Fourier Transform II: Eigenfunctions and frequency responses of LTI systems.

I. Complex exponentials as eigen-functions to LTI systems

A. Frequency domain view of the eigenfunction property

We have seen that complex exponentials are eigenfunctions of LTI systems. In other words, consider an LTI system with input $x(t)$, output $y(t)$ and impulse response $h(t)$. If the input is $x(t) = Ae^{j\omega_0 t}$, then $y(t)$ equals $x(t)$ times a (possibly complex) scale factor. What should that scale-factor be? For your reference, we repeat the table of common Fourier transform pairs and properties here.

$x(t)$	$X(\omega)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$\delta(t-t_0)$	$e^{-j\omega t_0}$
$\cos(\omega_0 t)$	$\pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)$
$\sin(\omega_0 t)$	$\frac{\pi}{j}\delta(\omega-\omega_0)-\frac{\pi}{j}\delta(\omega+\omega_0)$
$\sum_{k=-\infty}^{\infty}\delta(t-kT)$	$\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi}{T}k\right)$

TABLE I

TABLE OF FOURIER TRANSFORM PAIRS

We know that the output $y(t) = x * h(t)$. Moreover, we know that $Y(\omega) = X(\omega)H(\omega)$ by the CT convolution theorem. Let's consider a representative $H(\omega)$ as illustrated in Figure 1. If $x(t) = Ae^{j\omega_0 t}$, then $X(\omega) = \frac{2A}{\pi}\delta(\omega-\omega_0)$. We can then find $Y(\omega)$ as shown in Figure 1. Since $X(\omega)$ is non-zero only at $\omega = \omega_0$, $Y(\omega) = X(\omega)H(\omega)$ cannot have non-zero values anywhere except at $\omega = \omega_0$. Moreover, as the figure illustrates $Y(\omega) = 2\pi AH(\omega_0)\delta(\omega-\omega_0)$. The inverse transform of $Y(\omega)$ is thus $y(t) = H(\omega_0)Ae^{j\omega_0 t}$. Thus, the complex exponential is an eigenfunction of LTI systems. Note that $H(\omega_0)$ is a constant here as it equals the function $H(\omega)$ evaluated at a particular value, $\omega = \omega_0$.

$y(t)$	$Y(\omega)$
$x * h(t)$	$X(\omega)H(\omega)$
$x(t)h(t)$	$\frac{1}{2\pi} X * H(\omega)$
$e^{j\omega_0 t}x(t)$	$X(\omega - \omega_0)$
$x(t - t_0)$	$e^{-j\,\omega\, t_0}\, X(\omega)$
$\frac{d}{dt}x(t)$	$j\omega\, X(\omega)$

TABLE II

TABLE OF FOURIER TRANSFORM PROPERTIES

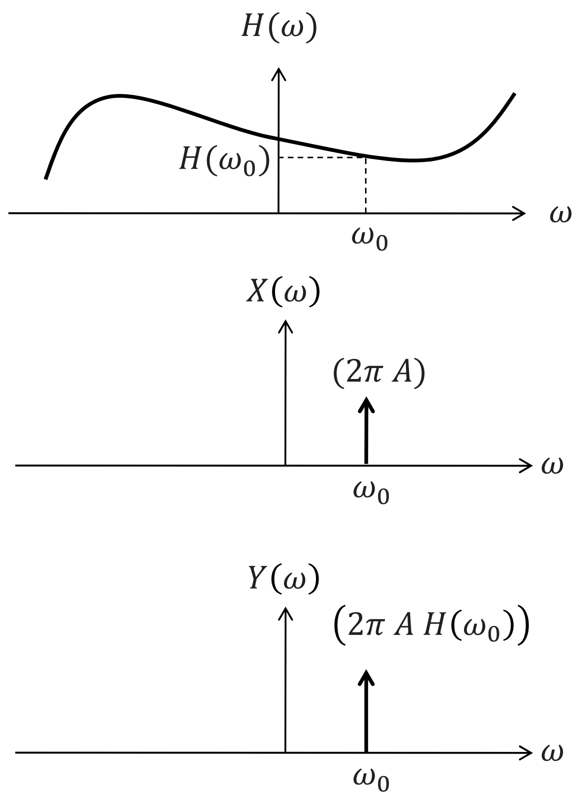

Fig. 1. Illustration of eigenfunctions in the frequency domain

B. LTI systems cannot produce new frequencies

Another similar property of LTI systems which is a consequence of the convolution theorem is the fact that LTI systems cannot produce new frequencies. Consider an LTI system with input

$x(t)$, output $y(t)$ and impulse response $h(t)$. The output $y(t)$ cannot contain any frequencies that are not present in the input signal, as the product $X(\omega)H(\omega)$ has to be zero at all values of ω where $X(\omega) = 0$, as illustrated in Figure 2.

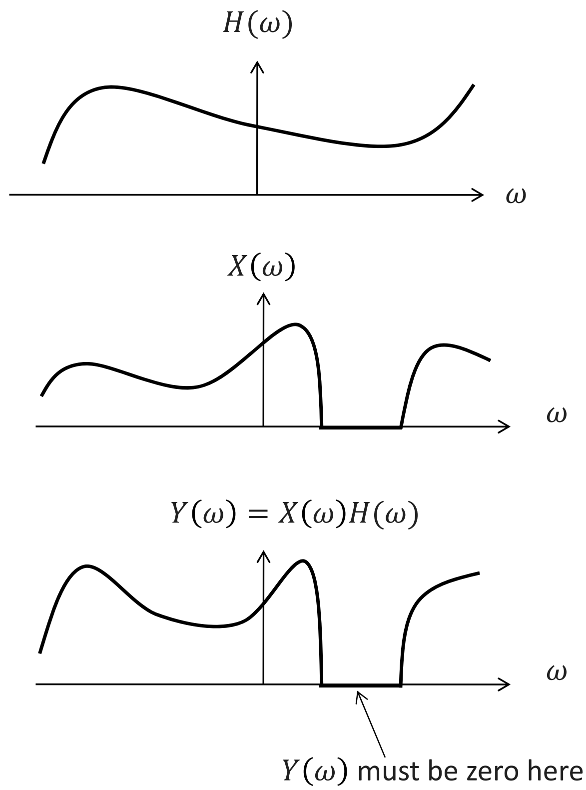

Fig. 2. LTI systems cannot produce new frequencies

Example:

The fact that LTI systems cannot produce new frequencies is used in many different applications. One such application is in testing for non-linearities in amplifiers. In an ideal world, amplifiers are completely linear. In other words, the output of an amplifier will simply be a scaled version of the input. If the input is $x(t)$, then the output should be $y(t) = Gx(t)$ where G is the gain of the amplifier. In reality however, amplifiers have more complicated characteristics. For instance an amplifier gain curve may look like $G(x)$ as illustrated in Figure 3.

If the input signal lives in the linear range of the amplifier, then the behavior of the amplifier

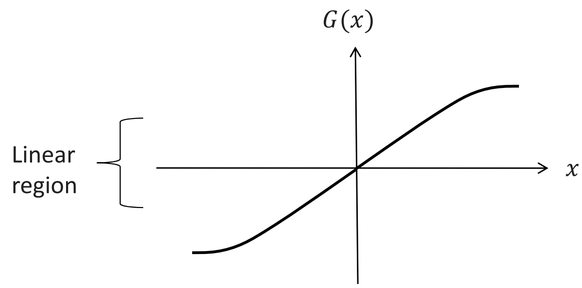

Fig. 3. Amplifier gain curve

is well modeled as $y(t) = Gx(t)$. However if the input $x(t)$ lives in in the non-linear range of the amplifier, the amplifier cannont be modeled as being linear and the system will exhibit non-LTI behavior. One way we can test if an amplifier is linear in the range of amplitudes we are interested in is to drive the amplifier with a signal $A\cos(\omega_c t)$. The input signal has only two frequency components, at ω_c and $-\omega_c$. If we the output of the amplifier has components at other frequencies, then we know that the amplifier is not well modeled as being linear when the amplitude is A.

II. Frequency response of LTI systems

As we have seen before, the unit impulse response of an LTI system completely characterizes the system. The Fourier transform of the impulse response is also a complete characterization of the system as the mapping between a function and its CTFT is one-to-one. The Fourier transform of the impulse response of a system is called the frequency-response of the system as it tells us how the system modifies an input signal at a particular frequency. As we saw in the previous section, if the input signal to an LTI system has a component at only one frequency, i.e. $x(t) = Ae^{j\omega_0 t}$, $H(\omega_0)$ tells us how much that frequency gets scaled by the system. We additionally note that the frequency response of a system can be obtained by taking the ratio of the output to input frequency responses since

$$Y(\omega) = X(\omega)H(\omega) \tag{1}$$

$$\frac{Y(\omega)}{X(\omega)} = H(\omega) \tag{2}$$

To illustrate with a tangible example, consider an RC circuit as shown below. We can view this circuit as an LTI system with input signal $v_{in}(t)$ and output signal $v_{out}(t)$.

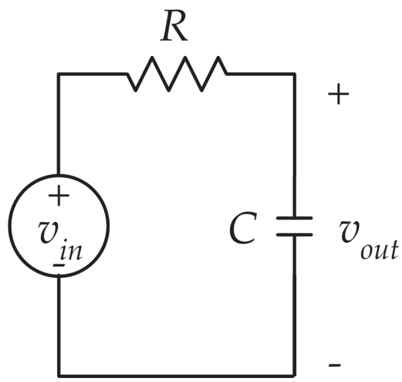

Fig. 4. RC Circuit.

Recall that the current through the capacitor is given by

$$i_C(t) = C\frac{d}{dt}v_{out}(t) \tag{3}$$

Additionally recall that the voltage going around the circuit should sum to zero (this is sometimes called Kirchoff's voltage law). In other words,

$$v_{in}(t) = v_R(t) + v_{out}(t) \tag{4}$$

Where $v_R(t)$ is the voltage across the resistor. Since $v_R(t) = Ri_c(t)$, we have

$$v_{in}(t) = Ri_c(t) + v_{out}(t) \tag{5}$$

$$= RC\frac{d}{dt}v_{out}(t) + v_{out}(t) \tag{6}$$

Taking the Fourier transform of each term in the above expression, and using the property in the last entry of Table II, we have

$$V_{in}(\omega) = j\omega RCV_{out}(\omega) + V_{out}(\omega)$$

$$H(\omega) = \frac{V_{out}(\omega)}{V_{in}(\omega)} = \frac{1}{j\omega RC + 1} \tag{7}$$

To understand the behavior of the system, we can look at the magnitude of the frequency response of the system. The magnitude of the ratio of complex numbers is the ratio of the magnitudes of the numerator and the numerator. Hence

$$|H(\omega)| = \frac{1}{|j\omega RC + 1|} = \frac{1}{\sqrt{\omega^2 (RC)^2 + 1}}\,. \tag{8}$$

As $\omega \to 0$, we have $|H(\omega)| \to 1$ and as $\omega \to \infty$, we have $|H(\omega)| \to 0$. These properties indicate that this system acts as a low-pass filter, which is consistent with the intuition that the capacitor will allow high frequency components to pass through it. However, explicitly analyzing the frequency response gives us a more detailed characterization of the system.

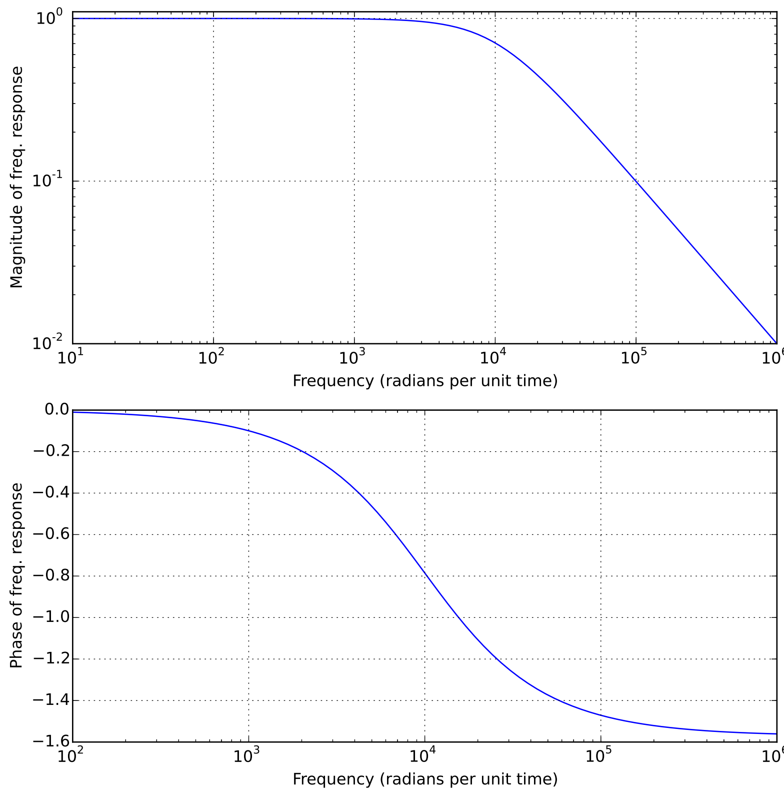

Fig. 5. RC low pass filter frequency response.

To illustrate, lets consider a system with $R = 100k\Omega$ and $C = 1nF$. The magnitude and phase of the frequency response of this system is plotted on a log-log scale for the magnitude and linear-log scale for phase in Figure 5. As expected, this system is a low-pass filter. When

$\omega = \frac{1}{RC} = 10^4$, the magnitude of the frequency response is

$$|H(\omega)| = \frac{1}{\sqrt{2}}\,. \tag{9}$$

This frequency is sometimes called the cut-off frequency for the low-pass filter. It is the frequency above which the system scales signals by a factor $\frac{1}{\sqrt{2}}$ or smaller. The phase of the filter can be found most easily by expressing the frequency response in polar form

$$H(\omega) = \frac{1}{\sqrt{(\omega RC)^2 + 1} e^{j \tan^{-1}\left(\frac{\omega RC}{1}\right)}} = \frac{1}{\sqrt{(\omega RC)^2 + 1}} e^{-j \tan^{-1}(\omega RC)}\,. \tag{10}$$

At $\omega = \frac{1}{RC}$, we have the angle equaling $-\frac{\pi}{4} \approx -0.8$, as we can see in the figure.