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Systematic and generic construction of shape functions for p-adaptive meshes of multidimensional finite elements

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ABSTRACT

This paper presents a methodology for generating high-order shape functions for the complete family of finite elements. The geometric entities presented are the point, line, triangle, quadrilateral, tetrahedron, pyramid, prism, and hexahedron. The shape functions constructed are hierarchical and generate continuous approximation spaces. The order of interpolation of the shape function can be determined independently for each edge, face, and volume. The space of interpolation of order *p* for each element is complete. This definition allows for the construction of hybrid meshes with any combination of these elements, including elements of distinct dimension and/or interpolation order. The reader will observe that our systematic way of constructing shape functions can be extended to elements of higher-dimension.

1. Introduction

This work presents a definition of shape functions for the finite elements. Shape functions are defined for topologies of one, two, and three space dimensions. A sequence f_n of polynomial functions is used in the construction of the shape functions. Any set of polynomial functions can be used, for example, Legendre, Jacobi, or Chebyshev polynomials. The continuity of the shape functions defined here do not depend on the selected set of polynomials (see [1]). For a revision of the properties of these polynomials the reader is referred to [10].

One of the key concepts of our study is the partitioning of an element by its sides. The sides of the element are its vertices, edges, faces and interior volume. The computational domain is formed as the union of these sides. Each side is a topologically open set (see [1,4]). The p-adaptive version of the finite element method consists of being able to attribute *p* distinct orders to different sides of the mesh.

A p-adaptive version of the finite element method for mixed element geometries has been developed by Karniadakis [9] and Shephard [6]. In their work [5,9,12] the authors considered a set of hierarchical functions over triangles and tetrahedrons. The functions are defined on the reference element of the triangle by projecting the geometry of the quadrilateral master element onto the geometry of the triangular master element and projecting the geometry of the hexahedron master element onto the geometry of the tetrahedron master element. This approach allows the expressions of the functions to be written as a tensorial product. The functions on the quadrilateral are constructed using Jacobi polynomials and are defined in terms of area coordinates or volume coordinates. The functions are associated with the vertices, edges, faces, and interior volume and preserve the continuity of the interpolation space on the mesh. The polynomial form is preserved for both the triangle and tetrahedron elements.

Different approaches for higher-order constructions have been presented by Nigam and Philipps [13] and Schöberl and Zaglmayr [14]. In [13] the authors present a systematic method for generating higher-order shape functions for pyramid elements, using mapping functions similar to those presented in [5]. Nigam and Philipps do not present numerical simulations exemplifying the use of their element combined with other elements. In [14] the complete set of elements is presented, with the exception of the pyramid element. The construction of shape functions in their work follows the systematic method presented in [5]. Both of these works emphasize the construction of $H(div)$ and $H(curl)$ spaces.

Unlike Shephard's work [6], the parametric space used here does not depend on singular maps of the reference elements and they do not use area coordinates [3,8]. Therefore the basic sequences of the shape functions are different. There are no restrictions on the orientation of the elements in the mesh to guarantee continuity of the approximation space. In his doctoral

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thesis, Cedric [1] showed the completeness of the interpolation space for each one of the elements, for any order of interpolation, independent of the generating sequence f_n. It was demonstrated that the set of functions of order p defined on the element contains a set of linearly independent polynomials of degree p and that this set is complete. The notion of completeness of the polynomials set associated with the element implies that the set of shape functions associated with an element of order p can represent any polynomial of degree less than or equal to p on the elementary domain.

If the generating functions f_n are hierarchical, then the polynomials constructed will inherit this property [11], meaning that the polynomial set of order $p+N, N \geqslant 0$, for each fixed $p \geqslant 0$, contains the polynomial set of order p, $\forall N$. This property applies independently to shape functions associated with edge, face, and volume topologies.

The shape functions are defined by the product of two functions. The first function has fixed degree $\Psi(E,S)$, called the blending function [2,6,7], and does not depend on the polynomial order of the element. The second function, denoted by $\Phi(S)$, varies as the degree of the shape function:

$$N = \Psi(E,S)\Phi(S). \tag{1}$$

In the current case the blending function Ψ is constructed with a specific combination of the vertex functions. The function $\Phi(S)$ corresponds to a product of generating functions f_n.

2. Description of the geometry and nomenclature

In this section the geometry of the master elements is defined, as are the nomenclature associated with the shape functions and associated linear shape functions.

2.1. Elementary geometry

The mathematical definition of geometry for the mentioned elements is given in Eqs. (2)–(8) for line elements and the triangle, quadrilateral, tetrahedron, pyramid, prism, and hexahedron, respectively:

$$L = \{x \in \Re / -1 \leqslant x \leqslant +1\} \subset \Re, \tag{2}$$

$$T = \{(x,y) \in \Re^2 / 0 \leqslant x \leqslant +1,\ 0 \leqslant y \leqslant 1-x\} \subset \Re^2, \tag{3}$$

$$Q = \{(x,y) \in \Re^2 / -1 \leqslant x,y \leqslant +1\} \subset \Re^2, \tag{4}$$

$$Te = \{(x,y,z) \in \Re^3 / 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1-x,\ 0 \leqslant z \leqslant 1-x-y\} \subset \Re^3, \tag{5}$$

$$Py = \{(x,y,z)/z-1 \leqslant x \leqslant 1-z, z-1 \leqslant y \leqslant 1-z,\ 0 \leqslant z \leqslant 1\} \subset \Re^3, \tag{6}$$

$$Pr = \{(x,y,z) \in \Re^3 / 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1-x,\ -1 \leqslant z \leqslant 1\} \subset \Re^3, \tag{7}$$

$$H = \{(x,y,z) \in \Re^3 / -1 \leqslant x,y,z \leqslant +1\} \subset \Re^3. \tag{8}$$

2.2. Partitioning of the element topology

Each reference element is the union of open sets of points associated with the topology of the element and topologies of smaller dimension consisting of the "boundaries" of the element. For instance, a triangle is the union of its interior, three sets associated with a one-dimensional topology, and three points.

Each topology is called a *side* of an element. A line element has 3 sides, 2 endpoints and the set of interior points; a triangle element has 7 sides, a quadrilateral element 9 sides, a hexahedron element 27 sides, etc.

Table 1
Types of element and their identifiers.

Superscript	Element type	Dimension
$1d$	Line	1
tr	Triangle	2
q	Quadrilateral	2
te	Tetrahedron	3
py	Pyramid	3
pr	Prism	3
h	Hexahedron	3

Table 2
Functions and their associated subscripts.

Subscript	Function	Dimension
c	vertex	1,2,3
r	edge	1,2,3
f	face	2,3
v	volume	3

2.3. Nomenclature for the shape functions

The shape functions associated with each element is based on the following definitions:

- A unique polynomial of degree 1 is associated with each vertex of the element.[1]
- The definition of an extension function associated with each side (side blending function).
- The choice of a sequence of family of one-dimensional polynomials (Lagrange, Chebyshev, Jacobi).

The shape functions are associated with topological entities, or sides of the elements, that is, vertices, edges, faces, and interior volume. They will be denoted as vertex functions, edge functions, face functions, and volume functions. The following nomenclature is introduced: In Table 1 the superscript indicates the type of element. In Table 2 the subscript indicates the side associated with the shape function.

The integer p denotes the interpolation order of the element. The term *polynomial order* is preferred to the expression *polynomial degree* because the interpolation space is not necessarily of complete degree. The interpolation space associated with the pyramid can be used as an example since it contains rational terms.

The example $\Psi^{1d}_{r0(n)}$ denotes a shape function associated with the edge of an element. The edges are numbered from $r0$ to r_k. The index n indicates that edge r_n is the nth function associated with the edge. Another example, $\Psi^{q}_{f(n1,n2)}$, indicates that the function of the quadrilateral element is associated with its face. The indexes $n1$ and $n2$ indicate that two polynomials from the selected orthogonal sequence are used.

2.4. Orthogonal sequence: Chebyshev polynomials

It is possible to use any orthogonal sequence, such as of Legendre, Jacobi, Hermite, and Laguerre polynomials, to generate the shape functions following the presented strategy. Further samples

[1] Considering more than one shape function would amount to an approximation space similar to that obtained by the extended finite element method.

have used Chebyshev polynomials with a simple recurrence formula to get the polynomials f_n, as is given in the following:

$$f_n(\xi) : \xi \in [-1,1], \quad n = 0,1,2,3\ldots, \tag{9}$$
$$f_0 = 1,$$
$$f_1 = \xi,$$
$$f_n = 2\xi f_{n-1} - f_{n-2}\ n \geqslant 2.$$

Among others properties, these polynomials are complete and orthogonal in the interval [−1,1] with respect to the weight function $\sqrt{1-x^2}$. A more detailed study of the properties of these polynomials is found in [10].

3. General properties of shape functions

To obtain an approximation space of continuous functions the following properties are required:

(1) Each elementary shape function is associated with a single side.
(2) The number of shape functions associated with sides of non-null dimension is greater or equal to zero.
(3) A function associated with a side of dimension n is nonzero on this side and zero on all other sides of dimension $\leqslant n$.
(4) The functions of the element are associated with its sides.

These essential properties guarantee that the approximation space will be continuous.

The functions associated with vertices ψ_i correspond to the Lagrange functions. Their definition is such that their value is 1 on the vertex c_i with which they are associated and 0 on all other vertices of the element, as written here:

$$\psi_i(c_j) = \delta_{ij}, \quad \sum_{k=1}^{n} \psi_k = 1. \tag{10}$$

Properties 1 through 4 above are obtained by taking the product of the linear combination of two or more vertex functions. The obtained functions are commonly called blending functions. There is exactly one blending function associated with each side. Subsequently the degree of the polynomials is increased by multiplying the blending function with a polynomial extension function.

The choice of blending functions is not unique. A parametrized family of blending functions will be presented.

4. Continuity of the interpolation space

The continuity of functions associated with the sides of neighboring elements is obtained using the following concepts:

- Use of properties 1 through 4 (see Section 3) to construct shape functions.
- Transformations between the parametric spaces of the sides of neighboring elements.
- Use of the polynomials of the orthogonal sequence whose arguments are obtained by an affine transformation applied to the master element coordinates.

If a side A of an element is included in the closure of a higher-dimension side B, then a point in the parametric space of side A can be uniquely transformed in a point of side B, denoted T_{AB}. This transformation, being injective, has a left inverse T_{BA}^{-L}, such that $T_{BA}^{-L}T_{AB} = I_{AA}$.

A linear projection can be defined with the sides B and A, projecting the points of the interior of side B to points on side A. These projections, which are essential in the definition of the basis functions, will be documented for each element topology.

Finally, a coordinate system needs to be associated with element sides. When two neighboring elements share a side, they will also share the shape functions associated with this side and its coordinate system. The global interpolation space will be continuous only if the shape functions associated with a common side are continuous. This continuity is ensured by associating global axes with the topological space of the sides that the elements have in common. These global axes are uniquely determined by the order of the global identifiers associated with the nodes of the mesh, which correspond to the vertices of the elements. For an edge, the positive direction is that one that goes from the vertex with the smaller identifier to the vertex with the larger identifier. For a face two axes define a local coordinate system. The origin of the system corresponds to the vertex with smallest identifier of the face, vertex v_0. The first axis is defined by the vertex v_0 and the next smallest vertex of the face which is connected to the origin by an edge. The other axis is constructed by the vertices v_0 and the third lower vertex of the face connected to the origin by an edge. Similar procedures can be developed for volumes.

4.1. Line element

The reference element of the line, defined in Eq. (2), has two vertices $c0$ and $c1$ and one edge $r0$ joining them. The functions of vertices are widely known:

$$\psi_{c0}^{1d}(\xi) = \frac{1-\xi}{2}, \tag{11}$$
$$\psi_{c1}^{1d}(\xi) = \frac{1+\xi}{2}. \tag{12}$$

The edge functions are defined by

$$T_{r0} = T_r^{1d}(id_0, id_1), \tag{13}$$
$$\hat{\xi}_{r0} = T_{r0}(\xi), \tag{14}$$
$$\psi_{r0}^{bl} = \alpha^{1d}\psi_{c0}^{1d}(\xi)\psi_{c1}^{1d}(\xi), \tag{15}$$
$$\psi_{r0(n)}^{1d}(\xi) = \psi_{r0}^{bl}(\xi)f_n(\hat{\xi}_{r0}), \quad n = 0,1,\ldots,p-2. \tag{16}$$

Function T_{r0} is an affine transformation which transforms a point on the element side to a point on the correspondingly oriented side. In the current situation this transformation can be either $\hat{\xi} = \xi$ or $\hat{\xi} = -\xi$, depending on the order of the identifiers associated with the vertices. The blending function, in this case, corresponds to the product of the two functions of the vertex, Eq. (15). The coefficient α^{1d} is a scaling factor associated with one-dimensional edges.

4.2. Triangle element

The triangular reference element has seven sides, three vertices $c0, c1$, and $c2$, three edges $r0, r1$, and $r2$, and one face $f0$. The local enumeration and respective coordinates are given by 0 – (0,0), 1 – (1,0), and 2 – (0,1). Each of these sides have a particular geometry and associated parametric space.

The functions of the vertices correspond to the traditional shape functions,

$$\psi_{c0}^{tr}(\xi,\eta) = 1 - \xi - \eta, \tag{17}$$
$$\psi_{c1}^{tr}(\xi,\eta) = \xi, \tag{18}$$
$$\psi_{c2}^{tr}(\xi,\eta) = \eta. \tag{19}$$

The corresponding generating blending functions are

$$\psi_{r0}^{bl} = \alpha^{1d}\psi_{c0}^{tr}(\xi,\eta)\psi_{c1}^{tr}(\xi,\eta), \tag{20}$$
$$\psi_{r1}^{bl} = \alpha^{1d}\psi_{c1}^{tr}(\xi,\eta)\psi_{c2}^{tr}(\xi,\eta), \tag{21}$$
$$\psi_{r2}^{bl} = \alpha^{1d}\psi_{c2}^{tr}(\xi,\eta)\psi_{c0}^{tr}(\xi,\eta), \tag{22}$$
$$\psi_{f0}^{bl} = \alpha^{tr}\psi_{c0}^{tr}(\xi,\eta)\psi_{c1}^{tr}(\xi,\eta)\psi_{c2}^{tr}(\xi,\eta). \tag{23}$$

The transformations between edges and face parameters are defined as

$$T_{r0} = T^{1d}(id_0, id_1), \tag{24}$$
$$T_{r1} = T^{1d}(id_1, id_2), \tag{25}$$
$$T_{r2} = T^{1d}(id_2, id_0), \tag{26}$$
$$T_{f0} = T^{tr}(id_0, id_1, id_2), \tag{27}$$

where the transformation T_{f0} is defined in Section 5.

The projections of the interior points to the edges and face are defined as $\xi_{r0} = \eta + 2\xi - 1, \xi_{r1} = \eta - \xi, \xi_{r2} = 1 - \xi - 2\eta$, and $(\xi_{f0}, \eta_{f0}) = (\xi, \eta)$.

The shape functions of the triangle edges are presented in Eqs. (28)–(30):

$$\psi^{tr}_{r0(n)}(\xi,\eta) = \psi^{bl}_{r0}(\xi,\eta) f_n(T_{r0}(\xi_{r0})), \tag{28}$$
$$\psi^{tr}_{r1(n)}(\xi,\eta) = \psi^{bl}_{r1}(\xi,\eta) f_n(T_{r1}(\xi_{r1})), \tag{29}$$
$$\psi^{tr}_{r2(n)}(\xi,\eta) = \psi^{bl}_{r2}(\xi,\eta) f_n(T_{r2}(\xi_{r2})), \quad n = 0, 1, \ldots, p-2, \tag{30}$$
$$(\hat{\xi}_{f0}, \hat{\eta}_{f0}) = T_{f0}(\xi,\eta), \tag{31}$$
$$\psi^{tr}_{f0(n1,n2)}(\xi,\eta) = \psi^{bl}_{f0}(\xi,\eta) f_{n1}(2\hat{\xi}_{f0} - 1) f_{n2}(2\hat{\eta}_{f0} - 1), \quad 0 \leqslant n1 + n2 \leqslant p - 3. \tag{32}$$

The transformation T_{f0} is a homogeneous transformation. The parameters $(\hat{\xi}_{f0}, \hat{\eta}_{f0})$ are the projections of an interior point to the coordinate system associated with the side $f0$.

4.3. Quadrilateral element

The quadrilateral element has four vertices $c0, c1, c2$, and $c3$, four edges $r0$, $r1$, $r2$, and $r3$, and one face $f0$. The parametric space of the quadrilateral element is $[-1, 1]^2$. The functions of the vertices are bilinear and are given in Eqs. (33)–(36).

The vertices are enumerated counterclockwise:

$$\psi^q_{c0}(\xi,\eta) = \psi^{1d}_{c0}(\xi)\psi^{1d}_{c0}(\eta), \tag{33}$$
$$\psi^q_{c1}(\xi,\eta) = \psi^{1d}_{c1}(\xi)\psi^{1d}_{c0}(\eta), \tag{34}$$
$$\psi^q_{c2}(\xi,\eta) = \psi^{1d}_{c1}(\xi)\psi^{1d}_{c1}(\eta), \tag{35}$$
$$\psi^q_{c3}(\xi,\eta) = \psi^{1d}_{c0}(\xi)\psi^{1d}_{c1}(\eta). \tag{36}$$

The corresponding generating blending functions are

$$\psi^{bl}_{r0} = \alpha^{1d}(\psi^q_{c0}(\xi,\eta) + \beta^q\psi^q_{c3}(\xi,\eta))\psi^q_{c1}(\xi,\eta), \tag{37}$$
$$\psi^{bl}_{r1} = \alpha^{1d}(\psi^q_{c1}(\xi,\eta) + \beta^q\psi^q_{c0}(\xi,\eta))\psi^q_{c2}(\xi,\eta), \tag{38}$$
$$\psi^{bl}_{r2} = \alpha^{1d}(\psi^q_{c2}(\xi,\eta) + \beta^q\psi^q_{c1}(\xi,\eta))\psi^q_{c3}(\xi,\eta), \tag{39}$$
$$\psi^{bl}_{r3} = \alpha^{1d}(\psi^q_{c3}(\xi,\eta) + \beta^q\psi^q_{c2}(\xi,\eta))\psi^q_{c0}(\xi,\eta), \tag{40}$$
$$\psi^{bl}_{f0} = \alpha^q\psi^q_{c0}(\xi,\eta)\psi^q_{c2}(\xi,\eta). \tag{41}$$

The constant β^q varies between 0 and 1 and determines the extension of the blending functions of the edges to the interior of the element. When $\beta^q = 1$, the extension of the edge functions to the interior of the element is linear. The constant α^q defines a scaling factor for the quadratic bubble functions.

The transformations between the edge and face parameters are defined as $T_{r0} = T^{1d}(id_0, id_1), T_{r1} = T^{1d}(id_1, id_2)$, $T_{r2} = T^{1d}(id_2, id_3)$, $T_{r_3} = T^{1d}(id_3, id_0)$, and $T_{f0} = T^q(id_0, id_1, id_2, id_3)$, where the transformation T_{f0} is defined in Section 5.

The projections of an interior point to the parameter space associated with the edges and face are

$\xi_{r0} = \xi, \xi_{r1} = \eta, \xi_{r2} = -\xi, \xi_{r_3} = -\eta$, and $(\xi_{f0}, \eta_{f0}) = (\xi, \eta)$.

The resulting edge functions are

$$\psi^q_{r0(n)}(\xi,\eta) = \psi^{bl}_{r0}(\xi,\eta) f_n(T_{r0}(\xi_{r0})), \tag{42}$$
$$\psi^q_{r1(n)}(\xi,\eta) = \psi^{bl}_{r1}(\xi,\eta) f_n(T_{r1}(\xi_{r1})), \tag{43}$$
$$\psi^q_{r2(n)}(\xi,\eta) = \psi^{bl}_{r2}(\xi,\eta) f_n(T_{r2}(\xi_{r2})), \tag{44}$$
$$\psi^q_{r3(n)}(\xi,\eta) = \psi^{bl}_{r_3}(\xi,\eta) f_n(T_{r_3}(\xi_{r3})), \quad n = 0, 1, \ldots, p-2. \tag{45}$$

The face functions are defined as

$$(\hat{\xi}_{f0}, \hat{\eta}_{f0}) = T_{f0}(\xi_{f0}, \eta_{f0}), \tag{46}$$
$$\psi^q_{f0(n1,n2)}(\xi,\eta) = \psi^{bl}_{f0} f_{n1}(\hat{\xi}_{f0}) f_{n2}(\hat{\eta}_{f0}), \quad 0 \leqslant n1, n2 \leqslant p - 2 \tag{47}$$

The number of functions of the quadrilateral of order p is $(p+1)^2$, that is, the four vertex functions plus the $p-1$ functions for each edge and the $(p-1)^2$ face functions.

4.4. Tetrahedron element

The geometric tetrahedron has four vertices, six edges, and four faces. The parametric space of the tetrahedron element is defined in Eq. (5). The local orientation of the entities of the tetrahedron is sketched in Fig. 1.

The local enumeration of the entities of the tetrahedron determines the local coordinate systems and the projections of interior points to the faces and edges.

The vertex functions are calculated using the following equations:

$$\psi^{te}_{c0}(\xi,\eta,\zeta) = 1 - \xi - \eta - \zeta, \tag{48}$$
$$\psi^{te}_{c1}(\xi,\eta,\zeta) = \xi, \tag{49}$$
$$\psi^{te}_{c2}(\xi,\eta,\zeta) = \eta, \tag{50}$$
$$\psi^{te}_{c2}(\xi,\eta,\zeta) = \zeta. \tag{51}$$

The corresponding generating blending functions are

$$\psi^{bl}_{r0} = \alpha^{1d}\psi^{te}_{c0}(\xi,\eta)\psi^{te}_{c1}(\xi,\eta), \tag{52}$$
$$\psi^{bl}_{r1} = \alpha^{1d}\psi^{te}_{c1}(\xi,\eta)\psi^{te}_{c2}(\xi,\eta), \tag{53}$$
$$\psi^{bl}_{r2} = \alpha^{1d}\psi^{te}_{c2}(\xi,\eta)\psi^{te}_{c0}(\xi,\eta), \tag{54}$$

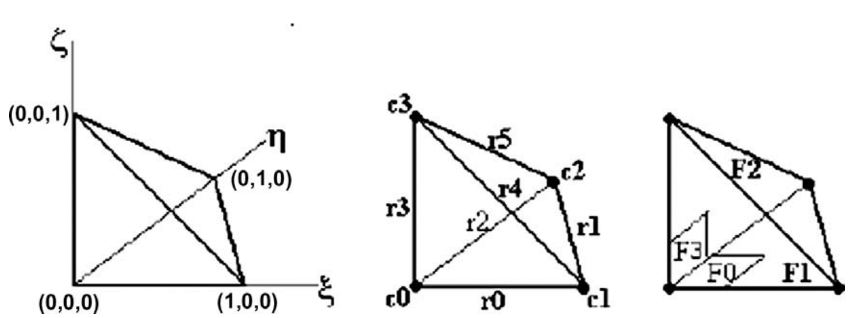

Fig. 1. Geometry of the reference tetrahedron.

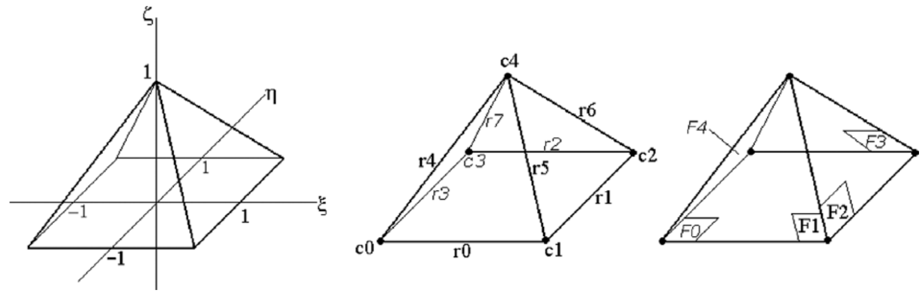

Fig. 2. Parametric space of the pyramid element.

$$\psi^{bl}_{r3}=\alpha^{1d}\psi^{te}_{c3}(\xi,\eta)\psi^{te}_{c0}(\xi,\eta), \tag{55}$$

$$\psi^{bl}_{r4}=\alpha^{1d}\psi^{te}_{c1}(\xi,\eta)\psi^{te}_{c3}(\xi,\eta), \tag{56}$$

$$\psi^{bl}_{r5}=\alpha^{1d}\psi^{te}_{c2}(\xi,\eta)\psi^{te}_{c3}(\xi,\eta), \tag{57}$$

$$\psi^{bl}_{f0}=\alpha^{tr}\psi^{te}_{c0}(\xi,\eta)\psi^{te}_{c1}(\xi,\eta)\psi^{te}_{c2}(\xi,\eta), \tag{58}$$

$$\psi^{bl}_{f1}=\alpha^{tr}\psi^{te}_{c0}(\xi,\eta)\psi^{te}_{c1}(\xi,\eta)\psi^{te}_{c3}(\xi,\eta), \tag{59}$$

$$\psi^{bl}_{f2}=\alpha^{tr}\psi^{te}_{c1}(\xi,\eta)\psi^{te}_{c2}(\xi,\eta)\psi^{te}_{c3}(\xi,\eta), \tag{60}$$

$$\psi^{bl}_{f3}=\alpha^{tr}\psi^{te}_{c0}(\xi,\eta)\psi^{te}_{c2}(\xi,\eta)\psi^{te}_{c3}(\xi,\eta), \tag{61}$$

$$\psi^{bl}_{v0}=\alpha^{te}\psi^{te}_{c0}(\xi,\eta)\psi^{te}_{c1}(\xi,\eta)\psi^{te}_{c2}(\xi,\eta)\psi^{te}_{c3}(\xi,\eta). \tag{62}$$

The transformations between the edge and face parameters are defined as $T_{r0}=T^{1d}(id_0,id_1),\ T_{r1}=T^{1d}(id_1,id_2),\ T_{r2}=T^{1d}(id_2,id_0),\ T_{r_3}=T^{1d}(id_3,id_0),\ T_{r_4}=T^{1d}(id_1,id_3),\ T_{r_5}=T^{1d}(id_2,id_3),\ T_{f0}=T^{tr}(id_0,id_1,id_2),\ T_{f_1}=T^{tr}(id_0,id_1,id_3),\ T_{f_2}=T^{tr}(id_1,id_2,id_3)$, and $T_{f_3}=T^{tr}(id_0,id_2,id_3)$, where the affine transformations T^{1d} and T^{tr} are defined in Section 5.

The projections of the interior points to the edges and face are $\xi_{r0}=2\xi+\eta+\zeta-1, \xi_{r1}=-\xi+\eta, \xi_{r2}=-\xi-2\eta-\zeta+1, \xi_{r_3}=\xi+\eta+2\zeta-1, \xi_{r_4}=-\xi+\zeta, \xi_{r_5}=-\eta+\zeta, (\xi_{f0},\eta_{f0})=(\xi,\eta), (\xi_{f_1},\eta_{f_1})=(\xi,\zeta), (\xi_{f_2},\eta_{f_2})=(-\frac{1}{3}\xi+\frac{2}{3}\eta-\frac{1}{3}\zeta+\frac{1}{3},-\frac{2}{3}\xi-\frac{2}{3}\eta+\frac{2}{3}\zeta+\frac{1}{3})$, and $(\xi_{f_3},\eta_{f_3})=(\eta,\zeta)$.

The edge and face functions are defined as

$$\hat{\xi}_{r_i}=T_{r_i}(\xi_{r_i}), \tag{63}$$

$$(\hat{\xi}_{f_i},\hat{\eta}_{f_i})=T_{f_i}(\xi_{f_i},\eta_{f_i}), \tag{64}$$

$$\psi^{te}_{ri(n)}(\xi,\eta)=\psi^{bl}_{r_i}(\xi,\eta)f_n(2\hat{\xi}_{r_i}-1),\quad 0\leqslant i\leqslant 6, \tag{65}$$

$$\psi^{te}_{f_i(n1,n2)}(\xi,\eta,\zeta)=\psi^{bl}_{f_i}(\xi,\eta,\zeta)f_{n1}(2\hat{\xi}_{f_i}-1)f_{n2}(2\hat{\eta}_{f_i}-1),\quad 0\leqslant i\leqslant 3. \tag{66}$$

The volume functions of the tetrahedron are given by

$$\psi^{te}_{v_0(n1,n2,n3)}(\xi,\eta,\zeta)=\psi^{bl}_{v_0}(\xi,\eta,\zeta)f_{n1}(2\xi-1)f_{n2}(2\eta-1)f_{n3}(2\zeta-1), \tag{67}$$

where $n1+n2+n3\leqslant p-3$.

The interpolation space of the hierarchical tetrahedron of order p is determined by four vertex functions, $p-1$ edge functions, $\frac{(p-2)(p-1)}{2}$ face functions, and $\sum_{i=0}^{p-4}\frac{(i+1)(i+2)}{2}$ functions associated with the interior. The total number of shape functions is $\sum_{i=0}^{p}\frac{(i+1)(i+2)}{2}$.

4.5. Pyramid element

The parametric space of the pyramid and the definition of its vertices, Edges, and faces are shown in Fig. 2. Fig. 2 also presents the local enumeration that defines the parametric spaces of the sides.

The vertex functions must be such that they are linear on each edge and compatible with the functions of neighboring elements (as well as the vertex). The defined functions satisfy these properties but are not linear on the element. Two auxiliary functions are defined:

$$T_0(c,t)=\frac{(1-t)-c}{2(1-t)}, \tag{68}$$

$$T_1(c,t)=\frac{(1-t)+c}{2(1-t)}. \tag{69}$$

Then the vertex functions are given by

$$\psi^{pi}_{c0}(\xi,\eta,\zeta)=T_0(\xi,\zeta)T_0(\eta,\zeta)(1-\zeta), \tag{70}$$

$$\psi^{pi}_{c1}(\xi,\eta,\zeta)=T_1(\xi,\zeta)T_0(\eta,\zeta)(1-\zeta), \tag{71}$$

$$\psi^{pi}_{c2}(\xi,\eta,\zeta)=T_1(\xi,\zeta)T_1(\eta,\zeta)(1-\zeta), \tag{72}$$

$$\psi^{pi}_{c3}(\xi,\eta,\zeta)=T_0(\xi,\zeta)T_1(\eta,\zeta)(1-\zeta), \tag{73}$$

$$\psi^{pi}_{c4}(\xi,\eta,\zeta)=\zeta. \tag{74}$$

The functions are bilinear when restricted to horizontal planes. The functions are linear along lines that pass through the superior vertex of the pyramid, in particular along the edges that include the vertex $c4$. The edge functions are defined by Eqs. (74)–(81).

The corresponding generating blending functions are

$$\psi^{bl}_{r0}=\alpha^{1d}(\psi^{pi}_{c0}(\xi,\eta,\zeta)+\beta^{q}\psi^{pi}_{c3}(\xi,\eta,\zeta))\psi^{pi}_{c1}(\xi,\eta,\zeta), \tag{75}$$

$$\psi^{bl}_{r1}=\alpha^{1d}(\psi^{pi}_{c1}(\xi,\eta,\zeta)+\beta^{q}\psi^{pi}_{c0}(\xi,\eta,\zeta))\psi^{pi}_{c2}(\xi,\eta,\zeta), \tag{76}$$

$$\psi^{bl}_{r2}=\alpha^{1d}(\psi^{pi}_{c2}(\xi,\eta,\zeta)+\beta^{q}\psi^{pi}_{c1}(\xi,\eta,\zeta))\psi^{pi}_{c3}(\xi,\eta,\zeta), \tag{77}$$

$$\psi^{bl}_{r3}=\alpha^{1d}(\psi^{pi}_{c3}(\xi,\eta,\zeta)+\beta^{q}\psi^{pi}_{c2}(\xi,\eta,\zeta))\psi^{pi}_{c0}(\xi,\eta,\zeta), \tag{78}$$

$$\psi^{bl}_{r4}=\alpha^{1d}\psi^{pi}_{c0}(\xi,\eta,\zeta)\psi^{pi}_{c4}(\xi,\eta,\zeta), \tag{79}$$

$$\psi^{bl}_{r5}=\alpha^{1d}\psi^{pi}_{c1}(\xi,\eta,\zeta)\psi^{pi}_{c4}(\xi,\eta,\zeta), \tag{80}$$

$$\psi^{bl}_{r6}=\alpha^{1d}\psi^{pi}_{c2}(\xi,\eta,\zeta)\psi^{pi}_{c4}(\xi,\eta,\zeta), \tag{81}$$

$$\psi^{bl}_{r7}=\alpha^{1d}\psi^{pi}_{c3}(\xi,\eta,\zeta)\psi^{pi}_{c4}(\xi,\eta,\zeta), \tag{82}$$

$$\psi^{bl}_{f0}=\alpha^{q}\psi^{pi}_{c0}(\xi,\eta,\zeta)\psi^{pi}_{c2}(\xi,\eta,\zeta), \tag{83}$$

$$\psi^{bl}_{f1}=\alpha^{tr}\psi^{pi}_{c0}(\xi,\eta,\zeta)\psi^{pi}_{c1}(\xi,\eta,\zeta)\psi^{pi}_{c4}(\xi,\eta,\zeta), \tag{84}$$

$$\psi^{bl}_{f2}=\alpha^{tr}\psi^{pi}_{c1}(\xi,\eta,\zeta)\psi^{pi}_{c2}(\xi,\eta,\zeta)\psi^{pi}_{c4}(\xi,\eta,\zeta), \tag{85}$$

$$\psi^{bl}_{f3}=\alpha^{tr}\psi^{pi}_{c3}(\xi,\eta,\zeta)\psi^{pi}_{c2}(\xi,\eta,\zeta)\psi^{pi}_{c4}(\xi,\eta,\zeta), \tag{86}$$

$$\psi^{bl}_{f3}=\alpha^{tr}\psi^{pi}_{c0}(\xi,\eta,\zeta)\psi^{pi}_{c3}(\xi,\eta,\zeta)\psi^{pi}_{c4}(\xi,\eta,\zeta), \tag{87}$$

$$\psi^{bl}_{v0}=\alpha^{pi}\psi^{pi}_{c0}(\xi,\eta,\zeta)\psi^{pi}_{c2}(\xi,\eta,\zeta)\psi^{pi}_{c4}(\xi,\eta,\zeta). \tag{88}$$

The vertex enumeration which determines the transformations between the edge and face parameters are defined as $T^{1d}(id_0,id_1)$, $T^{1d}(id_1,id_2)$, $T^{1d}(id_2,id_3)$, $T^{1d}(id_3,id_0)$, $T^{1d}(id_0,id_4)$, $T^{1d}(id_1,id_4)$, $T^{1d}(id_2,id_4)$, $T^{1d}(id_3,id_4)$, $T^{q}(id_0,id_1,id_2,id_3)$, $T^{tr}(id_0,id_1,id_4)$, $T^{tr}(id_1,id_2,id_4)$, $T^{tr}(id_3,id_2,id_4)$, and $T^{tr}(id_3,id_0,id_4)$, where the transformations for the faces are defined in Section 5.

The projections of the interior points to the edges and faces are $\xi_{r0}=\xi,\xi_{r1}=\eta,\xi_{r2}=-\xi,\xi_{r_3}=-\eta,\xi_{r_4}=\frac{1}{2}\xi+\frac{1}{2}\eta+\zeta,\xi_{r_5}=-\frac{1}{2}\xi+\frac{1}{2}\eta+\zeta,\xi_{r_6}=-\frac{1}{2}\xi-\frac{1}{2}\eta+\zeta,\xi_{r_5}=\frac{1}{2}\xi-\frac{1}{2}\eta+\zeta,(\xi_{f0},\eta_{f0})=(\xi,\eta),(\xi_{f_1},\eta_{f_1})=(\frac{1}{2}\xi-\frac{1}{4}\eta-\frac{1}{4}\zeta+\frac{1}{4},\frac{1}{2}\eta+\frac{1}{2}\zeta+\frac{1}{2}),(\xi_{f_2},\eta_{f_2})=(\frac{1}{4}\xi+\frac{1}{2}\eta-\frac{1}{4}\zeta+\frac{1}{4},-\frac{1}{2}\xi+$

$\frac{1}{2}\zeta+\frac{1}{2}), (\xi_{f_3},\eta_{f_3}) = (\frac{1}{2}\xi+\frac{1}{4}\eta-\frac{1}{4}\zeta+\frac{1}{4}, -\frac{1}{2}\eta+\frac{1}{2}\zeta+\frac{1}{2})$ and $(\xi_{f_4},\eta_{f_4}) = (-\frac{1}{4}\xi+\frac{1}{2}\eta-\frac{1}{4}\zeta+\frac{1}{4}, \frac{1}{2}\xi+\frac{1}{2}\zeta+\frac{1}{2})$.

Fig. 3 presents the direction of the projections from the interior points to the edges.

The edge and shape functions are defined as

$$\hat{\xi}_{r_i} = T_{r_i}(\xi_{r_i}), \tag{89}$$

$$(\hat{\xi}_{f_i}, \hat{\eta}_{f_i}) = T_{f_i}(\xi_{f_i}, \eta_{f_i}), \tag{90}$$

$$n\psi^{pi}_{ri}(\xi,\eta,\zeta) = \psi^{bl}_{r_i}(\xi,\eta,\zeta) f_n(\hat{\xi}_{r_i}), \quad 0 \leqslant i \leqslant 7,\ n = 0,1,\ldots,p-2, \tag{91}$$

$$\psi^{pi}_{f0(n1,n2)}(\xi,\eta,\zeta) = \psi^{bl}_{f0}(\xi,\eta,\zeta) f_{n1}(\hat{\xi}_{f0}) f_{n2}(\hat{\eta}_{f0}), \quad 0 \leqslant n1+n2 \leqslant p-2, \tag{92}$$

$$\psi^{pi}_{f1_i(n1,n2)}(\xi,\eta,\zeta) = \psi^{bl}_{f_i}(\xi,\eta,\zeta) f_{n1}(\hat{\xi}_{f_i}) f_{n2}(\hat{\eta}_{f_i}), \quad 1 \leqslant i \leqslant 4\ \ 0 \leqslant n1+n2 \leqslant p-3 \tag{93}$$

The volume functions are

$$\psi^{pi}_{v0(n1,n2,n3)}(\xi,\eta,\zeta) = \psi^{bl}_{v_0}(\xi,\eta,\zeta) f_{n1}(\xi) f_{n2}(\eta) f_{n3}(2\zeta-1), \quad 0 \leqslant n1+n2+n3 \leqslant p-3 \tag{94}$$

The hierarchical pyramid of order p has five functions associated with the vertices, $(p-1)$ edge functions, $\frac{(p-1)p}{2}$ face functions for the quadrilateral face, $\frac{(p-2)(p-1)}{2}$ face functions for each triangular face, and $\sum_{i=0}^{p-3}\frac{(i+1)(i+2)}{2}$ volume functions. The total number of functions is $\sum_{i=0}^{p}\frac{(i+1)(i+2)}{2}+p^2$. In [1] it was demonstrated that the pyramid of order p contains the complete set of polynomials degree p in three dimensions. This element also contains exactly p^2 rational functions.

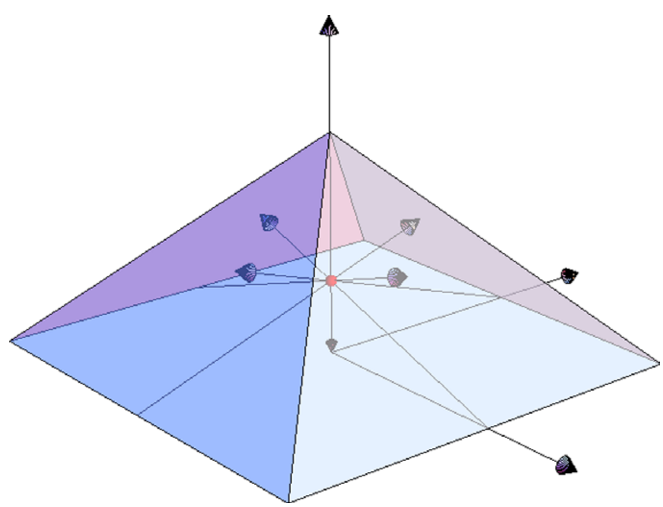

Fig. 3. Direction of the projections of interior points to edges.

4.6. Prism element

The parametric space and the reference element of the prism are shown in Fig. 4a. The vertex functions are given in Eqs. (94)–(99). These functions are obtained by the tensor product of the shape functions of the triangular element of the plane (ξ,η) with the shape functions of the linear element in the variable ζ:

$$\psi^{pr}_{c0}(\xi,\eta,\zeta) = \psi^{1d}_{c0}(\xi,\eta)\psi^{1d}_{c0}(\zeta) = \frac{1}{2}(1-\xi-\eta)(1-\zeta), \tag{95}$$

$$\psi^{pr}_{c1}(\xi,\eta,\zeta) = \psi^{1d}_{c1}(\xi,\eta)\psi^{1d}_{c0}(\zeta) = \frac{1}{2}\xi(1-\zeta), \tag{96}$$

$$\psi^{pr}_{c2}(\xi,\eta,\zeta) = \psi^{1d}_{c2}(\xi,\eta)\psi^{1d}_{c0}(\zeta) = \frac{1}{2}\eta(1-\zeta), \tag{97}$$

$$\psi^{pr}_{c3}(\xi,\eta,\zeta) = \psi^{1d}_{c0}(\xi,\eta)\psi^{1d}_{c1}(\zeta) = \frac{1}{2}(1-\xi-\eta)(1+\zeta), \tag{98}$$

$$\psi^{pr}_{c4}(\xi,\eta,\zeta) = \psi^{1d}_{c1}(\xi,\eta)\psi^{1d}_{c1}(\zeta) = \frac{1}{2}\xi(1+\zeta), \tag{99}$$

$$\psi^{pr}_{c5}(\xi,\eta,\zeta) = \psi^{1d}_{c2}(\xi,\eta)\psi^{1d}_{c1}(\zeta) = \frac{1}{2}\eta(1+\zeta). \tag{100}$$

The corresponding generating blending functions are

$$\psi^{bl}_{r0} = \alpha^{1d}(\psi^{pr}_{c0}(\xi,\eta,\zeta) + \beta^q\psi^{pr}_{c3}(\xi,\eta,\zeta))\psi^{pi}_{c1}(\xi,\eta,\zeta), \tag{101}$$

$$\psi^{bl}_{r1} = \alpha^{1d}(\psi^{pr}_{c1}(\xi,\eta,\zeta) + \beta^q\psi^{pr}_{c4}(\xi,\eta,\zeta))\psi^{pi}_{c2}(\xi,\eta,\zeta), \tag{102}$$

$$\psi^{bl}_{r2} = \alpha^{1d}(\psi^{pr}_{c0}(\xi,\eta,\zeta) + \beta^q\psi^{pr}_{c3}(\xi,\eta,\zeta))\psi^{pr}_{c2}(\xi,\eta,\zeta), \tag{103}$$

$$\psi^{bl}_{r3} = \alpha^{1d}(\psi^{pr}_{c0}(\xi,\eta,\zeta)\psi^{pr}_{c3}(\xi,\eta,\zeta) + \beta^q(\psi^{pr}_{c0}(\xi,\eta,\zeta)\psi^{pr}_{c5}(\xi,\eta,\zeta) + \psi^{pr}_{c1}(\xi,\eta,\zeta)\psi^{pr}_{c3}(\xi,\eta,\zeta))), \tag{104}$$

$$\psi^{bl}_{r4} = \alpha^{1d}(\psi^{pr}_{c1}(\xi,\eta,\zeta)\psi^{pr}_{c4}(\xi,\eta,\zeta) + \beta^q(\psi^{pr}_{c2}(\xi,\eta,\zeta)\psi^{pr}_{c4}(\xi,\eta,\zeta) + \psi^{pr}_{c1}(\xi,\eta,\zeta)\psi^{pr}_{c3}(\xi,\eta,\zeta))), \tag{105}$$

$$\psi^{bl}_{r5} = \alpha^{1d}(\psi^{pr}_{c2}(\xi,\eta,\zeta)\psi^{pr}_{c5}(\xi,\eta,\zeta) + \beta^q(\psi^{pr}_{c2}(\xi,\eta,\zeta)\psi^{pr}_{c4}(\xi,\eta,\zeta) + \psi^{pr}_{c0}(\xi,\eta,\zeta)\psi^{pr}_{c5}(\xi,\eta,\zeta))), \tag{106}$$

$$\psi^{bl}_{r6} = \alpha^{1d}(\psi^{pr}_{c3}(\xi,\eta,\zeta)\psi^{pr}_{c4}(\xi,\eta,\zeta) + \beta^q\psi^{pr}_{c3}(\xi,\eta,\zeta)\psi^{pr}_{c1}(\xi,\eta,\zeta)), \tag{107}$$

$$\psi^{bl}_{r7} = \alpha^{1d}(\psi^{pr}_{c4}(\xi,\eta,\zeta)\psi^{pr}_{c5}(\xi,\eta,\zeta) + \beta^q\psi^{pr}_{c4}(\xi,\eta,\zeta)\psi^{pr}_{c2}(\xi,\eta,\zeta)), \tag{108}$$

$$\psi^{bl}_{r8} = \alpha^{1d}(\psi^{pr}_{c5}(\xi,\eta,\zeta)\psi^{pr}_{c3}(\xi,\eta,\zeta) + \beta^q\psi^{pr}_{c5}(\xi,\eta,\zeta)\psi^{pr}_{c0}(\xi,\eta,\zeta)), \tag{109}$$

$$\psi^{bl}_{f0} = \alpha^{tr}((\psi^{pr}_{c0}(\xi,\eta,\zeta) + \beta^q\psi^{pr}_{c3}(\xi,\eta,\zeta))(\psi^{pr}_{c1}(\xi,\eta,\zeta) + \beta^q\psi^{pr}_{c4}(\xi,\eta,\zeta))\psi^{pr}_{c2}(\xi,\eta,\zeta)), \tag{110}$$

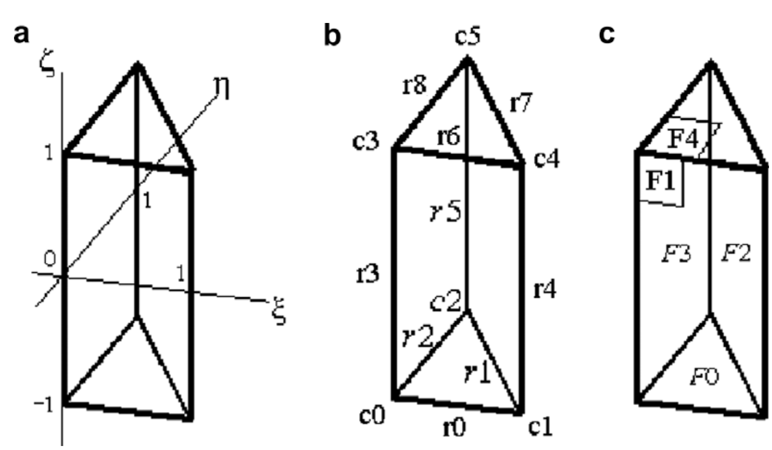

Fig. 4. Definition of the geometry and entities of the prism.

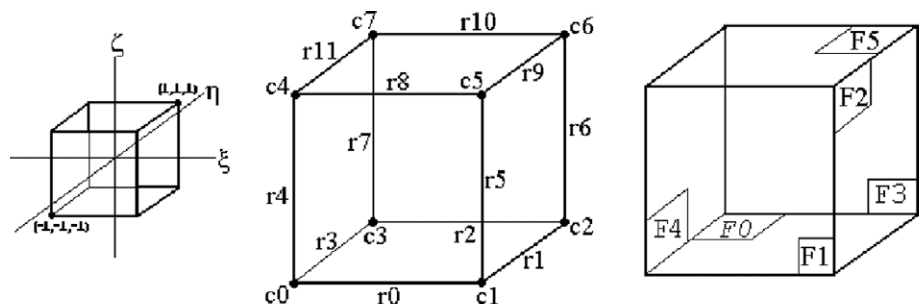

Fig. 5. Geometric configuration of the reference hexahedron.

$$\psi^{bl}_{f1} = \alpha^q \psi^{pr}_{c0}(\xi,\eta,\zeta)\psi^{pr}_{c4}(\xi,\eta,\zeta), \tag{111}$$
$$\psi^{bl}_{f2} = \alpha^q \psi^{pi}_{c1}(\xi,\eta,\zeta)\psi^{pi}_{c5}(\xi,\eta,\zeta), \tag{112}$$
$$\psi^{bl}_{f3} = \alpha^q \psi^{pi}_{c2}(\xi,\eta,\zeta)\psi^{pi}_{c3}(\xi,\eta,\zeta), \tag{113}$$
$$\psi^{bl}_{f4} = \alpha^{tr}((\psi^{pr}_{c3}(\xi,\eta,\zeta) + \beta^q \psi^{pr}_{c0}(\xi,\eta,\zeta))(\psi^{pr}_{c4}(\xi,\eta,\zeta) + \beta^q \psi^{pr}_{c1}(\xi,\eta,\zeta))\psi^{pr}_{c5}(\xi,\eta,\zeta)), \tag{114}$$
$$\psi^{bl}_{v0} = \alpha^{pr}(\psi^{pr}_{c0}(\xi,\eta,\zeta)\psi^{pr}_{c4}(\xi,\eta,\zeta)(\psi^{pr}_{c2}(\xi,\eta,\zeta) + \psi^{pr}_{c5}(\xi,\eta,\zeta))). \tag{115}$$

The identifiers of the edges and faces which subsequently define the transformations between edge and face parameters are defined as $r0(id_0,id_1), r1(id_1,id_2), r2(id_2,id_0), r_3(id_0,id_3), r_4(id_1,id_4),\ r_5(id_2,id_5), r_6(id_3,id_4), r_7(id_4,id_5), r_8(id_5,id_3), f0(id_0,\ id_1,id_2), f_1(id_0,id_1,id_4,id_3), f_2(id_1,id_2,id_5,id_4), f_3(id_0,id_2,id_5,id_3)$, and $f_4(id_3,id_4,id_5)$.

The projections of interior points to edges and faces are:

$\xi_{r0} = 2\xi+\eta-1, \xi_{r1} = -\xi+\eta, \xi_{r2} = -\xi+2\eta+1, \xi_{r_3} = \zeta, \xi_{r_4} = \zeta, \xi_{r_5} = \ \zeta, \xi_{r_6} = 2\xi+\eta-1, \xi_{r_7} = -\xi+\eta, \xi_{r_8} = -\xi+2\eta+1, (\xi_{f0},\eta_{f0}) = (\xi,\eta), (\xi_{f_1},\eta_{f_1}) = (2\xi-1,\zeta), (\xi_{f_2},\eta_{f_2}) = (-\xi+\eta-1,\zeta), (\xi_{f_3},\eta_{f_3}) = (2\eta-1,\zeta)$, and $(\xi_{f_4},\eta_{f_4}) = (\xi,\eta)$.

The edge and face functions are

$$\hat{\xi}_{r_i} = T_{r_i}(\xi_{r_i}), \tag{116}$$
$$(\hat{\xi}_{f_i},\hat{\eta}_{f_i}) = T_{f_i}(\xi_{f_i},\eta_{f_i}). \tag{117}$$

The edge functions are written as

$$\psi^{pr}_{ri}(\xi,\eta,\zeta) = \psi^{bl}_{r_i}(\xi,\eta,\zeta)f_n(\hat{\xi}_{r_i}), \quad 0 \leqslant i \leqslant 8, \quad n = 0,1,\ldots,p-2, \tag{118}$$

$$\psi^{pr}_{f0}(\xi,\eta,\zeta) = \psi^{bl}_{f0}(\xi,\eta,\zeta)f_{n1}(2\hat{\xi}_{f0}-1)f_{n2}(2\hat{\eta}_{f0}-1), \quad 0 \leqslant n1+n2 \leqslant p-3, \tag{119}$$

$$\psi^{pr}_{f1}(\xi,\eta,\zeta) = \psi^{bl}_{f0}(\xi,\eta,\zeta)f_{n1}(\hat{\xi}_{f_1})f_{n2}(\hat{\eta}_{f_1}), \tag{120}$$
$$\psi^{pr}_{f2}(\xi,\eta,\zeta) = \psi^{bl}_{f_2}(\xi,\eta,\zeta)f_{n1}(\hat{\xi}_{f_2})f_{n2}(\hat{\eta}_{f_2}), \tag{121}$$

$$\psi^{pr}_{f3}(\xi,\eta,\zeta) = \psi^{bl}_{f_3}(\xi,\eta,\zeta)f_{n1}(\hat{\xi}_{f_3})f_{n2}(\hat{\eta}_{f_3}), \quad 0 \leqslant n1,n2 \leqslant p-2, \tag{122}$$

$$\psi^{pr}_{f4}(\xi,\eta,\zeta) = \psi^{bl}_{f_4}(\xi,\eta,\zeta)f_{n1}(2\hat{\xi}_{f_3}-1)f_{n2}(2\hat{\eta}_{f_3}-1), \quad 0 \leqslant n1+n2 \leqslant p-3. \tag{123}$$

The volume functions are

$$\psi^{pr}_{v0}(\xi,\eta,\zeta) = \psi^{bl}_{v0}(\xi,\eta,\zeta)f_{n1}(2\xi-1)f_{n2}(2\eta-1)f_{n3}(\zeta), \quad 0 \leqslant n1+n2 \leqslant p-3,\ 0 \leqslant n3 \leqslant p-2. \tag{124}$$

The prism of order p has six vertex functions, $p-1$ functions for each edge, $\frac{(p-2)(p-1)}{2}$ functions for each triangular face, $(p-1)^2$ for each quadrilateral face, and $\frac{(p-2)(p-1)}{2}(p-1)$ functions for the volume. The total number of functions is $\frac{(p+12)^2(p+1)}{2}$.

4.7. Hexahedron element

The parametric space of the hexahedron element is $[-1,1]^3$, as shown in Fig. 5. The vertices are numbered from 0 to 7, the edges are numbered from 8 to 19, and the faces are numbered from 20 to 25. Identifier 26 is associated with the interior volume.

The vertex functions are

$$\psi^h_{c0}(\xi,\eta,\zeta) = \psi^{1d}_{c0}(\xi)\psi^{1d}_{c0}(\eta)\psi^{1d}_{c0}(\zeta), \tag{125}$$
$$\psi^h_{c1}(\xi,\eta,\zeta) = \psi^{1d}_{c1}(\xi)\psi^{1d}_{c0}(\eta)\psi^{1d}_{c0}(\zeta), \tag{126}$$
$$\psi^h_{c2}(\xi,\eta,\zeta) = \psi^{1d}_{c1}(\xi)\psi^{1d}_{c1}(\eta)\psi^{1d}_{c0}(\zeta), \tag{127}$$
$$\psi^h_{c3}(\xi,\eta,\zeta) = \psi^{1d}_{c0}(\xi)\psi^{1d}_{c1}(\eta)\psi^{1d}_{c0}(\zeta), \tag{128}$$
$$\psi^h_{c4}(\xi,\eta,\zeta) = \psi^{1d}_{c0}(\xi)\psi^{1d}_{c0}(\eta)\psi^{1d}_{c1}(\zeta), \tag{129}$$
$$\psi^h_{c5}(\xi,\eta,\zeta) = \psi^{1d}_{c1}(\xi)\psi^{1d}_{c0}(\eta)\psi^{1d}_{c1}(\zeta), \tag{130}$$
$$\psi^h_{c6}(\xi,\eta,\zeta) = \psi^{1d}_{c1}(\xi)\psi^{1d}_{c1}(\eta)\psi^{1d}_{c1}(\zeta), \tag{131}$$
$$\psi^h_{c7}(\xi,\eta,\zeta) = \psi^{1d}_{c0}(\xi)\psi^{1d}_{c1}(\eta)\psi^{1d}_{c1}(\zeta). \tag{132}$$

The corresponding generating blending functions are

$$\psi^{bl}_{r0} = \alpha^{1d}(\psi^h_{c0}(\xi,\eta,\zeta)\psi^h_{c1}(\xi,\eta,\zeta) + \beta^q(\psi^h_{c0}(\xi,\eta,\zeta)\psi^h_{c2}(\xi,\eta,\zeta) + \psi^h_{c1}(\xi,\eta,\zeta)\psi^h_{c4}(\xi,\eta,\zeta))), \tag{133}$$
$$\psi^{bl}_{r1} = \alpha^{1d}(\psi^h_{c1}(\xi,\eta,\zeta)\psi^h_{c2}(\xi,\eta,\zeta) + \beta^q(\psi^h_{c1}(\xi,\eta,\zeta)\psi^h_{c3}(\xi,\eta,\zeta) + \psi^h_{c2}(\xi,\eta,\zeta)\psi^h_{c5}(\xi,\eta,\zeta))), \tag{134}$$
$$\psi^{bl}_{r2} = \alpha^{1d}(\psi^h_{c2}(\xi,\eta,\zeta)\psi^h_{c3}(\xi,\eta,\zeta) + \beta^q(\psi^h_{c2}(\xi,\eta,\zeta)\psi^h_{c0}(\xi,\eta,\zeta) + \psi^h_{c3}(\xi,\eta,\zeta)\psi^h_{c6}(\xi,\eta,\zeta))), \tag{135}$$
$$\psi^{bl}_{r3} = \alpha^{1d}(\psi^h_{c3}(\xi,\eta,\zeta)\psi^h_{c0}(\xi,\eta,\zeta) + \beta^q(\psi^h_{c3}(\xi,\eta,\zeta)\psi^h_{c1}(\xi,\eta,\zeta) + \psi^h_{c0}(\xi,\eta,\zeta)\psi^h_{c7}(\xi,\eta,\zeta))), \tag{136}$$
$$\psi^{bl}_{r4} = \alpha^{1d}(\psi^h_{c0}(\xi,\eta,\zeta)\psi^h_{c4}(\xi,\eta,\zeta) + \beta^q(\psi^h_{c0}(\xi,\eta,\zeta)\psi^h_{c7}(\xi,\eta,\zeta) + \psi^h_{c1}(\xi,\eta,\zeta)\psi^h_{c4}(\xi,\eta,\zeta))), \tag{137}$$
$$\psi^{bl}_{r5} = \alpha^{1d}(\psi^h_{c1}(\xi,\eta,\zeta)\psi^h_{c5}(\xi,\eta,\zeta) + \beta^q(\psi^h_{c1}(\xi,\eta,\zeta)\psi^h_{c4}(\xi,\eta,\zeta) + \psi^h_{c2}(\xi,\eta,\zeta)\psi^h_{c5}(\xi,\eta,\zeta))), \tag{138}$$
$$\psi^{bl}_{r6} = \alpha^{1d}(\psi^h_{c2}(\xi,\eta,\zeta)\psi^h_{c6}(\xi,\eta,\zeta) + \beta^q(\psi^h_{c2}(\xi,\eta,\zeta)\psi^h_{c5}(\xi,\eta,\zeta) + \psi^h_{c3}(\xi,\eta,\zeta)\psi^h_{c6}(\xi,\eta,\zeta))), \tag{139}$$
$$\psi^{bl}_{r7} = \alpha^{1d}(\psi^h_{c3}(\xi,\eta,\zeta)\psi^h_{c7}(\xi,\eta,\zeta) + \beta^q(\psi^h_{c3}(\xi,\eta,\zeta)\psi^h_{c6}(\xi,\eta,\zeta) + \psi^h_{c0}(\xi,\eta,\zeta)\psi^h_{c7}(\xi,\eta,\zeta))), \tag{140}$$
$$\psi^{bl}_{r8} = \alpha^{1d}(\psi^h_{c4}(\xi,\eta,\zeta)\psi^h_{c5}(\xi,\eta,\zeta) + \beta^q(\psi^h_{c4}(\xi,\eta,\zeta)\psi^h_{c6}(\xi,\eta,\zeta) + \psi^h_{c1}(\xi,\eta,\zeta)\psi^h_{c4}(\xi,\eta,\zeta))), \tag{141}$$
$$\psi^{bl}_{r9} = \alpha^{1d}(\psi^h_{c5}(\xi,\eta,\zeta)\psi^h_{c6}(\xi,\eta,\zeta) + \beta^q(\psi^h_{c5}(\xi,\eta,\zeta)\psi^h_{c7}(\xi,\eta,\zeta) + \psi^h_{c2}(\xi,\eta,\zeta)\psi^h_{c5}(\xi,\eta,\zeta))), \tag{142}$$
$$\psi^{bl}_{r10} = \alpha^{1d}(\psi^h_{c6}(\xi,\eta,\zeta)\psi^h_{c7}(\xi,\eta,\zeta) + \beta^q(\psi^h_{c6}(\xi,\eta,\zeta)\psi^h_{c4}(\xi,\eta,\zeta) + \psi^h_{c3}(\xi,\eta,\zeta)\psi^h_{c6}(\xi,\eta,\zeta))), \tag{143}$$
$$\psi^{bl}_{r11} = \alpha^{1d}(\psi^h_{c7}(\xi,\eta,\zeta)\psi^h_{c4}(\xi,\eta,\zeta) + \beta^q(\psi^h_{c7}(\xi,\eta,\zeta)\psi^h_{c5}(\xi,\eta,\zeta) + \psi^h_{c0}(\xi,\eta,\zeta)\psi^h_{c7}(\xi,\eta,\zeta))), \tag{144}$$

$$\psi_{f0}^{bl} = \alpha^q(\psi_{c0}^h(\xi,\eta,\zeta)\psi_{c2}^h(\xi,\eta,\zeta) + \beta^q\psi_{c0}^h(\xi,\eta,\zeta)\psi_{c6}^h(\xi,\eta,\zeta)), \tag{145}$$

$$\psi_{f1}^{bl} = \alpha^q(\psi_{c0}^h(\xi,\eta,\zeta)\psi_{c5}^h(\xi,\eta,\zeta) + \beta^q\psi_{c0}^h(\xi,\eta,\zeta)\psi_{c6}^h(\xi,\eta,\zeta)), \tag{146}$$

$$\psi_{f2}^{bl} = \alpha^q(\psi_{c1}^h(\xi,\eta,\zeta)\psi_{c6}^h(\xi,\eta,\zeta) + \beta^q\psi_{c0}^h(\xi,\eta,\zeta)\psi_{c6}^h(\xi,\eta,\zeta)), \tag{147}$$

$$\psi_{f3}^{bl} = \alpha^q(\psi_{c2}^h(\xi,\eta,\zeta)\psi_{c7}^h(\xi,\eta,\zeta) + \beta^q\psi_{c0}^h(\xi,\eta,\zeta)\psi_{c6}^h(\xi,\eta,\zeta)), \tag{148}$$

$$\psi_{f4}^{bl} = \alpha^q(\psi_{c3}^h(\xi,\eta,\zeta)\psi_{c4}^h(\xi,\eta,\zeta) + \beta^q\psi_{c0}^h(\xi,\eta,\zeta)\psi_{c6}^h(\xi,\eta,\zeta)), \tag{149}$$

$$\psi_{f5}^{bl} = \alpha^q(\psi_{c4}^h(\xi,\eta,\zeta)\psi_{c6}^h(\xi,\eta,\zeta) + \beta^q\psi_{c0}^h(\xi,\eta,\zeta)\psi_{c6}^h(\xi,\eta,\zeta)), \tag{150}$$

$$\psi_{v0}^{bl} = \alpha^{pr}(\psi_{c0}^{pr}(\xi,\eta,\zeta)\psi_{c4}^{pr}(\xi,\eta,\zeta)(\psi_{c2}^{pr}(\xi,\eta,\zeta) + \psi_{c5}^{pr}(\xi,\eta,\zeta))). \tag{151}$$

The identifiers of the edges and faces are $r0(id_0,id_1)$, $r1(id_1,id_2)$, $r2(id_2,id_3)$, $r_3(id_3,id_0)$, $r_4(id_0,id_4)$, $r_5(id_1,id_5)$, $r_6(id_2,id_6)$, $r_7(id_3,\,id_7)$, $r_8(id_4,\,id_5)$, $r_9(id_5,\,id_6)$, $r10(id_6,\,id_7)$, $r11(id_7,\,id_4)$, $f0(id_0, id_1,id_2,id_3)$, $f_1(id_0,id_1,id_5,id_4)$, $f_2(id_1,id_2,id_6,id_5)$, $f_3(id_3,id_2,id_6,id_7)$, $f_4(id_0,id_3,id_7,id_4)$, and $f_5(id_4,id_5,id_6,id_7)$.

The projections of the interior points to the edges and faces are $\xi_{r0} = \xi, \xi_{r1} = \eta, \xi_{r2} = -\xi, \xi_{r_3} = -\eta, \xi_{r_4} = \zeta, \xi_{r_5} = \zeta, \xi_{r_6} = \zeta, \xi_{r_7} = \zeta, \xi_{r_8} = \xi, \xi_{r_9} = \eta, \xi_{r10} = -\xi, \xi_{r11} = -\eta, (\xi,\eta)_{f0} = (\xi,\eta)$, $(\xi,\eta)_{f_1} = (\xi,\zeta)$, $(\xi,\eta)_{f_2} = (\eta,\zeta), (\xi,\eta)_{f_3} = (\xi,\zeta), (\xi,\eta)_{f_4} = (\eta,\zeta)$, and $(\xi,\eta)_{f_5} = (\xi,\eta)$.

The edge, face and volume functions are defined as

$$\psi_{r_i(n)}^h(\xi,\eta,\zeta) = \psi_{r_i}^{bl}(\xi,\eta,\zeta)f_n(\hat{\xi}_{r_i}), \quad 0 \leqslant i \leqslant 11, \quad n = 0,1,\ldots,p-2, \tag{152}$$

$$\psi_{f_i(n1,n2)}^h(\xi,\eta,\zeta) = \psi_{f_i}^{bl}(\xi,\eta,\zeta)f_{n1}(\hat{\xi}_{f_i})f_{n2}(\hat{\eta}_{f_i}), \quad 0 \leqslant i \leqslant 5, \quad 0 \leqslant n1,n2 \leqslant p-2, \tag{153}$$

$$\psi_{v0(n1,n2,n3)}^h(\xi,\eta,\zeta) = \psi_{f_i}^{bl}(\xi,\eta,\zeta)f_{n1}(\xi)f_{n2}(\eta)f_{n3}(\zeta), \quad 0 \leqslant n1,n2,n3 \leqslant p-2. \tag{154}$$

5. Affine transformations

In order to make the shape functions continuous on the interface between The elements, the shape functions associated with the sides of an element are defined in function of a parameter space associated with the side. This parameter space may not correspond to the natural parameter space of the side of the element. Therefore affine transformations are defined which map the point associated with the parameter space of the side of the element to the parameter space of the side itself.

$$(\hat{\xi},\hat{\eta}) = T_i^{tr}(\xi,\eta), \tag{155}$$

$$(\hat{\xi},\hat{\eta}) = T_i^{q}(\xi,\eta). \tag{156}$$

Figs. 6 and 7 document the choice of axes that define the transformation presented in Eqs. 154 and 155. Numerical values of these transformations are documented in Tables 3 and 4.

6. Modular implementation in an object-oriented program

The current definition of the elements and functions presented here were implemented in an environment of scientific programming, developed at LabMeC, a laboratory of the Department of Structures of the College of Civil Engineering, State University of Campinas, Unicamp/FEC/DES, São Paulo, Brazil. This programming environment, called PZ, is object oriented and written in C++. The classes which define the approximation space are independent of the global finite element structure, allowing them to be used in other programming environments.

The approximation functions are implemented in two class structures: Topology and Shape. The classes within the Topology directory define the parametric space of each element and the correspondence between the parametric spaces of neighboring sides. The classes within the Shape directory are derived from their counterparts in the Topology directory and implement the projection of interior points to the sides of the element.

The source code is freely available. It has been extensively tested in a variety of finite element approximations. The location of our server is subject to change; therefore the authors recommend e-mailing the first author to gain access to the code.

7. Numerical sample

The proposed strategy has been implemented in a finite element program which has been used for the numerical simulation of several kinds of problems, such as flow analysis in porous media, elasticity, Euler flow, and beams and shells. In order to test the validity of the shape functions, a mesh consisting of the complete set of element geometries was generated such as illustrated in Fig. 8. The order of the shape functions was varied from $p = 1$ to $p = n$. The approximation space was projected in L^2 and H^1 on a three-dimensional polynomial of the same order with randomly

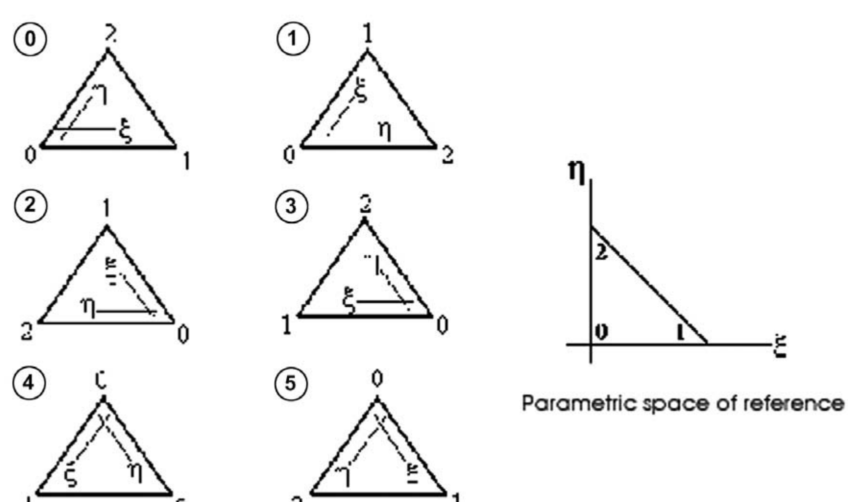

Fig. 6. Axes definition of the triangle faces.

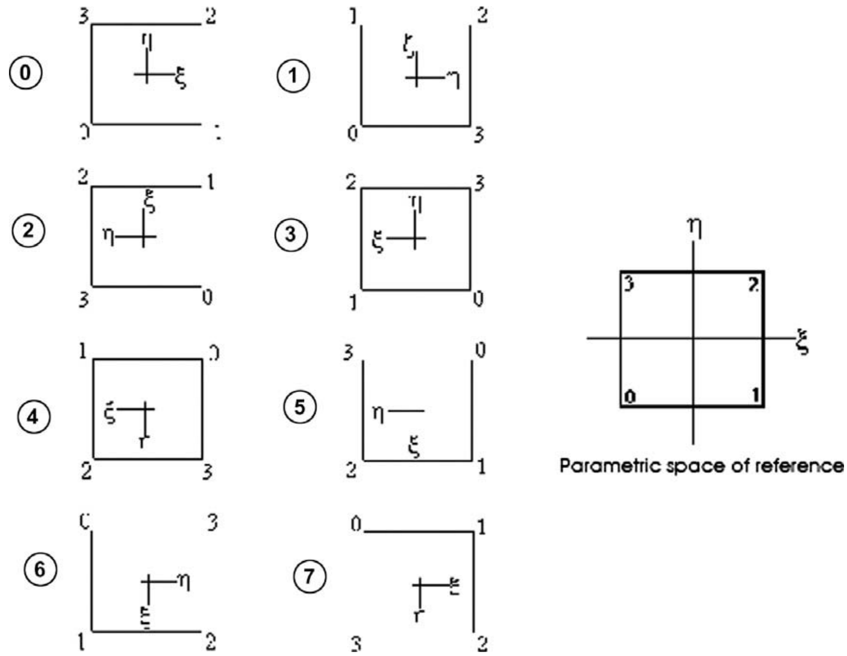

Fig. 7. Axes definition for the quadrilateral faces.

Table 3
Parametric transformations for triangle faces.

No.	Transformation
0	$\begin{bmatrix}\hat{\varepsilon}\\ \hat{\eta}\end{bmatrix} = \begin{bmatrix}-1 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\varepsilon\\ \eta\end{bmatrix} + \begin{bmatrix}1\\ 0\end{bmatrix}$
1	$\begin{bmatrix}\hat{\varepsilon}\\ \hat{\eta}\end{bmatrix} = \begin{bmatrix}-1 & -1\\ 0 & 1\end{bmatrix}\begin{bmatrix}\varepsilon\\ \eta\end{bmatrix} + \begin{bmatrix}1\\ 0\end{bmatrix}$
2	$\begin{bmatrix}\hat{\varepsilon}\\ \hat{\eta}\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\varepsilon\\ \eta\end{bmatrix}$
3	$\begin{bmatrix}\hat{\varepsilon}\\ \hat{\eta}\end{bmatrix} = \begin{bmatrix}0 & 1\\ -1 & -1\end{bmatrix}\begin{bmatrix}\varepsilon\\ \eta\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}$
4	$\begin{bmatrix}\hat{\varepsilon}\\ \hat{\eta}\end{bmatrix} = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\varepsilon\\ \eta\end{bmatrix}$
5	$\begin{bmatrix}\hat{\varepsilon}\\ \hat{\eta}\end{bmatrix} = \begin{bmatrix}1 & 0\\ -1 & -1\end{bmatrix}\begin{bmatrix}\varepsilon\\ \eta\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}$

Table 4
Parametric transformations for quadrilateral faces.

No.	Transformation
0	$\begin{bmatrix}\hat{\varepsilon}\\ \hat{\eta}\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\varepsilon\\ \eta\end{bmatrix}$
1	$\begin{bmatrix}\hat{\varepsilon}\\ \hat{\eta}\end{bmatrix} = \begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\varepsilon\\ \eta\end{bmatrix}$
2	$\begin{bmatrix}\hat{\varepsilon}\\ \hat{\eta}\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\varepsilon\\ \eta\end{bmatrix}$
3	$\begin{bmatrix}\hat{\varepsilon}\\ \hat{\eta}\end{bmatrix} = \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\varepsilon\\ \eta\end{bmatrix}$
4	$\begin{bmatrix}\hat{\varepsilon}\\ \hat{\eta}\end{bmatrix} = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\varepsilon\\ \eta\end{bmatrix}$
5	$\begin{bmatrix}\hat{\varepsilon}\\ \hat{\eta}\end{bmatrix} = \begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix}\begin{bmatrix}\varepsilon\\ \eta\end{bmatrix}$
6	$\begin{bmatrix}\hat{\varepsilon}\\ \hat{\eta}\end{bmatrix} = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\varepsilon\\ \eta\end{bmatrix}$
7	$\begin{bmatrix}\hat{\varepsilon}\\ \hat{\eta}\end{bmatrix} = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}\varepsilon\\ \eta\end{bmatrix}$

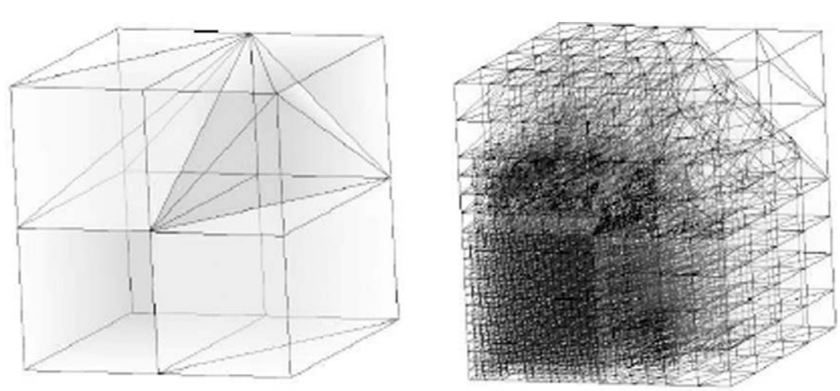

Fig. 8. Validation model: original mesh and a refined mesh.

generated coeficients. For each simulation the error between the projected solution and the exact solution was within roundoff error.

8. Extensions

The main idea of this contribution is to present higher-order interpolation functions for the complete family of finite elements up to three-dimensions. The development of the approximation spaces is systematic and has been documented for each element. The authors believe it should be possible to develop a higher level of abstraction so that the projections from the interior nodes to the sides can be generically computed. This would open the way to defining higher-dimensional approximation spaces for prismatic and pyramidal finite elements.

9. Conclusions

Higher-order shape functions were defined to construct finite element approximation spaces in zero, one, two, and three-dimensions. The approximation spaces are continuous regardless of the orientation of the elements. Elements of distinct dimension can be superposed while still generating continuous functions. The definition allows the implementation of p-adaptive meshes with arbitrary degree p. In our approach an element is seen as the union of open sets of points, associated with the sides of the element. A different order of approximation can be associated with each side.

The definition of the approximation spaces is consistent in that a one-dimensional element can be superposed on two-dimensional or three-dimensional elements. Two-dimensional elements can be superposed on three-dimensional elements. This greatly simplifies the implementation of boundary conditions (i.e., boundary conditions are integrals over surfaces of lower dimension).

The separation of generating blending functions from the approximation space itself allows one to define linear extensions within the element, and to define the approximation space independently of the sequence of orthogonal polynomials.

Having developed the previously described shape functions, it seems possible to generalize the concept to higher-dimensional elements. This is left as future work.

Readers can have access to the C++ source code which implements the previously described functions and which is independent of the finite element code.

References

[1] Cedric M.A. Ayala Bravo. Um sistema de refinamento h-p adaptativo utilizando elementos finitos hierárquicos multidimensional, Ph.D. Thesis, Departamento de MecGnica Computacional, Universidade Estadual de Campinas, Campinas, Brazil, 2000.
[2] J. Fish, R. Guttal, Recent advances in the p-version of the finite element method for shells, Comput. Syst. Eng. 6 (3) (1995) 195–211.
[3] Michael S. Floater, Mean value coordinates, Comput. Aided Geomet. Des. 20 (2003) 19–27.
[4] L.F. Demkowicz, J.T. Oden, Applied Functional Analysis, New York, London, Tokyo, 1996.
[5] T.C. Warburton, I. Lomtev, Y. Du, S.J. Sherwin, G.E. Karniadakis, Galerkin and discontinuous Galerkin spectral/hp methods, Comput. Meth. Appl. Mech. Engrg. 175 (3–4) (1999) 343–359.
[6] J.E. Flaherty, M.S. Shephard, S. Dey, A straightforward structure to construct shape functions for variable p-order meshes, Comput. Meth. Appl. Mech. Engrg. 147 (1997) 209–233.
[7] J.L. Jarvis, S. Campion, An investigation of the implementation of the p-version finite element method, Finite Elem. Anal. Des. 23 (1996) 1–21.
[8] M. Morandi Cecchi, S. De Marchi, The polynomial approximation in the finite element method, J. Comput. Appl. Math. 57 (1995) 99–114.
[9] G.E. Karniadakis, S.J. Sherwin, A new triangular and tetrahedral basis for high-order finite elements, Int. J. Numer. Meth. Engrg. 38 (3775) (1995).
[10] D.R. Houser, S.M. Vijayakar, H.R. Busby, Finite element analysis of quasi-prismatic bodies using Chebyshev polynomnials, Int. J. Numer. Meth. Engrg. 24 (1987) 1461–1477.
[11] B.A. Szabo, A.G. Peano, Hierarchic Finite Elements, in: Finite Element Handbook, McGraw-Hill, New York, 1987, pp. 2.227–2.233.
[12] S.J. Sherwin, T.C. Warburton, G.E. Karniadakis, Basis functions for triangular and quadrilateral high-order elements, SIAM J. Sci. Comput. 20 (5) (1999) 1671–1693.
[13] N. Nigam, J. Philipps, Higher-order finite elements on pyramids. <http://arxiv.org/abs/math.NA/0610206>.
[14] J. Shöberl, S. Zaglmayr, High order nedelec elements with local complete sequence properties, Int. J. Comput. Math. Electr. Electr. 24 (2) (2005) 374–384.