Arruda-Boyce Material UMAT Details

Fangda Cui

Using the notion $J = \det \mathbf{F}$ as is done in the Abaqus documentation, with

$$\mathbf{B}_{\mathrm{dis}} = J^{-2/3}\mathbf{B}, \mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}, (\mathbf{B}_{\mathrm{dis}})_0 = \mathbf{B}_{\mathrm{dis}} - \frac{1}{3}\mathrm{tr}\mathbf{B}_{\mathrm{dis}}\mathbf{I}$$

The strain energy function for compressible Arruda-Boyce material is given by:

$$\rho\psi = G\sqrt{N}\left[\bar{\beta}\bar{\lambda}_c - \sqrt{N}\ln\left(\frac{\sinh\bar{\beta}}{\bar{\beta}}\right)\right] + \frac{K}{2}\left(\frac{J^2-1}{2} - \ln J\right) \quad G = nRT$$

where the parameters are given as:

$$\bar{\beta} = \mathcal{L}^{-1}\left(\bar{\lambda}_c / \sqrt{N}\right), \bar{\lambda}_c = \sqrt{\bar{I}_1 / 3}$$

The invariants of stretch tensors are given as:

$$\bar{I}_1 = J^{-\frac{2}{3}} I_1, \; I_1 = tr(\mathbf{B}) = tr(\mathbf{C}) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$

We ignore bulk modulus term momentarily, and then the Cauchy stress tensor is given by

$$\begin{aligned}
\mathbf{T} &= \frac{2}{J}\rho\mathbf{F}\frac{\partial\psi}{\partial\mathbf{C}}\mathbf{F}^T \\
&= \frac{2}{J}\mathbf{F}\left\{\frac{\partial W}{\partial\bar{I}_1}\frac{\partial\bar{I}_1}{\partial\mathbf{C}}\right\}\mathbf{F}^T \qquad Note: \rho\psi = W \\
&= \frac{2}{J}G\sqrt{N}\mathbf{F}\left\{\frac{\partial\bar{\beta}}{\partial\bar{I}_1}\lambda_c + \bar{\beta}\frac{\partial\bar{\lambda}_c}{\partial\bar{I}_1} - \sqrt{N}\frac{\bar{\beta}}{\sinh\bar{\beta}}\frac{\left(\partial\sinh\bar{\beta}/\partial\bar{I}_1\right)\bar{\beta} - \sinh\bar{\beta}\left(\partial\bar{\beta}/\partial\bar{I}_1\right)}{\bar{\beta}^2}\right\}\frac{\partial\bar{I}_1}{\partial\mathbf{C}}\mathbf{F}^T \\
&= \frac{2}{J}G\sqrt{N}\mathbf{F}\left\{\frac{\partial\bar{\beta}}{\partial\bar{I}_1}\lambda_c + \bar{\beta}\frac{\partial\bar{\lambda}_c}{\partial\bar{I}_1} - \sqrt{N}\frac{\cosh\bar{\beta}}{\sinh\bar{\beta}}\frac{\partial\bar{\beta}}{\partial\bar{I}_1} + \sqrt{N}\frac{1}{\bar{\beta}}\frac{\partial\bar{\beta}}{\partial\bar{I}_1}\right\}\frac{\partial\bar{I}_1}{\partial\mathbf{C}}\mathbf{F}^T \\
&= \frac{2}{J}G\sqrt{N}\mathbf{F}\left\{\left[\lambda_c - \sqrt{N}\left(\coth\bar{\beta} - \frac{1}{\bar{\beta}}\right)\right]\frac{\partial\bar{\beta}}{\partial\bar{I}_1} + \bar{\beta}\frac{\partial\bar{\lambda}_c}{\partial\bar{I}_1}\right\}\frac{\partial\bar{I}_1}{\partial\mathbf{C}}\mathbf{F}^T \qquad Note: \mathcal{L}\left(\bar{\beta}\right) = \coth\bar{\beta} - \frac{1}{\bar{\beta}}
\end{aligned}$$

$$
\begin{aligned}
\mathbf{T} &= \frac{2}{J} G\sqrt{N}\mathbf{F}\left\{\bar{\beta}\frac{\partial\bar{\lambda}_c}{\partial\bar{I}_1}\frac{\partial\bar{I}_1}{\partial\mathbf{C}}\right\}\mathbf{F}^T \\
&= \frac{2}{J}\frac{G}{3}\sqrt{N}\mathbf{F}\left\{\frac{1}{2}\frac{1}{\bar{\lambda}_c}\mathcal{L}^{-1}\left(\frac{\bar{\lambda}_c}{\sqrt{N}}\right)\mathbf{I}\left(-\frac{1}{3}J^{-2/3}\mathbf{C}^{-1}I_1 + J^{-2/3}\mathbf{I}\right)\right\}\mathbf{F}^T \qquad Note: \frac{\partial J}{\partial \mathbf{C}} = \frac{J}{2}\mathbf{C}^{-1} \\
&= J^{-1}\frac{G}{3}\frac{\sqrt{N}}{\bar{\lambda}_c}\mathcal{L}^{-1}\left(\frac{\bar{\lambda}_c}{\sqrt{N}}\right)\left(\mathbf{B}_{\text{dis}} - \frac{1}{3}tr(\mathbf{B}_{\text{dis}})\mathbf{I}\right) \\
&= J^{-1}\frac{G}{3}\frac{\sqrt{N}}{\bar{\lambda}_c}\mathcal{L}^{-1}\left(\frac{\bar{\lambda}_c}{\sqrt{N}}\right)\left(\mathbf{B}_{\text{dis}}\right)_0
\end{aligned}
$$

Now bring back the bulk modulus term, we get

$$
\mathbf{T} = J^{-1}\left[\frac{G}{3}\frac{\sqrt{N}}{\bar{\lambda}_c}\mathcal{L}^{-1}\left(\frac{\bar{\lambda}_c}{\sqrt{N}}\right)\left(\mathbf{B}_{\text{dis}}\right)_0 + K\frac{J^2-1}{2}\mathbf{I}\right]
$$

Here we use the following approximation for inverse Langevin function

$$
\mathcal{L}^{-1}(\text{x}) = x\frac{3-x^2}{1-x^2}
$$

Furthermore, for nearly incompressible materials we can use the following approximation

$$
\ln J \approx \frac{J^2-1}{2} \;\; when \;\; J \approx 1
$$

Yield the Cauchy stress tensor can be simplified into

$$
\mathbf{T} = J^{-1}\left\{\frac{G}{3}\left[\frac{3-\left(\frac{\bar{\lambda}_c}{\sqrt{N}}\right)^2}{1-\left(\frac{\bar{\lambda}_c}{\sqrt{N}}\right)^2}\right]\left(\mathbf{B}_{\text{dis}}\right)_0 + K(\ln J)\mathbf{I}\right\}
$$

Cauchy stress tensor can be written alternatively using index notations as:

$$
T_{ij} = J^{-1}\left\{\frac{G}{3}\left[\frac{3-\left(\frac{\bar{\lambda}_c}{\sqrt{N}}\right)^2}{1-\left(\frac{\bar{\lambda}_c}{\sqrt{N}}\right)^2}\right]\left[J^{-2/3}\left(B_{ij} - \frac{1}{3}tr(\mathbf{B})\delta_{ij}\right)\right] + K(\ln J)\delta_{ij}\right\}
$$

In Abaqus, total-form constitutive laws follow:

$$\delta(J\mathbf{T}) = J(\mathbb{C}:\delta\mathbf{D}) + \delta\mathbf{W}\mathbf{T} - \mathbf{T}\delta\mathbf{W}$$

In this case we use the approximation

$$\delta(JT_{ij}) = J\mathbb{C}_{ijkl}\,\mathrm{sym}(\delta F_{km}F_{ml}^{-1})$$

Ignoring the symmetry of $\delta\mathbf{D}$ momentarily we have

$$\delta(JT_{ij})F_{lm} = J\mathbb{C}_{ijkl}\,\delta F_{km}$$

$$\mathbb{C}_{ijkl} = J^{-1}\frac{\delta(JT_{ij})}{\delta F_{km}}F_{lm}$$

And enforcing symmetry again, implies

$$\mathbb{C}_{ijkl} = \frac{1}{2}J^{-1}\left(\frac{\delta(JT_{ij})}{\delta F_{km}}F_{lm} + \frac{\delta(JT_{ij})}{\delta F_{lm}}F_{km}\right)$$

By using the Cauchy stress tensor, we get

$$\begin{aligned}\frac{\delta(JT_{ij})}{\delta F_{km}}F_{lm} = {} & \frac{G}{3}\left\{\frac{\dfrac{\partial\left[(-\frac{\bar{\lambda}_c}{\sqrt{N}})^2\right]}{\partial F_{km}}\left[1-(\frac{\bar{\lambda}_c}{\sqrt{N}})^2\right] - \left[3-(\frac{\bar{\lambda}_c}{\sqrt{N}})^2\right]\dfrac{\partial\left[(-\frac{\bar{\lambda}_c}{\sqrt{N}})^2\right]}{\partial F_{km}}}{\left[1-(\frac{\bar{\lambda}_c}{\sqrt{N}})^2\right]^2}\left[B_{dis,ij} - \frac{1}{3}tr(\mathbf{B}_{dis})\delta_{ij}\right]\right\}F_{lm} \\ & + \frac{G}{3}\left\{\frac{3-(\frac{\bar{\lambda}_c}{\sqrt{N}})^2}{1-(\frac{\bar{\lambda}_c}{\sqrt{N}})^2}\frac{\partial\left[B_{dis,ij} - \frac{1}{3}tr(\mathbf{B}_{dis})\delta_{ij}\right]}{\partial F_{km}}\right\}F_{lm} \\ & + K\frac{\partial(\ln J)}{\partial F_{km}}F_{lm}\delta_{ij}\end{aligned}$$

$$\begin{aligned}\frac{\partial\left[-(\frac{\bar{\lambda}_c}{\sqrt{N}})^2\right]}{\partial F_{km}} &= -\frac{1}{N}\frac{\partial(\bar{\lambda}_c)^2}{\partial F_{km}} \\ &= -\frac{1}{N}2\bar{\lambda}_c\frac{1}{2\bar{\lambda}_c}\frac{1}{3}\frac{\partial\left[J^{-2/3}\,\mathrm{tr}(\mathbf{B})\right]}{\partial F_{km}} \\ &= \frac{2}{9N}J^{-2/3}F_{mk}^{-1}tr(\mathbf{B}) - \frac{2}{3N}J^{-2/3}F_{km}\end{aligned}$$

$$\frac{\delta(JT_{ij})}{\delta F_{km}}F_{lm}=\frac{G}{3}\left\{\frac{-2\left[\frac{2}{9N}J^{-2/3}F_{mk}^{-1}tr(\mathbf{B})-\frac{2}{3N}J^{-2/3}F_{km}\right]}{\left[1-(\frac{\bar{\lambda}_c}{\sqrt{N}})^2\right]^2}\right\}\left[B_{dis,ij}-\frac{1}{3}tr(\mathbf{B}_{dis})\delta_{ij}\right]F_{lm}$$

$$+\frac{G}{3}\left\{\frac{\partial\left[B_{dis,ij}-\frac{1}{3}tr(\mathbf{B}_{dis})\delta_{ij}\right]}{\partial F_{km}}\right\}\frac{3-(\frac{\bar{\lambda}_c}{\sqrt{N}})^2}{1-(\frac{\bar{\lambda}_c}{\sqrt{N}})^2}F_{lm}+K\frac{\partial(\ln J)}{\partial F_{km}}F_{lm}\delta_{ij}$$

$$=\frac{G}{3}J^{-2/3}\left\{\frac{\left[-\frac{4}{9N}F_{mk}^{-1}tr(\mathbf{B})+\frac{4}{3N}F_{km}\right]}{\left[1-(\frac{\bar{\lambda}_c}{\sqrt{N}})^2\right]^2}\right\}\left[B_{dis,ij}-\frac{1}{3}tr(\mathbf{B}_{dis})\delta_{ij}\right]F_{lm}$$

$$+\frac{G}{3}J^{-2/3}\frac{3-(\frac{\bar{\lambda}_c}{\sqrt{N}})^2}{1-(\frac{\bar{\lambda}_c}{\sqrt{N}})^2}\left\{\delta_{ik}F_{jm}+F_{im}\delta_{jk}-\frac{2}{3}\delta_{ij}F_{km}-\frac{2}{3}B_{ij}F_{mk}^{-1}+\frac{2}{9}tr(\mathbf{B})\delta_{ij}F_{mk}^{-1}\right\}F_{lm}$$

$$+K\delta_{ij}F_{mk}^{-1}F_{lm}$$

$$\frac{\delta(JT_{ij})}{\delta F_{lm}}F_{km}=\frac{G}{3}\left\{\frac{-2\left[\frac{2}{9N}J^{-2/3}F_{ml}^{-1}tr(\mathbf{B})-\frac{2}{3N}J^{-2/3}F_{lm}\right]}{\left[1-(\frac{\bar{\lambda}_c}{\sqrt{N}})^2\right]^2}\right\}\left[B_{dis,ij}-\frac{1}{3}tr(\mathbf{B}_{dis})\delta_{ij}\right]F_{km}$$

$$+\frac{G}{3}\left\{\frac{\partial\left[B_{dis,ij}-\frac{1}{3}tr(\mathbf{B}_{dis})\delta_{ij}\right]}{\partial F_{lm}}\right\}\frac{3-(\frac{\bar{\lambda}_c}{\sqrt{N}})^2}{1-(\frac{\bar{\lambda}_c}{\sqrt{N}})^2}F_{km}+K\frac{\partial(\ln J)}{\partial F_{lm}}F_{km}\delta_{ij}$$

$$=\frac{G}{3}J^{-2/3}\left\{\frac{\left[-\frac{4}{9N}F_{ml}^{-1}tr(\mathbf{B})+\frac{4}{3N}F_{lm}\right]}{\left[1-(\frac{\bar{\lambda}_c}{\sqrt{N}})^2\right]^2}\right\}\left[B_{dis,ij}-\frac{1}{3}tr(\mathbf{B}_{dis})\delta_{ij}\right]F_{km}$$

$$+\frac{G}{3}J^{-2/3}\frac{3-(\frac{\bar{\lambda}_c}{\sqrt{N}})^2}{1-(\frac{\bar{\lambda}_c}{\sqrt{N}})^2}\left\{\delta_{il}F_{jm}+F_{im}\delta_{jl}-\frac{2}{3}\delta_{ij}F_{lm}-\frac{2}{3}B_{ij}F_{ml}^{-1}+\frac{2}{9}tr(\mathbf{B})\delta_{ij}F_{ml}^{-1}\right\}F_{km}$$

$$+K\delta_{ij}F_{ml}^{-1}F_{km}$$

$$\mathbb{C}_{ijkl}=\frac{1}{2}J^{-1}(\frac{\delta(JT_{ij})}{\delta F_{km}}F_{lm}+\frac{\delta(JT_{ij})}{\delta F_{lm}}F_{km})$$

$$=\frac{1}{2}J^{-1}\frac{2G}{3}\frac{\frac{4}{3N}B_{dis,kl}-\frac{4}{9N}tr(\mathbf{B}_{\mathrm{dis}})\delta_{lk}}{\left[1-(\frac{\bar{\lambda}_c}{\sqrt{N}})^2\right]^2}\left[B_{dis,ij}-\frac{1}{3}tr(\mathbf{B}_{dis})\delta_{ij}\right]$$

$$+\frac{1}{2}J^{-1}\frac{G}{3}\frac{3-(\frac{\bar{\lambda}_c}{\sqrt{N}})^2}{1-(\frac{\bar{\lambda}_c}{\sqrt{N}})^2}\left[\delta_{ik}B_{dis,jl}+\delta_{il}B_{dis,jk}+B_{dis,ik}\delta_{jl}+B_{dis,il}\delta_{jk}-\frac{4}{3}\delta_{ij}B_{dis,kl}-\frac{4}{3}B_{dis,ij}\delta_{lk}+\frac{4}{9}tr(\mathrm{B}_{\mathrm{dis}})\delta_{ij}\delta_{kl}\right]$$

$$+\frac{1}{2}J^{-1}2K\delta_{ij}\delta_{kl}$$

$$\mathbb{C}_{ijkl}=\frac{G}{3J}\frac{\frac{4}{3N}B_{dis,kl}-\frac{4}{9N}tr(\mathbf{B}_{\mathrm{dis}})\delta_{lk}}{\left[1-(\frac{\bar{\lambda}_c}{\sqrt{N}})^2\right]^2}\left[B_{dis,ij}-\frac{1}{3}tr(\mathbf{B}_{dis})\delta_{ij}\right]$$

$$+\frac{G}{3J}\frac{3-(\frac{\bar{\lambda}_c}{\sqrt{N}})^2}{1-(\frac{\bar{\lambda}_c}{\sqrt{N}})^2}\left[\frac{1}{2}\left(\delta_{ik}B_{dis,jl}+\delta_{il}B_{dis,jk}+B_{dis,ik}\delta_{jl}+B_{dis,il}\delta_{jk}\right)-\frac{2}{3}\delta_{ij}B_{dis,kl}-\frac{2}{3}B_{dis,ij}\delta_{lk}+\frac{2}{9}tr(\mathrm{B}_{\mathrm{dis}})\delta_{ij}\delta_{kl}\right]$$

$$+\frac{K}{J}\delta_{ij}\delta_{kl}$$