Euler Totient Function $\phi(n)$

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1 Defintion

Euler Totient Function $\phi(n)$ is defined for a natural number n as the count of the natural numbers which are less than n and are co prime to n.

$\phi(n) = \#$ of numbers co-prime with n.

```
int Phi(int N) {
        int cnt = 0;
        for(int i = 1; i <= N; ++i)
                if(GCD(i, N) == 1)
                        cnt++;

        return cnt;
}
```

Time Complexity $\mathcal{O}(n \log n)$

$\phi(p) = p - 1$ where p is a prime number.

$\phi(p^x) = ?$

$\phi(p^x) = p^x -$ Number of integers not co-prime with p

Observations - All multiples of p can never be co-prime with p^x

How many such numbers are there?

$\frac{p^x}{p}$

Hint for above : Number of divisors in a range from 1 to 100 of 3 are?

$\phi(p^x) = p^x - (\dfrac{p^x}{p})$

$\phi(p^x) = p^{x-1}(p-1)$

2 Multiplicative

An arithmetic fucntion $f(x)$ is called multiplicative if

$f(N * M) = f(N) * f(M)$ where $\gcd(N, M) = 1$

Let $f(x)$ is multiplicative,

to evaluate $f(N)$ where

$N = p_1^{x_1} * p_2^{x_2} * \cdots * p_k^{x_k}$

$d(N) = (x_1 + 1) * (x_2 + 1) * \cdots * (x_k + 1)$

where $d(N)$ denotes number of divisors of a number

$d(N) = d(p_1^{x_1}) * d(p_2^{x_2}) * \cdots * d(p_k^{x_k})$

$d(p_1^{x_1}) = (x_1 + 1)$

$\phi(p^x) = p^{x-1}(p-1)$

$\therefore \phi(N) = p_1^{x_1-1} \times p_2^{x_2-1} \times p_3^{x_3-1} \times \cdots \times p_k^{x_k-1}$

Above formula is not useful for computation as it involves exponents of prime factors rewriting this as

$$\phi(N) = p_1^{x_1}\left(1-\frac{1}{p_1}\right)\times p_2^{x_2}\left(1-\frac{1}{p_2}\right)\times p_3^{x_3}\left(1-\frac{1}{p_3}\right)\times\cdots\times p_k^{x_k}\left(1-\frac{1}{p_k}\right)$$

$$\phi(N) = p_1^{x1}\times p_2^{x_2}\times p_3^{x_3}\times\cdots\times p_k^{x_k}\times\left(\frac{p_1-1}{p_1}\right)\times\left(\frac{p_2-1}{p_2}\right)\times\left(\frac{p_3-1}{p_3}\right)\times\cdots\times\left(\frac{p_k-1}{p_k}\right)$$

$$\phi(N) = N\times\left(\frac{p_1-1}{p_1}\right)\times\left(\frac{p_2-1}{p_2}\right)\times\left(\frac{p_3-1}{p_3}\right)\times\cdots\times\left(\frac{p_k-1}{p_k}\right)$$