Discrete to Continuous Time

I. Continuous Time Signals

So far, we have almost exclusively dealt with signals that are discretized in time. In other words, these signals are defined only at discrete values of the time variable. On a computer, signals are represented simply as a sequence of numbers stored in memory. When these signals are played through a speaker for example, the discrete-time samples are converted into a continuous voltage which is then used to drive a speaker. On the other hand, when we record a sound wave through a microphone on the computer, a continuous voltage that comes out of the microphone is sampled every T_s time units to produce a discretized version of the continuous waveform. T_s here is known as the sampling period, and $f_s = \frac{1}{T_s}$ is known as the sampling frequency, or frame-rate.

A commonly used notation to distinguish between discrete-time signals and continuous time signals is to represent discrete time signals using square brackets and continuous functions using parenthesis. Thus a discrete-time signal would be represented as $x[n]$ and a continuous-time signal would be represented as $x(t)$. Henceforth, we shall allow for both negative and positive time values here. Time-zero here simply refers to an arbitrary point in time so there is no issue with allowing for negative times. Moreover, we shall allow signals to be infinitely long here (at least in theory). If $x[n]$ is a discrete-time signal which is the result of sampling $x(t)$ every T_s time units, we can write

$$x[n] = x(nT_s) \qquad \text{for } n = \cdots, -2, -1, 0, 1, 2, \cdots \tag{1}$$

Such a conversion could take place in an analog to digital converter such as the ones that some of you may have used in PoE to read the continuous output of a voltage divider. The sampling operation is illustrated in Figure 1.

One question that one may have is how small does the sampling period T_s or equivalently how large does the sampling or frame rate $f_s = \frac{1}{T_s}$ need to be? The sampling rate directly impacts hardware costs in a number of ways. Generally, higher sampling rates require more expensive

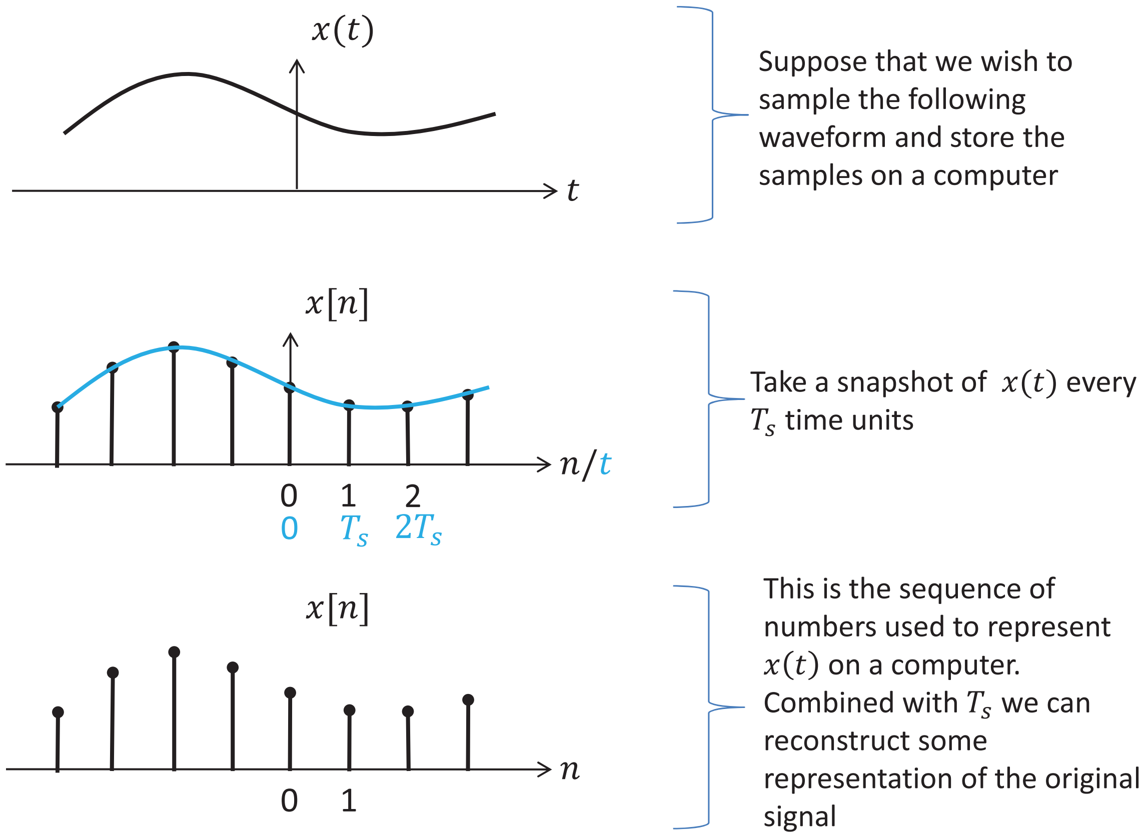

Fig. 1. Sampling a continuous-time signal

analog to digital or digital to analog converters. Moreover, higher sampling frequencies will require us to store many more samples per unit time resulting in larger memory requirements, and faster computation or processing times as any computations involving the sampled signal will have to be done over a larger number of samples if the sampling rate is higher.

Based on our discussions earlier in the course, we already have the idea that the sampling rate should be greater than twice the maximum frequency that is present in the signal to ensure that the original signal can be reconstructed from the samples unambiguously.

The fact that one can unambiguously, and in theory perfectly, reconstruct a signal that is sampled at a rate higher than twice the maximum frequency contained in that signal is almost magical. In the next couple of weeks, we shall learn about why this is the case by examining the frequency content of continuous time signals and how the sampling operation effects the information contained in the signal. This will be followed by an analysis of how to construct a continuous waveform from a discrete signal.

Three potential approaches to constructing a continuous wave from discrete samples are illustrated in Figure 2, along with their resulting frequency content.

The topmost plot on the left illustrates the discrete-time signal, and the three plots below that illustrate different approaches to converting the discrete-time signal into continuous waveforms which are plotted in blue. In the first approach, each sampled value is held until the next sampling time when it is updated (this approach is sometimes called a zero-order hold). In the second approach, the samples are linearly interpolated by connecting a straight line between consecutive samples.

In the third approach, a sinc function, defined as

$$\mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t} \tag{2}$$

is placed at the location of each sample and the height of the sinc function is scaled to the value of the sample. The zeros of the sinc function are aligned with all other sampling times. The interpolated waveform is the sum of all the different sinc functions. In the figure, the sinc functions that are used to construct the continuous waveform are shown in grey.

Note that given the discrete-time samples, any of the three continuous waveforms are valid continuous-time representations of the discrete sequence. This is because if you sample the continuous waveform in Figure 2, you will produce the same discrete sequence.

The third approach however, leads to the smoothest waveform, and has the most concentrated frequency content as illustrated in the second column of plots. To provide a better comparison, Figure 3 shows the spectra of the three different interpolation approaches. It is evident from this figure that the sinc-based interpolation has the narrowest spectrum.

In the next couple of weeks, we shall see why this is the case, and we shall see why the sinc function provides the most frequency-efficient interpolation of the discrete-time sequence.

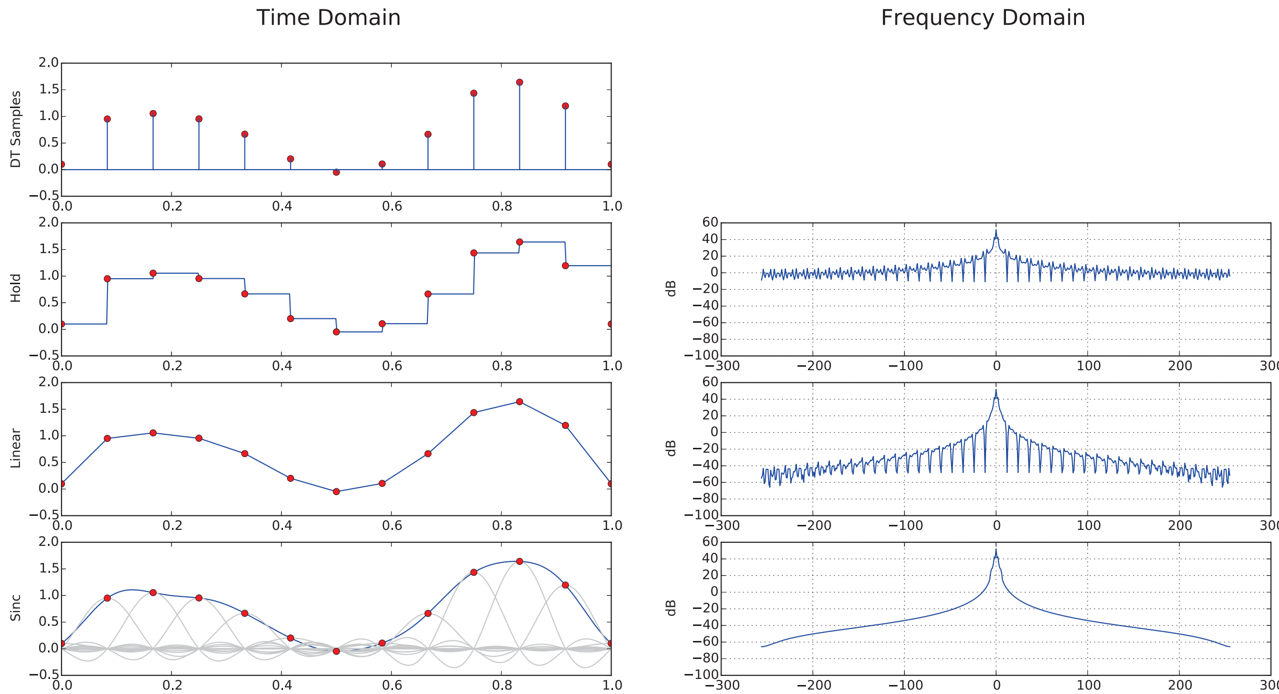

Fig. 2. Constructing a continuous time signal from discrete-time samples.

We shall revisit the sampling operation in greater detail soon after we have developed some tools to characterize continuous time signals. These tools include the continuous-time Fourier transform (CTFT) and continuous-time analogs (pun intended) to many of the properties that we have found useful for analyzing discrete-time systems such as the convolution theorem, and impulse and frequency responses. Moving forward, we shall apply these techniques to model and analyze communcations systems.

In order to develop the CTFT, we shall first develop the Fourier series which is a technique for representing continuous-time periodic signals as a sum of complex exponentials. But before we jump into the Fourier series, we shall develop some general tools to analyze continuous-time signals and systems, starting with the continuous-time impulse and the impulse response of continuous-time LTI systems.

Exercises:

1. Write an expression for a sinc function that equals 1 at $t = 0$, and equals zero for every other integer multiple of T_s. Verify your answer by plotting a graph of this function.
2. Write an equation for the continuous representation of a discrete-time sequence $x[n]$ with

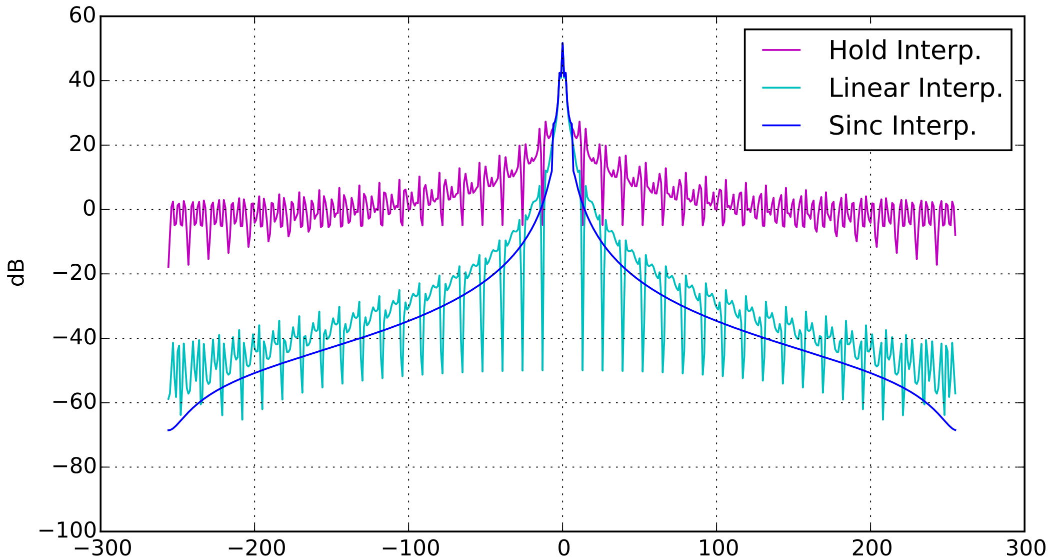

Fig. 3. Spectra associated with different continuous reconstructions of the same DT signal

sample period T_s, and interpolation using the sinc function.

II. The continuous-time unit impulse and impulse responses

A. The CT impulse

In discrete-time, the impulse response of an LTI system provides a complete characterization of that system. In DT, the unit impulse was defined as follows

$$\delta[n] = \begin{cases} 1 \text{ if } n = 0, \\ 0 \text{ otherwise.} \end{cases} \tag{3}$$

In continuous time, the unit impulse cannot be defined so simply. If we defined the impulse as simply having a value of one at $t = 0$, and zero for all other times, then its integral would be zero, which means that there is no energy in the signal. Therefore, this input would not produce any response in a system. Instead, we have to be careful to define the impulse in such a way that it has a nonzero integral, and therefore, energy.

We define the unit impulse as the limit of a rectangle with unit area, as the width of the rectangle goes to zero. In other words, start with a function $b(t)$ as shown in Figure 4. We then

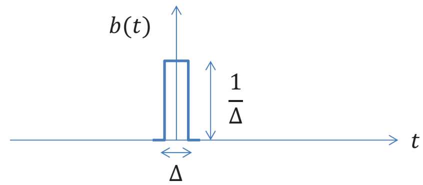

Fig. 4. Unit area box used to define a CT unit impulse.

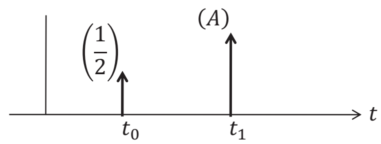

Fig. 5. Impulses at t_0 and t_1. The figure illustrates $\frac{1}{2}\delta(t - t_0) + A\delta(t - t_1)$.

define the CT unit impulse as

$$\delta(t) = \lim_{\Delta \to 0} b(t) \tag{4}$$

Hence, the unit impulse is a function that exists over an infinitesimal time duration and has infinite height, but unit area. The unit impulse is used both for mathematical convenience and as model for a short-duration signal such as a gunshot.

The impulse function can be used to analyze the sampling operation as we can view the sampling of a continuous waveform as a multiplication of the waveform by a periodic train of impulses. We shall see more of this soon.

Graphically, we typically represent impulses using arrows, with the area of the impulse given in parenthesis, as shown in Figure II-A. Note that a *unit* impulse is an impulse with area one.

One property of the unit impulse that will come in handy is the sifting, or picking property of the unit impulse given as follows

$$\int_{-\infty}^{\infty} x(t)\delta(t - t_0)dt = x(t_0) \tag{5}$$

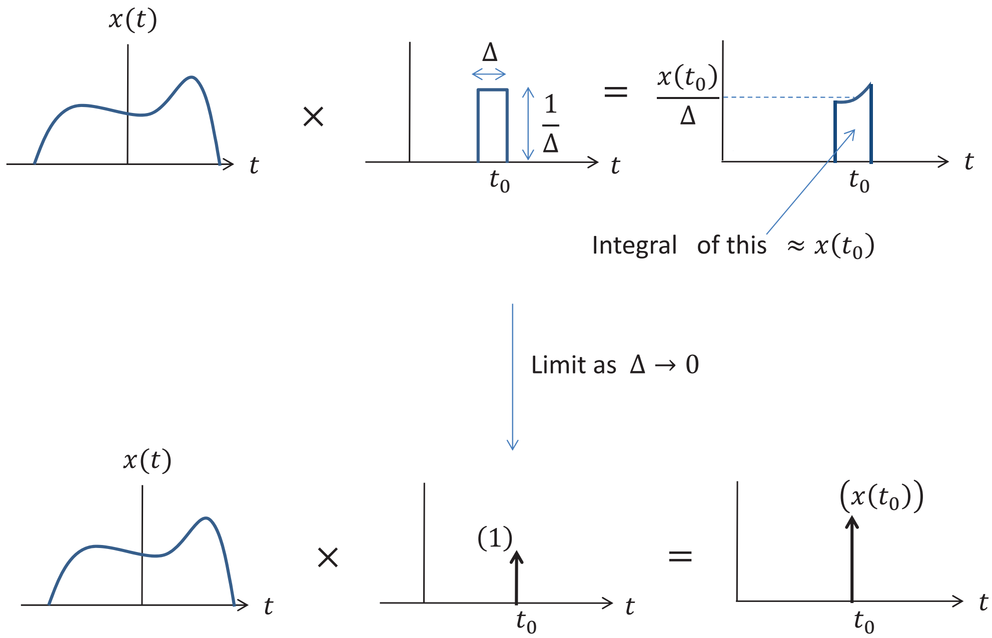

Fig. 6. Sifting property of the unit impulse.

This property is illustrated in Figure II-A. In the top sub-figure we see a generic function $x(t)$ multiplied by our rectangle $b(t)$. The area of the product of these two functions is approximately equal to $x(t_0)$ as indicated in the figure. If we take the limit as $\Delta \to 0$, this approximation becomes exact as shown in the bottom sub-figure.

B. *Impulse response and convolution*

If we stick a unit impulse into an LTI system, the output of the system is referred to as the impulse response of the system. Suppose that the impulse response of a CT system was $h(t)$. Then the output of that system in response to the input signal $x(t)$ is

$$y(t) = x * h(t) = h * x(t) \,, \tag{6}$$

where the CT convolution is defined as follows

$$p * q(t) = \int_{-\infty}^{\infty} p(\tau) q(t - \tau) d\tau = q * p(t) \tag{7}$$

This relationship is proved in the Appendix if you are interested, but the derivation is optional.

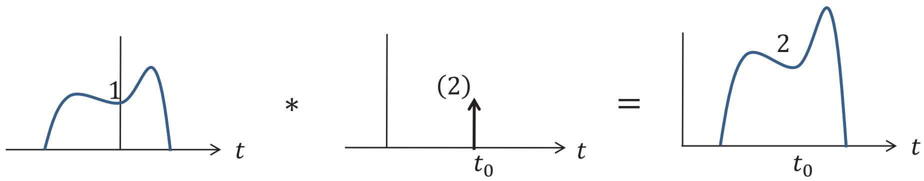

Fig. 7. Convolving a function with an impulse.

It is instructive to compare this expression with the discrete-time convolution of length N which was defined as

$$f * g[n] = \sum_{k=0}^{N-1} f[k]g[n-k]\,. \tag{8}$$

The differences are that the time index now goes from $-\infty$ to ∞, and the summation over the time index k becomes an integral with respect to the time variable t.

C. *Properties of the CT Impulse*

Convolving any function with an impulse moves that function to be centered around that impulse, and scales that function by the area under that impulse. In other words

$$x * (A\delta(t-t_0)) = Ax(t-t_0)\,. \tag{9}$$

This property is illustrated in Figure 7.

Moreover the convolution is a linear operation since it is essentially an integral. The linearity of the convolution implies that

$$x * (p+q)(t) = x * p(t) + x * q(t)\,.$$

Exercises:

1. Earlier in the course, you saw how the recorded audio signal of a gun being fired in a shooting range can be convolved (not convoluted!) with a violin recording to approximate how the violin would sound if played in a shooting range. Please explain this using what you know about the impulse and impulse responses.

2. Consider a simple model of an echo channel. Suppose that the output of the echo channel is $y(t)$ and the input is $x(t)$, and the input and output are related as follows:

$$y(t) = \frac{1}{2}x(t) + \frac{1}{4}x(t-10)\,.$$

Explain why it is reasonable to call this an echo channel and find an expression for, and sketch the impulse response of this system.

APPENDIX

We can see why the output of an LTI system is the convolution of the input and the impulse response as follows. First let's start with the narrow box that is shown in Figure 4. Suppose that if the input to our LTI system is $b(t)$, then the output is $h_b(t)$.

- If the input to our LTI system is $Ab(t)$, then the output of the LTI system is $Ah_b(t)$.
- If the input to our system is $Ab(t-t_a)$, then the output to the system will be $Ah_b(t-t_a)$.
- If the input to our system is $Ab(t-t_a)+Cb(t-t_c)$, then the output to our system will be $Ah_b(t-t_a)+Ch_b(t-t_b)$.
- If the input to our system is $\Delta x(k\Delta)b(t-k\Delta)$, then the output will be $\Delta x(k\Delta)h_b(t-k\Delta)$.
- If the input to our system is the following:

$$\sum_{k=-\infty}^{\infty} \Delta x(k\Delta)b(t-k\Delta) \tag{10}$$

  then the output will be

$$\sum_{k=-\infty}^{\infty} \Delta x(k\Delta)h_b(t-k\Delta). \tag{11}$$

- The signal described in (10) can be visualized by the example given in Figure 8. The dashed lines represent a generic function $x(t)$. The signal described in (10) is the sum of the areas of the boxes in Figure 8.
- Next, we take the limit as $\Delta \to 0$. Recalling how integration is defined, we have the following

$$\lim_{\Delta\to 0}\sum_{k=-\infty}^{\infty} \Delta x(k\Delta)b(t-k\Delta) = \int_{-\infty}^{\infty} x(\tau)\lim_{\Delta\to 0} b(t-\tau)d\tau \tag{12}$$

$$= \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \tag{13}$$

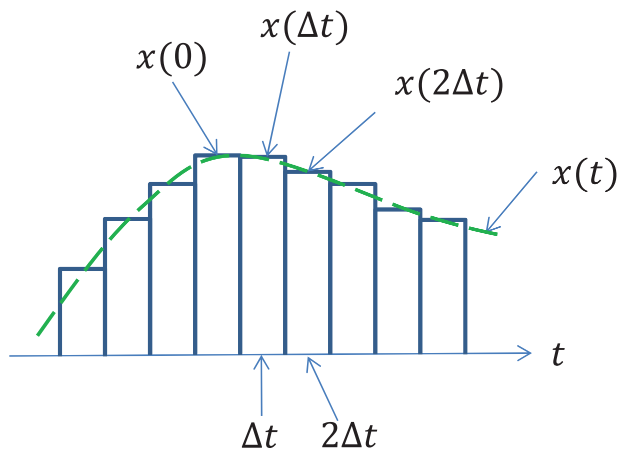

Fig. 8. Decomposing continous-time signal into sum of scaled, shifted rectangles.

By invoking the sifting property of the unit impulse, we have

$$\lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} \Delta x(k\Delta) b(t - k\Delta) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau = x(t) \tag{14}$$

- By a similar argument, the output of the system is given by

$$\lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} \Delta x(k\Delta) h_b(t - k\Delta) = \int_{-\infty}^{\infty} x(\tau) \lim_{\Delta \to 0} h_b(t-\tau)d\tau \tag{15}$$

$$= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \tag{16}$$

- Hence, if the input to the LTI system is $x(t)$, then the output is $x * h(t)$.