Some definitions/small proofs

Some important definitions and proofs for the final. Some proofs are fleshed out in detail. Others are not. Counter-examples are provided usually.

This is basically what the course focused on when Stanley Alama taught it in Fall 2018.

(Warning: My references to points by number aren't perfect because I went back and added a bunch of things later, so things may be out of order in places.)

Sequences

This was the first unit. It focuses on the properties of sequences of real numbers.

1. The triangle inequality states that $|x + y| \leq |x| + |y|$. This is also referred to as the Cauchy-Schwarz inequality.

2. The Archimidean property states that for every $x \in \mathbb{R}$ there exists an $n \in \mathbb{N}$ such that $n > x$.

3. Supremum is defined as the least upper bound for a set. That is for a non-empty set $S \subseteq \mathbb{R}$, we have that $x \leq \sup S$ for all $x \in S$.

4. Infimum is defined as the greatest lower bound for a set. It's mathematical definition is similar to that of the supremum.

5. The empty set is the set with no elements. It's supremum is $-\infty$ and it's infimum is ∞. This is important for later definitions.

6. The limit of a sequence is the value a sequence converges to. It must be finite. Formally, x is the limit of a sequence (x_n) if for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $|x_n - x| < \epsilon$ for every $n \geq N$. Some more limit properties are outlined below:

   (a) If two sequences $(x_n), (y_n)$ converge to x, y respectively, then $(x + y)_n$ converges to (x+y). Just take N to be $\max(n_x(\epsilon/2), n_y(\epsilon/2))$.

(b) Similarly the product of two sequences converges to the product of limits. The same is true for quotients, provided that the denominator is not 0. These can again be established via the triangle inequality.

7. The Squeeze Theorem states that if $(l_n) \leq (x_n) \leq (u_n)$, and $(l_n) \to 0$ and $(u_n) \to 0$, then $(x_n) \to 0$.

8. A sequence is divergent if it does not converge. Consider the sequence $(1, -1, 1, -1...) = (-1)^n$.

9. A sequence is said to be properly divergent if for all $\alpha \in \mathbb{R}$, we have that there exists an $N \in \mathbb{N}$ so that $x_n > \alpha$ if $n \geq N$. We note that n properly diverges but $n \sin n$ does not since it oscillates.

10. A sequence is convergent if all of its subsequences converge to the same limit. Take N to be $\max n_1, n_2, ...$ where n_i corresponds to the $n(\epsilon)$ of each subsequence. The proof follows naturally.

11. The $\limsup (x_n)$ is the supremum of the subsequential limit points of the sequence. The $\liminf$ is the infimum. Consider the sequence $(1, -1, 1, -1, ...)$. We have $\limsup = 1$, and $\liminf = -1$. We note that $\pm\infty$ are not formally limit points, but they are $\limsup/\liminf$.

12. A sequence is monotone increasing if for all x_n in (x_n) we have that $x_{n+1} \geq x_n$. We say that the sequence is strictly increasing if the $\geq$ is replaced with $>$. A similar definition holds for monotone decreasing functions. Traditionally, we establish monotonicity using induction.

13. The Monotone Convergence Theorem (MCT) states that every bounded monotone sequence converges to the supremum or infimum depending on whether it is increasing or decreasing.

14. The Monotone Sequence Theorem states that every bounded sequence of real numbers has a monotone increasing/decreasing subsequence.

15. Bolzano-Weirstrass Theorem states that every bounded sequence of real numbers has a convergent subsequence. This proof is of note. For each $n \in \mathbb{N}$ we take an $x_n > u - 1/n$, where u is the supremum. It's easy to establish that this sequence converges to u. See the first question on the first test. A similar construction can be given for the infimum.

Alternatively, we can combine the MCT with the MST, and the result follows.

16. A sequence is Cauchy if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ so that $|x_n - x_m| < \epsilon$ if $n, m \geq N$. Every Cauchy sequence is convergent, and every convergent sequence is Cauchy.

Series

This was a brief unit on sums of sequences of real numbers. This was formally the second unit, but can also be considered a sub-unit of the first.

1. A series is a sum of a sequence. An infinite series is the sum of all the terms of a sequence. We denote an infinite sum of a sequence by $\sum^{\infty}(x_n)$.

2. We say that an infinite sequence converges if its partial sums converge. Cauchy convergence also applies to sequences of partial sums.

3. We give some important theorems involving convergence:

   (a) If a series converges, the its underlying sequence converges to 0. This is a useful criterion for checking whether a series is for sure divergent. A proof is as follows:

   If a series converges then we have that $|s_n - s_m| < \epsilon$ for all $m, n \geq N$. We note that $x_n = s_n - s_{n-1}$. And so we have $|x_n| < \epsilon$. It follows that (x_n) goes to 0 by definition of the limit of a sequence.

   (b) A series $\sum_{n=1} a_n$ is absolutely convergent if $\sum_{n=1} |a_n|$ converges. If a series converges absolutely, then it converges. This follows from the fact that the series is always less than or equal to the absolute series. So really, this is a corollary. We give proof of its associated lemma/theorem next.

   (c) Suppose a series is absolutely convergent. Then we have $|x_m + x_{m+1} + ... + x_n| = |x_m| + |x_{m+1}| + ... + |x_n| < \epsilon$ by Cauchy criterion for convergence. But then the result follows naturally, since otherwise, it's less than or equal to for the same $N(\epsilon)$.

(d) Another useful theorem is the Comparison Test. Suppose $0 < x_n < y_n$ for all $n \geq K$ for some K. Also, suppose the series of y_n converges. Then the series of x_n will converge too. Take N to be $\max(K, n_y)$ and the result follows by the Cauchy criterion. Divergence can also be established in this manner.

(e) Note that alternating series are convergent if the underlying sequence converges absolutely to 0. For example consider

$$\frac{(-1)^n}{\sqrt{n}}.$$

The proof has to do with manipulating the partial sums.

Functions

This is probably the most important unit. It establishes the well-known results of calculus rigorously.

1. A function, $f(x)$, is said to converge to a limit L at c if for every $\epsilon > 0$ there exists a $\delta > 0$ so that $|f(x) - L| < \epsilon$ whenever $|x - c| < \delta$. This is the famous ϵ-δ formulation of the limit of a function.

2. We also say that a function has limit L at c if regardless of how we approach c, $f(x)$ approaches L. This let's us think of functions the way we think of sequences. Some examples are given below:

   (a) Consider the signum function $x/|x|$, which is defined everywhere except $x = 0$. We will show that it does not converge at $x = 0$. Consider the sequences $(x_n) = 1/n$ and $(y_n) = -1/n$. Both converge to 0 but $f(x_n)$ goes to 1 while $f(y_n)$ goes to -1. Visually this makes perfect sense.

   (b) Consider the function $f(x) = \cdot \sin 1/x$. Consider the sequence $\frac{1}{n\pi}$, and the sequence $\frac{1}{(2n-1)\pi}$. Both sequences go to zero but the function goes to 0 for the first one but $(1, -1, ...)$ for the other one, and so the function diverges.

   (c) Also consider the function $\cos 1/x$, which is similar to the sine case.

(d) We can even use the sequential criterion to determine that a limit does indeed exist. For instance consider the function

$$\frac{1}{1-x}.$$

The claim is that it goes to -1 at 2. So suppose that we have a sequence going to 2. We note that the limit of a function of a sequence is the function of the limit of a sequence, so the result follows.

3. The limit theorems follow naturally from this kind of sequential definition of the limit of the function. They are given below with some proofs:

   (a) If two functions f, g converge to L, M at c, then we have that $f+g$ converges to $M+L$. We give the sequential proof. Just take any sequence (x_n) going to c, then we have $f(x_n), g(x_n)$ go to L, M. It follows from the properties of sequences that their sum goes to $L+M$. We can give a different proof in terms of functions alone, by taking the minimum of the $\delta(\epsilon)$'s.

   (b) Products go to products.

   (c) Quotients go to quotients.

4. We can hence formulate the Squeeze Theorem for functions. We have that if $f(x) \leq g(x) \leq h(x)$, and the former and latter converge to L at a point c, then so too does $g(x)$. Some examples are given:

   (a) Consider the function $(1/x)\sin 1/x$. We already established that $\sin 1/x$ doesn't converge on its own. But, we note that $-(1/x) \leq (1/x)\sin 1/x \leq (1/x)$. The result that the function goes to 0 at 0 follows by the Squeeze Theorem.

   (b) The same is true for the corresponding cosine function.

5. We say that a function is continuous at c if $\lim_{x\to c} f(x) = f(c)$. Or in other words $\lim_{x\to c} f(x) - f(c) = 0$. Graphically, this means you can draw the function on a specified interval where it's continuous without ever lifting your hand off the page. Now we prove some stuff about continuity:

(a) Suppose a function is continuous on the interval $[a, b]$, and that $f(c) > 0$, with $c \in [a, b]$. There is some interval in $[a, b]$, namely, $A = (c - \delta, c + \delta)$, where $f(x) > 0$. The proof is fairly straightforward. If f is continuous on $[a, b]$ it follows that $\lim_{x \to c} f(x) - f(c) = 0$. So $|f(x) - f(c)| < \epsilon$ when $|x - c| < \delta$. Letting $f(c) = L > 0$, and letting $\epsilon < L$, the result follows naturally. This is a useful way for thinking about continuity.

(b) An interesting function (with regards to continuity) is

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ -x & \text{if } x \in \mathbb{R} \end{cases}.$$

This function is continuous nowhere, but its absolute value is continuous everywhere.

(c) Sums of continuous functions are continuous. Products of continuous functions are continuous. And so on.

6. The following are some results of continuity which I give without proof (though I do discuss the rationality behind them):

(a) The Boundedness Theorem states that if f is continuous on a closed interval $[a, b]$, that it is continuous. This basically says that I can't have a discontinuity on the interval if I'm continuous, which makes perfect sense, since otherwise the function wouldn't be continuous.

(b) The Maximum-Minimum Theorem states that if f is continuous on a closed interval $[a, b]$, it has an absolute maximum and absolute minimum on this interval.

(c) The Bisection Theorem states that if $f(a) < 0$ and $f(b) > 0$, and f is continuous on $[a, b]$, then there exists $c \in (a, b)$, with $f(c) = 0$. We do give a proof of this.

Define $c = \sup\{x : f(x) < 0\}$. We'll show that $f(c) = 0$. Suppose $f(c) > 0$, then c could not be the supremum of this set. The same is true for the case where $f(c) > 0$. The result will follow. A similar proof holds for the case where $f(a) > 0$ and $f(b) < 0$.

Again, this is a fairly intuitive notion. If we have to draw a line from a to b, without picking up our pen, we have to cross the x-axis at some point.

(d) Bolzano's Intermediate Value Theorem states that if f is continuous on $[a, b]$ and $f(a) < k < f(b)$, that there is a c such that $f(c) = k$. The proof follows from the proof of the Bisection Theorem. Let $g(x) = f(x) - k$. Then we have $g(a) = f(a) - k$, and $g(b) = f(b) - k$. So we have $g(a) < 0 < g(b)$. Then by the Bisection Theorem, since f is continuous (so that g is continuous too), we have that there is a c so that $g(c) = f(c) - k = 0$, so that $f(c) = k$.

(e) The Preservation of Intervals Theorem says that if I is any interval where f is continuous, then so too is $f(I)$.

7. I actually do some practice problems for this section, since some of them can be tricky:

(a) Let I be a closed bounded interval for which $f(x) > 0$ everywhere with f continuous. Show that there exists an $\alpha > 0$, with $f(x) \geq \alpha$. Clearly there is a minimum by the Maximum-Minimum Theorem. The fact that the minimum is greater than 0 follows by 5(a).

(b) We'll show that $x = \cos x$ has a root on $[0, \pi/2]$. Let $g(x) = x - \cos x$. We note that $g(0) = -1$, and that $g(\pi/2) = \pi/2$. And the result follows by the Bisection Theorem.

(c) Suppose that there exists a y so that $|f(y)| \leq (1/2)|f(x)|$, for all x, y in some continuous interval I. We'll show that there exists a point c so $f(c) = 0$. Suppose there isn't one. Then $|f(x)| > 0$ for all x in the interval. It follows by 7(a), that there's a minimum/maximum then. But then this contradicts the hypothesis, and we're done. There is probably another way to do this, but contradiction always seems like the cleanest method to proceed for me.

(d) We can also show that every polynomial of odd degree has at least one real root. This follows from the fact that for every odd polynomial, there exists x so that $f(x) < 0$ and there exists x' so $f(x') > 0$. The result follows from Bisection Theorem.

8. Uniform continuity is stronger than continuity. It states the following. For all $\epsilon > 0$, there is a δ so that $|f(x) - f(u)| < \epsilon$ whenever $|x - u| < \delta$, where x, u are not fixed.

9. We can phrase this in terms of sequences. Basically, if there are two sequences going to x, u, then the functions of those sequences must go to the same point. This also provides a non-uniform continuity criterion. For example, consider the function $f(x) = 1/x$. We show this is continuous on the interval (a, ∞), where $a > 0$. We have that for $|1/x - 1/u| = |u - x|/xu \leq |u - x|/a^2$. So we can pick our δ to be $a^2\epsilon$. It's not however, uniformly continuous on $(0, \infty)$. Consider the sequences $1/n$ and $1/(n+1)$. Both go to 0 as n goes to infinity, but the functions of them go to 1, not 0.

10. For the case of $1/x^2$, we can just consider $1/\sqrt{n}$ and $1/\sqrt{n+1}$, the same result follows.

11. If a function is continuous on a closed bounded interval, then it is uniformly continuous. This follows since there is a maximum/minimum value for the function by the Maximum-Minimum Theorem. The result is fairly straightforward from there.

12. We say that f is Lipschitz continuous if for all x, u there exists k, so we have $|f(x) - f(u)| \leq k|x - u|$. For instance, x is Lipschitz continuous (though that is fairly trivial). We can show that $\sqrt{x}$ is not Lipschitz continuous on $[0, 1]$. This is fairly obvious upon inspection (just consider what happens to the denominator).

13. A function is monotone increasing if $f(x) \leq f(y)$ if $x < y$. A similar construction can be provided for monotone decreasing functions, and strictly increasing/decreasing functions.

14. Now we can talk about Riemann integrability:

    (a) A partition P is a division of some interval into smaller segments.

    (b) We define the following:

    $$m_i = m_i(P) = \inf \{f(x) : x \in i\}$$

    $$M_i = M_i(P) = \sup \{f(x) : x \in i\}$$

    Here, i is some sub-interval.

(c) Now, we can define the lower and upper sums of a partition:

$$L_P = \sum \text{len}(i) m_i$$

$$U_P = \sum \text{len}(i) M_i$$

Basically the idea is that we're underestimating the integral with the lower sum and vice versa with the upper sum of any given partition.

(d) We'll prove a lemma involving partitions. If Q is a refinement of P, it follows that $L_P \leq L_Q \leq U_Q \leq L_P$. The proof is simple. Suppose Q just adds one sub-interval to P. So we split $[x_i, x_{i+1}]$ into $[x_i, y]$ and $[y, x_{i+1}]$. By a property of the supremum it follows that $\sup f(x)$ on I is greater than or equal to the $\sup f(x)$ on either of the sub-sub-intervals. The result follows with a bit of basic factoring. A similar result holds for the infimum and m_i.

(e) We say that f is Riemann integrable on $[a, b]$ if $\sup L_P = \inf L_Q$.

(f) Alternatively, we may use Darboux's criterion, which says that a function is Riemann integrable if for all $\epsilon > 0$ there exists P so $U_P - L_P < \epsilon$.

(g) If a function is continuous on a closed bounded interval it is Riemann integrable on that interval. This follows as any function continuous on a closed interval, is uniformly convergent by the Maximum-Minimum Theorem.

(h) Sums/products of Riemann integrable functions on a domain are Riemann integrable on the same domain.

15. We prove a theorem involving monotone functions and integrability:

(a) Monotone functions have at most a countable number of discontinuities. I don't give a proof, but you can kind of see it since the most discontinuous function you can have, splits the rationals and the reals.

(b) It follows that monotone functions are Riemann integrable (by Exercise B from Practice Problems 8).

16. Finally we move to differentiability of a function. A function f is said to be differentiable if

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

exists.

17. We have $f(x) - f(c) = \frac{f(x)-f(c)}{x-c}(x - c)$. Take the limit of both sides, we have $\lim\limits_{x \to c} = 0$ since the derivative exists. The result follows.

18. The value of the derivative at an extremum is 0. Suppose $f(c)$ is an maximum of $[a, b]$. Suppose $f'(c) > 0$. Then there exists a neighborhood where $\frac{f(x)-f(c)}{x-c} > 0$. Supposing that $x > c$ it follows that $f(x) - f(c) > 0$. But this makes no sense, since c is the maximum. A similar proof is given for the minimum.

19. A corollary of this is as follows: If f is a constant function, then its derivative is 0 everywhere. This is because every point is extremal.

20. Rolle's Theorem is also a corollary of this. Suppose that $f(a) = 0 = f(b)$ where $I = [a, b]$, and f is continuous on I, and differentiable on (a, b). We have that there exists $f(c)$ so $f(c) = 0$. This follows as a result of the Maximum-Minimum Theorem. There is some maximum/minimum on (a, b) and so, the result follows by (18).

21. The Mean Value Theorem (MVT) is just an extension of Rolle's Theorem. Suppose, f is continuous on $[a, b]$ and differentiable on (a, b), then there exists c, so $f'(c)(b - a) = f(b) - f(a)$. This is really just a restatement of the Fundamental Theorem of Calculus. The proof involves Rolle's Theorem, since they're really the same thing. I won't bother with it. But graphically, if we twist our heads, so the point $f'(c) = 0$, we just have Rolle's Theorem.

22. I do some problems involving MVT, since they can be tricky.

    (a) We'll prove $|\sin x - \sin y| \leq |x - y|$ for all $x, y \in \mathbb{R}$. We note that the sine function is continuous and differentiable everywhere. We have by MVT then, that there exists $c \in (x, y)$ so $\sin x - \sin y = \cos c(x - y)$. It follows that $-(x - y) \leq \sin x - \sin y \leq (x - y)$, and the result follows naturally.

(b) We can do a similar thing for the natural logarithm function. We can show that

$$\frac{x-1}{x} < \ln x < x - 1.$$

We note that the natural logarithm function is continuous and differentiable on $(0, \infty)$, and has derivative $1/x$. Now consider the interval $[1, x]$. We have $\ln x - \ln 1 = \ln x = (1/c)(x-1)$. On the interval $(1, x)$ this is bounded above by $(x-1)$ and below by $(x-1)/x$. The result follows.

(c) Suppose $f(0) = g(0) = 0$, and $f'(x) \leq g'(x)$ for all x. Show that $f(x) \leq g(x)$. Using MVT we have that $f(x) = f'(c)x$. Same is true for $g(x)$. So the result follows. We can do this for the most general case where $f(a) = g(a)$.

23. If a function is monotone increasing and differentiable and continuous on some interval, it follows that $f'(c) \geq 0$ for all $c \in [a, b]$. Suppose $f'(c) < 0$. It follows that there's some interval where $\frac{f(x)-f(c)}{x-c} < 0$. So for $x - c > 0$ we have that $f(x) - f(c) < 0$, but f is monotone increasing. Strictly increasing/decreasing functions have similar properties.

24. Let's prove a version of L'Hospital's Rule using MVT. It states the following: Assume f, g are both in C^1 for some interval, and suppose $f(x_0) = g(x_0) = 0$, but $g'(x_0) \neq 0$. Then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{f'(x_0)}{g'(x_0)}.$$

By the MVT we have that on the interval (x, x_0) that there exists $f(x) - f(x_0) = f'(c)(x - x_0)$. Rewriting then, we have

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(c_1)(x-x_0) + f(x_0)}{g'(c_2)(x-x_0) + g(x_0)} = \lim_{x \to x_0} \frac{f'(c_1)}{g'(c_2)}.$$

So we have by Squeeze Theorem that $f'(c_1) = f'(x_0)$, and same for g. The result follows.

The other L'Hospital's Rule states the same thing for infinities.

25. More generally, this follows from Taylor's Theorem which states the following: For any function, $f \in C^{n+1}$, on $I = (a, b)$ we have that if $x, x_0 \in I$, that there exists c between the two so that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + ... + \frac{f^{(n)}(x_0)(x - x_0)^n}{n!} + \frac{f^{(n+1)}(c)(x - x_0)^{n+1}}{(n+1)!}.$$

26. We can prove some inequalities using Taylor's Theorem. For example, we can show that $1 + (1/2)x - (1/8)x^3 \leq \sqrt{1+x} \leq 1 + (1/2)x$, on the interval $(0, \infty)$. Pick $x_0 = 0$, then expand about the point $x > 0$ up to the third degree. We have

$$f(x) = 1 + (1/2)x + (1/8)x^2 + (3/48)(1+c)^{-5/2}x^3.$$

We note that the third term is always positive, (and the second term always negative), and the result follows.

27. We can also prove L'Hospital's Rules with Taylor's Theorem.

28. Having completed differentiability and integrability, we move the the Fundamental Theorem of Calculus (FTC). The first, FTC states: Given a continuous/differentiable function, f, we have that,

$$f(b) - f(a) = \int_a^b f \, dx,$$

if f' is Riemann integrable.

29. The second FTC states: If g is continuous on $[a, b]$, then there is a function $f(x)$ so $f(x) = \int g(t) \, dt$, with $f'(x) = g(x)$.

Sequences of functions

This unit examines sequences of real functions. It is useful in the study of applied mathematics (i.e. differential equations, partial differential equations, etc.).

1. A sequence of functions $\{f_n\}$ converges pointwise if the following is true: There exists f, so that for each x and all $\epsilon > 0$, there exists N so $|f_n - f| < \epsilon$ if $n \geq N$. This is really similar to the definition of the limit of a sequence.

2. A sequence of functions $\{f_n\}$ converges uniformly if the following is true: For all $\epsilon > 0$ there exists N so that for all x we have $|f_n - f| < \epsilon$ if $n \geq N$. Basically, this can be interpreted as saying the error everywhere is less than ϵ for some choice of n. Assuming, the functions are continuous, it's really like saying that the integrals converge to the same one (we'll state this as a theorem later).

3. We use a sequential criterion to establish when a sequence of functions does not converge uniformly. **Need to finish this.**

4. We introduce the concept of the supremum norm:

   (a) The supremum norm, denoted $||f||_A$, is defined for an interval A on which f is bounded, and is equivalent to $\sup |f(x)|$ for $x \in A$.

   (b) We say that f_n converges in the supremum norm to f if $||f_n - f|| \to 0$. Basically, what this means, is that the supremum of this difference goes to 0. For example, consider $f_n = x/n$. We pick an interval on which this function is bounded, say $[0, 1]$. We have that $\sup |x/n - 0| = 1/n$ on this interval, which does indeed converge to 0 as $n \to \infty$.

   As another example, consider, x^2/n on $[0, 8]$. We have that $\sup |x^2/n - 0| = 64/n$, which also goes to 0. The idea is basically that the maximum difference between the two functions is going to 0, which is what we want really.

   (c) We note that convergence in the supremum norm is equivalent to uniform convergence. A proof is as follows: Suppose f_n converges to f in the supremum norm. We have $\sup f_n - f \to 0$. But this means that $|f_n - f|$ goes to 0 to by Squeeze Theorem for the same N. A formal proof is as follows:

   Suppose f_n converges in the supremum norm to f. We have that $\lim_{n\to\infty} \sup |f_n - f| = 0$. But what this really means is that there exists an N for every $\epsilon > 0$ so that if $n \geq N$ then $|\sup |f_n - f|| < \epsilon$. But we note that if $|\sup |f_n - f|| < \epsilon$, we have that $|f_n - f| < \epsilon$ and the result follows. Going in the opposite direction is similar.

5. We prove some theorems involving sequences of functions (a couple are stated without proof):

   (a) If $\{f_n\}$ is a sequence of continuous functions on some interval A, then if $f_n \to f$ uniformly, f is continuous on A. A proof is as follows:

   If f_n is continuous on A we have for any $x_0 \in A$ that for all $\epsilon > 0$ that there exists a $\delta > 0$ so that $|f(x) - f(x_0)| < \epsilon/3$ if $|x - x_0| < \delta$. Furthermore, since f_n converges uniformly to f so that for all $\epsilon > 0$, there exists N so $|f_n(x_0) - f(x_0| < \epsilon/3$ when $n \geq N$. We note that $|f(x) - f(x_0)| = |f(x) - f_N(x) + f_N(x) + f_N(x_0) - f_N(x_0) - f(x_0)|$. By the triangle inequality, we have

   $$|f(x)-f(x_0)| \leq |f_N(x)-f(x)|+|f_N(x)-f_N(x_0)|+|f_N(x_0)-f(x_0)|.$$

   But each of the three terms is less than $\epsilon/3$ – the first by uniform convergence, the second by continuity, and the third again by uniform convergence. The result follows.

   This is a criterion for establishing uniform convergence of a sequence of functions. If f is not continuous, but the sequence f_n is on an interval A, then there cannot be uniform convergence. For example, consider the function,

   (b) If $\{f_n\}$ is a sequence of continuous (and hence Riemann integrable) functions on some interval A, and $f_n \to f$ uniformly, then f is Riemann integrable on A. Moreover,

   $$\int_a^b f\,dx = \lim_{n\to\infty} \int_a^b f_n\,dx.$$

   This is a criterion for determining whether or not a sequence of functions uniformly converge or not. For example, consider the sequence of functions, $g_n = n^2x^n(1-x)$ on $[0,1]$. We have that this function converges pointwise to 0 on the interval. We can do this using L'Hospital's Rule. We write it in the form n^2/x^{-n}, which is indeterminate. Differentiating gives $2n/-nx^{-n-1} = -2/x^{-(n+1)} = -2x^{n+1}$, which definitely converges to 0 on $(0,1)$ as $n \to \infty$. The cases at $0, 1$ are trivial. We note that this function is also continuous on the interval and hence Riemann integrable. We have that if it was uniformly convergent, the limit of the integrals should equate to the integral of the limit. We note however that we end up with an expression of the form $0 = 1$, which is invalid, and so this function does not converge uniformly.

We give a proof: Suppose f_n converges uniformly to f. Then there is an N so that $|f_n - f| < \epsilon/(b-a)$. We want to show that there exists N so that $|\int f_n - f| < \epsilon$. The result follows by a property of the Riemann integral.

(c) This final one is more involved. We state it without proof. If $\{f_n\}$ is a sequence of differentiable functions, and $\{f_n'\}$ converges uniformly to g, and for fixed x_0 $\{f_n\}$ converges, we have f_n converges uniformly to f and $f' = g$.

6. Here are some problems on sequences of functions:

(a) We show that $f_n = x^n$ converges uniformly on $[0, \mu]$ with $\mu < 1$. We have $|x^n - 0| \leq \mu^n$. So pick $N > \log_\mu \epsilon$.

(b) We show that $\sin(x + 1/n)$ converges uniformly to $\sin x$. We note by MVT that $\sin(x + 1/n) - \sin x = (1/n)\cos c$. Taking the absolute value, the result follows.

(c) Suppose that $\{f_n\}$ converges uniformly to f on some closed interval, and that it is continuous on this interval and hence Riemann integrable. For all continuous functions g, show that $\int f_n g$ converges to $\int fg$. We want

$$\left| \int_a^b f_n g - fg \right| = \left| \int_a^b g(f_n - f) \right| < \epsilon.$$

Since f_n converges uniformly to f, there's an N so $|f_n - f| < \epsilon/M(b-a)$, where $|g| < M$ by the Boundedness Theorem. We have then by a property of Riemann integrals that $\int_a^b |g||f_n - f| < \epsilon$, for the same N and we're done.

7. Finally, we talk about series of sequences of functions. The three theorems we discussed apply to series, since it's just a sequence of partial sums.

8. We talk about power series. We have that $f(x) = \sum a_n x^n$ is a power series. This converges for radius of convergence $1/\limsup a_n^{1/n}$. the proof involves bringing a_n inside the power term.

   (a) We can also compute the radius of convergence via $\lim_{n\to\infty} |a_n/a_{n+1}|$.

   (b) Shifting a power series doesn't change the radius of convergence, only the interval of convergence.

   (c) The Cauchy-Hadamard Theorem states that the series converges for all $|x| < R$ where R is the radius of convergence.

   (d) The derivative of a power series has the same radius of convergence.

9. Also, recall the Weirstrass M-Test which states that if I can find an M_n so that $|f_n(x)| < M_n$ for all n, and series M_n converges, then $f_n(x)$ converges uniformly.

   (a) Consider $\frac{1}{x^2+n^2}$. This is always less than or equal to $1/n^2$. This series converges, so the result follows.