STATISTICAL MODELLING

PRACTICAL XII
SOLUTIONS

XII.1 Let $\psi = E[\mathbf{Y}] = \mathbf{X}_{\mathrm{T}}\tau = (\mathbf{1}_b \otimes \mathbf{I}_t)\tau$, $\mathbf{M}_{\mathrm{BU}} = \mathbf{I}_b \otimes \mathbf{I}_t = \mathbf{I}_n$, $\mathbf{M}_{\mathrm{B}} = \mathbf{I}_b \otimes \frac{1}{t}\mathbf{J}_t$, $\mathbf{M}_{\mathrm{T}} = \frac{1}{b}\mathbf{J}_b \otimes \mathbf{I}_t$, $\mathbf{M}_{\mathrm{G}} = \frac{1}{b}\mathbf{J}_b \otimes \frac{1}{t}\mathbf{J}_t$, $\mathbf{Q}_{\mathrm{B}} = \mathbf{M}_{\mathrm{B}} - \mathbf{M}_{\mathrm{G}}$, $\mathbf{Q}_{\mathrm{T}} = \mathbf{M}_{\mathrm{T}} - \mathbf{M}_{\mathrm{G}}$ and $\mathbf{Q}_{\mathrm{BU_{Res}}} = \mathbf{M}_{\mathrm{BU}} - \mathbf{M}_{\mathrm{B}} - \mathbf{M}_{\mathrm{T}} + \mathbf{M}_{\mathrm{G}}$. Prove that $\mathbf{Q}_{\mathrm{B}}\mathbf{M}_{\mathrm{BU}} = \mathbf{Q}_{\mathrm{B}}$ and $\mathbf{Q}_{\mathrm{B}}\mathbf{M}_{\mathrm{B}} = \mathbf{Q}_{\mathrm{B}}$.

As $\mathbf{M}_{\mathrm{BU}} = \mathbf{I}_b \otimes \mathbf{I}_t = \mathbf{I}_n$*, clearly* $\mathbf{Q}_{\mathrm{B}}\mathbf{M}_{\mathrm{UP}} = \mathbf{Q}_{\mathrm{B}}\mathbf{I}_n = \mathbf{Q}_{\mathrm{B}}$*.*

$$\begin{aligned}
\mathbf{Q}_{\mathrm{B}}\mathbf{M}_{\mathrm{B}} &= (\mathbf{M}_{\mathrm{B}} - \mathbf{M}_{\mathrm{G}})\mathbf{M}_{\mathrm{B}} \\
&= \mathbf{M}_{\mathrm{B}} - \mathbf{M}_{\mathrm{G}}\mathbf{M}_{\mathrm{B}} \\
&= \mathbf{M}_{\mathrm{B}} - \left(\tfrac{1}{b}\mathbf{J}_b \otimes \tfrac{1}{t}\mathbf{J}_t\right)\left(\mathbf{I}_b \otimes \tfrac{1}{t}\mathbf{J}_t\right) \\
&= \mathbf{M}_{\mathrm{B}} - \left(\tfrac{1}{b}\mathbf{J}_b \otimes \tfrac{1}{t}\mathbf{J}_t\right) \text{ as } \tfrac{1}{b}\mathbf{J}_b\mathbf{I}_b = \tfrac{1}{b}\mathbf{J}_b \text{ and } \tfrac{1}{b}\mathbf{J}_b\tfrac{1}{b}\mathbf{J}_b = \tfrac{1}{b}\mathbf{J}_b \\
&= \mathbf{M}_{\mathrm{B}} - \mathbf{M}_{\mathrm{G}} \\
&= \mathbf{Q}_{\mathrm{B}}
\end{aligned}$$

XII.2 Derive an expression, in terms of the other **Q** matrices, for the matrix of the quadratic form for the Residual sum of squares $\mathbf{Y}'\mathbf{Q}_{\mathrm{U_{Res}}}\mathbf{Y}$ for a two-factor factorial experiment laid out using a CRD. Using this expression, derive an expression in terms of **M** matrices.

$$\begin{aligned}
\mathbf{Q}_{\mathrm{U_{Res}}} &= \mathbf{Q}_{\mathrm{U}} - \mathbf{Q}_{\mathrm{A}} - \mathbf{Q}_{\mathrm{B}} - \mathbf{Q}_{\mathrm{AB}} \\
&= (\mathbf{M}_{\mathrm{U}} - \mathbf{M}_{\mathrm{G}}) - (\mathbf{M}_{\mathrm{A}} - \mathbf{M}_{\mathrm{G}}) - (\mathbf{M}_{\mathrm{B}} - \mathbf{M}_{\mathrm{G}}) - (\mathbf{M}_{\mathrm{AB}} - \mathbf{M}_{\mathrm{A}} - \mathbf{M}_{\mathrm{B}} + \mathbf{M}_{\mathrm{G}}) \\
&= \mathbf{M}_{\mathrm{U}} - \mathbf{M}_{\mathrm{AB}}
\end{aligned}$$

Suppose that the observations are in standard order for A then B and finally the *r* reps of each combination. Give direct-product expressions for the matrices $\mathbf{M}_{\mathrm{U}}$, $\mathbf{M}_{\mathrm{AB}}$, $\mathbf{M}_{\mathrm{A}}$, $\mathbf{M}_{\mathrm{B}}$ and $\mathbf{M}_{\mathrm{G}}$ using rule X.1 and the fact that A, B and a dummy Reps factor would uniquely index the units.

$\mathbf{M}_{\mathrm{U}} = \mathbf{I}_a \otimes \mathbf{I}_b \otimes \mathbf{I}_r = \mathbf{I}_n$, $\mathbf{M}_{\mathrm{AB}} = \mathbf{I}_a \otimes \mathbf{I}_b \otimes \frac{1}{r}\mathbf{J}_r$, $\mathbf{M}_{\mathrm{A}} = \mathbf{I}_a \otimes \frac{1}{b}\mathbf{J}_b \otimes \frac{1}{r}\mathbf{J}_r$, $\mathbf{M}_{\mathrm{B}} = \frac{1}{a}\mathbf{J}_a \otimes \mathbf{I}_b \otimes \frac{1}{r}\mathbf{J}_r$ and $\mathbf{M}_{\mathrm{G}} = \frac{1}{a}\mathbf{J}_a \otimes \frac{1}{b}\mathbf{J}_b \otimes \frac{1}{r}\mathbf{J}_r$.

Prove that $\mathbf{Q}_{\mathrm{U_{Res}}}$ is symmetric and idempotent and that the Residual degrees of freedom are equal to $ab(r-1)$.

As $\mathbf{M}_{\mathrm{U}} = \mathbf{I}_a \otimes \mathbf{I}_b \otimes \mathbf{I}_r = \mathbf{I}_n$ *and* $\mathbf{M}_{\mathrm{AB}} = \mathbf{I}_a \otimes \mathbf{I}_b \otimes \frac{1}{r}\mathbf{J}_r$*, we have from lemma XII.1 that these two matrices are symmetric and idempotent.*

Hence, symmetry is proved as follows:

$$\begin{aligned}\mathbf{Q}'_{\mathrm{U_{Res}}} &= \left(\mathbf{M}_{\mathrm{U}} - \mathbf{M}_{\mathrm{AB}}\right)' \\ &= \mathbf{M}_{\mathrm{U}} - \mathbf{M}_{\mathrm{AB}} \\ &= \mathbf{Q}_{\mathrm{U_{Res}}}\end{aligned}$$

and, as clearly $\mathbf{M}_{\mathrm{U}}\mathbf{M}_{\mathrm{AB}} = \mathbf{M}_{\mathrm{AB}}\mathbf{M}_{\mathrm{U}} = \mathbf{M}_{\mathrm{AB}}$*, idempotency is proved as follows:*

$$\begin{aligned}\mathbf{Q}_{\mathrm{U_{Res}}}\mathbf{Q}_{\mathrm{U_{Res}}} &= \left(\mathbf{M}_{\mathrm{U}} - \mathbf{M}_{\mathrm{AB}}\right)\left(\mathbf{M}_{\mathrm{U}} - \mathbf{M}_{\mathrm{AB}}\right) \\ &= \mathbf{M}_{\mathrm{U}} - \mathbf{M}_{\mathrm{AB}} - \mathbf{M}_{\mathrm{AB}} + \mathbf{M}_{\mathrm{AB}}\mathbf{M}_{\mathrm{AB}} \\ &= \mathbf{M}_{\mathrm{U}} - \mathbf{M}_{\mathrm{AB}} - \mathbf{M}_{\mathrm{AB}} + \mathbf{M}_{\mathrm{AB}} \\ &= \mathbf{M}_{\mathrm{U}} - \mathbf{M}_{\mathrm{AB}} \\ &= \mathbf{Q}_{\mathrm{U_{Res}}}\end{aligned}$$

As $\mathbf{Q}_{\mathrm{U_{Res}}}$ *is idempotent the Residual degrees of freedom are given by* $trace\left(\mathbf{Q}_{\mathrm{U_{Res}}}\right)$*. Now, using lemma XII.4,*

$$\begin{aligned}trace\left(\mathbf{Q}_{\mathrm{U_{Res}}}\right) &= trace\left(\mathbf{M}_{\mathrm{U}} - \mathbf{M}_{\mathrm{AB}}\right) \\ &= trace\left(\mathbf{M}_{\mathrm{U}}\right) - trace\left(\mathbf{M}_{\mathrm{AB}}\right) \\ &= abr - ab \\ &= ab\left(r-1\right)\end{aligned}$$

XII.3 For an RCBD, let $\psi = E\left[\mathbf{Y}\right] = \mathbf{X}_{\mathrm{T}}\tau = \left(\mathbf{1}_b \otimes \mathbf{I}_t\right)\tau$, $\mathbf{V} = \sigma^2_{\mathrm{BU}}\mathbf{I}_n + \sigma^2_{\mathrm{B}}\left(\mathbf{I}_b \otimes \mathbf{J}_t\right) = \sigma^2_{\mathrm{BU}}\mathbf{M}_{\mathrm{BU}} + t\sigma^2_{\mathrm{B}}\mathbf{M}_{\mathrm{B}}$, $SS_{\mathrm{B}} = \mathbf{Y}'\mathbf{Q}_{\mathrm{B}}\mathbf{Y} = \mathbf{Y}'\left(\mathbf{M}_{\mathrm{B}} - \mathbf{M}_{\mathrm{G}}\right)\mathbf{Y}$, $SS_{\mathrm{T}} = \mathbf{Y}'\mathbf{Q}_{\mathrm{T}}\mathbf{Y} = \mathbf{Y}'\left(\mathbf{M}_{\mathrm{T}} - \mathbf{M}_{\mathrm{G}}\right)\mathbf{Y}$ and $SS_{\mathrm{BU_{Res}}} = \mathbf{Y}'\mathbf{Q}_{\mathrm{BU_{Res}}}\mathbf{Y} = \mathbf{Y}'\left(\mathbf{Q}_{\mathrm{BU}} - \mathbf{Q}_{\mathrm{T}}\right)\mathbf{Y} = \mathbf{Y}'\left(\mathbf{M}_{\mathrm{BU}} - \mathbf{M}_{\mathrm{B}} - \mathbf{M}_{\mathrm{T}} + \mathbf{M}_{\mathrm{G}}\right)\mathbf{Y}$.

Prove that

$$E\left[SS_{\mathrm{T}}/\left(t-1\right)\right] = \sigma^2_{\mathrm{BU}} + q_{\mathrm{T}}\left(\psi\right),$$

where $q_{\mathrm{T}}\left(\psi\right) = \sum_{j=1}^{t} b\left(\tau_j - \overline{\tau}_.\right)^2 \Big/ \left(t-1\right)$, $\overline{\tau}_. = \sum_{j=1}^{t}\tau_j \big/ t$, τ_j is the jth element of the t-vector τ, b is the number of blocks and t is the number of treatments.

From theorem XII.2 we have that

$$E\left[\frac{SS_{\mathrm{T}}}{(t-1)}\right]=E\left[\mathbf{Y}'\mathbf{Q}_{\mathrm{T}}\mathbf{Y}\right]\Big/(b-1)$$

$$=\left\{trace\left(\mathbf{Q}_{\mathrm{T}}\left\{\sigma^2_{\mathrm{BU}}\mathbf{M}_{\mathrm{BU}}+t\sigma^2_{\mathrm{B}}\mathbf{M}_{\mathrm{B}}\right\}\right)+\left(\mathbf{X}_{\mathrm{T}}\tau\right)'\mathbf{Q}_{\mathrm{T}}\left(\mathbf{X}_{\mathrm{T}}\tau\right)\right\}\Big/\{t-1\}$$

Now $\mathbf{Q}_{\mathrm{T}}\mathbf{M}_{\mathrm{BU}}=\mathbf{Q}_{\mathrm{T}}$ *and in the proof of theorem XII.6 we showed that* $\mathbf{M}_{\mathrm{T}}\mathbf{M}_{\mathrm{B}}=\mathbf{M}_{\mathrm{G}}\mathbf{M}_{\mathrm{B}}=\mathbf{M}_{\mathrm{G}}$ *so that*

$$\begin{aligned}\mathbf{Q}_{\mathrm{T}}\mathbf{M}_{\mathrm{B}}&=\left(\mathbf{M}_{\mathrm{T}}-\mathbf{M}_{\mathrm{G}}\right)\mathbf{M}_{\mathrm{B}}\\&=\mathbf{M}_{\mathrm{T}}\mathbf{M}_{\mathrm{B}}-\mathbf{M}_{\mathrm{G}}\mathbf{M}_{\mathrm{B}}\\&=\mathbf{M}_{\mathrm{G}}-\mathbf{M}_{\mathrm{G}}\\&=\mathbf{0}\end{aligned}$$

Further from theorem XII.6 $trace\left(\mathbf{Q}_{\mathrm{T}}\right)=t-1$ *and using lemma XII.5,*

$$\begin{aligned}\mathbf{X}'_{\mathrm{T}}\mathbf{Q}_{\mathrm{T}}\mathbf{X}_{\mathrm{T}}&=\mathbf{X}'_{\mathrm{T}}\mathbf{M}_{\mathrm{T}}\mathbf{X}_{\mathrm{T}}-\mathbf{X}'_{\mathrm{T}}\mathbf{M}_{\mathrm{G}}\mathbf{X}_{\mathrm{T}}\\&=b\mathbf{I}_t-\tfrac{b}{t}\mathbf{J}_t\\&=b\left(\mathbf{I}_t-\tfrac{1}{t}\mathbf{J}_t\right)\end{aligned}$$

where $\mathbf{I}_t-\frac{1}{t}\mathbf{J}_t$ *is symmetric and idempotent.*

Hence,

$$\begin{aligned}E\left[\frac{SS_{\mathrm{T}}}{(t-1)}\right]&=E\left[\mathbf{Y}'\mathbf{Q}_{\mathrm{T}}\mathbf{Y}\right]\Big/(b-1)\\&=\left\{trace\left(\mathbf{Q}_{\mathrm{T}}\left\{\sigma^2_{\mathrm{BU}}\mathbf{M}_{\mathrm{BU}}+t\sigma^2_{\mathrm{B}}\mathbf{M}_{\mathrm{B}}\right\}\right)+\boldsymbol{\psi}'\mathbf{Q}_{\mathrm{T}}\boldsymbol{\psi}\right\}\Big/\{t-1\}\\&=\left\{trace\left(\mathbf{Q}_{\mathrm{T}}\left\{\sigma^2_{\mathrm{BU}}\mathbf{M}_{\mathrm{BU}}+t\sigma^2_{\mathrm{B}}\mathbf{M}_{\mathrm{B}}\right\}\right)+\left(\mathbf{X}_{\mathrm{T}}\tau\right)'\mathbf{Q}_{\mathrm{T}}\left(\mathbf{X}_{\mathrm{T}}\tau\right)\right\}\Big/\{t-1\}\\&=\left\{\sigma^2_{\mathrm{BU}}trace\left(\mathbf{Q}_{\mathrm{T}}\right)+\mathbf{0}+\tau'\mathbf{X}'_{\mathrm{T}}\mathbf{Q}_{\mathrm{T}}\mathbf{X}_{\mathrm{T}}\tau\right\}\Big/\{t-1\}\\&=\left\{\sigma^2_{\mathrm{BU}}(t-1)+\tau'b\left(\mathbf{I}_t-\tfrac{1}{t}\mathbf{J}_t\right)\tau\right\}\Big/\{t-1\}\\&=\sigma^2_{\mathrm{BU}}+\left\{b\left(\left(\mathbf{I}_t-\tfrac{1}{t}\mathbf{J}_t\right)\tau\right)'\left(\left(\mathbf{I}_t-\tfrac{1}{t}\mathbf{J}_t\right)\tau\right)\right\}\Big/\{t-1\}\end{aligned}$$

But $\left(\mathbf{I}_t-\frac{1}{t}\mathbf{J}_t\right)\tau=\tau-\overline{\tau}_{.}\mathbf{1}_t$ *and the quantity inside the curly braces is b times the sum of squares of the elements of this vector. That is* $b\left(\left(\mathbf{I}_t-\frac{1}{t}\mathbf{J}_t\right)\tau\right)'\left(\left(\mathbf{I}_t-\frac{1}{t}\mathbf{J}_t\right)\tau\right)\Big/(t-1)=b\sum_{j=1}^{t}\left(\tau_j-\overline{\tau}_{.}\right)^2\Big/(t-1)=q_{\mathrm{T}}(\psi)$ *where* $\overline{\tau}_{.}=\sum_{j=1}^{t}\tau_j\Big/t$.

XII.4 For a Latin square design in which Rows and Columns are random the maximal models are $E[\mathbf{Y}] = \mathbf{X}_{\mathrm{T}}\boldsymbol{\tau}$, $\mathbf{V} = \sigma^2_{\mathrm{RC}}(\mathbf{I}_t \otimes \mathbf{I}_t) + \sigma^2_{\mathrm{R}}(\mathbf{I}_t \otimes \mathbf{J}_t) + \sigma^2_{\mathrm{C}}(\mathbf{J}_t \otimes \mathbf{I}_t) = \sigma^2_{\mathrm{RC}}\mathbf{M}_{\mathrm{RC}} + t\sigma^2_{\mathrm{R}}\mathbf{M}_{\mathrm{R}} + t\sigma^2_{\mathrm{C}}\mathbf{M}_{\mathrm{C}}$ where $\mathbf{M}_{\mathrm{RC}}$, $\mathbf{M}_{\mathrm{R}}$ and $\mathbf{M}_{\mathrm{C}}$ are symmetric and idempotent. Prove that the test statistic for testing the hypothesis that $\sigma^2_{\mathrm{R}} = 0$,

$$F_{(t-1),\,(t-1)(t-2)} = \frac{s^2_{\mathrm{R}}}{s^2_{\mathrm{RC}_{\mathrm{Res}}}} = \frac{\mathbf{Y}'\mathbf{Q}_{\mathrm{R}}\mathbf{Y}/(t-1)}{\mathbf{Y}'\mathbf{Q}_{\mathrm{RC}_{\mathrm{Res}}}\mathbf{Y}/(t-1)(t-2)},$$

is distributed as Snedecor's F with $(t-1)$ and $(t-1)(t-2)$ degrees of freedom.

You are given that $\mathbf{M}_{\mathrm{R}}\mathbf{X}_{\mathrm{T}} = \mathbf{M}_{\mathrm{C}}\mathbf{X}_{\mathrm{T}} = \mathbf{M}_{\mathrm{G}}\mathbf{X}_{\mathrm{T}} = \frac{1}{t}\mathbf{1}_t \otimes \mathbf{J}_t$, $\mathbf{M}_{\mathrm{T}}\mathbf{X}_{\mathrm{T}} = \mathbf{X}_{\mathrm{T}}$ and the product of any nonidentical pair of $\mathbf{M}_{\mathrm{R}}$, $\mathbf{M}_{\mathrm{C}}$, $\mathbf{M}_{\mathrm{T}}$ and $\mathbf{M}_{\mathrm{G}}$ is equal to $\mathbf{M}_{\mathrm{G}}$. Also $\mathbf{Q}_{\mathrm{R}}$ and $\mathbf{Q}_{\mathrm{RC}_{\mathrm{Res}}}$ are idempotent with $trace(\mathbf{Q}_{\mathrm{T}}) = t-1$ and $trace(\mathbf{Q}_{\mathrm{RC}_{\mathrm{Res}}}) = (t-1)(t-2)$.

We have to show that

a) $E[\mathbf{Q}_{\mathrm{R}}\mathbf{Y}] = E[\mathbf{Q}_{\mathrm{RC}_{\mathrm{Res}}}\mathbf{Y}] = \mathbf{0}$ *and* $E[\mathbf{Y}'\mathbf{Q}_{\mathrm{R}}\mathbf{Y}/(t-1)] = E[\mathbf{Y}'\mathbf{Q}_{\mathrm{RC}_{\mathrm{Res}}}\mathbf{Y}/\{(t-1)(t-2)\}] = \sigma^2_{\mathrm{RC}}$ *under the null expectation and variation models so that theorem XII.3 can be invoked to conclude that* $(1/\sigma^2_{\mathrm{RC}})\mathbf{Y}'\mathbf{Q}_{\mathrm{R}}\mathbf{Y}$ *and* $(1/\sigma^2_{\mathrm{RC}})\mathbf{Y}'\mathbf{Q}_{\mathrm{RC}_{\mathrm{Res}}}\mathbf{Y}$ *follow chi-square distributions;*

b) $\mathbf{Y}'\mathbf{Q}_{\mathrm{R}}\mathbf{Y}$ *and* $\mathbf{Y}'\mathbf{Q}_{\mathrm{RC}_{\mathrm{Res}}}\mathbf{Y}$ *are independent quadratic forms as outlined in theorem XII.4;*

c) *then theorem XII.5 can be invoked to obtain the distribution of the F test statistic.*

Firstly,

$$\begin{aligned} E[\mathbf{Q}_{\mathrm{R}}\mathbf{Y}] &= \mathbf{Q}_{\mathrm{R}}E[\mathbf{Y}] \\ &= \mathbf{Q}_{\mathrm{R}}\mathbf{X}_{\mathrm{T}}\boldsymbol{\tau} \\ &= (\mathbf{M}_{\mathrm{R}} - \mathbf{M}_{\mathrm{G}})\mathbf{X}_{\mathrm{T}}\boldsymbol{\tau} \\ &= (\mathbf{M}_{\mathrm{R}}\mathbf{X}_{\mathrm{T}} - \mathbf{M}_{\mathrm{G}}\mathbf{X}_{\mathrm{T}})\boldsymbol{\tau} \\ &= \left(\tfrac{1}{t}\mathbf{1}_t \otimes \mathbf{J}_t - \tfrac{1}{t}\mathbf{1}_t \otimes \mathbf{J}_t\right)\boldsymbol{\tau} \\ &= \mathbf{0} \end{aligned}$$

and

$$
\begin{aligned}
E\left[\mathbf{Q}_{\mathrm{RC}_{\mathrm{Res}}}\mathbf{Y}\right] &= \mathbf{Q}_{\mathrm{RC}_{\mathrm{Res}}}E\left[\mathbf{Y}\right] \\
&= \mathbf{Q}_{\mathrm{RC}_{\mathrm{Res}}}\mathbf{X}_{\mathrm{T}}\boldsymbol{\tau} \\
&= \left(\mathbf{Q}_{\mathrm{RC}} - \mathbf{Q}_{\mathrm{T}}\right)\mathbf{X}_{\mathrm{T}}\boldsymbol{\tau} \\
&= \left(\mathbf{M}_{\mathrm{RC}} - \mathbf{M}_{\mathrm{R}} - \mathbf{M}_{\mathrm{C}} - \mathbf{M}_{\mathrm{T}} + 2\mathbf{M}_{\mathrm{G}}\right)\mathbf{X}_{\mathrm{T}}\boldsymbol{\tau} \\
&= \left(\mathbf{X}_{\mathrm{T}} - \tfrac{1}{t}\mathbf{1}_t \otimes \mathbf{J}_t - \tfrac{1}{t}\mathbf{1}_t \otimes \mathbf{J}_t - \mathbf{X}_{\mathrm{T}} + 2\tfrac{1}{t}\mathbf{1}_t \otimes \mathbf{J}_t\right)\boldsymbol{\tau} \\
&= \mathbf{0}
\end{aligned}
$$

As $\mathbf{Q}_{\mathrm{R}}\mathbf{M}_{\mathrm{RC}} = \mathbf{Q}_{\mathrm{R}}$, $\sigma^2_{\mathrm{R}} = 0$, $\mathbf{Q}_{\mathrm{R}}\mathbf{M}_{\mathrm{C}} = \left(\mathbf{M}_{\mathrm{R}} - \mathbf{M}_{\mathrm{G}}\right)\mathbf{M}_{\mathrm{C}} = \mathbf{M}_{\mathrm{G}} - \mathbf{M}_{\mathrm{G}} = \mathbf{0}$ *and* $\mathbf{Q}_{\mathrm{R}}\mathbf{X}_{\mathrm{T}} = \mathbf{0}$ *(see above for latter),*

$$
\begin{aligned}
&E\left[\mathbf{Y}'\mathbf{Q}_{\mathrm{R}}\mathbf{Y}\big/(t-1)\right] \\
&= \left\{trace\left(\mathbf{Q}_{\mathrm{R}}\left\{\sigma^2_{\mathrm{RC}}\mathbf{M}_{\mathrm{RC}} + t\sigma^2_{\mathrm{R}}\mathbf{M}_{\mathrm{R}} + t\sigma^2_{\mathrm{C}}\mathbf{M}_{\mathrm{C}}\right\}\right) + \left(\mathbf{X}_{\mathrm{T}}\boldsymbol{\tau}\right)'\mathbf{Q}_{\mathrm{R}}\left(\mathbf{X}_{\mathrm{T}}\boldsymbol{\tau}\right)\right\}\Big/\left\{t-1\right\} \\
&= \left\{\sigma^2_{\mathrm{RC}}trace\left(\mathbf{Q}_{\mathrm{T}}\right) + \mathbf{0} + \mathbf{0} + \boldsymbol{\tau}'\mathbf{X}'_{\mathrm{T}}\mathbf{Q}_{\mathrm{R}}\mathbf{X}_{\mathrm{T}}\boldsymbol{\tau}\right\}\Big/\left\{t-1\right\} \\
&= \left\{\sigma^2_{\mathrm{RC}}\left(t-1\right) + 0\right\}\Big/\left\{t-1\right\} \\
&= \sigma^2_{\mathrm{RC}}
\end{aligned}
$$

Also, $\mathbf{Q}_{\mathrm{RC}_{\mathrm{Res}}}\mathbf{M}_{\mathrm{RC}} = \mathbf{Q}_{\mathrm{RC}_{\mathrm{Res}}}$,
$\mathbf{Q}_{\mathrm{RC}_{\mathrm{Res}}}\mathbf{M}_{\mathrm{C}} = \left(\mathbf{M}_{\mathrm{RC}} - \mathbf{M}_{\mathrm{R}} - \mathbf{M}_{\mathrm{C}} - \mathbf{M}_{\mathrm{T}} + 2\mathbf{M}_{\mathrm{G}}\right)\mathbf{M}_{\mathrm{C}} = \mathbf{M}_{\mathrm{C}} - \mathbf{M}_{\mathrm{G}} - \mathbf{M}_{\mathrm{C}} - \mathbf{M}_{\mathrm{G}} + 2\mathbf{M}_{\mathrm{G}} = \mathbf{0}$ *and* $\mathbf{Q}_{\mathrm{RC}_{\mathrm{Res}}}\mathbf{X}_{\mathrm{T}} = \mathbf{0}$,

$$
\begin{aligned}
&E\left[\mathbf{Y}'\mathbf{Q}_{\mathrm{RC}_{\mathrm{Res}}}\mathbf{Y}\big/\left\{(t-1)(t-2)\right\}\right] \\
&= \left\{\begin{matrix} trace\left(\mathbf{Q}_{\mathrm{RC}_{\mathrm{Res}}}\left\{\sigma^2_{\mathrm{RC}}\mathbf{M}_{\mathrm{RC}} + t\sigma^2_{\mathrm{R}}\mathbf{M}_{\mathrm{R}} + t\sigma^2_{\mathrm{C}}\mathbf{M}_{\mathrm{C}}\right\}\right) \\ + \left(\mathbf{X}_{\mathrm{T}}\boldsymbol{\tau}\right)'\mathbf{Q}_{\mathrm{RC}_{\mathrm{Res}}}\left(\mathbf{X}_{\mathrm{T}}\boldsymbol{\tau}\right) \end{matrix}\right\}\Bigg/\left\{(t-1)(t-2)\right\} \\
&= \left\{\sigma^2_{\mathrm{RC}}trace\left(\mathbf{Q}_{\mathrm{RC}_{\mathrm{Res}}}\right) + \mathbf{0} + \mathbf{0} + \boldsymbol{\tau}'\mathbf{X}'_{\mathrm{T}}\mathbf{Q}_{\mathrm{RC}_{\mathrm{Res}}}\mathbf{X}_{\mathrm{T}}\boldsymbol{\tau}\right\}\Big/\left\{(t-1)(t-2)\right\} \\
&= \left\{\sigma^2_{\mathrm{RC}}\left\{(t-1)(t-2)\right\} + 0\right\}\Big/\left\{(t-1)(t-2)\right\} \\
&= \sigma^2_{\mathrm{RC}}
\end{aligned}
$$

Secondly, to show that $\mathbf{Y}'\mathbf{Q}_{\mathrm{R}}\mathbf{Y}$ *and* $\mathbf{Y}'\mathbf{Q}_{\mathrm{RC}_{\mathrm{Res}}}\mathbf{Y}$ *are independent quadratic forms we have to show that* $\mathbf{Q}_{\mathrm{R}}$ *and* $\mathbf{Q}_{\mathrm{RC}_{\mathrm{Res}}}$ *meet two of the three conditions outlined in theorem XII.4. As given above,* $\mathbf{Q}_{\mathrm{R}}$ *and* $\mathbf{Q}_{\mathrm{RC}_{\mathrm{Res}}}$ *are idempotent so that condition 1 is met. For condition 3, we require that* $\mathbf{Q}_{\mathrm{R}}\mathbf{Q}_{\mathrm{RC}_{\mathrm{Res}}} = \mathbf{0}$*. Now,*

$$\begin{aligned}
\mathbf{Q}_{\mathrm{R}}\mathbf{Q}_{\mathrm{RC}_{\mathrm{Res}}} &= \left(\mathbf{M}_{\mathrm{R}} - \mathbf{M}_{\mathrm{G}}\right)\left(\mathbf{M}_{\mathrm{RC}} - \mathbf{M}_{\mathrm{R}} - \mathbf{M}_{\mathrm{C}} - \mathbf{M}_{\mathrm{T}} + 2\mathbf{M}_{\mathrm{G}}\right) \\
&= \left(\mathbf{M}_{\mathrm{R}}\mathbf{M}_{\mathrm{RC}} - \mathbf{M}_{\mathrm{R}}\mathbf{M}_{\mathrm{R}} - \mathbf{M}_{\mathrm{R}}\mathbf{M}_{\mathrm{C}} - \mathbf{M}_{\mathrm{R}}\mathbf{M}_{\mathrm{T}} + 2\mathbf{M}_{\mathrm{R}}\mathbf{M}_{\mathrm{G}}\right) \\
&\qquad - \left(\mathbf{M}_{\mathrm{G}}\mathbf{M}_{\mathrm{RC}} - \mathbf{M}_{\mathrm{G}}\mathbf{M}_{\mathrm{R}} - \mathbf{M}_{\mathrm{G}}\mathbf{M}_{\mathrm{C}} - \mathbf{M}_{\mathrm{G}}\mathbf{M}_{\mathrm{T}} + 2\mathbf{M}_{\mathrm{G}}\mathbf{M}_{\mathrm{G}}\right) \\
&= \left(\mathbf{M}_{\mathrm{R}} - \mathbf{M}_{\mathrm{R}} - \mathbf{M}_{\mathrm{G}} - \mathbf{M}_{\mathrm{G}} + 2\mathbf{M}_{\mathrm{G}}\right) - \left(\mathbf{M}_{\mathrm{G}} - \mathbf{M}_{\mathrm{G}} - \mathbf{M}_{\mathrm{G}} - \mathbf{M}_{\mathrm{G}} + 2\mathbf{M}_{\mathrm{G}}\right) \\
&= \mathbf{0} - \mathbf{0} \\
&= \mathbf{0}
\end{aligned}$$

Thirdly, theorem XII.5 means that as $\left(1/\sigma^2_{\mathrm{RC}}\right)\mathbf{Y}'\mathbf{Q}_{\mathrm{R}}\mathbf{Y}$ *and* $\left(1/\sigma^2_{\mathrm{RC}}\right)\mathbf{Y}'\mathbf{Q}_{\mathrm{RC}_{\mathrm{Res}}}\mathbf{Y}$ *are distributed as independent chi-squares with degrees of freedom* $(t-1)$ *and* $(t-1)(t-2)$*, respectively, then*

$$F_{(t-1),\,(t-1)(t-2)} = \frac{\left(1/\sigma^2_{\mathrm{RC}}\right)\mathbf{Y}'\mathbf{Q}_{\mathrm{R}}\mathbf{Y}\big/(t-1)}{\left(1/\sigma^2_{\mathrm{RC}}\right)\mathbf{Y}'\mathbf{Q}_{\mathrm{RC}_{\mathrm{Res}}}\mathbf{Y}\big/(t-1)(t-2)} = \frac{\mathbf{Y}'\mathbf{Q}_{\mathrm{R}}\mathbf{Y}\big/(t-1)}{\mathbf{Y}'\mathbf{Q}_{\mathrm{RC}_{\mathrm{Res}}}\mathbf{Y}\big/(t-1)(t-2)} = \frac{s^2_{\mathrm{R}}}{s^2_{\mathrm{RC}_{\mathrm{Res}}}}$$

follows an F distribution with $(t-1)$ *and* $(t-1)(t-2)$ *degrees of freedom.* ■