Commutator algebra

An often asked question is how to handle commutator algebra with Cadabra. This requires a few steps which are perhaps not entirely transparent to a new user, hence the following simple example.

In this notebook, we will verify the invariance of the two quadratic Casimirs of the Poincaré algebra. That is, we will verify that

$$[J_{\mu\nu}, P^2] = 0\,, \quad \text{and} \quad [J_{\mu\nu}, W^2] = 0\,, \tag{1}$$

where $P^2 = P_\mu P_\mu$ is the momentum squared and $W^2 = W_\mu W_\mu$ is the square of

$$W_\mu = \epsilon_{\mu\nu\rho\sigma} P_\nu J_{\rho\sigma}\,. \tag{2}$$

We first make some straightforward property assignments: declaration of indices, declaration of the operators P_μ and $J_{\mu\nu}$ and the fact that they do not commute, and so on.

```
{\mu,\nu,\rho,\sigma,\lambda,\kappa,\alpha,\beta,\gamma,\xi}::Indices
{\mu,\nu,\rho,\sigma,\lambda,\kappa,\alpha,\beta,\gamma,\xi}::Integer(0..3).
\eta_{\mu\nu}::Metric.
\delta{#}::KroneckerDelta.
e_{\mu\nu\lambda\rho}::EpsilonTensor(delta=\delta).
J_{\mu\nu}::AntiSymmetric.
J_{\mu\nu}::SelfNonCommuting.
{ J_{\mu\nu}, P_{\mu} }::NonCommuting.
```

```
Assigning property Integer to \mu, \nu, \rho, \sigma, \lambda, \kappa, \alpha, \beta, \gamma,
Assigning list property Indices to \mu, \nu, \rho, \sigma, \lambda, \kappa, \alpha, \beta, \ga
Assigning property Metric to \eta.
Assigning property KroneckerDelta to \delta.
Assigning property EpsilonTensor to e.
Assigning property AntiSymmetric to J.
Assigning property SelfNonCommuting to J.
Assigning list property NonCommuting to J, P.
```

We also need one more somewhat non-intuitive property. This one tells the program that the operators P_μ and $J_{\mu\nu}$ are to be kept inside commutators (they cannot be factored out). Cadabra does not (yet) deduce this from the `NonCommuting` property information, so you have to add this by hand.

```
{J_{\mu\nu}, P_{\mu}}::Depends(\commutator).
```

```
Assigning property Depends to J, P.
```

In order to simplify expressions automatically, we set up a few default rules which are to be executed after each command. These simply get rid of metric and Kronecker delta factors, canonicalise the indices and collect equal terms.

```
::PostDefaultRules( @@eliminate_metric!(%), @@eliminate_kr!(%),
                    @@canonicalise!(%), @@rename_dummies!(%), @@collect_terms!(%) ).
```

```
Assigning property PostDefaultRules to .
```

The Poincaré algebra

We now input the rules which define the Poincaré algebra. These are simply substitution rules, to be used later in explicit substition commands.

```
poincare:= { \commutator{J_{\mu\nu}}{P_{\rho}}
                       -> \eta_{\mu\rho} P_{\nu} - \eta_{\nu\rho} P_{\mu},
             \commutator{J_{\mu\nu}}{J_{\rho\sigma}}
                       -> \eta_{\mu\rho} J_{\nu\sigma}
                                            - \eta_{\mu\sigma} J_{\nu\rho}
                                            - \eta_{\nu\rho} J_{\mu\sigma}
                                            + \eta_{\nu\sigma} J_{\mu\rho} };
```

$$poincare := \{[J_{\mu\nu}, P_\rho] \to (\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu), \quad [J_{\mu\nu}, J_{\rho\sigma}] \to (\eta_{\mu\rho}J_{\nu\sigma} - \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\nu\rho}J_{\mu\sigma} + \eta_{\nu\sigma}J_{\mu\rho})\};$$

The P^2 Casimir

We know that P^2 is a Casimir, so the following should vanish:

```
\commutator{J_{\mu\nu}}{ P_{\rho}P_{\rho} };
```

$$2 := [J_{\mu\nu}, P_\rho P_\rho];$$

Write out the commutator of products as a sum of commutators (this works because we have declared `\commutator` as a `Derivative`).

```
@prodrule!(%);
```

$$2 := [J_{\mu\nu}, P_\rho]P_\rho + P_\rho[J_{\mu\nu}, P_\rho];$$

We can now rewrite the commutators, distribute the products and eliminate the metric to obtain the desired result:

```
@substitute!(%)( @(poincare) );
```

$$2 := (\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu)P_\rho + P_\rho(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu);$$

```
@distribute!(%);
```

$$2 := 0;$$

The W^2 Casimir

Now we do the same thing with W^2, the other Poincaré Casimir…

```
\commutator{J_{\mu\nu}}{W_\mu W_\mu};
```

$$3 := [J_{\mu\nu}, W_\rho W_\rho];$$

```
@substitute!(%)( W_\mu -> e_{\mu\nu\lambda\rho} P_\nu J_{\lambda\rho} );
```

$$3 := [J_{\mu\nu}, (-1)\, e_{\rho\sigma\lambda\kappa} P_\rho J_{\sigma\lambda} e_{\kappa\alpha\beta\gamma} P_\alpha J_{\beta\gamma}];$$

```
@prodrule!(%);
```

$$\begin{aligned} 3 := &-[J_{\mu\nu}, e_{\rho\sigma\lambda\kappa}] P_\rho J_{\sigma\lambda} e_{\kappa\alpha\beta\gamma} P_\alpha J_{\beta\gamma} - e_{\rho\sigma\lambda\kappa} [J_{\mu\nu}, P_\rho] J_{\sigma\lambda} e_{\kappa\alpha\beta\gamma} P_\alpha J_{\beta\gamma} \\ &- e_{\rho\sigma\lambda\kappa} P_\rho [J_{\mu\nu}, J_{\sigma\lambda}] e_{\kappa\alpha\beta\gamma} P_\alpha J_{\beta\gamma} - e_{\rho\sigma\lambda\kappa} P_\rho J_{\sigma\lambda} [J_{\mu\nu}, e_{\kappa\alpha\beta\gamma}] P_\alpha J_{\beta\gamma} \\ &- e_{\rho\sigma\lambda\kappa} P_\rho J_{\sigma\lambda} e_{\kappa\alpha\beta\gamma} [J_{\mu\nu}, P_\alpha] J_{\beta\gamma} - e_{\rho\sigma\lambda\kappa} P_\rho J_{\sigma\lambda} e_{\kappa\alpha\beta\gamma} P_\alpha [J_{\mu\nu}, J_{\beta\gamma}]; \end{aligned}$$

We get rid of the vanishing commutators by using:

```
@unwrap!(%);
```

$$\begin{aligned} 3 := &-e_{\rho\sigma\lambda\kappa} [J_{\mu\nu}, P_\rho] J_{\sigma\lambda} e_{\kappa\alpha\beta\gamma} P_\alpha J_{\beta\gamma} - e_{\rho\sigma\lambda\kappa} P_\rho [J_{\mu\nu}, J_{\sigma\lambda}] e_{\kappa\alpha\beta\gamma} P_\alpha J_{\beta\gamma} \\ &- e_{\rho\sigma\lambda\kappa} P_\rho J_{\sigma\lambda} e_{\kappa\alpha\beta\gamma} [J_{\mu\nu}, P_\alpha] J_{\beta\gamma} - e_{\rho\sigma\lambda\kappa} P_\rho J_{\sigma\lambda} e_{\kappa\alpha\beta\gamma} P_\alpha [J_{\mu\nu}, J_{\beta\gamma}]; \end{aligned}$$

```
@substitute!(%)( @(poincare) );
```

$$\begin{aligned} 3 := &-e_{\rho\sigma\lambda\kappa} (\eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu) J_{\sigma\lambda} e_{\kappa\alpha\beta\gamma} P_\alpha J_{\beta\gamma} \\ &- e_{\rho\sigma\lambda\kappa} P_\rho (-\eta_{\mu\sigma} J_{\lambda\nu} - \eta_{\lambda\mu} J_{\nu\sigma} + \eta_{\nu\sigma} J_{\lambda\mu} + \eta_{\lambda\nu} J_{\mu\sigma}) e_{\kappa\alpha\beta\gamma} P_\alpha J_{\beta\gamma} \\ &- e_{\rho\sigma\lambda\kappa} P_\rho J_{\sigma\lambda} e_{\kappa\alpha\beta\gamma} (\eta_{\alpha\mu} P_\nu - \eta_{\alpha\nu} P_\mu) J_{\beta\gamma} \\ &- e_{\rho\sigma\lambda\kappa} P_\rho J_{\sigma\lambda} e_{\kappa\alpha\beta\gamma} P_\alpha (-\eta_{\beta\mu} J_{\gamma\nu} + \eta_{\gamma\mu} J_{\beta\nu} + \eta_{\beta\nu} J_{\gamma\mu} - \eta_{\gamma\nu} J_{\beta\mu}); \end{aligned}$$

```
@distribute!(%);
```

$$\begin{aligned} 3 := &-e_{\mu\rho\sigma\lambda} P_\nu J_{\rho\sigma} e_{\lambda\kappa\alpha\beta} P_\kappa J_{\alpha\beta} + e_{\nu\rho\sigma\lambda} P_\mu J_{\rho\sigma} e_{\lambda\kappa\alpha\beta} P_\kappa J_{\alpha\beta} + 2\, e_{\mu\rho\sigma\lambda} P_\rho J_{\nu\sigma} e_{\lambda\kappa\alpha\beta} P_\kappa J_{\alpha\beta} \\ &- 2\, e_{\nu\rho\sigma\lambda} P_\rho J_{\mu\sigma} e_{\lambda\kappa\alpha\beta} P_\kappa J_{\alpha\beta} - e_{\mu\rho\sigma\lambda} P_\nu J_{\kappa\alpha} e_{\rho\kappa\alpha\beta} P_\beta J_{\sigma\lambda} + e_{\nu\rho\sigma\lambda} P_\mu J_{\kappa\alpha} e_{\rho\kappa\alpha\beta} P_\beta J_{\sigma\lambda} \\ &- 2\, e_{\mu\rho\sigma\lambda} P_\rho J_{\kappa\alpha} e_{\sigma\kappa\alpha\beta} P_\beta J_{\nu\lambda} + 2\, e_{\nu\rho\sigma\lambda} P_\rho J_{\kappa\alpha} e_{\sigma\kappa\alpha\beta} P_\beta J_{\mu\lambda}; \end{aligned}$$

```
@epsprod2gendelta!(%);
```

$$
\begin{aligned}
3 \ := {} & 6\,P_{\nu}J_{\rho\sigma}\delta_{\mu\lambda\rho\kappa\sigma\alpha}P_{\lambda}J_{\kappa\alpha} - 6\,P_{\mu}J_{\rho\sigma}\delta_{\nu\lambda\rho\kappa\sigma\alpha}P_{\lambda}J_{\kappa\alpha} + 12\,P_{\rho}J_{\nu\sigma}\delta_{\mu\rho\sigma\lambda\kappa\alpha}P_{\kappa}J_{\lambda\alpha} \\
& - 12\,P_{\rho}J_{\mu\sigma}\delta_{\nu\rho\sigma\lambda\kappa\alpha}P_{\kappa}J_{\lambda\alpha} + 6\,P_{\nu}J_{\rho\sigma}\delta_{\mu\rho\lambda\sigma\kappa\alpha}P_{\alpha}J_{\lambda\kappa} - 6\,P_{\mu}J_{\rho\sigma}\delta_{\nu\rho\lambda\sigma\kappa\alpha}P_{\alpha}J_{\lambda\kappa} \\
& - 12\,P_{\rho}J_{\sigma\lambda}\delta_{\mu\rho\kappa\sigma\alpha\lambda}P_{\kappa}J_{\nu\alpha} + 12\,P_{\rho}J_{\sigma\lambda}\delta_{\nu\rho\kappa\sigma\alpha\lambda}P_{\kappa}J_{\mu\alpha};
\end{aligned}
$$

The Kronecker delta with more than two indices is a generalised one; we break it up into normal ones by using:

```
@breakgendelta!(%);
```

$$
\begin{aligned}
3 \ := {} & 6\,P_{\nu}J_{\rho\sigma}(\frac{1}{3}\,\delta_{\lambda\sigma}\delta_{\kappa\rho}\delta_{\alpha\mu} - \frac{1}{3}\,\delta_{\lambda\sigma}\delta_{\kappa\mu}\delta_{\alpha\rho} + \frac{1}{3}\,\delta_{\lambda\mu}\delta_{\kappa\sigma}\delta_{\alpha\rho})P_{\alpha}J_{\kappa\lambda} \\
& - 6\,P_{\mu}J_{\rho\sigma}(\frac{1}{3}\,\delta_{\lambda\sigma}\delta_{\kappa\rho}\delta_{\alpha\nu} - \frac{1}{3}\,\delta_{\lambda\sigma}\delta_{\kappa\nu}\delta_{\alpha\rho} + \frac{1}{3}\,\delta_{\lambda\nu}\delta_{\kappa\sigma}\delta_{\alpha\rho})P_{\alpha}J_{\kappa\lambda} \\
& + 12\,P_{\rho}J_{\nu\sigma}(\frac{1}{3}\,\delta_{\lambda\kappa}\delta_{\alpha\sigma}\delta_{\mu\rho} - \frac{1}{3}\,\delta_{\lambda\kappa}\delta_{\alpha\mu}\delta_{\rho\sigma} + \frac{1}{3}\,\delta_{\lambda\sigma}\delta_{\kappa\rho}\delta_{\alpha\mu})P_{\kappa}J_{\alpha\lambda} \\
& - 12\,P_{\rho}J_{\mu\sigma}(\frac{1}{3}\,\delta_{\lambda\kappa}\delta_{\alpha\sigma}\delta_{\nu\rho} - \frac{1}{3}\,\delta_{\lambda\kappa}\delta_{\alpha\nu}\delta_{\rho\sigma} + \frac{1}{3}\,\delta_{\lambda\sigma}\delta_{\kappa\rho}\delta_{\alpha\nu})P_{\kappa}J_{\alpha\lambda} \\
& + 6\,P_{\nu}J_{\rho\sigma}(\frac{1}{3}\,\delta_{\lambda\kappa}\delta_{\alpha\sigma}\delta_{\mu\rho} - \frac{1}{3}\,\delta_{\lambda\kappa}\delta_{\alpha\rho}\delta_{\mu\sigma} + \frac{1}{3}\,\delta_{\lambda\mu}\delta_{\kappa\sigma}\delta_{\alpha\rho})P_{\lambda}J_{\alpha\kappa} \\
& - 6\,P_{\mu}J_{\rho\sigma}(\frac{1}{3}\,\delta_{\lambda\kappa}\delta_{\alpha\sigma}\delta_{\nu\rho} - \frac{1}{3}\,\delta_{\lambda\kappa}\delta_{\alpha\rho}\delta_{\nu\sigma} + \frac{1}{3}\,\delta_{\lambda\nu}\delta_{\kappa\sigma}\delta_{\alpha\rho})P_{\lambda}J_{\alpha\kappa} \\
& - 12\,P_{\rho}J_{\sigma\lambda}(\frac{1}{3}\,\delta_{\kappa\lambda}\delta_{\alpha\sigma}\delta_{\mu\rho} - \frac{1}{3}\,\delta_{\kappa\lambda}\delta_{\alpha\rho}\delta_{\mu\sigma} + \frac{1}{3}\,\delta_{\kappa\rho}\delta_{\alpha\lambda}\delta_{\mu\sigma})P_{\alpha}J_{\nu\kappa} \\
& + 12\,P_{\rho}J_{\sigma\lambda}(\frac{1}{3}\,\delta_{\kappa\lambda}\delta_{\alpha\sigma}\delta_{\nu\rho} - \frac{1}{3}\,\delta_{\kappa\lambda}\delta_{\alpha\rho}\delta_{\nu\sigma} + \frac{1}{3}\,\delta_{\kappa\rho}\delta_{\alpha\lambda}\delta_{\nu\sigma})P_{\alpha}J_{\mu\kappa};
\end{aligned}
$$

The rest is straightforward. Note that it works only because of all the `PostDefaultRules`.

```
@distribute!(%);
```

$$3 \ := 0;$$