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ON LATTICES ADMITTING UNIQUE LAGRANGE INTERPOLATIONS†

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Abstract. In this paper generalizations of the classical Lagrange interpolation formula to n-dimensional spaces are discussed. It simplifies and improves upon certain results of some recent authors.

1. Introduction. In connection with the finite element method, Lagrange interpolation in R^n has recently been studied by several authors [1], [2], [4], [5]. In [2] and [4], formulas of interpolating polynomials were used and error estimates were discussed. In [4], Nicolaides established the existence and uniqueness of a Lagrange-type interpolating polynomial of n variables when the nodes form a "kth order principal lattice of an n-simplex". This is a generalization of a result of Ciarlet and Wagschal [2]. The purpose of this paper is to simplify and generalize Nicolaides' result to all lattices admitting unique interpolations. We first give an easy geometric characterization GC for the nodes in R^n which ensures the existence of the unique Lagrange-type interpolant. A simple type of lattices, called "natural lattices", is then introduced and shown to satisfy the condition GC. We then show that the "principal lattices" defined in [3] possess the same characterization. By means of suitable transformations of the nodes, called "lattice-transformations", new kinds of lattices satisfying GC are obtained. The characterization GC is then further generalized to GC(V), where V is a certain matrix, such that a lattice admits unique interpolation if and only if it satisfies GC(V) for some matrix V.

2. Geometric characterization of the nodes. In the following a *node* means a point in R^n, a *lattice* means a set of nodes, and a polynomial always means a polynomial in n variables. Let k (>0) be the degree of a polynomial. The number of terms of such a polynomial is then given by $N=\binom{n+k}{k}$.

DEFINITION 1. Let $X=\{x_1, \cdots, x_m\}$ be a lattice of m distinct nodes in R^n. We say that *X admits interpolations of degree $\leqq k$* if and only if for any $f: X\to C$ (C is the set of complex numbers), there exists a complex polynomial P of degree $\leqq k$ such that $P(x_i)=f(x_i)$ for all $i=1, \cdots, m$. If, for each f, P is uniquely determined, then we say that *X admits a unique interpolation of degree $\leqq k$.*

We shall make use of the following facts:

(a) X admits a unique interpolation of degree $\leqq k$ iff $m=N$ and X is not a subset of any algebraic surface of degree $\leqq k$.

(b) If $m=N$ and X admits interpolations of degree $\leqq k$, then X admits a unique interpolation of degree $\leqq k$.

(c) Suppose that the same conditions as in (b) are satisfied. Then, given any $f: X\to C$, the interpolating polynomial P may be put in the form

$$P(x)=\sum_{i=1}^{N} p_i(x)f(x_i), \qquad x\in R^n, \tag{1}$$

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where p_i are polynomials satisfying the following properties:

(i) Each p_i depends on X only and not on f.

(ii) $p_i(x_j) = \delta_{ij}$.

(iii) Each p_i is a real polynomial of degree exactly $= k$.

(iv) Each p_i has no factor of the form u^2 where u is a real polynomial of degree $\geqq 1$.

Statements (a), (c)(i) and (ii) are well-known (e.g. see [3]). (b) follows from the theory of systems of linear equations. (c)(iii) and (iv) follow from the uniqueness of the interpolating polynomial.

To prove (c)(iii), we first show that each p_r is a real polynomial. Let $p_r = A_r + iB_r$ where A_r and B_r are real polynomials of degree $\leqq k$. Then by (ii), $A_r(x_r) = 1$ and $A_r(x_s) = 0$ for all $r \neq s$. By the uniqueness of the interpolating polynomial, we have $p_r = A_r$ which is a real polynomial. We now show that p_r is of degree exactly $= k$. Suppose that p_r is of degree $< k$. Let $S_r(t) = 0$ be the equation of any hyperplane in R^n which does not contain x_r. Thus, $S_r(x_r) \neq 0$ and S_r is a real polynomial of first degree. Consider the polynomial T_r defined by $T_r(t) = (S_r(t)/S_r(x_r))p_r(t)$ for $t \in R^n$. T_r is of degree $\leqq k$ which also satisfies $T_r(x_s) = \delta_{rs}$. Thus T_r is also an interpolating polynomial whose degree is one greater than that of p_r. This contradicts the uniqueness of the interpolating polynomial.

To prove (c)(iv), let $p_i(t) = U_i(t) \cdot u_i^2(t)$ where u_i is a real polynomial of degree $\geqq 1$. Then consider the polynomial V_i defined by $V_i(t) = U_i(t) \cdot u_i(t)$ for $t \in R^n$. For $i \neq j$, we have $0 = p_i(x_j) = U_i(x_j)u_i^2(x_j) = V_i(x_j)u_i(x_j)$. It follows that $V_i(x_j) = 0$ or $u_i(x_j) = 0$. However, both cases imply that $V_i(x_j) = 0$. Furthermore, $1 = p_i(x_i) = V_i(x_i)u_i(x_i)$ implies that $V_i(x_i) \neq 0$. Thus the polynomial defined by $W_i(t) = V_i(t)/V_i(x_i)$ is of degree strictly less than that of p_i and W_i satisfies $W_i(x_j) = \delta_{ij}$. As before this is again a contradiction to the uniqueness of the interpolating polynomial.

Statement (a) should enable us to decide whether a subset X of R^n will admit a unique interpolation. However, it is usually difficult to see whether the N nodes lie on an algebraic surface of degree $\leqq k$ or not, especially when the dimension n or the degree k is high. This problem is somewhat simplified when we notice that a lattice admitting a unique interpolation possesses a geometric characterization which enables us to write down the unique interpolating polynomial immediately. This geometric characterization, which we shall refer to later as Condition GC, is described as follows:

Condition GC *for a lattice* $X = \{x_1, \cdots, x_N\}$ *of* N *nodes of* R^n: Corresponding to each node x_i, there exist k distinct hyperplanes $G_{i1}, G_{i2}, \cdots, G_{ik}$ such that (i) x_i does not lie on any of these hyperplanes, and (ii) all the other nodes in X lie on at least one of these hyperplanes.

(i) and (ii) may be combined and stated mathematically as follows:

$$x_j \in \bigcup_{l=1}^{k} G_{il} \Leftrightarrow i \neq j \tag{2}$$

for all $i, j = 1, 2, \cdots, N$.

THEOREM 1. *Let* X *be a lattice of* N *nodes in* R^n. *If* X *satisfies Condition* GC, *then* X *admits a unique interpolation of degree* $\leqq k$. *Furthermore, for each* $i = 1, \cdots, N$, *the real polynomial* p_i *in* (1) *is a product of real polynomials of first degree.*

In this case, we may write $p_i = u_{i1} \cdots u_{ik}$ *where* $u_{ij}(t) = 0$ $(j = 1, \cdots, k)$ *is the equation of the hyperplane* G_{ij} *given in Condition* GC.

The converse is also true: If X admits a unique interpolation of degree $\leqq k$ *and if furthermore, for each* $i = 1, \cdots, N$, *the real polynomial* p_i *in* (1) *is a product of real polynomials of first degree, then the lattice X must satisfy Condition* GC.

Proof. Suppose that X satisfies Condition GC. Then the equation of each hyperplane G_{ij} determines, up to a constant multiple, a polynomial u_{ij}. Write $\pi_i = u_{i1} \cdots u_{ik}$. Condition GC implies that $\pi_i(x_i) \neq 0$ and that the polynomial

$$P(x) = \sum_{i=1}^{N} \frac{\pi_i(x)}{\pi_i(x_i)} f(x_i)$$

is the interpolating polynomial of degree $\leqq k$ for any f. Thus X admits interpolations (hence also a unique interpolation) of degree $\leqq k$.

Conversely, suppose that p_i is a product of polynomials of first degree. Since p_i is of degree exactly $= k$, it must have exactly k such factors $u_{i1}, \cdots, u_{ik}$. For each $i = 1, \cdots, N; j = 1, \cdots, k$, let the hyperplane whose equation is $u_{ij}(x) = 0$ be denoted by G_{ij}. For $r \neq s$, $G_{ir} \neq G_{is}$, otherwise u_{ir}^2 would be a factor of p_i. By the property (ii) of p_i, we see that Condition GC is satisfied.

If X satisfies Condition GC, then, by the uniqueness of interpolating polynomials, we obtain, for each $x_i \in X$, one and only one set of k hyperplanes G_{il} $(l = 1, \cdots, k)$ such that equation (2) is satisfied. Each hyperplane G_{ij} $(j = 1, \cdots, k)$ is called a *hyperplane associated with the node* x_i. The set of all hyperplanes associated with all the nodes of X will be denoted by Γ_X.

3. Natural lattices.

DEFINITION 2. Let k be any positive integer. Suppose that there exist $M \equiv k + n$ distinct hyperplanes $H_1, \cdots, H_M$ in R^n such that the intersection of any n distinct hyperplanes chosen from $H_1, \cdots, H_M$ is a point and different choices give different points. Then the set of all the above points is called the *k-th order natural lattice in R^n generated by* $H_1, \cdots, H_M$.

Two-dimensional examples of natural lattices of various orders are given in Fig. 1, where we have a series of lattices of general pattern. The straight lines are the hyperplanes generating the corresponding lattices. These figures indicate a general method of constructing the series, which also motivates the proof of Theorem 2.

In general kth order natural lattices exist for all values of k. To prove this we need a lemma. We first observe that any hyperplane in R^n has an equation of the form $a \cdot x = c$ where $a(\neq 0) \in R^n$, $c \in R$ and $a \cdot x$ is the usual dot-product of a and x.

LEMMA 1. *For each* $i = 1, 2, \cdots, n$, *let* $a_i \in R^n$, $c_i \in R$ *and let* H_i *be the hyperplane having the equation* $a_i \cdot x_i = c_i$. *Then* $H_1, \cdots, H_n$ *intersect at a point if and only if* $a_1, \cdots, a_n$ *are linearly independent.*

This lemma follows from the fact that the system $a_i \cdot x_i = c_i$ $(i = 1, \cdots, n)$ of n linear equations in x has exactly one solution iff $a_1, \cdots, a_n$ are linearly independent.

THEOREM 2. *For each n and each k, there always exists a set of $n + k$ hyperplanes in R^n generating a k-th order natural lattice.*

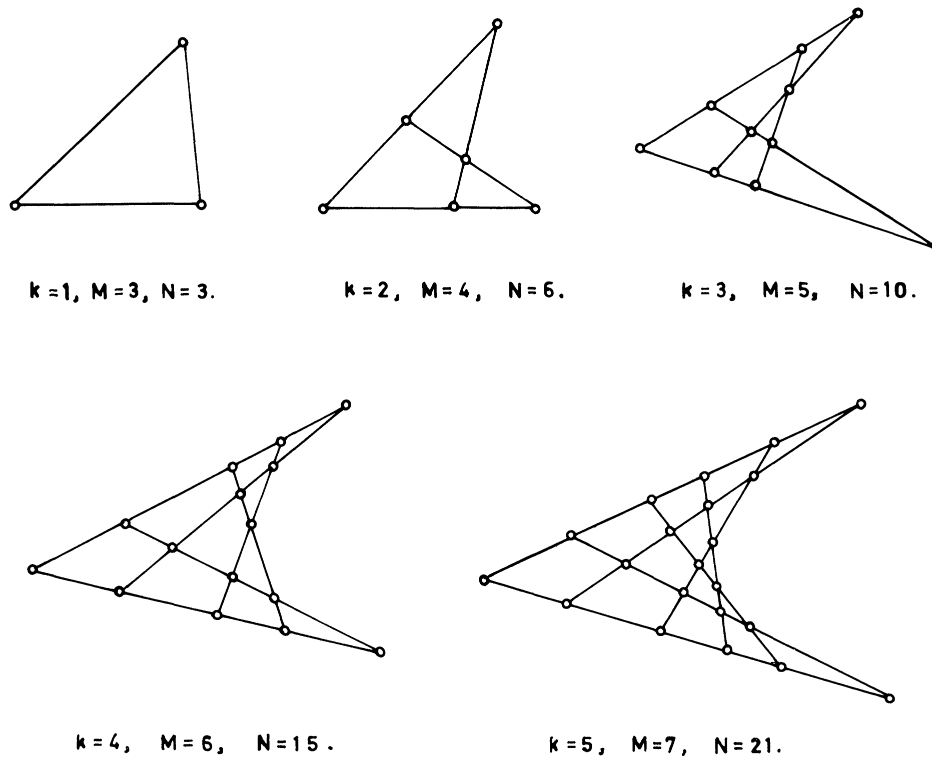

FIG. 1

Proof. We shall prove this theorem by induction on k. Let $e_1, \cdots, e_n$ be the canonical basis of R^n. Then the $n+1$ hyperplanes having equations $e_i \cdot x = 0$ $(i = 1, \cdots, n)$ and $(e_1 + e_2 + \cdots + e_n) \cdot x = 1$ clearly generate a first order natural lattice.

Suppose that there exist $M \equiv n + k$ hyperplanes $H_1, \cdots, H_M$ which generate a kth order natural lattice X. Let the equation of H_i be $a_i \cdot x = c_i$ $(i = 1, \cdots, k)$. Choose a vector a_{M+1} so that it is not a linear combination of any $(n-1)$ vectors chosen from $a_1, \cdots, a_M$. Then any n vectors chosen from $a_1, \cdots, a_{M+1}$ are linearly independent. Let c be an arbitrary real number and H_{M+1} be the hyperplane having equation $a_{M+1} \cdot x = c$. Then by Lemma 1, the intersection of any n distinct hyperplanes chosen from $H_1, \cdots, H_{M+1}$ is a point. By choosing c sufficiently large, we can have

$$H_{M+1} \cap X = \varnothing. \tag{3}$$

This implies that different choices of n distinct hyperplanes from $H_1, \cdots, H_{M+1}$ give different points of intersection. For, if one of the two choices does not contain H_{M+1}, then by induction assumption and by (3), the two choices give different points. If both choices contain H_{M+1}, we let them be

$$\{F_1, \cdots, F_{n-1}, H_{M+1}\} \quad \text{and} \quad \{G_1, \cdots, G_{n-1}, H_{M+1}\}.$$

Since the two choices are different, there is a $k=1,\cdots$, or $n-1$ such that $F_k, G_1, \cdots, G_{n-1}$ are distinct. If

$$F_1 \cap \cdots \cap F_{n-1} \cap H_{M+1} = G_1 \cap \cdots \cap G_{n-1} \cap H_{M+1} = \{z\},$$

say, then $z \in F_k \cap G_1 \cap \cdots \cap G_{n-1}$ and hence z is a point of X as well as a point in H_{M+1}. This contradicts (3). Hence $H_1, \cdots, H_{M+1}$ generate a $(k+1)$st order natural lattice.

THEOREM 3. *Every k-th order natural lattice satisfies Condition* GC.

Proof. Clearly any kth order natural lattice has $N=\binom{n+k}{k}$ nodes. Let X be a kth order natural lattice generated by $M \equiv n+k$ hyperplanes $H_1, \cdots, H_M$. Let $x \in X$. Then x is the intersection of n hyperplanes chosen from $H_1, \cdots, H_M$. Without loss of generality, we assume that $\{x\} = H_1 \cap \cdots \cap H_n$. We shall show that $H_{n+1}, \cdots, H_M$ are the k hyperplanes associated with the node x.

If x belongs to one of the hyperplanes in $\{H_{n+1}, \cdots, H_M\}$, say, H_{n+1}, then $x \in H_1 \cap \cdots \cap H_n = H_1 \cap \cdots \cap H_{n-1} \cap H_{n+1}$. This contradicts the condition imposed on natural lattices. Hence x does not belong to any of the hyperplanes $H_{n+1}, \cdots, H_M$. If $y \in X$ and $y \neq x$, then $\{y\} = H_{i_1} \cap \cdots \cap H_{i_n}$ where $i_1, \cdots, i_n$ are n distinct numbers in $\{1, \cdots, M\}$. By the condition imposed on natural lattices, $\{H_{i_1}, \cdots, H_{i_n}\} \neq \{H_1, \cdots, H_n\}$. Thus there exists $j=1, \cdots$, or n such that $y \in H_{i_j} \notin \{H_1, \cdots, H_n\}$ and hence y belong to one of the hyperplanes $H_{n+1}, \cdots, H_M$. The theorem is thus proved.

Some properties of natural lattices which follow directly from the proof of Theorem 3 are given in the following corollary.

COROLLARY 1. *Let X be a k-th order natural lattice generated by the hyperplanes $H_1, \cdots, H_M$ $(M=n+k)$. Then*

(a) $\{H_1, \cdots, H_M\} = \Gamma_X$, *the set of hyperplanes associated with the nodes of X. Hence every natural lattice is generated by exactly one set of hyperplanes.*

(b) *For each $i=1, \cdots, M$, if $z \in X$ but $z \notin H_i$ then H_i is a hyperplane associated with the node z.*

4. Principal lattice of an n-simplex. In [4], Nicolaides defines the kth order principal lattice of an n-simplex Δ. We give the definition here and introduce some convenient symbols.

Let Δ be a nondegenerate n-simplex in R^n. Once we have given an order to the vertices, we have a barycentric coordinate function $\Lambda: R^n \to R^{n+1}$ with $\Lambda(x) = (\lambda_1(x), \cdots, \lambda_{n+1}(x))$. $(\sum_{i=1}^{n+1} \lambda_i = 1.)$

DEFINITION 3. The set of all points x of R^n such that $\Lambda(x)$ is of the form

$$\Lambda(x) = \frac{1}{k}(s_1, \cdots, s_{n+1}),$$

where s_p $(p=1, \cdots, n+1)$ are nonnegative integers $\leqq k$ is called the *k-th order principal lattice* of Δ and will be denoted by B.

For each $p=1, \cdots, n+1$; $r=0, 1, \cdots, k-1$, we denote the set $\{x \in R^n : \lambda_p(x) = r/k\}$ by H_{pr}. Since $\lambda_p(x) - r/k$ is a polynomial of first degree, H_{pr} is a hyperplane in R^n. Clearly, the hyperplanes H_{pr}, $p=1, \cdots, n+1$; $r=0, 1, \cdots, k-1$, are distinct. We denote the set of these hyperplanes by H. B and

H depend on Δ and k but not on the way we ordered the vertices of Δ. Furthermore, B contains exactly N elements.

THEOREM 4. *The principal lattice B satisfies Condition* GC *and* $H=\Gamma_B$.

Proof. Let $x\in B$. Then $\Lambda(x)=k^{-1}(s_1,\cdots,s_{n+1})$, where s_p $(p=1,\cdots,n+1)$ are nonnegative integers $\leqq k$. For each $p=1,\cdots,n+1$, such that $s_p>0$, we select exactly s_p hyperplanes $H_{pr}(0\leqq r<s_p)$ from H. Since $s_1+\cdots+s_{n+1}=k$, we have exactly k hyperplanes, $H_{p0}, H_{p1},\cdots,H_{p,s_p-1}$; $p=1,\cdots,n+1$. Since $\Lambda_p(x)=s_p/k$, x does not lie on any of these hyperplanes. Suppose $y\in B$ but $y\neq x$. Let $\Lambda(y)=k^{-1}(t_1,\cdots,t_{n+1})$. Since $\sum s_i=\sum t_i=k$ and $\Lambda(x)\neq\Lambda(y)$ there exists p such that $t_p<s_p$. Thus $y\in H_{pt_p}$ which is one of the hyperplanes selected. It follows that B satisfies Condition GC. The fact that $H=\Gamma_B$ is obvious.

COROLLARY 2. *If $B=\{x_1,\cdots,x_N\}$ is a principal lattice of an n-simplex Δ, then B admits a unique interpolation with*

$$p_i(t)=\prod_{\substack{s=1\\ \lambda_s(x_i)>0}}^{n+1}\ \prod_{r=0}^{k\lambda_s(x_i)-1}\left(\lambda_s(t)-\frac{r}{k}\right)\Big/\left(\lambda_s(x_i)-\frac{r}{k}\right),$$

where λ_s $(s=1,\cdots,n+1)$ is the s-th barycentric coordinate function with respect to a fixed ordering of vertices of Δ.

This corollary is one of the main theorems in Nicolaides' paper [4]. We feel that our approach here is different and simpler.

5. Lattice transformations. The next question is to ask whether there exist other types of lattices of N nodes which satisfy Condition GC but which are neither principal lattices nor natural lattices. The answer is in the affirmative and in the following we give a method of constructing some new lattices. In brief, we shall transform, by a suitable transformation, a lattice which satisfies Condition GC. We shall denote by Π the set of all hyperplanes in R^n.

DEFINITION 4. Let X be a lattice of N nodes which satisfies Condition GC and let $\Phi: X\to R^n$, $\Psi:\Gamma_X\to\Pi$ be two mappings. The ordered pair (Φ,Ψ) is said to be a *lattice-transformation on X* if the following conditions are satisfied:

(a) Φ is injective, and

(b) for all $x\in X$, and all $G\in\Gamma_X$,

$$x\in G\Leftrightarrow\Phi(x)\in\Psi(G). \tag{4}$$

Φ and Ψ are called *node-transformation* and *hyperplane-transformation associated with the lattice-transformation* respectively.

THEOREM 5. *If (Φ,Ψ) is a lattice-transformation on a lattice X which satisfies Condition* GC, *then $\Phi[X]$ also satisfies Condition* GC *and* $\Gamma_{\Phi[X]}=\Psi[\Gamma_X]$.

Proof. Let $X=\{x_1,\cdots,x_N\}$ and $\Gamma_X=\{G_{il}: i=1,\cdots,N;\ l=1,\cdots,k\}$ where the G_{il} satisfy condition (2).

Since Φ is injective, $\Phi[X]$ contains N points. Let $y_i=\Phi(x_i)$, for all $i=1,\cdots,N$. For $i=1,\cdots,N$; $l=1,\cdots,k$, $\Psi(G_{il})$ are hyperplanes. Furthermore,

$$j\neq i\Leftrightarrow x_j\in\bigcup_{l=1}^{k}G_{il}\qquad\text{(by (2))}$$

$$\Leftrightarrow y_j\in\bigcup_{l=1}^{k}\Psi(G_{il})\ \ \text{(by (4))}.$$

Hence $\Phi[X]$ satisfies Condition GC and $\Gamma_{\Phi[X]} = \Psi[\Gamma_X]$.

Examples. In the following examples, $n = 2$, $k = 3$, $N = 10$.

1. Figure 2 represents a principal lattice of a 2-simplex Δ whose vertices are ordered as (x_1, x_2, x_3). The hyperplanes in H are shown. By Theorem 4 it satisfies Condition GC. For example, H_{10}, H_{20}, H_{21} are the hyperplanes associated with the node x_4.

2. Figure 3 represents a lattice with 10 nodes represented by $y_1, \cdots, y_{10}$. Let B denote the principal lattice in Fig. 2. We define the node-transformation $\Phi: B \to R^n$ by $\Phi(x_i) = y_i$ for all $i = 1, \cdots, 10$. The hyperplane-transformation Ψ is defined in an obvious way, e.g., Ψ carries the line containing x_5, x_9, x_{10} to the line containing y_5, y_9, y_{10}. (Φ, Ψ) is then a lattice transformation and the lattice in Fig. 3 satisfies Condition GC.

3. We may verify easily that the lattice Z of 10 nodes in Fig. 4 satisfies Condition GC. Note that z is a node lying on 4 distinct lines in Γ_Z. Since there is no node in B of Fig. 2 lying on 4 lines in Γ_B, we conclude that B cannot be transformed onto Z by means of a lattice transformation.

The last example shows that a principal lattice can be transformed, by a lattice transformation, to one which is not a principal lattice. Also, by considering the

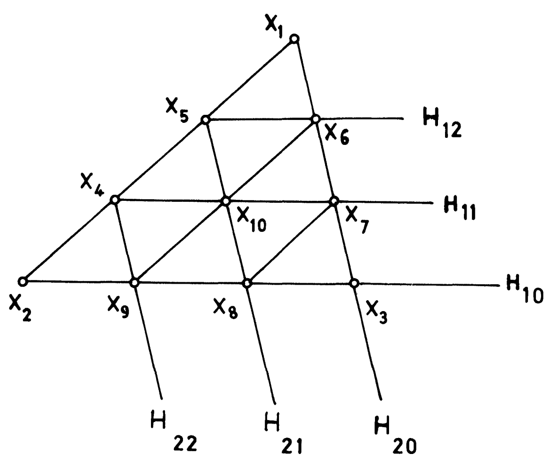

FIG. 2

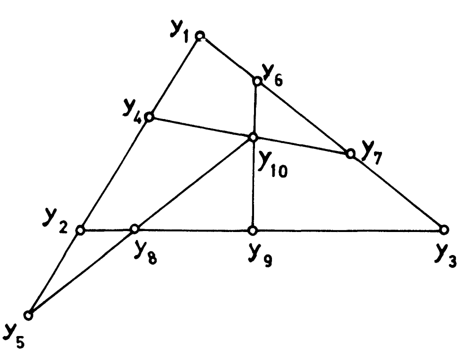

FIG. 3

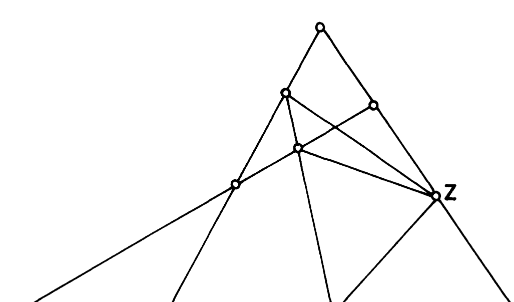

FIG. 4

number of generating hyperplanes, we see that a principal lattice cannot be transformed to a natural lattice by any lattice transformation, and vice versa. However, the following fact is obvious, and we state it as a theorem.

THEOREM 6. *A natural lattice X is always transformed to another natural lattice by a lattice transformation on X.*

6. Generalizations. In this section, a real surface of zero degree is regarded as the empty set $\varnothing$. We denote by $\mathcal{V}$ the set of all $N\times k$ real matrices $V=(v_{ij})$ such that v_{ij} are nonnegative integers and $v_{i1}+\cdots+v_{ik}=k$ for all $i=1,2,\cdots,N$.

Given $V=(v_{ij})\in\mathcal{V}$, we ask if there is a lattice $X=\{x_1,\cdots,x_N\}$ satisfying the following condition:

Condition GC(V): For each $i=1,\cdots,N$, there exist real surfaces $S_{i1},\cdots,S_{ik}$ of degrees $v_{i1},\cdots,v_{ik}$ respectively such that

$$x_j\in\bigcup_{l=1}^{k}S_{il}\Leftrightarrow i\neq j \tag{5}$$

for all $j=1,\cdots,N$.

If X satisfies the above condition, the set $\{S_{il}: i=1,\cdots,N; l=1,\cdots,k\}$ will be denoted by Γ_X.

Remarks. (a) If both V and U belong to $\mathcal{V}$, clearly Condition GC(V) and Condition GC(U) are equivalent if V may be obtained from U by permuting the rows of U and permuting the entries in each row of U.

(b) If E is the $N\times k$ matrix whose entries are all equal to 1, then Condition GC(E) is reduced to Condition GC discussed previously.

(c) There are various examples of lattices X satisfying Condition GC(V) where V are matrices different from E.

(d) For $n=2$, $k=2$, examples of $N\times k$ matrices V can be found for which no lattice of N nodes satisfies Condition GC(V).

The following is a generalization of Theorem 1. Its proof is elementary and is similar to that of Theorem 1.

THEOREM 7. *Let X be a lattice of N nodes in R^n. If X satisfies Condition* GC(V) *for some $V\in\mathcal{V}$, then X admits a unique interpolation of degree $\leqq k$. Furthermore, for each $i=1,\cdots,N$, the real polynomial p_i in* (1) *is of the form: $p_i=u_{i1}u_{i2}\cdots u_{ik}$ where $u_{ij}(t)=0$ is the equation of the surface S_{ij} given in Condition* GC(V).

The converse is also true: Suppose that X admits a unique interpolation of degree $\leqq k$ and that, for each $i=1,\cdots,N$, the real polynomial p_i is a product of irreducible real polynomials: $p_i=u_{i1}u_{i2}\cdots u_{ik}$ (in order to have k factors, some u_{ij} are allowed to be identically equal to 1). *Let $V=(v_{ij})$ where v_{ij} is the degree of u_{ij}. Then $V\in\mathcal{V}$ and X satisfies* GC(V).

DEFINITION 5. Let $V\in\mathcal{V}$ and let X be a lattice satisfying Condition GC(V). Let $\Phi: X\to R^n$, $\Psi: \Gamma_X\to\Sigma$ (Σ is the set of real surfaces of degrees $0,1,2,\cdots$) be two mappings. The ordered pair (Φ,Ψ) is called a *generalized lattice-transformation on X* if

(a) Φ and Ψ are injective,

(b) for every $S\in\Gamma_X$, S is of degree $d\Rightarrow\Psi(S)$ is of degree d, and

(c) for all $S \in \Gamma_X$ and all $x \in X$,

$$x \in S \Leftrightarrow \Phi(x) \in \Psi(S).$$

The following theorem corresponds to Theorem 5. The proof is also elementary.

THEOREM 8. *If X satisfies Condition* GC(V) *and* (Φ, Ψ) *is a generalized lattice transformation on X, then $\Phi[X]$ satisfies Condition* GC(V) *with* $\Gamma_{\Phi[X]} = \Psi[\Gamma_X]$.

7. A note on the classification of the lattices admitting unique interpolations. Let $\mathscr{L}$ denote the set of all lattices of N nodes in R^n admitting unique interpolations. For each $N \times k$ matrix $V \in \mathscr{V}$, we denote by $\mathscr{L}_V$ the set of lattices satisfying Condition GC(V). Then

$$\mathscr{L} = \bigcup \{\mathscr{L}_V \colon V \in \mathscr{V}, \mathscr{L}_V \neq \emptyset\}.$$

Two distinct sets $\mathscr{L}_U$ and $\mathscr{L}_V$ are disjoint. For if X is a lattice in $\mathscr{L}_U \cap \mathscr{L}_V$, then X satisfies Condition GC(U) as well as GC(V). By the uniqueness of the interpolating polynomial, U must be a matrix which may be obtained by rearranging the rows of V and rearranging the entries in each row of V. Thus Condition GC(U) is equivalent to Condition GC(V), i.e. $\mathscr{L}_U = \mathscr{L}_V$. It follows that the nonempty $\mathscr{L}_V$'s ($V \in \mathscr{V}$) form a partition of $\mathscr{L}$.

Furthermore, we may define, within each nonempty $\mathscr{L}_V$, an equivalence relation $\sim$ so that $X \sim Y$ iff X can be transformed to Y by a generalized lattice transformation. Then $\sim$ induces a partition on each nonempty $\mathscr{L}_V$.

Thus we have indicated a method of classification of $\mathscr{L}$. However, some obvious questions arising from this classification still remain unsolved. For example, (i) For what V is $\mathscr{L}_V$ nonempty? (ii) How many equivalence classes (induced by $\sim$) are there in a given nonempty $\mathscr{L}_V$?

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