STATISTICAL MODELLING

PRACTICAL XI
SOLUTIONS

XI.1 The analysis for an example involving a 4×4 Latin Square was derived in section VI.B, *The Latin square example*. In particular, the expectation and variation models were written symbolically as

$$\psi = \text{E[Y]} = \text{Additive and}$$
$$\text{var[Y]} = \text{Driver + Car + Driver}\wedge\text{Car}$$

Write down matrix expressions for these models, the expectation model involving indicator-variable (**X**) matrices and the variation model involving summation (**S**) matrices and variance components. Also, give the variance matrix in terms of direct products of **I** and **J** matrices, assuming that the observations are in standard order for Driver then Car.

$$E[\mathbf{Y}] = \mathbf{X}_{\text{A}}\boldsymbol{\alpha}$$
$$\mathbf{V} = \sigma^2_{\text{D}}\mathbf{S}_{\text{D}} + \sigma^2_{\text{C}}\mathbf{S}_{\text{C}} + \sigma^2_{\text{DC}}\mathbf{S}_{\text{DC}} = \sigma^2_{\text{D}}\mathbf{I}_4 \otimes \mathbf{J}_4 + \sigma^2_{\text{C}}\mathbf{J}_4 \otimes \mathbf{I}_4 + \sigma^2_{\text{DC}}\mathbf{I}_4 \otimes \mathbf{I}_4$$

What will be the estimator of the expected values under this model? Which theorems justify your answer?

The estimator of the expected values will be $\hat{\boldsymbol{\psi}} = \mathbf{X}\hat{\boldsymbol{\theta}} = \overline{\mathbf{A}}_{n\times 1} = \mathbf{M}_{\text{A}}\mathbf{Y}$. *Theorem XI.17 tells us that the GLS estimators will equal the OLS estimators with the expectation model* $E[\mathbf{Y}] = \mathbf{X}_{\text{A}}\boldsymbol{\alpha}$. *Corollary XI.1 tells us that this estimator is the vector of means for the generalized factor that in this case is just the Additive means.*

XI.2 Suppose that for a particular linear model

$$\mathbf{X} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix},\ \mathbf{X}'\mathbf{X} = \begin{bmatrix} 2 & 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 & 1 \\ 1 & 1 & 1 & 3 & 0 \\ 1 & 1 & 1 & 0 & 3 \end{bmatrix} \text{ and } \mathbf{y} = \begin{bmatrix} 19 \\ 29 \\ 14 \\ 24 \\ 12 \\ 22 \end{bmatrix}$$

a) Find a generalized inverse for **X'X** and use it to find a solution to the normal equations by computing $\left(\mathbf{X}'\mathbf{X}\right)^{-}\mathbf{X}'\mathbf{y}$. Obtain the fitted values.

*Since **X** (and **X'X**) appears to be of rank 4, a generalized inverse for **X'X** can be obtained by omitting any row and column of the matrix and inverting the reduced matrix. Since **X'X** is of the form.*

$$\begin{bmatrix} t\mathbf{I}_b & \mathbf{J}_{b\times t} \\ \mathbf{J}_{t\times b} & b\mathbf{I}_t \end{bmatrix}$$

with $t = 2$ and $b = 3$, the generalized inverse when the last row and column are omitted can be obtained from theorem V.7:

$$\left(\mathbf{X}'\mathbf{X}\right)^- = \begin{bmatrix} \frac{1}{t}\mathbf{I}_b + \frac{(t-1)}{bt}\mathbf{J}_b & -\frac{1}{b}\mathbf{J}_{b\times(t-1)} & \mathbf{0}_{b\times 1} \\ -\frac{1}{b}\mathbf{J}_{(t-1)\times b} & \frac{1}{b}\left(\mathbf{I}_{(t-1)} + \mathbf{J}_{(t-1)}\right) & \mathbf{0}_{(t-1)\times 1} \\ \mathbf{0}_{1\times b} & \mathbf{0}_{1\times(t-1)} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}\mathbf{I}_3 + \frac{1}{6}\mathbf{J}_3 & -\frac{1}{3}\mathbf{J}_{3\times 1} & \mathbf{0}_{3\times 1} \\ -\frac{1}{3}\mathbf{J}_{1\times 3} & \frac{1}{3}\left(\mathbf{I}_1 + \mathbf{J}_1\right) & \mathbf{0}_{1\times 1} \\ \mathbf{0}_{1\times 3} & \mathbf{0}_{1\times 1} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{3} & \frac{1}{6} & \frac{1}{6} & -\frac{1}{3} & 0 \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & -\frac{1}{3} & 0 \\ \frac{1}{6} & \frac{1}{6} & \frac{2}{3} & -\frac{1}{3} & 0 \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now $\mathbf{X}'\mathbf{y} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 19 \\ 29 \\ 14 \\ 24 \\ 12 \\ 22 \end{bmatrix} = \begin{bmatrix} 48 \\ 38 \\ 34 \\ 45 \\ 75 \end{bmatrix}$

The solution to the normal equations is

$$\left(\mathbf{X}'\mathbf{X}\right)^- \mathbf{X}'\mathbf{y} = \begin{bmatrix} \frac{2}{3} & \frac{1}{6} & \frac{1}{6} & -\frac{1}{3} & 0 \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & -\frac{1}{3} & 0 \\ \frac{1}{6} & \frac{1}{6} & \frac{2}{3} & -\frac{1}{3} & 0 \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 48 \\ 38 \\ 34 \\ 45 \\ 75 \end{bmatrix} = \begin{bmatrix} 29 \\ 24 \\ 22 \\ -10 \\ 0 \end{bmatrix}$$

The fitted values are

$$\mathbf{X}\left(\mathbf{X}'\mathbf{X}\right)^- \mathbf{X}'\mathbf{y} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 29 \\ 24 \\ 22 \\ -10 \\ 0 \end{bmatrix} = \begin{bmatrix} 19 \\ 29 \\ 14 \\ 24 \\ 12 \\ 22 \end{bmatrix}$$

*That is, the fitted values are equal to the observed **y** and the model fits the data exactly. This would not necessarily always be the case since the model is of rank 4 and there are 6 observations — it is only when the rank of the model equals the number of observations that the model must fit the data exactly.*

b) Find a second generalized inverse for **X'X** by deleting its first row and column. Use it to find a solution to the normal equations and to obtain the fitted values.

*A second generalized inverse for **X'X** can be obtained by omitting any other row and column of the matrix and inverting the reduced matrix. We omit the first row and column and then use the formula for the inverse of a partitioned matrix:*

if $\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}' & \mathbf{D} \end{bmatrix}$ *then*

$$\mathbf{M}^{-1} = \left(\mathbf{M}^{-1}\right)' = \begin{bmatrix} \mathbf{U} & \mathbf{V} \\ \mathbf{V}' & \mathbf{W} \end{bmatrix} = \begin{bmatrix} \mathbf{A}^{-1} - \mathbf{VB}'\mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{BW} \\ -\mathbf{WB}'\mathbf{A}^{-1} & \left(\mathbf{D} - \mathbf{B}'\mathbf{A}^{-1}\mathbf{B}\right)^{-1} \end{bmatrix}$$

In our case, $\boldsymbol{A} = 2\boldsymbol{I}_2$, $\boldsymbol{B} = \boldsymbol{J}_2$ *and* $\boldsymbol{D} = 3\boldsymbol{I}_2$ *so* $\boldsymbol{A}^{-1} = ½\boldsymbol{I}_2$ *and*

$$\mathbf{W} = \left(\mathbf{D} - \mathbf{B}'\mathbf{A}^{-1}\mathbf{B}\right)^{-1} = \left(3\mathbf{I}_2 - \tfrac{1}{2}\mathbf{J}_2\mathbf{I}_2\mathbf{J}_2\right)^{-1} = \left(3\mathbf{I}_2 - \mathbf{J}_2\right)^{-1} = \tfrac{1}{3}\left(\mathbf{I}_2 + \mathbf{J}_2\right).$$

$$\mathbf{V} = -\mathbf{A}^{-1}\mathbf{BW} = -\tfrac{1}{2}\mathbf{I}_2\mathbf{J}_2\tfrac{1}{3}\left(\mathbf{I}_2 + \mathbf{J}_2\right) = -\tfrac{1}{6}\left(\mathbf{J}_2 + 2\mathbf{J}_2\right) = -\tfrac{1}{2}\mathbf{J}_2$$

$$\mathbf{U} = \mathbf{A}^{-1} - \mathbf{VB}'\mathbf{A}^{-1} = \tfrac{1}{2}\mathbf{I}_2 - \left(-\tfrac{1}{2}\mathbf{J}_2\right)\mathbf{J}_2\tfrac{1}{2}\mathbf{I}_2 = \tfrac{1}{2}\mathbf{I}_2 + \tfrac{1}{2}\mathbf{J}_2 = \tfrac{1}{2}\left(\mathbf{I}_2 + \mathbf{J}_2\right)$$

Consequently,

$$\left(\mathbf{X}'\mathbf{X}\right)^{-} = \begin{bmatrix} 0 & \mathbf{0}_{1\times2} & \mathbf{0}_{1\times2} \\ \mathbf{0}_{2\times1} & \tfrac{1}{2}\left(\mathbf{I}_2 + \mathbf{J}_2\right) & -\tfrac{1}{2}\mathbf{J}_2 \\ \mathbf{0}_{2\times1} & -\tfrac{1}{2}\mathbf{J}_2 & \tfrac{1}{3}\left(\mathbf{I}_2 + \mathbf{J}_2\right) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \tfrac{1}{2} & -\tfrac{1}{2} & -\tfrac{1}{2} \\ 0 & \tfrac{1}{2} & 1 & -\tfrac{1}{2} & -\tfrac{1}{2} \\ 0 & -\tfrac{1}{2} & -\tfrac{1}{2} & \tfrac{2}{3} & \tfrac{1}{3} \\ 0 & -\tfrac{1}{2} & -\tfrac{1}{2} & \tfrac{1}{3} & \tfrac{2}{3} \end{bmatrix}$$

The solution to the normal equations is

$$\left(\mathbf{X}'\mathbf{X}\right)^{-}\mathbf{X}'\mathbf{y} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \tfrac{1}{2} & -\tfrac{1}{2} & -\tfrac{1}{2} \\ 0 & \tfrac{1}{2} & 1 & -\tfrac{1}{2} & -\tfrac{1}{2} \\ 0 & -\tfrac{1}{2} & -\tfrac{1}{2} & \tfrac{2}{3} & \tfrac{1}{3} \\ 0 & -\tfrac{1}{2} & -\tfrac{1}{2} & \tfrac{1}{3} & \tfrac{2}{3} \end{bmatrix} \begin{bmatrix} 48 \\ 38 \\ 34 \\ 45 \\ 75 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ -7 \\ 19 \\ 29 \end{bmatrix}$$

The fitted values are

$$\mathbf{X}\left(\mathbf{X}'\mathbf{X}\right)^{-}\mathbf{X}'\mathbf{y} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -5 \\ -7 \\ 19 \\ 29 \end{bmatrix} = \begin{bmatrix} 19 \\ 29 \\ 14 \\ 24 \\ 12 \\ 22 \end{bmatrix}$$

*That is, the fitted values are equal to the observed **y** and the model fits the data exactly.*

c) How do the solutions and fitted values obtained in the previous parts compare?

The solutions differ but the fitted values are the same.

XI.3 From theorem XI.8 we know that the expected values for the simple linear model with $\boldsymbol{\psi} = E[\mathbf{Y}] = \mathbf{X}\theta$, where $\mathbf{X}$ is $n \times q$ of rank $m \leq q$, is estimable. Use theorem XI.11 to show that $\text{var}[\hat{\boldsymbol{\psi}}] = \sigma^2 \mathbf{X}'(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}$. (Hint: note that $\psi_i = \mathbf{x}_i'\boldsymbol{\theta}$ where $\mathbf{x}_i$ is the q-vector that is the *i*th row of $\mathbf{X}$.)

From theorem XI.11,

$$\text{var}\left[\mathbf{x}_i'\hat{\boldsymbol{\theta}}\right] = \sigma^2 \mathbf{x}_i'(\mathbf{X}'\mathbf{X})^{-}\mathbf{x}_i$$

and so putting the n elements of ψ *together it is clear that*

$$\text{var}[\hat{\boldsymbol{\psi}}] = \sigma^2 \mathbf{X}'(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}$$

XI.4 Consider the following expectation models for a two-factor factorial experiment:

$$E[Y_{ik\ell}] = (\alpha\beta)_{k\ell},\ E[Y_{ik\ell}] = \alpha_k + \beta_\ell,\ E[Y_{ik\ell}] = \alpha_k,\ E[Y_{ik\ell}] = \beta_\ell \text{ and } E[Y_{ik\ell}] = \mu.$$

a) What is the estimator of the expected values for the first model?

The is the estimator of the expected values for the first model is an n-vector whose elements are $\overline{Y}_{k\ell}$, *the* $\overline{Y}_{k\ell}$ *for a particular element being the mean corresponding to the levels of A and B that the unit received. This is because the model involves a single-term that corresponds to the generalized factor A∧B and corollary XI.1 applies.*

b) For which models are the individual parameters estimable? Give reasons for your answer.

All models except the second consist of an $\boldsymbol{X}$ *matrix for a single, generalized factor. In proving theorem XI.2 it is shown that such matrices are of full rank and so, as pointed out in section XI.B e), the individual parameters of these models are estimable. The second model is like the model for the maximal RCBD and is not of full rank as the sum of both* $\boldsymbol{X}_A$ *and* $\boldsymbol{X}_B$ *sum to a vector of ones.*

XI.5 An experiment was conducted to study the effects of temperature on the life (in hours) of a component. An RCBD was employed with five ovens forming the blocks. Four temperatures were randomly assigned to four runs within each oven. The following results were recorded:

		Temperature (degrees)			
		200	300	400	500
Oven	I	340	324	307	274
	II	361	338	312	281
	III	346	328	298	276
	IV	358	332	315	285
	V	343	321	294	269

This data was analysed as part of exercise IV.4 and the following analysis of variance table obtained.

```
> RCBDComponent.aov <- aov(Life ~ Oven + Temperature + Error(Oven/Run),
    RCBDComponent)
> summary(RCBDComponent.aov, split = list(Temperature = list(L = 1, Q = 2)))
Error: Oven
     Df Sum of Sq Mean Sq
Oven  4     845.3 211.325

Error: Run %in% Oven
                 Df Sum of Sq  Mean Sq  F Value     Pr(F)
     Temperature  3  14610.60  4870.20  365.493 0.0000000
  Temperature: L  1  14544.36 14544.36 1091.509 0.0000000
  Temperature: Q  1     64.80    64.80    4.863 0.0476871
       Residuals 12    159.90    13.32
```

Assume that Ovens and Runs are random factors and that Temperatures is a fixed factor. What are the expected mean squares under the maximal model in these circumstances?

a) Assume that Ovens and Runs are random factors and that Temperatures is a fixed factor. What are the expected mean squares under the maximal model in these circumstances?

Source	df	MSq	E[MSq]
Oven	4	s_{O}^2	$\sigma_{\mathrm{OR}}^2 + 4\sigma_{\mathrm{O}}^2$
Run[Oven]	15		
Temperature	3	s_{T}^2	$\sigma_{\mathrm{OR}}^2 + q_{\mathrm{T}}(\psi)$
Residual	12	s_{OR}^2	σ_{OR}^2
Total	19		

b) Give the expressions for the variance matrix in terms of the variance components and the summation and mean-operator matrices, for this example.

$$\begin{aligned}\mathbf{V} &= \sigma_{\mathrm{O}}^2\mathbf{S}_{\mathrm{O}} + \sigma_{\mathrm{OR}}^2\mathbf{S}_{\mathrm{OR}} \\ &= 4\sigma_{\mathrm{O}}^2\mathbf{M}_{\mathrm{O}} + \sigma_{\mathrm{OR}}^2\mathbf{M}_{\mathrm{OR}}\end{aligned}$$

c) What are the estimates of the variance components?

$s_{\mathrm{OR}}^2 = \hat{\sigma}_{\mathrm{OR}}^2 = 13.32$, *and* $s_{\mathrm{O}}^2 = \widehat{\sigma_{\mathrm{OR}}^2 + 4\sigma_{\mathrm{O}}^2} = 211.325$ *so that* $\hat{\sigma}_{\mathrm{O}}^2 = \left(s_{\mathrm{O}}^2 - s_{\mathrm{OR}}^2\right)/4 = (211.325 - 13.32)/4 = 49.50$.

XI.6 Let $\mathbf{Y}$ be a normally distributed random vector representing a random sample with $E[\mathbf{Y}] = \mathbf{X}\boldsymbol{\theta}$ and $\text{var}[\mathbf{Y}] = \mathbf{V} = \sigma^2\mathbf{I}_n$ where $\mathbf{X}$ is an $n \times q$ matrix of full rank, $\boldsymbol{\theta}$ is a $q \times 1$ vector of unknown parameters and $n \geq q$. Prove that the maximum likelihood estimator of σ^2 is given by

$$\tilde{\sigma}_n^2 = \frac{\left(\mathbf{Y} - \mathbf{X}\tilde{\boldsymbol{\theta}}\right)'\left(\mathbf{Y} - \mathbf{X}\tilde{\boldsymbol{\theta}}\right)}{n} = \frac{\tilde{\boldsymbol{\varepsilon}}'\tilde{\boldsymbol{\varepsilon}}}{n}$$

Note that you can use the log likelihood function, ℓ, from Theorem X.17 so that your first step will be to simplify the expression for $\text{var}[\mathbf{Y}] = \mathbf{V} = \sigma^2\mathbf{I}_n$. The you will need to maximize the log likelihood with respect to σ^2.

The maximum likelihood estimate of σ^2 is given by $\partial\ell / \partial\sigma^2 = 0$ as follows:

$$\begin{aligned}\frac{\partial\ell}{\partial\sigma^2} &= \frac{\partial\left\{-\frac{n}{2}\ln\left(2\pi\sigma^2\right) - \frac{1}{2\sigma^2}\left(\mathbf{y} - \mathbf{X}\boldsymbol{\theta}\right)'\left(\mathbf{y} - \mathbf{X}\boldsymbol{\theta}\right)\right\}}{\partial\sigma^2} \\ &= -\left\{\left(\frac{n}{2}\right)\left(\frac{2\pi}{2\pi\sigma^2}\right) - \left(\frac{1}{2\left[\sigma^2\right]^2}\right)\left(\mathbf{y} - \mathbf{X}\boldsymbol{\theta}\right)'\left(\mathbf{y} - \mathbf{X}\boldsymbol{\theta}\right)\right\} \\ &= 0\end{aligned}$$

which implies

$$\left(\frac{n}{2\sigma^2}\right) = \left(\frac{1}{2\left[\sigma^2\right]^2}\right)\left(\mathbf{y} - \mathbf{X}\boldsymbol{\theta}\right)'\left(\mathbf{y} - \mathbf{X}\boldsymbol{\theta}\right)$$

so that the estimate is given by

$$\tilde{\sigma}^2 = \frac{\left(\mathbf{y} - \mathbf{X}\boldsymbol{\theta}\right)'\left(\mathbf{y} - \mathbf{X}\boldsymbol{\theta}\right)}{n}$$

so that the estimator is given by substituting $\mathbf{Y}$ and $\tilde{\boldsymbol{\theta}}$ for $\mathbf{y}$ and $\boldsymbol{\theta}$ to obtain

$$\tilde{\sigma}_n^2 = \frac{\left(\mathbf{Y} - \mathbf{X}\tilde{\boldsymbol{\theta}}\right)'\left(\mathbf{Y} - \mathbf{X}\tilde{\boldsymbol{\theta}}\right)}{n} = \frac{\tilde{\boldsymbol{\varepsilon}}'\tilde{\boldsymbol{\varepsilon}}}{n}$$

Is this estimator unbiased?

Theorem XI.19 tells us that the estimator is biased.

XI.7 We have that for the RCBD, provided the observations are ordered on Blocks then Treatments, $\mathbf{M}_{\mathrm{BU}} = \mathbf{I}_n$ and $\mathbf{M}_{\mathrm{T}} = t^{-1}\mathbf{I}_b \otimes \mathbf{J}_t$, $\mathbf{M}_{\mathrm{B}} = b^{-1}\mathbf{J}_b \otimes \mathbf{I}_t$ and $\mathbf{M}_{\mathrm{B}} = b^{-1}\mathbf{J}_b \otimes \mathbf{J}_t$. Also, $\mathbf{Q}_{\mathrm{G}} = \mathbf{M}_{\mathrm{G}}$, $\mathbf{Q}_{\mathrm{B}} = \mathbf{M}_{\mathrm{B}} - \mathbf{M}_{\mathrm{G}}$ and $\mathbf{Q}_{\mathrm{BU}} = \mathbf{M}_{\mathrm{BU}} - \mathbf{M}_{\mathrm{B}}$.

a) Prove that $\mathbf{M}_{\mathrm{T}}\mathbf{M}_{\mathrm{B}} = \mathbf{M}_{\mathrm{B}}\mathbf{M}_{\mathrm{T}} = \mathbf{M}_{\mathrm{G}}$ and that the product of $\mathbf{M}_{\mathrm{G}}$ with any of the other **M** is $\mathbf{M}_{\mathrm{G}}$.

$$\begin{aligned}
\mathbf{M}_{\mathrm{T}}\mathbf{M}_{\mathrm{B}} &= \left(t^{-1}\mathbf{I}_b \otimes \mathbf{J}_t\right)\left(b^{-1}\mathbf{J}_b \otimes \mathbf{I}_t\right) \\
&= (bt)^{-1}\mathbf{J}_b \otimes \mathbf{J}_t \\
&= \mathbf{M}_{\mathrm{B}}\mathbf{M}_{\mathrm{T}} \\
&= \mathbf{M}_{\mathrm{G}}
\end{aligned}$$

Now $\mathbf{M}_{\mathrm{G}} = (bt)^{-1}\mathbf{J}_b \otimes \mathbf{J}_t$ *so that multiplying it by a matrix that has* ***I*** *in either position will leave a* ***J*** *in the same position. Multiplying it by a matrix with a* $\frac{1}{k}\mathbf{J}_k$*, where* $k = b$ or t*, in either position will also leave that matrix in the same position as* $\frac{1}{k}\mathbf{J}_k \frac{1}{k}\mathbf{J}_k = \frac{1}{k^2}k\mathbf{J}_k = \frac{1}{k}\mathbf{J}_k$*.*

b) Prove that $\mathbf{Q}_{\mathrm{T}}\mathbf{M}_{\mathrm{B}} = \mathbf{M}_{\mathrm{B}}\mathbf{Q}_{\mathrm{T}} = \mathbf{Q}_{\mathrm{T}}\mathbf{M}_{\mathrm{G}} = \mathbf{M}_{\mathrm{G}}\mathbf{Q}_{\mathrm{T}} = \mathbf{0}$.

$$\mathbf{Q}_{\mathrm{T}}\mathbf{M}_{\mathrm{B}} = \left(\mathbf{M}_{\mathrm{T}} - \mathbf{M}_{\mathrm{G}}\right)\mathbf{M}_{\mathrm{B}} = \mathbf{M}_{\mathrm{G}} - \mathbf{M}_{\mathrm{G}} = \mathbf{0} = \mathbf{M}_{\mathrm{B}}\mathbf{Q}_{\mathrm{T}}$$ *and*

$$\mathbf{Q}_{\mathrm{T}}\mathbf{M}_{\mathrm{G}} = \left(\mathbf{M}_{\mathrm{T}} - \mathbf{M}_{\mathrm{G}}\right)\mathbf{M}_{\mathrm{G}} = \mathbf{M}_{\mathrm{G}} - \mathbf{M}_{\mathrm{G}} = \mathbf{0} = \mathbf{M}_{\mathrm{G}}\mathbf{Q}_{\mathrm{T}}$$