Hierarchical High Order Finite Element Approximation Spaces for H(div) and H(curl)

Denise De Siqueira, Philippe R.B. Devloo, and Sônia M. Gomes

Abstract The aim of this paper is to present a systematic procedure for the construction of a hierarchy of high order finite element approximations for H(div) and H(curl) spaces based on quadrilateral and triangular elements with rectilinear edges. The principle is to chose appropriate vector fields, based on the geometry of each element, which are multiplied by an available set of H^1 hierarchical scalar basic functions. We show that the resulting local vector bases can be combined to obtain continuous normal or tangent components on the elements interfaces, properties that characterize piecewise polynomial functions in H(div) or H(curl), respectively.

1 Introduction

In applications of mixed methods, the mathematical analysis uses constantly H(div) and H(curl) spaces, and approximations of them are required [1]. The main characteristic of piecewise polynomial H(div) functions is the continuity of the normal components over the interface of the elements, while H(curl) functions require continuous tangencial components. There are several papers in the literature where the techniques employed in the construction of finite element spaces for H(div) and H(curl) are based on De Rham Diagram (e.g., [2, 4, 5]).

In the present paper we present a different approach. Instead of De Rham Diagram, we use the geometry of the elements to construct appropriate vector fields which are multiplied by hierarchical H^1 conforming scalar functions developed in [3]. Using this systematic procedure, hierarchical vector bases are defined for quadrilateral and triangular elements. There are those basic functions that are

D. De Siqueira (✉) and S.M. Gomes
Unicamp, IMECC, Brasil
e-mail: dsiqueira@ime.unicamp.br, soniag@ime.unicamp.br

P.R.B. Devloo
Unicamp, FEC, Brasil
e-mail: phil@fec.unicamp.br

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associated to the edges, whose normal (or tangencial) components on the edges of the element are expressed in terms of the H^1 scalar basis functions corresponding to them. There are also other basis vector functions which are internal to the element, whose normal (or tangencial) components vanish over all edges. Therefore, H(div) (or H(curl)) conforming spaces can be created by simply imposing that the sum of the multiplying coefficients associated with the edge vector functions of neighboring elements is zero.

The finite element spaces obtained by our proposed procedure differs from the ones derived via De Rham diagrams. For instance, in a quadrilateral element, the current approximating spaces have dimension $(p+1)^2$, while using De Rham diagrams [5] the dimension is $2(p+2)(p+1)$. The main difference occurs in the number of internal basic functions, which sum p^2-1 in the current proposal, and $2p(p+1)$ in finite element spaces constructed in [5].

The outline of the paper is the following. Section 2 is dedicated to the construction of the H(div) approximation spaces of any degree p. We present the H^1 hierarchical scalar basic functions, both for quadrilateral and triangular functions. For both cases, the appropriate vector fields are presented and the resulting hierarchical vector bases are defined, and their principal properties are described. In Sect. 3 the H(curl) case is briefly considered, since in bidimensional regions this setting is derived from the H(div) case by simply rotating the corresponding vector field by $\pi/2$. The conclusions of the present paper are presented in Sect. 4.

2 H(div) Approximation Spaces

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary $\partial\Omega$. The purpose in this section is to construct approximations of the H(div) space

$$H\,(div;\Omega) = \left\{\overrightarrow{\varphi} \in L^2\,(\Omega)^2 : div\left(\overrightarrow{\varphi}\right) \in L^2\,(\Omega)\right\}. \tag{1}$$

by piecewise polynomials of high degree based on a partition $\mathscr{T}_h$ of Ω formed by polygonal elements (triangular or quadrilateral). For each $K \in \mathscr{T}_h$, $\mathscr{V}$ is the set of vertices a_k, $\mathscr{E}$ is the set of edges l_k, and C is the surface element . If $\mathscr{P}_p(K)$ denotes the polynomial space of degree at most p on K, the aim is to construct subspaces $V(\mathscr{T}_h) \subset H\,(div;\Omega)$ of the form

$$V(\mathscr{T}_h) = \left\{\overrightarrow{\varphi} : \overrightarrow{\varphi}|_K \in \mathscr{P}_p(K) \times \mathscr{P}_p(K), \forall\, K \in \mathscr{T}_h\right\}. \tag{2}$$

In order to built from $V(\mathscr{T}_h)$ an approximation of $H\,(div;\Omega)$, it will be necessary to impose continuity of the normal components $\overrightarrow{\varphi}.\overrightarrow{\eta}$ at the interfaces of the elements.

2.1 H(div): Quadrilateral Elements

Let $\hat{K} = \{(\xi,\eta) : -1 \leq \xi, \eta \leq 1\}$ be the master element with vertices $a_0 = (-1,-1)$, $a_1 = (1,-1)$, $a_2 = (1,1)$ and $a_3 = (-1,1)$. The edges $l_k, k = 0, \cdots 3$ correspond to the sides linking the vertices a_k to $a_{k+1(mod 4)}$.

In [3], a hierarchy of finite element subspaces in $H^1(\Omega)$ is constructed, where the basic functions in $\hat{K}$ are classified by:

- 4 vertex functions

$$\varphi^{a_0}(\xi,\eta) = \frac{(1-\xi)}{2}\frac{(1-\eta)}{2}, \quad \varphi^{a_1}(\xi,\eta) = \frac{(1+\xi)}{2}\frac{(1-\eta)}{2} \tag{3}$$

$$\varphi^{a_2}(\xi,\eta) = \frac{(1+\xi)}{2}\frac{(1+\eta)}{2}, \quad \varphi^{a_3}(\xi,\eta) = \frac{(1-\xi)}{2}\frac{(1+\eta)}{2} \tag{4}$$

  Note that the value of φ^{a_k} is one at a_k and zero at the other vertices.
- $4(p-1)$ edge functions

$$\begin{aligned}
\varphi^{l_0,n}(\xi,\eta) &= \varphi^{a_0}(\xi,\eta)[\varphi^{a_1}(\xi,\eta) + \varphi^{a_2}(\xi,\eta)]f_n(\xi),\\
\varphi^{l_1,n}(\xi,\eta) &= \varphi^{a_1}(\xi,\eta)[\varphi^{a_2}(\xi,\eta) + \varphi^{a_3}(\xi,\eta)]f_n(\eta),\\
\varphi^{l_2,n}(\xi,\eta) &= \varphi^{a_2}(\xi,\eta)[\varphi^{a_3}(\xi,\eta) + \varphi^{a_0}(\xi,\eta)]f_n(-\xi),\\
\varphi^{l_3,n}(\xi,\eta) &= \varphi^{a_3}(\xi,\eta)[\varphi^{a_0}(\xi,\eta) + \varphi^{a_1}(\xi,\eta)]f_n(-\eta),
\end{aligned}$$

  where f_n are the Chebychev polynomials of degree n, $n = 0, 1, \cdots, p-2$. The edge functions $\varphi^{l_k,n}$ vanish on all edges $l_m, m \neq k$;
- $(p-1)^2$ surface functions

$$\varphi^{C,n_0,n_1}(\xi,\eta) = \varphi^{a_0}(\xi,\eta)\varphi^{a_2}(\xi,\eta)f_{n_0}(\xi)f_{n_1}(\eta), \tag{5}$$

  with $0 \leq n_0, n_1 \leq p-2$. These functions are zero on all edges.

Let us consider a set of eighteen vectors $\overrightarrow{v}_m$, as indicated in Fig. 1, satisfying the properties

1. $\overrightarrow{v}_{2+3k} = \overrightarrow{\eta}_k$ is the outward unit normal, and $\overrightarrow{v}_{k+12}$ is tangent to l_k.
2. for $m = 3k$, $\overrightarrow{v}_m \cdot \overrightarrow{v}_{m+1} = \overrightarrow{v}_m \cdot \overrightarrow{v}_{m+2} = \overrightarrow{v}_{m+1} \cdot \overrightarrow{v}_{m+2} = 1$.
3. on the surface element, v_{16} and $\overrightarrow{v}_{17}$ are orthogonal vectors $\overrightarrow{v}_{16} \perp \overrightarrow{v}_{17}$.

We propose the construction of a family of vector functions by multiplication this vector field by the hierarchical scalar basis according to the following procedure:

$4(p+1)$ **edge vector functions**

$$\begin{aligned}
k = 0: &\quad \overrightarrow{\varphi}^{l_0,a_0} = \varphi^{a_0}\overrightarrow{v_0}, \quad \overrightarrow{\varphi}^{l_0,a_1} = \varphi^{a_1}\overrightarrow{v}_1, \quad \overrightarrow{\varphi}^{l_0,n} = \varphi^{l_0,n}\overrightarrow{v}_2 &(6)\\
k = 1: &\quad \overrightarrow{\varphi}^{l_1,a_1} = \varphi^{a_1}\overrightarrow{v_3}, \quad \overrightarrow{\varphi}^{l_1,a_2} = \varphi^{a_2}\overrightarrow{v}_4, \quad \overrightarrow{\varphi}^{l_1,n} = \varphi^{l_1,n}\overrightarrow{v}_5 &(7)\\
k = 2: &\quad \overrightarrow{\varphi}^{l_2,a_2} = \varphi^{a_2}\overrightarrow{v_6}, \quad \overrightarrow{\varphi}^{l_2,a_3} = \varphi^{a_3}\overrightarrow{v}_7, \quad \overrightarrow{\varphi}^{l_2,n} = \varphi^{l_2,n}\overrightarrow{v}_8 &(8)\\
k = 3: &\quad \overrightarrow{\varphi}^{l_3,a_3} = \varphi^{a_3}\overrightarrow{v_9}, \quad \overrightarrow{\varphi}^{l_3,a_0} = \varphi^{a_0}\overrightarrow{v}_{10}, \quad \overrightarrow{\varphi}^{l_3,n} = \varphi^{l_3,n}\overrightarrow{v}_{11} &(9)
\end{aligned}$$

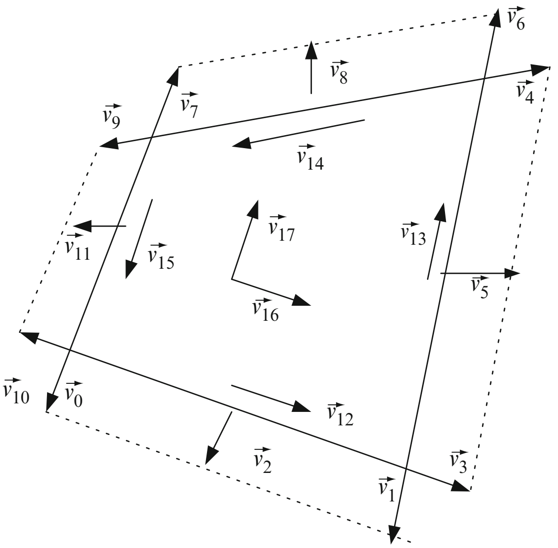

Fig. 1 Vector field for H(div)-quadrilateral elements

Observe that the vector functions associated to the edge l_0 satisfy

$$\overrightarrow{\varphi}^{l_0,a_0} \cdot \overrightarrow{\eta_0} = \varphi^{a_0} \in \mathscr{P}_1(K), \quad \overrightarrow{\varphi}^{l_0,a_1} \cdot \overrightarrow{\eta_0} = \varphi^{a_1} \in \mathscr{P}_1(K), \quad \overrightarrow{\varphi}^{l_0,n} \cdot \overrightarrow{\eta}_0 = \varphi^{l_0,n} \in \mathscr{P}_n(K). \tag{10}$$

Similar results hold for the vectors functions associated to l_k, $k = 1, 2$ and 3

$$\overrightarrow{\varphi}^{l_k,a_j} \cdot \overrightarrow{\eta_k} = \varphi^{a_j} \in \mathscr{P}_1(K), \quad \text{for } j = k, k+1(mod 4), \quad \overrightarrow{\varphi}^{l_k,n} \cdot \overrightarrow{\eta}_k = \varphi^{l_k,n} \in \mathscr{P}_n(K). \tag{11}$$

$2(p^2 - 1)$**internal vector functions**

To complete the space, we add three types of functions

$$\overrightarrow{\varphi}_1^{C,n_0,n_1} = \varphi^{C,n_0,n_1} \overrightarrow{v}_{16}, \quad \overrightarrow{\varphi}_2^{C,n_0,n_1} = \varphi^{C,n_0,n_1} \overrightarrow{v}_{17}, \quad \text{and} \quad \overrightarrow{\varphi}_3^{l_k,n} = \varphi^{l_k,n} \overrightarrow{v}_{k+12}. \tag{12}$$

The normal components of these internal vector functions vanishes at all edges.

The numbers of edge and internal vector functions sums $2(p+1)^2$, coinciding with the dimension of $V_K = \mathscr{P}_p(K) \times \mathscr{P}_p(K)$.

2.2 H(div): Triangular Elements

Consider the master triangular element $\hat{K} = \{(\xi, \eta) : 0 \leq \xi \leq 1, 0 \leq \eta \leq 1 - \xi\}$, with vertices $a_0 = (0, 0)$, $a_1 = (1, 0)$ and $a_2 = (0, 1)$, and edges $l_k, k = 0, 1, 2$

linking the vertex a_k to $a_{k+1(mod\,3)}$. For the hierarchy of finite element subspaces in $H^1(\Omega)$ constructed in [3], the basic functions are classified by:

- 3 vertex functions

$$\varphi^{a_0}(\xi,\eta) = 1-\xi-\eta, \quad \varphi^{a_1}(\xi,\eta) = \xi, \quad \varphi^{a_2}(\xi,\eta) = \eta \tag{13}$$

  that have unit value on the corresponding vertex and zero on the other ones;
- $3(p-1)$ edge functions

$$\varphi^{l_0,n}(\xi,\eta) = \varphi^{a_0}(\xi,\eta)\varphi^{a_1}(\xi,\eta) f_n(\eta+2\xi-1), \tag{14}$$
$$\varphi^{l_1,n}(\xi,\eta) = \varphi^{a_1}(\xi,\eta)\varphi^{a_2}(\xi,\eta) f_n(\eta-\xi), \tag{15}$$
$$\varphi^{l_2,n}(\xi,\eta) = \varphi^{a_2}(\xi,\eta)\varphi^{a_0}(\xi,\eta) f_n(1-\xi-2\eta) \tag{16}$$

- $\frac{(p-2)(p-1)}{2}$ surface functions

$$\varphi^{C,n_0,n_1}(\xi,\eta) = \varphi^{a_0}(\xi,\eta)\varphi^{a_1}(\xi,\eta)\varphi^{a_2}(\xi,\eta) f_{n_0}(2\xi-1) f_{n_1}(2\eta-1) \tag{17}$$

  with $0 \le n_0 + n_1 \le p-3$.

Consider a field of 14 vectors associated to a triangular element, as illustrated in Fig. 2. These vectors satisfy the properties

1. $\vec{v}_{2+3k} = \vec{\eta}_k$ is the outward unit normal, and $\vec{v}_{k+9}$ is tangent to the edge l_k.
2. for $m = 3k$, $\vec{v}_m \cdot \vec{v}_{m+1} = \vec{v}_m \cdot \vec{v}_{m+2} = \vec{v}_{m+1} \cdot \vec{v}_{m+2} = 1$.
3. $\vec{v}_{12} \perp \vec{v}_{13}$.

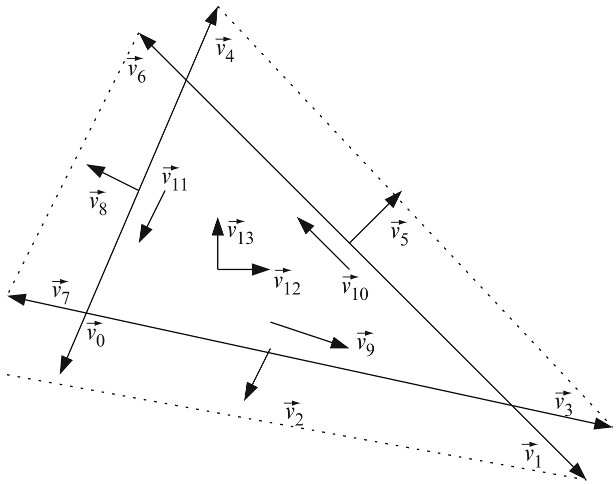

Fig. 2 Vector field for H(div)-triangular elements

As in the quadrilateral case, we introduce the vector functions associated to the edges

$$k = 0: \qquad \vec{\varphi}^{l_0,a_0} = \varphi^{a_0}\,\vec{v}_0, \quad \vec{\varphi}^{l_0,a_1} = \varphi^{a_1}\,\vec{v}_1, \quad \vec{\varphi}^{l_0,n} = \varphi^{l_0,n}\,\vec{v}_2 \tag{18}$$

$$k = 1: \qquad \vec{\varphi}^{l_1,a_1} = \varphi^{a_1}\,\vec{v}_3, \quad \vec{\varphi}^{l_1,a_2} = \varphi^{a_2}\,\vec{v}_4, \quad \vec{\varphi}^{l_1,n} = \varphi^{l_1,n}\,\vec{v}_5 \tag{19}$$

$$k = 2: \qquad \vec{\varphi}^{l_2,a_2} = \varphi^{a_2}\,\vec{v}_6, \quad \vec{\varphi}^{l_2,a_3} = \varphi^{a_3}\,\vec{v}_7, \quad \vec{\varphi}^{l_2,n} = \varphi^{l_2,n}\,\vec{v}_8 \tag{20}$$

and internal vector functions

$$\vec{\varphi}_1^{C,n_0,n_1} = \varphi^{C,n_0,n_1}\,\vec{v}_{12} \quad \vec{\varphi}_2^{C,n_0,n_1} = \varphi^{C,n_0,n_1}\,\vec{v}_{13} \quad \vec{\varphi}_3^{l_k,n} = \varphi^{l_k,n}\,\vec{v}_{9+k}. \tag{21}$$

Again, the normal components of the vector functions associated to the edge l_k are given by

$$\vec{\varphi}^{l_k,a_j}\cdot\vec{\eta_k} = \varphi^{a_j} \in \mathscr{P}_1(K), \quad \text{for } j = k, k+1(mod 3), \quad \vec{\varphi}^{l_k,n}\cdot\vec{\eta}_k = \varphi^{l_k,n} \in \mathscr{P}_n(K), \tag{22}$$

and the normal components of the internal vector functions vanish at all edges. Furthermore, for triangular elements the total number of edge and internal vector functions is $(p+1)(p+2)$, also coinciding with the dimension of $V_K = \mathscr{P}_p(K) \times \mathscr{P}_p(K)$.

Having defined the two set of hierarchical vector functions in V_K, both for quadrilateral and triangular elements, it remains to verify that they indeed form bases for V_K. Furthermore, if they are supposed to be combined to span H(div) spaces $V(\mathscr{T}_h)$, we need to show that the normal components on the elements interfaces are continuous.

Theorem 1. *The edge and internal vector functions defined in (6–12), for quadrilateral elements, and in formulae (18–21) for triangular elements, form a hierarchical basis for V_K.*

Proof. Since the cardinality of such bases coincide with the dimension of V_K, it only remains to prove their linear independency. Let us consider the linear combination

$$\sum_{l_k\in\mathscr{E}}\sum_{j=k}^{k+1(mod 4)} \alpha_j^k \vec{\varphi}^{l_k,a_j} + \sum_{l_k\in\mathscr{E}}\sum_{n=0}^{p-2} \beta_n^k \vec{\varphi}^{l_k,n} + \sum_{m=1}^{2}\sum_{n_0=0}^{p-2}\sum_{n_1=0}^{p-2} \gamma_m^{n_0,n_1} \vec{\varphi}_m^{C,n_0,n_1} + \sum_{l_k\in\mathscr{E}}\sum_{n=0}^{p-2} \mu_n^k \vec{\varphi}_3^{l_k,n} = 0.$$

Restricting to the edge l_k and doing the inner product with $\vec{\eta}_k$ we obtain

$$\sum_{j=k}^{k+1(mod 4)} \alpha_j^k \varphi^{a_j} + \sum_{n=0}^{p-2} \beta_n^k \varphi^{l_k,n} = 0.$$

Using the linear independency of φ^{a_j} and $\varphi^{l_k,n}$, we conclude that $\alpha_j^k = \beta_j^k = 0$. Next, considering the tangencial component associate to l_k we obtain that, restricted to this edge,

$$\sum_{n=0}^{p-2} \mu_n^k \varphi^{l_k,n} = 0,$$

implying that $\mu_n^k = 0$. Finally, doing the inner product with ν_{16}, and then with ν_{17}, we obtain

$$\sum_{n_0=0}^{p-2} \sum_{n_1=0}^{p-2} \gamma_1^{n_0,n_1} \varphi_1^{C_{n_0,n_1}} = \sum_{n_0=0}^{p-2} \sum_{n_1=0}^{p-2} \gamma_2^{n_0,n_1} \varphi_2^{C_{n_0,n_1}} = 0$$

to get $\gamma_m^{n_0,n_1} = 0$, and conclude the proof. □

Theorem 2. *Using the hierarchical vector bases defined by (6–12), for quadrilateral elements, and in formulae (18–21) for triangular elements, H(div)-conforming spaces $V(\mathscr{T}_h)$ can be created by imposing that the sum of the multiplying coefficients associated with the edge vector functions of neighboring elements is zero .*

Proof. Let $\overrightarrow{\varphi} \in V(\mathscr{T}_h)$, and K_i and K_j be two elements that share a commom edge l_k. To verify that the quantity $\overrightarrow{\varphi} \cdot \overrightarrow{\eta}_k$ is continuous across the edge l_k, taking into account the the internal vector functions have vanishing normal components on all edges, it is only necessary to verify whether the contributions of the normal components of the edge vector functions associated to l_k can be made compatible. But for these functions we have already seen that $\overrightarrow{\varphi}^{l_k,a_j} \cdot \overrightarrow{\eta_k} = \varphi^{a_j}$, and $\overrightarrow{\varphi}^{l_k,n} \cdot \overrightarrow{\eta_k} = \varphi^{l_k}$. Therefore, since the outward unit norm changes its sign from K_i to K_j , for the normal component to be continuous it is sufficient that the sum of the coefficients multiplying the edge functions is zero. □

3 H(curl) Approximation Spaces

Now we turn to the question concerning the construction of approximations of the H(curl) space

$$H\left(\mathbf{curl}; \Omega\right) = \left\{\overrightarrow{v} \in L^2\left(\Omega\right)^2 : \mathbf{curl}\left(\overrightarrow{v}\right) \in L^2\left(\Omega\right)^2\right\} \tag{23}$$

In order to construct piecewise polynomial vector subspaces $V(\mathscr{T}_h) \subset H\left(\mathbf{curl}; \Omega\right)$ of the form (2), it will be necessary to impose continuity of the tangencial components at the interfaces of the elements K. Similar to the H(div) case, a systematic procedure consists in first choosing an appropriate vector field, based on the geometry of the elements, and then multiply it by the set of H^1 hierarchical scalar basic functions. In order to guarantee the continuity of the tangencial components of the functions $V(\mathscr{T}_h)$ on the interfaces of the elements, such H(curl) vector field can be obtained by a $\pi/2$ rotation of the H(div) vector field, as shown in Fig. 3.

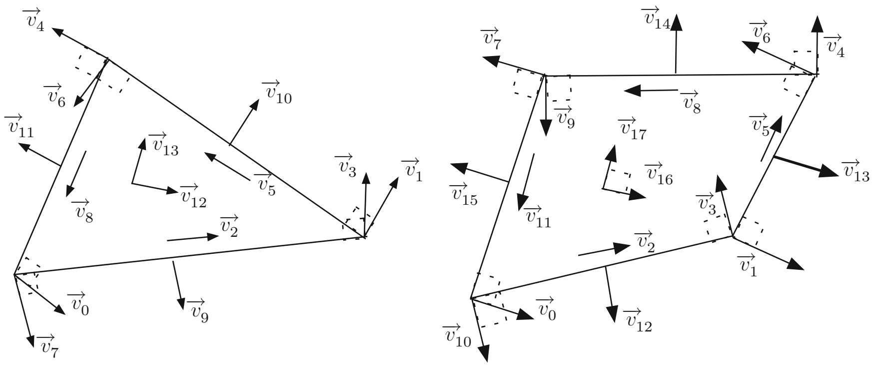

Fig. 3 Vector field for H(curl) 2D elements

4 Conclusion

We present a systematic procedure to construct hierarchical bases for H(div) and H(curl) approximation spaces which are consistent with the proprieties of these spaces. The geometrical properties of the elements are strongly used for the definition of vector fields, which are multiplied by consistent H^1 scalar functions already developed, to obtain continuity of the normal or tangencial components of the resulting vector functions. As perspectives, we plan to extend the construction of H(div) and H(curl) approximation space for elements with curvilinear boundaries and to study the stability of the mixed finite element method using these bases.

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