# Thermodynamic Costs of Turing Machines (Kolchinsky and Wolpert 2020)

Daniel Briseno

February 12, 2021

# Context of the Paper

## Prior work on Thermodynamics of Information Processing

- Landauer cost of erasing a bit: $kT \ln 2$ (1961)
- Logically reversible computations can be performed with no heat or entropy production (1973)
- Informal argument for minimum cost of $x \mapsto y$ (1989 - 2019)
- Development of non-equilibrium statistical physics
  - Trajectory-based and stochastic thermodynamics (2013-2015)
- Thermodynamic costs of specific implementations of Turing Machines (TM)(2015-2019)

# Purpose of the Paper

## Thermodynamic costs of computation

- Extends results to general class of TM
- Analyzes the thermodynamic costs of $f : \mathbb{N} \nrightarrow \mathbb{N}$ on a physical implementation of a TM $M$
- Logical properties of $f$ and $M$ impose constraints on thermodynamic costs.
- Result might generalize to any implementation of a TM

# Heat From Information

## The Second Law of Thermodynamics

- Qualitatively, heat cannot flow from a cooler object to a hotter object.
- More Quantitatively, there exists a thermodynamic variable $S$, called the entropy, such that:

  $$0 \leq S_f - S_0 + \Delta Q/T$$

  where the RHS of the inequality is called Entropy Production (EP).
- An increase in "order" comes at the cost of producing Heat.

# Heat From Information

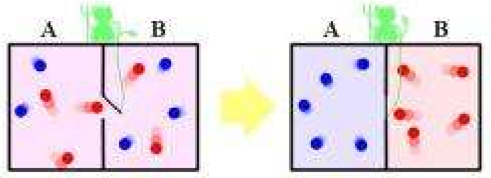



Figure: Maxwell's Demon

*"... if we conceive of a being whose faculties are so sharpened that he can follow every molecule in its course, such a being, whose attributes are as essentially finite as our own, would be able to do what is impossible to us."*

– James Clerk Maxwell

# Heat From Information

## Maxwell's Demon

- Small door can, in principle be opened with 0 EP.
- The location of each gas particle can be measured with 0 EP.
- Key lies in the demon's memory.
  - The Demon must have finite memory.
  - No matter how well-prepared, eventually the Demon must over-write (erase) one of its memory cells (Bennett 1982).

# Heat From Information

## Landauer Cost

- The Boltzmann Entropy of a single information-carrying bit is $k\ln(2)$.
- The Boltzmann Entropy of a bit that has been erased is 0.
- By the second law:

$$
\begin{aligned}
0 &\leq S_f - S_0 + \Delta Q/T \\
\implies 0 &\leq 0 - k\ln(2) + \Delta Q/T \\
\implies kT\ln(2) &\leq \Delta Q
\end{aligned}
$$

and we obtain the Landauer cost.

# Entropy

**Measure of Disorder... Or Energy... Or Information**

- Boltzmann Entropy: $S_B = k\ln(w)$
- Gibbs Entropy: $S_G = -k\sum_i P_i \ln(P_i)$
- Shannon Entropy: $\mathrm{H} = -\sum_i P_i \ln(P_i)$

# Entropy

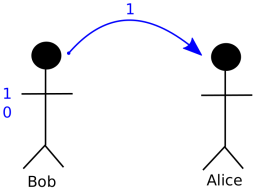



Figure: A tale as old as time.

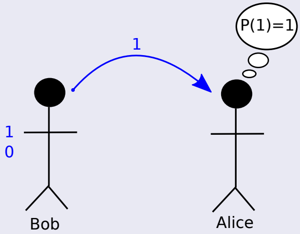
## Information Content

- How much information does Bob's message give Alice?
- More quantitatively, given $\Gamma := \{0, 1\}$

$$I : \Gamma \to \mathbb{R}$$

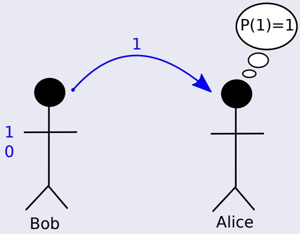
# Entropy

## Case 1



# Entropy

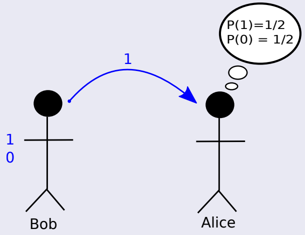
## Case 1 cont.




- Bob's message provides no information.
- $I(1) = 0$

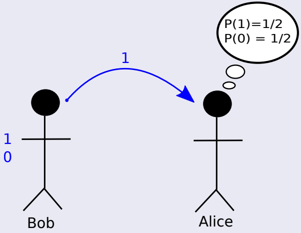
# Entropy

## Case 2



# Entropy

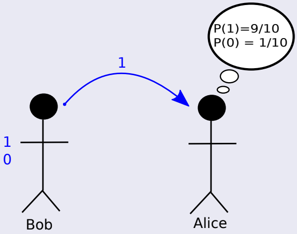
## Case 2 cont.




- Bob's message provides one bit of information
- $0 < I(1) = 1\text{bit}$

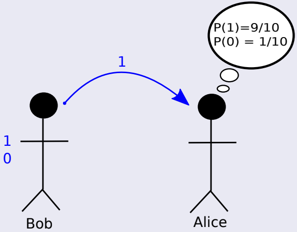
# Entropy

## Case3



# Entropy

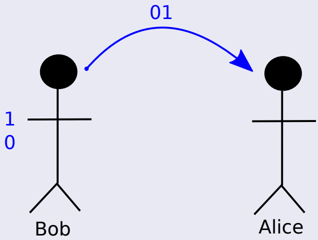
## Case 3




- Bob's message removes some uncertainty
- Not as informative as in case 2
- $0 = I^{(1)}(1) < I^{(3)}(1) < I^{(2)}(1) = 1\text{bit}$

# Entropy

## Case 4




- Decision to send 0 independent from decision to send 1
- $I(0,1) = I(0) + I(1)$

# Entropy

## Three Conditions

For any $x, y \in \Gamma$, the following three axioms must hold:

1. $P(x) = 1 \implies I(x) = 0$
2. $P(x) < P(y) \implies I(y) < I(x)$
3. $I(xy) = I(x) + I(y)$

# Entropy

## Information Content, or Surprisal

Single function which can satisfy these three axioms (Shannon and Weaver n.d.)

$$I(x) = -\log_b(P_x)$$

Where we write $P_x = P(x)$ for notational simplicity.

- This value is called the **Surprisal**, or **Information Content**
- $b$ sets our units of information

# Entropy

## The Bit: $b = 2$

- If $P(1) = P(2) = 1/2$, then $I(1) = 1$ bit.

$$
\begin{aligned}
I_2(1) &= -\log_2(P_1) \\
&= -\log_2\left(\frac{1}{2}\right) \\
&= 1
\end{aligned}
$$

# Entropy

**The nat: $b = e$**

- Recall the Boltzmann Entropy $S_B$:

$$S_B = k \ln(w)$$

- $P(w_0) = 1/w$. Thus

$$\begin{aligned} S_B &= -k \ln(P_{w_0}) \\ &= k I_e(w_0) \end{aligned}$$

# Entropy

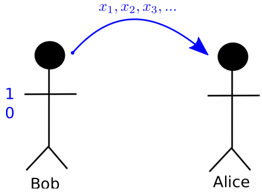
## Unit Conversions

All $I_b$ are related by the logarithm change-of-base formula:

$$I_c = \frac{1}{\log_b c} I_b$$

the factor $\frac{1}{\log_b c}$ can be thought of as a conversion factor.

# Entropy



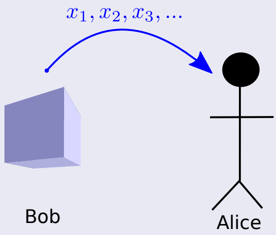

## Shannon Entropy

- Single message replaced with message random variable $X$
- How informative is Bob to Alice?
- Expected value of Suprisal:

$$\begin{aligned}\mathrm{H}(X) &= -\sum_{i=1}^{2} P_i \log_2(P_i) \\ &= \mathbb{E}[I_2(X)]\end{aligned}$$

# Entropy

## Gibbs Entropy




- We replace the message alphabet $\Gamma$ with a finite state-space $\chi$

$$S_G = -k \sum_{x \in \chi} P_x \ln(P_x)$$
$$= k\mathbb{E}[I_e(X)]$$

# Entropy

## Types of Entropies Revisited

For a system with state-space $\chi$, $x \in \chi$, and random variable $X$ with support $\chi$:

- Information Content: $I_b(x) = -\log_b(P_x)$
- Boltzmann Entropy: $S_B = kI_e(w_0)$
- Shannon Entropy: $\mathrm{H} = \mathbb{E}[I_b(X)]$
- Gibbs Entropy: $S_B = k\mathbb{E}[I_e(X)] = k\,\mathrm{H}_e$

# Entropy

## Maxwell's Demon Revisited

- Boltzmann entropy of a information-carrying bit:

$$
\begin{aligned}
w_0 &= 1 \\
kI_e(1) &= -k\ln(P_1) = k\ln(2)
\end{aligned}
$$

- Boltzmann entropy of a bit overwritten with 0:

$$
\begin{aligned}
w_0 &= 0 \\
kI_e(0) &= 0
\end{aligned}
$$

- Second law implies:

$$
\begin{aligned}
0 &\leq 0 - k\ln(2) + \Delta Q/T \\
\implies kT\ln(2) &\leq \Delta Q
\end{aligned}
$$

# CS Background

## Turing Machines

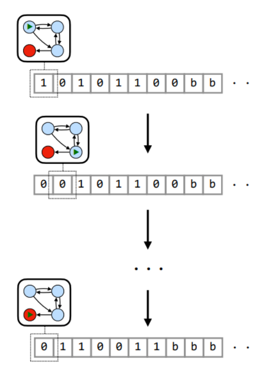



Figure: Graphical representation of a TM

# CS Background

## Turing Machines

Formal definition of a Turing Machine:

- A Turing machine $M$ is a 4-tuple $M = (Q, \Sigma, q_0, \delta)$ where:
  - $Q$ is a finite nonempty set of states.
  - $\Sigma$ is a finite nonempty set of symbols.
  - $q_0 \in Q$ is the initial state of $M$
  - $\delta : (Q \times \Sigma) \nrightarrow (\Sigma \times \{L, R\} \times Q)$ is a partial transition function determining the symbol written on the tape, the movement of the read-write head, and the next state of the $M$.

# CS Background

### Additional Assumptions on TM $M$

1. $\Sigma = \{0, 1, b\}$
2. If and when $M$ halts on an input, the tape will contain an output string $s \in \{0, 1\}^*$ followed by all blank symbols, and the pointer will be set to the start of the tape.

Assumptions do not affect the computational capabilities of $M$.

# CS Background

## Turing Machines as Partial Functions

Any computation performed by a TM $M$ can be represented as

$$\phi_M : \{0,1\}^* \nrightarrow \{0,1\}^*$$

and $\phi_M(x) = y$ indicates that $M$ started with input program $x$ yields the output string $y$.

## Universal TM

There exist Universal Turing Machines (UTM) such that given a UTM $U$ and any TM $M$, there exists an interpreter program $\sigma_{U,M}$ such that

$$\phi_U(\sigma_{U,M}, x) = \phi_M(x)$$

# CS Background

### The Print function, an example

- There exists a FPGA (Field-programmable-gate-array) which constitutes a circuit capable of parroting back any binary string fed to it.
- This computer is also capable of printing out a binary string.

# CS Background

## The Print function, an example

- $\phi_D$ UTM which models my personal computer
- $\phi_M$ TM which models a "print" FPGA
- $x$ binary string which $\phi_M$ can print
- $\sigma_{D,M}$ interpreter program
- $(\sigma_{D,M}, x)$ input to my UTM $\phi_D$

# CS Background

### TM input

- The input to a TM is not generally the input to a program
- The input to a TM is the program itself, along with any necessary parameters

# CS Background

## Computability

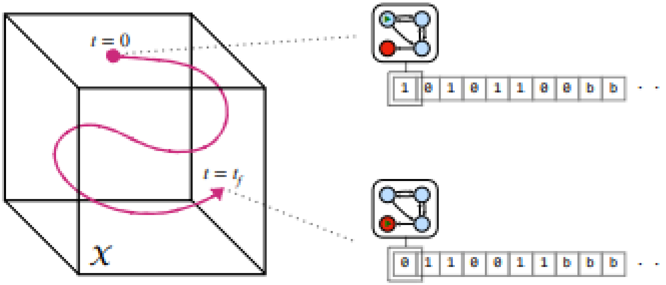
- Church Turing Thesis: A function can be calculated by a sequence of formal operations if and only if it is computable by a Turing Machine.
- Physical Church Turing Thesis: Any function implemented by a physical process can also be implemented by a Turing Machine

# Realizations of a TM

## Realizations and Computable Realizations

- **Physical Realization**: A physical process consistent with the laws of thermodynamics and whose dynamics correspond to the input-output map of a TM $M$
- **Computable Realization**: A physical realization of a TM $M$ whose generated heat on an input program $x$ can be determined by a computable function

# Realizations of a TM



# Algorithmic Information Theory

## Kolmogorov Complexity

The Kolmogorov complexity $K_U$ of a bitstring $x$ is the length of the shortest input program that when given to a UTM $U$ can produce $x$ as an output:

$$K_U(x) := \min_{z:\phi_U(z)=x} \ell(z)$$

- Measure of amount of information in $x$

# Algorithmic Information Theory

## Kolmogorov Complexity of Bitstring $x$

$$K_U(x) := \min_{z:\phi_U(z)=x} \ell(z)$$

## Kolmogorov Complexity of a Computable Function $f$

$$K_U(f) := \min_{M:\phi_M=f} \ell(\sigma_{U,M})$$

## Conditional Kolmogorov Complexity of $x$ Given Bitstring $y$

$$K_U(x|y) = \min_{z:\phi_U(z,y)=x} \ell(z)$$

# Algorithmic Information Theory

## Invariance Theorem

For distinct UTM $U$, $U'$:

$$K_{U'}(x) = K_U(x) + O(1)$$

Thus, $U$ is usually omitted and we write $K(x)$ for Kolmogorov complexity of $x$

# Algorithmic Information Theory

## Incompressible string $x$

If $x$ is incompressible, then

$$K(x) = \ell(\text{print } x)$$

- Any program capable of producing $x$ must contain $x$ explicitly
- $x$ is "maximally dense" with information

## Highly compressible string $\pi$

$$K(\pi) \leq \ell\left(6\sin^{-1}\left(\frac{1}{2}\right)\right) < \ell(\text{print } \pi)$$

# Algorithmic Information Theory

## Input Distributions

- Input string $x$ as random variable with probability distribution $p_X$
- Important example: coin flipping distribution of TM $M$

$$m_X^{\text{coin}}(x) := \begin{cases} 2^{-\ell(x)} & \text{if } x \in \text{dom } \phi_M \\ 0 & \text{otherwise} \end{cases}$$

- With normalizing constant $\Omega_M := \sum_{x \in \text{ dom } \phi_M} 2^{-\ell(x)}$

$$p_X^{\text{coin}}(x) = m_X^{\text{coin}}(x)/\Omega_M$$

## Shannon Entropy of Distribution $p_X$

$$S(p_X) = -\sum_{x \in X} p_X(x) \ln p_X(x)$$

- Measure of amount of information in $p_X$
- $\ln \frac{1}{p_X}$: "surprisal", how unexpected, and hence informative, is $x$?
- $p_X(x)$: how often do we receive surprise $\ln p_X$

**Entropy Production (EP)**

The expected EP, written $\Sigma(p_X)$ of a physical process with initial state distribution $p_X$ and final state distribution $p_Y$ is:

$$\Sigma(p_X) = S(p_Y) - S(p_X) + \langle Q \rangle_{p_X} / kT$$

Thermodynamically reversible processes have $\Sigma(p_X) = 0$. EP is always nonnegative.

# References

[1] C. H. Bennett, "The thermodynamics of computation—a review", en, International Journal of Theoretical Physics **21**, 905–940 (1982).

[2] A. Kolchinsky and D. H. Wolpert, "Thermodynamic costs of turing machines", Physical Review Research **2**, `10.1103/physrevresearch.2.033312` (2020).

[3] C. Shannon and W. Weaver, "The Mathematical Theory of Communication", en, 131.