Chapter 4. Bayesian Analysis (C6)

♠ Admissibility of Bayes Rules

• **R-Better**

A decision rule δ_1 is R-better than a decision rule δ_2 if

$$R(\theta, \delta_1) \leq R(\theta, \delta_2) \text{ for all } \theta$$

and

$$R(\theta, \delta_1) < R(\theta, \delta_2) \text{ for some } \theta$$

• **Admissible and Inadmissible**

A decision rule δ is *admissible* if there exists no R-better decision rule. A decision rule δ is *inadmissible* if there does exist an R-better decision rule.

$\diamondsuit$ Main Results on Admissibility of Bayes Rules

Theorem 1: *Assume that Θ is discrete (say $\Theta = \{\theta_1, \theta_2, \dots\}$) and that the prior, π, gives positive probability to each $\theta_i \in \Theta$. A Bayes rule δ^π, with respect to π, is then admissible.*

Proof: If δ^π is inadmissible, then there exists a rule δ with

$$R(\theta_i, \delta) \leq R(\theta_i, \delta^\pi) \text{ for all } i,$$

with strict inequality for, say, θ_k. Hence,

$$r(\pi, \delta) = \sum_{i=1}^{\infty} R(\theta_i, \delta)\pi(\theta_i) < \sum_{i=1}^{\infty} R(\theta_i, \delta^\pi)\pi(\theta_i) = r(\pi, \delta^\pi),$$

the inequality being strict since $R(\theta_k, \delta) < R(\theta_k, \delta^\pi)$, $\pi(\theta_k) > 0$, and $r(\pi, \delta^\pi)$. This contradicts the fact that δ^π is Bayes. Therefore, δ^π must be admissible. $\square$

Theorem 2: *If a Bayes rule is unique, it is admissible.*

Proof: Let δ^π denote a Bayes rule with respect to π. If δ^π is inadmissible, then there exists a rule δ with

$$R(\theta, \delta) \leq R(\theta, \delta^\pi) \text{ for all } \theta,$$

with strict inequality for some θ. Hence,

$$r(\pi, \delta) = \int_\Theta R(\theta, \delta) dF^\pi(\theta)$$
$$\leq \int_\Theta R(\theta, \delta^\pi) dF^\pi(\theta) = r(\pi, \delta^\pi).$$

Thus, δ is Bayes, which contradicts the fact that a Bayes rule is unique. Therefore, δ^π must be admissible. □

Theorem 3: *Assume that the risk functions $R(\theta, \delta)$ are continuous for all decision rules δ. Assume also that the prior π gives positive probability to any open subset of Θ. Then a Bayes rule with respect to π is admissible.*

Proof: Let δ^π denote a Bayes rule with respect to π. If δ^π is inadmissible, then there exists a rule δ with

$$R(\theta, \delta) \leq R(\theta, \delta^\pi) \text{ for all } \theta,$$

with strict inequality for some θ. Let θ_0 denote a value of θ so that

$$R(\theta_0, \delta) < R(\theta_0, \delta^\pi).$$

Since $R(\theta, \delta)$ and $R(\theta_0, \delta^\pi)$ are continuous, there exists $\epsilon > 0$ so that for all $\theta \in I_{\theta_0} = (\theta_0 - \epsilon, \theta_0 + \epsilon)$,

$$R(\theta, \delta) < R(\theta, \delta^\pi).$$

Thus,

$$
\begin{aligned}
r(\pi,\delta) &= \int_{\Theta} R(\theta,\delta)dF^{\pi}(\theta) \\
&= \int_{\Theta - I_{\theta_0}} R(\theta,\delta)dF^{\pi}(\theta) + \int_{I_{\theta_0}} R(\theta,\delta)dF^{\pi}(\theta) \\
&< \int_{\Theta - I_{\theta_0}} R(\theta,\delta^{\pi})dF^{\pi}(\theta) + \int_{I_{\theta_0}} R(\theta,\delta^{\pi})dF^{\pi}(\theta) \\
&= \int_{\Theta} R(\theta,\delta^{\pi})dF^{\pi}(\theta) = r(\pi,\delta^{\pi}),
\end{aligned}
$$

which contradicts the fact that δ^{π} is Bayes. $\square$

Example 1: Assume $X \sim N(\theta,1)$ and $\theta \sim N(0,1)$, and

$$L(\theta,a) = \exp\left\{\frac{3\theta^2}{4}\right\}(\theta - a)^2.$$

It is easy to show that the posterior distribution $\pi(\theta|x)$ is $N\left(\frac{x}{2},\frac{1}{2}\right)$.

The Bayes rule is

$$\delta^{\pi} = \frac{\int_{-\infty}^{\infty} \exp\left\{\frac{3\theta^2}{4}\right\} \theta\pi(\theta|x)d\theta}{\int_{-\infty}^{\infty} \exp\left\{\frac{3\theta^2}{4}\right\} \pi(\theta|x)d\theta}$$

$$= \frac{\int_{-\infty}^{\infty} \exp\left\{\frac{3\theta^2}{4}\right\} \theta \exp\left\{-(\theta - \boldsymbol{x}/2)^2\right\} d\theta}{\int_{-\infty}^{\infty} \exp\left\{\frac{3\theta^2}{4}\right\} \exp\left\{-(\theta - \boldsymbol{x}/2)^2\right\} d\theta}$$

$$= \int_{-\infty}^{\infty} \theta \frac{1}{\sqrt{2\pi}\sqrt{2}} \exp\left\{-\frac{(\theta - 2x)^2}{4}\right\} d\theta = 2x.$$

Thus, the risk function for δ^{π} is given by

$$R(\theta, \delta^{\pi}) = \int_{-\infty}^{\infty} L(\theta, \delta^{\pi}) f(x|\theta)dx$$

$$= \exp\left\{\frac{3\theta^2}{4}\right\} \int_{-\infty}^{\infty} (\theta - 2x)^2 f(x|\theta)dx$$

$$= 4\exp\left\{\frac{3\theta^2}{4}\right\} \int_{-\infty}^{\infty} (x - \theta/2)^2 f(x|\theta)dx$$

$$= 4\exp\left\{\frac{3\theta^2}{4}\right\} [1 + (\theta - \theta/2)^2] = \exp\left\{\frac{3\theta^2}{4}\right\} [4 + \theta^2]$$

$$> \exp\left\{\frac{3\theta^2}{4}\right\} (1) = R(\theta, \delta_1),$$

where $\delta_(x) = x$. Thus δ^{π} is seriously inadmissible.

Note that $r(\pi, \delta^\pi) = E^\pi[R(\theta, \delta^\pi)] = \infty$, and indeed it can be shown that $r(\pi, \delta^*) = \infty$ for all $\delta^* \in \mathcal{D}^*$, where

$$\mathcal{D}^* = \{\text{all randomized decision rules } \delta^* : \\ R(\theta, \delta^*) < \infty \text{ for all } \theta \in \Theta\}.$$

♢ Stein Estimation

Let $\boldsymbol{X} = (X_1, X_2, \ldots, X_p)' \sim N_p(\boldsymbol{\theta}, I_p)$, where $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_p)'$. It is desired to estimate $\boldsymbol{\theta}$ under sum-of-squares error loss

$$L(\boldsymbol{\theta}, \boldsymbol{a}) = \sum_{i=1}^{p} (\theta_i - a_i)^2.$$

Since $\boldsymbol{\theta}$ is a location parameter, the noninformative prior density $\pi(\boldsymbol{\theta}) = 1$ is deemed appropriate. It is easy to see that the (formal) posterior density of $\boldsymbol{\theta}$ given $\boldsymbol{x}$ is then a $N_p(\boldsymbol{x}, I_p)$ density. The generalized Bayes estimator of $\boldsymbol{\theta}$ is the mean of the posterior under sum-of-squares error loss, or indeed any quadratic loss, so

$$\boldsymbol{\delta}^0(\boldsymbol{x}) = \boldsymbol{x} = (x_1, x_2, \ldots, x_p)'$$

is the generalized Bayes estimator. In fact, $\boldsymbol{\delta}^0(\boldsymbol{x})$ is also the maximum likelihood estimate of $\boldsymbol{\theta}$.

The risk of $\boldsymbol{\delta}^0(\boldsymbol{x})$ is

$$R(\boldsymbol{\theta}, \boldsymbol{\delta}^0) = \sum_{i=1}^{p} E[(\theta_i - X_i)^2] = p.$$

We are led to the following theorem, which is due to James and Stein (1960).

Theorem 4: *Assume $p \geq 3$. Consider the estimator*

$$\boldsymbol{\delta}^c(\boldsymbol{x}) = \left(1 - \frac{c}{||\boldsymbol{x}||^2}\right)\boldsymbol{x}, \quad 0 < c < 2(p-2).$$

Then $\boldsymbol{\delta}^c(\boldsymbol{x})$ is R-better than $\boldsymbol{\delta}^0(\boldsymbol{x}) = \boldsymbol{x}$ under sum-of-squares error loss.

Proof: Let $\boldsymbol{\delta}^c = (\delta_1^c(\boldsymbol{x}), \delta_2^c(\boldsymbol{x}), \ldots, \delta_p^c(\boldsymbol{x}))'$.

$$R(\boldsymbol{\theta}, \delta^c) = E[||\delta^c(\boldsymbol{x}) - \boldsymbol{\theta}||^2] = \sum_{i=1}^{n} E[(\delta_i^c(\boldsymbol{x}) - \theta_i)^2]$$

$$= \sum_{i=1}^{p} E_{\boldsymbol{\theta}} \left\{ \left[(x_i - \theta_i) - \frac{c x_i}{||\boldsymbol{x}||^2} \right]^2 \right\}$$

$$= \sum_{i=1}^{p} E_{\boldsymbol{\theta}} \left[(x_i - \theta_i)^2 - 2c \frac{x_i(x_i - \theta_i)}{||\boldsymbol{x}||^2} + \frac{c^2 x_i^2}{||\boldsymbol{x}||^4} \right]$$

$$= p - 2c \sum_{i=1}^{p} E_{\boldsymbol{\theta}} \left[\frac{x_i(x_i - \theta_i)}{||\boldsymbol{x}||^2} \right] + c^2 E_{\boldsymbol{\theta}} \left[\frac{1}{||\boldsymbol{x}||^2} \right].$$

Now consider

$$
\begin{aligned}
&E_{\boldsymbol{\theta}}\left[\frac{x_1(x_1-\theta_1)}{||\boldsymbol{x}||^2}\right] \\
&= \int_{R^{p-1}} \int_{-\infty}^{\infty} \frac{x_1}{||\boldsymbol{x}||^2}(x_1-\theta_1)e^{-\frac{1}{2}(x_1-\theta_1)^2}\,dx_1 \\
&\quad \times (2\pi)^{-p/2} \exp\left\{-\frac{1}{2}\sum_{i=2}^{p}(x_i-\theta_i)^2\right\} dx_2 \cdots dx_p.
\end{aligned}
$$

Note that

$$
\int (x_1-\theta_1)e^{-\frac{1}{2}(x_1-\theta_1)^2}\,dx_1 = -\exp\left\{-\frac{1}{2}(x_1-\theta_1)^2\right\}
$$

and

$$
\frac{d}{dx_1}\left(\frac{x_1}{||\boldsymbol{x}||^2}\right) = \frac{1}{||\boldsymbol{x}||^2} - \frac{2x_1^2}{||\boldsymbol{x}||^4}.
$$

Thus,

$$\int_{-T}^{T} \frac{x_1}{||\boldsymbol{x}||^2}(x_1 - \theta_1)e^{-\frac{1}{2}(x_1-\theta_1)^2} dx_1$$

$$= \left[\frac{x_1}{||\boldsymbol{x}||^2} \left(-\exp\left\{ -\frac{1}{2}(x_1 - \theta_1)^2 \right\} \right) \right]_{-T}^{T}$$

$$- \int_{-T}^{T} \left[\frac{1}{||\boldsymbol{x}||^2} - \frac{2x_1^2}{||\boldsymbol{x}||^4} \right] \left(-\exp\left\{ -\frac{1}{2}(x_1 - \theta_1)^2 \right\} \right) dx_1$$

$$=0 + \int_{-T}^{T} \left[\frac{1}{||\boldsymbol{x}||^2} - \frac{2x_1^2}{||\boldsymbol{x}||^4} \right] \exp\left\{ -\frac{1}{2}(x_1 - \theta_1)^2 \right\} dx_1$$

$$\longrightarrow \int_{-\infty}^{\infty} \left[\frac{1}{||\boldsymbol{x}||^2} - \frac{2x_1^2}{||\boldsymbol{x}||^4} \right] \exp\left\{ -\frac{1}{2}(x_1 - \theta_1)^2 \right\} dx_1$$

as $T \to \infty$. Thus,

$$E_{\boldsymbol{\theta}} \left[\frac{x_i(x_i - \theta_i)}{||\boldsymbol{x}||^2} \right] = E_{\boldsymbol{\theta}} \left[\frac{1}{||\boldsymbol{x}||^2} - \frac{2x_i^2}{||\boldsymbol{x}||^4} \right]$$

and hence

$$
\begin{aligned}
& R(\boldsymbol{\theta}, \boldsymbol{\delta}^c) \\
=& p - 2c \sum_{i=1}^{p} E_{\boldsymbol{\theta}} \left[\frac{1}{||\boldsymbol{x}||^2} - \frac{2x_i^2}{||\boldsymbol{x}||^4} \right] + c^2 E_{\boldsymbol{\theta}} \left[\frac{1}{||\boldsymbol{x}||^2} \right] \\
=& p - 2c E_{\boldsymbol{\theta}} \left[\frac{p}{||\boldsymbol{x}||^2} - \frac{2}{||\boldsymbol{x}||^2} \right] + c^2 E_{\boldsymbol{\theta}} \left[\frac{1}{||\boldsymbol{x}||^2} \right] \\
=& p - E_{\boldsymbol{\theta}} \left[\frac{c[2(p-2) - c]}{||\boldsymbol{x}||^2} \right] \\
<& p = R(\boldsymbol{\theta}, \boldsymbol{\delta}^0)
\end{aligned}
$$

for $0 < c < 2(p-2)$. The above result shows that $\boldsymbol{x}$ is inadmissible for $p \geq 3$. $\square$

♢ Main Results on Admissibility of Generalized Bayes Rules

As with formal Bayes rules (see Example 1), generalized Bayes rules need not be admissible (see James and Stein Estimate). One situation in which a generalized Bayes rule, δ, can be easily shown to be admissible, is when the loss is positive and

$$r(\pi, \delta) = \int_{\Theta} R(\theta, \delta) dF^{\pi}(\theta) < \infty.$$

Unfortunately, it is rather rare to have $r(\pi, \delta) < \infty$ for improper π. When $r(\pi, \delta) = \infty$, "natural" generalized Bayes rules can be inadmissible, as the James and Stein estimate is as well as the following example shows.

Example 2: Assume $X \sim \mathcal{G}(\alpha, \beta)$ ($\alpha > 1$ known) is observed, and it is desired to estimate β under squared-error loss. Since β is a scale parameter, it is felt that the noninformative prior density $\pi(\beta) = \beta^{-1}$ should be used. The (formal) posterior density of β given x is then

$$\pi(\beta|x) = \frac{f(x|\beta)\pi(\beta)}{\int_0^\infty f(x|\beta)\pi(\beta)d\beta} = \frac{\beta^{-\alpha}e^{-x/\beta}\beta^{-1}}{\int_0^\infty \beta^{-\alpha}e^{-x/\beta}\beta^{-1}d\beta},$$

which is recognizable as an $\mathcal{IG}(\alpha, x^{-1})$ density. Since the loss is squared-error, the generalized Bayes rule, $\delta^0(x) = x/(\alpha - 1)$, is the mean of the posterior (see page 561 of the text book).

Consider the risk of $\delta_c(x) = cx$. Clearly,

$$\begin{aligned} R(\beta, \delta_c) &= E_\beta[cx - \beta)^2] \\ =&E_\beta\left\{[c(x - \alpha\beta) + (c\alpha - 1)\beta]^2\right\} \\ =&c^2\alpha\beta^2 + (c\alpha - 1)^2\beta^2 \\ =&\beta^2[c^2\alpha + (c\alpha - 1)^2]. \end{aligned}$$

Differentiating with respect to c and setting equal to zero shows that the value of c minimizing this expression is unique and is given by

$$c_0 = (\alpha + 1)^{-1}.$$

It follows that if $c \neq c_0$, then

$$R(\beta, \delta_{c_0}) < R(\beta, \delta_c)$$

for all β, showing in particular that δ^0 (which is δ_c with $c = (\alpha - 1)^{-1}$) is inadmissible. Indeed the ratio of risks of δ^0 and δ_{c_0} is

$$\frac{R(\beta, \delta^0)}{R(\beta, \delta_{c_0})} = \frac{\alpha(\alpha - 1)^{-2} + (\alpha/(\alpha - 1) - 1)^2}{\alpha(\alpha + 1)^{-2} + (\alpha/(\alpha + 1) - 1)^2} = \frac{(\alpha + 1)^2}{(\alpha - 1)^2}.$$

For small α, δ^0 has significantly worse risk than δ_{c_0}.

Now, a natural question is: when is a generalized Bayes rule admissible?

For example, when $x \sim N(\theta, 1)$ and $\pi(\theta) = 1$. Then $\pi(\theta|x)$ is $N(x, 1)$. Thus, $\delta(x) = x$ is a generalized Bayes rule under squared-error loss $L(\theta, a) = (\theta - a)^2$. Is $\delta = x$ admissible? The answer is YES. But, we need to use the Blyth theorem to show this.

Before we introduce the Blyth theorem, we first consider some preliminary concepts.

Definition 1: A class $\mathcal{C}$ of decision rules is said to be *essentially complete* if, for any decision rule δ not in $\mathcal{C}$, there is a decision rule $\delta' \in \mathcal{C}$ which is R-better than or R-equivalent to δ.

Definition 2: A class $\mathcal{C}$ of decision rules is said to be *complete* if, for any decision rule δ not in $\mathcal{C}$, there is a decision rule $\delta' \in \mathcal{C}$ which is R-better than δ.

Definition 3: A class $\mathcal{C}$ of decision rules is said to be *minimal complete* if $\mathcal{C}$ is complete and if no proper subset of $\mathcal{C}$ is complete.

Note 1:

(1) Complete class $\Longrightarrow$ essentially complete.

(2) A complete class admits inadmissible rules.

(3) A minimal complete class does not admit any inadmissible rules.

Lemma 1: *A complete class must contain all admissible decision rules.*

Lemma 2: *If an admissible decision rule δ is not in an essentially complete class $\mathcal{C}$, then there must exist a decision rule δ' in $\mathcal{C}$ which is R-equivalent to δ.*

Lemma 3: *If a minimal complete class $\mathcal{C}$ exists, it is exactly the class of admissible decision rules.*

Theorem 5: *The class of randomized decision rules based on sufficient statistics is essentially complete.*

The proof directly follows from Theorem 1 on page 2-6.

Theorem 6: *Assume $\mathcal{A}$ is a convex subset of R^m. For each $\theta \in \Theta$, the loss function $L(\theta, a)$ is a convex function in a.*

(i) The class of nonrandomized decision rule is essentially complete.

(ii) The class of nonrandomized rules base on sufficient statistics is essentially complete.

The proof follows from Theorems 3 and 4 on pages 2-14 and 2-15.

Theorem 7: *Suppose $\Theta \subset R^m$ and that $L(\theta, a)$ is a bounded function which is continuous in θ for each $a \in \mathcal{A}$. Suppose also that X has a density $f(x|\theta)$ which is continuous in θ for each $x \in \mathcal{X}$. Then all decision rules have continuous risk functions.*

The most restrictive condition in Theorem 7 is the condition that the loss function be bounded. When Θ is unbounded, many standard losses will not satisfy this condition. The boundedness condition can be relaxed, as shown in the following theorem.

Theorem 8: *Suppose that $\mathcal{X}$, Θ, and $\mathcal{A}$ are subsets of R^1, with $\mathcal{A}$ being closed, and that the distribution of X has monotone likelihood ratio. Suppose also that $f(x|\theta)$ is continuous in θ for each $x \in \mathcal{X}$, and that the loss function $L(\theta, a)$ is such that*

(a) *$L(\theta, a)$ is continuous in θ for each $a \in \mathcal{A}$;*

(b) *$L(\theta, a)$ is nonincreasing in a for $a \leq \theta$ and is nondecreasing in a for $a \geq \theta$;*

(c) *there exist functions $K_1(\theta_1, \theta_2)$ and $K_2(\theta_1, \theta_2)$ on $\Theta \times \Theta$ which are bounded on all bounded subsets of $\Theta \times \Theta$, and such that*

$$L(\theta_2, a) \leq K_1(\theta_1, \theta_2) L(\theta_1, a) + K_2(\theta_1, \theta_2)$$

for all $a \in \mathcal{A}$.

Then the decision rules with continuous, finite valued risk functions form a complete class.

Example 3: Let $X \sim N(\theta, 1)$ and assume that it is desired to estimate θ under squared-error loss. The only condition in Theorem 8 which is not obviously satisfied is Condition (c). To check this condition, note that

$$\begin{aligned}(\theta_1 - a)^2 =& ([\theta_2 - \theta_1] + [\theta_1 - a])^2 \\ \leq& 2(\theta_2 - \theta_1)^2 + 2(\theta_1 - a)^2.\end{aligned}$$

Hence Condition (c) is satisfied with

$$K_1(\theta_1, \theta_2) = 2 \text{ and } K_2(\theta_1, \theta_2) = 2(\theta_2 - \theta_1)^2.$$

It can be concluded that the decision rules with continuous risk functions form a complete class.

Theorem 9 (Blyth, 1951): *Consider a decision problem in which* Θ *is a nondegenerate convex subset of Euclidean space (i.e., Theta has positive Lebesgue measure), and in which the decision rules with continuous risk functions form a complete class. Then an estimator* δ_0 *(with a continuous risk function) is admissible if there exists a sequence* $\{\pi_n\}$ *of (generalized) priors such that*

(a) *the Bayes risks* $r(\pi_n, \delta_0)$ *and* $r(\pi_n, \delta^n)$ *are finite for all* n*, where* δ^n *is the Bayes rule with respect to* π_n*;*

(b) *for any nondegenerate convex set* $C \subset \Theta$*, there exists a* $K > 0$ *and an integer* N *such that, for* $n \geq N$*,*
$$\int_C dF^{\pi_n}(\theta) \geq K;$$

(c) $\lim_{n\to\infty}[r(\pi_n, \delta_0) - r(\pi_n, \delta^n)] = 0.$

Proof: Suppose δ_0 is not admissible. Then there exists a decision rule δ' such that

$$R(\theta, \delta') \leq R(\theta, \delta_0),$$

with strict inequality for some θ, say, θ_0. Since the rules with continuous risk functions form a complete class, it can be assumed that δ' has a continuous risk function. Since $R(\theta, \delta_0)$ is also continuous, it follows that there exist constants $\epsilon_1 > 0$ and $\epsilon_2 > 0$ such that

$$R(\theta, \delta') < R(\theta, \delta_0) - \epsilon_1$$

for

$$\theta \in C = \{\theta : \ |\theta - \theta_0) < \epsilon_2\}.$$

Using this, Conditions (a) and (b), and the fact that

$$r(\pi_n, \delta^n) \leq r(\pi_n, \delta'),$$

it can be concluded that for $n \geq N$,

$$\begin{aligned} r(\pi_n, \delta_0) - r(\pi_n, \delta^n) \geq & r(\pi_n, \delta_0) - r(\pi_n, \delta') \\ = & E^{\pi_n}[R(\theta, \delta_0) - R(\theta, \delta')] \\ \geq & \int_C [R(\theta, \delta_0) - R(\theta, \delta')] dF^{\pi_n}(\theta) \end{aligned}$$

$$\geq \epsilon_1 \int_C dF^{\pi_n}(\theta) \geq \epsilon_1 K.$$

This contradicts Condition (c) of the theorem. Hence δ_0 must be admissible. $\square$

Example 4 (Blyth (1951): Suppose that $X \sim N(\theta, 1)$ and that it is desired to estimate θ under squared-error loss. We seek to prove that the usual estimator, $\delta_0(x) = x$, is admissible.

The conditions of Theorem 9 will clearly be satisfied for this situation (see Example 3), once a suitable sequence $\{\pi_n\}$ is found. A convenient choice for π_n is the unnormalized normal density

$$\pi_n(\theta) = (2\pi)^{-1/2} \exp\left\{-\frac{\theta^2}{2n}\right\}.$$

If C is a nondegenerate convex subset of Θ, then clearly

$$\int_C \pi_n(\theta) d\theta \geq \int_C \pi_1(\theta) d\theta = K > 0,$$

so that Condition (b) of Theorem 9 is satisfied for this choice of the π_n.

It can be shown that the Bayes rule with respect to π_n is

$$\delta^n = \frac{n}{1+n} x$$

and its corresponding risk function is

$$R(\theta, \delta^n) = E_\theta \left[(\theta - \frac{n}{1+n} x)^2 \right] = \frac{n^2 + \theta^2}{(n+1)^2},$$

so that the Bayes risk is

$$\begin{aligned}
r(\pi_n, \delta^n) &= \int_{-\infty}^{\infty} R(\theta, \delta^n) \pi_n(\theta) d\theta \\
&= \int_{-\infty}^{\infty} \frac{n^2 + \theta^2}{(n+1)^2} \sqrt{n} \frac{\pi_n(\theta)}{\sqrt{n}} d\theta \\
&= \frac{n^2 + n}{(n+1)^2} \sqrt{n} = \frac{n\sqrt{n}}{n+1}.
\end{aligned}$$

Similarly, we obtain $r(\pi_n, \delta^0) = \sqrt{n}$. Hence, we have verified Condition (a) of Theorem 9. Finally,

$$\begin{aligned}
&\lim_{n \to \infty} [r(\pi_n, \delta_0) - r(\pi_n, \delta^n)] \\
&= \lim_{n \to \infty} \left[\sqrt{n} \left(1 - \frac{n}{n+1} \right) \right] \\
&= \lim_{n \to \infty} \left[\frac{\sqrt{n}}{n+1} \right] = 0.
\end{aligned}$$

Condition (c) of Theorem 9 is thus satisfied, and it can be concluded that δ_0 is admissible.