00STATISTICAL MODELLING

XII. Justifying the ANOVA-based hypothesis test

In chapter VI we gave a procedure for determining an ANOVA table. In this chapter we look at the theory that justifies this procedure. In particular, we examine the basis of the sums of squares, degrees of freedom, expected mean squares and the distribution of the F statistic.

XII.A The sources for an ANOVA

There are various methods for determining the sources to include in an analysis of variance table. Perhaps the most popular is to try to identify the situation that is closest to yours in a textbook. This has the disadvantage that, if your experiment is not exactly the same as one in a textbook, then you might use an analysis from the textbook that does not take into account the difference(s).

The approach that we have used is based on dividing the factors into unrandomized and randomized factors as follows:

1. identify the unrandomized and randomized factors;
2. determine the structure formulae that describe the crossing and nesting relations a) amongst the unrandomized factors and b) amongst the randomized factors (and in some cases between the randomized and some unrandomized factors);
3. expand the structure formulae to obtain two sets of sources;
4. form the lines in the ANOVA table by a) listing the unrandomized sources in the table, b) entering the randomized sources indented under the appropriate unrandomized sources, and c) adding Residuals for unrandomized sources where there are degrees of freedom left over.

This approach extracts the key features of the experiment arising from the randomization and captures these in the structure formulae for use in determining the terms in the analysis. The analysis of variance table produced using this method displays the confounding arising from the randomization, something that traditional tables do not do.

XII.B The sums of squares for an ANOVA

Derivation of the sums of squares, given the sources in the ANOVA, is based on the **S** matrices. For a structure formula, one must identify the generalized factors corresponding to the sources derived from that formula. For each generalized factor there is an **S** that specifies the units that are related in having the same level of that generalized factor. For example, $\mathbf{S}_{\text{Blocks}}$ has a one for every pair of units that occur in the same block. Generally we have two sets of relationship (**S**) or, equivalently, mean-operator $\left(\mathbf{M} = \frac{1}{n/f}\mathbf{S} = \frac{1}{g}\mathbf{S}\right)$ matrices: one for the unrandomized factors and one for the randomized factors. (Note that relationship matrices are also called summation matrices.)

Now a set of such matrices forms the basis for an algebra that is called a relationship algebra. Associated with a set of **S**s is a set of **Q**s, again one **Q** for each generalized factor, that are the idempotents of the relationship algebra and provide the quadratic-form matrices for the analysis. Indeed under certain conditions, that are met by all our examples, a set of **Q**s derived from a structure formula forms a complete set of mutually-orthogonal idempotents (CSMOI — see definition XI.10).

So we form the sums of squares of the projections of the data vector into the subspaces projected onto by the **Q**s. Rule VI.6 can be used to obtain expressions for **Q**s in terms of **M**s. The following rule, that generalizes Rule XI.1, can be used to obtain expressions for the **S** and **M** matrices as the direct products of **I** and **J** matrices.

Rule XII.1: The **S** for a generalized factor, formed as the subset of a set of *s* equally-replicated factors that uniquely index the units, is the direct product of *s* matrices, provided the *s* factors are arranged in standard order. Taking the *s* factors in the sequence specified by the standard order, for each factor in the generalized factor include an **I** matrix in the direct product and **J** matrices for those that are not. The order of a matrix in the direct product is equal to the number of levels of the corresponding factor. The **M** matrix is the same direct product of **I** and **J** matrices, except that each **J** is multiplied by the reciprocal of its order. ■

In particular this rule would apply to the factors from the unrandomized structure. We have been claiming that all the **M**s are symmetric and idempotent. We prove this useful result for those that can be expressed as the direct product of **I** and **J** matrices in the next lemma.

Lemma XII.1: Let **M** be the direct product of **I** and **J** matrices, the latter multiplied by the reciprocal of their order, as prescribed in rule XII.1. Then **M** is symmetric and idempotent.

Proof: Since $(\mathbf{A}\otimes\mathbf{B})' = \mathbf{A}'\otimes\mathbf{B}'$ and **I** and **J** are both symmetric, **M** must also be symmetric.

Since $(\mathbf{A}\otimes\mathbf{B})(\mathbf{C}\otimes\mathbf{D}) = \mathbf{AC}\otimes\mathbf{BD}$ and in particular $(\mathbf{A}\otimes\mathbf{B})(\mathbf{A}\otimes\mathbf{B}) = \mathbf{A}^2\otimes\mathbf{B}^2$, $\mathbf{M}^2$ must be the direct product of **I** and the square of **J** matrices, with each **J** multiplied by the reciprocal of its order. But $\frac{1}{q}\mathbf{J}_q\frac{1}{q}\mathbf{J}_q = \frac{1}{q^2}\mathbf{J}_q\mathbf{J}_q = \frac{1}{q^2}q\mathbf{J}_q = \frac{1}{q}\mathbf{J}_q$. Consequently, $\mathbf{M}^2$ and **M** will be the same direct product of **I** and **J** matrices, the latter multiplied by the reciprocal of their order. ■

XII.C Degrees of freedom of the sums of squares for an ANOVA

Definition XII.1: The degrees of freedom of a sum of squares is the rank of the idempotent of its quadratic form. That is the degrees of freedom of $\mathbf{Y}'\mathbf{QY}$ is given by $rank(\mathbf{Q})$. ■

The following definition and lemma establish that the trace of an idempotent is the same as rank and hence its degrees of freedom.

Definition XII.2: The trace of a square matrix is the sum of its diagonal elements. ■

Lemma XII.2: For **B** idempotent, $rank(\mathbf{B}) = trace(\mathbf{B})$.

Proof: The proof is based on the following facts:

- the trace of a matrix is equal to the sum of its eigenvalues;
- the rank of a matrix is equal to the number of nonzero eigenvalues; and
- the eigenvalues of an idempotent can be proved to be either 1 or zero so that the sum and number of nonzero eigenvalues are equal. ■

Clearly, the degrees of freedom can be established by determining the trace of a matrix so that the following results on the traces of matrices will be useful.

Lemma XII.3: Let c be a scalar and **A**, **B** and **C** be matrices. Then, when the appropriate operations are defined, we have

- $trace(\mathbf{A}') = trace(\mathbf{A})$
- $trace(c\mathbf{A}) = c \times trace(\mathbf{A})$
- $trace(\mathbf{A}+\mathbf{B}) = trace(\mathbf{A}) + trace(\mathbf{B})$
- $trace(\mathbf{AB}) = trace(\mathbf{BA})$
- $trace(\mathbf{ABC}) = trace(\mathbf{CAB}) = trace(\mathbf{BCA})$
- $trace(\mathbf{A}\otimes\mathbf{B}) = trace(\mathbf{A})trace(\mathbf{B})$
- $trace(\mathbf{A}'\mathbf{A}) = 0$ if and only if $\mathbf{A} = \mathbf{0}$. ■

Now each **Q** matrix is a linear combination of **M** matrices so that the $trace(\mathbf{Q})$ will be the same linear combination of the traces of the **M** matrices. In the next lemma we prove that the trace of an **M** is equal to the number of levels in its generalized factor, when the corresponding summation matrix is direct product of **I** and **J** matrices.

Lemma XII.4: Let $\mathbf{M}_{\mathrm{F}}$ be a mean operator matrix for a generalized factor F that has f levels each replicated $\frac{n}{f} = g$ times and $\mathbf{S}_{\mathrm{F}}$ be the corresponding summation matrix that is direct product of **I** and **J** matrices. Then $trace(\mathbf{M}_{\mathrm{F}}) = f$.

Proof: Now $trace(\mathbf{M}_{\mathrm{F}}) = \frac{1}{g} trace(\mathbf{S}_{\mathrm{F}})$. But $\mathbf{S}_{\mathrm{F}}$ is the direct product of **I** and **J** matrices. However, $trace(\mathbf{A} \otimes \mathbf{B}) = trace(\mathbf{A}) trace(\mathbf{B})$ and the traces of **I** and **J** matrices are equal to their orders. As the product of the orders of the **I** and **J** matrices for any **S** must be n, $trace(\mathbf{S}_{\mathrm{F}}) = n$ and so $trace(\mathbf{M}_{\mathrm{F}}) = \frac{1}{g} trace(\mathbf{S}_{\mathrm{F}}) = \frac{n}{g} = f$. ■

XII.D Expected mean squares for an ANOVA

As has been previously stated, the expected values of the mean squares are just the average or mean value of the mean squares under sampling from a population whose **Y** variables behave as described by the linear model. That is, it is the true mean value of the mean square and it depends on the model parameters. To derive the expected values, we note that the general form of a mean square is a quadratic form divided by a degrees of freedom, $\mathbf{Y}'\mathbf{QY}/\nu$. So we first establish an expression for the expectation of any quadratic form.

Theorem XII.1: Let **Y** be an $n \times 1$ vector of random variables with

$$E[\mathbf{Y}] = \boldsymbol{\psi} \text{ and } \operatorname{var}[\mathbf{Y}] = \mathbf{V},$$

where $\boldsymbol{\psi}$ is a $n \times 1$ vector of expected values and **V** is an $n \times n$ matrix.

Let **A** an $n \times n$ matrix of real numbers.

Then

$$E[\mathbf{Y}'\mathbf{AY}] = trace(\mathbf{AV}) + \boldsymbol{\psi}'\mathbf{A}\boldsymbol{\psi}.$$

Proof: Firstly note that, as $\mathbf{Y}'\mathbf{AY}$ is a scalar, $\mathbf{Y}'\mathbf{AY} = trace(\mathbf{Y}'\mathbf{AY})$

and recall that the trace of matrix products are cyclically commutative.

Thus, $E[\mathbf{Y}'\mathbf{AY}] = E[trace(\mathbf{Y}'\mathbf{AY})] = E[trace(\mathbf{AYY}')]$.

Now, for an $n \times n$ matrix **Z** $E[trace(\mathbf{Z})] = E\left[\sum_{i=1}^{n} z_{ii}\right] = \sum_{i=1}^{n} E[z_{ii}] = trace(E[\mathbf{Z}])$.

Also, lemma XI.5 states that $E[\mathbf{AY}] = \mathbf{A}E[\mathbf{Y}]$ and this can be extended to $E[\mathbf{AYY}'] = \mathbf{A}E[\mathbf{YY}']$.

Hence, $E\left[\mathit{trace}(\mathbf{AYY}')\right] = \mathit{trace}\left(E[\mathbf{AYY}']\right) = \mathit{trace}\left(\mathbf{A}E[\mathbf{YY}']\right)$.

Now we require an expression for $E[\mathbf{YY}']$ which we obtain by considering definition I.7 of $\mathbf{V} = E\left[(\mathbf{Y}-\boldsymbol{\psi})(\mathbf{Y}-\boldsymbol{\psi})'\right]$. From this we obtain

$$\begin{aligned}\mathbf{V} &= E\left[(\mathbf{Y}-\boldsymbol{\psi})(\mathbf{Y}'-\boldsymbol{\psi}')\right] \qquad \text{(lemma I.1)}\\ &= E\left[\mathbf{YY}' - \mathbf{Y}\boldsymbol{\psi}' - \boldsymbol{\psi}\mathbf{Y}' + \boldsymbol{\psi}\boldsymbol{\psi}'\right]\\ &= E[\mathbf{YY}'] - E[\mathbf{Y}\boldsymbol{\psi}'] - E[\boldsymbol{\psi}\mathbf{Y}'] + E[\boldsymbol{\psi}\boldsymbol{\psi}'].\end{aligned}$$

Now, $E[\boldsymbol{\psi}] = \boldsymbol{\psi}$ as the elements of $\boldsymbol{\psi}$ are population quantities and their average values in the population are the values to which they are equal. That is, they are constants with respect to expectation.

Thus

$$E[\mathbf{Y}\boldsymbol{\psi}'] = E[\mathbf{Y}]\boldsymbol{\psi}' = \boldsymbol{\psi}\boldsymbol{\psi}' = E[\boldsymbol{\psi}\mathbf{Y}'].$$

Hence, $\mathbf{V} = E[\mathbf{YY}'] - E[\mathbf{Y}\boldsymbol{\psi}'] - E[\boldsymbol{\psi}\mathbf{Y}'] + E[\boldsymbol{\psi}\boldsymbol{\psi}'] = E[\mathbf{YY}'] - \boldsymbol{\psi}\boldsymbol{\psi}'$ so that $E[\mathbf{YY}'] = \mathbf{V} + \boldsymbol{\psi}\boldsymbol{\psi}'$.

This leads to

$$\begin{aligned}E[\mathbf{Y}'\mathbf{AY}] &= \mathit{trace}\left(\mathbf{A}E[\mathbf{YY}']\right)\\ &= \mathit{trace}\left(\mathbf{A}(\mathbf{V} + \boldsymbol{\psi}\boldsymbol{\psi}')\right)\\ &= \mathit{trace}(\mathbf{AV}) + \mathit{trace}(\mathbf{A}\boldsymbol{\psi}\boldsymbol{\psi}')\\ &= \mathit{trace}(\mathbf{AV}) + \mathit{trace}(\boldsymbol{\psi}'\mathbf{A}\boldsymbol{\psi})\\ &= \mathit{trace}(\mathbf{AV}) + \boldsymbol{\psi}'\mathbf{A}\boldsymbol{\psi}.\end{aligned}$$

■

This extends to yield the following theorem for the expected value of a mean square.

Theorem XII.2: Let $\mathbf{Y}$ be an $n \times 1$ vector of random variables with

$$E[\mathbf{Y}] = \boldsymbol{\psi} \text{ and } \operatorname{var}[\mathbf{Y}] = \mathbf{V},$$

where $\boldsymbol{\psi}$ is a $n \times 1$ vector of expected values and $\mathbf{V}$ is an $n \times n$ matrix.

Let $\mathbf{Y}'\mathbf{QY}/\nu$ be the mean square where $\mathbf{Q}$ is an $n \times n$ symmetric, idempotent matrix and $\nu = \mathit{trace}(\mathbf{Q})$ is the degrees of freedom of the sums of squares.

Then

$$E[\mathbf{Y}'\mathbf{QY}/\nu] = \left(\mathit{trace}(\mathbf{QV}) + \boldsymbol{\psi}'\mathbf{Q}\boldsymbol{\psi}\right)/\nu.$$

Proof: Since ν is a constant, $E[\mathbf{Y}'\mathbf{QY}/\nu] = E[\mathbf{Y}'\mathbf{QY}]/\nu$ the result follows straightforwardly using theorem XII.1. ■

So to derive the expected mean square for a particular source under a specific model, one substitutes the **Q** matrix for the source and the ψ and **V** for the model into the expression given by theorem XII.2.

XII.E The distribution of the F statistics for an ANOVA

Each F statistic in an ANOVA is used to assess whether or not a null hypothesis is likely to be true by determining if the value of our test statistic is unlikely when the null hypothesis is true. To do this we need to establish the sampling distribution of the test statistic when the null hypothesis is true. The sampling distribution of the test statistic is the distribution that would be obtained under repeated sampling of the population. To construct this sampling distribution you would have to find a population in which you knew that the null hypothesis was true, take many samples from the population, for each sample compute the test statistic and construct the distribution from the computed test statistics. Not very practical! However, the following theorems will help. Note that the test statistic is of the following general form

$$F_{\nu_1,\,\nu_2} = \frac{s_1^2}{s_2^2} = \frac{\mathbf{Y}'\mathbf{Q}_1\mathbf{Y}/\nu_1}{\mathbf{Y}'\mathbf{Q}_2\mathbf{Y}/\nu_2}.$$

It is the ratio of two mean squares each of which is a quadratic form divided by its degrees of freedom. We will in the following three theorems:

1. Give the distribution of a quadratic form.
2. Establish the relationship between several quadratic forms.
3. Obtain the distribution of the ratio of two independent quadratic forms.

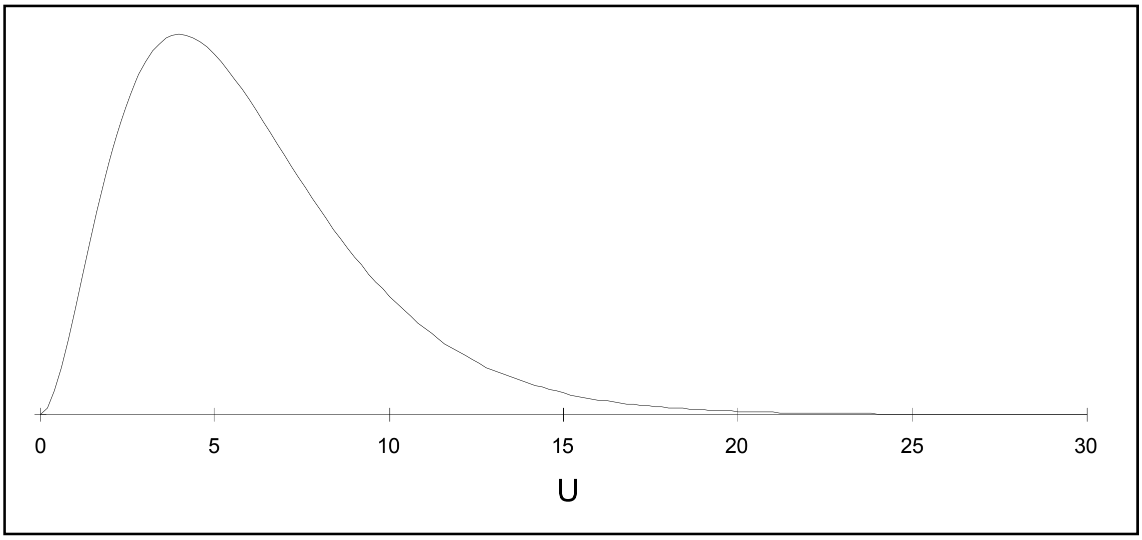
Theorem XII.3: Let **A** be an $n \times n$ symmetric matrix of rank ν and **Y** be an $n \times 1$ normally distributed random vector with $E[\mathbf{AY}] = \mathbf{0}$, $\mathrm{var}[\mathbf{Y}] = \mathbf{V}$ and $E[\mathbf{Y}'\mathbf{AY}/\nu] = \lambda$. Then $(1/\lambda)\mathbf{Y}'\mathbf{AY}$ follows a chi-squared distribution with ν degrees of freedom if and only if **A** is idempotent.

Proof: not given ■

The chi-square probability distribution function for the random variable U is:

$$\chi_\nu^2(u) = \left(\frac{1}{\Gamma(n/2)2^{\nu/2}}\right)u^{(\nu-2)/2}e^{-u/2}, \qquad 0 < u < \infty.$$

Probability distribution function for χ_6^2



Theorem XII.4: Let $\mathbf{Y}$ be an $n \times 1$ normally distributed random vector with $E[\mathbf{Y}] = \mathbf{X\theta}$ and $\text{var}[\mathbf{Y}] = \mathbf{V}$. Let $\mathbf{Y}'\mathbf{A}_1\mathbf{Y}, \mathbf{Y}'\mathbf{A}_2\mathbf{Y}, \ldots, \mathbf{Y}'\mathbf{A}_h\mathbf{Y}$ be a collection of h quadratic forms where, for each $i = 1, 2, \ldots, h$,

$$\mathbf{A}_i \text{ is symmetric, of rank } \nu_i,\ E[\mathbf{A}_i\mathbf{Y}] = \mathbf{0} \text{ and } E[\mathbf{Y}'\mathbf{A}_i\mathbf{Y}/\nu_i] = \lambda_i .$$

If any two of the following three statements are true (because any two implies the other),

1. All $\mathbf{A}_i$ are idempotent
2. $\sum_{i=1}^{h} \mathbf{A}_i$ is idempotent
3. $\mathbf{A}_i\mathbf{A}_j = \mathbf{0}, \quad i \neq j$

then, not only does $(1/\lambda_i)\mathbf{Y}'\mathbf{A}_i\mathbf{Y}$, for each i, follow a chi-squared distribution with ν_i degrees of freedom as Theorem XII.3 establishes, but the $\mathbf{Y}'\mathbf{A}_i\mathbf{Y}$ are independent for $i \neq j$ and $\sum_{i=1}^{h} \nu_i = \nu$ where ν denotes the rank of $\sum_{i=1}^{h} \mathbf{A}_i$.

Proof: not given ■

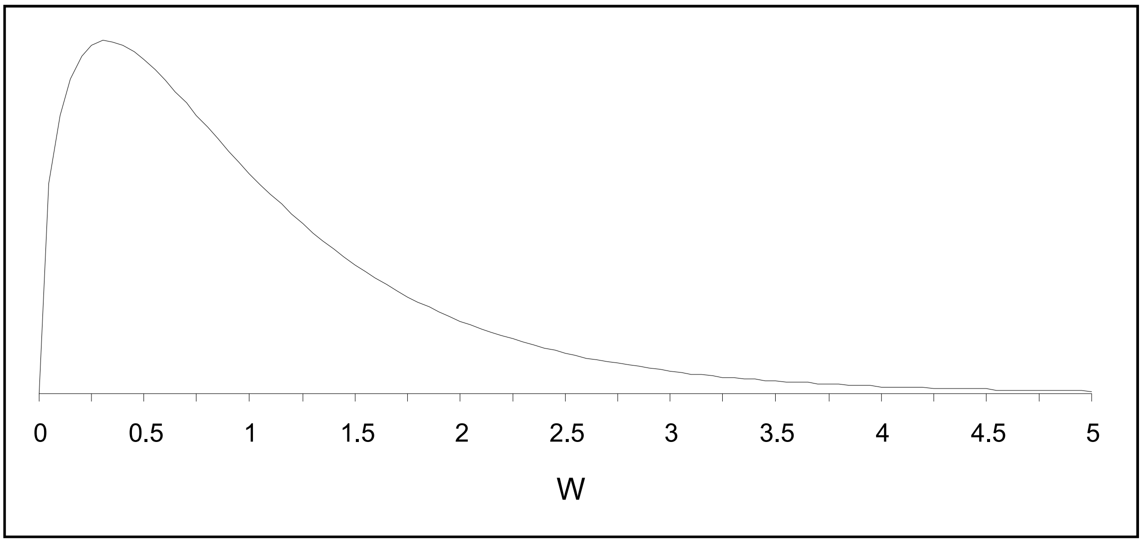
Theorem XII.5: Let U_1 and U_2 be two random variables distributed as chi-squares with ν_1 and ν_2 degrees of freedom. Then, provided U_1 and U_2, are independent, the random variable $W = \dfrac{U_1/\nu_1}{U_2/\nu_2}$ is distributed as Snedecor's F with ν_1 and ν_2 degrees of freedom.

Proof: not given ■

The F probability distribution function for the random variable W is:

$$F_{\nu_1,\nu_2}(w)=\left(\frac{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)\left(\frac{r_1}{r_2}\right)^{r_1/2}}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)}\right)w^{(\nu_1-2)/2}\left(1+\left(\frac{\nu_1}{\nu_2}\right)w\right)^{-(\nu_1+\nu_2)/2},\quad 0<w<\infty.$$

Probability distribution function for $F_{3,46}$



XII.F Application of theory for ANOVA-based hypothesis test to an example

a) The sources for an ANOVA
b) The sums of squares for an ANOVA
c) Degrees of freedom of the sums of squares for an ANOVA
d) Expected mean squares for an ANOVA
e) The distribution of the F statistics for an ANOVA

Example XII.1 Randomized Complete Block Design

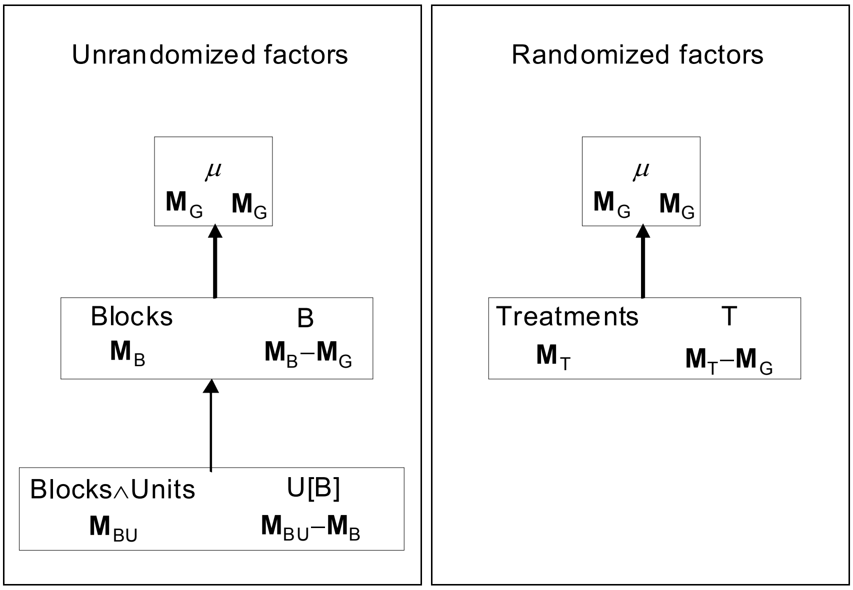
a) The sources for an ANOVA

1. For the RCBD, the unrandomized factors are Blocks and Units and the randomized factors are Treatments.
2. The unrandomized structure formula is *b* Blocks/*t* Units and the randomized structure formula *t* Treatments.
3. The unrandomized sources are Blocks and Units[Blocks] and the randomized source is Treatments.
4. The sources for the ANOVA table for this experiment are:

Source
Blocks
Units[Blocks]
Treatments
Residual
Total

b) The sums of squares for an ANOVA

The unrandomized generalized factors are (G), Blocks and Blocks∧Units and the randomized generalized factors are (G) and Treatments. So the unrandomized **Q**s are $\mathbf{Q}_{\mathrm{G}}$, $\mathbf{Q}_{\mathrm{B}}$ and $\mathbf{Q}_{\mathrm{BU}}$ and the randomized **Q**s are $\mathbf{Q}_{\mathrm{G}}$ and $\mathbf{Q}_{\mathrm{T}}$. Since each of these sets is derived from a structure formula consisting of only nesting and crossing operators, they each form a complete set of mutually orthogonal idempotents (CSMOI). The Hasse diagrams of generalized-factor marginalities, that includes expressions for the **Q**s in terms of the **M**s, are as follows:



If the data is ordered in standard order for Blocks and Treatments and if Units are given the same numbering as Treatments (this does not affect the analysis), rule XII.1 yields:

$$\mathbf{M}_{\mathrm{BU}} = \mathbf{I}_b \otimes \mathbf{I}_t = \mathbf{I}_n,\ \mathbf{M}_{\mathrm{B}} = \mathbf{I}_b \otimes \tfrac{1}{t}\mathbf{J}_t \text{ and } \mathbf{M}_{\mathrm{G}} = \tfrac{1}{b}\mathbf{J}_b \otimes \tfrac{1}{t}\mathbf{J}_t;$$

$$\mathbf{M}_{\mathrm{T}} = \tfrac{1}{b}\mathbf{J}_b \otimes \mathbf{I}_t \text{ and } \mathbf{M}_{\mathrm{G}} = \tfrac{1}{b}\mathbf{J}_b \otimes \tfrac{1}{t}\mathbf{J}_t.$$

Latter requires the inclusion of a factor indexing the replicates, in this case Blocks with *b* levels, This factor never occurs in the generalized factors arising from the randomized factors and so always contributes $\tfrac{1}{b}\mathbf{J}_b$ to the direct product.

So the ANOVA table with sums of squares added is:

Source	SSq
Blocks	$\mathbf{Y}'\mathbf{Q}_{\mathrm{B}}\mathbf{Y} = \mathbf{Y}'\left(\mathbf{M}_{\mathrm{B}} - \mathbf{M}_{\mathrm{G}}\right)\mathbf{Y}$
Units[Blocks]	$\mathbf{Y}'\mathbf{Q}_{\mathrm{BU}}\mathbf{Y} = \mathbf{Y}'\left(\mathbf{M}_{\mathrm{BU}} - \mathbf{M}_{\mathrm{B}}\right)\mathbf{Y}$
Treatments	$\mathbf{Y}'\mathbf{Q}_{\mathrm{T}}\mathbf{Y} = \mathbf{Y}'\left(\mathbf{M}_{\mathrm{T}} - \mathbf{M}_{\mathrm{G}}\right)\mathbf{Y}$
Residual	$\mathbf{Y}'\mathbf{Q}_{\mathrm{BU}_{\mathrm{Res}}}\mathbf{Y} = \mathbf{Y}'\left(\mathbf{Q}_{\mathrm{BU}} - \mathbf{Q}_{\mathrm{T}}\right)\mathbf{Y}$ $= \mathbf{Y}'\left(\mathbf{M}_{\mathrm{BU}} - \mathbf{M}_{\mathrm{B}} - \mathbf{M}_{\mathrm{T}} + \mathbf{M}_{\mathrm{G}}\right)\mathbf{Y}$
Total	$\mathbf{Y}'\mathbf{Q}_{\mathrm{U}}\mathbf{Y} = \mathbf{Y}'\left(\mathbf{M}_{\mathrm{BU}} - \mathbf{M}_{\mathrm{G}}\right)\mathbf{Y}$

c) Degrees of freedom of the sums of squares for an ANOVA

The following theorem establishes that the degrees of freedom of the sums of squares are as given in the analysis of variance table.

Theorem XII.6: Let quadratic forms for the sums of squares be

$$\mathbf{Y}'\mathbf{Q}_{\mathrm{U}}\mathbf{Y} = \mathbf{Y}'\left(\mathbf{M}_{\mathrm{BU}} - \mathbf{M}_{\mathrm{G}}\right)\mathbf{Y},$$

$$\mathbf{Y}'\mathbf{Q}_{\mathrm{B}}\mathbf{Y} = \mathbf{Y}'\left(\mathbf{M}_{\mathrm{B}} - \mathbf{M}_{\mathrm{G}}\right)\mathbf{Y},$$

$$\mathbf{Y}'\mathbf{Q}_{\mathrm{BU}}\mathbf{Y} = \mathbf{Y}'\left(\mathbf{M}_{\mathrm{BU}} - \mathbf{M}_{\mathrm{B}}\right)\mathbf{Y},$$

$$\mathbf{Y}'\mathbf{Q}_{\mathrm{T}}\mathbf{Y} = \mathbf{Y}'\left(\mathbf{M}_{\mathrm{T}} - \mathbf{M}_{\mathrm{G}}\right)\mathbf{Y} \text{ and}$$

$$\mathbf{Y}'\mathbf{Q}_{\mathrm{BU}_{\mathrm{Res}}}\mathbf{Y} = \mathbf{Y}'\left(\mathbf{Q}_{\mathrm{BU}} - \mathbf{Q}_{\mathrm{T}}\right)\mathbf{Y} = \mathbf{Y}'\left(\mathbf{M}_{\mathrm{BU}} - \mathbf{M}_{\mathrm{B}} - \mathbf{M}_{\mathrm{T}} + \mathbf{M}_{\mathrm{G}}\right)\mathbf{Y}$$

where $\mathbf{M}_{\mathrm{BU}} = \mathbf{I}_b \otimes \mathbf{I}_t = \mathbf{I}_n$, $\mathbf{M}_{\mathrm{B}} = \mathbf{I}_b \otimes \frac{1}{t}\mathbf{J}_t$, $\mathbf{M}_{\mathrm{T}} = \frac{1}{b}\mathbf{J}_b \otimes \mathbf{I}_t$ and $\mathbf{M}_{\mathrm{G}} = \frac{1}{b}\mathbf{J}_b \otimes \frac{1}{t}\mathbf{J}_t$. Their degrees of freedom are n–1, b–1, b(t–1), t–1 and (b–1)(t–1), respectively, where n is the number of observations, b is the number of blocks and t is the number of treatments.

Proof: First we establish that all **Q** matrices are symmetric and idempotent so that we can utilise lemma XII.2 to conclude that the ranks of the **Q** matrices are equal to their traces. Now $\mathbf{Q}_{\mathrm{B}}$, $\mathbf{Q}_{\mathrm{BU}}$ and $\mathbf{Q}_{\mathrm{T}}$ are symmetric and idempotent as they are in the complete sets of mutually-orthogonal idempotents. It remains to demonstrate that $\mathbf{Q}_{\mathrm{U}}$ and $\mathbf{Q}_{\mathrm{BU}_{\mathrm{Res}}}$ are symmetric and idempotent.

Firstly, from lemma I.1 $\left(c\mathbf{A} + d\mathbf{B}\right)' = c\mathbf{A}' + d\mathbf{B}'$ and we have from lemma XII.1 that the **M**s are symmetric and idempotent so that $\mathbf{Q}_{\mathrm{U}}' = \left(\mathbf{M}_{\mathrm{BU}} - \mathbf{M}_{\mathrm{G}}\right)' = \mathbf{M}_{\mathrm{BU}}' - \mathbf{M}_{\mathrm{G}}' = \mathbf{M}_{\mathrm{BU}} - \mathbf{M}_{\mathrm{G}} = \mathbf{Q}_{\mathrm{U}}$ and $\mathbf{Q}_{\mathrm{U}}\mathbf{Q}_{\mathrm{U}} = \left(\mathbf{M}_{\mathrm{BU}} - \mathbf{M}_{\mathrm{G}}\right)\left(\mathbf{M}_{\mathrm{BU}} - \mathbf{M}_{\mathrm{G}}\right) = \mathbf{M}_{\mathrm{BU}}\mathbf{M}_{\mathrm{BU}} - \mathbf{M}_{\mathrm{BU}}\mathbf{M}_{\mathrm{G}} - \mathbf{M}_{\mathrm{G}}\mathbf{M}_{\mathrm{BU}} + \mathbf{M}_{\mathrm{G}}\mathbf{M}_{\mathrm{G}}$. Now $\mathbf{M}_{\mathrm{BU}} = \mathbf{I}_n$ so that $\mathbf{Q}_{\mathrm{U}}\mathbf{Q}_{\mathrm{U}} = \mathbf{M}_{\mathrm{BU}} - \mathbf{M}_{\mathrm{G}} - \mathbf{M}_{\mathrm{G}} + \mathbf{M}_{\mathrm{G}} = \mathbf{M}_{\mathrm{BU}} - \mathbf{M}_{\mathrm{G}} = \mathbf{Q}_{\mathrm{U}}$. That is, $\mathbf{Q}_{\mathrm{U}}$ is symmetric and idempotent.

Secondly, $\mathbf{Q}'_{\mathrm{BU_{Res}}} = \left(\mathbf{M}_{\mathrm{BU}} - \mathbf{M}_{\mathrm{B}} - \mathbf{M}_{\mathrm{T}} + \mathbf{M}_{\mathrm{G}}\right)' = \mathbf{M}_{\mathrm{BU}} - \mathbf{M}_{\mathrm{B}} - \mathbf{M}_{\mathrm{T}} + \mathbf{M}_{\mathrm{G}}$ as each $\mathbf{M}$ is symmetric. Further,

$$\begin{aligned}
\mathbf{Q}^2_{\mathrm{BU_{Res}}} &= \left(\mathbf{M}_{\mathrm{BU}} - \mathbf{M}_{\mathrm{B}} - \mathbf{M}_{\mathrm{T}} + \mathbf{M}_{\mathrm{G}}\right)\left(\mathbf{M}_{\mathrm{BU}} - \mathbf{M}_{\mathrm{B}} - \mathbf{M}_{\mathrm{T}} + \mathbf{M}_{\mathrm{G}}\right)\\
&= \left(\mathbf{M}_{\mathrm{BU}}\mathbf{M}_{\mathrm{BU}} - \mathbf{M}_{\mathrm{BU}}\mathbf{M}_{\mathrm{B}} - \mathbf{M}_{\mathrm{BU}}\mathbf{M}_{\mathrm{T}} + \mathbf{M}_{\mathrm{BU}}\mathbf{M}_{\mathrm{G}}\right) - \left(\mathbf{M}_{\mathrm{B}}\mathbf{M}_{\mathrm{BU}} - \mathbf{M}_{\mathrm{B}}\mathbf{M}_{\mathrm{B}} - \mathbf{M}_{\mathrm{B}}\mathbf{M}_{\mathrm{T}} + \mathbf{M}_{\mathrm{B}}\mathbf{M}_{\mathrm{G}}\right)\\
&\quad - \left(\mathbf{M}_{\mathrm{T}}\mathbf{M}_{\mathrm{BU}} - \mathbf{M}_{\mathrm{T}}\mathbf{M}_{\mathrm{B}} - \mathbf{M}_{\mathrm{T}}\mathbf{M}_{\mathrm{T}} + \mathbf{M}_{\mathrm{T}}\mathbf{M}_{\mathrm{G}}\right) + \left(\mathbf{M}_{\mathrm{G}}\mathbf{M}_{\mathrm{BU}} - \mathbf{M}_{\mathrm{G}}\mathbf{M}_{\mathrm{B}} - \mathbf{M}_{\mathrm{G}}\mathbf{M}_{\mathrm{T}} + \mathbf{M}_{\mathrm{G}}\mathbf{M}_{\mathrm{G}}\right)\\
&= \left(\mathbf{M}_{\mathrm{BU}} - \mathbf{M}_{\mathrm{B}} - \mathbf{M}_{\mathrm{T}} + \mathbf{M}_{\mathrm{G}}\right) - \left(\mathbf{M}_{\mathrm{B}} - \mathbf{M}_{\mathrm{B}} - \mathbf{M}_{\mathrm{B}}\mathbf{M}_{\mathrm{T}} + \mathbf{M}_{\mathrm{B}}\mathbf{M}_{\mathrm{G}}\right)\\
&\quad - \left(\mathbf{M}_{\mathrm{T}} - \mathbf{M}_{\mathrm{T}}\mathbf{M}_{\mathrm{B}} - \mathbf{M}_{\mathrm{T}} + \mathbf{M}_{\mathrm{T}}\mathbf{M}_{\mathrm{G}}\right) + \left(\mathbf{M}_{\mathrm{G}} - \mathbf{M}_{\mathrm{G}}\mathbf{M}_{\mathrm{B}} - \mathbf{M}_{\mathrm{G}}\mathbf{M}_{\mathrm{T}} + \mathbf{M}_{\mathrm{G}}\right).
\end{aligned}$$

Now

$$\begin{aligned}
\mathbf{M}_{\mathrm{B}}\mathbf{M}_{\mathrm{T}} &= \left(\mathbf{I}_b \otimes \tfrac{1}{t}\mathbf{J}_t\right)\left(\tfrac{1}{b}\mathbf{J}_b \otimes \mathbf{I}_t\right) = \tfrac{1}{b}\mathbf{J}_b \otimes \tfrac{1}{t}\mathbf{J}_t = \mathbf{M}_{\mathrm{G}} = \mathbf{M}_{\mathrm{T}}\mathbf{M}_{\mathrm{B}}\\
\mathbf{M}_{\mathrm{B}}\mathbf{M}_{\mathrm{G}} &= \left(\mathbf{I}_b \otimes \tfrac{1}{t}\mathbf{J}_t\right)\left(\tfrac{1}{b}\mathbf{J}_b \otimes \tfrac{1}{t}\mathbf{J}_t\right) = \tfrac{1}{b}\mathbf{J}_b \otimes \tfrac{1}{t}\mathbf{J}_t = \mathbf{M}_{\mathrm{G}} = \mathbf{M}_{\mathrm{G}}\mathbf{M}_{\mathrm{B}}\\
\mathbf{M}_{\mathrm{T}}\mathbf{M}_{\mathrm{G}} &= \left(\tfrac{1}{b}\mathbf{J}_b \otimes \mathbf{I}_t\right)\left(\tfrac{1}{b}\mathbf{J}_b \otimes \tfrac{1}{t}\mathbf{J}_t\right) = \tfrac{1}{b}\mathbf{J}_b \otimes \tfrac{1}{t}\mathbf{J}_t = \mathbf{M}_{\mathrm{G}} = \mathbf{M}_{\mathrm{G}}\mathbf{M}_{\mathrm{T}}
\end{aligned}$$

and so

$$\begin{aligned}
\mathbf{Q}^2_{\mathrm{BU_{Res}}} &= \left(\mathbf{M}_{\mathrm{BU}} - \mathbf{M}_{\mathrm{B}} - \mathbf{M}_{\mathrm{T}} + \mathbf{M}_{\mathrm{G}}\right) - \left(\mathbf{M}_{\mathrm{B}} - \mathbf{M}_{\mathrm{B}} - \mathbf{M}_{\mathrm{B}}\mathbf{M}_{\mathrm{T}} + \mathbf{M}_{\mathrm{B}}\mathbf{M}_{\mathrm{G}}\right)\\
&\quad - \left(\mathbf{M}_{\mathrm{T}} - \mathbf{M}_{\mathrm{T}}\mathbf{M}_{\mathrm{B}} - \mathbf{M}_{\mathrm{T}} + \mathbf{M}_{\mathrm{T}}\mathbf{M}_{\mathrm{G}}\right) + \left(\mathbf{M}_{\mathrm{G}} - \mathbf{M}_{\mathrm{G}}\mathbf{M}_{\mathrm{B}} - \mathbf{M}_{\mathrm{G}}\mathbf{M}_{\mathrm{T}} + \mathbf{M}_{\mathrm{G}}\right)\\
&= \left(\mathbf{M}_{\mathrm{BU}} - \mathbf{M}_{\mathrm{B}} - \mathbf{M}_{\mathrm{T}} + \mathbf{M}_{\mathrm{G}}\right) - \left(\mathbf{M}_{\mathrm{B}} - \mathbf{M}_{\mathrm{B}} - \mathbf{M}_{\mathrm{G}} + \mathbf{M}_{\mathrm{G}}\right)\\
&\quad - \left(\mathbf{M}_{\mathrm{T}} - \mathbf{M}_{\mathrm{G}} - \mathbf{M}_{\mathrm{T}} + \mathbf{M}_{\mathrm{G}}\right) + \left(\mathbf{M}_{\mathrm{G}} - \mathbf{M}_{\mathrm{G}} - \mathbf{M}_{\mathrm{G}} + \mathbf{M}_{\mathrm{G}}\right)\\
&= \mathbf{M}_{\mathrm{BU}} - \mathbf{M}_{\mathrm{B}} - \mathbf{M}_{\mathrm{T}} + \mathbf{M}_{\mathrm{G}}\\
&= \mathbf{Q}_{\mathrm{BU_{Res}}}.
\end{aligned}$$

Consequently lemma XII.2 applies to the **Q**s so that the ranks of all the **Q** matrices are equal to their trace.

We also note that using lemma XII.4 we have that

$$\begin{aligned}
trace\left(\mathbf{M}_{\mathrm{G}}\right) &= 1\\
trace\left(\mathbf{M}_{\mathrm{B}}\right) &= b\\
trace\left(\mathbf{M}_{\mathrm{T}}\right) &= t\\
trace\left(\mathbf{M}_{\mathrm{BU}}\right) &= bt = n.
\end{aligned}$$

Since, from lemma XII.3, $trace\left(c\mathbf{A} + d\mathbf{B}\right) = c \times trace\left(\mathbf{A}\right) + d \times trace\left(\mathbf{B}\right)$, we have

$$\begin{aligned}
trace\left(\mathbf{Q}_{\mathrm{U}}\right) &= trace\left(\mathbf{M}_{\mathrm{BU}} - \mathbf{M}_{\mathrm{G}}\right) = n - 1\\
trace\left(\mathbf{Q}_{\mathrm{B}}\right) &= trace\left(\mathbf{M}_{\mathrm{B}} - \mathbf{M}_{\mathrm{G}}\right) = b - 1\\
trace\left(\mathbf{Q}_{\mathrm{BU}}\right) &= trace\left(\mathbf{M}_{\mathrm{BU}} - \mathbf{M}_{\mathrm{B}}\right) = bt - b = b\left(t - 1\right)\\
trace\left(\mathbf{Q}_{\mathrm{T}}\right) &= trace\left(\mathbf{M}_{\mathrm{T}} - \mathbf{M}_{\mathrm{G}}\right) = t - 1\\
trace\left(\mathbf{Q}_{\mathrm{BU_{Res}}}\right) &= trace\left(\mathbf{M}_{\mathrm{BU}} - \mathbf{M}_{\mathrm{B}} - \mathbf{M}_{\mathrm{T}} + \mathbf{M}_{\mathrm{G}}\right) = bt - b - t + 1 = \left(b - 1\right)\left(t - 1\right).
\end{aligned}$$

■

So the ANOVA table with degrees of freedom added is:

Source	df	SSq
Blocks	b-1	$\mathbf{Y}'\mathbf{Q}_{\mathrm{B}}\mathbf{Y} = \mathbf{Y}'\left(\mathbf{M}_{\mathrm{B}} - \mathbf{M}_{\mathrm{G}}\right)\mathbf{Y}$
Units[Blocks]	b(t-1)	$\mathbf{Y}'\mathbf{Q}_{\mathrm{BU}}\mathbf{Y} = \mathbf{Y}'\left(\mathbf{M}_{\mathrm{BU}} - \mathbf{M}_{\mathrm{B}}\right)\mathbf{Y}$
Treatments	t-1	$\mathbf{Y}'\mathbf{Q}_{\mathrm{T}}\mathbf{Y} = \mathbf{Y}'\left(\mathbf{M}_{\mathrm{T}} - \mathbf{M}_{\mathrm{G}}\right)\mathbf{Y}$
Residual	(b-1)(t-1)	$\mathbf{Y}'\mathbf{Q}_{\mathrm{BU_{Res}}}\mathbf{Y} = \mathbf{Y}'\left(\mathbf{Q}_{\mathrm{BU}} - \mathbf{Q}_{\mathrm{T}}\right)\mathbf{Y}$ $= \mathbf{Y}'\left(\mathbf{M}_{\mathrm{BU}} - \mathbf{M}_{\mathrm{B}} - \mathbf{M}_{\mathrm{T}} + \mathbf{M}_{\mathrm{G}}\right)\mathbf{Y}$
Total	n-1	$\mathbf{Y}'\left(\mathbf{Q}_{\mathrm{U}}\right)\mathbf{Y} = \mathbf{Y}'\left(\mathbf{M}_{\mathrm{BU}} - \mathbf{M}_{\mathrm{G}}\right)\mathbf{Y}$

d) Expected mean squares for an ANOVA

We now prove the E[MSq]s for the case of Blocks random. But first we have a useful lemma.

Lemma XII.5: Let $\psi = E\left[\mathbf{Y}\right] = \mathbf{X}_{\mathrm{T}}\tau = \left(\mathbf{1}_b \otimes \mathbf{I}_t\right)\tau$, $\mathbf{M}_{\mathrm{BU}} = \mathbf{I}_b \otimes \mathbf{I}_t = \mathbf{I}_n$, $\mathbf{M}_{\mathrm{B}} = \mathbf{I}_b \otimes \tfrac{1}{t}\mathbf{J}_t$, $\mathbf{M}_{\mathrm{T}} = \tfrac{1}{b}\mathbf{J}_b \otimes \mathbf{I}_t$, $\mathbf{M}_{\mathrm{G}} = \tfrac{1}{b}\mathbf{J}_b \otimes \tfrac{1}{t}\mathbf{J}_t$, $\mathbf{Q}_{\mathrm{B}} = \mathbf{M}_{\mathrm{B}} - \mathbf{M}_{\mathrm{G}}$, $\mathbf{Q}_{\mathrm{T}} = \mathbf{M}_{\mathrm{T}} - \mathbf{M}_{\mathrm{G}}$ and $\mathbf{Q}_{\mathrm{BU_{Res}}} = \mathbf{M}_{\mathrm{BU}} - \mathbf{M}_{\mathrm{B}} - \mathbf{M}_{\mathrm{T}} + \mathbf{M}_{\mathrm{G}}$. Then,

$$\mathbf{Q}_{\mathrm{G}}\mathbf{X}_{\mathrm{T}} = \tfrac{1}{t}\left(\mathbf{1}_b \otimes \mathbf{J}_t\right)$$
$$\mathbf{Q}_{\mathrm{B}}\mathbf{X}_{\mathrm{T}} = \mathbf{0}$$
$$\mathbf{Q}_{\mathrm{BU_{Res}}}\mathbf{X}_{\mathrm{T}} = \mathbf{0}.$$

and $\mathbf{X}_{\mathrm{T}}'\mathbf{Q}_{\mathrm{G}}\mathbf{X}_{\mathrm{T}} = \tfrac{b}{t}\mathbf{J}_t$

Proof: First we have

$$\mathbf{M}_{\mathrm{G}}\mathbf{X}_{\mathrm{T}} = \tfrac{1}{bt}\left(\mathbf{J}_b \otimes \mathbf{J}_t\right)\left(\mathbf{1}_b \otimes \mathbf{I}_t\right) = \tfrac{1}{t}\left(\mathbf{1}_b \otimes \mathbf{J}_t\right)$$
$$\mathbf{M}_{\mathrm{B}}\mathbf{X}_{\mathrm{T}} = \tfrac{1}{t}\left(\mathbf{I}_b \otimes \mathbf{J}_t\right)\left(\mathbf{1}_b \otimes \mathbf{I}_t\right) = \tfrac{1}{t}\left(\mathbf{1}_b \otimes \mathbf{J}_t\right)$$
$$\mathbf{M}_{\mathrm{T}}\mathbf{X}_{\mathrm{T}} = \tfrac{1}{b}\left(\mathbf{J}_b \otimes \mathbf{I}_t\right)\left(\mathbf{1}_b \otimes \mathbf{I}_t\right) = \left(\mathbf{1}_b \otimes \mathbf{I}_t\right)$$
$$\mathbf{M}_{\mathrm{BU}}\mathbf{X}_{\mathrm{T}} = \left(\mathbf{I}_b \otimes \mathbf{I}_t\right)\left(\mathbf{1}_b \otimes \mathbf{I}_t\right) = \left(\mathbf{1}_b \otimes \mathbf{I}_t\right)$$

so that

$$\mathbf{Q}_{\mathrm{G}}\mathbf{X}_{\mathrm{T}} = \mathbf{M}_{\mathrm{G}}\mathbf{X}_{\mathrm{T}} = \tfrac{1}{t}\left(\mathbf{1}_b \otimes \mathbf{J}_t\right)$$
$$\mathbf{Q}_{\mathrm{B}}\mathbf{X}_{\mathrm{T}} = \left(\mathbf{M}_{\mathrm{B}} - \mathbf{M}_{\mathrm{G}}\right)\mathbf{X}_{\mathrm{T}} = \tfrac{1}{t}\left(\mathbf{1}_b \otimes \mathbf{J}_t\right) - \tfrac{1}{t}\left(\mathbf{1}_b \otimes \mathbf{J}_t\right) = \mathbf{0}$$
$$\mathbf{Q}_{\mathrm{BU_{Res}}}\mathbf{X}_{\mathrm{T}} = \left(\mathbf{M}_{\mathrm{BU}} - \mathbf{M}_{\mathrm{B}} - \mathbf{M}_{\mathrm{T}} + \mathbf{M}_{\mathrm{G}}\right)\mathbf{X}_{\mathrm{T}} = \left(\mathbf{1}_b \otimes \mathbf{I}_t\right) - \tfrac{1}{t}\left(\mathbf{1}_b \otimes \mathbf{J}_t\right) - \left(\mathbf{1}_b \otimes \mathbf{I}_t\right) + \tfrac{1}{t}\left(\mathbf{1}_b \otimes \mathbf{J}_t\right) = \mathbf{0}.$$

Finally,

$$\mathbf{X}_{\mathrm{T}}'\mathbf{Q}_{\mathrm{G}}\mathbf{X}_{\mathrm{T}} = \tfrac{1}{t}\left(\mathbf{1}_b \otimes \mathbf{I}_t\right)'\left(\mathbf{1}_b \otimes \mathbf{J}_t\right) = \tfrac{b}{t}\mathbf{J}_t.$$

■

Theorem XII.7: Let $\psi = E[\mathbf{Y}] = \mathbf{X}_{\text{T}}\tau = (\mathbf{1}_b \otimes \mathbf{I}_t)\tau$, $\mathbf{V} = \sigma^2_{\text{BU}}\mathbf{I}_n + \sigma^2_{\text{B}}(\mathbf{I}_b \otimes \mathbf{J}_t)$, $SS_{\text{B}} = \mathbf{Y}'\mathbf{Q}_{\text{B}}\mathbf{Y} = \mathbf{Y}'(\mathbf{M}_{\text{B}} - \mathbf{M}_{\text{G}})\mathbf{Y}$, $SS_{\text{T}} = \mathbf{Y}'\mathbf{Q}_{\text{T}}\mathbf{Y} = \mathbf{Y}'(\mathbf{M}_{\text{T}} - \mathbf{M}_{\text{G}})\mathbf{Y}$ and $SS_{\text{BU}_{\text{Res}}} = \mathbf{Y}'\mathbf{Q}_{\text{BU}_{\text{Res}}}\mathbf{Y} = \mathbf{Y}'(\mathbf{Q}_{\text{BU}} - \mathbf{Q}_{\text{T}})\mathbf{Y} = \mathbf{Y}'(\mathbf{M}_{\text{BU}} - \mathbf{M}_{\text{B}} - \mathbf{M}_{\text{T}} + \mathbf{M}_{\text{G}})\mathbf{Y}$.

Then, the expected mean squares are

$$E\left[SS_{\text{B}}/(b-1)\right] = \sigma^2_{\text{BU}} + t\sigma^2_{\text{B}}$$

$$E\left[SS_{\text{T}}/(t-1)\right] = \sigma^2_{\text{BU}} + q_{\text{T}}(\psi),$$

$$E\left[SS_{\text{BU}_{\text{Res}}}/\{(b-1)(t-1)\}\right] = \sigma^2_{\text{BU}}$$

where $q_{\text{T}}(\psi) = \sum_{j=1}^{t} b\left(\tau_j - \bar{\tau}_.\right)^2 \Big/ (t-1)$, $\bar{\tau}_. = \sum_{j=1}^{t}\tau_j \big/ t$, τ_j is the jth element of the t-vector τ, b is the number of blocks and t is the number of treatments.

Proof: First note that we can write $\mathbf{V} = \sigma^2_{\text{BU}}\mathbf{M}_{\text{BU}} + t\sigma^2_{\text{B}}\mathbf{M}_{\text{B}}$.

For $E\left[SS_{\text{B}}/(b-1)\right]$, we use theorem XII.2 and that it can be shown that $\mathbf{Q}_{\text{B}}\mathbf{M}_{\text{BU}} = \mathbf{Q}_{\text{B}}$ and $\mathbf{Q}_{\text{B}}\mathbf{M}_{\text{B}} = \mathbf{Q}_{\text{B}}$ to obtain the following expressions:

$$\begin{aligned}
E\left[\frac{SS_{\text{B}}}{(b-1)}\right] &= E\left[\mathbf{Y}'\mathbf{Q}_{\text{B}}\mathbf{Y}\right]/(b-1) \\
&= \left\{trace\left(\mathbf{Q}_{\text{B}}\left\{\sigma^2_{\text{BU}}\mathbf{M}_{\text{BU}} + t\sigma^2_{\text{B}}\mathbf{M}_{\text{B}}\right\}\right) + (\mathbf{X}_{\text{T}}\tau)'\mathbf{Q}_{\text{B}}(\mathbf{X}_{\text{T}}\tau)\right\} \Big/ \{b-1\} \\
&= \left\{\sigma^2_{\text{BU}}trace(\mathbf{Q}_{\text{B}}\mathbf{M}_{\text{BU}}) + t\sigma^2_{\text{B}}trace(\mathbf{Q}_{\text{B}}\mathbf{M}_{\text{B}}) + (\mathbf{X}_{\text{T}}\tau)'\mathbf{Q}_{\text{B}}(\mathbf{X}_{\text{T}}\tau)\right\} \Big/ \{b-1\} \\
&= \left\{\sigma^2_{\text{BU}}trace(\mathbf{Q}_{\text{B}}) + t\sigma^2_{\text{B}}trace(\mathbf{Q}_{\text{B}}) + \tau'\mathbf{X}'_{\text{T}}\mathbf{Q}_{\text{B}}\mathbf{X}_{\text{T}}\tau\right\} \Big/ \{b-1\} \\
&= \left\{\left(\sigma^2_{\text{BU}} + t\sigma^2_{\text{B}}\right)trace(\mathbf{Q}_{\text{B}}) + \tau'\mathbf{X}'_{\text{T}}\mathbf{Q}_{\text{B}}\mathbf{X}_{\text{T}}\tau\right\} \Big/ \{b-1\}.
\end{aligned}$$

Now from theorem XII.6 we have that $trace(\mathbf{Q}_{\text{B}}) = (b-1)$ and from lemma XII.5 $\mathbf{Q}_{\text{B}}\mathbf{X}_{\text{T}} = \mathbf{0}$. Hence, the expected mean square is

$$\begin{aligned}
E\left[SS_{\text{B}}/(b-1)\right] &= \left\{\left(\sigma^2_{\text{BU}} + t\sigma^2_{\text{B}}\right)trace(\mathbf{Q}_{\text{B}}) + \tau'\mathbf{X}'_{\text{T}}\mathbf{Q}_{\text{B}}\mathbf{X}_{\text{T}}\tau\right\} \Big/ \{b-1\} \\
&= \left\{\left(\sigma^2_{\text{BU}} + t\sigma^2_{\text{B}}\right)(b-1) + 0\right\} \Big/ \{b-1\} \\
&= \sigma^2_{\text{BU}} + t\sigma^2_{\text{B}}.
\end{aligned}$$

The proof that $E\left[SS_{\text{T}}/(t-1)\right] = \sigma^2_{\text{BU}} + q_{\text{T}}(\psi)$ is left as an exercise for you.

Now, for $E\left[SS_{\mathrm{BU_{Res}}}\big/\{(b-1)(t-1)\}\right]$, again we use theorem XII.2 and also that it can be shown that $\mathbf{M}_{\mathrm{BU}}\mathbf{Q}_{\mathrm{BU_{Res}}}=\mathbf{Q}_{\mathrm{BU_{Res}}}\mathbf{M}_{\mathrm{BU}}=\mathbf{Q}_{\mathrm{BU_{Res}}}$ and $\mathbf{M}_{\mathrm{B}}\mathbf{Q}_{\mathrm{BU_{Res}}}=\mathbf{Q}_{\mathrm{BU_{Res}}}\mathbf{M}_{\mathrm{B}}=\mathbf{0}$ so that we have

$$
\begin{aligned}
&E\left[SS_{\mathrm{BU_{Res}}}\big/\{(b-1)(t-1)\}\right]\\
&=E\left[\mathbf{Y}'\mathbf{Q}_{\mathrm{BU_{Res}}}\mathbf{Y}\right]\big/\{(b-1)(t-1)\}\\
&=\frac{\left\{trace\left(\mathbf{Q}_{\mathrm{BU_{Res}}}\left\{\sigma^2_{\mathrm{BU}}\mathbf{M}_{\mathrm{BU}}+t\sigma^2_{\mathrm{B}}\mathbf{M}_{\mathrm{B}}\right\}\right)+\left(\mathbf{X}_{\mathrm{T}}\boldsymbol{\tau}\right)'\mathbf{Q}_{\mathrm{BU_{Res}}}\left(\mathbf{X}_{\mathrm{T}}\boldsymbol{\tau}\right)\right\}}{\{(b-1)(t-1)\}}\\
&=\frac{\left\{\sigma^2_{\mathrm{BU}}trace\left(\mathbf{Q}_{\mathrm{BU_{Res}}}\right)+0+\boldsymbol{\tau}'\mathbf{X}'_{\mathrm{T}}\mathbf{Q}_{\mathrm{BU_{Res}}}\mathbf{X}_{\mathrm{T}}\boldsymbol{\tau}\right\}}{\{(b-1)(t-1)\}}.
\end{aligned}
$$

Now from theorem XII.6 we have that $trace\left(\mathbf{Q}_{\mathrm{BU_{Res}}}\right)=(b-1)(t-1)$ and from lemma XII.5 $\mathbf{Q}_{\mathrm{BU_{Res}}}\mathbf{X}_{\mathrm{T}}=\mathbf{0}$ so that

$$
\begin{aligned}
&E\left[SS_{\mathrm{BU_{Res}}}\big/\{(b-1)(t-1)\}\right]\\
&=\frac{\left\{\sigma^2_{\mathrm{BU}}trace\left(\mathbf{Q}_{\mathrm{BU_{Res}}}\right)+\boldsymbol{\tau}'\mathbf{X}'_{\mathrm{T}}\mathbf{Q}_{\mathrm{BU_{Res}}}\mathbf{X}_{\mathrm{T}}\boldsymbol{\tau}\right\}}{\{(b-1)(t-1)\}}\\
&=\frac{\left\{\sigma^2_{\mathrm{BU}}(b-1)(t-1)+0\right\}}{\{(b-1)(t-1)\}}\\
&=\sigma^2_{\mathrm{BU}}
\end{aligned}
$$

as claimed. ■

So the ANOVA table with expected mean squares added is:

Source	df	SSq	E[MSq]
Blocks	b-1	$\mathbf{Y}'\mathbf{Q}_{\mathrm{B}}\mathbf{Y}=\mathbf{Y}'\left(\mathbf{M}_{\mathrm{B}}-\mathbf{M}_{\mathrm{G}}\right)\mathbf{Y}$	$\sigma^2_{\mathrm{BU}}+t\sigma^2_{\mathrm{B}}$
Units[Blocks]	b(t-1)	$\mathbf{Y}'\mathbf{Q}_{\mathrm{BU}}\mathbf{Y}=\mathbf{Y}'\left(\mathbf{M}_{\mathrm{BU}}-\mathbf{M}_{\mathrm{B}}\right)\mathbf{Y}$	
Treatments	t-1	$\mathbf{Y}'\mathbf{Q}_{\mathrm{T}}\mathbf{Y}=\mathbf{Y}'\left(\mathbf{M}_{\mathrm{T}}-\mathbf{M}_{\mathrm{G}}\right)\mathbf{Y}$	$\sigma^2_{\mathrm{BU}}+q_{\mathrm{T}}\left(\psi\right)$
Residual	(b-1)(t-1)	$\mathbf{Y}'\mathbf{Q}_{\mathrm{BU_{Res}}}\mathbf{Y}=\mathbf{Y}'\left(\mathbf{Q}_{\mathrm{BU}}-\mathbf{Q}_{\mathrm{T}}\right)\mathbf{Y}$ $=\mathbf{Y}'\left(\mathbf{M}_{\mathrm{BU}}-\mathbf{M}_{\mathrm{B}}-\mathbf{M}_{\mathrm{T}}+\mathbf{M}_{\mathrm{G}}\right)\mathbf{Y}$	σ^2_{BU}
Total	n-1	$\mathbf{Y}'\left(\mathbf{Q}_{\mathrm{U}}\right)\mathbf{Y}=\mathbf{Y}'\left(\mathbf{M}_{\mathrm{BU}}-\mathbf{M}_{\mathrm{G}}\right)\mathbf{Y}$	

e) The distribution of the F statistics for an ANOVA

We next derive the sampling distribution of the F-statistics for testing treatment and block differences. Note that the model under the null hypothesis of no treatment effects is $E\left[\mathbf{Y}\right]=\mathbf{X}_{\mathrm{G}}\mu$ and $\mathbf{V}=\sigma_{\mathrm{BU}}^{2}\mathbf{I}_{n}+\sigma_{\mathrm{B}}^{2}\left(\mathbf{I}_{b}\otimes\mathbf{J}_{t}\right)$, while under the null hypothesis of no block variation is $E\left[\mathbf{Y}\right]=\mathbf{X}_{\mathrm{T}}\boldsymbol{\tau}$ and $\mathbf{V}=\sigma_{\mathrm{BU}}^{2}\mathbf{I}_{n}$. The following theorems involve establishing the sampling distribution of the F-statistics under these models.

Theorem XII.8: Let $\boldsymbol{\psi}=E\left[\mathbf{Y}\right]=\mathbf{X}_{\mathrm{G}}\mu=\mathbf{1}_{b}\otimes\mathbf{1}_{t}\mu$, $\mathbf{V}=\sigma_{\mathrm{BU}}^{2}\mathbf{I}_{n}+\sigma_{\mathrm{B}}^{2}\left(\mathbf{I}_{b}\otimes\mathbf{J}_{t}\right)$, $s_{\mathrm{T}}^{2}=\mathbf{Y}'\mathbf{Q}_{\mathrm{T}}\mathbf{Y}/(t-1)$ and $s_{\mathrm{BU_{Res}}}^{2}=\mathbf{Y}'\mathbf{Q}_{\mathrm{BU_{Res}}}\mathbf{Y}/\left\{(b-1)(t-1)\right\}$ where $\mathbf{Q}_{\mathrm{T}}=\mathbf{M}_{\mathrm{T}}-\mathbf{M}_{\mathrm{G}}$, $\mathbf{Q}_{\mathrm{BU_{Res}}}=\mathbf{M}_{\mathrm{BU}}-\mathbf{M}_{\mathrm{B}}-\mathbf{M}_{\mathrm{T}}+\mathbf{M}_{\mathrm{G}}$, $\mathbf{M}_{\mathrm{BU}}=\mathbf{I}_{b}\otimes\mathbf{I}_{t}=\mathbf{I}_{n}$, $\mathbf{M}_{\mathrm{B}}=\mathbf{I}_{b}\otimes\frac{1}{t}\mathbf{J}_{t}$, $\mathbf{M}_{\mathrm{T}}=\frac{1}{b}\mathbf{J}_{b}\otimes\mathbf{I}_{t}$ and $\mathbf{M}_{\mathrm{G}}=\frac{1}{b}\mathbf{J}_{b}\otimes\frac{1}{t}\mathbf{J}_{t}$. Then, the ratio of these two mean squares, given by

$$F_{(t-1),\,(b-1)(t-1)}=\frac{s_{\mathrm{T}}^{2}}{s_{\mathrm{BU_{Res}}}^{2}}$$

is distributed as a Snedecor's F with $(t-1)$ and $(b-1)(t-1)$ degrees of freedom.

Proof: We have to show that

1. $E\left[\mathbf{Q}_{\mathrm{T}}\mathbf{Y}\right]=E\left[\mathbf{Q}_{\mathrm{BU_{Res}}}\mathbf{Y}\right]=\mathbf{0}$ and $E\left[\mathbf{Y}'\mathbf{Q}_{\mathrm{T}}\mathbf{Y}/(t-1)\right]=E\left[\mathbf{Y}'\mathbf{Q}_{\mathrm{BU_{Res}}}\mathbf{Y}/\left\{(b-1)(t-1)\right\}\right]=\sigma_{\mathrm{BU}}^{2}$, under the expectation and variation models above, so that theorem XII.3 can be invoked to conclude that $\left(1/\sigma_{\mathrm{BU}}^{2}\right)\mathbf{Y}'\mathbf{Q}_{\mathrm{T}}\mathbf{Y}$ and $\left(1/\sigma_{\mathrm{BU}}^{2}\right)\mathbf{Y}'\mathbf{Q}_{\mathrm{BU_{Res}}}\mathbf{Y}$ follow a chi-square distribution;
2. the quadratic forms $\mathbf{Y}'\mathbf{Q}_{\mathrm{T}}\mathbf{Y}$ and $\mathbf{Y}'\mathbf{Q}_{\mathrm{BU_{Res}}}\mathbf{Y}$ are independent as outlined in theorem XII.4;
3. then theorem XII.5 can be invoked to obtain the distribution of the F test statistic.

Firstly,

$$\begin{aligned}
E\left[\mathbf{Q}_{\mathrm{T}}\mathbf{Y}\right]&=\mathbf{Q}_{\mathrm{T}}E\left[\mathbf{Y}\right]\\
&=\mathbf{Q}_{\mathrm{T}}\mathbf{X}_{\mathrm{G}}\mu\\
&=\left(\mathbf{M}_{\mathrm{T}}-\mathbf{M}_{\mathrm{G}}\right)\mathbf{X}_{\mathrm{G}}\mu\\
&=\left(\tfrac{1}{b}\mathbf{J}_{b}\otimes\mathbf{I}_{t}-\tfrac{1}{b}\mathbf{J}_{b}\otimes\tfrac{1}{t}\mathbf{J}_{t}\right)\mathbf{1}_{b}\otimes\mathbf{1}_{t}\mu\\
&=\left(\mathbf{1}_{b}\otimes\mathbf{1}_{t}-\mathbf{1}_{b}\otimes\mathbf{1}_{t}\right)\mu\\
&=\mathbf{0}
\end{aligned}$$

and

$$
\begin{aligned}
E\left[\mathbf{Q}_{\mathrm{BU_{Res}}}\mathbf{Y}\right] &= \mathbf{Q}_{\mathrm{BU_{Res}}}E\left[\mathbf{Y}\right] \\
&= \mathbf{Q}_{\mathrm{BU_{Res}}}\mathbf{X}_{\mathrm{G}}\mu \\
&= \left(\mathbf{M}_{\mathrm{BU}} - \mathbf{M}_{\mathrm{B}} - \mathbf{M}_{\mathrm{T}} + \mathbf{M}_{\mathrm{G}}\right)\mathbf{X}_{\mathrm{G}}\mu \\
&= \left(\mathbf{I}_b \otimes \mathbf{I}_t - \mathbf{I}_b \otimes \tfrac{1}{t}\mathbf{J}_t - \tfrac{1}{b}\mathbf{J}_b \otimes \mathbf{I}_t - \tfrac{1}{b}\mathbf{J}_b \otimes \tfrac{1}{t}\mathbf{J}_t\right)\mathbf{1}_b \otimes \mathbf{1}_t\mu \\
&= \left(\mathbf{1}_b \otimes \mathbf{1}_t - \mathbf{1}_b \otimes \mathbf{1}_t - \mathbf{1}_b \otimes \mathbf{1}_t + \mathbf{1}_b \otimes \mathbf{1}_t\right)\mu \\
&= \mathbf{0}.
\end{aligned}
$$

Also, using theorem XII.2 and as $\mathbf{Q}_{\mathrm{T}}\mathbf{M}_{\mathrm{BU}} = \mathbf{Q}_{\mathrm{T}}\mathbf{I}_n = \mathbf{Q}_{\mathrm{T}}$, $\mathbf{Q}_{\mathrm{T}}\mathbf{M}_{\mathrm{B}} = \left(\mathbf{M}_{\mathrm{T}} - \mathbf{M}_{\mathrm{G}}\right)\mathbf{M}_{\mathrm{B}} = \mathbf{M}_{\mathrm{G}} - \mathbf{M}_{\mathrm{G}} = \mathbf{0}$ ($\mathbf{M}_{\mathrm{B}}\mathbf{M}_{\mathrm{T}} = \mathbf{M}_{\mathrm{T}}\mathbf{M}_{\mathrm{B}} = \mathbf{M}_{\mathrm{B}}\mathbf{M}_{\mathrm{G}} = \mathbf{M}_{\mathrm{G}}\mathbf{M}_{\mathrm{B}} = \mathbf{M}_{\mathrm{T}}\mathbf{M}_{\mathrm{G}} = \mathbf{M}_{\mathrm{G}}\mathbf{M}_{\mathrm{T}} = \mathbf{M}_{\mathrm{G}}$ from the proof of theorem XII.6), $\mathbf{Q}_{\mathrm{T}}\mathbf{X}_{\mathrm{G}} = \mathbf{0}$ (see above in this proof), and $trace\left(\mathbf{Q}_{\mathrm{T}}\right) = t - 1$ (from theorem XII.6), we have

$$
\begin{aligned}
E\left[\mathbf{Y}'\mathbf{Q}_{\mathrm{T}}\mathbf{Y}/(t-1)\right] &= \left\{trace\left(\mathbf{Q}_{\mathrm{T}}\left\{\sigma^2_{\mathrm{BU}}\mathbf{M}_{\mathrm{BU}} + t\sigma^2_{\mathrm{B}}\mathbf{M}_{\mathrm{B}}\right\}\right) + \left(\mathbf{X}_{\mathrm{G}}\mu\right)'\mathbf{Q}_{\mathrm{T}}\left(\mathbf{X}_{\mathrm{G}}\mu\right)\right\}\Big/\{t-1\} \\
&= \left\{\sigma^2_{\mathrm{BU}}trace\left(\mathbf{Q}_{\mathrm{T}}\right) + 0 + \mu\mathbf{X}'_{\mathrm{G}}\mathbf{Q}_{\mathrm{T}}\mathbf{X}_{\mathrm{G}}\mu\right\}\Big/\{t-1\} \\
&= \left\{\sigma^2_{\mathrm{BU}}(t-1) + 0\right\}\Big/\{t-1\} \\
&= \sigma^2_{\mathrm{BU}}.
\end{aligned}
$$

Now, using theorem XII.2 and as $\mathbf{Q}_{\mathrm{BU_{Res}}}\mathbf{M}_{\mathrm{BU}} = \mathbf{Q}_{\mathrm{BU_{Res}}}\mathbf{I}_n = \mathbf{Q}_{\mathrm{BU_{Res}}}$, $\mathbf{Q}_{\mathrm{BU_{Res}}}\mathbf{M}_{\mathrm{B}} = \left(\mathbf{M}_{\mathrm{BU}} - \mathbf{M}_{\mathrm{B}} - \mathbf{M}_{\mathrm{T}} + \mathbf{M}_{\mathrm{G}}\right)\mathbf{M}_{\mathrm{B}} = \mathbf{M}_{\mathrm{B}} - \mathbf{M}_{\mathrm{B}} - \mathbf{M}_{\mathrm{G}} + \mathbf{M}_{\mathrm{G}} = \mathbf{0}$ ($\mathbf{M}_{\mathrm{B}}\mathbf{M}_{\mathrm{T}} = \mathbf{M}_{\mathrm{T}}\mathbf{M}_{\mathrm{B}} = \mathbf{M}_{\mathrm{B}}\mathbf{M}_{\mathrm{G}} = \mathbf{M}_{\mathrm{G}}\mathbf{M}_{\mathrm{B}} = \mathbf{M}_{\mathrm{T}}\mathbf{M}_{\mathrm{G}} = \mathbf{M}_{\mathrm{G}}\mathbf{M}_{\mathrm{T}} = \mathbf{M}_{\mathrm{G}}$ from the proof of theorem XII.6), $\mathbf{Q}_{\mathrm{BU_{Res}}}\mathbf{X}_{\mathrm{G}} = \mathbf{0}$ (see above in this proof), and $trace\left(\mathbf{Q}_{\mathrm{BU_{Res}}}\right) = (b-1)(t-1)$ (from theorem XII.6), we have

$$
\begin{aligned}
E\left[\mathbf{Y}'\mathbf{Q}_{\mathrm{BU_{Res}}}\mathbf{Y}/\{(b-1)(t-1)\}\right] &= \left\{\begin{array}{c} trace\left(\mathbf{Q}_{\mathrm{BU_{Res}}}\left\{\sigma^2_{\mathrm{BU}}\mathbf{M}_{\mathrm{BU}} + t\sigma^2_{\mathrm{B}}\mathbf{M}_{\mathrm{B}}\right\}\right) \\ + \left(\mathbf{X}_{\mathrm{G}}\mu\right)'\mathbf{Q}_{\mathrm{BU_{Res}}}\left(\mathbf{X}_{\mathrm{G}}\mu\right) \end{array}\right\}\Big/\{(b-1)(t-1)\} \\
&= \left\{\sigma^2_{\mathrm{BU}}trace\left(\mathbf{Q}_{\mathrm{BU_{Res}}}\right) + 0 + \mu\mathbf{X}'_{\mathrm{G}}\mathbf{Q}_{\mathrm{BU_{Res}}}\mathbf{X}_{\mathrm{G}}\mu\right\}\Big/\{(b-1)(t-1)\} \\
&= \left\{\sigma^2_{\mathrm{BU}}\{(b-1)(t-1)\} + 0\right\}\Big/\{(b-1)(t-1)\} \\
&= \sigma^2_{\mathrm{BU}}.
\end{aligned}
$$

Secondly, to show that $\mathbf{Y}'\mathbf{Q}_{\mathrm{T}}\mathbf{Y}$ and $\mathbf{Y}'\mathbf{Q}_{\mathrm{BU_{Res}}}\mathbf{Y}$ are independent quadratic forms we have to show that $\mathbf{Q}_{\mathrm{T}}$ and $\mathbf{Q}_{\mathrm{BU_{Res}}}$ meet two of the three conditions outlined in theorem XII.4. As outlined in the proof of theorem XII.6, $\mathbf{Q}_{\mathrm{T}}$ and $\mathbf{Q}_{\mathrm{BU_{Res}}}$ are idempotent so that condition 1 is met. For condition 3, we require that $\mathbf{Q}_{\mathrm{T}}\mathbf{Q}_{\mathrm{BU_{Res}}} = \mathbf{0}$. Now,

$$\begin{aligned}\mathbf{Q}_\mathrm{T}\mathbf{Q}_{\mathrm{BU}_\mathrm{Res}} &= \left(\mathbf{M}_\mathrm{T}-\mathbf{M}_\mathrm{G}\right)\left(\mathbf{M}_\mathrm{BU}-\mathbf{M}_\mathrm{B}-\mathbf{M}_\mathrm{T}+\mathbf{M}_\mathrm{G}\right)\\ &= \left(\mathbf{M}_\mathrm{T}\mathbf{M}_\mathrm{BU}-\mathbf{M}_\mathrm{T}\mathbf{M}_\mathrm{B}-\mathbf{M}_\mathrm{T}\mathbf{M}_\mathrm{T}+\mathbf{M}_\mathrm{T}\mathbf{M}_\mathrm{G}\right)-\left(\mathbf{M}_\mathrm{G}\mathbf{M}_\mathrm{BU}-\mathbf{M}_\mathrm{G}\mathbf{M}_\mathrm{B}-\mathbf{M}_\mathrm{G}\mathbf{M}_\mathrm{T}+\mathbf{M}_\mathrm{G}\mathbf{M}_\mathrm{G}\right)\\ &= \left(\mathbf{M}_\mathrm{T}-\mathbf{M}_\mathrm{G}-\mathbf{M}_\mathrm{T}+\mathbf{M}_\mathrm{G}\right)-\left(\mathbf{M}_\mathrm{G}-\mathbf{M}_\mathrm{G}-\mathbf{M}_\mathrm{G}+\mathbf{M}_\mathrm{G}\right)\\ &= \mathbf{0}.\end{aligned}$$

Thirdly, theorem XII.5 means that, as $\left(1/\sigma^2_\mathrm{BU}\right)\mathbf{Y}'\mathbf{Q}_\mathrm{T}\mathbf{Y}$ and $\left(1/\sigma^2_\mathrm{BU}\right)\mathbf{Y}'\mathbf{Q}_{\mathrm{BU}_\mathrm{Res}}\mathbf{Y}$ are distributed as independent chi-squares with degrees of freedom $(t-1)$ and $(b-1)(t-1)$, respectively, then

$$F=\frac{\left(1/\sigma^2_\mathrm{BU}\right)\mathbf{Y}'\mathbf{Q}_\mathrm{T}\mathbf{Y}\big/(t-1)}{\left(1/\sigma^2_\mathrm{BU}\right)\mathbf{Y}'\mathbf{Q}_{\mathrm{BU}_\mathrm{Res}}\mathbf{Y}\big/(b-1)(t-1)}=\frac{\mathbf{Y}'\mathbf{Q}_\mathrm{T}\mathbf{Y}\big/(t-1)}{\mathbf{Y}'\mathbf{Q}_{\mathrm{BU}_\mathrm{Res}}\mathbf{Y}\big/(b-1)(t-1)}$$

follows an F distribution with $(t-1)$ and $(b-1)(t-1)$ degrees of freedom. ■

Theorem XII.9: Let $\psi=E[\mathbf{Y}]=\mathbf{X}_\mathrm{T}\tau=\left(\mathbf{1}_b\otimes\mathbf{I}_t\right)\tau$, $\mathbf{V}=\sigma^2_\mathrm{BU}\mathbf{I}_n$, $s^2_\mathrm{B}=\mathbf{Y}'\mathbf{Q}_\mathrm{B}\mathbf{Y}\big/(b-1)$ and $s^2_{\mathrm{BU}_\mathrm{Res}}=\mathbf{Y}'\mathbf{Q}_{\mathrm{BU}_\mathrm{Res}}\mathbf{Y}\big/\left\{(b-1)(t-1)\right\}$ where $\mathbf{Q}_\mathrm{B}=\mathbf{M}_\mathrm{B}-\mathbf{M}_\mathrm{G}$, $\mathbf{Q}_{\mathrm{BU}_\mathrm{Res}}=\mathbf{M}_\mathrm{BU}-\mathbf{M}_\mathrm{B}-\mathbf{M}_\mathrm{T}+\mathbf{M}_\mathrm{G}$, $\mathbf{M}_\mathrm{BU}=\mathbf{I}_b\otimes\mathbf{I}_t=\mathbf{I}_n$, $\mathbf{M}_\mathrm{B}=\mathbf{I}_b\otimes\frac{1}{t}\mathbf{J}_t$, $\mathbf{M}_\mathrm{T}=\frac{1}{b}\mathbf{J}_b\otimes\mathbf{I}_t$ and $\mathbf{M}_\mathrm{G}=\frac{1}{b}\mathbf{J}_b\otimes\frac{1}{t}\mathbf{J}_t$. Then, the ratio of these two mean squares, given by

$$F_{(b-1),\,(b-1)(t-1)}=\frac{s^2_\mathrm{B}}{s^2_{\mathrm{BU}_\mathrm{Res}}}$$

is distributed as a Snedecor's F with $(b-1)$ and $(b-1)(t-1)$ degrees of freedom.

Proof: parallels that for the treatment differences. ■

XII.G Summary

In this chapter we have:

- Described the theory underlying entries in an ANOVA table:
  1. the sources are derived form the experimental structure that is based on the division of the factors into unrandomized and randomized factors;
  2. the sums of squares for the analysis are the quadratic forms whose matrices (**Q**s) are the idempotents of the relationship algebras corresponding to the two structure formulae derived from the unrandomized and randomized factors; they represent the squared lengths of the orthogonal projections of the data vector onto the spaces of the idempotents; rules for obtaining expressions for the **Q**s in terms of the mean operator matrices (**M**s) were given in chapter VI and a rule for obtaining direct product expressions for the summation or relationship matrices (**S**s), and hence the **M**s, is given in this chapter;
  3. the degrees of freedom are the ranks of the idempotent spaces;
  4. the expected mean squares were proven to be given by the following general expression $E\left[\mathbf{Y}'\mathbf{Q}\mathbf{Y}/\nu\right]=\left(trace(\mathbf{QV})+\boldsymbol{\psi}'\mathbf{Q}\boldsymbol{\psi}\right)\big/\nu$ that depends on the models

through $\boldsymbol{\psi}$ and $\mathbf{V}$, and on the matrix of its quadratic form ($\mathbf{Q}$) and the rank of $\mathbf{Q}$ (ν, also known as the degrees of freedom of its sum of squares) ;

5. the F statistics follow the F distribution because each of them is the ratio of two independent mean squares that, under the null hypothesis, have the same expectation.

- This theory was applied to the randomized complete block design.

XII.H Exercises

XII.1 Let $\psi = E[\mathbf{Y}] = \mathbf{X}_{\mathrm{T}}\tau = (\mathbf{1}_b \otimes \mathbf{I}_t)\tau$, $\mathbf{M}_{\mathrm{BU}} = \mathbf{I}_b \otimes \mathbf{I}_t = \mathbf{I}_n$, $\mathbf{M}_{\mathrm{B}} = \mathbf{I}_b \otimes \frac{1}{t}\mathbf{J}_t$, $\mathbf{M}_{\mathrm{T}} = \frac{1}{b}\mathbf{J}_b \otimes \mathbf{I}_t$, $\mathbf{M}_{\mathrm{G}} = \frac{1}{b}\mathbf{J}_b \otimes \frac{1}{t}\mathbf{J}_t$, $\mathbf{Q}_{\mathrm{B}} = \mathbf{M}_{\mathrm{B}} - \mathbf{M}_{\mathrm{G}}$, $\mathbf{Q}_{\mathrm{T}} = \mathbf{M}_{\mathrm{T}} - \mathbf{M}_{\mathrm{G}}$ and $\mathbf{Q}_{\mathrm{BU}_{\mathrm{Res}}} = \mathbf{M}_{\mathrm{BU}} - \mathbf{M}_{\mathrm{B}} - \mathbf{M}_{\mathrm{T}} + \mathbf{M}_{\mathrm{G}}$. Prove that $\mathbf{Q}_{\mathrm{B}}\mathbf{M}_{\mathrm{BU}} = \mathbf{Q}_{\mathrm{B}}$ and $\mathbf{Q}_{\mathrm{B}}\mathbf{M}_{\mathrm{B}} = \mathbf{Q}_{\mathrm{B}}$.

XII.2 Derive an expression, in terms of the other $\mathbf{Q}$ matrices, for the matrix of the quadratic form for the Residual sum of squares $\mathbf{Y}'\mathbf{Q}_{\mathrm{U}_{\mathrm{Res}}}\mathbf{Y}$ for a two-factor factorial experiment laid out using a CRD. Using this expression, derive an expression in terms of $\mathbf{M}$ matrices.

Suppose that the observations are in standard order for A then B and finally the *r* reps of each combination. Give direct-product expressions for the matrices $\mathbf{M}_{\mathrm{U}}$, $\mathbf{M}_{\mathrm{AB}}$, $\mathbf{M}_{\mathrm{A}}$, $\mathbf{M}_{\mathrm{B}}$ and $\mathbf{M}_{\mathrm{G}}$ using rule X.1 and the fact that A, B and a dummy Reps factor would uniquely index the units.

Prove that $\mathbf{Q}_{\mathrm{U}_{\mathrm{Res}}}$ is symmetric and idempotent and that the Residual degrees of freedom are equal to $ab(r-1)$.

XII.3 For an RCBD, let $\psi = E[\mathbf{Y}] = \mathbf{X}_{\mathrm{T}}\tau = (\mathbf{1}_b \otimes \mathbf{I}_t)\tau$, $\mathbf{V} = \sigma^2_{\mathrm{BU}}\mathbf{I}_n + \sigma^2_{\mathrm{B}}(\mathbf{I}_b \otimes \mathbf{J}_t) = \sigma^2_{\mathrm{BU}}\mathbf{M}_{\mathrm{BU}} + t\sigma^2_{\mathrm{B}}\mathbf{M}_{\mathrm{B}}$, $SS_{\mathrm{B}} = \mathbf{Y}'\mathbf{Q}_{\mathrm{B}}\mathbf{Y} = \mathbf{Y}'(\mathbf{M}_{\mathrm{B}} - \mathbf{M}_{\mathrm{G}})\mathbf{Y}$, $SS_{\mathrm{T}} = \mathbf{Y}'\mathbf{Q}_{\mathrm{T}}\mathbf{Y} = \mathbf{Y}'(\mathbf{M}_{\mathrm{T}} - \mathbf{M}_{\mathrm{G}})\mathbf{Y}$ and $SS_{\mathrm{BU}_{\mathrm{Res}}} = \mathbf{Y}'\mathbf{Q}_{\mathrm{BU}_{\mathrm{Res}}}\mathbf{Y} = \mathbf{Y}'(\mathbf{Q}_{\mathrm{BU}} - \mathbf{Q}_{\mathrm{T}})\mathbf{Y} = \mathbf{Y}'(\mathbf{M}_{\mathrm{BU}} - \mathbf{M}_{\mathrm{B}} - \mathbf{M}_{\mathrm{T}} + \mathbf{M}_{\mathrm{G}})\mathbf{Y}$.

Prove that

$$E\left[SS_{\mathrm{T}}/(t-1)\right] = \sigma^2_{\mathrm{BU}} + q_{\mathrm{T}}(\psi),$$

where $q_{\mathrm{T}}(\psi) = \sum_{j=1}^{t} b\left(\tau_j - \bar{\tau}_.\right)^2 \Big/ (t-1)$, $\bar{\tau}_. = \sum_{j=1}^{t}\tau_j \Big/ t$, τ_j is the *j*th element of the *t*-vector τ, *b* is the number of blocks and *t* is the number of treatments.

XII.4 For a Latin square design in which Rows and Columns are random the maximal models are $E[\mathbf{Y}] = \mathbf{X}_{\mathrm{T}}\tau$, $\mathbf{V} = \sigma^2_{\mathrm{RC}}(\mathbf{I}_t \otimes \mathbf{I}_t) + \sigma^2_{\mathrm{R}}(\mathbf{I}_t \otimes \mathbf{J}_t) + \sigma^2_{\mathrm{C}}(\mathbf{J}_t \otimes \mathbf{I}_t) = \sigma^2_{\mathrm{RC}}\mathbf{M}_{\mathrm{RC}} + t\sigma^2_{\mathrm{R}}\mathbf{M}_{\mathrm{R}} + t\sigma^2_{\mathrm{C}}\mathbf{M}_{\mathrm{C}}$ where $\mathbf{M}_{\mathrm{RC}}$, $\mathbf{M}_{\mathrm{R}}$ and $\mathbf{M}_{\mathrm{C}}$ are symmetric and idempotent. Prove that the test statistic for testing the hypothesis that $\sigma^2_{\mathrm{R}} = 0$,

$$F_{(t-1),\,(t-1)(t-2)} = \frac{s^2_{\mathrm{R}}}{s^2_{\mathrm{RC}_{\mathrm{Res}}}} = \frac{\mathbf{Y}'\mathbf{Q}_{\mathrm{R}}\mathbf{Y}/(t-1)}{\mathbf{Y}'\mathbf{Q}_{\mathrm{RC}_{\mathrm{Res}}}\mathbf{Y}/(t-1)(t-2)},$$

is distributed as Snedecor's F with $(t-1)$ and $(t-1)(t-2)$ degrees of freedom.

You are given that $\mathbf{M}_{\mathrm{R}}\mathbf{X}_{\mathrm{T}} = \mathbf{M}_{\mathrm{C}}\mathbf{X}_{\mathrm{T}} = \mathbf{M}_{\mathrm{G}}\mathbf{X}_{\mathrm{T}} = \frac{1}{t}\mathbf{1}_t \otimes \mathbf{J}_t$, $\mathbf{M}_{\mathrm{T}}\mathbf{X}_{\mathrm{T}} = \mathbf{X}_{\mathrm{T}}$ and the product of any nonidentical pair of $\mathbf{M}_{\mathrm{R}}$, $\mathbf{M}_{\mathrm{C}}$, $\mathbf{M}_{\mathrm{T}}$ and $\mathbf{M}_{\mathrm{G}}$ is equal to $\mathbf{M}_{\mathrm{G}}$. Also $\mathbf{Q}_{\mathrm{R}}$ and $\mathbf{Q}_{\mathrm{RC}_{\mathrm{Res}}}$ are idempotent with $\mathit{trace}(\mathbf{Q}_{\mathrm{T}}) = t-1$ and $\mathit{trace}\left(\mathbf{Q}_{\mathrm{RC}_{\mathrm{Res}}}\right) = (t-1)(t-2)$.