IMPLEMENTATION OF HIGH ORDER EDGE AND FACE ELEMENTS

CONTENTS

1. PRELIMINARY

1.1. **Simplicial Lattice.** A multi-index is an array of non-negative integers, denoted by an array of multi-index of length $n+1$:

$$\boldsymbol{\alpha} = (\alpha_0, \alpha_1, ..., \alpha_n).$$

The order of the multi-index is $|\boldsymbol{\alpha}| = \sum_{i=0}^{n} \alpha_i$ and factorial is $\boldsymbol{\alpha}! = \prod_{i=0}^{n}(\alpha_i!)$. The simplicial lattice is the set of all multi-indexs of length $n+1$ and order k and denoted by $\mathbb{T}_k^n$, i.e.,

\mid

$$\mathbb{T}_k^n = \{\boldsymbol{\alpha} | \boldsymbol{\alpha} \in \mathbb{N}^{n+1}, |\boldsymbol{\alpha}| = k\}$$

and follow [–], we call it ....

The elements in $\mathbb{T}_k^n$ can be linearly indexed by the dictionary ordering. For example, for an element $\boldsymbol{\alpha}$ in $\mathbb{T}_k^2$, the index is given by the mapping:

$$R_2(\boldsymbol{\alpha}) = \frac{(\alpha_1+\alpha_2)(\alpha_1+\alpha_2+1)}{2} + \alpha_2.$$

For an element $\mathbb{T}_k^3$ in $\boldsymbol{\alpha}$, the mapping is:

$$\begin{aligned} R_3(\boldsymbol{\alpha}) = & \frac{(\alpha_1+\alpha_2+\alpha_3)(\alpha_1+\alpha_2+\alpha_3+1)(\alpha_1+\alpha_2+\alpha_3+2)}{6} \\ & + \frac{(\alpha_2+\alpha_3)(\alpha_2+\alpha_3+1)}{2} + \alpha_3. \end{aligned}$$

Below we present the first four lattice nodes with $k = 4$:

$$\begin{array}{c} 0: \\ 1: \\ 2: \\ 3: \end{array} \begin{pmatrix} 4 & 0 & 0 \\ 3 & 1 & 0 \\ 3 & 0 & 1 \\ 2 & 2 & 0 \end{pmatrix}.$$

x

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1.2. **Interpolation points on a simplex.** Define a set of points on an n-simplex $T = \operatorname{conv}\{\boldsymbol{x}_0, \boldsymbol{x}_1 ..., \boldsymbol{x}_n\}$

$$\mathcal{X}_T = \{\boldsymbol{x}_{\boldsymbol{\alpha}} | \boldsymbol{x}_{\boldsymbol{\alpha}} = \frac{1}{k}\sum_{i=0}^{n} \alpha_i \boldsymbol{x}_i, \ \boldsymbol{\alpha} \in \mathbb{T}_k^n\}.$$

This set is called the set of interpolation points with degree k on T. Denote by $\mathring{\mathcal{X}}_T$ the interpolation points in the interior of T. The ordering of $\mathcal{X}_T$ is also given by $R(\boldsymbol{\alpha})$. Note that the ordering map $R(\boldsymbol{\alpha})$ is only a local ordering of the interpolation points on a n-simplex. In Section 5, we will discuss the global ordering of all interpolation points on the triangulation composed of simplexes.

indexing

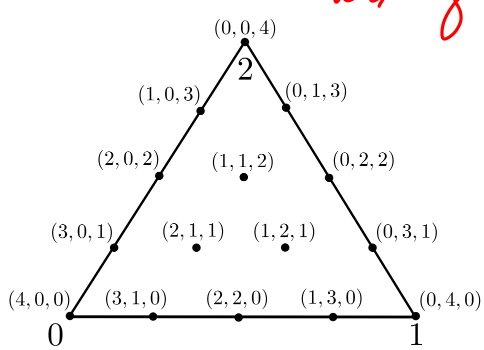


FIGURE 1. The interpolation points with $k = 4$ on the triangle and their corresponding multi-indexs.

sub

1.3. **Sub-simpl~~icial lattices.~~** Following [2], we let $\Delta(T)$ denote all the subsimplices of T, while $\Delta_\ell(T)$ denotes the set of subsimplices of dimension ℓ, for $0 \le \ell \le n$. The cardinality of $\Delta_\ell(T)$ is $\begin{pmatrix} n+1 \\ \ell+1 \end{pmatrix}$. Elements of $\Delta_0(T) = \{\mathrm{v}_0, \ldots, \mathrm{v}_n\}$ are $n+1$ vertices of T and $\Delta_n(T) = T$.

For a sub-simplex $f \in \Delta_\ell(T)$, we will overload the notation f for both the geometric simplex and the algebraic set of indices. Namely $f = \{f(0), \dots, f(\ell)\} \subseteq \{0, 1, \dots, n\}$ and

$$f = \mathrm{Convex}(\mathrm{v}_{f(0)}, \dots, \mathrm{v}_{f(\ell)}) \in \Delta_\ell(T)$$

is the ℓ-dimensional simplex spanned by the vertices $\mathrm{v}_{f(0)}, \dots, \mathrm{v}_{f(\ell)}$.

If $f \in \Delta_\ell(T)$, then $f^* \in \Delta_{n-\ell-1}(T)$ denotes the sub-simplex of T opposite to f. When treating f as a subset of $\{0, 1, \dots, n\}$, $f^* \subseteq \{0, 1, \dots, n\}$ so that $f \cup f^* = \{0, 1, \dots, n\}$, i.e., f^* is the complement of set f. Geometrically,

$$f^* = \mathrm{Convex}(\mathrm{v}_{f^*(1)}, \dots, \mathrm{v}_{f^*(n-\ell)}) \in \Delta_{n-\ell-1}(T)$$

$T_n^\ell(f)$, embedding.

is the $(n-\ell-1)$-dimensional simplex spanned by vertices not contained in f.

We reserve capital F for an $(n-1)$-dimensional face of T.

1.4. **Bubble polynomial.** The Bernstein representation of polynomial of degree k on a simplex T is

$$\mathbb{P}_k(T) := \{\lambda^{\boldsymbol{\alpha}} = \lambda_0^{\alpha_0}\lambda_1^{\alpha_1}\dots\lambda_n^{\alpha_n}, \alpha \in \mathbb{T}_k^n\}.$$

The bubble polynomial of f is a polynomial of degree $\ell + 1$:

$$b_f := \lambda_f = \lambda_{f(0)}\lambda_{f(1)}\dots\lambda_{f(\ell)} \in \mathbb{P}_{\ell+1}(f).$$

The following result can be found in [3]. For completeness, we include here.

Lemma 1.1. *Let $f \in \Delta_\ell(T)$. For $e \in \Delta_m(T)$ with $m \le \ell$ and $e \neq f$ when $m = \ell$, then $b_f|_e = 0$. And $b_f|_F = 0$ for faces $F \in \Delta_{n-1}(T)$, $f \not\subseteq F$.*

Proof. Take any $e \in \Delta_m(T)$ with $m \le \ell$ and $e \neq f$ when $m = \ell$. We claim $f \cap e^* \neq \varnothing$. Assume $f \cap e^* = \varnothing$. Then $\Delta_0(f^*) \cup \Delta_0(e) = \{0, 1, \dots, n\}$ and thus $f \subseteq e$ which contradicts with either $m < \ell$ or $e \neq f$.

As $f \cap e^* \neq \varnothing$, then b_f contains λ_i for some $i \in e^*$ and consequently $b_f|_e = 0$.

Similarly $f \not\subseteq F$ implies b_f contains λ_i for some $i \notin F$ and consequently $b_f|_F = 0$. □

1.5. **Triangulation.** Let Ω be a polyhedral domain in R^n, $n \ge 1$. A geometric triangulation $\mathcal{T}$ of Ω is a set of n-simplices such that

$$\cup_{T\in\mathcal{T}} T = \Omega, \quad \mathring{T}_0 \cap \mathring{T}_1 = \emptyset,\ \forall\, T_0, T_1 \in \mathcal{T}, T_0 \neq T_1.$$

conforming

$\oplus\, T_n^\ell(f)$

The interpolation points on $\mathcal{T}$ is $\mathcal{X} = \cup_{T\in\mathcal{T}} \mathcal{X}_T$

2. Geometric Decompositions of Lagrange elements

$\mathcal{X} = \mathcal{X}(f)$

2.1. **Geometric decomposition.** For the polynomial space $\mathbb{P}_k(T)$ with $k \ge 1$ on an n-dimensional simplex T, we have the following geometric decomposition of Lagrange element [2, (2.6)]. ~~The integral at a vertex is understood as the function value at that vertex and $\mathbb{P}_k(\mathrm{v}) = \mathbb{R}$. A proof of the unisolvence can be found in [3].~~

duplication.

~~**Theorem 2.1.**~~

$$\mathbb{P}_k(T) = \bigoplus_{\ell=0}^n \bigoplus_{f\in\Delta_\ell(T)} b_f \mathbb{P}_{k-(\ell+1)}(f). \tag{1}$$

local to global index mapping.

~~*The function $u \in \mathbb{P}_k(T)$ is uniquely determined by DoFs*~~

$$\cancel{\int_f u\,p\,\mathrm{d}s \quad \forall\, p \in \mathbb{P}_{k-(\ell+1)}(f), f \in \Delta_\ell(T), \ell = 0, 1, \dots, n.} \tag{2}$$

Introduce a bubble polynomial space on each sub-simplex as

$$\mathbb{B}_k(f) := b_f \mathbb{P}_{k-(\ell+1)}(f), \quad f \in \Delta_\ell(T), 1 \le \ell \le n.$$

It is called a bubble space as

$$\operatorname{tr}_f^{\mathrm{grad}} u := u|_{\partial f} = 0, \quad u \in \mathbb{B}_k(f).$$

Then ~~the decomposition (1) can be written as~~ we have another

$$\mathbb{P}_k(T) = \mathbb{P}_1(T) \oplus \bigoplus_{\ell=1}^{n} \bigoplus_{f\in\Delta_\ell(T)} \mathbb{B}_k(f). \tag{3}$$

That is a degree k polynomial can be decomposed into a linear polynomial plus bubbles on edges, faces, and all sub-simplexes.

The geometric decomposition can be naturally extended to vector Lagrange elements. For $k \ge 1$, define

$$\mathbb{B}_k^n(f) := b_f \mathbb{P}_{k-(\ell+1)}(f) \otimes \mathbb{R}^n.$$

Clearly we have

$$\mathbb{P}_k^n(T) = \mathbb{P}_1^n(T) \oplus \bigoplus_{\ell=1}^{n} \bigoplus_{f\in\Delta_\ell(T)} \mathbb{B}_k^n(f). \tag{4}$$

For an $f \in \Delta_\ell(T)$, we choose a $t-n$ coordinate $\{\boldsymbol{t}_i^f, \boldsymbol{n}_j^f, i = 1, \dots, l, j = 1, \dots n-l\}$ to write the decomposition into tangential and normal components. Define

$$\mathscr{T}^f := \cancel{\left\{\sum_{i=1}^{\ell} c_i \boldsymbol{t}_i^f : c_i \in \mathbb{R}\right\}}, \quad \mathscr{N}^f := \cancel{\left\{\sum_{i=1}^{n-\ell} c_i \boldsymbol{n}_i^f : c_i \in \mathbb{R}\right\}}. \qquad \text{span}$$

Thus $\mathscr{T}^f$ is the tangent plane of f and $\mathscr{N}^f$ is the normal plane. When $\ell = 0$, i.e., for vertices, no tangential component and for $\ell = n$, no normal component. Restricted to an ℓ-dim sub-simplex $f \in \Delta_\ell(T)$, define

$$\mathbb{B}_k^\ell(f) := \mathbb{B}_k(f) \otimes \mathscr{T}^f,$$

which is a space of ℓ-dimensional vectors on the tangential space with vanishing trace $\operatorname{tr}^{\mathrm{grad}}$ ~~(no definition)~~ on ∂f. We then write the decomposition (4) as

$$\mathbb{P}_k^n(T) = \mathbb{P}_1^n(T) \oplus \bigoplus_{\ell=1}^{n} \bigoplus_{f\in\Delta_\ell(T)} \left[\mathbb{B}_k^\ell(f) \oplus (\mathbb{B}_k(f) \otimes \mathscr{N}^f)\right]. \tag{5}$$

2.2. **Lagrange interpolation basis functions.** ~~First~~ we give the formulas of Lagrange interpolation basis functions of arbitrary order on n-simplex.

Lemma 2.2 (Lagrange interpolation basis functions)**.** *The basis functions of k-th order Lagrange finite element space on T can be explicitly written as:*

$$\phi_{\boldsymbol{\alpha}}(\boldsymbol{x}) = \frac{1}{\boldsymbol{\alpha}!} \prod_{i=0}^{n} \prod_{j=0}^{\alpha_i - 1} (k\lambda_i(\boldsymbol{x}) - j), \quad \boldsymbol{\alpha} \in \mathbb{T}_k^n,$$

with the DoFs defined as the function value at the interpolation points:

$$u(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathcal{X}_T.$$

It is straightforward to verify the duality of the basis and DoFs

$$\phi_{\boldsymbol{\alpha}}(\boldsymbol{x}_\beta) = \delta_{\boldsymbol{\alpha}}^{\boldsymbol{\beta}} = \begin{cases} 1 & \text{if } \boldsymbol{\alpha} = \boldsymbol{\beta} \\ 0 & \text{otherwise} \end{cases}.$$

Note that since the basis functions are uniquely determined by the interpolation points, their numbering rules can naturally adopt the numbering rules of interpolation points.

Given a triangulation $\mathcal{T}_h$ and degree k, define the set of interpolation points on $\mathcal{T}_h$

$$\mathcal{X} = \bigcup_{T\in\mathcal{T}_h} \mathcal{X}_T.$$

Definition 2.3 (DoFs of Lagrange finite element on $\mathcal{T}_h$)**.** *For the Lagrange finite element space $\mathscr{L}(\mathcal{T}_h)$, define the DoFs in $\mathscr{L}(\mathcal{T}_h)$ as: for any $\boldsymbol{x} \in \mathcal{X}$, the value of u at $\boldsymbol{x}$*

$$u(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathcal{X}.$$

Theorem. (⟶) Lagrange

3. Geometric Decompositions of Face elements

Pf. Conform

Define $H(\operatorname{div}, \Omega) := \{\boldsymbol{v} \in L^2(\Omega; \mathbb{R}^n) : \operatorname{div} \boldsymbol{v} \in L^2(\Omega)\}$. The trace operator for $H(\operatorname{div}, K)$ space

$$\operatorname{tr}^{\operatorname{div}} \boldsymbol{v} = \boldsymbol{n} \cdot \boldsymbol{v}|_{\partial K}.$$

Given a triangulation $\mathcal{T}_h$ and a piecewise polynomial function $\boldsymbol{u}$, it is well known that for $\boldsymbol{u} \in H(\operatorname{div}, \Omega)$ if and only if $\operatorname{tr}^{\operatorname{div}} \boldsymbol{u}|_F = \boldsymbol{n}_F \cdot \boldsymbol{u}$ is continuous for all faces $F \in \mathcal{F}_h$. A $H(\operatorname{div})$-conforming finite element is thus also called a face element.

3.1. **Geometric decomposition.** Define the polynomial div bubble space

$$\mathbb{B}_k(\operatorname{div}; T) = \ker(\operatorname{tr}^{\operatorname{div}}) \cap \mathbb{P}_k(T; \mathbb{R}^n).$$

Recall that $B^\ell(f)$ tangential and $f|_F = 0$ . . .

It is easy to verify that $\mathbb{B}_k^\ell(f) \subset \mathbb{B}_k(\operatorname{div}; T)$ for $k \geq 2, \dim f \geq 1$. In [3], we have proved that the div- bubble polynomial space has the following decomposition.

Lemma 3.1. *For $k \geq 2$,*

$$\mathbb{B}_k(\operatorname{div}; T) = \bigoplus_{\ell=1}^{n} \bigoplus_{f\in\Delta_\ell(T)} \mathbb{B}_k^\ell(f).$$

Notice that as no tangential plane on vertices, there is no div-bubble associated to vertices and thus the degree of a div-bubble polynomial is greater than or equal to 2. Next we present two geometric decompositions of a div-element.

Lemma 3.2. *For $k \geq 1$, we have*

$$\mathbb{P}_k^n(T) = \mathbb{P}_1^n(T) \oplus \bigoplus_{\ell=1}^{n-1} \bigoplus_{f\in\Delta_\ell(T)} (\mathbb{B}_k(f) \otimes \mathscr{N}^f) \oplus \mathbb{B}_k(\operatorname{div}; T), \tag{6}$$

$$\mathbb{P}_k^n(T) = \bigoplus_{F\in\Delta_{n-1}(T)} \left(\mathbb{P}_1(F) \oplus \bigoplus_{\ell=1}^{n-1} \bigoplus_{f\in\Delta_\ell(F)} \mathbb{B}_k(f)\boldsymbol{n}_F\right) \oplus \mathbb{B}_k(\operatorname{div}; T). \tag{7}$$

Proof. The first decomposition (6) is a rearrangement of (4) by merging the tangential component $\mathbb{B}_k^\ell(f)$ into the bubble space. Next we prove the decomposition (7).

For an ℓ-dimensional sub-simplex $f \in \Delta_\ell(T)$, the $n-\ell$ face normal vectors $\{\boldsymbol{n}_F : F \in \Delta_{n-1}(T), f \subseteq F\}$ form a basis of $\mathscr{N}^f$. Therefore we have

$$\mathbb{B}_k(f) \otimes \mathscr{N}^f = \bigoplus_{F\in\Delta_{n-1}(T), f\subseteq F} \mathbb{B}_k(f)\boldsymbol{n}_F.$$

~~Then we have~~ and (6) becomes

$$\mathbb{P}_k^n(T) = \mathbb{P}_1^n(T) \oplus \bigoplus_{\ell=1}^{n-1} \bigoplus_{f\in\Delta_\ell(T)} \bigoplus_{F\in\Delta_{n-1}(T), f\subseteq F} \mathbb{B}_k(f)\boldsymbol{n}_F \oplus \mathbb{B}_k(\operatorname{div}; T), \tag{8}$$

At vertices, we have

$$\begin{aligned}
\mathbb{P}_1^n(T) &= \bigoplus_{\texttt{v}\in\Delta_0(T)} \operatorname{span}\{\lambda_{\texttt{v}}\} \otimes \mathscr{N}^{\texttt{v}} \\
&= \bigoplus_{\texttt{v}\in\Delta_0(T)} \bigoplus_{F\in\Delta_{n-1}(T), \texttt{v}\in\Delta_0(F)} \operatorname{span}\{\lambda_{\texttt{v}} \boldsymbol{n}_F\} \\
&= \bigoplus_{F\in\Delta_{n-1}(T)} \bigoplus_{\texttt{v}\in\Delta_0(F)} \operatorname{span}\{\lambda_{\texttt{v}} \boldsymbol{n}_F\} \\
&= \bigoplus_{F\in\Delta_{n-1}(T)} \mathbb{P}_1(F)\boldsymbol{n}_F.
\end{aligned}$$

The decomposition (7) holds by swapping the ordering of f and F in the direct sum. $\square$

In decomposition (6), we single out $\mathbb{P}_1^n(T)$ to emphasize a div-conforming element can be obtained by adding div-bubble and normal component on sub-simplexes starting from edges. In (7), we group all normal components facewisely which leads to the classical BDM element.

Lemma 3.3 (BDM element). *The following* DoFs

$$\int_f (\boldsymbol{v}\cdot\boldsymbol{n}_F)\, p \,\mathrm{d}s, \quad f\in\Delta_\ell(T), F\in\Delta_{n-1}(T), f\subseteq F, \tag{9}$$
$$p\in\mathbb{P}_{k-(\ell+1)}(f), \ell=0,\ldots,n-1,$$

$$\int_f (\boldsymbol{v}\cdot\boldsymbol{t}_i^f)\, p \,\mathrm{d}s, \quad f\in\Delta_\ell(T), i=1,\ldots,\ell, \tag{10}$$
$$p\in\mathbb{P}_{k-(\ell+1)}(f), \ell=1,\ldots,n,$$

define a $H(\operatorname{div})$*-conforming space* $V_h=\{\boldsymbol{v}_h\in H(\operatorname{div};\Omega):\boldsymbol{v}_h|_T\in\mathbb{P}_k^n(T)\ \ \forall\, T\in\mathcal{T}_h\}$.

For BDM element, by the decomposition of Lagrange element, cf. Theorem 2.1, the face DoFs for the trace $\boldsymbol{v}\cdot\boldsymbol{n}_F$ can be merged into

$$\int_F (\boldsymbol{v}\cdot\boldsymbol{n}_F)\, p \,\mathrm{d}s, \quad p\in\mathbb{P}_k(F), \tag{11}$$

and the div bubble DoFs can be merged into one

$$\int_T \boldsymbol{v}\cdot\boldsymbol{p}\,\mathrm{d}x \quad \boldsymbol{p}\in\mathbb{B}_k(\operatorname{div},T).$$

But DoF (9)-(10) is more friendly for the implementation. One can start from the vector Lagrange element and split into tangential and normal components.

Based on the decomposition (6), we may impose the continuity at vertex to determine $\mathbb{P}_1^n(T)$ which is known as Sternberg's element [6].

Lemma 3.4 (Stenberg element). *The following DoFs*

$$\boldsymbol{v}(v), \quad v\in\Delta_0(\mathcal{T}_h),$$
$$\int_f (\boldsymbol{v}\cdot\boldsymbol{n}_F)\, p \,\mathrm{d}s, \quad f\in\Delta_\ell(T), F\in\Delta_{n-1}(T), f\subseteq F,$$
$$\int_f (\boldsymbol{v}\cdot\boldsymbol{t}_i^f)\, p \,\mathrm{d}s, \quad f\in\Delta_\ell(T), i=1,\ldots,\ell,$$
$$p\in\mathbb{P}_{k-(\ell+1)}(f), \ell=1,\ldots,n,$$

define a $H(\operatorname{div})$*-conforming finite element space being continuous at vertex in* $\Delta_0(\mathcal{T}_h)$.

In general, for each sub-simplex, we can choose either a local or global normal basis and obtain variants of Sternberg element. For each $f\in\Delta_\ell(\mathcal{T}_h)$, we choose a $t-n$ coordinate $\{\boldsymbol{t}_1^f,\ldots,\boldsymbol{t}_\ell^f,\boldsymbol{n}_1^f,\ldots,\boldsymbol{n}_{n-\ell}^f\}$. If a basis vector $\boldsymbol{t}_i^f$ or $\boldsymbol{n}_i^f$ depends only on f not on element T containing f, we call it *global* and otherwise *local*. For local $t-n$ basis, the corresponding DoFs are different for different T containing f. For a global normal basis, the corresponding DoFs depends only on f and thus the vector is continuous on the normal plane of f.

!

Lemma 3.5. *Let $-1 \le k \le n-1$. For each $f \in \Delta_\ell(\mathcal{T}_h)$ with $\ell \le k$, we choose $n-\ell$ normal vectors $\{\boldsymbol{n}_1^f, \ldots, \boldsymbol{n}_{n-\ell}^f\}$ depending only on f. Then the* DoFs

$$\int_f \boldsymbol{v} \cdot \boldsymbol{n}_i^f \, p \, \mathrm{d}s, \quad p \in \mathbb{P}_{k-(\ell+1)}(f), f \in \Delta_\ell(\mathcal{T}_h),$$
$$i = 1, \ldots, n-\ell, \ \ell = 0, \ldots, k,$$
$$\int_f (\boldsymbol{v} \cdot \boldsymbol{n}_F) \, p \, \mathrm{d}s, \quad p \in \mathbb{P}_{k-(\ell+1)}(f), f \in \Delta_\ell(T),$$
$$F \in \Delta_{n-1}(T), f \subseteq F, \ \ell = k+1, \ldots, n-1,$$
$$\int_T \boldsymbol{v} \cdot \boldsymbol{p} \, \mathrm{d}x, \quad \boldsymbol{p} \in \mathbb{B}_k(\mathrm{div}, T), T \in \mathcal{T}_h,$$

will determine a space $V^k \subset H(\mathrm{div}, \Omega)$ which is continuous on the normal plane of f for all $f \in \Delta_\ell(\mathcal{T}_h)$ with $\ell \le k$.

When $k = 0$, it is the original Stenberg element [6], i.e., only continuous at vertices. When $k = n-2$, it is the generalization of Christiansen-Hu-Hu element [4]. We allow $k = -1$ to include the BDM element. We refer to [3] for the proof on the unisolvence. of these elements

In the following we will present construct . . .

3.2. **BDM element on triangular meshes.** Let T be a triangle, for an interpolation point $\boldsymbol{x} \in \mathcal{X}_T$, we shall choose a frame $\{\boldsymbol{e}_{\boldsymbol{x}}^0, \boldsymbol{e}_{\boldsymbol{x}}^1\}$ at $\boldsymbol{x}$ and its dual frame $\{\hat{\boldsymbol{e}}_{\boldsymbol{x}}^0, \hat{\boldsymbol{e}}_{\boldsymbol{x}}^1\}$, i.e.

$$(\boldsymbol{e}_{\boldsymbol{x}}^i, \hat{\boldsymbol{e}}_{\boldsymbol{x}}^j) = \delta_i^j$$

as follows. All normal and tangential vectors are unit vectors.

(1) If $\boldsymbol{x} \in \Delta_0(T)$, assuming the two adjacent edges are e_0 and e_1, then

$$\boldsymbol{e}_{\boldsymbol{x}}^0 = \boldsymbol{n}_{e_0}, \quad \boldsymbol{e}_{\boldsymbol{x}}^1 = \boldsymbol{n}_{e_1}, \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^0 = \frac{\boldsymbol{t}_{e_1}}{\boldsymbol{t}_{e_1} \cdot \boldsymbol{n}_{e_0}}, \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^1 = \frac{\boldsymbol{t}_{e_0}}{\boldsymbol{t}_{e_0} \cdot \boldsymbol{n}_{e_1}}.$$

(2) If $\boldsymbol{x} \in \mathring{\mathcal{X}}_e, e \in \Delta_1(T)$, then

$$\boldsymbol{e}_{\boldsymbol{x}}^0 = \boldsymbol{n}_e, \quad \boldsymbol{e}_{\boldsymbol{x}}^1 = \boldsymbol{t}_e, \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^0 = \boldsymbol{n}_e, \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^1 = \boldsymbol{t}_e.$$

local or global?

(3) If $\boldsymbol{x} \in \mathring{\mathcal{X}}_T$, then

$$\boldsymbol{e}_{\boldsymbol{x}}^0 = (1,0), \quad \boldsymbol{e}_{\boldsymbol{x}}^1 = (0,1), \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^0 = (1,0), \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^1 = (0,1).$$

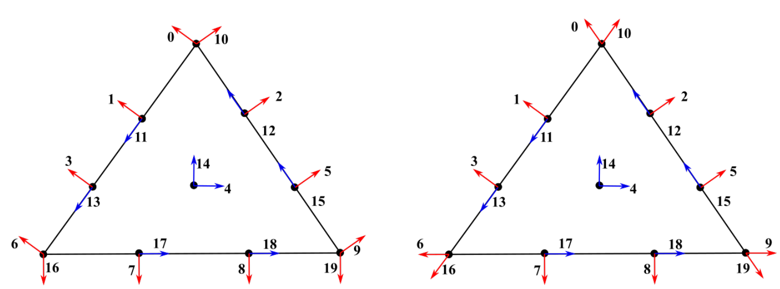


FIGURE 2. The left figure shows $\{\boldsymbol{e}_0, \boldsymbol{e}_1\}$ at each interpolation point, the right figure shows $\{\hat{\boldsymbol{e}}_0, \hat{\boldsymbol{e}}_1\}$ at each interpolation point

Lemma 3.6 (BDM element on triangle)**.** *A polynomial function* $\boldsymbol{u} \in \mathbb{P}_k^2(T)$ *can be uniquely determined by the DoFs:*

$$\boldsymbol{u}(\boldsymbol{x}_{\boldsymbol{\alpha}}) \cdot \boldsymbol{e}^i_{\boldsymbol{x}}, \quad \boldsymbol{x}_{\boldsymbol{\alpha}} \in \mathcal{X}_T, i = 0, 1.$$

The basis function of k*th order BDM element space on* T *can be explicitly written as:*

$$\boldsymbol{\phi}^i_{\boldsymbol{\alpha}}(\boldsymbol{x}) = \phi_{\boldsymbol{\alpha}}(\boldsymbol{x})\hat{\boldsymbol{e}}^i_{\boldsymbol{x}_{\boldsymbol{\alpha}}}, \quad \boldsymbol{\alpha} \in \mathbb{T}^2_k, i = 0, 1$$

Proof. We have the duality

$$\boldsymbol{\phi}^i_{\boldsymbol{\alpha}}(\boldsymbol{x}_\beta) \cdot \boldsymbol{e}^j_{\boldsymbol{x}_\beta} = \delta^j_i \delta^{\boldsymbol{\beta}}_{\boldsymbol{\alpha}}.$$

$\square$

Please correct the lemma below.

Lemma 3.7 (BDM space on a triangulation)**.** *The following* DoFs

$$\boldsymbol{u}(\boldsymbol{x}) \cdot \underline{\boldsymbol{n}_e}, \quad \boldsymbol{x} \in \mathcal{X}_e, e \in \Delta_1(\mathcal{T}_h) \tag{12}$$

global.

$$\boldsymbol{u}(\boldsymbol{x})|_T \cdot \boldsymbol{t}_e, \quad \boldsymbol{x} \in \mathring{\mathcal{X}}_e, e \in \Delta_1(T), T \in \Delta_2(\mathcal{T}_h) \tag{13}$$

$$\boldsymbol{u}(\boldsymbol{x}) \cdot (1, 0), \quad \boldsymbol{u}(\boldsymbol{x}) \cdot (0, 1), \quad \boldsymbol{x} \in \mathring{\mathcal{X}}_T, T \in \Delta_2(\mathcal{T}_h) \tag{14}$$

define a $H(\operatorname{div})$*-conforming space* $V_h = \{\boldsymbol{v}_h \in H(\operatorname{div}; \Omega) : \boldsymbol{v}_h|_T \in \mathbb{P}^2_k(T) \ \ \forall\, T \in \mathcal{T}_h\}$.

We call (12) as DoF on edge, (13) and (14) as DoF in cell.

3.3. **BDM element on tetrahedron mesh** es **.** Let T be a tetrahedron, for an interpolation point $\boldsymbol{x} \in \mathcal{X}_T$, we shall choose a frame $\{\boldsymbol{e}^0_{\boldsymbol{x}}, \boldsymbol{e}^1_{\boldsymbol{x}}, \boldsymbol{e}^2_{\boldsymbol{x}}\}$ at $\boldsymbol{x}$ and its dual frame $\{\hat{\boldsymbol{e}}^0_{\boldsymbol{x}}, \hat{\boldsymbol{e}}^1_{\boldsymbol{x}}, \hat{\boldsymbol{e}}^2_{\boldsymbol{x}}\}$ as follows:

(1) For a vertex $\boldsymbol{x} \in \Delta_0(T)$, let adjacent edges of $\boldsymbol{x}$ be e_0, e_1, e_2, and the adjacent faces be f_0, f_1, f_2, satisfying $f_i \cap e_i = \boldsymbol{x}$, then

$$\boldsymbol{e}^0_{\boldsymbol{x}} = \boldsymbol{n}_{f_0}, \quad \boldsymbol{e}^1_{\boldsymbol{x}} = \boldsymbol{n}_{f_1}, \quad \boldsymbol{e}^2_{\boldsymbol{x}} = \boldsymbol{n}_{f_2}.$$

$$\hat{\boldsymbol{e}}^0_{\boldsymbol{x}} = \frac{\boldsymbol{t}_{e_0}}{\boldsymbol{t}_{e_0} \cdot \boldsymbol{n}_{f_0}}, \quad \hat{\boldsymbol{e}}^1_{\boldsymbol{x}} = \frac{\boldsymbol{t}_{e_1}}{\boldsymbol{t}_{e_1} \cdot \boldsymbol{n}_{f_1}}, \quad \hat{\boldsymbol{e}}^2_{\boldsymbol{x}} = \frac{\boldsymbol{t}_{e_2}}{\boldsymbol{t}_{e_2} \cdot \boldsymbol{n}_{f_2}}.$$

(2) If $\boldsymbol{x} \in \mathring{\mathcal{X}}_e, e \in \Delta_1(T)$ with adjacent faces f_0, f_1, then

$$\boldsymbol{e}^0_{\boldsymbol{x}} = \boldsymbol{n}_{f_0}, \quad \boldsymbol{e}^1_{\boldsymbol{x}} = \boldsymbol{n}_{f_1}, \quad \boldsymbol{e}^2_{\boldsymbol{x}} = \boldsymbol{t}_e,$$

$$\hat{\boldsymbol{e}}^0_{\boldsymbol{x}} = \frac{\boldsymbol{n}_{f_1} \times \boldsymbol{t}_e}{\boldsymbol{n}_{f_0} \cdot (\boldsymbol{n}_{f_1} \times \boldsymbol{t}_e)}, \quad \hat{\boldsymbol{e}}^1_{\boldsymbol{x}} = \frac{\boldsymbol{n}_{f_0} \times \boldsymbol{t}_e}{\boldsymbol{n}_{f_1} \cdot (\boldsymbol{n}_{f_0} \times \boldsymbol{t}_e)}, \quad \hat{\boldsymbol{e}}^2_{\boldsymbol{x}} = \boldsymbol{t}_e$$

(3) If $\boldsymbol{x} \in \mathring{\mathcal{X}}_f$, with $f \in \Delta_2(T), e \in \partial f$, then

$$\boldsymbol{e}^0_{\boldsymbol{x}} = \boldsymbol{n}_f, \quad \boldsymbol{e}^1_{\boldsymbol{x}} = \boldsymbol{t}_e \times \boldsymbol{n}_f, \quad \boldsymbol{e}^2_{\boldsymbol{x}} = \boldsymbol{t}_e.$$

$$\hat{\boldsymbol{e}}^0_{\boldsymbol{x}} = \boldsymbol{n}_f, \quad \hat{\boldsymbol{e}}^1_{\boldsymbol{x}} = \boldsymbol{t}_e \times \boldsymbol{n}_f, \quad \hat{\boldsymbol{e}}^2_{\boldsymbol{x}} = \boldsymbol{t}_e.$$

(4) If $\boldsymbol{x} \in \mathring{\mathcal{X}}_T$, then

$$\boldsymbol{e}^0_{\boldsymbol{x}} = (1, 0, 0), \quad \boldsymbol{e}^1_{\boldsymbol{x}} = (0, 1, 0), \quad \boldsymbol{e}^2_{\boldsymbol{x}} = (0, 0, 1).$$

$$\hat{\boldsymbol{e}}^0_{\boldsymbol{x}} = (1, 0, 0), \quad \hat{\boldsymbol{e}}^1_{\boldsymbol{x}} = (0, 1, 0), \quad \hat{\boldsymbol{e}}^2_{\boldsymbol{x}} = (0, 0, 1).$$

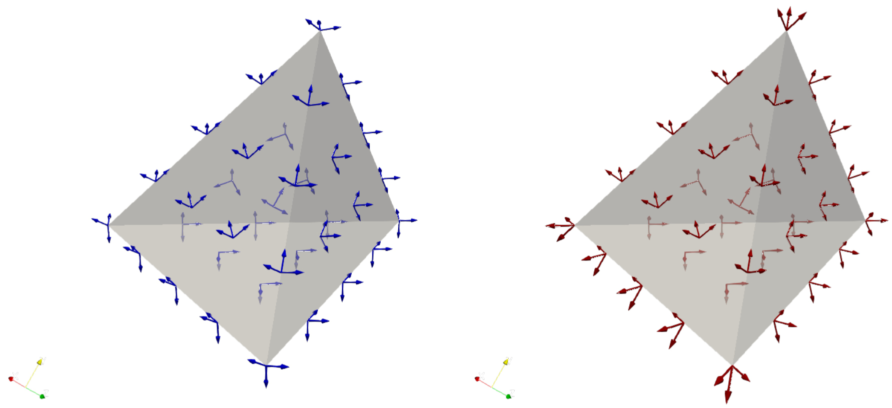


FIGURE 3. The left figure shows $\{\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2\}$ at each interpolation point, the right figure shows $\{\hat{\boldsymbol{e}}_0, \hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2\}$ at each interpolation point.

Lemma 3.8 (BDM element on a tetrahedron)**.** *A polynomial function $\boldsymbol{u} \in \mathbb{P}_k^3(T)$ can be uniquely determined by the DoFs:*

$$\boldsymbol{u}(\boldsymbol{x}_{\boldsymbol{\alpha}}) \cdot \boldsymbol{e}^i_{\boldsymbol{x}}, \quad \boldsymbol{x}_{\boldsymbol{\alpha}} \in \mathcal{X}_T, i = 0, 1, 2.$$

The basis function of kth order BDM element space on T can be explicitly written as:

$$\boldsymbol{\phi}^i_{\boldsymbol{\alpha}}(\boldsymbol{x}) = \phi_{\boldsymbol{\alpha}}(\boldsymbol{x})\hat{\boldsymbol{e}}^i_{\boldsymbol{x}_{\boldsymbol{\alpha}}}, \quad \boldsymbol{\alpha} \in \mathbb{T}^3_k, i = 0, 1, 2$$

Lemma 3.9 (BDM element on tetrahedron mesh)**.** *Choice of n_f, t_f etc For $f \in \Delta_2(\mathcal{T}_h)$, let t_f^0 is the normal tangent of the first edge of f, $t_f^1 = t_f^0 \times n_f$, the following* DoFs

n_F is global and t is local

$$\boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{n}_f, \quad \boldsymbol{x} \in \mathcal{X}_f, f \in \Delta_2(\mathcal{T}), \tag{15}$$

$$\boldsymbol{u}(\boldsymbol{x})|_T \cdot \boldsymbol{t}_f^0, \quad \boldsymbol{u}(\boldsymbol{x})|_T \cdot \boldsymbol{t}_f^1, \quad \boldsymbol{x} \in \mathring{\mathcal{X}}_F, F \in \Delta_2(T), T \in \Delta_3(\mathcal{T}_h) \tag{16}$$

$$\boldsymbol{u}(\boldsymbol{x})|_T \cdot \boldsymbol{t}_e, \quad \boldsymbol{x} \in \mathring{\mathcal{X}}_e, \ e \in \Delta_1(T), T \in \Delta_3(\mathcal{T}_h) \tag{17}$$

$$\boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{e}^i_{\boldsymbol{x}} \quad \boldsymbol{x} \in \mathring{\mathcal{X}}_T, i = 0, 1, 2, \ T \in \Delta_3(\mathcal{T}_h) \tag{18}$$

define a $H(\mathrm{div})$-conforming space $V_h = \{\boldsymbol{v}_h \in H(\mathrm{div}; \Omega) : \boldsymbol{v}_h|_T \in \mathbb{P}_k^3(T), \ \forall\, T \in \mathcal{T}_h\}$.

Proof. Verify it is $H(\mathrm{div})$-conforming. □

We call (15) as DoF on face, (16), (17) and (18) as DoF in cell. Correct the equation number

4. GEOMETRIC DECOMPOSITIONS OF EDGE ELEMENTS

In this section we shall present geometric decompositions of $H(\mathrm{curl}\,)$-conforming finite element space in arbitrary dimension.

4.1. **Differential operator** curl **and its trace.** Denote by $\mathbb{S}$ and $\mathbb{K}$ the subspace of symmetric matrices and skew-symmetric matrices of $\mathbb{R}^{n\times n}$, respectively. For a smooth vector function $\boldsymbol{v}$, define

$$\mathrm{curl}\, \boldsymbol{v} := 2\,\mathrm{skw}(\mathrm{grad}\, \boldsymbol{v}) = \mathrm{grad}\, \boldsymbol{v} - (\mathrm{grad}\, \boldsymbol{v})^{\intercal},$$

which is a skew-symmetric matrix function. Discuss 2D and 3D cases. In 3D, define mskw and identity it as a vector and in 2D, it is a scalar.

Given a face $F \in \Delta_{n-1}(T)$, define the trace operator of curl as

$$\operatorname{tr}_F^{\operatorname{curl}} \boldsymbol{v} = 2\operatorname{skw}(\boldsymbol{v}\boldsymbol{n}_F^{\intercal})|_F = (\boldsymbol{v}\boldsymbol{n}_F^{\intercal} - \boldsymbol{n}_F\boldsymbol{v}^{\intercal})|_F,$$

and the tangential part of $\boldsymbol{v}$ as

$$\Pi_F \boldsymbol{v} = \boldsymbol{v}|_F - (\boldsymbol{v}|_F \cdot \boldsymbol{n}_F)\boldsymbol{n}_F = \sum_{i=1}^{n-1} (\boldsymbol{v}|_F \cdot \boldsymbol{t}_{F,i})\boldsymbol{t}_{F,i}.$$

As we treat $\operatorname{curl} \boldsymbol{v}$ as a matrix, so is the trace $\operatorname{tr}_F^{\operatorname{curl}} \boldsymbol{v}$. The tangential component of $\boldsymbol{v}$ is a vector. Their relation is given in the following lemma.

Lemma 4.1. *For face $F \in \Delta_{n-1}(T)$, we have*

$$\operatorname{tr}_F^{\operatorname{curl}} \boldsymbol{v} = 2\operatorname{skw}\left((\Pi_F\boldsymbol{v})\boldsymbol{n}_F^{\intercal}\right), \quad \Pi_F\boldsymbol{v} = (\operatorname{tr}_F^{\operatorname{curl}} \boldsymbol{v})\boldsymbol{n}_F. \tag{19}$$

the decomposition

Proof. By $\boldsymbol{v}|_F = \Pi_F \boldsymbol{v} + (\boldsymbol{v}|_F \cdot \boldsymbol{n}_F)\boldsymbol{n}_F$,

$$\operatorname{tr}_F^{\operatorname{curl}} \boldsymbol{v} = 2\operatorname{skw}\left((\Pi_F\boldsymbol{v})\boldsymbol{n}_F^{\intercal} + (\boldsymbol{v}|_F \cdot \boldsymbol{n}_F)\boldsymbol{n}_F\boldsymbol{n}_F^{\intercal}\right) = 2\operatorname{skw}\left((\Pi_F\boldsymbol{v})\boldsymbol{n}_F^{\intercal}\right),$$

which implies the first identify. Then by $\boldsymbol{n}_F^{\intercal}\boldsymbol{n}_F = 1$ and $(\Pi_F\boldsymbol{v})^{\intercal}\boldsymbol{n}_F = 0$,

$$(\operatorname{tr}_F^{\operatorname{curl}} \boldsymbol{v})\boldsymbol{n}_F = \left((\Pi_F\boldsymbol{v})\boldsymbol{n}_F^{\intercal} - \boldsymbol{n}_F(\Pi_F\boldsymbol{v})^{\intercal}\right)\boldsymbol{n}_F = \Pi_F\boldsymbol{v},$$

i.e. the second identify holds. $\square$

Thanks to (19), the tangential part $\Pi_F\boldsymbol{v}$ and the tangential trace $(\operatorname{tr}_F^{\operatorname{curl}} \boldsymbol{v})$ are equivalent. We define

$$(\operatorname{tr}^{\operatorname{curl}} \boldsymbol{v})|_F = \operatorname{tr}_F^{\operatorname{curl}} \boldsymbol{v}, \quad F \in \Delta_{n-1}(T).$$

∂T

4.2. **Geometric decompositions.** Define the polynomial bubble space for the curl operator as

$$\mathbb{B}_k(\operatorname{curl}; T) = \ker(\operatorname{tr}^{\operatorname{curl}}) \cap \mathbb{P}_k(T; \mathbb{R}^n).$$

For Lagrange bubble $\mathbb{B}_k^n(T)$, all components of the vector vanish on ∂T and thus vanish on all sub-simplex with dimension $\leq n-1$. For $\boldsymbol{u} \in \mathbb{B}_k(\operatorname{curl}; T)$, only the tangential component vanishes, which will imply $\boldsymbol{u}$ vanishes on sub-simplex s with dimension less than or equal to $n-2$.

Lemma 4.2. *For $\boldsymbol{u} \in \mathbb{B}_k(\operatorname{curl}; T)$, it holds $\mathbf{u}|_f = \mathbf{0}$ for all $f \in \Delta_\ell(T), 0 \leq \ell \leq n-2$. Consequently $\boldsymbol{u} \in \bigoplus_{F \in \Delta_{n-1}(T)} \mathbb{B}_k^n(F)$.*

Proof. It suffices to consider a sub-simplex $f \in \Delta_{n-2}(T)$. Let $F_1, F_2 \in \Delta_{n-1}(T)$ such that $f = F_1 \cap F_2$. By $\operatorname{tr}_{F_i}^{\operatorname{curl}} \boldsymbol{u} = \mathbf{0}$ for $i = 1, 2$, we have

$$(\boldsymbol{u} \cdot \boldsymbol{t}_i^f)|_f = 0, \quad (\boldsymbol{u} \cdot \boldsymbol{n}_{F_1,f})|_f = (\boldsymbol{u} \cdot \boldsymbol{n}_{F_2,f})|_f = 0 \quad \text{for } i = 1, \ldots, n-2,$$

where $\boldsymbol{n}_{F_i,f}$ is a normal vector f sitting on F_i. As $\operatorname{span}\{\boldsymbol{t}_1^f, \ldots, \boldsymbol{t}_{n-2}^f, \boldsymbol{n}_{F_1,f}, \boldsymbol{n}_{F_2,f}\} = \mathbb{R}^n$, we acquire $\boldsymbol{u}|_f = \mathbf{0}$. By the property of face bubbles, we conclude $\boldsymbol{u}$ is a linear combination of $n-1$ face bubbles. $\square$

typical as $b_T = 0$

Obviously $\mathbb{B}_k^n(T) \subset \mathbb{B}_k(\operatorname{curl}; T)$. As $\operatorname{tr}^{\operatorname{curl}}$ contains the tangential component only, the normal component $\mathbb{B}_k(F)\boldsymbol{n}_F$ is also a curl bubble. The following result says their sum is precisely all curl bubble polynomials.

Theorem 4.3. *For $k \geq 1$, it holds that*

$$\mathbb{B}_k(\operatorname{curl}; T) = \mathbb{B}_k^n(T) \oplus \bigoplus_{F \in \Delta_{n-1}(T)} \mathbb{B}_k(F) \boldsymbol{n}_F, \tag{20}$$

and

$$\operatorname{tr}^{\operatorname{curl}} : \mathbb{P}_1(T) \oplus \bigoplus_{\ell=1}^{n-2} \bigoplus_{f \in \Delta_\ell(T)} \mathbb{B}_k^n(f) \oplus \bigoplus_{F \in \Delta_{n-1}(T)} \mathbb{B}_k^{n-1}(F) \to \operatorname{tr}^{\operatorname{curl}} \mathbb{P}_k^n(T) \tag{21}$$

is a bijection.

Proof. It is obvious that

$$\mathbb{B}_k^n(T) \oplus \bigoplus_{F \in \Delta_{n-1}(T)} \mathbb{B}_k(F) \boldsymbol{n}_F \subseteq \mathbb{B}_k(\operatorname{curl}, T).$$

copy here.

Then apply the trace operator to the decomposition (4) to conclude that the map $\operatorname{tr}^{\operatorname{curl}}$ in (21) is onto.

Now we prove it is also injective. Take a function $\boldsymbol{u} \in \mathbb{P}_1(T) \oplus \bigoplus_{\ell=1}^{n-2} \bigoplus_{f \in \Delta_\ell(T)} \mathbb{B}_k^n(f) \oplus \bigoplus_{F \in \Delta_{n-1}(T)} \mathbb{B}_k^{n-1}(F)$ and $\operatorname{tr}^{\operatorname{curl}} \boldsymbol{u} = \boldsymbol{0}$. By Lemma 4.2, we can assume $\boldsymbol{u} = \sum_{F \in \Delta_{n-1}(T)} \boldsymbol{u}_k^F$ with $\boldsymbol{u}_k^F \in \mathbb{B}_k^{n-1}(F)$. Take $F \in \Delta_{n-1}(T)$. We have $\boldsymbol{u}|_F = \boldsymbol{u}_k^F|_F \in \mathbb{B}_k^{n-1}(F)$. Hence $(\boldsymbol{u}_k^F \cdot \boldsymbol{t})|_F = (\boldsymbol{u} \cdot \boldsymbol{t})|_F = 0$ for any $\boldsymbol{t} \in \mathscr{T}^F$, which results in $\boldsymbol{u}_k^F = \boldsymbol{0}$. Therefore $\boldsymbol{u} = \boldsymbol{0}$.

Once we have proved the map tr in (21) is bijection, we conclude (20) from the decomposition (4). $\square$

We will use curl_f to denote the curl operator restricted to a sub-simplex f with $\dim f \geq 1$. For $f \in \Delta_\ell(T)$, $\ell = 2, \ldots, n-1$, by applying Theorem 4.3 to f, we have

$$\mathbb{B}_k(\operatorname{curl}_f; f) = \mathbb{B}_k^\ell(f) \oplus \bigoplus_{e \in \partial f} \mathbb{B}_k(e) \boldsymbol{n}_{f,e}, \tag{22}$$

where $\boldsymbol{n}_{f,e}$ is a normal vector of e but parallel to f. Notice that the curl_f-bubble function is defined for $\ell \geq 2$ not including edges. Indeed, for an edge e and a vertex v of e, $\boldsymbol{n}_{e,\mathrm{v}}$ is $\boldsymbol{t}_e$ or $-\boldsymbol{t}_e$ where the sign depends on the orientation. Then for $\ell = 1$

when when

$$\mathbb{B}_k(e) \boldsymbol{t}_e \oplus \bigoplus_{\mathrm{v} \in \partial e} \operatorname{span}\{\lambda_{\mathrm{v}} \boldsymbol{n}_{e,\mathrm{v}}\} = \mathbb{P}_k(e) \boldsymbol{t}_e.$$

which is no longer a bubble function on e.

Lemma 4.4. *For $k \geq 1$, we have*

$$\mathbb{P}_k^n(T) = \mathbb{P}_1^n(T) \oplus \bigoplus_{e \in \Delta_1(T)} \mathbb{B}_k(e) \boldsymbol{t}_e \oplus \bigoplus_{\ell=2}^n \bigoplus_{f \in \Delta_\ell(T)} \mathbb{B}_k(\operatorname{curl}_f; f), \tag{23}$$

$$\mathbb{P}_k^n(T) = \bigoplus_{e \in \Delta_1(T)} \mathbb{P}_k(e) \boldsymbol{t}_e \oplus \bigoplus_{\ell=2}^n \bigoplus_{f \in \Delta_\ell(T)} \mathbb{B}_k(\operatorname{curl}_f; f). \tag{24}$$

Proof. ~~Decomposition (23) is a re-arrangement of components in the decomposition (4).~~ For $e \in \Delta_{\ell-1}(T)$, the face normal vectors $\{\boldsymbol{n}_{f,e} : f \in \Delta_\ell(T), e \subseteq f\}$ form a basis of $\mathscr{N}^e$. So we have

$$\mathbb{B}_k(e) \otimes \mathscr{N}^e = \bigoplus_{f \in \Delta_\ell(T), e \subseteq f} \mathbb{B}_k(e) \boldsymbol{n}_{f,e}.$$

Then shift the normal component one level up and use (22) to get the decomposition (23). At a vertex

We then distribute the n-component of vector function value at the vertices to the n edges connected to this vertex

$$\mathbb{P}_1^n(T) \oplus \bigoplus_{e \in \Delta_1(T)} \mathbb{B}_k(e) \boldsymbol{t}_e = \bigoplus_{e \in \Delta_1(T)} (\mathbb{P}_1(e) \boldsymbol{t}_e \oplus \mathbb{B}_k(e) \boldsymbol{t}_e) = \bigoplus_{e \in \Delta_1(T)} \mathbb{P}_k(e) \boldsymbol{t}_e.$$

$\{t_e\}$ …

Thus (24) holds. $\square$

Decomposition (23) is the counterpart of (3) for Lagrange elements. In decomposition (24), the linear vector polynomial $\mathbb{P}_1^n(T)$ is redistributed to edges and the second kind Nédélec element [5] can be derived. To emphasize the dependence on edges, in the following DoFs, we shall use e instead f for a generic sub-simplex.

Lemma 4.5 (Local Nédélec element)**.** *For $e \in \Delta_\ell(T)$, let $\{\boldsymbol{t}_i^e, i = 1, \ldots, \ell\}$ be a basis of the tangent plane of e and choose $\{\boldsymbol{n}_{f,e} : f \in \Delta_{\ell+1}(T), e \subseteq f\}$ as the basis of $\mathscr{N}^e$. The shape function space $\mathbb{P}_k^n(T)$ is uniquely determined by the DoFs*

$$\boldsymbol{v} \cdot \boldsymbol{t}_e(\mathrm{v}), \quad e \in \Delta_1(T), \mathrm{v} \in \partial e, \tag{25}$$

$$\int_e (\boldsymbol{v} \cdot \boldsymbol{t}_i^e)\, p \,\mathrm{d}s, \quad p \in \mathbb{P}_{k-(\ell+1)}(e), e \in \Delta_\ell(T), \quad i = 1, \ldots, \ell, \ell = 1, \ldots, n-1, \tag{26}$$

$$\int_e (\boldsymbol{v} \cdot \boldsymbol{n}_{f,e})\, p \,\mathrm{d}s, \quad p \in \mathbb{P}_{k-(\ell+1)}(e), e \in \Delta_\ell(T), \quad f \in \Delta_{\ell+1}(T), e \subseteq f, \ell = 1, \ldots, n-1, \tag{27}$$

$$\int_T \boldsymbol{v} \cdot \boldsymbol{p} \,\mathrm{d}x \quad \boldsymbol{p} \in \mathbb{P}_{k-(n+1)}^n(T). \tag{28}$$

Proof. First of all, by the geometric decomposition (4) of $\mathbb{P}_k^n(T)$ and Theorem 4.3, the number of DoFs is equal to the dimension of the shape function space.

Assume $\boldsymbol{v} \in \mathbb{P}_k^n(T)$ and all the DoFs (25)-(28) vanish. Since $\{\boldsymbol{t}_e, e \in \Delta_1(T), \mathrm{v} \in \partial e\}$ is a basis of $\mathbb{R}^n$, $\{(\boldsymbol{v} \cdot \boldsymbol{t}_e)(\mathrm{v}), e \in \Delta_1(T), \mathrm{v} \in \partial e\}$ will determine the vector $\boldsymbol{v}(\mathrm{v})$. Thus vanishing (25) implies $\boldsymbol{v}$ is zero at vertices. In general, $\{\boldsymbol{n}_{f,e} : f \in \Delta_{\ell+1}(T), e \subseteq f\}$ forms a basis of $\mathscr{N}^e$. DoF (27) is equivalent to

$$\int_e (\boldsymbol{v} \cdot \boldsymbol{n}_i^e)\, p \,\mathrm{d}s, \quad p \in \mathbb{P}_{k-(\ell+1)}(e), e \in \Delta_\ell(T), i = 1, \ldots, n-\ell, \ \ell = 1, \ldots, n-1,$$

which together with vanishing DoF (26) implies

$$\int_f \boldsymbol{v} \cdot \boldsymbol{p} \,\mathrm{d}s = 0, \quad \boldsymbol{p} \in \mathbb{P}_{k-(\ell+1)}(f; \mathbb{R}^n), f \in \Delta_\ell(T), \ell = 1, \ldots, n-1.$$

It follows from the uni-solvence of Lagrange element that $\boldsymbol{v}|_{\partial T} = \boldsymbol{0}$, i.e. $\boldsymbol{v} \in \mathbb{B}_k^n(T)$. Finally $\boldsymbol{v} = \boldsymbol{0}$ is an immediate result of the vanishing DoF (28). $\square$

Lemma 4.6 (Nédélec space)**.** *Choice of $\boldsymbol{t}_e$ and $\boldsymbol{n}_{f,e}$. The following DoFs*

$$\boldsymbol{v} \cdot \boldsymbol{t}_e(\mathrm{v}), \quad e \in \Delta_1(\mathcal{T}_h), \mathrm{v} \in \partial e, \tag{29}$$

$$\int_e (\boldsymbol{v} \cdot \boldsymbol{t}_i^e)\, p \,\mathrm{d}s, \quad p \in \mathbb{P}_{k-(\ell+1)}(e), e \in \Delta_\ell(\mathcal{T}_h), \quad i = 1, \ldots, \ell, \ell = 1, \ldots, n-1, \tag{30}$$

$$\int_e (\boldsymbol{v} \cdot \boldsymbol{n}_{f,e})\, p \,\mathrm{d}s, \quad p \in \mathbb{P}_{k-(\ell+1)}(e), , e \in \Delta_\ell(\mathcal{T}_h), \quad f \in \Delta_{\ell+1}(\mathcal{T}_h), e \subseteq f, \ell = 1, \ldots, n-1, \tag{31}$$

$$\int_T \boldsymbol{v} \cdot \boldsymbol{p} \,\mathrm{d}x \quad \boldsymbol{p} \in \mathbb{P}_{k-(n+1)}^n(T), T \in \mathcal{T}_h \tag{32}$$

define a $H(\mathrm{curl})$-conforming space $V_h = \{\boldsymbol{v}_h \in H(\mathrm{curl}; \Omega) : \boldsymbol{v}_h|_T \in \mathbb{P}_k^n(T) \quad \forall\, T \in \mathcal{T}_h\}$.

Proof. On each element T, DoFs (29)-(32) will determine a function in $\mathbb{P}_k^n(T)$ by Lemma 4.5. For $F \in \Delta_{n-1}(\mathcal{T}_h)$, DoFs (29)-(31) restricted to F are

$$\boldsymbol{v} \cdot \boldsymbol{t}_e(\mathrm{v}), \quad e \in \Delta_1(F), \mathrm{v} \in \partial e,$$

$$\int_e (\boldsymbol{v} \cdot \boldsymbol{t}_i^e)\, p \,\mathrm{d}s, \quad p \in \mathbb{P}_{k-(\ell+1)}(e), e \in \Delta_\ell(F),$$
$$i = 1, \ldots, \ell, \ell = 1, \ldots, n-1,$$

$$\int_e (\boldsymbol{v} \cdot \boldsymbol{n}_{f,e})\, p \,\mathrm{d}s, \quad e \in \Delta_\ell(F), f \in \Delta_{\ell+1}(F), e \subseteq f,$$
$$p \in \mathbb{P}_{k-(\ell+1)}(e), \ell = 1, \ldots, n-2.$$

Since $\{\boldsymbol{t}_1^e, \ldots, \boldsymbol{t}_\ell^e, \boldsymbol{n}_{f,e}, f \in \Delta_{\ell+1}(F), e \subseteq f\}$ spans the tangent plane $\mathscr{T}^F$, by the unisolvence of the Lagrange element, these DoFs will determine the trace $\mathrm{tr}_F^{\mathrm{curl}}\, \boldsymbol{v}$ on F independent of the elements containing F and thus the function is $H(\mathrm{curl}\,;\Omega)$-conforming. $\square$

The vertex DoF can be merged into the edge DoF and result in the classical DoF

$$\int_e \boldsymbol{v} \cdot \boldsymbol{t}^e\, p \,\mathrm{d}s \quad p \in \mathbb{P}_k(e), e \in \Delta_1(\mathcal{T}_h).$$

It can be also used to define an $H(\mathrm{curl}\,;\Omega)$-conforming finite element space with vertex continuity [4].

m m

In general, given an integer $-1 \leq k \leq n-1$, we can split the DoFs: for $\ell \leq k$, it is Lagrange and for $\ell > k$, it is Nédélec. It returns to the vector Lagrange element when $k = n-1$, and Nédélec element for $k = -1$.

Lemma 4.7. *Choice of $\mathbf{t}_e$ and $\boldsymbol{n}_{f,e}$. Some are local and some are global. The following* DoFs

degree k?

$$\int_e \boldsymbol{v} \cdot \boldsymbol{p} \,\mathrm{d}s, \quad \boldsymbol{p} \in \mathbb{P}_{k-(\ell+1)}^n(e), e \in \Delta_\ell(\mathcal{T}_h), \ell = 0, \ldots, k, \tag{33}$$

$$\int_e (\boldsymbol{v} \cdot \boldsymbol{t}_i^e)\, p \,\mathrm{d}s, \quad p \in \mathbb{P}_{k-(\ell+1)}(e), e \in \Delta_\ell(\mathcal{T}_h), \tag{34}$$
$$i = 1, \ldots, \ell, \ell = k+1, \ldots, n-1,$$

$$\int_e (\boldsymbol{v} \cdot \boldsymbol{n}_{f,e})\, p \,\mathrm{d}s, \quad p \in \mathbb{P}_{k-(\ell+1)}(e), e \in \Delta_\ell(\mathcal{T}_h), \tag{35}$$
$$f \in \Delta_{\ell+1}(\mathcal{T}_h), e \subseteq f, \ell = k+1, \ldots, n-1,$$

$$\int_T \boldsymbol{v} \cdot \boldsymbol{p} \,\mathrm{d}x \quad \boldsymbol{p} \in \mathbb{P}_{k-(n+1)}^n(T), T \in \mathcal{T}_h, \tag{36}$$

define a $H(\mathrm{curl}\,)$-conforming space $V_h = \{\boldsymbol{v}_h \in H(\mathrm{curl}\,;\Omega) \cap C^0(\Delta_k(\mathcal{T}_h)) : \boldsymbol{v}_h|_T \in \mathbb{P}_k^n(T) \;\; \forall\, T \in \mathcal{T}_h\}$, where $H(\mathrm{curl}\,;\Omega) \cap C^0(\Delta_{-1}(\mathcal{T}_h)) = H(\mathrm{curl}\,;\Omega)$.

Proof. Clearly the number of DoFs (33)-(36) equals to the number of DoFs (29)-(32). DoF (33) determines DoFs (29)-(31) for $\ell = 0, \ldots, k$. Then we conclude the result from Lemma 4.6. $\square$

With vertex DoFs, we can choose the basis of the edge element based on those of the Lagrange element, which is related to the geometric decomposition (23). The $H(\mathrm{curl}\,)$-conforming space V_h defined in Lemma 4.6 is same as the second kind Nédélec element space, V_h in Lemma 4.7 with $k = 0$ same as the $H(\mathrm{curl}\,)$-conforming space in [4], while DoFs (29)-(32) are different from those in [5, 4]. And the corresponding geometric decomposition (23) is also different from those in [2, 1]. The geometric decomposition (23) enable the use of Lagrange basis.

meshes

4.3. **Nédélec element on a triangle.** Let T be a triangle, for any $\boldsymbol{x} \in \mathcal{X}_T$, define a frame $\{\boldsymbol{e}_{\boldsymbol{x}}^0, \boldsymbol{e}_{\boldsymbol{x}}^1\}$ at $\boldsymbol{x}$ and its dual frame $\{\hat{\boldsymbol{e}}_{\boldsymbol{x}}^0, \hat{\boldsymbol{e}}_{\boldsymbol{x}}^1\}$ as follows:

(1) If $\boldsymbol{x} \in \Delta_0(T)$, the two adjacent edges are e_0 and e_1, where the local indexing of e_0 is greater than that of e_1, then

$$\boldsymbol{e}_{\boldsymbol{x}}^0 = \boldsymbol{t}_{e_0}, \quad \boldsymbol{e}_{\boldsymbol{x}}^1 = \boldsymbol{t}_{e_1}. \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^0 = \frac{\boldsymbol{n}_{e_1}}{\boldsymbol{n}_{e_1} \cdot \boldsymbol{t}_{e_0}}, \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^1 = \frac{\boldsymbol{n}_{e_0}}{\boldsymbol{n}_{e_0} \cdot \boldsymbol{t}_{e_1}}.$$

(2) If $\boldsymbol{x} \in \mathring{\mathcal{X}}_e, e \in \Delta_1(T)$, then

$$\boldsymbol{e}_{\boldsymbol{x}}^0 = \boldsymbol{t}_e, \quad \boldsymbol{e}_{\boldsymbol{x}}^1 = \boldsymbol{n}_e, \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^0 = \boldsymbol{t}_e, \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^1 = \boldsymbol{n}_e.$$

em If $\boldsymbol{x} \in \mathring{\mathcal{X}}_T$, then

$$\boldsymbol{e}_{\boldsymbol{x}}^0 = (1,0), \quad \boldsymbol{e}_{\boldsymbol{x}}^1 = (0,1), \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^0 = (1,0), \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^1 = (0,1).$$

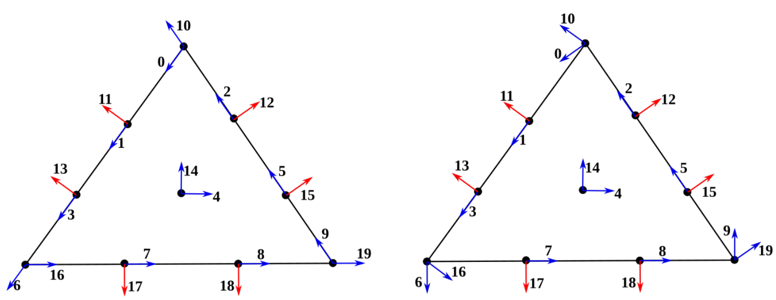


FIGURE 4. The left figure shows $\{\boldsymbol{e}_0, \boldsymbol{e}_1\}$ at each interpolation point, the right figure shows $\{\hat{\boldsymbol{e}}_0, \hat{\boldsymbol{e}}_1\}$ at each interpolation point.

Lemma 4.8 (Nédélec element on a triangle)**.** *Any function $\boldsymbol{u} \in \mathbb{P}_k^2(T)$ can be uniquely determined by the DoFs:*

$$\boldsymbol{u}(\boldsymbol{x}_{\boldsymbol{\alpha}}) \cdot \boldsymbol{e}_{\boldsymbol{x}_{\boldsymbol{\alpha}}}^i, \quad \boldsymbol{x}_{\boldsymbol{\alpha}} \in \mathcal{X}_T, i = 0, 1.$$

The basis function of 2-dim kth order Nédélec element space on T can be explicitly written as:

$$\boldsymbol{\phi}_{\boldsymbol{\alpha}}^i(\boldsymbol{x}) = \phi_{\boldsymbol{\alpha}}(\boldsymbol{x})\hat{\boldsymbol{e}}_{\boldsymbol{x}_{\boldsymbol{\alpha}}}^i, \quad \boldsymbol{\alpha} \in \mathbb{T}_k^2, i = 0, 1$$

Define te, ne.
te is global
ne is local

Lemma 4.9 (Nédélec space on a triangulation)**.** *The following* DoFs

$$\boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{t}_e, \quad \boldsymbol{x} \in \mathcal{X}_e, e \in \Delta_1(\mathcal{T}_h) \tag{37}$$

$$\boldsymbol{u}(\boldsymbol{x})|_T \cdot \boldsymbol{n}_e, \quad \boldsymbol{x} \in \mathring{\mathcal{X}}_e, \ e \in \Delta_1(T), T \in \Delta_2(\mathcal{T}_h) \tag{38}$$

$$\boldsymbol{u}(\boldsymbol{x}) \cdot (1,0), \quad \boldsymbol{u}(\boldsymbol{x}) \cdot (0,1), \quad \boldsymbol{x} \in \mathring{\mathcal{X}}_T, T \in \Delta_2(\mathcal{T}_h) \tag{39}$$

defines a $H(\mathrm{curl})$-conforming space $V_h = \{\boldsymbol{v}_h \in H(\mathrm{curl}\,;\Omega) : \boldsymbol{v}_h|_T \in \mathbb{P}_k^2(T)\, \forall T \in \mathcal{T}_h\}$.

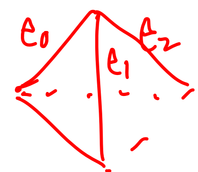
We call (37) as DoF on edge, (38) and (39) as DoF in cell. ~~Correct the equation number~~

4.4. **Nédélec element on a tetrahedron mesh.** Let T be a tetrahedron, for any $\boldsymbol{x} \in \mathcal{X}_T$, define a frame $\{\boldsymbol{e}_{\boldsymbol{x}}^0, \boldsymbol{e}_{\boldsymbol{x}}^1, \boldsymbol{e}_{\boldsymbol{x}}^2\}$ at $\boldsymbol{x}$ and its dual frame $\{\hat{\boldsymbol{e}}_{\boldsymbol{x}}^0, \hat{\boldsymbol{e}}_{\boldsymbol{x}}^1, \hat{\boldsymbol{e}}_{\boldsymbol{x}}^2\}$ as follows:

(1) If $\boldsymbol{x} \in \Delta_0(T)$ and adjacent edges of $\boldsymbol{x}$ are e_0, e_1, e_2, adjacent faces of $\boldsymbol{x}$ are f_0, f_1, f_2, where f_0, f_1, f_2 are sorted by local indexing from large to small, statify $f_i \cap e_i = \boldsymbol{x}$, then

$$\boldsymbol{e}_{\boldsymbol{x}}^0 = \boldsymbol{t}_{e_0}, \quad \boldsymbol{e}_{\boldsymbol{x}}^1 = \boldsymbol{t}_{e_1}, \quad \boldsymbol{e}_{\boldsymbol{x}}^2 = \boldsymbol{t}_{e_2}.$$

$$\hat{\boldsymbol{e}}_{\boldsymbol{x}}^0 = \frac{\boldsymbol{n}_{f_0}}{\boldsymbol{n}_{f_0} \cdot \boldsymbol{t}_{e_0}}, \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^1 = \frac{\boldsymbol{n}_{f_1}}{\boldsymbol{n}_{f_1} \cdot \boldsymbol{t}_{e_1}}, \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^2 = \frac{\boldsymbol{n}_{f_2}}{\boldsymbol{n}_{f_2} \cdot \boldsymbol{t}_{e_2}}.$$



$f_0 \cap e_i = x$ for all $i = 0, 1, 2$.

(2) If $\boldsymbol{x} \in \mathring{\mathcal{X}}_e, e \in \Delta_1(T)$ and adjacent faces are f_0, f_1 then

$$\boldsymbol{e}_{\boldsymbol{x}}^0 = \boldsymbol{t}_e, \quad \boldsymbol{e}_{\boldsymbol{x}}^1 = \boldsymbol{n}_{f_0} \times \boldsymbol{t}_e, \quad \boldsymbol{e}_{\boldsymbol{x}}^2 = \boldsymbol{n}_{f_1} \times \boldsymbol{t}_e.$$

$$\hat{\boldsymbol{e}}_{\boldsymbol{x}}^0 = \boldsymbol{t}_e, \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^1 = \frac{\boldsymbol{n}_{f_1}}{\boldsymbol{n}_{f_1} \cdot (\boldsymbol{n}_{f_0} \times \boldsymbol{t}_e)}, \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^2 = \frac{\boldsymbol{n}_{f_0}}{\boldsymbol{n}_{f_0} \cdot (\boldsymbol{n}_{f_1} \times \boldsymbol{t}_e)}.$$

(3) If $\boldsymbol{x} \in \mathring{\mathcal{X}}_f, f \in \Delta_2(T)$, the first edge of f is e, then

$$\boldsymbol{e}_{\boldsymbol{x}}^0 = \boldsymbol{t}_e, \quad \boldsymbol{e}_{\boldsymbol{x}}^1 = \boldsymbol{t}_e \times \boldsymbol{n}_f, \quad \boldsymbol{e}_{\boldsymbol{x}}^2 = \boldsymbol{n}_f.$$

$$\hat{\boldsymbol{e}}_{\boldsymbol{x}}^0 = \boldsymbol{t}_e, \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^1 = \boldsymbol{t}_e \times \boldsymbol{n}_f, \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^2 = \boldsymbol{n}_f.$$

(4) If $\boldsymbol{x} \in \mathring{\mathcal{X}}_T$, then

$$\boldsymbol{e}_{\boldsymbol{x}}^0 = (1,0,0), \quad \boldsymbol{e}_{\boldsymbol{x}}^1 = (0,1,0), \quad \boldsymbol{e}_{\boldsymbol{x}}^2 = (0,0,1).$$

$$\hat{\boldsymbol{e}}_{\boldsymbol{x}}^0 = (1,0,0), \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^1 = (0,1,0), \quad \hat{\boldsymbol{e}}_{\boldsymbol{x}}^2 = (0,0,1).$$

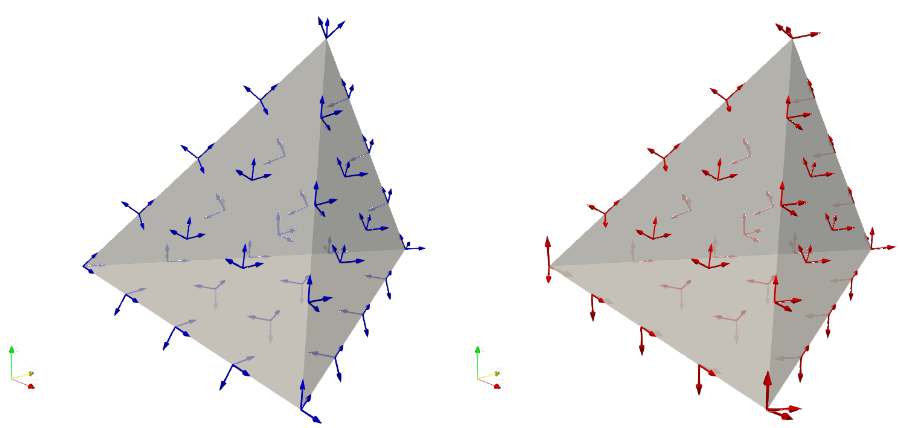


FIGURE 5. The left figure shows $\{\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2\}$ at each interpolation point, the right figure shows $\{\hat{\boldsymbol{e}_0}, \hat{\boldsymbol{e}_1}, \hat{\boldsymbol{e}_2}\}$ at each interpolation point.

$\hat{e}_i$

Lemma 4.10 (Nédélec element on a tetrahedron). *A polynomial function $\boldsymbol{u} \in \mathbb{P}_k^3(T)$ can be uniquely determined by the DoFs:*

$$\boldsymbol{u}(\boldsymbol{x}_{\boldsymbol{\alpha}}) \cdot \boldsymbol{e}_{\boldsymbol{x}_{\boldsymbol{\alpha}}}^i, \quad \boldsymbol{x}_{\boldsymbol{\alpha}} \in \mathcal{X}_T, i = 0, 1, 2.$$

The basis function of kth order Nédélec element space on T can be explicitly written as:

$$\boldsymbol{\phi}_{\boldsymbol{\alpha}}^i(\boldsymbol{x}) = \phi_{\boldsymbol{\alpha}}(\boldsymbol{x}) \hat{\boldsymbol{e}}_{\boldsymbol{x}_{\boldsymbol{\alpha}}}^i, \quad \boldsymbol{\alpha} \in \mathbb{T}_k^3, i = 0, 1, 2$$

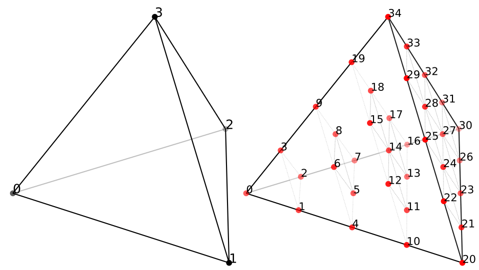


FIGURE 6. The numbering rules of tetrahedron's vertices and interpolation points for $k = 4$ on it.

Define local and global t-n basis

Lemma 4.11 (Nédélec space on tetrahedron mesh). as ~~*Please check and correct the following DoFs*~~ *For $f \in \Delta_2(\mathcal{T}_h)$, let $\boldsymbol{t}_f^0$ is the normal tangent of the first edge of f, $\boldsymbol{t}_f^1 = \boldsymbol{t}_f^0 \times \boldsymbol{n}_f$, the following* DoFs

$$\boldsymbol{u} \cdot \boldsymbol{t}_e(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathcal{X}_e, e \in \Delta_1(\mathcal{T}_h) \tag{40}$$

$$\boldsymbol{u}(\boldsymbol{x})|_f \cdot (\boldsymbol{n}_f \times \boldsymbol{t}_e), \quad \boldsymbol{x} \in \mathring{\mathcal{X}}_e,\ e \in \Delta_1(f), f \in \Delta_2(\mathcal{T}_h) \tag{41}$$

$$\boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{t}_f^0, \quad \boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{t}_f^1, \quad \boldsymbol{x} \in \mathring{\mathcal{X}}_f, f \in \Delta_2(\mathcal{T}_h) \tag{42}$$

$$\boldsymbol{u}(\boldsymbol{x})|_T \cdot \boldsymbol{n}_f, \quad \boldsymbol{x} \in \mathring{\mathcal{X}}_f,\ f \in \Delta_2(T), T \in \Delta_3(\mathcal{T}_h) \tag{43}$$

$$\boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{e}_{\boldsymbol{x}}^i \quad \boldsymbol{x} \in \mathring{\mathcal{X}}_T, i = 0, 1, 2, T \in \Delta_3(\mathcal{T}_h) \tag{44}$$

defines a $H(\text{curl})$-conforming space $V_h = \{\boldsymbol{v}_h \in H(\text{curl};\Omega) : \boldsymbol{v}_h|_T \in \mathbb{P}_k^3(T),\ \forall T \in \mathcal{T}_h\}$.

Proof. Proof it is $H(\text{curl})$-conforming □

We call (40) as DoF on edge, (41) and (42) as DoF in face, (43) and (44) as DoF in cell. Correct equation number

5. ~~THE~~ MANAGEMENT OF GLOBAL DOFS

The management of global DoFs is the key to the impelmentation of finite element algorithm. The key to the management of global DoFs is to determine the global index number of each local DoF on each simplex. In this section, we will discuss the global numbering rules for the Lagrange, BDM and Nédélec finite element spaces.

§ Local to global number

For Lagrange finite element space, the management of global index number of DoFs is equivalent to the management of global index number of the interpolation points. Based on the global index number of interpolation points, we can easily obtain the global index number of DoFs of BDM and Nédélec spaces.

Lemma 2.2 only gives the expression of the basis functions on a single n-simplex. Theoretically, we need to continuously "splice" the basis functions locallly defined on each n-simplex to form the global basis function on the triangulation. However, in implementation, we only need to give each interpolation point on the triangulation a globally unique index number. But it is not a very obvious thing, because the adjacent simplexes in the triangulation will share some interpolation points on the their shared sub-simplexes.

5.1. **Lagrange finite element space.** Let's take the tetrahedral mesh as an example to illustrate ~~discuss~~ the global numbering problem of the interpolation points. Fig 5.1 shows the local numbering rules of tetrahedron vertices and the interpolation points for $k = 4$ on it. Note that the local ordering of the four vertices of a tetrahedron satisfies the right-hand rule, and the interpolation points follow the ordering map $R(\boldsymbol{\alpha})$.

We can use the following two ~~2D~~ arrays to represent a tetrahedral mesh $\mathcal{T}_h$.

- `node` with shape `(NN, 3)`, and `node[i, j]` be the j-th component of the i-th node's Cartesian coordinate.
- `cell` with shape `(NC, 4)`, and `cell[i, j]` be the global index number of j-th vertex of i-th cell, which is the row number of `node`.

Furthermore, given a tretrahedron cell `[0, 1, 2, 3]`, we define its local edges and faces as following:

- `SEdge = [(0, 1), (0, 2), (0, 3), (1, 2), (1, 3), (2, 3)]`
- `SFace = [(1, 2, 3), (0, 2, 3), (0, 1, 3), (0, 1, 2)]`
- `OFace = [(1, 2, 3), (0, 3, 2), (0, 1, 3), (0, 2, 1)]`

Here we have defined two kind of local faces. The prefix `S` means sorted, and the prefix `O` means outer normal direction. Note that `SFace[i, :]` or `OFace[i, :]` represents the local face opposite to the i-th vertex.

By the `unique` ~~algorithm for array~~ command, we can construct the following 2D arrays from `cell`, `SEdge` and `OFace`.

- `edge` with shape `(NE, 2)`, and `edge[i, j]` be the global index number of j-th vertex of i-th edge, which is the row number of `node`.
- `face` with shape `(NF, 3)`, and `face[i, j]` be the global index number of j-th vertex of i-th face, which is the row number of `node`.
- `cell2edge` with shape `(NC, 6)`, and `cell2edge[i, j]` be the global index number of j-th edge of i-th cell, which is the row number of `edge`.
- `cell2face` with shape `(NC, 4)`, and `cell2face[i, j]` be the global index number of j-th face of i-th cell, which is the row number of `face`.

Note that the normal direction determined by the ordering of the three vertices of `OFace` is exactly the outer normal direction of the tetrahedron, which is used to ensure that the outer normal direction of boundary `face` will point to the outside of the mesh. And `SFace` is used to determine the global index number of the interpolation points on the face of each cell.

According to the numbering of node, edge, face, and cell, we can get a numbering rules for the interpolation points on $\mathcal{T}_h$. Let k be the degree of the Lagrange finite element space. The numbers of interpolation points on each cell is

$$
ldof = \frac{(k+1)(k+2)(k+3)}{6},
$$

and the total number of interpolation points on $\mathcal{T}_h$ is

not defined

$$
gdof = NN + n_e \cdot NE + n_f \cdot NF + n_c \cdot NC,
$$

where

$$
n_e = k-1, \quad n_f = \frac{(k-2)(k-1)}{2}, \quad n_c = \frac{(k-3)(k-2)(k-1)}{6}.
$$

We first set a global numbering rule for all interpolation points. For the interpolation points that coincide with nodes, their global index numbers are set as $0, 1, \cdots, NN-1$. When $k > 1$, for the interpolation points that inside each edges, their global index numbers

vertices

are set as

Confusing

$$
\begin{array}{r}
0: \\
1: \\
\vdots \\
NE-1:
\end{array}
\begin{bmatrix}
NN+0\cdot n_e & \cdots & NN+1\cdot n_e-1 \\
NN+1\cdot n_e & \cdots & NN+2\cdot n_e-1 \\
\vdots & \vdots & \vdots \\
NN+(NE-1)\cdot n_e & \cdots & NN+NE\cdot n_e-1
\end{bmatrix}
$$

When $k>2$, for the interpolation points that inside each face, their global index numbers are set as

$$
\begin{array}{r}
0: \\
1: \\
\vdots \\
NF-1:
\end{array}
\begin{bmatrix}
NN+NE\cdot n_e+0\cdot n_f & \cdots & NN+NE\cdot n_e+1\cdot n_f-1 \\
NN+NE\cdot n_e+1\cdot n_f & \cdots & NN+NE\cdot n_e+2\cdot n_f-1 \\
\vdots & \vdots & \vdots \\
NN+NE\cdot n_e+(NF-1)\cdot n_f & \cdots & NN+NE\cdot n_e+NF\cdot n_f-1
\end{bmatrix}
$$

font

When $k>3$, the global index numbers of the interpolation points that inside each cell can be setted in a similar way. Note that the local ordering of interpolation points inside a face or cell is still determined by the ordering map $R(\boldsymbol{\alpha})$.

We next discuss in detail how to generate a two-dimensional array named `cell2ipoint` with shape `(NC, ldof)`. Let's consider the j-th interpolation point of the i-th cell. Our task is to get its unique global index number and store it into `cell2ipoint[i, j]`.

Note that, by the simplicial lattices set $\mathbb{T}_k^n$ and the ordering map $R(\boldsymbol{\alpha})$, one can easily know the position of each interpolation point relative to the simplex and its sub-simplexes.

If j-th interpolation point is coincide with one vertex or inside a cell, one can easily set value of `cell2ipoint[i, j]` by the global numbering rule mentioned earlier. But when j-th interpolation point is inside one edge or one face of i-th cell, it is not an easy ~~thing~~ to determine its global index number.

Without loss of generality, suppose the j-th interpolation point is inside 0-th local face of i-th cell, and let `[m0, m1, m2, m3]` be its corresponding multi-index array. One can verify that `m0` must be 0, and other ones must be not 0, and

$$
\begin{aligned}
j =& \frac{(m1+m2+m3)(m1+m2+m3+1)(m1+m2+m3+2)}{6} \\
&+ \frac{(m2+m3)(m2+m3+1)}{2} + m3.
\end{aligned}
$$

Next we can get two kind of array representations of the face with global index number `cell2face[i, 0]`, and one is the local representation and another is the global representation.

```
0. LFace = cell[i, SFace[0, :]]
1. GFace = face[cell2face[i, 0], :]
```

Note that `LFace` and `GFace` contain the same vertex numbers, but in different order. Array `m=[m1, m2, m3]` correspond one-to-one with the three vertices of `LFace`. Next we need to reorder `m` so that it corresponds one-to-one to the vertices of `GFace`. We can finish this task by argument sorting.

```
0. i0 = argsort(argsort(GFace))
1. i1 = argsort(LFace)
2. i2 = i1[i0]
3. m = m[i2]
```

Based on the reordered `m=[m1, m2, m3]`, we can get local index l of j-th interpolation point on the global face `f = cell2face[i, 0]`,

$$\begin{aligned} l =& \frac{(m2-1+m3-1)(m2-1+m3-1+1)}{2} + m3 - 1. \\ =& \frac{(m2+m3-2)(m2+m3-1)}{2} + m3 - 1. \end{aligned}$$

Note that we only need to consider the index number of interpolation points inside the face. Finally, we can compute the global index number of j-th interpolation point which is inside 0-th local face of i-th cell.

$$J = NN + n_e \cdot NE + n_f \cdot f + l,$$

where

$$n_e = k-1, \quad n_f = \frac{(k-2)(k-1)}{2}.$$

Remark 5.1. *Here We just discussed the global numbering problem of the interpolation points inside the cell faces. However, the method to solve the problem is very general, which is applicable to the interpolation points inside the cell edges and even the interpolation point numbering problem of arbitrary dimensional simplex meshes.*

In the next two sections, let's discuss the management of DoFs for BDM and Nédélec finite element space based on the 2D array `cell2ipoint`.

5.2. **BDM and Nédélec finite element space.** First, we want to emphasize that the management of DoFs is to manage the continuity of finite element space.

The BDM and Nédélec are vector finite element spaces, which define DoFs of vector type by defining a vector frame on each interpolation point. At the same time, they define their vector basis functions by combining the Lagrange basis function and the dual frame of the DoFs.

Alternatively, we can say each DoF in BDM or Nédélec sapce corresponds to a unique interpolation point p and a unique vector $\boldsymbol{e}$, and each basis funcion also corresponds to a unique Lagrange basis function which is defined on p and a vector $\boldsymbol{e}'$ which is the dual vector of $\boldsymbol{e}$.

The management of DoFs is essentially a counting problem. First of all, we need to set global and local numbering rules for all DoFs. move early

We can globally divide the DoFs into shared and unshared among simplexes. The DoFs shared among simplexes can be further divided into on-edge and on-face according to the dimension of the sub-simplex where the DoFs locate. Note that, for BDM and Nédélec space there are no DoFs shared on nodes. And for 3D BDM space there are no DoFs shared on edges. So the global numberring rule is similar with the Lagrange interpolation points. First count the shared DoFs on each edge according to the order of the edges, then count the shared DoFs on each face according to the order of the faces, and finally count the unshared DoFs in the cell. On each edge or face, the DoFs' order can follow the order of the interpolation points.

By the global numbering rule, we also can get a (an) array named `dof2vector` with shape `(gdof, GD)`, where `gdof` is the number of global DoFs and `GD` represent geometry dimensions. And `dof2vector[i, :]` store the vector of the i-th DoF.

Next we need to set a local numbering rules and generate a array `cell2dof` with shape `(NC, ldof)`, where `ldof` is the number of local DoFs on each cell. Note that each DoF was determined by a intepolation point and a vector. And for each interpolation point, there is a frame (including `GD` vectors) on it. Given a DoF on i-th cell, denote the local index number of its interpoltion point as p, and the local index number of its vector in the frame denote as q, then one can set a unique local index number j by p and q, for example

$$j = n \cdot q + p$$

where n is the number of interpolation points in i-th cell. Furthermore, we can compute the `cell2dof[i, j]` by the global index number `cell2ipoint[i, p]`, the sub-simplex that the interpolation point locate, and the global numbering rule.

Remark 5.2. *Note that the local and global number rules mentioned above are not unique. Furthermore, with the array* `cell2dof`*, the implementation of these higher-order finite element methods mentioned in this paper is not fundamentally different from the conventional finite element in terms of matrix vector assembly and boundary condition handling.*

6. Numerical Examples

6.1. **High Order Elements for Maxwell Equations.** Set $\Omega = [0, 1]^3$, consider the time harmonic problem:

(1)
$$\begin{cases} \nabla \times (\mu^{-1} \nabla \times \boldsymbol{E}) - \omega^2 \tilde{\epsilon} \boldsymbol{E} & = \boldsymbol{J} \quad \text{in} \quad \Omega \\ \nabla \times \boldsymbol{E} & = \boldsymbol{0} \quad \text{on} \quad \partial\Omega \end{cases},$$

~~To test the correctness of Nédélec space we choose electric field~~ $\boldsymbol{E} = (f, \sin(x) f, \cos(y) f)$, $f = (x^2 - x)^2 (y^2 - y)^2 (z^2 - z)^2$ $\omega = \tilde{\epsilon} = \mu = 1$, current source $\boldsymbol{J} = \nabla \times \nabla \times \boldsymbol{E} - \boldsymbol{E}$.

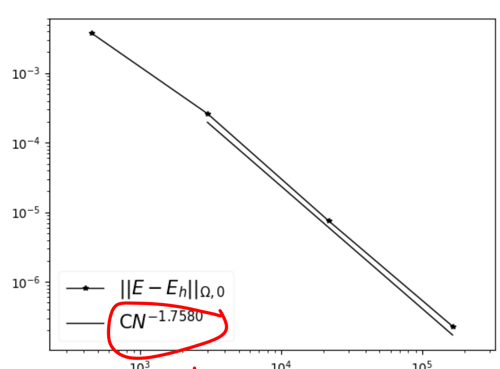
Variational problem: find $\boldsymbol{E} \in H_0(\text{curl}, \Omega)$, $\forall\, \boldsymbol{v} \in H_0(\text{crul}, \Omega)$ staify:

$$(\quad \int_\Omega \mu^{-1} (\nabla \times \boldsymbol{E}) \cdot (\nabla \times \boldsymbol{v}) \,\mathrm{d}\boldsymbol{x} - \int_\Omega \omega^2 \tilde{\epsilon} \boldsymbol{E} \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{J} \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x}$$

Finite element problem: find $\boldsymbol{E}_h \in ND_4(\mathcal{T}_h) \cap H_0(\text{curl}, \Omega), \forall\, \boldsymbol{v}_h \in ND_4(\mathcal{T}_h) \cap H_0(\text{curl}, \Omega)$ staify:

$$\int_\Omega \mu^{-1} (\nabla \times \boldsymbol{E}_h) \cdot (\nabla \times \boldsymbol{v}_h) \,\mathrm{d}\boldsymbol{x} - \int_\Omega \omega^2 \tilde{\epsilon} \boldsymbol{E}_h \cdot \boldsymbol{v}_h \,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{J} \cdot \boldsymbol{v}_h \,\mathrm{d}\boldsymbol{x} \tag{45}$$

The numerical results are shown in 6.1, we can see that $||\boldsymbol{E} - \boldsymbol{E}_h||_{0,\Omega} = O(h^5)$.



h

ω

mesh

solver.

numerical quadrature to compute the error

FIGURE 7. Error $||\boldsymbol{E} - \boldsymbol{E}_h||_{0,\Omega}$ of finite element methond (45).

$k = 2, 3, 4.$

6.2. **High Order Elements for Mixed Poisson.** Consider the Poisson problem:

$$\begin{cases} \boldsymbol{u} + \nabla p = 0 & \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u} = f & \text{in } \Omega, \\ p = g & \text{on } \partial\Omega, \end{cases}$$

where $\Omega = (0,1)^3$. To test the correctness of BDM space we set

$$\boldsymbol{u} = \begin{pmatrix} -\pi \sin(\pi x)\cos(\pi y)\cos(\pi z) \\ -\pi \cos(\pi x)\sin(\pi y)\cos(\pi z) \\ -\pi \cos(\pi x)\cos(\pi y)\sin(\pi z) \end{pmatrix}.$$

$$p = \cos(\pi x)\cos(\pi y)\cos(\pi z), \quad f = 3\pi^2 \cos(\pi x)\cos(\pi y)\cos(\pi z),$$

Variational problem: find $\boldsymbol{u} \in H(\mathrm{div}, \Omega)$, $p \in L^2(\Omega)$, $\forall \boldsymbol{v} \in H(\mathrm{div}, \Omega)$, $q \in L^2(\Omega)$ staify:

$$\int_\Omega \boldsymbol{u} \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} - \int_\Omega p \nabla \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} = -\int_{\partial\Omega} g(\boldsymbol{v} \cdot \boldsymbol{n}) \, \mathrm{d}\boldsymbol{s}$$
$$-\int_\Omega (\nabla \cdot \boldsymbol{u}) q \, \mathrm{d}\boldsymbol{x} = -\int_\Omega f q \, \mathrm{d}\boldsymbol{x}$$

Finite element problem: find $\boldsymbol{u}_h \in V_0, p_h \in V_1$, $\forall \boldsymbol{v}_h \in V_0, q_h \in V_1$ staify:

$$\int_\Omega \boldsymbol{u}_h \cdot \boldsymbol{v}_h \, \mathrm{d}\boldsymbol{x} - \int_\Omega p \nabla \cdot \boldsymbol{v}_h \, \mathrm{d}\boldsymbol{x} = -\int_{\partial\Omega} g(\boldsymbol{v}_h \cdot \boldsymbol{n}) \, \mathrm{d}\boldsymbol{s} \tag{46}$$

$$-\int_\Omega (\nabla \cdot \boldsymbol{u}_h) q_h \, \mathrm{d}\boldsymbol{x} = -\int_\Omega f q_h \, \mathrm{d}\boldsymbol{x} \tag{47}$$

The numerical results are shown in 6.2, we can see that $||\boldsymbol{u} - \boldsymbol{u}_h||_{\Omega,0} = O(h)^4$, $||p - p_h||_{\Omega,0} = O(h)^5$.

$0,\Omega$

can't be right

$u_h \in P_k$ h^{k+1}

$p_h \in P_{k-1}$. h^k

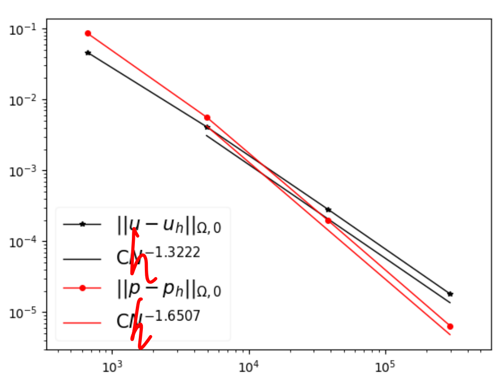


FIGURE 8. Error $||\boldsymbol{u} - \boldsymbol{u}_h||_{\Omega,0}$ and $||p - p_h||_{\Omega,0}$ of finite element method (46).

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