THE DESIGN AND MIXED-MODEL ANALYSIS OF EXPERIMENTS

PRACTICAL I SOLUTIONS

I.1 Suppose that Y is a random variable that represents the actual contents of a 1-lb can of coffee. The model proposed for the distribution of Y is the uniform distribution over the interval $[15.5, 17.0]$

$$f(y) = \frac{1}{1.5}, \quad 15.5 \le y \le 17.0$$

a) What is the probability that a can will contain less than 16oz?

The required probability is $\int_{15.5}^{16} f(y)\,dy = \int_{15.5}^{16} \frac{1}{1.5}\,dy = \left.\frac{y}{1.5}\right|_{15.5}^{16} = \frac{0.5}{1.5} = 0.3333$.

b) Find the population mean and standard deviation for these cans.

$$E[Y] = \int_{-\infty}^{\infty} yf(y)\,dy = \int_{15.5}^{17} y\frac{1}{1.5}\,dy = \int_{15.5}^{17} \frac{y}{1.5}\,dy = \left.\frac{y^2}{3}\right|_{15.5}^{17} = \frac{17^2 - 15.5^2}{3} = 16.25$$

$$\begin{aligned}
\operatorname{var}[Y] &= \int_{-\infty}^{\infty} (y-\psi_Y)^2 f(y)\,dy = \int_{15.5}^{17} (y-16.25)^2 \frac{1}{15}\,dy = \frac{1}{15}\int_{15.5}^{17} (y-16.25)^2\,dy \\
&= \frac{1}{15}\left.\frac{(y-16.25)^3}{3}\right|_{15.5}^{17} \\
&= \frac{1}{15}\left(\frac{(17-16.25)^3}{3} - \frac{(15.5-16.25)^3}{3}\right) \\
&= \frac{0.421875-(-0.421875)}{45} \\
&= 0.01875
\end{aligned}$$

I.2 For a continuous random variable Y, what is $E\left[3Y^2+2\right]$.

Using theorem I.2,

$$E\left[3Y^2+2\right] = 3E\left[Y^2\right]+2 \text{ where } E\left[Y^2\right] = \int_{-\infty}^{\infty} y^2 f(y)\,dy$$

I.3 Definition 1.5 shows that $\text{var}[Y]$ is a function of Y so that $\text{var}[aY+b]$ is also a function of Y. Prove that $\text{var}[aY+b]=a^2\,\text{var}[Y]$ where a and b are constants. (Hint: use theorem I.2.)

$$\begin{aligned}\text{var}[aY+b] &= E\left[\left(aY+b-E[aY+b]\right)^2\right]\\ &= E\left[\left(aY+b-aE[Y]-b\right)^2\right]\\ &= E\left[\left(aY-aE[Y]\right)^2\right]\\ &= E\left[a^2\left(Y-E[Y]\right)^2\right]\\ &= a^2E\left[\left(Y-E[Y]\right)^2\right]\\ &= a^2\,\text{var}[Y]\end{aligned}$$

I.4 What is the expected value of the sum of a sample of n observations from a continuous random variable?

We have a random sample $y_1,y_2,\ldots,y_n$ *that has a continuous multivariate probability distribution function* $f(y_1,y_2,\ldots,y_n)=f(y_1)f(y_2)\ldots f(y_n)$. *Let S be the random variable that is given by* $S=\sum_{i=1}^{n}Y_i$. *We require* $E[S]$.

In the notation of theorem I.6, $c_j=1$ *and* $u_j(Y_1,Y_2,\ldots,Y_n)=Y_j$. *Clearly,* $E\left[u_j(Y_1,Y_2,\ldots,Y_n)\right]=E\left[Y_j\right]=\psi$. *Now,*

$$E[S]=E\left[\sum_{i=1}^{n}Y_i\right]=\sum_{i=1}^{n}E[Y_i]=\sum_{i=1}^{n}E[Y_i]=\sum_{i=1}^{n}\psi=n\psi$$