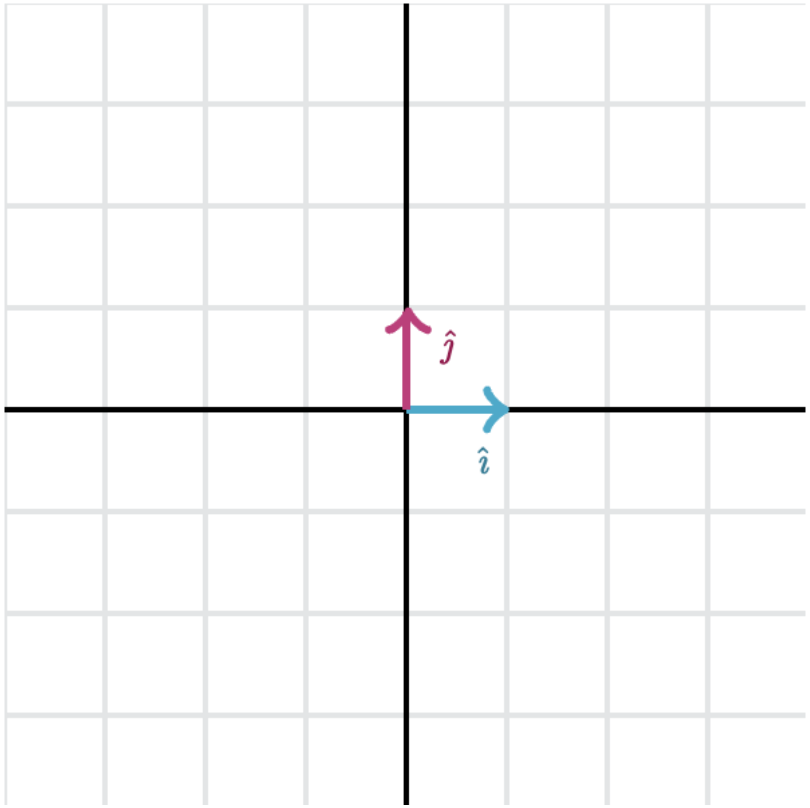
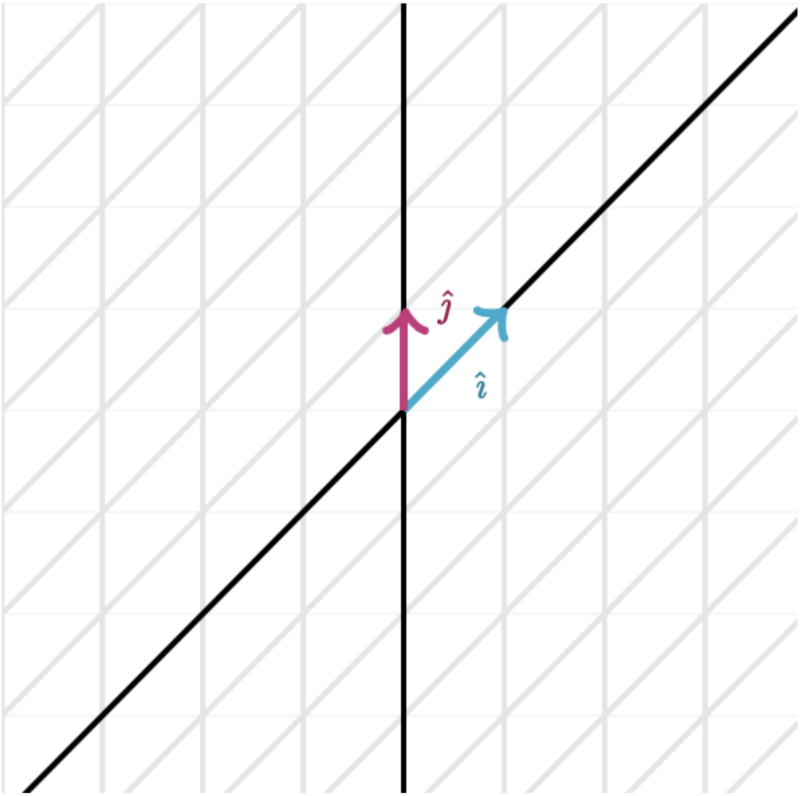
Content:

Matrix Multiplication

- As shown in the diagram below, consider a plane, along with the unit $\hat{i}$ and $\hat{j}$, which stands for $(1, 0)$ and $(0, 1)$



- Consider a matrix A = $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$
- This is how matrix acts upon grid (another name for the coordinate plane consisting of a space of small squares):
    - The columns of the matrix tell us where it moves the unit vectors $\hat{i}$ and $\hat{j}$, which again stands for $(1, 0)$ and $(0, 1)$.
    - The rest of the grid follows accordingly, always keeping grid lines parallel and evenly spaced. The origin stays frozen in place.
- That means A moves $\hat{i} \rightarrow (1, 1)$ and $\hat{j} \rightarrow (0, 1)$. This is how it looks like:



- The unit vector $\hat{j}$ didn't move because it started at $(0, 1)$. The unit vector $\hat{i}$ moved upward one unit, and this dragged the grid with it.
- Notice that there's a faint copy of the original lines in the background to help us stay oriented.
- Let's see the same process for one more matrix. Consider B = $\begin{bmatrix} 0 & -1 \\ -2 & 1 \end{bmatrix}$

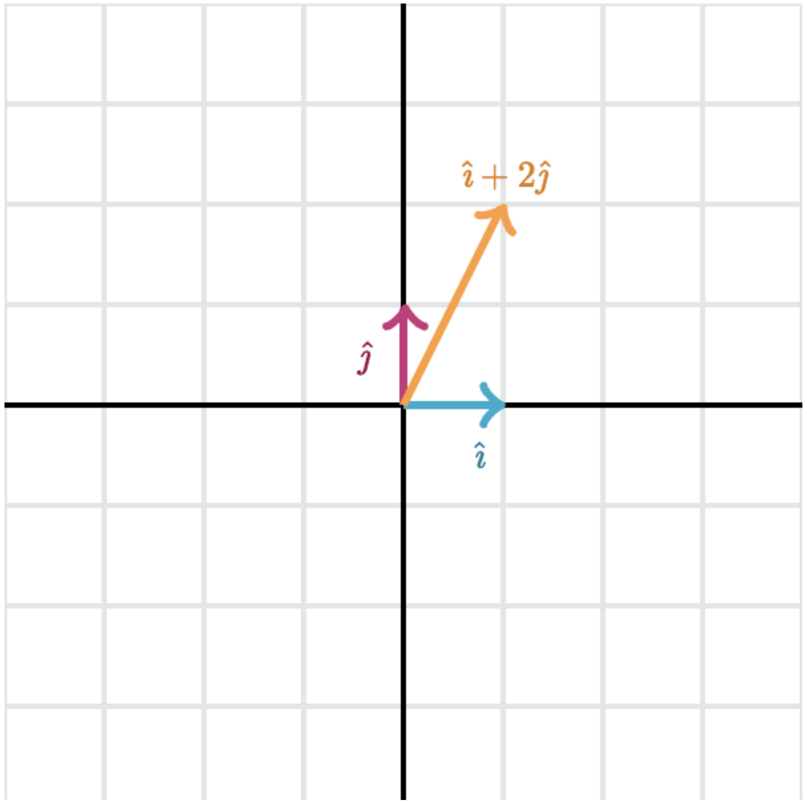
- We know B moves $\hat{i} \rightarrow (0, -2)$ and $\hat{j} \rightarrow (-1, 1)$. That looks like this:
- This is known as a **action of a matrix**. So, the action of a matrix is to move the entire grid.We can understand it by thinking about how it moves the unit vectors.
- These ideas extend into three dimensions as well. The third row of the matrix contains z-coordiantes for all the unit vectors, and the third column of the matrix tells us where $\hat{k}$ lands.

NOTE: This is an interactive tool that you can use to play around with matrices as movement Drag the vectors to make the grid move, and see the matrix that corresponds to the movement in the top left corner.

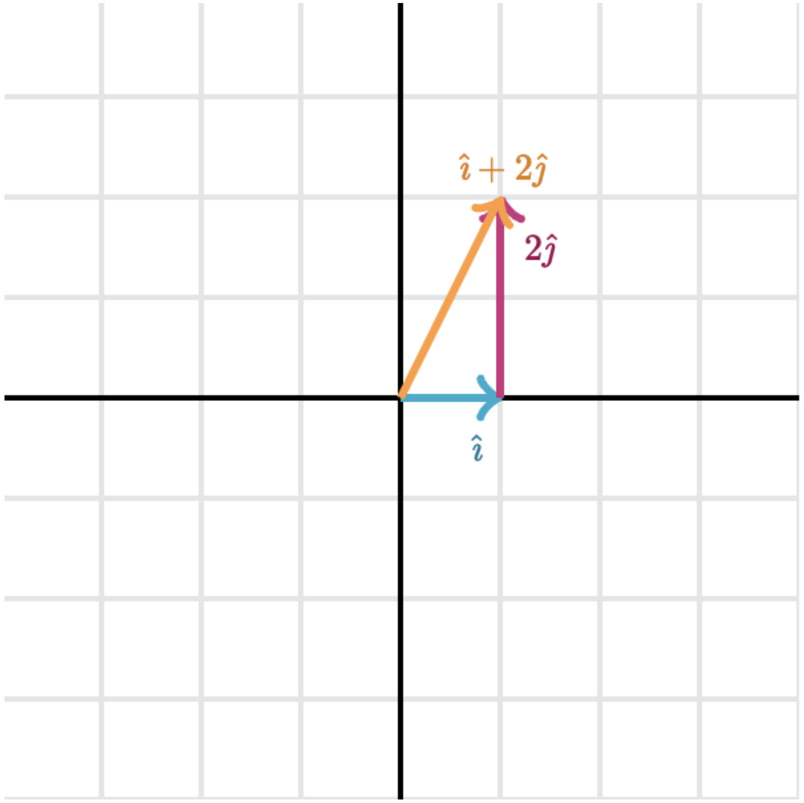
- Link: [Interactive tool]. ---> Click on **Spin-Off** button for movements.

Matrix-Vector Multiplication

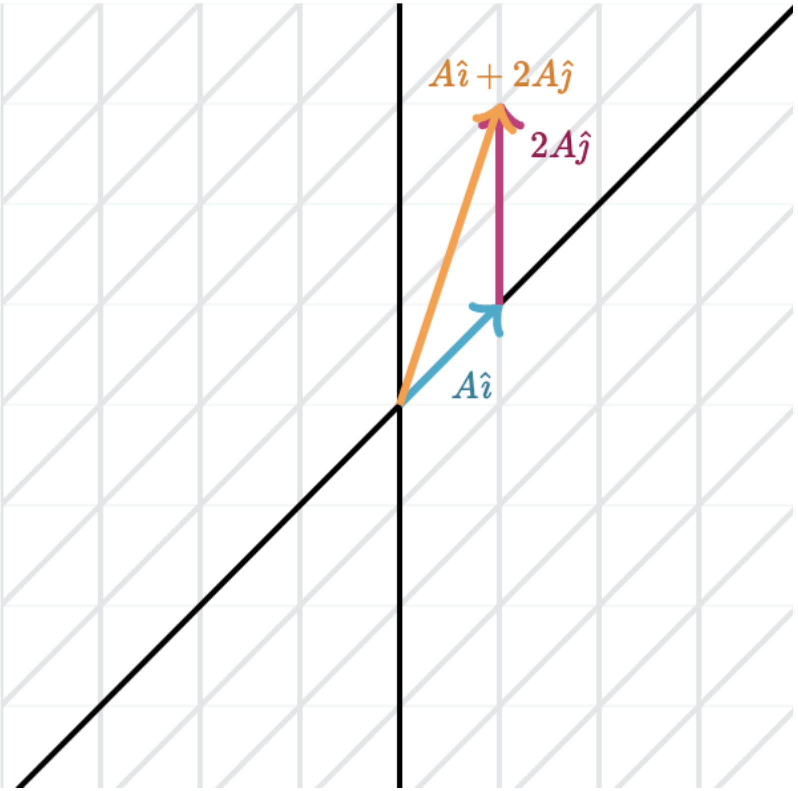
- Now that we know how a given matrix moves the unit vectors $\vec{i}$ and $\vec{j}$ but how can we find where a matrix moves any arbitrary vector?
- Let's consider a specific example using the first matrix from the previous section. A = $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$
- Now, How does A move the non-unit vector $(1, 2)$? Before anything, let's get a feel for this visually. First, the vector with no matrix movement:



- We represent vector $(1, 2)$ as a combination of the unit vectors by saying $(1, 2)$ = $1\vec{i} + 2\vec{j}$. It will look something like this:



- This combination remains the same after we apply A, but instead of using $\vec{i}$ and $\vec{j}$, we use the result of applying A to $\vec{i}$ and $\vec{j}$:



- Here's what the process looks like in symbols:

$$A\begin{bmatrix} 1 \\ 2 \end{bmatrix} = A\left(1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$= A\left(1\hat{\imath} + 2\hat{\jmath}\right)$$

$$= 1A\hat{\imath} + 2A\hat{\jmath}$$

$$= 1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

- The critical step is when we break $A(1\vec{i} = +2\vec{j})$ into $1A\vec{i} + 2A\vec{j}$.
- That is when we are able to represent where $(1, 2) lands in terms of where$\vec{i}$and$\vec{i}$ land.

- Like in the section above, the idea behind matrices moving vectors extends into three dimensions.
- We just decompose our vector into a sum of $\vec{i}$, $\vec{j}$ and $\vec{k}$, then we use where the matrix takes all these unit vectors to find where it takes our vector.

NOTE: Here's another Interactive tool to play around with matrices moving vectors.

Matrix-Matrix Multiplication

- Consider two matrices A and B, where;
  - $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$
  - $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
- The product AB just means apply BBB, then apply A.
- When we apply A second, we treat the transformed $\vec{i}$ and $\vec{j}$ as regular vectors getting moved by AAA the way we learned in the Matrix-vector multiplication.
- To calculate the end result, we follow $\vec{i}$ and $\vec{j}$. First B takes $\vec{i} \to (0, 1)$ and $\vec{j} \to (-1, 0)$.
- Second, we find where A takes these vectors:

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

- Putting these in a matrix, we have the product. Notice that our calculations are reflected in the visual above:

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

- To conclude, we can think of matrix multiplication as composing the movements each matrix represents. When we follow unit vectors along these movements, we can calculate the product.