SigSys PS06

Ruby Spring

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1 Guns and Violins

The recorded audio signal of a gun in a shooting range behaves like an impulse in a CT system. Thus we can say that we get an impulse response from shooting the gun, more generally called a transfer function, $h(t)$. Obviously the gun is not a perfect impulse, as a perfect impulse has infinite amplitude and infinitesimal width in time, but maybe it is good enough. In order to hear what a violin might sound like in the shooting range, we can simply convolve the impulse response with the signal of the violin, $x(t)$, like so:

$$y(t) = x * h(t)$$

where "$*$" is the operator that represents the convolution function:

$$x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

2 Echoes

If the output of an echo chamber, $y(t)$, is such that:

$$y(t) = \frac{1}{2}x(t-1) + \frac{1}{4}x(t-10)$$

then the impulse response of the system is such the input is an impulse, $\delta(t)$. So we substitute $x(t) = \delta(t)$ and the output becomes the transfer function $h(t)$:

$$h(t) = \frac{1}{2}\delta(t-1) + \frac{1}{4}\delta(t-10)$$

It is reasonable to say an echo chamber will repeatedly attenuate and shift any signal to a later point in time, and this is exactly what the echo function is doing. An approximate graph of the impulse response is below:

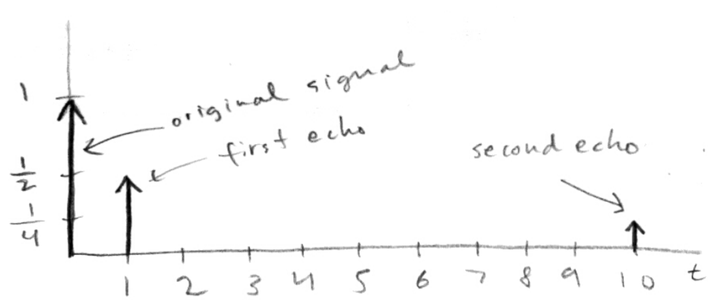

Figure 1: The semilog plot of both the collector and base current as a function of the base voltage.

3 Square Waves

3.1 Fourier representation of a square wave

To compute the coefficients C_k of the Fourier series representation of a square wave, the following equation is used:

$$C_k = \frac{1}{T}\int_0^T x(t)e^{\frac{-j2\pi}{T}kt}dt \text{ for } k = \mathbb{Z}$$

where T is the period, and $x(t)$ is the square wave signal. There is a special case that's easy to find when $k = 0$, so I'll do that first:

$$\begin{aligned}
C_0 &= \frac{1}{T}\int_0^T x(t)e^{\frac{-j2\pi}{T}0t}dt \\
C_0 &= \frac{1}{T}\int_0^T x(t)dt \\
C_0 &= \frac{1}{T}\frac{T}{2} \\
C_0 &= \frac{1}{2} \qquad (1)
\end{aligned}$$

Eq. 1 gives us the edge case of $k = 0$, which is also the average of $x(t)$.

Next we'll compute the coefficients for $k \neq 0$:

$$\begin{aligned}
C_k &= \frac{1}{T}\int_0^T x(t)e^{\frac{-j2\pi}{T}kt}dt \\
C_k &= \frac{1}{2}\int_0^T x(t)e^{\frac{-j2\pi}{T}kt}dt \\
C_k &= \frac{T}{j4\pi k}\left(1 - e^{-j2k\pi}\right) \\
C_k &= \frac{T}{j4\pi k}\left(1 - \cos(2\pi k) + j\sin(s\pi k)\right) \\
&\text{But } \sin(n\pi) = 0 \text{ for all integers } n \text{ is zero, so:} \\
C_k &= \frac{T}{j4\pi k}\left(1 - \cos(2\pi k)\right) \qquad (2)
\end{aligned}$$

Eq. 2 can be further analyzed by noticing how the cosine function responds to even and odd values of k. When k is even, we get:

$$C_k = \frac{T}{j4\pi k}(1 - 1) = 0$$

and when k is odd:

$$C_k = \frac{T}{j4\pi k}(1 + 1) = \frac{T}{j2\pi k}$$

.

I could now represent the signal $x(t)$ as a piecewise function, but it is really enough to know the piecewise function for C_k, because the rest is just trivial gymnastics. The piecewise function for C_k is:

$$C_k = \begin{cases} \frac{1}{2} & k = 0 \\ \frac{T}{j4\pi k}(1+1) = \frac{T}{j2\pi k} & k \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$$

3.2 Square Wave Fourier representation

Using the piecewise function for C_k obtained in the previous subsection in combination with the Fourier series summation, I wrote some code that plots the Fourier series representation of a square wave. My code can be found at https://github.com/bringsyrup/SigSys/blob/master/square.py

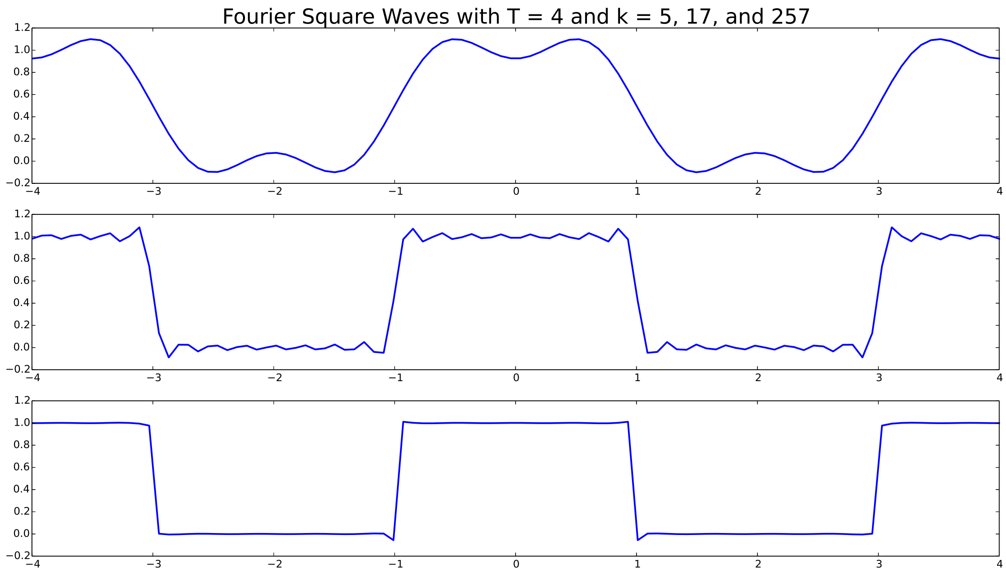

Figure 2: Square waves!

3.3 Fourier limitations at discontinuities

Even at $k = 257$, the Fourier series seems to be limited in it's ability to represent the instantaneous jump from the trough to the peak and vice versa. The Dirichelet Theorem states that the signal $x(t)$ will not converge if it contains infinite discontinuities, which is the case for a square wave. The theorem also requires a finite number of extreme over any given bounded interval, which is also obviously not the case (for example, over the range $0 < t < 2$ in Fig. 3, there are infinite points of extrema).

4 Triangle Waves

4.1 Shifted signals

The proof for this is pretty simple. Is a signal is periodic, it should have the same integram (area) over any interval of width T. In mathy representation:

$$C_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{\frac{-j2\pi}{T}kt}dt = \frac{1}{T}\int_{-T/2-W}^{T/2-W} x(t)e^{\frac{-j2\pi}{T}kt}dt$$

for any W (I actually figured this out earlier so I could make problem 3a easier).

Therefore it stands to reason that a shifted signal $x(t - T_o)$, which obviously has the same period T as the original signal $x(t)$, will have identical coefficients.

4.2 Fourier representation of a triangle wave

Building off of the coefficients for the triangle wave found in Siddhartan's notes:

$$C_k = \begin{cases} \frac{2}{\pi^2 k^2} & k \text{ is odd} \\ \frac{1}{2} & k = 0 \\ 0 & \text{otherwise} \end{cases}$$

Using the above function I wrote some code for a Fourier series representation of a triangle wave (code at https://github.com/bringsyrup/SigSys/blob/master/triangle.py). The results for $T = 1$ and $k = 50$ are below:

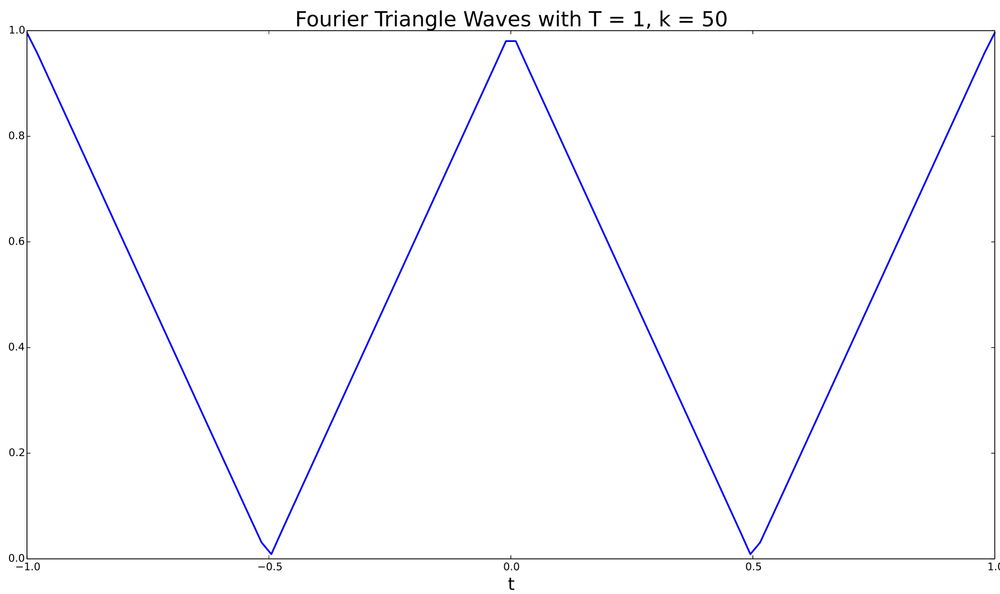

Figure 3: Triangle Wave with k = 100.