4 Written Problems (40 points)

1. Use LU factorization **only** to solve:

$$\begin{cases} 2x_1 + 2x_2 + 3x_3 + 4x_4 &= 8 \\ 2x_1 + 4x_2 + 9x_3 + 16x_4 &= 6 \\ 4x_1 + 8x_2 + 24x_3 + 63x_4 &= -26 \\ 6x_1 + 16x_2 + 51x_3 + 100x_4 &= 6 \end{cases}$$

$$A = \begin{bmatrix} 2 & 2 & 3 & 4 \\ 2 & 4 & 9 & 16 \\ 4 & 8 & 24 & 63 \\ 6 & 16 & 51 & 100 \end{bmatrix} \qquad b = \begin{bmatrix} 8 \\ 6 \\ -26 \\ 6 \end{bmatrix}$$

Using other approaches will result in partial credits only. Show all steps.

① REDUCE A TO ROW ECHELON FORM TO GET U.

$$A = \begin{bmatrix} 2 & 2 & 3 & 4 \\ \boxed{2} & 4 & 9 & 16 \\ \boxed{4} & 8 & 24 & 63 \\ \boxed{6} & 16 & 51 & 100 \end{bmatrix} \xrightarrow[\substack{R_3 \leftarrow R_3 - 2R_1 \\ R_4 \leftarrow R_4 - 3R_1}]{R_2 \leftarrow R_2 - R_1} \begin{bmatrix} 2 & 2 & 3 & 4 \\ 0 & 2 & 6 & 12 \\ 0 & \boxed{4} & 18 & 55 \\ 0 & \boxed{10} & 42 & 88 \end{bmatrix} \xrightarrow[R_4 \leftarrow R_4 - 5R_2]{R_3 \leftarrow R_3 - 2R_2} \begin{bmatrix} 2 & 2 & 3 & 4 \\ 0 & 2 & 6 & 12 \\ 0 & 0 & 6 & 31 \\ 0 & 0 & \boxed{12} & 28 \end{bmatrix} \xrightarrow{R_4 \leftarrow R_4 - 2R_3} \begin{bmatrix} 2 & 2 & 3 & 4 \\ 0 & 2 & 6 & 12 \\ 0 & 0 & 6 & 31 \\ 0 & 0 & 0 & -34 \end{bmatrix} = U$$

② DIVIDE THE ENTRIES IN EACH COLUMN BY THE PIVOT AT THE TOP TO GET L

Since A has 4 rows, L will be 4×4.
Take any remaining columns from I_4.

$$\begin{bmatrix} 2 \\ 2 \\ 4 \\ 6 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 10 \end{bmatrix} \begin{bmatrix} 6 \\ 12 \end{bmatrix} \longrightarrow L = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 2 & 2 & 1 & \\ 3 & 5 & 2 & \boxed{1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 3 & 5 & 2 & 1 \end{bmatrix}$$

③ Solve for $Ly = b$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 3 & 5 & 2 & 1 \end{bmatrix} y = \begin{bmatrix} 8 \\ 6 \\ -26 \\ 6 \end{bmatrix} \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 8 \\ 1 & 1 & 0 & 0 & 6 \\ 2 & 2 & 1 & 0 & -26 \\ 3 & 5 & 2 & 1 & 6 \end{array}\right] \longrightarrow \begin{aligned} x_1 &= 8 \\ x_1 + x_2 &= 6 \\ 2x_1 + 2x_2 + x_3 &= -26 \\ 3x_1 + 5x_2 + 2x_3 + x_4 &= 6 \end{aligned} \longrightarrow y = \begin{bmatrix} 8 \\ -2 \\ -38 \\ 68 \end{bmatrix}$$

④ Solve for $Ux = y$

$$\begin{bmatrix} 2 & 2 & 3 & 4 \\ 0 & 2 & 6 & 12 \\ 0 & 0 & 6 & 31 \\ 0 & 0 & 0 & -34 \end{bmatrix} x = \begin{bmatrix} 8 \\ -2 \\ -38 \\ 68 \end{bmatrix} \longrightarrow \left[\begin{array}{cccc|c} 2 & 2 & 3 & 4 & 8 \\ 0 & 2 & 6 & 12 & -2 \\ 0 & 0 & 6 & 31 & -38 \\ 0 & 0 & 0 & -34 & 68 \end{array}\right] \longrightarrow \begin{aligned} 2x_1 + 2x_2 + 3x_3 + 4x_4 &= 8 \\ 2x_2 + 6x_3 + 12x_4 &= -2 \\ 6x_3 + 31x_4 &= -38 \\ -34x_4 &= 68 \end{aligned} \longrightarrow x = \begin{bmatrix} 3 \\ -1 \\ 4 \\ -2 \end{bmatrix}$$

2. Let $\mathbf{A} = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 5 & 6 \\ 2 & 11 & 5 \end{bmatrix}$. Find the elementary matrices $\mathbf{E_1}, \mathbf{E_2}, \mathbf{E_3}$ such that

$\mathbf{E_3 E_2 E_1 A} = \mathbf{U}$. Then, using **only** the elementary matrices you found, calculate $\mathbf{L}$. Do not compute $\mathbf{L}$ based on your reduction from $\mathbf{A}$ to $\mathbf{U}$.

Note that $\mathbf{L}$ and $\mathbf{U}$ here refer to the LU factorization of $\mathbf{A}$. Show all steps.

$E_3E_2E_1A = U \longrightarrow A = LU \longrightarrow A = E_1E_2E_3U \longrightarrow L = E_1E_2E_3$

① Find the elementary matrices.

$$A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 5 & 6 \\ \boxed{2} & 11 & 5 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 - R_1} \begin{bmatrix} 2 & 2 & 2 \\ \boxed{2} & 5 & 6 \\ 0 & 9 & 3 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - R_1} \begin{bmatrix} 2 & 2 & 2 \\ 0 & 3 & 4 \\ 0 & \boxed{9} & 3 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 - 3R_2} \begin{bmatrix} 2 & 2 & 2 \\ 0 & 3 & 4 \\ 0 & 0 & -9 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

② Using the elementary matrices, calculate L.

$$E_1E_2E_3 = L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix}$$