SigSys PS07

Ruby Spring

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1 Dirac Comb Fourier transform

A train of unit impulses is given by:

$$p(t) = \sum_{k=-\infty}^{\infty} \delta\left(t - kT\right)$$

A sketch of this Dirac Comb is shown in Fig. 1 below.

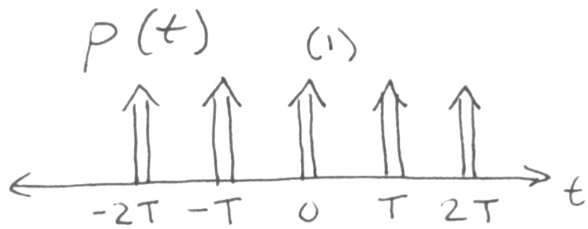

Figure 1: Dirac comb, or impulse train, of unit impulses with period T.

To find the Fourier series representation of this signal, we must first derive the coefficient C_k as follows:

$$C_k = \frac{1}{T}\int_{-T/2}^{T/2} x\left(t\right)e^{-i\omega_o kT}dt \tag{1}$$

Eq. 1 finds the Fourier series coefficient C_k. Notice that the derivative is taken from $-T/2 < t < T/2$, so $k = 0$ and $\delta(t - kT) = \delta(t)$. This allows the following derivation:

$$\begin{aligned}
C_k &= \frac{1}{T}\int_{-T/2}^{T/2} \delta\left(t - kT\right)e^{-i\omega_o kT}dt \\
&= \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)dt \\
&= \frac{1}{T}\int_{-T/2}^{T/2} 1dt \\
C_k &= \frac{1}{T} \qquad (2)
\end{aligned}$$

Plugging Eq. 2 into the Fourier series equation yields:

$$\delta(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty} e^{i\omega_o kt}$$

To find the Fourier transform $P(\omega)$ of this signal, we can use the following known transform:

$$\mathcal{F}\left\{e^{i\omega_o t}\right\} = 2\pi\delta(\omega - \omega_o)$$

knowing this we can write:

$$\begin{aligned}
P(\omega) &= \int_{-\infty}^{\infty} \delta(t)\, e^{-i\omega t} dt \\
&= \int_{-\infty}^{\infty} \frac{1}{T}\sum_{k=-\infty}^{\infty} e^{i\omega_o kt} e^{-i\omega t} dt \\
&= \frac{1}{T}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\omega_o kt} e^{-i\omega t} dt \\
&= \frac{1}{T}\sum_{k=-\infty}^{\infty} \mathcal{F}\left\{e^{i\omega_o kt}\right\} \\
&= \frac{1}{T}\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_o k) \\
P(\omega) &= \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta(\omega - \frac{2\pi}{T}k) \qquad (3)
\end{aligned}$$

A sketch of $P(\omega)$ is in Fig. 2 below.

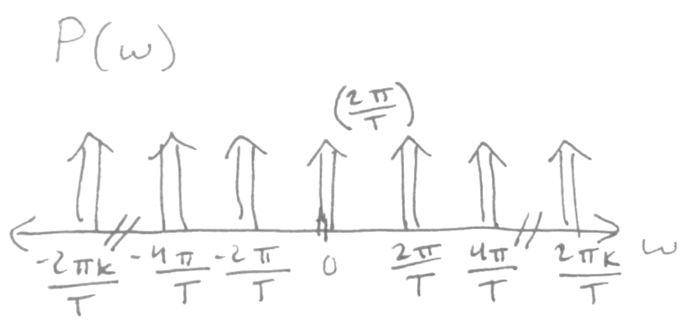

Figure 2: The Fourier transformed Dirac comb from Fig. 1. The transformed signal has impulses with period $\frac{2\pi}{T}$ and impulse area $A = \frac{2\pi}{T}$.

Say the period of $p(t)$ is denoted as T_t and the period of $P(w)$ is denoted as T_ω. Then increasing T has the effect of increasing T_t becuase $T_t = T$ and decreasing T_ω because $T_\omega = \frac{2\pi}{T}$. Decreasing T would have the opposite effect. This is as expected because increasing the period of a function in the time domain should decrease the period in the frequency domain because frequency is inversely related to the time domain period by

$$\omega = \frac{1}{T}$$

where T has units of $\frac{s}{cycle}$. It's confusing to think of a frequency domain period because the units are actually in $\frac{frequencies}{cycle} = \frac{cycle}{s*cycle} = s^{-1}$, which looks like another interpretation of frequency.

2 Ideal Low-pass filter

The frequency domain signal $H(\omega)$ is shown in Fig. 3 below.

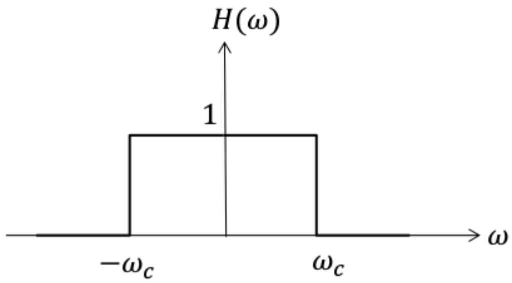

Figure 3

The inverse-Fourier transform of $H(\omega)$ is obtained as follows:

$$\begin{aligned} h(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)e^{j\omega t}d\omega \\ &= \frac{\omega_c}{\pi}\int_{-\infty}^{\infty} e^{j\omega t}d\omega \\ &= \frac{1}{j\pi t}\left(e^{j\omega_c t} - e^{-j\omega_c t}\right) \\ &= \frac{2\sin(\omega_c t)}{\pi t} \\ h(t) &= \frac{2\omega_c}{\pi}\text{sinc}(\frac{\omega_c}{\pi}t) \end{aligned} \tag{4}$$

Now suppose we have a signal that looks like Fig. 4 below.

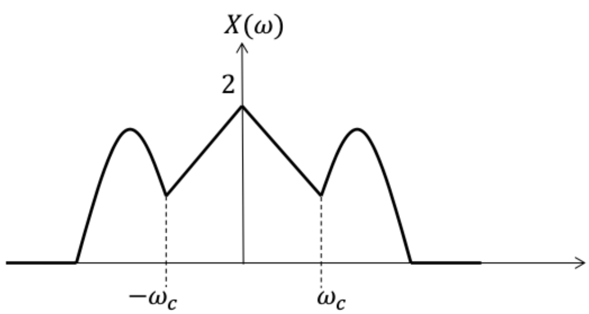

Figure 4

Applying $H(\omega)$ to this signal would result in some new signal, call it $Y(\omega)$. We know $Y(\omega) = H(\omega)X(\omega)$, so $Y(\omega)$ will look like the sketch in Fig. 5 below.

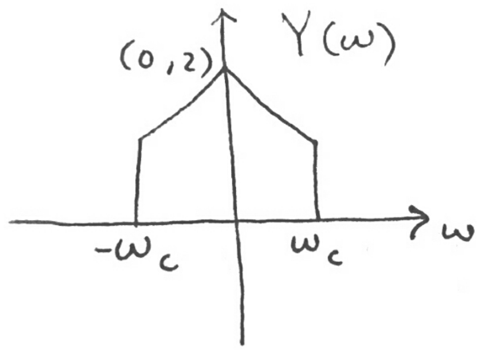

Figure 5: The resulting frequency domain signal from applying $H(\omega)$ to $X(\omega)$.

It is now clear that $H(\omega)$ is an ideal low-pass filter, where ω_c is the cutoff frequency. The transformed signal $X(\omega)$ is left alone (scaled by 1 and shifted by 0) when $|X(\omega)| \leq \omega_c$ and is completely cut (scaled by 0) otherwise.

The results from running SquareWaveFIlterExercise.ipnb with $\omega_c = 0.75\pi$ and $\omega_c = 1.75\pi$ are in Fig. 6 and Fig. 7 below.

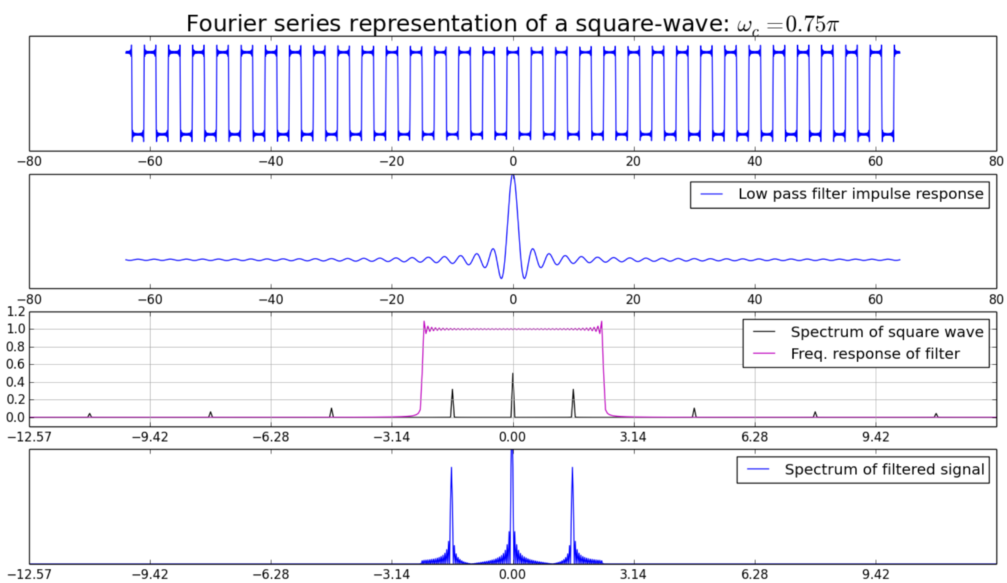

Figure 6

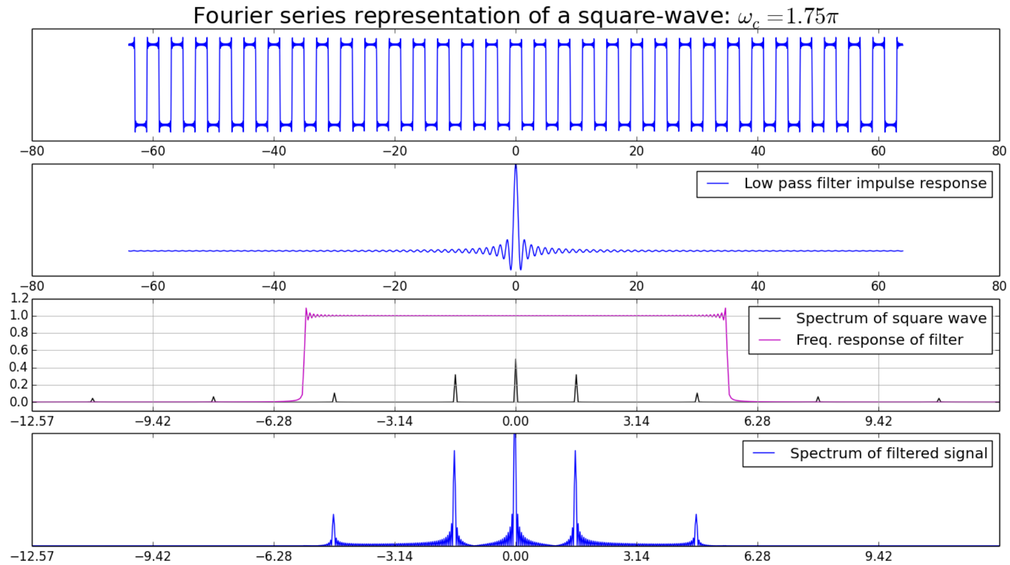

Figure 7

3 Weird time domain multiplication stuff

Taking a signal $x(t)$ that is limited to the range $[-\omega_M, \omega_M]$ and multiply it by $h(t) = \cos(\omega_c t$ produces the signal $y(t)$:

$$y(t) = x(t)\cos(\omega_c t)$$

where $\omega_c \gg \omega_M$ (the reason for this will become apparent shortly). Since we're multiplying the signals in the time domain, we must convolve them in the frequency domain and scale them by $\frac{1}{2\pi}$ (I'm actually not sure why scaling is necessary...):

$$\begin{aligned} Y(\omega) &= \frac{1}{2\pi}\left(X(\omega) * \mathcal{F}\{\cos(\omega_c t)\}\right) \\ Y(\omega) &= \frac{1}{2\pi}\left(X(\omega) * \pi\left(\delta(\omega - \omega_c) + \delta(\omega + \omega_c)\right)\right) \end{aligned} \tag{5}$$

Eq. 5 will therefore produce a frequency domain signal with two copies of $X(\omega)$, scaled by $\frac{1}{2}$ and shifted by $\pm\omega_c$. A sketch of $Y(\omega)$ is in Fig. 8 below.

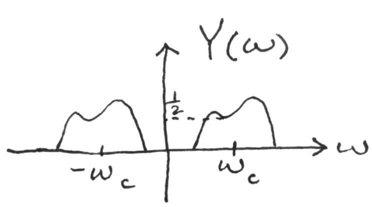

Figure 8: Shifted and scaled copies of $X(\omega)$ by convolution with a Fourier-transformed cosine signal

It is now clear that if $\omega_c \le \omega_M$, the two copies would overlap (adding together) and we would be unable to recover the original signal.