An Introduction to The Linear Complementarity Problem

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Brief Motivation – The Twenty Minutes Disclaimer

Many problems can be stated as complementarity problems

- Contact Force Problems
- Human Motion, Inverse Kinematics
- Space-Time Optimization
- And many more...

For interactive applications “we” want fast, robost and stable methods for computing solutions.

The Complementarity Problem

If given $x, y \in \mathbb{R}$ where

$$x \geq 0$$
$$y \geq 0$$

and

$$x > 0 \Rightarrow y = 0$$
$$y > 0 \Rightarrow x = 0$$

Then we have a complementarity problem. If for some $a, b \in \mathbb{R}$

$$y = ax + b$$

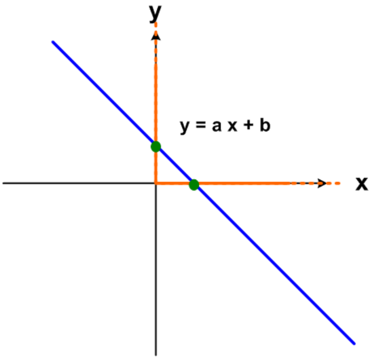
We have a Linear Complementarity Problem (LCP).

The Linear Complementarity Problem

More compact notation

$$x \geq 0$$
$$ax + b \geq 0$$
$$x(ax + b) = 0$$

The Geometry



How many #Solutions of LCP?

Hint: Try to examine signs of a and b

	$b < 0$	$b = 0$	$b > 0$
$a < 0$			
$a = 0$			
$a > 0$			

Answer of # Solutions

Verify this using geometry

	$b < 0$	$b = 0$	$b > 0$
$a < 0$	0	1	2
$a = 0$	0	∞	1
$a > 0$	1	1	1

Going to Higher Dimensions

Let $b, x \in \mathbb{R}^n$ and $\mathbf{A} \in \mathbb{R}^{n \times n}$ so $y = \mathbf{A}x + b$,

$$\begin{aligned} x_i &\geq 0 \quad \forall i \in [1..n] \\ (\mathbf{A}x + b)_i &\geq 0 \quad \forall i \in [1..n] \\ x_i(\mathbf{A}x + b)_i &= 0 \quad \forall i \in [1..n] \end{aligned}$$

In Matrix-Vector Notation

$$\begin{aligned} x &\geq 0 \\ (\mathbf{A}x + b) &\geq 0 \\ x^T(\mathbf{A}x + b) &= 0 \end{aligned}$$

How can we solve this?

Guessing A Solution

Given the index set $\mathcal{I} = \{1, \ldots, n\}$ define

$$\begin{aligned} \mathcal{F} &= \{i \quad | \quad i \in \mathcal{I} \quad \wedge \quad y_i > 0\} \\ \mathcal{A} &= \{i \quad | \quad i \notin \mathcal{F} \quad \wedge \quad y_i = 0\} \end{aligned}$$

Make "lucky" guess of $\mathcal{F}$ and $\mathcal{A}$,

$$\begin{bmatrix} y_{\mathcal{A}} \\ y_{\mathcal{F}} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{\mathcal{A}\mathcal{A}} & \mathbf{A}_{\mathcal{A}\mathcal{F}} \\ \mathbf{A}_{\mathcal{F}\mathcal{A}} & \mathbf{A}_{\mathcal{F}\mathcal{F}} \end{bmatrix} \begin{bmatrix} x_{\mathcal{A}} \\ x_{\mathcal{F}} \end{bmatrix} + \begin{bmatrix} b_{\mathcal{A}} \\ b_{\mathcal{F}} \end{bmatrix}$$

By assumption $y_{\mathcal{F}} > 0 \Rightarrow x_{\mathcal{F}} = 0$

$$\begin{bmatrix} 0 \\ y_{\mathcal{F}} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{\mathcal{A}\mathcal{A}} & \mathbf{A}_{\mathcal{A}\mathcal{F}} \\ \mathbf{A}_{\mathcal{F}\mathcal{A}} & \mathbf{A}_{\mathcal{F}\mathcal{F}} \end{bmatrix} \begin{bmatrix} x_{\mathcal{A}} \\ 0 \end{bmatrix} + \begin{bmatrix} b_{\mathcal{A}} \\ b_{\mathcal{F}} \end{bmatrix}$$

Verify if Guess was a Solution

So

$$\begin{bmatrix} 0 \\ y_{\mathcal{F}} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{\mathcal{A}\mathcal{A}} x_{\mathcal{A}} + b_{\mathcal{A}} \\ \mathbf{A}_{\mathcal{F}\mathcal{A}} x_{\mathcal{A}} + b_{\mathcal{F}} \end{bmatrix}$$

Compute

$$x_{\mathcal{A}} = -\mathbf{A}_{\mathcal{A}\mathcal{A}}^{-1} b_{\mathcal{A}}$$

Verify

$$x_{\mathcal{A}} \geq 0$$

Compute

$$y_{\mathcal{F}} = \mathbf{A}_{\mathcal{F}\mathcal{A}} x_{\mathcal{A}} + b_{\mathcal{F}}$$

Verify

$$y_{\mathcal{F}} > 0$$

How Many Guesses?

We only need

$$\mathbf{A}^{-1}_{\mathcal{A}\mathcal{A}}$$

Hopefully

$$\| \mathcal{A} \| \ll n$$

Cool this will be fast!

How many guesses do we need?

Answer: Non-Polynomial Complexity

Worst case time complexity of guessing

$$\mathcal{O}(n^3 2^n)$$

Not computational very efficient!

Use a Splitting Method

Use the splitting

$$\mathbf{A} = \mathbf{M} - \mathbf{N}$$

then

$$\begin{aligned} \mathbf{M}x - \mathbf{N}x + b &\geq 0 \\ x &\geq 0 \\ (x)^T(\mathbf{M}x - \mathbf{N}x + b) &= 0 \end{aligned}$$

Use Discretization $\Rightarrow$ Fixed Point Formulation

Create a sequence of sub problems

$$
\begin{aligned}
\mathbf{M}x^{k+1} + c^k &\geq 0 \\
x^{k+1} &\geq 0 \\
(x^{k+1})^T(\mathbf{M}x^{k+1} + c^k) &= 0
\end{aligned}
$$

where

$$c^k = b - \mathbf{N}x^k$$

Use Minimum Map Reformulation

Given sub problem

$$
\begin{aligned}
\mathbf{M}x^{k+1} + c^k &\geq 0 \\
x^{k+1} &\geq 0 \\
(x^{k+1})^T(\mathbf{M}x^{k+1} + c^k) &= 0
\end{aligned}
$$

Same as (Why?)

$$
\underbrace{\min(x^{k+1}, \mathbf{M}x^{k+1} + c^k)}_{H(x^{k+1})} = 0
$$

A root search problem: $H(x^{k+1}) = 0$.

The Minimum Map Formulation

Say $a, b \in \mathbb{R}$ are complementary

$$a > 0 \Rightarrow b = 0$$
$$b > 0 \Rightarrow a = 0$$

Let us look at the minimum map

$\min(a, b)$	$a > 0$	$a = 0$	$a < 0$
$b > 0$	$+$	0	$-$
$b = 0$	0	0	$-$
$b < 0$	$-$	$-$	$-$

Same solutions as complementarity problem.

More Clever Manipulation

So

$$\min(x^{k+1}, \mathbf{M}x^{k+1} + c^k) = 0$$

Subtract x^{k+1}

$$\min(0, \mathbf{M}x^{k+1} + c^k - x^{k+1}) = -x^{k+1}$$

Multiply by minus one

$$\underbrace{\max(0, -\mathbf{M}x^{k+1} - c^k + x^{k+1})}_{F(x^{k+1})} = x^{k+1}$$

A fixed point formulation: $F(x^{k+1}) = x^{k+1}$.

Do A Case-by-Case Analysis

So

$$\max(0, -\mathbf{M}x^{k+1} - c^k + x^{k+1}) = x^{k+1}$$

If

$$(-\mathbf{M}x^{k+1} - c^k + x^{k+1})_i \leq 0$$

Then

$$x_i^{k+1} = 0$$

Else

$$(-\mathbf{M}x^{k+1} - c^k + x^{k+1})_i > 0$$

and

$$(x^{k+1} - \mathbf{M}x^{k+1} - c^k)_i = x_i^{k+1}$$

That is

$$(\mathbf{M}x^{k+1})_i = -c_i^k$$

Putting it Together

For suitable choice of **M**

$$(\mathbf{M}x^{k+1})_i = -c_i^k \Rightarrow x_i^{k+1} = (-\mathbf{M}^{-1}c^k)_i$$

Back-substitution of $c^k = b - \mathbf{N}x^k$ we have

$$\left(\mathbf{M}^{-1}\left(\mathbf{N}x^k - b\right)\right)_i = x_i^{k+1}$$

Insert in fixed point formulation

$$\underbrace{\max\left(0, \left(\mathbf{M}^{-1}\left(\mathbf{N}x^k - b\right)\right)\right)}_{G(x^k)} = x^{k+1}$$

Closed form solution for sub problem: $x^{k+1} \leftarrow G(x^k)$.

Final Iterative Scheme

Given x^1 set $k = 1$

Step 1 Compute

$$z^k = \left(\mathbf{M}^{-1}\left(\mathbf{N}x^k - b\right)\right)$$

Step 2 Compute

$$x^{k+1} = \max(0, z^k)$$

Step 3 If convergene then return x^{k+1} otherwise goto Step 1

Lessons Learned...

- The Definition of a LCP formulation
- A geometric interpretation of the LCP
- A pivoting method (guessing) for solving a LCP
- A fixed point reformulation of LCPs
- A root search reformulation of LCPs
- An iterative projection method for solving a LCP

Connection to Linear Programming (LP) Problems

First some algebra on $y = \mathbf{A}x + b$,

$$\mathbf{I}y = \mathbf{A}x + b$$

$$\begin{bmatrix} \mathbf{I} & -\mathbf{A} \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} = b$$

Make guesses $\mathcal{F} = \{i | y_i \geq 0\}$ and $\mathcal{A} = \{i | x_i \geq 0\}$ so $\mathcal{F} \cap \mathcal{A} = \emptyset$ and $\mathcal{F} \cup \mathcal{A} = \{1, \ldots, n\}$

$$\underbrace{\begin{bmatrix} \mathbf{I}_{\cdot\mathcal{F}} & -\mathbf{A}_{\cdot\mathcal{A}} \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} y_{\mathcal{F}} \\ x_{\mathcal{A}} \end{bmatrix}}_{z} = b$$

Verify if LP

$$\mathbf{M}z = b \quad \text{subject to} \quad z \geq 0$$

Has a solution (same as b in positive cone of $\mathbf{M}$).

Connection to Quadratic Programming (QP) Problems

Consider the minimization problem

$$x^* = \min_{x \geq 0} \frac{1}{2} x^T \mathbf{A} x + b$$

where $\mathbf{A}$ is symmetric. First order optimality (KKT) conditions

$$\begin{aligned} \mathbf{A}x + b - \mathbf{I}y &= 0 \\ x &\geq 0 \\ y &\geq 0 \\ y^T x &= 0 \end{aligned}$$

Same as LCP problem.