Indraprastha Institute of Information Technology, Delhi
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Computer Science and Engineering

1D Higher Order Finite Element Method

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MTH598 - Numerical Partial Differential Equations

Course Project Report

INDRAPRASTHA INSTITUTE *of* INFORMATION TECHNOLOGY DELHI

Contents

1 Outline

In class, we derived the Finite Element Method for the linear 1D case with the help of the Poisson equation. In this course project, I have studied the following -

- Concepts of shape functions, matrix assembly and the 1D Linear FEM in detail.
- Lagrange $\mathbb{P}_2$ basis elements - their motivation, usage, practical implementation and convergence
- Extension of the general FEM to the Lagrange $\mathbb{P}_k$ basis element case

After presenting the topics that I have understood, I have also implemented the Linear and the Quadratic 1D FEMs on the same problem and compared the two solutions - visually, and with the help of L^2 error.

The content in my report is heavily based on my reference chapter provided for this project - *Chapter 7 - Finite Element Approximation*

2 Lagrange $\mathbb{P}_1$ basis

Consider the following boundary value poisson problem, defined on the domain $\Omega =]0, 1[$ -
Given $f \in L^2(\Omega)$ and $c \in L^\infty(\Omega)$ find the function u, such that,

$$\begin{cases} -u''(x) = f(x), x \in \Omega \\ u(0) = u(1) = 0 \end{cases}$$

We define the following terms -

- Mesh - A set of points $(x_j)_{0 \leq j \leq N+1}$
- Interval K_j - $K_j = [x_j, x_{j+1}]$
- h (The size parameter) - Assuming that we have a uniform mesh, $h = |x_{j+1} - x_j|$

The definitions of V_h^1 and $V_{0,h}^1$ hold as per the class discussion.

2.1 Development of the Linear basis

Every function $v_h \in V_h^1$ can be uniquely determined by its value at the mesh verticies, such that -

$$v_h(x) = \sum_{j=0}^{N+1} v_h(x_j)\varphi_j(x), \forall x \in \Omega$$

where $(\varphi_j)_{0 \leq j \leq N+1}$ is the basis of the shape functions φ_j in each interval $[x_j, x_{j+1}]$ defined as follows -

$$\varphi_j(x) = \begin{cases} \frac{x - xj - 1}{h}, x \in [x_{j-1}, x_j] \\ \frac{x_{j+1} - x}{h}, x \in [x_j, x_{j+1}] \\ 0, \ elsewhere \end{cases}$$

We also notice that $\varphi_j(x_i) = \delta_{ij}$ where δ_{ij} is the Kronecker's delta.

A cartoon representation of Basis functions for the space V_h^1 can be seen in Figure 1.

We can now represent $(\varphi_j)_{0 \leq j \leq N+1}$ using two functions, that we term as the Shape functions -

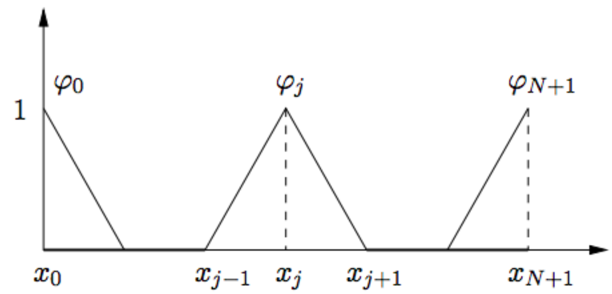

Figure 1: Basis functions for the space V_h^1

$$\omega_0(x) = 1 - x, \qquad \omega_1(x) = x$$

Thus, the Basis function φ_j can now be re-written as -

$$\varphi_j(x) = \begin{cases} \omega_1\left(\frac{x - xj-1}{x_j - xj-1}\right), x \in [x_{j-1}, x_j] \\ \omega_0\left(\frac{x - x_j}{x_{j+1} - xj}\right), x \in [x_j, x_{j+1}] \\ 0, \ elsewhere \end{cases}$$

and $\varphi_{N+1}(x) = \omega_1((x - x_N)/h)$.

This method of representing the basis functions help us in assembling the matrix A of our linear system that we will solve. This method also helps in translating our ideas from the Lagrange $\mathbb{P}_1$ basis to the Lagrange $\mathbb{P}_k$ basis.

2.2 Linear FEM

The weak formulation of the internal approximation, as done in class, consists of finding a $u_h \in V_{0,h}^1$ such that -

$$\int_\Omega u_h^{'}(x) v_h^{'}(x) dx = \int_\Omega f(x) v_h(x) dx, \qquad \forall v_h \in V_{0,h}^1$$

Let $u_h(x_j)_{1 \leq j \leq N}$ be the approximate value of the exact solution at the mesh vertex x_j, which leads to the approximate problem as follows -
Find $u_h(x_i) \forall i = 1, \ldots, N$ such that -

$$\sum_{j=1}^{N}\Big(\int_{\Omega}\varphi_j^{'}(x)\varphi_i^{'}(x)dx\Big)u_h(x_j)=\int_{\Omega}f(x)\varphi_i(x)dx, \qquad \forall v_h\in V_{0,h}^{1}$$

This formulation is now reduced to solving the $\mathbb{R}^N$ linear system -

$$A_hU_h=F_h$$

,

where $U_h = u_h(x_j)_{1\le j\le N}$, $F_h = (\int_{\Omega} f(x)\varphi_i(x)dx)_{1\le i\le N}$ and the matrix $A_h = (\int_{\Omega}\varphi_j^{'}(x)\varphi_i^{'}(x)dx)_{1\le i,j\le N}$

2.2.1 Finding Coefficients of A_h

Since the shape functions φ_j are defined in such a way that they have a small support, it can be intuitively seen that the matrix A_h will have most of its values are zeroes. A_h will have non-zero values only at three consecutive indices - $j-1, j, j+1$. Hence, the structure of the matrix A_h is a tri-diagonal matrix.

Now, we can see that -

$$\varphi_j^{'}(x)=\begin{cases}1, \quad x\in[x_{j-1},x_j]\\ -1, \quad x\in[x_j,x_{j+1}]\\ 0, elsewhere\end{cases}$$

Hence, we have the non-zero coefficients of A_h as -

$$a_{j,j}=\int_{x_{j-1}}^{x_{j+1}}(\varphi_j^{'}(x))^2dx=\frac{2}{h}$$

$$a_{j-1,j}=\int_{x_{j-1}}^{x_j}\varphi_j^{'}(x)\varphi_{j-1}^{'}(x)dx=\frac{-1}{h}$$

$$a_{j,j+1}=\int_{x_j}^{x_{j+1}}\varphi_j^{'}(x)\varphi_{j+1}^{'}(x)dx=\frac{-1}{h}$$

Now these $a_{j,j}, a_{j-1,j}, a_{j,j+1}$ are node contributions,i.e, they are based on the global indices $j-1, j, j+1$. Instead of analysing global indices, we can analyse individual elements $K_j = [x_j, x_{j+1}]$.

Let us analyse the element K_j. In this element, we can see that -

$$\varphi_j^{'}|K_j = \frac{-1}{x_{j+1} - x_j} = \frac{-1}{h}$$

$$\varphi_{j+1}^{'}|K_j = \frac{1}{x_{j+1} - x_j} = \frac{1}{h}$$

$\therefore$ we arrive at an elementary matrix EK_j, which is defined as follows -

$$EK_j = \begin{bmatrix} k_{11}^j & k_{12}^j \\ k_{21}^j & k_{22}^j \end{bmatrix} = \frac{1}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

2.2.2 Matrix Assembly for A_h

Once we have arrived at the elementary matrices for A_h, we would be able to assemble the entire A_h matrix. The pseudocode for the same is as follows -

Algorithm 1 Matrix Assembly for A_h for 1D Linear FEM

```
for k = 1, ..., N+1 do // loop over all elements
    for i=1,2 do // local loop
        for j=1,2 do
            ig = k+i-2 // global indices
            jg = k+j-2
            A(ig,jg) = A(ig, jg) + EK(i,j)
        end for
    end for
end for
```

The intution for this matrix assembly is as follows - We take a `2x2` elementary matrix. We initialise A_h with zeros. We slide the `2x2` elementary matrix diagonally, while adding the overlapping elements. Once this sliding is over, our A_h is succesfully constructed.

2.2.3 Finding Coefficients of F_h

For the vector $F_h \in \mathbb{R}^n$, we get -

$$f_i = \sum_{k=0}^{N} \int_{x_k}^{x_{k+1}} f(x)\varphi_i(x)dx$$

Since $f(x) = \sum_{j=1}^{N-1} f_j \varphi_j(x)$, we get -

$$f_i = \sum_{k=0}^{N} \int_{x_k}^{x_{k+1}} \sum_{j=1}^{N-1} f_j \varphi_j(x)\varphi_i(x)dx$$

which reduces to finding the integrals -

$$\int_{x_k}^{x_{k+1}} \varphi_j(x)\varphi_i(x)dx$$

To solve the above integral, we use the **Trapeze Formula**, which is as follows -

$$\int_{x_k}^{x_{k+1}} \theta(x)dx = \frac{x_{k+1} - x_k}{2}(\theta(x_{k+1}) + \theta(x_k))$$

Thus we get that $f_j = hf(x_j)$. We can thus construct the entire $F_h \in \mathbb{R}^n$. Thus, the final solution U_h is obtained by solving the linear system $A_h U_h = F_h$ using any solver.

3 Lagrange $\mathbb{P}_2$ basis

We motivate the use of Lagrange $\mathbb{P}_2$ basis by indicating that higher-order FEMs approximate smooth functions better than linear FEMs. At the same time, less regular functions are better approximated by lower-degree FEMs.

3.1 Development of the Quadratic basis

We let the previous definitions of the Mesh and the Intervals as is. The domain remains the same - $\Omega =]0, 1[$. We define two spaces - V_h^2 and $V_{0,h}^2$ as follows -

$$V_h^2 = \{v_h \in C^0([0, 1]), v_h|K_j \in \mathbb{P}_2, 0 \leq j \leq N\}$$

$$V_{0,h}^2 = \{v_h \in V_h^2 \texttt{ such that } v_h(0) = v_h(1) = 0\}$$

Every function $v_h \in V_h^2$ is uniquely determined by its values at the mesh vertices $(x_j)_{0\leq j\leq N+1}$ and now, also at the midpoints $(x_{j+\frac{1}{2}})_{0\leq j\leq N} = (x_j + \frac{h}{2})_{0\leq j\leq N}$.

Therefore we have -

$$v_h(x) = \sum_{j=0}^{N+1} v_h(x_j)\varphi_j(x) + \sum_{j=0}^{N} v_h(x_{j+\frac{1}{2}})\varphi_{j+\frac{1}{2}}(x), \qquad \forall x \in \Omega$$

where $(\varphi_j)_{0\leq j\leq N+1}$ is the basis function defined as -

$$\varphi_j(x) = \phi\big(\frac{x - x_j}{h}\big), 0 \leq j \leq N+1$$

and

$$\varphi_{j+\frac{1}{2}}(x) = \psi\big(\frac{x - x_{j+\frac{1}{2}}}{h}\big), 0 \leq j \leq N$$

where,

$$\phi(x) = \begin{cases} (1+x)(1+2x), -1 \leq x \leq 0 \\ (1-x)(1-2x), 0 \leq x \leq 1 \\ 0, |x| > 1 \end{cases} \quad and \quad \psi(x) = \begin{cases} 1 - 4x^2, |x| \leq \frac{1}{2} \\ 0, |x| > \frac{1}{2} \end{cases}$$

We can thus notice that, $\varphi_i(x_j) = \delta_{ij}$, $\varphi_i(x_{j+\frac{1}{2}}) = 0$, $\varphi_{i+\frac{1}{2}}(x_j) = 0$ and $\varphi_{i+\frac{1}{2}}(x_{j+\frac{1}{2}}) = \delta_{ij}$

A cartoon representation of Basis functions for the space V_h^2 can be seen in Figure 2.

3.2 Quadratic FEM

The weak formulation of the internal approximation of the boundary-valued poisson problem now consists of finding a $u_h \in V_{0,h}^2$ such that -

$$\int_\Omega u_h^{'}(x)v_h^{'}(x)dx = \int_\Omega f(x)v_h(x)dx$$

As we know that, we can write $u_h(x)$ as

$$u_h(x) = \sum_{j=0}^{N+1} u_h(x_j)\varphi_j(x) + \sum_{j=0}^{N} u_h(x_{j+\frac{1}{2}})\varphi_{j+\frac{1}{2}}(x), \qquad \forall x \in \Omega$$

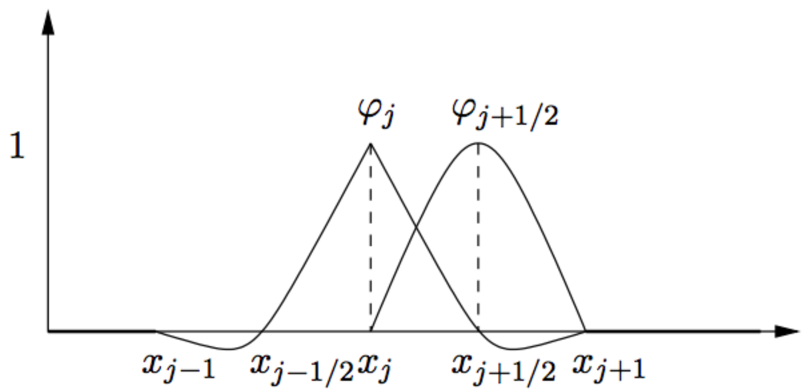

Figure 2: Basis functions for the space V_h^2

Thus, we apply an index manipulation to use $(x_{\frac{k}{2}})_{1\leq k\leq 2N+1}$ for the mesh points and $(\varphi_{\frac{k}{2}})_{1\leq k\leq 2N+1}$ for the basis elements. We thus have -

$$u_h(x) = \sum_{k=1}^{2N+1} u_h(x_{\frac{k}{2}})\varphi_{\frac{k}{2}}(X)$$

We thus lead to a $\mathbb{R}^{2N+1}$ linear system -

$$A_h U_h = F_h$$

where $U_h = (u_h(x_{\frac{k}{2}}))_{1\leq k\leq 2N+1}$, $A_h = (\int_\Omega \varphi'_{\frac{k}{2}}(x)\varphi'_{\frac{l}{2}}(x)dx)_{1\leq k,l\leq 2N+1}$ and $F_h = \int_\Omega f(x)\varphi_{\frac{k}{2}}(x)dx$

3.2.1 Finding Coefficients of A_h

Since the shape functions φ_j are defined in such a way that they have a small support, it can be intuitively seen that the matrix A_h will have most of its values as zeroes.

We now apply a *change of variables for* $t \in [-1, 1]$ such that -

$$x = x_{j+1} + \frac{x_{j+2} - x_j}{2}t = x_{j+1} + \frac{h}{2}t$$

Using this change of variable, we arrive at 3 shape functions, defined as -

$$\omega_{-1}(t) = \frac{t(t-1)}{2} \; and \;\; \omega_0(t) = -(t-1)(t+1) \; and \;\; \omega_1(t) = \frac{t(t+1)}{2}$$

and thus, we can re-write the basis functions with the help of the shape functions -

$$\phi(x) = \begin{cases} \omega_1(\frac{x-x_{j+1}}{h}2), -1 \leq x \leq 0 \\ \omega_{-1}(\frac{x-x_{j+1}}{h}2), 0 \leq x \leq 1 \\ 0, |x| > 1 \end{cases} \quad and \;\; \psi(x) = \begin{cases} \omega_0(\frac{x-x_{j+1}}{h}2), |x| \leq \frac{1}{2} \\ 0, |x| > \frac{1}{2} \end{cases}$$

Also, we can see that $\frac{dt}{dx} = \frac{2}{h}$ and $\frac{d\varphi_j}{dx} = \frac{d\omega_k}{dt}\frac{dt}{dx}$, where -

$$\frac{d\omega_{-1}}{dt}(t) = \frac{2t-1}{2} \; and \;\; \frac{d\omega_0}{dt}(t) = -2t \; and \;\; \frac{d\omega_1}{dt}(t) = \frac{2t+1}{2}$$

Thus, we get our elementary matrix EK_j as -

$$EK_j = \frac{1}{3h}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}$$

3.2.2 Matrix Assembly for A_h

Once we have arrived at the elementary matrices for A_h, we would be able to assemble the entire A_h matrix. The pseudocode for the same is in Pseudocode 2.

3.2.3 Finding Coefficients of F_h

Using the previous approach for finding the coefficients of F_h, we obtain -

$$f_i = \sum_{j=0}^{2N} f_j \Big(\sum_{k=1}^{N} \int_{x_{2k-2}}^{x_{2k}} \varphi_j(x)\varphi_i(x)dx \Big)$$

Thus, we need to compute the integral $\int_{x_{2k-2}}^{x_{2k}} \varphi_j(x)\varphi_i(x)dx$

Algorithm 2 Matrix Assembly for A_h for 1D Quadratic FEM

```
for k = 1, ..., N+1 do // loop over all elements
    for i=1,2,3 do // local loop
        for j=1,2,3 do
            ig = 2k+i-3 // global indices
            jg = 2k+j-3
            A(ig,jg) = A(ig, jg) + EK(i,j)
        end for
    end for
end for
```

This can be calculated using Quadratures, which have been used (giving due credit) from the sources in the internet, as I am not yet familiar with the concept. $\therefore$ we have -

$$\begin{bmatrix} f_i^k \\ f_{i+1}^k \\ f_{i+2}^k \end{bmatrix} = \frac{h}{30} \begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} f_i \\ f_{i+1} \\ f_{i+2} \end{bmatrix}$$

Thus, after assembling A_h and F_h, we can simply use a linear solver to find U_h

3.3 Convergence of Lagrange $\mathbb{P}_2$ FEM

We will state here, without the proof -

$$\|r_h u\|^2_{H^1_0(\Omega)} \leq \frac{h^4}{4} \|u'''\|_{L^2(\Omega)}$$

where $r_h u = u - w_h$ and $w_h(x_i) = u(x_i)$

Thus, we can see that the Lagrange $\mathbb{P}_2$ FEM converges, i.e, we have -

$$\lim_{h \to 0} \|u - u_h\|_{H^1(\Omega)} = 0$$

Thus, we can note that the convergence rate of the $\mathbb{P}_2$ FEM is better than that of the $\mathbb{P}_1$, provided that f is sufficiently smooth.

4 A note on the Lagrange $\mathbb{P}_k$ basis

We can extend our previous idea of Quadratic FEMs to all polynimial functions which degree $k \geq 1$.

We define the two spaces - V_h^k and $V_{0,h}^k$ as follows -

$$V_h^k = \{v_h \in C^0([0,1]), v_h|K_j \in \mathbb{P}_k, 0 \leq j \leq N\}$$

$$V_{0,h}^k = \{v_h \in V_h^k \texttt{ such that } v_h(0) = v_h(1) = 0\}$$

Every function $v_h \in V_h^k$ is uniquely determined by $k+1$ points in an interval K_j. Apart from the mesh verticties, we add new additions verticies in the line segment of the interval K_j which are given by -

$$y_{j,l} = x_j + \frac{l}{k}h, \quad 0 \leq l \leq k-1$$

and $y_{N+1,0} = x_{N+1}$.

Also, we have the basis functions defined as follows -

$$\varphi_{j,l}(y_{j',l'}) = \delta_{jj'}\delta_{ll'}$$

Thus, by the approaches detailed earlier, we can assemble A_h and F_h to solve the linear system $A_hU_h = F_h$.

For the convergence of the $\mathbb{P}_k$ FEM, we have (without proof) -

$$\lim_{h \to 0} \|u - u_h\|_{H^1(\Omega)} = 0$$