Thermodynamic Costs of Turing Machines (Kolchinsky and Wolpert 2020)

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Context of the Paper

Prior work on Thermodynamics of Information Processing

- Landauer cost of erasing a bit: $kT \ln 2$ (1961)
- Logically reversible computations can be performed with no heat or entropy production (1973)
- Informal argument for minimum cost of $x \mapsto y$ (1989 - 2019)
- Development of non-equilibrium statistical physics
    - Trajectory-based and stochastic thermodynamics (2013-2015)
- Thermodynamic costs of specific implementations of Turing Machines (TM)(2015-2019)

Purpose of the Paper

Thermodynamic costs of computation

- Extends results to general class of TM
- Analyzes the thermodynamic costs of $f : \mathbb{N} \nrightarrow \mathbb{N}$ on a physical implementation of a TM M
- Logical properties of f and M impose constraints on thermodynamic costs.
- Result might generalize to any implementation of a TM

CS Background

Turing Machines

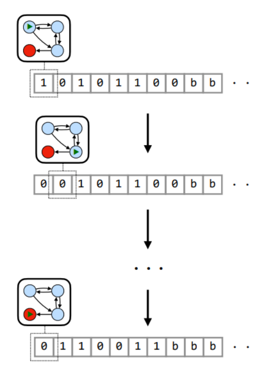


Figure: Graphical representation of a TM

CS Background

Turing Machines

Formal definition of a Turing Machine:

- A Turing machine M is a 4-tuple $M = (Q, \Sigma, q_0, \delta)$ where:
  - Q is a finite nonempty set of states.
  - Σ is a finite nonempty set of symbols.
  - $q_0 \in Q$ is the initial state of M
  - $\delta : (Q \times \Sigma) \nrightarrow (\Sigma \times \{L, R\} \times Q)$ is a partial transition function determining the symbol written on the tape, the movement of the read-write head, and the next state of the M.

CS Background

Additional Assumptions on TM M

1. $\Sigma = \{0, 1, b\}$
2. If and when M halts on an input, the tape will contain an output string $s \in \{0, 1\}^*$ followed by all blank symbols, and the pointer will be set to the start of the tape.

Assumptions do not affect the computational capabilities of M.

CS Background

Turing Machines as Partial Functions

Any computation performed by a TM M can be represented as

$$\phi_M : \{0,1\}^* \nrightarrow \{0,1\}^*$$

and $\phi_M(x) = y$ indicates that M started with input program x yields the output string y.

Universal TM

There exist Universal Turing Machines (UTM) such that given a UTM U and any TM M, there exists an interpreter program $\sigma_{U,M}$ such that

$$\phi_U(\sigma_{U,M}, x) = \phi_M(x)$$

CS Background

Computability

- Church Turing Thesis: A function can be calculated by a sequence of formal operations if and only if it is computable by a Turing Machine.
- Physical Church Turing Thesis: Any function implemented by a physical process can also be implemented by a Turing Machine

Realizations of a TM

Realizations and Computable Realizations

- **Physical Realization**: A physical process consistent with the laws of thermodynamics and whose dynamics correspond to the input-output map of a TM M
- **Computable Realization**: A physical realization of a TM M whose generated heat on an input program x can be determined by a computable function

Algorithmic Information Theory

Kolmogorov Complexity

The Kolmogorov complexity K_U of a bitstring x is the length of the shortest input program that when given to a UTM U can produce x as an output:

$$K_U(x) := \min_{z:\phi_U(z)=x} \ell(z)$$

- Measure of amount of information in x

Algorithmic Information Theory

Kolmogorov Complexity of Bitstring x

$$K_U(x) := \min_{z:\phi_U(z)=x} \ell(z)$$

Kolmogorov Complexity of a Computable Function f

$$K_U(f) := \min_{M:\phi_M=f} \ell(\sigma_{U,M})$$

Conditional Kolmogorov Complexity of x Given Bitstring y

$$K_U(x|y) = \min_{z:\phi_U(z,y)=x} \ell(z)$$

Algorithmic Information Theory

Invariance Theorem

For distinct UTM U, U':

$$K_{U'}(x) = K_U(x) + O(1)$$

Thus, U is usually omitted and we write $K(x)$ for Kolmogorov complexity of x

Algorithmic Information Theory

Incompressible string x

If x is incompressible, then

$$K(x) = \ell(\text{print } x)$$

- Any program capable of producing x must contain x explicitly
- x is "maximally dense" with information

Highly compressible string π

$$K(\pi) \leq \ell\left(6\sin^{-1}\left(\frac{1}{2}\right)\right) < \ell(\text{print } \pi)$$

Algorithmic Information Theory

Input Distributions

- Input string x as random variable with probability distribution p_X
- Important example: coin flipping distribution of TM M

$$m_X^{\text{coin}}(x) := \begin{cases} 2^{-\ell(x)} & \text{if } x \in \text{dom } \phi_M \\ 0 & \text{otherwise} \end{cases}$$

- With normalizing constant $\Omega_M := \sum_{x \in \text{ dom } \phi_M} 2^{-\ell(x)}$

$$p_X^{\text{coin}}(x) = m_X^{\text{coin}}(x) / \Omega_M$$

Algorithmic Information Theory

Shannon Entropy of Distribution p_X

$$S(p_X) = -\sum_{x \in X} p_X(x) \ln p_X(x)$$

- Measure of amount of information in p_X
- $\ln \frac{1}{p_X}$: "surprisal", how unexpected, and hence informative, is x?
- $p_X(x)$: how often do we receive surprise $\ln p_X$

Algorithmic Information Theory

Entropy Production (EP)

The expected EP, written $\Sigma(p_X)$ of a physical process with initial state distribution p_X and final state distribution p_Y is:

$$\Sigma(p_X) = S(p_Y) - S(p_X) + \langle Q \rangle_{p_X}/kT$$

Thermodynamically reversible processes have $\Sigma(p_X) = 0$. EP is always nonnegative.

Physical Setup

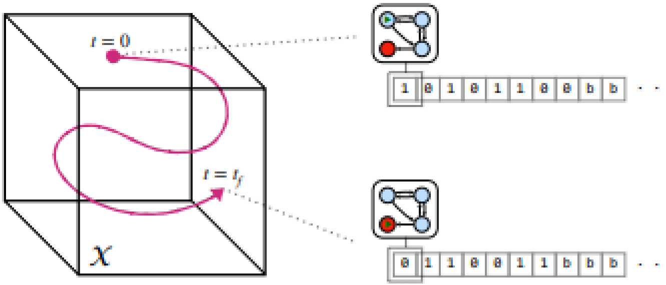
System under consideration

The authors consider a physical system which:

- has a countable state-space $\mathcal{X}$
- is connected to a work reservoir and a heat bath at temperature T. The bath is taken to be in a Boltzmann distribution.
- evolves according to a driving protocol in the time interval $[0, t_f]$.

In this scenario, the heat function $Q(x)$ is defined as the expected amount of heat transferred from the system to the heat bath assuming that the system began in state x.

Physical Setup



Physical Setup

System under consideration

The joint Hamiltonian of the system is

$$H_X^t(x) + H_B(b) + H_{\mathsf{int}}(x, b)$$

If $p_B(b)$ is the initial distribution of the bath and $p'_{B|X}$ is the final distribution, then $Q(x)$ is more formally defined as:

$$Q(x) = \langle H_B \rangle_{p'_{B|x}} - \langle H_B \rangle_{p_B}$$

Physical Setup

Realization Formally Defined

A physical process is a **realization** of a partial function $f : \mathcal{X} \nrightarrow \mathcal{X}$ if the conditional probability of the system's final state given the initial state follows:

$$p_{Y|X}(y|x) = \delta(f(x), y)$$

Realization of a TM Defined

- Recall that a TM M can be written as a partial function $\phi_M : \{0,1\} \nrightarrow \{0,1\}$
- A physical process is a realization of a TM M if it is a realization of ϕ_M.

Prop. 1

Proposition 1

Given a countable set $\mathcal{X}$ and partial functions $f : \mathcal{X} \nrightarrow \mathcal{X}$ and $G : \mathcal{X} \nrightarrow \mathbb{R}$, the following are equivalent:

1. For all p_X with $\operatorname{supp} p_X \subseteq \operatorname{dom} f$

$$\langle G \rangle_{p_X} + S[p_{f(X)}] - S(p_X) \geq 0$$

2. For all $y \in \operatorname{img} f$

$$\sum_{x : f(x) = y} e^{-G(x)} \leq 1$$

3. There exists a realization of f coupled to a heat bath at temperature T whose heat function Q obeys

$$Q(x)/kT = G(x) \qquad\qquad \forall x \in \operatorname{dom} f$$

Prop.1 As Generalization of Landauer Cost

Take $x \in \{0, 1\}$ to be a random bit determined by a coin toss, and f as the bit-erasing operation $f(x) = 0$. Then:

$$p_X(x) = \frac{1}{2}$$
$$p_{f(X)}(y) = \begin{cases} 0 & \text{if } y = 1 \\ 1 & \text{if } y = 0 \end{cases}$$

Then for any $G(x) = Q(x)/kT$, condition 1 implies:

$$\begin{aligned} \langle G \rangle_{p_X} + S[p_{f(X)}] - S(p_X) &\geq 0 \\ \implies \langle G \rangle_{p_X} &\geq S(p_X) \\ \implies \langle G \rangle_{p_X} &\geq \ln 2 \end{aligned}$$

Prop.1 As Generalization of Landauer Cost

We would like to characterize the cost of an arbitrary bit deletion, so taking G to be identical for inputs $\{0, 1\}$

$$G(x) \geq \ln 2$$

and using equivalent condition 3 from Proposition 1 we recover the Landauer cost of a bit deletion:

$$\begin{aligned} Q(x)/kT &\geq \ln 2 \\ \implies Q(x) &\geq kT \ln 2 \end{aligned}$$

Realizations of TM

Realizations of TM Used in Analysis

- **Coin-Flipping Realization**: thermodynamically reversible when inputs are sampled from coin-flipping distribution
- **Dominating Realization**: produces less heat than any computable realization of a TM

Coin-Flipping Realization

Input Distribution

$$m_X^{\text{coin}}(x) := \begin{cases} 2^{-\ell(x)} & \text{if } x \in \text{dom } \phi_M \\ 0 & \text{otherwise} \end{cases}$$

$$p_X^{\text{coin}}(x) = m_X^{\text{coin}}(x)/\Omega_M$$

Output Distribution

$$m_Y^{\text{coin}}(y) = \sum_{x:\phi_M(x)=y} 2^{-\ell(x)}$$

$$p_Y^{\text{coin}}(y) = m_Y(y)/\Omega_M$$

Coin-Flipping Realization

Associated Heat Function of Coin-Flipping Realization for TM M

Can be shown that

$$G(x) = -\ln p_X^{\mathsf{coin}}(x) + \ln p_Y^{\mathsf{coin}}[\phi_M(x)]$$

Satisfies condition 2 of Prop 1. Thus, multiplying by kT and using definitions of p_X^{coin} and p_Y^{coin}:

$$\begin{aligned} Q_{\mathsf{coin}}(x) &= kT\{-\ln p_X^{\mathsf{coin}}(x) + \ln p_Y^{\mathsf{coin}}[\phi_M(x)]\} \\ &= kT \ln\{\ell(x) + \log_2 m_Y[\phi_M(x)]\} \end{aligned}$$

Coin-Flipping Realization

Zero Entropy Production

$$Q_{\text{coin}}(x) = kT\{-\ln p_X^{\text{coin}} + \ln p_Y^{\text{coin}}[\phi_M(x)]\}$$

Using

$$\langle Q_{\text{coin}}\rangle_{p_X} = \sum_{x\in X} p_X(x)Q(x)$$

We can verify that:

$$\langle Q_{\text{coin}}\rangle_{p_X} = kT\{S(p_X^{\text{coin}}) - S(p_Y^{\text{coin}})\}$$
$$\implies \Sigma(p_X^{\text{coin}}) = S(p_Y^{\text{coin}}) - S(p_X^{\text{coin}}) + S(p_X^{\text{coin}}) - S(p_Y^{\text{coin}}) = 0$$

Coin-Flipping Realization

Associated Heat Function of Coin-Flipping Realization for TM M

$$Q_{\mathsf{coin}}(x) = kT \ln\{\ell(x) + \log_2 m_Y[\phi_M(x)]\}$$

Recall definition of m_Y:

$$m_Y^{\mathsf{coin}}(y) = \sum_{x:\phi_M(x)=y} 2^{-\ell(x)}$$

- $\log_2 m_Y[\phi_M(x)]$ minimal for logically reversible ϕ_M.
- Q_{coin} minimal for short and logically reversible input programs.

Coin-Flipping Realization

Levin's Coding Theorem for UTM

$$-\log_2 m_Y(y) = K(y) + O(1)$$

Heat Function for UTM

$$Q_{\mathsf{coin}}(x) = kT \ln 2\{\ell(x) - K[\phi_M(x)]\} + O(1)$$

- Q_{coin} achieves its minimum value when x is the shortest program capable of producing $\phi_U(x)$ (always true if ϕ_U is reversible).

$$\min_{x:\phi_U(x)=y} Q_{\mathsf{coin}}(x) = O(1)$$

Coin-Flipping Realization

Expected Heat of Coin-Flipping Distribution

Recall that

$$\langle Q_{\text{coin}} \rangle_{p_X} = kT\{S(p_X^{\text{coin}}) - S(p_Y^{\text{coin}})\}$$

- Difference of entropies is infinite
- Implies infinite expected heat
- Implies infinite expected length of input programs and infinite expected runtime

Coin-Flipping Realization

Initial Distribution for Minimum Expected Heat

Input distribution can be varied to minimize $Q(x)$ in a UTM:

$$p_X^{\mathsf{min}}(x) = \delta(x_0, x)$$
$$Q_{\mathsf{coin}}(x_0) = \min_{x \in X} Q(x) = O(1)$$
$$\langle Q_{\mathsf{coin}} \rangle_{p_X^{\mathsf{min}}} = O(1)$$

But then EP is no longer 0:

$$\Sigma(p_X^{\mathsf{min}}) = S(p_Y^{\mathsf{min}}) - S(p_X^{\mathsf{min}}) + O(1) > 0$$

Dominating Realization

Heat Function for Dominating Realization of TM M

Can be shown that $G(x) = \ln 2K[x|\phi_M(x)]$ satisfies condition 2 of Prop 1. Thus

$$Q_{\mathsf{dom}} = kT \ln 2K[x|\phi_M(x)]$$

is the heat function for a realization, called the *dominating realization*, of TM M.

- Inputs generating a lot of heat are large and incompressible, and ϕ_M is non-invertible for that input
- Inputs generating little heat are those for which ϕ_M is invertible
  - For these inputs, $Q(x) = O(1)$

Dominating Realization

Non-Computability

- Dominating realization is not computable
- It is upper semi-computable
  - Can be obtained in limit by sequence of increasingly efficient computable realizations $Q_n(x)$
  - Converges on $Q_{\mathsf{dom}}(x)$ from above

Dominating Realization

Efficiency of Dominating Realization

Q_{dom} is optimal in the sense that for any other *computable* realization with heat function $Q(X)$:

$$Q(x) \geq Q_{\text{dom}} - kT[\ln 2K(Q/kT) + K(\phi_M)] + O(1)$$

- Q_{dom} is minimal up to a negative constant.
  - For $Q(x) \leq Q_{\text{dom}}$, ϕ_M has to have high complexity, or Q has to have high complexity
- The above inequality only holds true for computable realizations. Thus it is not necessarily true that $Q_{\text{dom}} \leq Q_{\text{coin}} + O(1)$

Dominating Realization

Heat VS. Complexity Trade-off

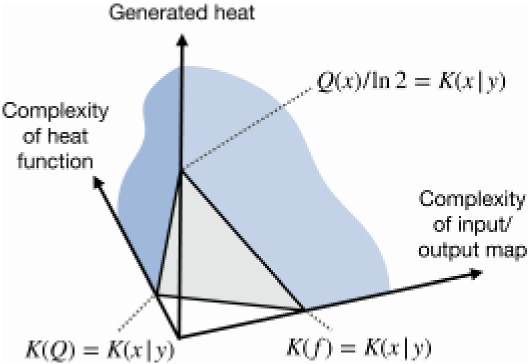
$$Q(x) \geq Q_{\mathsf{dom}} - kT[\ln 2K(Q/kT) + K(\phi_M)] + O(1)$$

Using $Q_{\mathsf{dom}} = kT \ln 2K[x|\phi_M(x)]$ and re-arraigning gives:

$$Q(x)/\ln 2 + K(Q) + K(f) \geq K(x|y) + O(1)$$

- Every computation mapping x to y comes with a "cost" of $K(x|y)$
- Cost can be paid by generating heat, having a high complexity heat function, or having a high complexity mapping f

Heat Vs. Complexity Trade-off



Heat VS. Complexity Trade-off

Example: Erasing a Bitstring

$$Q(x)/\ln 2 + K(Q) + K(f) \geq K(x|y) + O(1)$$

- Consider the an example where f erases a long and incompressible bitstring x.
- $x \mapsto y$ comes with an intrinsic cost of $K(x|y) = K(x) \approx \ell(x)$

Heat VS. Complexity Trade-off

Generate a Lot of Heat

Take f to be

$$f(x') = \text{`}000...000' \, \forall x'$$

- f has low complexity
- Using dominating implementation, $Q(x)/\ln 2 = K(x|y) = K(x) \approx \ell(x)$
  - Heat function has low complexity
  - x long and incompressible implies high heat generation

Heat Vs. Complexity Trade-off

Have a High Complexity Heat Function

Can be shown that the following heat function satisfies conditions of Prop.1 for dominating realization of $f(x') = \text{`}000...000'$

$$Q(x') := \begin{cases} Q_{\mathsf{dom}}(x') & x' \notin \{x, \text{`}000...000'\} \\ Q_{\mathsf{dom}}(\text{`}000...000') & x' = x \\ Q_{\mathsf{dom}}(x) & x' = \text{`}000...000' \end{cases}$$

- Generates little heat
- Low complexity f
- x hard-coded into Q implies high complexity heat function

Heat Vs. Complexity Trade-off

Have a High Complexity Mapping

Consider the logically reversible map:

$$f(x') := \begin{cases} x' & x \notin \{x, \text{`}000...000'\} \\ \text{`}000...000' & x = x' \\ x & x' = \text{`}000...000' \end{cases}$$

- Logically reversible maps can be carried out with 0 heat generation
- 0 heat generation would imply minimally complex heat map
- x hard-coded into f implies high complexity mapping

Physical Church-Turing Thesis

Significance of Physical Church Turing Thesis

- Current conclusions only apply to computable realizations
- In principle, non-computable realizations of TM could exist
- Validity of Church-Turing Thesis would imply any physical realization of a TM must follow thermodynamic constraints shown in paper

Conclusion

- Proposition 1 allows us to relate logical properties of a TM to its thermodynamic properties.
- Coin-flipping realization gives a highly thermodynamically reversible case
  - Infinite expected heat for zero EP input distribution
  - Heat minimizing input distribution implies nonzero EP
- Dominating realization gives lower bound on heat production for any computable realization
  - Upper semicomputable
  - The inequality $Q(x)/\ln 2 + K(Q) + K(f) \geq K(x|y) + O(1)$ allows us to decompose intrinsic cost of mapping $x \mapsto y$ into complexity of heat function, complexity of mapping, and heat production.

References

[1] A. Kolchinsky and D. H. Wolpert, "Thermodynamic costs of turing machines", Physical Review Research **2**, 10.1103/physrevresearch.2.033312 (2020).