Implementation of

GEOMETRIC DECOMPOSITION OF HIGH ORDER EDGE AND FACE ELEMENTS

CONTENTS

1. PRELIMINARY

1.1. Notation.

1.1.1. *Multi-index.* A multi-index is an array of non-negative integers, denoted by an array of multi-index of length $n+1$:

$$\boldsymbol{\alpha} = (\alpha_0, \alpha_1, ..., \alpha_n).$$

simplicial lattice.

The order of the multi-index is $|\boldsymbol{\alpha}| = \sum_{i=0}^{n} \alpha_i$. Denote by $\mathcal{A}_n^k$ the set of all multi-indexs of length $n+1$ and order k.

$\mathbb{T}^k_n$

$$\mathcal{A}_n^k = \{\boldsymbol{\alpha} | \boldsymbol{\alpha} \in \mathbb{N}^{n+1}, |\boldsymbol{\alpha}| = k\}$$

The elements in $\mathcal{A}_n^k$ are ordered by dictionary, e.g., the elements in $\mathcal{A}_2^4$ are ordered as follows:

$$\begin{array}{r}0:\\1:\\2:\\3:\\4:\\5:\\6:\\7:\\8:\\9:\\10:\\11:\\12:\\13:\\14:\end{array}\begin{pmatrix}4&0&0\\3&1&0\\3&0&1\\2&2&0\\2&1&1\\2&0&2\\1&3&0\\1&2&1\\1&1&2\\1&0&3\\0&4&0\\0&3&1\\0&2&2\\0&1&3\\0&0&4\end{pmatrix}$$

For an element $\boldsymbol{\alpha}$ in $\mathcal{A}_2^k$, by mapping:

$$R_2(\boldsymbol{\alpha}) = \frac{(\alpha_1+\alpha_2)(\alpha_1+\alpha_2+1)}{2} + \alpha_2$$

can compute the indexing of $\boldsymbol{\alpha}$ in $\mathcal{A}_2^k$. For an element $\mathcal{A}_3^k$ in $\boldsymbol{\alpha}$, the ~~by~~ mapping: is given by

$$R_3(\boldsymbol{\alpha}) = \frac{(\alpha_1+\alpha_2+\alpha_3)(\alpha_1+\alpha_2+\alpha_3+1)(\alpha_1+\alpha_2+\alpha_3+2)}{6} + \frac{(\alpha_2+\alpha_3)(\alpha_2+\alpha_3+1)}{2} + \alpha_3$$

can compute the indexing of $\boldsymbol{\alpha}$ in $\mathcal{A}_3^k$. barycentric coordinate.

1.1.2. *Interpolation points on a simplex.* Define a set of points on an n-simplex $T = \mathrm{conv}\{\boldsymbol{x}_0, \boldsymbol{x}_1 ..., \boldsymbol{x}_n\}$ λ. $\alpha \to \lambda = \frac{\alpha}{k}$.

$\chi(\alpha)$

$$\mathcal{X}_T = \{\boldsymbol{x}_{\boldsymbol{\alpha}} | \boldsymbol{x}_{\boldsymbol{\alpha}} = \frac{1}{k}\sum_{i=0}^{n} \alpha_i \boldsymbol{x}_i,\ \boldsymbol{\alpha} \in \mathcal{A}_n^k\}.$$

This set is called the set of k-th interpolation points on T. Also we denote by $\mathring{\mathcal{X}}_T$ the k-th interpolation point in the interior of ~~inside~~ T, $\mathring{\mathcal{X}}_T \subset \mathcal{X}_T$.

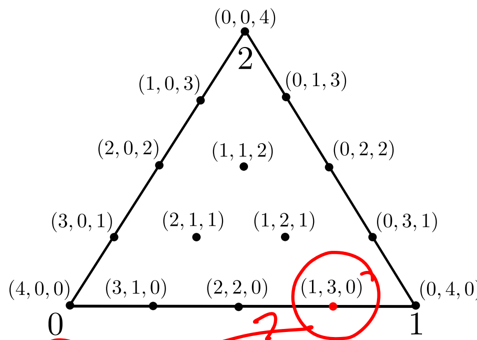
By definition, each interpolation point in $\mathcal{X}_T$ corresponds to a multi-index in $\mathcal{A}_n^k$, and naturally, the order of the interpolation points in $\mathcal{X}_T$ is induced by the order of their corresponding multi-index. That is, if an interpolated point corresponds to a multi-index of $\boldsymbol{\alpha}$, then its indexing is $R(\boldsymbol{\alpha})$.

1.2. **Barycentric coordinate and Bernstein basis.** $b_f = \lambda_f$. Property of bubble b_f.

Lemma 1.1. *(1)* $b_f|_e = 0$ *for all* e *with* $\dim e \leq \dim f$.
(2) $b_f|_{\mathring{F}} \neq 0$ *for all* F *with* $\dim F > \dim f$.

Through the Bernstein form, a polynomial defined on a face f can be naturally extended to the whole simplex.

subsimplex.



? not consistent with the table

FIGURE 1. The 4-th interpolation points on the triangle and their corresponding multi-indexs.

2. GEOMETRIC DECOMPOSITIONS OF LAGRANGE ELEMENTS

For the polynomial space $\mathbb{P}_r(T)$ with $r \geq 1$ on an n-dimensional simplex T, we have the following decomposition of Lagrange element [2, (2.6)].

Theorem 2.1.

$$\mathbb{P}_r(T) = \bigoplus_{\ell=0}^{n} \bigoplus_{f \in \Delta_\ell(T)} b_f \mathbb{P}_{r-(\ell+1)}(f). \tag{1}$$

The function $u \in \mathbb{P}_r(T)$ is uniquely determined by DoFs

$$\int_f u\, p \,\mathrm{d}s \quad \forall\, p \in \mathbb{P}_{r-(\ell+1)}(f), f \in \Delta_\ell(T), \ell = 0, 1, \ldots, n. \tag{2}$$

The integral at a vertex is understood as the function value at that vertex and $\mathbb{P}_k(\mathrm{v}) = \mathbb{R}$. A proof of the unisolvence can be found in [4].

Introduce a bubble polynomial space on each sub-simplex as

$$\mathbb{B}_r(f) := b_f \mathbb{P}_{r-(\ell+1)}(f), \quad f \in \Delta_\ell(T), 1 \leq \ell \leq n.$$

It is called a bubble space as

$$\mathrm{tr}_f^{\mathrm{grad}}\, u := u|_{\partial f} = 0, \quad u \in \mathbb{B}_r(f).$$

~~The notation $\mathbb{B}_r(f)$ can be naturally extended to vertices: $\mathbb{B}_r(\mathrm{v}) = \mathrm{span}\{\lambda_{\mathrm{v}}\}$~~. Then the decomposition (1) can be written as

$$\mathbb{P}_r(T) = \mathbb{P}_1(T) \oplus \bigoplus_{\ell=1}^{n} \bigoplus_{f \in \Delta_\ell(T)} \mathbb{B}_r(f). \tag{3}$$

That is a degree r polynomial can be decomposed into a linear polynomial plus bubbles on edges, faces, and all sub-simplexes.

Remark 2.2. *For spaces associated to vertices, it can be changed to*

$$\mathrm{span}\{\lambda_i^r, i = 0, 1, \ldots, n\},$$

which is consistent with the Bernstein representation λ^α.

The geometric decomposition can be naturally extended to vector Lagrange elements. For $r \geq 1$, define

$$\mathbb{B}_r^n(f) := b_f \mathbb{P}_{r-(\ell+1)}(f) \otimes \mathbb{R}^n.$$

Clearly we have

$$\mathbb{P}_r^n(T) = \mathbb{P}_1^n(T) \oplus \bigoplus_{\ell=1}^{n} \bigoplus_{f \in \Delta_\ell(T)} \mathbb{B}_r^n(f). \tag{4}$$

For any $f \in \Delta_\ell(T)$,

w We choose a $t - n$ coördinate to write the decomposition into tangential and normal components. Define

$$\mathscr{T}^f := \left\{ \sum_{i=1}^{\ell} c_i \boldsymbol{t}_i^f : c_i \in \mathbb{R} \right\}, \quad \mathscr{N}^f := \left\{ \sum_{i=1}^{n-\ell} c_i \boldsymbol{n}_i^f : c_i \in \mathbb{R} \right\}.$$

Thus $\mathscr{T}^f$ is the tangent plane of f and $\mathscr{N}^f$ is the normal plane.
When $\ell = 0$, i.e., for vertices, no tangential component and for $\ell = n$, no normal component. We then write the decomposition (5) as

$$\mathbb{P}_r^n(T) = \mathbb{P}_1^n(T) \oplus \bigoplus_{\ell=1}^{n} \bigoplus_{f \in \Delta_\ell(T)} \left[\cancel{(\mathbb{B}_r(f) \otimes \mathscr{T}^f)} \oplus (\mathbb{B}_r(f) \otimes \mathscr{N}^f) \right]. \tag{5}$$

Restricted to an ℓ-dim sub-simplex $f \in \Delta_\ell(T)$, define $\mathbb{B}_r^\ell(f)$

$$\mathbb{B}_r^\ell(f) := \mathbb{B}_r(f) \otimes \mathscr{T}^f,$$

which is a space of ℓ-dimensional vectors on the tangential space with vanishing trace $\operatorname{tr}^{\text{grad}}$ on ∂f.

3. Geometric Decompositions of Face element

define $H(\text{div})$ space face element

~~The trace operator for $H(\text{div}, K)$ space~~

$$\cancel{\operatorname{tr}^{\text{div}} : H(\text{div}, K) \to H^{-1/2}(\partial K)}$$

Let ~~is a continuous extension of $\operatorname{tr}^{\text{div}} \boldsymbol{v} = \boldsymbol{n} \cdot \boldsymbol{v}|_{\partial K}$ defined on smooth functions. We then focus on the restriction of the trace operator to the polynomial space~~. Define the polynomial div bubble space

$$\mathbb{B}_r(\text{div}; T) = \ker(\operatorname{tr}^{\text{div}}) \cap \mathbb{P}_r(T; \mathbb{E}^n).$$

It is easy to verify that $\mathbb{B}_r^\ell(f) \subset \mathbb{B}_r(\text{div}; T)$ for $r \geq 2, \dim f \geq 1$. In [4], we have proved that the div- bubble polynomial space has the following decomposition.

Lemma 3.1. *For $r \geq 2$,*

$$\mathbb{B}_r(\text{div}; T) = \bigoplus_{\ell=1}^{n} \bigoplus_{f \in \Delta_\ell(T)} \mathbb{B}_r^\ell(f).$$

Notice that as no tangential plane on vertices, there is no div-bubble associated to vertices. Th~~e lowest dimensional sub-simplex is an edge and t~~ T hus the degree of div-bubble polynomial is ≥ 2. And the volume Lagrange bubble $\mathbb{B}_r^n(T)$ has no contribution to the trace. Next we present ~~a~~ two geometric decompositions of div-element.

Lemma 3.2. *For $r \geq 1$, we have*

$$\mathbb{P}_r^n(T) = \mathbb{P}_1^n(T) \oplus \bigoplus_{\ell=1}^{n-1} \bigoplus_{f \in \Delta_\ell(T)} (\mathbb{B}_r(f) \otimes \mathscr{N}^f) \oplus \mathbb{B}_r(\text{div}; T), \tag{6}$$

$[P_1(F)$

$$\mathbb{P}_r^n(T) = \bigoplus_{F \in \Delta_{n-1}(T)} \bigoplus_{\ell=0}^{n-1} \bigoplus_{f \in \Delta_\ell(F)} \mathbb{B}_r(f) \boldsymbol{n}_F] \oplus \mathbb{B}_r(\text{div}; T). \tag{7}$$

all

Proof. The first H(div) decomposition (6) is a rearrangement of (5) by merging the tangential components into the bubble space. Next we prove the decomposition (7).

For an ℓ-dimensional sub-simplex $f \in \Delta_\ell(T)$, the $n - \ell$ face normal vectors $\{\boldsymbol{n}_F : F \in \Delta_{n-1}(T), f \subseteq F\}$ form a basis of $\mathscr{N}^f$. Therefore we have

$$\mathbb{B}_r(f) \otimes \mathscr{N}^f = \bigoplus_{F \in \Delta_{n-1}(T), f \subseteq F} \mathbb{B}_r(f) \boldsymbol{n}_F.$$

Then we have

$$\mathbb{P}_r^n(T) = \mathbb{P}_1^n(T) \oplus \bigoplus_{\ell=1}^{n-1} \bigoplus_{f \in \Delta_\ell(T)} \bigoplus_{F \in \Delta_{n-1}(T), f \subseteq F} \mathbb{B}_r(f) \boldsymbol{n}_F \oplus \mathbb{B}_r(\text{div}; T), \tag{8}$$

The decomposition (7) holds from (8) by swapping the ordering of f and F in the direct sum except the component $\mathbb{P}_1^n(T)$ associated to vertices. At vertices, we have

$$\begin{aligned}\mathbb{P}_1^n(T) &= \bigoplus_{\mathrm{v}\in\Delta_0(T)}\lambda_{\mathrm{v}}\otimes\mathscr{N}^{\mathrm{v}} = \bigoplus_{\mathrm{v}\in\Delta_0(T)}\bigoplus_{F\in\Delta_{n-1}(T),\mathrm{v}\in\Delta_0(F)}\mathrm{span}\{\lambda_{\mathrm{v}}\boldsymbol{n}_F\}\\ &= \bigoplus_{F\in\Delta_{n-1}(T)}\bigoplus_{\mathrm{v}\in\Delta_0(F)}\mathrm{span}\{\lambda_{\mathrm{v}}\boldsymbol{n}_F\}. = \bigoplus P_1(F)\, n_F\end{aligned}$$

we use $\{ n_F \}$ as the basis ---

Thus decomposition (7) follows. $\square$

In decomposition (6), we single out $\mathbb{P}_1^n(T)$ to emphasize a div-conforming element can be obtained by adding div-bubble and normal component on sub-simplexes starting from edges. In (7), we group all normal components facewisely which leads to the classical BDM element.

Lemma 3.3 (BDM element)**.** *The following* DoFs

$$\int_f (\boldsymbol{v}\cdot\boldsymbol{n}_F)\,p\,\mathrm{d}s, \quad f\in\Delta_\ell(T), F\in\Delta_{n-1}(T), f\subseteq F, \tag{9}$$
$$p\in\mathbb{P}_{r-(\ell+1)}(f), \ell=0,\ldots,n-1,$$
$$\int_f (\boldsymbol{v}\cdot\boldsymbol{t}_i^f)\,p\,\mathrm{d}s, \quad f\in\Delta_\ell(T), i=1,\ldots,\ell, \tag{10}$$
$$p\in\mathbb{P}_{r-(\ell+1)}(f), \ell=1,\ldots,n,$$

H( )

define a div-conforming space $V_h = \{\boldsymbol{v}_h\in H(\mathrm{div};\Omega) : \boldsymbol{v}_h|_T\in\mathbb{P}_r^n(T) \ \ \forall T\in\mathcal{T}_h\}$.

For BDM element, by the decomposition of Lagrange element, cf. Theorem 2.1, the face DoFs can be merged into

$$\int_F (\boldsymbol{v}\cdot\boldsymbol{n}_F)\,p\,\mathrm{d}s, \quad p\in\mathbb{P}_r(F), \tag{11}$$

and the div bubble DoFs can be merged into one

$$\int_T \boldsymbol{v}\cdot\boldsymbol{p}\,\mathrm{d}x \quad \boldsymbol{p}\in\mathbb{B}_r(\mathrm{div},T).$$

But DoF (9)-(10) is more friendly for the implementation. One can start from the vector Lagrange element and split into tangential and normal components.

Based on the decomposition (6), we may impose the continuity at vertex to determine $\mathbb{P}_1^n(T)$ which is known as Sternberg's element [7].

Lemma 3.4 (Stenberg element)**.** *The following DoFs*

$$\boldsymbol{v}(v), \quad v\in\Delta_0(\mathcal{T}_h),$$
$$\int_f (\boldsymbol{v}\cdot\boldsymbol{n}_F)\,p\,\mathrm{d}s, \quad f\in\Delta_\ell(T), F\in\Delta_{n-1}(T), f\subseteq F,$$
$$p\in\mathbb{P}_{r-(\ell+1)}(f), \ell=1,\ldots,n-1,$$
$$\int_T \boldsymbol{v}\cdot\boldsymbol{p}\,\mathrm{d}x \quad \boldsymbol{p}\in\mathbb{B}_r(\mathrm{div},T).$$

define a div-conforming finite element space being continuous at vertex in $\Delta_0(\mathcal{T}_h)$.

In general, for each sub-simplex, we can choose either a local or global normal basis and obtain variants of Sternberg element.

explain

Lemma 3.5. *Let $-1 \le k \le n-1$. For each $f \in \Delta_\ell(\mathcal{T}_h)$ with $\ell \le k$, we choose $n-\ell$ normal vectors $\{\boldsymbol{n}_1^f, \dots, \boldsymbol{n}_{n-\ell}^f\}$. Then the* DoFs

remove.

$$
\begin{aligned}
\boldsymbol{v}(\mathtt{v}) \quad & \mathtt{v} \in \Delta_0(\mathcal{T}_h),\\
\int_f \boldsymbol{v}\cdot\boldsymbol{n}_i^f\, p \,\mathrm{d}s, \quad & p \in \mathbb{P}_{r-(\ell+1)}(f), f \in \Delta_\ell(\mathcal{T}_h),\\
& i = 1,\dots,n-\ell,\ \ell = 1,\dots,k,\\
\int_f (\boldsymbol{v}\cdot\boldsymbol{n}_F)\, p \,\mathrm{d}s, \quad & p \in \mathbb{P}_{r-(\ell+1)}(f), f \in \Delta_\ell(T),\\
& F \in \Delta_{n-1}(T), f \subseteq F,\ \ell = k+1,\dots,n-1,\\
\int_T \boldsymbol{v}\cdot\boldsymbol{p}\,\mathrm{d}x, \quad & \boldsymbol{p} \in \mathbb{B}_r(\mathrm{div}, T), T \in \mathcal{T}_h,
\end{aligned}
$$

will determine a space $V^r \subset H(\mathrm{div},\Omega)$ which is continuous on the normal plane of f for all $f \in \Delta_\ell(\mathcal{T}_h)$ with $\ell \le k$.

When $k = 0$, it is the original Stenberg element [7], i.e., only continuous at vertices. When $k = n-2$, it is the generalization of Christiansen-Hu-Hu element [5]. We allow $k = -1$ to include the BDM element. We refer to [4] for the proof on the unisolvence.

4. Geometric Decompositions of Edge elements

Overview.

4.1. curl operator and trace.

Denote by $\mathbb{S}$ and $\mathbb{K}$ the subspace of symmetric matrices and skew-symmetric matrices of $\mathbb{R}^{n\times n}$, respectively. For vector function $\boldsymbol{v}$, let

$$
\mathrm{curl}\,\boldsymbol{v} := 2\,\mathrm{skw}(\mathrm{grad}\,\boldsymbol{v}) = \mathrm{grad}\,\boldsymbol{v} - (\mathrm{grad}\,\boldsymbol{v})^{\intercal},
$$

which is or skew-symmetric. matrix. In 3D, define mskw and identify curl v as a vector.

Given a face $F \in \mathcal{F}^1(K)$, define the trace operator of curl as

$$
\mathrm{tr}_F^{\mathrm{curl}}\,\boldsymbol{v} = 2\,\mathrm{skw}(\boldsymbol{v}\boldsymbol{n}_F^{\intercal})|_F = (\boldsymbol{v}\boldsymbol{n}_F^{\intercal} - \boldsymbol{n}_F\boldsymbol{v}^{\intercal})|_F,
$$

and the tangential part of $\boldsymbol{v}$ as

$$
\Pi_F\boldsymbol{v} = \boldsymbol{v}|_F - (\boldsymbol{v}|_F\cdot\boldsymbol{n}_F)\boldsymbol{n}_F = \sum_{i=1}^{n-1}(\boldsymbol{v}|_F\cdot\boldsymbol{t}_{F,i})\boldsymbol{t}_{F,i}.
$$

As we treat curl v as a matrix, so is $\mathrm{tr}_F^{\mathrm{curl}} v$. But $\Pi_F v$ is a vector. Their relation is given . . . .

Lemma 4.1. *For face $F \in \mathcal{F}^1(K)$, we have*

$$
\mathrm{tr}_F^{\mathrm{curl}}\,\boldsymbol{v} = 2\,\mathrm{skw}\left((\Pi_F\boldsymbol{v})\boldsymbol{n}_F^{\intercal}\right), \quad \Pi_F\boldsymbol{v} = (\mathrm{tr}_F^{\mathrm{curl}}\,\boldsymbol{v})\boldsymbol{n}_F. \tag{12}
$$

Proof. By $\boldsymbol{v}|_F = \Pi_F\boldsymbol{v} + (\boldsymbol{v}|_F\cdot\boldsymbol{n}_F)\boldsymbol{n}_F$,

$$
\mathrm{tr}_F^{\mathrm{curl}}\,\boldsymbol{v} = 2\,\mathrm{skw}\left((\Pi_F\boldsymbol{v})\boldsymbol{n}_F^{\intercal} + (\boldsymbol{v}|_F\cdot\boldsymbol{n}_F)\boldsymbol{n}_F\boldsymbol{n}_F^{\intercal}\right) = 2\,\mathrm{skw}\left((\Pi_F\boldsymbol{v})\boldsymbol{n}_F^{\intercal}\right),
$$

which implies the first identify. Then by $\boldsymbol{n}_F^{\intercal}\boldsymbol{n}_F = 1$ and $(\Pi_F\boldsymbol{v})^{\intercal}\boldsymbol{n}_F = 0$,

$$
(\mathrm{tr}_F^{\mathrm{curl}}\,\boldsymbol{v})\boldsymbol{n}_F = \left((\Pi_F\boldsymbol{v})\boldsymbol{n}_F^{\intercal} - \boldsymbol{n}_F(\Pi_F\boldsymbol{v})^{\intercal}\right)\boldsymbol{n}_F = \Pi_F\boldsymbol{v},
$$

i.e. the second identify holds. $\square$

Thanks to (12), the tangential part $\Pi_F\boldsymbol{v}$ and the tangential trace $(\mathrm{tr}_F^{\mathrm{curl}}\,\boldsymbol{v})$ are equivalent.

4.2. **Geometric Decompositions ~~of Edge elements~~.** Define the polynomial bubble space for the curl operator as

$$\mathbb{B}_r(\mathrm{curl}\,;T) = \ker(\mathrm{tr}^{\mathrm{curl}}_{F}) \cap \mathbb{P}_r(T;\mathbb{E}^n).$$

For Lagrange bubble $\mathbb{B}_r^n(T)$, all components of the vector vanish on ∂T and thus vanish on all sub-simplex with dimension $\leq n-1$. For $\boldsymbol{u} \in \mathbb{B}_r(\mathrm{curl}\,;T)$, only the tangential component vanishes, ~~i.e., $\boldsymbol{u}\times\boldsymbol{n}$~~ $\Pi_F u$ $|_{\partial T} = 0$ which will imply $\boldsymbol{u}$ vanishes on sub-simplex with dimension less than or equal to $n-2$.

Lemma 4.2. *For $\boldsymbol{u} \in \mathbb{B}_r(\mathrm{curl}\,;T)$, it holds $\boldsymbol{u}|_f = \boldsymbol{0}$ for all $f \in \Delta_\ell(T), 0 \leq \ell \leq n-2$. Consequently $\boldsymbol{u} \in \bigoplus_{F\in\Delta_{n-1}(T)} \mathbb{B}_r^n(F)$.*

Proof. It suffices to consider $f \in \Delta_{n-2}(T)$. Let $F_1, F_2 \in \Delta_{n-1}(T)$ such that $f = F_1 \cap F_2$. By $\mathrm{tr}^{\mathrm{curl}}_{F_i}\,\boldsymbol{u} = \boldsymbol{0}$ for $i = 1, 2$, we have

$$(\boldsymbol{u}\cdot\boldsymbol{t}_i^f)|_f = 0, \quad (\boldsymbol{u}\cdot\boldsymbol{n}_{F_1,f})|_f = (\boldsymbol{u}\cdot\boldsymbol{n}_{F_2,f})|_f = 0 \qquad \text{for } i = 1,\ldots,n-2,$$

where $\boldsymbol{n}_{F_i,f}$ is a normal vector f sitting on F_i. As $\mathrm{span}\{\boldsymbol{t}_1^f,\ldots,\boldsymbol{t}_{n-2}^f,\boldsymbol{n}_{F_1,f},\boldsymbol{n}_{F_2,f}\} =$ $\mathbb{R}^n$ $\mathbb{E}^n$, we acquire $\boldsymbol{u}|_f = \boldsymbol{0}$. By the property of face bubbles, we conclude $\boldsymbol{u}$ is a linear combination of $n-1$ face bubbles. $\square$

Obviously $\mathbb{B}_r^n(T) \subset \mathbb{B}_r(\mathrm{curl}\,;T)$. As $\mathrm{tr}^{\mathrm{curl}}_{F}$ contains the tangential component only, the normal component $\mathbb{B}_r(F)\boldsymbol{n}_F$ is also a curl bubble. The following result says their sum is precisely all curl bubble polynomials.

Theorem 4.3. *For $r \geq 1$, it holds that*

$$\mathbb{B}_r(\mathrm{curl}\,;T) = \mathbb{B}_r^n(T) \oplus \bigoplus_{F\in\Delta_{n-1}(T)} \mathbb{B}_r(F)\boldsymbol{n}_F, \tag{13}$$

and

$$\mathrm{tr}^{\mathrm{curl}}_{F} : \mathbb{P}_1(T) \oplus \bigoplus_{\ell=1}^{n-2} \bigoplus_{f\in\Delta_\ell(T)} \mathbb{B}_r^n(f) \oplus \bigoplus_{F\in\Delta_{n-1}(T)} \mathbb{B}_r^{n-1}(F) \to \mathrm{tr}^{\mathrm{curl}}_{F}\,\mathbb{P}_r^n(T) \tag{14}$$

is a bijection.

Proof. It is obvious that

$$\mathbb{B}_r^n(T) \oplus \bigoplus_{F\in\Delta_{n-1}(T)} \mathbb{B}_r(F)\boldsymbol{n}_F \subseteq \mathbb{B}_r(\mathrm{curl}\,, T).$$

Then apply the trace operator to the decomposition (5) to conclude that the map $\mathrm{tr}^{\mathrm{curl}}_{F}$ in (14) is onto.

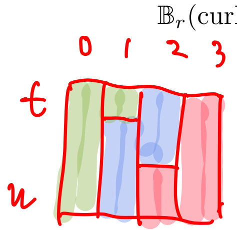
Now we prove it is also injective. Take a function $\boldsymbol{u} \in \mathbb{P}_1(T) \oplus \bigoplus_{\ell=1}^{n-2} \bigoplus_{f\in\Delta_\ell(T)} \mathbb{B}_r^n(f) \oplus \bigoplus_{F\in\Delta_{n-1}(T)} \mathbb{B}_r^{n-1}(F)$ and $\mathrm{tr}^{\mathrm{curl}}_{F}\,\boldsymbol{u} = \boldsymbol{0}$. By Lemma 4.2, we can assume $\boldsymbol{u} = \sum_{F\in\Delta_{n-1}(T)} \boldsymbol{u}_r^F$ with $\boldsymbol{u}_r^F \in \mathbb{B}_r^{n-1}(F)$. Take $F \in \Delta_{n-1}(T)$. We have $\boldsymbol{u}|_F = \boldsymbol{u}_r^F|_F \in \mathbb{B}_r^{n-1}(F)$. Hence $(\boldsymbol{u}_r^F\cdot\boldsymbol{t})|_F = (\boldsymbol{u}\cdot\boldsymbol{t})|_F = 0$ for any $\boldsymbol{t} \in \mathscr{T}^F$, which results in $\boldsymbol{u}_r^F = \boldsymbol{0}$. Therefore $\boldsymbol{u} = \boldsymbol{0}$.

Once we have proved the map tr in (14) is bijection, we conclude (13) from the decomposition (5). $\square$

Similar to the previous case, the normal component of the edge $e \in \partial f$ will also contribute to the bubble functions. For $f \in \Delta_\ell(T), \ell = 2,\ldots,n-1$, by applying Theorem 4.3 to f, we have → curl_f $n_{f,e}$.

$$\mathbb{B}_r(\mathrm{curl}_f; f) = \mathbb{B}_r^\ell(f) \oplus \bigoplus_{e\in\partial f} \mathbb{B}_r(e)\boldsymbol{n}_{f,e}.$$

Fig.



Here curl_f operator is the exterior derivative of an 1-form on f. The curl_f-bubble function is defined for $\ell \geq 2$ not including edges. Indeed, for $\mathrm{v} \in \Delta_0(T)$, $\boldsymbol{n}_{e,\mathrm{v}}$ is $\boldsymbol{t}_e$ or $-\boldsymbol{t}_e$ where the sign depends on the orientation. Then for $\ell = 1$

$$\mathbb{B}_r(e)\boldsymbol{t}_e \oplus \bigoplus_{\mathrm{v}\in\partial e}\mathbb{B}_r(\mathrm{v})\boldsymbol{n}_{e,\mathrm{v}} = \mathbb{P}_r(e)\boldsymbol{t}_e.$$

which is no longer a bubble function.

Lemma 4.4. *For $r \geq 1$, we have*

$$\mathbb{P}_r^n(T) = \mathbb{P}_1^n(T) \oplus \bigoplus_{e\in\Delta_1(T)}\mathbb{B}_r(e)\boldsymbol{t}_e \oplus \bigoplus_{\ell=2}^n\bigoplus_{f\in\Delta_\ell(T)}\mathbb{B}_r(\operatorname{curl}_f; f), \tag{15}$$

$$\mathbb{P}_r^n(T) = \bigoplus_{e\in\Delta_1(T)}\mathbb{P}_r(e)\boldsymbol{t}_e \oplus \bigoplus_{\ell=2}^n\bigoplus_{f\in\Delta_\ell(T)}\mathbb{B}_r(\operatorname{curl}_f; f). \tag{16}$$

Proof. It is a re-arrangement of components in the decomposition (5). For $e \in \Delta_{\ell-1}(T)$, the face normal vectors $\{\boldsymbol{n}_{f,e} : f \in \Delta_\ell(T), e \subseteq f\}$ form a basis of $\mathscr{N}^e$. So we have

$$\mathbb{B}_r(e) \otimes \mathscr{N}^e = \bigoplus_{f\in\Delta_\ell(T), e\subseteq f}\mathbb{B}_r(e)\boldsymbol{n}_{f,e}.$$

Then shift the normal component one level up to get the decomposition (15).

We then distribute the n-component of vector function value at the vertices to the n edges connected to this vertex

$$\mathbb{P}_1^n(T) \oplus \bigoplus_{e\in\Delta_1(T)}\mathbb{B}_r(e)\boldsymbol{t}_e = \bigoplus_{e\in\Delta_1(T)}\left(\mathbb{P}_1(e)\boldsymbol{t}_e \oplus \mathbb{B}_r(e)\boldsymbol{t}_e\right).$$

Thus (16) holds. $\square$

Decomposition (15) is the counterpart of (3) for Lagrange elements. In decomposition (16), the linear polynomial component is redistributed to edges and the second kind Nédélec element [6] can be derived from a special $t - n$ basis. To emphasize the dependence on edges, we shall use e instead f. in the DoFs

Lemma 4.5 (Local Nédélec element)**.** *For $e \in \Delta_\ell(T)$, let $\{\boldsymbol{t}_i^e, i = 1, \dots, \ell\}$ be a basis of the tangent plane of e and choose $\{\boldsymbol{n}_{f,e} : f \in \Delta_{\ell+1}(T), e \subseteq f\}$ as the basis of $\mathscr{N}^e$. The shape function space $\mathbb{P}_r^n(T)$ is uniquely determined by the DoFs*

$$\boldsymbol{v}\cdot\boldsymbol{t}_e(\mathrm{v}), \quad e \in \Delta_1(T), \mathrm{v} \in \partial e, \tag{17}$$

$$\int_e (\boldsymbol{v}\cdot\boldsymbol{t}_i^e)\, p \,\mathrm{d}s, \quad p \in \mathbb{P}_{r-(\ell+1)}(e), e \in \Delta_\ell(T), \quad i = 1, \dots, \ell, \ell = 1, \dots, n-1, \tag{18}$$

$$\int_e (\boldsymbol{v}\cdot\boldsymbol{n}_{f,e})\, p \,\mathrm{d}s, \quad p \in \mathbb{P}_{r-(\ell+1)}(e), e \in \Delta_\ell(T), \quad f \in \Delta_{\ell+1}(T), e \subseteq f, \ell = 1, \dots, n-1, \tag{19}$$

$$\int_T \boldsymbol{v}\cdot\boldsymbol{p}\,\mathrm{d}x \quad \boldsymbol{p} \in \mathbb{P}_{r-(n+1)}^n(T). \tag{20}$$

Proof. First of all, by the geometric decomposition (5) of $\mathbb{P}_r^n(T)$ and Theorem 4.3, the number of DoFs is equal to the dimension of the shape function space.

Assume $\boldsymbol{v} \in \mathbb{P}_r^n(T)$ and all the DoFs (17)-(20) vanish. Since $\{\boldsymbol{t}_e, e \in \Delta_1(T), \mathrm{v} \in \partial e\}$ is a basis of $\mathbb{E}^n$, $\{(\boldsymbol{v}\cdot\boldsymbol{t}_e)(\mathrm{v}), e \in \Delta_1(T), \mathrm{v} \in \partial e\}$ will determine the vector $\boldsymbol{v}(\mathrm{v})$. Thus vanishing (17) implies $\boldsymbol{v}$ is zero at vertices. In general, $\{\boldsymbol{n}_{f,e} : f \in \Delta_{\ell+1}(T), e \subseteq f\}$ forms a basis of $\mathscr{N}^e$. DoF (19) is equivalent to

$$\int_e (\boldsymbol{v}\cdot\boldsymbol{n}_i^e)\, p \,\mathrm{d}s, \quad p \in \mathbb{P}_{r-(\ell+1)}(e), e \in \Delta_\ell(T), i = 1, \dots, n-\ell,\ \ell = 1, \dots, n-1,$$

which together with vanishing DoF (18) implies

$$\int_f \boldsymbol{v}\cdot\boldsymbol{p}\,\mathrm{d}s = 0, \quad \boldsymbol{p}\in\mathbb{P}_{r-(\ell+1)}(f;\mathbb{R}^n), f\in\Delta_\ell(T), \ell=1,\dots,n-1.$$

It follows from the uni-solvence of Lagrange element that $\boldsymbol{v}|_{\partial T}=\boldsymbol{0}$, i.e. $\boldsymbol{v}\in\mathbb{B}_r^n(T)$. Finally $\boldsymbol{v}=\boldsymbol{0}$ is an immediate result of the vanishing DoF (20). □

Lemma 4.6 (Nédélec space)**.** *The following* DoFs

How to choose $n_{f,e}$?

(21) $$\boldsymbol{v}\cdot\boldsymbol{t}_e(\mathrm{v}), \quad e\in\Delta_1(\mathcal{T}_h), \mathrm{v}\in\partial e,$$

(22) $$\int_e (\boldsymbol{v}\cdot\boldsymbol{t}_i^e)\,p\,\mathrm{d}s, \quad p\in\mathbb{P}_{r-(\ell+1)}(e), e\in\Delta_\ell(\mathcal{T}_h), \quad i=1,\dots,\ell, \ell=1,\dots,n-1,$$

(23) $$\int_e (\boldsymbol{v}\cdot\boldsymbol{n}_{f,e})\,p\,\mathrm{d}s, \quad p\in\mathbb{P}_{r-(\ell+1)}(e),, e\in\Delta_\ell(\mathcal{T}_h), \quad f\in\Delta_{\ell+1}(\mathcal{T}_h), e\subseteq f, \ell=1,\dots,n-1,$$

(24) $$\int_T \boldsymbol{v}\cdot\boldsymbol{p}\,\mathrm{d}x \quad \boldsymbol{p}\in\mathbb{P}^n_{r-(n+1)}(T), T\in\mathcal{T}_h$$

define a curl-conforming space $V_h=\{\boldsymbol{v}_h\in H(\mathrm{curl}\,;\Omega):\boldsymbol{v}_h|_T\in\mathbb{P}_r^n(T)\ \ \forall\, T\in\mathcal{T}_h\}$.

Proof. On each element T, DoFs (21)-(24) will determine a function in $\mathbb{P}_r^n(T)$ by Lemma 4.5. For $F\in\Delta_{n-1}(\mathcal{T}_h)$, DoFs (21)-(23) restricted to F are

more explanation

$$\boldsymbol{v}\cdot\boldsymbol{t}_e(\mathrm{v}), \quad e\in\Delta_1(F), \mathrm{v}\in\partial e,$$

$$\int_e (\boldsymbol{v}\cdot\boldsymbol{t}_i^e)\,p\,\mathrm{d}s, \quad p\in\mathbb{P}_{r-(\ell+1)}(e), e\in\Delta_\ell(F), \quad i=1,\dots,\ell, \ell=1,\dots,n-1,$$

$$\int_e (\boldsymbol{v}\cdot\boldsymbol{n}_{f,e})\,p\,\mathrm{d}s, \quad e\in\Delta_\ell(F), f\in\Delta_{\ell+1}(F), e\subseteq f, \quad p\in\mathbb{P}_{r-(\ell+1)}(e), \ell=1,\dots,n-2.$$

$\Pi_F v$

Since $\{\boldsymbol{t}_1^e,\dots,\boldsymbol{t}_\ell^e,\boldsymbol{n}_{f,e}, f\in\Delta_{\ell+1}(F), e\subseteq f\}$ spans the tangent plane $\mathscr{T}^F$, by the unisolvence of the Lagrange element, these DoFs will determine the trace $\mathrm{tr}^{\mathrm{curl}}_F\boldsymbol{v}$ on F independent of the elements containing F and thus the function is $H(\mathrm{curl}\,;\Omega)$-conforming. □

The vertex DoF can be merged into the edge DoF and result in the classical DoF

$$\int_e \boldsymbol{v}\cdot\boldsymbol{t}^e\,p\,\mathrm{d}s \quad p\in\mathbb{P}_r(e), e\in\Delta_1(\mathcal{T}_h).$$

It can be also used to define an $H(\mathrm{curl}\,;\Omega)$-conforming finite element space with vertex continuity [5].

In general, given an integer $-1\le k\le n-1$, we can split the DoFs: for $\ell\le k$, it is Lagrange and for $\ell>k$, it is Nédélec. It returns to the vector Lagrange element when $k=n-1$, and Nédélec element for $k=-1$.

Lemma 4.7. *The following* DoFs $n_{f,e}.$

do we need such generality?

$$\int_e \boldsymbol{v}\cdot\boldsymbol{p}\,\mathrm{d}s,\quad \boldsymbol{p}\in\mathbb{P}^n_{r-(\ell+1)}(e), e\in\Delta_\ell(\mathcal{T}_h), \ell=0,\ldots,k, \tag{25}$$

$$\int_e (\boldsymbol{v}\cdot\boldsymbol{t}_i^e)\,p\,\mathrm{d}s,\quad p\in\mathbb{P}_{r-(\ell+1)}(e), e\in\Delta_\ell(\mathcal{T}_h), \quad i=1,\ldots,\ell, \ell=k+1,\ldots,n-1, \tag{26}$$

$$\int_e (\boldsymbol{v}\cdot\boldsymbol{n}_{f,e})\,p\,\mathrm{d}s,\quad p\in\mathbb{P}_{r-(\ell+1)}(e), e\in\Delta_\ell(\mathcal{T}_h), \quad f\in\Delta_{\ell+1}(\mathcal{T}_h), e\subseteq f, \ell=k+1,\ldots,n-1, \tag{27}$$

$$\int_T \boldsymbol{v}\cdot\boldsymbol{p}\,\mathrm{d}x\quad \boldsymbol{p}\in\mathbb{P}^n_{r-(n+1)}(T), T\in\mathcal{T}_h, \tag{28}$$

define a curl-conforming space $V_h = \{\boldsymbol{v}_h \in H(\mathrm{curl};\Omega)\cap C^0(\Delta_k(\mathcal{T}_h)) : \boldsymbol{v}_h|_T \in \mathbb{P}^n_r(T)\ \ \forall\, T\in\mathcal{T}_h\}$*, where* $H(\mathrm{curl};\Omega)\cap C^0(\Delta_{-1}(\mathcal{T}_h)) = H(\mathrm{curl};\Omega)$.

Proof. Clearly the number of DoFs (25)-(28) equals to the number of DoFs (21)-(24). DoF (25) determines DoFs (21)-(23) for $\ell = 0,\ldots,k$. Then we conclude the result from Lemma 4.6. □

With vertex DoFs, we can choose the basis of the edge element based on those of the Lagrange element, which is related to the geometric decomposition (15). The curl-conforming space V_h defined in Lemma 4.6 is same as the second kind Nédélec element space, V_h in Lemma 4.7 with $k = 0$ same as the curl-conforming space in [5] , while DoFs (21)-(24) are different from those in [6, 5]. And the corresponding geometric decomposition (15) is also different from those in [2, 1]. The geometric decomposition (15) enable the use of Lagrange basis.

5. High order Lagrange Finite Element ?

In this section we introduce higher-order node-type Lagrange elements and define high-dimensional BDM elements and second kind Nédélec elements based on the aforementioned geometric decomposition.

5.1. **Lagrange Element.** merge to Section 2.

Lemma 5.1 (Lagrange finite element on simplex)**.** *The function in* $\mathbb{P}_k(T)$ *is uniquely determined by the DoFs:*

$$u(\boldsymbol{x}),\quad \boldsymbol{x}\in\mathcal{X}_T$$

The basis function of kth order Lagrange finite element space on T *can be explicitly written as:*

$$\phi_{\boldsymbol{\alpha}}(\boldsymbol{x}) = \frac{1}{\boldsymbol{\alpha}!}\prod_{i=0}^{n}\prod_{j=0}^{\alpha_i-1}(k\lambda_i(\boldsymbol{x})-j),\quad \boldsymbol{\alpha}\in\mathcal{A}_n^k$$

Definition 5.2 (DoFs of Lagrange Finite Element on $\mathcal{T}_h$)**.** *For the Lagrange element space* $\mathscr{L}(\mathcal{T}_h)$*, define the dofs in* $\mathscr{L}(\mathcal{T}_h)$ *as: for any* $\boldsymbol{x}\in\mathcal{X}$*, the value of* u *at* $\boldsymbol{x}$

$$u(\boldsymbol{x}),\quad \boldsymbol{x}\in\mathcal{X}.$$

Define $(e_x^i, \hat{e}_x^j) = \delta_{ij}$

t_e, n_e: unit tangential (... ...).

5.2. BDM element on triangle mesh.

merge into section 3.

Definition 5.3. *Let T be a triangle, for any $\boldsymbol{x} \in \mathcal{X}_T$, define a frame $\{\boldsymbol{e}_{\boldsymbol{x}}^0, \boldsymbol{e}_{\boldsymbol{x}}^1\}$ at $\boldsymbol{x}$ and its dual frame $\{\hat{\boldsymbol{e}}_{\boldsymbol{x}}^0, \hat{\boldsymbol{e}}_{\boldsymbol{x}}^1\}$ as follows:*

- (1) *If $\boldsymbol{x} \in \Delta_0(T)$, the two adjacent edges are e_0 and e_1, where the local indexing of e_0 is greater than that of e_1, then*

$$\boldsymbol{e}_{\boldsymbol{x}}^0 = \boldsymbol{n}_{e_0}, \quad \boldsymbol{e}_{\boldsymbol{x}}^1 = \boldsymbol{n}_{e_1}, \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^0 = \frac{\boldsymbol{t}_{e_1}}{\boldsymbol{t}_{e_1} \cdot \boldsymbol{n}_{e_0}}, \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^1 = \frac{\boldsymbol{t}_{e_0}}{\boldsymbol{t}_{e_0} \cdot \boldsymbol{n}_{e_1}}.$$

- (2) *If $\boldsymbol{x} \in \mathring{\mathcal{X}}_e, e \in \Delta_1(T)$, then*

$$\boldsymbol{e}_{\boldsymbol{x}}^0 = \boldsymbol{n}_e, \quad \boldsymbol{e}_{\boldsymbol{x}}^1 = \boldsymbol{t}_e, \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^0 = \boldsymbol{n}_e, \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^1 = \boldsymbol{t}_e.$$

- (3) *If $\boldsymbol{x} \in \mathring{\mathcal{X}}_T$, then*

$$\boldsymbol{e}_{\boldsymbol{x}}^0 = (1,0), \quad \boldsymbol{e}_{\boldsymbol{x}}^1 = (0,1), \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^0 = (1,0), \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^1 = (0,1).$$

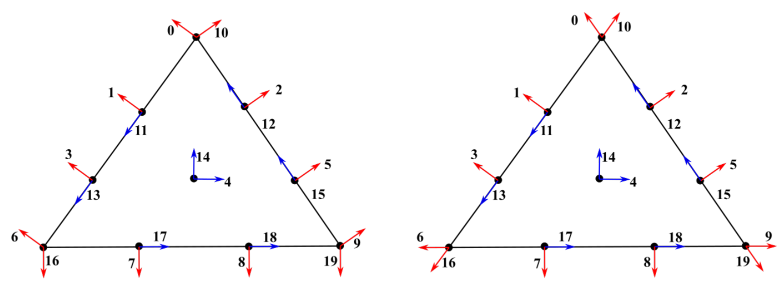


FIGURE 2. The left figure shows $\{\boldsymbol{e}_0, \boldsymbol{e}_1\}$ at each interpolation point, the right figure shows $\{\hat{\boldsymbol{e}}_0, \hat{\boldsymbol{e}}_1\}$ at each interpolation point

Lemma 5.4 (BDM element on triangle)**.** *Any function $\boldsymbol{u} \in \mathbb{P}_k^2(T)$ can be uniquely determined by the DoFs:*

$$\boldsymbol{u}(x) \cdot \boldsymbol{e}_{\boldsymbol{x}}^i, \quad \boldsymbol{x} \in \mathcal{X}_T, i = 0, 1.$$

The basis function of kth order BDM element space on T can be explicitly written as:

$$\boldsymbol{\phi}_{\boldsymbol{\alpha}}^i(\boldsymbol{x}) = \phi_{\boldsymbol{\alpha}}(\boldsymbol{x}) \hat{\boldsymbol{e}}_{\boldsymbol{x}_{\boldsymbol{\alpha}}}^i, \quad \boldsymbol{\alpha} \in \mathcal{A}_2^k, i = 0, 1$$

t_e is local n_e is global.

Lemma 5.5 (BDM space)**.** *The following* DoFs

$$\underline{\boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{t}_e, \quad \boldsymbol{x} \in \mathcal{X}_e} \tag{29}$$

? incorrect.

$$\boldsymbol{u}(\boldsymbol{x})|_T \cdot \boldsymbol{n}_e, \quad \boldsymbol{x} \in \mathring{\mathcal{X}}_e, e \in \Delta_1(T) \tag{30}$$

$$\boldsymbol{u}(\boldsymbol{x}) \cdot (1,0), \quad \boldsymbol{u}(\boldsymbol{x}) \cdot (0,1), \quad \boldsymbol{x} \in \mathring{\mathcal{X}}_T \tag{31}$$

H( )

define a div-conforming space $V_h = \{\boldsymbol{v}_h \in H(\mathrm{div}; \Omega) : \boldsymbol{v}_h|_T \in \mathbb{P}_r^2(T) \ \ \forall T \in \mathcal{T}_h\}$.

We call (29) as DoF on edge, (30) and (31) as DoF in cell.

5.3. **BDM element on tetrahedron mesh.**

Definition 5.6. *Let T be a tetrahedron, for any $\boldsymbol{x} \in \mathcal{X}_T$, define a frame $\{\boldsymbol{e}_{\boldsymbol{x}}^0, \boldsymbol{e}_{\boldsymbol{x}}^1, \boldsymbol{e}_{\boldsymbol{x}}^2\}$ at $\boldsymbol{x}$ and its dual frame $\{\hat{\boldsymbol{e}}_{\boldsymbol{x}}^0, \hat{\boldsymbol{e}}_{\boldsymbol{x}}^1, \hat{\boldsymbol{e}}_{\boldsymbol{x}}^2\}$ as follows:*

- *If $\boldsymbol{x} \in \Delta_0(T)$, adjacent edges of $\boldsymbol{x}$ are e_0, e_1, e_2, the adjacent faces of $\boldsymbol{x}$ are f_0, f_1, f_2, where f_0, f_1, f_2 are sorted by local indexing from large to small satisfying $f_i \cap e_i = \boldsymbol{x}$, then*

$$\boldsymbol{e}_{\boldsymbol{x}}^0 = \boldsymbol{n}_{f_0}, \quad \boldsymbol{e}_{\boldsymbol{x}}^1 = \boldsymbol{n}_{f_1}, \quad \boldsymbol{e}_{\boldsymbol{x}}^2 = \boldsymbol{n}_{f_2}.$$

$$\hat{\boldsymbol{e}}_{\boldsymbol{x}}^0 = \frac{\boldsymbol{t}_{e_0}}{\boldsymbol{t}_{e_0} \cdot \boldsymbol{n}_{f_0}}, \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^1 = \frac{\boldsymbol{t}_{e_1}}{\boldsymbol{t}_{e_1} \cdot \boldsymbol{n}_{f_1}}, \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^2 = \frac{\boldsymbol{t}_{e_2}}{\boldsymbol{t}_{e_2} \cdot \boldsymbol{n}_{f_2}}.$$

- *If $\boldsymbol{x} \in \mathring{\mathcal{X}}_e, e \in \Delta_1(T)$ and adjacent faces are f_0, f_1 then*

$$\boldsymbol{e}_{\boldsymbol{x}}^0 = \boldsymbol{n}_{f_0}, \quad \boldsymbol{e}_{\boldsymbol{x}}^1 = \boldsymbol{n}_{f_1}, \quad \boldsymbol{e}_{\boldsymbol{x}}^2 = \boldsymbol{t}_e,$$

$$\hat{\boldsymbol{e}}_{\boldsymbol{x}}^0 = \frac{\boldsymbol{n}_{f_1} \times \boldsymbol{t}_e}{\boldsymbol{n}_{f_0} \cdot (\boldsymbol{n}_{f_1} \times \boldsymbol{t}_e)}, \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^1 = \frac{\boldsymbol{n}_{f_0} \times \boldsymbol{t}_e}{\boldsymbol{n}_{f_1} \cdot (\boldsymbol{n}_{f_0} \times \boldsymbol{t}_e)}, \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^2 = \boldsymbol{t}_e$$

- *If $\boldsymbol{x} \in \mathring{\mathcal{X}}_f, f \in \Delta_2(T)$, the first edge of f is e, then*

$$\boldsymbol{e}_{\boldsymbol{x}}^0 = \boldsymbol{n}_f, \quad \boldsymbol{e}_{\boldsymbol{x}}^1 = \boldsymbol{t}_e \times \boldsymbol{n}_f, \quad \boldsymbol{e}_{\boldsymbol{x}}^2 = \boldsymbol{t}_e.$$

$$\hat{\boldsymbol{e}}_{\boldsymbol{x}}^0 = \boldsymbol{n}_f, \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^1 = \boldsymbol{t}_e \times \boldsymbol{n}_f, \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^2 = \boldsymbol{t}_e.$$

- *If $\boldsymbol{x} \in \mathring{\mathcal{X}}_T$, then*

$$\boldsymbol{e}_{\boldsymbol{x}}^0 = (1, 0, 0), \quad \boldsymbol{e}_{\boldsymbol{x}}^1 = (0, 1, 0), \quad \boldsymbol{e}_{\boldsymbol{x}}^2 = (0, 0, 1).$$

$$\hat{\boldsymbol{e}}_{\boldsymbol{x}}^0 = (1, 0, 0), \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^1 = (0, 1, 0), \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^2 = (0, 0, 1).$$

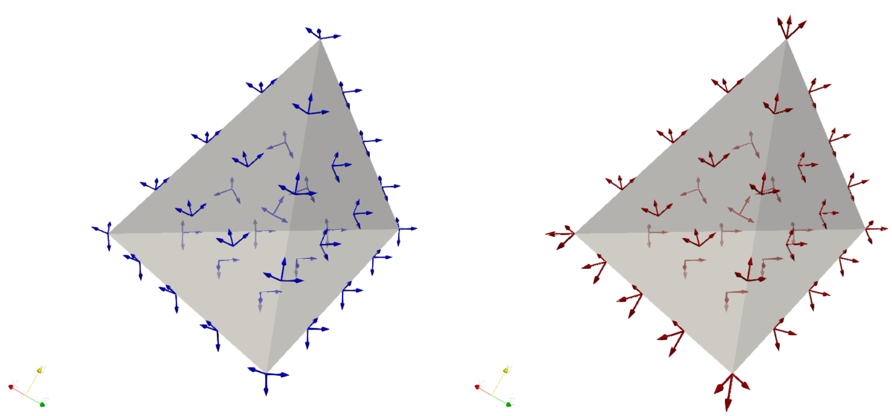


FIGURE 3. The left figure shows $\{\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2\}$ at each interpolation point, the right figure shows $\{\hat{\boldsymbol{e}_0}, \hat{\boldsymbol{e}_1}, \hat{\boldsymbol{e}_2}\}$ at each interpolation point

Lemma 5.7 (BDM element on tetrahedron)**.** *Any function $\boldsymbol{u} \in \mathbb{P}_k^3(T)$ can be uniquely determined by the DoFs:*

$$\boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{e}_{\boldsymbol{x}}^i, \quad \boldsymbol{x} \in \mathcal{X}_T, i = 0, 1, 2.$$

The basis function of kth order BDM element space on T can be explicitly written as:

$$\boldsymbol{\phi}_{\boldsymbol{\alpha}}^i(\boldsymbol{x}) = \phi_{\boldsymbol{\alpha}}(\boldsymbol{x}) \hat{\boldsymbol{e}}_{\boldsymbol{x}_{\boldsymbol{\alpha}}}^i, \quad \boldsymbol{\alpha} \in \mathcal{A}_3^k, i = 0, 1, 2$$

choice of te and tf are local.

Lemma 5.8 (BDM element on tetrahedron mesh)**.** *The following* DoFs

$$\boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{n}_f, \quad \boldsymbol{x} \in \mathcal{X}_f, f \in \Delta_2(T) \tag{32}$$

$$\boldsymbol{u}(\boldsymbol{x})|_T \cdot \boldsymbol{t}_f, \quad \boldsymbol{u}(\boldsymbol{x})|_T \cdot (\boldsymbol{n}_f \times \boldsymbol{t}_f), \quad \boldsymbol{x} \in \mathring{\mathcal{X}}_f, f \in \Delta_2(T) \tag{33}$$

$$\boldsymbol{u}(\boldsymbol{x})|_T \cdot \boldsymbol{t}_e, \quad \boldsymbol{x} \in \mathring{\mathcal{X}}_e, \ e \in \Delta_1(T) \tag{34}$$

$$\boldsymbol{u}(\boldsymbol{x}) \cdot (1, 0, 0), \quad \boldsymbol{u}(\boldsymbol{x}) \cdot (0, 1, 0), \quad \boldsymbol{u}(\boldsymbol{x}) \cdot (0, 0, 1), \quad \boldsymbol{x} \in \mathring{\mathcal{X}}_T \tag{35}$$

define a curl-conforming space $V_h = \{\boldsymbol{v}_h \in H(\mathrm{div}; \Omega) : \boldsymbol{v}_h|_T \in \mathbb{P}_k^3(T), \ \forall\, T \in \mathcal{T}_h\}$.

We call (32) as DoF on face, (33), (34) and (35) as DoF in cell.

5.4. Nédélec element on triangle.

merge to Section 4.

Definition 5.9. *Let T be a triangle, for any $\boldsymbol{x} \in \mathcal{X}_T$, define a frame $\{\boldsymbol{e}_{\boldsymbol{x}}^0, \boldsymbol{e}_{\boldsymbol{x}}^1\}$ at $\boldsymbol{x}$ and its dual frame $\{\hat{\boldsymbol{e}}_{\boldsymbol{x}}^0, \hat{\boldsymbol{e}}_{\boldsymbol{x}}^1\}$ as follows:*

- *If $\boldsymbol{x} \in \Delta_0(T)$, the two adjacent edges are e_0 and e_1, where the local indexing of e_0 is greater than that of e_1, then*

$$\boldsymbol{e}_{\boldsymbol{x}}^0 = \boldsymbol{t}_{e_0}, \quad \boldsymbol{e}_{\boldsymbol{x}}^1 = \boldsymbol{t}_{e_1}. \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^0 = \frac{\boldsymbol{n}_{e_1}}{\boldsymbol{n}_{e_1} \cdot \boldsymbol{t}_{e_0}}, \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^1 = \frac{\boldsymbol{n}_{e_0}}{\boldsymbol{n}_{e_0} \cdot \boldsymbol{t}_{e_1}}.$$

- *If $\boldsymbol{x} \in \mathring{\mathcal{X}}_e, e \in \Delta_1(T)$, then*

$$\boldsymbol{e}_{\boldsymbol{x}}^0 = \boldsymbol{t}_e, \quad \boldsymbol{e}_{\boldsymbol{x}}^1 = \boldsymbol{n}_e, \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^0 = \boldsymbol{t}_e, \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^1 = \boldsymbol{n}_e.$$

  em If $\boldsymbol{x} \in \mathring{\mathcal{X}}_T$, then

$$\boldsymbol{e}_{\boldsymbol{x}}^0 = (1, 0), \quad \boldsymbol{e}_{\boldsymbol{x}}^1 = (0, 1), \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^0 = (1, 0), \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^1 = (0, 1).$$

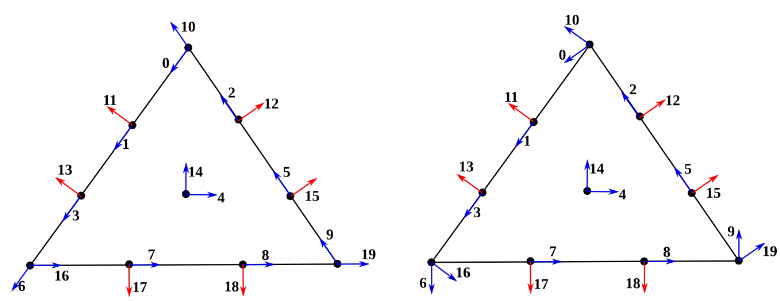


FIGURE 4. The left figure shows $\{\boldsymbol{e}_0, \boldsymbol{e}_1\}$ at each interpolation point, the right figure shows $\{\hat{\boldsymbol{e}_0}, \hat{\boldsymbol{e}_1}\}$ at each interpolation point

Lemma 5.10 (Nédélec element on a triangle)**.** *Any function $\boldsymbol{u} \in \mathbb{P}_k^2(T)$ can be uniquely determined by the DoFs:*

$$\boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{e}_{\boldsymbol{x}}^i, \quad \boldsymbol{x} \in \mathcal{X}_T, i = 0, 1.$$

The basis function of 2-dim kth order Nédélec element space on T can be explicitly written as:

$$\boldsymbol{\phi}_{\boldsymbol{\alpha}}^i(\boldsymbol{x}) = \phi_{\boldsymbol{\alpha}}(\boldsymbol{x})\hat{\boldsymbol{e}}_{\boldsymbol{x}_{\boldsymbol{\alpha}}}^i, \quad \boldsymbol{\alpha} \in \mathcal{A}_2^k, i = 0, 1$$

Lemma 5.11 (Nédélec space on $\mathcal{T}_h$)**.** *The following* DoFs

$$\boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{t}_e, \quad \boldsymbol{x} \in \mathcal{X}_e, \tag{36}$$

$$\boldsymbol{u}(\boldsymbol{x})|_T \cdot \boldsymbol{n}_e, \quad \boldsymbol{x} \in \mathring{\mathcal{X}}_e, \ T \in \Delta_2(e) \tag{37}$$

$$\boldsymbol{u}(\boldsymbol{x}) \cdot (1, 0), \quad \boldsymbol{u}(\boldsymbol{x}) \cdot (0, 1), \quad \boldsymbol{x} \in \mathring{\mathcal{X}}_T \tag{38}$$

defines a curl-conforming space $V_h = \{\boldsymbol{v}_h \in H(\mathrm{curl}\,;\Omega) : \boldsymbol{v}_h|_T \in \mathbb{P}_r^2(T)\, \forall T \in \mathcal{T}_h\}$.

We call (36) as DoF on edge, (37) and (38) as DoF in cell.

5.5. **Nédélec element on tetrahedron mesh.**

Definition 5.12. *Let T be a tetrahedron, for any $\boldsymbol{x} \in \mathcal{X}_T$, define a frame $\{\boldsymbol{e}_{\boldsymbol{x}}^0, \boldsymbol{e}_{\boldsymbol{x}}^1, \boldsymbol{e}_{\boldsymbol{x}}^2\}$ at $\boldsymbol{x}$ and its dual frame $\{\hat{\boldsymbol{e}}_{\boldsymbol{x}}^0, \hat{\boldsymbol{e}}_{\boldsymbol{x}}^1, \hat{\boldsymbol{e}}_{\boldsymbol{x}}^2\}$ as follows:*

- *If $\boldsymbol{x} \in \Delta_0(T)$ and adjacent edges of $\boldsymbol{x}$ are e_0, e_1, e_2, adjacent faces of $\boldsymbol{x}$ are f_0, f_1, f_2, where f_0, f_1, f_2 are sorted by local indexing from large to small, statify $f_i \cap e_i = \boldsymbol{x}$, then*

$$\boldsymbol{e}_{\boldsymbol{x}}^0 = \boldsymbol{t}_{e_0}, \quad \boldsymbol{e}_{\boldsymbol{x}}^1 = \boldsymbol{t}_{e_1}, \quad \boldsymbol{e}_{\boldsymbol{x}}^2 = \boldsymbol{t}_{e_2}.$$

$$\hat{\boldsymbol{e}}_{\boldsymbol{x}}^0 = \frac{\boldsymbol{n}_{f_0}}{\boldsymbol{n}_{f_0} \cdot \boldsymbol{t}_{e_0}}, \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^1 = \frac{\boldsymbol{n}_{f_1}}{\boldsymbol{n}_{f_1} \cdot \boldsymbol{t}_{e_1}}, \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^2 = \frac{\boldsymbol{n}_{f_2}}{\boldsymbol{n}_{f_2} \cdot \boldsymbol{t}_{e_2}}.$$

- *If $\boldsymbol{x} \in \mathring{\mathcal{X}}_e, e \in \Delta_1(T)$ and adjacent faces are f_0, f_1 then*

$$\boldsymbol{e}_{\boldsymbol{x}}^0 = \boldsymbol{t}_e, \quad \boldsymbol{e}_{\boldsymbol{x}}^1 = \boldsymbol{n}_{f_0} \times \boldsymbol{t}_e, \quad \boldsymbol{e}_{\boldsymbol{x}}^2 = \boldsymbol{n}_{f_1} \times \boldsymbol{t}_e.$$

$$\hat{\boldsymbol{e}}_{\boldsymbol{x}}^0 = \boldsymbol{t}_e, \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^1 = \frac{\boldsymbol{n}_{f_1}}{\boldsymbol{n}_{f_1} \cdot (\boldsymbol{n}_{f_0} \times \boldsymbol{t}_e)}, \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^2 = \frac{\boldsymbol{n}_{f_0}}{\boldsymbol{n}_{f_0} \cdot (\boldsymbol{n}_{f_1} \times \boldsymbol{t}_e)}.$$

- *If $\boldsymbol{x} \in \mathring{\mathcal{X}}_f, f \in \Delta_2(T)$, the first edge of f is e, then*

$$\boldsymbol{e}_{\boldsymbol{x}}^0 = \boldsymbol{t}_e, \quad \boldsymbol{e}_{\boldsymbol{x}}^1 = \boldsymbol{t}_e \times \boldsymbol{n}_f, \quad \boldsymbol{e}_{\boldsymbol{x}}^2 = \boldsymbol{n}_f.$$

$$\hat{\boldsymbol{e}}_{\boldsymbol{x}}^0 = \boldsymbol{t}_e, \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^1 = \boldsymbol{t}_e \times \boldsymbol{n}_f, \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^2 = \boldsymbol{n}_f.$$

- *If $\boldsymbol{x} \in \mathring{\mathcal{X}}_T$, then*

$$\boldsymbol{e}_{\boldsymbol{x}}^0 = (1, 0, 0), \quad \boldsymbol{e}_{\boldsymbol{x}}^1 = (0, 1, 0), \quad \boldsymbol{e}_{\boldsymbol{x}}^2 = (0, 0, 1).$$

$$\hat{\boldsymbol{e}}_{\boldsymbol{x}}^0 = (1, 0, 0), \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^1 = (0, 1, 0), \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^2 = (0, 0, 1).$$

Lemma 5.13 (Nédélec element on tetrahedron)**.** *Any function $\boldsymbol{u} \in \mathbb{P}_k^3(T)$ can be uniquely determined by the DoFs:*

$$\boldsymbol{u}(x) \cdot \boldsymbol{e}_{\boldsymbol{x}}^i, \quad \boldsymbol{x} \in \mathcal{X}_T, i = 0, 1, 2.$$

The basis function of kth order Nédélec element space on T can be explicitly written as:

$$\boldsymbol{\phi}_{\boldsymbol{\alpha}}^i(\boldsymbol{x}) = \phi_{\boldsymbol{\alpha}}(\boldsymbol{x})\hat{\boldsymbol{e}}_{\boldsymbol{x}_{\boldsymbol{\alpha}}}^i, \quad \boldsymbol{\alpha} \in \mathcal{A}_3^k, i = 0, 1, 2$$

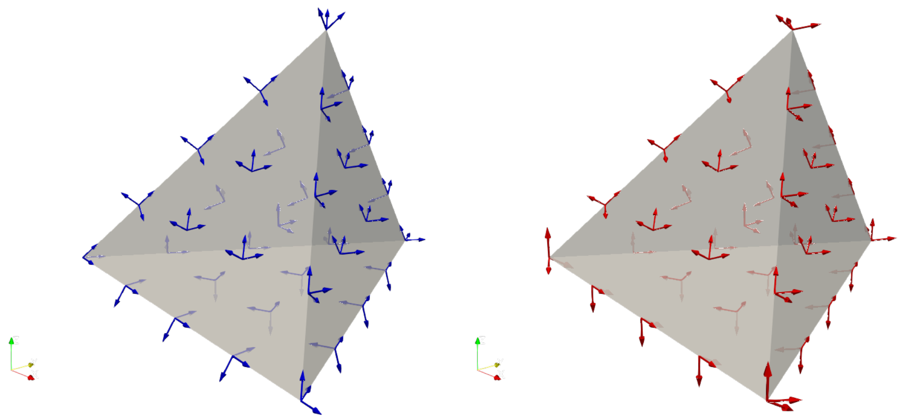


FIGURE 5. The left figure shows $\{\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2\}$ at each interpolation point, the right figure shows $\{\hat{\boldsymbol{e}_0}, \hat{\boldsymbol{e}_1}, \hat{\boldsymbol{e}_2}\}$ at each interpolation point

Lemma 5.14 (Nédélec space on tetrahedron mesh)**.** *The following* DoFs

$$\boldsymbol{u} \cdot \boldsymbol{t}_e(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathcal{X}_e,\ f \in \Delta_2(e) \tag{39}$$

$$\boldsymbol{u}(\boldsymbol{x})|_f \cdot (\boldsymbol{n}_f \times \boldsymbol{t}_e), \quad \boldsymbol{x} \in \mathring{\mathcal{X}}_e,\ f \in \Delta_2(e) \tag{40}$$

$$\boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{t}_f, \quad \boldsymbol{u}(\boldsymbol{x}) \cdot (\boldsymbol{n}_f \times \boldsymbol{t}_f), \quad \boldsymbol{x} \in \mathring{\mathcal{X}}_f \tag{41}$$

$$\boldsymbol{u}(\boldsymbol{x})|_T \cdot \boldsymbol{n}_f, \quad \boldsymbol{x} \in \mathring{\mathcal{X}}_f,\ T \in \Delta_3(f) \tag{42}$$

$$\boldsymbol{u}(\boldsymbol{x}) \cdot (1,0,0), \quad \boldsymbol{u}(\boldsymbol{x}) \cdot (0,1,0), \quad \boldsymbol{u}(\boldsymbol{x}) \cdot (0,0,1), \quad \boldsymbol{x} \in \mathring{\mathcal{X}}_T \tag{43}$$

defines a curl-conforming space $V_h = \{\boldsymbol{v}_h \in H(\mathrm{curl}\,;\Omega) : \boldsymbol{v}_h|_T \in \mathbb{P}_k^3(T),\ \forall T \in \mathcal{T}_h\}$.

We call (39) as DoF on edge, (40) and (41) as DoF in face, (42) and (43) as DoF in cell.

6. IMPLEMENTATION

The process of finite element method is:

$$\begin{array}{ccccc} & \text{Calculation of basis function on element} & & \text{global number of dof on element} & \\ & \downarrow & & \downarrow & \\ \text{Mesh} & \longrightarrow & \text{Assembly cell matrix} & \longrightarrow & \text{Assembly matrix} \end{array}$$

6.1. **Mesh.** A mesh consists of node, edge, face, cells, and these are called entities of the mesh. To describe a mesh two types of information are required: 1. Geometric information: the location of the entities of a mesh. 2. Topological information: the adjacencies between the entities of mesh.

For the common linear simp~~lex~~icial mesh, the convex hull of the vertices of the simplex is itself, so in a linear simplex mesh, only the node positions need to be known for geometric information, and the vertices of the simplex must be known for topological information.

For the convenience of description, the following notation~~s~~ are made.

1 NN, NE, NF, NC denote the number of nodes, edges, faces, and cells in the mesh, respectively.
2 GD, TD are the geometric dimension and topological dimension of the mesh, respectively.

In this paper, we use the array `node` of shape NN*GD to describe the node positions in the mesh, and the array `cell` of shape NC*(TD+1) to describe the topology of the mesh, where `node[i]` denotes the coordinate of the i-th node in the mesh, and `cell[i, j]` denotes the global indexing of the j-th vertex of the i-th cell in the mesh. In addition, we use k+1 vertices to describe a k-simplex, e.g. `[2, 4, 5]` denotes a triangle with the 2nd, 4th, and 5th nodes in the mesh as vertices, while different vertex orders represent the same simplex. but with different orientation

Take a tetrahedral mesh as an example, for a tetrahedron define its local edges and local faces:

```
localEdge = [[0, 1], [0, 2], [0, 3], [1, 2], [1, 3], [2, 3]]
localFace = [[1, 2, 3], [0, 2, 3], [0, 1, 3], [0, 1, 2]]
```

where `0, 1, 2, 3` is the local indexing of the cell vertex. For example:

```
[cell[i, localEdge[j, 0]], cell[i, localEdge[j, 1]]]
```

denote the j-th edge of the i-th cell in mesh. For a face, define its local edges:

```
localEdgeofFace = [[1, 2], [0, 2], [0, 1]]
```

where `0, 1, 2` is the local indexing of the face vertices.

According to the method in [3], we can obtain the edge set `edge`, face set `face` and the adjacency relations `cell2edge, face2cell, cell2face` between simplex of different dimensions, where `X2Y[i, j]` denotes the global indexing of the j-th `Y` adjacent to the i-th `X`.

The global vertex order of edges and faces in the mesh is the column order of edge, face, and the local vertex order is recorded in `localFace, localEdge`. The global vertex order of faces and edges may be different from the local vertex order, and two arrays `face2edgesign, cell2faceorder` are used here to record this difference.

For edges, there are two kinds of vertex order, the local vertex order is either the same as the global vertex order or different, using the array `face2edgesign` to record the relationship between the local vertex order and the global vertex order for each edge of each cell. `face2edgesign[i, j] = 0` means the local vertex order of the jth edge of the i-th cell is different from the global and `face2edgesign[i, j] = 1` means that the local vertex order of the j-th edge of the i-th cell is the same as the global one.

For faces, there are 6 types of vertex order, noted as:

```
faceorder = [[0, 1, 2], [0, 2, 1], [1, 0, 2],
             [1, 2, 0], [2, 0, 1], [2, 1, 0]],
```

using the array `cell2faceorder` to record the difference between the local vertex order of the cell's face and the global vertex order, e.g. `cell2faceorder[i, j] = k`, means that the relationship between the global vertex order and local vertex order of j-th face of the i-th cell is `faceorder[k]`.

For a face, a different global and local vertex order also leads to a change in the relationship between the indexing sizes of the local edges.

Use the array `faceordersign` to describe whether different vertex orders lead to a change in the local indexing of the two edges adjacent to a point: `faceordersign[i, j] = 1`, which means that the i-th vertex order leads to a change in the local indexing of the edges on both sides of the j-th vertex, otherwise there is no change.Clearly

```
faceordersign = [[0, 0, 0], [1, 0, 0], [0, 0, 1],
```

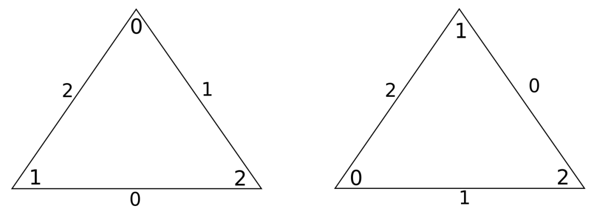


FIGURE 6. The left side is the local vertex order, the right side is the global vertex order, and the order relationship is `[1, 0, 2]`, where only the local indexing of the two edges around the local 2nd point is changed

```
[1, 1, 0], [0, 1, 1], [1, 1, 1]]
```

6.2. **Interpolation point.**

6.2.1. *Correspondence of interpolation points between simplex and subsimplex.* Obviously, the interpolation point on the boundary of a simplex is also the interpolation point of the subsimplex where it is located, as in the Figure2 where the red interpolation point is the interpolation point of the triangle and also the interpolation point of the 2nd side of the triangle. For the triangle `[0, 1, 2]` we can define the array `e2fp`

$$\text{e2fp}[i] = \{R_2(\boldsymbol{\alpha})|\boldsymbol{\alpha} \in \mathcal{A}_2^p,\ \alpha_{2-i} = 0\}$$

satisfying `e2fp[i, j] < e2fp[i, j+1]`. `e2fp[i, j]` denotes the local indexing of the j-th interpolation point of the i-th edge of the triangle. For example, for 4 interpolation points.

```
e2fp = [[10, 11, 12, 13, 14], [0, 2, 5, 9, 14], [0, 1, 3, 6, 10]]
```

Similarly, for a tetrahedron one can define the array `f2cp`

$$\text{f2cp}[i] = \{R_3(\boldsymbol{\alpha})|\boldsymbol{\alpha} \in \mathcal{A}_3^p,\ \alpha_{3-i} = 0\}$$

satisfying `f2cp[i, j] < f2cp[i, j+1]`. `f2cp[i, j]` denotes the indexing of the j-th interpolation point of the i-th face of the tetrahedron among all interpolation points of the tetrahedron.

In addition, for the convenience of description, we use an array `v2fp`

$$\text{v2fp} = [0, n_f - p - 1, n_f - 1]$$

to denote the local indexing of the interpolation points at the three vertices of the face.

6.2.2. *Correspondence of interpolation points between two simplexs.* If two simplexs have the same vertices, but in different order, this will result in the same interpolation points, but each interpolation point has a different index, here we only consider the vertex order of faces:

For faces, there are 6 kinds of vertex order, if two faces `f0, f1` have the same vertices and the vertex order relation is `faceorder[i]`, take the interpolation point on `f1` with multi-index $\boldsymbol{\alpha}$, then the multi-index of this interpolation point in `f0` is

$$\boldsymbol{\alpha}' = \{\alpha_{\text{faceorder}[i,0]}, \alpha_{\text{faceorder}[i,1]}, \alpha_{\text{faceorder}[i,2]}\}$$

Define the array `f2fp`

$$\text{f2fp[i]} = \{R_2(\boldsymbol{\alpha}')|\alpha'_j = \alpha_{\text{faceorder}[i,j]},\ j = 0, 1, 2;\ \boldsymbol{\alpha} \in \mathcal{A}_2^p\}$$

Then the j-th interpolation point of `f1` is the `f2fp[i, j]` -th interpolation point of `f0`.

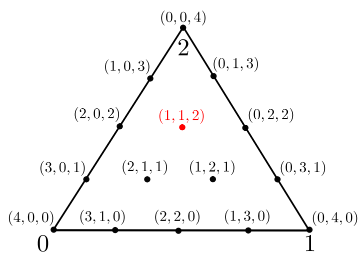


FIGURE 7. The multi-index of the interpolated points on the two triangles correspond, where the multi-index in the left triangle is `(1, 1, 2)` the multi-index in the right triangle is `(2, 1, 1)`

6.3. **DoFs management.** The dof are defined at the interpolation points of edges, faces and cells, and each interpolation point has multiple dof, which are indexed first on the edges, then on the faces and finally on the cells. The global indexing of the dof on a simplex is recorded using the arrays `edge2dof, face2dof, cell2dof`, where `X2dof[i, j]` is the global indexing of the j-th dof on the i-th `X`. So the process of managing degrees of freedom is the process of generating these 3 arrays.

If the dofs are defined on the edge and face, then the dof on the edge are easy to index and `edge2dof` is easy to get, but for the dof on the face, they are partly defined in the face and partly defined on the edge to share with the adjacent face, the dof defined on the face need to be indexed, while the dof defined on the edge already have indexs and need to be obtained from `edge2dof`, the process of obtaining them needs to get the local indexing and global indexing of the dof defined on the edge, the details are discussed below.

In this paper, we discuss p finite elements, the times of interpolation points is p, noting

$$n_e = p+1, n_f = \frac{(p+1)(p+2)}{2}, n_c = \frac{(p+1)(p+2)(p+3)}{6}$$

are the number of interpolation points on the edges, faces, and cells, respectively.

6.3.1. *BDM element dof on triangle mesh.* The dof of BDM element on triangle mesh are defined on the edges and faces.
1. `edge2dof` The number of dof on each edge is n_e, define the local indexing of dof at the i-th interpolation point on a edge as i. Thus, for global indexing,

$$\text{edge2dof}(e, j) = en_e + j, \quad j \in \{0, 1, 2, \cdots, p\}.$$

2. `face2dof` Each face has $2n_f$ dofs, of which there are $3n_e$ dofs on the boundary, $\mathring{n}_f = 2n_f - 3n_e$ internal dofs.

There are two dof at each interpolation point. Define the local indexing of j-th dof at the i-th interpolation point on the face is $i + jn_f$.

For a face with index f, the global indexing of the three edges is e_0, e_1, e_2. The three local edges are E_0, E_1, E_2, and the boundary dof of f are the normal dof on E_0, E_1, E_2. In the following we discuss the local and global indexing of dof on E_i.

The local indexing of the two dof on the face at the k interpolation point is $\{k, k + n_f\}$ by the above definition. Denote by L_{ij} the local indexing of the j-th dof on E_i, Then L_{ij} can be written as follows:

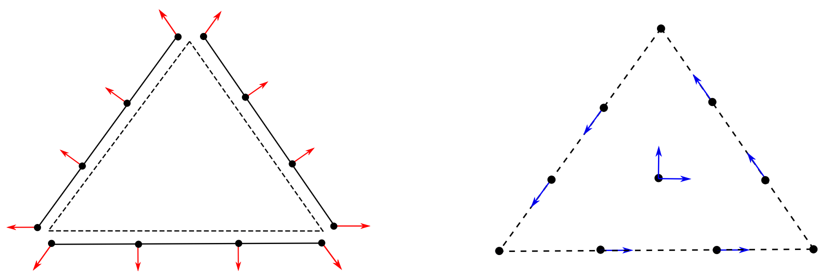


FIGURE 8. shared dof(left) and internal dof(right)

The local indexing of dof on E_0 is

$$L_{0j} = \begin{cases} \text{e2fp}[0,\text{j}] + n_f & j \in \{0, p\} \\ \text{e2fp}[0,\text{j}] & other \end{cases}$$

The local indexing of dof on E_1 is

$$L_{1j} = \begin{cases} \text{e2fp}[1,\text{j}] + n_f & j = 0 \\ \text{e2fp}[1,\text{j}] & other \end{cases}$$

The local indexing of dof on E_2 is

$$L_{2j} = \text{e2fp}[2,\text{j}] \quad j \in \{0, 1, \cdots, p\}.$$

Now we obtain the global indexing of the dof on E_i. e_i and E_i are the same edge, denote by G_{ij} the local index of the j-th dof on E_i, when e_i and E_i are oriented in the same direction:

$$G_{ij} = \text{edge2dof}(e_i, j), \quad j \in \{0, 1, 2, \cdots, p\}$$

when the directions are opposite:

$$G_{ij} = \text{edge2dof}(e_i, p - j), \quad j \in \{0, 1, 2, \cdots, p\}$$

Let $K = \{0, 1, \cdots, 2n_f - 1\} - \cup_{i=0}^{2} L_i$ be the local indexing of the internal dof, Thus, for global indexing,

$$\text{face2dof}(f, L_{ij}) = G_{ij}, \quad i \in \{0, 1, 2\},\ j \in \{0, 1, 2, \cdots, p\}$$
$$\text{face2dof}(f, K_i) = NEn_e + f\mathring{n}_f + i, \quad i \in \{0, 1, \cdots, \mathring{n}_f - 1\}$$

6.3.2. *BDM element dof on tetrahedron mesh.* The dof of BDM element on tetrahedron mesh are defined on the faces and cells.

1. `face2dof` The number of dof on each face is $2n_f$, define the local indexing of dof at the i-th interpolation point on a face as i. Thus, for global indexing,

$$\text{face2dof}(f, j) = fn_f + j, \quad j \in \{0, 1, 2, \cdots, 2n_f - 1\}$$

2. `cell2dof` Each face has $3n_c$ dofs, of which there are $4n_f$ dofs on the boundary, $\mathring{n}_c = 3n_c - 4n_f$ internal dofs.

There are three dof at each interpolation point. Define the local indexing of j-th dof at the i-th interpolation point on the face is $i + jn_c$.

For a cell with index c, the global indexing of the four faces is f_0, f_1, f_2, f_3. The four local faces are F_0, F_1, F_2, F_3, and the boundary dof of c are the normal dof on F_0, F_1, F_2, F_3. In the following we discuss the local and global indexing of dof on F_i.

The local indexing of the three dofs on the cell at the k-th interpolation point is $\{k, k+n_c, k+2n_c\}$ by the above definition. Denote by L_{ij} the local indexing of the j-th dof on F_i, Then L_{ij} can be written as follows:

The local indexing of dof on F_0 is

$$L_{0j} = \begin{cases} \text{f2cp}(0,j) + n_c & j \in \text{e2fp} \\ \text{f2cp}(0,j) & \text{other} \end{cases}$$

The local indexing of dof on F_1 is

$$L_{1j} = \begin{cases} \text{f2cp}(1,j) + n_c & j \in \text{e2fp}(1) \cup \text{e2lp}(2) \\ \text{f2cp}(1,j) & \text{other} \end{cases}$$

The local indexing of dof on F_2 is

$$L_{2j} = \begin{cases} \text{f2cp}(2,j) + n_c & j \in \text{e2fp}(2) \\ \text{f2cp}(2,j) & \text{other} \end{cases}$$

The local indexing of dof on F_3 is

$$L_{3j} = \text{f2cp}(3,j)$$

Now we obtain the global indexing of the dof on F_i. f_i and E_i are the same face, Their vertex correspondence is $\text{t} = \text{cell2faceorder[c, i]}$, denote by G_{ij} the local index of the j-th dof on F_i, then

$$G_{ij} = \text{face2dof}(f_i, \text{f2fp[t, j]}), \quad j \in \{0, 1, 2, \cdots, n_f - 1\}$$

Let $K = \{0, 1, \cdots, 3n_c - 1\} - \cup_{i=0}^{3} L_i$, K is the local indexing of the internal dof, Thus, for global indexing,

$$\text{cell2dof}(c, L_{ij}) = G_{ij}, \quad i \in 0, 1, 2,\ j \in 0, 1, 2, \cdots, n_f - 1$$
$$\text{cell2dof}(c, K_i) = NF * n_f + c * \mathring{n}_c + i, \quad i \in \{0, 1, \cdots, \mathring{n}_c - 1\}$$

6.3.3. *Nédélec element dof on tetrahedron mesh.* The dof of Nédélec element on tetrahedron mesh are defined on the edges, faces and cells. The dof on the edges and faces are indexed in the same way as the 2-dimensional Nédélec elements, so only additional indexing of the dof on the cell is needed

`cell2dof` Each face has $3n_c$ dofs, of which there are $\mathring{n}_c = \frac{(p-1)(p-2)(p-3)}{2} + 2(p-1)(p-2)$ internal dofs, $3n_c - \mathring{n}_c$ dofs on the boundary.

There are three dof at each interpolation point. Define the local indexing of j-th dof at the i-th interpolation point on the face is $i + jn_c$.

For a cell with index c, the global indexing of the four faces is f_0, f_1, f_2, f_3. The four local faces are F_0, F_1, F_2, F_3, and the boundary dof of c are the normal dof on F_0, F_1, F_2, F_3. In the following we discuss the local and global indexing of dof on F_i.

The local indexing of the three dofs on the cell at the k-th interpolation point is $\{k, k+n_c, k+2n_c\}$ by the above definition. Denote by L_{ij} the local indexing of the j-th dof on F_i, Then L_{ij} can be written as follows:

The local indexing of dof on F_0 is

$$L_{0j} = \begin{cases} \text{f2cp}(0,j) + n_c & j \in \text{v2fp} \\ \text{f2cp}(0,j) & \text{other} \end{cases} \quad , j \in \{0, 1, \cdots, n_f - 1\}$$

$$L_{0j} = \begin{cases} \text{f2cp}(0, j - n_f) + 2n_c & j - n_f \in \text{e2fp} \\ \text{f2cp}(0, j - n_f) + n_c & \text{other} \end{cases} \quad , j \in \{n_f, n_f + 1, \cdots, 2n_f - 1\}$$

The local indexing of dof on F_1 is

$$L_{1j} = \begin{cases} \mathrm{f2cp}(1,j) + n_c & j = \mathrm{v2fp}[0] \\ \mathrm{f2cp}(1,j) & \text{other} \end{cases} \quad , j \in \{0, 1, \cdots, n_f - 1\}$$

$$L_{1j} = \begin{cases} \mathrm{f2cp}(1, j - n_f) + 2n_c & j - n_f \in \mathrm{e2fp}[1] \cup \mathrm{e2fp}[2] \\ \mathrm{f2cp}(1, j - n_f) + n_c & \text{other} \end{cases} \quad , j \in \{n_f, n_f+1, \cdots, 2n_f-1\}$$

The local indexing of dof on F_2 is

$$L_{2j} = \mathrm{f2cp}(2, j), \quad j \in \{0, 1, \cdots, n_f - 1\}$$

$$L_{2j} = \begin{cases} \mathrm{f2cp}(2, j - n_f) + 2n_c & j - n_f \in \mathrm{e2fp}[2] \\ \mathrm{f2cp}(2, j - n_f) + n_c & \text{other} \end{cases} \quad , j \in \{n_f, n_f + 1, \cdots, 2n_f - 1\}$$

The local indexing of dof on F_3 is

$$L_{3j} = \mathrm{f2cp}(3, j), \quad j \in \{0, 1, \cdots, n_f - 1\}$$

$$L_{3j} = \mathrm{f2cp}(3, j - n_f) + n_c, \quad j \in \{n_f, n_f + 1, \cdots, 2n_f - 1\}$$

Now we obtain the global indexing of the dof on F_i. f_i and E_i are the same face, Their vertex correspondence is $\mathrm{t} = \mathrm{cell2faceorder}[\mathrm{c}, \mathrm{i}]$, denote by G_{ij} the local index of the j-th dof on F_i, then

$$G_{ij} = \mathrm{face2dof}(f_i, \mathrm{f2fp}[\mathrm{t}, \mathrm{j}]), \quad j \in \{0, 1, 2, \cdots, n_f - 1\} - \mathrm{v2fd}$$

$$G_{ij} = \mathrm{face2dof}(f_i, \mathrm{f2fp}[\mathrm{t}, \mathrm{j} - \mathrm{n_f}]), \quad j - n_f \in \{0, 1, 2, \cdots, n_f - 1\} - \mathrm{v2fd}$$

$$G_{i,\mathrm{v2fd[j]}} = \begin{cases} \mathrm{face2dof}(f_i, \mathrm{f2fp}[\mathrm{t}, \mathrm{v2fd}[\mathrm{j}]]), & \mathrm{faceordersign}[\mathrm{i}, \mathrm{j}] = 0 \\ \mathrm{face2dof}(f_i, \mathrm{f2fp}[\mathrm{t}, \mathrm{v2fd}[\mathrm{j}] + n_f), & \mathrm{faceordersign}[\mathrm{i}, \mathrm{j}] = 1 \end{cases} \quad j \in \{0, 1, 2\}$$

$$G_{i,\mathrm{v2fd[j]}+n_f} = \begin{cases} \mathrm{face2dof}(f_i, \mathrm{f2fp}[\mathrm{t}, \mathrm{v2fd}[\mathrm{j}] + n_f), & \mathrm{faceordersign}[\mathrm{i}, \mathrm{j}] = 0 \\ \mathrm{face2dof}(f_i, \mathrm{f2fp}[\mathrm{t}, \mathrm{v2fd}[\mathrm{j}]]), & \mathrm{faceordersign}[\mathrm{i}, \mathrm{j}] = 1 \end{cases} \quad j \in \{0, 1, 2\}$$

Let $K = \{0, 1, \cdots, 3n_c - 1\} - \cup_{i=0}^{3} L_i$, K is the local indexing of the internal dof, Thus, for global indexing,

$$\mathrm{cell2dof}(c, L_{ij}) = G_{ij}, \quad i \in \{0, 1, 2\},\ j \in \{0, 1, 2, \cdots, n_f - 1\}$$

$$\mathrm{cell2dof}(c, K_i) = NEn_e + NF\mathring{n}_f + c * \mathring{n}_c + i, \quad i \in \{0, 1, \cdots, \mathring{n}_c - 1\}$$

6.4. **Calculation of basis function on element.**

7. De Rham Complexes

8. High Order Elements for Maxwell Equations

Numerical Examples.

Consider the time harmonic problem:

$$\begin{cases} \nabla \times (\mu^{-1} \nabla \times \boldsymbol{E}) - \omega^2 \tilde{\epsilon} \boldsymbol{E} & = \boldsymbol{J} \quad in \ \Omega \\ \nabla \times \boldsymbol{E} & = \boldsymbol{0} \quad on \ \partial\Omega \end{cases},$$

where $\Omega = [0, 1]^3$, $\boldsymbol{E} = (f, \sin(x)f, \cos(y)f)$, $f = (x^2 - x)^2(y^2 - y)^2(z^2 - z)^2$, $\omega = \tilde{\epsilon} = \mu = 1$, $\boldsymbol{J} = \nabla \times \nabla \times \boldsymbol{E} - \boldsymbol{E}$

Variational problem: find $\boldsymbol{E} \in H(\mathrm{curl}, \Omega), \forall\, \boldsymbol{v} \in H(\mathrm{crul}, \Omega)$ staify:

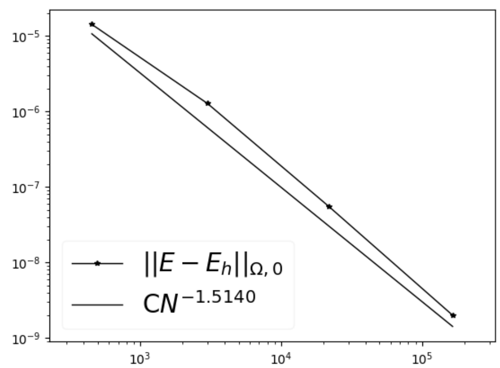
$$\int_\Omega \mu^{-1} (\nabla \times \boldsymbol{E}) \cdot (\nabla \times \boldsymbol{v}) \, \mathrm{d}\boldsymbol{x} - \int_\Omega \omega^2 \tilde{\epsilon} \boldsymbol{E} \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{J} \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x}$$

Finite element problem: find $\boldsymbol{E}_h \in Nedelec^4(\mathcal{T}_h), \forall\, \boldsymbol{v}_h \in Nedelec^4(\mathcal{T}_h)$ staify:

$$\int_\Omega \mu^{-1}(\nabla \times \boldsymbol{E}_h) \cdot (\nabla \times \boldsymbol{v}_h)\, \mathrm{d}\boldsymbol{x} - \int_\Omega \omega^2 \tilde{\epsilon} \boldsymbol{E}_h \cdot \boldsymbol{v}_h\, \mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{J} \cdot \boldsymbol{v}_h\, \mathrm{d}\boldsymbol{x}$$

TABLE 1. Error table

DoF	455	3010	21740	164920
$\|\boldsymbol{E} - \boldsymbol{E_h}\|_{\boldsymbol{\Omega,0}}$	$1.4243e-05$	1.2755e-06	5.5699e-08	2.0062e-09
Order	–	3.48	4.52	4.8



9. HIGH ORDER ELEMENTS FOR MIXED POISSON

Consider the Poisson problem:

$$\begin{cases} \boldsymbol{u} + \nabla p = 0 & in\ \Omega \\ \nabla \cdot \boldsymbol{u} = f & in\ \Omega \\ p = g & on\ \partial\Omega \end{cases}$$

where $\Omega = [0,1]^3$

$$\boldsymbol{u} = \begin{pmatrix} -\pi \sin(\pi x)\cos(\pi y)\cos(\pi z) \\ -\pi \cos(\pi x)\sin(\pi y)\cos(\pi z) \\ -\pi \cos(\pi x)\cos(\pi y)\sin(\pi z) \end{pmatrix}.$$

$$p = \cos(\pi x)\cos(\pi y)\cos(\pi z), \quad f = 3\pi^2 \cos(\pi x)\cos(\pi y)\cos(\pi z),$$

Variational problem: find $\boldsymbol{u} \in H(\mathrm{div}, \Omega)$, $p \in L^2(\Omega)$, $\forall \boldsymbol{v} \in H(\mathrm{div}, \Omega)$, $w \in L^2(\Omega)$ staify:

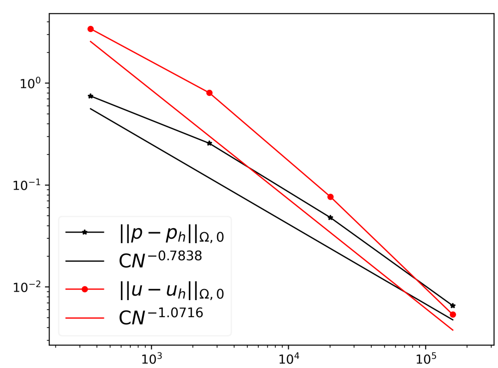
$$\begin{cases} \int_\Omega \boldsymbol{u} \cdot \boldsymbol{v}\, \mathrm{d}\boldsymbol{x} - \int_\Omega p \nabla \cdot \boldsymbol{v}\, \mathrm{d}\boldsymbol{x} = -\int_{\partial\Omega} g(\boldsymbol{v} \cdot \boldsymbol{n})\, \mathrm{d}\boldsymbol{s} \\ -\int_\Omega (\nabla \cdot \boldsymbol{u}) w\, \mathrm{d}\boldsymbol{x} = -\int_\Omega f w\, \mathrm{d}\boldsymbol{x} \end{cases}$$

Finite element problem: find $\boldsymbol{u}_h \in V_0, p_h \in V_1, \forall \boldsymbol{v}_h \in V_0, w_h \in V_1$ staify:

$$\begin{cases} \int_\Omega \boldsymbol{u}_h \cdot \boldsymbol{v}_h \,\mathrm{d}\boldsymbol{x} - \int_\Omega p \nabla \cdot \boldsymbol{v}_h \,\mathrm{d}\boldsymbol{x} = -\int_{\partial\Omega} g(\boldsymbol{v}_h \cdot \boldsymbol{n}) \,\mathrm{d}\boldsymbol{s} \\ \qquad\qquad -\int_\Omega (\nabla \cdot \boldsymbol{u}_h) w_h \,\mathrm{d}\boldsymbol{x} = -\int_\Omega f w_h \,\mathrm{d}\boldsymbol{x} \end{cases}$$

TABLE 2. Error table

DoF	360	2640	20160	157440
$\lVert \boldsymbol{p} - \boldsymbol{p_h} \rVert_{\boldsymbol{\Omega,0}}$	7.4565e-01	2.5656e-01	4.7964e-02	6.5568e-03
Order	–	1.54	2.42	2.87
$\lVert \boldsymbol{u} - \boldsymbol{u_h} \rVert_{\boldsymbol{\Omega,0}}$	3.4054e+00	8.0262e-01	7.6813e-02	5.3623e-03
Order	–	2.09	3.39	3.84



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