Master's Thesis

Mathematical Modelling of High-Frequency Data

Brieuc Antoine Dit Urban
Yves Leconte
Ivan Svatko

Supervisor : Raphaël Maillet

2022 — 2023

Contents

Acknowledgements

We would like to thank Raphaël Maillet for providing us with the opportunity to work on such an interesting topic, as well as for his unwavering dedication and support. This engaging research work presented a unique chance for us to combine our knowledge of stochastic calculus and mathematical statistics. Furthermore, we are grateful to Julien Stoehr for allowing us to embark on this project, further enriching our experience and fostering our growth in the field.

Introduction

Context. In 1900, Louis Bachelier defended his thesis titled "Theory of Speculation"[Bac00] in front of Henri Poincaré. In his paper, he assumed that financial asset prices could only be described by a Brownian motion. Since then, numerous researchers contributed to this theory as in 1973, when Fisher Black and Myron Scholes published their famous paper "The Pricing of Options and Corporate Liabilities" [BS73], which is still considered as one of the major contributions to mathematization of Finance. Robert Cox Merton was one of the first to understand the potential of it and so he soon afterwards published his paper "Theory of Rational Option Pricing" [Mer73] in which he derived an explicit formula for both call and put options. Those results opened huge possibilities for quantitative finance as it allowed market agents to exactly price complex financial products as derivatives and options, and thus to foster exchanges on financial markets. Nevertheless, since 2000, the advent of electronic trading and rapid advancements in data processing capabilities have led mathematicians to question whether the assumptions made by previous researchers still hold for the analysis of high-frequency financial markets. In 2010, Yacine Aït-Sahalia and Jean Jacod published a paper titled "Is Brownian motion necessary to model High-Frequency Data ?" [ASJ10]. Inspired by the work of Yacine Aït-Sahalia and Jean Jacod, in this paper we delve into the realm of mathematical modelling of high-frequency data.

Motivation. Our goal is to study estimations of a volatility coefficient σ to determine statistical procedures to test whether the trajectory of a process X representing financial data over $[0, T]$ can be described by a Brownian diffusion in increasingly complex models.
In the first section, we review some fundamental stochastic calculus results that will be used in the remainder of the paper. In the second section, we define the general parametric framework of the paper. In the third section, we study the problem in a particular and idealistic case where σ is constant, providing a complete statistical procedure for this framework. In the fourth section, we extend our model to the case where σ is a deterministic function dependent on time or a stochastic process. We also give a complete statistical procedure for this new framework when σ is deterministic.

Within the time-dependent framework, we derive estimators and Central Limit Theorems for both integrated and spot-volatility.

1 Preliminary results

We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $B = (B_t,\ t \geq 0)$ a process defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We let $\mathbb{F} = (\mathcal{F}_s, s \geq 0)$ be a filtration such that the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F})$ is complete. In this section we give elements of stochastic calculus that are often used in the following. We will begin with a definition of an $\mathbb{F}$-brownian motion.

Definition 1 (Brownian motion in $\mathbb{R}$)**.** *If the $\mathbb{F}$-adapted process B is such that:*

(Continuity) B has continuous trajectories a.s.;

(Independence of the increments) For all $0 \leq s \leq t,\ B_t - B_s$ is independent from $\mathcal{F}_s$;

(Stationarity of the increments) For all $0 \leq s \leq t,\ B_t - B_s \sim \mathcal{N}(0, t-s)$;

we say that B is an $\mathbb{F}$-Brownian motion.

Remark 1. *If $B_0 = 0$ then the Brownian motion is said to be standard.*

From now on, we consider that the process B is a one-dimensional standard Brownian motion.

Definition 2 (Variation of order p)**.** *For $t \geq 0$ and $\Delta := \{t_0 = 0 \leq t_1 \leq ... \leq t_n = t\}$ a subdivision of the set $[0,t]$, we denote $|\Delta| := \sup_i \{t_i - t_{i-1}\}$. For any function g and any real number $p \geq 1$, we define*

$$V_t^{(p)}(g, \Delta) := \sum_{i=1}^{n} |g(t_i) - g(t_{i-1})|^p.$$

We say that g has finite variation of order p if

$$\sup_{\Delta \in \mathcal{P}} V_t^{(p)}(g, \Delta) < \infty$$

where $\mathcal{P}$ is the is set of all possible subdivisions of $[0,t]$.

Proposition 1. *A Brownian motion has an infinite total variation for all $t > 0$.*

Proof. We are going to show that $\sup_{|\Delta| \to 0} V_t^{(p)}(B, \Delta) = \infty$ where B is a Brownian motion and Δ is a uniform subdivision of $[0,t]$. We have that :

$$\begin{aligned}
\sup_{\Delta \in \mathcal{P}} V_t^{(p)}(B, \Delta) &= \sup_{\Delta \in \mathcal{P}} \sum_{i=1}^{n} \left| B_{t_i} - B_{t_{i-1}} \right| \\
&\geq \sum_{i=1}^{n} \left| B_{T\frac{i}{n}} - B_{T\frac{i-1}{n}} \right| \\
&= \frac{1}{\sqrt{\frac{n}{T}}} \sum_{i=1}^{n} \left| \sqrt{\frac{n}{T}} \left(B_{T\frac{i}{n}} - B_{T\frac{i-1}{n}} \right) \right| \\
&= \sqrt{n} \frac{\sqrt{T}}{n} \sum_{i=1}^{n} \left| \sqrt{\frac{n}{T}} \left(B_{T\frac{i}{n}} - B_{T\frac{i-1}{n}} \right) \right| \quad .
\end{aligned}$$

But, by the stationarity of the increments of the Brownian motion, we know that for all $i \in \{1, \ldots, n\}$,

$$B_{T\frac{i}{n}} - B_{T\frac{i-1}{n}} \sim \mathcal{N}(0, \frac{T}{n})$$
$$i.e. \quad \sqrt{\frac{n}{T}} \left(B_{T\frac{i}{n}} - B_{T\frac{i-1}{n}} \right) \sim \mathcal{N}(0,1)\,.$$

Thus, by independence of the increments, we can apply the law of large numbers :

$$\frac{1}{n} \sum_{i=1}^{n} \left| \sqrt{\frac{n}{T}} \left(B_{T\frac{i}{n}} - B_{T\frac{i-1}{n}} \right) \right| \longrightarrow E(|X|) > 0 \quad a.s$$

where $X \sim \mathcal{N}(0,1)$, then

$$\sqrt{n} \frac{\sqrt{T}}{n} \sum_{i=1}^{n} \left| \sqrt{\frac{n}{T}} \left(B_{T\frac{i}{n}} - B_{T\frac{i-1}{n}} \right) \right| \longrightarrow \infty \quad a.s\,.$$

□

Proposition 2. *For all $t \geq 0$,*

$$\lim_{|\Delta| \to 0} \mathbb{E} \left[\left| V_t^{(2)}(B, \Delta) - t \right|^2 \right] = 0$$

i.e.

$$\lim_{\sup_i |t_i - t_{i-1}| \to 0} \mathbb{E} \left[\left(\sum_{i=1}^{\infty} \left| B_{t_i} - B_{t_{i-1}} \right|^2 - t \right)^2 \right] = 0.$$

Proof. We begin by noticing that

$$\sum_{i=1}^{n} (t_i - t_{i-1}) = t$$

Then

$$\mathbb{E} \left[\left(\sum_{i=1}^{\infty} \left| B_{t_i} - B_{t_{i-1}} \right|^2 - t \right)^2 \right] = \mathbb{E} \left[\left(\sum_{i=1}^{\infty} \left| B_{t_i} - B_{t_{i-1}} \right|^2 - (t_i - t_{i-1}) \right)^2 \right].$$

Expanding the right hand side and using linearity of the expectation, we obtain

$$\begin{aligned}
&\mathbb{E} \left[\left| \sum_{i=1}^{n} \left(\left(B_{t_i} - B_{t_{i-1}} \right)^2 - (t_i - t_{i-1}) \right) \right|^2 \right] \\
&= \sum_{i=1}^{n} \mathbb{E} \left[\left(\left(B_{t_i} - B_{t_{i-1}} \right)^2 - (t_i - t_{i-1}) \right)^2 \right] \\
&+ 2 \sum_{k<i} \mathbb{E} \Big[\left(\left(B_{t_i} - B_{t_{i-1}} \right)^2 - (t_i - t_{i-1}) \right) \\
&\qquad \times \left(\left(B_{t_k} - B_{t_{k-1}} \right)^2 - (t_k - t_{i-1}) \right) \Big] \\
&= \sum_{i=1}^{n} \mathbb{E} \left[\left(\left(B_{t_i} - B_{t_{i-1}} \right)^2 - (t_i - t_{i-1}) \right)^2 \right],
\end{aligned}$$

where in the last equality we used independence of the increments of Brownian motion and the fact that for any $s < t$: $B_t - B_s \sim \mathcal{N}(0, t-s)$.

Let $\xi_k \coloneqq B_{t_k} - B_{t_{k-1}}$, for $k \in \{1,..,n\}$, then

$$\sum_{k=1}^{n} \mathbb{E}\left[\left(\left(B_{t_k} - B_{t_{k-1}}\right)^2 - (t_k - t_{k-1})\right)^2\right] = \sum_{k=1}^{n} \mathbb{E}\left[\xi_k^4\right] - 2\sum_{k=1}^{n}(t_k - t_{k-1})\mathbb{E}\left[\xi_k^2\right] + \sum_{k=1}^{n}(t_k - t_{k-1})^2. \quad (1)$$

By definition of ξ_k we know that $\xi_k \sim \mathcal{N}(0, \sigma_k^2)$ with $\sigma_k^2 \coloneqq t_k - t_{k-1}$. Then $\phi_{\xi_k} : \ u \in \mathbb{R} \mapsto \mathbb{E}e^{iu\xi_k}$ is the characteristic function of ξ_k and

$$\phi_{\xi_k}(u) = e^{-\frac{\sigma_k^2 u^2}{2}}.$$

Then $\mathbb{E}[\xi_k^4] = i^{-4}\phi_{\xi_k}^{(4)}(0)$. Computing the derivatives yields the following:

$$i^{-4}\phi_{\xi_k}^{(4)}(u) = \left(\sigma_k^8 u^4 - 6\sigma_k^6 u^2 + 3\sigma_k^4\right) e^{-\frac{\sigma_k^2 u^2}{2}}.$$

Thus,

$$\mathbb{E}[\xi_k^4] = 3\sigma_k^4 = 3(t_k - t_{k-1})^2.$$

Hence, from Equation (1) we obtain

$$\sum_{i=1}^{n} \mathbb{E}\left[\left(\left(B_{t_i} - B_{t_{i-1}}\right)^2 - (t_i - t_{i-1})\right)^2\right] = 2\sum_{i=1}^{n}(t_i - t_{i-1})^2.$$

Whence we derive following inequality

$$\begin{aligned}\sum_{i=1}^{n} \mathbb{E}\left[\left(\left(B_{t_i} - B_{t_{i-1}}\right)^2 - (t_i - t_{i-1})\right)^2\right] &= 2\sum_{i=1}^{n}(t_i - t_{i-1})^2 \\ &\leq 2\sup_i |t_i - t_{i-1}| \sum_{i=1}^{n}(t_i - t_{i-1}) \\ &= 2\,|\Delta|\,t.\end{aligned}$$

We conclude by taking the limit as $|\Delta| \to 0$. □

Remark 2. *From the last inequality of the proof we can derive the speed of convergence of quadratic variation in L^2. Notably, by taking a uniform partition $t_i = ti/n$ for $i \in \{0, \cdots, n\}$ we have that $V_t^{(2)}(B, \Delta)$ converges to t in L^2 at rate $n^{-1/2}$.*

Next, we will define some key notions from stochastic calculus that will be important thereafter. From now on, we let $[0,T]$ be a fixed time interval on which our model evolves.

We consider the following spaces:

$$\mathcal{L}^2([0,T]) = \left\{ (H_t,\ t \in [0,T])\,,\ \text{an } \mathbb{F}\text{-adapted process} \;\middle|\; \mathbb{E}\left[\int_0^T H_u^2 du\right] < \infty \right\}, \quad (2)$$

which is the space of square integrable adapted processes and

$$\mathcal{M}^2([0,T]) = \left\{ (M_t,\ t \in [0,T])\,,\ \mathbb{F}\text{-martingale} \;\middle|\; \forall t \in [0,T],\ \mathbb{E}[M_t^2] < \infty \right\}, \quad (3)$$

the space of square integrable martingales. If $[0,T]$ is not specified we assume that $\mathcal{M}^2$ is the space of square integrable martingales that evolve with $t \geq 0$.

Proposition 3 (Wiener's Integral)**.** *Let $f : [0,T] \mapsto \mathbb{R}$ a map in $L^2([0,T], dx)$ i.e. deterministic. The process $\left(\int_0^t f(s)dBs,\ t \in [0,T]\right)$ is called a Wiener integral, which satisfies*

$$\int_0^t f(s)dB_s \sim \mathcal{N}\left(0, \int_0^t f^2(s)ds\right). \tag{4}$$

Proof. Admitted, see [BTTT16]. □

To construct the Itô integral one often starts by defining it for a class of elementary processes.

Definition 3 (Elementary process)**.** *We call the process H elementary if it can be defined as follows:*

$$H_t(\omega) = \sum_{i=1}^{n} \theta_i \mathbb{1}_{]t_{i-1},t_i]},\ \forall t \in]0,T],$$

where $0 = t_0 < \cdots < t_n = T$ is a partition of $[0,T]$ and $(\theta_i)_{1 \leq i \leq n}$ is a bounded sequence of $\mathcal{F}_{t_{i-1}}$-measurable random variables.

Definition 4 (Itô integral for elementary process)**.** *Let H be an elementary process as defined above. Then, the we define the Itô stochastic integral*

$$\int_0^t H_u\, dB_u \coloneqq \sum_{i=1}^{n} \theta_{i-1}\left(B_{t_i \wedge t} - B_{t_{i-1} \wedge t}\right), \qquad t \in [0,T].$$

Then, for $0 \leq s \leq t \leq T$ we define

$$\int_s^t H_u\, dB_u \coloneqq \int_0^t H_u\, dB_u - \int_0^s H_u\, dB_u.$$

It can be shown that elementary processes can be used to approximate processes in $\mathcal{L}^2([0,T])$. For a detailed construction of the Itô integral one can refer to [Øks03] and [KS98].

The following Proposition allows us to define the Itô integral in $\mathcal{L}^2[0,T]$.

Proposition 4 (Itô integral in $\mathcal{L}^2([0,T])$)**.** *There exists a unique linear application $\mathcal{I}$, which for any process $H \in \mathcal{L}^2([0,T])$, $\mathcal{I}[H] \in \mathcal{M}^2([0,T])$ which coincides with the integral defined above on the class of elementary processes*

$$\mathcal{I}\left[H^e\right]_\cdot = \int_0^\cdot H_u\, dB_u, \qquad \text{for } H^e \text{ elementary}$$

and such that the isometry property is verified. Meaning that: $\forall H \in \mathcal{L}^2([0,T]), \forall t \in [0,T]$

$$\mathbb{E}\left[\mathcal{I}\left[H\right]_t^2\right] = \mathbb{E}\left[\int_0^t H_s^2 ds\right]. \tag{5}$$

We denote $\int_0^t H_s dB_s \coloneqq \mathcal{I}[H]_t$ for $H \in \mathcal{L}^2([0,T])$. We call $\mathcal{I}$ Itô stochastic integral.

Proof. Admitted, see [BTTT16]. □

Definition 5 (CÀDLÀG)**.** *We say that a real function has the càdlàg property if it is right continuous and has has left limits everywhere on it's domain.*

Definition 6. *A real valued process X defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F})$ is called a semimartingale if it can be decomposed as $X_t = M_t + A_t$ where M is a local martingale and A is a càdlàg adapted process of locally finite total variation.*

2 General framework

From now on, we will build a statistical procedure to test the presence of a Brownian motion in a given stochastic process $X = (X_t,\ t \in [0,T])$ defined as

$$X_t = X_0 + \int_0^t b(s, X_s)\, ds + \int_0^t \sigma_s dB_s, \tag{6}$$

where B is an $\mathbb{F}$-Brownian motion, σ is the volatility associated to it and b is the drift component of X. We remark that the last term on the right-hand side is an Itô stochastic integral as defined above.

The idea is that if $\sigma_s = 0 \quad a.s$ for $s \in [0,T]$, then we can conclude the Brownian motion has no significance in dynamics of X. This leads us to the construction of a parametric model $(\Omega, \mathcal{B}, \mathbb{P}_\sigma, \sigma \in \Theta)$ where realisations of X are in $\Omega = \mathcal{C}([0,T],\mathbb{R})$, $\mathcal{B}$ is the borel set associated to it and $\mathbb{P}_\sigma$ depends of the drift. The set of parameters is

$$\Theta = \{\sigma := s \mapsto \sigma_s\, , \sigma \in \mathcal{C}([0,T],\mathbb{R}_+)\}$$

and, unless mentioned otherwise, our null and alternative hypothesis, respectively, are

$$H_0 : \int_0^T \sigma_s^2\, ds > 0 \quad a.s \qquad \text{against} \qquad H_1 : \sigma_s = 0 \quad a.s\,.$$

Throughout the paper, we are going to consider various assumptions over b and σ, however, here are the two assumptions that we will consider to be always be true.

Assumption (A2.1). *$b : [0,T] \times \mathbb{R} \mapsto \mathbb{R}$ is a Riemann integrable in time and measurable function satisfying*

$$|b(t,x) - b(t,y)| \le C\,|x-y|\,, \qquad x,y \in \mathbb{R},\ t \in [0,T],$$

where C is some real constant.

Assumption (A2.2). *$\sigma = (\sigma_t,\ t \in [0,T])$ is an $\mathbb{F}$-adapted process, independent from $\mathbb{F}$-Brownian motion B.*

These assumptions alone don't necessarily imply the existence and uniqueness of a solution once we have a stochastic differential equation and σ depends on time. These cases will be treated in the section dedicated dedicated to the time-varying volatility.

To approach this statistical problem we will build an estimator for volatility that can later be used to construct a test statistic. We assume that the process is observed on a finite time interval $[0,T]$ fixed in advance and we have access to it's sampled values $(X_{t_i})_{0\le i\le n}$ at times $0 = t_0 \le t_1 \le \cdots \le t_n = T$ for $n \in \mathbb{N}^*$ that create a partition of $[0,T]$.

The key idea that we will explore lies in comparing quadratic variation of stochastic integrals with respect to Brownian motion with the one of regular, in a certain sense, functions. We will estimate the quadratic variation with help of realized volatility

$$\hat{\sigma}_n^2 := \frac{1}{T} \sum_{i=1}^{n} \left(X_{t_i} - X_{t_{i-1}}\right)^2 .$$

The previous section explored the link between this object and the quadratic variation of Brownian motion. In fact, this estimator turns out to be extremely versatile and we will use it to estimate the integrated volatility on $[0,T]$ as well as the spot-volatility σ_t at a given $t \in [0,T]$.

We will mostly focus on the two possible approaches to testing our hypothesis: one focuses on finding the asymptotic distribution for $\hat{\sigma}_n^2$ and the other studies the convergence of a ratio of estimators of volatility at different sampling rates.

Remark 3. *Throughout this paper, we will call an estimator consistent if it converges in probability to the estimated value. This property is sometimes referred to as weak consistency.*

3 Testing volatility's degeneracy in the case of discrete observations of a process with constant volatility

From now on, we will build a statistical procedure to test if the volatility σ^2 is degenerate, given observations of a process $X = (X_t,\ t \in [0,T])$.

Let us formally define a statistical model that will be used throughout this section.

Assumption (A3). *By now we are going to consider the following process:*

$$X_t = X_0 + \int_0^t b_s ds + \sigma B_t \quad \forall t \in [0,T]\,,\, T > 0$$

Where $s \mapsto b_s$ is a deterministic and Lipschitz continuous function, $\sigma \geq 0$ is constant and B is a Brownian motion defined over the same interval.
This process evolves in the parametric model $(\Omega, \mathcal{B}, \mathbb{P}_\sigma, \sigma \in \Theta)$ where realisations are in $\Omega = \mathcal{C}([0,T], \mathbb{R})$ and $\mathcal{B}$ is the borel set associated to it.
The set of parameters is $\Theta = \{\sigma \in \mathbb{R}_+\}$ and the process follows a Gaussian law of parameter σ : $\mathbb{P}_\sigma = \mathcal{N}(\int_0^T b_s\, ds, \sigma^2 T)$.

Finally, in this section we assume that our observations come from a uniform partition of $[0,T]$, defined by $t_i = Ti/n,\ i \in \{1, \cdots, n\}$ for some $n \in \mathbb{N}^*$.

We will approach our goal step by step starting with a simplified case where b is null.

Assumption (A3.1). *Under Assumption (**A3**) we suppose additionally that $b_s = 0$ for all $s \in [0,T]$.*

Now we build a consistent estimator of σ^2 based on the properties the quadratic variation of Brownian Motion.

Proposition 5. *Under (**A3.1**), for all $t \in [0,T]$,*

$$\sum_{i=1}^{n} (X_{t_i} - X_{t_{i-1}})^2 \to \sigma^2 T,$$

in L^2 as n goes to infinity.

Proof. We showed in Proposition (2) for $\{0 = t_0 \leq ... \leq t_n = T\}$ that

$$\sum_{i=1}^{n} (B_{t_i} - B_{t_{i-1}})^2 \to T$$

in L^2 as n goes to infinity, so under (**A3.1**)

$$\begin{aligned}\sum_{i=1}^{n} (X_{t_i} - X_{t_{i-1}})^2 &= \sigma^2 \sum_{i=1}^{n} (B_{t_i} - B_{t_{i-1}})^2 \\ &\to \sigma^2 T,\end{aligned}$$

in L^2 as $n \to \infty$. □

Given the model in Assumption (**A3.1**) our estimator will be positive if and only if σ is positive. This leads to a straightforward test.

Proposition 6. *Under (**A3.1**) we can consider the test statistic*

$$\hat{\sigma}_n^2 := \frac{1}{T} \sum_{i=1}^{n} (X_{t_i} - X_{t_{i-1}})^2, \tag{7}$$

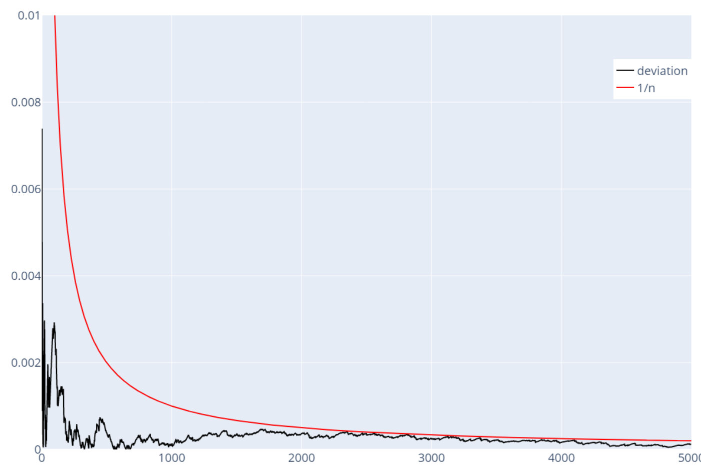
where $\hat{\sigma}_n^2 \to \sigma^2$ in L^2 as $n \to \infty$, which gives us a test for

$$H_0 : \sigma^2 > 0 \text{ vs } H_1 : \sigma^2 = 0$$

from the rejection region $\mathcal{R} := \{\hat{\sigma}_n^2 = 0\}$.

We remark that our test deals with a rather simplistic model. Under H_1 our process is degenerate and is equal to 0 on the whole interval. At the same time, under H_0 our estimator is based on a sum of squared independent Gaussian random variables that for which the probability of them simultaneously being equal to 0 is null. Considering this, probabilities of Type 1 and Type 2 errors are equal to 0.

Finally, we mention a practical application of Remark (2). We can see in the plot below the error of convergence of the estimator (7) towards σ^2 if we take $\sigma^2 = 0.01$, observing simulated high-frequency data of 5000 observations in a time-frame from 0 to 1 second (i.e. an observation every 0.002 seconds if we consider seconds as our time scale, for instance).



Error of estimation of σ^2 using $\hat{\sigma}_n^2$ from Equation (7) considering $X_t = \sigma B_t$, $t \in [0,1]$, on 5000 simulated observations with $\sigma^2 = 0.01$.

We now return to the original assumption (**A3**) to see if a similar idea still works. We demonstrate in the next proposition that the estimator defined in (7) remains consistent in presence of a deterministic drift.

Proposition 7. *Let us assume without loss of generality that $T = 1$. Then, under (**A3**) we have that*

$$\hat{\sigma}_n^2 := \sum_{i=1}^{n} (X_{t_i} - X_{t_{i-1}})^2 \to \sigma^2 \tag{8}$$

in L^2 as $n \to \infty$. The intuition is that the drift part tends towards 0 at higher speed than the Brownian part .

Proof. We have that

$$\sum_{i=1}^{n} (X_{t_i} - X_{t_{i-1}})^2 = \sum_{i=1}^{n} \left[\left(\int_{t_{i-1}}^{t_i} b_s ds \right)^2 + 2\sigma (B_{t_i} - B_{t_{i-1}}) \int_{t_{i-1}}^{t_i} b_s ds + \sigma^2 (B_{t_i} - B_{t_{i-1}})^2 \right]$$

As b_s is a Lipschitz continuous function over $[0,1]$, it is also continuous in this set. Thus, as this set is compact, b_s is bounded. Let $M > 0$ such that $|b_s| < M$ $\forall s \in [0,1]$ we have

$$\sum_{i=1}^{n} \left(\int_{t_{i-1}}^{t_i} b_s ds \right)^2 \leq \sum_{i=1}^{n} \left(\frac{M}{n} \right)^2 = \frac{M^2}{n} \to 0$$

as $n \to \infty$. For the second part we have that:

$$
\begin{aligned}
E\left[\left|2\sigma \sum_{i=1}^{n}(B_{t_i}-B_{t_{i-1}})\int_{t_{i-1}}^{t_i} b_s ds\right|^2\right] &= E\left[4\sigma^2 \sum_{i=1}^{n}(B_{t_i}-B_{t_{i-1}})^2\left(\int_{t_{i-1}}^{t_i} b_s ds\right)^2\right] \\
&\leq 4\sigma^2 \frac{M}{n^2} E\left[\sum_{i=1}^{n}(B_{t_i}-B_{t_{i-1}})^2\right] \\
&= 4\sigma^2 \frac{M}{n^3} E\left[\sum_{i=1}^{n} Z_i^2\right] \\
&= 4\sigma^2 \frac{M}{n^3} n \\
&= 4\sigma^2 \frac{M}{n^2} \to 0
\end{aligned}
$$

as $n \to \infty$. The Z_i are iid standard normal random variables and we used the independence of the increments, the fact that $E[B_{t_i} - B_{t_{i-1}}] = 0$ and the basic properties of the chi-squared law. Finally, as we already know that in L^2

$$
\sum_{i=1}^{n} \sigma^2 (B_{t_i}-B_{t_{i-1}})^2 \to \sigma^2,
$$

we have

$$
\sum_{i=1}^{n} (X_{t_i}-X_{t_{i-1}})^2 \to \sigma^2 .
$$

$\square$

It is easy to check that in this case the previous test does not work as type 1 error is no longer 0, moreover, it depends on n. Thus, we are interested in constructing an asymptotic test. To do this we proceed to check whether the Central limit theorem holds in our case. The analysis of asymptotic behaviour provided in Proposition (7) features a term with convergence rate of $n^{-1/2}$, and thus does not allow us to immediately verify the CLT. Thus, we proceed by studying a more precise development of $\left(X_{t_i} - X_{t_{i-1}}\right),\ i \in \{1, ..., n\}$.

For X as defined above, we consider the following result.

Theorem 1. *Under (**A3**), the estimator $\hat{\sigma}_n^2$, defined in (8), converges in distribution to a normal random variable:*

$$
\sqrt{n}\left(\hat{\sigma}_n^2 - \sigma^2\right) \to \mathcal{N}\left(0, 2\sigma^4\right), \qquad n \to +\infty \tag{9}
$$

Proof. The following proof is an adaptation of the proof provided in [GC20, pp. 28–29]. We begin by using the definition of X:

$$
\begin{aligned}
\sqrt{n}\left(\hat{\sigma}_n^2 - \sigma^2\right) =& \sqrt{n}\left(\frac{1}{T}\sum_{i=1}^{n}\left(X_{t_i}-X_{t_{i-1}}\right)^2 - \sigma^2\right) \\
=& \sqrt{n}\left(\frac{1}{T}\sum_{i=1}^{n}\left(\int_{t_{i-1}}^{t_i} b_s\, ds + \sigma\left(B_{t_i}-B_{t_{i-1}}\right)\right)^2 - \sigma^2\right) \\
=& \sqrt{n}\frac{\sigma^2}{T}\left(\sum_{i=1}^{n}\left(B_{t_i}-B_{t_{i-1}}\right)^2 - 1\right) + \sqrt{n}\frac{2}{T}\sum_{i=1}^{n}\sigma\left(B_{t_i}-B_{t_{i-1}}\right)\int_{t_{i-1}}^{t_i} b_s\, ds \\
& + \sqrt{n}\frac{1}{T}\sum_{i=1}^{n}\left(\int_{t_{i-1}}^{t_i} b_s\, ds\right)^2 = I_1 + I_2 + I_3.
\end{aligned}
$$

We will study each term separately. Using independence and normality of increments of the Brownian motion we apply the central limit theorem to I_1 to conclude that

$$\sqrt{n}\frac{\sigma^2}{T}\left(\sum_{i=1}^{n}\left(B_{t_i}-B_{t_{i-1}}\right)^2-1\right)\sim\sqrt{n}\left(\frac{\sigma^2}{n}\sum_{i=1}^{n}Z_i^2-1\right)\to\mathcal{N}\left(0,2\sigma^4\right),$$

where $(Z_i)_i$ for $i\in\{1,\cdots,n\}$ are independent copies of standard Gaussian random variable and $\mathrm{Var}Z_i^4=2$, which can be deduced, for instance, from differentiating the characteristic function as we did in Proposition (2).

Using Lipschitz continuity of b we get the following asymptotic evaluation for I_3:

$$\sqrt{n}\frac{1}{T}\sum_{i=1}^{n}\left(\int_{t_{i-1}}^{t_i}b_s\,ds\right)^2=\mathcal{O}\left(\frac{1}{\sqrt{n}}\right),\qquad n\to+\infty,$$

denoting that the limit of the absolute value of ratio of left-hand side and right-hand side is bounded by a constant. Then, let us rewrite I_2 as follows:

$$\begin{aligned}I_2=&\sqrt{n}\frac{2}{T}\sum_{i=1}^{n}\sigma\left(B_{t_i}-B_{t_{i-1}}\right)\int_{t_{i-1}}^{t_i}\left(b_s-b_{t_{i-1}}+b_{t_{i-1}}\right)\,ds\\=&\sqrt{n}\frac{2}{T}\sum_{i=1}^{n}\sigma\left(B_{t_i}-B_{t_{i-1}}\right)\int_{t_{i-1}}^{t_i}\left(b_s-b_{t_{i-1}}\right)\,ds+\sqrt{n}\frac{2}{T}\frac{T}{n}\sum_{i=1}^{n}\sigma\left(B_{t_i}-B_{t_{i-1}}\right)b_{t_{i-1}}.\end{aligned}$$

Then

$$\frac{2}{\sqrt{n}}\sum_{i=1}^{n}\sigma\left(B_{t_i}-B_{t_{i-1}}\right)b_{t_{i-1}}=\frac{2\sigma}{\sqrt{n}}\left(\int_0^T b_s\,dB_s+o_{\mathbb{P}}(1)\right),$$

where $o_{\mathbb{P}}(1)$ vanishes in probability at $n\to+\infty$. Using Lipschitz continuity of b, we conclude that this term converges to 0 in probability.

Since Brownian motion is α-Holder continuous for some $0<\alpha<1/2$ and b is Lipschitz continuous, we can find $M\in\mathbb{R}_+$ such that

$$\left|\sqrt{n}\frac{2}{T}\sum_{i=1}^{n}\sigma\left(B_{t_i}-B_{t_{i-1}}\right)\int_{t_{i-1}}^{t_i}\left(b_s-b_{t_{i-1}}\right)\,ds\right|\le\sqrt{n}\frac{2\sigma}{T}\sum_{i=1}^{n}M\left|t_i-t_{i-1}\right|^{\alpha}\frac{\left(t_i-t_{i-1}\right)^2}{2}=\frac{MT^{\alpha+1}\sigma}{n^{\alpha+1/2}}\text{ a.s.}$$

Thus, I_1 converges in distribution to a Gaussian random variable and both I_2 and I_3 converge in probability to 0. We conclude by applying Slutsky's theorem that the following convergence in distribution holds

$$\sqrt{n}\left(\hat{\sigma}_n^2-\sigma^2\right)\to\mathcal{N}\left(0,2\sigma^4\right),\qquad n\to+\infty.$$

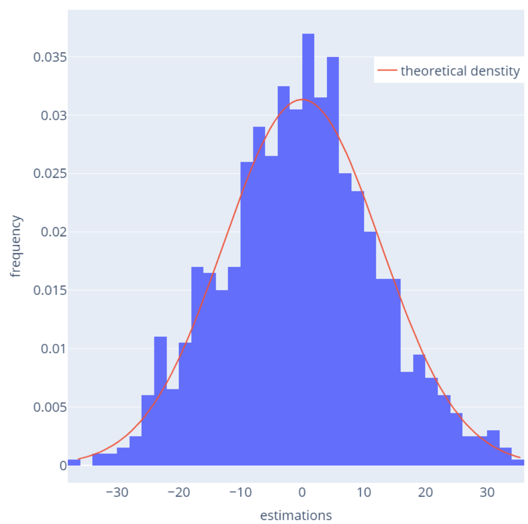
□

We will now illustrate our results with empirical tests.

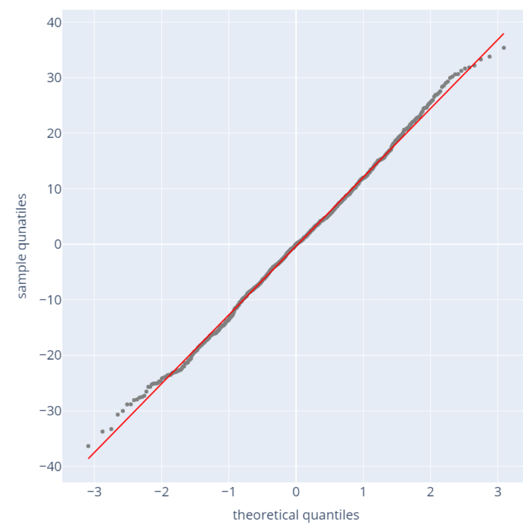
For this purpose, we took $b:\ s\mapsto s$ and $\sigma=3$. We fixed the time interval $[0,1]$ and constructed realisations of $\hat{\sigma}^2_{1000}$, meaning that each estimator was built using $n=1000$ observations of the process X. The histogram and the quantile-quantile plot are given for 1000 centered and normalised estimators.

Using the various results from above, we will build a statistical procedure to test if the volatility of the process X defined as $X_t=X_0+\int_0^t b_s ds+\sigma B_t$ is degenerate, with $t\in[0,T]$.
In practice, testing if the volatility is degenerate or not depends on how low we allow the volatility to be. As a result, we will use a threshold $\rho>0$ in the null hypothesis, which happens to be very convenient to build the test. This time, let us denote

$$H_0:\sigma>\rho\text{ vs }H_1:\sigma=0.$$



Histogram of 1000 estimations



Quantile-quantile plot of 1000 estimations

Proposition 8. *From the estimator defined in Proposition (7) and the CLT from Theorem (1), we have under (**A3**) the following convergence in distribution*

$$T_n \coloneqq \sqrt{\frac{n}{2}}\left(1-\frac{\sigma^2}{\hat{\sigma}_n^2}\right) \to \mathcal{N}(0,1), \qquad n \to +\infty. \tag{10}$$

We can build a consistent test which rejects the null hypothesis when $\hat{\sigma}_n^2$ is close to 0. We have the following test of size α,

$$\Phi(X_1,...,X_n) = \mathbb{1}_{\left\{\hat{\sigma}_n^2 < \frac{\rho}{1-q_\alpha\sqrt{\frac{n}{2}}}\right\}} \tag{11}$$

with q_α the quantile α of a $\mathcal{N}(0,1)$.

Proof. From the Proposition (7) we conclude that $\sigma_n^2 \to \sigma^2$ in probability. Whence, the result (10) follows immediately by applying the continuity theorem and Slutsky's theorem.
Given the hypothesis H_0: $\sigma > \rho$ vs H_1: $\sigma = 0$ for $\rho > 0$, we want to build a rejection of the form

$$\mathcal{R}_\alpha = \left\{\hat{\sigma}_n^2 < \kappa_\alpha\right\}$$

with $\kappa_\alpha > 0$ such that

$$\sup_{\sigma^2>\rho} \mathbb{P}_{\sigma^2}\left(\hat{\sigma}_n^2 < \kappa_\alpha\right) = \alpha$$

to have a test of size α. We have

$$\begin{aligned}
\mathbb{P}_{\sigma^2}\left(\hat{\sigma}_n^2 < \kappa_\alpha\right) =& \mathbb{P}_{\sigma^2}\left(-\frac{\sigma^2}{\hat{\sigma}_n^2} < -\frac{\sigma^2}{\kappa_\alpha}\right) \\
=& \mathbb{P}_{\sigma^2}\left(\sqrt{\frac{n}{2}}\left(1-\frac{\sigma^2}{\hat{\sigma}_n^2}\right) < \sqrt{\frac{n}{2}}\left(1-\frac{\sigma^2}{\kappa_\alpha}\right)\right) \\
=& F\left(\sqrt{\frac{n}{2}}\left(1-\frac{\sigma^2}{\kappa_\alpha}\right)\right)
\end{aligned}$$

with F the cdf (cumulative distribution function) of a $\mathcal{N}(0,1)$.
As $\sigma^2 \longmapsto F\left(\sqrt{\frac{n}{2}}\left(1-\frac{\sigma^2}{\kappa_\alpha}\right)\right)$ is decreasing on $(0,+\infty)$ we deduce that

$$\sup_{\sigma^2>\rho} \mathbb{P}_{\sigma^2}\left(\hat{\sigma}_n^2 < \kappa_\alpha\right) = F\left(\sqrt{\frac{n}{2}}\left(1-\frac{\rho}{\kappa_\alpha}\right)\right) = \alpha.$$

From this last Equation we obtain that

$$\begin{aligned}
\sqrt{\frac{n}{2}}\left(1-\frac{\rho}{\kappa_\alpha}\right) =& q_\alpha \\
\Longleftrightarrow \frac{\rho}{\kappa_\alpha} =& 1-q_\alpha\sqrt{\frac{2}{n}} \\
\Longleftrightarrow \kappa_\alpha =& \frac{\rho}{1-q_\alpha\sqrt{\frac{2}{n}}}
\end{aligned}$$

with q_α the quantile α of a $\mathcal{N}(0,1)$.
As a result, we obtain the following rejection region,

$$\mathcal{R}_\alpha = \left\{\hat{\sigma}_n^2 < \frac{\rho}{1-q_\alpha\sqrt{\frac{2}{n}}}\right\}$$

which gives us a test of size α. Moreover when $n \to \infty$,

$$\mathbb{P}_{\sigma^2=0}\left(\hat{\sigma}_n^2 < \frac{\rho}{1-q_\alpha\sqrt{\frac{2}{n}}}\right) \longrightarrow \mathbb{P}(0<\rho) = 1.$$

Thus, the test is consistent. $\square$

4 Model with Time-varying Volatility

In this section we increase the complexity of our model by assuming that the volatility changes either in a deterministic or in a stochastic way. Thus, X is assumed to satisfy Equation (6) with assumptions on b and σ specified below. To study this model we focus on two possible approaches at estimating the volatility. Firstly, we will build a consistent estimator for integrated volatility giving a complete statistical procedure to test if it is degenerate when σ it is deterministic. Then we will build and study the convergence of a consistent estimator of the spot-volatility, meaning an estimator of σ taken at a given time $t \in [0, T]$.

4.1 Estimation of stochastic integrated volatility

To set the general framework for the whole subsection we remind the Equation (6)

$$X_t = X_0 + \int_0^t b\left(s, X_s\right) ds + \int_0^t \sigma_s dB_s, \qquad t \in [0, T],\ T > 0.$$

In this subsection, unless otherwise is stated, B is an $\mathbb{F}$-Brownian motion, $X_0 = 0$ almost surely and we work with observations from the uniform partition of $[0, T]$ given by $t_i = Ti/n$ for $i \in \{1, \cdots, n\}$, $n \in \mathbb{N}$.

Let us specify the following assumptions about b and σ.

Assumption (A4.1). *$b : [0, T] \times \mathbb{R} \mapsto \mathbb{R}$ is a Riemann integrable in time and measurable function satisfying*

$$|b(t, x) - b(t, y)| \leq C\,|x - y|\,, \qquad x, y \in \mathbb{R},\ t \in [0, T],$$

where $C > 0$ is some real constant.

We remark that we don not put any additional assumptions on continuity of b in time.

Assumption (A4.2). *$\sigma = (\sigma_t)_{t\in[0,T]}$ is a non-negative càdlàg $\mathbb{F}$-adapted process, independent from $\mathbb{F}$-Brownian motion B, such that*

$$|\sigma_t(\omega)| \leq D, \qquad t \in [0, T],\ \omega \in \Omega,$$

where $D > 0$ is some real constant.

As a direct consequence of σ being càdlàg, and thus having countable discontinuities, and uniformly bounded we have that under (**A4.2**)

$$\mathbb{E}\left[\int_0^T \sigma_s^2\, ds\right] < +\infty \qquad \text{and} \qquad \mathbb{E}\left[\int_0^T \sigma_s^4\, ds\right] < +\infty.$$

The assumptions while needed for the specific results below are natural in the sense that they are required to ensure existence and uniqueness of X as a solution of the stochastic differential equation (SDE). The conditions below are required when working with SDEs and can be found in [Cha20, p.108]

(Lipschitz regularity). There exists a real constant $C > 0$ such that for all $t \geq 0$, $\omega \in \Omega$ and $x, y \in \mathbb{R}$,

$$|\sigma(t, \omega, x) - \sigma(t, \omega, y)| \leq C\,|x - y| \quad \text{and} \quad |b(t, \omega, x) - b(t, \omega, y)| \leq C\,|x - y|\,;$$

(Progressive measurability). For all $t \geq 0$ and $x \in \mathbb{R}$ the following maps are measurable with respect to $\mathcal{B}([0, t]) \otimes \mathcal{F}_t$

$$(u, \omega) \in [0, t] \times \Omega \mapsto \sigma(u, \omega, x) \quad \text{and} \quad (u, \omega) \in [0, t] \times \Omega \mapsto b(u, \omega, x),$$

where $\mathcal{B}([0, T])$ is a Borel sigma algebra, built on $[0, t]$;

(Square integrability). For all $t \geq 0$ and $x \in \mathbb{R}$,

$$\mathbb{E}\left[\int_0^t |\sigma(u,\cdot,x)|^2 \, du\right] < +\infty \quad \text{and} \quad \mathbb{E}\left[\int_0^t |b(u,\cdot,x)|^2 \, du\right] < +\infty.$$

The questions of existence and unicity of solutions of SDEs are explored in [Cha20, pp. 107–133]. We will cite the result that establishes existence, unicity and continuity of X in time.

Proposition 9. *Let $X_0 = 0$ almost surely. Then there exists an $\mathbb{F}$-adapted and continuous process X that solves the following SDE*

$$X_t = \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dB_s, \qquad t \geq 0.$$

Furthermore, X is unique up to undistiguishability and

$$\mathbb{E}\left[\int_0^t |X_s|^2 \, ds\right] < +\infty \qquad t \geq 0.$$

Proof. This is a classical result. This version with the conditions above is detailed in [Cha20, pp. 109–111]. □

We remark that the assumptions of our model are less general. Namely, we always restrict t to $[0,T]$ and choose a deterministic b. Then, our σ being adapted and right continuous satisfies the condition of progressive measurability on $[0,T]$. The integrability of b^2 and σ^2 in t and ω follows from the assumptions of this section. Finally, we mention that our decision to take σ càdlàg is inspired by the models described in [ASJ14], especially in Chapters 3 and 6.

Thus, we place ourselves under the conditions of the Proposition above. To proceed towards our goals for this subsection we will need a series of preliminary results.

First, we will focus on convergence of the quadratic variation of Itô stochastic integrals. More precisely, let $\mathcal{I}[H]$ be an Itô integral as defined above. Then, for all $t \in [0,T]$ we have the following limit in probability:

$$V_t^{(2)}(I[H], \Delta) \longrightarrow \int_0^t |H_s|^2 \, ds, \qquad |\Delta| \to 0 \tag{12}$$

Our approach is specifically based on the martingale property of the Itô integral.

Definition 7. *We define a space of continuous square-integrable martingales started at* 0*:*

$$c\mathcal{M}_0^2([0,T]) = \left\{ M \in \mathcal{M}^2([0,T]) \ : \ M_0 = 0 \ \textit{a.s and } M \textit{ has continuous trajectories a.s.} \right\}$$

As before if $[0,T]$ is not specified we assume that $c\mathcal{M}_0^2$ is the space of square integrable continuous martingales issued from 0 that evolve with $t \geq 0$.

Proposition 10. *Ito integral as defined above is a $c\mathcal{M}_0^2([0,T])$ process.*

Proof. This is a classical result, based on approximation with elementary processes and on application of Doob martingale inequality. Detailed proof can be found in [Øks03]. □

Now, let us remind a known theorem for square integrable continuous martingales.

Proposition 11 (Doob-Meyer decomposition)**.** *Let $M = (M_t)_{t\geq 0} \in c\mathcal{M}_0^2$. Then*

- *There exists a unique a.s. non-decreasing continuous process $\langle M \rangle = (\langle M_t \rangle)_{t \geq 0}$ with $\langle M \rangle_0 = 0$ a.s, such that: $\left(M_t^2 - \langle M \rangle_t\right)_{t\geq 0}$ is a martingale.*
- *$V_t^{(2)}(M)$ exists for all $t \geq 0$ and $V_t^{(2)}(M) = \langle M \rangle_t$.*

Proof. This is a classical theorem. The detailed proof this particular version of the Proposition can be found for example in [Lal14]. □

We want to apply this result to our stochastic integral. The following Lemma establishes it for a simple class of processes used to construct the Itô integral.

Lemma 1. *Let H be an elementary process. Then, for all $0 \leq s \leq t \leq T$*

$$\mathbb{E}\left[\left(\int_s^t H_u \, dBu\right)^2 \middle| \mathcal{F}_s\right] = \mathbb{E}\left[\int_s^t |H_u|^2 \, dBu \middle| \mathcal{F}_s\right].$$

Proof. The property can be directly checked using the definition of elementary process 3. Detailed version is available in [KS98]. □

Proposition 12. *Let $\mathcal{I}[H]_t = \int_0^t H_s \, dB_s$ for $t \in [0, T]$ be an Itô integral as defined above. Then $D = (D_t)_{t \geq 0}$, such that:*

$$D_t := \left(\left(\int_0^t H_s \, dB_s\right)^2 - \int_0^t |H_s|^2 \, ds\right)_{t \geq 0}$$

is an $\mathbb{F}$-martingale.

Proof. Integrability follows from the Itô isometry. Then, since H is $\mathbb{F}$-adapted and the Itô integral is an $\mathbb{F}$-martingale, and thus it is $\mathbb{F}$-adapted, D is $\mathbb{F}$ adapted.

Finally, for $0 \leq s \leq t \leq T$

$$\begin{aligned}
\mathbb{E}\left[D_t \mid \mathcal{F}_s\right] &= \mathbb{E}\left[\left(\int_0^t H_u \, dB_u\right)^2 - \int_0^t |H_u|^2 \, du \middle| \mathcal{F}_s\right] \\
&= \mathbb{E}\left[\left(\int_s^t H_u \, dB_u\right)^2 + \left(\int_0^s H_u \, dB_u\right)^2\right. \\
&\qquad \left. + 2\left(\int_s^t H_u \, dB_u\right)\left(\int_0^s H_u \, dB_u\right) - \int_s^t |H_u|^2 \, du - \int_0^s |H_u|^2 \, du \middle| \mathcal{F}_s\right] \\
&= \left(\int_0^s H_u \, dB_u\right)^2 - \int_0^s |H_u|^2 \, du \\
&\qquad + \mathbb{E}\left[\left(\int_s^t H_u \, dB_u\right)^2 + 2\left(\int_s^t H_u \, dB_u\right)\left(\int_0^s H_u \, dB_u\right) - \int_s^t |H_u|^2 \, du \middle| \mathcal{F}_s\right] \\
&= D_s + \mathbb{E}\left[\left(\int_s^t H_u \, dB_u\right)^2 - \int_s^t |H_u|^2 \, du \middle| \mathcal{F}_s\right] + 2\left(\int_0^s H_u \, dB_u\right)\mathbb{E}\left[\int_s^t H_u \, dB_u \middle| \mathcal{F}_s\right],
\end{aligned}$$

where we used the fact that Itô integral is $\mathbb{F}$-adapted as a martingale.

Applying the martingale property to the last term we write

$$2\left(\int_0^s H_u \, dB_u\right)\mathbb{E}\left[\int_s^t H_u \, dB_u \middle| \mathcal{F}_s\right] = 2\left(\int_0^s H_u \, dB_u\right)\mathbb{E}\left[\int_0^t H_u \, dB_u - \int_0^s H_u \, dB_u \middle| \mathcal{F}_s\right] = 0.$$

We want to show that the middle term is 0. We have already established this result in Lemma (1). Now we will generalise it.

Let $\left(\phi^{(n)}\right)_{n \in \mathbb{N}}$ be a sequence of elementary processes, such that

$$\lim_{n \to +\infty} \mathbb{E}\left[\int_s^t \left|\phi_u^{(n)} - H_u\right|^2 du\right] = 0.$$

Then $L^2 - \lim_{n\to+\infty} \left(\mathcal{I}\left[\phi^{(n)}\right]_t - \mathcal{I}\left[\phi^{(n)}\right]_s\right) = \mathcal{I}\left[H\right]_t - \mathcal{I}\left[H\right]_s$. This result follows from construction of the Itô integral (detailed construction can be found, for example, in [Øks03] or [KS98]).

Thus, for $A \in \mathcal{F}_s$ by a consequence of L^2-convergence

$$\mathbb{E}\left[\mathbb{1}_A \left(\int_s^t H_u\, dB_u\right)^2\right] = \lim_{n\to+\infty} \mathbb{E}\left[\mathbb{1}_A \left(\int_s^t \phi_u^{(n)}\, dB_u\right)^2\right].$$

Applying the results of Lemma (1) we write

$$\lim_{n\to+\infty} \mathbb{E}\left[\mathbb{1}_A \left(\int_s^t \phi_u^{(n)}\, dB_u\right)^2\right] = \lim_{n\to+\infty} \mathbb{E}\left[\mathbb{1}_A \int_s^t \left|\phi_u^{(n)}\right|^2\, du\right].$$

Using the definition of the sequence $\left(\phi^{(n)}\right)_{n\in\mathbb{N}}$ we get

$$\lim_{n\to+\infty} \mathbb{E}\left[\mathbb{1}_A \int_s^t \left|\phi_u^{(n)}\right|^2\, du\right] = \mathbb{E}\left[\mathbb{1}_A \int_s^t \left|H_u\right|^2\, du\right].$$

Hence,

$$\mathbb{E}\left[\left(\int_s^t H_u\, dB_u\right)^2 \,\middle|\, \mathcal{F}_s\right] = \mathbb{E}\left[\int_s^t \left|H_u\right|^2\, du \,\middle|\, \mathcal{F}_s\right],$$

which allows us to conclude that

$$\mathbb{E}\left[\, D_t \mid \mathcal{F}_s\right] = D_s.$$

We, therefore, conclude that D is indeed an $\mathbb{F}$-martingale. □

Proposition 13. *Let $\mathcal{I}$ be an Ito integral as defined above. Then,*

$$\langle \mathcal{I}\left[H\right]_\cdot \rangle_t = \int_0^t \left|H_s\right|^2\, ds,$$

and

$$V_T^{(2)}\left(\mathcal{I}\left[H\right], \Delta\right) \longrightarrow \int_0^T \left|H_s\right|^2\, ds, \qquad |\Delta| \to 0,$$

where the convergence holds in probability.

Proof. Both equalities follow immediately follow after applying Proposition (11) to the result of Proposition (12). □

The following lemmas will help us deal with the influence of the stochastic drift in our model.

Lemma 2. *Under **(A4.1)** and **(A4.2)** we have*

$$\mathbb{E}\left[\left|X_t\right|^2\right] < \infty, \qquad t \in [0,T] \tag{13}$$

and

$$\mathbb{E}\left[\left|X_t - X_s\right|^2\right] \longrightarrow 0 \qquad \text{for } t,s \in [0,T] \text{ as } |t-s| \to 0. \tag{14}$$

Proof. To demonstrate Equation (13)we notice that

$$\begin{aligned}\mathbb{E}\left[\left|X_t\right|^2\right] &= \mathbb{E}\left[\left|\int_0^t b\left(s, X_s\right)\, ds + \int_0^t \sigma_s\, dB_s\right|^2\right]\\ &\leq 2\mathbb{E}\left[\left(\int_0^t b\left(s, X_s\right)\, ds\right)^2 + \left(\int_0^t \sigma_s\, dB_s\right)^2\right].\end{aligned}$$

Using a consequence of Lipschitz continuity of b and the fact that $X_0 = 0$ almost surely, we can find $L > 0$ such that

$$2\mathbb{E}\left[\left(\int_0^t b\left(s, X_s\right)\, ds\right)^2 + \left(\int_0^t \sigma_s\, dB_s\right)^2\right] \leq 2\mathbb{E}\left[\left(\int_0^t L\left(1 + |X_s|\right)\, ds\right)^2 + \left(\int_0^t \sigma_s\, dB_s\right)^2\right].$$

Then, by Cauchy-Schwartz inequality and Itô isometry and Assumption (**A4.2**)

$$\begin{aligned}2\mathbb{E}\left[\left(\int_0^t L\left(1 + |X_s|\right)\, ds\right)^2 + \left(\int_0^t \sigma_s\, dB_s\right)^2\right] &\leq 2\left(T\mathbb{E}\left[\int_0^t L^2\left(1 + |X_s|\right)^2\, ds\right] + \mathbb{E}\left[\int_0^T \sigma_s^2\, ds\right]\right) \\ &\leq 2\left(2T\left(TL^2 + L^2\mathbb{E}\left[\int_0^t |X_s|^2\, ds\right]\right) + C\right),\end{aligned}$$

where $C > 0$ is some constant.

Then we apply Fubini-Tonelli theorem to get

$$\begin{aligned}2\left(2T\left(TL^2 + L^2\mathbb{E}\left[\int_0^t |X_s|^2\, ds\right]\right) + C\right) &= 4T^2L^2 + 4TL^2\int_0^t \mathbb{E}\left[|X_s|^2\right]\, ds + 2C \\ &= A + B\int_0^t \mathbb{E}\left[|X_s|^2\right]\, ds,\end{aligned}$$

where $A, B > 0$ are some constants. Thus, we have

$$\mathbb{E}\left[|X_t|^2\right] \leq A + B\int_0^t \mathbb{E}\left[|X_s|^2\right]\, ds, \qquad t \in [0, T].$$

We apply Gronwall Lemma to obtain

$$\mathbb{E}\left[|X_t|^2\right] \leq Ae^{Bt}, \qquad t \in [0, T],$$

which allows us to conclude the first part of the proof.

Now let us assume without loss of generality that $s \leq t$. To demonstrate Equation (14) we follow the same steps as before to obtain the following inequality

$$\mathbb{E}\left[|X_t - X_s|^2\right] \leq 4L^2\left(t - s\right)\left(\left(t - s\right) + \int_s^t \mathbb{E}\left[|X_u|^2\right]\, du\right) + 2\mathbb{E}\left[\int_s^t \sigma_u^2\, du\right].$$

Then,

$$\mathbb{E}\left[|X_t - X_s|^2\right] \leq 4L^2\left(t - s\right)^2\left(1 + \sup_{u\in[0,T]}\mathbb{E}\left[|X_u|^2\right]\right) + 2\mathbb{E}\left[\int_s^t \sigma_u^2\, du\right].$$

We apply the first part of the Lemma to the first term and the monotone convergence theorem to the second one. Thus, the right-hand side vanishes as $|t - s| \to 0$. □

The next Lemma is inspired by the decomposition from [GC20, pp. 25–29] that we used to demonstrate Theorem (1). Here we expand on this idea to make it work in a more complicated scenario.

Lemma 3. *Under (**A4.1**) and (**A4.2**) we have the following convergence in probability*

$$\frac{\sqrt{n}}{T}\sum_{i=1}^n\left(\left(\int_{t_{i-1}}^{t_i} b(s, X_s)\, ds\right)^2 + 2\int_{t_{i-1}}^{t_i} b(s, X_s)\, ds\int_{t_{i-1}}^{t_i} \sigma_s\, dB_s\right) \longrightarrow 0, \qquad n \to +\infty.$$

Proof. We write the following decomposition

$$\begin{aligned}
\frac{\sqrt{n}}{T}\sum_{i=1}^{n}\left(\left(\int_{t_{i-1}}^{t_i} b(s,X_s)\,ds\right)^2 + 2\int_{t_{i-1}}^{t_i} b(s,X_s)\,ds\int_{t_{i-1}}^{t_i}\sigma_s\,dB_s\right)& \\
= \frac{\sqrt{n}}{T}\sum_{i=1}^{n}\left(\int_{t_{i-1}}^{t_i} b(s,X_s)\,ds\right)^2 + \frac{2\sqrt{n}}{T}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\big(b(s,X_s)-b(s,X_{t_{i-1}})\big)\,ds\int_{t_{i-1}}^{t_i}\sigma_s\,dB_s& \\
+\frac{2\sqrt{n}}{T}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} b(t_{i-1},X_{t_{i-1}})\,ds\int_{t_{i-1}}^{t_i}\sigma_s\,dB_s = I_1+I_2+I_3.&
\end{aligned}$$

For I_1 we have by Cauchy-Schwartz

$$\mathbb{E}\left[|I_1|\right] \le \frac{\sqrt{n}}{T}\sum_{i=1}^{n}\frac{T}{n}\mathbb{E}\left[\int_{t_{i-1}}^{t_i} b^2(s,X_s)\,ds\right].$$

Then, by Assumption (**A4.1**) there exists a constant $L>0$ such that

$$\begin{aligned}
\le \frac{\sqrt{n}}{T}\sum_{i=1}^{n}\frac{T}{n}\mathbb{E}\left[\int_{t_{i-1}}^{t_i} |b(s,X_s)|^2\,ds\right] &\le \frac{\sqrt{n}}{T}\sum_{i=1}^{n}\frac{T}{n}\mathbb{E}\left[\int_{t_{i-1}}^{t_i} |L(1+|X_s|)|^2\,ds\right]\\
&\le \frac{2\sqrt{n}}{T}\sum_{i=1}^{n}\frac{T}{n}\left(L^2\frac{T}{n}+L^2\mathbb{E}\left[\int_{t_{i-1}}^{t_i}|X_s|^2\,ds\right]\right).
\end{aligned}$$

Then, by Fubini-Tonelli

$$\begin{aligned}
\frac{2\sqrt{n}}{T}\sum_{i=1}^{n}\frac{T}{n}\left(L^2\frac{T}{n}+L^2\mathbb{E}\left[\int_{t_{i-1}}^{t_i}|X_s|^2\,ds\right]\right) &\le \frac{2\sqrt{n}}{T}\sum_{i=1}^{n}\left(\frac{T}{n}\right)^2\left(L^2+L^2\sup_{s\in[0,T]}\mathbb{E}\left[|X_s|^2\right]\right)\\
&= \frac{2TL^2}{\sqrt{n}}\left(1+\sup_{s\in[0,T]}\mathbb{E}\left[|X_s|^2\right]\right).
\end{aligned}$$

Thus, by Lemma (2) and by non-negativity of I_1, we have the convergence in probability:

$$I_1 \longrightarrow 0, \qquad n\to+\infty.$$

To control the second term we use the Cauchy-Schwartz inequality

$$\begin{aligned}
\mathbb{E}\left[|I_2|\right] &= \mathbb{E}\left[\left|\frac{2\sqrt{n}}{T}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\big(b(s,X_s)-b(t_{i-1},X_{t_{i-1}})\big)\,ds\int_{t_{i-1}}^{t_i}\sigma_s\,dB_s\right|\right]\\
&\le \frac{2\sqrt{n}}{T}\sum_{i=1}^{n}\mathbb{E}\left[\left|\int_{t_{i-1}}^{t_i}\big(b(s,X_s)-b(t_{i-1},X_{t_{i-1}})\big)\,ds\right|\left|\int_{t_{i-1}}^{t_i}\sigma_s\,dB_s\right|\right]\\
&\le \frac{2\sqrt{n}}{T}\sum_{i=1}^{n}\sqrt{\mathbb{E}\left[\left|\int_{t_{i-1}}^{t_i}\big(b(s,X_s)-b(t_{i-1},X_{t_{i-1}})\big)\,ds\right|^2\right]}\sqrt{\mathbb{E}\left[\left|\int_{t_{i-1}}^{t_i}\sigma_s\,dB_s\right|^2\right]}\\
&= \frac{2\sqrt{n}}{T}\sum_{i=1}^{n}\sqrt{\mathbb{E}\left[\left|\int_{t_{i-1}}^{t_i}\big(b(s,X_s)-b(t_{i-1},X_{t_{i-1}})\big)\,ds\right|^2\right]}\sqrt{\mathbb{E}\left[\int_{t_{i-1}}^{t_i}\sigma_s^2\,ds\right]},
\end{aligned}$$

where the last line is due to the Itô isometry.

Since the final sum has non-negative terms we can directly apply the Cauchy-Schwartz inequality again

$$\mathbb{E}\left[|I_2|\right] \le \frac{2\sqrt{n}}{T}\sqrt{\sum_{i=1}^{n}\mathbb{E}\left[\left|\int_{t_{i-1}}^{t_i}\left(b(s,X_s)-b(t_{i-1},X_{t_{i-1}})\right)\,ds\right|^2\right]}\sqrt{\sum_{i=1}^{n}\mathbb{E}\left[\int_{t_{i-1}}^{t_i}\sigma_s^2\,ds\right]}.$$

Then, we apply the Cauchy-Schwartz inequality Lipschitz continuity of b and the Fubini-Tonelli theorem

$$\begin{aligned}\mathbb{E}\left[|I_2|\right] &\le \frac{2\sqrt{n}}{T}\sqrt{\sum_{i=1}^{n}\left(\frac{T}{n}\right)^2\sup_{\substack{0\le t\le s\le T,\\|t-s|\le T/n}}\mathbb{E}\left[L^2\left(X_s-X_t\right)^2\right]}\sqrt{\mathbb{E}\left[\int_0^T\sigma_s^2\,ds\right]}\\ &= 2T\sqrt{L^2\sup_{\substack{0\le t\le s\le T,\\|t-s|\le T/n}}\mathbb{E}\left[\left(X_s-X_t\right)^2\right]}\sqrt{\mathbb{E}\left[\int_0^T\sigma_s^2\,ds\right]},\end{aligned}$$

where $L>0$ is some constant. Thus, applying Lemma (2) and Assumption (**A4.2**) we have the convergence in probability:

$$I_2 \longrightarrow 0, \qquad n\to+\infty.$$

Finally, let us study the last remaining term.

$$I_3 = \frac{2\sqrt{n}}{T}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}b(t_{i-1},X_{t_{i-1}})\,ds\int_{t_{i-1}}^{t_i}\sigma_s\,dB_s = \frac{2\sqrt{n}}{T}\sum_{i=1}^{n}\frac{T}{n}b(t_{i-1},X_{t_{i-1}})\left(\mathcal{I}[\sigma]_{t_i}-\mathcal{I}[\sigma]_{t_{i-1}}\right).$$

The sum on the right-hand side of the last equation reminds us of Itô-Riemann sums. Naturally, the theory of stochastic integrals extends way beyond integration with respect to the Brownian motion. We will use a version of a stochastic integrals for continuous semi martingales issued from the origin. One can refer to the Theorem 6.3.5 from [Cha20, pp. 85–86] for details. By Proposition (10) $\mathcal{I}[\sigma]\in c\mathcal{M}_0^2([0,T])$, and thus it is a continuous semimartingale. Furthermore, the process $b=(b(t,X_t),\ t\in[0,T])$ has continuous paths by Proposition (9) and is $\mathbb{F}$-adapted. Thus, the conditions of the mentioned theorem are satisfied and we have

$$\frac{2\sqrt{n}}{T}\sum_{i=1}^{n}\frac{T}{n}b(t_{i-1},X_{t_{i-1}})\left(\mathcal{I}[\sigma]_{t_i}-\mathcal{I}[\sigma]_{t_{i-1}}\right) = \frac{2}{\sqrt{n}}\left(\int_0^T b(s,X_s)\,d\mathcal{I}[\sigma]_s + o_{\mathbb{P}}(1)\right), \tag{15}$$

where $o_{\mathbb{P}}(1)$ vanishes in probability as $n\to+\infty$. To demonstrate that the integral on the right hand side is almost surely finite we can check the following condition

$$\mathbb{E}\left[\int_0^T b^2\left(s,X_s\right)\,d\langle\mathcal{I}[\sigma].\rangle_s\right] < +\infty. \tag{16}$$

By Propositon (13) we have

$$\langle\mathcal{I}[\sigma].\rangle_t = \int_0^t\sigma_s^2\,ds, \qquad t\in[0,T].$$

Thus, uniform boundedness of σ in $\omega\in\Omega$ and $t\in[0,T]$ together with Lemma (2) and Fubini-Tonelli yields Equation (16). Whence, using Theorem 6.2.1 from [Cha20, p. 82] we conclude that the integral is finite almost surely and the right hand side of Equation (15) tends to 0 in probability as $n\to+\infty$. This allows us to conclude. $\square$

Now we are ready to focus on the main results of this subsection. To build a consistent estimator of integrated volatility we want to use the same idea as before. Using the results above we expect the drift term to be asymptotically negligible. The following proposition formalises this intuition.

Theorem 2. *Under (**A4.1**) and (**A4.2**) we have the following convergence in probability:*

$$\hat{\sigma}_n^2 = \frac{1}{T}\sum_{i=1}^{n}\left(X_{t_i} - X_{t_{i-1}}\right)^2 \longrightarrow \frac{1}{T}\int_0^T \sigma_s^2\, ds, \qquad n \to +\infty.$$

Proof. We consider the following decomposition:

$$\begin{aligned}\frac{1}{T}\sum_{i=1}^{n}\left(X_{t_i} - X_{t_{i-1}}\right)^2 =& \frac{1}{T}\sum_{i=1}^{n}\left(\int_{t_{i-1}}^{t_i} b\left(s, X_s\right)\, ds + \int_{t_{i-1}}^{t_i}\sigma_s\, dB_s\right)^2 \\ =& \frac{1}{T}\sum_{i=1}^{n}\left(\left(\int_{t_{i-1}}^{t_i} b\left(s, X_s\right)\, ds\right)^2 + 2\int_{t_{i-1}}^{t_i} b\left(s, X_s\right)\, ds\int_{t_{i-1}}^{t_i}\sigma_s\, dB_s\right) \\ &+ \frac{1}{T}\sum_{i=1}^{n}\left(\int_{t_{i-1}}^{t_i}\sigma_s\, dBs\right)^2 = I_1 + I_2.\end{aligned}$$

We notice that by Lemma (3) I_1 converges in probability to 0. Now, using the Proposition (13), I_2 converges to the quadratic variation of an Itô integral in probability

$$\frac{1}{T}\sum_{i=1}^{n}\left(\int_{t_{i-1}}^{t_i}\sigma_s\, dB_s\right)^2 \longrightarrow \frac{1}{T}\int_0^T \sigma_s^2\, ds, \qquad n \to +\infty,$$

which allows us to conclude. □

To build a testing procedure we want to demonstrate a counterpart of the Central limit theorem in this setting. In presence of a stochastic drift the result is less straightforward. We will discuss several possible cases, starting from the simplest one where σ is deterministic.

Theorem 3. *We assume (**A4.1**), (**A4.2**) and that σ is deterministic, thus, a trivial case of (**A4.2**). Similarly to the consequence of the assumptions above, we have*

$$\int_0^T \sigma_s^2\, ds < +\infty, \qquad \int_0^T \sigma_s^4\, ds < +\infty.$$

Then we have the convergence in distribution:

$$\sqrt{n}\left(\sum_{i=1}^{n}\left(X_{t_i} - X_{t_{i-1}}\right)^2 - \int_0^T \sigma_s^2\, ds\right) \longrightarrow \mathcal{N}\left(0,\ 2\int_0^T \sigma_s^4\, ds\right), \qquad n \to +\infty.$$

Proof. We start with a decomposition similar to the one in Proposition (2).

$$\begin{aligned}\sqrt{n}\left(\sum_{i=1}^{n}\left(X_{t_i} - X_{t_{i-1}}\right)^2 - \int_0^T \sigma_s^2\, ds\right) =& \sqrt{n}\sum_{i=1}^{n}\left(\left(\int_{t_{i-1}}^{t_i} b\left(s, X_s\right)\, ds\right)^2 \right.\\ &\left. +2\int_{t_{i-1}}^{t_i} b\left(s, X_s\right)\, ds\int_{t_{i-1}}^{t_i}\sigma_s\, dB_s\right) \\ &+ \sqrt{n}\sum_{i=1}^{n}\left(\left(\int_{t_{i-1}}^{t_i}\sigma_s\, dBs\right)^2 - \int_{t_{i-1}}^{t_i}\sigma_s^2\, ds\right) = I_1 + I_2.\end{aligned}$$

We have already established that I_1 converges in probability to 0 as $n \to +\infty$ in Lemma (3). Now, let us focus on I_2. The following section of the proof is described in [ASJ14]. We define random variables $(\xi_i^n),\ i \in \{1, \cdots, n\}$ with

$$\xi_i^n = \sqrt{n} \left(\int_{t_{i-1}}^{t_i} \sigma_s \, dBs \right)^2 - \sqrt{n} \int_{t_{i-1}}^{t_i} \sigma_s^2 \, ds.$$

We remark that since σ is deterministic and the increments of B are independent, the variables $(\xi_i^n),\ i \in \{1, \cdots, n\}$ are row-wise independent within each row n. However, they are clearly not identically distributed.

Based on the properties of the Wiener integral as described in Definition (3), we have

$$\mathbb{E}\left[\xi_i^n\right] = 0, \qquad i \in \{1, \cdots, n\}.$$

We, thus, have a triangular array of row-wise independent centered random variables. To establish convergence distribution to the Gaussian law $\mathcal{N}(0, V)$ we use the Lindeberg condition for the Central limit theorem. Namely, we verify if the following limit relationships hold

$$\sum_{i=1}^{n} \mathbb{E}\left[(\xi_i^n)^2\right] \longrightarrow V, \qquad\qquad \sum_{i=1}^{n} \mathbb{E}\left[(\xi_i^n)^4\right] \longrightarrow 0, \tag{17}$$

as $n \to +\infty$. This implies the Lindeberg condition for triangular arrays of row-wise independent random variables.

Let us verify these conditions directly, using the properties of Wiener integrals. First, we have

$$\begin{aligned}
\mathbb{E}\left[(\xi_i^n)^2\right] &= n\mathbb{E}\left[\left(\int_{t_{i-1}}^{t_i} \sigma_s \, dBs \right)^4 - 2 \left(\int_{t_{i-1}}^{t_i} \sigma_s \, dBs \right)^2 \int_{t_{i-1}}^{t_i} \sigma_s^2 \, ds + \left(\int_{t_{i-1}}^{t_i} \sigma_s^2 \, ds \right)^2 \right] \\
&= n \left(3 \left(\int_{t_{i-1}}^{t_i} \sigma_s^2 \, dBs \right)^2 - 2 \left(\int_{t_{i-1}}^{t_i} \sigma_s^2 \, ds \right)^2 + \left(\int_{t_{i-1}}^{t_i} \sigma_s^2 \, ds \right)^2 \right) \\
&= 2n \left(\int_{t_{i-1}}^{t_i} \sigma_s^2 \, ds \right)^2.
\end{aligned}$$

In the first equality we had to calculate the fourth moment of a Gaussian random variable. To do this we can, for example, differentiate the characteristic function as we did in Proposition (2).

Then, by càdlàg property of σ and by properties of the Riemann integral we have

$$\sum_{i=1}^{n} \mathbb{E}\left[(\xi_i^n)^2\right] \longrightarrow 2 \int_0^T \sigma_s^4 \, ds.$$

Using the same approach we get

$$\mathbb{E}\left[(\xi_i^n)^4\right] = 60n^2 \left(\int_{t_{i-1}}^{t_i} \sigma_s^2 \, ds \right)^4.$$

Then,

$$0 \leq \sum_{i=1}^{n} \mathbb{E}\left[(\xi_i^n)^4\right] \leq 60 \sum_{i=1}^{n} \frac{n^2 T^4}{n^4} \sup_{s \in [0,T]} |\sigma_s|^2 = \frac{60 T^4}{n} \sup_{s \in [0,T]} |\sigma_s|^2.$$

Thus,

$$\sum_{i=1}^{n} \mathbb{E}\left[(\xi_i^n)^4\right] \longrightarrow 0, \qquad n \to +\infty.$$

Hence the conditions are satisfied and this concludes the proof. □

When σ is stochastic and independent of B a similar result holds.

Theorem 4. *Under (**A4.1**) and (**A4.2**) we have the following convergence in distribution*

$$\sqrt{n}\left(\sum_{i=1}^{n}\left(X_{t_i}-X_{t_{i-1}}\right)^2-\int_0^T \sigma_s^2\, ds\right)\longrightarrow Y, \qquad n\to+\infty,$$

where Y follows the mixed Gaussian law $\int \mu(dx)\mathcal{N}(0,x)$, where μ is the law of $2\int_0^T \sigma_s^4\, ds$.

Proof. We follow the same steps as in the previous proof to establish that the drift term is asymptotically negligible. For the term that converges in distribution we admit the result. Detailed demonstration is available in [ASJ14]. $\square$

To successfully use this result we need one additional step. Namely, we want to obtain convergence in distribution to a Gaussian law that does not depend on σ. Unlike the previous model with constant volatility the estimation of variance of the limiting Gaussian under the current assumptions is not immediate. We have to build a consistent estimator for the so-called quarticity

$$2\int_0^T \sigma_s^4\, ds.$$

This problem requires careful consideration. For now we will let $b\equiv 0$ on $[0,T]$. Thus, our model simplifies to

$$X_t=\int_0^t \sigma_s dB_s, \qquad t\in[0,T],\ T>0.$$

Then, we have the following results.

Theorem 5. *Under the assumptions of Proposition (3) (σ is deterministic) and with $b\equiv 0$ on $[0,T]$ we have the following convergence in distribution:*

$$\sqrt{\frac{n}{V_T^n}}\left(\sum_{i=1}^{n}\left(X_{t_i}-X_{t_{i-1}}\right)^2-\int_0^T \sigma_s^2\, ds\right)\longrightarrow \mathcal{N}(0,1), \qquad n\to+\infty,$$

where

$$V_T^n := \frac{2n}{3}\sum_{i=1}^{n}\left(X_{t_i}-X_{t_{i-1}}\right)^4. \tag{18}$$

Proof. The idea for this proof is given in [ASJ14]. We will outline the main steps here.

Our goal is to demonstrate that we have the convergence in probability:

$$V_T^n\longrightarrow 2\int_0^T \sigma_s^4\, ds, \qquad n\to+\infty.$$

To do this we show that we have the following convergence in distribution:

$$\sqrt{n}\left(V_T^n-2\int_0^T \sigma_s^4\, ds\right)\longrightarrow \mathcal{N}(0,\ \Sigma) \qquad n\to+\infty$$

by using condition (17) applied to

$$\xi_i^n=n^{\frac{3}{2}}\left(\left(X_{t_i}-X_{t_{i-1}}\right)^4-3\left(\int_{t_{i-1}}^{t_i}\sigma_s^2\, ds\right)^2\right).$$

Now, since $1/\sqrt{n}\to 0$ as $n\to+\infty$, we deduce Equation (5) from convergence in distribution and application of Slutsky's therorem. Whence, another direct application of Slutsky's theorem allows us to conclude. $\square$

Similarly we give the version of this proposition for stochastic volatility.

Theorem 6. *Under* $(\boldsymbol{A4.2})$ *and with* $b \equiv 0$ *on* $[0,T]$ *we have the convergence in distribution:*

$$\sqrt{\frac{n}{V_T^n}}\left(\sum_{i=1}^{n}\left(X_{t_i}-X_{t_{i-1}}\right)^2-\int_0^T \sigma_s^2\,ds\right)\longrightarrow \mathcal{N}\left(0,1\right), \qquad n\to+\infty,$$

where V_T^n *is given by Equation (18).*

Proof. We admit this result. Details are available in [ASJ14]. □

Remark 4. *We notice that in Proposition (6) we are no longer dealing with mixed Gaussian law as a limiting distribution.*

Propositions (5) and (6) allow us to construct asymptotic confidence intervals for the integrated volatility. For $\alpha \in]0,1[$ we have a confidence interval of asymptotic size $1-\alpha$ is given by

$$\left[\sum_{i=1}^{n}\left(X_{t_i}-X_{t_{i-1}}\right)^2 - q_{1-\frac{\alpha}{2}}\sqrt{\frac{V_T^n}{n}}\ ;\ \sum_{i=1}^{n}\left(X_{t_i}-X_{t_{i-1}}\right)^2 + q_{1-\frac{\alpha}{2}}\sqrt{\frac{V_T^n}{n}}\right],$$

where $q_{1-\frac{\alpha}{2}}$ is a $\left(1-\frac{\alpha}{2}\right)$-quantile of $\mathcal{N}(0,1)$.

4.2 A second method to test the presence of diffusion

From the results of Theorems (2) and (3), we can build a test of the nullity of σ when it is deterministic. This part strongly relies on the paper [ASJ10].

First we create a test statistic that has the advantage to converge towards two different quantities respectively over the null and the alternative hypothesis:

$$H_0 : \frac{1}{T}\int_0^T \sigma_s^2\,ds > 0 \quad a.s \qquad \text{against} \qquad H_1 : \sigma_s = 0 \quad a.s\,.$$

The idea is to take a ratio of test statistics defined in Theorem (2) with uniform observations for both but with fewer observations at the numerator.

Proposition 14. *We assume* $(\boldsymbol{A4.1})$, $(\boldsymbol{A4.2})$, *and we consider* σ_s *to be deterministic. In order to differentiate the test statistic obtained from the one defined in theorem (2), we set a new notation.*

For $n \in \mathbb{N}^*$ *and* $\ell \in \{2,...,6\}$, *lets denote* $RV(n,\frac{\ell}{n})$ *the realized variance of our process, i.e:*

$$RV\left(n,\frac{\ell}{n}\right)=\frac{1}{T}\sum_{i=1}^{\lfloor\frac{n}{\ell}\rfloor}\left(X_{\ell\frac{i}{n}}-X_{\ell\frac{i-1}{n}}\right)^2.$$

Then, we have the convergence in probability:

$$S_n^\ell := \frac{RV\left(n,\frac{\ell}{n}\right)}{RV\left(n,\frac{1}{n}\right)}\longrightarrow\begin{cases}1\ under\,H_0\\ \ell\ under\,H_1\end{cases}, \quad n\to+\infty.$$

We observe that S_n^ℓ *isolate each assumption better when* ℓ *is big, however if* ℓ *is too big we lose information and* $RV\left(n,\frac{\ell}{n}\right)$ *converges too slowly, that is why we usually take* $\ell<7$

Proof. Under H_0, we have shown in Theorem (2) the convergence in probability

$$RV\left(n,\frac{1}{n}\right)\to\frac{1}{T}\int_0^T\sigma_s^2\,ds, \quad n\to+\infty.$$

With the same setting, it is direct that for all $\ell \in \{2, ..., 6\}$,

$$RV\left(n, \frac{\ell}{n}\right) \to \frac{1}{T}\int_0^T \sigma_s^2\, ds, \quad n \to +\infty \quad \textit{in probability}.$$

Thus, for $\sigma > 0$, by Slutsky's theorem, we have well $S_n^\ell \longrightarrow 1, \quad n \to +\infty$ in probability.

We demonstrate the convergence over H_1, when $b(s, X_s)$ is a polynomial function in s. For other cases it is admitted.

If we note d the degree of b, then there exists $a_j \in \mathbb{R}$ for all $j \in \{1, ..., d-1\}$ and $a_d \in \mathbb{R}^*$ such that :

$$b_s = \sum_{j=1}^{d} a_j s^j \quad \forall s \in \mathbb{R}^+.$$

Thus we have:

$$\begin{aligned}
RV\left(n, \frac{\ell}{n}\right) &= \sum_{i=1}^{\lfloor \frac{n}{\ell} \rfloor} \left(\int_{\ell \frac{i-1}{n}}^{\ell \frac{i}{n}} \sum_{j=1}^{d} a_j s^j\, ds \right)^2 \\
&= \sum_{i=1}^{\lfloor \frac{n}{\ell} \rfloor} \left(\left[\sum_{j=1}^{d} \frac{a_j}{j+1} x^{j+1} \right]_{\ell \frac{i-1}{n}}^{\ell \frac{i}{n}} \right)^2 \\
&= \sum_{i=1}^{\lfloor \frac{n}{\ell} \rfloor} \left(\sum_{j=1}^{d} \frac{a_j}{j+1} \left(\ell \frac{i}{n}\right)^{j+1} - \sum_{j=1}^{d} \frac{a_j}{j+1} \left(\ell \frac{i-1}{n}\right)^{j+1} \right)^2 \\
&= \sum_{i=1}^{\lfloor \frac{n}{\ell} \rfloor} \left(\sum_{j=1}^{d} \frac{a_j}{j+1} \left(\frac{\ell}{n}\right)^{j+1} \left(i^{j+1} - (i-1)^{j+1}\right) \right)^2 \\
&= \frac{a_d}{d+1} \left(\frac{\ell}{n}\right)^{2d+2} \sum_{i=1}^{\lfloor \frac{n}{\ell} \rfloor} i^{2d} + o(i^{2d}) \\
&= C\frac{\ell}{n} + o\left(\frac{\ell}{n}\right)
\end{aligned}$$

where C is a strictly positive constant. Thus we have well $S_n^\ell \longrightarrow \ell, \quad n \to +\infty$.

$\square$

Proposition 15. *Over the same assumptions than Proposition (14) and under H_0, we have the following convergence in distribution:*

$$\sqrt{n}(S_n^\ell - 1) \longrightarrow \mathcal{N}\left(0, \Sigma_\ell^2\right) \qquad n \to +\infty$$

where Σ_ℓ^2 can be explicitly determined in function of ℓ and independently of any parameter of the model:

$$\Sigma_\ell^2 = 2(\ell+1) - 4m_\ell$$

with $m_\ell = \mathbb{E}\left[Z^2\left(Z + \sqrt{\ell-1}Z'\right)^2\right]$ and Z, Z' two independent standard Gaussian variables.

Proof. For $\ell \in \{2, ..., 6\}$, according to the Theorem 2 and the Lemma 1 of [ASJ10], adapted to the Theorem (3) we have the following convergence in distribution when $n \to +\infty$:

$$\sqrt{n}\left(RV\left(n, \frac{\ell}{n}\right) - \frac{1}{T}\int_0^T \sigma_s^2\, ds\, ,\, RV\left(n, \frac{1}{n}\right) - \frac{1}{T}\int_0^T \sigma_s^2\, ds \right) \to \mathcal{N}(0, \Sigma')$$

where

$$\Sigma' = \begin{pmatrix} 2\ell \int_0^T \sigma_s^4 \, ds & 2\sigma^4 m_\ell \\ 2\sigma^4 m_\ell & 2\int_0^T \sigma_s^4 \, ds \end{pmatrix}.$$

So we can use the the delta method with the function $g : (x, y) \mapsto \frac{x}{y}$. Its Jacobian matrix in (x, y) is determined by:

$$Dg(x, y) = \left(\frac{1}{y}, -\frac{x}{y^2}\right).$$

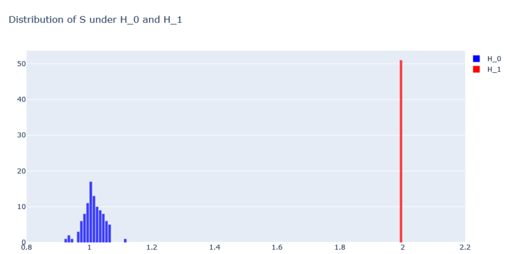
Thus we have the following convergence in distribution:

$$\sqrt{n}\left[g\left(RV\left(n, \frac{\ell}{n}\right), RV\left(n, \frac{1}{n}\right)\right) - g\left(\frac{1}{T}\int_0^T \sigma_s^2 \, ds, \frac{1}{T}\int_0^T \sigma_s^2 \, ds\right)\right]$$

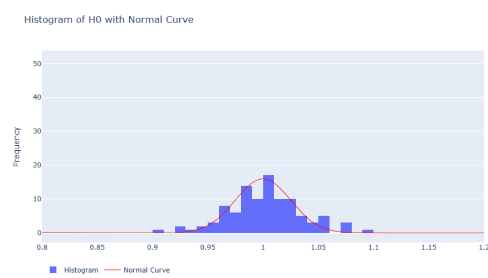
$$\longrightarrow \mathcal{N}\left(0\,,\, \left(\frac{1}{\frac{1}{T}\int_0^T \sigma_s^2 \, ds}, -\frac{1}{\frac{1}{T}\int_0^T \sigma_s^2 \, ds}\right) \Sigma' \left(\frac{1}{\frac{1}{T}\int_0^T \sigma_s^2 \, ds}, -\frac{1}{\frac{1}{T}\int_0^T \sigma_s^2 \, ds}\right)^T\right)$$

$$i.e. \quad \sqrt{n}(S_n^l - 1) \longrightarrow \mathcal{N}(0, \Sigma_l^2), \quad n \to +\infty.$$

□



Distribution of S_n^k respectively under the null and the alternative hypothesis with $n = 100$, $\ell = 2$ and $b_s = 3s + 4$.



Distribution of S_n^k under the null with $n = 100$, $\ell = 2$ and $b_s = 3s^2 + 2s + 5$ compared to $\mathcal{N}(1, \Sigma_2^2)$.

This convergence allow us the build the test for the significance of the volatility in X.

Proposition 16. *Over the same assumptions than Proposition (14) and under H_0, we can build a unilateral test asymptotically of size α and consistent for every alpha.*

$$\Phi(X_1, ..., X_n) = \mathbb{1}_{S_n^\ell \in \mathcal{R}} \tag{19}$$

where the rejection region is

$$\mathcal{R}_\alpha = \left]1 + q_{1-\alpha}\frac{\Sigma_\ell}{\sqrt{n}}, \; +\infty\right[$$

with $q_{1-\alpha}$ the quantile $1 - \alpha$ of $\mathcal{N}(0, 1)$.

Proof. According to Proposition (15), we have that $\frac{\sqrt{n}}{\Sigma_\ell}\left(S_n^\ell - 1\right) \to \mathcal{N}(0, 1)$ asymptotically in distribution.

Thus, asymptotically, we have that:

$$\mathbb{P}\left(\frac{\sqrt{n}}{\Sigma_\ell}\left(S_n^\ell - 1\right) > q_{1-\alpha}\right) = \alpha$$

$$i.e \quad \mathbb{P}\left(S_n^\ell > 1 + q_{1-\alpha}\frac{\Sigma_\ell}{\sqrt{n}}\right) = \alpha$$

this gives us the test of size α. Moreover, when $n \to +\infty$,

$$\mathbb{P}_{\sigma=0}\left(S_n^\ell > 1 + q_{1-\alpha}\frac{\Sigma_\ell}{\sqrt{n}}\right) \longrightarrow \mathbb{P}(\ell > 1) = 1$$

Thus the test is consistent. □

4.3 Estimation of Spot-Volatility

In this section we study the estimation of the volatility at a given time, using [ASJ14]. In the following, Δ_n represents the size of the time-intervals, so we can consider $\Delta_n = 1/n$ instance.

Assumption (A4.4). *We have $X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dB_s$ continuous with b_t locally bounded, σ_t deterministic càdlàg for all $t \in [0,T]$.*

The main idea that will drive the thought process of the following estimation of the spot-volatility is that we first consider the estimator of the integrated volatility $\widehat{C}(\Delta_n)_t = \sum_{i=1}^{[t/\Delta_n]}(\Delta_i^n X)^2$. From this estimator of the integrated volatility, we will consider its rate of change to build an estimator of the spot-volatility. To do so, we will require a sequence s_n going to 0 to build this new estimator $\left(\widehat{C}(\Delta_n)_{t+s_n} - \widehat{C}(\Delta_n)_t\right)/s_n$. We choose to denote $s_n = k_n\Delta_n$.

Assumption (A4.5). *Suppose $k_n \geq 1$ such that $k_n \to \infty$ with $k_n\Delta_n \to 0$ when $n \to \infty$.*

Proposition 17. *Suppose (**A4.4**) and (**A4.5**). We can estimate the value of the volatility at a given time $t \in [0,T]$. We have the following convergence in probability as $n \to \infty$,*

$$\widehat{c}_t^n = \frac{1}{k_n\Delta_n}\sum_{i=[t/\Delta_n]+1}^{[t/\Delta_n]+k_n}(\Delta_i^n X)^2 \longrightarrow \sigma_t^2$$

where $\Delta_i^n X = X_{i\Delta_n} - X_{(i-1)\Delta_n}$.

Proof. Using a particular case of the estimator of the realized volatility as defined in (6.4) from [ASJ14], with X continuous, we get the following estimator,

$$\widehat{C}(\Delta_n)_t = \sum_{i=1}^{[t/\Delta_n]}(\Delta_i^n X)^2$$

such that $\widehat{C}(\Delta_n)_t \longrightarrow \int_0^t \sigma_s^2 ds$ in probability.

We get that in probability for $s > 0, \widehat{C}(\Delta_n)_{t+s} - \widehat{C}(\Delta_n)_t \longrightarrow \int_t^{t+s}\sigma_r^2 dr$. Moreover,

$$\left|\frac{1}{s}\int_t^{t+s}\sigma_r^2 dr - \sigma_t^2\right| \leq \frac{1}{s}\int_t^{t+s}\left|\sigma_r^2 - \sigma_t^2\right| dr \leq \sup_{r\in[t,t+s]}\left|\sigma_r^2 - \sigma_t^2\right|.$$

So as σ is right continuous, $\lim\limits_{s\to 0}\frac{1}{s}\int_t^{t+s}\sigma_r^2 dr = \sigma_t^2$. It follows that if we take $s_n \longrightarrow 0$ as $n \to \infty$, we have the following convergence in probability

$$\widehat{c}_t^n := \frac{\widehat{C}(\Delta_n)_{t+s_n} - \widehat{C}(\Delta_n)_t}{s_n} \longrightarrow \sigma_t^2.$$

More precisely, as $t \mapsto \widehat{C}(\Delta_n)_t$ is constant on each interval $[i\Delta_n, (i+1)\Delta_n)$ we can use $s_n = k_n\Delta_n$ for $k_n \geq 1$ integer such that $k_n \to \infty$ and $k_n\Delta_n \to 0$ as $n \to \infty$. As a result, we have in probability

$$\hat{c}_t^n = \frac{1}{k_n\Delta_n}\sum_{i=[t/\Delta_n]+1}^{[t/\Delta_n]+k_n}(\Delta_i^n X)^2 \longrightarrow \sigma_t^2.$$

□

As the drift disappears asymptotically, we can consider a process X without drift from now on. In the plot below, the estimator $\widehat{c}_t^n$ from Proposition (17) is converging towards σ_t^2 taking $k_n = \sqrt{n}$, $\Delta_n = 1/n$ and $s \mapsto \sin(s), s \geq 0$ as the volatility function. We simulate trajectories with a dynamic $dX_t = \sigma_t dB_t$ with σ_t defined as mentioned previously and B a simulated Brownian motion.

Estimations of the spot-volatility using $\widehat{c}_t^n$, $k_n = \sqrt{n}$, $\Delta_n = 1/n$ considering $dX_t = \sigma_t dB_t$ with $\sigma_. : s \mapsto \sin(s), s \geq 0$.

Proposition 18. *Suppose (**A4.4**) and (**A4.5**). It follows that for all $i \geq 1$*

$$\Delta_i^n X \sim \mathcal{N}\left(0, \int_{(i-1)\Delta_n}^{i\Delta_n} \sigma_s^2 ds\right).$$

Proof. Indeed, we have that

$$\begin{aligned}\Delta_i^n X =& X_{i\Delta_n} - X_{(i-1)\Delta_n} \\ =& \int_{(i-1)\Delta_n}^{i\Delta_n} \sigma_s dB_s.\end{aligned}$$

We do have a Wiener integral as $(\sigma_s)_{s\geq 0}$ is deterministic that is square-integrable, thus, $\Delta_i^n X$ is Gaussian. Moreover using Proposition (4),

$$\begin{aligned}\mathbb{V}ar\left(\Delta_i^n X\right) =& \mathbb{V}ar\left(\int_{(i-1)\Delta_n}^{i\Delta_n} \sigma_s dB_s\right) \\ =& \mathbb{E}\left[\left(\int_{(i-1)\Delta_n}^{i\Delta_n} \sigma_s dB_s\right)^2\right] \\ =& \mathbb{E}\left[\int_{(i-1)\Delta_n}^{i\Delta_n} \sigma_s^2 ds\right].\end{aligned}$$

As σ_s is assumed to be deterministic, we have that

$$\mathbb{V}ar(\Delta_i^n X) = \int_{(i-1)\Delta_n}^{i\Delta_n} \sigma_s^2 ds.$$

□

Now we can study the error of estimation, when $t = 0$ as the argument remains the same for any $t \in [0, T]$. We can split the error between a deterministic error and a statistical error to study more precisely the error of estimation of σ_t^2.

Proposition 19. *Suppose (**A4.4**) and (**A4.5**). Let us denote $E_n = \widehat{c}_0^n - \sigma_0^2$ the estimation error. We have that*

$$E_n = D_n + S_n$$

where

$$D_n = \frac{1}{k_n} \sum_{i=1}^{k_n} \alpha_i^n - \sigma_0^2 = \frac{1}{k_n \Delta_n} \int_0^{k_n \Delta_n} (\sigma_s^2 - \sigma_0^2) ds$$

and

$$S_n = \frac{1}{k_n} \sum_{i=1}^{k_n} Z_i^n$$

with $(Z_i^n)_{i \in \{1,\dots,k_n\}}$ independent, centered, with respective variances $2(\alpha_i^n)^2$ with $\alpha_i^n = \frac{1}{\Delta_n} \int_{(i-1)\Delta_n}^{i\Delta_n} \sigma_s^2 ds$.

Proof. We have

$$\begin{aligned} D_n + S_n &= \frac{1}{k_n} \sum_{i=1}^{k_n} \alpha_i^n + \frac{1}{k_n} \sum_{i=1}^{k_n} Z_i^n - \sigma_0^2 \\ &= \frac{1}{k_n \Delta_n} \sum_{i=1}^{k_n} \Delta_n \left(\alpha_i^n + Z_i^n\right) - \sigma_0^2. \end{aligned}$$

Moreover, as

$$\begin{aligned} E_n &= \hat{c}_0^n - \sigma_0^2 \\ &= \frac{1}{k_n \Delta_n} \sum_{i=1}^{k_n} (\Delta_i^n X)^2 - \sigma_0^2, \end{aligned}$$

we can identify that if we choose $Z_i^n = \frac{1}{\Delta_n}(\Delta_i^n X)^2 - \alpha_i^n$ we do obtain $E_n = D_n + S_n$ such that $\mathbb{E}[Z_i^n] = 0$ and

$$\begin{aligned} \mathbb{V}ar(Z_i^n) &= \frac{1}{\Delta_n^2} \mathbb{V}ar\left((\Delta_i^n X)^2\right) \\ &= \frac{1}{\Delta_n^2} \left(\mathbb{E}\left[(\Delta_i^n X)^4\right] - \mathbb{E}\left[(\Delta_i^n X)^2\right]^2\right) \\ &= \frac{1}{\Delta_n^2} \left(3\left(\int_{(i-1)\Delta_n}^{i\Delta_n} \sigma_s^2 ds\right)^2 - \left(\int_{(i-1)\Delta_n}^{i\Delta_n} \sigma_s^2 ds\right)^2\right) \\ &= 2(\alpha_i^n)^2. \end{aligned}$$

□

To be able to study the rate of convergence of the error D_n, and thus of E_n, we need $(\sigma_s)_{s \geq 0}$ to be smooth enough. The following assumption on the smoothness of σ^2 will be used to control the rate of convergence of the deterministic part of the error of estimation.

Assumption (A4.6). *Suppose $(\sigma_s^2)_{s \geq 0}$ is 1/2-Hölder.*

Proposition 20. *Suppose (**A4.4**),(**A4.5**) and (**A4.6**). We have the following convergences,*

$$\sqrt{k_n} S_n \longrightarrow \mathcal{N}(0, 2\sigma_0^4) := Z_0$$

in distribution, and as $n \to \infty$

$$D_n \longrightarrow 0.$$

As $(\sigma_s^2)_{s\geq 0}$ is $1/2$-Hölder then there exists $A > 0$ such that

$$|D_n| \leq A(k_n\Delta_n)^{1/2}.$$

Proof. We have

$$\begin{aligned}
S_n &= \frac{1}{k_n}\sum_{i=1}^{k_n} Z_i^n \\
&= \frac{1}{k_n}\sum_{i=1}^{k_n} \frac{1}{\Delta_n}\left(\Delta_i^n X\right)^2 - \frac{1}{\Delta_n}\int_{(i-1)\Delta_n}^{i\Delta_n} \sigma_s^2 ds \\
&= \frac{1}{k_n\Delta_n}\sum_{i=1}^{k_n}\left(\int_{(i-1)\Delta_n}^{i\Delta_n} \sigma_s dB_s\right)^2 - \int_{(i-1)\Delta_n}^{i\Delta_n} \sigma_s^2 ds.
\end{aligned}$$

From there, we can see that we are in a similar setting which we already encountered in the proof of Theorem (3) explained in part 3.1.2 of [ASJ14] if we denote $\xi_i^n = \frac{1}{k_n\Delta_n}\left(\left(\int_{(i-1)\Delta_n}^{i\Delta_n} \sigma_s dB_s\right)^2 - \int_{(i-1)\Delta_n}^{i\Delta_n} \sigma_s^2 ds\right)$, as we also have σ deterministic. The key difference between the current framework and the previous one is that we do not sum integrals from $t_0 = 0$ to $t_n = T$ anymore. Instead we obtain

$$\mathbb{E}\left[(\xi_i^n)^2\right] = \frac{2}{k_n\Delta_n}\left(\int_{(i-1)\Delta_n}^{i\Delta_n} \sigma_s^2 ds\right)^2.$$

Which yields similarly as in Theorem (3), and using that $\frac{1}{k_n\Delta_n}\int_0^{k_n\Delta_n} \sigma_s^4 ds \to \sigma_0^4$ as n goes to ∞,

$$\sum_{i=1}^{k_n} \mathbb{E}\left[(\xi_i^n)^2\right] \to 2\sigma_0^4$$

allowing us to also establish a convergence in distribution to the Gaussian $\mathcal{N}\left(0, 2\sigma_0^4\right)$ using once again the Lindeberg CLT. Now, using that $k_n\Delta_n \longrightarrow 0$ with $k_n \longrightarrow \infty$ when $n \to \infty$, we get

$$\begin{aligned}
\alpha_i^n &= \frac{1}{\Delta_n}\int_{(i-1)\Delta_n}^{i\Delta_n} \sigma_s^2 ds \\
&\longrightarrow \sigma_0^2.
\end{aligned}$$

Now let's show that $D_n \longrightarrow 0$ as $n \to \infty$.

$$\begin{aligned}
D_n &= \frac{1}{k_n}\sum_{i=1}^{k_n} \alpha_i^n - \sigma_0^2 \\
&= \frac{1}{k_n\Delta_n}\sum_{i=1}^{k_n}\int_{(i-1)\Delta_n}^{i\Delta_n} \sigma_s^2 ds - \sigma_0^2 \\
&= \frac{1}{k_n\Delta_n}\int_0^{k_n\Delta_n} \sigma_s^2 ds - \sigma_0^2 \\
&\longrightarrow \sigma_0^2 - \sigma_0^2 = 0.
\end{aligned}$$

So we do have $D_n \longrightarrow 0$ as $n \to \infty$. As $(\sigma_s^2)_{s\geq 0}$ is $1/2$-Hölder, we have that

$$
\begin{aligned}
|D_n| &= \frac{1}{k_n\Delta_n}\left|\int_0^{k_n\Delta_n}\left(\sigma_s^2-\sigma_0^2\right)ds\right| \\
&\leq \frac{1}{k_n\Delta_n}\int_0^{k_n\Delta_n}\left|\sigma_s^2-\sigma_0^2\right|ds \\
&\leq \frac{1}{k_n\Delta_n}\int_0^{k_n\Delta_n}As^{1/2}ds \\
&= \frac{2}{3}A\left(k_n\Delta_n\right)^{1/2}.
\end{aligned}
$$

Thus, D_n converges at rate $(k_n\Delta_n)^{-1/2}$.

□

We have D_n and S_n converging respectively at speed $(k_n\Delta_n)^{-1/2}$ and $\sqrt{k_n}$ towards 0. When we have $A^{-1} \leq k_n\sqrt{\Delta_n} \leq A$ then as $|D_n| \leq A\sqrt{k_n\Delta_n}$ we have S_n and D_n converging at speed $\sqrt{k_n}$. However, when we have $k_n\sqrt{\Delta_n} \to 0$ as n goes to ∞ then S_n dominates D_n. Notice that having $A^{-1} \leq k_n\sqrt{\Delta_n} \leq A$ is equivalent to $k_n\sqrt{\Delta_n} \to \beta$ with $\beta \in (0,\infty)$.

Remark 5. *From [ASJ14] we have that the above Theorem and the following comment stay true when we have, as a condition of smoothness on σ^2,*

$$
\mathbb{E}\left(\left|\sigma_{S+s}^2-\sigma_S^2\right| |\mathcal{F}_S\right) \leq Ks^{1/2}
$$

for K constant and S a stopping time. As a result, this is true in particular if we take σ^2 an Itô semimartingale (see [ASJ18]).

The optimal rate of convergence is $(\Delta_n)^{-1/4}$ which is obtained if we take $k_n \sim (\Delta_n)^{-1/2}$. Indeed we have that for C constant

$$
\begin{aligned}
\left(\mathbb{E}\left[\left|\widehat{c}_0^n-\sigma_0^2\right|^2\right]\right)^{1/2} &\leq \left(\mathbb{E}\left[\left|\widehat{c}_0^n-\frac{1}{k_n\Delta_n}\int_0^{k_n\Delta_n}\sigma_s^2ds\right|^2\right]+\mathbb{E}\left[\left|\frac{1}{k_n\Delta_n}\int_0^{k_n\Delta_n}\sigma_s^2ds-\sigma_0^2\right|^2\right]\right)^{1/2} \\
&\leq \left(\frac{1}{k_n}\mathbb{E}\left[\left|\sqrt{k_n}\left(\widehat{c}_0^n-\frac{1}{k_n\Delta_n}\int_0^{k_n\Delta_n}\sigma_s^2ds\right)\right|^2\right]+D_n^2\right)^{1/2} \\
&\leq \left(\frac{C}{k_n}+k_n\Delta_n\right)^{1/2} \\
&\leq C\left(k_n\right)^{-1/2}+\left(k_n\Delta_n\right)^{1/2}.
\end{aligned}
$$

So we choose $k_n \in \underset{h}{\operatorname{argmin}}((h\Delta_n)^{1/2}+h^{-1/2})$, thus, $k_n \sim (\Delta_n)^{-1/2}$. As a result, if we take $\Delta_n = 1/n$ then it follows that we should consider $k_n = \sqrt{n}$.

Remark 6. *We notice that the estimator we defined in Proposition (17) is a kernel estimator,*

$$
\widehat{c}_t^n = \frac{1}{k_n\Delta_n}\sum_{i\geq 1}\phi\left(\frac{t-i\Delta_n}{k_n\Delta_n}\right)\left(\Delta_i^nX\right)^2
$$

*taking $\phi(x) = \mathbb{1}_{(0,1]}(x)$. Under (**A4.6**) we get the minimax rate of convergence $1/\Delta_n^{1/4}$, which is consistent with the results from above.*

Theorem 7. *Under (**A4.4**),(**A4.5**) and (**A4.6**), for all $t \geq 0$, suppose there exists $\beta \in [0, \infty)$ such that $k_n\sqrt{\Delta_n} \to \beta$. We have the following convergence in law,*

$$\sqrt{k_n}(\widehat{c}_t^n - \sigma_t^2) \longrightarrow Z_t + \beta Z_t'$$

with Z_t' limit of $\sqrt{k_n}D_n$.

Proof. See Theorem 13.3.3 in [JP12]. □

Remark 7. *If $\beta = 0$ then we have the particular convergence $\sqrt{k_n}(\widehat{c}_t^n - \sigma_t^2) \longrightarrow Z_t$ with $Z_t := \mathcal{N}(0, 2\sigma_t^4)$. We can deduce a CLT from this convergence, and thus a confidence interval. Trying to build a test to see if the spot-volatility is degenerate for all t is not very convenient, as the result would be the same as if we were trying to build a testing procedure for the integrated volatility, which is easier to work with.*

Proposition 21. *If $\beta = 0$ in the setting of Theorem (7), we have the following confidence interval of size $1 - \alpha$ for σ_t^2 when $t \in [0, T]$,*

$$IC_{1-\alpha} = \left[\widehat{c}_t^n \pm \widehat{c}_t^n \sqrt{\frac{2}{k_n}} q_{1-\alpha/2}\right],$$

with $q_{1-\alpha/2}$ the quantile of size $1 - \alpha/2$ of $\mathcal{N}(0, 1)$.

Proof. Suppose $\beta = 0$, from Theorem (7), using both Slutsky's theorem and the continuity theorem we get the following convergence in distribution

$$\frac{1}{\widehat{c}_t^n}\sqrt{\frac{k_n}{2}}(\widehat{c}_t^n - \sigma_t^2) \to \mathcal{N}(0, 1).$$

Then we have for $\alpha \in [0, 1]$

$$\begin{aligned} 1 - \alpha &= \mathbb{P}\left(-q_{1-\alpha/2} \leq \sqrt{\frac{k_n}{2}} \frac{1}{\widehat{c}_t^n}(\widehat{c}_t^n - \sigma_t^2) \leq q_{1-\alpha/2}\right) \\ &= \mathbb{P}\left(-\widehat{c}_t^n\left(1 - q_{1-\alpha/2}\sqrt{\frac{2}{k_n}}\right) \leq \sigma_t^2 \leq \widehat{c}_t^n\left(1 - q_{1-\alpha/2}\sqrt{\frac{2}{k_n}}\right)\right), \end{aligned}$$

with $q_{1-\alpha/2}$ quantile of size $1 - \alpha/2$ of $\mathcal{N}(0, 1)$. So we have $IC_{1-\alpha} = \left[\widehat{c}_t^n \pm \widehat{c}_t^n \sqrt{\frac{2}{k_n}} q_{1-\alpha/2}\right]$. □

5 Conclusion

In this paper we studied estimations of the volatility to determine statistical procedures to test whether the trajectory of a process X representing financial data can be described by a Brownian diffusion in increasingly complex models. We began with models featuring a constant volatility, for which we provided a complete statistical procedure to test the significance of the volatility associated to the Brownian motion in X. We then considered more realistic models with time-varying volatility, which presented greater challenges.

In this setting, we proved various results that enabled us to build useful statistical tools, including estimators and Central Limit Theorems when studying both integrated and spot-volatility. We provided a more subtle testing procedure for this framework considering a deterministic volatility. We also examined the optimal convergence rate of our spot-volatility estimator.

While we were able to build a statistical procedure for time-varying models, it would be interesting to extend our results to even more complex models. A natural idea could be to consider a model with state-depending volatility, leading to a non-parametric (rather than semi-parametric) problem. This idea has been explored in the literature by Danielle Florens-Zmirou ([FZ93]), Marc Hoffmann ([Hof97])and other authors. Another approach could involve adding jumps, which are often relevant in financial data. Such processes may be Markovian (e.g., Poisson processes) or non-Markovian (e.g., Hawkes processes). Finally, incorporating market microstructure effects like transaction fees can result in more realistic models and, consequently, stronger statistical testing procedures. Jean Jacod and Yacine Aït-Sahalia have extensively studied these considerations in [ASJ14].

References

[ASJ10] Yacine Aït-Sahalia and Jean Jacod. Is Brownian motion necessary to model high-frequency data? *The Annals of Statistics*, 38(5):3093 – 3128, 2010.

[ASJ14] Yacine Aït-Sahalia and Jean Jacod. *High-Frequency Financial Econometrics*. Princeton University Press, Princeton, 2014.

[ASJ18] Yacine Aït-Sahalia and Jean Jacod. Semimartingale: Itô or not ? *Stochastic Processes and their Applications*, 128(1):233–254, 2018.

[Bac00] Louis Bachelier. *Théorie de la Spéculation*. Annales Scientifiques de l'E.N.S 3ème série, 1900.

[BS73] Fisher Black and Myron Scholes. *The Pricing of Options and Corporate Liabilities*. Journal of Political Economy, 1973.

[BTTT16] Imen Ben Tahar, José Trashorras, and Gabriel Turinici. *Eléments de calcul stochastique pour l'évaluation et la couverture des actifs dérivés: avec exercices corrigés, travaux pratiques et études de cas*. Ellipses, 2016.

[Cha20] Djalil Chafaï. Introduction to stochastic calculus. `https://djalil.chafai.net/docs/M2/m2-stochastic-calculus-course-2020-2021.pdf`, 2020.

[FZ93] Danielle Florens-Zmirou. On estimating the diffusion coefficient from discrete observations. *Journal of Applied Probability*, 30(4):790–804, 1993.

[GC20] Valentine Genon-Catalot. Cours de statistique des diffusions. `https://helios2.mi.parisdescartes.fr/~genon/Cstadiff_0_6_10_03_20.pdf`, 2020.

[Hof97] Marc Hoffmann. Estimation non-paramétrique du coefficient de diffusion pour une perte lp. *Comptes Rendus de l'Académie des Sciences - Series I - Mathematics*, 324(4):475–480, 1997.

[JP12] Jean Jacod and Philip Protter. *Discretization of Processes*. Springer Berlin Heidelberg, 2012.

[KS98] Ioannis Karatzas and Steven E. Shreve. *Brownian Motion and Stochastic Calculus*. Springer New York, 1998.

[Lal14] Jean-Claude Laleuf. *Processus et intégrales stochastiques: Cours et exercices corrigés*. Ellipses, 2014.

[Mer73] Robert Cox Merton. *Theory of Rational Option Pricing*. The Bell Journal of Economic and Management Science, 1973.

[Øks03] Bernt Øksendal. *Stochastic Differential Equations*. Springer Berlin Heidelberg, 2003.