Fourier Transform III: Applications to communications systems

I. Introduction

One application for the frequency domain techniques we have developed so far is in analyzing and designing communications systems. We shall find the tables of Fourier transforms and their properties useful and so we have repeated them here.

$x(t)$	$X(\omega)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$\delta(t-t_0)$	$e^{-j\omega t_0}$
$\cos(\omega_0 t)$	$\pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)$
$\sin(\omega_0 t)$	$\frac{\pi}{j}\delta(\omega-\omega_0)-\frac{\pi}{j}\delta(\omega+\omega_0)$
$\sum_{k=-\infty}^{\infty}\delta(t-kT)$	$\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi}{T}k\right)$

TABLE I

TABLE OF FOURIER TRANSFORM PAIRS

$y(t)$	$Y(\omega)$
$x*h(t)$	$X(\omega)H(\omega)$
$x(t)h(t)$	$\frac{1}{2\pi}X*H(\omega)$
$e^{j\omega_0 t}x(t)$	$X(\omega-\omega_0)$
$x(t-t_0)$	$e^{-j\,\omega\,t_0}\,X(\omega)$
$\frac{d}{dt}x(t)$	$j\omega\,X(\omega)$

TABLE II

TABLE OF FOURIER TRANSFORM PROPERTIES

II. Amplitude modulation

A. Basic amplitude modulation

The Fourier transform and frequency-domain analysis play a major role in the design and analysis of communications systems, in particular, wireless communications systems. In radio frequency (RF) systems, antennas can only radiate efficiently when the physical dimensions of the antenna are comparable to the wave length of the signal that you wish to transmit through the antenna. The frequency of an electromagnetic wave f and its wave-length λ are related by $\lambda = c/f$, where c is the speed of light. This means that for antennas of a reasonable size, signals need to be at very high frequencies. For instance at a frequency of 2.4 GHz (which is the frequency used by many wifi systems), the wavelength is approximately 12.5cm.

On the other hand, many signals that we wish to communicate wirelessly such as audio signals have bandwidths that are much lower. For instance, humans typically cannot hear signals above 20kHz, so audio signals are band limited to about 20kHz. Therefore, in order to communicate signals such as audio signals, efficiently through reasonably sized antennas, we need a way to translate signals from lower to higher frequencies and vice-versa.

A simplified, but surprisingly accurate, model of many wireless communications channels is shown in Figure 3. The channel (which includes the antennas at the transmitter and recevier, as well as the physical propagation channel) is modeled as a LTI system with additive noise, where the transmitted signal is $x(t)$, the channel frequency response is $H(w)$, $n(t)$ is the additive noise and $y(t)$ is the received signal. We can write the received signal $y(t)$ and its Fourier transform $Y(\omega)$ as follows

$$y(t) = x * h(t) + n(t) \tag{1}$$

$$Y(\omega) = X(\omega)H(\omega) + N(\omega) \tag{2}$$

Since the channel includes antennas at the transmitter and receiver which are designed to operate over a range of frequencies, the channel has a band-pass characteristic. Note that in this model, we have assumed that the pass-band of the channel relatively flat, which is a decent model in certain communications systems but not all. In this course, we shall not explicitly analyze the effects of the additive noise.

Suppose that we wish to communicate a signal $m(t)$ through the communications channel as shown in Figure 3. In order to propagate it through the channel, we need to translate $m(t)$,

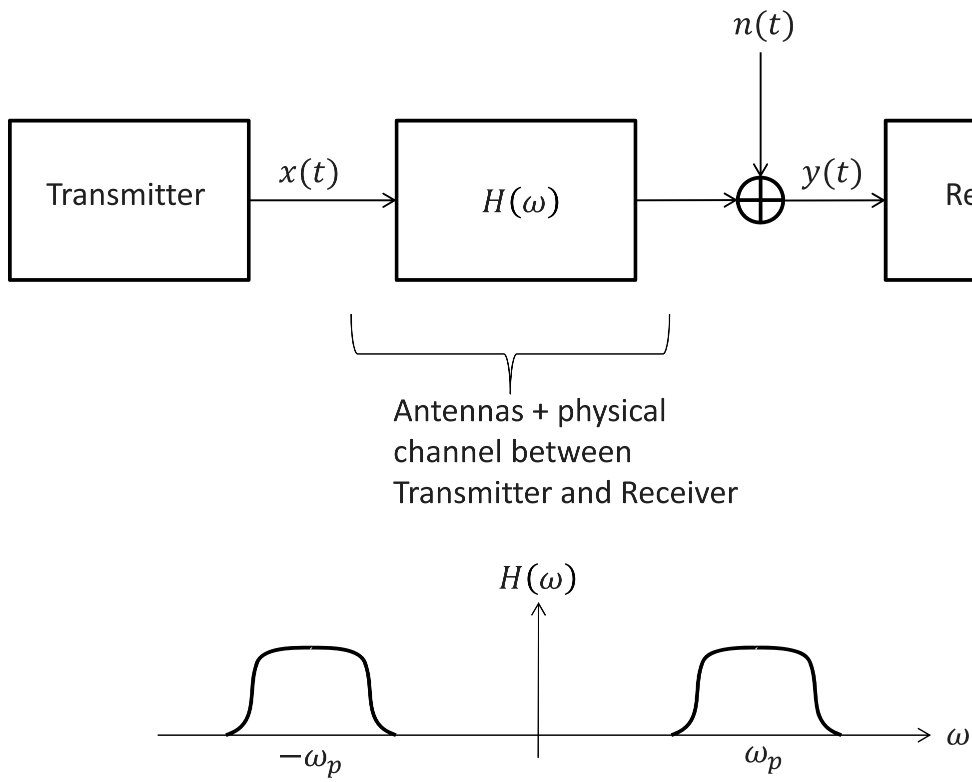

Fig. 1. Simplified radio channel

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which is known as a *base-band* signal, to higher frequencies to enable propagation through the channel. One way to do this is to multiply $m(t)$ with a cosine at a high frequency such that the modulated signal $x(t) = m(t)\cos(\omega_c t)$ is in the pass-band of the channel. This is called amplitude modulation, because the amplitude of the cosine is changed by the message signal that we wish to communicate.

The product of a base-band signal with a cosine at a high frequency (called a carrier), translates signals in the frequency domain because multiplication in time becomes convolution in frequency

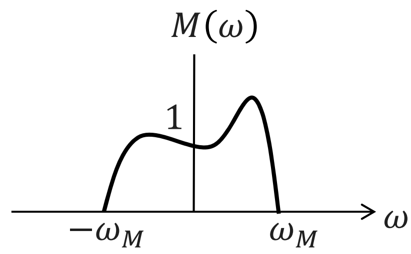

Fig. 2. Spectrum of $M(\omega)$

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(along with a division by 2π), as illustrated below.

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\begin{aligned}
x(t) &= m * \cos(\omega_c t) \\
X(\omega) &= \frac{1}{2\pi} M(\omega) * [\pi\delta(\omega - \omega_c) + \pi\delta(\omega + \omega_c)] \\
&= \frac{1}{2} M(\omega - \omega_c) + \frac{1}{2} M(\omega + \omega_c) \qquad (3)
\end{aligned}
$$

Neglecting the effects of the noise $n(t)$, if the portion of the spectrum of $X(\omega)$ that contains the signal is in the pass band of the channel, and the frequency response of the channel is approximately flat in the pass band, the received signal is approximately equal to a scaled version of the transmitted signal.

Suppose that the channel is flat in the pass band, ω_c is chosen appropriately and the noise is negligible, then the received signal is

$$y(t) = Ax(t) = Am(t)\cos(\omega_c t) \qquad (4)$$

How can we recover $m(t)$ (or a scaled version of it) from $y(t)$? We already saw that multiplying signals by cosines can move things around in frequency. Using the same idea, consider the

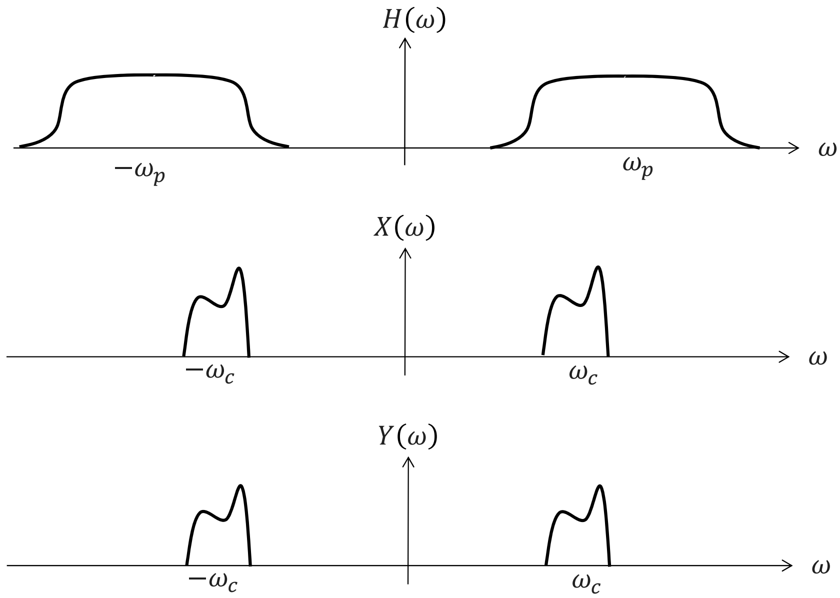

Fig. 3. Transmitted and received signals in a radio channel

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following

$$y_c(t) = y(t)\cos(\omega_c t) = Am(t)\cos^2(\omega_c t). \tag{5}$$

$$Y_c(\omega) = \frac{A}{2}\left[M(\omega-\omega_c)+M(\omega+\omega_c)\right] * \left[\pi\delta(\omega-\omega_c)+\pi\delta(\omega+\omega_c)\right] \tag{6}$$

$$= \frac{1}{2\pi}\times\frac{A}{2}\left[M(\omega-\omega_c)+M(\omega+\omega_c)\right] * \left[\pi\delta(\omega-\omega_c)+\pi\delta(\omega+\omega_c)\right] \tag{7}$$

$$= \frac{A}{4}\left[M(\omega-2\omega_c)+2M(\omega)+M(\omega+2\omega_c)\right] \tag{8}$$

Figure 4 illustrates this operation. Note that the convolution of $Y(\omega)$ with the impulse at ω_c produces two scaled copies of $M(\omega)$. One centered at $2\omega_c$ and the other at zero, since convolution with the impulse at ω_c moves $Y(\omega)$ to be centered around ω_c. Similarly, convolution of $Y(\omega)$ with the impulse at $-\omega_c$ produces a scaled copy of $M(\omega)$ at $2\omega_c$ and at zero. Since two copies of $M(\omega)$ are added together at the origin, the height of the spectrum at the origin is twice that of the two copies at $\pm 2\omega_c$. As is apparent from the figure, we can recover $M(\omega)$ from $Y_c(\omega)$ by applying a low-pass filter.

Thus, we can design an Amplitude Modulation (AM) radio system as shown in Figure 5,

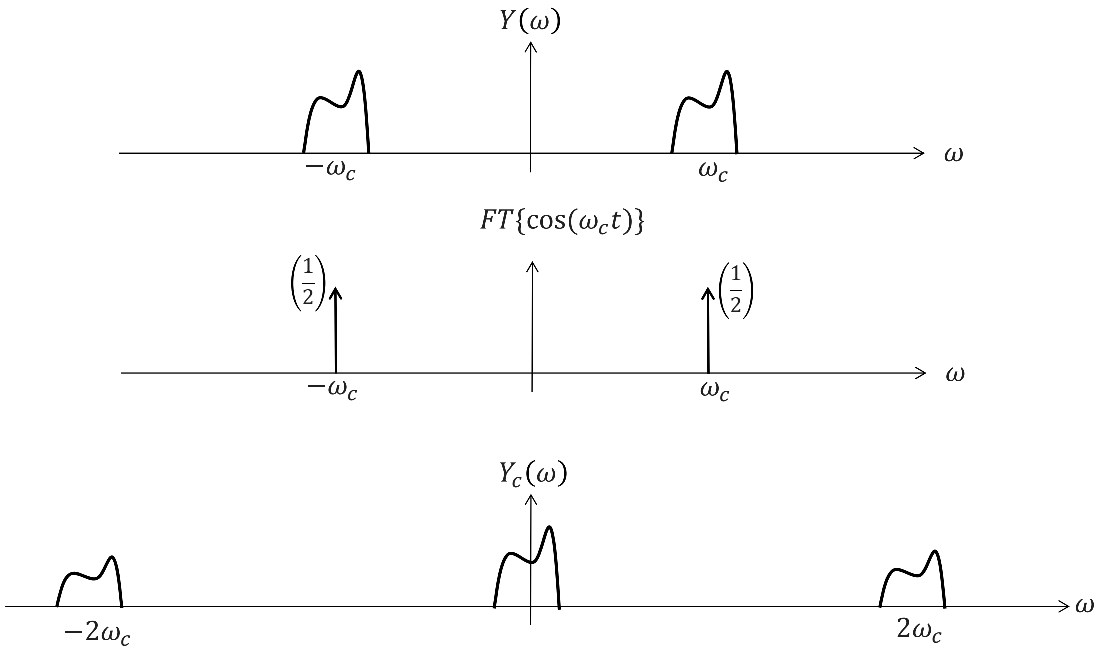

Fig. 4. Demodulation

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where $\tilde{m}(t)$ is a scaled version of $m(t)$, provided that the channel is flat in the pass band, ω_c is chosen appropriately and the noise is negligible. Note that we can assume $\phi = 0$ for now. We shall see the effects of $\phi \neq 0$ later. The process of amplitude modulation and demodulation can also be analyzed in the time domain. Note that

$$y_c(t) = Am(t)\cos^2(\omega_c t)$$

If you apply the trignometric identity $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$, we have

$$\begin{aligned} y_c(t) &= Am(t)\left[\frac{1}{2} + \frac{1}{2}\cos(2\omega_c t)\right] \\ &= \frac{A}{2}m(t) + \frac{A}{2}m(t)\cos(2\omega_c t) \end{aligned}$$

The term $\frac{A}{2}m(t)\cos(2\omega_c t)$ is a high frequency term as it is the product of a relatively low frequency signal $m(t)$ and a high frequency cosine. This term will thus be removed by a low-pass filter, leaving a scaled version of $m(t)$.

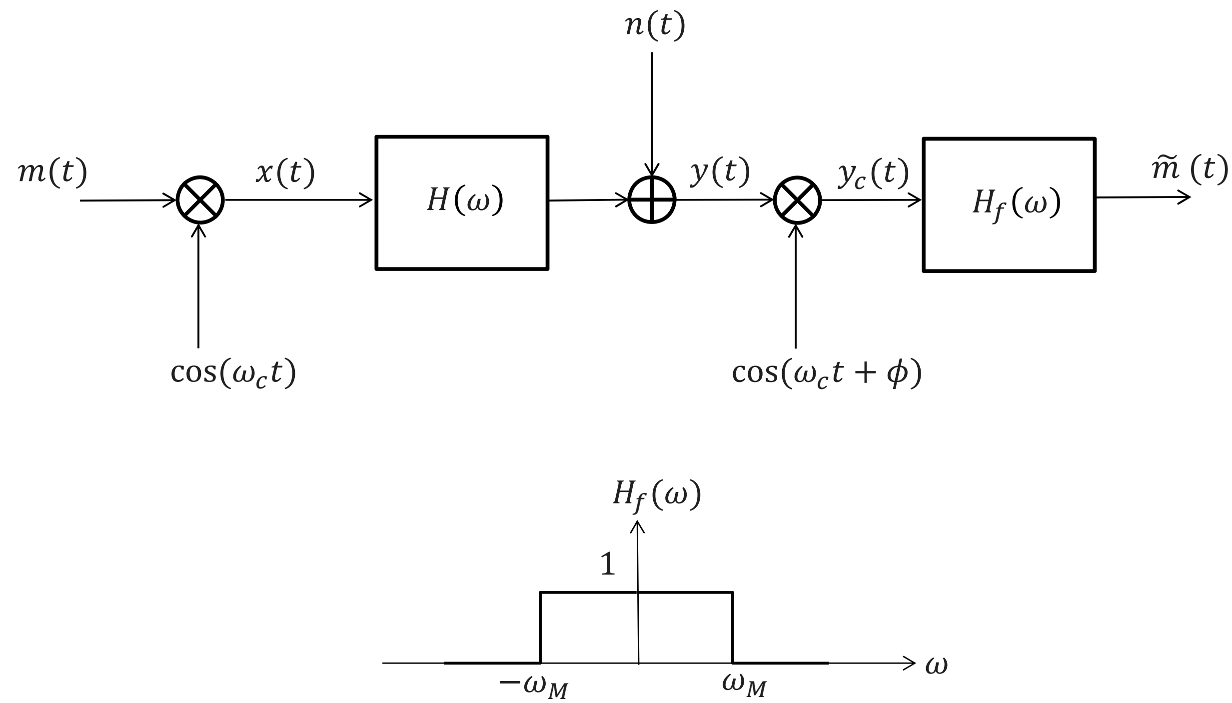

Fig. 5. Amplitude-Modulation System

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B. Phase-offsets

Since the transmitter and receiver are on separate devices, the cosine used for demodulating the signal will in general have a phase offset compared to the cosine used for modulating the signal. This phase offset, which we shall denote by ϕ here, reduces the amplitude of the demodulated signal as the following analysis shows. Consider Figure 5, with $\phi \neq 0$ in and $n(t) = 0$. We then have

$$y_c(t) = Am(t)\cos(\omega_c t)\cos(\omega_c t + \phi) \tag{9}$$

Applying the trigonometric identity $\cos(\alpha)\cos(\beta) = \frac{1}{2}\cos(\alpha - \beta) + \cos(\alpha + \beta)$, we have

$$y_c(t) = Am(t)\frac{1}{2}\cos(\phi) + Am(t)\frac{1}{2}\cos(\phi)\cos(2\omega_c t + \phi) \tag{10}$$

The last term in the expression above is a high frequency term, centered at $\pm 2\omega_c$ which will be removed by the low-pass filter leaving us with

$$\tilde{m}(t) = \cos(\phi)\frac{A}{2}m(t) \tag{11}$$

Thus, the amplitude of the signal is reduced by a factor equaling the cosine of the phase offset between the transmitter and receiver, which is problematic if $\phi \approx \frac{\pi}{2}$, for instance. There are several algorithms out there to correct for this timing offset and they called are timing synchronization algorithms. If we implement this system numerically on a computer with lots of computation power, we could use a brute-force approach. We can perform multiple versions of the demodulation each with a different phase offset and pick the one which results in the strongest signal coming out of the low-pass filter for instance.

III. Pulse-Amplitude Modulation (PAM) for digital communications

A. Binary-PAM

In the previous section, we saw how we can communicate an arbitrary waveform through a band-pass channel. The signal $m(t)$ can e.g. represent an analog audio signal, such as that used in AM radio. We can also encode digital information in the waveform $m(t)$.

A sequence of 1s and 0s can be encoded using pulses of height V and -V respectively. This is called Pulse-Amplitude Modulation (PAM). Consider a rectangular pulse of width W_p and height 1 as shown in Figure 6.

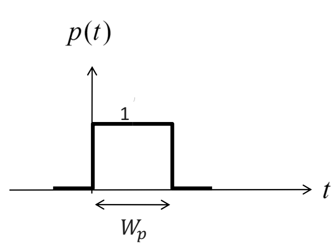

Fig. 6. Single pulse

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Suppose that we wish to transmit a sequence of bits b_k, $k = \cdots, 0, 1, 2, \cdots$. We can then construct a waveform $m(t)$ as follows

$$m(t) = \sum_{k=-\infty}^{\infty} (2b_k - 1) V p(t - kT_s)\,. \tag{12}$$

Based on this definition, $m(t)$ is a sequence of pulses $p(t)$ separated by T_s, whose heights are either V or $-V$. Note that each pulse here is called a symbol, which is the reason for the notation for the symbol period T_s. Figure 7 illustrates $m(t)$ which results from the sequence $b_0 = 1, b_1 = 0, b_2 = 0$.

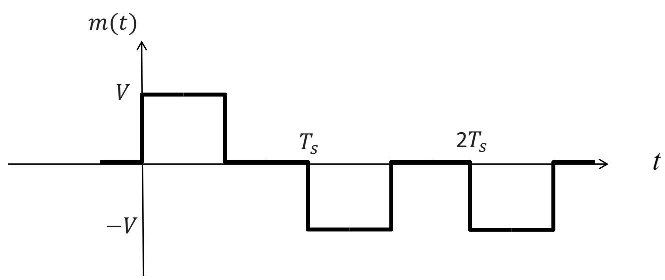

Fig. 7. Example PAM signal

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Suppose that this signal is modulated, and demodulated through the communications channel illustrated in Figure 5, with additive noise. The received signal may look like that illustrated in Figure 8. Note that there could be a longer delay, and the possiblity that the signal gets flipped as well.

A simple way to decode this signal is to sample it every T_s time units and take the sign of the resulting sample. Note that more sophisitcated techniques which involve averaging nearby samples can also be used here.

B. Binary-Phase-Shift Keyeing (BPSK)

In the PAM example given above, the pulses are separated by some amount of time. The reason for separating the pulses in time is that real channels typically spread the pulses out in time. The gaps between the pulses are added to ensure that earlier pulses die out by the

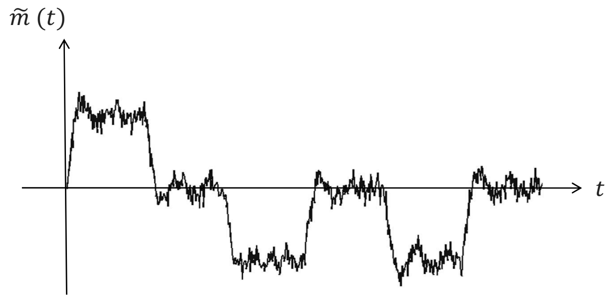

Fig. 8. Example demodulated PAM signal

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time a new pulse comes along. The phenomenon of pulses interfering with each other is called inter-symbol interference (ISI) and is treated in more detail in digital communications courses.

When the symbol period T_p equals the pulse width W_p, the pulses line-up next to each other. Systems that have this property are called binary-phase-shift-keyeing (BPSK) systems. The reason for this terminology is that the bits are encoded in the phase of the carrier signal as illustrated in Figure 9.

BPSK systems are attractive in several ways including simpler implementation since one could implement this with an oscillator circuit if the phase of the oscillator is controllable. Moreover, the receiver could look for phase changes whic h are high frequency events as is evident from Figure 9.

Exercises:

1) **Frequency-Division-Multiple Access (FDMA)** Consider a signal $y(t) = x_1(t)\cos(\omega_1 t) + x_2(t)\cos(\omega_2 t)$. Suppose that $X_1(\omega) = 0$ and $X_2(\omega) = 0$ if $|\omega| > \omega_M$. Assume that $\omega_1 \gg \omega_M$ and $\omega_2 \gg \omega_M$, and $\omega_1 + 2\omega_M < \omega_2$. Suppose that $X_1(\omega)$ and $X_2(\omega)$ are given by the following:
   a. Please sketch $Y(\omega)$.
   b. Please sketch the Fourier transforms of $y(t)\cos(\omega_1 t)$ and $y(t)\cos(\omega_2 t)$.
   c. Using appropriate sketches, describe how you can recover $x_1(t)$ and $x_2(t)$ from $y(t)$. We can view $y(t)$ as the received AM radio signal from two different AM transmitters,

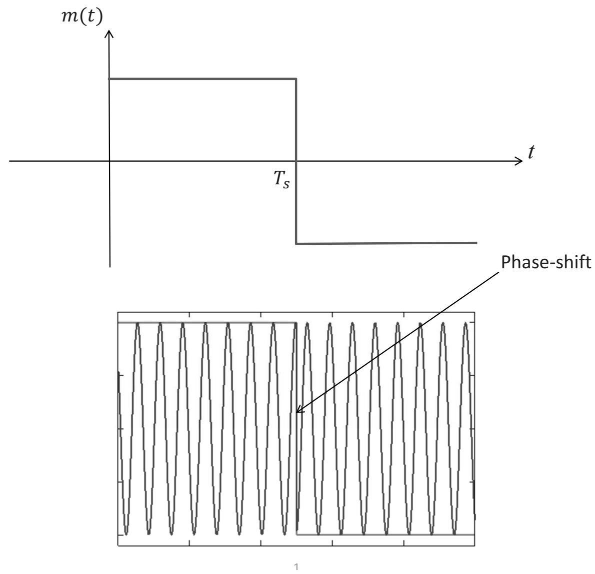

Fig. 9. BPSK signal

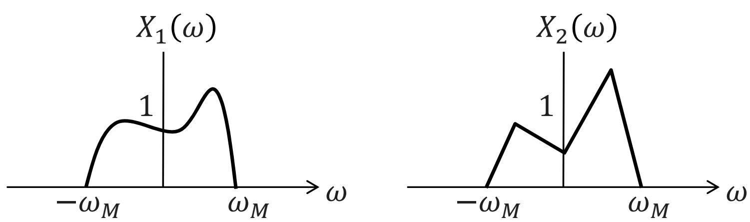

Fig. 10. $X_1(\omega)$ and $X_2(\omega)$.

one using the frequency ω_1 and ω_2.

2) Consider the signals $x_1(t)$ and $x_2(t)$ from the previous part. Let $y(t) = x_1(t)\cos(\omega_c t) + x_2(t)\sin(\omega_c t)$. Show how you can recover $x_1(t)$ from $y(t)\cos(\omega_c t)$ and $x_2(t)$ from $y\sin(\omega_c t)$.