

Euler Totient Function $\phi(n)$

June 5, 2020

1 Defintion

Euler Totient Function $\phi(n)$ is defined for a natural number n as the count of the natural numbers which are less than n and are co prime to n .

$\phi(n) = \#$ of numbers co-prime with n .

```
int Phi(int N) {  
    int cnt = 0;  
    for(int i = 1; i <= N; ++i)  
        if(GCD(i, N) == 1)  
            cnt++;  
  
    return cnt;  
}
```

Time Complexity $\mathcal{O}(n \log n)$

$\phi(p) = p - 1$ where p is a prime number.

$\phi(p^x) = ?$

$\phi(p^x) = p^x -$ Number of integers not co-prime with p

Observations - All multiples of p can never be co-prime with p^x

How many such numbers are there?

$$\frac{p^x}{p}$$

Hint for above : Number of divisors in a range from 1 to 100 of 3 are?

$$\phi(p^x) = p^x - \left(\frac{p^x}{p}\right)$$

$$\phi(p^x) = p^{x-1}(p-1)$$

2 Multiplicative

An arithmetic function $f(x)$ is called multiplicative if

$$f(N * M) = f(N) * f(M) \text{ where } \gcd(N, M) = 1$$

Let $f(x)$ is multiplicative,

to evaluate $f(N)$ where

$$N = p_1^{x_1} * p_2^{x_2} * \dots * p_k^{x_k}$$

$$d(N) = (x_1 + 1) * (x_2 + 1) * \dots * (x_k + 1)$$

where $d(N)$ denotes number of divisors of a number

$$d(N) = d(p_1^{x_1}) * d(p_2^{x_2}) * \dots * d(p_k^{x_k})$$

$$d(p_1^{x_1}) = (x_1 + 1)$$

$$\phi(p^x) = p^{x-1}(p-1)$$

$$\therefore \phi(N) = p_1^{x_1-1} \times p_2^{x_2-1} \times p_3^{x_3-1} \times \dots \times p_k^{x_k-1}$$

Above formula is not useful for computation as it involves exponents of prime factors

rewriting this as

$$\begin{aligned}
\phi(N) &= p_1^{x_1} \left(1 - \frac{1}{p_1}\right) \times p_2^{x_2} \left(1 - \frac{1}{p_2}\right) \times p_3^{x_3} \left(1 - \frac{1}{p_3}\right) \times \cdots \times p_k^{x_k} \left(1 - \frac{1}{p_k}\right) \\
\phi(N) &= p_1^{x_1} \times p_2^{x_2} \times p_3^{x_3} \times \cdots \times p_k^{x_k} \times \left(\frac{p_1 - 1}{p_1}\right) \times \left(\frac{p_2 - 1}{p_2}\right) \times \left(\frac{p_3 - 1}{p_3}\right) \times \\
&\quad \cdots \times \left(\frac{p_k - 1}{p_k}\right) \\
\phi(N) &= N \times \left(\frac{p_1 - 1}{p_1}\right) \times \left(\frac{p_2 - 1}{p_2}\right) \times \left(\frac{p_3 - 1}{p_3}\right) \times \cdots \times \left(\frac{p_k - 1}{p_k}\right)
\end{aligned}$$