Journal of Computational and Applied Mathematics 240 (2013) 204–214

Contents lists available at SciVerse ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

A new procedure for the construction of hierarchical high order Hdiv and Hcurl finite element spaces

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ARTICLE INFO

Article history:
Received 30 January 2012
Received in revised form 18 September 2012

Keywords:
Mixed finite element formulations
Hierarchical finite element bases
Hdiv approximating spaces

ABSTRACT

This paper considers a systematic procedure for the construction of a hierarchy of high order finite element approximation for Hdiv and Hcurl spaces based on triangular and quadrilateral partitions of bidimensional domains. The principle is to choose an appropriate set of vectors, based on the geometry of each element, which are multiplied by an available set of H^1 hierarchical scalar basic functions. This strategy produces vector basis functions with continuous normal or tangent components on the elements interfaces, properties that characterise functions in Hdiv or Hcurl, respectively. We also present a numerical study to evaluate the correct balancedness of the resulting Hdiv spaces of degree k and L^2 spaces of degree $k-1$ on the resolution of the mixed formulation for a Steklov eigenvalue problem.

1. Introduction

In the formulation of mixed methods, vector functional spaces of type Hdiv or Hcurl, and approximations of them, are required [1]. The main characteristic of Hdiv functions is the continuity of the normal components over element interfaces of a partition of the domain, while Hcurl functions require continuous tangential components.

Different techniques for the construction of Hdiv and Hcurl finite element spaces have been used in the literature, since the classical settings by [2,3,1,4]. In some contexts the vector basis functions are constructed on the master element and then they are transformed to the elements of the partition. More recently, formulations for the construction of hierarchical high order spaces by [5–7] are based on the properties of the De Rham complex and require the computations of gradients of scalar functions in H^1-conforming spaces. There is also the recent isogeometric analysis [8], where Hcurl subspaces can be defined in terms of suitable B-spline spaces satisfying the De Rham diagram.

In the present paper we present a different approach, introduced in [9], where the principle consists of constructing an appropriate set of vectors, based on the geometric characteristics of the elements, which are multiplied by hierarchical H^1 conforming scalar functions. This systematic procedure facilitates the computational implementation, avoiding the difficulties of consistently transforming vector fields from the master element or the computation of gradients. Using this methodology, hierarchical vector bases are defined for quadrilateral and triangular elements. There are those basic vector functions that are associated to the edges, whose normal (or tangential) components on the edges of the elements are expressed in terms of the H^1 scalar basis functions corresponding to them. There are also other basis vector functions which are internal to the element, whose normal (or tangential) components vanish over all edges. Therefore, Hdiv (or Hcurl) conforming spaces can be created by simply imposing that the sum of the multiplying coefficients associated with the edge vector functions of neighbouring elements is zero. A procedure for constructing p-adaptive H^1 spaces has been presented in [10], which is adopted here.

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doi:10.1016/j.cam.2012.09.026

The development of p or hp adaptive approximation spaces for mixed formulations present a challenge in terms of balancedness (sometimes referred as compatibility) of approximation spaces of the primal and dual variables. The stability of the method depends strongly on this property, that can be expressed by the Ladyzenskaya–Babuska–Brezzi (LBB) condition (or *Inf-Sup* condition). If the approximation space for the dual variable is too poor, the lack of restraints on the primal variable may cause the solution to oscillate. Such oscillations are known as spurious modes. If, on the other hand, the approximation space of the dual variable is too rich, the problem of the primal variable is over restrained, leading to a phenomenon known as locking.

Although the LBB condition is mathematically elegant, it is difficult to use as a practical tool for analysing the compatibility of spaces generated in hp-adaptive contexts. In [1], it is shown that the compatibility of the spaces can be numerically analysed in terms of the eigenvalues associated with the restraint matrix. In [11], a compatibility analysis is also based on the behaviour of the eigenvalues of the matrix of a mixed finite element approximation. In neither of these publications, are objective values given to compare with.

In this paper, we propose to analyse the balancedness of the approximation spaces of the primal and dual variables by observing the evolution of numerical approximations of an eigenvalue Steklov problem. The advantage of this approach is that analytical and/or high precision eigenvalues can be computed. These values can be used for comparison with numerically obtained values. If the spaces are not balanced, spurious modes may appear as artificial eigenvalues, which do not correspond to Steklov eigenvalues. Locking may also happen, which is recognised by the inability of the approximation to represent the first eigenmodes and corresponding eigenvalues. If the spaces are well balanced, it is expected that the eigenvalue problem will be approximated with optimal rates of convergence.

The outline of the paper is as follows. Section 2 is dedicated to the construction of the hierarchy of Hdiv approximation spaces, of any degree k, as introduced in [9]. In Section 3 the Hcurl case is briefly considered, since in bidimensional regions this setting is derived from the Hdiv case by simply rotating the corresponding set of vectors by $\pi/2$. In Section 4, a mixed formulation for an eigenvalue Steklov problem is considered. Results of numerical experiments are reported in Section 5, in order to verify the balancedness of the finite element spaces developed in Section 2, combined with L^2 approximations spaces, when applied to the mixed formulation of the previous section. Finally, the conclusions and perspectives of the paper are presented in Section 6.

2. Hdiv approximation spaces

Let $\Omega \subset \mathbb{R}^2$ be a bounded open domain with Lipschitz boundary $\partial\Omega$, and $\mathcal{T}_h$ a partition of Ω formed by polygonal elements (triangular or quadrilateral). The purpose in this section is to describe a new construction of approximation subspaces of piecewise functions based on $\mathcal{T}_h$ of the Hdiv space

$$Hdiv(\Omega) = \left\{\overrightarrow{\varphi} \in L^2(\Omega)^2 : div(\mathbf{v}) \in L^2(\Omega)\right\}. \tag{1}$$

It is well known that the functions in Hdiv are characterised by having continuous normal components across the interfaces of the elements [1, p. 95]. Precisely, given two elements $K^i,\ K^j \in \mathcal{T}_h$, with a common boundary interface $f^{ij} = \partial K^i \cap \partial K^j$, then for $\overrightarrow{v} \in Hdiv(\Omega)$

$$\overrightarrow{v} \cdot \overrightarrow{\boldsymbol{\eta}}^i|_{f^{ij}} + \overrightarrow{v} \cdot \overrightarrow{\boldsymbol{\eta}}^j|_{f^{ij}} = 0, \tag{2}$$

where $\overrightarrow{\boldsymbol{\eta}}^i$ and $\overrightarrow{\boldsymbol{\eta}}^j$ are the outward unit normal vectors to ∂K^i and ∂K^j, respectively.

The procedure suggested in [9] for the construction of piecewise functions in $Hdiv(\Omega)$, based on $\mathcal{T}_h$, considers hierarchical H^1-conforming basis functions φ [10], and a set of vectors $\overrightarrow{v}$ properly constructed, according to each element geometry. Thus, Hdiv basis functions are defined by taking vectorial functions of the form

$$\overrightarrow{\varphi} := \varphi\,\overrightarrow{v}.$$

The next sections describe this procedure for constructing quadrilateral and triangular elements, by first revising the definition of H^1-conforming bases, then setting the appropriate set of vectors, and finally specifying the vector basis functions for $Hdiv(\Omega)$.

2.1. Quadrilateral elements

2.1.1. H^1-conforming basis functions

Hierarchical H^1-conforming bases in [10] are defined by means of parametrised geometric transformations of scalar bases defined on a master element.

Let $\hat{K} = \{(\xi,\eta) : -1 \le \xi, \eta \le 1\}$ be the quadrilateral master element with vertices $\hat{a}_0 = (-1,-1)$, $\hat{a}_1 = (1,-1)$, $\hat{a}_2 = (1,1)$ and $\hat{a}_3 = (-1,1)$. The edges $\hat{l}_m$, $m = 0,\ldots,3$, correspond to the sides linking the vertices $\hat{a}_m$ to $\hat{a}_{m+1 (\mathrm{mod}\ 4)}$. Let $\mathcal{Q}_k(\hat{K})$ be the space of polynomials of maximum degree k in each variable.

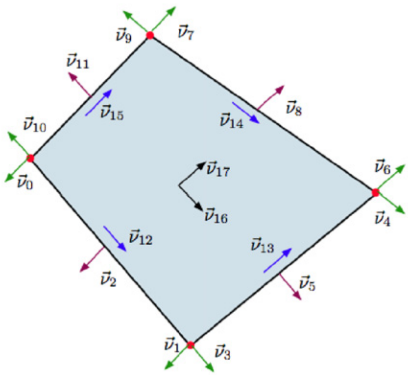

Fig. 1. Set of vectors used in the definition of Hdiv finite element spaces for quadrilateral elements.

A hierarchy of polynomial bases of degree k for $\mathcal{Q}_k(\hat{K})$ is constructed, which are classified by:

- 4 vertex functions $\hat{\varphi}^{\hat{a}_m}$:

$$\hat{\varphi}^{\hat{a}_0}(\xi,\eta) = \frac{(1-\xi)}{2}\frac{(1-\eta)}{2}, \qquad \hat{\varphi}^{\hat{a}_1}(\xi,\eta) = \frac{(1+\xi)}{2}\frac{(1-\eta)}{2},$$
$$\hat{\varphi}^{\hat{a}_2}(\xi,\eta) = \frac{(1+\xi)}{2}\frac{(1+\eta)}{2}, \qquad \hat{\varphi}^{\hat{a}_3}(\xi,\eta) = \frac{(1-\xi)}{2}\frac{(1+\eta)}{2},$$

such that $\varphi^{\hat{a}_m}(\hat{a}_s) = \delta_{ms}$.

- $4(k-1)$ edge functions

$$\hat{\varphi}^{\hat{l}_0,n}(\xi,\eta) = \hat{\varphi}^{\hat{a}_0}(\xi,\eta)[\hat{\varphi}^{\hat{a}_1}(\xi,\eta) + \hat{\varphi}^{\hat{a}_2}(\xi,\eta)]f_n(\xi),$$
$$\hat{\varphi}^{\hat{l}_1,n}(\xi,\eta) = \hat{\varphi}^{\hat{a}_1}(\xi,\eta)[\hat{\varphi}^{\hat{a}_2}(\xi,\eta) + \hat{\varphi}^{\hat{a}_3}(\xi,\eta)]f_n(\eta),$$
$$\hat{\varphi}^{\hat{l}_2,n}(\xi,\eta) = \hat{\varphi}^{\hat{a}_2}(\xi,\eta)[\hat{\varphi}^{\hat{a}_3}(\xi,\eta) + \hat{\varphi}^{\hat{a}_0}(\xi,\eta)]f_n(-\xi),$$
$$\hat{\varphi}^{\hat{l}_3,n}(\xi,\eta) = \hat{\varphi}^{\hat{a}_3}(\xi,\eta)[\hat{\varphi}^{\hat{a}_0}(\xi,\eta) + \hat{\varphi}^{\hat{a}_1}(\xi,\eta)]f_n(-\eta),$$

where f_n are the Chebychev polynomials of degree n, $n = 0, 1, \ldots, k-2$. The edge functions $\hat{\varphi}^{\hat{l}_m,n}$ vanish on all edges $\hat{l}_s$, for $s \neq m$;

- $(k-1)^2$ internal functions

$$\varphi^{\hat{K},n_0,n_1}(\xi,\eta) = \varphi^{a_0}(\xi,\eta)\varphi^{a_2}(\xi,\eta)f_{n_0}(\xi)f_{n_1}(\eta),$$

with $0 \leq n_0, n_1 \leq k-2$. These functions vanish on all edges and vertices.

For a generic element $K \in \mathcal{T}_h$, traditional geometric transformations are used to map these functions to create vertex, edge and internal element shape functions in K. Using the same arguments as in textbooks on finite elements, H^1-conforming spaces can be created by imposing that the multiplying coefficients associated with vertex element shape functions of elements sharing the same node are identical. Furthermore, choosing an appropriate orientation of the sides, the multiplying coefficients associated with edge element shape functions of neighbouring elements should coincide. Internal shape functions are continuous because their traces are zero on the edges.

2.1.2. Set of vectors

The next step is the construction of a set of vectors $\vec{v} = \{\vec{v}_j,\ j = 0, \ldots, 17\}$ verifying the next properties, as illustrated in Fig. 1:

(i) $\vec{v}_{2+3m} = \vec{\eta}_m$ is the outward unit normal, and $\vec{v}_{m+12}$ is tangent to the edge l_m, for $m = 0, 1, 2, 3$.
(ii) For $m = 3\,s$, with $s = 0, 1, 2, 3$, $\vec{v}_m \cdot \vec{v}_{m+1} = \vec{v}_m \cdot \vec{v}_{m+2} = \vec{v}_{m+1} \cdot \vec{v}_{m+2} = 1$.
(iii) On the surface of the element, $\vec{v}_{16}$ and $\vec{v}_{17}$ are orthogonal vectors $\vec{v}_{16} \perp \vec{v}_{17}$.

2.1.3. Hdiv basis functions

We propose the construction of a family of vector functions by the multiplication of the hierarchical scalar H^1 basis by properly chosen vectors $\vec{v}$, according to the following procedure:

$4(k+1)$ edge vector functions

$$\begin{array}{llll} m=0: & \overrightarrow{\varphi}^{l_0,a_0}=\varphi^{a_0}\,\overrightarrow{v_0}, & \overrightarrow{\varphi}^{l_0,a_1}=\varphi^{a_1}\,\overrightarrow{v}_1, & \overrightarrow{\varphi}^{l_0,n}=\varphi^{l_0,n}\,\overrightarrow{v}_2,\\ m=1: & \overrightarrow{\varphi}^{l_1,a_1}=\varphi^{a_1}\,\overrightarrow{v_3}, & \overrightarrow{\varphi}^{l_1,a_2}=\varphi^{a_2}\,\overrightarrow{v}_4, & \overrightarrow{\varphi}^{l_1,n}=\varphi^{l_1,n}\,\overrightarrow{v}_5,\\ m=2: & \overrightarrow{\varphi}^{l_2,a_2}=\varphi^{a_2}\,\overrightarrow{v_6}, & \overrightarrow{\varphi}^{l_2,a_3}=\varphi^{a_3}\,\overrightarrow{v}_7, & \overrightarrow{\varphi}^{l_2,n}=\varphi^{l_2,n}\,\overrightarrow{v}_8,\\ m=3: & \overrightarrow{\varphi}^{l_3,a_3}=\varphi^{a_3}\,\overrightarrow{v_9}, & \overrightarrow{\varphi}^{l_3,a_0}=\varphi^{a_0}\,\overrightarrow{v}_{10}, & \overrightarrow{\varphi}^{l_3,n}=\varphi^{l_3,n}\,\overrightarrow{v}_{11}. \end{array} \tag{3}$$

Observe that these edge vector functions satisfy

$$\overrightarrow{\varphi}^{l_m,a_s}\cdot\overrightarrow{\eta_m}=\varphi^{a_s},\qquad \overrightarrow{\varphi}^{l_m,n}\cdot\overrightarrow{\eta}_m=\varphi^{l_m,n}. \tag{4}$$

$2(k^2-1)$ internal vector functions

To complete the space, we add three types of internal functions

$$\overrightarrow{\varphi}_1^{K,n_0,n_1}=\varphi^{K,n_0,n_1}\overrightarrow{v}_{16},\qquad \overrightarrow{\varphi}_2^{K,n_0,n_1}=\varphi^{K,n_0,n_1}\overrightarrow{v}_{17},\quad \text{and}\quad \overrightarrow{\varphi}_3^{l_k,n}=\varphi^{l_k,n}\overrightarrow{v}_{k+12}. \tag{5}$$

The normal components of these internal vector functions vanish at all edges. For quadrilateral elements, the total number of edge and internal vector functions is $2(k+1)^2$.

2.2. Triangular elements

2.2.1. H^1-conforming basis functions

Consider the master triangular element $\hat{K}=\{(\xi,\eta):0\le\xi\le 1, 0\le\eta\le 1-\xi\}$, with vertices $\hat{a}_0=(0,0)$, $\hat{a}_1=(1,0)$ and $\hat{a}_2=(0,1)$, and edges $\hat{l}_m$, $m=0,1,2$ linking the vertex $\hat{a}_m$ to $\hat{a}_{m+1(mod3)}$. Let $\mathcal{P}_k(\hat{K})$ be the space of polynomials of total degree k.

The hierarchy of polynomial bases for $\mathcal{P}_k(\hat{K})$ constructed in [10] are classified as follows:

- 3 vertex functions

  $$\hat{\varphi}^{\hat{a}_0}(\xi,\eta)=1-\xi-\eta,\qquad \hat{\varphi}^{\hat{a}_1}(\xi,\eta)=\xi,\qquad \hat{\varphi}^{\hat{a}_2}(\xi,\eta)=\eta,$$

  such that $\varphi^{\hat{a}_m}(\hat{a}_s)=\delta_{ms}$.
- $3(k-1)$ edge functions

  $$\begin{aligned}\hat{\varphi}^{\hat{l}_0,n}(\xi,\eta)&=\varphi^{\hat{a}_0}(\xi,\eta)\varphi^{\hat{a}_1}(\xi,\eta)f_n(\eta+2\xi-1),\\ \hat{\varphi}^{\hat{l}_1,n}(\xi,\eta)&=\varphi^{\hat{a}_1}(\xi,\eta)\varphi^{\hat{a}_2}(\xi,\eta)f_n(\eta-\xi),\\ \hat{\varphi}^{\hat{l}_2,n}(\xi,\eta)&=\varphi^{\hat{a}_2}(\xi,\eta)\varphi^{\hat{a}_0}(\xi,\eta)f_n(1-\xi-2\eta),\end{aligned}$$

  where f_n are the Chebychev polynomials of degree n, $n=0,1,\dots,k-2$. The edge functions $\hat{\varphi}^{\hat{l}_m,n}$ vanish on all edges $\hat{l}_s$, for $s\neq m$;
- $\frac{(k-2)(k-1)}{2}$ internal functions

  $$\varphi^{\hat{K},n_0,n_1}(\xi,\eta)=\varphi^{\hat{a}_0}(\xi,\eta)\varphi^{\hat{a}_1}(\xi,\eta)\varphi^{\hat{a}_2}(\xi,\eta)f_{n_0}(2\xi-1)f_{n_1}(2\eta-1),$$

  with $0\le n_0+n_1\le k-3$, which vanish on all edges and vertices.

As in the quadrilateral element case, finite element bases for subspaces of $H^1(\Omega)$ can be defined for triangular partitions $\mathcal{T}_h$ by using appropriate parametrised geometric transformations to map these functions to vertex, edge and internal element shape functions in each element $K\in\mathcal{T}_h$.

2.2.2. Set of vectors

Consider a set of fourteen vectors associated to a generic triangular element $K\in\mathcal{T}_h$, as illustrated in Fig. 2, satisfying the properties

(i) $\overrightarrow{v}_{2+3m}=\overrightarrow{\eta}_m$ is the outward unit normal, and $\overrightarrow{v}_{m+9}$ is tangent to the edge l_m, $m=0,\ 1,\ 2$.
(ii) for $m=3\,s$, with $s=0,\ 1\,2$, $\overrightarrow{v}_m\cdot\overrightarrow{v}_{m+1}=\overrightarrow{v}_m\cdot\overrightarrow{v}_{m+2}=\overrightarrow{v}_{m+1}\cdot\overrightarrow{v}_{m+2}=1$.
(iii) $\overrightarrow{v}_{12}\perp\overrightarrow{v}_{13}$.

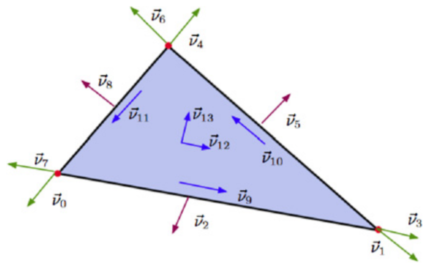

Fig. 2. Set of vectors used in the definition of Hdiv finite element spaces for triangular elements.

2.2.3. Hdiv basis functions

We introduce the vector functions associated to the edges l_m of a triangular element $K \in \mathcal{T}_h$:

$$\begin{array}{llll} m=0: & \vec{\varphi}^{\,l_0,a_0} = \varphi^{a_0}\,\vec{v}_0, & \vec{\varphi}^{\,l_0,a_1} = \varphi^{a_1}\,\vec{v}_1, & \vec{\varphi}^{\,l_0,n} = \varphi^{l_0,n}\,\vec{v}_2, \\ m=1: & \vec{\varphi}^{\,l_1,a_1} = \varphi^{a_1}\,\vec{v}_3, & \vec{\varphi}^{\,l_1,a_2} = \varphi^{a_2}\,\vec{v}_4, & \vec{\varphi}^{\,l_1,n} = \varphi^{l_1,n}\,\vec{v}_5, \\ m=2: & \vec{\varphi}^{\,l_2,a_2} = \varphi^{a_2}\,\vec{v}_6, & \vec{\varphi}^{\,l_2,a_3} = \varphi^{a_3}\,\vec{v}_7, & \vec{\varphi}^{\,l_2,n} = \varphi^{l_2,n}\,\vec{v}_8, \end{array} \tag{6}$$

having normal components

$$\vec{\varphi}^{\,l_m,a_j} \cdot \vec{\eta}_m = \varphi^{a_j}, \qquad \vec{\varphi}^{\,l_m,n} \cdot \vec{\eta}_m = \varphi^{l_m,n}, \tag{7}$$

and internal vector functions

$$\vec{\varphi}_1^{\,K,n_0,n_1} = \varphi^{K,n_0,n_1}\,\vec{v}_{12} \qquad \vec{\varphi}_2^{\,K,n_0,n_1} = \varphi^{K,n_0,n_1}\,\vec{v}_{13} \qquad \vec{\varphi}_3^{\,l_m,n} = \varphi^{l_m,n}\,\vec{v}_{9+m} \tag{8}$$

whose normal components vanish at all edges. For triangular elements, the total number of edge and internal vector functions is $(k+1)(k+2)$.

Having defined the two sets of hierarchical vector functions, both for quadrilateral and triangular elements, the next theorem states that they indeed are formed by linearly independent functions.

Theorem 1. *Let K be an element of the partition $\mathcal{T}_h$. The edge and internal vector functions defined in Eqs. (3) and (5), for quadrilateral elements, or in formulae (6) and (8), for triangular elements, form a linearly independent set of vector functions in $Hdiv(K)$.*

Proof. Let $\mathcal{V}$ and $\mathcal{E}$ be the sets formed by the vertices of K. Let us consider the linear combination

$$0 = \sum_{l_m\in\mathcal{E}}\left[\sum_{a_s\in\mathcal{V}} \alpha_{m,s}\vec{\varphi}^{\,l_m,a_s} + \sum_{n=0}^{k-2}\beta_{m,n}\vec{\varphi}^{\,l_m,n}\right] + \sum_{s=1}^{2}\sum_{n_0=0}^{k-2}\sum_{n_1=0}^{k-2}\gamma_{s,n_0,n_1}\vec{\varphi}_s^{\,K,n_0,n_1} + \sum_{l_m\in\mathcal{E}}\sum_{n=0}^{k-2}\mu_{m,n}\vec{\varphi}_3^{\,l_m,n}.$$

Restricting this expression to the edge l_r, and taking the inner product with $\vec{\eta}_r$, we obtain

$$\sum_{s=r}^{r+1}\alpha_{r,s}\varphi^{a_r} + \sum_{n=0}^{k-2}\beta_{r,n}\varphi^{l_r,n} = 0,$$

where $r+1$ is understood (mod 4), in the case of quadrilateral elements, or (mod 3), for triangular elements. Using the linear independence of the scalar basis functions, we conclude that $\alpha_{r,s} = \beta_{r,n} = 0$. Next, considering the tangential component to the edge l_r of the remaining terms, we obtain that, restricted to this edge,

$$\sum_{n=0}^{k-2}\mu_{r,n}\varphi^{l_r,n} = 0,$$

implying that $\mu_{r,n} = 0$. Finally, the linear combination is reduced to the terms

$$\sum_{s=1}^{2}\sum_{n_0=0}^{k-2}\sum_{n_1=0}^{k-2}\gamma_{s,n_0,n_1}\vec{\varphi}_s^{\,K,n_0,n_1}.$$

Doing inner products of this expression with $\vec{v}_{16}$ and $\vec{v}_{17}$, in the quadrilateral case (or with $\vec{v}_{12}$ and $\vec{v}_{13}$, for triangles), we obtain

$$\sum_{n_0=0}^{k-2}\sum_{n_1=0}^{k-2}\gamma_{1,n_0,n_1}\varphi_1^{K,n_0,n_1} = \sum_{n_0=0}^{k-2}\sum_{n_1=0}^{k-2}\gamma_{2,n_0,n_1}\varphi_2^{K,n_0,n_1} = 0,$$

implying that $\gamma_{1,n_0,n_1} = \gamma_{2,n_0,n_1} = 0$. Since the scalar basis functions belong to $H^1(K)$, we conclude that the corresponding set of vector functions form a basis for a subspace of Hdiv(K). □

Finally, we arrive at the goal of the construction of Hdiv subspaces $V(\mathcal{T}_h)$ formed by vector functions $\overrightarrow{\varphi}$ such that:

1. The restrictions $\overrightarrow{\varphi}|_K \in Hdiv(K)$ belong to the finite element spaces spanned by the vector bases defined by Eqs. (3) and (5), for quadrilateral elements, and in formulae (6) and (8), for triangular elements.
2. The Hdiv compatibility condition (1) is satisfied.

Theorem 2. *Using the hierarchical vector bases defined by Eqs. (3) and (5), for quadrilateral elements, and in formulae (6) and (8), for triangular elements, Hdiv-conforming spaces $V(\mathcal{T}_h)$ can be created with vector functions $\overrightarrow{\varphi}$ by imposing that $\overrightarrow{\varphi}|_K \in Hdiv(K)$, if and only if the sum of the multiplying coefficients associated with the edge vector functions of neighbouring elements is zero.*

Proof. Let K_i and K_j be two elements that share a common edge $l^{i,j}$, and consider $\overrightarrow{\varphi}^i = \overrightarrow{\varphi}|_K^i$ and $\overrightarrow{\varphi}^j = \overrightarrow{\varphi}|_K^j$ be functions defined in each element by means of the corresponding finite element bases. For instance, consider the expansion

$$\overrightarrow{\varphi}^i = \sum_{l_m^i \in \mathcal{E}^i} \left[\sum_{a_m^i \in \mathcal{V}^i} \alpha_{m,s}^i \overrightarrow{\varphi}^{l_m^i, a_m^i} + \sum_{n=0}^{k-2} \beta_{m,n}^i \overrightarrow{\varphi}^{l_m^i, n} \right] + \sum_{s=1}^{2} \sum_{n_0=0}^{k-2} \sum_{n_1=0}^{k-2} \gamma_{s,n_0,n_1}^i \overrightarrow{\varphi}_m^{K^i, n_0, n_1} + \sum_{l_m^i \in \mathcal{E}^i} \sum_{n=0}^{k-2} \mu_{m,n}^i \overrightarrow{\varphi}_3^{l_m^i, n}.$$

Suppose that $l^{i,j} = l_r^i$, with vertices a_r^i and a_{r+1}^i (where $r+1$ is understood (mod 4), in the case of quadrilateral elements, or (mod 3), for triangular elements). Considering that the normal components of all edge vector functions $\overrightarrow{\varphi}^{l_m^i, n}$, for $m \neq r$, and internal vector functions $\overrightarrow{\varphi}_1^{K^i, n_0, n_1}$, $\overrightarrow{\varphi}_2^{K^i, n_0, n_1}$ and $\overrightarrow{\varphi}_3^{l_m^i, n}$, vanish on l_r, and recalling the properties (4) and (7), it follows that

$$\overrightarrow{\varphi}^i \cdot \boldsymbol{\eta}^i|_{l^{ij}} = \sum_{s=r}^{r+1} \alpha_{r,s}^i \varphi^{a_s^i}|_{l^{ij}} + \sum_{n=0}^{k-2} \beta_{r,n}^i \varphi^{l_r^i, n}|_{l^{ij}}.$$

Similarly, suppose that $l^{i,j} = l_q^j$, with vertices a_q^j and a_{q+1}^j, such that

$$\overrightarrow{\varphi}^j \cdot \boldsymbol{\eta}^j|_{l^{ij}} = \sum_{s=q}^{q+1} \alpha_{q,s}^j \varphi^{a_s^j}|_{l^{ij}} + \sum_{n=0}^{k-2} \beta_{q,n}^j \varphi^{l_q^j, n}|_{l^{ij}}.$$

Consider the case where $a_q^j = a_r^i$ and $a_{q+1}^j = a_{r+1}^i$. On the edge l^{ij}, the H^1 compatibility of the scalar basis functions implies the trace equalities $\varphi^{a_r^i}|_{l^{ij}} = \varphi^{a_q^j}|_{l^{ij}}$, $\varphi^{a_{r+1}^i}|_{l^{ij}} = \varphi^{a_{q+1}^j}|_{l^{ij}}$ and $\varphi^{l_r^i, n}|_{l^{ij}} = \varphi^{l_q^j, n}|_{l^{ij}}$. Therefore,

$$\overrightarrow{\varphi}^i \cdot \boldsymbol{\eta}^i|_{l^{ij}} + \overrightarrow{\varphi}^j \cdot \boldsymbol{\eta}^j|_{l^{ij}} = (\alpha_{r,r}^i + \alpha_{q,q}^j)\varphi^{a_r^i}|_{l^{ij}} + (\alpha_{r,r+1}^i + \alpha_{q,q+1}^j)\varphi^{a_{r+1}^i}|_{l^{ij}} + \sum_{n=0}^{k-2} (\beta_{r,n}^i + \beta_{q,n}^j)\varphi^{l_r^i, n}|_{l^{ij}}.$$

Consequently, the continuity of the normal component on l^{ij} holds if and only if $(\beta_{r,n}^i + \beta_{q,n}^j) = 0$ and $(\alpha_{r,r}^i + \alpha_{q,q}^j) = (\alpha_{r,r+1}^i + \alpha_{q,q+1}^j) = 0$. The other case, where $a_q^j = a_{r+1}^i$ and $a_{q+1}^j = a_r^i$, can be analysed in a similar way. □

3. Hcurl approximation spaces

Now we turn to the construction of approximations of the Hcurl space

$$Hcurl\,(\Omega) = \left\{ \overrightarrow{v} \in L^2(\Omega)^2 : \mathbf{curl}\left(\overrightarrow{v}\right) \in L^2(\Omega)^2 \right\}.$$

In order to construct subspaces $V(\mathcal{T}_h) \subset Hcurl\,(\Omega)$, with vector functions defined piece wisely on the partition $\mathcal{T}_h$, it is necessary to impose continuity of the tangential components at the interfaces of the elements. Similarly to the Hdiv case, a systematic procedure consists in first choosing an appropriate set of vectors, based on the geometry of the elements, and then multiplying it by the set of H^1 hierarchical scalar basis functions. In order to guarantee the continuity of the tangential components of the functions in $V(\mathcal{T}_h)$ on the interfaces of bidimensional elements, such Hcurl set of vectors can be obtained by a $\pi/2$ rotation of the Hdiv set of vectors, as shown in Fig. 3.

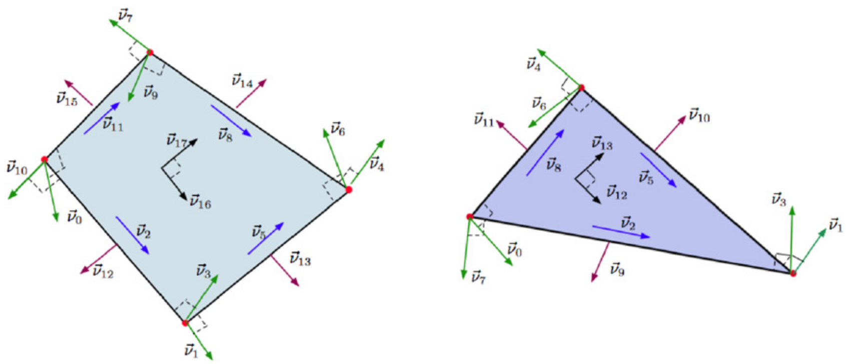

Fig. 3. Set of vectors used in the definition of Hcurl vector basis functions for 2D triangular and quadrilateral elements.

Table 1
Eigenvalues and eigenfunctions for the Steklov problem associated to $\xi_i \in [0, 2\pi[$.

i	ξ_i	λ_i	$p_i(x, y)$
1	0.9375520343559806	0.688252742336267	$\cosh(\xi_1 x)\sin(\xi_1 y)$
2	0	1	xy
3	2.365020372431352	2.323637753431723	$\cosh(\xi_3 x)\cos(\xi_3 y)$
4	2.347045566487087	2.390389205105817	$\sinh(\xi_4 x)\cos(\xi_4 y)$
5	3.927378719118806	3.924333023244752	$\cosh(\xi_5 x)\sin(\xi_5 y)$
6	3.926602312047919	3.929654506780184	$\sinh(\xi_6 x)\sin(\xi_6 y)$
7	5.497803919000836	5.497619468368826	$\cosh(\xi_7 x)\cos(\xi_7 y)$
8	5.497770367437734	5.497954835510735	$\sinh(\xi_8 x)\cos(\xi_8 y)$

4. The Steklov problem

We consider the Steklov problem: find $\lambda \in \mathbb{R}$ and p such that

$$\begin{aligned} &\triangle p = 0 \quad \text{in } \Omega, \\ &\frac{\partial p}{\partial \eta} = \lambda p \quad \text{in } \partial\Omega. \end{aligned} \tag{9}$$

This is an eigenvalue problem relating the flux $\frac{\partial p}{\partial \eta}$ and the solution p over the boundary of the domain.

For the square domain $\Omega = (-1, 1) \times (-1, 1)$, the eigenvalue problem (9) can be solved using separation of variables. There is the eigenvalue $\lambda = 1$, with single multiplicity, for which $p(x, y) = xy$ is an eigenfunction. All the other eigenvalues $\lambda \neq 1$ are positive, and have double multiplicity, associated to linearly independent eigenfunctions of the form $p(x, y) = w_1(x)w_2(y)$ and $q(x, y) = w_2(x)w_1(y)$. The possibilities for $w_1(x)$ are $\cosh(\xi x)$ or $\sinh(\xi x)$, while $w_2(y)$ should be of the form $\cos(\xi y)$ or $\sin(\xi y)$. Consequently, the eigenvalues have the form $\lambda = \frac{\xi \cosh(\xi)}{\sinh(\xi)}$, or $\lambda = \frac{\xi \sinh(\xi)}{\cosh(\xi)}$, where $\xi > 0$ should be searched as the solutions of $\coth(\xi) = \cot(\xi)$, or $\tanh(\xi) = \cot(\xi)$, in regions where $\tan(\xi) > 0$. On the other hand, where $\tan(\xi) < 0$, then we have solve $\coth(\xi) = -\tan(\xi)$, or $\tanh(\xi) = -\tan(\xi)$.

The values of ξ_i occurring on the interval $[0, 2\pi[$, the corresponding eigenvalues λ_i, and their eigenfunctions $p_i(x, y)$, are shown in Table 1. For simplicity, the associated dual eigenfunctions $q_i(x, y)$ are not included.

4.1. Mixed formulation for the Steklov problem

Consider the modified statement of problem (9): Find $\lambda \in \mathbb{R}$ and $(\mathbf{u}, p)$ such that

$$\begin{cases} \mathbf{u} = \nabla p \quad \text{in } \Omega, \\ div(\mathbf{u}) = 0 \quad \text{in } \Omega, \\ \dfrac{\partial p}{\partial \boldsymbol{\eta}} = \lambda\, p \quad \text{in } \partial\Omega. \end{cases} \tag{10}$$

Observe that for $p \in H^1(\Omega)$, the boundary condition implies that $\nabla p \cdot \boldsymbol{\eta} \in H^{1/2}(\partial\Omega)$.

Problem (10) can be expressed in weak form as: Find $(\mathbf{u}, p) \in Hdiv(\Omega) \times L^2(\Omega)$ such that $\mathbf{u} \cdot \boldsymbol{\eta} \in H^{1/2}(\partial\Omega)$ and

$$\begin{cases} a(\mathbf{u}, \overrightarrow{v}) + b(\overrightarrow{v}, p) = \lambda^{-1}\langle \mathbf{u}, \overrightarrow{v} \rangle \quad \forall \overrightarrow{v} \in Hdiv(\Omega), \\ b(\overrightarrow{v}, w) = 0 \quad \forall w \in L^2(\Omega), \end{cases} \tag{11}$$

where $a(\mathbf{u}, \overrightarrow{v}) = \int_\Omega \overrightarrow{v} \cdot \mathbf{u}\, dx$, $b(\overrightarrow{v}, p) = \int_\Omega p\, div(\overrightarrow{v})\, dx$, and $\langle \mathbf{u}, \overrightarrow{v} \rangle$ represents the the duality of $H^{1/2}(\partial\Omega)$ and $H^{-1/2}(\partial\Omega)$.

According to [12], if $V(\mathcal{T}_h) \subset Hdiv\ (\Omega)$ and $Y(\mathcal{T}_h) \subset L^2\ (\Omega)$ are finite element subspaces, with degree of approximations k and $k-1$, respectively, and satisfying the *Inf-Sup* condition, then eigenvalues $\lambda_{i,h}$ and eigenfunctions $\mathbf{u}_{i\,h}$ approximations of mixed formulations of this kind of eigenvalue problem verify the a priori error estimates

$$|\lambda_i - \lambda_{i\,h}| \leq Ch^{2(k+1)}, \tag{12}$$

$$\|\mathbf{u}_i - \mathbf{u}_{i\,h}\|_{L^2(\Omega)} \leq Ch^{k+1}. \tag{13}$$

The purpose of the next section is to test numerically whether the Hdiv-conforming subspaces $V(\mathcal{T}_h)$, constructed in Section 2, can be safely used in a discrete version of the mixed formulation (11), expecting to get optimal convergence rates for the eigenvalues.

5. Numerical experiments

Initially, the numerical experiments consider partitions $\mathcal{T}_h$ formed by triangular elements, obtained by diagonal subdivision of a uniform quadrilateral mesh, with step size $h = 2^{1-s}$, $s = 1, 2, 3$ e 4. The Hdiv-conforming subspaces $V(\mathcal{T}_h)$ are taken of type $\vec{\mathcal{P}}_k$, indicating the total degree k of the scalar basis used in their construction. The finite element spaces $Y(\mathcal{T}_h) \subset L^2\ (\Omega)$ are defined by piecewise polynomials in $\mathcal{P}_{k-1}$, without any continuity restriction over element interfaces. For this setting of type $\vec{\mathcal{P}}_k\ \mathcal{P}_{k-1}$, the property $divV(\mathcal{T}_h) = Y(\mathcal{T}_h)$ holds. Next, quadrilateral partitions and approximation spaces of type $\vec{\mathcal{Q}}_k\ \mathcal{Q}_{k-1}$ are considered.

In matrix form, the discrete version of problem (11) can be written as a generalised eigenvalue problem

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_h \\ p_h \end{pmatrix} = \lambda_h^{-1} \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_h \\ p_h \end{pmatrix},$$

where the matrices A, B and C correspond to discrete versions of the forms $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ $\langle\cdot,\cdot\rangle$, respectively, $\mathbf{u}_h$ and p_h being the flux and pressure degrees of freedom. The following considerations with respect to the solution of this algebraic system are in order.

Given the degrees of freedom organised in the form $(\mathbf{u}_{0h}, p_{0h}, \mathbf{u}_{1h})^T$, where $\mathbf{u}_{0h}$ and $\mathbf{u}_{1h}$ refer to internal and boundary fluxes, respectively, and p_{0h} indicates internal pressure, the generalised eigenvalue problem can be expressed in the form [13]

$$\left(\begin{array}{cc|c} A_{00} & B_{00}^T & A_{01}^T \\ B_{00} & 0 & 0 \\ \hline A_{10} & 0 & A_{11}^T \end{array}\right) \left(\begin{array}{c} \mathbf{u}_{0h} \\ p_{0h} \\ \hline \mathbf{u}_{1h} \end{array}\right) = \lambda^{-1} \left(\begin{array}{cc|c} 0 & 0 & 0 \\ \hline 0 & 0 & C_{11} \end{array}\right) \left(\begin{array}{c} \mathbf{u}_0 \\ p_0 \\ \hline \mathbf{u}_1 \end{array}\right). \tag{14}$$

Then, static condensation is applied by eliminating the internal degrees of freedom $\mathbf{u}_{0h}$ and p_{0h}, to get the condensed system

$$S\,\mathbf{u}_1 = \lambda_h^{-1}\, C_{11}\, \mathbf{u}_{1h}, \tag{15}$$

where the Schur complement S is given by the formula

$$S = \begin{pmatrix} A_{11} & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} A_{10} & B_{00}^T \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A_{00} & B_{00}^T \\ B_{00} & 0 \end{pmatrix}^{-1} \begin{pmatrix} A_{01}^T \\ 0 \end{pmatrix}. \tag{16}$$

In all experiments, the matrices are assembled and computed in the object oriented environment PZ (www.labmec.org.br//pz/documentacao). Then, the condensed eigenvalue problem (15) is solved by using the software *Mathematica*.

5.1. Triangular elements

The behaviour of the errors on the approximations $\lambda_{i\,h}$, for the eigenvalues shown in Table 1, are illustrated in Fig. 4, for finite element spaces based on triangular partitions. It is clear that, for approximation spaces of type $\vec{\mathcal{P}}_k\ \mathcal{P}_{k-1}$, $k = 1, \ldots, 4$, the expected $O(h^{2(k+1)})$ optimal rates of convergence are reached, suggesting that these $\vec{\mathcal{P}}_k\ \mathcal{P}_{k-1}$ approximation spaces are well balanced. Observe that better accuracy is reached for smaller eigenvalues. Furthermore, the curves for the pairs of closer eigenvalues $\{\lambda_{3\,h}, \lambda_{4\,h}\}$, $\{\lambda_{5\,h}, \lambda_{6\,h}\}$, and $\{\lambda_{7\,h}, \lambda_{8\,h}\}$ almost coincide.

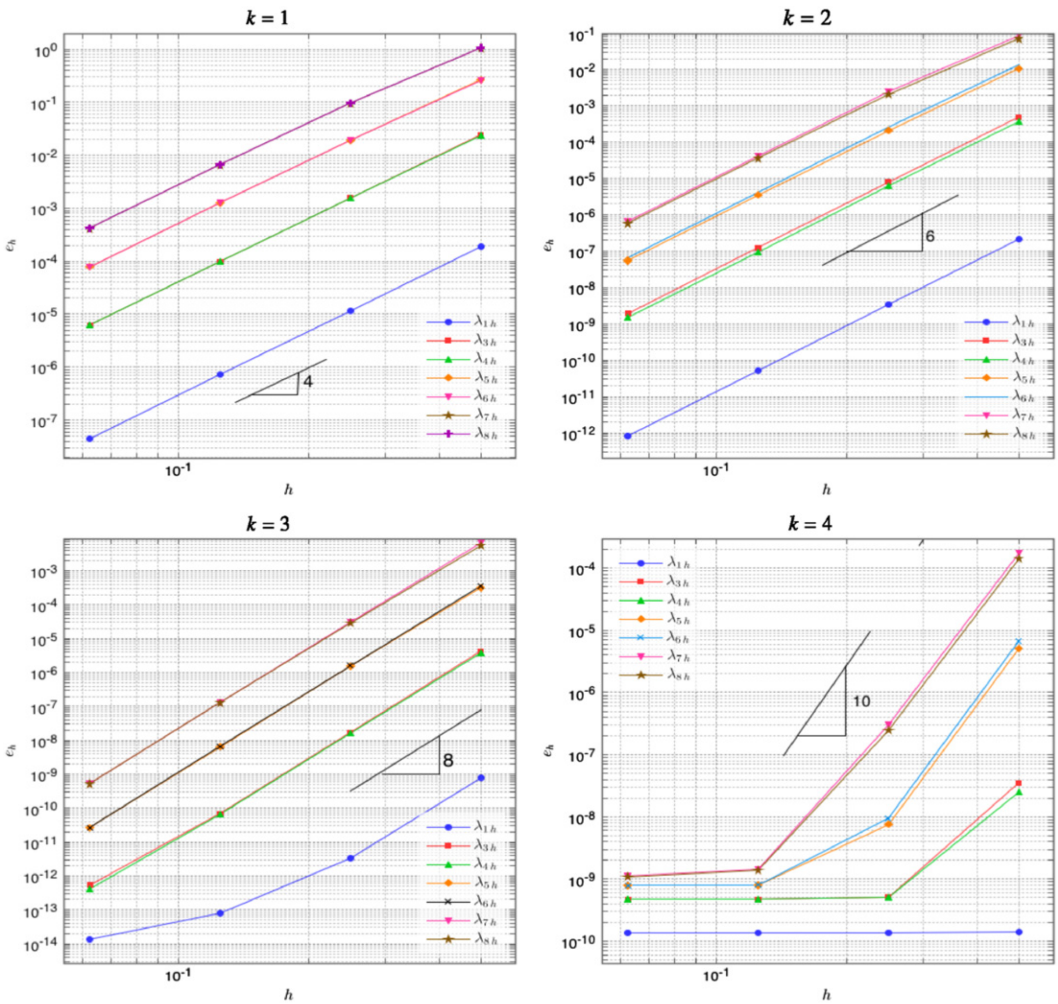

Fig. 4. Convergence of the approximations λ_{ih} using approximating spaces of type $\vec{\mathcal{P}}_k\, \mathcal{P}_{k-1}$ on triangular meshes.

5.2. Quadrilateral elements

For approximation spaces of type $\vec{\mathcal{Q}}_k\, \mathcal{Q}_{k-1}$, $k \geq 1$, based on quadrilateral partitions, optimal rates of convergence could not be obtained. For $k = 1$, $\vec{\mathcal{P}}_1\, \mathcal{P}_0$ is known to be the kind of approximation spaces that do not satisfy the *Inf-sup* condition. However, as shown in Fig. 5 (top left side), the method resulted in stability, but with a sub-optimal rate of convergence $O(h^2)$.

Unlike the triangular case with approximation spaces of type $\vec{\mathcal{P}}_k\, \mathcal{P}_{k-1}$, the approximation spaces of type $\vec{\mathcal{Q}}_k\, \mathcal{Q}_{k-1}$ for quadrilateral meshes do not verify the property $div V(\mathcal{T}_h) = Y(\mathcal{T}_h)$, for $k \geq 2$. To eliminate this drawback, we propose to form subspaces $V(\mathcal{T}_h)^* \subset V(\mathcal{T}_h)$, to get $div V(\mathcal{T}_h)^* = Y(\mathcal{T}_h)$.

Since $V(\mathcal{T}_h)$ is constructed by vector functions of type $\overrightarrow{\varphi} := \varphi\, \overrightarrow{v}$, where $\overrightarrow{v}$ is a set of vectors, and φ is a scalar function in $H^1(\Omega)$, then $div(\overrightarrow{\varphi}) = \overrightarrow{v} \cdot \nabla\varphi$. For scalar functions $\varphi \in \mathcal{Q}_k(K)$, $\nabla\varphi$ is a combination of terms of the form $(\xi^{i-1}\eta^j, \xi^j\eta^{i-1})$ with $i = 1, \ldots, k$ and $j = 0, \ldots, k$. Thus, we propose to keep only the vector basis functions constructed from scalar basis functions involving terms of the form $(\xi^r\eta^s, \xi^s\eta^r)$ with $s < r$ and $r, s \leq k$. Thus, we eliminate the vector basis functions constructed from the following scalar basis functions:

(i) 4 edge functions of degree k: $\varphi^{l_m,k}$, with $m = 0, \ldots, 3$.
(ii) $2(k+1)$ internal functions: two internal functions of maximum order in both directions, namely, $\varphi_1^{K,k-2,k-2}$ and $\varphi_2^{K,k-2,k-2}$, and $2k$ internal functions of maximum order in only one direction, $\varphi_1^{K,k-3,k-2}$ and $\varphi_2^{K,k-2,k-3}$.

Therefore, $4k$ edge vector functions, and $2(k-1)k$ internal vector functions remain, forming a subspace $V(\mathcal{T}_h)^* \subset V(\mathcal{T}_h)$ satisfying $div V(\mathcal{T}_h)^* = Y(\mathcal{T}_h)$. We denote by $\vec{\mathcal{Q}}_k^*\, \mathcal{Q}_{k-1}$, $k \geq 2$, this kind of approximation space for $Hdiv(\Omega)$ and $L^2(\Omega)$.

The errors on the approximations λ_{ih}, using approximations of type $\vec{\mathcal{Q}}_k^*\, \mathcal{Q}_{k-1}$, $k = 2, 3$ and 4 are illustrated in Fig. 5. For $k = 1$, the optimal rate of convergence was not obtained (would be 4), demonstrating the importance of using

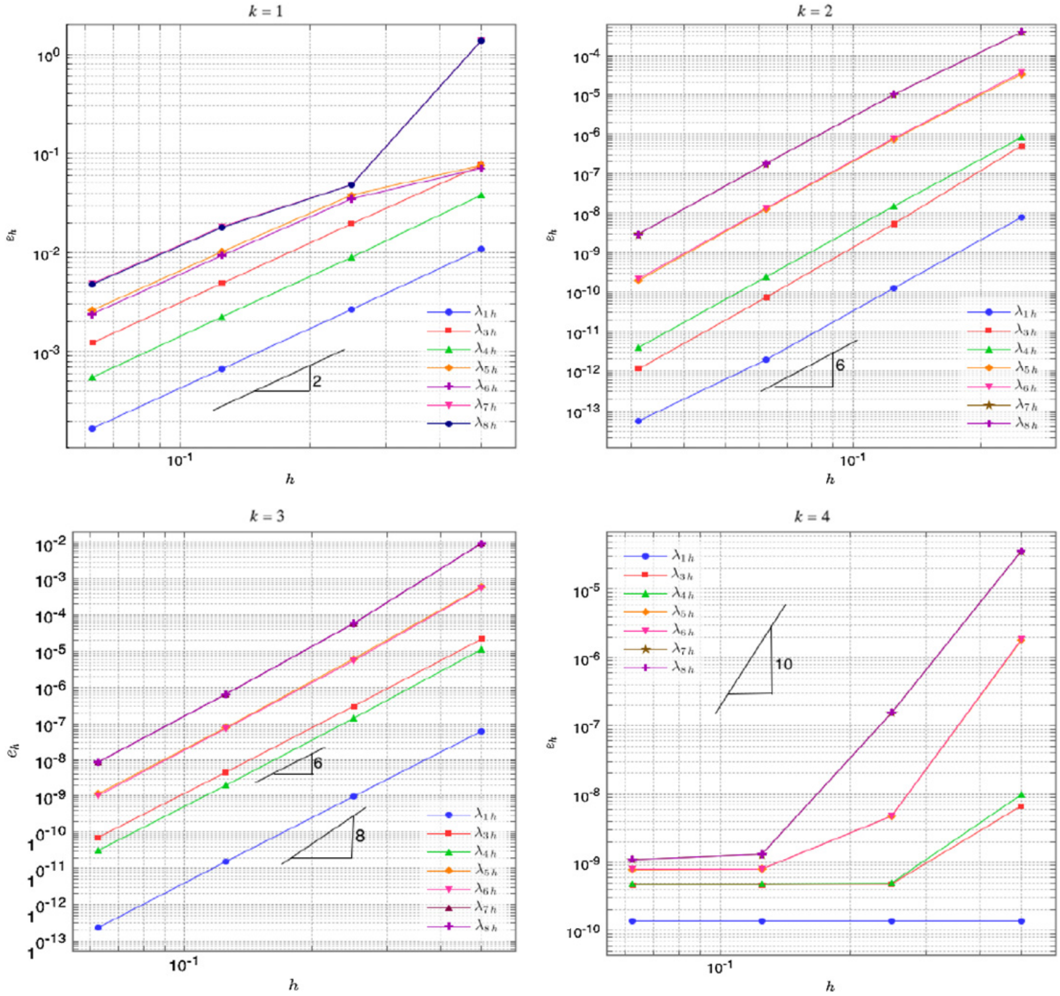

Fig. 5. Convergence of the approximations λ_{ih} using approximating spaces of type $\mathbf{Q}_1 Q_0$ (top left), and $\mathbf{Q}_k^* Q_{k-2}$, for $k = 2, 3$ and 4, on quadrilateral meshes.

compatible approximations spaces. Surprisingly, the rate of convergence for $k = 2$ is $O(h^6)$. This result is better than expected because the flux space $\vec{\mathcal{Q}}_2^*$ does not contain the complete family of polynomial functions of order 2. For $k = 3$ and $k = 4$ the rates of convergence correspond to the expected theoretical rate $O(h^{2(k)})$ because the polynomial space $\vec{\mathcal{Q}}_k{}^*$ only contains the complete polynomial functions of $k-1$. All results confirm that the $\vec{\mathcal{Q}}_k{}^* \, \mathcal{Q}_{k-1}$ approximation spaces are well balanced.

6. Conclusions

This paper is dedicated to a systematic procedure to construct consistent hierarchical finite element approximation spaces $V(\mathcal{T}_h)$ for $Hdiv\,(\Omega)$ or $Hcurl\,(\Omega)$. To obtain vector functions with continuous normal or tangential components, the approach is to use geometrical properties of the elements for the definition of a set of vectors, which are multiplied by consistent H^1 scalar bases functions already developed.

Having in mind applications of such spaces in mixed formulations, we present a numerical study for a Steklov eigenvalue problem, using approximation spaces $V(\mathcal{T}_h)$ and $Y(\mathcal{T}_h)$ for dual and primal variables $(\mathbf{u}, p) \in Hdiv\,(\Omega) \times L^2\,(\Omega)$, respectively. For approximations of type $\vec{\mathcal{P}}_k \, \mathcal{P}_{k-1}$, based on triangular meshes, for which the property $divV(\mathcal{T}_h) = Y(\mathcal{T}_h)$ holds, optimal $O(h^{2(k+1)})$ rates of convergence of the approximated eigenvalues are reached, suggesting that these pair of spaces are well balanced.

However, for quadrilateral elements, using approximation spaces of type $\vec{\mathcal{Q}}_k \, \mathcal{Q}_{k-1}$, the results are not consistent. In order to get subspaces $V(\mathcal{T}_h)^* \subset V(\mathcal{T}_h)$, such that $divV(\mathcal{T}_h)^* = Y(\mathcal{T}_h)$, some basic functions are removed. Using these reduced type of approximation $\vec{\mathcal{Q}}_k^* \, \mathcal{Q}_{k-1}$, convergence is recovered with very closer to optimal rates, for $k = 2, 3$ and 4.

Acknowledgment

The authors thankfully acknowledges financial support from the Brazilian National Agency of Petroleum, Natural Gas and Biofuels (ANP - PETROBRAS). P. Devloo and S. M. Gomes thankfully acknowledges financial support from CNPq - the Brazilian Research Council.

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