Fourier Transform I: Introduction and Basic Properties

I. Introduction

As we have seen before, representing signals as sums of complex exponentials is a very powerful technique for understanding and using Linear, Time-Invariant (LTI) systems because complex exponentials are eigenfunctions of LTI systems. We have already seen how periodic, discrete-time signals can be represented as sums of complex exponentials through the IDFT/DFT. We then saw how we can represent continuous-time (CT) periodic signals as sums of complex exponentials through the Fourier series. Here, we expand the class of signals that we can represent using a linear combination of complex exponentials to non-periodic CT signals. These signals will be represented using a dense sum (i.e. an integral) of complex exponentials.

These tools will then set the stage for us to derive the sampling theorem (i.e. a signal can be perfectly reconstructed from its sampled representation if the sampling frequency is greater than twice the maximum frequency present in the signal original signal). We can also use these techniques to analyze the input-output behavior of systems that are represented as linear, constant-coeffiecient differential equations, which is a very useful class of systems.

We shall represent the CTFT of a signal $x(t)$ using $X(\omega)$, and use this convention throughout, e.g. the CTFT of $q(t)$ is $Q(\omega)$ etc.

Most of the properties/features of discrete-time signals and Fourier transforms carry over to the continuous time with natural changes that one might expect, (e.g. summations become integrals). The CTFT and inverse CTFT are:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \tag{1}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega \tag{2}$$

Notice that $x(t)$ is expressed as a dense sum (i.e. an integral) of scaled complex exponentials, together with a scaling of $\frac{1}{2\pi}$. Note that if you think of a frequency as the number of radians that a complex number rotating around on the complex plane goes through per unit time, then $e^{j\,\omega\,t}$ is a pure frequency component at frequency ω radians per unit time. Thus the ICTFT expresses

$x(t)$ as a dense sum of pure frequency components, where the frequency component of ω radians per unit time is weighted by $X(\omega)$.

The derivation of the CTFT starting from the Fourier series in given in Section II. The basic idea is to start with the Fourier series representation of a periodic signal and then take the period to infinity to produce a non-periodic signal. Note that a signal and its Fourier transform have a one-to-one mapping. In other words, if you know the FT of the signal, then in principle, you know the entire signal.

II. Fourier series to the continuous-time Fourier transform

Suppose that we have a non-periodic signal $x(t)$ where we shall assume that $x(t) = 0$ for $|t| > \frac{T}{2}$. Note that we shall be taking T to infinity later so this is not a real restriction. We shall then construct a periodic extension of $x(t)$ which we shall call $\hat{x}(t)$ which is given by

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(t - kT)\,. \tag{3}$$

Figure 1 illustrates an example of this.

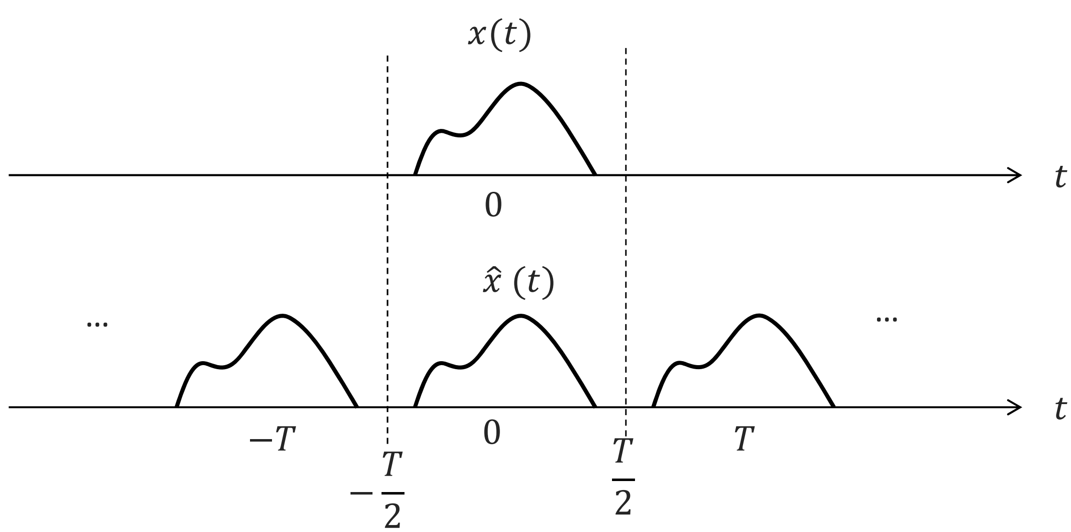

Fig. 1. A signal and its periodic extension.

The general strategy is to represent the periodic signal $\hat{x}(t)$ using a Fourier series. Then, we shall take the period of the periodic signal to infinity which results in a non-periodic signal.

First, lets define $\omega_0 = \frac{2\pi}{T}$, i.e. ω_0 is the fundamental frequency of $\hat{x}(t)$, in radians per unit time. We can then write the Fourier series representation of $\hat{x}(t)$ as follows:

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} C_k \, e^{j\,\omega_0\,k\,t} \tag{4}$$

Usiing the expression for the Fourier series coefficients, and noting that $\omega_0 = \frac{2\pi}{T}$,

$$C_k = \frac{\omega_0}{2\,\pi} \int_{-\frac{T}{2}}^{\frac{T}{2}} \hat{x}(t) e^{-j\,\omega_0\,k\,t} \, dt. \tag{5}$$

Since $x(t) = \hat{x}(t)$ for $T/2 < t < T/2$, we can write the previous expression as

$$C_k = \frac{\omega_0}{2\,\pi} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) e^{-j\,\omega_0\,k\,t} \, dt \tag{6}$$

Additionally, since $x(t)$ is zero when $t > \frac{T}{2}$ or $t < \frac{T}{2}$, we can write the above expression as

$$C_k = \frac{\omega_0}{2\,\pi} \int_{-\infty}^{\infty} x(t) e^{-j\,\omega_0\,k\,t} \, dt. \tag{7}$$

Next, let's define a function $X(\omega)$ as follows:

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\,\omega\,t} \, dt \,. \tag{8}$$

Comparing (7) and (8), observe that

$$C_k = \frac{\omega_0}{2\pi} X(\omega_0 k) \,. \tag{9}$$

We can think of the C_k's as a sampling of the function $X(\omega)$ where samples are taken every ω_0, and scaled by $\frac{\omega_0}{2\pi}$.

Next, we substitute (9) into (4) to get

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} \frac{\omega_0}{2\pi} X(\omega_0 k) \, e^{j\,\omega_0\,kt} \,. \tag{10}$$

Now define a new function $Q(\omega) = \frac{1}{2\pi} X(\omega) e^{j\omega t}$. A representative plot of $Q(\omega)$ is shown as a bold line in Figure 2.

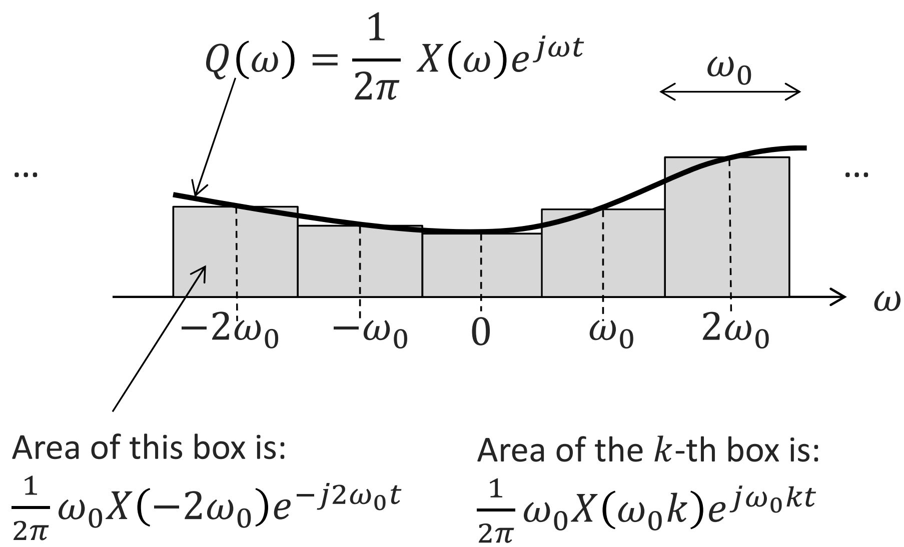

Fig. 2. Illustration of Fourier series representation as a sum of areas of rectangles

The height of the k-th grey box in Figure 2 is $Q(\omega_0 k) = \frac{1}{2\pi} X(\omega_0 k) e^{j\omega_0 kt}$. The area of the k-th grey box is $\frac{1}{2\pi}\omega_0 X(k\omega_0) e^{j\omega_0 kt}$. Since

$$\hat{x}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \omega_0 X(\omega_0 k)\, e^{j\,\omega_0\,kt}\,, \tag{11}$$

$\hat{x}(t)$ is the sum of areas of all the grey boxes in Figure 2.

Next, we take $T \to \infty$. Since $\omega_0 = \frac{2\pi}{T}$, $T \to \infty$ means $\omega_0 \to 0$. Since the width of the boxes in Figure 2 is ω_0, as $\omega_0 \to 0$, the summation above converges to the integral of the function $Q(\omega)$. In other words

$$\begin{aligned}
\lim_{T\to\infty} \hat{x}(t) &= \lim_{T\to\infty} \sum_{k=-\infty}^{\infty} \frac{1}{2\pi}\omega_0 X(\omega_0 k)\, e^{j\,\omega_0\,kt} \\
&= \lim_{\omega_0\to 0} \sum_{k=-\infty}^{\infty} \frac{1}{2\pi}\omega_0 X(\omega_0 k)\, e^{j\,\omega_0\,kt} \\
&= \int_{-\infty}^{\infty} Q(\omega) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t} d\omega
\end{aligned} \tag{12}$$

Finally, we note that as $T \to \infty$, $\hat{x}(t) \to x(t)$. Hence we have,

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t} d\omega \tag{13}$$

A. *Fourier transform pairs*

Some cannonical signals and their Fourier transforms are given in Table I.

$x(t)$	$X(\omega)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$\delta(t-t_0)$	$e^{-j\omega t_0}$
$\cos(\omega_0 t)$	$\pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)$
$\sin(\omega_0 t)$	$\frac{\pi}{j}\delta(\omega-\omega_0)-\frac{\pi}{j}\delta(\omega+\omega_0)$
$\sum_{k=-\infty}^{\infty}\delta(t-kT)$	$\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi}{T}k\right)$

TABLE I

TABLE OF FOURIER TRANSFORM PAIRS

These transform pairs can be computed using direct integration. Another commonly encountered transform pair is given in the following example. Let the signal $x(t)$ be given by

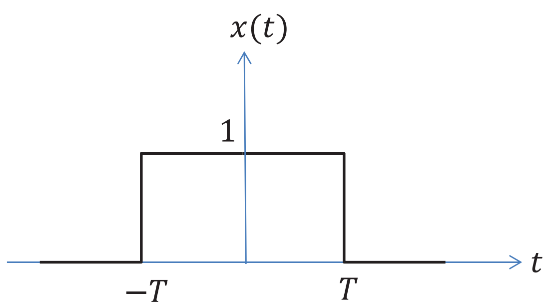

Fig. 3. Rectangular signal in time.

$y(t)$	$Y(\omega)$
$x * h(t)$	$X(\omega)H(\omega)$
$x(t)h(t)$	$\frac{1}{2\pi} X * H(\omega)$
$e^{j\omega_0 t} x(t)$	$X(\omega - \omega_0)$
$x(t - t_0)$	$e^{-j\,\omega\,t_0}\, X(\omega)$
$\frac{d}{dt} x(t)$	$j\omega\, X(\omega)$

TABLE II

TABLE OF FOURIER TRANSFORM PROPERTIES

The Fourier transform of this signal is

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\,\omega\,t}\, dt \tag{14}$$

$$= \int_{-T}^{T} e^{-j\,\omega\,t}\, dt \tag{15}$$

$$= \frac{1}{-j\omega} \left[e^{-j\,\omega\,t} \right]_{-T}^{T} \tag{16}$$

$$= \frac{2}{\omega} \left(\frac{1}{2j} e^{j\,\omega\,T} - \frac{1}{2j} e^{-j\,\omega\,T} \right) \tag{17}$$

$$= \frac{2 \sin(\omega\, T)}{\omega} \tag{18}$$

B. Fourier transform properties

Some properties of the CTFT are given in Table II. It is assumed here that the Fourier transform of $x(t)$, $y(t)$ and $h(t)$ are $X(\omega)$, $Y(\omega)$, and $H(\omega)$ respectively. Note that these properties can all be derived starting from the definitions of the CTFT and ICTFT and using standard manipulations such as a change of variables. To illustrate, consider the fourth property in the table.

$$\text{FT}\left\{ x(t - t_0) \right\} = \int_{-\infty}^{\infty} x(t - t_0) e^{-j\omega t} dt\,, \tag{19}$$

where $\text{FT}\{x(t-t_0)\}$ here denotes the Fourier transform of $x(t-t_0)$. Make a change of variables $\tau = t - t_0$, for which $dt = d\tau$:

$$\text{FT}\{x(t-t_0)\} = \int_{-\infty}^{\infty} x(\tau) e^{-j\omega(\tau+t_0)} d\tau \tag{20}$$

$$= \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} e^{-j\omega t_0} d\tau \tag{21}$$

$$= e^{-j\omega t_0} \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} d\tau \tag{22}$$

$$= e^{-j\omega t_0} X(\omega) . \tag{23}$$

The first property in the table is of course the continuous-time analog to the convolution theorem you saw earlier in the class. It is particularly useful in characterizing the input-output properties of LTI systems. Since we know that the output of an LTI system is the input signal convolved with the impulse response, we can say the following:

$$y(t) = x * h(t) = h * x(t) \tag{24}$$

$$Y(\omega) = H(\omega) X(\omega) \tag{25}$$

This property can be derived as follows.

$$y(t) = x * h(t) = \int_{-\infty}^{\infty} x(q)\, h(t-q)\, dq \tag{26}$$

Taking the Fourier transform of $y(t)$, we have

$$Y(\omega) = \int_{-\infty}^{\infty} y(t) e^{-j\,\omega\, t}\, dt \tag{27}$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(q)\, h(t-q)\, dq \right] e^{-j\,\omega\, t}\, dt \tag{28}$$

Exchanging the order of the integral above, we have

$$Y(\omega) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(q)\, h(t-q)\, e^{-j\,\omega\, t}\, dt \right] dq \tag{29}$$

Since $x(q)$ is not a function of t, we can pull it out of the inner-most integral.

$$Y(\omega) = \int_{-\infty}^{\infty} x(q) \left[\int_{-\infty}^{\infty} h(t-q)\, e^{-j\,\omega\, t}\, dt \right] dq \tag{30}$$

Invoking the time-shift property (which we just derived in (23)), observe that

$$\int_{-\infty}^{\infty} h(t-q)\, e^{-j\,\omega\, t}\, dt = e^{-j\omega q} H(\omega)$$

If we substitute the previous expression into (30), we have

$$\begin{aligned} Y(\omega) &= \int_{-\infty}^{\infty} x(q)\, e^{-j\omega q}\, H(\omega)\, dq \\ &= H(\omega) \int_{-\infty}^{\infty} x(q)\, e^{-j\omega q}\, dq \\ &= H(\omega) X(\omega) \end{aligned} \tag{31}$$

Exercises:

1. Consider a train of unit impulses separated by T time units, given by the following expression

$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT) \tag{32}$$

   a. Sketch a representation of $p(t)$.
   b. Find the Fourier series representation of $p(t)$ with an infinite number of terms.
   c. Let a function $x(t)$ be represented as a Fourier series with an infinite number of terms as follows:

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{j\frac{2\pi}{T}k\,t}. \tag{33}$$

   Find $X(\omega)$, in terms of C_k.
   d. Using your answer to the previous two parts, find $P(\omega)$.
   e. Sketch $P(\omega)$. How does changing T affect $p(t)$ and $P(\omega)$? Is this what you would expect?

2. Consider an LTI system with an impulse response $h(t)$, input signal $x(t)$ and output $y(t)$. It is known that $H(\omega)$ is the following.

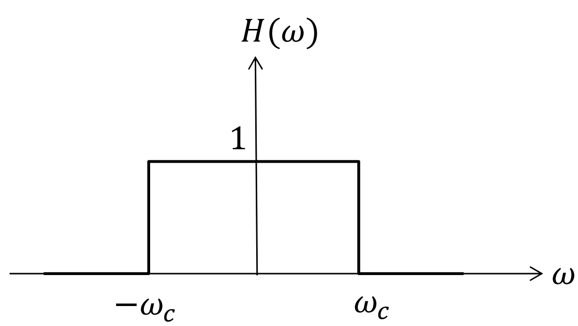

Fig. 4. Ideal low pass filter

a. Find $h(t)$.

b. Suppose that $X(\omega)$ is the following. Please sketch $Y(\omega)$. (Note here that the shape of

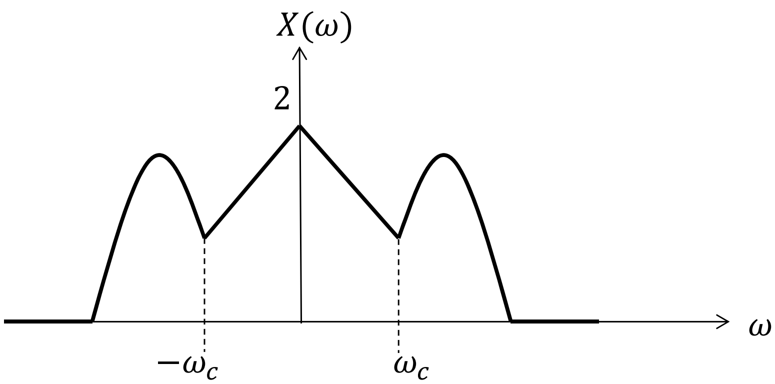

Fig. 5. Spectrum of a representative signal.

$X(\omega)$ was deliberately made to illustrate a point).

c. Explain why this LTI system is known as an ideal low-pass filter with cut-off frequency ω_c.

3. Consider a signal $x(t)$ which is band-limited to the range $[-f_M, f_M]$. In other words, $X(w) = 0$ for $\omega < -f_M$ and $\omega > f_M$. Suppose that $X(\omega)$ is given in Figure 6. Let $y(t) =$

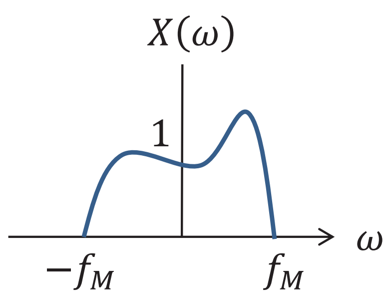

Fig. 6. $X(\omega)$.

$x(t)\cos(\omega_0 t)$. Please sketch the Fourier transform of $y(t)$, $Y(\omega)$.