Braid groups

Summer Graduate School
MSRI-CMO: Geometric Group Theory
FECHA

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Braids

Definition

An *n-braid* is a collection of n disjoint strings.

Braids

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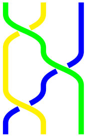
Examples



3-braid

id

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Braid groups

The n-strand braid group B_n has the presentation:

$$B_n = \langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| \geq 2, \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}\rangle.$$

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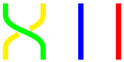
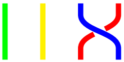
Generators of B_4

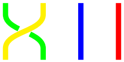


σ_1

σ_2

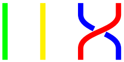


σ_3



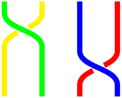
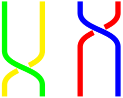
σ_1^{-1}

σ_2^{-1}



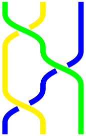
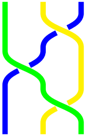
σ_3^{-1}

Relations

 $=$ 

$\sigma_1\sigma_3$ $\quad$ $\sigma_3\sigma_1$

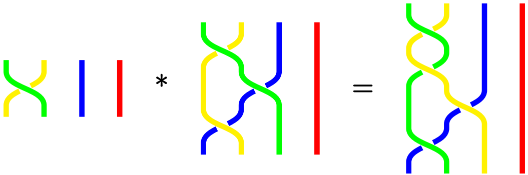
$$\sigma_i\sigma_j = \sigma_j\sigma_i$$
if $|i-j| \geq 2$

 $=$ 

$\sigma_1\sigma_2\sigma_1$ $\quad$ $\sigma_2\sigma_1\sigma_2$

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$$

Multiplication (Concatenation)



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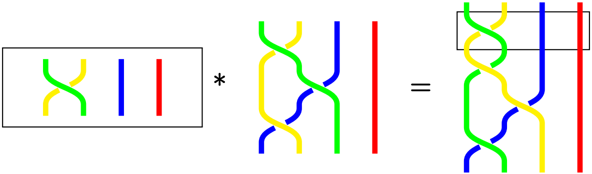

The word problem

Two words $\beta_1, \beta_2 \in B_n$ represent the same braid $\Leftrightarrow \beta_1\beta_2^{-1} = 1$.

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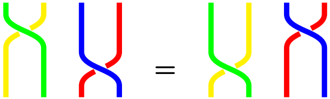

$\sigma_1\sigma_3 \qquad \sigma_3\sigma_1$

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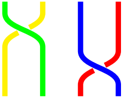

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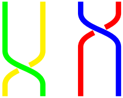
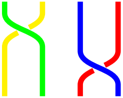


$\sigma_1\sigma_3$ $\sigma_3\sigma_1$

$\Leftrightarrow$



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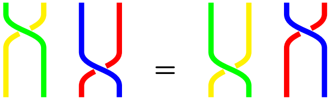

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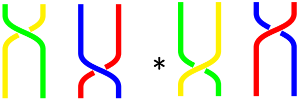

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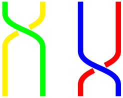
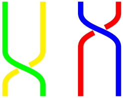
 $=$ 

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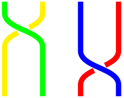
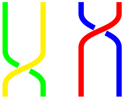
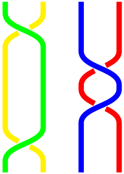
 $*$  $=$ 

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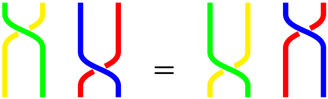

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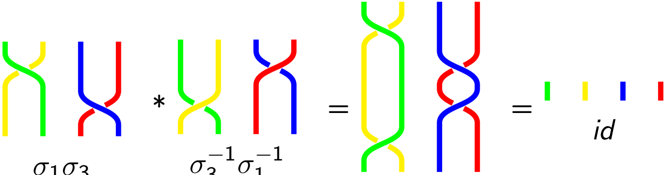
$\Leftrightarrow$



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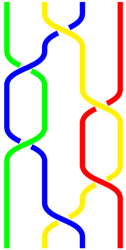
Pure braids



Pure braid

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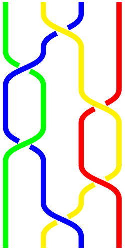
Pure braids



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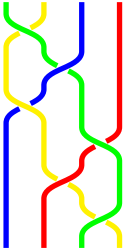
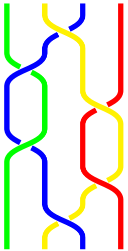


Not pure

Pure braids



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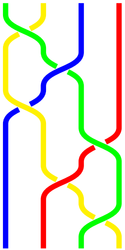

Not pure

Theorem

The pure braid group P_n is biorderable for all $n \geq 1$.

Mapping class groups

Definition

Let $\mathcal{S}$ an oriented compact surface, possibly with boundary, and $\mathcal{P}$ be a finite set of distinguished interior points of $\mathcal{S}$.

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Let $\mathcal{S}$ an oriented compact surface, possibly with boundary, and $\mathcal{P}$ be a finite set of distinguished interior points of $\mathcal{S}$.
The mapping clas group $\mathcal{MCG}(\mathcal{S},\mathcal{P})$ of the surface $\mathcal{S}$ relative to $\mathcal{P}$ is the group of all isotopy classes of orientation-preserving homeomorphisms $\psi : \mathcal{S} \to \mathcal{S}$ satisfying $\psi_{|\partial S} = id$ and $\psi(\mathcal{P}) = \mathcal{P}$.

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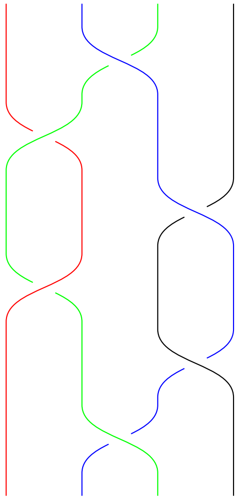
Proposition

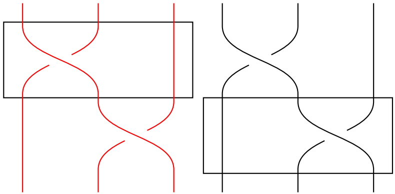
There is an isomorphism $B_n \cong \mathcal{MCG}(D_n)$, where D_n is the disk D^2 with n regularly spaced points.

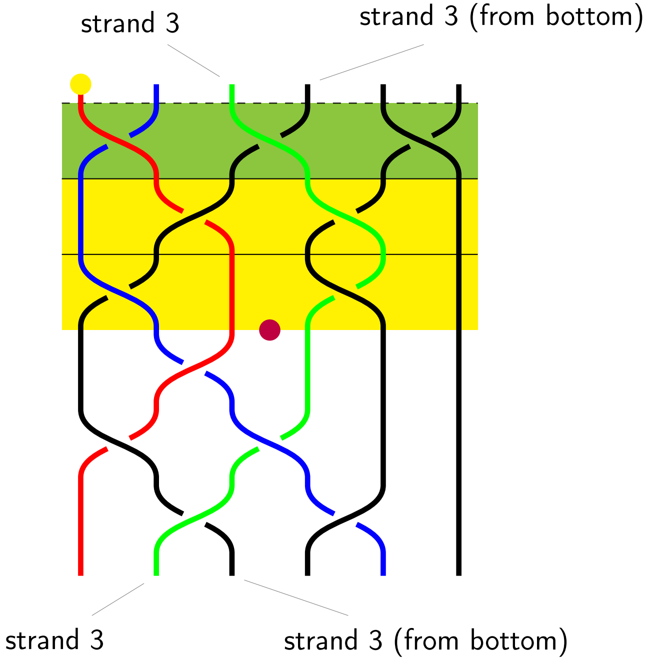
Order in B_n

σ-positive
subword property

Proposition






strand 3
strand 3 (from bottom)
strand 3
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Thanks.