Fourier Series

I. Complex exponentials and and the differentiation operator

Most continuous periodic signals (in fact all practically interesting continuous periodic signals) can be represented as a sum of complex exponentials using the Fourier series. We have already seen how representing signals as linear combinations of complex exponentials can be very powerful. This idea also extends to continuous time. In particular, the complex exponential is an eigenfunction for the differentiation operator, as mentioned in class before.

In other words, given a complex exponential at a frequency ω_0, i.e. $Ae^{j\omega_0 t}$, its derivative is proportional to $Ae^{j\omega_0 t}$, because

$$\frac{d}{dt}Ae^{j\omega_0 t} = Aj\omega_0 e^{j\omega_0 t}\,. \tag{1}$$

Thus, we can easily compute the derivative of a function described by a sum of complex exponentials, which is interesting and quite powerful. It gets better, however.

Observe that the derivative of the complex exponential is itself a complex exponential. Therefore, complex exponentials are eigenfunctions of multiple derivatives as well. This is because

$$\frac{d^2}{dt^2}Ae^{j\omega_0 t} = A(j\omega_0)^2 e^{j\omega_0 t} \tag{2}$$

$$\frac{d^3}{dt^3}Ae^{j\omega_0 t} = A(j\omega_0)^3 e^{j\omega_0 t} \tag{3}$$

$$\frac{d^n}{dt^n}Ae^{j\omega_0 t} = A(j\omega_0)^n e^{j\omega_0 t}\,. \tag{4}$$

Moreover, differentiation is a linear operation. What this means is that we can apply the above property to equations like the following

$$y(t) = \frac{d^2}{dt^2}x(t) + 3\frac{d}{dt}x(t) + 2x(t)\,. \tag{5}$$

If $x(t) = Ae^{j\omega_0 t}$, we have

$$y(t) = \left[(j\omega_0)^2 + 3(j\omega_0) + 2\right] Ae^{j\omega_0 t} \tag{6}$$

In other words, complex exponentials are eigenfunctions of systems described by differential equations of the form in (5). It turns out, (and we will see more of this later in the course), that complex exponentials are also eigenfunctions of systems described by more general differential equations as well, i.e. ones where the y and x terms all have derivatives.

II. Fourier series

In this section, we shall see how the Fourier series can be used to represent continuous-time periodic signals as linear combinations of complex exponentials. In subsequent sections, we will use the Fourier series to derive the continuous-time Fourier transform (CTFT), which will enable us to analyze the frequency content of continuous-time signals, or to put it in a different way, express continuous-time signals as dense sums (i.e. integrals) of complex exponentials. If you wish to see how the Fourier series can be derived from the DFT, you can read Appendix A, but that section is optional, and this derivation is not normally covered in Signals and Systems courses.

Consider a continuous-time signal $x(t)$ which is periodic with period $T > 0$. In other words,

$$x(t+T) = x(t). \tag{7}$$

We further assume that T is the fundamental period, i.e. it is the smallest positive T for which the above equation is true.

Then we can write an approximation to $x(t)$ as a sum of $2K+1$ scaled, complex exponentials as follows:

$$\tilde{x}_K(t) = \sum_{k=-K}^{K} C_k e^{j\frac{2\pi}{T}k\,t}\,. \tag{8}$$

Thus, $\tilde{x}_K(t)$ is a sum of $2K+1$ complex exponentials whose frequencies are integer multiples of $\frac{2\pi}{T}$.

The coefficients C_k, which are how much the complex exponentials get weighted by can be found using

$$C_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\, e^{-j\frac{2\pi}{T}k\,t}\, dt. \tag{9}$$

A derivation for the the above expression is given in Section III. You are expected to follow this derivation, in particular the part about the orthogonality of the complex exponentials, but will not have to replicate the derivation.

The Fourier series approximation of $x(t)$ is such that as $K \to \infty$, i.e. if we use an increasing number of complex exponentials to approximate $x(t)$, we have the following

$$\int_{-T/2}^{T/2} \left|x(t) - \tilde{x}_K(t)\right|^2 \, dt \to 0. \tag{10}$$

The term $|x(t) - \tilde{x}_K(t)|^2$ represents the squared error between $x(t)$ and its Fourier series approximation, $\tilde{x}_K(t)$. What does this mean in practical systems? In many systems such as electrical systems, signals are represented by voltages or currents. If we assume that a signal such as $x(t)$ is a voltage applied across some resistance, then $x(t)^2$ is proportional to power, and its time-integral is its energy. Therefore,

$$\int_{-T/2}^{T/2} |x(t) - \tilde{x}_K(t)|^2 \, dt$$

represents the energy over one period, in the error signal between $x(t)$ and $\tilde{x}_K(t)$. Thus, (10) indicates that the error between $x(t)$ and $\tilde{x}_K(t)$ goes to zero. So for all practical purposes $\tilde{x}_K(t)$ is equal to $x(t)$ when K is large[1].

Example:

Here we examine the Fourier series representation of a triangle wave. Consider a triangle wave with fundamental period T as depicted in Figure 1. In the range $-T/2 < t \leq T/2$, the equation for this triangle wave is

$$x(t) = \frac{2}{T}\,|t| \tag{11}$$

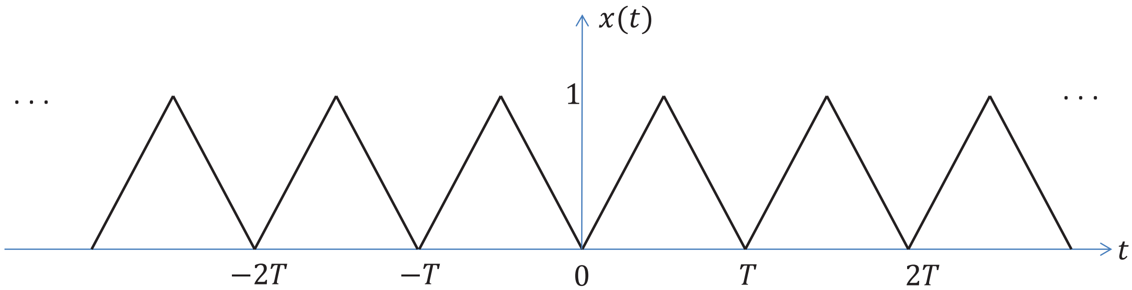

Fig. 1. Triangle wave with period T.

[1]Note that there are certain mathematical conditions that $x(t)$ needs to satisfy for it to have a Fourier series representation. These conditions are called the Dirichelet conditions (you can google this if you like), but all interesting signals will satisfy these conditions. So moving forward, we shall just assume that $x(t)$ satisfies these conditions.

The k-th Fourier series coefficient for $x(t)$ can be found using (9):

$$C_k = \frac{1}{T}\int_{-T/2}^{T/2} \frac{2}{T}\,|t|\,e^{-j\frac{2\pi}{T}\,k\,t}\,dt$$

After some gymnastics with integration (or by using a symbolic math software package), we find that

$$C_k = \begin{cases} -\frac{2}{\pi^2\,k^2} \text{ if } k \text{ is odd} \\ \frac{1}{2} \text{ if } k = 0 \\ 0 \text{ otherwise.} \end{cases} \tag{12}$$

Fourier-series representations of the the triangle wave with different number of terms are illustrated in Figure 2. Notice that with just 11 terms in the Fourier series representation, we obtain a relatively accurate representation of the triangle wave.

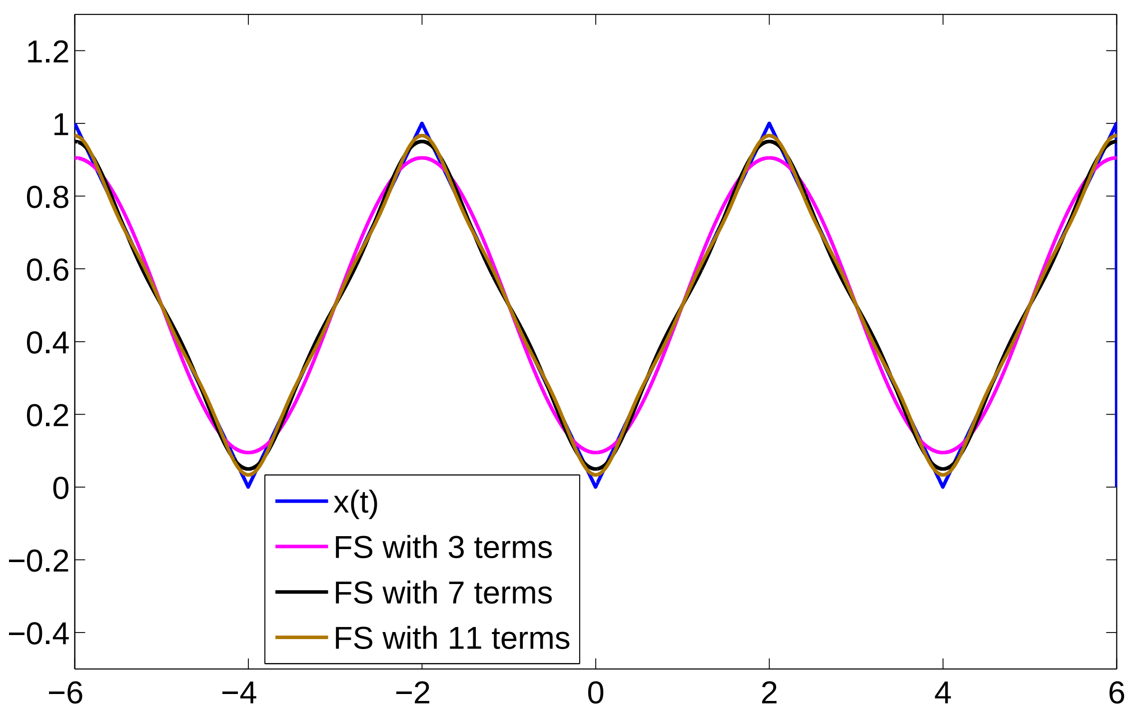

Fig. 2. Fourier series representation of triangle wave.

The Fourier series tells us that a periodic signal with a fundamental period T is really made up of a bunch of complex exponentials whose frequencies are integer multiples of $\frac{2\pi}{T}$. We have already seen that describing discrete-time signals as sums of complex exponentials is a powerful technique to analyze the behavior of LTI systems. Similar techniques can be used to analyze LTI systems when the inputs are continuous-time periodic signals.

III. The Fourier series coefficients

You saw how a periodic, continuous-time signal $x(t)$ with period T can be represented as a sum of complex exponentials. If we assume that it is indeed possible to represent $x(t)$ in this fashion, we can say that

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{j\frac{2\pi}{T}k\,t}. \tag{13}$$

Let m be some integer and multiply both sides by $\frac{1}{T}e^{-j\frac{2\pi}{T}m\,t}$. This gives us

$$\begin{aligned}\frac{1}{T}x(t)e^{-j\frac{2\pi}{T}m\,t} &= \sum_{k=-\infty}^{\infty} C_k e^{j\frac{2\pi}{T}k\,t}\,\frac{1}{T}e^{-j\frac{2\pi}{T}m\,t} \\ &= \frac{1}{T}\sum_{k=-\infty}^{\infty} C_k e^{j\frac{2\pi}{T}(k-m)\,t} \qquad (14)\end{aligned}$$

Now integrate both sides with respect to t.

$$\begin{aligned}\frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-j\frac{2\pi}{T}m\,t}\,dt &= \frac{1}{T}\int_{-T/2}^{T/2}\sum_{k=-\infty}^{\infty} C_k e^{j\frac{2\pi}{T}(k-m)\,t}\,dt \\ &= \sum_{k=-\infty}^{\infty}\frac{C_k}{T}\int_{-T/2}^{T/2} e^{j\frac{2\pi}{T}(k-m)\,t}\,dt \qquad (15)\end{aligned}$$

Now, let's consider the integral on the right hand side of the summation above. When $k \neq m$, we have

$$\begin{aligned}\int_{-T/2}^{T/2} e^{j\frac{2\pi}{T}(k-m)\,t}\,dt &= \frac{1}{j\frac{2\pi}{T}(k-m)}\left[e^{j\frac{2\pi}{T}(k-m)\,t}\right]_{-T/2}^{T/2} \qquad (16) \\ &= \frac{1}{j\frac{2\pi}{T}(k-m)}\left[e^{j\frac{2\pi}{T}(k-m)(T/2)} - e^{j\frac{2\pi}{T}(k-m)(-T/2)}\right] \qquad (17) \\ &= \frac{1}{\frac{\pi}{T}(k-m)}\left[\frac{1}{2j}e^{j\pi(k-m)} - \frac{1}{2j}e^{-j\pi(k-m)}\right] \qquad (18) \\ &= \frac{1}{\frac{\pi}{T}(k-m)}\sin(\pi(k-m)) = 0 \qquad (19)\end{aligned}$$

The second last step follows from the identity $\frac{1}{2j}e^{j\theta} - \frac{1}{2j}e^{-j\theta} = \sin(\theta)$, and the last step is because a sine function equals zero at integer multiples of π.

When $k = m$ on the other hand, we have

$$\int_{-T/2}^{T/2} e^{j\frac{2\pi}{T}(m-m)\,t}\,dt = \int_{-T/2}^{T/2} 1\,dt = T. \tag{20}$$

Together with (19), the previous equation implies that

$$\frac{1}{T}\int_{-T/2}^{T/2} e^{j\frac{2\pi}{T}\ell\,t}\,dt = \begin{cases} 1 \text{ if } \ell = 0, \\ 0 \text{ otherwise.} \end{cases} \tag{21}$$

The expression above is a result of a fundamental property that complex exponentials at frequencies that are integer multiples of $\frac{2\pi}{T}$ are orthogonal to each other.

If we inspect the summation on the right hand side of (15), we notice that all terms in the summation where $k \neq m$ are zero because of (19). This leaves the $k = m$ term as the only surviving term in the summation which leads to

$$\begin{aligned} \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-j\frac{2\pi}{T}m\,t}\,dt &= \frac{1}{T}\int_{-T/2}^{T/2} \sum_{k=-\infty}^{\infty} C_k e^{j\frac{2\pi}{T}(k-m)\,t}\,dt \\ &= \frac{C_m}{T}\int_{-T/2}^{T/2} e^{j\frac{2\pi}{T}(m-m)\,t}\,dt \qquad (22) \\ &= C_m\,. \qquad (23) \end{aligned}$$

The last equation is the expression for the m-th Fourier series coefficient.

Exercises:

1. Find the Fourier series representation for the square wave in Figure 3. You may find the identity $\sin(\theta) = \frac{1}{2j}e^{j\theta} - \frac{1}{2j}e^{-j\theta}$ useful here.

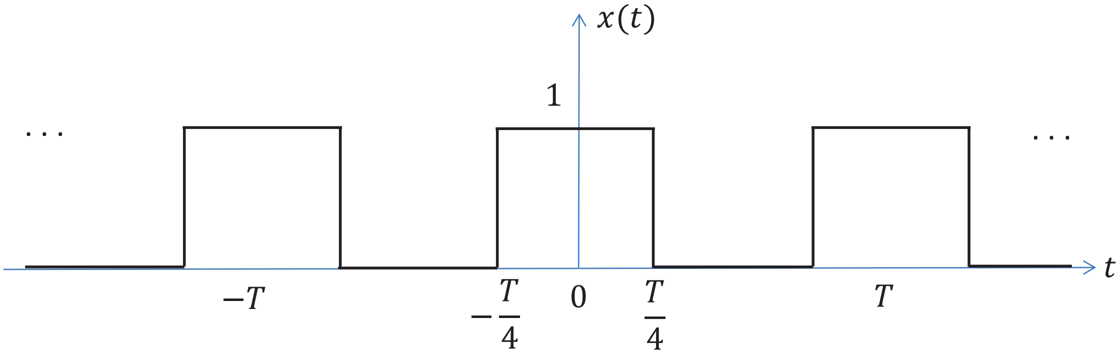

Fig. 3. Square wave with period T.

2. Using a computer, plot the Fourier series representation of the square wave in the previous part with fundamental period $T = 4$, and for 5, 17, and 257 terms in the Fourier series. For clarity, you should plot them in separate subplots/plots. Note that the function $\frac{\sin(\pi x)}{\pi x}$ is called the *sinc* function and is available in the numpy package in python.
3. Using your understanding of the Fourier series, describe how you can produce a cosine wave from the square wave in Figure 3, using appropriate filters. You can assume that you have access to ideal filters here, i.e. you can filter out signals perfectly. What frequencies of cosines can you produce using this approach? How might you apply this idea to generate a cosine signal from the output of a microcontroller (e.g. an Arduino) for instance?
4. Suppose that $x(t)$ is a periodic signal with fundamental period T, and has a Fourier series representation with coefficients C_k. Consider a new signal, $y(t) = x(t - T_1)$, where $|T_1| < T$. Thus $y(t)$ is a delayed version of $x(t)$. Find the Fourier series coefficients for $y(t)$ in terms of C_k. Feel free to use symbolic math software to solve this if you wish.
5. Using your answer above, find the Fourier series coefficients for the triangle wave in Figure 4. Verify that your answer is correct by modifying and running the code for the Fourier series of the triangle wave that you used in class.

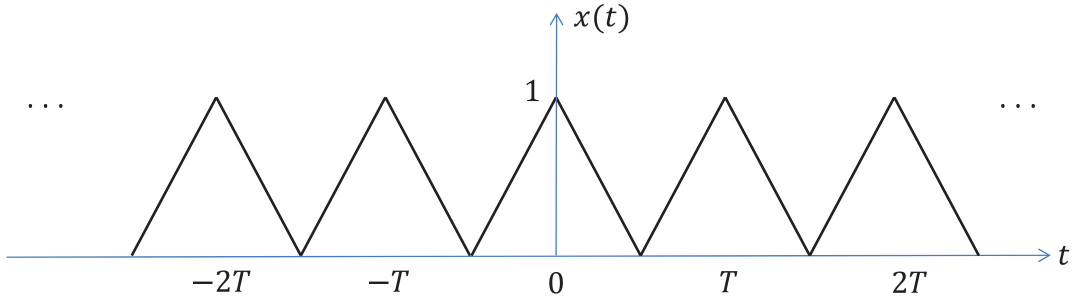

Fig. 4. Triangle wave with period T.

Appendix

A. Derivation of the Fourier Series

One can derive the Fourier series starting from the DFT. As we saw in a previous homework, the Discrete-Fourier Transform (DFT) can be written in summation form as follows

$$X_k = \sum_{n=0}^{N-1} x[n] e^{-j\,\frac{2\pi}{N}\,k\,n}\,. \tag{24}$$

The Inverse-DFT can also be written in summation form as follows

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X_k e^{j\,\frac{2\pi}{N}\,k\,n}\,. \tag{25}$$

Observe that X_k and $x[n]$ are periodic with period N. This periodicity can be seen as follows

$$\begin{aligned} X_{k+N} &= \sum_{n=0}^{N-1} x[n] e^{-j\,\frac{2\pi}{N}\,(k+N)\,n} = \sum_{n=0}^{N-1} x[n] e^{-j\,\frac{2\pi}{N}\,k\,n} e^{-j\,\frac{2\pi}{N}\,N\,n} \\ &= \sum_{n=0}^{N-1} x[n] e^{-j\,\frac{2\pi}{N}\,k\,n} e^{-j 2\,\pi\,n} = X_k \end{aligned} \tag{26}$$

since $e^{-j2\pi n} = 1$ for all integers n. Similarly,

$$x[n+N] = \sum_{k=0}^{N-1} X_k e^{j\,\frac{2\pi}{N}\,k\,(n+N)} = x[n]\,. \tag{27}$$

Assuming that N is even, we can write

$$x[n] = \sum_{k=0}^{N/2-1} X_k e^{j\,\frac{2\pi}{N}\,k\,n} + \sum_{k=N/2}^{N-1} X_k e^{j\,\frac{2\pi}{N}\,k\,n} \tag{28}$$

$$= \sum_{k=0}^{N/2-1} X_k e^{j\,\frac{2\pi}{N}\,k\,n} + \sum_{k=N/2}^{N-1} X_{k-N} e^{j\,\frac{2\pi}{N}\,k\,n} \tag{29}$$

The last equation is a result of the fact that X_k is periodic with period N. Next, we make a substition $\ell = k - N$, which implies that $k = \ell + N$.

$$x[n] = \sum_{k=0}^{N/2-1} X_k e^{j\,\frac{2\pi}{N}\,k\,n} + \sum_{\ell=-N/2}^{-1} X_\ell e^{j\,\frac{2\pi}{N}\,(\ell+N)\,n} \tag{30}$$

$$= \sum_{k=0}^{N/2-1} X_k e^{j\,\frac{2\pi}{N}\,k\,n} + \sum_{\ell=-N/2}^{-1} X_\ell e^{j\,\frac{2\pi}{N}\,\ell\,n} e^{j\,2\,\pi\,n} \tag{31}$$

$$= \sum_{k=0}^{N/2-1} X_k e^{j\,\frac{2\pi}{N}\,k\,n} + \sum_{\ell=-N/2}^{-1} X_\ell e^{j\,\frac{2\pi}{N}\,\ell\,n} \tag{32}$$

$$= \sum_{k=-N/2}^{N/2-1} X_k e^{j\,\frac{2\pi}{N}\,k\,n} \tag{33}$$

Now consider a continuous-time signal $x(t)$ which is periodic with a fundamental period T. In other words:

$$x(t) = x(t + T) \tag{34}$$

for all t. We further assume that T is the smallest positive T for which the above equation is true. Suppose that $x(t)$ is sampled with a sampling period of T_s, such that

$$x[n] = x(nT_s)\,. \tag{35}$$

Suppose that there are N samples per period T, i.e. $T/T_s = N$. At times that are integer multiples of T_s, writing $t = nT_s$, we can use the IDFT equation (33) to express $x(t)$ in terms of the DFT coefficients X_k as follows:

$$x(t) = x[n] = \frac{1}{N} \sum_{k=-N/2}^{N/2-1} X_k e^{j\,\frac{2\pi}{N}\,k\,\left(\frac{t}{T_s}\right)}\,. \tag{36}$$

Since N samples are taken per period T and the sampling period is T_s, we have $NT_s = T$, which yields:

$$x(t) = x[n] = \sum_{k=-N/2}^{N/2-1} \frac{1}{N} X_k e^{j\,\frac{2\pi}{T}\,k\,t} \tag{37}$$

The equation above gives us the values of $x(t)$ for $t = nT_s$. In other words, the above equation can be used to evaluate $x(t)$ at times that are integer multiples of the sampling period. Now if we take $T_s \to 0$ and $n \to \infty$ such that $t = nT_s$, we can use (37) to evaluate $x(t)$ for any value of t. Taking the limit of (37) as $N \to \infty$, we have

$$x(t) = \sum_{k=-\infty}^{\infty} \lim_{N\to\infty} \left(\frac{1}{N} X_k\right) e^{j\,\frac{2\pi}{T}\,k\,t}\,. \tag{38}$$

Thus, a periodic signal with period T can be represented by an infinite sum of complex exponentials with frequencies that are integer multiples of $\frac{2\pi}{T}$, which is what the Fourier series representation of $x(t)$ is.

At this point, it is unclear what the coefficients of the complex exponentials should be. One approach we could take is to try to evaluate

$$\lim_{N\to\infty} \frac{1}{N} X_k\,. \tag{39}$$

However a simpler approach using only direct integration is given in the main body of this document.