Thermodynamic Costs of Turing Machines (Kolchinsky and Wolpert 2020)

Daniel Briseno

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Context of the Paper

Prior work on Thermodynamics of Information Processing

- Landauer cost of erasing a bit: $kT \ln 2$ (1961)
- Logically reversible computations can be performed with no heat or entropy production (1973)
- Informal argument for minimum cost of $x \mapsto y$ (1989 - 2019)
- Development of non-equilibrium statistical physics
  - Trajectory-based and stochastic thermodynamics (2013-2015)
- Thermodynamic costs of specific implementations of Turing Machines (TM)(2015-2019)

Purpose of the Paper

Thermodynamic costs of computation

- Extends results to general class of TM
- Analyzes the thermodynamic costs of $f : \mathbb{N} \nrightarrow \mathbb{N}$ on a physical implementation of a TM M
- Logical properties of f and M impose constraints on thermodynamic costs.
- Result might generalize to any implementation of a TM

Background – Computer Science

Turing Machines

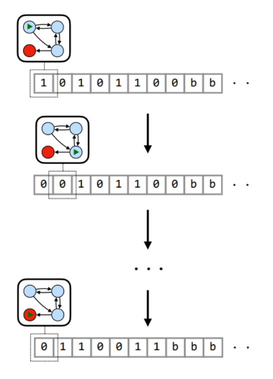


Figure: Graphical representation of a TM

Background – Computer Science

Additional Assumptions on TM M

1. $\Sigma = \{0, 1, b\}$
2. If and when M halts on an input, the tape will contain an output string $s \in \{0, 1\}^*$ followed by all blank symbols, and the pointer will be set to the start of the tape.

Assumptions do not affect the computational capabilities of M.

Background – Computer Science

Turing Machines as Partial Functions

Any computation performed by a TM M can be represented as

$$\phi_M : \{0,1\}^* \nrightarrow \{0,1\}^*$$

and $\phi_M(x) = y$ indicates that M started with input program x yields the output string y.

Universal TM

There exist Universal Turing Machines (UTM) such that given a UTM U and any TM M, there exists an interpreter program $\sigma_{U,M}$ such that

$$\phi_U(\sigma_{U,M}, x) = \phi_M(x)$$

Background – Computer Science

Computability

- Church Turing Thesis: A function can be calculated by a sequence of formal operations if and only if it is computable by a Turing Machine.
- Physical Church Turing Thesis: Any function implemented by a physical process can also be implemented by a Turing Machine

Background – Algorithmic Information Theory

Kolmogorov Complexity

The Kolmogorov complexity K_U of a bitstring x is the length of the shortest input program that when given to a UTM U can produce x as an output:

$$K_U(x) := \min_{z:\phi_U(z)=x} \ell(z)$$

- Measure of amount of information in x

Background – Algorithmic Information Theory

Kolmogorov Complexity of Bitstring x

$$K_U(x) := \min_{z:\phi_U(z)=x} \ell(z)$$

Kolmogorov Complexity of a Computable Function f

$$K_U(f) := \min_{M:\phi_M=f} \ell(\sigma_{U,M})$$

Conditional Kolmogorov Complexity of x Given Bitstring y

$$K_U(x|y) = \min_{z:\phi_U(z,y)=x} \ell(z)$$

Background – Algorithmic Information Theory

Invariance Theorem

For distinct UTM U, U':

$$K_{U'}(x) = K_U(x) + O(1)$$

Thus, U is usually omitted and we write $K(x)$ for Kolmogorov complexity of x

Background – Input Distributions

Coin-Flipping Distribution

- Input string x as random variable with probability distribution p_X
- Important example: coin flipping distribution of TM M

$$m_X^{\text{coin}}(x) := \begin{cases} 2^{-\ell(x)} & \text{if } x \in \text{dom } \phi_M \\ 0 & \text{otherwise} \end{cases}$$

- With normalizing constant $\Omega_M := \sum_{x \in \text{dom } \phi_M} 2^{-\ell(x)}$

$$p_X^{\text{coin}}(x) = m_X^{\text{coin}}(x) / \Omega_M$$

Background – Entropy

Shannon Entropy of Distribution p_X

$$S(p_X) = -\sum_{x \in X} p_X(x) \ln[p_X(x)]$$

- Measure of amount of information in p_X
- $-\ln[p_X(x)]$: "surprisal", how unexpected, and hence informative, is x?
- $p_X(x)$: how often do we receive surprise $\ln p_X$

Background – Entropy

Entropy Production (EP)

The expected EP, written $\Sigma(p_X)$ of a physical process with initial state distribution p_X and final state distribution p_Y is:

$$\Sigma(p_X) = S(p_Y) - S(p_X) + \langle Q \rangle_{p_X}/kT$$

Thermodynamically reversible processes have $\Sigma(p_X) = 0$. EP is always nonnegative.

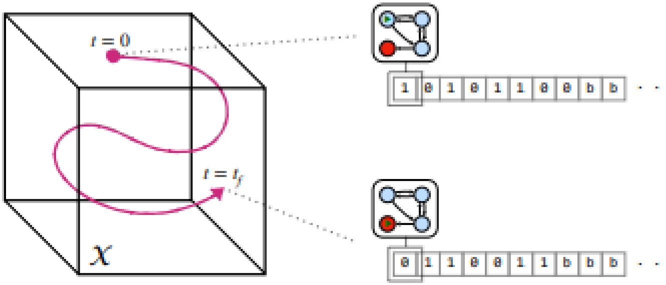
Physical Setup

Physical System Under Consideration

- Countable state-space $\mathcal{X}$
- Connected to a work reservoir
- Connected to heat bath at temperature T
  - Bath in a Boltzmann distribution
- System evolves according to driving protocol during time interval $[0, t_f]$

The heat function $Q(x)$ is the expected thermal energy transferred from the system to the heat bath.

Physical Setup



Physical Setup

System under consideration

The joint Hamiltonian of the system is

$$H_X^t(x) + H_B(b) + H_{\mathsf{int}}(x, b)$$

If $p_B(b)$ is the initial distribution of the bath and $p'_{B|X}$ is the final distribution, then:

$$Q(x) = \langle H_B \rangle_{p'_{B|x}} - \langle H_B \rangle_{p_B}$$

Realizations of a TM

Realization Formally Defined

- Let $p_{Y|X}$ be the conditional probability distribution of the system's final state y given initial state x
- A physical process is a **realization** of a partial function $f : \mathcal{X} \nrightarrow \mathcal{X}$ if
$$p_{Y|X}(y|x) = \delta(f(x), y)$$

Realization of a TM M Formally Defined

- TM M can be written as a partial function $\phi_M : \{0,1\} \nrightarrow \{0,1\}$
- A physical process is a realization of a TM M if it is a realization of ϕ_M