Cavity Radiation and the Ultraviolet Catastrophy

Reyleigh-Jeans: $\rho_T(\nu) = \frac{8\pi\nu k_B T}{c^3}d\nu$

Stefans Law: $R_T = \sigma T^4$, $\sigma = \frac{2\pi^5 k^4}{15c^2 h^3}$,

Wien's Displacement: $\nu_{\max} \propto T$

RJ is good fit at low frequencies, but assumption of equipartition is wrong, total energy is constant with atoms taking discrete energies in a Boltzmann distribution.

$\varepsilon = kT$ and $P(\varepsilon) = \frac{e^{\frac{\varepsilon}{k_B T}}}{k_B T}$

$\Delta\varepsilon = h\nu$ so $\bar{\varepsilon} = \frac{h\nu}{e^{\frac{h\nu}{k_B T}}-1}$

Plank's Radiation Law:

$\rho_T(\nu) = \frac{8\pi\nu^3}{c^3}\frac{h\nu}{e^{\frac{h\nu}{k_B T}}-1}$

Photon energy: $E = h\nu$, $\hbar = \frac{h}{2\pi}$ Fine Structure constant: $\alpha = \frac{e^2}{4\pi\epsilon_0 \hbar c}$, Compton wavelength $\lambda_C = \frac{h}{m_e c}$

Newtonian assumes time is the same in all frames for all observers, special relativity assumes the speed of light is the same for all observers and frames. $\gamma = \frac{1}{\sqrt{1-\beta}}$ and $\beta = \frac{V}{c}$

Invariant interval: $\Delta s = c^2\Delta t^2 - \Delta x^2 > 0 \implies$ time like, $< 0 \implies$ space like, $= 0 \implies$ light like.

Four vector: $(c\Delta t, \Delta x, \Delta y, \Delta z) = (c\Delta t, \Delta\mathbf{r})$

Inner Product: $A \cdot B = A_0 B_0 - \mathbf{A} \cdot \mathbf{B} \implies (\Delta s)^2 = c^2\Delta t^2 - \Delta x^2$, the inner product of the space-time four vector is the invariant interval.

Momentum, $\mathbf{p} = (\gamma mc, \gamma m\mathbf{u})$ and $p = ((\frac{E}{c}), \mathbf{p})$ where $E = \gamma mc^2 = mc^2 + KE$

de Broglie: $E = h\nu$ so, $p = \frac{h}{\lambda}$ Phase velocity: $\omega = \frac{dx_n}{dt} = \frac{\nu}{\kappa}$ Group velocity: $g = \frac{d\nu}{d\kappa}$
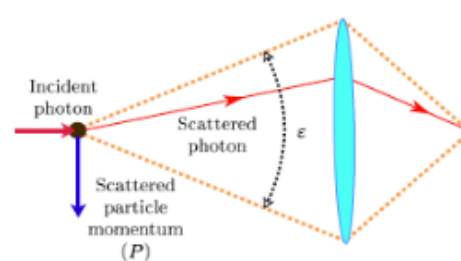
Energy-momentum Diagram:

Mass shell: $E^2 + |p|^2 c^2 = m^2 c^4$

Conservation of Energy: $\sum E_i = \sum E_f$

Conservation of Momentum: $\sum p_i = \sum p_f$

Heisenberg's Microscope: To consider qualitatively the uncertainty principle, we will look at how we measure things, and how the measurement affects atomic particles.

Habibi et al. 2016, J. Opt 18, 084004

The light we observe has interacted off the incident proton, and has some uncertainty in x and movement (momentum).

Diffraction limit: $\Delta x = \frac{\lambda}{\sin(\epsilon/2)}$ (this is our uncertainty in x)

There is also the uncertainty in momentum: $\Delta p_x = \frac{h}{\lambda}[2\sin(\epsilon/2)]$, so the uncertainty product is constant: $\Delta p_x \Delta x = 2h$

Philosophy of Quantum Physics:

- Quantum is probabilistic in nature.

- There is no objective reality, making a measurement is the same thing as an interaction between two unobserved particles. These interactions collapse the superimposed wave forms, to the single outcome.

- Because of this, there is an uncertainty introduced.

Schrodinger's Equation

$i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r}, t) = \hat{H}\Psi(\mathbf{r}, t)$

This is just application of the Hamiltonian (energy) operator to the wave function of the particle.

$\bar{H} = \frac{\mathbf{p}^2}{2m} + V = \frac{\hbar^2}{2m}(\frac{\partial}{\partial\mathbf{r}})^2 + V$

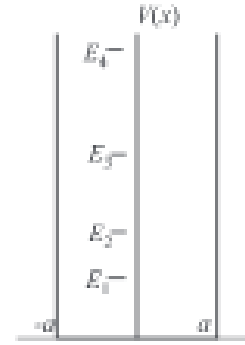Splits into $R(\mathbf{r})$ and $T(t)$, and the time solution: $T(t) = \frac{i\hbar}{E}e^{-iEt/\hbar}$

$-\frac{\hbar^2}{2m}\frac{1}{\psi}\frac{d^2\psi}{d\mathbf{r}^2} + V = E$

This is linear, meaning importantly that superposition of solutions and separation of variables are possible.

Normalisation $\int P d\tau = 1$
$\implies \int \Psi^*\Psi d\tau = 1$

Infinite Square Well:

Boundary conditions, 1. Must be continuous and single valued everywhere. $\int\Psi d\tau = 1$ 3. $\nabla\Psi$ continuous where V is continuous.

Try a sinusoidal solution: $\Psi(x) = A\cos kx + B\cos kx$, where $k = (\frac{2mE}{\hbar})$ but where $V(x) = \infty$ the particle cannot exist, is $\Psi(\pm a) = A\cos ka + B\sin ka = A\cos ka + B\sin(-ka) = 0$ so we have solutions for $A = 0$ and $B = 0$ consisting of only odd, or only even solutions in n. These solutions give the general case:

$$\sum_{n=1,3,5\ldots}^{\infty} A_n\cos\left(\frac{n\pi}{2a}x\right) + \sum_{n=2,4,6\ldots}^{\infty} B_n\cos\left(\frac{n\pi}{2a}x\right)$$

Which scales by factor: $A_n = B_n = a^{-1/2}$ and has associated with energy eigenvalues, $E_n = \frac{\hbar^2\pi^2 n^2}{8ma^2}$.

Finite square well, same solution for inside well, but the outside is the decay equation: $\Psi(x) = Ce^{\pm\kappa x}$, which results in two sets of equations, the one relating to continuous case and the continuous gradient case.

So the solutions become: $k\tan ka = \kappa = -k\cot ka$ where $k = \sqrt{\frac{2mE}{\hbar^2}}$ and $\kappa = \sqrt{\frac{2m}{\hbar^2}(V_0 - E)}$, solved graphic/numerically.

Wave function sketching rules come from this solution, k is proportional to the well depth and is related to energy, so shallow well high amplitude, low frequency. The height of the sides determines decay rate κ. The number of nodes is $n-1$, parity of this is parity of the function. **Barrier tunnelling**, is the ability for particles to pass through an area of potential of higher energy than they possess. Related to the decay in a finite square well. A barrier of appreciable width b has tunneling probablility, $\frac{|F|^2}{|A|^2} = \frac{16k^2\kappa^2}{k^2+\kappa^2}e^{-2\kappa b} = \frac{16E(V_0-E)}{V_0^2}e^{-2\kappa b}$

which is a rate, $\Gamma = \frac{n}{m}\hbar k\frac{|F|^2}{|A|^2}$.

Example: for β and α decay, there is

$P_{\text{trans}} = \exp\left[\int_{r_{\text{nuc}}}^{r_{\text{end}}}\frac{-2m}{\hbar^2}(V(r)-E)\right]$

$\Gamma = \frac{n}{m}\hbar k\frac{16E(V_0-E)}{V_0^2}e^{-2\kappa b}$
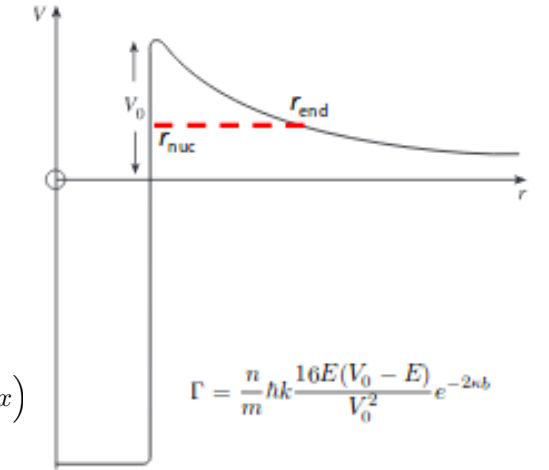
Operators: $\hat{A}\Psi = a\Psi$

Commutator: $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$

Momentum: $\hat{p}_z = -i\hbar\frac{\partial}{\partial z}$, Energy: $\hat{T}_z = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2}$ Angular Momentum: $\hat{L}_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$, Potential Energy: $\hat{V}_z = V(z)$, Hamiltonian: $\hat{H}_z = \hat{T}_z + \hat{V}_z$

Complex Conjugates: $\hat{Q}^* f^* = \left(\hat{Q}f\right)^*$ Hermitian operators: $\int f\hat{Q}g d\tau = \int g\hat{Q}^* f d\tau$ with real eigenvalues, and orthonormal eigenvectors, $\int \phi_n^*\phi_m d\tau = \delta_{nm}$ scaled so that: $\int |\phi_n|^2 d\tau = 1$

Coefficients: $a_n = \int \phi_n^* \psi d\tau$

1. A quantum system is described by a wave function (and nothing else)
2. Measurements return one of the eigenvalues of an operator associated with that measurement.
3. The operators representing dynamical quantities are exactly analogous to the dynamical variables of classical mechanics.
4. The probability of a particular outcome equals the squared modulus of the overlap between the wave function and the eigenfunction associated with the outcome.
5. As the result of a measurement, the system is left in the corresponding eigenstate associated with the measurement outcome.

Uncertainty:

Observations are compatible if they are governed by the same set of eigenfunctions. $\hat{Q}' = (\hat{Q} - \langle \hat{Q} \rangle)$ so the deviation compared to the expected result is defined. This means that the standard deviation is the expectation value of this. $\Delta q^2 = \int \psi^* \hat{Q}' \hat{Q}' \psi d\tau$ so the deviation product: $\Delta q \Delta r \geq \frac{1}{2} \left| \int \psi^* \left[\hat{Q}, \hat{R} \right] \psi d\tau \right|^2$

Degeneracy:

If $\hat{Q}\phi_n = p_n\phi_n$ then $q_n \neq q_m$ for all $n \neq m$. Is the requirement that must be modified. If $\hat{Q}$ acting on ϕ_1 and ϕ_2 are degenerate in the eigenvalue q and commute with $\hat{R}$. Then there is some linear combination of these eigenvectors that equate to the eigenvector of $\hat{R}$, ϕ_n'. ie: $\hat{R}\phi_n' = r_n\phi_n' = r_n(C_1\phi_1 + C_2\phi_2)$

Ladder Operators:

$(\hat{P}^2 + \hat{X}^2) \overset{?}{=} (\hat{P} + i\hat{X})(\hat{P} - i\hat{X})$ so let $\hat{A} = (\hat{P} - i\hat{X})$ and $\hat{A}^\dagger = (\hat{P} + i\hat{X})$ so $\hat{H} = \frac{1}{2}(\hat{A}^\dagger \hat{A} + 1) = \frac{1}{2}(\hat{A}\hat{A}^\dagger - 1)$ So the commutators, $\left[\hat{A}, \hat{H}\right] = \hat{A}\frac{1}{2}(\hat{A}^\dagger \hat{A} + 1) - \frac{1}{2}(\hat{A}\hat{A}^\dagger - 1)\hat{A} = \hat{A}$ So when the commutator operates on the wave function, $\hat{H}\hat{A}^\dagger \psi = (E_n - 1)\psi$ So this is the lowering operator and $\hat{H}\hat{A}\psi_0 = \frac{1}{2}\psi_0$
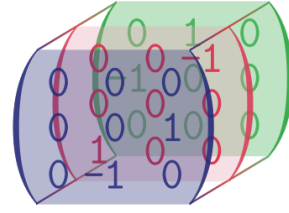
Angular Momentum:

$\left[\hat{L}_x, \hat{L}_y\right] = \hat{Y}\hat{P}(\hat{P}_z\hat{Z} - \hat{Z}\hat{P}_z) + \hat{X}\hat{P}_y(\hat{Z}\hat{P}_z - \hat{P}_z\hat{Z}) = i\hbar\hat{L}_z$, or in general: $\left[\hat{L}_i, \hat{L}_j\right] = i\hbar\varepsilon_{ijk}\hat{L}_k$ where levi cevita: $\epsilon_{xyz} = 1$

$\epsilon_{xxz} = 0$
$\epsilon_{yxz} = -1$

$\epsilon_{ijk} =$

$\hat{L}_i$ and $\hat{L}_j$, $\hat{P}_j$ or $\hat{X}_j$ are incompatible, but $\hat{L}^2$ and $\hat{L}_i$ or $\hat{H}$ are compatible.

Spherical Coordinates:

$\frac{\partial}{\partial x} = \sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial \theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial \phi}$
$\frac{\partial}{\partial y} = \sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial \theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial \phi}$
$\frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$

So the grad operator becomes: $\boldsymbol{\nabla} = r_0\frac{\partial}{\partial r} + \frac{1}{r}\theta_0\frac{\partial}{\partial \theta} + \frac{1}{r\sin\theta}\phi_0\frac{\partial}{\partial \phi}$
$\hat{L} = -i\hbar \boldsymbol{r} \times \boldsymbol{\nabla} = -i\hbar\left(\phi_0\frac{\partial}{\partial \theta} - \frac{1}{\sin\theta}\theta_0\frac{\partial}{\partial \phi}\right)$
$\hat{L}_z = -i\hbar\frac{\partial}{\partial \phi}$
$\hat{L}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial \phi^2}\right]$

Polar Schrodinger:

$$-\frac{\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial\psi}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial \phi^2}\right] + V(r)\psi = E\psi$$

Which has solutions:

$Y_{lm}(\theta, \phi) = (-1)^m\left[\frac{(2l+1)(l-|m|)!}{4\phi(l+|m|)!}\right]^{\frac{1}{2}}e^{im\phi}$

Which normalised become:

$Y_{00} = \left(\frac{1}{4\pi}\right)^{1/2}$

$Y_{10} = \left(\frac{3}{4\pi}\right)^{1/2}\cos\theta$

$Y_{1\pm1} = \mp\left(\frac{3}{4\pi}\right)^{1/2}\sin\theta\, e^{\pm i\phi}$
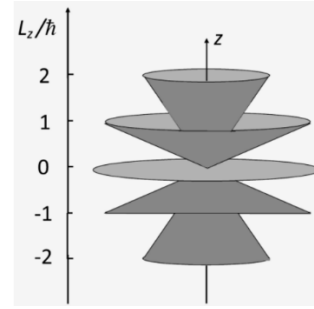
$Y_{20} = \left(\frac{5}{16\pi}\right)^{1/2}(3\cos^2\theta - 1)$

$Y_{2\pm1} = \mp\left(\frac{15}{8\pi}\right)^{1/2}\cos\theta\sin\theta e^{\pm i\phi}$

$Y_{2\pm2} = \left(\frac{15}{32\pi}\right)^{1/2}\sin^2\theta e^{\pm 2i\phi}$

l and m are important because they show the eigenvalues of $\hat{L}^2$ and $\hat{L}_z$ respectively.

Angular Momentum as Vectors:

The Bohr Magneton $\mu_B \equiv \frac{\mu\hbar}{2m_e}$ yielding a $\Delta E = \mu B m\hbar$

Ladder operators and Angular Moment:

$\hat{L}_+ = \hat{L}_x + i\hat{L}_y$ and $\hat{L}_- = \hat{L}_x - i\hat{L}_y$ and their commutator, $\left[\hat{L}_+, \hat{L}_-\right] = 2\hbar\hat{L}_z$, and they don't commute with $\hat{L}_z$.

Using this operator: $\hat{L}_z\hat{L}_+\phi = (m+1)\hbar\phi$ but the raising and lowering operators commute with $\hat{L}^2$ so the eigenfunction/total angular momentum is unchanged.

Semi-classical μ_B:

$\mu_l = \sqrt{l(l+1)}g_l\mu_b$ and $\mu_{l_z} = -g_l\mu_b m_l$

Larmor precession: $\omega = \frac{g_l\mu_b}{\hbar}B$ **Particle Spin:** $S = \sqrt{s(s+1)}\hbar = \frac{\sqrt{3}\hbar}{2}$, so $S_z = m_s\hbar$, where $m_s = \pm\frac{1}{2}$, so the moments: $\mu_s = -\frac{e}{m}S$ and $\mu_{S_z} = -2m_s\mu_b$ These new operators commute with the old angular momentum operators. Hence they introduce new degeneracies in to the equations.

Matrix Eigenvalue equation:

$$\begin{pmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = q\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

The Hermitian requirement means, $Q_{mn} = Q_{nm}^*$ and $[Q] = [Q^\dagger]$.

$$\begin{vmatrix} Q_{11} - q & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} - q & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} - q \end{vmatrix} = 0$$

Are the solutions.

Pauli Spin Matrices:

$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ $\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

So the spin operator becomes, $\hat{S}_x = \frac{1}{2}\hbar\sigma_x$, etc. and the angular momentum operators become, $\hat{L}^2 = 2\hbar^2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\hat{L}_z = \hbar\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, the raising operators, $\hat{L}_+ = \begin{bmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{bmatrix}$ and $\hat{L}_- = \hbar\begin{bmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix}$

Dirac Notation:

$Q_{12} = \int \psi_1^*\hat{Q}\psi_2 d\tau$ **Operator** $\hat{Q}\phi_n = q_n\phi_n$

$Q_{12} = \langle 1|\hat{Q}|2\rangle$ **Dirac** $\hat{Q}|n\rangle = q|n\rangle$

Spin Orbit Interaction:

By considering the Biot-Savart Law, gives a potential difference due to the internal **B** field. $U_{LS} = -\mu_s \cdot B = \frac{1}{2m^2c^2r}\frac{dV(r)}{dr}S \cdot L$ The interaction means they are not independent, resulting a new quantum number $\hat{J} = \sqrt{j(j+1)}\hbar$, and $\hat{J}_z = m_j\hbar$, which in the z direction is, $\hat{J}_z = m_l + m_s$. This gives a degeneracy the energies that sum to the total J,

$$|j, m_j\rangle = \sum_{m_l}\sum_{m_s}C_{j,m_j,m_l,m_s}|m_l\rangle|m_s\rangle$$

The largest value of this is $\hat{J}_z = \hat{L}_z + \hat{S}_z$, ie $j = l \pm \frac{1}{2}$. and the separations, $\langle E \rangle = \frac{\hbar^2}{4m^2c^2r}\left(\frac{1}{r}\frac{dV(r)}{dr}\right)^*$

Zeeman Effect:

The spin-orbit Hamiltonian, $\hat{H}' = \frac{1}{4m_e^2c^2r}\frac{\partial V}{\partial r} + \frac{\mu_b}{\hbar}B_0(\hat{L}_z + 2\hat{S}_z)$ in no external field, this becomes simply, $(\hat{J}^2 - \hat{L}^2 - \hat{S}^2) = [j(j+1) - l(l+1) - s(s+1)]\hbar^2$. When the magnetic field is much larger, the energy shifts become, $\delta E = \mu_b B(m_l + 2m_s)$

Perturbation theory:

To the first order: $\hat{H} = (\hat{H}_0 + \lambda\hat{H}) \implies (\hat{H}_0 + \lambda\hat{H})\psi = E\psi$ The eigenfunction's are then $\psi_n = \phi_{0n} + \lambda\phi_{1n} + \lambda^2\phi_{2n} + \ldots$ and the energy values: $E_n = E_{0n} + \lambda E_{1n} + \lambda^2 E_{2n} + \ldots$

So the first order perturbation: $\hat{H}'\phi_{0n} + \hat{H}_0\phi_{1n} = E_{0n}\phi_{1n} + E_{1n}\phi_{0n}$ so the n'th order correction is the sum of the unperturbed system and the n'th order perturbation, then the energy is $E\uparrow$ and $\phi\downarrow$.

$\lambda E_n^{(1)} = \langle \lambda H'\rangle = \langle \phi_n|\lambda\hat{H}'|\phi_n\rangle$

Second order correction: $E_n^{(2)} = \sum_{k\neq n}\frac{|\langle\phi_k|\lambda\hat{H}'|\phi_n\rangle|^2}{E_{0n}-E_{0k}}$ This second order shift will result in a spreading out of these levels.

$\psi_n = \phi_n + \sum_{k\neq n}\frac{\langle\phi_k|\hat{H}'|\psi_n\rangle}{E_{0n}-E_{0k}}\phi_k$

Doesn't work for degeneracies. $E - E = 0$

Sudden Approximation:

$\hat{H} = \begin{cases}\hat{H}_1 & ,t<0 \\ \hat{H}_2 & ,t\geq 0\end{cases}$

Fermi's golden rule:

for transition rates between states m and n, $\Gamma_{mn} = \frac{2\pi}{\hbar}\delta(E_n - E_m)|\hat{H}''|^2$ the delta function is cancelled in the integration of over a range of energies,

$\int\Gamma_{mn}\rho(E_n)dE_n = \frac{2\pi}{\hbar}\rho(E_m)|\hat{H}''|^2$

Ehrenfest Theorem:

$\frac{\partial\langle\hat{Q}\rangle}{\partial t} = \frac{i}{\hbar}\left\langle[\hat{H},\hat{Q}]\right\rangle + \left\langle\frac{\partial\hat{Q}}{\partial t}\right\rangle$ Ehrenfest states that the rate of change of the expectation value of a physical quantity has two factors, the commutator with the Hamiltonian, and the expectation value of the time derivative.

EG. $\frac{\partial\langle\hat{X}\rangle}{\partial t} = \frac{\langle\hat{P}\rangle}{m}$

Cross Section:

Differential cross section is the fraction of

incident particles scattered per unit solid angle, $\frac{d\sigma}{d\Omega}$

Classically, for a hard sphere: $\frac{d\sigma(\theta)}{d\Omega} = \frac{R^2}{4}$ and $\sigma = \int\frac{d\sigma(\theta)}{d\Omega}d\Omega = \pi R^2$

In quantum, $\Gamma = \frac{\hbar k}{mL}$ and the particle probability, $\Gamma(x_2) - \Gamma(x_1) = \frac{\partial P}{\partial t} = \frac{d}{dt}\left(\int_{x_1}^{x_2}\psi^*\psi dx\right) = \int_{x_1}^{x_2}\left(\psi^*\frac{d\psi}{dt} + \frac{d\psi^*}{dt}\psi\right)dt$

So the flux is, $\Gamma(x) = -\frac{i\hbar}{2m}\left(\psi^*\frac{\partial\psi}{\partial t} - \frac{\partial\psi^*}{\partial t}\psi\right)$

Born Approximation:

The scatterer is a perturbation to the incident wave is an elastic scattering example, with transition rate, $W = \frac{2\pi}{\hbar^2}|V_{k_1,k_0}|^2 g(\omega)$, where $V_{k_1 k_0} = \int\psi_f^* V(r)\psi_i d\tau$ and $dg = $

$\frac{mk[Volume]}{8\pi^3\hbar}d\Omega$ describes the density of states.

Differential cross section for a free particle and a radial perturbing potential, $\frac{d\sigma}{d\Omega}(\theta,\phi) = \left(\frac{m}{2\pi\hbar^2}\right)^2\left|\int V(r)e^{-iK\cdot r}d\tau\right|^2$

Partial Wave:

Now we care about the phase shifts induced by the scatterer and the angular momentum of the particle. $dN = \frac{\hbar k}{mV}|f(\theta)|^2 d\Omega \implies \frac{d\sigma}{d\Omega} = |f(\theta)|^2$ so the total cross section: $\sigma = 2\pi\int_0^\pi|f(\theta)|^2\sin\theta d\Omega$. Which for our radial electric potential is, $\frac{4\pi}{k^2}\sum(2l+1)\sin^2\delta_l$ over all l, however δ_l is hard to get, except for when $l = 0$.

Examples:

Question 1 :

(a) (i) Explain in your own words the origin of the Heisenberg uncertainty principle.

The uncertainty principle describes how we perceive our surroundings through interaction. One example is Heisenberg's microscope, which observes a photon after it is deflected off an atomic particle. There are a range of initial particle positions and momenta that will yield us observing a photon. To observe the universe without affecting it would be impractical and not useful.

(ii) Explain why a localised free particle will in general spread out with time.

A localised free particle will spread out with time as the uncertainty in it's position grows with respect to it's uncertainty of momentum. If we have a particle with some momentum $\rho_0 \pm \Delta\rho$ and it is localised to some position $x_0 \pm \Delta x$ and time zero, then we expect it to be at some time t_0 there will be an uncertainty in position $\Delta x + \Delta\rho \times t_0$, hence we expect the particle wave function to spread-out.

(iii) Given a wavefunction $\psi(x) = cx^{-1}e^{-\alpha x^2/2}$, where c and α are constants, calculate Δx.

The standard deviation is the uncertainty in x. ie,

$\sigma = \Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$

$x^2 = \int\psi^*(x)x^2\psi(x)dx = \int c^2 x^{-2}e^{-\alpha x^2}x^2 dx = \frac{c^2}{\alpha}$

$\langle x\rangle^2 = \left(\int\psi^2 x dt\right)^2 = \int c^2 x^{-1}e^{-\alpha x^2}dx = 0$

$\therefore \Delta x = c\alpha^{-1/2}$

(iv) Suppose the wave function of a particle is given by

$$\psi(x) = Ae^{ikx} + Be^{-ikx}.$$

What flux does this represent?

The components of the wavefunction Ae^{ikx} and Be^{-ikx} are associated with standing waves with constant amplitude, hence there is no flux caused by total wave function as there is no velocity.

(v) The relation between the wavelength and the frequency in a waveguide is given by $\lambda = c(\nu^2 - \nu_0^2)^{-1/2}$. What is the group velocity of such waves?

The group velocity is: $g = \frac{d\nu}{d\kappa}$

So from Q: $\frac{\lambda}{c} = (\nu^2 - \nu_0^2)^{-1/2}$ and $\kappa = \frac{1}{\lambda}$

$\therefore \frac{\kappa^2}{c^2} = \nu^2 - \nu_0^2 \therefore \nu = \frac{\sqrt{\kappa^2 - \nu^2 c^2}}{c} = \frac{\sqrt{\kappa^2 - \kappa_0^2}}{c}$

$\therefore g = \frac{2\kappa c}{\sqrt{\kappa^2 - \nu^2 c^2}}$

(b) (i) Show that the $n = 1$ wavefunction for a harmonic oscillator given by

$$\psi(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\sqrt{\frac{2m\omega}{\hbar}}x\,e^{-m\omega x^2/2\hbar}$$

satisfies the Schrödinger wave equation and obtain the energy eigenvalue.

(ii) Show that this wavefunction has definite parity. Is it odd or even?

(iii) Show that an eigenvalue expression does not exist for the momentum operator.

i. from the schrodinger equation, $-\frac{\hbar^2}{2m}\frac{1}{\psi}\frac{d^2\psi}{dr^2} + V = E$

Solve and find E as a function of constants.

(a) A semi-classical analysis uses the 'old quantum mechanics' of the Wilson-Sommerfeld model, where

$$\oint p_q dq = nh$$

for each classical degree of freedom q and associated momentum p_q.

(i) Show that, for a classical electron with $E < V_0$, the model gives $E_n = h^2 n^2/(8mr_0^2)$, where m is the mass of the electron.

$\therefore E = \frac{p^2}{2m} \implies p = \sqrt{2mE}$

$\int p dq = nh \implies \int p dq + \int -p dq = 2\int\sqrt{2mE}dx$

$2\sqrt{2mE}r_0 = nh \therefore E = \frac{h^2 n^2}{8mr_0^2}$

q2b:

$X_*^2 \to X_2 + \gamma$

$E^2 - c^2 p^2 = m^2 c^4$

$\gamma \implies m = 0 \therefore E^2 = c^2 p^2$

Sketch the vector sum.

Show $E_\gamma = \frac{1}{2}\frac{(m_*^2 - m^2)c^4}{m_* c^2}$

Conservation of energy: $E_* = E_m + E_\gamma$

Convervation of momentum: $p_* = p_m + p_\gamma = 0$

So from mass shell and squaring above: $m_*^2 c^4 - 0^2 c^2 = E_m^2 + E_\gamma^2 + 2E_m E_\gamma - c^2(p_m^2 + p_\gamma^2 - 2p_\gamma^2)$

Rearrange for $2E_m E_\gamma = m_*^2 c^4 - (E_m^2 - c^2 p_m^2) - (E_\gamma^2 - c^2 p_\gamma^2) - 2c^2 p_\gamma^2$ which are the 2 mass shells, $\therefore = m_*^2 c^4$ and $= m^2 c^4$

$\therefore 2E_m E_\gamma = m_*^2 c^4 - m^2 c^4 - 2p_\gamma$

Sub E_m from conservation of energy. Solve for E_γ