Neo-Hookean Material UMAT Details

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Using the notation $J = \det \mathbf{F}$ as is done in the Abaqus documentation, with

$$\mathbf{B}_{\mathrm{dis}} = J^{-2/3}\mathbf{B}, \mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}, (\mathbf{B}_{\mathrm{dis}})_0 = \mathbf{B}_{\mathrm{dis}} - \frac{1}{3}\mathrm{tr}\mathbf{B}_{\mathrm{dis}}\mathbf{I}$$

$\mathbf{B}_{\mathrm{dis}}$ is the distortional part of the left Cauchy-Green tensor, meaning without any volumetric contributions. The stored energy function for compressible Neo-Hookean material with one family of fiber is given by:

$$\rho\psi = C_1(\overline{I}_1 - 3) + D_1(J-1)^2 \quad C_1 = \frac{G}{2}, D_1 = \frac{K}{2}$$

The Cauchy stress is (The volumetric term is fine for nearly incompressible simulation.):

$$\mathbf{T} = J^{-1}[G(\mathbf{B}_{\mathrm{dis}})_0 + K(\ln J)\mathbf{I}] \qquad Note: \ln J \text{ for } J(J-1)$$

For **total-form constitutive laws**, we have:

$$\delta(J\mathbf{T}) = J(\mathbb{C} : \delta\mathbf{D}) + \delta\mathbf{W}\mathbf{T} - \mathbf{T}\delta\mathbf{W}$$

In this case, we use the approximation:

$$\delta(JT_{ij}) = J\mathbb{C}_{ijkl}\,\mathrm{sym}(\delta F_{km}F_{ml}^{-1})$$

Ignoring the symmetry of $\delta\mathbf{D}$, we have:

$$\delta(JT_{ij})F_{lm} = J\mathbb{C}_{ijkl}\,\delta F_{km}$$

$$\mathbb{C}_{ijkl} = J^{-1}\frac{\delta(JT_{ij})}{\delta F_{km}}F_{lm}$$

And enforcing symmetry again, implies:

$$\mathbb{C}_{ijkl} = \frac{1}{2}J^{-1}\left(\frac{\delta(JT_{ij})}{\delta F_{km}}F_{lm} + \frac{\delta(JT_{ij})}{\delta F_{lm}}F_{km}\right)$$

For this particular material model, we have:

$$\frac{\partial (JT_{ij})}{\partial F_{km}} = G\frac{\partial (J^{-\frac{2}{3}}[B_{ij}-\frac{1}{3}\mathrm{tr}(\mathbf{B})\delta_{ij}])}{\partial F_{km}} + K\frac{\partial (\ln J)}{\partial F_{km}}\delta_{ij}$$

$$= G\frac{\partial (J^{-\frac{2}{3}})}{\partial F_{km}}[B_{ij}-\frac{1}{3}\mathrm{tr}(\mathbf{B})\delta_{ij}] + GJ^{-\frac{2}{3}}\frac{\partial ([B_{ij}-\frac{1}{3}\mathrm{tr}(\mathbf{B})\delta_{ij}])}{\partial F_{km}} + K\frac{\partial (\ln J)}{\partial F_{km}}\delta_{ij}$$

$$= -\frac{2}{3}GJ^{-\frac{5}{3}}JF^{-1}_{mk}[B_{ij}-\frac{1}{3}\mathrm{tr}(\mathbf{B})\delta_{ij}] + GJ^{-\frac{2}{3}}(\frac{\partial B_{ij}}{\partial F_{km}} - \frac{1}{3}\frac{\partial\,\mathrm{tr}(\mathbf{B})}{\partial F_{km}}\delta_{ij}) + K\frac{\partial (\ln J)}{\partial F_{km}}\delta_{ij} \qquad \mathrm{Note:}\ \frac{\partial J}{\partial \mathbf{F}} = J\mathbf{F}^{\text{-T}}$$

$$= GJ^{-\frac{2}{3}}[-\frac{2}{3}\mathrm{B}_{ij}\mathrm{F}^{-1}_{mk} + \frac{2}{9}\mathrm{tr}(\mathbf{B})\delta_{ij}F^{-1}_{mk}] + GJ^{-\frac{2}{3}}(\frac{\partial F_{im}}{\partial F_{km}}F_{jm} + F_{im}\frac{\partial F_{jm}}{\partial F_{km}} - \frac{1}{3}\frac{\partial B_{ii}}{\partial F_{km}}\delta_{ij}) + K\frac{\partial (\ln J)}{\partial F_{km}}\delta_{ij}$$

$$= GJ^{-\frac{2}{3}}[-\frac{2}{3}\mathrm{B}_{ij}\mathrm{F}^{-1}_{mk} + \frac{2}{9}\mathrm{tr}(\mathbf{B})\delta_{ij}F^{-1}_{mk}] + GJ^{-\frac{2}{3}}(\delta_{ik}F_{jm} + F_{im}\delta_{jk} - \frac{1}{3}(\frac{\partial F_{im}}{\partial F_{km}}F_{im} + F_{im}\frac{\partial F_{im}}{\partial F_{km}})\delta_{ij}) + K\frac{1}{J}JF^{-1}_{mk}\delta_{ij}$$

$$= GJ^{-\frac{2}{3}}(\delta_{ik}F_{jm} + F_{im}\delta_{jk} - \frac{2}{3}B_{ij}\,\mathrm{F}^{-1}_{mk} - \frac{2}{3}\delta_{ij}F_{km} + \frac{2}{9}\mathrm{tr}(\mathbf{B})\delta_{ij}F^{-1}_{mk}) + K\delta_{ij}F^{-1}_{mk}$$

With similar derivation process, we get:

$$\frac{\partial (JT_{ij})}{\partial F_{lm}} = G\frac{\partial (J^{-\frac{2}{3}}[B_{ij}-\frac{1}{3}\mathrm{tr}(\mathbf{B})\delta_{ij}])}{\partial F_{lm}} + K\frac{\partial (\ln J)}{\partial F_{lm}}\delta_{ij}$$

$$= G\frac{\partial (J^{-\frac{2}{3}})}{\partial F_{lm}}[B_{ij}-\frac{1}{3}\mathrm{tr}(\mathbf{B})\delta_{ij}] + GJ^{-\frac{2}{3}}\frac{\partial ([B_{ij}-\frac{1}{3}\mathrm{tr}(\mathbf{B})\delta_{ij}])}{\partial F_{lm}} + K\frac{\partial (\ln J)}{\partial F_{lm}}\delta_{ij}$$

$$= -\frac{2}{3}GJ^{-\frac{5}{3}}JF^{-1}_{ml}[B_{ij}-\frac{1}{3}\mathrm{tr}(\mathbf{B})\delta_{ij}] + GJ^{-\frac{2}{3}}(\frac{\partial B_{ij}}{\partial F_{lm}} - \frac{1}{3}\frac{\partial\,\mathrm{tr}(\mathbf{B})}{\partial F_{lm}}\delta_{ij}) + K\frac{\partial (\ln J)}{\partial F_{lm}}\delta_{ij} \qquad \mathrm{Note:}\ \frac{\partial J}{\partial \mathbf{F}} = J\mathbf{F}^{\text{-T}}$$

$$= GJ^{-\frac{2}{3}}[-\frac{2}{3}\mathrm{B}_{ij}\mathrm{F}^{-1}_{ml} + \frac{2}{9}\mathrm{tr}(\mathbf{B})\delta_{ij}F^{-1}_{ml}] + GJ^{-\frac{2}{3}}(\frac{\partial F_{im}}{\partial F_{lm}}F_{jm} + F_{im}\frac{\partial F_{jm}}{\partial F_{lm}} - \frac{1}{3}\frac{\partial B_{ii}}{\partial F_{lm}}\delta_{ij}) + K\frac{\partial (\ln J)}{\partial F_{lm}}\delta_{ij}$$

$$= GJ^{-\frac{2}{3}}[-\frac{2}{3}\mathrm{B}_{ij}\mathrm{F}^{-1}_{ml} + \frac{2}{9}\mathrm{tr}(\mathbf{B})\delta_{ij}F^{-1}_{ml}] + GJ^{-\frac{2}{3}}(\delta_{il}F_{jm} + F_{im}\delta_{jl} - \frac{1}{3}(\frac{\partial F_{im}}{\partial F_{lm}}F_{im} + F_{im}\frac{\partial F_{im}}{\partial F_{lm}})\delta_{ij}) + K\frac{1}{J}JF^{-1}_{ml}\delta_{ij}$$

$$= GJ^{-\frac{2}{3}}(\delta_{il}F_{jm} + F_{im}\delta_{jl} - \frac{2}{3}B_{ij}\,\mathrm{F}^{-1}_{ml} - \frac{2}{3}\delta_{ij}F_{lm} + \frac{2}{9}\mathrm{tr}(\mathbf{B})\delta_{ij}F^{-1}_{ml}) + K\delta_{ij}F^{-1}_{ml}$$

Sum up above equations, we can get:

$$\frac{\delta(JT_{ij})}{\delta F_{km}}F_{lm}=GJ^{-\frac{2}{3}}(\delta_{ik}F_{jm}+F_{im}\delta_{jk}-\frac{2}{3}B_{ij}\,\mathrm{F}_{mk}^{-1}-\frac{2}{3}\delta_{ij}F_{km}+\frac{2}{9}\mathrm{tr}(\mathbf{B})\delta_{ij}F_{mk}^{-1})F_{lm}+K\delta_{ij}F_{mk}^{-1}F_{lm}$$

$$=GJ^{-\frac{2}{3}}(\delta_{ik}B_{jl}+B_{il}\delta_{jk}-\frac{2}{3}B_{ij}\delta_{kl}-\frac{2}{3}\delta_{ij}B_{kl}+\frac{2}{9}\mathrm{tr}(\mathbf{B})\delta_{ij}\delta_{kl})+K\delta_{ij}\delta_{kl}$$

$$\frac{\delta(JT_{ij})}{\delta F_{lm}}F_{km}=GJ^{-\frac{2}{3}}(\delta_{il}F_{jm}+F_{im}\delta_{jl}-\frac{2}{3}B_{ij}\,\mathrm{F}_{ml}^{-1}-\frac{2}{3}\delta_{ij}F_{lm}+\frac{2}{9}\mathrm{tr}(\mathbf{B})\delta_{ij}F_{ml}^{-1})F_{km}+K\delta_{ij}F_{ml}^{-1}F_{km}$$

$$=GJ^{-\frac{2}{3}}(\delta_{il}B_{jk}+B_{ik}\delta_{jl}-\frac{2}{3}B_{ij}\delta_{kl}-\frac{2}{3}\delta_{ij}B_{kl}+\frac{2}{9}\mathrm{tr}(\mathbf{B})\delta_{ij}\delta_{kl})+K\delta_{ij}\delta_{kl}$$

Finally, we get:

$$\mathbb{C}_{ijkl}=\frac{1}{2}J^{-1}(\frac{\delta(JT_{ij})}{\delta F_{km}}F_{lm}+\frac{\delta(JT_{ij})}{\delta F_{lm}}F_{km})$$

$$=\frac{1}{2}J^{-1}[GJ^{-\frac{2}{3}}(\delta_{ik}B_{jl}+B_{il}\delta_{jk}-\frac{2}{3}B_{ij}\delta_{kl}-\frac{2}{3}\delta_{ij}B_{kl}+\frac{2}{9}\mathrm{tr}(\mathbf{B})\delta_{ij}\delta_{kl})+K\delta_{ij}\delta_{kl}$$

$$+GJ^{-\frac{2}{3}}(\delta_{il}B_{jk}+\mathrm{B}_{ik}\,\delta_{jl}-\frac{2}{3}B_{ij}\delta_{kl}-\frac{2}{3}\delta_{ij}B_{kl}+\frac{2}{9}\mathrm{tr}(\mathbf{B})\delta_{ij}\delta_{kl})+K\delta_{ij}\delta_{kl}]$$

$$=\frac{G}{J}[\frac{1}{2}(\delta_{ik}B_{dis,jl}+\delta_{jk}B_{dis,il}+\delta_{il}B_{dis,jk}+\delta_{jl}B_{dis,ik})-\frac{2}{3}\delta_{ij}B_{dis,kl}-\frac{2}{3}B_{dis,ij}\delta_{kl}+\frac{2}{9}(\mathrm{tr}\,\mathbf{B}_{dis})\delta_{ij}\delta_{kl}]+\frac{K}{J}\delta_{ij}\delta_{kl}$$