Personal Statement and research summary

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My name is Brinda Venkataramani and I am a graduate student in the Department of Mathematics and Statistics at McMaster University. My research focus is on set theory and my supervisor is Dr. Patrick Speissegger. In particular, I am interested in disproving a claim made by Bertrand Russell in a 1905 letter to Phillip Jourdain [1]. He asserts the following:

Proposition 1 (Partition Principle (PP))**.** *Let x, y be sets. If $f : x \to y$ is a surjection, then there exists $g : y \to x$ an injection.*

At first glance, the proposition is deceptively similar to the Axiom of Choice (AC), and Russell for the most part in the body of his letter assumes their equivalence. However, it has been shown that the problem of showing equivalence of the two over the Zermelo-Fraenkel (ZF) axioms is non-trivial, and remains one of the oldest open problems in set theory. While it is easy to show AC $\implies$ PP, the reverse implication is much harder due to the inability to define a notion of cardinality for arbitrary sets in the absence of AC.

I first encountered this problem as an undergraduate student in an introductory course on mathematical logic. Initially as a graduate student however, I was interested in reverse mathematics and computable set theory, and shelved the problem for a while. I revisited it later after finally settling on a problem in logic I wanted to work on for my Master's thesis.

In my thesis, I make the following propositions and conjectures toward the hopeful resolution of this problem:

Proposition 2. *The partition principle is equivalent to the axiom of choice if and only if it is possible to construct a choice-like injection $h : y \to x$ given the existence of an injection $g : y \to x$ using only constructions from ZF, where choice-like refers to a function which acts as an inverse on a surjection $f : x \to y$.*

Conjecture 3. *It is not possible to construct a choice-like injection $h : y \to x$ given the existence of a single injection $g : y \to x$.*

Since the existence of ω is independent of AC, it may be possible to at least assert under the assumption of PP, that for infinite sets x, y with f a surjection from x to y, that at least ω-many injections exist from y to x [2].

When considered through the lens of infinitary combinatorics, the problem takes on an almost pigeon-hole problem like nature. In particular, one may ask the question of "how many" injections must exist from y to x in order to guarantee one is choice-like.

We are hopeful that the conjecture may be proven using the forcing techniques outlined in Kunen's and Jech's texts, and this is currently being explored[3]. The biggest problem I have personally encountered so far in my work has been the lack of progress on this problem. Indeed a cursory search on arXiv, jstor, and other platforms yields only around three papers on the problem – the most recent of which was published in 1995 by Masasi Higasikawa [4]. The other most recent paper is by Bernhard Banaschewski, which gives

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[1] See [Mar16] and [Gra72].

[2] See [Hal17].

[3] It is my hope that this makes it into the body of my thesis. Also see [Jec06] and [Kun83].

[4] See [Hig95].

an outline of related open problems [5]. However, the lack of progress made on the problem has also been motivating for me as a student; first, because it reminds me that there are goals to work towards when I am feeling discouraged, and second because it reminds me that I have just started out on my mathematical journey and that there are many more things to learn.

For my doctoral work I would hope to continue addressing these problems. I also have interests in reverse mathematics and computability theory. Namely I am interested in problems involving the classification of problems by their difficulty to solve.

I have also worked in *o*-minimality in the past, and have experience with scientific computing. I have a soft spot for the mathematical foundations of physics, and my non-academic interests include running and bouldering.

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[5] See [Ban90].