr{C}$:



where the top and bottom rows are exact. By the previous construction, we have two connecting morphisms $\partial_1 : \ker c_1 \to \text{coker } a_1$ and $\partial_2 : \ker c_2 \to \text{coker } a_2$.

We can also use the previous constructions to show that the restriction $r|_{\ker c_1}$ is a morphism from $\ker c_1$ to $\ker c_2$ and that there exists a unique morphism $\psi : \text{coker } a_1 \to \text{coker } a_2$ such that the following diagram commutes:

$$\begin{tikzcd} A_1' \arrow[r, "\pi_{a_1}"] \arrow[dr, "\pi_{a_2} \circ p'"'] & \text{coker } a_1 \arrow[d, dashed, "\psi"] \\ & \text{coker } a_2 \end{tikzcd}$$

To obtain naturality of the connecting morphism, it suffices to show that $\partial_2 \circ r|_{\ker c_1} = \psi \circ \partial_1$.

To show: (m) $\partial_2 \circ r|_{\ker c_1} = \psi \circ \partial_1$.

(m) Assume that $c \in \ker c_1$. Then,

$$\partial_2(r|_{\ker c_1}(c)) = \epsilon_{r(c),u_2} + \text{im } a_2$$

where $u_2 \in \beta_2^{-1}(\{r(c)\})$ and $\alpha_2'(\epsilon_{r(c),u_2}) = b_2(u_2)$.

On the other hand, we have

$$\psi(\partial_1(c)) = \psi(\epsilon_{c,u_1} + \text{im } a_1) = p'(\epsilon_{c,u_1}) + \text{im } a_2$$

where $u_1 \in \beta_1^{-1}(\{c\})$ and $\alpha_1'(\epsilon_{c,u_1}) = b_1(u_1)$. Now, it suffices to show that $p'(\epsilon_{c,u_1}) - \epsilon_{r(c),u_2} \in \text{im } a_2$.

To show: (ma) $p'(\epsilon_{c,u_1}) - \epsilon_{r(c),u_2} \in \text{im } a_2$.

(ma) We begin by applying α_2'. By commutativity of the diagram, we have

$$\begin{aligned} \alpha_2'(p'(\epsilon_{c,u_1}) - \epsilon_{r(c),u_2}) &= \alpha_2'(p'(\epsilon_{c,u_1})) - \alpha_2'(\epsilon_{r(c),u_2}) \\ &= (q' \circ \alpha_1')(\epsilon_{c,u_1}) - \alpha_2'(\epsilon_{r(c),u_2}) \\ &= (q' \circ \alpha_1')(\epsilon_{c,u_1}) - b_2(u_2) \\ &= (q' \circ b_1)(u_1) - b_2(u_2) \\ &= (b_2 \circ q)(u_1) - b_2(u_2) = b_2(q(u_1) - u_2). \end{aligned}$$

We claim that $q(u_1) - u_2 \in \ker \beta_2$. Indeed, we compute directly that

$$\begin{aligned} \beta_2(q(u_1) - u_2) &= (\beta_2 \circ q)(u_1) - \beta_2(u_2) \\ &= (r \circ \beta_1)(u_1) - \beta_2(u_2) \\ &= r(c) - r(c) = 0. \end{aligned}$$

By exactness, $\ker \beta_2 = \text{im } \alpha_2$. So, there exists $z \in A_2$ such that $\alpha_2(z) = q(u_1) - u_2$. Finally, we claim that $a_2(z) = p'(\epsilon_{c,u_1}) - \epsilon_{r(c),u_2}$.

By commutativity of the diagram, we have $(b_2 \circ \alpha_2)(z) = (\alpha'_2 \circ a_2)(z)$. But,

$$
\begin{aligned}
b_2(\alpha_2(z)) &= b_2(q(u_1) - u_2) \\
&= \alpha'_2(p'(\epsilon_{c,u_1}) - \epsilon_{r(c),u_2}).
\end{aligned}
$$

By exactness, α'_2 is injective. Since $\alpha'_2(p'(\epsilon_{c,u_1}) - \epsilon_{r(c),u_2}) = \alpha'_2(a_2(z))$, $a_2(z) = p'(\epsilon_{c,u_1}) - \epsilon_{r(c),u_2}$.

Therefore, $p'(\epsilon_{c,u_1}) - \epsilon_{r(c),u_2} \in \text{im } a_2$.

(m) Part (ma) demonstrates that $\partial_2(r|_{\ker c_1}(c)) = \psi(\partial_1(c))$. Since $c \in \ker c_1$ was arbitrary, we must have $\partial_2 \circ r|_{\ker c_1} = \psi \circ \partial_1$. This finally completes the proof of the Snake lemma. $\square$