Olin College of Engineering
ENGR2410 – Signals and Systems

Impedance, Laplace and controls introduction

Standard forms

Higher order derivatives or terms in a polynomial should not have coefficients:

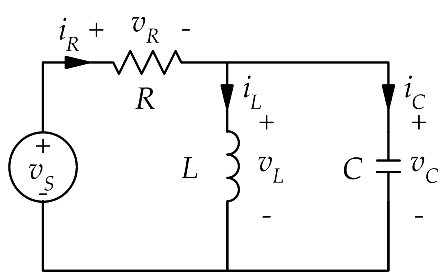
$$\ddot{v}_{out} + 2\alpha\dot{v}_{out} + \omega_0^2 v_{out} = \omega_0^2 v_{in}$$

$$H(s) = \frac{\omega_0^2}{s^2 + 2\alpha s + \omega_0^2}$$

$$H(s) = \frac{1/\tau}{s + 1/\tau}$$

When doing algebra, identify knowns terms, group and keep them together, and write them in the typical order: RC, LC, L/R. This will help you immensely.

Circuits as LTI systems



Circuit diagrams are graphical ways to represent differential equations. For example, the circuit above must satisfy:

- Voltage laws

$$v_S(t) = v_R(t) + v_L(t)$$

$$v_L(t) = v_C(t)$$

- Current laws

$$i_R(t) = i_L(t) + i_C(t)$$

- Device laws

$$v_R(t) = Ri_R(t)$$

$$v_L(t) = L\frac{di_L}{dt}$$

$$i_C(t) = C\frac{dv_C}{dt}$$

All these equations are linear, as defined before. This implies that if either *any* function is an exponential, for example,

$$v_R(t) = V_R e^{st}$$

then *all* other functions must also be exponentials, including

$$v_L(t) = V_L e^{st}$$
$$v_C(t) = V_C e^{st}$$

As a result, the amplitudes of all these exponential functions also satisfy KVL and KCL!

$$v_S(t) = V_R(t) + v_L(t)$$
$$V_S\cancel{e^{st}} = V_R\cancel{e^{st}} + V_L\cancel{e^{st}}$$
$$V_S = V_R + V_L$$

More importantly, these exponential amplitudes satisfy any differential device laws as *algebraic* equations:

$$v_C(t) = V_C e^{st}$$

$$i_C(t) = I_C e^{st}$$

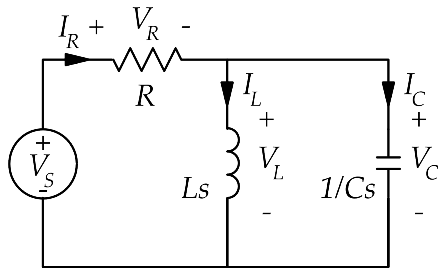
$$i_C = C\dot{v}_C(t)$$

$$I_C\cancel{e^{st}} = CV_C s\cancel{e^{st}}$$

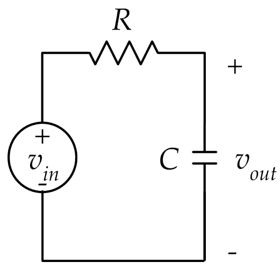
$$Z_C = \frac{V_C}{I_C} = \frac{1}{Cs}$$

Z_C is called *impedance*. The impedance of a resistor R is $Z_R = R$, and the impedance of an inductor is $Z_L = Ls$.

Equivalent circuit using impedances and exponential amplitudes



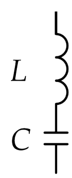
Impedances can be treated as resistors, including all the usual resistor techniques: series, parallel, voltage dividers, current dividers and so forth.



$$V_{out} = \frac{\frac{1}{Cs}}{R + \frac{1}{Cs}} V_{in}$$

$$H(s) = \frac{V_{out}}{V_{in}} = \frac{\frac{1}{RC}}{s + \frac{1}{RC}}$$

Impedances can also be computed for combinations of elements.



$$Z_{eq} = Ls + \frac{1}{Cs} = \frac{LCs^2 + 1}{Cs} = \frac{s^2 + \frac{1}{LC}}{\frac{s}{L}}$$

In particular, the impedance at the resonant frequency $\omega_0 = 1/\sqrt{LC}$ of each element is

$$Z_L = jL\omega = jL\frac{1}{\sqrt{LC}} = j\sqrt{\frac{L}{C}}$$

$$Z_C = -j\frac{1}{C\omega} = -j\frac{\sqrt{LC}}{C} = -j\sqrt{\frac{L}{C}}$$

and $Z_L + Z_C = 0$ at resonance.

Laplace transform

In general, e^{st} is an eigenfunction of LTI systems and $H(s)$ is the associated eigenvalue.

$$e^{st} \longrightarrow \boxed{h(t)} \longrightarrow H(s)e^{st}$$

$$H(s)e^{st} = e^{st} * h(t) = \int_{-\infty}^{\infty} h(t')e^{s(t-t')}dt' = e^{st} \underbrace{\int_{-\infty}^{\infty} h(t')e^{-st'}dt'}_{H(s)}$$

$H(s)$ is the *Laplace transform*[1] of $h(t)$,

$$H(s) \triangleq \mathscr{L}\{h(t)\} = \int_{-\infty}^{\infty} h(t)e^{-st}dt$$

The Fourier transform is the Laplace transform when $s = j\omega$,

$$H(s)|_{s=j\omega} = \mathscr{F}\{h(t)\}$$

For example, if $h(t) = e^{-t/\tau}u(t)$,

$$H(s) = \int_{0}^{\infty} e^{-t/\tau}e^{-st}dt = \frac{1}{-(s+1/\tau)}\left[e^{-(s+1/\tau)t}\right]_0^{\infty} = \frac{1}{s+1/\tau}, \quad \mathrm{Re}\{s\} > -1/\tau$$

Since the integral only converges if $\mathrm{Re}\{s\} > -1/\tau$, the Laplace transform exists only if this condition holds. This condition can be represented as a region in the *s-plane* of all possible complex values of s, known as *Region of convergence (ROC)*.

In general, $H(s)$ can be represented in the s-plane. Points where $H(s) = 0$ are zeros, usually labeled as "o", and points where $H(s) \to \infty$ are poles, usually labeled as "x". In our examples, $H(s)$ has a single pole at $s = -1/\tau$, and the ROC is the half-plane to the right of this pole. The boundary of any ROC always has at least one pole.

The vertical (imaginary) axis is where $s = j\omega$. Thus, the Fourier transform is a "slice" of $H(s)$ along this line. In our example, if the pole is in the left-half plane (LHP), where $\mathrm{Re}\{s\} < 0$, the Fourier transform exists since it is inside the ROC. In this case, $h(t)$ approaches 0 as $t \to \infty$. On the other hand, if the pole is on the right-half plane (RHP), $\tau < 0$ and $h(t)$ approaches ∞ as $t \to \infty$. In this case, the Laplace transform does not converge when $s = j\omega$ and the Fourier transform does not exist.

Given the close relationship between the Laplace and Fourier transforms, most properties of the Fourier transform are also true for the Laplace transform. In particular, the Laplace transform is also linear, $Y(s) = H(s)X(s)$, and impedances are as expected if $s = j\omega$.

[1] Pedantic terminology note: This is more specifically referred to as the *bilateral* Laplace transform to distinguish it from the *unilateral* Laplace transform, where the limits range from 0^- to ∞.

The page of Peter Mathys at `http://ecee.colorado.edu/~mathys/ecen2420/notes/FilterPlots.html` shows the relationship between the frequency response of filters and their associated pole-zero diagram using the s-plane.

Properties

Given that

$$e^{-at}u(t) \quad \overset{\mathscr{L}}{\Longleftrightarrow} \quad \frac{1}{s+a}, \quad \mathrm{Re}\{s\} > -a$$

If $a = 0$,

$$u(t) \quad \overset{\mathscr{L}}{\Longleftrightarrow} \quad \frac{1}{s}, \quad \mathrm{Re}\{s\} > 0$$

As a result,

$$\mathscr{L}^{-1}\left\{\frac{X(s)}{s}\right\} = u(t) * x(t) = \int_{-\infty}^{\infty} x(t')u(t-t')dt' = \int_{-\infty}^{t} x(t')dt'$$

Therefore,

$$\int_{-\infty}^{t} x(t')dt' \quad \overset{\mathscr{L}}{\Longleftrightarrow} \quad \frac{X(s)}{s}$$

Similarly,

$$\dot{x(t)} \quad \overset{\mathscr{L}}{\Longleftrightarrow} \quad sX(s)$$

Analysis of proper rational transfer functions

Systems of the form

$$\dddot{y} + a_2\ddot{y} + a_1\dot{y} + a_0 y = \ddot{x} + b_1\dot{x} + b_0 x$$

can be transformed to

$$s^3Y + a_2s^2Y + a_1sY + a_0Y = s^2X + b_1sX + b_0X$$

$$H(s) = \frac{Y}{X} = \frac{s^2 + b_1s + b_0}{s^3 + a_2s^2 + a_1s + a_0} = \frac{(s+z_1)(s+z_2)}{(s+p_1)(s+p_2)(s+p_3)}$$

Note that if $H(s)$ is real, any poles or zeros that are not real must have their complex conjugate. The transfer function $H(s)$ can be expanded into several fractions such that

$$H(s) = \frac{A_1}{s+p_1} + \frac{A_2}{s+p_2} + \frac{A_3}{s+p_3}$$

Therefore,

$$h(t) = A_1 e^{-p_1 t} u(t) + A_2 e^{-p_2 t} u(t) + A_3 e^{-p_3 t} u(t)$$

Thus, *the poles of a system correspond to its natural response.*

Also, the frequency response can be interpreted using vectors from each zero and pole to the $j\omega$ axis,

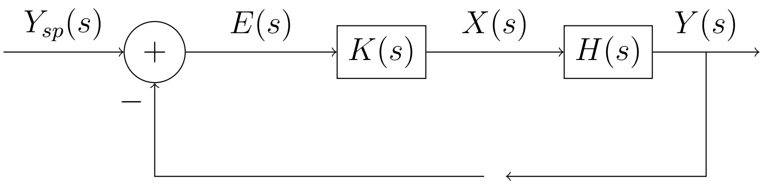
$$H(j\omega) = \frac{(j\omega + z_1)(j\omega + z_2)}{(j\omega + p_1)(j\omega + p_2)(j\omega + p_3)}$$

As a rule of thumb, real poles decrease the slope by 1 when $\omega = p$ as ω increases, and decrease the phase by $\pi/2$ at the same time. Real zeros increase the slope and phase correspondingly. Complex poles have "double" effect: the slope decreases by 2 and the phase by π when $\omega = \mathrm{Im}\{p\}$.

Control

Control systems are used typically for *tracking*, where some output y must "follow" some set point y_{sp}, or to *stabilize* the dynamics of a system by moving any poles from the left-half plane to the right-half plane such that $h(t)$ is bounded as $t \to \infty$.

In the system below, the set point y_{sp} is compared to the output y, and the resulting error e is fed into a controller $K(s)$ that then drives the plant $H(s)$ with x.



The overall transfer function is

$$K(\underbrace{Y_{sp} - HX}_{error}) = X$$

such that

$$\frac{X}{Y_{sp}} = \frac{K}{1 + KH}$$

Since $Y = HX$,

$$\frac{Y}{Y_{sp}} = \frac{KH}{1 + KH}$$

This is *Black's formula*.

In control systems, we typically care about the step response, since it shows the response of the system when the set point is changed. In particular, the step response has a *settling time* until it reaches the new set point, the final value might have an *offset*, or a *DC gain* not equal to one, and some *overshoot* beyond the set point. We typically want to decrease all these as much as possible without making the system unstable or sensitive to external disturbances and system variations.

The *final value theorem* states that

$$\lim_{s \to 0} sX(s) = x(\infty)$$

Similarly, the *initial value theorem* states that

$$\lim_{s \to \infty} sX(s) = x(0)$$

For any system $H(s)$, the step response is $u(t) * h(t) = \mathscr{F}^{-1}\left\{\frac{1}{s}H(s)\right\}$. We can find the DC gain using the final value theorem such that

$$\text{DC gain} = \lim_{s\to 0} \cancel{s} \cdot \underbrace{\frac{1}{\cancel{s}}H(s)}_{\textit{step response}} = \lim_{s\to 0} H(s)$$

Proportional control

Proportional control is letting $K(s) = K_p$. For example, assume $H(s)$ is a first order system

$$H(s) = \frac{1/\tau}{s + 1/\tau}$$

If we let $K(s) = K_p$, the overall transfer function becomes

$$\frac{Y(s)}{Y_{sp}(s)} = \frac{K_p \frac{1/\tau}{s+1/\tau}}{1 + K_p \frac{1/\tau}{s+1/\tau}} = \frac{K_p/\tau}{s + (K_p + 1)/\tau}$$

The DC gain is $K_p/(K_p + 1)$ and the equivalent time constant is $\tau/(K_p + 1)$. It seems that choosing an arbitrarily high K_p would make the DC gain closer to 1 and reduce the settling time. However, any real system will have some delay. This can be modeled as a system with step response $\delta(t - t_0)$. The transfer function of this delay system is e^{-st_0}. Including this delay in the system yields

$$\frac{Y(s)}{Y_{sp}(s)} = \frac{K_p e^{-st_0}/\tau}{s + (K_p e^{-st_0} + 1)/\tau}$$

Since $t_0 \ll 1$, we can approximate the delay as $e_{-st_0} \approx 1 - st_0$. Substituting this approximation into the transfer function yields

$$\frac{Y(s)}{Y_{sp}(s)} \approx \frac{K_p(1 - st_0)/\tau}{s + [K_p(1 - st_0) + 1]/\tau} = \frac{K_p(1 - st_0)/\tau}{s(1 - K_p t_0/\tau) + (K_p + 1)/\tau}$$

The resulting pole is

$$s = \frac{-(K_p + 1)/\tau}{(1 - K_p t_0/\tau)}$$

which means the system will become unstable if $s > 0$, or

$$1 < K_p t_0/\tau \quad \Rightarrow \quad K_p > \tau/t_0$$