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ON A CLASS OF FINITE ELEMENTS GENERATED BY LAGRANGE INTERPOLATION*

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Abstract. Functions which are continuous in a closed triangulated region and behave like polynomials in each triangle of the triangulation are important in the finite element method. The present paper is concerned with certain local aspects of functions of this type, relating to their existence, uniqueness, construction and approximation properties. The results are obtained for the general case of n-dimensional space.

1. Introduction. In the paper [8], Zlamal has studied, in connection with the finite element method, a class of quadratic polynomials defined by their values at the vertices and midpoints of the sides of a given triangle in the plane. He also mentioned a class of cubic polynomials [8, p. 398] defined by their values at the points of the principal lattice (see below) of a triangle. A recent paper [1] extends a number of the results of [8] to polynomials defined in R_n. In the present paper, we approach the problems of the existence, uniqueness and representation of polynomials of arbitrary degree in Euclidean spaces of arbitrary dimension from the standpoint of interpolation theory. The polynomials already mentioned and considered in [1] and [8] are special cases of the polynomials considered here.

The contents of the paper are as follows: in §§ 2 and 3 we shall demonstrate the existence and uniqueness of a class of interpolation polynomials when the data are given at the points of a principal lattice of a simplex in n-space. The method of proof is capable of substantial generalization with regard to the placing of these points. In § 4, we give an explicit representation for the interpolation polynomial, and in § 5, we shall obtain upper bounds on the error of interpolation of sufficiently smooth functions, which parallel known results for the case $n = 1$. A few remarks about the application of these results are given in § 6.

2. Simplexes and polynomials. Throughout, $\bar{S}_n$ will mean a simplex whose $n + 1$ vertices X_i, $i = 0, 1, \cdots, n$, are situated in R_n, and whose n measure is not zero. As is well known, if $x \in R_n$, it is uniquely representable in the barycentric coordinates

$$x = \sum_{i=0}^{n} \lambda_i X_i, \qquad \sum_{i=0}^{n} \lambda_i = 1. \tag{2.1}$$

Moreover, if X_{k_i}, $i = 0, \cdots, q$, are any $q + 1$ members of X_i, then

$$\Pi_q = \left\{ x \in R_n | x = \sum_{i=0}^{q} \lambda_i X_{k_i}, \ \sum_{i=0}^{q} \lambda_i = 1 \right\} \tag{2.2}$$

is a hyperplane of dimension q in R_n. In particular, if $q = n - 1$, then Π_{n-1} is that hyperplane containing the face of the simplex which does not contain vertex X_{k_n}. Let $X_k = (X_{1k}, \cdots, X_{nk})^T$. Then (2.1) may be written as

$$x_l = \sum_{j=0}^{n} \lambda_j X_{lj}, \qquad \sum_{i=0}^{n} \lambda_i = 1, \qquad l = 1, 2, \cdots, n, \tag{2.3}$$

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which is a system of $n + 1$ linear equations from which the vector $(\lambda_0, \cdots, \lambda_n)^T$ corresponding to any given $(x_1, \cdots, x_n)^T$ may be determined. Equation (2.3) defines λ_i as a linear function of x, which we take in the form $\lambda_i(x) = (u^{(i)}, x) - b^{(i)}$, where the $u^{(i)}$ and $b^{(i)}$ can be easily found from (2.3). It follows from what was said above that the hyperplane $\lambda_k(x) \equiv (u^{(k)}, x) - b^{(k)} = 0$ contains the face of the simplex not containing vertex X_k. We shall use this later.

Let m be a fixed positive integer and let Z_m denote the set of numbers $\{0, 1/m, 2/m, \cdots, 1\}$. With $\bar{S}_n$ we associate the discrete point set $B(m, n)$ which is defined as

$$B(m, n) = \left\{x \in R_n | x = \sum_{i=0}^{n} \lambda_i X_i, \lambda_i \in Z_m, \sum_{i=0}^{n} \lambda_i = 1\right\}.$$

$B(m, n)$ is called *the* (m*-th order*) *principal lattice of* $\bar{S}_n$. It is easy to prove that $B(m, n)$ contains $\binom{m+n}{m}$ members.

The class of polynomials to be considered are those in the form

$$P_{r,s} = \sum_{|\alpha| \leqq n} a_{\alpha_1 \cdots \alpha_s} x_1^{\alpha_1} \cdots x_s^{\alpha_s}, \tag{2.4}$$

where α_i, $i = 1, \cdots, s$, is a nonnegative integer, $|\alpha| = \sum_{i=1}^{s} \alpha_i$ and $(x_1, \cdots, x_s)$ is an s-dimensional vector. The class of polynomials (2.4) of degree r and with $x \in R_s$ will be denoted by $V_{r,s}$. If $P \in V_{r,s}$, then $R(P, \Pi_q)$ will mean the restriction of P to the hyperplane Π_q, where Π_q is

$$\left\{x \in R_s | x = \sum_{i=0}^{q} \lambda_i X_{K_i}, \sum_{i=0}^{q} \lambda_i = 1\right\}$$

for $q + 1$ vertices X_{k_i}, $i = 0, \cdots, q$. Also, $V_{r,s}(\Pi_q)$ will mean the class of functions obtained by restricting each element of $V_{r,s}$ to Π_q.

Important properties of (2.4) are obtained in the following lemmas.

LEMMA 2.1. *Let* $Q \in V_{m,n}(\Pi_{n-d})$ *for some given* Π_{n-d}. *Then* $Q \in V_{m,n-d}$.

Proof. By definition, Q is obtained by restricting a certain element P of $V_{m,n}$ to Π_{n-d}. Select $n - d + 1$ points X_i, $i = 0, \cdots, n - d$, in Π_{n-d} in such a way that each $x \in \Pi_{n-d}$ is uniquely representable as

$$x = \sum_{i=0}^{n-d} \lambda_i X_i, \qquad \sum_{i=0}^{n-d} \lambda_i = 1.$$

For such an x,

$$\begin{aligned} P(x) = P(x(\lambda)) &= \sum_{|\alpha| \leqq m} a_{\alpha_1 \cdots \alpha_n} x_1^{\alpha_1} \cdots x_n^{\alpha_n} \\ &= \sum_{|\alpha| \leqq m} a_{\alpha_1 \cdots \alpha_n} \left(X_{10} + \sum_{i=1}^{n-d} \lambda_i (X_{1i} - X_{10})\right)^{\alpha_1} \\ &\qquad \left(X_{n0} + \sum_{i=1}^{n-d} \lambda_i (X_{ni} - X_{n0})\right)^{\alpha_n}, \end{aligned} \tag{2.5}$$

where $X_i = (X_{1i}, \cdots, X_{ni})^T$. It is clear that the quantity on the right is a polynomial in the λ_i, and that this polynomial is of degree m. Since also the λ_i appear symmetrically in (2.5), we can write it as

$$\sum_{|\alpha| \leq m} a'_{\alpha_1 \cdots \alpha_{n-d}} \lambda_1^{\alpha_1} \cdots \lambda_{n-d}^{\alpha_{n-d}}$$

on rearrangement. But the λ_i, $i = 1, \cdots, n - d$, can be chosen arbitrarily and thus the lemma is established.

The intuitive content of this result is that, roughly speaking, a polynomial of type (2.4) retains its characteristics if its domain of definition is limited to linear manifolds in R_s.

The next two lemmas, interesting in themselves, are needed for the proof of Theorem 2.1. This theorem may be regarded as a generalization of the fact that if $\alpha_1, \alpha_2, \cdots, \alpha_j$ are distinct zeros of a polynomial P of degree m defined in R_1, then $P = (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_j)Q$, where Q has degree $m - j$. A theorem of this type which holds for far more general functions than the polynomials considered here is proved by the use of integral representations in [6, p. 733]. However, it is felt that the present treatment, essentially algebraic in character, reveals more clearly the nature of the result with respect to polynomials.

We shall denote by $\sigma_{n-d}(x_0, \rho)$ an open sphere of radius $\rho > 0$ and center $x_0 \in \Pi_{n-d}$ and lying entirely in Π_{n-d}.

LEMMA 2.2. *Let $P \in V_{m,n}$ and $P(x) \equiv 0$ for all $x \in \sigma_{n-d}(x_0, \rho)$. Then for each $y \in \Pi_{n-d}$, $P(y) = 0$.*

Proof. Let $x_1 \in \Pi_{n-d}$. Since Π_{n-d} is convex, it contains the segment $\overline{x_0 x_1}$. Consider

$$\Pi_1 = \{x | x = (1 - \mu)x_0 + \mu x_1\}.$$

By Lemma 2.1, $R(P, \Pi_1) \in V_{m,1}$. Selecting $m + 1$ points in $\Pi_1 \cap \sigma_{n-d}(x_0, \rho)$ defined by μ_i, $i = 0, 1, \cdots, m$, we see that $R(P, \Pi_1) \equiv 0$. Since x_1 is arbitrary, the lemma follows.

LEMMA 2.3. *Let $P_1 \in V_{m,n}$, and suppose that for any fixed but arbitrary hyperplane Π_{n-1} we have $R(P_1, \Pi_{n-1}) = 0$. Then $P_1(x)$ can be written in the form*

$$P_1(x) = \lambda(x) \cdot P(x),$$

where $\lambda(x) \in V_{1,n}$ and for each $y \in \Pi_{n-1}$, $\lambda(y) = 0$.

Proof. For the proof, we introduce into R_n the simplex $\bar{S}_n$, where vertices $X_1, \cdots, X_n$ lie in Π_{n-1} and the distance from X_0 to Π_{n-1} is positive. For any $x \in R_n$ there is the unique representation

$$x = \sum_{i=0}^{n} \lambda_i X_i, \qquad \sum_{i=0}^{n} \lambda_i = 1$$

and so

$$\begin{aligned} P_1(x) &= \sum_{j=0}^{m} \sum_{|\alpha|=j} a_{\alpha_1 \cdots \alpha_n} \left(\sum_{i=0}^{n} \lambda_i X_{1i} \right)^{\alpha_1} \cdots \left(\sum_{i=0}^{n} \lambda_i X_{ni} \right)^{\alpha_n} \\ &= \sum_{j=0}^{n} \left(\sum_{i=0}^{m} \lambda_i \right)^{m-j} \sum_{|\alpha|=j} a_{\alpha_1 \cdots \alpha_n} \left(\sum_{i=0}^{n} \lambda_i X_{1i} \right)^{\alpha_1} \cdots \left(\sum_{i=0}^{n} \lambda_i X_{ni} \right)^{\alpha_n} \\ &= \sum_{|\alpha|=m} b_{\alpha_0 \cdots \alpha_n} \lambda_0^{\alpha_0} \cdots \lambda_n^{\alpha_n}, \end{aligned}$$

where $b_{\alpha_0 \cdots \alpha_n}$, $(|\alpha| = m)$ are certain combinations of the components of the X_i and the $a_{\alpha_1 \cdots \alpha_n}$.

Now let $y \in \Pi_{n-1}$; then

$$y = \sum_{i=0}^{n} \mu_i X_i, \qquad \sum_{i=0}^{n} \mu_i = 1, \qquad \mu_0 = 0.$$

For such a y, P_1 takes the form

$$\begin{aligned} P_1(y) &= \sum_{|\alpha|=m} b_{\alpha_0 \cdots \alpha_n} \mu_0^{\alpha_0} \cdots \mu_n^{\alpha_n} \\ &= \sum_{|\alpha|=m} b_{0\alpha_1 \cdots \alpha_n} \mu_1^{\alpha_1} \cdots \mu_n^{\alpha_n}. \end{aligned}$$

But this is zero, since $P_1(y) = 0$ for $y \in \Pi_{n-1}$, so that

$$\sum_{|\alpha|=m} b_{0\alpha_1 \cdots \alpha_n} \mu_1^{\alpha_1} \cdots \mu_n^{\alpha_n} = 0.$$

We shall now consider the homogeneous polynomial

$$\sum_{|\alpha|=m} b_{0\alpha_1 \cdots \alpha_n} \zeta_1^{\alpha_1} \cdots \zeta_n^{\alpha_n}$$

and we shall show that, whatever the ζ_i, it is zero. Indeed, for $\sum_{i=1}^{n} \zeta_i = 1$ this statement is already true. The general case follows simply from this when $\sum_{i=1}^{n} \zeta_i \neq 0$. For then,

$$\sum_{|\alpha|=m} b_{0\alpha_1 \cdots \alpha_n} \zeta_1^{\alpha_1} \cdots \zeta_n^{\alpha_n} = s^m \sum_{|\alpha|=m} b_{0\alpha_1 \cdots \alpha_n} (\zeta_1/s)^{\alpha_1} \cdots (\zeta_n/s)^{\alpha_n},$$

where $s = \sum_{i=1}^{n} \zeta_i$. If $s = 0$, then a continuity argument is used.

Now write, generally,

$$\begin{aligned} \sum_{|\alpha|=m} b_{\alpha_0 \cdots \alpha_n} \lambda_0^{\alpha_0} \cdots \lambda_n^{\alpha_n} &= \sum_{|\alpha|=m} b_{0\alpha_1 \cdots \alpha_n} \lambda_1^{\alpha_1} \cdots \lambda_n^{\alpha_n} \\ &\quad + \lambda_0 \sum_{|\beta|=m-1} b'_{\beta_0 \cdots \beta_n} \lambda_0^{\beta_0} \cdots \lambda_n^{\beta_n} \\ &= \lambda_0 \sum_{|\beta|=m-1} b'_{\beta_0 \cdots \beta_n} \lambda_0^{\beta_0} \cdots \lambda_n^{\beta_n} \end{aligned}$$

in view of what we just proved. But as we saw at the beginning of this section, the λ_i's can be regarded as (linear) functions of position and therefore,

$$P_1(x) = \lambda_0(x) \sum_{|\beta|=m-1} b'_{\beta_0 \cdots \beta_n} \lambda_0(x)^{\beta_0} \cdots \lambda_n(x)^{\beta_n}.$$

This is the statement of the lemma. We should point out that the function $\lambda(x)$ of the lemma is determined only up to an arbitrary multiplicative constant, since if $\lambda(y) = 0$, so does $K\lambda(y) = 0$.

THEOREM 2.1. *Let positive integers m, s be given, $s \leqq m$, and let $\Pi_{n-1,j}$, $j = 0, 1, \cdots, s-1$, be s distinct hyperplanes. Let there be given s functions, $\eta_j(x)$, $j = 0, 1, \cdots, s-1$, linear in x and such that $\eta_j(x) = 0$ for all $x \in \Pi_{n-1,j}$. Suppose $P_1 \in V_{m,n}(x)$ and that $R(P_1, \Pi_{n-1,j}) = 0, j = 0, 1, \cdots, s-1$. Then*

$$P_1(x) = \left(\prod_{i=0}^{s-1} \eta_i(x) \right) \cdot P_2(x),$$

where $P_2 \in V_{m-s,n}$.

Proof. The proof is by induction. Assume that the theorem is true for some $1 \leqq t < s$ so that P_1 vanishes in the hyperplanes $\Pi_{n-1,i}, i = 0, 1, \cdots, t-1$, and is representable as

$$P_1(x) = P_3(x) \cdot \left(\prod_{i=0}^{t-1} \eta_i(x) \right), \qquad P_3 \in V_{m-t,n}.$$

Consider $\Pi_{n-1,t}$. By hypothesis, $\Pi_{n-1,t}$ contains a $\sigma(x_0, \rho)$ not containing any point of the intersection $\Pi_{n-1,0} \cap \Pi_{n-1,1} \cap \cdots \cap \Pi_{n-1,t}$. Let $x_1 \in \sigma(x_0, \rho)$. Then

$$P(x_1) = P_3(x_1) \cdot \left(\prod_{i=0}^{t-1} \eta_i(x_1) \right) = 0$$

and therefore, $P_3(x_1) = 0$. By Lemma 2.2, $R(P_3, \Pi_{n-1,t}) \equiv 0$, and by Lemma 2.3,

$$P_3(x) = P_4(x)\eta_t(x).$$

Therefore, we have proved that

$$P_1(x) = P_4(x) \cdot \left(\prod_{i=0}^{t} \eta_i(x) \right).$$

But again by Lemma 2.3, the lemma is true for $t = 1$ and so the lemma is proved, by mathematical induction. The corollary follows easily.

COROLLARY. *Let $P \in V_{m,n}$ and vanish in $s > m$ distinct hyperplanes $\Pi_{n-1,i}$, $i = 0, 1, \cdots, s-1$. Then $P \equiv 0$.*

Proof. Select any m hyperplanes, say $\Pi_{n-1,i}, i = 0, 1, \cdots, m-1$. Then

$$P(x) = C_1 \cdot \prod_{i=0}^{m} \eta_i(x), \qquad C_1 = \text{const.}$$

Since P vanishes in a further hyperplane, C_1 must be zero and the result is proved.

Before continuing the development, let us indicate an extension of this theorem. Considering again R_1, we recall that if a polynomial $P(x)$ has the form $(x-\alpha)^s Q(x)$, then P and its derivatives up to $(s-1)$th order vanish at α, and conversely. If we replace $(x-\alpha)$ by $\eta_i(x)$, we can make a related statement in the general case. Since this result is not required for our present purposes, we shall omit the rigorous statement and proof.

3. Interpolation. The material in § 2 can now be used to prove the following theorem.

THEOREM 3.1. *Let $x \in R_n$ and let $\bar{S}_n$ be given. Let the points of the principal m-th order lattice of $\bar{S}_n$ be ordered and called $q_i, i = 1, 2, \cdots, \binom{m+n}{m}$. Suppose that $f_i, i = 1, 2, \cdots, \binom{m+n}{m}$, are given arbitrary numbers. Then there is exactly one $P \in V_{m,n}$ satisfying $P(q_i) = f_i$ for $i = 1, 2, \cdots, \binom{m+n}{m}$.*

Proof. Put another way, the theorem asserts the unique solvability of the interpolation problem in the class (2.4) and in $B(m, n)$.

To prove this result, we use induction on n. Let us note the following important fact: Let $0 \leqq j \leqq m$ be a fixed integer, and consider the hyperplane

$$\Pi_{n-1,j} = \left\{x | x = (j/m)X_0 + \sum_{i=1}^{n} \lambda_i X_i,\ j/m + \sum_{i=1}^{n} \lambda_i = 1\right\}. \tag{3.1}$$

Speaking geometrically, this is a certain hyperplane parallel to that face of the simplex opposite vertex X_0. Its intersections with the edges of $\bar{S}_n$, $n + 1$ of them, are the lattice points defined by

$$\lambda_0 = j/m, \qquad \lambda_i = 1 - j/m, \qquad i = 1, 2, \cdots, n,$$

i.e., if these points are Y_i, $i = 1, \cdots, m$, then

$$Y_i = (j/m)X_0 + (1 - j/m)X_i, \qquad i = 1, 2, \cdots, n. \tag{3.2}$$

These n points generate an $(n - 1)$-dimensional simplex, $\bar{S}_{n-1}$. We shall show that the lattice $B(m - j, n - 1)$ in $\bar{S}_{n-1}$ coincides with that part of $B(m, n)$ of $\bar{S}_n$ which lies in $\Pi_{n-1,j}$. Indeed, $B(m - j, n - 1)$ is

$$\left\{x | x = \sum_{i=1}^{n} \mu_i Y_i, \mu_i = \frac{I_i}{m - j}, 0 \leqq I_i \leqq m - j, \sum_{i=1}^{n} \mu_i = 1\right\},$$

where I_i is an integer, so that $x \in B(m - j, n - 1)$ implies, by (3.2),

$$\begin{aligned} x &= \sum_{i=1}^{n} \mu_i[(j/m)X_0 + (1 - j/m)X_i] \\ &= (j/m)X_0 + \sum_{i=1}^{n} \mu_i(1 - j/m)X_i \\ &= (j/m)X_0 + \sum_{i=1}^{n} \frac{I_i}{m - j}\left(\frac{m - j}{m}\right)X_i \\ &= (j/m)X_0 + \sum_{i=1}^{n} \frac{I_i}{m}X_i. \end{aligned}$$

This is, by definition, a point of $B(m, n)$. It is easy then to see that $B(m - j, n - 1)$ and that part of $B(m, n)$ in $\Pi_{n-1,j}$ are identical. The proof uses this fact. We shall now show that the homogeneous interpolation problem $\left(\text{i.e., } f_i = 0, i = 1, \cdots, \binom{m + n}{m}\right)$ has only the zero solution. Let $1 \leqq N < n$ and suppose that the homogeneous problem in $B(m, N)$ has any solution $P_0 \in V_{m,N}$. Then $Q_0 = R(P_0, \Pi_{N-1,0}) \in V_{m,N-1}$. But Q_0 vanishes at the lattice points of $\Pi_{N-1,0}$ and by the induction hypothesis, the only element of $V_{m,N-1}$ with this property is the zero element, and thus, $Q_0 \equiv 0$. By Theorem 2.1 then

$$P_0(x) = P_1(x) \cdot \eta_0(x),$$

where $\eta_0(x)$ is as given in that theorem and $P_1 \in V_{m-1,N}$. Now consider $\Pi_{N-1,1}$.

Since $\Pi_{N-1,1} \cap \Pi_{N-1,0} \equiv \varnothing$, we can apply a similar reasoning again to P_1 and $\Pi_{N-1,1}$ to get

$$P_1(x) = P_2(x) \cdot \eta_1(x), \qquad P_2 \in V_{m-2,N},$$

and continuing this, we eventually arrive at

$$P_0(x) = K \prod_{i=0}^{m-1} \eta_i(x), \qquad K = \text{const.};$$

it remains to consider $\Pi_{N-1,m}$. Now $P_0(x)$ must vanish at X_0 and this means that $K \equiv 0$. Therefore, if the homogeneous problem has a solution, that solution is necessarily zero. Since zero is a solution it is unique. The homogeneous problem is uniquely solvable for the N-dimensional problem if the $(N-1)$-dimensional problem is. Obviously for $N = 1$ we have this result, and the theorem is proved.

It follows immediately from the above proof that if the numbers f_i are themselves the values at the points of $B(m, n)$ of some $P' \in V_{m,n}$, then the interpolant P coincides with P'.

Note that P has $\binom{m+n}{m}$ coefficients.

4. Representation results. So far we have not given any method for finding the interpolation polynomial. In the present part, we shall give explicit formulas which will enable it to be found. In fact, we shall construct $\binom{m+n}{m}$ functions $l_i(x)$ such that

$$l_i(p_j) = \delta_{ij}, \qquad i, j = 1, 2, \ldots, \binom{m+n}{m}, \tag{4.1}$$

where p_j are points of $B(m, n)$, and δ_{ij} is the Kronecker delta. Consider any vertex, say X_0 of $\bar{S}_n$, anc consider the parallel hyperplanes $\Pi_{n-1,j}$, $j = 0, \cdots, m-1$, of (3.1). Let $\eta_0^{(j)}(x)$ be linear and such that

$$\eta_0^{(j)}(x) = 0 \quad \text{for all } x \in \Pi_{n-1,j}.$$

Thus $\eta_0^{(j)}(x) = 0$ is the equation of the hyperplane $\Pi_{n-1,j}$: moreover, $\eta_0^{(0)}$ differs by only a multiplicative constant from $\eta_0(x)$ of Theorem 2.1. We shall associate a similar set of functions with each vertex. Thus with X_i is associated $\eta_i^{(j)}(x)$, for $i = 0, 1, \cdots, n$, and $\eta_i^{(j)}(x) = 0$ whenever $x \in \Pi_{n-1,j}^{(i)}$.

Assume now that $\bar{x}$ is any definite member of $B(m, n)$,

$$\bar{x} = \sum_{i=0}^{n} \bar{\lambda}_i X_i, \qquad \sum_{i=0}^{n} \bar{\lambda}_i = 1,$$

where $\bar{\lambda}_i = a_i/m$ and a_i is a nonnegative integer. By definition of the $\eta_i^{(j)}(x)$, we have $\eta_i^{(a_i)}(\bar{x}) = 0$, $i = 0, 1, \cdots, n$. This is because $\bar{x}$ lies at the intersection of the hyperplanes $\lambda_k(x) = a_k/m$, $k = 0, 1, \cdots, n$, and so lies in each of them: therefore, since $\lambda_k(x) - a_k/m = c \cdot \eta_k^{(a_k)}(x)$ for some constant c, $\eta_i^{(a_i)}(\bar{x}) \equiv 0$.

Consider the expression

$$P_{\bar{x}} = \prod_{i=0}^{n}\left(\prod_{j=0}^{a_i-1} \eta_i^{(j)}(x)\right), \qquad \eta_i^{(-1)} \equiv 1, \text{ for all } i, \tag{4.2}$$

and observe that $P_{\bar{x}} \in V_{m,n}$.

Let $\bar{y}(\neq \bar{x}) \in B(m, n)$ and

$$\bar{y} = \sum_{i=0}^{n} \bar{\mu}_i x_i, \quad \sum_{i=0}^{n} \bar{\mu}_i = 1, \quad \mu_i = b_i/m, \quad i = 0, 1, \cdots, n.$$

As with $\bar{x}$, we have $\eta_i^{(b_i)}(\bar{y}) = 0$, $i = 0, 1, \cdots, n$. Now, there is at least one index K, $0 \leqq K \leqq n$, for which $b_K - a_K \leqq -1$. For if not, since b_j and a_j are integers, $b_j - a_j \geqq 0$ for each j and this contradicts $\sum_{j=0}^{n}(a_j - b_j) = 0$, since $\bar{x} \neq \bar{y}$. So, let K be such that $b_K \leqq a_K - 1$. Now consider

$$\begin{aligned} P_{\bar{x}}(\bar{y}) &= \prod_{i=0}^{n}\left(\prod_{j=0}^{a_i-1} \eta_i^{(j)}(\bar{y})\right) \\ &= \prod_{\substack{i=0\\ i\neq k}}^{n}\left(\prod_{j=0}^{a_i-1} \eta_i^{(j)}(\bar{y})\right) \cdot \prod_{j=0}^{a_K-1} \eta_K^{(j)}(\bar{y}). \end{aligned}$$

We have just shown that $b_K \leqq a_K - 1$, and therefore the factor $\eta_K^{(b_K)}(\bar{y})$ appears in the rightmost product. But this is zero, as we have seen, so $P_{\bar{x}}(\bar{y}) = 0$. Thus, $P_{\bar{x}}(x)$ vanishes whenever $x(\neq \bar{x}) \in B(m, n)$. It follows that the function

$$\frac{P_{\bar{x}}(x)}{P_{\bar{x}}(\bar{x})} \tag{4.3}$$

satisfies the condition (4.1) and is the desired function. Since $\bar{x}$ was any point of $B(m, n)$, the entire family (4.1) can be generated in this way.

We shall write (4.3) in a more useful way. We take $\eta_i^{(0)}(x) \equiv \lambda_i(x)$; then, as already indicated, $\eta_i^{(j)}(x) = \lambda_i(x) - j/m$. Inserting this in (4.3) and (4.1), we get

$$\begin{aligned} l_{\bar{x}}(x) &= (a_0!a_1! \cdots a_r!)^{-1} \prod_{i=0}^{n} \prod_{j=0}^{a_i-1} [m\lambda_i(x) - j] \\ &= \frac{1}{a!} \prod_{i=0}^{n} \prod_{j=0}^{a_i-1} [m\lambda_i(x) - j] \end{aligned} \tag{4.4}$$

after carrying out some manipulations. Expressions which are essentially identical to these were given in [7], where they were postulated as possessing certain formal properties in common with the one-dimensional Lagrange polynomials. For the case $n = 2$ they are also given in [5].

Finally, since the interpolation process described yields, for any $0 \leqq m_1 \leqq m$ and $P \in V_{m_1,n}$, an alternative representation for P, as follows from a consideration of the difference between P and its interpolant, we can get certain algebraic identities from the relation

$$P(x) = \sum_{B(m,n)} l_i(x)P(q_i), \qquad q_i \in B(m, n).$$

In particular,

$$\sum_{B(m,n)} l_i(x) = 1,$$

which is well known in the case $n = 1$.

5. Errors. In this article, on the basis of an exact representation of the interpolation error, we shall obtain error estimates of the kind

$$|f(x) - \sum_{B(m,n)} l_i(x) f_i| \leqq K M_{m+1} h^{m+1},$$

where h is the largest linear dimension of the simplex, M_{m+1} is an upper bound on the $(m + 1)$th derivatives of f, and where K depends only on m and n. No attempt is made to get precise values for K, but the results are of value in studying the behavior of the error as $h \to 0$. The method used in obtaining the bounds differs essentially from those used in [1] and [8]. In [8], a bound of the type indicated is derived for the case $m = n = 2$, while in [1], the cases $m = 1, 2$ with n arbitrary are considered. These authors also obtained bounds on the error of interpolation of the first derivatives. This problem, which is of interest in the field of partial differential equations, is not discussed here. It is hoped to consider it in a later paper.

Let $f \in C^{(m+1)}(\bar{S}_n)$ and let p and q be any two points of $\bar{S}_n$. With these two points is associated a function of onè variable, $\phi_{qp}(\lambda)$, defined by

$$\phi_{qp}(\lambda) = f(q + \lambda(p - q)),$$

and for $p_i \in B(m, n)$, we shall write $\phi_{pp_i} = \phi_i$. Further, write

$$R_{m+1,i} = \frac{1}{m!} \int_0^1 \phi_i^{(m+1)}(\lambda)(1 - \lambda)^m \, d\lambda.$$

If $p_i \in B(m, n)$, then that mth degree polynomial taking the value 1 at p_i and zero at every other point of $B(m, n)$ will be denoted by $l_i(x)$.

THEOREM 5.1. *Let $p \in \bar{S}_n$. With the above notations there holds*

$$f(p) = \sum_{B(m,n)} [f(p_i) - R_{m+1,i}] l_i(p) \tag{5.1}$$

and

$$|f(p)| \leqq C_{m,n,1} \max_i |f(p_i)| + C_{m,n,2} M_{m+1} h^{m+1}, \tag{5.2}$$

where $|D^i f| \leqq M_{m+1}$ in $\bar{S}_n$, $|i| = m + 1$, h is the greatest linear dimension in the simplex, and $C_{m,n,j}$, $j = 1, 2$, depend on m and n, but not on $\bar{S}_n$.

Proof. Let $q \in \bar{S}_n$ and consider $\phi_{qp}(\lambda)$. By Taylor's theorem,

$$\phi_{qp}(1) = \sum_{k=0}^{m} \frac{1}{k!} \phi_{qp}^{(k)}(0) + \frac{1}{m!} \int_0^1 \phi_{qp}^{(m+1)}(\lambda)(1 - \lambda)^m \, d\lambda. \tag{5.3}$$

Let $(p - q)_k$ be the kth component of $(p - q)$. Then (5.3) is equivalent to

$$f(p) = \sum_{k=0}^{m} \frac{1}{k!} \sum_{|i|=k} \frac{\partial^k f(q)}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}} (p - q)_1^{i_1} \cdots (p - q)_n^{i_n} + \frac{1}{m!} \int_0^1 \phi_{qp}^{(m+1)}(\lambda)(1 - \lambda)^m \, d\lambda. \tag{5.4}$$

The first of the two terms on the right is a polynomial in the components of p of degree m. Put it equal to $T_m(p, q)$. By the remark at the end of Theorem 3.1, we can represent $T_m(p, q)$ in the form

$$T_m(p, q) = \sum_{B(m,n)} T_m(p_i, q) l_i(p),$$

and by (5.4),

$$T_m(p_i, q) = f(p_i) - \frac{1}{m!} \int_0^1 \phi_{qp_i}^{(m+1)}(\lambda)(1 - \lambda)^m \, d\lambda.$$

Substituting these into (5.4), and taking q (hitherto arbitrary) to be equal to p, we get

$$f(p) = \sum_{B(m,n)} [f(p_i) - R_{m+1,i}] l_i(p),$$

which proves (5.1).

From this follows immediately that

$$|f(p)| \leqq \left(\max_i |f_i| + \max_i |R_{m+1,i}| \right) \sum_{B(m,n)} |l_i(p)|. \tag{5.5}$$

But it is clear from (4.4) that for $x \in \bar{S}_n$, each $l_{\bar{x}}(x)$ can be bounded from above by a number dependent only on m and n, and not on $\bar{S}_n$. Thus,

$$\sum_{B(m,n)} |l_i(p)| \leqq K_1(m, n). \tag{5.6}$$

Moreover,

$$\phi_i^{(m+1)}(\lambda) = \sum_{|i|=m+1} \frac{\partial^{m+1} f(p + \lambda(p_i - p))}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}} (p_i - p)_1^{i_1} \cdots (p_i - p)_n^{i_n},$$

where

$$|R_{m+1,i}| \leqq \frac{1}{m!} \max_{\lambda \in (0,1)} |\phi_i^{(m+1)}(\lambda)| \cdot \left| \int_0^1 (1 - \lambda)^m \, d\lambda \right|.$$

Evidently,

$$|\phi_i^{(m+1)}(\lambda)| \leqq K_1'(m, n) M_{m+1} h^{m+1},$$

and carrying out the last integration above, we have

$$|R_{m+1,i}| \leqq K_2(m, n) M_{m+1} h^{m+1}. \tag{5.7}$$

From (5.5)–(5.7), it follows immediately that

$$|f(p)| \leqq \{\max_i |f_i| + K_2 M_{m+1} h^{m+1}\} K_1,$$

which is equivalent to (5.2), and the theorem is proved.

In the case $n = 1$, (5.1) reduces to the error formula of Kowalewski [2, p. 71], sometimes obtained from Peano kernel arguments. To get a bound on the interpolation error, we have only to apply the theorem to the error itself, which vanishes at the points of $B(m, n)$.

6. Applications. We shall not prove any results in this section, but merely point out two areas of numerical analysis where the previous results are applicable. The first of these is the finite element method. From the results of §§ 2 and 3, we can infer that if Ω_n is a triangulated region of R_n, with a polyhedral boundary, then finite element trial functions which are continuous on $\bar{\Omega}_n$ can always be constructed; these functions will be members of the Sobolev space $W^{1,2}(\Omega)$. In each simplex of the triangulation, they reduce to an mth degree polynomial which is given by an expression consisting of a linear combination of polynomials of the type (4.4). For a detailed discussion of this and related topics, the reader is referred to [8] and [9].

A second application is to the problem of numerical integration in a simplex. This application is based on the fact that a closed expression is known for the integrals of monomials over a simplex. It is only necessary to integrate expressions (4.4) to generate the weights. This program is carried out in [7] for a range of values of m and n. We can integrate (5.1) to get error bounds.

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