GEOMETRIC DECOMPOSITION AND EFFICIENT IMPLEMENTATION OF HIGH ORDER FACE AND EDGE ELEMENTS

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ABSTRACT. This paper delves into the world of high-order curl and div elements within finite element methods, providing valuable insights into their geometric properties, indexing management, and practical implementation considerations. It first explores the decomposition of Lagrange finite elements. The discussion then extends to $H(\mathrm{div})$-conforming finite elements and $H(\mathrm{curl})$-conforming finite element spaces by choosing different frames at different sub-simplex. The required tangential continuity or normal continuity will be imposed by appropriate choices of the tangential and normal basis. The paper concludes with a focus on efficient indexing management strategies for degrees of freedom, offering practical guidance to researchers and engineers. It serves as a comprehensive resource that bridges the gap between theory and practice in the realm of high-order curl and div elements in finite element methods, which are vital for solving vector field problems and understanding electromagnetic phenomena.

1. INTRODUCTION

This paper introduces node-based basis functions for high-order finite elements, with a specific focus on Lagrange, BDM (Brezzi-Douglas-Marini) [**?**, **?**, **?**], and second-kind Nédélec elements [**?**, **?**]. These finite elements are subspaces of the function spaces H^1, $H(\mathrm{div})$, and $H(\mathrm{curl})$, respectively, with shape functions being $\mathbb{P}_k^n$, where k signifies the polynomial degree and n denotes the geometric dimension. It is worth noting that due to varying definitions of trace spaces, these finite elements exhibit distinct characteristics.

In the realm of Lagrange finite elements, nodal basis functions stand out for their simplicity and ease of computation. In contrary, constructing basis functions for BDM and second-kind Nédélec elements is a more intricate endeavor. Traditional approaches involve the Piola transformation, where basis functions are first devised on a reference element and subsequently mapped to the actual element using either covariant (to preserve tangential continuity, in the case of second-kind Nédélec elements) or contravariant (to maintain normal continuity, for BDM elements) Piola transformation. Detailed explanations of this approach can be found in [**?**, **?**], and implementation is in open-source software such as MFEM [**?**] and Fenics [**?**].

Arnold, Falk and Winther, in [**?**], introduced a geometric decomposition of polynomial differential forms of various orders. Basis functions based on Bernstein polynomials were proposed, paving the way for subsequent advancements. In [**?**, **?**], basis functions founded on Bernstein polynomials were explored, accompanied by fast algorithms for matrix assembly. Additionally, hierarchical basis functions for $H(\mathrm{curl})$-conforming finite elements were introduced in [**?**, **?**, **?**, **?**]. Hierarchical geometric decomposition of

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$H(\operatorname{div})$-conforming finite elements in two and three dimensions is discussed in [?, ?, ?], and hierarchical geometric decomposition of $H(\operatorname{curl})$-conforming finite elements in two dimensions can also be found in [?, ?].

While these methods offer valuable insights, they can be quite complex. Researchers have thus ventured into simpler approaches. In [?], a method multiplying scalar nodal finite element methods by vectors was introduced, resulting in $H(\operatorname{div})$ and $H(\operatorname{curl})$ conforming finite elements that exhibit continuity on both vertices and edges. However, further investigations revealed that using this $H(\operatorname{curl})$-conforming finite element discretization could yield spurious solutions for the harmonic Maxwell equations. As a solution, an approach based on macroelements was proposed in [?], ensuring full continuity on both edges and vertices. Nevertheless, nodal basis functions for BDM elements and second-kind Nédélec elements remained a challenge.

The key to constructing basis functions for BDM and second-kind Nédélec elements lies in ensuring trace continuity on the element's boundary. For BDM elements, this continuity is normal, while for second-kind Nédélec elements, it is tangential. We first elucidate the basis of Lagrange elements, viewing them as dual to degrees of freedom formed by values at interpolation points. This concept is readily extended to vector polynomial spaces, where each interpolation point defines a frame encompassing tangential and normal directions. This allows unique determination of vector polynomial functions, and continuity of either tangential or normal components leads to the degrees of freedom of BDM or second-kind Nédélec elements. The dual basis functions for these elements can be explicitly derived based on the basis functions of Lagrange elements.

An essential aspect of practical finite element implementations is the management of local Degrees of Freedom (DoFs), ensuring proper mapping to global DoFs. This is critical for maintaining the correct continuity between elements and correctly integrating matrices and vectors across the system. Remarkably, there is limited literature addressing degree of freedom management for high-order finite elements. In this paper, we delve into the issue of global indexing for interpolation points in arbitrary-degree Lagrange finite elements. We systematically discuss degree of freedom management for Lagrange, BDM, and second-kind Nédélec elements, with the aim of simplifying the implementation of high-order finite element methods.

The subsequent sections of this paper are organized as follows. Section 2 elucidates foundational concepts such as simplicial lattices, interpolation points, bubble polynomials, and triangulation. Sections 3, 4, and 5 delve into the construction of basis functions for Lagrange, BDM, and second-kind Nédélec elements, respectively. Section 6 addresses the management of degrees of freedom, a critical aspect of effective matrix assembly. Finally, Section 7 presents two numerical examples, solving the mixed Poisson problem and Maxwell problem using BDM and second-kind Nédélec elements, respectively, to validate the correctness of the proposed basis function construction method.

2. Preliminaries

In this section, we provide essential foundations for our study. We introduce multi-indices, simplicial lattices, and interpolation points, which play a crucial role in our design of finite element methods. We explain sub-simplices and their relations. Furthermore, we introduce the concept of bubble polynomials.

2.1. **Simplicial lattice.** A multi-index of length $n+1$ is an array of non-negative integers:

$$\boldsymbol{\alpha} = (\alpha_0, \alpha_1, \dots, \alpha_n), \quad \alpha_i \in \mathbb{N}, i = 0, \dots, n.$$

The degree or sum of the multi-index is $|\boldsymbol{\alpha}| = \sum_{i=0}^{n} \alpha_i$ and factorial is $\boldsymbol{\alpha}! = \prod_{i=0}^{n}(\alpha_i!)$. The set of all multi-indices of length $n+1$ and degree k will be called a simplicial lattice and denoted by $\mathbb{T}_k^n$, i.e.,

$$\mathbb{T}_k^n = \{\boldsymbol{\alpha} \in \mathbb{N}^{n+1} : |\boldsymbol{\alpha}| = k\}$$

The elements in $\mathbb{T}_k^n$ can be linearly indexed by the dictionary ordering R_n:

$$R_n(\boldsymbol{\alpha}) = \sum_{i=1}^{n} \binom{n+1-i}{\alpha_i + \alpha_{i+1} + \cdots + \alpha_n + n - i}.$$

Notice that α_0 is not used in the calculation of $R_n(\boldsymbol{\alpha})$. For example, for an element $\boldsymbol{\alpha}$ in $\mathbb{T}_k^2$, the index is given by the mapping:

$$R_2(\boldsymbol{\alpha}) = \frac{(\alpha_1 + \alpha_2)(\alpha_1 + \alpha_2 + 1)}{2} + \alpha_2.$$

Below we present the first four lattice nodes in $R_2(\boldsymbol{\alpha})$ with $k = 4$:

$$\begin{array}{c} 0: \\ 1: \\ 2: \\ 3: \end{array} \begin{pmatrix} 4 & 0 & 0 \\ 3 & 1 & 0 \\ 3 & 0 & 1 \\ 2 & 2 & 0 \end{pmatrix}.$$

2.2. **Interpolation points.** Let $\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ be $n+1$ points in $\mathbb{R}^n$ and

$$T = \mathrm{Convex}(\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) = \left\{ \sum_{i=0}^{n} \lambda_i \boldsymbol{x}_i : 0 \le \lambda_i \le 1, \sum_{i=0}^{n} \lambda_i = 1 \right\},$$

be an n-simplex, where $\boldsymbol{\lambda} = (\lambda_0, \ldots, \lambda_n)$ is called the barycentric coordinate. We can have a geometric embedding of the algebraic set $\mathbb{T}_k^n$ as follows:

$$\mathcal{X}_T = \left\{ \boldsymbol{x}_{\boldsymbol{\alpha}} = \frac{1}{k} \sum_{i=0}^{n} \alpha_i \boldsymbol{x}_i : \boldsymbol{\alpha} \in \mathbb{T}_k^n \right\},$$

which is called the set of interpolation points with degree k on T; see Fig. 1 for $k = 4$ and $n = 2$. Given $\boldsymbol{\alpha} \in \mathbb{T}_k^n$, the barycentric coordinate of $\boldsymbol{\alpha}$ is given by

$$\boldsymbol{\lambda}(\boldsymbol{\alpha}) = (\alpha_0, \alpha_1, \ldots, \alpha_n)/k.$$

The ordering of $\mathcal{X}_T$ is also given by $R_n(\boldsymbol{\alpha})$. Note that the indexing map $R_n(\boldsymbol{\alpha})$ is only a local ordering of the interpolation points on one n-simplex. In Section 6, we will discuss the global indexing of all interpolation points on the triangulation composed of simplexes.

By this geometric embedding, we can apply operators to the geometric simplex T. For example, $\mathcal{X}_{\mathring{T}}$ denotes the set of interpolation points in the interior of T and $\mathcal{X}_{\partial T}$ is the set of interpolation points on the boundary of T.

2.3. **Sub-simplexes and sub-simplicial lattices.** Following [**?**], let $\Delta(T)$ denote all the subsimplices of T, while $\Delta_\ell(T)$ denotes the set of subsimplices of dimension ℓ for $\ell = 0, \ldots, n$. Elements of $\Delta_0(T) = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_n\}$ are $n+1$ vertices of T and $\Delta_n(T) = T$. The capital letter F is reserved for an $(n-1)$-dimensional face of T and $F_i \in \Delta_{n-1}(T)$ denotes the face opposite to $\boldsymbol{x}_i$ for $i = 0, 1, \ldots, n$.

For a sub-simplex $f \in \Delta_\ell(T)$, we will overload the notation f for both the geometric simplex and the algebraic set of indices. Namely $f = \{f(0), \ldots, f(\ell)\} \subseteq \{0, 1, \ldots, n\}$ and

$$f = \mathrm{Convex}(\boldsymbol{x}_{f(0)}, \ldots, \boldsymbol{x}_{f(\ell)}) \in \Delta_\ell(T)$$

is the ℓ-dimensional simplex spanned by the vertices $\boldsymbol{x}_{f(0)}, \ldots, \boldsymbol{x}_{f(\ell)}$.

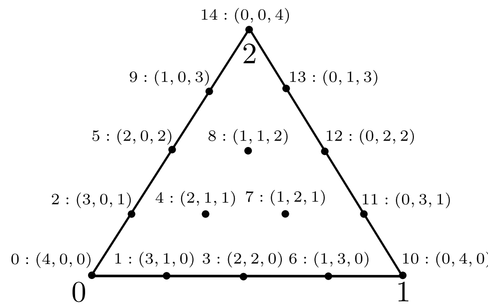

FIGURE 1. The interpolation points associated to $\mathbb{T}_4^2$ and their indices and corresponding multi-indexs.

If $f \in \Delta_\ell(T)$, then $f^* \in \Delta_{n-\ell-1}(T)$ denotes the sub-simplex of T opposite to f. When treating f as a subset of $\{0, 1, \dots, n\}$, $f^* \subseteq \{0, 1, \dots, n\}$ so that $f \cup f^* = \{0, 1, \dots, n\}$, i.e., f^* is the complement of set f. Geometrically,

$$f^* = \mathrm{Convex}(\boldsymbol{x}_{f^*(1)}, \dots, \boldsymbol{x}_{f^*(n-\ell)}) \in \Delta_{n-\ell-1}(T)$$

is the $(n-\ell-1)$-dimensional simplex spanned by vertices not contained in f.

Given a sub-simplex $f \in \Delta_\ell(T)$, through the geometric embedding $f \hookrightarrow T$, we define the prolongation/extension operator $E : \mathbb{T}_k^\ell \to \mathbb{T}_k^n$ as follows:

$$E(\alpha)_{f(i)} = \alpha_i, i = 0, \dots, \ell, \quad \text{and } E(\alpha)_j = 0, j \notin f. \tag{1}$$

For example, assume $f = \{1, 3, 4\}$, then for $\alpha = (\alpha_0, \alpha_1, \alpha_2) \in \mathbb{T}_k^\ell(f)$, the extension $E(\alpha) = (0, \alpha_0, 0, \alpha_1, \alpha_2, \dots, 0)$. With a slight abuse of notation, for a node $\alpha_f \in \mathbb{T}_k^\ell(f)$, we still use the same notation $\alpha_f \in \mathbb{T}_k^n(T)$ to denote such extension. The geometric embedding $\boldsymbol{x}_{E(\alpha)} \in f$ which justifies the notation $\mathbb{T}_k^\ell(f)$ and its geometric embedding will be denoted by $\mathcal{X}_f$, which consists of interpolation points on f.

2.4. **Bubble polynomial.** The Bernstein representation of polynomial of degree k on a simplex T is

$$\mathbb{P}_k(T) := \mathrm{span}\{\lambda^{\boldsymbol{\alpha}} = \lambda_0^{\alpha_0}\lambda_1^{\alpha_1}\dots\lambda_n^{\alpha_n}, \alpha \in \mathbb{T}_k^n\}.$$

The bubble polynomial of f is a polynomial of degree $\ell + 1$:

$$b_f := \lambda_f = \lambda_{f(0)}\lambda_{f(1)}\dots\lambda_{f(\ell)} \in \mathbb{P}_{\ell+1}(f).$$

The following result can be found in [**?**]. For completeness, we include here.

Lemma 2.1. *Let $f \in \Delta_\ell(T)$ with $\ell \geq 1$. For $e \in \Delta_m(T)$ with $m \leq \ell$ and $e \neq f$, then $b_f|_e = 0$. And $b_f|_F = 0$ for face $F \in \Delta_{n-1}(T), f \not\subseteq F$.*

Proof. Take any $e \in \Delta_m(T)$ with $m \leq \ell$ and $e \neq f$ when $m = \ell$. We claim $f \cap e^* \neq \varnothing$. Assume $f \cap e^* = \varnothing$. Then $\Delta_0(f^*) \cup \Delta_0(e) = \{0, 1, \dots, n\}$ and thus $f \subseteq e$ which contradicts with either $m < \ell$ or $e \neq f$.

As $f \cap e^* \neq \varnothing$, then b_f contains λ_i for some $i \in e^*$ and consequently $b_f|_e = 0$.

Similarly $f \not\subseteq F$ implies b_f contains λ_i for some $i \notin F$ and consequently $b_f|_F = 0$. $\square$

2.5. **Triangulation.** Let Ω be a polyhedral domain in $\mathbb{R}^n$, $n \geq 1$. A geometric triangulation $\mathcal{T}_h$ of Ω is a set of n-simplices such that

$$\cup_{T\in\mathcal{T}_h} T = \Omega, \quad \mathring{T}_i \cap \mathring{T}_j = \emptyset,\ \forall\, T_i, T_j \in \mathcal{T}_h, T_i \neq T_j.$$

The subscript h denotes the diameter of each element and can be understood as a piecewise constant function on $\mathcal{T}_h$. A triangulation is conforming if the intersection of two simplexes are common lower dimensional sub-simplex. We shall restrict to conforming triangulations in this paper.

The interpolation points on a conforming triangulation $\mathcal{T}_h$ is

$$\mathcal{X} = \bigcup_{T\in\mathcal{T}_h} \mathcal{X}_T. \tag{2}$$

Note that a lot of duplications exists in (2). A direct sum of the interpolation set is given by

$$\mathcal{X} = \bigoplus_{\ell=0}^{n} \bigoplus_{f\in\Delta_\ell(\mathcal{T}_h)} \mathcal{X}_{\mathring{f}}, \tag{3}$$

where $\Delta_\ell(\mathcal{T}_h)$ denotes the set of ℓ-dimensional subsimplices of $\mathcal{T}_h$. In implementation, computation of local matrices on each simplex is based on (2) while to assemble a matrix equation, (3) should be used. As Lagrange element is globally continuous, the indexing of interpolation points on vertices, edges, faces should be unique and a mapping from the local index to the global index is needed. The index mapping from (2) to (3) will be discussed in Section 6.

3. Geometric Decompositions of Lagrange elements

In this section, we present a geometric decomposition for Lagrange finite elements on n-dimensional simplices. We introduce the concept of Lagrange interpolation basis functions, where function values at interpolation points serve as degrees of freedom.

3.1. **Geometric decomposition.** For the polynomial space $\mathbb{P}_k(T)$ with $k \geq 1$ on an n-dimensional simplex T, we have the following geometric decomposition of Lagrange element [**?**, (2.6)] and a proof can be found in [**?**].

Theorem 3.1 (Geometric decomposition of Lagrange element)**.** *For the polynomial space $\mathbb{P}_k(T)$ with $k \geq 1$ on an n-dimensional simplex T, we have the following decomposition*

$$\mathbb{P}_k(T) = \bigoplus_{\ell=0}^{n} \bigoplus_{f\in\Delta_\ell(T)} b_f \mathbb{P}_{k-(\ell+1)}(f). \tag{4}$$

The function $u \in \mathbb{P}_k(T)$ is uniquely determined by DoFs

$$\int_f u\, p\, \mathrm{d}s \quad p \in \mathbb{P}_{k-(\ell+1)}(f), f \in \Delta_\ell(T), \ell = 0, 1, \ldots, n.$$

Introduce the bubble polynomial space of degree k on a sub-simplex f as

$$\mathbb{B}_k(f) := b_f \mathbb{P}_{k-(\ell+1)}(f), \quad f \in \Delta_\ell(T), 1 \leq \ell \leq n.$$

It is called a bubble space as

$$\mathrm{tr}^{\mathrm{grad}}\, u := u|_{\partial f} = 0, \quad u \in \mathbb{B}_k(f).$$

Then we can write (4) as

$$\mathbb{P}_k(T) = \mathbb{P}_1(T) \oplus \bigoplus_{\ell=1}^{n} \bigoplus_{f\in\Delta_\ell(T)} \mathbb{B}_k(f). \tag{5}$$

That is a polynomial of degree k can be decomposed into a linear polynomial plus bubbles on edges, faces, and all sub-simplexes.

Based on a conforming triangulation $\mathcal{T}_h$, the k-th order Lagrange finite element space $V_k^{\mathrm{L}}(\mathcal{T}_h)$ is defined as

$$V_k^{\mathrm{L}}(\mathcal{T}_h) = \{v \in C(\Omega) : v|_T \in \mathbb{P}_k(T), T \in \mathcal{T}_h\},$$

and will have a geometric decomposition

$$V_k^{\mathrm{L}}(\mathcal{T}_h) = V_1^{\mathrm{L}}(\mathcal{T}_h) \oplus \bigoplus_{\ell=1}^{n} \bigoplus_{f \in \Delta_\ell(\mathcal{T}_h)} \mathbb{B}_k(f).$$

Here we extend the polynomial on f to each element T containing f by the Bernstein form and extension of multi-index; see $E(\alpha)$ defined in (1). Consequently the dimension of V_k^{L} is

$$V_k^{\mathrm{L}}(\mathcal{T}_h) = \sum_{\ell=0}^{n} |\Delta_\ell(\mathcal{T}_h)| \binom{k-1}{\ell},$$

where $|\Delta_\ell(\mathcal{T}_h)|$ is the cardinality of number of $\Delta_\ell(\mathcal{T}_h)$, i.e., the number of ℓ-dimensional simplices in $\mathcal{T}_h$.

The geometric decomposition can be naturally extended to vector Lagrange elements. For $k \geq 1$, define

$$\mathbb{B}_k^n(f) := b_f \mathbb{P}_{k-(\ell+1)}(f) \otimes \mathbb{R}^n.$$

Clearly we have

$$\mathbb{P}_k^n(T) = \mathbb{P}_1^n(T) \oplus \bigoplus_{\ell=1}^{n} \bigoplus_{f \in \Delta_\ell(T)} \mathbb{B}_k^n(f). \tag{6}$$

For an $f \in \Delta_\ell(T)$, we choose a $t-n$ coordinate $\{\boldsymbol{t}_i^f, \boldsymbol{n}_j^f, i = 1, \ldots, \ell, j = 1, \ldots, n-\ell\}$ so that

- $\mathscr{T}^f := \operatorname{spn}\{\boldsymbol{t}_1, \ldots, \boldsymbol{t}_\ell\}$, is the tangential plane of f;
- $\mathscr{N}^f := \operatorname{spn}\{\boldsymbol{n}_1, \ldots, \boldsymbol{n}_{n-\ell}\}$ is the normal plane of f.

When $\ell = 0$, i.e., for vertices, no tangential component, and for $\ell = n$, no normal component. We have the trivial decompositions

$$\mathbb{R}^n = \mathscr{T}^f \oplus \mathscr{N}^f, \quad \mathbb{B}_k^n(f) = \left[\mathbb{B}_k(f) \otimes \mathscr{T}^f\right] \bigoplus \left[\mathbb{B}_k(f) \otimes \mathscr{N}^f\right]. \tag{7}$$

Restricted to an ℓ-dimensional sub-simplex $f \in \Delta_\ell(T)$, define

$$\mathbb{B}_k^\ell(f) := \mathbb{B}_k(f) \otimes \mathscr{T}^f,$$

which is a space of ℓ-dimensional vectors on the tangential space with vanishing trace $\operatorname{tr}^{\mathrm{grad}}$ on ∂f. As a corollary of Theorem 3.1 and $t-n$ decomposition (7), we have the following decomposition of the vector Lagrange elements.

Corollary 3.2. *Let $k \geq 1$ be an integer. For each $f \in \Delta_\ell(T)$, we choose a $t-n$ basis $\{\boldsymbol{t}_i^f, \boldsymbol{n}_j^f, i = 1, \ldots, \ell, j = 1, \ldots, n-\ell\}$. Then a function $\boldsymbol{u} \in \mathbb{P}_k(T; \mathbb{R}^n)$ is uniquely determined by the DoFs*

$$\int_f (\boldsymbol{u} \cdot \boldsymbol{t}_i^f)\, p \,\mathrm{d}s, i = 1, \ldots, \ell, p \in \mathbb{P}_{k-(\ell+1)}(f), f \in \Delta_\ell(T), \ell = 1, \ldots, n, \tag{8a}$$

$$\int_f (\boldsymbol{u} \cdot \boldsymbol{n}_j^f)\, p \,\mathrm{d}s, j = 1, \ldots, n-\ell, p \in \mathbb{P}_{k-(\ell+1)}(f), f \in \Delta_\ell(T), \ell = 0, \ldots, n-1. \tag{8b}$$

When move to a triangulation $\mathcal{T}_h$, we shall call a basis of $\mathscr{T}^f$ or $\mathscr{N}^f$ is global if it depends only on f not the element T containing f. Otherwise it is called local and may vary in different elements.

3.2. **Lagrange interpolation basis functions.** Previously the DoFs are given by moments on sub-simplexes. Now we present a set of DoFs as function values on the interpolation points and give its dual basis for the k-th order Lagrange element on an n-simplex.

Lemma 3.3 (Lagrange interpolation basis functions)**.** *A basis function of k-th order Lagrange finite element space on T is:*

$$\phi_{\boldsymbol{\alpha}}(\boldsymbol{x}) = \frac{1}{\boldsymbol{\alpha}!}\prod_{i=0}^{n}\prod_{j=0}^{\alpha_i-1}(k\lambda_i(\boldsymbol{x})-j), \quad \boldsymbol{\alpha}\in\mathbb{T}_k^n,$$

with the DoFs defined as the function value at the interpolation points:

$$N_{\boldsymbol{\alpha}}(u) = u(\boldsymbol{x}_\alpha), \quad \boldsymbol{x}_\alpha\in\mathcal{X}_T.$$

Proof. It is straightforward to verify the duality of the basis and DoFs

$$N_{\boldsymbol{\beta}}(\phi_{\boldsymbol{\alpha}}) = \phi_{\boldsymbol{\alpha}}(\boldsymbol{x}_\beta) = \delta_{\boldsymbol{\alpha}}^{\boldsymbol{\beta}} = \begin{cases} 1 & \text{if } \boldsymbol{\alpha}=\boldsymbol{\beta} \\ 0 & \text{otherwise} \end{cases}.$$

As

$$|\mathbb{T}_k^n| = \binom{n+k}{k} = \dim\mathbb{P}_k(T),$$

$\{\phi_{\boldsymbol{\alpha}},\boldsymbol{\alpha}\in\mathbb{T}_k^n\}$ is a basis of $\mathbb{P}_k(T)$ and $\{N_{\boldsymbol{\alpha}},\boldsymbol{\alpha}\in\mathbb{T}_k^n\}$ is a basis of the dual space $\mathbb{P}_k^*(T)$. ☐

Given a triangulation $\mathcal{T}_h$ and degree k, recall that

$$\mathcal{X}_{\mathcal{T}_h} = \bigcup_{T\in\mathcal{T}_h}\mathcal{X}_T = \bigoplus_{\ell=0}^{n}\bigoplus_{f\in\Delta_\ell(\mathcal{T}_h)}\mathcal{X}_{\mathring{f}}.$$

Denote by

$$\mathbb{T}_k^n(\mathcal{T}_h) := \bigoplus_{\boldsymbol{x}_i\in\Delta_0(\mathcal{T}_h)}\mathbb{T}_k^0(\boldsymbol{x}_i)\oplus\bigoplus_{\ell=1}^{n}\bigoplus_{f\in\Delta_\ell(\mathcal{T}_h)}\mathbb{T}_k^\ell(\mathring{f}).$$

For a lattice $\alpha\in\mathbb{T}_k^\ell(\mathring{f})$, we use extension operator E defined in (1) to extend α to each simplex T containing f. We also extend the polynomial on f to T by the Bernstein form.

Theorem 3.4 (DoFs of Lagrange finite element on $\mathcal{T}_h$)**.** *A basis for the k-th Lagrange finite element space $V_k^{\mathrm{L}}(\mathcal{T}_h)$ is*

$$\{\phi_{\boldsymbol{\alpha}},\boldsymbol{\alpha}\in\mathbb{T}_k^n(\mathcal{T}_h)\}$$

with DoFs

$$N_{\boldsymbol{\alpha}}(u) = u(\boldsymbol{x}_\alpha), \quad \boldsymbol{x}_\alpha\in\mathcal{X}_{\mathcal{T}_h}.$$

Proof. For $F\in\Delta_{n-1}(\mathcal{T}_h)$, thanks to Lemma 3.3, $\phi_{\boldsymbol{\alpha}}|_F$ is uniquely determined by DoFs $\{u(\boldsymbol{x}_\alpha),\boldsymbol{x}_\alpha\in\mathcal{X}_F\}$ on face F, hence $\phi_{\boldsymbol{\alpha}}\in V_k^{\mathrm{L}}(\mathcal{T}_h)$. Clearly the cardinality of $\{\phi_{\boldsymbol{\alpha}},\boldsymbol{\alpha}\in\mathbb{T}_k^n(\mathcal{T}_h)\}$ is same as the dimension of space $V_k^{\mathrm{L}}(\mathcal{T}_h)$. Then we only need to show these functions are linearly independent, which follows from the fact $N_{\boldsymbol{\beta}}(\phi_{\boldsymbol{\alpha}}) = \phi_{\boldsymbol{\alpha}}(\boldsymbol{x}_\beta) = \delta_{\boldsymbol{\alpha}}^{\boldsymbol{\beta}}$ for $\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathbb{T}_k^n(\mathcal{T}_h)$. ☐

For an interpolation point $\boldsymbol{x}\in\mathcal{X}_T$, let $\{\boldsymbol{e}_{\boldsymbol{x}}^i, i=0,\ldots,n-1\}$ be a basis of $\mathbb{R}^n$, and its dual basis is denoted by $\{\hat{\boldsymbol{e}}_{\boldsymbol{x}}^i, i=0,\ldots,n-1\}$. When $\{\boldsymbol{e}_{\boldsymbol{x}}^i, i=0,\ldots,n-1\}$ is orthonormal, its dual basis is itself.

Lemma 3.5. *A polynomial function $\boldsymbol{u} \in \mathbb{P}_k^n(T)$ can be uniquely determined by the DoFs:*

$$N_{\boldsymbol{\alpha}}^i(\boldsymbol{u}) := \boldsymbol{u}(\boldsymbol{x}_{\boldsymbol{\alpha}}) \cdot \boldsymbol{e}_{\boldsymbol{x}_{\boldsymbol{\alpha}}}^i, \quad \boldsymbol{x}_{\boldsymbol{\alpha}} \in \mathcal{X}_T, i = 0, \ldots, n-1.$$

The basis function on T dual to this set of DoFs can be explicitly written as:

$$\boldsymbol{\phi}_{\boldsymbol{\alpha}}^i(\boldsymbol{x}) = \phi_{\boldsymbol{\alpha}}(\boldsymbol{x}) \hat{\boldsymbol{e}}_{\boldsymbol{x}_{\boldsymbol{\alpha}}}^i, \quad \boldsymbol{\alpha} \in \mathbb{T}_k^n, i = 0, \ldots, n-1.$$

Proof. It is straightforward to verify the duality

$$N_{\boldsymbol{\beta}}^j(\boldsymbol{\phi}_{\boldsymbol{\alpha}}^i) = \boldsymbol{\phi}_{\boldsymbol{\alpha}}^i(\boldsymbol{x}_\beta) \cdot \boldsymbol{e}_{\boldsymbol{x}_\beta}^j = \phi_{\boldsymbol{\alpha}}(\boldsymbol{x}_\beta) \hat{\boldsymbol{e}}_{\boldsymbol{x}_{\boldsymbol{\alpha}}}^i \cdot \boldsymbol{e}_{\boldsymbol{x}_\beta}^j = \delta_i^j \delta_{\boldsymbol{\alpha}}^{\boldsymbol{\beta}},$$

for $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{T}_k^n, i, j = 0, \ldots, n-1.$ □

4. Geometric Decompositions of Face Elements

Define $H(\operatorname{div}, \Omega) := \{\boldsymbol{v} \in L^2(\Omega; \mathbb{R}^n) : \operatorname{div} \boldsymbol{v} \in L^2(\Omega)\}$. For a subdomain $K \subseteq \Omega$, the trace operator for the div operator is

$$\operatorname{tr}_K^{\operatorname{div}} \boldsymbol{v} = \boldsymbol{n} \cdot \boldsymbol{v}|_{\partial K} \quad \text{for} \quad \boldsymbol{v} \in H(\operatorname{div}, \Omega),$$

where $\boldsymbol{n}$ denotes the outwards unit normal vector of ∂K. Given a triangulation $\mathcal{T}_h$ and a piecewise smooth function $\boldsymbol{u}$, it is well known that $\boldsymbol{u} \in H(\operatorname{div}, \Omega)$ if and only if $\boldsymbol{n}_F \cdot \boldsymbol{u}$ is continuous across all faces $F \in \Delta_{n-1}(\mathcal{T}_h)$, where $\boldsymbol{n}_F$ is a global normal vector of F. An $H(\operatorname{div})$-conforming finite element is thus also called a face element.

We shall consider Brezzi-Douglas-Marini (BDM) element [**?**] or second family of Nédélec face element [**?**, **?**, **?**] whose shape function space is full polynomial space $\mathbb{P}_k^n$.

4.1. **Geometric decomposition.** Define the polynomial div bubble space

$$\mathbb{B}_k(\operatorname{div}, T) = \ker(\operatorname{tr}_T^{\operatorname{div}}) \cap \mathbb{P}_k^n(T).$$

Recall that $\mathbb{B}_k^\ell(f) = \mathbb{B}_k(f) \otimes \mathscr{T}^f$ consists of bubble polynomials on the tangential plane of f. For $\boldsymbol{u} \in \mathbb{B}_k^\ell(f)$, as $\boldsymbol{u}$ is on the tangent plane, $\boldsymbol{u} \cdot \boldsymbol{n}_F = 0$ for $f \subseteq F$. When $f \not\subseteq F$, by Lemma 2.1, $b_f|_F = 0$. So $\mathbb{B}_k^\ell(f) \subseteq \mathbb{B}_k(\operatorname{div}, T)$ for $k \geq 2$ and $\dim f \geq 1$. In [**?**], we have proved that the div-bubble polynomial space has the following decomposition.

Lemma 4.1. *For $k \geq 2$,*

$$\mathbb{B}_k(\operatorname{div}, T) = \bigoplus_{\ell=1}^n \bigoplus_{f \in \Delta_\ell(T)} \mathbb{B}_k^\ell(f).$$

Notice that as no tangential plane on vertices, there is no div-bubble associated to vertices and consequently the degree of a div-bubble polynomial is greater than or equal to 2. Next we present two geometric decompositions of a div-element.

Theorem 4.2. *For $k \geq 1$, we have*

$$\mathbb{P}_k^n(T) = \mathbb{P}_1^n(T) \oplus \left(\bigoplus_{\ell=1}^{n-1} \bigoplus_{f \in \Delta_\ell(T)} (\mathbb{B}_k(f) \otimes \mathscr{N}^f)\right) \oplus \mathbb{B}_k(\operatorname{div}, T), \tag{9}$$

$$\mathbb{P}_k^n(T) = \bigoplus_{F \in \Delta_{n-1}(T)} (\mathbb{P}_k(F) \boldsymbol{n}_F) \oplus \mathbb{B}_k(\operatorname{div}, T). \tag{10}$$

Proof. The first decomposition (9) is a rearrangement of (6) by merging the tangential component $\mathbb{B}_k^\ell(f)$ into the bubble space $\mathbb{B}_k(\operatorname{div}, T)$.

Next we prove the decomposition (10). For an ℓ-dimensional, $0 \leq \ell \leq n-1$, sub-simplex $f \in \Delta_\ell(T)$, we choose the $n - \ell$ face normal vectors $\{\boldsymbol{n}_F : F \in \Delta_{n-1}(T), f \subseteq F\}$ as the basis of $\mathscr{N}^f$. Therefore we have

$$\mathbb{B}_k(f) \otimes \mathscr{N}^f = \bigoplus_{F \in \Delta_{n-1}(T), f \subseteq F} \mathbb{B}_k(f) \boldsymbol{n}_F.$$

Then (9) becomes

$$\mathbb{P}_k^n(T) = \mathbb{P}_1^n(T) \oplus \left(\bigoplus_{\ell=1}^{n-1} \bigoplus_{f \in \Delta_\ell(T)} \bigoplus_{F \in \Delta_{n-1}(T), f \subseteq F} \mathbb{B}_k(f) \boldsymbol{n}_F\right) \oplus \mathbb{B}_k(\operatorname{div}, T).$$

At a vertex v, we choose $\{\boldsymbol{n}_F : F \in \Delta_{n-1}(T), \mathrm{v} \subseteq F\}$ as the basis of $\mathbb{R}^n$ and write

$$\begin{aligned}\mathbb{P}_1^n(T) &= \bigoplus_{\boldsymbol{x}\in\Delta_0(T)}\bigoplus_{F\in\Delta_{n-1}(T),\boldsymbol{x}\in\Delta_0(F)}\operatorname{span}\{\lambda_{\boldsymbol{x}}\boldsymbol{n}_F\}\\ &= \bigoplus_{F\in\Delta_{n-1}(T)}\bigoplus_{\boldsymbol{x}\in\Delta_0(F)}\operatorname{span}\{\lambda_{\boldsymbol{x}}\boldsymbol{n}_F\}\\ &= \bigoplus_{F\in\Delta_{n-1}(T)}\mathbb{P}_1(F)\boldsymbol{n}_F.\end{aligned}$$

Then by swapping the ordering of f and F in the direct sum, i.e.,

$$\bigoplus_{\ell=1}^{n-1}\bigoplus_{f\in\Delta_\ell(T)}\bigoplus_{F\in\Delta_{n-1}(T),f\subseteq F} = \bigoplus_{F\in\Delta_{n-1}(T)}\bigoplus_{\ell=1}^{n-1}\bigoplus_{f\in\Delta_\ell(F)},$$

and using the decomposition (4) of Lagrange element, we obtain the decomposition (10). $\square$

In decomposition (9), we single out $\mathbb{P}_1^n(T)$ to emphasize a div-conforming element can be obtained by adding div-bubble and normal component on sub-simplexes starting from edges. In (10), we group all normal components facewisely which leads to the classical BDM element.

As $H(\operatorname{div},\Omega)$-conforming elements require the normal continuity across each F, the normal vector $\boldsymbol{n}_F$ is chosen globally. Namely $\boldsymbol{n}_F$ depends on F only. It may coincide with the outwards or inwards normal vector for an element T containing F. On the contrary, all tangential basis for $\mathscr{T}^f$ are local and thus the tangential component are multi-valued and merged to the element-wise div bubble function $\mathbb{B}_k(\operatorname{div},T)$.

4.2. **Tangential-normal decomposition of face elements.** Given a triangulation $\mathcal{T}_h$, for each $F \in \Delta_{n-1}(\mathcal{T}_h)$, we choose a global normal vector $\boldsymbol{n}_F$, i.e., it depends on F only not the element T containing F. Given an $f \in \Delta_\ell(T)$, we choose $\{\boldsymbol{n}_F, f \subseteq F \in \partial T\}$ as the basis for its normal plane and an arbitrary basis for the tangential plane. We claim the resulting element is the BDM element.

Theorem 4.3. *Take $\mathbb{P}_k^n(T)$ as the shape function space. Given an $f \in \Delta_\ell(T)$, we choose $\{\boldsymbol{n}_F, f \subseteq F \in \partial T\}$ as the basis for its normal plane and an arbitrary basis $\{\boldsymbol{t}_i^f\}$ for the tangent plane. Then the DoFs*

$$\int_f (\boldsymbol{u}\cdot\boldsymbol{n}_F)\, p \,\mathrm{d}s,\quad p\in\mathbb{P}_{k-(\ell+1)}(f), F\in\Delta_{n-1}(T), f\subseteq F, \tag{11a}$$

$$\int_f (\boldsymbol{u}\cdot\boldsymbol{t}_i^f)\, p \,\mathrm{d}s,\quad p\in\mathbb{P}_{k-(\ell+1)}(f), i=1,\ldots,\ell, \tag{11b}$$

for all $f\in\Delta_\ell(T), \ell=0,1,\ldots,n$ are equivalent to the BDM element

$$\int_F \boldsymbol{v}\cdot\boldsymbol{n}_F p \,\mathrm{d}S,\quad p\in\mathbb{P}_k(F), F\in\Delta_{n-1}(T), \tag{12a}$$

$$\int_T \boldsymbol{v}\cdot\boldsymbol{p}\,\mathrm{d}x,\quad \boldsymbol{p}\in\mathbb{B}_k(\operatorname{div},T). \tag{12b}$$

Proof. The unisolvence of DoFs (11) for $\mathbb{P}_k^n(T)$ follows from Corollary 3.2 and switching $\{\boldsymbol{n}_1^f,\ldots,\boldsymbol{n}_{n-\ell}^f\}$ to $\{\boldsymbol{n}_F : F\in\Delta_{n-1}(T), f\subseteq F\}$.

By the geometric decomposition (4) of Lagrange element, we can merge DoFs in (11a) into DoF (12a). By Lemma 4.1, we can merge DoFs in (11b) into DoF (12b). $\square$

4.3. **A basis for the BDM element.** For the efficient implementation, we change the DoFs as the function values at interpolation nodes and combine with $t-n$ decomposition to present a nodal basis for the BDM element.

Given an $f \in \Delta_\ell(T)$, we choose $\{\boldsymbol{n}_F, F \in \Delta_{n-1}(T), f \subseteq F\}$ as the basis for its normal plane $\mathcal{N}^f$ and an arbitrary basis $\{\boldsymbol{t}_i^f, i = 1, \ldots, \ell\}$ for the tangential plane. Denote by $\{\hat{\boldsymbol{n}}_F \in \mathcal{N}^f, F \in \Delta_{n-1}(T), f \subseteq F\}$ the basis of $\mathcal{N}^f$ dual to $\{\boldsymbol{n}_F, F \in \Delta_{n-1}(T), f \subseteq F\}$, i.e.,

$$(\boldsymbol{n}_F, \hat{\boldsymbol{n}}_{F'}) = \delta_{(F,F')}, \quad F, F' \in \Delta_{n-1}(T), f \in F \cap F'.$$

Similarly choose a basis $\{\hat{\boldsymbol{t}}_i^f\}$ dual to $\{\boldsymbol{t}_j^f\}$.

For $f \in \Delta_\ell(T)$ and $e \in \partial f$, let $\boldsymbol{n}_{f,e}$ be an unit normal vector of e but tangential to f. When $\ell = 1$,f is an edge and e is a vertex. Then $\boldsymbol{n}_{f,e}$ is the edge vector of f. When $\ell = n$, choose $\boldsymbol{n}_{f,e}$ as the normal vector $\boldsymbol{n}$. Using these notations we can give an explicit expression of the dual basis $\{\hat{\boldsymbol{n}}_F \in \mathcal{N}^f, F \in \Delta_{n-1}(T), f \subseteq F\}$.

Lemma 4.4. *For $f \in \Delta_\ell(T)$,*

$$\hat{\boldsymbol{n}}_{F_i} = \frac{1}{\boldsymbol{n}_{f+i,f} \cdot \boldsymbol{n}_{F_i}} \boldsymbol{n}_{f+i,f} \quad i \in f^*, \tag{13}$$

where $f+i$ denotes the $(\ell+1)$-dimensional face in $\Delta_{\ell+1}(T)$ with vertices $f(0), \ldots, f(\ell)$ and i for $i \in f^*$.

Proof. Clearly $\boldsymbol{n}_{f+i,f} \in \mathscr{N}^f$ for $i \in f^*$. It suffices to prove

$$\boldsymbol{n}_{f+i,f} \cdot \boldsymbol{n}_{F_j} = 0 \quad \text{for } i, j \in f^*, i \neq j,$$

which follows from $\boldsymbol{n}_{f+i,f} \in \mathscr{T}^{f+i}$ and $f + i \subseteq F_j$. $\square$

For an interpolation point $\boldsymbol{x} \in \mathcal{X}_T$, we collect the $t-n$ basis together and write as $\{\boldsymbol{e}_{\boldsymbol{x}}^i, i = 0, \ldots, n-1\}$. We can always choose an orthonormal basis for the tangential plane $\mathscr{T}^f$ but for the normal plane $\mathscr{N}^f$ with basis $\{\boldsymbol{n}_F, F \in \Delta_{n-1}(T), f \subseteq F\}$, we use Lemma 4.4 to find its dual basis. For $\boldsymbol{x}_\alpha \in \mathcal{X}_{\mathring{f}}, f \in \Delta_\ell(T)$, by Lemma 3.5, the basis related to $\boldsymbol{x}_\alpha$ are

$$\phi_{\boldsymbol{\alpha}}(\boldsymbol{x})\boldsymbol{t}_i^f, i = 1, \ldots, l; \quad \frac{1}{\boldsymbol{n}_{f+i,f} \cdot \boldsymbol{n}_{F_i}} \phi_{\boldsymbol{\alpha}}(\boldsymbol{x})\boldsymbol{n}_{f+i,f}, i \in f^*.$$

By choosing a global normal basis in the sense that $\boldsymbol{n}_F$ depending only on F not the element containing F, we impose the continuity on the normal direction. We choose a local $\boldsymbol{t}^f$, i.e., for different element T containing f, $\phi_{\boldsymbol{\alpha}}(\boldsymbol{x})\boldsymbol{t}^f(T)$ is different, then no continuity is imposed for the tangential direction.

The novelty of our approach is that we only need a basis of Lagrange element which is well documented; see Section 3.2. Coupling with different $t-n$ decomposition at different sub-simplex, we obtain the classical face elements.

Define the global finite element space

$$\begin{aligned} V_h^{\mathrm{div}} := &\{\boldsymbol{u} \in L^2(\Omega; \mathbb{R}^n) : \boldsymbol{u}|_T \in \mathbb{P}_k(T; \mathbb{R}^n) \text{ for each } T \in \mathcal{T}_h, \\ &\text{all the DoFs (11a) across } F \in \Delta_{n-1}(\mathcal{T}_h) \text{ are single-valued}\}. \end{aligned}$$

By Theorem 4.3, $V_h^{\mathrm{div}} \subset H(\mathrm{div}, \Omega)$ and is equivalent to the BDM space.

Below we present explicit formulae in two and three dimensions.

4.3.1. *BDM element on triangular meshes.* Let T be a triangle, for an interpolation point $\boldsymbol{x} \in \mathcal{X}_T$, we shall choose a frame $\{\boldsymbol{e}_{\boldsymbol{x}}^0, \boldsymbol{e}_{\boldsymbol{x}}^1\}$ at $\boldsymbol{x}$ and its dual frame $\{\hat{\boldsymbol{e}}_{\boldsymbol{x}}^0, \hat{\boldsymbol{e}}_{\boldsymbol{x}}^1\}$, i.e.

$$(\boldsymbol{e}_{\boldsymbol{x}}^i, \hat{\boldsymbol{e}}_{\boldsymbol{x}}^j) = \delta_i^j$$

as follows where all normal and tangential vectors are unit vectors.

(1) If $\boldsymbol{x} \in \Delta_0(T)$, assuming the two adjacent edges are e_0 and e_1, then

$$\boldsymbol{e}_{\boldsymbol{x}}^0 = \boldsymbol{n}_{e_0}, \quad \boldsymbol{e}_{\boldsymbol{x}}^1 = \boldsymbol{n}_{e_1}, \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^0 = \frac{\boldsymbol{t}_{e_1}}{\boldsymbol{t}_{e_1} \cdot \boldsymbol{n}_{e_0}}, \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^1 = \frac{\boldsymbol{t}_{e_0}}{\boldsymbol{t}_{e_0} \cdot \boldsymbol{n}_{e_1}}.$$

(2) If $\boldsymbol{x} \in \mathcal{X}_{\mathring{e}}, e \in \Delta_1(T)$, then

$$\boldsymbol{e}_{\boldsymbol{x}}^0 = \boldsymbol{n}_e, \quad \boldsymbol{e}_{\boldsymbol{x}}^1 = \boldsymbol{t}_e, \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^0 = \boldsymbol{n}_e, \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^1 = \boldsymbol{t}_e.$$

(3) If $\boldsymbol{x} \in \mathcal{X}_{\mathring{T}}$, then

$$\boldsymbol{e}_{\boldsymbol{x}}^0 = (1,0), \quad \boldsymbol{e}_{\boldsymbol{x}}^1 = (0,1), \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^0 = (1,0), \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^1 = (0,1).$$

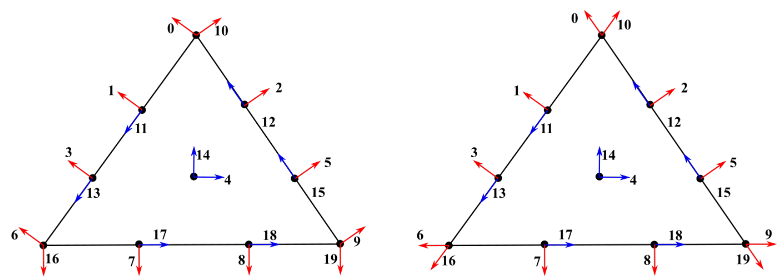

FIGURE 2. The left figure shows $\{\boldsymbol{e}_0, \boldsymbol{e}_1\}$ at each interpolation point, the right figure shows $\{\hat{\boldsymbol{e}}_0, \hat{\boldsymbol{e}}_1\}$ at each interpolation point.

Notice the dual basis on vertices is consistent with the general formulae (13) as $\boldsymbol{n}_{e,\mathrm{v}}$ is the edge vector by definition.

4.3.2. *BDM element on tetrahedron meshes.* Let T be a tetrahedron, for an interpolation point $\boldsymbol{x} \in \mathcal{X}_T$, we shall choose a frame $\{\boldsymbol{e}_{\boldsymbol{x}}^0, \boldsymbol{e}_{\boldsymbol{x}}^1, \boldsymbol{e}_{\boldsymbol{x}}^2\}$ at $\boldsymbol{x}$ and its dual frame $\{\hat{\boldsymbol{e}}_{\boldsymbol{x}}^0, \hat{\boldsymbol{e}}_{\boldsymbol{x}}^1, \hat{\boldsymbol{e}}_{\boldsymbol{x}}^2\}$ as follows:

(1) For a vertex $\boldsymbol{x} \in \Delta_0(T)$, let adjacent edges of $\boldsymbol{x}$ be e_0, e_1, e_2, and the adjacent faces be F_0, F_1, F_2, satisfying $F_i \cap e_i = \boldsymbol{x}$, then

$$\boldsymbol{e}_{\boldsymbol{x}}^0 = \boldsymbol{n}_{F_0}, \quad \boldsymbol{e}_{\boldsymbol{x}}^1 = \boldsymbol{n}_{F_1}, \quad \boldsymbol{e}_{\boldsymbol{x}}^2 = \boldsymbol{n}_{F_2},$$

$$\hat{\boldsymbol{e}}_{\boldsymbol{x}}^0 = \frac{\boldsymbol{t}_{e_0}}{\boldsymbol{t}_{e_0} \cdot \boldsymbol{n}_{F_0}}, \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^1 = \frac{\boldsymbol{t}_{e_1}}{\boldsymbol{t}_{e_1} \cdot \boldsymbol{n}_{F_1}}, \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^2 = \frac{\boldsymbol{t}_{e_2}}{\boldsymbol{t}_{e_2} \cdot \boldsymbol{n}_{F_2}}.$$

(2) If $\boldsymbol{x} \in \mathcal{X}_{\mathring{e}}, e \in \Delta_1(T)$ with adjacent faces F_0, F_1, then

$$\boldsymbol{e}_{\boldsymbol{x}}^0 = \boldsymbol{n}_{F_0}, \quad \boldsymbol{e}_{\boldsymbol{x}}^1 = \boldsymbol{n}_{F_1}, \quad \boldsymbol{e}_{\boldsymbol{x}}^2 = \boldsymbol{t}_e,$$

$$\hat{\boldsymbol{e}}_{\boldsymbol{x}}^0 = \frac{\boldsymbol{n}_{F_1} \times \boldsymbol{t}_e}{\boldsymbol{n}_{F_0} \cdot (\boldsymbol{n}_{F_1} \times \boldsymbol{t}_e)}, \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^1 = \frac{\boldsymbol{n}_{F_0} \times \boldsymbol{t}_e}{\boldsymbol{n}_{F_1} \cdot (\boldsymbol{n}_{F_0} \times \boldsymbol{t}_e)}, \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^2 = \boldsymbol{t}_e.$$

(3) If $\boldsymbol{x} \in \mathcal{X}_{\mathring{F}}$ with $F \in \Delta_2(T), e \in \partial F$, then

$$\boldsymbol{e}_{\boldsymbol{x}}^0 = \boldsymbol{n}_F, \quad \boldsymbol{e}_{\boldsymbol{x}}^1 = \boldsymbol{t}_e \times \boldsymbol{n}_F, \quad \boldsymbol{e}_{\boldsymbol{x}}^2 = \boldsymbol{t}_e,$$

$$\hat{\boldsymbol{e}}_{\boldsymbol{x}}^0 = \boldsymbol{n}_F, \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^1 = \boldsymbol{t}_e \times \boldsymbol{n}_F, \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^2 = \boldsymbol{t}_e.$$

(4) If $\boldsymbol{x} \in \mathcal{X}_{\mathring{T}}$, then

$$\boldsymbol{e}_{\boldsymbol{x}}^0 = (1,0,0), \quad \boldsymbol{e}_{\boldsymbol{x}}^1 = (0,1,0), \quad \boldsymbol{e}_{\boldsymbol{x}}^2 = (0,0,1),$$

$$\hat{\boldsymbol{e}}_{\boldsymbol{x}}^0 = (1,0,0), \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^1 = (0,1,0), \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^2 = (0,0,1).$$

The dual basis on edges is also consistent with (13) as $\boldsymbol{n}_{F_1} \times \boldsymbol{t}_e$ is $\boldsymbol{n}_{F_1,e}$.

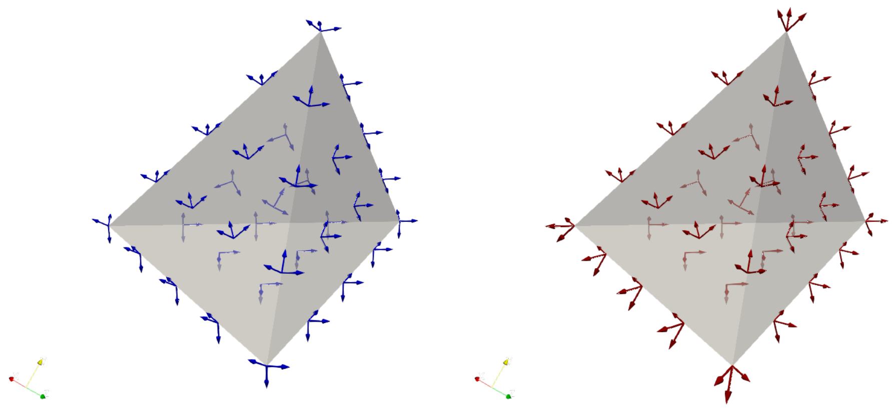

FIGURE 3. The left figure shows $\{\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2\}$ at each interpolation point, the right figure shows $\{\hat{\boldsymbol{e}}_0, \hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2\}$ at each interpolation point.

5. GEOMETRIC DECOMPOSITIONS OF EDGE ELEMENTS

In this section we present geometric decompositions of $H(\operatorname{curl})$-conforming finite element space on an n-dimensional simplex. We first generalize the differential operator to n dimensions and study its trace which motivates our decomposition. We then give an explicit basis dual to function values at interpolation points.

5.1. **Differential operator and its trace.** Denote by $\mathbb{S}$ and $\mathbb{K}$ the subspace of symmetric matrices and skew-symmetric matrices of $\mathbb{R}^{n\times n}$, respectively. For a smooth vector function $\boldsymbol{v}$, define

$$\operatorname{curl} \boldsymbol{v} := 2\operatorname{skw}(\operatorname{grad} \boldsymbol{v}) = \operatorname{grad} \boldsymbol{v} - (\operatorname{grad} \boldsymbol{v})^{\intercal},$$

which is a skew-symmetric matrix function. In two dimensions, for $\boldsymbol{v} = (v_1, v_2)^{\intercal}$,

$$\operatorname{curl} \boldsymbol{v} = \begin{pmatrix} 0 & \partial_{x_2} v_1 - \partial_{x_1} v_2 \\ \partial_{x_1} v_2 - \partial_{x_2} v_1 & 0 \end{pmatrix} = \operatorname{mskw}(\operatorname{rot} \boldsymbol{v}),$$

where $\operatorname{mskw} u := \begin{pmatrix} 0 & -u \\ u & 0 \end{pmatrix}$ and $\operatorname{rot} \boldsymbol{v} := \partial_{x_1} v_2 - \partial_{x_2} v_1$. In three dimensions, for $\boldsymbol{v} = (v_1, v_2, v_3)^{\intercal}$,

$$\operatorname{curl} \boldsymbol{v} = \begin{pmatrix} 0 & \partial_{x_2} v_1 - \partial_{x_1} v_2 & \partial_{x_3} v_1 - \partial_{x_1} v_3 \\ \partial_{x_1} v_2 - \partial_{x_2} v_1 & 0 & \partial_{x_3} v_2 - \partial_{x_2} v_3 \\ \partial_{x_1} v_3 - \partial_{x_3} v_1 & \partial_{x_2} v_3 - \partial_{x_3} v_2 & 0 \end{pmatrix} = \operatorname{mskw}(\nabla \times \boldsymbol{v}),$$

where $\operatorname{mskw} \boldsymbol{u} := \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix}$ with $\boldsymbol{u} = (u_1, u_2, u_3)^\intercal$. Hence we can identify $\operatorname{curl} \boldsymbol{v}$ as scalar $\operatorname{rot} \boldsymbol{v}$ in two dimensions, and vector $\nabla \times \boldsymbol{v}$ in three dimensions. In general $\operatorname{curl} \boldsymbol{u}$ is a skew-symmetric matrix.

Define Sobolev space

$$H(\operatorname{curl}, \Omega) := \{\boldsymbol{v} \in L^2(\Omega; \mathbb{R}^n) : \operatorname{curl} \boldsymbol{v} \in L^2(\Omega; \mathbb{K})\}.$$

Given a face $F \in \Delta_{n-1}(T)$, define the trace operator of curl as

$$\operatorname{tr}_F^{\operatorname{curl}} \boldsymbol{v} = 2 \operatorname{skw}(\boldsymbol{v}\boldsymbol{n}_F^\intercal)|_F = (\boldsymbol{v}\boldsymbol{n}_F^\intercal - \boldsymbol{n}_F \boldsymbol{v}^\intercal)|_F.$$

We define $\operatorname{tr}^{\operatorname{curl}}$ as a piecewise defined operator as

$$(\operatorname{tr}^{\operatorname{curl}} \boldsymbol{v})|_F = \operatorname{tr}_F^{\operatorname{curl}} \boldsymbol{v}, \quad F \in \Delta_{n-1}(T).$$

For a vector $\boldsymbol{v} \in \mathbb{R}^n$ and an $n-1$ dimensional face F, the tangential part of $\boldsymbol{v}$ is

$$\Pi_F \boldsymbol{v} := \boldsymbol{v}|_F - (\boldsymbol{v}|_F \cdot \boldsymbol{n}_F)\boldsymbol{n}_F = \sum_{i=1}^{n-1} (\boldsymbol{v}|_F \cdot \boldsymbol{t}_{F,i})\boldsymbol{t}_{F,i},$$

where $\{\boldsymbol{t}_{F,i}, i = 1, \ldots, n-1\}$ is an orthonormal basis of F. As we treat $\operatorname{curl} \boldsymbol{v}$ as a matrix, so is the trace $\operatorname{tr}_F^{\operatorname{curl}} \boldsymbol{v}$. The tangential component of $\boldsymbol{v}$ is a vector. Their relation is given in the following lemma.

Lemma 5.1. *For face $F \in \Delta_{n-1}(T)$, we have*

$$\operatorname{tr}_F^{\operatorname{curl}} \boldsymbol{v} = 2 \operatorname{skw}\big((\Pi_F \boldsymbol{v})\boldsymbol{n}_F^\intercal\big), \quad \Pi_F \boldsymbol{v} = (\operatorname{tr}_F^{\operatorname{curl}} \boldsymbol{v})\boldsymbol{n}_F. \tag{14}$$

Proof. By the decomposition $\boldsymbol{v}|_F = \Pi_F \boldsymbol{v} + (\boldsymbol{v}|_F \cdot \boldsymbol{n}_F)\boldsymbol{n}_F$,

$$\operatorname{tr}_F^{\operatorname{curl}} \boldsymbol{v} = 2 \operatorname{skw}\big((\Pi_F \boldsymbol{v})\boldsymbol{n}_F^\intercal + (\boldsymbol{v}|_F \cdot \boldsymbol{n}_F)\boldsymbol{n}_F\boldsymbol{n}_F^\intercal\big) = 2 \operatorname{skw}\big((\Pi_F \boldsymbol{v})\boldsymbol{n}_F^\intercal\big),$$

which implies the first identity. Then by $\boldsymbol{n}_F^\intercal \boldsymbol{n}_F = 1$ and $(\Pi_F \boldsymbol{v})^\intercal \boldsymbol{n}_F = 0$,

$$(\operatorname{tr}_F^{\operatorname{curl}} \boldsymbol{v})\boldsymbol{n}_F = \big((\Pi_F \boldsymbol{v})\boldsymbol{n}_F^\intercal - \boldsymbol{n}_F(\Pi_F \boldsymbol{v})^\intercal\big)\boldsymbol{n}_F = \Pi_F \boldsymbol{v},$$

i.e. the second identity holds. $\square$

Thanks to (14), the vanishing tangential part $\Pi_F \boldsymbol{v}$ and the vanishing tangential trace $(\operatorname{tr}_F^{\operatorname{curl}} \boldsymbol{v})$ are equivalent.

Lemma 5.2. *Let $\boldsymbol{v} \in L^2(\Omega; \mathbb{R}^n)$ and $\boldsymbol{v}|_T \in H^1(T; \mathbb{S})$ for each $T \in \mathcal{T}_h$. Then $\boldsymbol{v} \in H(\operatorname{curl}, \Omega)$ if and only if $\Pi_F \boldsymbol{v}\,|_{T_1} = \Pi_F \boldsymbol{v}\,|_{T_2}$ for all interior face $F \in \Delta_{n-1}(\mathcal{T}_h)$, where T_1 and T_2 are two elements sharing F.*

Proof. It is an immediate result of Lemma 5.1 in [**?**] and (14). $\square$

5.2. **Geometric decompositions.** Define the polynomial bubble space for the curl operator as

$$\mathbb{B}_k(\operatorname{curl}, T) = \ker(\operatorname{tr}^{\operatorname{curl}}) \cap \mathbb{P}_k^n(T).$$

For Lagrange bubble $\mathbb{B}_k^n(T)$, all components of the vector vanish on ∂T and thus vanish on all sub-simplex with dimension $\leq n-1$. For $\boldsymbol{u} \in \mathbb{B}_k(\operatorname{curl}, T)$, only the tangential component vanishes, which will imply $\boldsymbol{u}$ vanishes on sub-simplex with dimension less than or equal to $n-2$.

Lemma 5.3. *For $\boldsymbol{u} \in \mathbb{B}_k(\operatorname{curl}, T)$, it holds $\boldsymbol{u}|_f = \boldsymbol{0}$ for all $f \in \Delta_\ell(T), 0 \leq \ell \leq n-2$. Consequently $\mathbb{B}_k(\operatorname{curl}, T) \subset \mathbb{B}_k^n(T) \oplus \bigoplus_{F \in \Delta_{n-1}(T)} \mathbb{B}_k^n(F)$.*

Proof. It suffices to consider a sub-simplex $f \in \Delta_{n-2}(T)$. Let $F_1, F_2 \in \Delta_{n-1}(T)$ such that $f = F_1 \cap F_2$. By $\operatorname{tr}_{F_i}^{\operatorname{curl}} \boldsymbol{u} = \mathbf{0}$ for $i = 1, 2$, we have $\Pi_{F_i} \boldsymbol{u} = 0$ and consequently

$$(\boldsymbol{u} \cdot \boldsymbol{t}_i^f)|_f = 0, \quad (\boldsymbol{u} \cdot \boldsymbol{n}_{F_1,f})|_f = (\boldsymbol{u} \cdot \boldsymbol{n}_{F_2,f})|_f = 0 \quad \text{ for } i = 1, \dots, n-2,$$

where $\boldsymbol{n}_{F_i,f}$ is a normal vector f sitting on F_i. As $\operatorname{span}\{\boldsymbol{t}_1^f, \dots, \boldsymbol{t}_{n-2}^f, \boldsymbol{n}_{F_1,f}, \boldsymbol{n}_{F_2,f}\} = \mathbb{R}^n$, we acquire $\boldsymbol{u}|_f = \mathbf{0}$. By the property of face bubbles, we conclude $\boldsymbol{u}$ is a linear combination of $n-1$ face bubbles. $\square$

Obviously $\mathbb{B}_k^n(T) \subset \mathbb{B}_k(\operatorname{curl}, T)$. As $\operatorname{tr}^{\operatorname{curl}}$ contains the tangential component only, the normal component $\mathbb{B}_k(F)\boldsymbol{n}_F$ is also a curl bubble. The following result says their sum is precisely all curl bubble polynomials.

Theorem 5.4. *For $k \geq 1$, it holds that*

$$\mathbb{B}_k(\operatorname{curl}, T) = \mathbb{B}_k^n(T) \oplus \bigoplus_{F \in \Delta_{n-1}(T)} \mathbb{B}_k(F)\boldsymbol{n}_F, \tag{15}$$

and

$$\operatorname{tr}^{\operatorname{curl}} : \mathbb{P}_1^n(T) \oplus \bigoplus_{\ell=1}^{n-2} \bigoplus_{f \in \Delta_\ell(T)} \mathbb{B}_k^n(f) \oplus \bigoplus_{F \in \Delta_{n-1}(T)} \mathbb{B}_k^{n-1}(F) \to \operatorname{tr}^{\operatorname{curl}} \mathbb{P}_k^n(T) \tag{16}$$

is a bijection.

Proof. It is obvious that

$$\mathbb{B}_k^n(T) \oplus \bigoplus_{F \in \Delta_{n-1}(T)} \mathbb{B}_k(F)\boldsymbol{n}_F \subseteq \mathbb{B}_k(\operatorname{curl}, T).$$

Then apply the trace operator to the decomposition (6) to conclude that the map $\operatorname{tr}^{\operatorname{curl}}$ in (16) is onto.

Now we prove it is also injective. Take a function $\boldsymbol{u} \in \mathbb{P}_1^n(T) \oplus \bigoplus_{\ell=1}^{n-2} \bigoplus_{f \in \Delta_\ell(T)} \mathbb{B}_k^n(f) \oplus \bigoplus_{F \in \Delta_{n-1}(T)} \mathbb{B}_k^{n-1}(F)$ and $\operatorname{tr}^{\operatorname{curl}} \boldsymbol{u} = \mathbf{0}$. By Lemma 5.3, we can assume $\boldsymbol{u} = \sum_{F \in \Delta_{n-1}(T)} \boldsymbol{u}_k^F$ with $\boldsymbol{u}_k^F \in \mathbb{B}_k^{n-1}(F)$. Take $F \in \Delta_{n-1}(T)$. We have $\boldsymbol{u}|_F = \boldsymbol{u}_k^F|_F \in \mathbb{B}_k^{n-1}(F)$. Hence $(\boldsymbol{u}_k^F \cdot \boldsymbol{t})|_F = (\boldsymbol{u} \cdot \boldsymbol{t})|_F = 0$ for any $\boldsymbol{t} \in \mathscr{T}^F$, which results in $\boldsymbol{u}_k^F = \mathbf{0}$. Therefore $\boldsymbol{u} = \mathbf{0}$.

Once we have proved the map tr in (16) is bijection, we conclude (15) from the decomposition (6). $\square$

Lemma 5.5. *The* curl *bubble space* $\mathbb{B}_k(\operatorname{curl}, T)$ *is uniquely determined by*

$$\int_F (\boldsymbol{u} \cdot \boldsymbol{n}_F)\, p \,\mathrm{d}s, \quad p \in \mathbb{P}_{k-n}(F), F \in \partial T, \tag{17}$$

$$\int_T \boldsymbol{u} \cdot \boldsymbol{q} \,\mathrm{d}x, \quad \boldsymbol{q} \in \mathbb{B}_k^n(T). \tag{18}$$

Proof. Due to the decomposition (15), the number of (17) and (18) equals $\dim \mathbb{B}_k(\operatorname{curl}, T)$. Assume $\boldsymbol{u} \in \mathbb{B}_k(\operatorname{curl}, T)$, and (17)-(18) vanish. By the decomposition (15), $(\boldsymbol{u} \cdot \boldsymbol{n}_F)|_F \in \mathbb{B}_k(F)$, then the vanishing (17) implies $(\boldsymbol{u} \cdot \boldsymbol{n}_F)|_F = 0$. Thus $\boldsymbol{u} \in \mathbb{B}_k^n(T)$, which together with the vanishing (18) yields $\boldsymbol{u} = \mathbf{0}$. $\square$

We will use curl_f to denote the curl operator restricted to a sub-simplex f with $\dim f \geq 1$. For $f \in \Delta_\ell(T), \ell = 2, \dots, n-1$, by applying Theorem 5.4 to f, we have

$$\mathbb{B}_k(\operatorname{curl}_f, f) = \mathbb{B}_k^\ell(f) \oplus \bigoplus_{e \in \partial f} \mathbb{B}_k(e)\boldsymbol{n}_{f,e}. \tag{19}$$

Notice that the curl_f-bubble function is defined for $\ell \geq 2$ not including edges. Indeed, for an edge e and a vertex $\boldsymbol{x}$ of e, $\boldsymbol{n}_{e,\boldsymbol{x}}$ is $\boldsymbol{t}_e$ if $\boldsymbol{x}$ is the ending vertex of e and $-\boldsymbol{t}_e$ otherwise. Then for $\ell = 1$

$$\mathbb{B}_k(e)\boldsymbol{t}_e \oplus \bigoplus_{\boldsymbol{x} \in \partial e} \operatorname{span}\{\lambda_{\boldsymbol{x}} \boldsymbol{n}_{e,\boldsymbol{x}}\} = \mathbb{P}_k(e)\boldsymbol{t}_e.$$

which is no longer a bubble function on e.

Theorem 5.6. *For $k \geq 1$, we have*

$$\mathbb{P}_k^n(T) = \mathbb{P}_1^n(T) \oplus \bigoplus_{e\in\Delta_1(T)} \mathbb{B}_k(e)\boldsymbol{t}_e \oplus \bigoplus_{\ell=2}^n \bigoplus_{f\in\Delta_\ell(T)} \mathbb{B}_k(\operatorname{curl}_f, f), \tag{20}$$

$$\mathbb{P}_k^n(T) = \bigoplus_{e\in\Delta_1(T)} \mathbb{P}_k(e)\boldsymbol{t}_e \oplus \bigoplus_{\ell=2}^n \bigoplus_{f\in\Delta_\ell(T)} \mathbb{B}_k(\operatorname{curl}_f, f). \tag{21}$$

Proof. For $e \in \Delta_{\ell-1}(T)$, the face normal vectors $\{\boldsymbol{n}_{f,e} : f \in \Delta_\ell(T), e \subseteq f\}$ form a basis of $\mathscr{N}^e$. So we have

$$\mathbb{B}_k(e) \otimes \mathscr{N}^e = \bigoplus_{f\in\Delta_\ell(T), e\subseteq f} \mathbb{B}_k(e)\boldsymbol{n}_{f,e}.$$

In decomposition (6) shift the normal component one level up:

$$\begin{aligned}&\bigoplus_{\ell=2}^n \bigoplus_{e\in\Delta_{\ell-1}(T)} \big[\mathbb{B}_k^{\ell-1}(e) \bigoplus_{e\subseteq f, f\in\Delta_\ell(T)} \mathbb{B}_k(e)\boldsymbol{n}_{f,e}\big]\\ =&\bigoplus_{e\in\Delta_1(T)} \mathbb{B}_k(e)\boldsymbol{t}_e \oplus \bigoplus_{\ell=2}^n \bigoplus_{f\in\Delta_\ell(T)} \big[\mathbb{B}_k^{\ell}(f) \bigoplus_{e\subseteq f, e\in\Delta_{\ell-1}(T)} \mathbb{B}_k(e)\boldsymbol{n}_{f,e}\big],\end{aligned}$$

where we switch the sum of e and f. Then by the characterization (19) of $\mathbb{B}_k(\operatorname{curl}_f, f)$, we get the decomposition (20).

We then distribute the n-component of vector function value at a vertex to the n edges connected to this vertex. For a vertex $\boldsymbol{x}$ of T, there are n edges of T containing this vertex. We use n edge vectors as a basis of $\mathbb{R}^n$ and thus

$$\mathbb{P}_1^n(T) \oplus \bigoplus_{e\in\Delta_1(T)} \mathbb{B}_k(e)\boldsymbol{t}_e = \bigoplus_{e\in\Delta_1(T)} (\mathbb{P}_1(e)\boldsymbol{t}_e \oplus \mathbb{B}_k(e)\boldsymbol{t}_e) = \bigoplus_{e\in\Delta_1(T)} \mathbb{P}_k(e)\boldsymbol{t}_e.$$

Thus (21) holds. $\square$

Decomposition (20) is the counterpart of (5) for Lagrange elements. In decomposition (21), the linear vector polynomial $\mathbb{P}_1^n(T)$ is redistributed to edges and the second kind Nédélec element [**?**] can be derived.

5.3. Tangential-Normal decomposition of second family of Nédélec element.

For $e \in \Delta_{\ell-1}(T)$, the basis $\{\boldsymbol{n}_{f,e} : f \in \Delta_\ell(T), e \subseteq f\}$ of $\mathscr{N}^e$ is dual to the basis $\{\boldsymbol{n}_F : F \in \Delta_{n-1}(T), e \subseteq F\}$; see Lemma 4.4.

Theorem 5.7. *Take $\mathbb{P}_k^n(T)$ as the shape function space. Given an $e \in \Delta_{\ell-1}(T)$, we choose $\{\boldsymbol{n}_{f,e}, e \subseteq f \in \Delta_\ell(T)\}$ as the basis for the normal plane $\mathscr{N}^e$ and a basis $\{\boldsymbol{t}_i^e\}$ of $\mathscr{T}^e$. Then the DoFs*

$$\int_e (\boldsymbol{u}\cdot\boldsymbol{t}_i^e)\, p \,\mathrm{d}s, \quad i = 1,\ldots,\ell-1, p \in \mathbb{P}_{k-\ell}(e), e \in \Delta_{\ell-1}(T), \ell = 2,\ldots,n, \tag{22a}$$

$$\int_e (\boldsymbol{u}\cdot\boldsymbol{n}_{f,e})\, p \,\mathrm{d}s, \quad f \in \Delta_\ell(T), e \subseteq f, p \in \mathbb{P}_{k-\ell}(e), e \in \Delta_{\ell-1}(T), \ell = 1,\ldots,n, \tag{22b}$$

$$\int_T \boldsymbol{u}\cdot\boldsymbol{q} \,\mathrm{d}x, \quad \boldsymbol{q} \in \mathbb{B}_k^n(T) \tag{22c}$$

are equivalent to the second-kind Nédélec element

$$\int_e \boldsymbol{u}\cdot\boldsymbol{t}\, p \,\mathrm{d}s, \quad p \in \mathbb{P}_k(e), e \in \Delta_1(T), \tag{23a}$$

$$\int_f \boldsymbol{u}\cdot\boldsymbol{p} \,\mathrm{d}s, \quad \boldsymbol{p} \in \mathbb{B}_k(\operatorname{curl}_f, f), f \in \Delta_\ell(T), \ell = 2,\ldots,n-1, \tag{23b}$$

$$\int_T \boldsymbol{u}\cdot\boldsymbol{p} \,\mathrm{d}x, \quad \boldsymbol{p} \in \mathbb{B}_k(\operatorname{curl}, T). \tag{23c}$$

Proof. The unisolvence of DoFs (22) for $\mathbb{P}_k^n(T)$ follows from Corollary 3.2 by switching $\{\boldsymbol{n}_1^e, \ldots, \boldsymbol{n}_{n-\ell+1}^e\}$ to $\{\boldsymbol{n}_{f,e}, e \subseteq f \in \Delta_\ell(T)\}$.

By rearranging e and f, DoFs (22) can be rewritten as

$$\int_e \boldsymbol{u} \cdot \boldsymbol{t}\, p \,\mathrm{d}s, \quad p \in \mathbb{P}_k(e), e \in \Delta_1(T), \tag{24a}$$

$$\int_f (\boldsymbol{u} \cdot \boldsymbol{t}_i^f)\, p \,\mathrm{d}s, \quad p \in \mathbb{P}_{k-(\ell+1)}(f), i = 1, \ldots, \ell, \tag{24b}$$

$$\int_e (\boldsymbol{u} \cdot \boldsymbol{n}_{f,e})\, p \,\mathrm{d}s, \quad p \in \mathbb{P}_{k-\ell}(e), e \in \partial f \tag{24c}$$

for $f \in \Delta_\ell(T), \ell = 2, \ldots, n$. By (15) and (19), DoFs (24) are equivalent to DoFs (23). $\square$

For $e \in \Delta_{\ell-1}(T)$, we redistribute the normal DoFs (8b) on e to the tangential part (22b) of f for $e \subseteq f \in \Delta_\ell(T)$.

Remark 5.8. The DoFs of the second kind Nédélec element in [**?**, **?**] are

$$\begin{aligned}
&\int_e \boldsymbol{u} \cdot \boldsymbol{t}\, p \,\mathrm{d}s, \quad p \in \mathbb{P}_k(e), e \in \Delta_1(T),\\
&\int_f \boldsymbol{u} \cdot \boldsymbol{p} \,\mathrm{d}s, \quad \boldsymbol{p} \in \mathbb{P}_{k-\ell}^\ell(f) + (\Pi_f \boldsymbol{x})\mathbb{P}_{k-\ell}(f), f \in \Delta_\ell(T), \ell = 2, \ldots, n-1,\\
&\int_T \boldsymbol{u} \cdot \boldsymbol{p} \,\mathrm{d}x, \quad \boldsymbol{p} \in \mathbb{P}_{k-n}^n(T) + \boldsymbol{x}\mathbb{P}_{k-n}(T).
\end{aligned}$$

There is an isomorphism between $\mathbb{P}_{k-\ell}^\ell(f) + (\Pi_f \boldsymbol{x})\mathbb{P}_{k-\ell}(f)$ and $\mathbb{B}_k(\mathrm{curl}_f, f)$ for $f \in \Delta_\ell(T)$ with $\ell = 2, \ldots, n$, that is $\mathbb{B}_k(\mathrm{curl}_f, f)$ is uniquely determined by DoF

$$\int_f \boldsymbol{u} \cdot \boldsymbol{p} \,\mathrm{d}s, \quad \boldsymbol{p} \in \mathbb{P}_{k-\ell}^\ell(f) + (\Pi_f \boldsymbol{x})\mathbb{P}_{k-\ell}(f),$$

whose proof can be found in Lemma 4.7 in [**?**]. $\square$

Given an $e \in \Delta_{\ell-1}(T_h)$, we choose a global $\{\boldsymbol{n}_{f,e}, e \subseteq f \in \Delta_\ell(T_h)\}$ as the basis for the normal plane $\mathscr{N}^e$ and a global basis $\{\boldsymbol{t}_i^e\}$ of $\mathscr{T}^e$. Define the global finite element space

$$\begin{aligned}
V_h^{\mathrm{curl}} := \{&\boldsymbol{u} \in L^2(\Omega; \mathbb{R}^n) : \boldsymbol{u}|_T \in \mathbb{P}_k(T; \mathbb{R}^n) \text{ for each } T \in \mathcal{T}_h,\\
&\text{all the DoFs (22a) across } e \in \Delta_\ell(\mathcal{T}_h) \text{ are single-valued for } \ell = 1, \ldots, n-1,\\
&\text{all the DoFs (22b) across } f \in \Delta_\ell(\mathcal{T}_h) \text{ are single-valued for } \ell = 1, \ldots, n-1\}.
\end{aligned}$$

Since DoFs (22) are equivalent to DoFs (23), we have $V_h^{\mathrm{curl}} \subset H(\mathrm{curl}, \Omega)$.

Lemma 5.9. *We have $V_h^{\mathrm{curl}} \subset H(\mathrm{curl}, \Omega)$.*

Proof. For an $F \in \Delta_{n-1}(T)$, DoFs (22) related to $\Pi_F \boldsymbol{u}$ are

$$\int_e (\Pi_F \boldsymbol{u} \cdot \boldsymbol{t}_i^e)\, p \,\mathrm{d}s, \quad i = 1, \ldots, \ell-1, p \in \mathbb{P}_{k-\ell}(e), e \in \Delta_{\ell-1}(F), \ell = 2, \ldots, n,$$

$$\int_e (\Pi_F \boldsymbol{u} \cdot \boldsymbol{n}_{f,e})\, p \,\mathrm{d}s, \quad f \in \Delta_\ell(F), e \subseteq f, p \in \mathbb{P}_{k-\ell}(e), e \in \Delta_{\ell-1}(F), \ell = 1, \ldots, n-1,$$

thanks to DoFs (24), which uniquely determine $\Pi_F \boldsymbol{u}$. $\square$

5.4. **A basis for the second-kind Nédélec element.**

Theorem 5.10. *Given an $e \in \Delta_{\ell-1}(T)$, we choose $\{\boldsymbol{n}_{f,e}, e \subseteq f \in \Delta_\ell(T)\}$ as the basis for the normal plane $\mathscr{N}^e$ and a basis $\{\boldsymbol{t}_i^e\}$ of $\mathscr{T}^e$. Then the DoFs*

$$\boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{t}_i^e, \quad \boldsymbol{x} \in \mathcal{X}_{\mathring{e}}, i = 1, \ldots, \ell-1, e \in \Delta_{\ell-1}(T), \ell = 2, \ldots, n, \tag{25a}$$

$$\boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{n}_{f,e}, \quad \boldsymbol{x} \in \mathcal{X}_{\mathring{e}}, f \in \Delta_\ell(T), e \subseteq f, e \in \Delta_{\ell-1}(T), \ell = 1, \ldots, n, \tag{25b}$$

$$\boldsymbol{u}(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathcal{X}_{\mathring{T}} \tag{25c}$$

are uni-solvent for space $\mathbb{P}_k^n(T)$. And V_h^{curl} is same as the global finite element space

$\{\boldsymbol{u} \in L^2(\Omega; \mathbb{R}^n) : \boldsymbol{u}|_T \in \mathbb{P}_k(T; \mathbb{R}^n)$ *for each* $T \in \mathcal{T}_h$,

all the DoFs (25a) *across* $e \in \Delta_\ell(\mathcal{T}_h)$ *are single-valued for* $\ell = 1, \ldots, n-1$,

all the DoFs (25b) *across* $f \in \Delta_\ell(\mathcal{T}_h)$ *are single-valued for* $\ell = 1, \ldots, n-1\}$.

Proof. Apply Theorem 5.7 and the nodal DoFs (simplicial lattices) of Lagrange element. □

By Lemma 4.4, the dual basis of $\{\boldsymbol{n}_{f+i,f}\}_{i \in f^*}$ are $\{\frac{1}{\boldsymbol{n}_{f+i,f} \cdot \boldsymbol{n}_{F_i}} \boldsymbol{n}_{F_i}\}_{i \in f^*}$. For $\boldsymbol{x}_\alpha \in \mathcal{X}_{\mathring{f}}, f \in \Delta_\ell(T)$, by Lemma 3.5, the basis related to $\boldsymbol{x}_\alpha$ are

$$\phi_{\boldsymbol{\alpha}}(\boldsymbol{x}) \boldsymbol{t}_i^f, i = 1, \ldots, \ell; \quad \frac{1}{\boldsymbol{n}_{f+i,f} \cdot \boldsymbol{n}_{F_i}} \phi_{\boldsymbol{\alpha}}(\boldsymbol{x}) \boldsymbol{n}_{F_i}, i \in f^*.$$

Again we present explicit formulae in two and three dimensions.

5.4.1. *Second-kind Nédélec element on triangular meshes.* Let T be a triangle, for lattice point $\boldsymbol{x}$ located in different sub-simplices, we shall choose different frame $\{\boldsymbol{e}_{\boldsymbol{x}}^0, \boldsymbol{e}_{\boldsymbol{x}}^1\}$ at $\boldsymbol{x}$ and its dual frame $\{\hat{\boldsymbol{e}}_{\boldsymbol{x}}^0, \hat{\boldsymbol{e}}_{\boldsymbol{x}}^1\}$ as follows:

(1) If $\boldsymbol{x} \in \Delta_0(T)$, assume the two adjacent edges are e_0 and e_1, then
$$\boldsymbol{e}_{\boldsymbol{x}}^0 = \boldsymbol{t}_{e_0}, \quad \boldsymbol{e}_{\boldsymbol{x}}^1 = \boldsymbol{t}_{e_1}, \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^0 = \frac{\boldsymbol{n}_{e_1}}{\boldsymbol{n}_{e_1} \cdot \boldsymbol{t}_{e_0}}, \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^1 = \frac{\boldsymbol{n}_{e_0}}{\boldsymbol{n}_{e_0} \cdot \boldsymbol{t}_{e_1}}.$$

(2) If $\boldsymbol{x} \in \mathcal{X}_{\mathring{e}}, e \in \Delta_1(T)$, then
$$\boldsymbol{e}_{\boldsymbol{x}}^0 = \boldsymbol{t}_e, \quad \boldsymbol{e}_{\boldsymbol{x}}^1 = \boldsymbol{n}_e, \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^0 = \boldsymbol{t}_e, \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^1 = \boldsymbol{n}_e.$$

(3) If $\boldsymbol{x} \in \mathcal{X}_{\mathring{T}}$, then
$$\boldsymbol{e}_{\boldsymbol{x}}^0 = (1,0), \quad \boldsymbol{e}_{\boldsymbol{x}}^1 = (0,1), \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^0 = (1,0), \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^1 = (0,1).$$

5.4.2. *Second-kind Nédélec element on tetrahedron meshes.* Let T be a tetrahedron, for any $\boldsymbol{x} \in \mathcal{X}_T$, define a frame $\{\boldsymbol{e}_{\boldsymbol{x}}^0, \boldsymbol{e}_{\boldsymbol{x}}^1, \boldsymbol{e}_{\boldsymbol{x}}^2\}$ at $\boldsymbol{x}$ and its dual frame $\{\hat{\boldsymbol{e}}_{\boldsymbol{x}}^0, \hat{\boldsymbol{e}}_{\boldsymbol{x}}^1, \hat{\boldsymbol{e}}_{\boldsymbol{x}}^2\}$ as follows:

(1) If $\boldsymbol{x} \in \Delta_0(T)$ and adjacent edges of $\boldsymbol{x}$ are e_0, e_1, e_2, adjacent faces of $\boldsymbol{x}$ are f_0, f_1, f_2, then
$$\boldsymbol{e}_{\boldsymbol{x}}^0 = \boldsymbol{t}_{e_0}, \quad \boldsymbol{e}_{\boldsymbol{x}}^1 = \boldsymbol{t}_{e_1}, \quad \boldsymbol{e}_{\boldsymbol{x}}^2 = \boldsymbol{t}_{e_2},$$
$$\hat{\boldsymbol{e}}_{\boldsymbol{x}}^0 = \frac{\boldsymbol{n}_{f_0}}{\boldsymbol{n}_{f_0} \cdot \boldsymbol{t}_{e_0}}, \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^1 = \frac{\boldsymbol{n}_{f_1}}{\boldsymbol{n}_{f_1} \cdot \boldsymbol{t}_{e_1}}, \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^2 = \frac{\boldsymbol{n}_{f_2}}{\boldsymbol{n}_{f_2} \cdot \boldsymbol{t}_{e_2}}.$$

(2) If $\boldsymbol{x} \in \mathcal{X}_{\mathring{e}}, e \in \Delta_1(T)$ and adjacent faces are f_0, f_1 then
$$\boldsymbol{e}_{\boldsymbol{x}}^0 = \boldsymbol{t}_e, \quad \boldsymbol{e}_{\boldsymbol{x}}^1 = \boldsymbol{n}_{f_0} \times \boldsymbol{t}_e, \quad \boldsymbol{e}_{\boldsymbol{x}}^2 = \boldsymbol{n}_{f_1} \times \boldsymbol{t}_e,$$
$$\hat{\boldsymbol{e}}_{\boldsymbol{x}}^0 = \boldsymbol{t}_e, \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^1 = \frac{\boldsymbol{n}_{f_1}}{\boldsymbol{n}_{f_1} \cdot (\boldsymbol{n}_{f_0} \times \boldsymbol{t}_e)}, \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^2 = \frac{\boldsymbol{n}_{f_0}}{\boldsymbol{n}_{f_0} \cdot (\boldsymbol{n}_{f_1} \times \boldsymbol{t}_e)}.$$

figures/nedelec.png

FIGURE 4. The left figure shows $\{\boldsymbol{e}_0, \boldsymbol{e}_1\}$ at each interpolation point, the right figure shows $\{\hat{\boldsymbol{e}}_0, \hat{\boldsymbol{e}}_1\}$ at each interpolation point.

(3) If $\boldsymbol{x} \in \mathcal{X}_{\mathring{f}}$, $f \in \Delta_2(T)$, the first edge of f is e, then

$$\boldsymbol{e}_{\boldsymbol{x}}^0 = \boldsymbol{t}_e, \quad \boldsymbol{e}_{\boldsymbol{x}}^1 = \boldsymbol{t}_e \times \boldsymbol{n}_f, \quad \boldsymbol{e}_{\boldsymbol{x}}^2 = \boldsymbol{n}_f,$$

$$\hat{\boldsymbol{e}}_{\boldsymbol{x}}^0 = \boldsymbol{t}_e, \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^1 = \boldsymbol{t}_e \times \boldsymbol{n}_f, \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^2 = \boldsymbol{n}_f.$$

(4) If $\boldsymbol{x} \in \mathcal{X}_{\mathring{T}}$, then

$$\boldsymbol{e}_{\boldsymbol{x}}^0 = (1,0,0), \quad \boldsymbol{e}_{\boldsymbol{x}}^1 = (0,1,0), \quad \boldsymbol{e}_{\boldsymbol{x}}^2 = (0,0,1),$$

$$\hat{\boldsymbol{e}}_{\boldsymbol{x}}^0 = (1,0,0), \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^1 = (0,1,0), \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^2 = (0,0,1).$$

6. INDEXING MANAGEMENT OF DEGREES OF FREEDOM

In Section 2, we have already discussed the dictionary indexing rule for interpolation points in each element. In this section, we will address the global indexing rules for Lagrange interpolation points, ensuring that interpolation points on a sub-simplex, shared by multiple elements, have a globally unique index. Given the one-to-one relationship between interpolation points and DoFs, this is equivalent to providing a indexing rule for global DoFs in the scalar Lagrange finite element space. Based on this, we will further discuss the indexing rules for DoFs in face and edge finite element spaces.

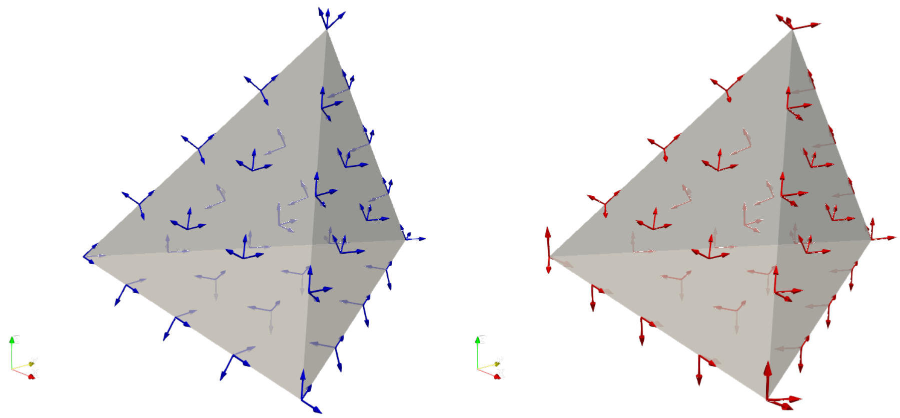

FIGURE 5. The left figure shows $\{\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2\}$ at each interpolation point, the right figure shows $\{\hat{\boldsymbol{e}}_0, \hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2\}$ at each interpolation point.

6.1. **Lagrange finite element space.** We begin by discussing the data structure of the tetrahedral mesh, denoted by $\mathcal{T}_h$. Let the numbers of nodes, edges, faces, and cells in $\mathcal{T}_h$ be represented as `NN`, `NE`, `NF`, and `NC`, respectively. We utilize two arrays to represent $\mathcal{T}_h$:

- `node` (shape: `(NN,3)`): `node[i,j]` represents the j-th component of the Cartesian coordinate of the i-th vertex.
- `cell` (shape: `(NC,4)`): `cell[i,j]` gives the global index of the j-th vertex of the i-th cell.

Given a tetrahedron denoted by `[0,1,2,3]`, we define its local edges and faces as:

- `SEdge = [(0,1),(0,2),(0,3),(1,2),(1,3),(2,3)];`
- `SFace = [(1,2,3),(0,2,3),(0,1,3),(0,1,2)];`
- `OFace = [(1,2,3),(0,3,2),(0,1,3),(0,2,1)];`

Here, we introduced two types of local faces. The prefix `S` implies sorting, and `O` indicates outer normal direction. Both `SFace[i,:]` and `OFace[i,:]` represent the face opposite to the i-th vertex but with varied ordering. The normal direction as determined by the ordering of the three vertices of `OFace` matches the outer normal direction of the tetrahedron. This ensures that the outer normal direction of a boundary `face` points outward from the mesh. Meanwhile, `SFace` aids in determining the global index of the interpolation points on the face. For an in-depth discourse on indexing, ordering, and orientation, we direct readers to `sc3` in iFEM [**?**].

Leveraging the `unique` algorithm for arrays, we can derive the following arrays from `cell`, `SEdge`, and `OFace`:

- `edge` (shape: `(NE,2)`): `edge[i,j]` gives the global index of the j-th vertex of the i-th edge.
- `face` (shape: `(NF,3)`): `face[i,j]` provides the global index of the j-th vertex of the i-th face.
- `cell2edge` (shape: `(NC,6)`): `cell2edge[i,j]` indicates the global index of the j-th edge of the i-th cell.

- `cell2face` (shape: `(NC,4)`): `cell2face[i,j]` signifies the global index of the j-th face of the i-th cell.

Having constructed the edge and face arrays and linked cells to them, we next establish indexing rules for interpolation points on $\mathcal{T}_h$. Let k be the degree of the Lagrange finite element space. The number of interpolation points on each cell is

$$\texttt{ldof} = \dim \mathbb{P}_k(T) = \frac{(k+1)(k+2)(k+3)}{6},$$

and the total number of interpolation points on $\mathcal{T}_h$ is

$$\texttt{gdof} = \texttt{NN} + n_e^k \cdot \texttt{NE} + n_f^k \cdot \texttt{NF} + n_c^k \cdot \texttt{NC},$$

where

$$n_e^k = k-1, \quad n_f^k = \frac{(k-2)(k-1)}{2}, \quad n_c^k = \frac{(k-3)(k-2)(k-1)}{6},$$

are numbers of interpolation points inside edge, face, and cell, respectively. We need an index mapping from $[0 : \texttt{ldof}-1]$ to $[0 : \texttt{gdof}-1]$. See Fig. 6 for an illustration of the local index and the global index of interpolation points.

The tetrahedron's four vertices are ordered according to the right-hand rule, and the interpolation points adhere to the dictionary ordering map $R_3(\boldsymbol{\alpha})$. As Lagrange element is globally continuous, the indexing of interpolation points on the boundary ∂T should be global. Namely a unique index for points on vertices, edges, faces should be used and a mapping from the local index to the global index is needed.

We first set a global indexing rule for all interpolation points. We sort the index by vertices, edges, faces, and cells. For the interpolation points that coincide with the vertices, their global index are set as $0, 1, \ldots, \texttt{NN}-1$. When $k > 1$, for the interpolation points that inside edges, their global index are set as

$$\begin{array}{c} 0 \\ 1 \\ \vdots \\ \texttt{NE}-1 \end{array} \begin{pmatrix} 0 \cdot n_e^k & \cdots & 1 \cdot n_e^k - 1 \\ 1 \cdot n_e^k & \cdots & 2 \cdot n_e^k - 1 \\ \vdots & & \vdots \\ (\texttt{NE}-1) \cdot n_e^k & \cdots & \texttt{NE} \cdot n_e^k - 1 \end{pmatrix} + \texttt{NN}.$$

Here recall that $n_e^k = k-1$ is the number of interior interpolation points on an edge. When $k > 2$, for the interpolation points that inside each face, their global index are set as

$$\begin{array}{c} 0 \\ 1 \\ \vdots \\ \texttt{NF}-1 \end{array} \begin{pmatrix} 0 \cdot n_f^k & \cdots & 1 \cdot n_f^k - 1 \\ 1 \cdot n_f^k & \cdots & 2 \cdot n_f^k - 1 \\ \vdots & & \vdots \\ (\texttt{NF}-1) \cdot n_f^k & \cdots & \texttt{NF} \cdot n_f^k - 1 \end{pmatrix} + \texttt{NN} + \texttt{NE} \cdot n_e^k,$$

where $n_f^k = (k-2)(k-1)/2$ is the number of interior interpolation points on f. When $k > 3$, the global index of the interpolation points that inside each cell can be set in a similar way.

Then we use the two-dimensional array named `cell2ipoint` of shape `(NC,ldof)` for the index map. On the j-th interpolation point of the i-th cell, we aim to determine its unique global index and store it in `cell2ipoint[i,j]`.

For vertices and cell interpolation points, the mapping is straightforward by the global indexing rule. Indeed `cell` is the mapping of the local index of a vertex to its global index. However, complications arise when the interpolation point is located within an edge or face due to non-unique representations of an edge and a face.

We use the more complicated face interpolation points as a typical example to illustrate the situation. Consider, for instance, the scenario where the j-th interpolation point lies within the 0-th local face F_0 of the i-th cell. Let α = `m = [m0,m1,m2,m3]` be its lattice point. Given that F_0 is opposite to vertex 0, we deduce that $\lambda_0|_{F_0} = 0$, which implies `m0` is 0. The remaining components of `m` are non-zero, ensuring that the point is interior to F_0.

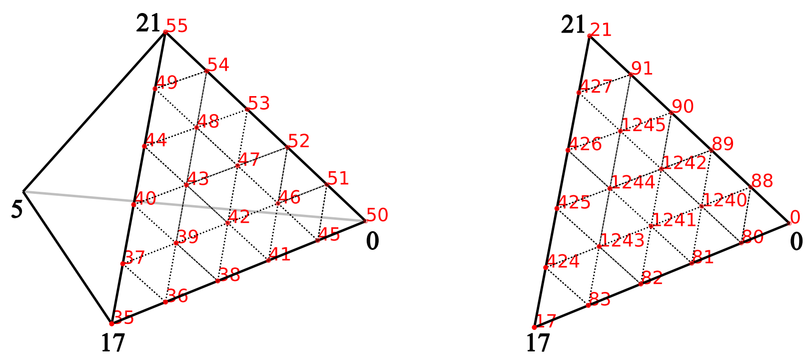

FIGURE 6. Local indexing (Left) and global indexing(Right) of DoFs for a face of tetrahedron, where the local vertex order is `[17, 0, 21]` and global vertex order is `[0, 17, 21]`. Due to the different ordering of vertices in local and global representation of the face, the ordering of the local indexing and the global indexing is different.

Two representations for the face with global index `cell2face[i,0]` are subsequently acquired:

- `LFace = cell[i,SFace[0,:]]` (local representation)
- `GFace = face[cell2face[i,0],:]` (global representation)

Although `LFace` and `GFace` comprise identical vertex numbers, their ordering differs. For example, `LFace = [5 6 10]` while `GFace = [10 6 5]`.

The array `m = [m1,m2,m3]` has a one-to-one correspondence with the vertices of `LFace`. To match this array with the vertices of `GFace`, a reordering based on argument sorting is performed:

```
i0 = argsort(argsort(GFace));
i1 = argsort(LFace);
i2 = i1[i0];
m = m[i2];
```

In the example of `LFace = [5 6 10]` while `GFace =[10 6 5]`, the input `m = [m1,m2,m3]` will be reordered to `m = [m3,m2,m1]`.

From the reordered `m = [m1,m2,m3]`, the local index ℓ of the j-th interpolation point on the global face `f = cell2face[i,0]` can be deduced:

$$
\begin{aligned}
\ell =& \frac{(\mathtt{m2}-1+\mathtt{m3}-1)(\mathtt{m2}-1+\mathtt{m3}-1+1)}{2}+\mathtt{m3}-1\\
=& \frac{(\mathtt{m2}+\mathtt{m3}-2)(\mathtt{m2}+\mathtt{m3}-1)}{2}+\mathtt{m3}-1.
\end{aligned}
$$

It's worth noting that the index of interpolation points solely within the face needs consideration. Finally, the global index for the j-th interpolation point within the 0-th local face of the i-th cell is:

$$\texttt{cell2ipoint[i, j]} = \texttt{NN} + n_e^k \cdot \texttt{NE} + n_f^k \cdot \texttt{f} + \ell.$$

Here, we provide a specific example. Consider a 5th-degree Lagrange finite element space on the `c`-th tetrahedron in a mesh depicted in Fig. 6. The vertices of this tetrahedron are `[5, 17, 0, 21]`, and its 0th face is denoted as `f`, with vertices `[0, 17, 21]`. Assuming that

$$\texttt{NN} + n_e^k \cdot \texttt{NE} + n_f^k \cdot \texttt{f} = 1240.$$

For the 39-th local DoF on tetrahedron, its corresponding multi-index is `[0, 3, 1, 1]` on cell and `[3, 1, 1]` on face. Since the local face is `[17, 0, 21]` and the global face is `[0, 17, 21]`, then `m = [1, 3, 1]` and ℓ =`3` thus

```
cell2ipoint[c, 39] = 1243.
```

Similar, for the 43-th local DoF on tetrahedron, `m = [1, 2, 2]` and ℓ=`4` thus

```
cell2ipoint[c, 43] = 1244.
```

Fig. 6 shows the correspondence between the local and the global indexing of DoFs for the cell.

In conclusion, we've elucidated the construction of global indexing for interpolation points inside cell faces. This method can be generalized for edges and, more broadly, for interior interpolation points of the low-dimensional sub-simplex of an n-dimensional simplex. Please note that for scalar Lagrange finite element spaces, the `cell2ipoint` array is the mapping array from local DoFs to global DoFs.

6.2. **BDM and second-kind Nédélec finite element space.** First, we want to emphasize that the management of DoFs is to manage the continuity of finite element space.

The BDM and second-kind Nédélec are vector finite element spaces, which define DoFs of vector type by defining a vector frame on each interpolation point. At the same time, they define their vector basis functions by combining the Lagrange basis function and the dual frame of the DoFs.

Alternatively, we can say each DoF in BDM or second-kind Nédélec sapce corresponds to a unique interpolation point p and a unique vector $\boldsymbol{e}$, and each basis funcion also corresponds to a unique Lagrange basis function which is defined on p and a vector $\boldsymbol{e}'$ which is the dual vector of $\boldsymbol{e}$.

The management of DoFs is essentially a counting problem. First of all, we need to set global and local indexing rules for all DoFs.

We can globally divide the DoFs into shared and unshared among simplexes. The DoFs shared among simplexes can be further divided into on-edge and on-face according to the dimension of the sub-simplex where the DoFs locate. Note that, for BDM and second-kind Nédélec space there are no DoFs shared on nodes. And for 3D BDM space there are no DoFs shared on edges. So the global numbering rule is similar with the Lagrange interpolation points. First count the shared DoFs on each edge according to the order of the edges, then count the shared DoFs on each face according to the order of the faces, and finally count the unshared DoFs in the cell. On each edge or face, the DoFs' order can follow the order of the interpolation points.

By the global indexing rule, we also can get a array named `dof2vector` with shape `(gdof, GD)`, where `gdof` is the number of global DoFs and `GD` represent geometry dimensions. And `dof2vector[i, :]` store the vector of the i-th DoF.

Next we need to set a local indexing rules and generate a array `cell2dof` with shape `(NC,ldof)`, where `ldof` is the number of local DoFs on each cell. Note that each DoF was determined by an interpolation point and a vector. And for each interpolation point, there is a frame (including `GD` vectors) on it. Given a DoF on i-th cell, denote the local index of its interpolation point as p, and the local index of its vector in the frame denote as q, then one can set a unique local index number j by p and q, for example

$$j = n \cdot q + p$$

where n is the number of interpolation points in i-th cell. Furthermore, we can compute the `cell2dof[i,j]` by the global index `cell2ipoint[i, p]`, the sub-simplex that the interpolation point locate, and the global indexing rule.

Remark 6.1. Note that the local and global number rules mentioned above are not unique. Furthermore, with the array `cell2dof`, the implementation of these higher-order finite element methods mentioned in this paper is not fundamentally different from the conventional finite element in terms of matrix vector assembly and boundary condition handling. □

7. NUMERICAL EXAMPLES

In this section, we numerically verify the 3-dimensional BDM elements basis and the second kind of Nédélec element basis using two test problems over the domain $\Omega = (0,1)^3$ partitioned into a structured tetrahedron mesh $\mathcal{T}_h$. All the algorithms and numerical examples are implemented based on the FEALPy package [**?**].

7.1. High Order Elements for Mixed Poisson.

Consider the Poisson problem:

$$\begin{cases} \boldsymbol{u} + \nabla p = 0 & \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u} = f & \text{in } \Omega, \\ p = g & \text{on } \partial\Omega. \end{cases}$$

The variational problem is : find $\boldsymbol{u} \in H(\mathrm{div}, \Omega)$, $p \in L^2(\Omega)$, such that

$$\begin{aligned} \int_\Omega \boldsymbol{u} \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} - \int_\Omega p \nabla \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} &= -\int_{\partial\Omega} g(\boldsymbol{v} \cdot \boldsymbol{n}) \, \mathrm{d}\boldsymbol{s}, \quad && \boldsymbol{v} \in H(\mathrm{div}, \Omega), \\ -\int_\Omega (\nabla \cdot \boldsymbol{u}) q \, \mathrm{d}\boldsymbol{x} &= -\int_\Omega f q \, \mathrm{d}\boldsymbol{x}, && q \in L^2(\Omega). \end{aligned}$$

Let $V_{h,k}^{\mathrm{div}}$ be the BDM space with degree k on $\mathcal{T}_h$ and piecewise k-degree polynomial space on $\mathcal{T}_h$ by V_{k-1}. The corresponding finite element method is: find $\boldsymbol{u}_h \in V_{h,k}^{\mathrm{div}}, p_h \in V_{k-1}$, staify:

$$\int_\Omega \boldsymbol{u}_h \cdot \boldsymbol{v}_h \, \mathrm{d}\boldsymbol{x} - \int_\Omega p \nabla \cdot \boldsymbol{v}_h \, \mathrm{d}\boldsymbol{x} = -\int_{\partial\Omega} g(\boldsymbol{v}_h \cdot \boldsymbol{n}) \, \mathrm{d}\boldsymbol{s} \quad \boldsymbol{v}_h \in V_{h,k}^{\mathrm{div}}, \tag{26}$$

$$-\int_\Omega (\nabla \cdot \boldsymbol{u}_h) q_h \, \mathrm{d}\boldsymbol{x} = -\int_\Omega f q_h \, \mathrm{d}\boldsymbol{x}, \qquad q_h \in V_{k-1}. \tag{27}$$

To test the correctness of BDM space we set

$$\boldsymbol{u} = \begin{pmatrix} -\pi \sin(\pi x) \cos(\pi y) \cos(\pi z) \\ -\pi \cos(\pi x) \sin(\pi y) \cos(\pi z) \\ -\pi \cos(\pi x) \cos(\pi y) \sin(\pi z) \end{pmatrix},$$

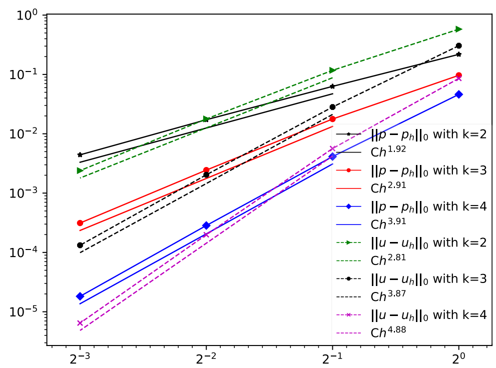

FIGURE 7. Errors $\|\boldsymbol{u}-\boldsymbol{u}_h\|_0$ and $\|p-p_h\|_0$ of finite element method (26) and (27) on uniformly refine mesh with $k=2,3,4$.

$$p = \cos(\pi x)\cos(\pi y)\cos(\pi z),$$
$$f = 3\pi^2\cos(\pi x)\cos(\pi y)\cos(\pi z).$$

The numerical results are shown in Fig. 7 and it is clear that

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_0 \le Ch^{k+1}, \quad \|p-p_h\|_0 \le Ch^k.$$

7.2. **High Order Elements for Maxwell Equations.** Consider the time harmonic problem:

$$\begin{cases} \nabla\times(\mu^{-1}\nabla\times\boldsymbol{E}) - \omega^2\tilde{\epsilon}\boldsymbol{E} &= \boldsymbol{J} \quad \text{in } \Omega, \\ \boldsymbol{n}\times\boldsymbol{E} &= \boldsymbol{0} \quad \text{on } \partial\Omega. \end{cases}$$

The variational problem is: find $\boldsymbol{E}\in H_0(\mathrm{curl},\Omega)$, satisfies:

$$\int_\Omega \mu^{-1}(\nabla\times\boldsymbol{E})\cdot(\nabla\times\boldsymbol{v})\,\mathrm{d}\boldsymbol{x} - \int_\Omega \omega^2\tilde{\epsilon}\boldsymbol{E}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{J}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} \quad \forall\,\boldsymbol{v}\in H_0(\mathrm{curl}\,,\Omega).$$

Let $\mathring{V}_{h,k}^{\mathrm{curl}} = V_{h,k}^{\mathrm{curl}} \cap H_0(\mathrm{curl}\,,\Omega)$, where $V_{h,k}^{\mathrm{curl}}$ is the edge element space defined in Theorem 5.10. The corresponding finite element method is: find $\boldsymbol{E}_h \in \mathring{V}_{h,k}^{\mathrm{curl}}$ s.t.

(28)
$$\int_\Omega \mu^{-1}(\nabla\times\boldsymbol{E}_h)\cdot(\nabla\times\boldsymbol{v}_h)\,\mathrm{d}\boldsymbol{x} - \int_\Omega \omega^2\tilde{\epsilon}\boldsymbol{E}_h\cdot\boldsymbol{v}_h\,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{J}\cdot\boldsymbol{v}_h\,\mathrm{d}\boldsymbol{x},\ \boldsymbol{v}_h\in\mathring{V}_{h,k}^{\mathrm{curl}}.$$

To test the convergence rate of second-kind Nédélec space, we choose

$$\omega = \tilde{\epsilon} = \mu = 1,$$
$$\boldsymbol{E} = (f, \sin(x)f, \sin(y)f),$$
$$f = (x^2-x)(y^2-y)(z^2-z),$$
$$\boldsymbol{J} = \nabla\times\nabla\times\boldsymbol{E} - \boldsymbol{E}.$$

The numerical results are shown in Fig. 8 and it is clear that

$$\|\boldsymbol{E}-\boldsymbol{E}_h\|_0 \le Ch^{k+1}, \quad \|\nabla\times(\boldsymbol{E}-\boldsymbol{E}_h)\|_0 \le Ch^k.$$

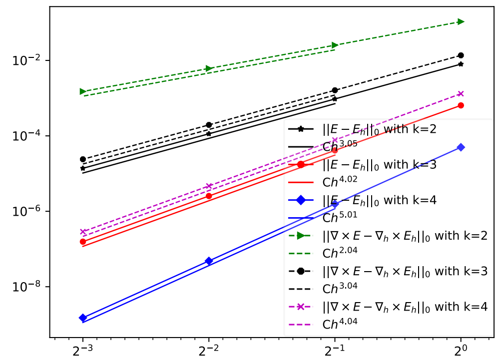

FIGURE 8. Errors $\|\boldsymbol{E} - \boldsymbol{E}_h\|_0$ and $\|\nabla \times (\boldsymbol{E} - \boldsymbol{E}_h)\|_0$ of finite element method (28) on uniformly refine mesh with $k = 2, 3, 4$.

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