DESIGN AND MIXED-MODEL ANALYSIS OF EXPERIMENTS

VIII. Sets of Latin squares

(Mead sec.8.1)

A problem that occurs with Latin Squares is that, because two sources of variation are isolated, the residual degrees of freedom are often small (< 10); thus, a reliable estimate of the uncontrolled variation is not obtained. To overcome this problem several squares can be used. However, there are several ways in which the squares can be repeated. In the case of Example VI.2, *Wheat samplers*, the Latin square could be repeated using:

1. using the same intervals and areas in each repeat or replicate;
2. using the same intervals but new areas (or the same areas but new intervals); or
3. using new areas and intervals.

We shall determine the analysis of variance for each of these cases. In determining the expected mean squares, it will be assumed that Area and Samplers are random factors and that Interval is a fixed factor.

In general, one can have as many squares as one likes. However, for space reasons I will only present layouts for 2 squares. However, I will give the degrees of freedom for the general case of r squares.

VIII.A Case 1 — same intervals and areas

This case involves a complete repetition of the experiment, say on consecutive mornings with the 4 time intervals being the same on the two days. There is no re-randomization of the square for the second day.

Layout (r=2)

	Day							
	1				2			
Area	1	2	3	4	1	2	3	4
Interval								
1	3	4	2	1	3	4	2	1
2	4	2	1	3	4	2	1	3
3	1	3	4	2	1	3	4	2
4	2	1	3	4	2	1	3	4

A. Description of pertinent features of the study

1. Observational unit – an area in an interval on a day
2. Response variable – Error
3. Unrandomized factors – Day, Interval, Area
4. Randomized factors – Samplers
5. Type of study – Sets of Latin Squares

B. The experimental structure

Structure	Formula
unrandomized	*2* Day**4* Interval**4* Area
randomized	*4* Samplers

For this structure to be appropriate requires that the same square without re-randomization be used for each day; otherwise, some factors would be nested.

C. Terms derived from the structure formulae

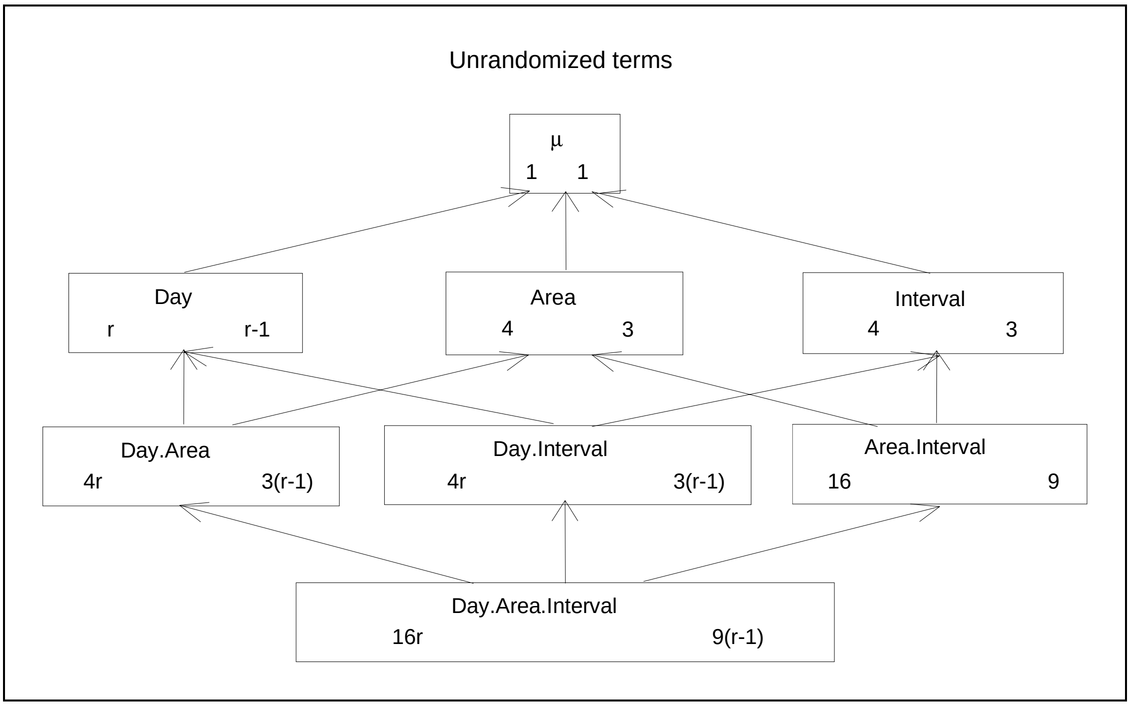
Day*Interval*Area = (Day+Interval+Day.Interval)*Area

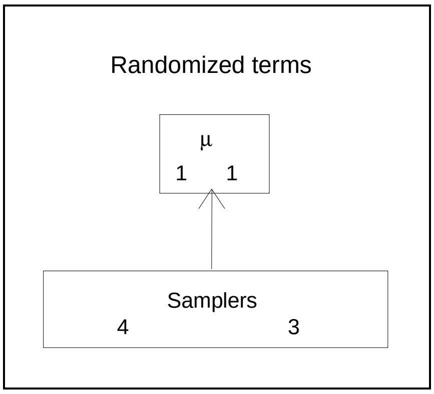
= Day+Interval+Day.Interval+Area

+ Day.Area+Interval.Area+Day.Interval.Area

Samplers = Samplers

D. Degrees of freedom





Alternatively, as all the factors in the unrandomized structure are crossed, the rule for a set of crossed factors can be used. That is, the degrees of freedom of any term can be calculated by taking the number of levels minus one for each factor in the term and multiplying these together. For example, since Day has 4 levels and Interval has r levels, the degrees of freedom of Day.Interval is $(4-1)(r-1) = 3(r-1)$.

E. The analysis of variance table

Source	df
Day	r–1
Int	3
Area	3
Day.Int	3(r–1)
Day.Area	3(r–1)
Int.Area	9
Samplers	3
Residual	6
Day.Int.Area	9(r–1)
Total	16r–1

F. Maximal expectation and variation models

Assume Days and Intervals are fixed factors and that Area and Samplers are random factors. Then the expectation terms are Day, Interval and Day.Interval. The variation terms are: Area, Day.Area, Interval.Area, Day.Interval.Area and Samplers.

The expectation model is

$$\text{E[Y]} = \text{Day.Interval}$$

and the variation model is

$$\text{var[Y]} = \text{Samplers + Area + Day.Area + Interval.Area + Day.Interval.Area}$$

Note that the expectation model does not include either Day or Interval as individual terms as these are marginal to Day.Interval. The expectation model allows for arbitrary differences between the day-interval combinations.

G. The expected mean squares.

The variation components are σ^2_{S}, σ^2_{A}, σ^2_{DA}, σ^2_{IA} and σ^2_{DIA}, respectively.

The multipliers of these components are 4*r*, 4*r*, 4, *r* and 1, respectively.

Source	df	E[MSq]			
Day	r–1	σ^2_{DIA}		$+4\sigma^2_{\mathrm{DA}}$	$+f_{\mathrm{D}}(\psi)$
Int	3	σ^2_{DIA}	$+r\sigma^2_{\mathrm{IA}}$		$+f_{\mathrm{I}}(\psi)$
Area	3	σ^2_{DIA}	$+r\sigma^2_{\mathrm{IA}}$	$+4\sigma^2_{\mathrm{DA}}$	$+4r\sigma^2_{\mathrm{A}}$
Day.Int	3(r–1)	σ^2_{DIA}			$+f_{\mathrm{DI}}(\psi)$
Day.Area	3(r–1)	σ^2_{DIA}		$+4\sigma^2_{\mathrm{DA}}$	
Int.Area	9				
Samplers	3	σ^2_{DIA}	$+r\sigma^2_{\mathrm{IA}}$		$+4r\sigma^2_{\mathrm{S}}$
Residual	6	σ^2_{DIA}	$+r\sigma^2_{\mathrm{IA}}$		
Day.Int.Area	9(r–1)	σ^2_{DIA}			
Total	16r–1				

VIII.B Case 2 — same areas different intervals

In this case the samplers repeat the experiment on a different occasion such that the intervals on one occasion are unconnected with those on the other. As a result the rows of the square, but not the columns, are rerandomized on the second occasion.

Layout (r=2)

	Occasion							
	1				2			
Area	1	2	3	4	1	2	3	4
Interval								
1	4	1	3	2	3	2	1	4
2	2	3	4	1	4	1	3	2
3	3	2	1	4	1	4	2	3
4	1	4	2	3	2	3	4	1

Note that the areas sampled by a sampler in the first interval on occasion 1 are the same as in the second interval of occasion 2; that is, the first row of the square on occasion 1 is the same as the second row on occasion 2.

A. Description of pertinent features of the study

1. Observational unit – an area in an interval
2. Response variable – Error
3. Unrandomized factors – Occasion, Interval, Area
4. Randomized factors – Samplers
5. Type of study – Sets of Latin Squares

B. The experimental structure

Structure	Formula
unrandomized	(*2* Occasion/*4* Interval)**4* Area
randomized	*4* Samplers

C. Terms derived from the structure formulae

(Occasion/Interval)*Area = (Occasion+Occasion.Interval)*Area

= Occasion+Occasion.Interval+Area+Occasion.Area
+Occasion.Interval.Area

Samplers = Samplers

D. Degrees of freedom

Left as an exercise

E. The analysis of variance table

Source	df
Occasion	r–1
Occasion.Int	3r
Area	3
Occasion.Area	3(r–1)
Occasion.Int.Area	9r
Samplers	3
Residual	9r–3
Total	16r–1

F. Maximal expectation and variation models

Assume Occasions and Intervals are fixed factors and that Area and Samplers are random factors. Then the expectation terms are: Occasion and

Occasion.Interval. The variation terms are: Samplers, Area, Occasion.Area, Occasion.Interval.Area.

The expectation model is

$$E[Y] = \text{Occasion.Interval}$$

and the variation model is

$$\text{var}[Y] = \text{Samplers + Area + Occasion.Area + Occasion.Interval.Area}$$

Note that the expectation model does not include Occasion as an individual term as it is marginal to Occasion.Interval.

G. The expected mean squares

The variation components are σ_{S}^2, σ_{A}^2, σ_{OA}^2 and σ_{OIA}^2, respectively.

The multipliers of these components are 4r, 4r, 4 and 1, respectively.

Source	df	E[MSq]		
Occasion	r–1	σ_{OIA}^2	$+4\sigma_{\mathrm{OA}}^2$	$+f_{\mathrm{O}}(\psi)$
Occasion.Int	3r	σ_{OIA}^2		$+f_{\mathrm{OI}}(\psi)$
Area	3	σ_{OIA}^2	$+4\sigma_{\mathrm{OA}}^2$	$+4r\sigma_{\mathrm{A}}^2$
Occasion.Area	3(r–1)	σ_{OIA}^2	$+4\sigma_{\mathrm{OA}}^2$	
Occasion.Int.Area	9r			
Samplers	3	σ_{OIA}^2		$+4r\sigma_{\mathrm{S}}^2$
Residual	9r–3	σ_{OIA}^2		
Total	16r–1			

VIII.C Case 3 — different intervals and areas

In this case, not only are the intervals on different occasions unconnected, but so are the areas as the areas used on the second occasion are completely different to those used on the first occasion. As a result the rows **and** columns of the square are rerandomized on the second occasion.

Layout (*r*=2)

	Occasion							
	1				2			
Area	1	2	3	4	1	2	3	4
Interval								
1	3	4	2	1	2	3	4	1
2	4	2	1	3	1	4	2	3
3	1	3	4	2	4	1	3	2
4	2	1	3	4	3	2	1	4

A. Description of pertinent features of the study

1. Observational unit – an area in an interval
2. Response variable – Error
3. Unrandomized factors – Occasion, Interval, Area
4. Randomized factors – Samplers
5. Type of study – Sets of Latin Squares

B. The experimental structure

Structure	Formula
unrandomized	*2* Occasion/(*4* Interval**4* Area)
randomized	*4* Samplers

C. Terms derived from the structure formulae

Occasion/(Interval*Area) = Occasion/(Interval+Area+Interval.Area)

= Occasion+Occasion.Interval+Occasion.Area +Occasion.Interval.Area

Samplers = Samplers

D. Degrees of freedom

Left as an exercise

E. The analysis of variance table

Source	df
Occasion	$r-1$
Occasion.Int	$3r$
Occasion.Area	$3r$
Occasion.Int.Area	$9r$
Samplers	3
Residual	$9r-3$
Total	$16r-1$

F. Maximal expectation and variation models

Assume Occasions and Intervals are fixed factors and that Area and Samplers are random factors. Then the expectation terms are: Occasion and Occasion.Interval. The variation terms are: Samplers, Occasion.Area, Occasion.Interval.Area.

The expectation model is

$$\mathrm{E}[\mathrm{Y}] = \text{Occasion.Interval}$$

and the variation model is

$$\mathrm{var}[\mathrm{Y}] = \text{Samplers} + \text{Occasion.Area} + \text{Occasion.Interval.Area}$$

Note that the expectation model does not include Occasion as an individual term as it is marginal to Occasion.Interval.

G. The expected mean squares.

The variation components are σ^2_{S}, σ^2_{OA} and σ^2_{OIA}, respectively.

The multipliers of these components are 4*r*, 4 and 1, respectively.

Source	df	E[MSq]		
Occasion	$r-1$	σ^2_{OIA}	$+4\sigma^2_{\mathrm{OA}}$	$+f_{\mathrm{O}}(\psi)$
Occasion.Int	$3r$	σ^2_{OIA}		$+f_{\mathrm{OI}}(\psi)$
Occasion.Area	$3r$	σ^2_{OIA}	$+4\sigma^2_{\mathrm{OA}}$	
Occasion.Int.Area	$9r$			
Samplers	3	σ^2_{OIA}		$+4r\sigma^2_{\mathrm{S}}$
Residual	$9r-3$	σ^2_{OIA}		
Total	$16r-1$			

VIII.D Summary of Latin square analyses

One square only

Unrandomized Structure	Area*Interval

Source	df
Area	3
Interval	3
Interval.Area	9
Samplers	3
Residual	6

Case 1

Same intervals and areas

Source	df
Day	*r*–1
Int	3
Area	3
Day.Int	3(*r*–1)
Day.Area	3(*r*–1)
Int.Area	9
Samplers	3
Residual	6
Day.Int.Area	9(*r*–1)
Total	16*r*–1

Case 2

Same areas different intervals

Source	df
Occ	*r*–1
Occ.Int	3*r*
Area	3
Occ.Area	3(*r*–1)
Occ.Int.Area	9*r*
Samplers	3
Residual	9*r*–3
Total	16*r*–1

Case 3

Different intervals and areas

Source	df
Occ	*r*–1
Occ.Int	3*r*
Occ.Area	3*r*
Occ.Int.Area	9*r*
Samplers	3
Residual	9*r*–3
Total	16*r*–1

Unrandomized structure

Case 1	Case 2	Case 3
2 Day**4* Interval**4* Area	(*2* Occ/*4* Interval)**4* Area	*2* Occ/(*4* Interval**4* Area)